# Thermal spin transport and electron-magnon interactions in easy-plane ferromagnets

*Thesis for a bachelor's degree in physics*

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**Abstract**

We investigate the dynamics of magnons in easy-plane ferromagnetic materials. We use the Boltzmann and the Gross-Pitaevskii equation to derive the transport coefficients and the hydrodynamic equations for both thermal and condensed magnon currents in ferromagnetic insulators. Assuming a linear thermal gradient, we calculate both these magnon currents for the normal state, in which all magnons are thermal, and for the superfluid state, in which our system obeys the two-fluid model. We also consider ferromagnetic conductors. We study electron-magnon interactions in order to construct the transport coefficients and the hydrodynamic equations for magnon currents and electric currents. To find these currents explicitly, we apply these general results to the particular situation in which the temperature gradient is linear. Finally, we calculate the Seebeck coefficient for the normal and the superfluid state.

# Contents

# Chapter 1

# Introduction

Almost 200 years ago Thomas Seebeck [1] discovered the thermoelectric effect, in which a junction of dissimilar metals produces an electric voltage when exposed to a temperature gradient. Today this phenomenon is better known as the Seebeck effect and has found a wide range of applications, including thermocouples, thermopiles, and energy sources for deep space exploration missions. More recently, Uchida *et al.* [2] observed that not only an electric voltage but also a spin voltage can be induced by a temperature gradient. In 2008 they reported their observations and called their newfound phenomenon the spin Seebeck effect.

Since then, the spin Seebeck effect has been a popular topic of research [3]. Flebus *et al.* [4] are currently studying the two-fluid model for the spin Seebeck effect in easy-plane magnetic insulators. Such an easy-plane magnetic insulator may undergo a phase transition between a magnon Bose-Einstein condensed and thus spin superfluid state, and a normal state. The two-fluid model, proposed independently by Tisza [5] and Landau [6], describes the behaviour of a mixture of a normal fluid and a superfluid. Flebus *et al.* have succesfully drawn up the hydrodynamic fundaments of both normal and condensed spin currents. Subsequently, they explicitly calculated these currents induced by a (linear) thermal gradient, i.e. by the spin Seebeck effect.

In this Thesis, we will broaden the research of Flebus *et al.* by considering conductors as well as insulators. In conductors the spin currents are accompanied by an electric current induced by the 'original' Seebeck effect. These currents influence one another through a phenomenon called magnon drag [7]. This causes the hydrodynamic description of a conductor to differ from that of an insulator.

This Thesis is divided in two parts. The first part, Chapter 2, will mostly be a reproduction of the work of Flebus *et al.* We will compose a Hamiltonian for our system, and transform it several times to write it in a more convenient form. After that we will use quantum kinetic theory, the Boltzmann and the Gross-Pitaevskii equations in particular, to lay the groundwork for the hydrodynamic description of spin currents in easy-plane magnetic insulators. We will end Chapter 2 with the explicit calculations of the normal and the condensed spin currents in the case of a linear thermal gradient.

For the second part, Chapter 3, we consider easy-plane ferromagnetic conductors, i.e. we 'add' free electrons to our system. This part is no reproduction and is thus to be regarded as new research. We begin Chapter 3 with a detailed study of electron-magnon interactions, from which we retrieve a new set of transport coefficients and hydrodynamic equations. Again, we will apply these general results to the specific situation in which we are dealing with a linear gradient in temperature. Finally, we calculate the Seebeck coefficient for the normal and for the superfluid state.

# Chapter 2

# Ferromagnetic insulators

## 2.1 Model

This section will give a basic outline of the model that we work in. First we will give short introductions on three of the main themes in this Thesis: superfluidity, the two-fluid model, and magnons. After that we will construct the Hamiltonian of an easy-plane ferromagnetic insulator, and use several transformations to write it in a more convenient form.

### 2.1.1 Superfluidity and the two-fluid model

A Bose-Einstein condensate is a state of matter of a gas of bosons, that occurs at temperatures close to absolute zero (temperatures below the critical temperature $T_c$, to be precise). In a Bose-Einstein condensate a macroscopic fraction of the particles occupies the same quantum state: the groundstate.

A Bose-Einstein condensate often behaves like a superfluid, i.e. a fluid with zero viscosity. This characteristic enables superfluids to ascend vertical walls and leak through microscopic holes. Because of these qualities, superfluids often appear to move in ways that defy the forces of gravity and surface tension.

The two-fluid model describes the behaviour of a mixture of a normal fluid and a superfluid. Here the normal fluid, which is viscous, carries all the thermal energy of the system, and the superfluid carries none. Most often, the two-fluid model is used to describe Helium-4, but its use can be extended to a variety of systems including superconductors, excitons, polaritons, and magnons.

### 2.1.2 Magnons

Imagine a lattice of spin particles in a ferromagnet with easy-plane anisotropy. When an external magnetic field is applied in the $-\hat{\boldsymbol{z}}$-direction, the spins will tend to align with it. However, when the magnetic field is weakened the spins will fluctuate from perfect alignment. This is an effect of the easy-plane anisotropy. Under the right conditions, these fluctuations can propagate through the lattice and create a spin wave. The quantized version of such a spin wave is a quasiparticle called a magnon [8].

Magnons are bosons, so for sufficiently low temperatures they undergo Bose-Einstein condensation. In this case, all condensed magnons occupy the single-particle groundstate. A condensate of magnons behaves like a superfluid: it has zero viscosity. For $T < T_c$, part of the magnons becomes a superfluid and the other part remains a normal fluid. Therefore, a Bose-Einstein condensate of magnons can be described by the two-fluid model.

### 2.1.3 Holstein-Primakoff transformation

We consider an easy-plane ferromagnetic insulator with a cubic lattice. An external magnetic field $B > 0$ is applied in the $-\hat{\boldsymbol{z}}$-direction, and hence our model is described by the Hamiltonian

$$\mathcal{H} = -\frac{J}{2\hbar^2}\sum_{\langle i,j\rangle}\mathbf{S}_i\cdot\mathbf{S}_j + \frac{K}{2\hbar^2}\sum_i (S_i^z)^2 + \frac{B}{\hbar}\sum_i S_i^z, \tag{2.1}$$

where $i$ and $j$ are positions on the lattice, and the notation $\langle i,j\rangle$ indicates that sites $i$ and $j$ are nearest neighbours. Furthermore, $J > 0$ is the strength of the exchange interactions between neighbouring spins, and $K > 0$ is a constant governing the strength of the easy-plane anisotropy. While this Hamiltonian describes our system quite good, it is not written in a convenient way to study the behaviour of magnons. To solve this inconvenience, we use the Holstein-Primakoff transformation [9]. This is a mapping from angular momentum operators (in this case $\mathbf{S}_i$) to the bosonic creation and annihilation operators $a_i^\dagger$ and $a_i$, which obey the commutation relation $[a_i, a_j^\dagger] = \delta_{i,j}$. The Holstein-Primakoff transformation is given by

$$S_i^+ = S_i^x + iS_i^y = \hbar a_i^\dagger\sqrt{2S - a_i^\dagger a_i}, \tag{2.2}$$

$$S_i^- = S_i^x - iS_i^y = \hbar a_i\sqrt{2S - a_i^\dagger a_i}, \tag{2.3}$$

$$S_i^z = \hbar(a_i^\dagger a_i - S), \tag{2.4}$$

and thus we can rewrite Eq. (2.1) to

$$\mathcal{H} = -\frac{J}{2\hbar^2}\sum_{\langle i,j\rangle}\left(\frac{1}{2}(S_i^+S_j^- + S_i^-S_j^+) + S_i^zS_j^z\right) + \frac{K}{2\hbar^2}\sum_i (S_i^z)^2 + \frac{B}{\hbar}\sum_i S_i^z. \tag{2.5}$$

Before we complete the Holstein-Primakoff transformation we use a zeroth-order Taylor expansion in powers of $1/S$ to simplify $S_i^+$ and $S_i^-$:

$$S_i^+ = \hbar a_i^\dagger\sqrt{2S}\sqrt{1 - \frac{a_i^\dagger a_i}{2S}} \approx \hbar a_i^\dagger\sqrt{2S}, \tag{2.6}$$

$$S_i^- = \hbar a_i\sqrt{2S}\sqrt{1 - \frac{a_i^\dagger a_i}{2S}} \approx \hbar a_i\sqrt{2S}. \tag{2.7}$$

To make this approximation accurate we must assume $1/S$ to be small, i.e. $S \gg 1$. Substitution of Eq. (2.6 - 2.7) in Eq. (2.5) yields

$$\mathcal{H} = -\frac{J}{2}\sum_{\langle i,j\rangle}(2S(a_i^\dagger a_j - a_i^\dagger a_i) + a_i^\dagger a_i a_j^\dagger a_j) + \frac{K}{2}\sum_i(-2Sa_i^\dagger a_i + a_i^\dagger a_i a_i^\dagger a_i) + B\sum_i a_i^\dagger a_i + \frac{KNS^2}{2} - 3JNS^2 - BNS, \tag{2.8}$$

where $N$ is the number of lattice sites. We neglect constant terms and use our assumption that $S \gg 1$ to let $1 - 2S \approx -2S$, to find the Bose-Hubbard Hamiltonian:

$$\mathcal{H} = -\frac{J}{2}\sum_{\langle i,j\rangle}(2Sa_i^\dagger a_j + a_i^\dagger a_j^\dagger a_i a_j) + \frac{K}{2}\sum_i a_i^\dagger a_i^\dagger a_i a_i + (6JS - KS + B)\sum_i a_i^\dagger a_i. \tag{2.9}$$

### 2.1.4 Fourier transformation

The Bose-Hubbard Hamiltonian labels magnons by their position in the lattice. In many situations it will prove more useful to label magnons by their momentum. To this end, we rewrite Eq. (2.9) using the following Fourier transformation for the creation and annihilation operators:

$$a_i^\dagger = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}_i} a_{\mathbf{k}}^\dagger, \tag{2.10}$$

$$a_i = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}_i} a_{\mathbf{k}}. \tag{2.11}$$

