Methods For The Renormalization Of The Top Quark

Master Thesis

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June 30, 2016

Abstract

The (automated) multiloop renormalization of the top quark mass and field in quantum chromodynamics is discussed, which involves calculating fermion propagator type Feynman diagrams. First a theoretical description is given of renormalization at multiloop order, after which the building blocks are discussed of an automated setup which calculates Feynman diagrams contributing to the quark propagator. Large sets of Feynman integrals are reduced to a small set of so-called master integrals using integration-by-parts identities.

The alpha-parametrization is introduced as a powerful way of turning Feynman integrals into scalar integrals. Mathematical techniques are studied to evaluate alpha-parametrized integrals either analytically or as Laurent series around $d = 4$ in terms of ϵ ($d = 4 - 2\epsilon$). To derive a Laurent series of a divergent integral one first has to do a finite integral expansion to rewrite the divergent integral in terms of finite integrals, which is possible for on-shell integrals. The poles of the divergent integral are captured in the coefficients in front of the finite integrals. Care has to be taken that the finite integral expansion does not contain 'spurious poles', which are poles that drop out due to non-trivial cancellations of the coefficients of the power series in ϵ of the finite integrals.

A new method called the 'projective trick' is introduced for deriving finite integral expansions, and rules of thumb are developed to derive finite integral expansions without spurious poles for the 2- and 3-loop on-shell integrals with one mass scale considered in this thesis. These finite integrals are then expanded as power series up to the desired order in ϵ, and the series coefficients are evaluated. It was possible for almost all integrals to evaluate the coefficients exactly using the Maple package HyperInt [1]. Alternatively, the coefficients can be calculated with a numerical integrator.

Acknowledgements

I would like to express my gratitude to my supervisor prof. dr. Eric Laenen for his insightful explanations and encouragement and for providing me with the freedom to explore topics in depth when they captured my interest. Furthermore I want to thank dr. Franz Herzog who was always available for discussions and who pointed me to the right papers multiple times. I also thank him for introducing me to much of the mathematics of Feynman integrals, on which I spent the last months of my thesis. Lastly my thanks go out to my parents for supporting me to make the most of my education.

Contents

I. Introduction

The Standard Model contains numerous free parameters for which numerical values cannot be derived within the theory itself, e.g. mass parameters and coupling constants. Only when comparing the predictions of the theory for certain processes to experimental results found in large colliders can these parameters be fitted to their experimental value. This in turn allows the theory to make predictions about different processes using the previously fitted values for the parameters. If the theory is consistent it should not matter which scattering process is used to measure the parameters in the Standard Model as the result should be the same.

Furthermore, any deviations in measurements or experimental observations which are not theoretically predicted give rise to new physics (defined as not yet described by the Standard Model). Hence to check consistency of the theoretical framework, or to test whether new physics is at play, high precision experiments and calculations are needed to check the significance of the measurements. In a sense the focus in this thesis is on making higher precision theoretical calculations, which can then be compared to high precision measurements later on. Calculations in quantum field theories are done perturbatively in the coupling constants and the order of the perturbative calculation increases with the number of loops in Feynman diagrams that describe interactions between particles. Calculating past leading order allows for higher precision calculations, but the calculations themselves become increasingly difficult to do.

Additional difficulties in quantum field theories are that they need to be renormalized to get finite results. When working with the 'bare' theory, which is the most straightforward theory that satisfies the required symmetries observed in nature, loop diagrams give divergent integrals. These infinities can be absorbed within the parameters in the theory in a consistent manner so that predictions for physical processes are finite. The process of making the theory finite is called renormalization, leading to the renormalized theory. Hence higher order perturbative calculations do not only involve calculating multiloop Feynman diagrams, but also require a proper bookkeeping to subtract infinities from these multiloop Feynman diagrams. The essence of the approach in this thesis is to discuss multiloop renormalization and to calculate renormalization constants in an automated manner, which then leads to a finite theory for which multiloop calculations can be done.

Because calculations in the Standard Model become so involved it is common to focus on subsets of the Standard Model. This thesis will discuss in particular the renormalization of the top quark (meaning the renormalization of the top quark mass and field) within pure quantum chromodynamics (excluding the rest of the Standard Model). This will involve calculating multiloop 2-point Feynman diagrams and doing the multiloop renormalization. Many techniques and concepts involved in that are also applicable to renormalization of other parameters in for example the complete Standard Model. For this reason the treatment will be tried to kept general wherever possible. The renormalization of the top quark is considerably more difficult than the renormalization of a simple scalar field 'toy-model' Lagrangian, but not so involved as to require a team of people to work on each Feynman diagram. A full renormalization will be done up to next-to-next-to leading order (NNLO), meaning up two loop Feynman diagrams. Furthermore parts of the N^3LO renormalization will be done.

A method will be presented for rewriting on-shell Feynman integrals into linear combinations of integrals which are finite as $\epsilon \to 0$, where $d = 4 - 2\epsilon$ is the complexified dimension in dimensional regularization. It turns out that in doing this one has to avoid creating so-called 'spurious poles' which are non-trivial integrals which evaluate to zero but show up in the Laurent expansion of the Feynman integral. Rules of thumb will be developed which generally seem to allow one to do finite integral expansions without spurious poles and with a manageable size of

finite integrals. The strength of a finite integral expansion lies in the fact that one can create power series of the finite integrals by expanding the integrand in ϵ. The series coefficients can then be evaluated numerically using general purpose numerical integration software up to the desired order in ϵ, and often also exactly using HyperInt [1].

II. Summary of quantum chromodynamics and conventions

In this section a short introduction is given to quantum chromodynamics (QCD). This introduction is aimed at the reader who is already familiar with non-abelian gauge theories like QCD. Some topics will be treated more extensively than others, especially those to which a typical graduate student might still have been underexposed. Furthermore the mathematical conventions used for the rest of the text will be established.

Quantum chromodynamics is the theory which describes the strong force. It is a non-abelian gauge theory with gauge group SU(3). The gluons belong to the bosonic gauge fields, and the quarks to the fermionic fields. Often the gauge group of QCD is generalized to SU(N_c) for arbitrary $N_c \in \mathbb{N}_{\geq 1}$. This will also be the case for the rest of this text. Every quark is then a SU(N_c) multiplet. The SU(N_c) index of the quark fields is sometimes called the color charge. Color charges are conserved, so a quark never changes color. There are $N_A = N_c^2 - 1$ gluon fields, as gluons live in the adjoint representation of the gauge group which has dimension $N_c^2 - 1$.

There are six quark flavours, historically named u (up), d (down), s (strange), c (charm), b (bottom), t (top). The top quark has a mass of 173.34 ± 0.76 GeV and is much heavier than the other quarks [2]. Because quarks are fermions there are 6 corresponding anti-quarks which have the same properties as the quarks but have opposite charges. The basic QCD Lagrangian has the typical form of a non-abelian gauge theory (a Yang-Mills theory):

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}_{f,i}(i\not{D}_{ij} - m_f\delta_{ij})\psi_{f,j} - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a \tag{1}$$

The index f sums over the 6 quark flavours (u, d, t, b, c, s), and the indices i, j are SU(N_c) indices. When not an index i refers to the imaginary constant. The covariant derivative is given by $D_{\mu,ij} \equiv \partial_\mu\delta_{ij} + igA^a_\mu T^a_{ij}$, with T^a_{ij} denoting the generators of the Lie-algebra su(N_c) in the fundamental representation and A^a being the gluon fields. The term $G^a_{\mu\nu}$ is the nonabelian field strength defined by $T^a G^a_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$. This leads explicitely to the form $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu$. The QCD Lagrangian written above comes with a certain set of conventions, which are made explicit next.

Coupling constant One is free to pick a different sign for the coupling constant g, and in some texts the sign differs from ours. This is not of much theoretical importance. As long as one is consistent within the theory itself (- note the definitions of the covariant derivative D_μ and field strength tensor G depend on the sign -) and takes care to check the definition of other authors when comparing results, no problems will arise. For example, some of the notation used in this thesis is inspired by ref. [3], but in this thesis the coupling constant is defined with a minus compared to the book mentioned. Sometimes the notation $\alpha_s \equiv \frac{g^2}{4\pi}$ is used in power series of the (top)-quark self-energy.

Metric signature and Dirac matrices The metric signature used in this thesis is $(+1, -1, -1, -1)$, i.e. the mostly minus convention. Furthermore, the following anticommutation relation is taken

for the Dirac matrices:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{2}$$

The Dirac equation is:

$$(i\partial\!\!\!/ - m)\psi = 0 \tag{3}$$

The quark propagator is proportional to the Green's function of the Dirac equation and takes the following form using our conventions:

$$S_{0,ij}(k\!\!\!/, m) = {}_i \xleftarrow{\quad k \quad} {}_j = \frac{i\delta_{ij}}{k\!\!\!/ - m} \tag{4}$$

(An additional factor $+i\epsilon$ should be added to the divisor to be rigorous, which is discussed later.) The quark propagator should have a pole for on-shell momenta. This means that $k^2 = m^2$ in the mostly minus convention. The commutation relation of eq. 2 gives $k\!\!\!/k\!\!\!/ = k^2$. This shows that:

$$S_{0,ij}(k\!\!\!/, m) = \frac{i\delta_{ij}}{k\!\!\!/ - m} \cdot \frac{k\!\!\!/ + m}{k\!\!\!/ + m} = i\delta_{ij}\frac{k\!\!\!/ + m}{k^2 - m^2} \tag{5}$$

Thus there is a pole for on-shell momenta. In other texts the metric might be defined with the mostly plus convention, in which case the on-shell condition is $k^2 = -m^2$. This alters the preceding definitions, so care has to be taken when comparing to the literature.

The su(N_c) color algebra The special unitary group, denoted SU(N_c), is a compact Lie group of $N_c \times N_c$ unitary matrices with determinant 1. Being a compact Lie group it has a Lie algebra associated to it, which will be denoted su(N_c). The basis elements of su(N_c) are denoted by T^a_{ij} and are Hermitian and traceless matrices, which satisfy the following commutation relations:

$$[T^a, T^b] = if^{abc}T^c \tag{6}$$

$$\{T^a, T^b\} = \frac{1}{3}\delta^{ab} + d^{abc}T^c \tag{7}$$

where the numbers f^{abc} are called the structure constants and the numbers d^{abc} are the symmetric symbols. For $N_c = 3$ it is often picked that $T^a = \lambda^a/2$ where the matrices λ^a are the Gell-Mann matrices. Consider a set of infinitesimal real parameters θ^a. The expression $U = 1 + i\theta^a T^a$ is an element of SU(N_c) up to order $\mathcal{O}(\theta^2)$:

$$UU^* \approx 1 + i\theta^a T^a - i\theta^a T^a + \mathcal{O}(\theta^a\theta^b) = 1 \tag{8}$$

$$\det(U) = e^{\mathrm{Tr}(\log(U))} = e^{\mathrm{Tr}(\log(1+i\theta^a T^a))} \approx e^{\mathrm{Tr}(i\theta^a T^a)} = e^0 = 1 \tag{9}$$

One can create general elements in the group by making an infinite series of infinitesimally small steps away from the identity element. Pick θ^a finite this time. Then one can write:

$$U(\theta^a) \equiv \lim_{n\to\infty}\left(1 + \frac{i\theta^a T^a}{n}\right)^n = e^{i\theta^a T^a} \tag{10}$$

Because SU(N_c) is a compact group the elements of eq. 10 cover the full group, which is motivated by observing that the number of generators T^a equals the dimension of SU(N_c):

- Using complex numbers writing down a general $N_c \times N_c$ matrix gives $2N_c^2$ free real parameters. The fact that matrices U in the group are unitary gives $U \cdot U^* = 1$, amounting to N_c^2 distinct equations, so one ends up with N_c^2 degrees of freedom. Fixing the determinant lastly leaves $N_c^2 - 1$ degrees of freedom.
- Similarly, fixing a general matrix to be hermitian means we have the same freedom as a real matrix, i.e N_c^2 real parameters. Lastly, tracelessness fixes one entry on the diagonal, which is real, so the degree of freedom is $N_c^2 - 1$ as well.

The elements T^a are called the infinitesimal generators of SU(N_c). In the evaluation of Feynman diagrams one will encounter contractions of structure constants and generators due to their presence in the Feynman rules of the vertices. The evaluation of these 'color factors' will be left to the algorithms developed in [4] of which an implementation exists in FORM. These algorithms make it possible to do a group independent reduction of group theory factors of Feynman diagrams in terms of group invariants. Explicit results for SU(N_c) color factors in the fundamental representation can be found afterwards by plugging in the invariants.

In the color factors of the Feynman diagrams that we will consider we will encounter the following terms:

$$C_F \delta_{ij} \equiv T^a_{ik} T^a_{kj} \qquad C_A \delta_{ab} \equiv f^{acd} f^{bcd} \qquad T_F \equiv \frac{N_c C_F}{N_c^2 - 1} \tag{11}$$

With the most common choice of generators $T^a = \lambda^a/2$ in the case $N_c = 3$ one has explicitely:

$$C_F = \frac{4}{3} \qquad C_A = 3 \qquad T_F = \frac{1}{2} \tag{12}$$

Feynman rules and gauge fixing The Feynman rules used in this thesis are taken from appendix D of ref. [3], but differ by a minus sign in the coupling constant. The resulting Feynman rules are stated in appendix B. These Feynman rules include a particular choice of gauge, explained next.

We cannot perform perturbation theory with the Lagrangian of eq. 1 without adding a gauge fixing term, because it is impossible to define the propagator for the gluon field otherwise. (The quadratic part of the gluon fields in the action has a zero eigenvector in momentum space, so it can not be inverted.) It should not matter how we pick the gauge, as we expect physical quantities to be gauge independent. However some choices of gauge can be more convenient than others, depending on the calculation that is done. In this thesis we pick the Lorentz gauge, sometimes also called the covariant gauge. With this choice of gauge the theory remains Lorentz invariant. But this choice of gauge makes it necessary to add a Faddeev-Popov ghost term to the Langrangian. The QCD-Lagrangian in the covariant gauge becomes:

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}_{f,i}(i\not{D}_{ij} - m_f \delta_{ij})\psi_{f,j} - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 + (\partial^\mu (\chi^a)^*) D^{ab}_\mu \chi^b \tag{13}$$

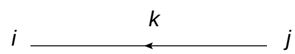
This result comes from ref. [3], page 83. with a sign flip on the coupling constant, and some small changes of notation. D^{ab}_μ refers to the covariant derivative in the adjoint representation:

$$D^{ab}_\mu = \delta^{ab}\partial_\mu + g f^{abc} A^c_\mu \tag{14}$$

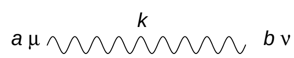
The parameter α is the gauge parameter. It can still be picked freely, so that we have a family of covariant gauges. For calculations by hand it is convenient to pick $\alpha = 1$, sometimes called the Feynman gauge choice. This choice rests on the assumption that the end result of a calculation

should be gauge-independent. In principle physical amplitudes should be gauge independent. However, leaving the gauge parameter in a calculation is a good check to see if the calculation went well. If it hasn't dropped out, a mistake might have been made. Furthermore, renormalization constants can depend on the gauge parameter.

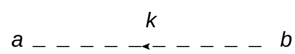
Feynman propagator prescription and Wick rotation The propagators in appendix A should actually have a factor $+i\epsilon$ included with $\epsilon > 0$ in the following ways:



$$= i\delta_{ij}\frac{\not{k} + m}{k^2 - m^2 + i\epsilon} \tag{15}$$



$$= -i\delta_{ab}\frac{d^{\mu\nu}(k)}{k^2 + i\epsilon} \tag{16}$$



$$= i\delta_{ab}\frac{1}{k^2 + i\epsilon} \tag{17}$$

This $+i\epsilon$ convention indicates we are calculating 'Feynman' propagators, meaning that the propagators correspond to the vacuum expectation value of a time-ordered product of the fields. This is quite extensively covered in (introductory) QFT courses and will not be further discussed here. There are nonetheless some quick ways to motivate adding this factor.

Firstly in the path-integral formulation one can look at the path-integral and note that it does not converge unless a factor $i\epsilon$ is added at the right place in the quadratic part of the fields in the action. This results in the factors $+i\epsilon$ in the propagators. Such an explanation is given in ref. [5].

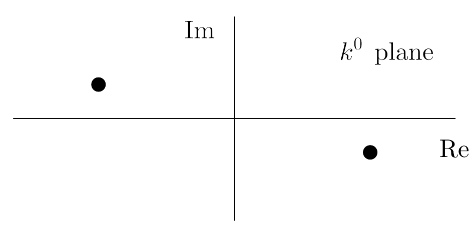
A more relevant observation for this thesis is that this factor is needed to properly define Feynman integrals, or in particular just the Fourier transform of a propagator in momentum space. The definition of the Fourier transform in this thesis is:

$$\phi(x) = \int \frac{d^d k}{(2\pi)^d}\phi(k)e^{-ikx} \tag{18}$$

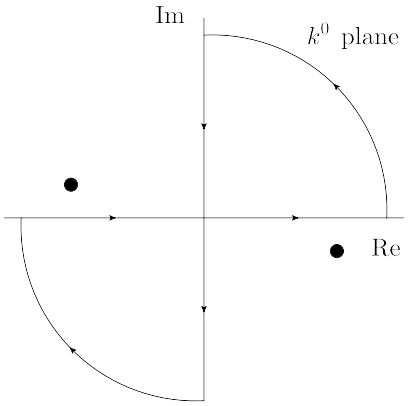
This form is needed for consistency with the Feynman rules that are used. In particular, applying the Dirac equation to the quark propagator in position space, expressed as an integral over the momentum representation, shows that the propagator is its Green's function. For this to work the exponent in the above Fourier transform should be $-ikx$ instead of $+ikx$. For the quark propagator, without the $i\epsilon$-prescription, one finds that the Fourier transform becomes:

$$S_0(x,m) = \int \frac{d^d k}{(2\pi)^d}\frac{ie^{-ikx}}{\not{k} - m} = \int \frac{d^d k}{(2\pi)^d}\frac{i(\not{k} + m)e^{-ikx}}{k^2 - m^2} \tag{19}$$

Taking a closer look at this expression it is seen that it is not well-defined, because there are poles at $(k^0)^2 - (k^i)^2 = m^2$. After adding the factor $+i\epsilon$ the poles are at the following positions in the k^0-plane:

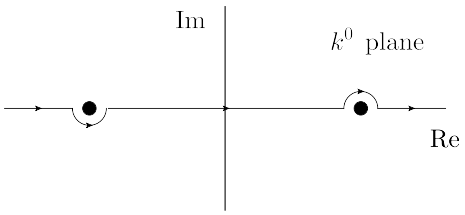


This makes it possible to evaluate integrals using a Wick rotation. A Wick rotation can be done by observing the following integration contour, in the limit $R \to \infty$ with R being the radius of the circle arcs:



Since the poles are not in the interior of the integration path the integral over this path should be zero, by Cauchy's theorem. Furthermore, the integrals over the circle's arcs go to zero for $R \to \infty$ rapidly because the exponential function will get a negative argument that scales with R on these arcs. Hence the integral on the real axis and on the imaginary axis can be identified. But the latter integral is over an Euclidean measure and therefore one can use standard Euclidean intuition to evaluate it. Furthermore, the poles do not play an important role anymore in the Wick-rotated integral so that the limit $\epsilon \to 0$ can be taken in that integral.

Adding the terms $+i\epsilon$ to the propagators is more easily thought of as an integration convention. In particular consider the following integration path:



When taking the limit $\epsilon \to 0$ the integral over this path is the same as the integral with the poles themselves shifted. This is seen by again doing a Wick rotation and noting that this integral is also equal to the Wick rotated one, where the integration is on the imaginary axis. In the rest of the thesis we will therefore not put the terms $+i\epsilon$ in the propagators of the integrals. Instead every integral in Minkowski space over an internal momentum parameter is implicitely understood to take the above path: One integrates underneath the left pole and above the pole on the right. The variable ϵ will in the future refer to a parameter that comes up in the dimensional regularization procedure.

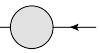
The gluon and ghost propagators are massless but the $+i\epsilon$ pre-scription still enforces two distinct poles in the k^0-plane strictly in the upper left and lower right quarters, even if the spatial part of the momentum is zero.

Lastly, note that in Feynman integrals the propagators contain linear combinations of internal and external momenta. In those cases the poles take the same positions but in the plane of the zeroth component of the composite momentum.

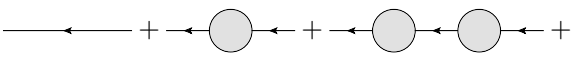
The full quark propagator The full quark propagator is defined as the quark propagator after including all quantum interactions. That means one has to theoretically calculate all quark 2-point diagrams for any number of loops to find the full quark propagator. This calculation is simplified

in a perturbative setting by recognizing that all such diagrams are expressed in the geometric series of so-called one particle irreducible diagrams.

By definition a diagram is one particle irreducible (1PI) if it can't be split in two by removing a single internal line. The observation can then be made that all one particle reducible diagrams can be composed from 1PI-diagrams. In particular, let the self-energy $-i\Sigma(\not{p}, m)$ represent the sum of all 1PI-diagrams, which we can diagrammatically represent in the following way:

$$-i\Sigma(\not{p}, m) =$$  (20)

The notation $\Sigma(\not{p}, m)$ indicates explicitely the dependence of the self-energy on the quark mass parameter and the external momentum. This will become important later on when renormalizing the theory, but is not always stated in this section. Note that, by looking at the Feynman rules, it is clear that the self-energy only depends on p^μ in the form $\not{p} = \gamma^\mu p_\mu$ (understanding that $p^2 = \not{p}\not{p}$). The following result holds for the full quark propagator, which is sometimes referred to as a Dyson sum:

$$S_{\text{full}}(\not{p}, m) =$$  (21)

The main motivation behind this expression is that all 1-particle reducible diagrams are composed of connected 1-particle irreducible diagrams. Note that diagrams with tadpoles are 1-particle reducible but are not included in the above sum. However, tadpole diagrams evaluate to zero.

One can recognize a geometric series in the diagrammatic formula stated above. This result is stated formulaically below. In doing this we don't write out color and Dirac indices, but it should be kept in mind we are dealing with matrices. The identity matrices in color space and Dirac space are thus implicitely added to scalar quantities.

$$\begin{aligned} S_{\text{full}} &= S + S(-i\Sigma)S + S(-i\Sigma)S(-i\Sigma)S + \dots \\ &= S\left(1 + (-i\Sigma)(S + S(-i\Sigma)S + S(-i\Sigma)S(-i\Sigma)S + \dots)\right) \\ &= S(1 + (-i\Sigma)S_{\text{full}}) \end{aligned} \tag{22}$$

$$\Rightarrow S_{\text{full}} = (S^{-1} + i\Sigma)^{-1} = (-i(\not{p} - m - \Sigma))^{-1} = \frac{i}{\not{p} - m - \Sigma} \tag{23}$$

In this notation a δ_{ij} is thus implicit ($S_{\text{full},ij} \propto \delta_{ij}$). The fractional notation that is used refers to an inverse matrix and $i/(\not{p} - m - \Sigma)$ is treated as a noncommutative quantity, because it is not proportional to the identity matrix in Dirac space.

The self-energy $\Sigma = \Sigma(\not{p}, m)$ can be decomposed in a Lorentz invariant manner. Two forms of decompositions are commonly seen:

$$\Sigma(\not{p}, m) = A(p^2, m^2)m + B(p^2, m^2)\not{p} \tag{24}$$

$$\Sigma(\not{p}, m) = \Sigma_1(p^2, m^2)m + (\not{p} - m)\Sigma_2(p^2, m^2) \tag{25}$$

They are clearly related by:

$$A(p^2, m^2) = \Sigma_1(p^2, m^2) - \Sigma_2(p^2, m^2) \tag{26}$$

$$B(p^2, m^2) = \Sigma_2(p^2, m^2) \tag{27}$$

The full gluon propagator Let $\mathrm{Gl}^{ab}_{\mu\nu}(p)$ denote the gluon propagator, and $\Pi^{ab}_{\mu\nu}$ the gluon self energy. The full gluon propagator is found by the geometric sum of the gluon self energy:

$$\begin{aligned}\mathrm{Gl}_{\text{full}} &= \mathrm{Gl} + \mathrm{Gl}(-i\Pi)\mathrm{Gl} + \dots \\ &= \mathrm{Gl}(1 + (-i\Pi)\mathrm{Gl}_{\text{full}}) \qquad (28)\\ \Rightarrow \mathrm{Gl}_{\text{full}} &= (\mathrm{Gl}^{-1} + i\Pi)^{-1} \qquad (29)\end{aligned}$$

The inverse gluon propagator can be found by inverting the propagator which is stated in the Feynman rules in app. B, or alternatively one identifies the quadratic matrix of the gluon field in the QCD Lagrangian, denoted K_A:

$$\mathcal{L}_{\text{quad. in A}} = \frac{1}{2} A^{\mu,a} K^{ab}_{A,\mu\nu} A^{\nu,b} \qquad (30)$$

It then follows from eq. 13 that:

$$K^{ab}_{A,\mu\nu} = \delta^{ab}\left(\eta^{\mu\nu}\Box - \partial^\mu\partial^\nu\left(1 - \frac{1}{\alpha}\right)\right) \qquad (31)$$

with $\Box = \partial^\mu\partial_\mu$. Taking the Fourier transform and putting the right prefactor gives the inverse gluon propagator in momentum space:

$$\mathrm{Gl}^{-1,ab}_{\mu\nu} = i\delta^{ab}\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\left(1 - \frac{1}{\alpha}\right)\right) \qquad (32)$$

The gluon self-energy satisfies a Ward-Takahashi identity (see ref. [3]) so that:

$$\begin{aligned}k^\mu k^\nu \Pi^{ab}_{\mu\nu}(k) &= 0 \qquad (33)\\ \Rightarrow \Pi^{ab}_{\mu\nu}(k) &= \delta_{ab}(k_\mu k_\nu - k^2\eta_{\mu\nu})\Pi(k^2) \qquad (34)\end{aligned}$$

Therefore it follows that:

$$(\mathrm{Gl}^{-1} + i\Pi)^{ab}_{\mu\nu} = i\delta^{ab}\left(\eta_{\mu\nu}k^2(1 - \Pi(k^2)) - k_\mu k_\nu\left(1 - \Pi(k^2) - \frac{1}{\alpha}\right)\right) \qquad (35)$$

The inverse of this matrix gives $\mathrm{Gl}_{\text{full}}$, leading to:

$$\mathrm{Gl}^{ab}_{\text{full},\mu\nu} = -i\frac{\delta^{ab}}{k^2}\left(\frac{\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}}{1 - \Pi(k^2)} + \alpha\frac{k_\mu k_\nu}{k^2}\right) \qquad (36)$$

Hence, only the transversal part of the gluon propagator is subject to higher-order corrections.

III. Dimensional regularization

In a quantum field theory it is generally not possible to find nonperturbative solutions to physical observables. For this reason they are calculated as perturbative series in terms of the coupling constants. It turns out that the coefficients in these expansions contain divergent integrals, which have to be dealt with.

To isolate divergences in a systematic fashion a regularization scheme is employed. The most successful one is dimensional regularization. The method was introduced by 't Hooft and Veltman (ref. [6]) and using it they showed that Yang-Mills theories with massive fields due to

spontaneous symmetry breaking are renormalizable, which led to their Nobel prize in physics in 1999.[1] Dimensional regularization has the advantage of preserving Lorentz invariance, and allows for precise analytical calculations. It is built on the observation that Feynman integrals which diverge in 4 dimensions can in fact converge for different dimensions. The idea is therefore to generalize Feynman integrals to arbitrary complex dimensions, by taking the analytic continuation of the results for integer dimensions. The dimension will be written as $d = 4 - 2\epsilon$, where ϵ is a complex number that gives the deviation from the true dimension.

Consider the integral in eq. 37, with d^dK being an Euclidean integral measure. (Lorentz measures in Feynman integrals can be made Euclidean by doing a Wick rotation.) One can observe the integral has a spherical symmetry. Changing variables to hyperspherical coordinates, the measure d^dK splits in an angular integral and a radial integral together with a factor K^{d-1}. Focus for now on the radial integral:

$$\int \frac{d^dK}{(2\pi)^d} \frac{1}{(K^2+m^2)^2} \propto \int_0^\infty dK \frac{K^{d-1}}{(K^2+m^2)^2} \tag{37}$$

If one calculates the indefinite integral close to infinity one has $K^2 \gg m^2$, so that the indefinite integral goes like $\log(K)$, which means the integral diverges logarithmically at infinity. However, for $d = 3$ we do have convergence, since the indefinite integral goes like $1/K$ at infinity which goes to zero. The idea is therefore to solve the integral for a region of d where the integral converges and to take the analytic continuation of this result. The divergence just seen is called an ultraviolet (UV)-divergence, which is a divergence that comes from the high-momentum region in the integral. Let's generalize the integral in eq. 37 by putting an arbitrary power $\lambda \in \mathbb{R}_{>0}$ on the propagator. We first pick d and λ so that $\lambda > d/2$, which makes the radial integral convergent.

$$\int \frac{d^dK}{(2\pi)^d} \frac{1}{(K^2+m^2)^\lambda} = \frac{1}{(2\pi)^d} \int d^{d-1}\Omega \int_0^\infty dK \frac{K^{d-1}}{(K^2+m^2)^\lambda} \tag{38}$$

The angular integral is just the area of a unit d-ball / the volume of the $d-1$ sphere, which gives:

$$\int d^{d-1}\Omega = \frac{2\pi^{\frac{d}{2}}}{(\frac{d}{2}-1)!}, \tag{39}$$

for integer d. We would like to generalize the result to arbitrary $d \in \mathbb{C}$ in some way. We can't geometrically define a sphere for non-integer dimension, but we can find a complex analytic function that generalizes the above integral to arbitrary values of d. This is done by replacing the factorial with the Gamma function (see eq. 418), which is the unique analytic extension of the factorial function. As a prescription that is part of the dimensional regularization procedure we will define the spherical integral for arbitrary dimensions to be this function. Thus we define:

$$\int d^{d-1}\Omega = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{40}$$

The (radial) K-integral can be evaluated in terms of the Euler Beta-function using eq. 430. We

[1] More specifically, the prize was awarded for 'elucidating the quantum structure of electroweak interactions in physics'.

work out the (radial) K integral to find:

$$\begin{aligned}\int_0^\infty dK \frac{K^{d-1}}{(K^2+m^2)^\lambda} &= (m^2)^{-\lambda} \int_0^\infty dK \frac{K^{d-1}}{\left(\frac{K^2}{m^2}+1\right)^\lambda} \\ &= \frac{1}{2}(m^2)^{d/2-\lambda} \cdot 2 \int_0^\infty dK \frac{(K^2)^{\frac{d-1}{2}}}{\left(K^2+1\right)^\lambda} \\ &= \frac{1}{2}(m^2)^{d/2-\lambda} B\big(d/2, \lambda - d/2\big) \text{ (for } \lambda > d/2) \end{aligned} \tag{41}$$

Putting all elements together we have:

$$\int \frac{d^dK}{(2\pi)^d} \frac{1}{(K^2+m^2)^\lambda} = \frac{B(d/2, \lambda - d/2)}{(4\pi)^{d/2}\Gamma(d/2)} (m^2)^{d/2-\lambda} \text{ (for } \lambda > d/2) \tag{42}$$

$$= \frac{\Gamma\left(\lambda - \frac{d}{2}\right)(m^2)^{d/2-\lambda}}{(4\pi)^{d/2}\Gamma(\lambda)} \tag{43}$$

Repeating the practice of analytically continuing results, eq. 42 can be extended to a larger domain $\lambda \in \mathbb{C}$, $d \in \mathbb{C}$. There are then poles at $\lambda - d/2 \in \mathbb{Z}_{\leq 0}$ and zeros at $\lambda \in \mathbb{Z}_{\leq 0}$, unless these conditions hold at the same time in which case the result is finite. Returning to the original example, let $\lambda = 2$ and $d = 4 - 2\epsilon$. One sees:

$$\begin{aligned}\int \frac{d^dK}{(2\pi)^d} \frac{1}{(K^2+m^2)^2} &= \frac{B(2-\epsilon, \epsilon)}{(4\pi)^{2-\epsilon}\Gamma(2-\epsilon)} (m^2)^{-\epsilon} \\ &= \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} (m^2)^{-\epsilon} \\ &\approx \frac{1}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma_E - \log\left(\frac{m^2}{4\pi}\right)\right) + \mathcal{O}(\epsilon) \end{aligned} \tag{44}$$

The following expansions are used in the last step:

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \tag{45}$$

$$x^\epsilon = e^{\epsilon \log(x)} \approx 1 + \epsilon \log(x) + \mathcal{O}(\epsilon^2) \tag{46}$$

The result in eq. 44 has a pole $1/\epsilon$. Hence in the limit $\epsilon \to 0$ we have $d \to 4$, but we also get back the original divergence. However, we gained an explicit characterization of the UV-divergence. It turns out that in so-called renormalizable quantum field theories the Lagrangian can be altered so that the divergent terms proportional to powers of $1/\epsilon$ are cancelled out in the calculations for certain quantities, especially those that correspond to observables. That part will be played by setting up a renormalized Lagrangian, treated in the next section.