We choose our cubic lattice to lie in the cartesian $(x, y, z)$-space, such that the vector denoting the location of any lattice site $i$ can be written as $\mathbf{x}_i = l_i a\hat{\boldsymbol{x}} + m_i a\hat{\boldsymbol{y}} + n_i a\hat{\boldsymbol{z}}$, where $a$ is the lattice constant and $l_i$, $m_i$, and $n_i$ are integers depending on $i$. Furthermore, $\mathbf{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}} + k_z\hat{\boldsymbol{z}}$ are the wavevectors spanning the momentum space. We use this Fourier transformation to obtain

$$\sum_i a_i^\dagger a_i = \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}, \tag{2.12}$$

$$\sum_i a_i^\dagger a_i^\dagger a_i a_i = \frac{1}{N}\sum_{\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{k}'''} a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}^\dagger a_{\mathbf{k}''} a_{\mathbf{k}'''}\ \delta_{\mathbf{k}+\mathbf{k}'-\mathbf{k}''-\mathbf{k}'''}, \tag{2.13}$$

$$\sum_{\langle i,j\rangle} a_i^\dagger a_j = 2\sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}(\cos(k_x a) + \cos(k_y a) + \cos(k_z a)), \tag{2.14}$$

$$\sum_{\langle i,j\rangle} a_i^\dagger a_j^\dagger a_i a_j = \frac{2}{N}\sum_{\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{k}'''} a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}^\dagger a_{\mathbf{k}''} a_{\mathbf{k}'''}\ \delta_{\mathbf{k}+\mathbf{k}'-\mathbf{k}''-\mathbf{k}'''} \times (\cos(k'_x a) + \cos(k'_y a) + \cos(k'_z a))(\cos(k'''_x a) + \cos(k'''_y a) + \cos(k'''_z a)). \tag{2.15}$$

In calculating these terms, we used the notion that a summation over neighbouring lattice sites $\langle i, j\rangle$ can be written as a summation over $i$ by considering all six nearest neighbours separately, i.e. $\mathbf{x}_j = (l_i \pm 1)a\hat{\boldsymbol{x}} + (m_i \pm 1)a\hat{\boldsymbol{y}} + (n_i \pm 1)a\hat{\boldsymbol{z}}$. Furthermore, we used the relation $\sum_{i'} e^{i\mathbf{k}\cdot\mathbf{x}_{i'}} = N\delta_{\mathbf{k}}$. Now we can rewrite Eq. (2.9) to

$$\mathcal{H} = \sum_{\mathbf{k}}(\hbar\omega_{\mathbf{k}} - KS + B)a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \sum_{\mathbf{k},\mathbf{k}',\mathbf{k}'',\mathbf{k}'''}\left(\frac{K}{2N}\right) a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}^\dagger a_{\mathbf{k}''} a_{\mathbf{k}'''}\ \delta_{\mathbf{k}+\mathbf{k}'-\mathbf{k}''-\mathbf{k}'''}, \tag{2.16}$$

neglecting the contribution of Eq. (2.15) in the domain of low energies. The term $\hbar\omega_{\mathbf{k}} = -2JS(\cos(k_x a) + \cos(k_y a) + \cos(k_z a) - 3)$ is interpreted as the energy of a magnon with wavevector $\mathbf{k}$. In the low-energy regime we can use a second-order Taylor expansion to write this energy as $\hbar\omega_{\mathbf{k}} = J_{\text{xc}}\mathbf{k}^2$, where $J_{\text{xc}} \equiv 3JSa^2$. It follows that magnons, observed as quasiparticles, have an effective mass of $m_{\text{eff}} = \hbar^2/(2J_{\text{xc}})$.

Now, we can transform our discrete Hamiltonian to a continuous one using

$$a_{\mathbf{k}}^{\dagger} = \frac{1}{\sqrt{V}} \int d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \psi^{\dagger}(\mathbf{x}), \tag{2.17}$$

$$a_{\mathbf{k}} = \frac{1}{\sqrt{V}} \int d\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, \psi(\mathbf{x}), \tag{2.18}$$

where $V = Na^3$ is the volume of the lattice. With the relation $\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} = V\delta_{\mathbf{x}}$, the transformation of Eq. (2.16) to the continuous form yields

$$\mathcal{H} = \int d\mathbf{x} \left[ \psi^{\dagger}(-J_{\text{xc}}\nabla^2 - \mu_{eq})\psi + \frac{Ka^3}{2}\psi^{\dagger}\psi^{\dagger}\psi\psi \right], \tag{2.19}$$

where $\mu_{eq} \equiv KS - B$ is interpreted as the effective chemical potential acquired by the magnons as a consequence of the competition between the external field and the anisotropy. At fixed temperature, this chemical potential tunes between condensed ($\mu_{eq} > 0$) and non-condensed ($\mu_{eq} < 0$) states.

## 2.2 Quantum kinetic theory

Now that we have a basic understanding of the behaviour of magnons in easy-plane ferromagnetic insulators, we will use quantum kinetics to research currents of magnons through these materials. For temperatures above the critical temperature for Bose-Einstein condensation, i.e. for $T > T_c$, there will be a single current of magnons $\mathbf{J}_{th}$, that we will refer to as the thermal spin current. For $T < T_c$, however, we will encounter the two-fluid model, in which the thermal spin current is accompanied by a current of condensed magnons $\mathbf{J}_c$ called the superfluid current.

The kinetics of a system like ours is characterized by two important equations: the Boltzmann equation and the Gross-Pitaevskii equation. We will begin this section by deriving these equations for the magnons in our system. After that, we will use them to find the transport coefficients of $\mathbf{J}_{th}$ and the hydrodynamic equations that govern the spin currents in our insulator.

### 2.2.1 Boltzmann and Gross-Pitaevskii equation

Analogous to Zaremba *et al.* [10] we find our Boltzmann equation to be the following differential equation of the magnon distribution function $f(\mathbf{k}, \mathbf{x}, t)$:

$$\frac{\partial f}{\partial t} + \frac{2J_{\text{xc}}\mathbf{k}}{\hbar} \cdot \nabla f = C_{12}[f] + C_{22}[f]. \tag{2.20}$$

Here $C_{12}$ and $C_{22}$ are the contributions to the collision integral. To be precise, $C_{12}$ represents the collision of a thermal and a condensed magnon that results in two thermal magnons and $C_{22}$ represents the collision of two thermal magnons, both of which are conserved in the collision. The collisions are visualized in Fig. 2.1 and Fig. 2.2.

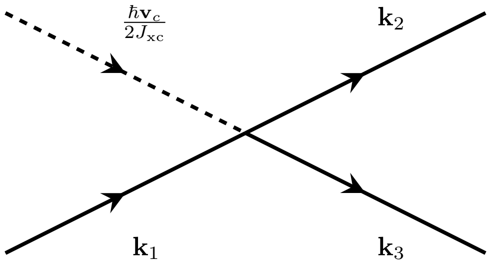


Figure 2.1: Feynman diagram representing $C_{12}$

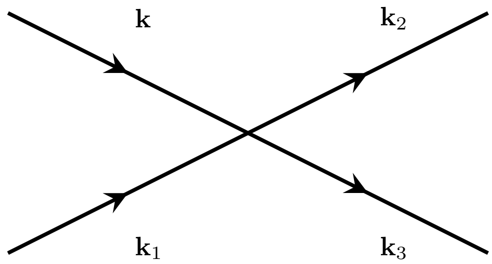


Figure 2.2: Feynman diagram representing $C_{22}$

The integral forms of $C_{12}$ and $C_{22}$ are [10]

$$C_{12}[f] = \frac{2g^2 n_c \hbar^5}{(2\pi)^2} \int d\mathbf{k}_1 \int d\mathbf{k}_2 \int d\mathbf{k}_3 \, \delta\left(\frac{\hbar \mathbf{v}_c}{2J_{\mathrm{xc}}} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3\right) \delta(\hbar\omega_c + \hbar\omega_{\mathbf{k}_1} - \hbar\omega_{\mathbf{k}_2} - \hbar\omega_{\mathbf{k}_3}) \\ \times [\delta(\mathbf{k} - \mathbf{k}_1) - \delta(\mathbf{k} - \mathbf{k}_2) - \delta(\mathbf{k} - \mathbf{k}_3)] \, [(1 + f_1) f_2 f_3 - f_1(1 + f_2)(1 + f_3)], \tag{2.21}$$

$$C_{22}[f] = \frac{2g^2 \hbar^2}{(2\pi)^5} \int d\mathbf{k}_2 \int d\mathbf{k}_3 \int d\mathbf{k}_4 \, \delta(\mathbf{k} + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \, \delta(\hbar\omega_{\mathbf{k}} + \hbar\omega_{\mathbf{k}_2} - \hbar\omega_{\mathbf{k}_3} - \hbar\omega_{\mathbf{k}_4}) \\ \times [(1 + f)(1 + f_2) f_3 f_4 - f f_2 (1 + f_3)(1 + f_4)]. \tag{2.22}$$

In these expressions $f_i = f(\mathbf{k}_i, \mathbf{x}, t)$ and $g \equiv Ka^3$. Furthermore, $n_c$ is the condensate density, $\mathbf{v}_c$ is the superfluid velocity, and $\hbar\omega_c$ is the energy of a condensed magnon. All of these quantities are functions of $\mathbf{x}$ and $t$. Since the effective mass of a magnon is $\hbar^2/(2J_{\mathrm{xc}})$, the term $\hbar \mathbf{v}_c/(2J_{\mathrm{xc}})$ can be regarded as the wavevector of a magnon in the condensate.