Now that we decide to keep $d \in \mathbb{C}$ a point of concern is how to pick the Dirac matrices, a complication that was left out of the previous example. Instead of interpreting them just as matrices we can consider them as objects that satisfy particular (commutation)-relations. We need these relations to be analogous to the ones that we usually use, and reduce to the usual relations for $d = 4$. As an example we might pick $\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 2^{d/2}\eta_{\mu\nu}$. But equally valid is $\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 4\eta_{\mu\nu}$.

In this thesis the following choices are made:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{47}$$

$$\gamma_\mu \gamma^\mu = d \tag{48}$$

$$\gamma_\mu \gamma_\nu \gamma^\mu = (2-d)\gamma_\nu \tag{49}$$

$$Tr[\gamma_\mu \gamma_\nu] = 4\eta_{\mu\nu} \tag{50}$$

These choices were made in accordance with ref. [3]. These conventions can affect the finite result that is obtained from dimensional regularization, and therefore it is important to be consistent. For example, suppose one uses that $\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 2^{d/2}\eta_{\mu\nu}$. Expanding this expression on ϵ gives a power series in ϵ. If from some other term there is an additional factor $1/\epsilon$, it becomes clear the finite part of the expansion is different from when the choice is $\mathrm{Tr}(\gamma_\mu \gamma_\nu) = 4\eta_{\mu\nu}$.

A last observation is that the coupling constant gains a dimensional dependence in the dimensional regularization scheme. First note that the action, $S = \int d^d x\, \mathcal{L}_{\mathrm{QCD}}(\vec{x}, t)$ should be of zero (mass) dimension, (for example because it appears in the exponent of the Feynman path integral.) Let $[\cdot]$ denote the mass dimension of some expression denoted with the dot. One has $[\mathcal{L}] = d$. From looking at the kinetic terms in eq. 1 it is seen that $[\psi_{f,i}] = 3/2 - \epsilon$ and $[A^a] = 1 - \epsilon$. In particular, together with the definition of g in the covariant derivative it becomes clear that $[g] = \epsilon$. It is reminded that the main procedure of quantum field theory is to expand observables in perturbative series of coupling constants. For such an expansion to make sense the coupling constant should be dimensionless. Therefore when doing dimensional regularization we will replace the coupling constant g by $\mu^\epsilon g$, such that g is indeed dimensionless, and the dimension is captured in a scale parameter μ. It will not be put explicitely in the Lagrangian, but will be added at the end of a calculation. The integral of eq. 44 can come from a one-loop diagram, which will have two vertex factors and thus be multiplied by $g^2\mu^{2\epsilon}$. Note that this gives:

$$\int \frac{d^d K}{(2\pi)^d} \frac{\mu^{2\epsilon} g^2}{(K^2 + m^2)^2} = g^2 \frac{\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}} \left(\frac{\mu^2}{m^2}\right)^\epsilon \tag{51}$$

$$\approx \frac{g^2}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma_E - \log\left(\frac{m^2}{4\pi\mu^2}\right)\right) + \mathcal{O}(\epsilon) \tag{52}$$

The overall result is dimensionless because the parameter μ^2 divides out the mass dimension of m^2. This will happen similarly for integrals arising from multiloop diagrams.

Go back to the massless case of eq. 37, and consider it with the propagator raised to an arbitrary power $\lambda > 0$. It turns out this integral can be set to zero. Requiring the dimensionally regulated integral to satisfy the usual additive property for integration on an interval, it is possible to split up the radial integral in some point between 0 and ∞, say 1, which leads to:

$$\int \frac{d^d K}{(2\pi)^d} \frac{1}{(K^2)^\lambda} = \frac{1}{(2\pi)^d} \int d^d\Omega \int_0^\infty dK \frac{1}{K^{2\lambda+1-d}} = \frac{1}{(2\pi)^d} \int d^d\Omega \left(\int_0^1 dK \frac{1}{K^{2\lambda-d+1}} + \int_1^\infty dK \frac{1}{K^{2\lambda-d+1}} \right) \tag{53}$$

The radial integrals converge for different domains of d. This is shown below:

$$\int_0^1 dK \frac{1}{K^{2\lambda-d+1}} = \left[\frac{1}{d-2\lambda} K^{d-2\lambda}\right]_0^1 = \frac{1}{d-2\lambda} \quad (\text{for } d > 2\lambda) \tag{54}$$

$$\int_1^\infty dK \frac{1}{K^{2\lambda-d+1}} = \left[\frac{1}{d-2\lambda} K^{d-2\lambda}\right]_1^\infty = -\frac{1}{d-2\lambda} \quad (\text{for } d < 2\lambda) \tag{55}$$

At $d = 2\lambda$ there is both an infrared divergence (meaning the integral diverges near the zero momentum region) and a UV-divergence. Nonetheless, the dimensional regularization of divergent integrals was introduced as an analytic continuation of the convergent result. Hence eqs. 54 and 55 are interpreted to hold for all $d \neq \lambda$. It is clear that these integrals then sum to 0, so that we have:

$$\int \frac{d^d K}{(2\pi)^d} \frac{1}{(K^2)^\lambda} = 0 \tag{56}$$

As a definition in dimensional regularization, 'scaleless' integrals are always taken to be zero, where the term 'scaleless' indicates that these integrals only depend on internal momenta and not on any external scales, like external momenta or masses. For the previous example this was explicitely shown by the cancellation of the UV- and IR-divergences.

A full motivation for this convention can be found in the literature. The convention does not lead to any inconsistencies and dimensionally regularized integrals obey the same result regardless of how they are calculated. For example, the following holds by doing a partial fractioning:

$$\int \frac{d^d k}{k^2(k^2 - m^2)} = \frac{1}{m^2} \left(\int \frac{d^d k}{k^2 - m^2} - \int \frac{d^d k}{k^2} \right) \tag{57}$$

(The integrals are taken over Minkowski space in this example.) It turns out that:

$$\int \frac{d^d k}{k^2(k^2 - m^2)} = -i(m^2)^{-\epsilon} \pi^{2-\epsilon} \Gamma(\epsilon - 1) = \frac{1}{m^2} \int \frac{d^d k}{k^2 - m^2} \tag{58}$$

Hence it must be that $\int \frac{d^d k}{k^2} = 0$, which is consistent with eq. 56.

Another important non-trivial result is that integration by parts in a dimensionally regulated integral can be applied while always setting the boundary terms to zero. An explanation for this using the alpha-parametrization is given in [7]. The alpha-parametrization is derived in this thesis in section IX. Integration by parts (IBP-) identities in dimensional regularization allow one to considerably simplify complicated Feynman integrals, which is discussed in section 7.6.

To conclude, a method was discussed to isolate divergences that arise in Feyman integrals. These divergences still need to be dealt with in some manner, which is done by renormalization of the quantum field theory.

IV. Renormalization

Renormalization involves passing from the original theory (called the 'bare' theory) to a renormalized theory, in order to remove divergences that are exposed using the regularization procedure. It will be seen that in essence this is done by rescaling parameters in the original theory, while letting the rescaling factors absorb infinities. More generally, renormalization involves just rescaling parameters in the theory and that can also be done for other reasons than removing divergences. Renormalization can be done in many ways, and depends on:

- Which (part of the) theory are we renormalizing?
- In which scheme are we doing the renormalization?

In this section we consider a single massive quark theory. QCD has of course 6 quark flavours, but it is easier to first do the analysis in a single quark theory. Remarks on how to extend the analysis

to a theory with multiple quarks will be given later on. Our goal will be to renormalize the quark self-energy diagrams, meaning that we'll make them finite. Renormalization is often introduced using counterterms which are new terms added to the Lagrangian. These counterterms do not alter the essence of the theory because they can be introduced by rescaling parameters like masses and fields in the theory, which will be seen soon.

The counterterms end up in the Feynman rules as vertices, and are added to diagrammatic calculations. They are defined in such a way that upon adding them the poles $1/\epsilon^k$ for $k \in \mathbb{Z}_{>0}$ from the regularization procedure are cancelled out. There is freedom in how to pick the counterterms, because the finite part of the counterterms is not constrained by the condition of cancelling divergences. The prescription for how to pick the finite parts of the counterterms is dictated by the so-called renormalization scheme. The minimal subtraction scheme (MS), the modified minimal subtraction scheme ($\overline{\text{MS}}$) and the on-shell scheme (OS) will be treated.

In the following section the renormalization constants Z_m and Z_ψ are introduced, by doing a rescaling of the quark field and mass in the bare theory. One will need to introduce additional renormalization constants in the theory to cancel divergences in other kinds of diagrams than the self-energy diagrams, but for our purposes this will be enough.

Make a change of notation and denote the quark mass as m_0 from now on, and the quark propagator as ψ_0. These quantities are called the bare quark mass and bare quark field since they come from the bare theory, which is called as such because it has not been renormalized yet. The basic QCD Lagrangian for a single quark, with bare mass m_0 and bare quark field ψ_0 is stated below:

$$\mathcal{L}_{\text{QCD}} = \bar{\psi}_{0,i}(i\partial\!\!\!/ - m_0)\delta_{ij}\psi_{0,j} - g\bar{\psi}_{0,i}T^a_{ij}\psi_{0,j}A\!\!\!/^a - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a \tag{59}$$

The covariant derivative $D\!\!\!/$ has been written out explicitely, exposing the term that gives rise to the quark-gluon vertex. The gauge fixing terms are not added as they play no explicit role in the kind of renormalization that will be done here. To find the renormalized theory the quark field and mass are rescaled in the following way:

$$\psi_0 \to \sqrt{Z_\psi}\psi_R \tag{60}$$

$$m_0 \to Z_m m_R \tag{61}$$

with ψ_R and m_R denoting the renormalized quark wave function and mass parameter respectively. The factors Z_ψ and Z_m are called the renormalization constants. While the aim is to remove the divergences of some calculations in the theory, the language will also be used that we are renormalizing the quark field and mass, which just refers to the rescaling done here. Under these rescalings the bare Lagrangian changes and we explicitely write out the result in the following ways.

$$\mathcal{L}_R^{\text{scaled}} \equiv Z_\psi\bar{\psi}_{R,i}(iD\!\!\!/_{ij} - Z_m m_R\delta_{ij})\psi_{R,j} - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu} \tag{62}$$

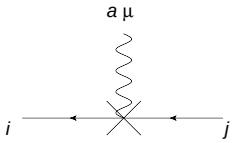
$$\begin{aligned}\mathcal{L}_R^{\text{counter}} \equiv{}& \bar{\psi}_{R,i}(i\partial\!\!\!/ - m_R)\delta_{ij}\psi_{R,j} - g\bar{\psi}_{R,i}T^a_{ij}\psi_{R,j}A\!\!\!/^a - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a - (Z_m Z_\psi - 1)m_R\bar{\psi}_R\psi_R + \\ & (Z_\psi - 1)\bar{\psi}_R(i\partial\!\!\!/)\psi_R - g(Z_\psi - 1)\bar{\psi}_{0,i}T^a_{ij}\psi_{0,j}A\!\!\!/^a\end{aligned} \tag{63}$$

In the equation for $\mathcal{L}_R^{\text{counter}}$, a part has been factored out that has the same form as eq. 59 but with the renormalized mass and field at the usual locations. The additional terms are the counterterms.

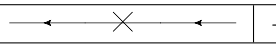
The Lagrangians $\mathcal{L}_R^{\text{scaled}}$ and $\mathcal{L}_R^{\text{counter}}$ are of course exactly equal to each other, but in the following analysis we will interpret their Feynman rules differently. It will then be shown using examples how those approaches are in fact equivalent, as they should be, and how to relate them.

4.1. The counterterm Lagrangian $\mathcal{L}_R^{\text{counter}}$

The counterterm Lagrangian $\mathcal{L}_R^{\text{counter}}$ is interpreted as a theory where the quark field is ψ_R, the quark mass is m_R, and which has additional counterterms. These counterterms are interpreted as vertices, so the quadratic counterterms in the Lagrangian are not part of the quark propagator. The counterterms are visually represented by a cross:

Counterterms		**Mathematical expression**
Mass counterterm		$-i(Z_m Z_\psi - 1)m_R$
Kinetic counterterm		$i(Z_\psi - 1)\not{p}$
Quark-gluon counterterm		$-ig(Z_\psi - 1)T^a_{ij}\gamma^\mu$

In this case the 2-point counterterms have been split in a kinematic counterterm, and a counterterm proportional to m_R. In some cases this is convenient for bookkeeping purposes. When stating the prefactors in the mathematical expressions of the counterterms one has to be careful to be consistent with the rest of the Feynman rules. With our Feynman rules every vertex term in the Lagrangian gains a factor i in the diagrammatic expression, and furthermore the kinetic counterterm gets an additional factor $-i$ that comes from the derivative term, which is seen by doing a Fourier transform of the quark field and taking the derivative. We will also often take both counterterms together into a single term:

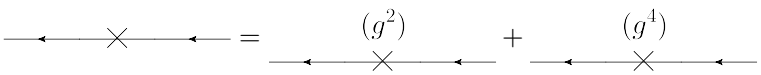
Combined counterterm		$-i(Z_m Z_\psi - 1)m_R + i(Z_\psi - 1)\not{p}$

Another thing to realize is that as one works perturbatively, one will want to expand the renormalization constants Z_m and Z_ψ in the coupling constant g. It turns out that the quark self-energy Σ_0 of the bare theory expands into even powers of g. For this reason Z_ψ and Z_m also expand into even powers of g. Furthermore, the leading constant terms are just equal to 1, because we don't need to alter the theory at leading order. (There are no divergent loop integrals at leading order.) As such we write that:

$$Z_m = 1 + \delta_m^{(g^2)} g^2 + \delta_m^{(g^4)} g^4 + \dots \tag{64}$$

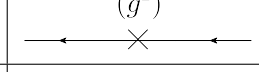
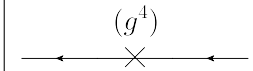
$$Z_\psi = 1 + \delta_\psi^{(g^2)} g^2 + \delta_\psi^{(g^4)} g^4 + \dots \tag{65}$$

In diagrammatic form we write that:

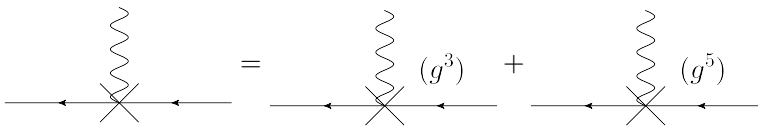


(66)

Leading to:

g^2 order counterterm		$g^2(-i(\delta_m^{(g^2)}+\delta_\psi^{(g^2)})m_R + i\delta_\psi^{(g^2)}\not{p})$
g^4 order counterterm		$g^4(-i(\delta_m^{(g^4)}+\delta_\psi^{(g^4)}+\delta_m^{(g^2)}\delta_\psi^{(g^2)})m_P + i\delta_\psi^{(g^4)}\not{p})$
...		

We continue in the same way for the quark-gluon counterterm:

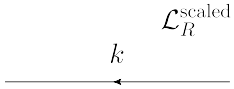


(67)

When there is mention of the 'renormalized' theory without further specification, for which quantities will be denoted with a subscript R, the theory $\mathcal{L}_R^{\text{counter}}$ will be meant, and not the theory $\mathcal{L}_R^{\text{scaled}}$ that is discussed next as an illustrative example of a different way to the renormalization.

4.2. The 'scaled' Lagrangian $\mathcal{L}^{\text{scaled}}$

The 'scaled' Lagrangian $\mathcal{L}^{\text{scaled}}$ is equivalent to the counterterm Lagrangian but in the interpretation of this theory we don't explicitely factor out the counterterms. Instead the rescalings are carried over into the Feynman rules. A quark line now corresponds to:



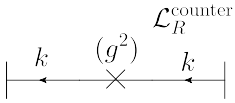
$$= \frac{1}{Z_\psi}\frac{i}{\not{p} - Z_m m_R} \tag{68}$$

and similarly each quark-gluon vertex gains a factor Z_ψ in the numerator. A calculation using Feynman diagrams in order of increasing loops will not give a proper perturbative expansion in this theory, but we can still do the calculation. The idea will then be to expand the Feynman diagrams at some step of their evaluation by writing out Z_ψ and Z_m as a series in g and expanding the whole expression in terms of g up to the desired order of the calculation. Such a calculation can be seen in the following example. In the theory $\mathcal{L}^{\text{scaled}}$ expand the quark propagator:

$$\frac{1}{Z_\psi}\frac{i}{\not{k} - Z_m m_R} \approx \frac{i}{\not{k} - m_R} + \frac{i}{(k^2 - m_R^2)^2}\Big[m_R k^2(\delta_m^{(g^2)} - \delta_\psi^{(g^2)}) - k^2\not{k}\delta_\psi^{(g^2)} + m_R^3(\delta_m^{(g^2)} + \delta_\psi^{(g^2)}) \tag{69}$$

$$+ m_R^2\not{p}(2\delta_m^{(g^2)} + \delta_\psi^{(g^2)})\Big]\, g^2 + \mathcal{O}(g^4) \tag{70}$$

Now look at the counterterm theory and calculate the diagram with two external quark propagators and a g^2-order counterterm vertex wedged in between. The result is:



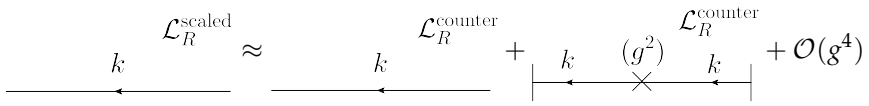
$$= g^2\frac{i}{\not{k} - m_R}\cdot\Big[-i(\delta_m^{(g^2)} + \delta_\psi^{(g^2)})m_R + i\delta_\psi^{(g^2)}\not{k}\Big]\cdot\frac{i}{\not{k} - m_R}$$

$$= g^2\frac{-1}{(k^2 - m_R^2)^2}\,[\not{k} + m_R]\cdot\Big[-i(\delta_m^{(g^2)} + \delta_\psi^{(g^2)})m_R + i\delta_\psi^{(g^2)}\not{k}\Big]\cdot[\not{k} + m_R]$$

$$= g^2\frac{i}{(k^2 - m_R^2)^2}\Big[m_R k^2(\delta_m^{(g^2)} - \delta_\psi^{(g^2)}) - k^2\not{k}\delta_\psi^{(g^2)} + m_R^3(\delta_m^{(g^2)} + \delta_\psi^{(g^2)})$$

$$+ m_R^2\not{p}(2\delta_m^{(g^2)} + \delta_\psi^{(g^2)})\Big] \tag{71}$$

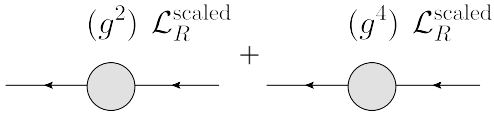
It then becomes clear that:



(72)

Note that at orders g^4 and higher on the right side there are terms with lower order counterterms concatenated with propagators in between. Equivalences like eq. 72 will hold in general. For example, say that one calculates the two-loop quark self-energy in the theory $\mathcal{L}_R^{\text{scaled}}$. The corresponding expression will have the following shape:

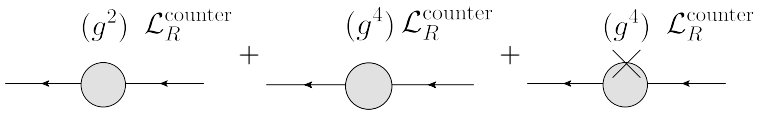
$$-i\Sigma_R^{\text{scaled,loops}\leq 2}(\not{p}, m) =$$



(73)

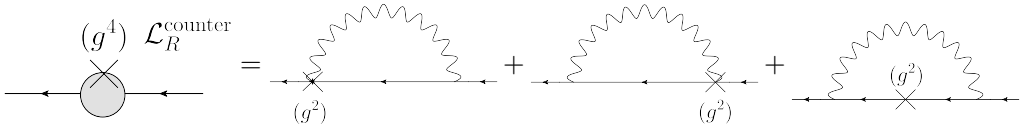
The two diagrams represent all of the 1-loop and 2-loop contributions using the Feynman rules for the scaled theory. These terms are not strictly proportional to g^2 and g^4 respectively, because the terms Z_ψ and Z_m have not been expanded out yet, but are full power series in g. Hence we are not doing a proper perturbative expansion. This changes when one expands the terms Z_ψ and Z_m up to order g^4 to get:

$$-i\Sigma_R^{\text{scaled,loops}\leq 2}(\not{p}, m_R) \approx$$



(74)

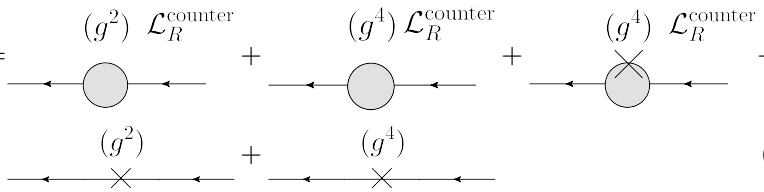
While the big O is put on the right, we are really expanding the expression on the left up to order g^4 and claim that this is exactly equal to the three terms on the right. The first two diagrams represent the bulk of the 1-loop and 2-loop contributions using the Feynman rules for the theory $\mathcal{L}_R^{\text{counter}}$, which are proportional to g^2 and g^4 respectively. The diagram on the right represents the contribution of one-loop diagrams with a counterterm insertion:



(75)

For the Lagrangian $\mathcal{L}_R^{\text{counter}}$ it holds that:

$$-i\Sigma_R^{\text{counter,loops}\leq 2}(\not{p}, m_R) =$$



(76)

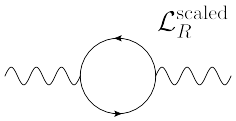
Thus expanding $\Sigma_R^{\text{scaled}}(\not{p}, m_R)$ almost gives us $\Sigma_R^{\text{counter}}(\not{p}, m)$ but we miss the 2-point counterterm vertices. This is a result of the fact that the quark propagator is not included in the self-energy, so in the theory $\mathcal{L}_R^{\text{scaled}}$ the counterterms do not show up after expansion. But the quark propagator is included in the Dyson sum of that theory, so to be consistent we also have to expand the quark propagator in the Dyson sum of $\mathcal{L}_R^{\text{scaled}}$, which has the exact same effect as including counterterms

in the self-energy. Hence the full quark propagator of both Lagrangians turns out exactly the same. One can make the following relation between the quark self-energies of the two theories:

$$-i\Sigma_R^{\text{scaled,loops}\leq N_{\text{loops}}}(\not{p}, m_R) \approx -i\Sigma_R^{\text{counter,loops}\leq N_{\text{loops}}}(\not{p}, m_R) - \sum_{\substack{k\geq 2 \\ k \text{ even}}}^{2N_{\text{loops}}} \quad + \mathcal{O}(g^{2N_{\text{loops}}+2})$$

(77)

There is another observation to make. Any (sub-)diagram in $\mathcal{L}_R^{\text{scaled}}$ theory without external quarks exhibits cancellation of all Z_ψ factors. For example, the below diagram has two vertices each yielding factors Z_ψ and two internal quark propagators each yielding factors Z_ψ^{-1}.



Thus, the factors Z_ψ cancel exactly. The quark self-energy $\Sigma_R^{\text{scaled}}(\not{p}, m_R)$ is proportional to Z_ψ, because the external quark lines are not counted, so that there isn't a complete cancellation leaving a single factor Z_ψ. This shows that:

$$\Sigma_R^{\text{scaled}}(\not{p}, m_R) = Z_\psi \Sigma_0(\not{p}, Z_m m_R) \tag{78}$$

Using eq. 77 this shows that:

$$-i\Sigma_R^{\text{counter,loops}\leq N_{\text{loops}}}(\not{p}, m_R) \approx -iZ_\psi\Sigma_0(\not{p}, m_R Z_m) + \sum_{\substack{k\geq 2 \\ k \text{ even}}}^{2N_{\text{loops}}} \quad + \mathcal{O}(g^{2N_{\text{loops}}+2})$$

(79)

In this equation the term $-iZ_\psi\Sigma_0(\not{p}, m_R Z_m)$ is expanded in g up to order $\mathcal{O}(g^{2N_{\text{loops}}+2})$ by plugging in the power series expansions of Z_ψ and Z_m. Hence if we have calculated the self-energy in the bare theory, we can always move to the renormalized theory using this relation. This method of doing the renormalization has the advantage that one does not explicitely have to consider new diagrams with counterterms, while expanding a series on a computer is relatively easy to do. We also see that:

$$S_{\text{full},R} = \frac{i}{Z_\psi \not{p} - Z_\psi Z_m m_P - \Sigma_R^{\text{scaled}}(\not{p}, m_R)} = \frac{1}{Z_\psi} \frac{i}{\not{p} - Z_m m_P - \Sigma_0(\not{p}, Z_m m_R)} = \frac{1}{Z_\psi} S_{\text{full},0} \tag{80}$$

4.3. Renormalization schemes

It has been explained in the previous part how to add new degrees of freedom to the bare theory to get the renormalized theory. How to use these degrees of freedom in order to actually remove unwanted divergences is dictated by the renormalization scheme that is being used.

4.3.1 Minimal subtraction scheme (MS)

In the MS-scheme we focus on removing only the divergent part that arises in the bare quark self energy $\Sigma_0(\not{p}, m_0)$, and we pick the finite part of the counterterms to be zero. Let coef$((\cdot), \epsilon^k)$

denote the ϵ^k coefficient of $(\cdot)$ as a Laurent series in ϵ. The MS-condition becomes:

$$0 = \text{coef}(-i\Sigma_R^{\text{counter}}(\not{p}, m_R), \epsilon^{-k}) \text{ for all } k \in \mathbb{Z}_{\geq 0} \tag{81}$$

One can use eq. 79 to calculate $\Sigma_R^{\text{counter}}(\not{p}, m_R)$ up to the desired order. This MS-condition has to be solved iteratively in orders of g. First one sets $N_{\text{loops}} = 1$. One can then solve for $\delta_\psi^{(g^2)}$ and $\delta_m^{(g^2)}$. Next one can set $N_{\text{loops}} = 2$, substituting the results for $\delta_\psi^{(g^2)}$ and $\delta_m^{(g^2)}$ in the one-loop diagrams with counterterm insertions, and solve for $\delta_\psi^{(g^4)}$ and $\delta_m^{(g^4)}$. The higher orders follow in similar manner. The finite parts of the counterterms, which are not proportional to an inverse power of ϵ are set to zero.

4.3.2 Modified minimal subtraction scheme ($\overline{\text{MS}}$)

It turns out that upon rescaling $\mu^2 \to \mu^2 \frac{e^{\gamma_E}}{4\pi}$ the power series expansion of Feynman diagrams becomes simpler. For example, a part of the quark self-energy at one-loop is given in eq. 112 and repeated below:

$$A(p^2, m_0^2) = \frac{\alpha_s \mu^{2\epsilon}}{(4\pi)^{1-\epsilon}} C_F \Gamma(\epsilon) (4 - 2\epsilon) \int_0^1 dx (m_0^2 x - p^2 x(1-x))^{-\epsilon} \tag{82}$$

Expanding up to order $\mathcal{O}(\epsilon)$ one finds the result of eq. 114. By first doing the rescaling of μ one finds the following (denoted with a tilde to distinguish from the usual result):

$$\tilde{A}(p^2, m_0^2) = \frac{C_F \alpha_s}{2\pi} \left(\frac{2}{\epsilon} - 1 - 2 \int_0^1 dx \log \left(\frac{m_0^2}{\mu^2} x - \frac{p^2}{\mu^2} x(1-x) \right) + \mathcal{O}(\epsilon) \right) \tag{83}$$

Note that the factors proportional to γ_E and $\log(4\pi)$ have disappeared. The $\overline{\text{MS}}$-scheme is roughly defined as the MS-scheme while also rescaling $\mu^2 \to \mu^2 \frac{e^{\gamma_E}}{4\pi}$. However, in practice μ is not actually rescaled but alternatively the finite parts of the counterterms are picked to cancel the terms coming from an implicit factor $(\frac{e^{\gamma_E}}{4\pi})^{-\epsilon}$ in the Feynman diagrams. At 1-loop this is done in eqs. 124 and 125.