Magnons are bosons, so close to equilibrium we can describe them using equilibrium Bose-Einstein statistics, i.e. we can take $f = n_{\mathrm{BE}}(\hbar\omega_{\mathbf{k}} - \mu(\mathbf{x}))$ and $f_i = n_{\mathrm{BE}}(\hbar\omega_{\mathbf{k}_i} - \mu(\mathbf{x}))$, where $n_{\mathrm{BE}}(\varepsilon) \equiv (e^{\beta\varepsilon} - 1)^{-1}$. With these distribution functions Eq. (2.22) becomes

$$C_{22}[f] = \frac{2g^2 \hbar^2}{(2\pi)^5} \int d\mathbf{k}_2 \int d\mathbf{k}_3 \int d\mathbf{k}_4 \, \delta(\mathbf{k} + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \delta(\hbar\omega_{\mathbf{k}} + \hbar\omega_{\mathbf{k}_2} - \hbar\omega_{\mathbf{k}_3} - \hbar\omega_{\mathbf{k}_4}) \\ \times \left[e^{\beta(\hbar\omega_{\mathbf{k}} + \hbar\omega_{\mathbf{k}_2} - 2\mu(\mathbf{x}))} - e^{\beta(\hbar\omega_{\mathbf{k}_3} + \hbar\omega_{\mathbf{k}_4} - 2\mu(\mathbf{x}))}\right] f f_2 f_3 f_4, \tag{2.23}$$

so evaluation over one of the integrals in combination with the second $\delta$-function yields $C_{22} = 0$. This reduces our Boltzmann equation (Eq. (2.20)) to

$$\frac{\partial f}{\partial t} + \frac{2J_{\mathrm{xc}} \mathbf{k}}{\hbar} \cdot \nabla f = C_{12}[f]. \tag{2.24}$$

Now that we have succesfully derived the Boltzmann equation for our system, it is time for the Gross-Pitaevskii equation. Using the same method as Zaremba *et al.* [10], we find it to be

$$i\hbar \frac{\partial \psi}{\partial t} = (-J_{\mathrm{xc}} \nabla^2 + g n_c) \psi. \tag{2.25}$$

Let us take a moment to consider the stationary solutions of this equation, i.e. solutions of the form $\psi = e^{-\frac{i}{\hbar}\mu_{eq} t} \psi_0$. Substitution in Eq. (2.25) yields $g n_c = \mu_{eq}$. This equality will prove to be quite useful later in this Thesis.

### 2.2.2 Transport coefficients

The spin current $\mathbf{J}_{th}$ is driven by parameters of the system, like the thermal magnon chemical potential $\mu(\mathbf{x})$ and the temperature $T(\mathbf{x})$. From now on we will work in the linear-response regime, in which the thermal spin current is written as

$$\mathbf{J}_{th}(\mathbf{x}) = -\frac{\sigma_s}{\hbar}\nabla\mu(\mathbf{x}) + L_{\mu Q}\frac{\nabla T(\mathbf{x})}{T}. \tag{2.26}$$

We will take some time to calculate the transport coefficients $\sigma_s$ and $L_{\mu Q}$ explicitly. In this calculation we assume that $\mathbf{J}_{th}(\mathbf{x})$ does not depend on $\mathbf{x}$ via higher orders of $\mu(\mathbf{x})$ and $T(\mathbf{x})$. To account for this mathematically, we will take $\mu(\mathbf{x}) \to 0$ and $T(\mathbf{x}) \to T$, whenever they appear in another form than $\nabla\mu(\mathbf{x})$ and $\nabla T(\mathbf{x})$ respectively.

The thermal spin current can be written as [11]

$$\mathbf{J}_{th}(\mathbf{x}) = \hbar\int \frac{d\mathbf{k}}{(2\pi)^3} f \frac{\partial(\hbar\omega_{\mathbf{k}})}{\partial(\hbar\mathbf{k})}. \tag{2.27}$$

This expression can be identified with Eq. (2.26) to yield $\sigma_s$ and $L_{\mu Q}$, when we choose a distribution function $f(\mathbf{k}, \mathbf{x}, t)$ satisfying the Boltzmann equation. However, we need to modify the Boltzmann equation in Eq. (2.24) a little before we can use it. Armaitis *et al.* [12] have shown that the collision term $C_{12}$ does not influence the transport coefficients, so for the time being we can leave it out. Now we add two relaxation terms [4], and obtain the following Boltzmann equation:

$$\frac{\partial f}{\partial t} + \frac{2J_{\mathrm{xc}}\mathbf{k}}{\hbar}\cdot\nabla f = -\frac{2\alpha k_B T}{\hbar}[f - n_{\mathrm{BE}}(\hbar\omega_{\mathbf{k}})] - \frac{1}{\tau}[f - n_{\mathrm{BE}}(\hbar\omega_{\mathbf{k}})]. \tag{2.28}$$

Here $\alpha$ is the Gilbert damping factor, $\tau$ is the thermalization time for number-conserving collisions and $k_B$ is the Boltzmann constant (not to be confused with the wavevector $\mathbf{k}$). To find the desired distribution function, we assume that the solution of the Boltzmann equation is time-independent and of the form $f(\mathbf{k}, \mathbf{x}, t) = n_{\mathrm{BE}}(\hbar\omega_{\mathbf{k}} - \mu(\mathbf{x})) + \delta f$. This ansatz leads to

$$\delta f = \frac{2\tau J_{\mathrm{xc}}}{2\alpha k_B T\tau + \hbar}\, n'_{\mathrm{BE}}(\hbar\omega_{\mathbf{k}})\, \mathbf{k}\cdot\left[\nabla\mu(\mathbf{x}) + \hbar\omega_{\mathbf{k}}\frac{\nabla T(\mathbf{x})}{T}\right], \tag{2.29}$$

so substitution in Eq. (2.27) gives

$$\mathbf{J}_{th}(\mathbf{x}) = \hbar\int \frac{d\mathbf{k}}{(2\pi)^3}\left[n_{\mathrm{BE}}(\hbar\omega_{\mathbf{k}}) + \frac{2\tau J_{\mathrm{xc}}}{2\alpha k_B T\tau + \hbar}\, n'_{\mathrm{BE}}(\hbar\omega_{\mathbf{k}})\, \mathbf{k}\cdot\left[\nabla\mu(\mathbf{x}) + \hbar\omega_{\mathbf{k}}\frac{\nabla T(\mathbf{x})}{T}\right]\right]\frac{\partial(\hbar\omega_{\mathbf{k}})}{\partial(\hbar\mathbf{k})}. \tag{2.30}$$

We already know that the energy $\hbar\omega_{\mathbf{k}}$ is quadratic in $\mathbf{k}$ and thus we obtain

$$\mathbf{J}_{th}(\mathbf{x}) = \frac{4\tau J_{\mathrm{xc}}^2}{2\alpha k_B T\tau + \hbar}\int \frac{d\mathbf{k}}{(2\pi)^3}\, n'_{\mathrm{BE}}(\hbar\omega_{\mathbf{k}})\left[(\mathbf{k}\cdot\nabla\mu(\mathbf{x}))\,\mathbf{k} + \hbar\omega_{\mathbf{k}}\left(\mathbf{k}\cdot\frac{\nabla T(\mathbf{x})}{T}\right)\mathbf{k}\right]. \tag{2.31}$$

This expression can be written in the form of Eq. (2.26) with

$$\sigma_s = -\frac{8\pi\tau J_{\mathrm{xc}}^2\hbar}{6\alpha k_B T\tau + 3\hbar}\int_0^\infty \frac{dk}{(2\pi)^3}\, n'_{\mathrm{BE}}(\hbar\omega_k)\, k^4 \tag{2.32}$$

and

$$L_{\mu Q} = \frac{8\pi\tau J_{\mathrm{xc}}^2}{6\alpha k_B T\tau + 3\hbar} \int_0^\infty \frac{dk}{(2\pi)^3}\, n'_{\mathrm{BE}}(\hbar\omega_k)\, \hbar\omega_k\, k^4. \tag{2.33}$$

These results express the transport coefficients in terms of the relaxation time, the Gilbert damping factor, and the temperature.

### 2.2.3 Hydrodynamic equations

The final tools that we need to fully understand the kinetics of magnons in easy-plane magnetic insulators are the hydrodynamic equations. These equations describe the evolution of $\mathbf{J}_{th}$ and $\mathbf{J}_c$ through the material and follow directly from the Boltzmann and the Gross-Pitaevskii equations found in Section 2.2.1.

First we take Eq. (2.24) and integrate it over $\mathbf{k}$:

$$\int \frac{d\mathbf{k}}{(2\pi)^3} \left[\frac{\partial f}{\partial t} + \frac{2J_{\mathrm{xc}}\mathbf{k}}{\hbar} \cdot \nabla f\right] = \int \frac{d\mathbf{k}}{(2\pi)^3}\, C_{12}[f]. \tag{2.34}$$

When we define

$$n_{th} = \int \frac{d\mathbf{k}}{(2\pi)^3} f \tag{2.35}$$

as the density of the thermal magnons and let

$$\Gamma_{12}[f] = \int \frac{d\mathbf{k}}{(2\pi)^3}\, C_{12}[f], \tag{2.36}$$

we obtain our first hydrodynamic equations as

$$\hbar\dot{n}_{th} = -\nabla \cdot \mathbf{J}_{th} + \hbar\Gamma_{12}[f]. \tag{2.37}$$

Now we look at the Gross-Pitaevskii equation, Eq. (2.25), with a dissipative term inserted [13]:

$$i\hbar\frac{\partial\psi}{\partial t} = (-J_{\mathrm{xc}}\nabla^2 + gn_c - \frac{i\hbar}{2n_c}\Gamma_{12}[f])\psi. \tag{2.38}$$

Using the substitutions $\psi = \sqrt{n_c}e^{i\theta}$ and $\nabla\theta = \hbar\mathbf{v}_c/(2J_{\mathrm{xc}})$, we find two more hydrodynamic equations. Up to linear order they read as

$$\hbar\dot{n}_c = -\nabla \cdot \mathbf{J}_c - \hbar\Gamma_{12}[f], \tag{2.39}$$

$$\hbar\dot{\mathbf{v}}_c = \frac{2J_{\mathrm{xc}}}{\hbar}\nabla(\mu_{eq} - \mu_c), \tag{2.40}$$

where $\mathbf{J}_c = \hbar n_c \mathbf{v}_c$ is the superfluid current and $\mu_c = gn_c - (J_{\mathrm{xc}}\nabla^2\sqrt{n_c})/\sqrt{n_c}$ is the condensate chemical potential.