4.3.3 On-shell scheme (OS)

In the on-shell scheme one is not only concerned with removing divergences but also makes claims for how the full renormalized propagator $S_{\text{full},R}$ should behave. The desired behavior is that $S_{\text{full},R}$ should have a pole at the renormalized mass ($\not{p} = m_R$) and that for momenta close to the pole, the full propagator should behave like a free propagator. Because the renormalized mass is taken to be at the pole, we use the notation $m_R = m_P$ where m_P is called the pole mass. Expanding around the pole mass gives:

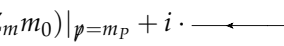
$$S_{R,\text{full}}(\not{p}, m_P) = \frac{i}{\not{p} - m_P - \Sigma_R(\not{p}, m_P)} \approx \frac{i}{(\not{p} - m_P)(1 - \frac{d}{d\not{p}} \Sigma_R(\not{p}, m_P)|_{\not{p}=m_P}) + \mathcal{O}(\not{p} - m_P)^2} \tag{84}$$

We read off that to have a pole at $\not{p} = m_P$ and to act like a free propagator around the pole the following conditions should hold:

$$\Sigma_R(\not{p}, m_P)|_{\not{p}=m_P} = 0 \tag{85}$$

$$\frac{d}{d\not{p}} \Sigma_R(\not{p}, m_P)|_{\not{p}=m_P} = 0 \tag{86}$$

And using some of our previous results the first on-shell condition gives:

$$\begin{aligned}\Sigma_R(\not{p}, m_P)|_{\not{p}=m_P} &= Z_\psi \Sigma_0(\not{p}, Z_m m_0)|_{\not{p}=m_P} + i \cdot \text{  }|_{\not{p}=m_P} \\ &= Z_\psi \Sigma_0(\not{p}, Z_m m_0)|_{\not{p}=m_P} + i\left(-i(Z_m Z_\psi - 1)m_P + i(Z_\psi - 1)\not{p}\right)|_{\not{p}=m_P} \\ &= Z_\psi \Sigma_0(\not{p}, Z_m m_0)|_{\not{p}=m_P} + Z_\psi m_P(Z_m - 1) = 0 \\ &\Rightarrow \Sigma_0(\not{p}, Z_m m_0)|_{\not{p}=m_P} = m_P(1 - Z_m) \end{aligned} \tag{87}$$

Thus the on-shell mass renormalization constant is given by:

$$Z_m = 1 - \frac{1}{m_P}\left(\Sigma_0(\not{p}, Z_m m_0)|_{\not{p}=m_P}\right) \tag{88}$$

To solve for Z_m one expands both sides of the equation perturbatively as a series in g and solves iteratively starting from 1-loop order (g^2) and moving up successively. The second on-shell condition gives in a similar manner:

$$\begin{aligned}\frac{d}{d\not{p}}\Sigma_R(\not{p}, m_P)|_{\not{p}=m_P} &= Z_\psi \frac{d}{d\not{p}}\Sigma_0(\not{p}, Z_m m_0)|_{\not{p}=m_P} + i\frac{d}{d\not{p}} \cdot \text{}|_{\not{p}=m_P} \\ &= Z_\psi \frac{d}{d\not{p}}\Sigma_0(\not{p}, Z_m m_0)|_{\not{p}=m_P} - (Z_\psi - 1) = 0 \end{aligned} \tag{89}$$

The wave function renormalization constant is therefore:

$$Z_\psi = \frac{1}{1 - \frac{d}{d\not{p}}\Sigma_0(\not{p}, Z_m m_0)|_{\not{p}=m_P}} \tag{90}$$

A trick for on-shell renormalization of the massive quark propagator A trick can be used that aides in bookkeeping and gets rid of tensor integrals in the on-shell calculation of the renormalization constants Z_ψ and Z_m [8], [9]. Firstly decompose the self-energy in the following way:

$$\Sigma_R(\not{p}, m_P) = m\Sigma_{R,1}(p^2, m_P^2) + (\not{p} - m_P)\Sigma_{R,2}(p^2, m_P^2) \tag{91}$$

Define the external momentum to be $p = (1+t)Q$, with $Q^2 = m_P^2$. Next construct $T_R(p^2, m_P^2)$:

$$\begin{aligned} T_R(p^2, m_P^2) &\equiv \mathrm{Tr}\left[\frac{\not{Q} + m_P}{4m_P^2}\Sigma_R(\not{p}, m_P)\right] \\ &= \frac{1}{4m_P^2}\mathrm{Tr}\left[(\not{Q} + m_P)(m_P\Sigma_{R,1}(p^2, m_P^2) + (\not{p} - m_P)\Sigma_{R,2}(p^2, m_P^2))\right] \\ &= \frac{1}{4m_P^2}\mathrm{Tr}\left[\not{Q}\not{p}\Sigma_{R,2}(p^2, m_P^2) + m_P^2(\Sigma_{R,1}(p^2, m_P^2) - \Sigma_{R,2}(p^2, m_P^2))\right] \\ &= \frac{1}{4m_P^2}(4Q^2(1+t)\Sigma_{R,2}(p^2, m_P^2) + 4m_P^2(\Sigma_{R,1}(p^2, m_P^2) - \Sigma_{R,2}(p^2, m_P^2))) \\ &= \Sigma_{R,1}(p^2, m_P^2) + t\Sigma_{R,2}(p^2, m_P^2) \end{aligned} \tag{92}$$

Expanding Σ_R up to first order in $\not{p} - m_P$ gives:

$$\begin{aligned}\Sigma_R(\not{p}, m_P) =& \Sigma_R(\not{p}, m_P)|_{\not{p}=m_P} + \frac{d}{d\not{p}}\Sigma_R(\not{p}, m_P)|_{\not{p}=m_P}(\not{p} - m_P) + \mathcal{O}((\not{p} - m_P)^2) \\ =& m_P\Sigma_{R,1}(p^2, m_P)|_{\not{p}=m_P} + \left[2m_P^2\frac{d}{dp^2}\Sigma_{R,1}(p^2, m_P^2)|_{\not{p}=m_P} + \Sigma_{R,2}(p^2, m_P^2)|_{\not{p}=m_P}\right](\not{p} - m_P) \\ &+ \mathcal{O}((\not{p} - m_P)^2) \end{aligned} \tag{93}$$

In the second line it has been used that:

$$\frac{d}{d\not{p}}\Sigma_{R,1}(p^2, m_P^2) = \frac{dp^2}{d\not{p}}\frac{d}{dp^2}\Sigma_{R,1}(p^2, m_P^2) = 2\not{p}\frac{d}{dp^2}\Sigma_{R,1}(p^2, m_P^2) \tag{94}$$

Setting the first two coefficients in the expansion of eq. 93 equal to zero amounts to the on-shell renormalization conditions. (See eqs. 85, 86) It turns out these terms can also be obtained from T_R, by expanding around $t = 0$. In particular, because we defined that $Q^2 = m_P^2$, the condition $t = 0$ is equivalent to $p^2 = m_P^2$. Thus one obtains the following expansions in t:

$$\begin{aligned}\Sigma_{R,1}(p^2, m_P^2) = \Sigma_{R,1}(Q^2(1+t)^2, m_P^2) &\approx \Sigma_{R,1}(p^2, m_P^2)|_{p^2=m_P^2} + t \cdot 2m_P^2 \frac{d}{dp^2}\Sigma_{R,1}(p^2, m_P^2)|_{p^2=m_P^2} + \mathcal{O}(t^2) \\ \Sigma_{R,2}(p^2, m_P^2) = \Sigma_{R,2}(Q^2(1+t)^2, m_P^2) &\approx \Sigma_{R,2}(p^2, m_P^2)|_{p^2=m_P^2} + \mathcal{O}(t)\end{aligned} \tag{95}$$

The dependence on p^2 on the left side of the equation is really a dependence on t. Plugging these expansions into the definition of T and not writing the momentum dependence anymore gives:

$$T_R(m_P^2) \approx \Sigma_{R,1}(p^2, m_P^2)|_{p^2=m_P^2} + t\left(2m_P^2\frac{d}{dp^2}\Sigma_{R,1}(p^2, m_P^2)|_{p^2=m_P^2} + \Sigma_{R,2}(p^2, m_P^2)|_{p^2=m_P^2}\right) + \mathcal{O}(t^2) \tag{96}$$

What is seen, is that the expansions of eqs. 93, 96 agree in the first two coefficients, and setting these coefficients to zero amounts to the on-shell renormalization conditions. By looking at eqs. 87, 89, we also see that:

$$Z_m = 1 - \text{coef}(T_0(Z_m^2 m_P^2), t^0) \tag{97}$$

$$Z_\psi = \frac{1}{1 - \text{coef}(T_0(Z_m^2 m_P^2), t^1)} \tag{98}$$

Here $T_0(m_0^2) = T_0(Z_m^2 m_P^2)$ refers to the quantity in the bare theory. These equations can be solved for iteratively by expanding in g. The inverse field renormalization constant Z_ψ^{-1} follows immediately once Z_m has been found.

The main advantage of doing the on-shell renormalization by constructing $T_0(Z_m^2 m_P^2)$ is that it is allows for a convenient bookkeeping and one can get rid of tensor integrals early on in the calculation. Furthermore eqs. 97, 98 take an easy form so that the renormalization is straightforward to do without employing counterterms.

However the method only works for the on-shell scheme. A more general form to find renormalization constants is to solve the renormalization conditions of the scheme that is used by starting from eq. 79.

4.4. Renormalization of other parameters

So far the treatment of renormalization has been confined to a rescaling of the quark field and mass for a single quark theory. To fully renormalize QCD at a certain loop order much more parameters of the theory should be rescaled/renormalized. This will not complicate the calculations much. Suppose we are still interested in calculating the quark self-energy but will also take into account rescalings of other parameters in the theory. We consider a few points:

- Rescaling of a field, say the gluon field $A_0^\mu = Z_A^{1/2} A_R^\mu$.

Since we are only looking at 2-point quark diagrams the scaling of the vertices will exactly cancel out the scaling of the gluon propagators. Hence, nothing changes for the calculation of the quark self-energy.

- Rescaling of a scalar parameter, say the coupling constant itself $g_0 = Z_g g_R$.

We can take any calculation in terms of the bare coupling constant, and simply rescale g_0 in that result and expand on g_R, with Z_g interpreted as a power series in g_R.

- Adding more quarks to the theory

We consider the top quark to be much heavier than the other quarks, so we don't add mass terms for the other quarks. In that case there is only a wave function renormalization for the other quarks and this drops out in top quark self-energy diagrams. In case we do not neglect masses for the other quarks there will be mass renormalization constants for these quarks as well. The calculation will become much more difficult in this case because one will need to calculate Feynman integrals with different masses.

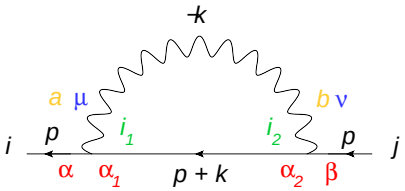
For the multiloop renormalization we will consider a model consisting of N_h massive quarks of the same mass and N_l 'light' quarks with no mass. At the end one can put $N_h = 1$ and $N_l = 5$ to get a model with a massive top quark and the masses of the other quarks neglected. It was discussed that the wave function renormalization constants of the massless particles drop out of the top quark self-energy diagrams. A renormalization of the coupling constant will still need to be done, but it is easy to reexpress Z_m and Z_ψ in the renormalized coupling constant as it amounts to a simple power series expansion after putting $g_0 = Z_g g_R$. Similarly, the wave function renormalization constant depends on the (bare) gauge parameter starting from 3-loop order. To include renormalization of the gauge parameter one can rescale $\alpha_{\text{bare}} = Z_\alpha \alpha_R$ and expand Z_α in terms of g_R. Of course to find Z_g and Z_α different types of Feynman diagrams will need to be considered than the quark propagator type.

V. Renormalization of the top quark at 1-loop order

5.1. Calculating the bare self-energy

The first order contribution to the self-energy is derived below in the bare theory. The only diagram contributing at one-loop level is:

$$-i\Sigma^{ij}_{\alpha\beta}(\not{p}, m) \;=$$



(99)

For completeness all indices have been written down explicitely:

i, i_1, i_2, j	SU(N_c) indices
$\alpha, \alpha_1, \alpha_2, \beta$	Spinor indices for Dirac algebra
a, b	Index for gauge group generator
μ, ν	Space-time indices

This diagram diverges in 4 dimensions and dimensional regularization is used to make the integral finite. From the Feynman rules in app. B one finds:

$$\Sigma^{ij}_{0,\alpha\beta}(\not{p}, m_0) = \mu^{2\epsilon} g^2 \int \frac{d^d k}{i(2\pi)^d} \left((\gamma^\mu)_{\alpha\alpha_1} \left(\frac{T^a_{ii_1} \delta_{i_1 i_2} \delta_{ab} T^b_{i_2 jj} d_{\mu\nu}(-k)}{((\not{p} + \not{k}) - m_0) k^2} \right)_{\alpha_1 \alpha_2} (\gamma^\nu)_{\alpha_2 \beta} \right) \tag{100}$$

Where $d_{\mu\nu}(k) = \eta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}$, and α denotes the gauge parameter (not to be confused with the index α used for the Dirac algebra, nor the strong coupling constant α_s.) To simplify the calculation we will pick the Feynman gauge ($\alpha = 1$). Evaluating the delta functions, it is seen the color factor equals the definition of C_F (eq. 11). We will use this relation and drop the color indices on $\Sigma_0(\not{p}, m_0)$, understanding that a δ_{ij} is implied. Similarly we will drop the indices for the Dirac algebra. Lastly note that:

$$\frac{1}{(\not{p} + \not{k}) - m} = \frac{m + (\not{p} + \not{k})}{(p+k)^2 - m^2} \tag{101}$$

This leads to:

$$\Sigma_0(\not{p}, m_0) = \mu^{2\epsilon} g^2 C_F \int \frac{d^d k}{i(2\pi)^d} \left(\frac{\gamma^\mu (m_0 + (\not{p} + \not{k}))\gamma_\mu}{((p+k)^2 - m_0^2)k^2} \right) \tag{102}$$

The diagram exhibits ultraviolet divergence, i.e. it diverges in the high momentum region: $\Sigma \sim \int d^d k \frac{\not{k}}{k^4} \sim \lim_{k\to\infty} k^{d+1-4} \Rightarrow$ lin. divergence in 4 dimensions. Using eqs. 48 and 49 we find that:

$$\Sigma_0(\not{p}, m_0) = \mu^{2\epsilon} g^2 C_F \int \frac{d^d k}{i(2\pi)^d} \left(\frac{dm + (2-d)(\not{p} + \not{k})}{((p+k)^2 - m_0^2)k^2} \right) \tag{103}$$

The well-known Feynman trick is used next:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2} \tag{104}$$

Leading to:

$$\begin{aligned} \Sigma_0(\not{p}, m_0) &= \mu^{2\epsilon} g^2 C_F \int_0^1 dx \int \frac{d^d k}{i(2\pi)^d} \frac{dm_0 + (2-d)(\not{p} + \not{k})}{(x((p+k)^2 - m_0^2) + (1-x)k^2)^2} \\ &= \mu^{2\epsilon} g^2 C_F \int_0^1 dx \int \frac{d^d k}{i(2\pi)^d} \frac{dm_0 + (2-d)(\not{p}(1-x) + \not{k})}{(k^2 - (m_0^2 - p^2(1-x))x)^2} \end{aligned} \tag{105}$$

where in the second step the internal momentum is shifted by $k \to k - xp$. This makes the divisor invariant under reflection $k \to -k$, so the $\not{k}$ term in the numerator can be dropped as it gives an odd integral. The k-integral can be put in a standard form by going to Euclidean space using a Wick rotation:

$$k_0 = iK_0 \tag{106}$$

$$k^2 = k_0^2 - k_i k_i = -K_0^2 - K_i K_i = -K^2 \tag{107}$$

$$d^d k = i d^d K \tag{108}$$

So that:

$$\Sigma_0(\not{p}, m_0) = \mu^{2\epsilon} g^2 C_F \int_0^1 dx \int \frac{d^d K}{(2\pi)^d} \frac{dm_0 + (2-d)\not{p}(1-x)}{(K^2 + L)^2} \tag{109}$$

with $L \equiv (m_0^2 - p^2(1-x))x$. This way one can use eq. 44 with m^2 replaced by L. Making use of the stated results, filling in $d = 4 - 2\epsilon$, and putting $g^2 = 4\pi\alpha_s$, which is a commonly used notation, the following result is found:

$$\Sigma_0(\not{p}, m_0) = \mu^{2\epsilon}\frac{\alpha_s}{(4\pi)^{1-\epsilon}}C_F\Gamma(\epsilon)\left((4-2\epsilon)m_0\int_0^1 dx(m_0^2x - p^2x(1-x))^{-\epsilon} + \not{p}(2\epsilon-2)\int_0^1 dx(m_0^2x - p^2x(1-x))^{-\epsilon}(1-x)\right) \tag{110}$$

It is convenient to factor out the kinematic part and the mass part:

$$\Sigma_0(\not{p}, m_0) \equiv A(p^2, m_0^2)m_0 + \not{p}B(p^2, m_0^2) \tag{111}$$

This yields:

$$A(p^2, m_0^2) = \mu^{2\epsilon}\frac{\alpha_s}{(4\pi)^{1-\epsilon}}C_F\Gamma(\epsilon)\,(4-2\epsilon)\int_0^1 dx(m_0^2x - p^2x(1-x))^{-\epsilon} \tag{112}$$

$$B(p^2, m_0^2) = \mu^{2\epsilon}\frac{\alpha_s}{(4\pi)^{1-\epsilon}}C_F\Gamma(\epsilon)\,(2\epsilon-2)\int_0^1 dx(m_0^2x - p^2x(1-x))^{-\epsilon}(1-x) \tag{113}$$

Expanding the expressions on ϵ gives in turn:

$$A(p^2, m_0^2) = \frac{\alpha_s}{4\pi}C_F\left(\frac{4}{\epsilon} - 2 - 4\gamma_E + 4\log(4\pi) - 4\int_0^1 dx\log\left(\frac{m_0^2x - p^2x(1-x)}{\mu^2}\right) + \mathrm{O}(\epsilon)\right) \tag{114}$$

$$B(p^2, m_0^2) = \frac{\alpha_s}{4\pi}C_F\left(-\frac{1}{\epsilon} + 1 + \gamma_E - \log(4\pi) + 2\int_0^1 dx\log\left(\frac{m_0^2x - p^2x(1-x)}{\mu^2}\right)(1-x) + \mathrm{O}(\epsilon)\right) \tag{115}$$

5.2. Renormalization

Using the dimensional regularization procedure the UV-divergent part of the quark self-energy at 1-loop has been isolated in the previous section. We will now renormalize the UV-divergences by going to the renormalized theory by adding a 2-point quark counterterm. This gives us:

$$-i\Sigma_R^{\text{1-loop}}(\not{p}, m_R) = -i\Sigma_0^{\text{1-loop}}(\not{p}, m_R) + g^2(-i(\delta_m^{(g^2)} + \delta_\psi^{(g^2)})m_R + i\delta_\psi^{(g^2)}\not{p}) \tag{116}$$

Minimal subtraction (MS) In the MS-scheme at 1-loop the counterterms are defined by the following relation, because there are only simple poles at one-loop:

$$\mathrm{coef}(-i\Sigma_R^{g^{\leq 2}}(\not{p}, m_R), \epsilon^{-1}) = 0 \tag{117}$$

This equation splits up into two independent equalities by factorizing the parts proportional to $\not{p}$ and to m_R, so that:

$$0 = \mathrm{coef}(-iA^{(g^2)} - i(\delta_m^{(g^2)} + \delta_\psi^{(g^2)}), \epsilon^{-1}) \tag{118}$$

$$0 = \mathrm{coef}(-iB^{(g^2)} + i\delta_\psi^{(g^2)}, \epsilon^{-1}) \tag{119}$$

Where it is defined that $A = g^2 A^{(g^2)} + \mathcal{O}(g^4)$. From the previous equations it follows that:

$$\delta_m^{(g^2)} = \text{coef}(-(A^{(g^2)} + B^{(g^2)}), \epsilon^{-1}) \tag{120}$$

$$\delta_\psi^{(g^2)} = \text{coef}(B^{(g^2)}, \epsilon^{-1}) \tag{121}$$

This finally leads to:

$$\delta_\psi^{(g^2)} g^2 = -\frac{\alpha_s}{4\pi} C_F \epsilon^{-1} \Rightarrow Z_\psi^{\text{MS}} = 1 - \frac{\alpha_s}{4\pi\epsilon} C_F + \mathcal{O}(g^4) \tag{122}$$

$$\delta_m^{(g^2)} g^2 = -\frac{3\alpha_s}{4\pi} C_F \epsilon^{-1} \Rightarrow Z_m^{\text{MS}} = 1 - \frac{3\alpha_s}{4\pi\epsilon} C_F + \mathcal{O}(g^4) \tag{123}$$

Modified minimal subtraction ($\overline{\text{MS}}$) In the modified MS scheme one also absorbs the terms γ_E and $\log(4\pi)$, that arose in eqns 114 and 115. This leads to a slightly different definition:

$$Z_\psi^{\overline{\text{MS}}} = 1 - \frac{\alpha_s}{4\pi} C_F \left(\epsilon^{-1} + \log(4\pi) - \gamma_E\right) + \mathcal{O}(g^4) \tag{124}$$

$$Z_m^{\overline{\text{MS}}} = 1 - \frac{3\alpha_s}{4\pi} C_F \left(\epsilon^{-1} + \log(4\pi) - \gamma_E\right) + \mathcal{O}(g^4) \tag{125}$$

On-shell scheme In the on-shell (subtraction) scheme the renormalized mass m_R is directly identified with the pole mass m_P, so that $m_R = m_P$. The on-shell renormalization conditions, eqs. 85, 86 tell us:

$$-i\,\Sigma_0^{g^{\leq 2}}(\not{p}, m_P)\Big|_{\not{p}=m_P} - ig^2\delta_m^{(g^2)} = 0 \tag{126}$$

$$-i\frac{d}{d\not{p}}\,\Sigma_0^{g^{\leq 2}}(\not{p}, m_P)\Big|_{\not{p}=m_P} + ig^2\delta_\psi^{(g^2)} = 0 \tag{127}$$

From this $\delta_m^{(g^2)}$ follows easily:

$$0 = m_P(B(m_P^2, m_P^2) - \delta_\psi^{(g^2)} g^2) + m_P(\delta_\psi^{(g^2)} g^2 + \delta_m^{(g^2)} g^2 + A(m_P^2, m_P^2)) \tag{128}$$

$$\Rightarrow \delta_m^{(g^2)} g^2 = -A(m_P^2, m_P^2) - B(m_P^2, m_P^2) \tag{129}$$

$$= -\frac{\alpha_s}{4\pi} C_F \left(\frac{3}{\epsilon} - 1 - 3\gamma_E + 3\log(4\pi) + \int_0^1 dx \log\left(\frac{m_P^2}{\mu^2} x^2\right)(-4 + 2(1-x)) + \mathcal{O}(\epsilon)\right) \tag{130}$$

Explicitely evaluating the integral gives us:

$$\begin{aligned} Z_m^{\text{OS}} &= 1 - \frac{\alpha_s}{4\pi} C_F \left(\frac{3}{\epsilon} - 1 - 3\gamma_E + 3\log(4\pi) + \left(5 - 3\log\left(\frac{m_P^2}{\mu^2}\right)\right) + \mathcal{O}(\epsilon)\right) \\ &= 1 - \frac{3\alpha_s}{4\pi} C_F \left(\frac{1}{\epsilon} + \frac{4}{3} - \gamma_E - \log\left(\frac{m_P^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon)\right) \end{aligned} \tag{131}$$

This is in correspondence with ref. [10], equation. 12.26 (b). Furthermore:

$$0 = B(m_P^2, m_P^2) - \delta_\psi^{(g^2)} g^2 + m_P \frac{d}{d\not{p}} B(p^2, m_P^2)|_{p^2=m_P^2} + m_P \frac{d}{d\not{p}} A(p^2, m_P^2)|_{p^2=m_P^2} \tag{132}$$

$$\Rightarrow \delta_\psi^{(g^2)} g^2 = B(m_P^2, m_P^2) + m_P \left(\frac{d}{d\not{p}} B(p^2, m_P^2)|_{p^2=m_P^2} + \frac{d}{d\not{p}} A(p^2, m_P^2)|_{p^2=m_P^2}\right) \tag{133}$$

If one tries to evaluate this expression using the expanded form of A and B in eqs. 114 and 115, one will end up with a divergence in the x integal. This is seen below:

$$\frac{d}{d\not{p}}\left(A(p^2,m_P^2)+B(p^2,m_P^2)\right)\Big|_{\not{p}=m_P} = -\int_0^1 dx\,\frac{\alpha_s C_F\left(x^2-1\right)}{\pi m x} \tag{134}$$

This divergence shows up because we were not allowed to take the derivative after first expanding in epsilon, because the derivative introduces different scaling behaviour as $x \to 0$ in the integrals in eqs. 112 and 113. Therefore one should first take the derivative on $\not{p}$ in the x-integral before expanding in terms of ϵ. Then we find that:

$$\frac{d}{d\not{p}}\left(A(p^2,m_P^2)+B(p^2,m_P^2)\right)\Big|_{\not{p}=m_P} = \frac{4\alpha_s}{(4\pi)^{1-\epsilon}}\mu^{2\epsilon} m C_F \Gamma(\epsilon)\int_0^1 dx\,\left((x-1)x(m^2x^2)^{-1-\epsilon}(x(\epsilon-1)-1)\epsilon\right) \tag{135}$$

The integral has been evaluated using Wolfram Mathematica which leads to:

$$\frac{d}{d\not{p}}\left(A(p^2,m_P^2)+B(p^2,m_P^2)\right)\Big|_{\not{p}=m_P} = -\frac{4\alpha_s}{(4\pi)^{1-\epsilon}}\mu^{2\epsilon} m C_F \Gamma(\epsilon)\frac{m^{-2(\epsilon+1)}(\epsilon-1)}{2(2\epsilon-1)} \tag{136}$$

The integration is in principle only allowed for $\mathrm{Re}(\epsilon) < 0$, but of course in dimensional regularization such constraints are relaxed and the analytic continuation is taken. However this does indicate that the integral is infrared (IR-)divergent. By looking at $d = 4 - 2\epsilon$ the integral is understood to converge when the dimension is higher than 4. This is typical for an infrared divergence, whereas for ultraviolet divergences one expects the integral to converge for dimensions lower than 4. Plugging eq. 136 into eq. 133 and expanding into terms of ϵ gives us:

$$\delta_\psi^{(g^2)} g^2 = \frac{\alpha_s}{4\pi} C_F\left(-\frac{3}{\epsilon} - 4 + 3\gamma_E + 3\log\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon)\right) \tag{137}$$

$$Z_\psi^{\mathrm{OS}} = 1 - \frac{3\alpha_s}{4\pi} C_F\left(\frac{1}{\epsilon} + \frac{4}{3} - \gamma_E - \log\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon)\right) \tag{138}$$

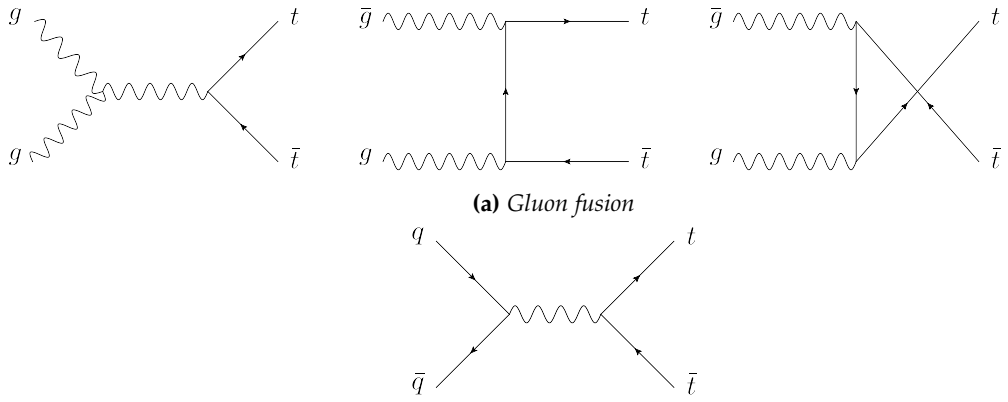
It is seen that $Z_\psi = Z_m$ in the on-shell renormalization scheme. These results are in correspondence with ref. [9]. (See eq. (6.29) in Grozin's book and expand it on ϵ.)

Note that the renormalization schemes were introduced as being only different in the UV-finite part. However, the divergent ϵ^{-1} term in the on-shell counterterm Z_ψ^{OS} is different than the divergent term in $Z_\psi^{\overline{\mathrm{MS}}}$. This can be interpreted in the sense that the IR-divergence that was observed in eq. 134 has now been absorbed together with the UV-divergence. In particular one might separate the divergent part in the following way:

$$Z_\psi^{\mathrm{OS}} = 1 - \frac{\alpha_s}{4\pi} C_F\left(\frac{1}{\epsilon} + \left(\frac{2}{\epsilon} + 4 - 3\gamma_E - 3\log\left(\frac{m^2}{4\pi\mu^2}\right)\right) + \mathcal{O}(\epsilon)\right) \tag{139}$$

The bracketed term containing a factor $2/\epsilon$ is then interpreted to be UV-finite but not IR-finite. This is done for example in [10].

Relation between $\overline{\mathrm{MS}}$ and on-shell mass Of theoretical interest is the relation between the $\overline{\mathrm{MS}}$- and on-shell mass of the quark, because it relates the perturbative series of the theories that have



(a) *Gluon fusion*

(b) *Quark-antiquark annihilation*

Figure 1: *Dominant diagrams for the production of a $t\bar{t}$ quark-antiquark pair in proton colliders*

been renormalized in these schemes. The result is a finite ratio, which is found by expanding up to order g^2 and gives:

$$\frac{m_P}{m_{\overline{\text{MS}}}} = 1 + \alpha_s C_F \frac{\left(3\log\left(m_P^2/\mu^2\right) - 4\right)}{4\pi} + \mathcal{O}(\alpha_s^2) \tag{140}$$

Picking the renormalization scale to be $\mu = m_P$ gets rid of the logarithm.

VI. Phenomenological context

The following section touches on a few topics within particle physics related to the renormalization of the top quark.

6.1. Experimental measurement of the top quark (mass)

The top quark was first observed in 1995 at the Fermilab $p\bar{p}$ Tevatron collider, while its existence was already suggested after the observation of the bottom quark in 1977 at Fermilab. The dominant diagrams for the production of a $t\bar{t}$ quark-antiquark pair in a $p\bar{p}$ collision are shown in fig. 1, (see ref. [11], [12]). The same processes also dominate at the LHC, which is a pp-collider, although in different ratios. The mean lifetime of the top quark is about 5×10^{-25}s, which is too short-lived for the top quark to form hadrons. Therefore the top quark behaves like a quasi-free quark. The dominant decay of the of the top quark in the SM is the decay into a W boson and a b quark, with a branching fraction of nearly 100%(see ref. [13]). Hence after production the $t\bar{t}$ pair is expected to undergo the following weak decay:

$$t\bar{t} \to W^+ b + W^- \bar{b} \tag{141}$$

The b, $\bar{b}$ quarks will each generally fragment into a jet of hadrons. Hence the $t\bar{t}$ final states can be classified according to the decay modes of the W bosons. The W's can decay to a lepton (in particular an electron or muon) and a corresponding neutrino, or into hadronic jets. It turns out that the most lucrative channel is where one of the W bosons decays into a lepton + neutrino while the other decays into a quark-antiquark pair, forming jets. For example, at the CMS detector

at the LHC such top quark pair events are selected. An approach of CMS for measuring the mass of the top quark using only the kinematic properties of its charged decay products is given in [14].

The experimental signature of the top quark pair production in this channel is thus composed of one lepton, four jets two of which come from bottom quarks, and missing transverse momentum due to the neutrino which is not detected. The branching fraction of the leptons + jets channel is about 29% which provides for good statistics. Generally spoken the top quark mass can be reconstructed by reconstructing the event kinematics up to ambiguities (for example due to the unmeasured neutrino).

A particular method is the 'template method'. It comes down to the following: make multiple event hypotheses for the ambiguities and for each hypothesis minimize a χ^2-fit by varying the top quark mass. The best fit is taken to correspond to the correct event hypothesis and gives a value for the top quark mass. Such an analysis is done for each event, and the combined measurements give an estimate of the top quark mass. A slightly more in-depth general explanation can be found in chapter 9.5 of [13] and the references therein. Furthermore a description of the use of the template method at Atlas is given in [15].

Future higher precision measurements of the top quark are expected from high-energy $e\bar{e}$ colliders, by looking at the threshold centre-of-mass energy region for $t\bar{t}$ production. (see ref. [16].) It turns out that the cross-section of top-quark pair production from $e\bar{e}$ interactions is highly sensitive to the top quark mass around the threshold of production. This is illustrated in fig 2, which gives the expected total cross-section for $t\bar{t}$ production at the threshold for different values of the top quark mass.

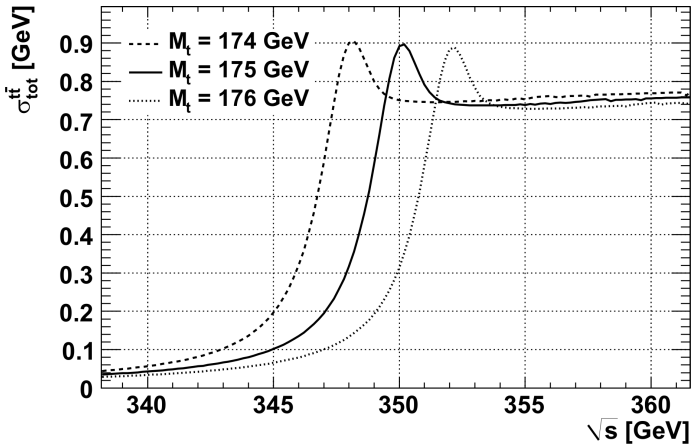


Figure 2: *Total cross-section for $t\bar{t}$ production at the threshold for different values of the top quark mass. (Figure taken from [16].)*

Both the position of the threshold as well as the peak height are seen to depend on the top quark mass. Therefore one can do a so-called threshold scan, where the cross-section for $t\bar{t}$ production is measured in the threshold region by varying the centre-of-mass energy of the collisions. One can then make a fit for the top quark mass by comparing the threshold scan to the theoretical prediction.

6.2. Interpretation of the measured top quark mass

The measured top mass is identified with the renormalized top quark mass, but the result depends on the renormalization scheme used. In principle the difference in value between schemes is small

and should become smaller as the theory is renormalized up to higher orders. This picture is however complicated by the non-perturbative behaviour of the theory. For example, an argument by Dyson states that QED cannot have convergent power series. The argument is roughly as follows. Suppose some quantity is expanded perturbatively in the elementary charge (the coupling constant of QED):

$$S(\alpha_e) \approx s_0 + s_1\alpha_e + s_2\alpha_e^2 + \dots \tag{142}$$

Where $\alpha_e = e^2/(4\pi)$ is the fine-structure constant. One can consider α_e to live in the complex plane and note that for the power series S to converge there must be some finite radius of convergence denoted r. In particular the series should then converge for $S(i\,\tilde{\alpha}_e)$ with $|\tilde{\alpha}_e| < r$. This however describes a theory wherein equal charges attract and opposite charges repel, which has an unstable vacuum and is not expected to converge. The conclusion is then that $r = 0$, meaning that the radius of convergence of QED is zero. When a series has zero radius of convergence it might however still be an asymptotic series, wherein only the first few terms of the power series expansions in the coupling constant improve the theoretical prediction and thereafter they worsen it. (The closer one takes α_e to zero the more terms one can take in the expansion until the error increases.)

Similarly there is question about the convergent behaviour of QCD. This can be shown by the presence of so-called renormalon ambiguities in the pole-mass. A short introduction to renormalons is given in one the following sections. Neglecting renormalons, the one-loop relation between the $\overline{MS}$- and pole-mass is given by eq. 140. In the literature one often finds this result with the explicit factor $C_F = 4/3$ (see eq. 12.) Using this relation one can convert between both renormalization schemes at one-loop order.

6.3. Asymptotic series

As was noted, it is suspected that perturbative series in the coupling constants of a quantum field theory, even after renormalization, are in fact asymptotic series. These kind of series are formalised in the definition below (see ref. [17]):

Definition. The power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to be asymptotic to the function $y(x)$ as $x \to x_0$ and we write $y(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$ for $x \to x_0$ if $y(x) - \sum_{n=0}^{N} a_n(x-x_0)^n \ll (x-x_0)^N$ as $x \to x_0$ for every N.

Note that the symbol $\ll$ indicates the following limit:

$$f(x) \ll g(x) \text{ as } x \to x_0 \iff \lim_{x\to x_0} \frac{f(x)}{g(x)} = 0 \tag{143}$$

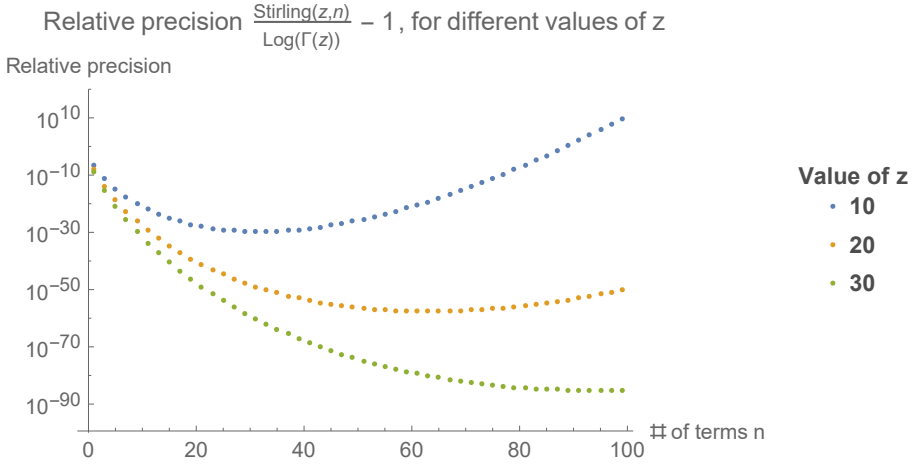
Thus asymptotic series of a function $f(x)$ satisfy the property that their partial sums can be made an arbitrarily good approximation of $f(x)$ when x tends to a particular point x_0, which might be infinity. This leaves open the possibility that the fully summed series is divergent. An often stated example is the Stirling series given below[2]:

$$\log\Gamma(z) = \frac{1}{2}\log(2\pi) + (z-\frac{1}{2})\log(z) - z + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)z^{2n-1}} \tag{144}$$

$$= \frac{1}{2}\log(2\pi) + (z-\frac{1}{2})\log(z) - z + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \dots \tag{145}$$

[2]This result has been taken from Wolfram Mathworld. The Stirling series can be found in many places of relevant literature.

The terms B_n are the Bernoulli numbers. While the series sums to infinity for any z, any truncated sum becomes a perfect approximation to $\log\Gamma(z)$ as $z \to \infty$. This is illustrated in the following graph[3]:



Even though the series ultimately diverges for each value of z, a very good approximation to the function sought is found by truncating the sum at some optimal point. Similar behaviour is expected for QCD. After including a significant number of loops, amplitudes are expected to start diverging from the physical values which are measured.

Often divergent asymptotic series can be assigned a finite value when one loosens their interpretation. It is possible to not interpret the series as an actual summation, but as a mathematical object called a 'formal' power series. Formal power series are commonly used to prove various properties of power series while sidestepping the question of convergence.

A few short examples of how to sum infinite series are given next, which are inspired by the PSI lectures on mathematical physics of Carl Bender, which are available online. One might define a general summation machine S for a formal series which has the following properties of associativity and linearity:

1. $S(a_0 + a_1 + a_2 + \ldots) = a_0 + S(a_1 + a_2 + \ldots)$
2. $S(\sum_{i=0}^{\infty}(\lambda_1 a_i + \lambda_2 b_i)) = \lambda_1 S(\sum_{i=0}^{\infty} a_i) + \lambda_2 S(\sum_{i=0}^{\infty} b_i)$

Note that commuting terms inside a sum is not generally allowed. That means the sum $a_0 + a_1 + \ldots$ is really to be interpreted as some infinite set of parameters $(a_0, a_1, \ldots)$ which is fed to the summation machine S. Using the above definitions, the formal geometric series gets its usual values for any complex value z. Say one takes $z = -2$ and looks at the sum $\sum_{i=0}^{\infty}(-2)^i$. One finds:

$$S\left(\sum_{i=0}^{\infty}(-2)^i\right) = 1 + S\left(\sum_{i=1}^{\infty}(-2)^i\right) = 1 - 2S\left(\sum_{i=0}^{\infty}2^i\right) \tag{146}$$

$$\Rightarrow S\left(\sum_{i=0}^{\infty}2^i\right) = \frac{1}{3} = \frac{1}{1+2} \tag{147}$$

There is a good reason for not generally allowing terms to be commuted within the series. Commutation of terms makes no difference when a series is convergent, but for a divergent sum

[3]This graph has been inspired by the treatment of the Stirling series on Wikipedia, at the time of writing.

different results are obtained when permuting terms. For example, the sum $\sum_{i=0}^{\infty}(-1)^i$ is equal to $1/2$ using the summation machine. On the other hand:

$$\begin{aligned} s &\equiv S(1+0-1+1+0-1+\ldots) \\ s-1 &= S(0-1+1+0-1+1+\ldots) \\ s-1 &= S(-1+1+0-1+1+0+\ldots) \end{aligned} \tag{148}$$

Adding these three equations together gives a term $S(0+0+\ldots)=0$ on the right side, so that one concludes $s=2/3$. This result is different from $\sum_{i=0}^{\infty}(-1)^i = 1/2$ even thought the sums look superficially similar.

What can be taken away from the short discussion is that finite values can sometimes be assigned to divergent sums, but one needs to be consistent in the method used. (Similarly one can use regularization and renormalization to remove infinities from a QFT, which gives unambiguous results as long as both are done consistently.)

A summation technique that satisfies at least the two properties of the summation machine described above is Borel summation, which is discussed next.

6.4. Borel summation

Consider a formal power series R in a coupling constant g of some theory. The series of R in g and its so-called Borel transform $\mathcal{B}(R)(g)$, named after Émile Borel, are given below:

$$R(g) = \sum_{n=-1}^{\infty} r_n g^{n+1} \qquad\qquad \mathcal{B}(R)(g) \equiv r_{-1}\delta(g) + \sum_{n=0}^{\infty} \frac{r_n}{n!} g^n \tag{149}$$

The Borel transform generally has a larger radius of convergence because the coefficients of the original series are divided by factorials. Suppose that the Borel transform converges for sufficiently small g, and that the following integral, which is called the Borel sum, exists:

$$\mathcal{B}_{\text{sum}}(R)(g) \equiv \int_0^{\infty} e^{-t/g}\mathcal{B}(R)(t)dt \tag{150}$$

According to [17], the Borel sum can be expanded for small g by explicitely plugging in the series of the Borel transform and integrating term-by-term which is justified by a result known as Watson's lemma which is also stated in [17]. This leads to:

$$\mathcal{B}_{\text{sum}}(R)(g) \sim r_{-1} + \sum_{n=0}^{\infty} \frac{r_n}{n!} \int_0^{\infty} e^{-t/g} t^n dt = \sum_{n=-1}^{\infty} r_n g^{n+1} \text{ as } g \to 0+ \tag{151}$$

The asymptotic relation follows from Watson's lemma and the last equality comes from applying repeated integration by parts on the integral:

$$\begin{aligned} \int_0^{\infty} e^{-t/g} t^n dt = \left[-g e^{-\frac{t}{g}} t^n\right]_0^{\infty} + ng \int_0^{\infty} e^{-t/g} t^{n-1} dt &= ng \int_0^{\infty} e^{-t/g} t^{n-1} dt \\ &= n(n-1)g^2 \int_0^{\infty} e^{-t/g} t^{n-2} dt \\ &= \ldots \\ &= \left(\int_0^{\infty} e^{-t/g} dt\right) n! g^n = n! g^{n+1} \end{aligned} \tag{152}$$

(Note that the Borel transform in [17] is stated slightly different from here. The Borel transform given here is chosen consistent with the conventions of [18].) In case the original power series R does not converge, but the Borel sum does, it can be interpreted as a summation machine.

6.5. Renormalons

By looking at certain unique infinite subsets of Feynman diagrams in perturbative series in QCD one can identify non-perturbative contributions. In particular one can consider diagrams with n massless quark bubble insertions. It turns out that integrating these diagrams over momentum space leads to $n!$-behaviour which can be dealt with by doing a Borel transform. In some cases it turns out however that there are poles in the positive real axis of the Borel plane which means the Borel sum can't be unambiguously defined without making a choice of integration convention around these poles. These poles are called renormalons, and were first considered by 't Hooft.

This will be illustrated with an example, which consists of n massless quark bubble insertions in the gluon propagator of the massive quark self-energy diagram. This is drawn in fig. 3. The following treatment of this type of diagram is based on ref. [18] and [19].

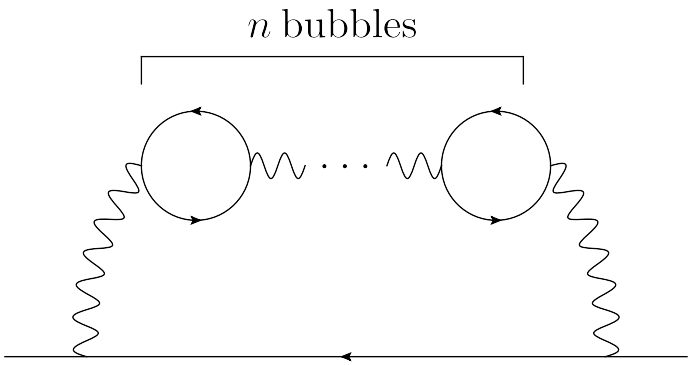
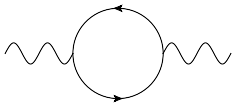


Figure 3: *n massless bubble insertions in the massive quark one-loop self-energy diagram.*

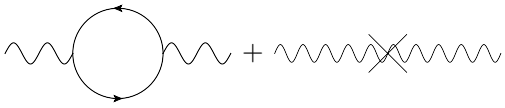
We will consider the sum over all n of the diagrams from fig. 3 and take a Borel transform of the sum before integrating over the internal momentum. Because the bubbles are all located in the gluon propagator it turns out one can define a 'modified' gluon propagator which is the Borel transform of the sum of gluon propagators with n massless quark bubble insertions. The Borel transform of the sum of diagrams in fig. 3 is then simply found by calculating the one-loop quark self-energy diagram but with the gluon propagator replaced by the 'modified' gluon propagator.

The motivation behind looking at massless quark bubbles is that they admit a simpler expansion in ϵ than massive bubbles. Each bubble will be considered to be renormalized in the $\overline{\text{MS}}$-scheme. From the gluon self-energy calculation in section 8.3 it follows that:



$$= \frac{ig^2 T_F N_l \delta_{ab}(k^\mu k^\nu - k^2 \eta^{\mu\nu})}{12\pi^2} \left(\frac{1}{\epsilon} + \left(\frac{5}{3} - \gamma_E + \log(4\pi) - \log(-k^2/\mu^2) \right) + \mathcal{O}(\epsilon) \right) \tag{153}$$

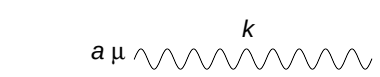
Renormalizing the bubble in the $\overline{\text{MS}}$-scheme leads to:



$$= \frac{ig^2 T_F N_l \delta_{ab}(k^\mu k^\nu - k^2 \eta^{\mu\nu})}{12\pi^2} \left(-\log(-k^2 e^{-5/3}/\mu^2) \right) \tag{154}$$

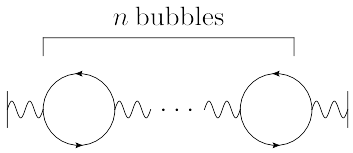
where the factors $\log(4\pi)$ and γ_E disappear by convention of the $\overline{\text{MS}}$-scheme. As the single exception in this thesis the counterterms will not be explicitely drawn in the next few equations,

but the bubbles are considered to be renormalized and hence finite. We consider the gluon propagator in the Landau gauge $\alpha = 0$. This means that:



$$= \frac{-i\delta_{ab}}{k^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \tag{155}$$

By concatenating numerous gluon propagators and massless renormalized bubbles one finds the below result:



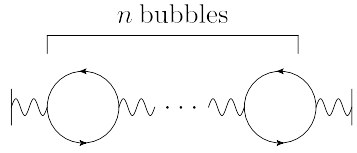
$$= \frac{i\delta_{ab}}{k^4}(k^\mu k^\nu - k^2\eta^{\mu\nu})\left(\frac{\alpha_s T_F N_l}{3\pi}\log(-k^2 e^{-5/3}/\mu^2)\right)^n \tag{156}$$

Note that the coupling constant has been replaced by $\alpha_s \equiv g^2/(4\pi)$, which is the more common notation in the literature, and that the external lines are included as this diagram will be inserted into another one. Next a somewhat hand-waving operation will be done. The reader is referred to [18] for a more in-depth analysis. We will replace N_l by $-3\beta_0/(4T_F)$, where $\beta_0 = \frac{11C_A}{3} - \frac{4T_F}{3}N_l$ is the first coefficient of the QCD β-function defined by:

$$\mu\frac{d}{d\mu}\alpha_s(\mu) = -\frac{\beta_0}{2\pi}\alpha_s^2(\mu) \tag{157}$$

One can think of this procedure as including some of the non-abelian behaviour of QCD into the bubble diagrams. In [18] this replacement is motivated by considering the calculation to be part of a $1/N_l$-expansion. In that case one can write that $\beta_0 \approx -\frac{4T_F}{3}N_l$ and solve for N_l in terms of β_0 which gives $N_l \approx -3\beta_0/(4T_F)$. However, in this approximation QCD loses the property of asymptotic freedom, so to restore the usual properties β_0 is taken with its full value.

We will keep indicating the expression with bubble diagrams so that we write:



$$= \frac{i\delta_{ab}}{k^4}(k^\mu k^\nu - k^2\eta^{\mu\nu})\left(-\frac{\alpha_s\beta_0}{4\pi}\log(-k^2 e^{-5/3}/\mu^2)\right)^n \tag{158}$$

$$\equiv \frac{i\delta_{ab}}{k^4}(k^\mu k^\nu - k^2\eta^{\mu\nu})r_n u^n \tag{159}$$

where $r_n = -\log(-k^2 e^{-5/3}/\mu^2)^n$ and $u = \alpha_s\beta_0/(4\pi)$. Next we take the Borel transform of the sum of all n-bubble contributions seen as a power series in u. Note that in eq. 149 the numbering between the coefficients and the power on the expansion parameter differs by 1. This was a convenient definition, because there will be an additional term α_s proportional to u coming from the two quark gluon vertices in fig. 3. Taking the Borel transform of the sum over n of the terms $r_n u^n$ in eq. 158 multiplied by an additional factor u gives:

$$\mathcal{B}\left(\sum_{n=0}^{\infty} r_n u^{n+1}\right)(u) = \sum_{n=0}^{\infty}\frac{r_n}{n!}u^n = \left(-k^2 e^{-5/3}/\mu^2\right)^{-u} \tag{160}$$

It is now possible to calculate the one-loop quark self-energy diagram with the 'modified' gluon propagator at the location of the usual one. One will need to take care that the factor g^2 coming from the quark gluon vertices has already been partly absorbed. In particular one has

$g^2 = -12\pi^2 u/(T_F N_l)$ and the factor u has already been absorbed in the Borel transform. Note that the pole mass satisfies:

$$m_P = m_{\overline{\text{MS}}} + \Sigma_{\overline{\text{MS}}}(\not{p}, m_{\overline{\text{MS}}})\Big|_{\not{p}=m_P} \tag{161}$$

Make a decomposition of the self-energy:

$$\Sigma_{\overline{\text{MS}}}(\not{p}, m_{\overline{\text{MS}}})\Big|_{\not{p}=m_P} = m_{\overline{\text{MS}}} A(m_P, m_{\overline{\text{MS}}}) + m_P B(m_P, m_{\overline{\text{MS}}}) \tag{162}$$

This leads to:

$$m_P = m_{\overline{\text{MS}}} \left(\frac{1 + A_{\overline{\text{MS}}}(m_P, m_{\overline{\text{MS}}})}{1 - B_{\overline{\text{MS}}}(m_P, m_{\overline{\text{MS}}})} \right) \tag{163}$$

Following the treatment in [18] we use that A and B are proportional to $1/N_l$, so that:

$$m_P = m_{\overline{\text{MS}}} \left(1 + A_{\overline{\text{MS}}}(m_P, m_{\overline{\text{MS}}}) + B_{\overline{\text{MS}}}(m_P, m_{\overline{\text{MS}}}) + \mathcal{O}\left(\frac{1}{N_l^2}\right) \right) \tag{164}$$

Note that we can plug eq. 164 into itself and expand up to order $\mathcal{O}\left(1/N_l^2\right)$ which then leads to:

$$m_P = m_{\overline{\text{MS}}} \left(1 + A_{\overline{\text{MS}}}(m_{\overline{\text{MS}}}, m_{\overline{\text{MS}}}) + B_{\overline{\text{MS}}}(m_{\overline{\text{MS}}}, m_{\overline{\text{MS}}}) + \mathcal{O}\left(\frac{1}{N_l^2}\right) \right) \tag{165}$$

Hence one can calculate A and B and put $\not{p} = m_P$. This leads to the following result, which is quoted from [18]:

$$m_{\text{pole}} = m_{\overline{\text{MS}}} \left(\delta(u) + \frac{C_F}{4\pi N_l} \left(\left(\frac{m_{\overline{\text{MS}}}^2}{\mu^2} \right)^{-u} e^{5u/3} 6(1-u) \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} - \frac{3}{u} + R_{\Sigma_1}(u) \right) \right) \tag{166}$$

The term $-3/u$ comes from renormalizing the whole renormalon diagram, and the term $R_{\Sigma_1}(u)$ is an entire function in the Borel plane which depends on the renormalization scheme. In the $\overline{\text{MS}}$-scheme it holds that $R_{\Sigma_1}(u) = -5/2 + 35u/24 + \mathcal{O}(u^2)$. The delta function $\delta(u)$ comes from transforming the constant 1 into Borel space. Looking at the eq. 168 there are seen to be poles at:

$$u \in \mathbb{Z}_{\leq 0} \cup \left\{ \frac{1}{2}, \frac{3}{2}, \frac{4}{2} \right\} \cup \left\{ \frac{5}{2} + k \,\middle|\, k \in \mathbb{Z}_{\geq 0} \right\} \tag{167}$$

In particular the pole mass has poles in u on the positive real axis which means the Borel sum cannot be unambiguously defined. The strongest renormalon is the $u = 1/2$ renormalon. One can choose to either integrate slightly above the pole or slightly below. Therefore the ambiguity in the pole mass can be estimated by the absolute value of $1/2$ times the integral around the pole at $u = 1/2$. Expanding around $u = 1/2$ yields:

$$m_{\text{pole}} \approx m_{\overline{\text{MS}}} \left(\delta(u) - \frac{\mu}{m_{\overline{\text{MS}}}} \frac{e^{5/6} C_F}{2\pi N_l \left(u - \frac{1}{2}\right)} \right) \tag{168}$$

One therefore finds that:

$$\text{ambiguity} \approx \frac{1}{2}\left| \oint \left(-\frac{\mu e^{5/6} C_F \exp\left(-\frac{4\pi u}{\beta_0 \alpha_s}\right)}{2\pi N_l \left(u - \frac{1}{2}\right)} \right) du \right| = \frac{e^{5/6} C_F}{2N_l} \mu e^{-\frac{2\pi}{\beta_0 \alpha_s}} = \frac{e^{5/6} C_F}{2N_l} \Lambda_{\text{QCD}} \tag{169}$$

$$\approx (170 - 180)\ \text{MeV} \tag{170}$$

where Λ_{QCD} is the QCD scale parameter, and the numerical estimate is taken from [18]. The integral around $u = 1/2$ was easily evaluated using the residue theorem.

6.6. Vacuum instability

From analysis of the running of the Higgs quartic coupling the possibility has been observed that the Higgs potential can contain a lower minimum than the electroweak (EW-) vacuum. This additional minimum is observed for particular regions of values of the Higgs mass and the top quark mass to which the potential is highly sensitive as a result of providing the dominant contributions to $\beta(\lambda) = d\lambda/d\log\mu$. The slope of the running coupling $\lambda(\mu)$ for a fixed value of $M_h = 125.7$ GeV and varying values of the top quark mass is shown in fig. 4, taken from [20].

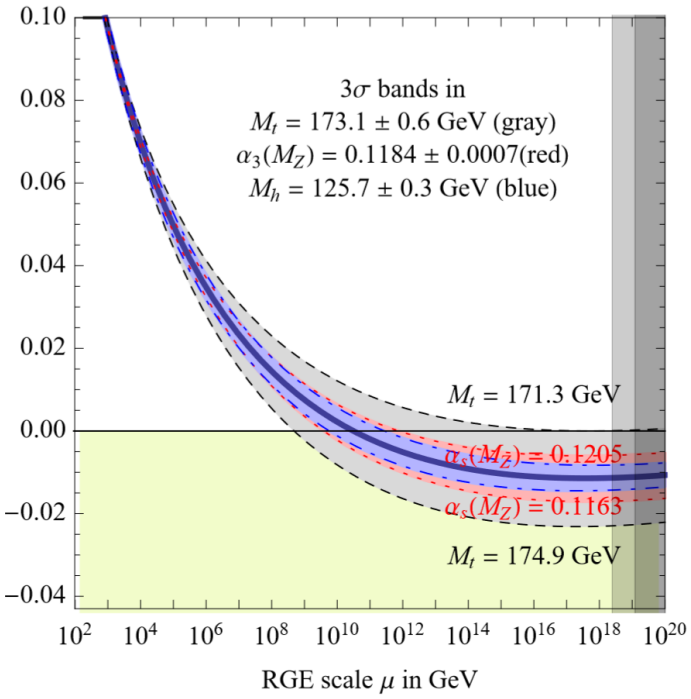


Figure 4: *Plot of the running quartic coupling $\lambda(\mu)$. (Figure taken from [20].)*

It is seen that the running coupling becomes negative at sufficiently high μ approaching the Planck scale, which leads to the Higgs potential becoming unstable. The reason for the unstability is that for high values of the Higgs field h the Higgs potential is dominated by the quartic term $\frac{1}{4}\lambda(\mu)h^4$. Furthermore, at large field values the coupling should be evaluated at a scale $\mu \sim h$. Hence this leads to the Higgs potential becoming deeper than the electroweak vacuum at sufficiently high scales. If our universe is indeed in a local minimum, it could move to the

lower global vacuum by either the occurence of extremely high energy processes crossing the potential barrier or as a result of quantum tunneling. If such an event were to happen a bubble residing in the new vacuum (with physics behaving very differently) would expand throughout space, effectively ending the state of the universe as we know it (and all life in it). Such an event is sometimes referred to as a 'vacuum collapse' event.

Because extremely high-energy particle collisions occur all the time in the universe, much larger than the energies obtainable at particle accelerators like the LHC, and because no vacuum collapse event has yet occured (because we would not be around to wonder about it), current accelerators do not present a risk of invoking such an event. However, through quantum tunneling a miniscule probability of decaying to the new vacuum state can add up to a close to certainty over time. For this reason it makes sense to look at the electroweak vacuum lifetime τ_{EW} defined as the expected mean lifetime of the electroweak vacuum. Furthermore let τ_{U} be the age of the universe. One might then distinguish the following cases:

1. Instability $\tau_{\text{EW}} < \tau_{\text{U}}$
2. Meta-stability $\tau_{\text{EW}} > \tau_{\text{U}}$
3. Complete stability $\tau_{\text{EW}} = \infty$

The contours of these three regions are plotted out in the (m_h, m_t)-plane in fig. 5, taken from [20].

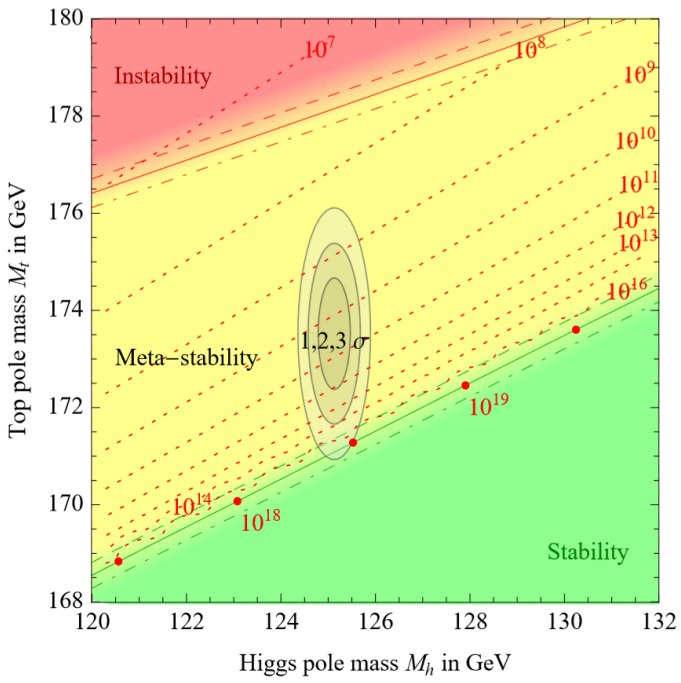


Figure 5: *Stability of the electroweak vacuum, plotted out in the (m_h, m_t)-plane. (Figure taken from [20].)*

It is seen that the current experimental measurements predict a metastable vacuum, in which vacuum collapse will happen at some point in the future. A precise experimental determination of the top quark mass gives a better window into the behaviour of the Higgs potential and as such the life-time of the electroweak vacuum. Furthermore, multiloop theoretical calculations yield a

more precise determination of the running quartic coupling and running top mass, and could alter the shape of the contour plot.

It should be noted that some significant assumptions are at play in determining the stability of the vacuum in the way described. In particular, in using the running coupling at very high scales it is assumed that the effects of new physics remaing negligible which might not be the case. Furthermore, around the Planck scale the effects of quantum gravity might also change the physical picture. Estimating the effects of these kinds of contributions can be done by making assumptions on the behaviour of the new physics which is outside the scope of this thesis. Nonetheless it is seen that high precision measurement and calculation of the top quark mass is a subject that can have significant phenomenological implications.