Two of our three hydrodynamic equations contain a term with the $\Gamma_{12}$-integral, defined in Eq. (2.36). It will prove fruitful to examine this term more closely. Again we use the fact that magnons are bosonic, i.e. $f_i = n_{\mathrm{BE}}(\hbar\omega_{\mathbf{k}_i} - \mu(\mathbf{x}))$, and so we find that close to equilibrium

$$\Gamma_{12}[f] = \frac{2g^2 n_c \hbar^5}{(2\pi)^5} \int d\mathbf{k} \int d\mathbf{k}_1 \int d\mathbf{k}_2 \int d\mathbf{k}_3\, \delta\left(\frac{\hbar\mathbf{v}_c}{2J_{\mathrm{xc}}} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3\right) \\ \times \delta(\hbar\omega_c + \hbar\omega_{\mathbf{k}_1} - \hbar\omega_{\mathbf{k}_2} - \hbar\omega_{\mathbf{k}_3}) \left[\delta(\mathbf{k} - \mathbf{k}_1) - \delta(\mathbf{k} - \mathbf{k}_2) - \delta(\mathbf{k} - \mathbf{k}_3)\right] \\ \times \left[e^{\beta(\hbar\omega_{\mathbf{k}_1} - \mu(\mathbf{x}))} - e^{\beta(\hbar\omega_{\mathbf{k}_2} + \hbar\omega_{\mathbf{k}_3} - 2\mu(\mathbf{x}))}\right] f_1 f_2 f_3. \tag{2.41}$$

Integration in combination with the properties of the second $\delta$-function leads to

$$\Gamma_{12}[f] = \left(1 - e^{\beta(\mu - \mu_c)}\right) n_c \Gamma, \tag{2.42}$$

where

$$\Gamma = -\frac{2g^2\hbar^5}{(2\pi)^5} \int d\mathbf{k} \int d\mathbf{k}_1 \int d\mathbf{k}_2 \int d\mathbf{k}_3\, \delta\left(\frac{\hbar\mathbf{v}_c}{2J_{\mathrm{xc}}} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3\right) \delta(\hbar\omega_c + \hbar\omega_{\mathbf{k}_1} - \hbar\omega_{\mathbf{k}_2} - \hbar\omega_{\mathbf{k}_3}) \\ \times \left[\delta(\mathbf{k} - \mathbf{k}_1) - \delta(\mathbf{k} - \mathbf{k}_2) - \delta(\mathbf{k} - \mathbf{k}_3)\right] \left[f_1(1 + f_2)(1 + f_3)\right]. \tag{2.43}$$

Assuming that the system is close to equilibrium, we simplify Eq. (2.42) even further with a first-order Taylor expansion in $\mu - \mu_c$:

$$\Gamma_{12}[f] = -\frac{\Gamma n_c(\mu - \mu_c)}{k_B T}. \tag{2.44}$$

With this approximation the hydrodynamic equations are given by

$$\hbar\dot{n}_{th} = -\nabla \cdot \mathbf{J}_{th} - \frac{\hbar\Gamma n_c(\mu - \mu_c)}{k_B T}, \tag{2.45}$$

$$\hbar\dot{n}_c = -\nabla \cdot \mathbf{J}_c + \frac{\hbar\Gamma n_c(\mu - \mu_c)}{k_B T}, \tag{2.46}$$

$$\hbar\dot{\mathbf{v}}_c = \frac{2J_{\mathrm{xc}}}{\hbar}\nabla(\mu_{eq} - \mu_c). \tag{2.47}$$

We have not yet accounted for Gilbert damping caused by lattice vibrations and other phenomena that cause magnons to decay, so we manually add a term to compensate for this flaw. When we take $\alpha$ as the Gilbert damping factor, we obtain

$$\hbar\dot{n}_{th} = -\nabla \cdot \mathbf{J}_{th} - \frac{3\alpha}{\Lambda_{th}^3}\delta\mu - \frac{\hbar\Gamma n_c(\mu - \mu_c)}{k_B T}, \tag{2.48}$$

$$\hbar\dot{n}_c = -\nabla \cdot \mathbf{J}_c + 2\alpha(\mu - \mu_c)n_c + \frac{\hbar\Gamma n_c(\mu - \mu_c)}{k_B T}, \tag{2.49}$$

$$\hbar\dot{\mathbf{v}}_c = \frac{2J_{\mathrm{xc}}}{\hbar}\nabla(\mu_{eq} - \mu_c), \tag{2.50}$$

where we define $\delta\mu(\mathbf{x}) \equiv \mu(\mathbf{x}) - \mu_{eq}$. Furthermore, $\Lambda_{th}^3 = \sqrt{4\pi J_{\mathrm{xc}}/(k_B T)}$ is the thermal De Broglie wavelength. Of course, we can also consider the total magnon density $n_s = n_{th} + n_c$, which leads to

$$\hbar\dot{n}_s(\mathbf{x}) = -\nabla \cdot (\mathbf{J}_c(\mathbf{x}) + \mathbf{J}_{th}(\mathbf{x})) + \alpha\left(2n_c(\mathbf{x}) - \frac{3}{\Lambda_{th}^3}\right)\delta\mu(\mathbf{x}) - 2\alpha n_c(\mathbf{x})\delta\mu_c(\mathbf{x}), \tag{2.51}$$

$$\hbar\dot{\mathbf{v}}_c(\mathbf{x}) = \frac{2J_{\mathrm{xc}}}{\hbar}\nabla(\mu_{eq} - \mu_c(\mathbf{x})), \tag{2.52}$$

with $\delta\mu_c(\mathbf{x}) \equiv \mu_c(\mathbf{x}) - \mu_{eq}$ defined analogous to $\delta\mu(\mathbf{x})$.

## 2.3 Applications

In this section, we will solve the hydrodynamic equations obtained in the previous section for a particular situation: we consider an easy-plane ferromagnetic insulator of length $L$ attached on both sides to a metallic reservoir that acts like a thermal bath. We choose our axes such that the side of the insulator with length $L$ is aligned with the $x$-axis. We assume one of the metallic reservoirs (the one at $x = 0$) to be set at temperature $T$ and the other at temperature $T+\Delta T$. Furthermore, we assume the temperature gradient in the insulator to be linear in the $\hat{\boldsymbol{x}}$-direction, i.e. $T(\mathbf{x}) = T[1 + \Delta T x/(TL)]$. Because this temperature gradient is time-independent, we consider the steady-state. Furthermore, we can forget about the vector properties of Eq. (2.51 - 2.52) because of the symmetry of our system. Under these assumptions we obtain

$$0 = -\nabla(J_c(x) + J_{th}(x)) + \alpha\left(2n_c(x) - \frac{3}{\Lambda_{th}^3}\right)\delta\mu(x) - 2\alpha n_c(x)\delta\mu_c(x), \tag{2.53}$$

$$0 = \frac{2J_{ex}}{\hbar}\nabla(\delta\mu(x) - \delta\mu_c(x)). \tag{2.54}$$

Before we start solving this set of equations, we notice that for $\alpha = 0$ Eq. (2.53) reduces to the conservation law of magnons. Since $\alpha = 0$ implies that there is no Gilbert damping, this is consistent with our expectations.

### 2.3.1 Transport in the normal state

For $T > T_c$ there is no Bose-Einstein condensate of magnons. This entails that $J_c = 0$, $n_c = 0$, and that we only use the hydrodynamic equation for the magnon density, and thus we find

$$0 = -\nabla J_{th}(x) - \frac{3\alpha}{\Lambda_{th}^3}\delta\mu(x). \tag{2.55}$$

To solve this equation, we substitute Eq. (2.26). By realizing that $\nabla\mu(x) = \nabla\delta\mu(x)$ and $\nabla^2 T(x) = 0$, we obtain a differential equation for $\delta\mu(x)$:

$$0 = \frac{\sigma_s}{\hbar}\nabla^2\delta\mu(x) - \frac{3\alpha}{\Lambda_{th}^3}\delta\mu(x). \tag{2.56}$$

Solving Eq. (2.56) leads to

$$J_{th}(x) = -\frac{\sigma_s}{\hbar\ell_1}(C_1 e^{\frac{x}{\ell_1}} - C_2 e^{-\frac{x}{\ell_1}}) + \frac{L_{\mu Q}}{L}\frac{\Delta T}{T}, \tag{2.57}$$

where $\ell_1 \equiv \sqrt{\sigma_s \Lambda_{th}^3/(3\alpha\hbar)}$ is the thermal magnon propagation length and $C_{1,2}$ are constants yet to be determined by the boundary conditions. We use the boundary conditions derived by Flebus *et al.* [4]:

$$J_{th}^{\mathcal{L}} = \frac{3g_{\mathcal{L}}^{\uparrow\downarrow}}{4\pi s\Lambda_{th}^3}(\mu_s^{\mathcal{L}} - \delta\mu^{\mathcal{L}}), \tag{2.58}$$

$$J_{th}^{\mathcal{R}} = \frac{3g_{\mathcal{R}}^{\uparrow\downarrow}}{4\pi s\Lambda_{th}^3}(\delta\mu^{\mathcal{R}} - \mu_s^{\mathcal{R}}). \tag{2.59}$$

In these expressions, $g_{\mathcal{L}}^{\uparrow\downarrow} = i\Im g_{\mathcal{L}}^{\uparrow\downarrow} + \Re g_{\mathcal{L}}^{\uparrow\downarrow}$ and $g_{\mathcal{R}}^{\uparrow\downarrow} = i\Im g_{\mathcal{R}}^{\uparrow\downarrow} + \Re g_{\mathcal{R}}^{\uparrow\downarrow}$ are the complex-valued spin-mixing conductances characterizing the left and right interface respectively and $s = S/a^3$ is the spin density. Furthermore, $\mu_s^{\mathcal{L}}$ and $\mu_s^{\mathcal{R}}$ are the spin accumulations in the boundaries of the metallic reservoirs that are connected to the ferromagnetic insulator, where the superscript $\mathcal{L}$ stands for the left and $\mathcal{R}$ for the right interface. We assume that the spin-mixing conductances are akin, so we take $g^{\uparrow\downarrow} \equiv g_{\mathcal{L}}^{\uparrow\downarrow} \sim g_{\mathcal{R}}^{\uparrow\downarrow}$ and neglect their real part. The expression that we obtain when we put our boundary conditions in Eq. (2.57), is too cumbersome to enter here.