VII. Introducing a setup for NNLO and N^{3}LO calculations

7.1. Introduction

The preceding sections have dealt with outlining the basics of QCD and the main topic of interest in this thesis: to do automated multiloop calculations for the renormalization of the top quark mass and field. To find the renormalization coefficients the self-energy has to be calculated up to the desired loop order. Therefore an automated setup has been developed for calculating 2-point 1 particle irreducible Feynman diagrams. The main ideas involved in the automated calculation will be discussed in the following section. The global outline of the calculation is given below:

Concept	Approach used
Graph generation	QGRAF
Plugging in Feynman amplitudes	Custom FORM code
Finding a minimal set of topologies	Custom Mathematica code
Reducing integrals by IBP-relations	FIRE5 & Mathematica
Finding equivalent master integrals	Custom Mathematica code
Evaluation of master integrals	Manual calculations, finite integrals expansions, HyperInt
Finding the renormalization constants	Custom mathematica calculation / code

A supervisor script is needed to connect all the parts together and communicate expressions between different programs and scripts. A Mathematica notebook is used for this purpose. While generally no explicit code will be given, the ideas and concepts will be treated so that the reader should be able to build a similar setup based on the ideas discussed here.

FORM Use will be made of the computer algebra system FORM, which is created and developed by dr. Jos Vermaseren at Nikhef and is widely used in particle physics. It has been in development since 1984. It is well suited for tensor algebra and contains numerous routines that aid in common particle physics calculations. For example it has inbuilt support for tensor algebra and Dirac traces in arbitrary dimension. Furthermore a package is available for expressing SU(N_c) color traces in Casimir invariants, described in [4]. For this reason FORM is perfectly suited to automate the evaluation of Feynman rules, to reduce products of Dirac matrices, and to evaluate the color algebra. At the time of writing, FORM can be found at its Nikhef homepage: `https://www.nikhef.nl/~form/`. An extensive manual with examples is available on the webpage and therefore no further discussion will be given on specific FORM code.

7.2. Graph generation

The first thing needed is a way to represent Feynman diagrams in a non-visual form, and to automatically generate all diagrams needed. In the automated setup developed for this thesis graph generation is done using a program called QGRAF, which is widely used in particle physics and is available on the author's website. A detailed explanation of the QGRAF algorithm is found in the paper [21], and the program comes bundled with a manual that explains all possible options. An alternative is the Mathematica package 'FeynArts' which has similar abilities but also comes with more baggage and custom notation in setting up the QCD model and processing the diagrams.

QGRAF is capable of presenting Feynman diagrams as a list of propagators and vertices and gives their connections by assigning numbers to the lines of vertices and heads and tails of propagators. Matching numbers indicate that the vertex and propagator are connected. Additionally it gives the prefactors of the diagrams, which consists of the symmetry factors and the minus signs from fermion loops.

QGRAF takes a style file, a model file and a configuration file as input. The model file specifies the propagators and whether they are fermionic or bosonic and also the vertices of the theory. The style file specifies the notation of QGRAF's output which allows it to yield output compatible with many different programs. Lastly a configuration file specifies the type of diagrams desired by specifying the external particles of the diagrams and a set of filters. These filters can be set to pick for example only 1-particle irreducible diagrams or diagrams without tadpoles.

The output of QGRAF should be imported in a computer algebra system (CAS) to plug in the Feynman rules. In my setup this is done using FORM, which is described later on. It was found convenient to specify the style file so that the output of QGRAF for each diagram is actually a FORM statement defining a local expression. The vertices and propagators are presented with a multiply statement between them and are given as symbolic functions of the external indices and momenta. These symbolic function are then substituted with Feynman rules in FORM using identify statements. The style file which has been used is given below:

```
<prologue>
<diagram>
Local Diag<diagram_index>  = <back>
(<sign>1)*
(<symmetry_factor>)*
<propagator_loop>qgraf<field>(<momentum>,g<field_index>,
<back>g<dual-field_index>,i<field_index>,
<back>i<dual-field_index>,a<field_index>,
<back>a<dual-field_index>,mu<field_index>,
<back>mu<dual-field_index>)*
<end><back>
<vertex_loop>qgrafv<ray_loop><field><end>(<ray_loop>
<back><momentum>,g<field_index>,i<field_index>,a<field_index>,mu<field_index>,
<back><end><back>)*
<end>1;
<epilogue>
<exit>
```

The QGRAF output for the 1-loop quark self-energy graph using our style file is given below:

```
Local Diag1  =
(+1)*
```

```
(1)*
qgrafu(-k1+p1,g1,g2,i1,i2,a1,a2,mu1,mu2)*
qgrafg(-k1,g3,g4,i3,i4,a3,a4,mu3,mu4)*
qgrafvUug(k1-p1,g2,i2,a2,mu2,p1,gn,i,b,nu,-k1,g3,i3,a3,mu3)*
qgrafvUug(-q1,g0,j,a,mu,-k1+p1,g1,i1,a1,mu1,k1,g4,i4,a4,mu4)*
1;
```

Note that the quark-gluon vertex carries two color indices i, j in the fundamental representation, a color index a in the adjoint representation and a spacetime index μ (see the Feynman rules in app. B.) The symbolic functions in QGRAF's output contain much more indices. What is done is that for each line the indices gx, ix, ax and mux are printed out, where x denotes the number belonging to the line. (The indices gx will be used for the components of Dirac matrices.) The indices which are not needed are simply ignored. For example the following 'id' statement in FORM amounts to plugging in the quark-gluon vertex:

```
id qgrafvUug(k1?,g1?,i1?,a1?,mu1?,k2?,g2?,i2?,a2?,mu2?,k3?,g3?,i3?,
        a3?,mu3?)=-i_*gs*gph(mu3,g1,g2)*T(i1,i2,a3);
```

The function gph is temporarily used to indicate a Dirac matrix with components g1 and g2. FORM has inbuilt support for Dirac matrices but considers them as noncommutative objects without indices. Hence, after all the propagators and vertices are plugged in some code is used to put the gph's in the right order of multiplication by ordering them based on their indices. Identify statements are then used to change the gph's into the inbuilt Dirac matrix notation of FORM denoted by _g.

Note that QGRAF outputs all diagrams in a single file. A small script has been written in the supervisor notebook that splits the QGRAF output into multiple files by searching for occurrences of ";", indicating the end of a FORM expression. Multiple instances of FORM are then run in parallel for each diagram, with no more instances active at the same time than the number of cores on the system.

7.3. First steps of the calculation

After plugging in the Feynman rules using FORM a number of simplifications are performed which are outlined next.

7.3.1 Simplifying the Dirac algebra

As was already seen during the 1-loop calculation one can make a decomposition of the quark self-energy in the following manner:

$$\Sigma(\not{p}, m) \equiv A(p^2, m^2)m + \not{p}B(p^2, m^2) \tag{171}$$

Because the trace over a single gamma matrix is zero the coefficients A and B can be expressed in terms of the self-energy on the left:

$$\mathrm{Tr}(\Sigma) = 4m \cdot A \quad \mathrm{Tr}(\not{p}\Sigma) = 4p^2 \cdot B \tag{172}$$

This means that one can express the self-energy of the quark in the following manner:

$$\Sigma = \frac{1}{4}\left(\mathrm{Tr}(\Sigma) + \not{p}\,\mathrm{Tr}(\not{p}\Sigma)/p^2\right) \tag{173}$$

Hence the self-energy can be calculated from the Dirac traces $\mathrm{Tr}(\Sigma)$ and $\mathrm{Tr}(\not{p}\Sigma)$. Calculating Dirac traces is inbuilt in FORM. Alternatively, if one doesn't use FORM, it is possible to prepare a table of trace identities of gamma matrices and lookup the traces in the table.

For a MS calculation eq. 173 can be used to find the self-energy. For an on-shell calculation the self-energy doesn't have to be directly computed but instead the quantity T_R of eq. 92 can be constructed which also involves doing a trace of the Dirac algebra.

7.3.2 Tracing over the color algebra

Because color charges are conserved the quark (full) propagator and self-energy are proportional to δ_{ij}. Therefore, after the Feynman rules are plugged in, the color factors can be traced over leading to:

$$\Sigma = \mathrm{Tr}_{\mathrm{col}}(\Sigma)\delta_{ij}/N_c \tag{174}$$

The algorithms from [4] and corresponding FORM package are used to write the color trace into Casimir elements.

7.3.3 Reducing tensor integrals

When calculating complicated Feynman integrals with external lines that carry space-time indices one will encounter tensor integrals. In principle this will not be the case in this thesis because the main focus lies on calculating the top quark self-energy, so that 2-point diagrams are considered that carry color indices but not space-time indices. It is therefore guaranteed that after doing a Dirac trace the self-energy contains only scalar integrals. It is however still of interest to shortly discuss how tensor integrals can be reduced to scalar integrals. Furthermore the gluon self-energy calculation is presented at 1-loop order which does require a reduction of tensor integrals. Consider a Lorentz invariant Feynman integral with a tensorial term t in the numerator composed of the internal momenta:

$$I^{\vec{\mu}}(p_1,\ldots,p_{h'}) = \prod_{i=1}^{h}\left(\int d^d k_i\right) t^{\vec{\mu}}(k_1,\ldots,k_h)s(k_1,\ldots,k_h,p_1,\ldots,p_{h'}) \tag{175}$$

The scalar term s can be a product of propagators (e.g a product of terms like $1/(k^2-m^2)$) which depends on h' external momenta p_i and h internal momenta k_i. There might also be dot products of momenta in s, but no tensorial terms, as these are assumed to be in $t^{\vec{\mu}}$. In particular, $\vec{\mu} = (\mu_1,\ldots,\mu_{n_t})$ is a vector of space-time indices. In principle some of these spacetime indices might be contracted. In the following treatment we consider each space-time index to be independent so that the Einstein summation convention is not working on any combination of the indices. The case where some of the indices are contracted follows from contracting the end result of the next discussion with Minkowski matrices.

Because the internal momenta are integrated out we can conclude the space-time indices must be transfered to tensors with $n_t = \sum_{i=1}^{h} n_i$ space-time indices which are formed from products of Minkowski matrices η and external momenta. We let S denote the set of all such tensors with n_t indices which are independent. Then we can write that:

$$I^{\vec{\mu}} = \sum_{s\in S} A_s\, s^{\vec{\mu}} \tag{176}$$

Here $\{A_s \,|\, s \in S\}$ is a set of scalars for which can be solved by doing contraction on both sides of eq. 176 for each element $s' \in S$. This yields $\#S$ equations and allows one to solve for all constants

A_s, which are scalar integrals. To make this more concrete, consider the simple case with internal momentum k, external momentum p, and the following integral:

$$\int d^d k\, k^\mu k^\nu s(k,p) = A p^\mu p^\nu + B\eta^{\mu\nu} \tag{177}$$

A system of equations is constructed by contracting with each element in $S = \{p^\mu p^\nu, \eta^{\mu\nu}\}$ to find:

$$\int d^d k\, (k\cdot p)^2 s(k,p) = A p^4 + B p^2 \tag{178}$$

$$\int d^d k\, k^2 s(k,p) = A p^2 + B d \tag{179}$$

The system is easily solved and yields:

$$A = \frac{1}{(d-1)p^4}\int d^k \left(d(k\cdot p)^2 - k^2 p^2\right) s(k,p) \tag{180}$$

$$B = \frac{1}{(d-1)p^2}\int d^d k \left(k^2 p^2 - (k\cdot p)^2\right) s(k,p) \tag{181}$$

Plugging this back into eq. 177 then gives the reduction of the tensor integral to scalar integrals. The dot products of momenta which reside in the prefactor of s and can also be in s itself can additionally be removed by solving for the dot products in terms of the propagators of the integral which is possible for complete topologies, a subject that is introduced in the following section.

The relations of eqs. 180 and 181 will be used to calculate the 1-loop gluon self-energy result which is given in section 8.3.

7.4. Finding a minimal set of topologies and rewriting dot-products

Code written in FORM is used to plug in the Feynman rules for each diagram using the output from QGRAF, and to do the simplifications outlined in the previous section. At this point every diagram is expressed as a sum of scalar Feynman integrals multiplied by some coefficients.

To discuss these Feynman integrals it is convenient to use the term 'massive propagator' to denote a factor $1/(k^2 - m^2)$ for some mass and some momentum and 'massless propagator' to denote a factor $1/k^2$ for some momentum. Because the QCD model in this thesis deals with a set of massive quarks of equal mass the term 'massive propagator' is not ambiguous.

It also turns out to be convenient to assign symbols to the denominators of each propagator, because these symbols will be easier to deal with algebraically. These symbols are usually denoted by D_i in this thesis and are called the denominator factors. Sometimes the denominator factors will be called 'propagators' as well when it does not lead to confusion. Example: in the integral below the (denominators of the) propagators are the terms $D_1 \equiv -k^2 + m^2$ and $D_2 \equiv -(k-p)^2$:

$$I_{\text{dot-sample}} = \int d^d k \frac{k\cdot p}{(-k^2+m^2)^{\lambda_1}(-(k-p)^2)^{\lambda_2}} = \int d^d k (k\cdot p) D_1^{-\lambda_1} D_2^{-\lambda_2} \tag{182}$$

The denominator factors are given with a minus sign on the momenta. This turns out to be the natural way of representing the propagators when taking the alpha-parametrization, which is derived in sec. IX. One can solve for $k\cdot p$ in terms of the denominator factors: $k\cdot p = \frac{1}{2}(-D_1 + D_2 + m^2 + p^2)$. If one uses this expression the integral can be rewritten as:

$$\begin{aligned} I_{\text{dot-sample}} &= \int d^d k \frac{k\cdot p}{(-k^2+m^2)^{\lambda_1}(-(k-p)^2)^{\lambda_2}} = \frac{1}{2}\int d^d k \frac{(-D_1 + D_2 + m^2 + p^2)}{D_1^{\lambda_1} D_2^{\lambda_2}} \\ &= -\frac{1}{2}\int d^d k \frac{1}{D_1^{\lambda_1 - 1} D_2^{\lambda_2}} + \frac{1}{2}\int d^d k \frac{1}{D_1^{\lambda_1} D_2^{\lambda_2 - 1}} + \frac{1}{2}(m^2+p^2)\int d^d k \frac{1}{D_1^{\lambda_1} D_2^{\lambda_2}} \end{aligned} \tag{183}$$

It is now seen that the sample integral is completely represented by integrals over the denominator factors. In general we want to consider only integrals over denominator factors with no dot products in the numerator, which will be called topologies. The name 'topology' comes from the fact that they admit diagrammatic representations. Define the following topology:

$$T^{1,2}(\lambda_1, \lambda_2) \equiv \int d^d k \, D_1^{-\lambda_1} D_2^{-\lambda_2} \tag{184}$$

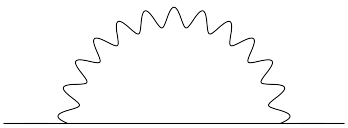
The first superscript index in $T^{1,2}$ indicates it is a 1-loop topology and the second index labels the topology with a number. In app. A the results for all integrals considered in this thesis are presented together with the numbering of the integrals. When the powers of the propagators are integral they are often put in the subscript:

$$T^{1,2}_{a_1,a_2} \equiv \int d^d k \, D_1^{-a_1} D_2^{-a_2} \text{ for } \lambda_i \in \mathbb{Z} \tag{185}$$

In app. A the term topology refers to Feynman integrals for general complex indices λ_i and the term integral is used for Feynman integrals with explicit values plugged in for the powers on the propagators. Note that the sample integral just considered can now be written in terms of the topology:

$$I_{\text{dot-sample}} = \frac{1}{2}\left(-T^{1,2}(\lambda_1 - 1, \lambda_2) + T^{1,2}(\lambda_1, \lambda_2 - 1) + (m^2 + p^2) T^{1,2}(\lambda_1, \lambda_2)\right) \tag{186}$$

The diagrammatic representation of this topology is just the 1-loop diagram that was considered in section V:



Diagrammatic representations of topologies are interpreted using simple Feynman rules where the massive propagators, denoted by solid lines, correspond to factors $1/(k^2 - m^2)$ and the massless propagators, denoted by wiggled lines, correspond to factors $1/k^2$, and where k of course denotes the momentum that flows on the line. The momenta that run through the diagram are not explicitely included in the drawing. By not adding the momenta to the diagram it is clear that the topology describes any equally valid choice of momenta over the propagators that satisfy the conservation conditions at the vertices.

We want to be able to define a minimal set of topologies such that all integrals in the calculation can be expressed in terms of these topologies. That means we should only consider topologies that are inequivalent with respect to coordinate transformations. For example, the integrals $\int d^d k (k^2 - m^2)^{-2}$ and $\int d^d k ((k-p)^2 - m^2)^{-4}$ clearly belong to the same topology, which is seen by shifting the second integral by $k \to k + p$. Hence these two topologies are considered to be the same and should not both appear in the set of topologies.

To be able to rewrite dot products in terms of propagators one needs to consider only so-called complete topologies. A topology is called complete when its propagators form a basis for all dot products of internal momenta with internal momenta, and dot products of external momenta with internal momenta. For the one-loop case the momenta are k, p, and the set of dot products is $\{k^2, k \cdot p\}$. Thus the topology $T^{1,2}$ is complete.

In general for h internal momenta and h' external momenta, the number of propagators of a complete topology N_{inv} is given by:

$$N_{\text{inva}} = \frac{h(h+1)}{2} + hh' \tag{187}$$

Hence the 2-loop 2-point quark self-energy calculation has complete topologies consisting of 5 propagators and the 3-loop calculation has complete topologies consisting of 9 propagators.

Now let's look at the number of independent topologies which are encountered in our calculations. In the 1-loop quark self-energy calculation there is only 1 contributing diagram. Hence there is only 1 topology which can be defined by hand. In the 2-loop calculation there are already 6 contributing diagrams, which belong to 2 independent complete topologies. Lastly, in the 3-loop calculation there are about 100 contributing diagrams, belonging to 11 complete topologies.

To find these independent topologies an automated approach is needed, because the number of shift-symmetries can become very large. The shift symmetries are all of the following forms:

$$\begin{pmatrix} k_1 \\ \dots \\ k_h \end{pmatrix} \to \mathbf{A}_{h\times(h+1)} \begin{pmatrix} k_1 \\ \dots \\ k_h \\ p_1 \end{pmatrix} \tag{188}$$

The matrix $\mathbf{A}_{h\times(h+1)}$ has entries in the set $\{-1, 0, 1\}$ and $\det(\mathbf{A}_{h\times h}) = \pm 1$, with $\mathbf{A}_{h\times h}$ denoting the upper $h \times h$ block corresponding to the internal momenta. The determinant has to be equal to ± 1 so that the absolute value of the determinant of the Jacobian of the transformation is 1. The number of matrices $\mathbf{A}_{h\times(h+1)}$ can get very large:

$$\#\mathbf{A}_{h\times(h+1)} = \begin{cases} 7 & \text{1-loop} \\ 361 & \text{2-loop} \\ 187921 & \text{3-loop} \end{cases} \tag{189}$$

Note that the integrals in diagrammatic calculations do not usually have a complete set of propagators. This means that these integrals can be mapped to a subset of many different complete topologies under some change of variables. Hence the set of distinct completed topologies to which the integrals of each diagram can be mapped is significantly smaller than the number of diagrams.

The following steps lay out how to find a minimal set of completed scalar topologies for the diagrams of the 2- and 3-loop calculation:

1. First store the (incomplete) topologies for all diagrams in a list, with entries (k, T) where k is the diagram number and T the corresponding topology. This can be done by looking solely at the QGRAF output, which contains a list of the propagators and the momenta that these propagators carry. Of course when the Feynman rules are plugged in and tensor integrals are reduced there will be many (sometimes thousands of) different integrals in the output of each diagram, but these integrals only differ in the powers on the denominator factors and not the set of denominator factors. (Some integrals might have a power of 0 on some denominator factors though.)

2. Sort the entries (k, T) in order of descending number of propagators so that the diagram and topology with the most propagators is the first entry.

3. Perform the following operations, described below in pseudo-code:

Definitions / initialization
$Q_{\text{rem(aining)}}$ is the ordered list of pairs (k, T) in descending order of #propagators. Let $Q_{\text{rem}}[1]$ denote the first pair in the ordered list.
$Q_{\text{com(pleted)}} = \emptyset$ is a set to be filled with entries (i, T) where i is the number of the completed topology T.
σ is a set of entries (k, i, s) denoting that diagram k belongs to complete topology (i, T) in Q_{com}, after the change of variables s is applied to the incomplete topology of k stored in Q_{rem}.
$S = \{\mathbf{A}_{h\times(h+1)} \in \mathbb{Z}^{h\times(h+1)} : |\det(\mathbf{A}_{h\times(h+1)})| = 1\}$ denotes the set of all allowable shift-symmetries.
$j = 1$ denotes the current completed topology.

Procedure
while (Q_{rem} is non-empty) {
    $(k, T) \leftarrow Q_{\text{rem}}[1]$;
    remove (k, T) from Q_{rem};

    if (number of propagators(T) < N_{inva})
        $T \leftarrow$ random completion of T;

    $Q_{\text{com}} \leftarrow Q_{\text{com}} \cup (j, T)$;
    $\sigma \leftarrow \sigma \cup (k, j, \mathbb{1})$;

    for $((k', T') \in Q_{\text{rem}})$ {
        If ($\exists s \in S$ so that T' under $s \subseteq T$) {
            $\sigma \leftarrow \sigma \cup (k', j, s)$;
            remove (k', T') from Q_{rem};
        }
    }

    $j \leftarrow j + 1$;
}

4 After the previous procedure has been evaluated Q_{rem} is empty, and Q_{com} contains the set of independent complete topologies. Lastly σ specifies which diagram belongs to which topology in Q_{com}.

Note that at some point in the procedure incompleted topologies are extended to complete topologies. Furthermore, the incompleted topologies of the diagrams contain only independent propagators with respect to the scalar invariants. Therefore a completed topology is found by picking some random new propagators so that the set of all propagators form a basis of the scalar invariants. The new propagators are taken massless for simplicity.

7.5. The Mathematica supervisor notebook

As was noted in the introduction, a Mathematica notebook is used to control and link parts of the calculation together. The first steps the Mathematica notebook performs are:

1. Call QGRAF and split the output into multiple files for each diagram.
2. From the QGRAF output find the set of independent topologies.

3. Update the momenta in the QGRAF output by shifting momenta of diagram k according to $s \in (k, i, s) \in \sigma$.

Next one wants to use FORM to perform the operations outlined earlier, like plugging in the Feynman rules. For this purpose a 'template'-file has been put together which contains FORM code but with a few placeholders. In particular there are placeholder for the diagram number, for the denominator factors, and for how to rewrite dot products into denominator factors.

The Mathematica supervisor notebook loads in the template file and for each diagram writes out a FORM file with a specific diagram number and where the substitutions of denominator factors and dot products are done in accordance with the topology that the diagram belongs to.

For example, if a 1-loop diagram is considered that happens to belong to the topology of eq. 185, the supervisor notebook generates the following FORM statements:

```
id k1.k1 = -D1 + mt^2;
id k1.p1 = (-D1 + D2 + mt^2 + p1.p1)/2;
```

These FORM statements are put at the right placeholder and a FORM file is created for the diagram considered. Furthermore Mathematica also automatically creates FORM code to do the expansions in t up to order $\mathcal{O}(t^2)$ of the quantity T_R defined in 92, in the case of the on-shell calculation. This expansion can be done by separately expanding each propagator which depends on t, and using a FORM initialization that states every term of order t^2 or higher should be dropped. For example:

$$\frac{1}{-(k-p)^2} \xrightarrow{p \to Q(1+t)} \frac{1}{-(k-Q(1+t))^2} = \frac{1}{-(k-Q)^2} + \frac{2tQ\cdot(Q-k)}{(-(k-Q)^2)^2} + \mathcal{O}(t^2) \tag{190}$$

Note that:

$$\begin{aligned} Q\cdot(Q-k) &= m^2 - \frac{1}{2}(-D_1 + D_2 + 2m^2) \\ &= \frac{1}{2}(D_1 - D_2) \end{aligned} \tag{191}$$

(With the denominator factors now defined with p replaced by Q, which satisfies $Q^2 = m^2$.) Hence the following form code is generated:

```
id Dinv2 = D2^(-1) + (t*(D1 - D2))/D2^2
```

So generally spoken, the supervisor notebook performs the following steps:

4. Generate FORM-code for each diagram

5. Invoke the FORM-code for each diagram in parallel

The FORM programs save their result in a new file, which is then loaded back into Mathematica. For every diagram k the resulting expressions look as follows:

$$\text{diag}(k) = \sum_{\vec{a}} c_{k,\vec{a}} T_{\vec{a}}^{\sigma(k)} \tag{192}$$

The sum is over powers of the propagators of the integrals in topology $T^{\sigma(k)}$, where $\sigma(k)$ is the number of the topology belonging to diagram k. The coefficients $c_{k,\vec{a}}$ may be zero if the integral is not expressed in the diagram.

If one solves for the integrals $T_{\vec{a}}^{\sigma(k)}$ expressed in each diagram k, the diagram has been fully calculated. However, the set of integrals is generally far too large to be able to calculate each integral separately. For this reason the set of integrals will need to be reduced somehow to a more manageable size, which is done by making use of integration by parts identities, which is explained next.

7.6. Reduction to master integrals using integration-by-parts relations

Some topologies can be solved analytically for arbitrary powers of the propagators. In those cases it doesn't matter how many integrals one has in the topology as one can evaluate them by plugging them into the general formula. However, generally the topologies are much too difficult to be solved in the general case.

In section III about dimensional regularization it was remarked that integration by parts (IBP) can be applied for dimensionally regularized integrals while always setting the boundary term to zero. It turns out that this can be used to reduce the powers of the propagators of complicated integrals. A few examples are discussed next, where IBP-identities are derived.

Example 1 Consider the following integral with $\lambda > 0$:

$$\int d^d k \frac{1}{(k^2 - m^2)^\lambda} \tag{193}$$

It holds that:

$$\begin{aligned} 0 = \int d^d k' \frac{d}{dk^\mu} \left[\frac{k^\mu}{(k^2 - m^2)^\lambda} \right]_{k=k'} &= d \int d^d k \frac{1}{(k^2 - m^2)^\lambda} - 2\lambda \int d^d k \frac{k^2}{(k^2 - m^2)^{\lambda+1}} \\ &= d \int d^d k \frac{1}{(k^2 - m^2)^\lambda} - 2\lambda \int d^d k \frac{(k^2 - m^2) + m^2}{(k^2 - m^2)^{\lambda+1}} \\ &= (d - 2\lambda) \int d^d k \frac{1}{(k^2 - m^2)^\lambda} - 2\lambda m^2 \int d^d k \frac{1}{(k^2 - m^2)^{\lambda+1}} \quad (194) \\ \Rightarrow \int d^d k \frac{1}{(k^2 - m^2)^{\lambda+1}} &= \frac{d - 2\lambda}{2\lambda m^2} \int d^d k \frac{1}{(k^2 - m^2)^\lambda} \quad (195) \end{aligned}$$

It turns out that higher powers of the propagator can be lowered by repeatedly aplying the above relation. The derivation fails when $\lambda = 0$. That means that if λ is a positive integer, $I(\lambda)$ can be expressed as a constant times $I(1)$, which can not be further reduced.

Example 2 In the following example it is shown that 'complicated' integrals with large powers on the denominator factors can be reached by generating IBP-relations from the ground up. Consider the following topology:

$$I_{a_1,a_2} \equiv \int d^d k \frac{1}{(-(k-p)^2)^{a_1} (-k^2 + m^2)^{a_2}} \tag{196}$$

One can state the following IBP-identities:

$$0 = \frac{d}{dk^\mu} k^\mu I_{0,1} = (2 - d) I_{0,1} - 2m^2 I_{0,2} \tag{197}$$

$$0 = \frac{d}{dk^\mu} k^\mu I_{1,1} = (3 - d) I_{1,1} + I_{2,1} \left(p^2 - m^2 \right) - 2m^2 I_{1,2} \tag{198}$$

$$0 = \frac{d}{dk^\mu} p^\mu I_{1,1} = I_{1,2} \left(-m^2 - p^2 \right) + I_{2,1} \left(p^2 - m^2 \right) - I_{0,2} \tag{199}$$

(Note that the derivative and vector on the left are of course taken inside of the integrals so a slight abuse of notation is used.) Solving for $I_{1,2}$, $I_{2,1}$, $I_{0,2}$ leads to:

$$I_{1,2} = -\frac{2(d-3)m^2 I_{1,1} + (d-2)I_{0,1}}{2m^2\left(m^2-p^2\right)} \tag{200}$$

$$I_{2,1} = \frac{(d-3)I_{1,1}\left(m^2+p^2\right) + (d-2)I_{0,1}}{\left(m^2-p^2\right)^2} \tag{201}$$

$$I_{0,2} = -\frac{(d-2)I_{0,1}}{2m^2} \tag{202}$$

These relations allow us to 'simplify' the integrals $I_{1,2}$, $I_{2,1}$, $I_{0,2}$. Now generate IBP-identities for these integrals:

$$0 = \frac{d}{dk^\mu}k^\mu I_{0,2} = (4-d)I_{0,2} - 4m^2 I_{0,3} \tag{203}$$

$$0 = \frac{d}{dk^\mu}k^\mu I_{1,2} = (5-d)I_{1,2} + I_{2,2}\left(p^2-m^2\right) - 4m^2 I_{1,3} + I_{2,1} \tag{204}$$

$$0 = \frac{d}{dk^\mu}k^\mu I_{2,1} = (4-d)I_{2,1} + I_{3,1}\left(2p^2-2m^2\right) - 2m^2 I_{2,2} \tag{205}$$

$$0 = \frac{d}{dk^\mu}p^\mu I_{1,2} = I_{1,3}\left(-2m^2-2p^2\right) + I_{2,2}\left(p^2-m^2\right) - 2I_{0,3} + I_{1,2} + I_{2,1} \tag{206}$$

Substituting the relations for $I_{1,2}$, $I_{2,1}$, $I_{0,2}$ and solving for $I_{0,3}$, $I_{1,3}$, $I_{2,2}$, $I_{3,1}$ leads to:

$$I_{0,3} = \frac{(d-4)(d-2)I_{0,1}}{8m^4} \tag{207}$$

$$I_{1,3} = \frac{(d-4)\left(4(d-3)m^4 I_{1,1} + (d-2)\left(3m^2-p^2\right)I_{0,1}\right)}{8m^4\left(m^2-p^2\right)^2} \tag{208}$$

$$I_{2,2} = -\frac{2(d-3)m^2\left((d-4)m^2+(d-6)p^2\right)I_{1,1} + (d-2)\left((2d-9)m^2-p^2\right)I_{0,1}}{2m^2\left(m^2-p^2\right)^3} \tag{209}$$

$$I_{3,1} = \frac{(d-5)(d-2)\left(m^2+p^2\right)I_{0,1} + (d-3)\left((d-4)m^4+2(d-6)m^2p^2+(d-4)p^4\right)I_{1,1}}{2\left(m^2-p^2\right)^4} \tag{210}$$

One can continue in the manner outlined until some desired integral is reached which is then fully expressed in the integrals $I_{0,1}$ and $I_{1,1}$ which are the so-called master integrals because they cannot be reduced any further. Note that the IBP-relations become increasingly complex the more are generated.