### 2.3.2 Transport in the superfluid state

Now that we have examined our system under conditions for which there is no condensate of magnons, we can look at temperatures $T < T_c$. In this case there are currents of both thermal and condensed magnons, i.e. our system can be described by the two-fluid model. For simplicity, we assume that $n_c$ is homogeneous in the material. In this case $\mu_c(x) = \mu_{eq}$, because of the equality we found at the end of Section 2.2.1. Consequently, $\delta\mu_c(x) = 0$ and Eq. (2.54) reduces to $\nabla\delta\mu(x) = 0$, i.e. thermal magnon chemical potential fluctuations are suppressed. It follows that $\delta\mu(x)$ is independent of $x$ (from now on we will just denote it by $\delta\mu$ to indicate this finding). Furthermore, the current of thermal magnons is constant throughout the material,

$$J_{th}(x) = \frac{L_{\mu Q}}{L}\frac{\Delta T}{T}, \tag{2.60}$$

so Eq. (2.53) yields

$$0 = -\nabla J_c(x) + \alpha\left(2n_c - \frac{3}{\Lambda_{th}^3}\right)\delta\mu. \tag{2.61}$$

Integrating Eq. (2.61) over $x$ returns

$$J_c(x) = \left[\alpha\left(2n_c - \frac{3}{\Lambda_{th}^3}\right)\delta\mu\right]x + C_3, \tag{2.62}$$

with $C_3$ the constant of integration. The boundary conditions derived by Flebus *et al.* [4] for this situation are

$$J_s^{\mathcal{L}} = \frac{g_{\mathcal{L}}^{\uparrow\downarrow}}{4\pi s}\left[\left(2n_c + \frac{3}{\Lambda_{th}^3}\right)\mu_s^{\mathcal{L}} + \left(2n_c - \frac{3}{\Lambda_{th}^3}\right)\delta\mu\right], \tag{2.63}$$

$$J_s^{\mathcal{R}} = -\frac{g_{\mathcal{R}}^{\uparrow\downarrow}}{4\pi s}\left[\left(2n_c + \frac{3}{\Lambda_{th}^3}\right)\mu_s^{\mathcal{R}} + \left(2n_c - \frac{3}{\Lambda_{th}^3}\right)\delta\mu\right], \tag{2.64}$$

where $J_s = J_c + J_{th}$ denotes the total spin current. We assume once more that the spin-mixing conductances at the interfaces are similar, i.e. $g^{\uparrow\downarrow} \equiv g_{\mathcal{L}}^{\uparrow\downarrow} \sim g_{\mathcal{R}}^{\uparrow\downarrow}$. Taking $J_c(0) + J_{th}(0) = J_s^{\mathcal{L}}$ and $J_c(L) + J_{th}(L) = J_s^{\mathcal{R}}$, we can determine the values of the constants $\delta\mu$ and $C_3$, and we find the superfluid current as

$$J_c(x) = \frac{g^{\uparrow\downarrow}\left(2n_c + \frac{3}{\Lambda_{th}^3}\right)}{4\pi s}\left[\mu_s^{\mathcal{L}} - \frac{g^{\uparrow\downarrow}(\mu_s^{\mathcal{L}} + \mu_s^{\mathcal{R}})}{2g^{\uparrow\downarrow} + 4\pi s\alpha L}\right] - \frac{\alpha g^{\uparrow\downarrow}(\mu_s^{\mathcal{L}} + \mu_s^{\mathcal{R}})\left(2n_c + \frac{3}{\Lambda_{th}^3}\right)}{2g^{\uparrow\downarrow} + 4\pi s\alpha L}x - \frac{L_{\mu Q}}{L}\frac{\Delta T}{T}. \tag{2.65}$$

In the absence of spin accumulation in the boundaries, i.e. for $\mu_s^{\mathcal{L}} = \mu_s^{\mathcal{R}} = 0$, Eq. (2.65) reduces to

$$J_c(x) = -\frac{L_{\mu Q}}{L}\frac{\Delta T}{T}. \tag{2.66}$$

Evidently, the condensate spin current flows opposite to the thermal spin current (Eq.(2.60)), and thus the total spin current is cancelled out.

# Chapter 3

# Ferromagnetic conductors

## 3.1 Electron-magnon interactions

When we consider a conductor instead of an insulator, we are confronted with free electrons in our system. As it turns out, these electrons interact with magnons, so our quantum kinetic calculations of Section 2.2 need to be altered.

We start by deriving the new Boltzmann equations: one for the (thermal) magnons and two for the electrons (a distinction is made between spin-up and spin-down electrons). After that, we will take a closer look at the Gilbert damping caused by electron-magnon interactions. With these results we will derive the transport coefficients for $\mathbf{J}_{th}$, $\mathbf{J}_{e,\uparrow}$, and $\mathbf{J}_{e,\downarrow}$, and a hydrodynamic equation for the electron density.

### 3.1.1 Boltzmann equations

We take $f = f(\mathbf{k}, \mathbf{x}, t)$, $h_\uparrow = h_\uparrow(\mathbf{p}, \mathbf{x}, t)$ and $h_\downarrow = h_\downarrow(\mathbf{p}, \mathbf{x}, t)$ as the distribution functions of the thermal magnons, the spin-up electrons and the spin-down electrons respectively. The Boltzmann equations for these distribution functions are then given by

$$\frac{\partial f}{\partial t} + \frac{2J_{\text{xc}}\mathbf{k}}{\hbar} \cdot \nabla f = \Gamma^f_{\text{me}} - \frac{2\alpha k_B T}{\hbar}[f - n_{\text{BE}}(\hbar\omega_{\mathbf{k}})] - \frac{1}{\tau_{\text{m}}}[f - n_{\text{BE}}(\hbar\omega_{\mathbf{k}})], \tag{3.1}$$

$$\frac{\partial h_\uparrow}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla h_\uparrow + \mathbf{E} \cdot \nabla_{\mathbf{p}} h_\uparrow = \Gamma^{h_\uparrow}_{\text{me}} - \frac{1}{\tau_\uparrow}[h_\uparrow - n_{\text{FD}}(\varepsilon_{\mathbf{p},\uparrow})], \tag{3.2}$$

$$\frac{\partial h_\downarrow}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla h_\downarrow + \mathbf{E} \cdot \nabla_{\mathbf{p}} h_\downarrow = \Gamma^{h_\downarrow}_{\text{me}} - \frac{1}{\tau_\downarrow}[h_\downarrow - n_{\text{FD}}(\varepsilon_{\mathbf{p},\downarrow})]. \tag{3.3}$$

Here $m$ is the electron mass, $\mathbf{E}$ is the electric field, and $\varepsilon_{\mathbf{p},\uparrow}$ and $\varepsilon_{\mathbf{p},\downarrow}$ are the energies of a spin-up and a spin-down electron. In terms of the momentum $\mathbf{p}$ these energies are given by $\varepsilon_{\mathbf{p},\uparrow} = \mathbf{p}^2/(2m) + \Delta/2$ and $\varepsilon_{\mathbf{p},\downarrow} = \mathbf{p}^2/(2m) - \Delta/2$ respectively, where $\Delta$ is simply defined as the difference between the two. Furthermore, $\tau_{\text{m}}$, $\tau_\uparrow$ and $\tau_\downarrow$ are the relaxation times for the particles in our system, and $n_{\text{FD}}(\varepsilon) \equiv (e^{\beta\varepsilon} + 1)^{-1}$ is the Fermi-Dirac distribution. The terms $\Gamma^f_{\text{me}}$, $\Gamma^{h_\uparrow}_{\text{me}}$ and $\Gamma^{h_\downarrow}_{\text{me}}$ represent the collision visualized in Fig. 3.1.

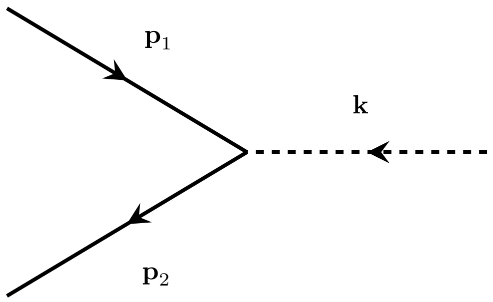


Figure 3.1: Feynman diagram representing $\Gamma_{\mathrm{me}}^{f,h_\uparrow,h_\downarrow}$

Here the continuous lines represent electrons and the dashed line represents a magnon. Without loss of generality, we assume that the electron with momentum $\mathbf{p}_1$ is spin-up and the one with momentum $\mathbf{p}_2$ is spin-down. Again we can write our collision terms as integrals:

$$\Gamma_{\mathrm{me}}^{f} = -\frac{2\pi}{\hbar}\int \frac{d\mathbf{p}_1\, d\mathbf{p}_2}{(2\pi)^6}\, \delta\big(\hbar\omega_{\mathbf{k}} + \varepsilon_{\mathbf{p}_1,\uparrow} - \varepsilon_{\mathbf{p}_2,\downarrow}\big) \left|V_{\mathbf{k},\mathbf{p}_1,\mathbf{p}_2}\right|^2 \times \left[f(\mathbf{k})\, h_\uparrow(\mathbf{p}_1)\,(1-h_\downarrow(\mathbf{p}_2)) - (1+f(\mathbf{k}))\,(1-h_\uparrow(\mathbf{p}_1))\, h_\downarrow(\mathbf{p}_2)\right], \tag{3.4}$$

$$\Gamma_{\mathrm{me}}^{h_\uparrow} = -\frac{2\pi\hbar^3}{\hbar}\int \frac{d\mathbf{k}\, d\mathbf{p}_2}{(2\pi)^6}\, \delta\big(\hbar\omega_{\mathbf{k}} + \varepsilon_{\mathbf{p}_1,\uparrow} - \varepsilon_{\mathbf{p}_2,\downarrow}\big) \left|V_{\mathbf{k},\mathbf{p}_1,\mathbf{p}_2}\right|^2 \times \left[f(\mathbf{k})\, h_\uparrow(\mathbf{p}_1)\,(1-h_\downarrow(\mathbf{p}_2)) - (1+f(\mathbf{k}))\,(1-h_\uparrow(\mathbf{p}_1))\, h_\downarrow(\mathbf{p}_2)\right], \tag{3.5}$$

$$\Gamma_{\mathrm{me}}^{h_\downarrow} = \frac{2\pi\hbar^3}{\hbar}\int \frac{d\mathbf{k}\, d\mathbf{p}_1}{(2\pi)^6}\, \delta\big(\hbar\omega_{\mathbf{k}} + \varepsilon_{\mathbf{p}_1,\uparrow} - \varepsilon_{\mathbf{p}_2,\downarrow}\big) \left|V_{\mathbf{k},\mathbf{p}_1,\mathbf{p}_2}\right|^2 \times \left[f(\mathbf{k})\, h_\uparrow(\mathbf{p}_1)\,(1-h_\downarrow(\mathbf{p}_2)) - (1+f(\mathbf{k}))\,(1-h_\uparrow(\mathbf{p}_1))\, h_\downarrow(\mathbf{p}_2)\right]. \tag{3.6}$$

The term $|V_{\mathbf{k},\mathbf{p}_1,\mathbf{p}_2}|^2$ denotes the strength of the electron-magnon coupling. These cumbersome expressions can be simplified using a method similar to the one used to make an approximation of $\Gamma_{12}$ in Section 2.2.3. If we now use the equilibrium functions for bosons and fermions, i.e. $f(\mathbf{k}) = n_{\mathrm{BE}}(\hbar\omega_{\mathbf{k}} - \mu(\mathbf{x}))$, $h_\uparrow(\mathbf{p}_1) = n_{\mathrm{FD}}(\varepsilon_{\mathbf{p}_1,\uparrow} - \mu_{e,\uparrow}(\mathbf{x}))$, and $h_\downarrow(\mathbf{p}_2) = n_{\mathrm{FD}}(\varepsilon_{\mathbf{p}_2,\downarrow} - \mu_{e,\downarrow}(\mathbf{x}))$, we obtain

$$\Gamma_{\mathrm{me}}^{f} = \frac{\mu(\mathbf{x}) + \mu_{e,\uparrow}(\mathbf{x}) - \mu_{e,\downarrow}(\mathbf{x})}{k_B T}\int d\mathbf{p}_1\, d\mathbf{p}_2\, \Gamma_{\mathrm{me}}, \tag{3.7}$$

$$\Gamma_{\mathrm{me}}^{h_\uparrow} = \frac{\mu(\mathbf{x}) + \mu_{e,\uparrow}(\mathbf{x}) - \mu_{e,\downarrow}(\mathbf{x})}{k_B T}\int d\mathbf{k}\, d\mathbf{p}_2\, \hbar^3\Gamma_{\mathrm{me}}, \tag{3.8}$$

$$\Gamma_{\mathrm{me}}^{h_\downarrow} = -\frac{\mu(\mathbf{x}) + \mu_{e,\uparrow}(\mathbf{x}) - \mu_{e,\downarrow}(\mathbf{x})}{k_B T}\int d\mathbf{k}\, d\mathbf{p}_1\, \hbar^3\Gamma_{\mathrm{me}}, \tag{3.9}$$

with

$$\Gamma_{\mathrm{me}} \equiv -\frac{\delta\big(\hbar\omega_{\mathbf{k}} + \varepsilon_{\mathbf{p}_1,\uparrow} - \varepsilon_{\mathbf{p}_2,\downarrow}\big)\left|V_{\mathbf{k},\mathbf{p}_1,\mathbf{p}_2}\right|^2 e^{\beta\varepsilon_{\mathbf{p}_2,\downarrow}}}{\hbar(2\pi)^5\left(e^{\beta\hbar\omega_{\mathbf{k}}} - e^{\beta\mu(\mathbf{x})}\right)\left(e^{\beta\varepsilon_{\mathbf{p}_1,\uparrow}} - e^{\beta\mu_{e,\uparrow}(\mathbf{x})}\right)\left(e^{\beta\varepsilon_{\mathbf{p}_2,\downarrow}} - e^{\beta\mu_{e,\downarrow}(\mathbf{x})}\right)}. \tag{3.10}$$

The terms $\mu_{e,\uparrow}(\mathbf{x})$ and $\mu_{e,\downarrow}(\mathbf{x})$ are the chemical potentials of the spin-up and spin-down electrons respectively.

### 3.1.2 Electron-induced Gilbert damping

In Chapter 2, we have accounted for Gilbert damping caused mainly by lattice vibrations. In conductors, however, the Gilbert damping is dominated by a far greater contribution: that of the electron-magnon interactions. Here we will neglect all other phenomena that cause Gilbert damping and calculate the Gilbert damping factor $\alpha$ for this specific contribution explicitly.

Taking the approximation $1 + f(\mathbf{k}) \approx f(\mathbf{k})$, we find the contribution of this Gilbert damping to the Boltzmann equation for magnons to be

$$\begin{aligned}\frac{df}{dt} = &-\frac{2\pi f(\mathbf{k})}{\hbar}\int \frac{d\mathbf{p}_1\, d\mathbf{p}_2}{(2\pi)^6}\, \delta\big(\hbar\omega_{\mathbf{k}} + \varepsilon_{\mathbf{p}_1,\uparrow} - \varepsilon_{\mathbf{p}_2,\downarrow}\big)\, \big|V_{\mathbf{k},\mathbf{p}_1,\mathbf{p}_2}\big|^2 \\ &\times \big[n_{\text{FD}}(\varepsilon_{\mathbf{p}_1,\uparrow} - \mu_{e,\uparrow}(\mathbf{x})) - n_{\text{FD}}(\varepsilon_{\mathbf{p}_2,\downarrow} - \mu_{e,\downarrow}(\mathbf{x}))\big]\,. \end{aligned} \tag{3.11}$$

We now assume $|V_{\mathbf{k},\mathbf{p}_1,\mathbf{p}_2}|^2$ to be independent of $\mathbf{k}$, and we use the equalities $1 = \int d\varepsilon\, \delta(\varepsilon - \varepsilon_{\mathbf{p}_1,\uparrow})$ and $1 = \int d\varepsilon'\, \delta(\varepsilon' - \varepsilon_{\mathbf{p}_2,\downarrow})$ to rewrite Eq. (3.11) to an integral over $\varepsilon$ and $\varepsilon'$:

$$\frac{df}{dt} = -\frac{2\pi f(\mathbf{k})}{\hbar}\int d\varepsilon\, d\varepsilon'\, \delta(\hbar\omega_{\mathbf{k}} + \varepsilon - \varepsilon')\, [n_{\text{FD}}(\varepsilon - \mu_{e,\uparrow}(\mathbf{x})) - n_{\text{FD}}(\varepsilon' - \mu_{e,\downarrow}(\mathbf{x}))]\, \xi(\varepsilon, \varepsilon'). \tag{3.12}$$

Here $\xi(\varepsilon, \varepsilon')$ is a function defined as

$$\xi(\varepsilon, \varepsilon') \equiv \int \frac{d\mathbf{p}_1\, d\mathbf{p}_2}{(2\pi)^6}\, \delta(\varepsilon - \varepsilon_{\mathbf{p}_1,\uparrow})\, \delta(\varepsilon' - \varepsilon_{\mathbf{p}_2,\downarrow})\, \big|V_{\mathbf{p}_1,\mathbf{p}_2}\big|^2\,. \tag{3.13}$$

Now we can evaluate the integral over $\varepsilon'$ in Eq. (3.12), and so we find

$$\frac{df}{dt} = -\frac{2\pi f(\mathbf{k})}{\hbar}\int d\varepsilon\, [n_{\text{FD}}(\varepsilon - \mu_{e,\uparrow}(\mathbf{x})) - n_{\text{FD}}(\hbar\omega_{\mathbf{k}} + \varepsilon - \mu_{e,\downarrow}(\mathbf{x}))]\, \xi(\varepsilon, \hbar\omega_{\mathbf{k}} + \varepsilon). \tag{3.14}$$

Under the assumption that $\mu_{e,\uparrow}(\mathbf{x}) = \mu_{e,\downarrow}(\mathbf{x})$, and that the magnon energy is low we use a first-order Taylor expansion and obtain

$$\frac{df}{dt} = 2\pi f(\mathbf{k})\, \omega_{\mathbf{k}} \int d\varepsilon\, n'_{\text{FD}}(\varepsilon - \mu_{e,\uparrow}(\mathbf{x}))\, \xi(\varepsilon, \hbar\omega_{\mathbf{k}} + \varepsilon). \tag{3.15}$$

Identification with the usual form of a Gilbert damping term in a Boltzmann equation yields

$$\alpha = -\pi \int d\varepsilon\, n'_{\text{FD}}(\varepsilon - \mu_{e,\uparrow}(\mathbf{x}))\, \xi(\varepsilon, \hbar\omega_{\mathbf{k}} + \varepsilon). \tag{3.16}$$

This is the Gilbert damping factor that we use in Eq. (3.1).

### 3.1.3 Transport coefficients and hydrodynamic equations

Here we will follow the method of Section 2.2.2 to find the transport coefficients for $\mathbf{J}_{th}$, $\mathbf{J}_{e,\uparrow}$, and $\mathbf{J}_{e,\downarrow}$. Now the driving forces are $\nabla\mu(\mathbf{x})$, $\nabla\mu_{e,\uparrow}(\mathbf{x})$, $\nabla\mu_{e,\downarrow}(\mathbf{x})$, $\mathbf{E}(\mathbf{x})$, and $\nabla T(\mathbf{x})$. Again, we will work in the linear-response regime and take $\mu(\mathbf{x}), \mu_{e,\uparrow}(\mathbf{x}), \mu_{e,\downarrow}(\mathbf{x}) \to 0$ and $T(\mathbf{x}) \to T$, whenever they appear in another form than in a single differential.