Example 3 Consider the following 2-loop topology,

$$I_{a_1,a_2,a_3,a_4,a_5} = \int\int \frac{d^dk_1 d^dk_2}{(-(k_1)^2)^{a_1}(-(k_1-p_1)^2)^{a_2}(-(k_2)^2)^{a_3}(-(k_2-p_1)^2)^{a_4}(-(k_1-k_2)^2)^{a_5}} \tag{211}$$

Then it holds that:

$$\frac{d}{dk_1^\mu}(k_1^\mu - k_2^\mu)I_{1,1,1,1,1} = 0 = (4+d)I_{1,1,1,1,1} - I_{1,2,1,0,1} + I_{1,2,1,1,0} - I_{2,1,0,1,1} + I_{2,1,1,1,0} \tag{212}$$

Making use of shift-symmetries to reduce 4 integrals to 2 one finds:

$$I_{1,1,1,1,1} = \frac{2(I_{1,2,1,1,0} - I_{1,2,1,0,1})}{d-4} \tag{213}$$

The integrals on the right hand side have only 4 contributing propagators which makes them easier to evaluate.

The Laporta algorithm The previous examples give an idea of the power of IBP-relations. For general multiloop topologies there is not a good strategy to reduce complicated integrals to master integrals but to generate all possible IBP-relations from the ground up and solve the linear system created for the integrals deemed the most complex, in a similar manner to example 2. This approach becomes computationally very expensive but is currently the most fruitful in state of the art N^3LO and N^4LO calculations. The algorithm in which this approach is fully worked out is called the Laporta algorithm, which is first described in [22].

The Laporta algorithm has been implemented in a number of widely used software packages. For this thesis the C++ version of the program FIRE5 has been used, which is described in [23]. Again the supervisor Mathematica notebook is used for automatically setting up the required configuration for FIRE for each topology and for invoking the program to run. Because FIRE5 runs separately on each topology what can happen is that the master integrals for different topologies are in fact the same under some change of variables. For this reason some code has been written in Mathematica that reduces the final set of master integrals using shift-symmetries as in eq. 188.

After applying the IBP-reduction the completely automated calculation provides us with an expression for each diagram k in terms of master integrals:

$$\text{diag}(k) = \sum_i c_{k,i} M^i \tag{214}$$

The sum runs over all the master integrals, denoted by M^i, and the $c_{k,i}$ are coefficients. The next step is to solve the master integrals to finish the calculation of each diagram. Generally every master integral will have to be solved in a case-by-case analysis. If numerical results are sufficient a close to automated method of evaluation is obtained by using the finite integral expansion method from section X. First results from the automated setup will be presented using the expressions for master integrals stated in app. A, which are derived using the methods from the sections afterwards.

VIII. Results from automated calculations

8.1. Quark 1-loop self-energy (general gauge)

Using the automated calculation the 1-loop self-energy for general gauge parameter α was derived, which gives the following result:

$$A(p^2, m_0^2) = \frac{\alpha_s C_F}{4\pi\epsilon}\left(\frac{(\alpha+3)}{\epsilon} - 2 + (\alpha+3)\left(-\gamma_E + \log(4\pi) - \int_0^1 dx \log\left(\frac{m_0^2 x - p^2 x(1-x)}{\mu^2}\right)\right) + \mathcal{O}(\epsilon)\right) \tag{215}$$

$$B(p^2, m_0^2) = \frac{C_F \alpha_s\, \alpha}{4\pi\epsilon}\left(-\frac{1}{\epsilon} + 1 + \gamma_E - \log(4\pi) + 2\int_0^1 dx \log\left(\frac{m_0^2 x - p^2 x(1-x)}{\mu^2}\right)(1-x) + \mathcal{O}(\epsilon)\right) \tag{216}$$

Interesting is to note that $B(p^2, m_0^2)$ is proportional to the gauge parameter. Plugging in $\alpha = 1$ reproduces eqs. 114, 115.

8.2. Quark 1-loop on-shell renormalization

The on-shell renormalization constants for the quark at 1-loop were already derived in the previous section for the convenient choice $\alpha = 1$. Using the setup that was developed it is now easy to check whether this choice was allowed. The quantity T_R from eq. 92 expanded to order $\mathcal{O}(t^2)$ evaluates to:

$$T_R^{\text{1-loop}}(m_P) = -g^2\mu^{2\epsilon}\frac{i\,C_F(d-2)(d-1)(t-1)I^{1,1}}{2(d-3)m_P^2} + \mathcal{O}(t^2) \tag{217}$$

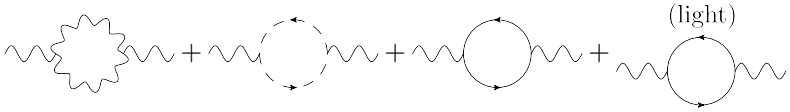
where $I^{1,1}$, defined in appendix A, is a massive tadpole integral. Using equations 97, 98 and expanding in ϵ shows that:

$$\delta_m^{(g^2)}g^2 = \delta_\psi^{(g^2)}g^2 = \frac{C_F\alpha_s}{4\pi}\left(-\frac{3}{\epsilon} - 4 + 3\gamma_E + 3\log\left(\frac{m_P^2}{4\pi\mu^2}\right)\right) + \mathcal{O}(\epsilon) \tag{218}$$

The result is in correspondence with equations 131, 137. It is remarked that the result here was derived for a general gauge parameter α which drops out. Hence the on-shell renormalization constants are gauge independent. It can not be claimed that the on-shell renormalization constants are always gauge independent and in [8] it's shown that the 3-loop on-shell wave function renormalization constant is in fact gauge dependent.

8.3. Gluon 1-loop self-energy

While the main concern of this thesis is to calculate the top quark self-energy it is a good effort to calculate the gluon self-energy at 1-loop and compare it to the literature. This is because already at 1-loop order all the vertices and propagators of the Feynman rules are expressed in the diagrams, except for the 4-gluon vertex which gives a so-called snail diagram which is trivially zero because it is proportional to a scaleless integral. The contributing diagrams at 1-loop order are given below:

$-i\Pi^{\mu\nu}(p) =$ 

(219)

We let $-i\Pi^{\mu\nu}_{\text{G,Gh,Q,QM}}$ be the gluon, ghost, quark and massless quark bubble diagrams. The master integrals turn out to be $I^{1,1}$, $I^{1,2}$ and $I^{1,3}$ from appendix A. The following expressions were found:

$$-i\Pi^{\mu\nu}_{\text{Q}}(p) = g^2\mu^{2\epsilon}\delta_{ab}N_h T_F\left(p^\mu p^\nu - p^2\eta^{\mu\nu}\right)\frac{2\left((d-2)p^2I^{1,3} + 2(d-2)I^{1,1} + 4m^2I^{1,3}\right)}{(d-1)p^2} \tag{220}$$

$$-i\Pi^{\mu\nu}_{\text{Qm}}(p) = g^2\mu^{2\epsilon}\delta_{ab}N_l T_F\left(p^\mu p^\nu - p^2\eta^{\mu\nu}\right)\frac{2(d-2)}{d-1}I^{1,2} \tag{221}$$

$$-i\Pi^{\mu\nu}_{\text{Gh}}(p) = g^2\mu^{2\epsilon}\delta_{ab}C_A\frac{\left((d-2)p^\mu p^\nu + p^2\eta^{\mu\nu}\right)}{4(d-1)}I^{1,2} \tag{222}$$

$$-i\Pi^{\mu\nu}_{\text{G}}(p) = g^2\mu^{2\epsilon}\delta_{ab}C_A I^{1,2}\frac{1}{8(d-1)}\left(\left(p^\mu p^\nu\right.\right.\left(4(\alpha(\alpha+5)-3) + (\alpha-1)(\alpha+7)d^2 + (17-\right.$$

$$\alpha(5\alpha+26))d\Big) - p^2\left(4(\alpha+5)\alpha+(\alpha-1)(\alpha+7)d^2+(19-\alpha(5\alpha+26))d-14\right)\eta^{\mu\nu})) \tag{223}$$

$$-i\Pi^{\mu\nu}(p) = g^2\mu^{2\epsilon}\delta_{ab}\left(p^\mu p^\nu - p^2\eta^{\mu\nu}\right)\frac{1}{8(d-1)p^2}\Bigg(16T_F\Big((d-2)p^2(I^{1,3}+I^{1,2})+2(d-2)I^{1,1}+ 4m^2I^{1,3}\Big)+C_AI^{1,2}p^2\left(4(\alpha(\alpha+5)-4)+(\alpha-1)(\alpha+7)d^2+(19-\alpha(5\alpha+26))d\right)\Bigg) \tag{224}$$

These diagrams can be expanded in ϵ by plugging in the explicit expressions for the master integrals. The results of this expansion has been compared up to finite order with an expansion of the results of appendix A of ref. [3] and was found to be in agreement.

This gives confidence that the calculation works as required. The analytical expressions take a slightly different form from those in stated in ref. [3], but it should be possible to use analytical relations to make both representations look the same. Note that in the comparison there is a minus difference, due to a difference in the definition of the self-energy. This is seen by comparing the expression for the full gluon propagator in eq. (2.5.165) of ref. [3] to ours in eq. 36.

The massless quark bubble diagram is used in the renormalon calculation, and its ϵ-expansion is desired. With the integral $I^{1,3}$ filled in one finds:

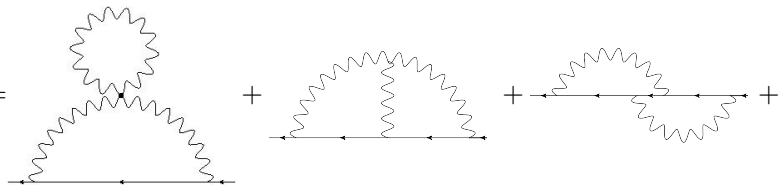
$$-i\Pi^{\mu\nu}_{2,massless}(p) = g^2\mu^{2\epsilon}\delta_{ab}T_F(p^\mu p^\nu - p^2\eta^{\mu\nu})\frac{4i(-p^2)^{-\epsilon}(\epsilon-1)\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{(4\pi)^{2-\epsilon}(2\epsilon-3)\Gamma(2-2\epsilon)} \tag{225}$$

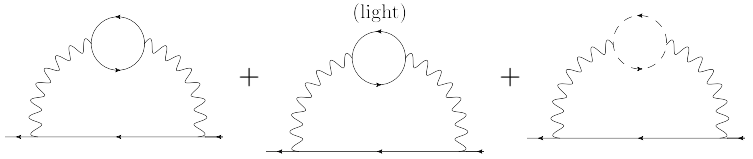
The expansion in ϵ up to order $\mathcal{O}(\epsilon)$ turns out to be:

$$-i\Pi^{\mu\nu}_{2,massless}(p) = \frac{ig^2T_F\delta_{ab}(p^\mu p^\nu - p^2\eta^{\mu\nu})}{12\pi^2}\left(\frac{1}{\epsilon}+\left(\frac{5}{3}-\gamma_{\mathrm{E}}-\log\left(\frac{-p^2}{4\pi\mu^2}\right)\right)+\mathcal{O}(\epsilon)\right) \tag{226}$$

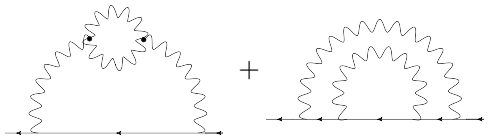
8.4. Quark field and mass renormalization at NNLO

There are eight 2-loop 1-particle irreducible diagrams in the quark self-energy. They are drawn below:

$-i\Sigma_0^{(g^4)}(\not{p},m) =$  (227)



(228)

 (229)

On-shell renormalization The on-shell 2-loop renormalization can be done by calculating $T_R(p^2, m_P^2)$ of eq. 92 and expanding up to order $\mathcal{O}(t^2)$. From the IBP-reduction one finds 3 master integrals: $I^{2,1}$, $I^{2,2}$ and $I^{2,3}$ from appendix A. The exact expressions of the diagrams in terms of the master integrals are too long to present here, and wouldn't be very insightful anyway. For this reason the renormalization constants are presented here directly as power expansions in ϵ. Following the conventions of [8] we define:

$$a_0 = \frac{\alpha_s^{(\text{bare})}}{\pi} \frac{\Gamma(1+\epsilon) m^{-2\epsilon}}{(4\pi)^{-\epsilon}} \tag{230}$$

The renormalization constants are presented as power series expansions in a_0. These expansions are significantly simpler than expansions in g^2 or α_s because the Γ functions are not expanded out which gets rids of logarithmic factors and terms like γ_E. Write the renormalization constants as follows:

$$Z_m^{\text{OS}} = 1 + a_0 C_F Z_m^{(1),\text{OS}} + a_0^2 C_F Z_m^{(2),\text{OS}} + a_0^3 C_F Z_m^{(3),\text{OS}} \tag{231}$$

$$Z_\psi^{\text{OS}} = 1 + a_0 C_F Z_\psi^{(1),\text{OS}} + a_0^2 C_F Z_\psi^{(2),\text{OS}} + a_0^3 C_F Z_\psi^{(3),\text{OS}} \tag{232}$$

The 1-loop coefficients are:

$$Z_m^{(1),\text{OS}} = Z_\psi^{(1),\text{OS}} = -\frac{3}{4\epsilon} - 1 - 2\epsilon - 4\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{233}$$

For the 2-loop coefficients it is convenient to group terms based on their color structure. Define the following:

$$Z_m^{(2),\text{OS}} = C_F d_1^{(2)} + C_A d_2^{(2)} + T_F N_l d_3^{(2)} + T_F N_h d_4^{(2)} \tag{234}$$

$$Z_\psi^{(2),\text{OS}} = C_F f_1^{(2)} + C_A f_2^{(2)} + T_F N_l f_3^{(2)} + T_F N_h f_4^{(2)} \tag{235}$$

Then the following results were found for the mass renormalization constant:

$$\begin{aligned} d_1^{(2)} = {} & \frac{9}{32\epsilon^2} + \frac{45}{64\epsilon} + \left(\frac{199}{128} - \frac{5\pi^2}{16} + \frac{1}{2}\pi^2 \log(2) - \frac{3\zeta_3}{4}\right) + \left(\frac{677}{256} - \frac{55\pi^2}{32} + \frac{\pi^4}{20} + 3\pi^2 \log(2) - \right. \\ & \left. \frac{3}{2}\pi^2 \log^2(2) - 6\zeta_{1,-3} - \frac{33\zeta_3}{4} + \frac{21}{2}\log(2)\zeta_3\right)\epsilon + \left(\frac{1167}{512} - \frac{1255\pi^2}{192} - \frac{\pi^4}{30} + 11\pi^2 \log(2) + \right. \\ & \left. \frac{7}{30}\pi^4 \log(2) - 9\pi^2 \log^2(2) + 3\pi^2 \log^3(2) - 36\zeta_{1,-3} + 36\log(2)\zeta_{1,-3} + 36\zeta_{1,1,-3} - \frac{297\zeta_3}{8} - \right. \\ & \left. \frac{3\pi^2\zeta_3}{2} + 63\log(2)\zeta_3 - \frac{63}{2}\log^2(2)\zeta_3 + 39\zeta_5\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \end{aligned} \tag{236}$$

$$\begin{aligned} d_2^{(2)} = {} & -\frac{11}{32\epsilon^2} - \frac{91}{64\epsilon} + \left(-\frac{605}{128} + \frac{\pi^2}{12} - \frac{1}{4}\pi^2 \log(2) + \frac{3\zeta_3}{8}\right) + \left(-\frac{3799}{256} + \frac{19\pi^2}{48} - \frac{\pi^4}{40} - \right. \\ & \left. \frac{3}{2}\pi^2 \log(2) + \frac{3}{4}\pi^2 \log^2(2) + 3\zeta_{1,-3} + \frac{13\zeta_3}{4} - \frac{21}{4}\log(2)\zeta_3\right)\epsilon + \left(-\frac{23269}{512} + \frac{125\pi^2}{96} - \frac{11\pi^4}{180} - \right. \\ & \left. \frac{11}{2}\pi^2 \log(2) - \frac{7}{60}\pi^4 \log(2) + \frac{9}{2}\pi^2 \log^2(2) - \frac{3}{2}\pi^2 \log^3(2) + 18\zeta_{1,-3} - 18\log(2)\zeta_{1,-3} - 18\zeta_{1,1,-3} + \right. \\ & \left. 13\zeta_3 + \frac{3\pi^2\zeta_3}{4} - \frac{63}{2}\log(2)\zeta_3 + \frac{63}{4}\log^2(2)\zeta(3) - \frac{39\zeta_5}{2}\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \end{aligned} \tag{237}$$

$$d_3^{(2)} = \frac{1}{8\epsilon^2} + \frac{7}{16\epsilon} + \left(\frac{45}{32} + \frac{\pi^2}{12}\right) + \left(\frac{279}{64} + \frac{7\pi^2}{24} + \zeta_3\right)\epsilon + \left(\frac{1701}{128} + \frac{15\pi^2}{16} + \frac{4\pi^4}{45} + \frac{7\zeta(3)}{2}\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{238}$$

$$d_4^{(2)} = \frac{1}{8\epsilon^2} + \frac{7}{16\epsilon} + \left(\frac{69}{32} - \frac{\pi^2}{6}\right) + \left(\frac{463}{64} - \frac{5\pi^2}{6} + \pi^2\log(2) - \frac{7\zeta_3}{2}\right)\epsilon + \left(\frac{3053}{128} - 3\pi^2 - \frac{7\pi^4}{90} + 5\pi^2\log(2) - 3\pi^2\log^2(2) - 12\zeta_{1,-3} - \frac{35\zeta_3}{2} + 21\log(2)\zeta_3\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{239}$$

The following was found for the wave function renormalization constant:

$$f_1^{(2)} = \frac{9}{32\epsilon^2} + \frac{51}{64\epsilon} + \left(\frac{433}{128} - \frac{13\pi^2}{16} + \pi^2\log(2) - \frac{3\zeta(3)}{2}\right) + \left(\frac{211}{256} - \frac{89\pi^2}{32} + \frac{\pi^4}{10} + \frac{23}{4}\pi^2\log(2) - 3\pi^2\log^2(2) - 12\zeta_{1,-3} - \frac{147\zeta_3}{8} + 21\log(2)\zeta_3\right)\epsilon + \left(\frac{4889}{512} - \frac{2321\pi^2}{192} - \frac{7\pi^4}{24} + \frac{41}{2}\pi^2\log(2) + \frac{7}{15}\pi^4\log(2) - \frac{69}{4}\pi^2\log^2(2) + 6\pi^2\log^3(2) - 69\zeta_{1,-3} + 72\log(2)\zeta_{1,-3} + 72\zeta_{1,1,-3} - \frac{513\zeta_3}{8} - 3\pi^2\zeta(3) + \frac{483}{4}\log(2)\zeta_3 - 63\log^2(2)\zeta_3 + 78\zeta_5\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{240}$$

$$f_2^{(2)} = -\frac{11}{32\epsilon^2} - \frac{101}{64\epsilon} + \left(-\frac{803}{128} + \frac{5\pi^2}{16} - \frac{1}{2}\pi^2\log(2) + \frac{3\zeta_3}{4}\right) + \left(-\frac{4241}{256} + \frac{41\pi^2}{48} - \frac{\pi^4}{20} - \frac{23}{8}\pi^2\log(2) + \frac{3}{2}\pi^2\log^2(2) + 6\zeta_{1,-3} + \frac{129\zeta_3}{16} - \frac{21}{2}\log(2)\zeta_3\right)\epsilon + \left(-\frac{30163}{512} + \frac{117\pi^2}{32} + \frac{11\pi^4}{240} - \frac{41}{4}\pi^2\log(2) - \frac{7}{30}\pi^4\log(2) + \frac{69}{8}\pi^2\log^2(2) - 3\pi^2\log^3(2) + \frac{69}{2}\zeta_{1,-3} - 36\log(2)\zeta_{1,-3} - 36\zeta_{1,1,-3} + \frac{205\zeta_3}{8} + \frac{3\pi^2\zeta_3}{2} - \frac{483}{8}\log(2)\zeta_3 + \frac{63}{2}\log^2(2)\zeta(3) - 39\zeta_5\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{241}$$

$$f_3^{(2)} = \frac{1}{8\epsilon^2} + \frac{9}{16\epsilon} + \left(\frac{59}{32} + \frac{\pi^2}{12}\right) + \left(\frac{369}{64} + \frac{3\pi^2}{8} + \zeta_3\right)\epsilon + \left(\frac{2259}{128} + \frac{59\pi^2}{48} + \frac{4\pi^4}{45} + \frac{9\zeta(3)}{2}\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{242}$$

$$f_4^{(2)} = \frac{1}{4\epsilon^2} + \frac{19}{48\epsilon} + \left(\frac{1139}{288} - \frac{\pi^2}{3}\right) + \left(\frac{20275}{1728} - \frac{19\pi^2}{12} + 2\pi^2\log(2) - 7\zeta_3\right)\epsilon + \left(\frac{450395}{10368} - \frac{67\pi^2}{12} - \frac{7\pi^4}{45} + \frac{19}{2}\pi^2\log(2) - 6\pi^2\log^2(2) - 24\zeta_{1,-3} - \frac{133\zeta_3}{4} + 42\log(2)\zeta_3\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{243}$$

Up to order ϵ^0 direct agreement with [8] can be read off. For the ϵ^1 coefficients the following relation gives the same result:

$$\zeta_{1,-3} = \frac{1}{48}\left(96a_4 + 84\zeta(3)\log(2) - \pi^4 + 4\log^4(2) - 4\pi^2\log^2(2)\right) \tag{244}$$

This was found by taking the difference between the result from [8] and our calculation and solving for $\zeta_{1,-3}$, assuming both results are correct. A quick numerical calculation, in which we calculate $\zeta_{1,-3}$ using eq. 451 gives:

$$\zeta_{1,-3} \approx 0.087785671568655302\ldots \tag{245}$$

Using that $a_4 \equiv \sum_{n=1}^{\infty} \frac{1}{2^n n^4}$ and numerically evaluating the right side of eq. 244 yields the same result up to very high order. (A quick check was done that gave agreement up to order 10^{-80} by setting the precision in Mathematica very high and calculating the zeta sum up to 200 terms, after which 20 Shanks transformations were taken to speed up convergence.) In the derivation of the 3-loop master integrals our results correspond with [24] when using eq. 244.

An extra coefficient of order ϵ^2 is presented here for the NNLO renomalization constants, compared to [8]. To find the expansion up to order ϵ^2 the master integral $I^{2,3}_{\text{OS}}$ had to be derived up to order ϵ^4, because it is multiplied by a term $1/\epsilon^2$ in the exact expression.

8.5. Partial results for field and mass renormalization at N^{3}LO

Using the automated setup a 3-loop calculation was performed. There are 100 diagrams contributing to the 3-loop self-energy. All diagrams were calculated in terms of master integrals and reduced using IBP-relations, although the ϵ-expansions of the renormalization constants have not been completely derived.

Therefore only partial results will be presented for the 3-loop renormalization constants. The 3-loop renormalization constants can be grouped based on color structure. Define the following:

$$\begin{aligned} Z_m^{(3),\text{OS}} &= C_F^2 d_1^{(3)} + C_F C_A d_2^{(3)} + C_A^2 d_3^{(3)} + C_F T_F N_l d_4^{(3)} + C_F T_F N_h d_5^{(3)} + C_A T_F N_l d_6^{(3)} + \\ &\quad C_A T_F N_h d_7^{(3)} + T_F^2 N_l N_h d_8^{(3)} + T_F^2 N_h^2 d_9^{(3)} + T_F^2 N_l^2 d_{10}^{(3)} \end{aligned} \tag{246}$$

$$\begin{aligned} Z_\psi^{(3),\text{OS}} &= C_F^2 f_1^{(3)} + C_F C_A f_2^{(3)} + C_A^2 f_3^{(3)} + C_F T_F N_l f_4^{(3)} + C_F T_F N_h f_5^{(3)} + C_A T_F N_l f_6^{(3)} + \\ &\quad C_A T_F N_h f_7^{(3)} + T_F^2 N_l N_h f_8^{(3)} + T_F^2 N_h^2 f_9^{(3)} + T_F^2 N_l^2 f_{10}^{(3)} \end{aligned} \tag{247}$$

The ϵ-expansions of the coefficients $d_8^{(3)}, d_9^{(3)}, d_{10}^{(3)}$ and $f_8^{(3)}, f_9^{(3)}, f_{10}^{(3)}$ have been derived which is done by considering diagrams with 2 quark bubbles. (Note that the diagram with two massive quark bubbles is not trivial as it cannot be done with successive integration.) There are 20 master integrals in the 3-loop calculation. Out of the 20 master integrals one needs 8 integrals to calculate the mentioned coefficients: $I^{3,8}, I^{3,9}, I^{3,13}, I^{3,14}, I^{3,15}, I^{3,16}, I^{3,18}$ and $I^{3,19}$ which are given in app. A. The results are shown below:

$$\begin{aligned} d_8^{(3)} &= -\frac{1}{18\epsilon^3} - \frac{17}{54\epsilon^2} + \frac{-304+9\pi^2}{162\epsilon} + \frac{1}{486}\bigg(-4064 - 9\pi^2(-26+36\log(2)) + 918\zeta_3\bigg) + \\ &\quad \bigg(-\frac{26582}{729} + \frac{64a_4}{3} - \frac{59\pi^4}{540} + \frac{8\log^4(2)}{9} + \frac{1}{162}\pi^2\bigg(349 - 828\log(2) + 288\log^2(2)\bigg) + \\ &\quad \frac{415\zeta_3}{27}\bigg)\epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \tag{248}$$

$$\begin{aligned} d_9^{(3)} &= -\frac{1}{36\epsilon^3} - \frac{17}{108\epsilon^2} + \frac{-\frac{385}{324} + \frac{\pi^2}{9}}{\epsilon} + \bigg(-\frac{5441}{972} + \pi^2\bigg(\frac{79}{135} - \frac{2\log(2)}{3}\bigg) + \frac{53\zeta_3}{18}\bigg) + \\ &\quad \bigg(-\frac{92294}{3645} + \frac{64a_4}{3} - \frac{11\pi^4}{72} + \frac{8\log^4(2)}{9} + \frac{2\pi^2\bigg(2918 - 3555\log(2) + 1125\log^2(2)\bigg)}{2025} + \\ &\quad \frac{4397\zeta_3}{270}\bigg)\epsilon + \mathcal{O}(\epsilon^2) \end{aligned} \tag{249}$$

$$d_{10}^{(3)} = -\frac{1}{36\epsilon^3} - \frac{17}{108\epsilon^2} + \frac{-223-18\pi^2}{324\epsilon} + \frac{1}{972}\bigg(-2687 - 306\pi^2 - 1026\zeta_3\bigg) +$$

$$\frac{1}{29160}\Big(-40140\pi^2-4077\pi^4-20(15728+8721\zeta_3)\Big)\epsilon+\mathcal{O}(\epsilon^2) \tag{250}$$

$$f_8^{(3)}=-\frac{1}{12\epsilon^3}-\frac{7}{18\epsilon^2}+\frac{-\frac{31}{9}+\frac{\pi^2}{6}}{\epsilon}+\Big(-\frac{1168}{81}-\frac{1}{6}\pi^2(-7+8\log(2))+4\zeta_3\Big)+\Big(\frac{128a_4}{3}-\frac{37\pi^4}{270}+\frac{4}{9}\pi^2\Big(11-23\log(2)+8\log^2(2)\Big)+\frac{2}{243}\Big(-8870+216\log^4(2)+3969\zeta_3\Big)\Big)\epsilon+\mathcal{O}(\epsilon^2) \tag{251}$$

$$f_9^{(3)}=-\frac{1}{12\epsilon^3}-\frac{5}{36\epsilon^2}+\frac{-\frac{131}{54}+\frac{2\pi^2}{9}}{\epsilon}+\Big(-\frac{6887}{648}+\pi^2\Big(\frac{11}{10}-\frac{4\log(2)}{3}\Big)+7\zeta_3\Big)+\Big(-\frac{1023397}{19440}+\frac{152a_4}{3}-\frac{47\pi^4}{120}+\frac{19\log^4(2)}{9}+\pi^2\Big(\frac{23263}{4050}-\frac{33\log(2)}{5}+\frac{17\log^2(2)}{9}\Big)+\frac{2847\zeta_3}{80}\Big)\epsilon+\mathcal{O}(\epsilon^2) \tag{252}$$

$$f_{10}^{(3)}=-\frac{1}{36\epsilon^3}-\frac{23}{108\epsilon^2}+\frac{-325-18\pi^2}{324\epsilon}+\frac{1}{972}\Big(-4025-414\pi^2-1026\zeta_3\Big)+\frac{1}{29160}\Big(-475780-58500\pi^2-4077\pi^4-235980\zeta_3\Big)\epsilon+\mathcal{O}(\epsilon^2) \tag{253}$$

The derived expansions have been compared with [8] up to finite order and are in agreement. The coefficient ϵ^1 is presented here as well. This is possible because the 3-loop master integrals have been plugged in up to order ϵ^2. In the exact calculation these master integrals are multiplied by a factor $1/\epsilon$. Some of the 3-loop integrals in app. A were calculated to an order smaller than ϵ^2. In those cases the missing coefficients up to order ϵ^2 were taken from the results in [24]. That means all the results are presented with $a_4 \equiv \sum_{n=1}^{\infty}\frac{1}{2^n n^4}$, instead of $\zeta_{1,-3}$.

IX. Parametrizing Feynman integrals using the alpha-representation

The evaluation of master integrals is the only part of the calculation that cannot be completely automated, although as will become clear later on some progress can be made by automating numerical evaluations of master integrals using so-called finite integral expansions.

The evaluation of master integrals is aided by moving to the so-called alpha-representation given in eq. 273. The derivation of the alpha-representation is treated in the following section. The alpha-parametrization is well known in literature and the treatment in the next section is inspired by [7] and [25], although these references do not provide a derivation in the amount of detail presented here.