First we need to find solutions of the Boltzmann equations, Eq. (3.1 - 3.3). For this we use first-order perturbation theory. To derive the zeroth-order solution, we use the following ansatzes:

$$f^{(0)}(\mathbf{k},\mathbf{x},t) = n_{\text{BE}}(\hbar\omega_{\mathbf{k}} - \mu(\mathbf{x})) + \delta f, \tag{3.17}$$

$$h_\uparrow^{(0)}(\mathbf{p},\mathbf{x},t) = n_{\text{FD}}(\varepsilon_{\mathbf{p},\uparrow} - \mu_{e,\uparrow}(\mathbf{x})) + \delta h_\uparrow, \tag{3.18}$$

$$h_\downarrow^{(0)}(\mathbf{p},\mathbf{x},t) = n_{\text{FD}}(\varepsilon_{\mathbf{p},\downarrow} - \mu_{e,\downarrow}(\mathbf{x})) + \delta h_\downarrow. \tag{3.19}$$

In zeroth-order we can ignore the $\Gamma_{\text{me}}$-terms in the Boltzmann equations, and so we find

$$\delta f = \frac{2\tau_{\text{m}} J_{\text{xc}}}{2\alpha k_B T \tau_{\text{m}} + \hbar}\, n'_{\text{BE}}(\hbar\omega_{\mathbf{k}})\, \mathbf{k}\cdot \left[\nabla\mu(\mathbf{x}) + \hbar\omega_{\mathbf{k}}\frac{\nabla T(\mathbf{x})}{T}\right], \tag{3.20}$$

$$\delta h_\uparrow = \frac{\tau_\uparrow}{m}\, n'_{\text{FD}}(\varepsilon_{\mathbf{p},\uparrow})\, \mathbf{p}\cdot \left[\nabla\mu_{e,\uparrow}(\mathbf{x}) + \varepsilon_{\mathbf{p},\uparrow}\frac{\nabla T(\mathbf{x})}{T} - \mathbf{E}(\mathbf{x})\right], \tag{3.21}$$

$$\delta h_\downarrow = \frac{\tau_\downarrow}{m}\, n'_{\text{FD}}(\varepsilon_{\mathbf{p},\downarrow})\, \mathbf{p}\cdot \left[\nabla\mu_{e,\downarrow}(\mathbf{x}) + \varepsilon_{\mathbf{p},\downarrow}\frac{\nabla T(\mathbf{x})}{T} - \mathbf{E}(\mathbf{x})\right]. \tag{3.22}$$

First-order perturbation theory now reveals $f^{(1)}$, $h_\uparrow^{(1)}$, and $h_\downarrow^{(1)}$ to be solutions of the following equations:

$$\frac{2J_{\text{xc}}\mathbf{k}}{\hbar}\cdot\nabla f^{(1)} = \Gamma^f_{\text{me}}\left[f^{(0)}, h_\uparrow^{(0)}, h_\downarrow^{(0)}\right] - \frac{2\alpha k_B T}{\hbar} f^{(1)} - \frac{1}{\tau_{\text{m}}} f^{(1)}, \tag{3.23}$$

$$\frac{\mathbf{p}}{m}\cdot\nabla h_\uparrow^{(1)} + \mathbf{E}\cdot\nabla_{\mathbf{p}} h_\uparrow^{(1)} = \Gamma^{h_\uparrow}_{\text{me}}\left[f^{(0)}, h_\uparrow^{(0)}, h_\downarrow^{(0)}\right] - \frac{1}{\tau_\uparrow} h_\uparrow^{(1)}, \tag{3.24}$$

$$\frac{\mathbf{p}}{m}\cdot\nabla h_\downarrow^{(1)} + \mathbf{E}\cdot\nabla_{\mathbf{p}} h_\downarrow^{(1)} = \Gamma^{h_\downarrow}_{\text{me}}\left[f^{(0)}, h_\uparrow^{(0)}, h_\downarrow^{(0)}\right] - \frac{1}{\tau_\downarrow} h_\downarrow^{(1)}. \tag{3.25}$$

We assume that the gradients on the left-hand sides of these equations are 0, and thus

$$f^{(1)} = \frac{\hbar\tau_{\text{m}}}{2\alpha k_B T\tau_{\text{m}} + \hbar}\,\Gamma^f_{\text{me}}\left[f^{(0)}, h_\uparrow^{(0)}, h_\downarrow^{(0)}\right], \tag{3.26}$$

$$h_\uparrow^{(1)} = \tau_\uparrow\,\Gamma^{h_\uparrow}_{\text{me}}\left[f^{(0)}, h_\uparrow^{(0)}, h_\downarrow^{(0)}\right], \tag{3.27}$$

$$h_\downarrow^{(1)} = \tau_\downarrow\,\Gamma^{h_\downarrow}_{\text{me}}\left[f^{(0)}, h_\uparrow^{(0)}, h_\downarrow^{(0)}\right]. \tag{3.28}$$

This concludes our first-order perturbatory calculations and we find the distribution functions to be $f = f^{(0)} + f^{(1)}$, $h_\uparrow = h_\uparrow^{(0)} + h_\uparrow^{(1)}$, and $h_\downarrow = h_\downarrow^{(0)} + h_\downarrow^{(1)}$. The currents of magnons and electrons are given by

$$\mathbf{J}_{th}(\mathbf{x}) = \hbar\int\frac{d\mathbf{k}}{(2\pi)^3}\, f\,\frac{\partial(\hbar\omega_{\mathbf{k}})}{\partial(\hbar\mathbf{k})}, \tag{3.29}$$

$$\mathbf{J}_{e,\uparrow}(\mathbf{x}) = -|e|\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\, h_\uparrow\,\frac{\partial\varepsilon_{\mathbf{p},\uparrow}}{\partial\mathbf{p}}, \tag{3.30}$$

$$\mathbf{J}_{e,\downarrow}(\mathbf{x}) = -|e|\int\frac{d\mathbf{p}}{(2\pi\hbar)^3}\, h_\downarrow\,\frac{\partial\varepsilon_{\mathbf{p},\downarrow}}{\partial\mathbf{p}}, \tag{3.31}$$

in which we substitute the distribution functions $f$, $h_\uparrow$, and $h_\downarrow$. This approach leads to

$$\mathbf{J}_{th}(\mathbf{x}) = -\frac{\sigma_s'}{\hbar}\nabla\mu(\mathbf{x}) - L_{D,\uparrow}\nabla\mu_{e,\uparrow}(\mathbf{x}) - L_{D,\downarrow}\nabla\mu_{e,\downarrow}(\mathbf{x}) - L_F\mathbf{E}(\mathbf{x}) + L_{\mu Q}'\frac{\nabla T(\mathbf{x})}{T}, \tag{3.32}$$

$$\mathbf{J}_{e,\uparrow}(\mathbf{x}) = -L_{D,1}'\nabla\mu(\mathbf{x}) - \sigma_{11}\nabla\mu_{e,\uparrow}(\mathbf{x}) - \sigma_{12}\nabla\mu_{e,\downarrow}(\mathbf{x}) - L_{F,1}'\mathbf{E}(\mathbf{x}) + L_{e,1}\frac{\nabla T(\mathbf{x})}{T}, \tag{3.33}$$

$$\mathbf{J}_{e,\downarrow}(\mathbf{x}) = -L_{D,2}'\nabla\mu(\mathbf{x}) - \sigma_{21}\nabla\mu_{e,\uparrow}(\mathbf{x}) - \sigma_{22}\nabla\mu_{e,\downarrow}(\mathbf{x}) - L_{F,2}'\mathbf{E}(\mathbf{x}) + L_{e,2}\frac{\nabla T(\mathbf{x})}{T}, \tag{3.34}$$

where

$$\sigma_s' = -\frac{8\pi\tau_\mathrm{m}J_\mathrm{xc}^2\hbar}{6\alpha k_BT\tau_\mathrm{m}+3\hbar}\int_0^\infty \frac{dk}{(2\pi)^3}\,n_\mathrm{BE}'(\hbar\omega_k)\,k^4 \times\left[1-\frac{2\pi\tau_\mathrm{m}}{2\alpha k_BT\tau_\mathrm{m}+\hbar}\int\frac{d\mathbf{p}_1\,d\mathbf{p}_2}{(2\pi)^6}\,\delta\big(\hbar\omega_k+\varepsilon_{\mathbf{p}_1,\uparrow}-\varepsilon_{\mathbf{p}_2,\downarrow}\big)\left|V_{k,\mathbf{p}_1,\mathbf{p}_2}\right|^2\left[n_\mathrm{FD}(\varepsilon_{\mathbf{p}_1,\uparrow})-n_\mathrm{FD}(\varepsilon_{\mathbf{p}_2,\downarrow})\right]\right]. \tag{3.35}$$

The other transport coefficients are similar in structure: cumbersome integrals over the wavevector of the magnons and the momenta of the (spin-up and spin-down) electrons. It would be ponderous to enter all 15 of them here, so we continue.

If we consider the total electric current, i.e. if we omit the distiction between spin-up and spin-down electrons, we are left with a much more compact system of transport-coefficients:

$$\mathbf{J}_{th}(\mathbf{x}) = -\frac{\sigma_s'}{\hbar}\nabla\mu(\mathbf{x}) - L_D\nabla\mu_e(\mathbf{x}) + L_{\mu Q}'\frac{\nabla T(\mathbf{x})}{T}, \tag{3.36}$$

$$\mathbf{J}_e(\mathbf{x}) = -L_D'\nabla\mu(\mathbf{x}) - \sigma\nabla\mu_e(\mathbf{x}) + L_e\frac{\nabla T(\mathbf{x})}{T}. \tag{3.37}$$

Here $\mu_e$ is the electro-chemical potential. So far, we have only considered thermal magnons. When we manually insert terms to account for the superfluid current of magnons, we obtain

$$\mathbf{J}_s(\mathbf{x}) = -\frac{\sigma_s'}{\hbar}\nabla\mu(\mathbf{x}) - L_D\nabla\mu_e(\mathbf{x}) + L_{\mu Q}'\frac{\nabla T(\mathbf{x})}{T} + \hbar n_c\mathbf{v}_c(\mathbf{x}), \tag{3.38}$$

$$\mathbf{J}_e(\mathbf{x}) = -L_D'\nabla\mu(\mathbf{x}) - \sigma\nabla\mu_e(\mathbf{x}) + L_e\frac{\nabla T(\mathbf{x})}{T} + \gamma n_c\mathbf{v}_c(\mathbf{x}), \tag{3.39}$$

where $\gamma$ is a dimensionless constant.

The electrons in our system are conserved. This fact is expressed in the following formula:

$$\dot{\rho}_e = -\nabla\cdot\mathbf{J}_e. \tag{3.40}$$

We add this conservation law to our system of hydrodynamic equations (Eq. (2.48 - 2.50)) to describe the dynamics of the electron current.