9.1. Derivation of the alpha-representation

A particular way to parametrize Feynman integrals is the alpha-representation, which is derived below. The main formula of interest rewrites a denominator into an exponent:

$$\frac{1}{D^a}=\frac{i^a}{\Gamma(a)}\int_0^\infty d\alpha\ \alpha^{a-1}\exp\left[-iD\alpha\right] \tag{254}$$

where $a \in \mathbb{N}_{\geq 1}$. This formula can be derived by starting with the integral and repeatedly using integration by parts relations. From the product rule it holds that:

$$\frac{d}{d\alpha}\left[\frac{1}{-iD}\alpha^{a-1}e^{-iD\alpha}\right]=\alpha^{a-1}e^{-iD\alpha}+\frac{i}{D}(a-1)\alpha^{a-2}e^{-iD\alpha} \tag{255}$$

Integrating the total derivative from 0 to ∞ gives zero since:

$$\int_0^\infty d\alpha \, \frac{d}{d\alpha}\left[\frac{1}{-iD}\alpha^{a-1}e^{-iD\alpha}\right] = \left[\frac{1}{-iD}\alpha^{a-1}e^{-iD\alpha}\right]_0^\infty = 0 \tag{256}$$

Thus one finds after repeatedly applying integration by parts:

$$\begin{aligned}\int_0^\infty d\alpha \, \alpha^{a-1}\exp\left[-iD\alpha\right] &= \int_0^\infty d\alpha \, \frac{1}{iD}(\alpha-1)\alpha^{a-2}\exp\left[-iD\alpha\right] \\ &= \dots \\ &= \frac{(a-1)(a-2)\dots 1}{(iD)^{a-1}}\int_0^\infty e^{-iD\alpha}d\alpha \\ &= \frac{(a-1)!}{(iD)^a} = \frac{\Gamma(a)}{(iD)^a}\end{aligned} \tag{257}$$

Rearranging the terms Γ and i leads to eq. 256. Next, consider a general Feynman integral topology $T_{a_1,\dots,a_n}$:

$$T_{a_1,\dots,a_n} = \int d^dk_1 \dots d^dk_h \frac{1}{D_1^{a_1}} \cdot \dots \cdot \frac{1}{D_n^{a_n}} \tag{258}$$

$$D_j \equiv -(D_{jj'}\tilde{P}_{j'})^2 + m_j^2 \tag{259}$$

(Note that a summation is implied for indices which occur twice. Occasionally the sums will be written out explicitely.) The notation $\tilde{P} = (k_1, \dots, k_h, p_1, \dots, p_{\tilde{h}})$ is used, where the k_j denote the internal momenta, the p_j denote the external momenta and the number of internal and external momenta are defined by $h, \tilde{h}$ respectively. Furthermore, the terms $D_{jj'}$ are coefficients of the momenta, expected to lie in $\{-1, 0, 1\}$, but it is not necessary to assume this. Hence the term $D_{jj'}\tilde{P}_{j'}$ simply denotes some linear combination of the momenta. Note that the propagators are defined with a negative sign on the momentum term. Repeatedly apply eq. 256 to get:

$$T_{a_1,\dots,a_n} = \frac{i^{a_1+\dots+a_n}}{\Gamma(a_1)\dots\Gamma(a_n)}\int_0^\infty d\alpha_1 \dots \int_0^\infty d\alpha_n \, \alpha_1^{a_1-1}\dots\alpha_n^{a_n-1}\int d^dk_1\dots d^dk_h \exp\left[-i\sum_{j=1}^n D_j\alpha_j\right] \tag{260}$$

Next make the following definition:

$$i\sum_j(\sum_{j'}D_{jj'}\tilde{P}_{j'})^2\alpha_j \equiv i\left[\sum_{jj'}A_{jj'}k_j\cdot k_{j'} + 2\sum_j(q_j\cdot k_j) + \sum_{jj'}C_{jj'}p_j\cdot p_{j'}\right] \tag{261}$$

The left hand expression is simply separated into three parts. The matrix A groups terms with dot products of internal momenta. The vector q groups dot products of external momenta with internal momenta. The matrix C groups the terms that only depend on external momenta. Note that the entries q_j contain external momenta, whereas the entries of A and C are scalars with respect to spacetime. Making use of the definition one finds:

$$\begin{aligned}T_{a_1,\dots,a_n} = \frac{i^{a_1+\dots+a_n}}{\Gamma(a_1)\dots\Gamma(a_n)}&\int_0^\infty d\alpha_1\dots\int_0^\infty d\alpha_n\,\alpha_1^{a_1-1}\dots\alpha_n^{a_n-1}\cdot \\ &\int d^dk_1\dots d^dk_h\exp\left[i(A_{ij}k_i\cdot k_j + 2q_i\cdot k_i + C_{ij}p_ip_j - m_i^2\alpha_i)\right]\end{aligned} \tag{262}$$

The momentum integrals can now be evaluated by recognizing a Gaussian integral, and using eq. 416, derived in app. C. The derivation of the Gaussian integral in the appendix holds for positive integer dimension, so for now we assume that dimensional regularization is not active.

A subtle point is that the integrals over the internal momenta that we are considering integrate around the poles of the Feynman propagators as outlined in section II in the paragraph on Feynman propagator prescription. It turns out that this does not affect the derivation of the Gaussian integral formula, so we can safely use the result of eq. 416. The motivation is as follows:

Focus on a particular pole in one of the propagators, and shift the momentum parameters so that the propagator considered depends on a single internal momentum parameter. Looking at eqs. 254 and 259 it is clear that the zeroth component of the momentum in the exponent has prefactor $(-i)\cdot(-1) = i$. Hence the integration region that is used for the Gaussian integral is that of fig. 9 in app. C and this is in the top-half plane while in the conventions discussed in section II the pole on the right is shifted down / we integrate slightly above the pole. So there is indeed no pole in the interior of fig. 9 and the usual derivation holds.

Hence, it follows that:

$$T_{a_1\dots a_n} = \frac{e^{\frac{i\pi}{2}\left[a_1+\dots+a_n+h(1-\frac{d}{2})\right]}}{\Gamma(a_1)\dots\Gamma(a_n)}\pi^{h\frac{d}{2}}\int_0^\infty d\alpha_1\dots\int_0^\infty d\alpha_n\,\alpha_1^{a_1-1}\dots\alpha_n^{a_n-1}\cdot \\ (\det A)^{-\frac{d}{2}}\exp\left[-iA^{-1}_{ij}q_i\cdot q_j + iC_{ij}p_i\cdot p_j - im_i^2\alpha_i\right] \tag{263}$$

To simplify this expression define the following polynomials:

$$\det A = U \tag{264}$$

$$i\frac{V}{U} = -iA^{-1}_{ij}q_i\cdot q_j + iC_{ij}p_i\cdot p_j \Rightarrow V = \left[-A^{-1}_{ij}q_i\cdot q_j + C_{ij}p_i\cdot p_j\right]U \tag{265}$$

The polynomials U and V are called the first and second Symanzik polynomials respectively. The topology is now written as:

$$T_{a_1\dots a_n} = \frac{e^{\frac{i\pi}{2}\left[a_1+\dots+a_n+h(1-\frac{d}{2})\right]}}{\Gamma(a_1)\dots\Gamma(a_n)}\pi^{h\frac{d}{2}}\int_0^\infty d\alpha_1\dots\int_0^\infty d\alpha_n\,\alpha_1^{a_1-1}\dots\alpha_n^{a_n-1}U^{-\frac{d}{2}}\exp\left[i\frac{V}{U} - im_i^2\alpha_i\right] \tag{266}$$

The exponent is simplified further by defining a third polynomial W:

$$W \equiv -V + U\sum_{j=1}^n m_j^2\alpha_j \Rightarrow -i\frac{W}{U} = i\frac{V}{U} - i\sum_{j=1}^n m_j^2\alpha_j \tag{267}$$

This leads to the following parametrization of $T_{a_1\dots a_n}$:

$$T_{a_1\dots a_n} = \frac{e^{\frac{i\pi}{2}\left[a_1+\dots+a_n+h(1-\frac{d}{2})\right]}}{\Gamma(a_1)\dots\Gamma(a_n)}\pi^{h\frac{d}{2}}\int_0^\infty d\alpha_1\dots\int_0^\infty d\alpha_n\,\alpha_1^{a_1-1}\dots\alpha_n^{a_n-1}U^{-\frac{d}{2}}\exp\left[-i\frac{W}{U}\right] \tag{268}$$

It turns out that the exponent can be removed by doing a change of variables. Let $\alpha_j = \eta\alpha'_j$ so that $\alpha'_1+\dots+\alpha'_n = 1$. The Jacobian of this transformation is η^{n-1}, which can be seen for example by explicitely letting $\alpha_n = \eta(1-\sum_{j=1}^{n-1}\alpha'_j)$ and calculating the Jacobian. Instead, we will introduce a term $\delta\left(1-\sum_{i=1}^n\alpha'_i\right)$, to rewrite T as:

$$T_{a_1\dots a_n} = \frac{e^{\frac{i\pi}{2}\left[a+h(1-\frac{d}{2})\right]}}{\Gamma(a_1)\dots\Gamma(a_n)}\pi^{h\frac{d}{2}}\int_0^\infty d\alpha'_1\dots\int_0^\infty d\alpha'_n\,\alpha_1'^{a_1-1}\dots\alpha_n'^{a_n-1}\delta\left(1-\sum_{i=1}^n\alpha'_i\right)\cdot \\ \int_0^\infty d\eta\eta^{a-1}U^{-\frac{d}{2}}\exp\left[-i\frac{W}{U}\right] \tag{269}$$

The definition $a \equiv a_1 + \dots a_n$ is used from now on. We would like to use the formulas from before, namely, eqs. 264, 265 and 267 for U, V and W, but we substituted $\alpha_j = \eta \alpha'_j$. Therefore we define:

$$\begin{aligned} U &= \eta^h U' \\ V &= \eta^{h+1} V' \\ W &= \eta^{h+1} W' \end{aligned} \tag{270}$$

Then eqs. 264, 265 and 267 apply for U', V', W' with α_j replaced by α'_j. This is seen from analyzing the degree of the polynomials. Let $\deg(P)$ be the degree of the monomials of the alpha-parameters in the polynomial P.

Firstly, $U = \det(A)$, and every nonzero entry in A carries a factor α_i for some index i. Furthermore A has dimension $h \times h$, so, $\deg(U) = h$. Similarly we have $\deg(q_i) = 1$ for nonzero q_i, $\deg(C_{ij}) = 1$ for nonzero C_{ij} and $\deg(A^{-1}_{ij}) = -1$ for nonzero A^{-1}_{ij}. From this and eq. 265 it is clear $\deg(V) = h + 1$. Lastly from eq. 267 it is then also seen $\deg(W) = h + 1$. Therefore it holds that:

$$\begin{aligned} T_{a_1 \dots a_n} = \frac{e^{\frac{i\pi}{2}\left[a + h(1-\frac{d}{2})\right]}}{\Gamma(a_1) \dots \Gamma(a_n)} \pi^{h\frac{d}{2}} \int_0^\infty d\alpha'_1 \dots \int_0^\infty d\alpha'_n \, \alpha_1'^{a_1 - 1} \dots \alpha_n'^{a_n - 1} \delta\left(1 - \sum_{i=1}^n \alpha'_i\right) \cdot \\ \int_0^\infty d\eta \eta^{a - 1 - \frac{hd}{2}} (U')^{-\frac{d}{2}} \exp\left[-i\eta \frac{W'}{U'}\right] \end{aligned} \tag{271}$$

The η-integral is easily evaluated by recognizing an integral representation of the Γ function (eq. 418):

$$\int_0^\infty d\eta \eta^{a - 1 - \frac{hd}{2}} \exp\left[-i\eta \frac{W'}{U'}\right] = \Gamma\left(a + \frac{hd}{2}\right) \left(i \frac{W'}{U'}\right)^{-a + \frac{hd}{2}} \tag{272}$$

Using this in eq. 271 leads to:

$$T_{a_1 \dots a_n} = \frac{\Gamma(a - \frac{hd}{2})}{\Gamma(a_1) \dots \Gamma(a_n)} (i\pi^{\frac{d}{2}})^h \int_0^\infty d\alpha_1 \dots \int_0^\infty d\alpha_n \, \delta\left(1 - \sum_{i=1}^n \alpha_i\right) \alpha_1^{a_1 - 1} \dots \alpha_n^{a_n - 1} U^{a - \frac{d}{2}(h+1)} W^{-a + \frac{hd}{2}} \tag{273}$$

This parametrization is what will be referred to as the alpha-representation of a topology $T_{a_1 \dots a_n}$ in this thesis. Sometimes eq. 269 is also referred to as the alpha-parametrization, or in other texts as the Schwinger parametrization.

The alpha-parametrization was derived assuming the dimension and powers on the propagators are integers and positive. By extending these to complex parameters an analytic continuation of Feynman integrals is defined by eq. 273. Such reasoning is for example used in [7]. Hence we will use the alpha-parametrization to deal with dimensionally regularized integrals as well as non-integer powers on propagators.

For the purposes of this thesis a function was written in Mathematica that takes in a set of propagators and powers and uses eqs. 261, 264, 265 and 267 to find the U and W polynomials. The alpha-representation is then easily retrieved from eq. 273.

An interesting observation is that, except for the delta function, eq. 273 is invariant under a rescaling of all alpha-parameters by some factor λ. This is a simple result which follows from the degrees of the polynomials in eqs. 270 and from looking at the powers which they are raised to in

eq. 273. Rescaling $\alpha_i \to \lambda\alpha_i$ ($\Rightarrow d\alpha_i \to \lambda d\alpha_i$) and counting the power of λ gives:

$$\deg_\lambda(T) = n + (a-n) + h\cdot\left(a - \frac{d}{2}(h+1)\right) + (h+1)\cdot\left(-a + \frac{hd}{2}\right) = 0 \tag{274}$$

where $\deg_\lambda(T)$ denotes the degree of T in λ. This type of rescaling does not preserve the delta function but one can do a similar rescaling such that the delta function is in fact preserved which is explained in the next section on the projective transform.

9.2. Projective transforms

The alpha-parametrization of Feynman integrals contains a factor $\delta(1-\sum_{i=1}^n \alpha_i)$ with n denoting the number of propagators. It turns out it is possible to rescale alpha-parameters while keeping the delta function in the same form, using a projective transform. This kind of transform was first considered in the context of Feynman integrals in [26]. Let $\lambda_i > 0$ and consider the following change of variables:

$$\alpha_i = \frac{\lambda_i\beta_i}{\sum_{j=1}^n \lambda_j\beta_j} \quad \text{for } i = 1,\dots,n \tag{275}$$

Starting from eq. 273 this leads to:

$$T_{a_1\dots a_n} = \frac{\Gamma(a-\frac{hd}{2})}{\Gamma(a_1)\dots\Gamma(a_n)}(i\pi^{\frac{d}{2}})^h \int_0^\infty d\beta_1 \dots \int_0^\infty d\beta_n\, \delta\left(1-\sum_{i=1}^n \beta_i\right) \lambda_1^{a_1}\dots\lambda_n^{a_n}\beta_1^{a_1-1}\dots\beta_n^{a_n-1}\cdot$$
$$U^{a-\frac{d}{2}(h+1)}\big|_{\alpha_i\to\lambda_i\beta_i} W^{-a+\frac{hd}{2}}\big|_{\alpha_i\to\lambda_i\beta_i} \tag{276}$$

The notation $U^{a-\frac{d}{2}(h+1)}\big|_{\alpha_i\to\lambda_i\beta_i}$ indicates that we take the usual U polynomial and replace the alpha-parameters according to $\alpha_i \to \lambda_i\beta_i$. Note that the delta function has retained its usual form with the alpha-parameters replaced by the new parameters. If we relabel the betas back to alpha-parameters we see that it is possible in the alpha-parametrization to simply rescale naively in the following manner: $d\alpha_i \to \lambda_i\, d\alpha_i$ and $\alpha_i \to \lambda_i\alpha_i$, while also keeping the delta function in its usual form.

It is not possible to directly do the transformation of eq. 275 inside the alpha-parametrization, because the Jacobian matrix is singular. To do the transformation one can first integrate out α_n leading to $\alpha_n \to \alpha_n(\vec{\alpha}) = 1-\sum_{i=1}^{n-1}\alpha_i$. The bracketed notation indicates α_n is interpreted as a function over the other alpha-parameters. Hence eq. 273 becomes:

$$T_{a_1\dots a_n} = \frac{\Gamma(a-\frac{hd}{2})}{\Gamma(a_1)\dots\Gamma(a_n)}(i\pi^{\frac{d}{2}})^h \int_0^1 d\alpha_1 \dots \int_0^{1-\sum_{i=0}^{n-2}\alpha_i} d\alpha_{n-1}\, \alpha_1^{a_1-1}\dots(\alpha_n(\vec{\alpha}))^{a_n-1} U^{a-\frac{d}{2}(h+1)} W^{-a+\frac{hd}{2}} \tag{277}$$

One also 'imagines' replacing α_n in the U and W polynomials by $\alpha_n(\vec{\alpha})$. Next apply the transform of eq. 275 on the variables $\alpha_1,\dots,\alpha_{n-1}$ but with $\beta_n(\vec{\beta}) = 1-\sum_{i=1}^{n-1}\beta_i$. Note that also:

$$\alpha_n(\vec{\alpha}) = 1 - \frac{\sum_{j=i}^{n-1}\lambda_i\beta_i}{\sum_{j=1}^{n-1}\lambda_j\beta_j + \lambda_n\beta_n(\vec{\beta})} \tag{278}$$

$$= \frac{\lambda_n\beta_n(\vec{\beta})}{\sum_{j=1}^{n-1}\lambda_j\beta_j + \lambda_n\beta_n(\vec{\beta})} \tag{279}$$

Hence, $\alpha_n(\vec{\alpha})$ transforms as expected. One can calculate the Jacobian determinant J of this change of coordinates which turns out to be:

$$|J| = \frac{\lambda_1 \cdot \ldots \cdot \lambda_n}{(\sum_{j=1}^{n-1} \lambda_j \beta_j + \lambda_n \beta_n(\vec{\beta}))^n} \tag{280}$$

Thus the Jacobian determinant gives the same result as naively rescaling the differentials according to $d\alpha_i \to d\beta_i \lambda_i / (\sum_{j=1}^{n-1} \lambda_j \beta_j + \lambda_n \beta_n(\vec{\beta}))$. It is known that under an equal rescaling of all the alpha-parameters and differentials the rescaling factor cancels out. For this reason the factor $(\sum_{j=1}^{n-1} \lambda_j \beta_j + \lambda_n \beta_n(\vec{\beta}))^{-1}$ cancels out. Therefore, after the change of variables one finds:

$$T_{a_1 \ldots a_n} = \frac{\Gamma(a - \frac{hd}{2})}{\Gamma(a_1) \ldots \Gamma(a_n)} (i\pi^{\frac{d}{2}})^h \lambda_1 \cdot \ldots \cdot \lambda_n \int_0^1 d\beta_1 \ldots \int_0^{1-\sum_{i=0}^{n-2} \beta_i} d\beta_{n-1} \, (\lambda_1 \beta_1)^{a_1 - 1} \ldots (\lambda_n \beta_n(\vec{\beta}))^{a_n - 1} \cdot$$
$$\cdot \, U^{a - \frac{d}{2}(h+1)} \big|_{\alpha_i \to \lambda_i \beta_i} W^{-a + \frac{hd}{2}} \big|_{\alpha_i \to \lambda_i \beta_i} \tag{281}$$

Note that the bounds on the integration region kept the same form with α_i replaced by β_i. This is seen as follows. Consider that $\alpha_i = 0$ for some $i = 1, \ldots, n-1$. Then we have:

$$0 = \frac{\lambda_i \beta_i}{\sum_{j=1}^{n-1} \lambda_j \beta_j + \lambda_n \beta_n(\vec{\beta})} \Leftrightarrow \beta_i = 0 \tag{282}$$

Next suppose that $\alpha_i = 1 - \sum_{1 \leq j < i} \alpha_j$ for some $i = 1, \ldots, n-1$. This also means that $\alpha_j = \beta_j = 0$ for all $i < j \leq n$. Plugging in the change of coordinates and multiplying out the denominators gives:

$$\begin{aligned} \frac{\lambda_i \beta_i}{\sum_{1 \leq j \leq i} \lambda_j \beta_j + \lambda_n (1 - \sum_{1 \leq j \leq i} \beta_j)} &= 1 - \frac{\sum_{1 \leq j < i} \lambda_i \beta_i}{\sum_{1 \leq j \leq i} \lambda_j \beta_j + \lambda_n (1 - \sum_{1 \leq j \leq i} \beta_j)} \\ \Rightarrow \; 0 &= \lambda_n (1 - \sum_{1 \leq j \leq i} \beta_j) \\ \Rightarrow \; \beta_i &= 1 - \sum_{1 \leq j < i} \beta_j \end{aligned} \tag{283}$$

This shows the bounds in eq. 281. Lastly because the terms $\beta_n(\beta)$ were not written out but kept in functional notation one can simply introduce a delta function $\delta(1 - \sum_i^n \beta_i)$ and change the integration domain back so that every parameter is integrated from 0 to infinity. This gives eq. 276.

Integrals with a factor $\delta(1 - \sum_{i=1}^n \alpha_i)$ and which are invariant under a projective transform with $\lambda_i = 1$ for all $i = 1, \ldots, n$ are called projective forms. Hence the alpha-parametrization of a Feynman integral is a projective form. Note that when a factor $\delta(1 - \sum_{i=1}^n \alpha_i)$ is present in a scalar integral one can also make the integral into a projective form by applying a projective transform with $\lambda_i = 1$.

9.3. The Cheng-Wu theorem

It turns out that projective forms like the alpha-parametrization obey an interesting integration theorem, which is called the Cheng-Wu theorem [27].

Cheng-Wu Theorem The delta function in the alpha-parametrization of a Feynman integral in the form of eq. 273 can be changed to $\delta(1 - \sum_{\alpha_i \in S} \alpha_i)$, where S denotes a non-empty subset of the alpha-parameters.

A simple proof of the Cheng-Wu theorem follows from eq. 268. To derive eq. 273 we rescaled the integration parameters by letting $\alpha_j = \eta \alpha'_j$ under the constraint $\sum_{i=1}^{n} \alpha'_j = 1$. Now instead let:

$$\alpha_j = \eta \alpha'_j \text{ for } j = 1, \ldots, n \tag{284}$$

$$\text{subject to } \sum_{\alpha'_i \in S} \alpha'_i = 1 \text{ with } S \subseteq \{\alpha'_1, \ldots, \alpha'_n\} \tag{285}$$

If one puts explicitely that $\alpha'_n = \eta(1 - \sum_{\alpha'_i \in S} \alpha'_i)$ the Jacobian J takes the following form:

$$J = \begin{pmatrix} \eta & 0 & \ldots & 0 & \alpha_1 \\ 0 & \eta & \ldots & 0 & \alpha_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \eta & \alpha_{n-1} \\ -\eta\delta_1 & -\eta\delta_2 & \ldots & -\eta\delta_{n-1} & 1 - \sum_{i \in S} \alpha_i \end{pmatrix} \tag{286}$$

where $\delta_i = 1$ if $i \in S$ and $\delta_i = 0$ otherwise. The determinant is found to be η^{n-1} by expanding the determinant along the bottom row using Laplace's formula. We can now proceed analogous to eqs. 271, 272 and 273 with the delta function replaced by $\delta(1 - \sum_{\alpha'_i \in S} \alpha'_i)$, which proofs the theorem.

Corollary In the alpha-parametrization in eq. 273 it is possible to set one integration parameter to 1, and integrate the other integration parameters from 0 to infinity.

The corollary can also be shown by doing a projective like transform:

$$\alpha_i = \frac{\beta_i}{\sum_{k=1, k \neq j}^{n} \beta_k + 1} \text{ for all } i \neq j \tag{287}$$

This transforms the δ-function into:

$$\delta\left(1 - \frac{\sum_{i=1, i \neq j}^{n} \beta_i}{\sum_{k=1, k \neq j}^{n} \beta_k + 1} - \alpha_j\right) = \delta\left(\frac{1}{\sum_{k=1, k \neq j}^{n} \beta_k + 1} - \alpha_j\right) \tag{288}$$

Integrating out α_j results in plugging in $\alpha_j = 1/(\sum_{k=1, k \neq j}^{n} \beta_k + 1)$. Furthermore, it turns out that the Jacobian determinant J of transformation 287 satisfies:

$$|J| = \left(\sum_{k=1, k \neq j}^{n} \beta_k + 1\right)^{-n} \tag{289}$$

Using the fact that the alpha-parametrization is a projective form, the denominator $\left(\sum_{k=1, k \neq j}^{n} \beta_k + 1\right)^{-1}$ completely cancels out. Hence one can effectively put one of the alpha-parameters to 1 and integrate the other parameters from 0 to ∞. It it stressed however that this only works for projective forms.

9.4. Probing divergences using power counting

A method is needed to quantify the divergent structure of Feynman integrals. Discussed next is a simple 'power counting' technique that archieves this goal. The discussion will be illustrated by looking at the scalar integral in the alpha-parametrization of $I^{2,2}_{\text{OS}}$:

$$I^{2,2}_{\text{part}} = \prod_{i=1}^{3}\left(\int_0^\infty d\alpha_i\right)(\alpha_1+\alpha_2)^{d-3}\,\alpha_3^{2d-6}\,(\alpha_1\alpha_2+\alpha_3\alpha_2+\alpha_1\alpha_3)^{3-\frac{3d}{2}}\,\delta\left(1-\sum_{i=1}^{3}\alpha_i\right) \tag{290}$$

There are no negative coefficients in the integrand. Therefore one can conclude that any divergences of the integral must be in regions where subsets of the alpha-parameters go to zero because those regions are the only ones where poles can occur. There cannot be divergences at infinity because the integration region is a finite 3-simplex by virtue of the delta function. We will let S denote the subset of alpha-parameters which go to zero simultaneously.

To probe whether a singularity exists in a given region one can rescale combinations of alpha-parameters by a common factor λ and factor out λ as much as possible. The differentials are also rescaled, so a proper coordinate transformation is done. The exponent of λ gives an indication of whether the integral is divergent in the region considered.

The delta function in the integral can be ignored as long as one sticks to regions with at least one alpha-parameter being nonzero so that we probe from within the integration domain. For example, one considers the change of variables $\alpha_1 \to \lambda\alpha_1$, $\alpha_2 \to \lambda\alpha_2$ to prope the region $S = \{\alpha_1, \alpha_2\}$ meaning that $\alpha_1 \to 0$, $\alpha_2 \to 0$.

Let $\text{div}(S)$ be the exponent of the scale-parameter λ after it has been factored out as much as possible. Factoring out λ is thought of in the following sense, illustrated by an example:

$$(\lambda\alpha_1+\lambda\alpha_2+\alpha_3)(\lambda\alpha_1+\lambda\alpha_2)\lambda\alpha_2 \xrightarrow{\text{factor }\lambda} \lambda^2(\lambda\alpha_1+\lambda\alpha_2+\alpha_3)(\alpha_1+\alpha_2)\alpha_2 \tag{291}$$

The following claims are made:

1. If $\text{div}(S) \leq 0$ the integral experiences a singularity at region S.
2. If $\text{div}(S) > 0$ the integral does not experience a singularity at region S

A rigorous proof of this claim will not be made here, but it is supported by common intuition on the behaviour of scalar integrals. Some examples of convergent and divergent integrals are given below:

1. The integral $\int_0^1 \frac{d\alpha_1}{\alpha_1}$ is divergent.
2. The integral $\int_0^1\int_0^1 \frac{d\alpha_1 d\alpha_2}{\alpha_1+\alpha_2}$ is convergent.
3. The integral $\int_0^1\int_0^1 \frac{d\alpha_1 d\alpha_2}{(\alpha_1+\alpha_2)^2}$ is divergent.

The first example satisfies $\text{div}(S = \{\alpha_1\}) = 0$ and is divergent. In the second example $\text{div}(S = \{\alpha_1, \alpha_2\}) = 1$ and the integral is convergent. In the third example one finds $\text{div}(S = \{\alpha_1, \alpha_2\}) = 0$ and there is a divergence.

The power counting method might miss a singularities because the assumption is made that the parameters considered go to zero at an equally fast rate. In principle one could expect a divergence to only pop up when some of the parameters go to zero at a faster rate than the others. These regions might additionally be found by rescaling parameters according to different powers of λ. In the integrals that were looked at in this thesis such behaviour was not observed. An example of an integral which looks superficially finite, but is divergent is given below:

4. The integral $\int_0^1 \int_0^1 d\alpha_1 d\alpha_2 \frac{\alpha_1 \alpha_2^2}{\left(\alpha_1^2+\alpha_2^4\right)^2}$ is divergent.

A divergence is observed upon rescaling $\alpha_1 \to \lambda^2 \alpha_1$, $\alpha_2 \to \lambda \alpha_2$.

The scalings $\mathrm{div}(S)$ for all subsets of alpha-parameters of the integral $I^{2,2}_{\text{part}}$ are shown below:

S	$\mathrm{div}(S)$	S	$\mathrm{div}(S)$
$\{\alpha_1\}$	1	$\{\alpha_1, \alpha_2\}$	ϵ
$\{\alpha_2\}$	1	$\{\alpha_1, \alpha_3\}$	$1-\epsilon$
$\{\alpha_3\}$	$3-4\epsilon$	$\{\alpha_2, \alpha_3\}$	$1-\epsilon$

Because we are interested in the limiting case $\epsilon \to 0$ the region $\alpha_1 \to 0$, $\alpha_2 \to 0$ falls in the case $\mathrm{div}(S) = 0$ and the other regions fall in the case $\mathrm{div}(S) > 0$. Hence the integral experiences a singularity at $S = \{\alpha_1, \alpha_2\}$ when $\epsilon \to 0$. It turns out one can remove such a divergent from the integral and perform a 'finite integral' expansion, where 'finite integral' refers to a set of integrals which are finite under the limit $\epsilon \to 0$. This will be clarified in the next section.

X. Finite integral expansions

10.1. Motivation

A method will be discussed to write divergent integrals as linear combinations of integrals which are finite under $\epsilon \to 0$. The coefficients multiplying the finite integrals will contain divergent factors ϵ^{-k}, for $k \geq 1$. As a trivial example, the Euler gamma function satisfies $\Gamma(1+x) = x\,\Gamma(x)$. Furthermore $\Gamma(x)$ admits a representation as an integral, given in eq. 418. One can therefore write:

$$\Gamma(\epsilon) = \frac{1}{\epsilon}\,\Gamma(1+\epsilon) \tag{292}$$

The left hand side is interpreted to be the divergent integral, whereas the right hand side is the finite integral expansion (in this case consisting of a single finite integral). Considering finite integrals is beneficial because one can do a power series expansion in ϵ inside of the integral which is not possible for integrals which diverge as $\epsilon \to 0$. For example, it is well known that the Laurent series in ϵ of $\Gamma(\epsilon)$ is given by:

$$\Gamma(\epsilon) \approx \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon) \tag{293}$$

Now try the integral representation of eq. 418 and expand in ϵ inside of the integral:

$$\Gamma(\epsilon) = \int_0^\infty x^{\epsilon-1} e^{-x}\,dx \approx \int_0^\infty \frac{e^{-x}}{x}\,dx + \epsilon \int_0^\infty \frac{e^{-x}\log(x)}{x}\,dx + \mathcal{O}(\epsilon^2) \tag{294}$$

The expansion is clearly not allowed as the pole $1/\epsilon$ is not present. Indeed, the expansion coefficients do not converge, so for example $\int_0^\infty \frac{e^{-x}}{x}\,dx$ is not a well-defined integral. Now instead consider a power series expansion inside the integral defining $\Gamma(1+\epsilon)$:

$$\Gamma(1+\epsilon) = \int_0^\infty x^\epsilon e^{-x}\,dx \approx \int_0^\infty e^{-x}\,dx + \epsilon \int_0^\infty e^{-x}\log(x)\,dx + \mathcal{O}(\epsilon^2) \tag{295}$$

$$= 1 - \epsilon\,\gamma_E + \mathcal{O}(\epsilon^2) \tag{296}$$

Using that $\Gamma(\epsilon) = \Gamma(1+\epsilon)/\epsilon$ one reproduces eq. 293.