## 3.2 Applications

Similarly to Section 2.3, we will solve the hydrodynamic equations of our system exactly. We will make the same assumptions (including the one of a linear thermal gradient in the $\hat{\boldsymbol{x}}$-direction). The only difference with the situation in the previous chapter is that we now consider a ferromagnetic conductor. We find the full hydrodynamic equations for the steady-state as

$$0 = -\nabla(J_c(x) + J_{th}(x)) + \alpha\left(2n_c(x) - \frac{3}{\Lambda_{th}^3}\right)\delta\mu(x) - 2\alpha n_c(x)\delta\mu_c(x), \tag{3.41}$$

$$0 = -\nabla J_e(x), \tag{3.42}$$

$$0 = \frac{2J_{ex}}{\hbar}\nabla(\delta\mu(x) - \delta\mu_c(x)). \tag{3.43}$$

Broadly, this section will follow the structure of Section 2.3. First we will calculate the currents $J_{th}$ and $J_e$ for $T > T_c$, and then we will calculate all of the currents in our system (including the superfluid current of magnons) for $T < T_c$. Finally, we will calculate the Seebeck coefficient for both states.

### 3.2.1 Transport in the normal state

For temperatures higher than the critical temperature for Bose-Einstein condensation, there is no current of condensed magnons. This enables us to simplify Eq. (3.41 - 3.43) to

$$0 = -\nabla J_{th}(x) - \frac{3\alpha}{\Lambda_{th}^3}\delta\mu(x), \tag{3.44}$$

$$0 = -\nabla J_e(x). \tag{3.45}$$

Substitution of our expressions for $J_{th}$ and $J_e$, Eq. (3.36) and Eq. (3.37) respectively, yields

$$\frac{\sigma_s'}{\hbar}\nabla^2\delta\mu(x) + L_D\nabla^2\mu_e(x) = \frac{3\alpha}{\Lambda_{th}^3}\delta\mu(x), \tag{3.46}$$

$$L_D'\nabla^2\delta\mu(x) + \sigma\nabla^2\mu_e(x) = 0, \tag{3.47}$$

which leads to

$$J_{th}(x) = \left(-\frac{\sigma_s'}{\hbar\ell_2} + \frac{L_D L_D'}{\sigma\ell_2}\right)\left(C_1 e^{\frac{x}{\ell_2}} - C_2 e^{-\frac{x}{\ell_2}}\right) - L_D C_3 + \frac{L_{\mu Q}'}{L}\frac{\Delta T}{T}, \tag{3.48}$$

$$J_e(x) = -\sigma C_3 + \frac{L_e}{L}\frac{\Delta T}{T}. \tag{3.49}$$

In these expressions, $\ell_2 \equiv \sqrt{\Lambda_{th}^3(\sigma\sigma_s' - \hbar L_D L_D')/(3\alpha\sigma\hbar)}$ is the thermal magnon propagation length and $C_{1,2,3}$ are constants that we will determine using the boundary conditions. For the spin current we use the same boundary conditions as before (Eq. (2.58 - 2.59)) and for the electric current we use $J_e^{\mathcal{L}} = J_e^{\mathcal{R}} = 0$. With these boundary conditions we determine the constants $C_{1,2,3}$. The electron current turns out to be $J_e(x) = 0$ and for the magnon current we find an expression that is too cumbersome to include in this Thesis.

### 3.2.2 Transport in the superfluid state

For $T < T_c$ we are dealing with the two-fluid model. Again, we assume that $n_c$ is homogeneous and so we find that $\delta\mu_c(x) = 0$. This reduces Eq. (3.41 - 3.43) to

$$0 = -\nabla(J_c(x) + J_{th}(x)) + \alpha\left(2n_c - \frac{3}{\Lambda_{th}^3}\right)\delta\mu, \tag{3.50}$$

$$0 = -\nabla J_e(x). \tag{3.51}$$

We substitute our expressions for the spin current and the electric current, Eq. (3.38) and Eq. (3.39), and obtain

$$\hbar n_c \nabla v_c(x) - L_D \nabla^2 \mu_e(x) = \alpha \left(2n_c - \frac{3}{\Lambda_{th}^3}\right) \delta\mu, \tag{3.52}$$

$$\gamma n_c \nabla v_c(x) - \sigma \nabla^2 \mu_e(x) = 0. \tag{3.53}$$

We solve this system of differential equations and find

$$v_c(x) = \frac{\sigma\alpha \left(2n_c - \frac{3}{\Lambda_{th}^3}\right)}{n_c\,(\hbar\sigma - \gamma L_D)}\, \delta\mu\, x + C_4, \tag{3.54}$$

$$\mu_e(x) = \frac{\gamma\alpha \left(2n_c - \frac{3}{\Lambda_{th}^3}\right)}{2\,(\hbar\sigma - \gamma L_D)}\, \delta\mu\, x^2 + C_5\, x + C_6, \tag{3.55}$$

which leads to

$$J_s(x) = \alpha \left(2n_c - \frac{3}{\Lambda_{th}^3}\right) \delta\mu\, x + \hbar n_c C_4 - L_D C_5 + \frac{L'_{\mu Q}}{L}\frac{\Delta T}{T}, \tag{3.56}$$

$$J_e(x) = \gamma n_c C_4 - \sigma C_5 + \frac{L_e}{L}\frac{\Delta T}{T}. \tag{3.57}$$

Again we take $J_e^{\mathcal{L}} = J_e^{\mathcal{R}} = 0$ as the boundary condition for the electric current and the boundary conditions derived by Flebus *et al.* [4] for the spin current. The latter are given by Eq. (2.63) and Eq. (2.64). With this approach, we find the currents in a ferromagnetic conductor as

$$J_s(x) = \frac{g^{\uparrow\downarrow}\left(2n_c + \frac{3}{\Lambda_{th}^3}\right)}{4\pi s}\left[\mu_s^{\mathcal{L}} - \frac{g^{\uparrow\downarrow}(\mu_s^{\mathcal{L}} + \mu_s^{\mathcal{R}})}{2g^{\uparrow\downarrow} + 4\pi s\alpha L}\right] - \frac{\alpha g^{\uparrow\downarrow}(\mu_s^{\mathcal{L}} + \mu_s^{\mathcal{R}})\left(2n_c + \frac{3}{\Lambda_{th}^3}\right)}{2g^{\uparrow\downarrow} + 4\pi s\alpha L}\, x, \tag{3.58}$$

$$J_e(x) = 0. \tag{3.59}$$

Note that this spin current is equal to the one that we found for a ferromagnetic insulator in the superfluid state. It follows that the spin current is zero in the absence of spin accumulation in the boundaries, i.e. for $\mu_s^{\mathcal{L}} = \mu_s^{\mathcal{R}} = 0$.

### 3.2.3 Seebeck coefficients

In both the normal and the superfluid state, we found that $J_e(x) = 0$ everywhere in our conductor. This is not surprising, since we chose our boundary conditions as $J_e^{\mathcal{L}} = J_e^{\mathcal{R}} = 0$. However, the absence of an electric current does not mean that there is no electro-chemical potential. We find

$$\mu_{e,n}(x) = -\frac{L'_D}{\sigma}\left(C_1 e^{\frac{x}{\ell_2}} + C_2 e^{-\frac{x}{\ell_2}}\right) + C_3\, x, \tag{3.60}$$

$$\mu_{e,s}(x) = \frac{\gamma\alpha \left(2n_c - \frac{3}{\Lambda_{th}^3}\right)}{2\,(\hbar\sigma - \gamma L_D)}\, \delta\mu\, x^2 + C_5\, x, \tag{3.61}$$

where $\mu_{e,n}$ and $\mu_{e,s}$ denote the electro-chemical potential in the normal state and in the superfluid state respectively. From these expressions we can calculate the Seebeck coefficient for both states:

$$S_n = -\frac{\mu_{e,n}(L) - \mu_{e,n}(0)}{T(L) - T(0)} = \frac{L'_D}{\sigma}\left(\frac{C_1}{\Delta T}\left(e^{\frac{L}{\ell_2}} - 1\right) + \frac{C_2}{\Delta T}\left(e^{-\frac{L}{\ell_2}} - 1\right)\right) - \frac{C_3}{\Delta T}\,L, \tag{3.62}$$

$$S_s = -\frac{\mu_{e,s}(L) - \mu_{e,s}(0)}{T(L) - T(0)} = -\frac{\gamma\alpha\left(2n_c - \frac{3}{\Lambda_{th}^3}\right)}{2\left(\hbar\sigma - \gamma L_D\right)}\,\frac{\delta\mu}{\Delta T}\,L^2 - \frac{C_5}{\Delta T}\,L. \tag{3.63}$$

We already derived the constants $C_{1,2,3,5}$ and $\delta\mu$ with the boundary conditions for $J_s$ and $J_e$ above. It follows that spin superfluidity in an easy-plane magnetic conductor can be measured as a change in the Seebeck voltage, and also that spin superfluidity results in a qualitative change of the dependence of this voltage on the system size.

# Chapter 4

# Conclusion

In this Thesis, we extensively studied spin currents in easy-plane ferromagnetic insulators and conductors and developed their hydrodynamic description. In insulators we found the dynamics of magnons to be described by Eq. (2.48 - 2.50) in combination with Eq. (2.26). In conductors, however, we had to take electron-magnon interactions into account. Doing so, we obtained an extra hydrodynamic equation as Eq. (3.40) with the magnon and electron currents given by Eq. (3.38) and Eq. (3.39). These are the central results of this Thesis.

We also made explicit calculations of spin currents and electric currents in both these materials under the assumption of a linear thermal gradient. Here we distinguished between the normal state and the superfluid state. An interesting result is that, in the superfluid state, there is no difference between the total spin current in conductors and in insulators. We also calculated the electric voltage and the associated Seebeck coefficient that the linear thermal gradient induces in easy-plane magnetic conductors in the normal state and in the superfluid state.

To find the explicit expressions of the magnon and the electron currents, we solved the hydrodynamic equations. In these calculations we worked in the linear-response regime. Of course, these approximations slightly misrepresent the reality. In future research the results could be obtained more precisely by solving the Boltzmann equation directly. A disadvantage of this approach is that the boundary conditions, that we used in this Thesis, would no longer be appropriate. Therefore, a new set of boundary conditions would need to be derived.

So far, spin superfluidity has been a purely theoretical phenomenon. Future experimental research could use our results to confirm the existence of this state of matter, as we found substantial changes in, for example, the voltage between the normal and the superfluid state. Nevertheless, a problem arises: we assumed a linear thermal gradient and started working in one dimension. Experiments concerning temperature gradients almost always take place in two or three dimensions, so for that purpose the calculations in Sections 2.3 and 3.2 would need to be redone, using a more realistic temperature gradient.

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