When doing power series expansions in ϵ of finite integrals each coefficient in the expansion can be calculated by a numerical integrator. This means that master integrals can be calculated numerically in an automated fashion if an automated method can be found for doing finite integral expansions of complicated Feynman integrals. Furthermore, the Maple package 'HyperInt' [1] will be used in this thesis to calculate a number of complicated 3-loop integrals. This program is basically a powerful symbolic integrator which can be used for calculating expansion coefficients in ϵ of finite Feynman integrals.

HyperInt includes some code to do finite integral expansions. The method used in HyperInt is outlined in [28], and has a similar effect as the 'projective trick' discussed in the following section. The aim in the following sections is to improve the analysis of finite integral expansions and the possibilities of HyperInt by looking at methods of getting rid of so-called 'spurious poles' which are poles that seem to show up, but are seen to be zero once their integrals are integrated. Furthermore we will then show how a finite integral expansion can be done automatically through recursive power counting and application of the projective trick.

10.2. The projective trick

Note that power counting the integral of $\Gamma(\epsilon)$ leads to the following expression:

$$\int_0^\infty \lambda^\epsilon x^{\epsilon-1} e^{-\lambda x}\, dx \tag{297}$$

so that $\mathrm{div}(S = \{x\}) = \epsilon$. Make the observation that the integral does not depend on λ. Hence it must hold that:

$$0 = \frac{d}{d\lambda}\left(\int_0^\infty \lambda^\epsilon x^{\epsilon-1} e^{-\lambda x}\, dx\right) = \epsilon \int_0^\infty \lambda^{\epsilon-1} x^{\epsilon-1} e^{-\lambda x}\, dx - \int_0^\infty \lambda^{\epsilon+1} x^\epsilon e^{-\lambda x}\, dx \tag{298}$$

Put $\lambda = 1$ on the right side to derive that:

$$\int_0^\infty x^{\epsilon-1} e^{-x}\, dx = \frac{1}{\epsilon}\int_0^\infty x^\epsilon e^{-x}\, dx \tag{299}$$

This reproduces eq. 292. It turns out that in a completely similar manner finite integral expansions are derived for non-trivial Feynman integrals. Let's first look at a slightly more complex example: the Euler beta function. Note that $B(\epsilon, \epsilon+1)$ is divergent for $\epsilon \to 0$. The divergence is isolated as follows:

$$B(\epsilon,\epsilon+1) = \frac{\Gamma(\epsilon)\Gamma(1+\epsilon)}{\Gamma(1+2\epsilon)} = \frac{1}{\epsilon}\cdot\frac{\Gamma(1+\epsilon)^2}{\Gamma(1+2\epsilon)} = \frac{(1+2\epsilon)}{\epsilon}\frac{\Gamma(1+\epsilon)^2}{\Gamma(2+2\epsilon)} = \frac{(1+2\epsilon)}{\epsilon}B(\epsilon+1,\epsilon+1) \tag{300}$$

Consider the integral representation of $B(\epsilon, \epsilon+1)$ in the form of eq. 431:

$$B(\epsilon,\epsilon+1) = \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2\, \alpha_1^{\epsilon-1}\alpha_2^\epsilon\, \delta(1-\alpha_1-\alpha_2) \tag{301}$$

The Beta function can also be written as a projective form by including a factor $(\alpha_1+\alpha_2)^{1-2\epsilon}$:

$$B(\epsilon,\epsilon+1) = \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2\, \alpha_1^{\epsilon-1}\alpha_2^\epsilon\, (\alpha_1+\alpha_2)^{1-2\epsilon}\, \delta(1-\alpha_1-\alpha_2) \tag{302}$$

From power counting the divergence is in the region $\alpha_1 \to 0$. The rescaling of α_1 by a factor λ can be done by actually performing a projective transform which keeps the delta function intact because the beta function has been stated as a projective form. This leads to an expression:

$$B(\epsilon, \epsilon+1) = \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \, \lambda^\epsilon \alpha_1^{\epsilon-1} \alpha_2^\epsilon \left(\lambda\alpha_1 + \alpha_2\right)^{1-2\epsilon} \delta(1-\alpha_1-\alpha_2) \tag{303}$$

Hence, $\text{div}(S = \{\alpha_1\}) = \epsilon$. Noting that $B(\epsilon, \epsilon+1)$ does not depend on λ leads to:

$$\begin{aligned} 0 &= \frac{d}{d\lambda} \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \, \lambda^\epsilon \alpha_1^{\epsilon-1} \alpha_2^\epsilon \left(\lambda\alpha_1 + \alpha_2\right)^{1-2\epsilon} \delta(1-\alpha_1-\alpha_2) \\ &= \epsilon \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \, \lambda^{\epsilon-1} \alpha_1^{\epsilon-1} \alpha_2^\epsilon \left(\lambda\alpha_1 + \alpha_2\right)^{1-2\epsilon} \delta(1-\alpha_1-\alpha_2) + \\ &\quad (1-2\epsilon) \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \, \lambda^\epsilon \alpha_1^\epsilon \alpha_2^\epsilon \left(\lambda\alpha_1 + \alpha_2\right)^{-2\epsilon} \delta(1-\alpha_1-\alpha_2) \end{aligned} \tag{304}$$

Putting $\lambda = 1$ gives that:

$$B(\epsilon, \epsilon+1) = \left(\frac{1-2\epsilon}{\epsilon}\right) B(\epsilon+1, \epsilon+1) \tag{305}$$

The same relation as in eq. 300 is now found and the divergence of the integral has been isolated.

To summarize, we have seen that power counting involves rescaling subsets of alpha-parameters by some factor λ and singularities can be found by looking at the exponent of λ after it has been factored out. It turns out, at least in the two cases just considered, that taking a derivative on λ leads to a relation where the original integral is expressed in new integrals where the divergence has disappeared in the region S.

Let's take a look again at the scalar integral in the alpha-parametrization of $I^{2,2}_{\text{OS}}$ in eq. 290. It was seen that there is one divergent region $S = \{\alpha_1, \alpha_2\}$. Apply the following projective transform:

$$\alpha_1 \to \frac{\lambda\alpha_1}{\lambda\alpha_1 + \lambda\alpha_2 + \alpha_3} \qquad \alpha_2 \to \frac{\lambda\alpha_2}{\lambda\alpha_1 + \lambda\alpha_2 + \alpha_3} \qquad \alpha_3 \to \frac{\alpha_3}{\lambda\alpha_1 + \lambda\alpha_2 + \alpha_3} \tag{306}$$

This leads to the following result, where we explicitely put $d = 4 - 2\epsilon$:

$$I^{2,2}_{\text{part}} = \prod_{i=1}^3 \left(\int_0^\infty d\alpha_i\right) \lambda^\epsilon \left(\alpha_1 + \alpha_2\right)^{1-2\epsilon} \alpha_3^{2-4\epsilon} \left(\alpha_1\alpha_2\lambda + \alpha_3\alpha_2 + \alpha_1\alpha_3\right)^{3(\epsilon-1)} \delta\left(1 - \sum_{i=1}^3 \alpha_i\right) \tag{307}$$

Taking a derivative on λ and putting $\lambda = 1$ yields:

$$I^{2,2}_{\text{part}} = \left(\frac{3}{\epsilon} - 3\right) \prod_{i=1}^3 \left(\int_0^\infty d\alpha_i\right) \alpha_1\alpha_2 \left(\alpha_1 + \alpha_2\right)^{1-2\epsilon} \alpha_3^{2-4\epsilon} \left(\alpha_2\alpha_3 + \alpha_1\alpha_2 + \alpha_1\alpha_3\right)^{3\epsilon-4} \delta\left(1 - \sum_{i=1}^3 \alpha_i\right) \tag{308}$$

The scalings $\text{div}(S)$ for all subsets of alpha-parameters of the integral on the right are:

S	$\text{div}(S)$	S	$\text{div}(S)$
$\{\alpha_1\}$	2	$\{\alpha_1, \alpha_2\}$	$1+\epsilon$
$\{\alpha_2\}$	2	$\{\alpha_1, \alpha_3\}$	$1-\epsilon$
$\{\alpha_3\}$	$3-4\epsilon$	$\{\alpha_2, \alpha_3\}$	$1-\epsilon$

Hence for all regions $\text{div}(S) > 0$ under the limit $\epsilon \to 0$, meaning that the integral on the right is finite. From now on the previous steps will be referred to as the projective trick. The projective trick hence refers to the following operations:

The projective trick on a projective form I and region S

1. Take a projective transform:

$$\alpha_i \to \frac{\lambda \alpha_i}{\lambda \left(\sum_{j \in S} \alpha_j\right) + \sum_{j \notin S} \alpha_j} \text{ for } i \in S \qquad \alpha_i \to \frac{\alpha_i}{\lambda \left(\sum_{j \in S} \alpha_j\right) + \sum_{j \notin S} \alpha_j} \text{ for } i \notin S \tag{309}$$

2. Let $\tilde{I}$ denote the integral I after applying the projective transform.

3. Factor λ out of $\tilde{I}$ as much as possible.

4. Write down the equation $0 = ((d/d\lambda)\tilde{I})|_{\lambda=1}$ and solve for I.

After applying the projective trick, the original projective form is rewritten into new integrals for which $\text{div}(S)$ has increased. (This even works if we already have $\text{div}(S) > 0$.) None of the other regions S' will have $\text{div}(S')$ lowered.

The projective trick was an idea of Franz Herzog at Nikhef who introduced it to me using the beta function as an example, and who invited me to look at applying it to more complex Feynman integrals. In collaboration we have looked at applying the projective trick to Feynman integrals and while doing this I have developed a few ideas, stated as rules of thumb, to deal with overcounted divergences which will be encountered in the following section.

The projective trick works as long as a nonzero power of λ can be factored out after applying the projective transform. Note that for a projective form the projective transform is really equivalent to rescaling $\alpha_i \to \lambda \alpha_i$ and $d\alpha_i \to \lambda d\alpha_i$ for $i \in S$ while not altering the delta function. If a nonzero power of λ has been factored out the derivative, the expression $((d/d\lambda)\tilde{I})|_{\lambda=1}$ will contain a term equal to the original integral, which is why step 4. is possible.

Note that all the other integrals that come from $0 = ((d/d\lambda)\tilde{I})|_{\lambda=1}$ have $\text{div}(S)$ improved. A proof for this is not given, but it is motivated with an example. Look at eq. 307. Taking a derivative on λ and using the product rule for derivatives, there is an integral containing a term

$$\frac{d}{d\lambda}\left(\alpha_1\alpha_2\lambda + \alpha_3\alpha_2 + \alpha_1\alpha_3\right)^{3\epsilon-3} = \alpha_1\alpha_2\left(\alpha_1\alpha_2\lambda + \alpha_3\alpha_2 + \alpha_1\alpha_3\right)^{3\epsilon-4} \xrightarrow{\lambda=1} \alpha_1\alpha_2\left(\alpha_1\alpha_2 + \alpha_3\alpha_2 + \alpha_1\alpha_3\right)^{3\epsilon-4} \tag{310}$$

Note that $(*) \equiv (\alpha_1\alpha_2 + \alpha_3\alpha_2 + \alpha_1\alpha_3)^{3\epsilon-3}$ contributes to the power counting of the regions $S = \{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_3\}$ and $\{\alpha_2, \alpha_3\}$ but not of the regions $S = \{\alpha_1\}$, $\{\alpha_2\}$ or $\{\alpha_3\}$. This is because every monomial in $(*)$ contains at least one of α_1, α_2 and α_3 but not all of them. Hence, when the exponent of $(\lambda\alpha_1\alpha_2 + \alpha_3\alpha_2 + \alpha_1\alpha_3)^{3\epsilon-3}$ is lowered by 1 due to the derivative this can only affect $\text{div}(\{\alpha_1, \alpha_2\})$, $\text{div}(\{\alpha_2, \alpha_3\})$ and $\text{div}(\{\alpha_1, \alpha_3\})$. However, because every monomial contains at least one of α_1, α_2 and α_3, the numerator term $d/d\lambda(\lambda\alpha_1\alpha_2 + \alpha_3\alpha_2 + \alpha_1\alpha_3)$ will contain at least one of the parameters in these regions so $\text{div}(\cdot)$ cannot be lowered for any of these regions. Lastly $\text{div}(\{\alpha_1, \alpha_2\})$ is net increased by 1: it lowers by 1 because $(\lambda\alpha_1\alpha_2 + \alpha_3\alpha_2 + \alpha_1\alpha_3)^{3\epsilon-3}$ loses 1 in the exponent, but then it gains 2 because of the term $\alpha_1\alpha_2$. Furthermore $\text{div}(\{\alpha_1\})$ and $\text{div}(\{\alpha_2\})$ turn out to increase as well.

To remove all the divergences in a Feynman integral the projective trick has to be applied recursively. This leads to the following algorithm:

The recursive projective trick

1. For all $S \subset \cup_{i=1}^{n}\{\alpha_i\}$ calculate $\mathrm{div}(S)$.
2. Take some region S which satisfies $\mathrm{div}(S) < 0$ if such a region exists, else we are done.
3. Apply the projective trick for region S.
4. For each new integral apply the recursive projective trick (go to step 1.)

The recursive projective trick terminates because $\mathrm{div}(S)$ is always increased when $\mathrm{div}(S) < 0$ and none of the other regions are lowered (but possibly increased.) Note that in the second step no directions are given on which region S to take. It turns out that it matters in some cases in which order the singular regions are removed.

All the previous cases have been rather simple. Complications may arise for more complicated integrals. Consider the scalar integral part of the Feynman parametrization of $I_{\mathrm{OS}}^{2,3}$ in app. A:

$$I_{\mathrm{part}}^{2,3} = \prod_{i=1}^{3}\left(\int_0^\infty d\alpha_i\right)(\alpha_1+\alpha_2)^{1-2\epsilon}(\alpha_1+\alpha_3)^{1-2\epsilon}(\alpha_2+\alpha_3)^{1-2\epsilon}(\alpha_2\alpha_3+\alpha_1\alpha_2+\alpha_1\alpha_3)^{-3+3\epsilon}\delta\left(1-\sum_{i=1}^{3}\alpha_i\right) \tag{311}$$

The scalings $\mathrm{div}(S)$ are given below:

S	$\mathrm{div}(S)$	S	$\mathrm{div}(S)$
$\{\alpha_1\}$	1	$\{\alpha_1,\alpha_2\}$	ϵ
$\{\alpha_2\}$	1	$\{\alpha_1,\alpha_3\}$	ϵ
$\{\alpha_3\}$	1	$\{\alpha_2,\alpha_3\}$	ϵ

It is seen that there are singularities in all regions S consisting of 2 alpha-parameters. It turns out that a Laurent expansion of eq. 311 has only a simple pole $1/\epsilon$ so the divergences are overcounted by the power counting procedure. Some intuition behind this is that the 3 regions do not overlap: when for example $(\alpha_1,\alpha_2) \to (0,0)$, it also holds that $\alpha_3 \to 1$ because of the delta function.

To keep the expressions simple the following notation is introduced:

$$\begin{aligned} T(a_1,a_2,a_3,a_4,a_5,a_6,a_7) = &\prod_{i=1}^{3}\left(\int_0^\infty d\alpha_i\right)\alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3}(\alpha_1+\alpha_2)^{a_4}(\alpha_1+\alpha_3)^{a_5}(\alpha_2+\alpha_3)^{a_6}\cdot \\ &\cdot(\alpha_2\alpha_3+\alpha_1\alpha_2+\alpha_1\alpha_3)^{a_7}\delta\left(1-\sum_{i=1}^{3}\alpha_i\right) \end{aligned} \tag{312}$$

In this notation $I_{\mathrm{part}}^{2,3}$ is written as:

$$I_{\mathrm{part}}^{2,3} = T(0,0,0,1-2\epsilon,1-2\epsilon,1-2\epsilon,-3+3\epsilon) \tag{313}$$

Some code in Mathematica was written to apply the recursive projective trick. Let's follow the steps. Putting $S = \{\alpha_1, \alpha_2\}$ it is found that:

$$\begin{aligned} I^{2,3}_{\text{part}} = &- \left(\frac{3(\epsilon-1)}{\epsilon}\right) T(1,1,0,1-2\epsilon,1-2\epsilon,1-2\epsilon,3\epsilon-4) - \\ &\left(\frac{1-2\epsilon}{\epsilon}\right) T(1,0,0,1-2\epsilon,-2\epsilon,1-2\epsilon,3\epsilon-3) - \\ &\left(\frac{1-2\epsilon}{\epsilon}\right) T(0,1,0,1-2\epsilon,1-2\epsilon,-2\epsilon,3\epsilon-3) \end{aligned} \tag{314}$$

The last integral on the right is equal to the second under the relabeling $\alpha_1 \leftrightarrow \alpha_2$. Consider the divergent regions of the two independent integrals:

S	$\text{div}(S)$	S	$\text{div}(S)$
$\{\alpha_1\}$	2	$\{\alpha_1,\alpha_2\}$	$1+\epsilon$
$\{\alpha_2\}$	2	$\{\alpha_1,\alpha_3\}$	ϵ
$\{\alpha_3\}$	1	$\{\alpha_2,\alpha_3\}$	ϵ

(a) *Scalings of* $T(1,1,0,1-2\epsilon,1-2\epsilon,1-2\epsilon,3\epsilon-4)$

S	$\text{div}(S)$	S	$\text{div}(S)$
$\{\alpha_1\}$	2	$\{\alpha_1,\alpha_2\}$	$1+\epsilon$
$\{\alpha_2\}$	1	$\{\alpha_1,\alpha_3\}$	ϵ
$\{\alpha_3\}$	1	$\{\alpha_2,\alpha_3\}$	ϵ

(b) *Scalings of* $T(1,0,0,1-2\epsilon,-2\epsilon,1-2\epsilon,3\epsilon-3)$

It is seen that both integrals still have 2 remaining divergent regions. Hence the projective trick is applied again on the resulting integrals for the remaining regions. Applying the projective trick two more times and writing only integrals which are independent under a relabeling of the alpha-parameters gives the following result:

$$\begin{aligned} I^{2,3}_{\text{part}} = &\left(\frac{12}{\epsilon^2} - \frac{48}{\epsilon} + 48\right) T(0,1,2,1-2\epsilon,-2\epsilon,-2\epsilon-1,3\epsilon-3) + \\ &\left(-\frac{18}{\epsilon^2} + \frac{54}{\epsilon} - 36\right) T(0,2,2,1-2\epsilon,1-2\epsilon,-2\epsilon-1,3\epsilon-4) + \\ &\left(-\frac{2}{\epsilon^3} + \frac{12}{\epsilon^2} - \frac{24}{\epsilon} + 16\right) T(1,1,1,-2\epsilon,-2\epsilon,-2\epsilon,3\epsilon-3) + \\ &\left(\frac{27}{\epsilon^3} - \frac{135}{\epsilon^2} + \frac{216}{\epsilon} - 108\right) T(1,1,2,1-2\epsilon,-2\epsilon,-2\epsilon,3\epsilon-4) + \\ &\left(-\frac{72}{\epsilon^3} + \frac{270}{\epsilon^2} - \frac{306}{\epsilon} + 108\right) T(1,2,2,1-2\epsilon,1-2\epsilon,-2\epsilon,3\epsilon-5) + \\ &\left(\frac{60}{\epsilon^3} - \frac{141}{\epsilon^2} + \frac{108}{\epsilon} - 27\right) T(2,2,2,1-2\epsilon,1-2\epsilon,1-2\epsilon,3\epsilon-6) \end{aligned} \tag{315}$$

All the integrals on the right are free of divergences. Expanding the finite integrals in powers of ϵ and using the inbuilt integrator of Wolfram Mathematica up to goal precision 7, gives us the following numerical result:

$$I^{2,3}_{\text{part}} \approx \frac{0\times10^{-8}}{\epsilon^3} + \frac{0\times10^{-7}}{\epsilon^2} + \frac{3.000000}{\epsilon} + 2.500000 + 7.184802\epsilon + 8.043951\epsilon^2 - 7.363772\epsilon^3 + \mathcal{O}(\epsilon^4) \tag{316}$$

There are terms proportional to ϵ^{-3} and ϵ^{-2} but these have vanishing coefficients up to the precision of the calculation. From an exact derivation of $I^{2,3}_{\text{part}}$ it can be shown these coefficients are exactly zero. These poles which are the result of non-trivial cancellations between coefficients of the finite integrals will be called 'spurious' poles. Spurious poles are quite undesirable for the following reasons:

1. One needs to calculate more coefficients than necessary.
2. The expansion coefficients in ϵ of finite Feynman integrals become increasingly complex the higher the order in ϵ. For a numerical calculation this decreases the accuracy. Furthermore, using HyperInt the expansion coefficients can be calculated exactly. However the computation time HyperInt needs to compute the next coefficient in ϵ grows exponentially. For complex 3-loop integrals HyperInt might take hours to calculate coefficients once it reaches some order. For example, suppose some divergent integral has been expanded in finite integrals and HyperInt takes an hour for the coefficient of order ϵ^p for some integer p. If a different finite integral expansion were done with k spurious poles one can expect the coefficient of order ϵ^{p-k} to already take a computation time of about an hour.

It turns out that using some rules of thumb spurious poles can be removed for all the integrals which have been considered in this thesis. Some intuition for this is developed next by again considering $I^{2,3}_{\text{part}}$.

A first attempt at removing spurious poles Observe that the delta function enforces $(\alpha_1 + \alpha_2 + \alpha_3 = 1)$. Multiply $I^{2,3}_{\text{part}}$ in the form of eq. 311 by $(\alpha_1 + \alpha_2 + \alpha_3)$. This leads to:

$$\begin{aligned} T(0,0,0,1-2\epsilon,1-2\epsilon,1-2\epsilon,-3+3\epsilon) &= T(1,0,0,1-2\epsilon,1-2\epsilon,1-2\epsilon,-3+3\epsilon)+ \\ &\quad T(0,1,0,1-2\epsilon,1-2\epsilon,1-2\epsilon,-3+3\epsilon)+ \\ &\quad T(0,0,1,1-2\epsilon,1-2\epsilon,1-2\epsilon,-3+3\epsilon) \\ &\quad = 3T(1,0,0,1-2\epsilon,1-2\epsilon,1-2\epsilon,-3+3\epsilon) \end{aligned} \tag{317}$$

The last step follows from the fact that all three integrals are the same under a relabeling of the alpha-parameters. Probing for divergences yields:

S	$\text{div}(S)$	S	$\text{div}(S)$
$\{\alpha_1\}$	2	$\{\alpha_1,\alpha_2\}$	$1+\epsilon$
$\{\alpha_2\}$	1	$\{\alpha_1,\alpha_3\}$	$1+\epsilon$
$\{\alpha_3\}$	1	$\{\alpha_2,\alpha_3\}$	ϵ

It is seen that there is only one divergent region left, so 2 overcounted divergences have disappeared. That this is possible was initially expected because the divergent regions were all non-overlapping in the integration domain. We want to remove the remaining divergent region using the projective trick. This is not directly possible because $T(1,0,0,1-2\epsilon,1-2\epsilon,1-2\epsilon,-3+3\epsilon)$ is not a projective form. The integral can be turned into a projective form by applying a projective transform with $\lambda_i = 1$ for all $i = 1,2,3$. The result is equivalent to what would have been obtained if we initially multiplied $I^{2,3}_{\text{part}}$ by a factor $1 = (\alpha_1 + \alpha_2 + \alpha_3)/(\alpha_1 + \alpha_2 + \alpha_3)$.

After the integral is turned back into a projective form one can apply the projective trick. In the resulting integrals we remove the factors $(\alpha_1 + \alpha_2 + \alpha_3)$ using the delta function so we can keep the notation of eq. 312. The result is:

$$\begin{aligned} T(0,0,0,1-2\epsilon,1-2\epsilon,1-2\epsilon,-3+3\epsilon) = \frac{3}{\epsilon}\Big(& T(0,0,1,2-2\epsilon,1-2\epsilon,1-2\epsilon,3\epsilon-3)+ \\ & (4\epsilon-2)T(0,1,1,1-2\epsilon,1-2\epsilon,-2\epsilon,3\epsilon-3)- \\ & 3(\epsilon-1)T(1,1,1,1-2\epsilon,1-2\epsilon,1-2\epsilon,3\epsilon-4)\Big) \end{aligned} \quad (318)$$

This finite integral expansion is considerably more dense than what was obtained before and there are no spurious poles. For numerical integration the above finite integral expansion is a reasonable result. It turns out that for exact evaluation using a program like HyperInt the finite integral expansion just derived is problematic. To explain this we first need to provide a few more words about HyperInt.

HyperInt HyperInt is a maple package, developed by dr. Erik Panzer, which is able to exactly calculate the coefficients of finite Feynman integrals seen as power series in ϵ. The algorithms behind HyperInt are described in [29] and [1]. The mathematics behind HyperInt is based on a general class of functions called hyperlogarithms. These functions are closely related to so-called multiple polylogarithms, stated in eq. 452. An introduction to the mathematics of harmonic polylogarithms and harmonic sums for use in calculating Feynman integrals can be found in [30] and [31]. The mathematics behind these polylogarithms is a very broad subject hence they will not be further discussed here.

There are however a few points to know about HyperInt to understand the following discussion.

1. HyperInt works by integrating Feynman integrals one integration (alpha-)parameter at a time.

2. For every integration step the partial integral has to be a hyperlogarithm in the next integration variable. This roughly means that at each integration step the polynomials occurring in the calculation should factor linearly with respect to the current integration variable. This condition is called the linearly reducible integration condition. A more precise formulation is given in [1].

3. HyperInt includes code for doing finite integral expansions in a similar manner to what has been described here (although it does not use the projective trick.) Namely, one applies power counting to probe for divergent regions, and for each divergent region a procedure is applied to remove the divergent region. In HyperInt this function is called 'dimregPartial'. However, HyperInt does not include a function for applying the recursive projective trick which recursively calls itself on resulting integrals until all divergences are removed. Furthermore, HyperInt includes no methods for dealing with spurious poles, which will be developed here.

4. HyperInt generally only works on Feynman integrals when using the corollary from section 9.3, meaning that one of the alpha-parameters is set to 1 and the other parameters are integrated from 0 to infinity. If one tries to integrate a Feynman integral using for example that $\alpha_n = 1 - \sum_{i=1}^{n-1} \alpha_i$ while adjusting the integration bounds accordingly to a $n-1$ simplex, the integral will usually not be linearly reducible.

It turns out that the finite integrals from eq. 318 cannot be evaluated using HyperInt. To evaluate them using HyperInt they first have to be turned back into projective forms by applying a projective transform with $\lambda_i = 1$ for $i = 1, \ldots, n$. Now one can use the corollary from section 9.3, hence put one of the variables to 1 and integrate the others from 0 to infinity. It turns out that now the integrals do not admit a linearly reducible integration order anymore, due to the presence of an additional factor $(\alpha_1 + \alpha_2 + 1)^{-1}$ (assuming we put $\alpha_3 = 1$.)

A better method of removing spurious poles From trial and error a better method of removing spurious poles was established which seems to preserve linear reducibility. Again take a look at $I^{2,3}_{\text{part}}$. Note that one can do the following kinds of extraction:

$$(\alpha_i + \alpha_j)^{1-2\epsilon} = (\alpha_i + \alpha_j)(\alpha_i + \alpha_j)^{-2\epsilon} \text{ for } (ij) \in \{(12), (13), (23)\} \tag{319}$$

If this extraction is done for all three terms one can write:

$$\begin{aligned} I^{2,3}_{\text{part}} = &\prod_{i=1}^{3} \left(\int_0^\infty d\alpha_i \right) \left[(\alpha_1 + \alpha_2)(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_3) \right] \cdot \\ &\left[(\alpha_1 + \alpha_2)^{-2\epsilon} (\alpha_1 + \alpha_3)^{-2\epsilon} (\alpha_2 + \alpha_3)^{-2\epsilon} \cdot (\alpha_2\alpha_3 + \alpha_1(\alpha_2 + \alpha_3))^{-3+3\epsilon} \delta \left(1 - \sum_{i=1}^{3} \alpha_i \right) \right] \end{aligned} \tag{320}$$

(The square brackets are added to group terms together in two parts and have no mathematical significance.) Note that both terms between the square brackets are symmetric under permutation of the parameters. Say that the left bracketed term is expanded into a sum and each resulting integral is written down separately. Then one finds:

$$\begin{aligned} I^{2,3}_{\text{part}} = \sum_{l(\alpha_i) \in L} \Bigg[&\prod_{i=1}^{3} \left(\int_0^\infty d\alpha_i \right) l(\alpha_i) \Bigg[(\alpha_1 + \alpha_2)^{-2\epsilon} (\alpha_1 + \alpha_3)^{-2\epsilon} (\alpha_2 + \alpha_3)^{-2\epsilon} \\ &(\alpha_2\alpha_3 + \alpha_1(\alpha_2 + \alpha_3))^{-3+3\epsilon} \delta \left(1 - \sum_{i=1}^{3} \alpha_i \right) \Bigg] \Bigg] \end{aligned} \tag{321}$$

with L being the set:

$$L = \left\{ \alpha_1^2\alpha_2,\ \alpha_1\alpha_2^2,\ \alpha_1^2\alpha_3,\ 2\alpha_1\alpha_2\alpha_3,\ \alpha_2^2\alpha_3,\ \alpha_1\alpha_3^2,\ \alpha_2\alpha_3^2 \right\} \tag{322}$$

Note that one is free to relabel the integration parameters in each integral in the sum and this will not affect the bracketed part, but only the term $l(\alpha_i)$. Using this fact one can rewrite many integrals and reduce the set L to:

$$L = \left\{ 6\alpha_1^2\alpha_2,\ 2\alpha_1\alpha_2\alpha_3 \right\} \tag{323}$$

The integral with '$l(\alpha_i) = 2\alpha_1\alpha_2\alpha_3$' is convergent which can be seen by power counting. The integral with the term '$l(\alpha_i) = 6\alpha_1^2\alpha_2$' has a remaining divergent region when $(\alpha_2, \alpha_3) \to (0, 0)$. This last region can be removed using the projective trick which gives a finite integral expansion without spurious poles:

$$\begin{aligned} I^{2,3}_{\text{part}} = &2T(1,1,1,-2\epsilon,-2\epsilon,-2\epsilon,3\epsilon-3) + 12T(2,1,1,-2\epsilon,-2\epsilon-1,-2\epsilon,3\epsilon-3) + \\ &12T(2,2,0,-2\epsilon-1,-2\epsilon,-2\epsilon,3\epsilon-3) - \frac{18(\epsilon-1)T(2,2,1,-2\epsilon,-2\epsilon,-2\epsilon,3\epsilon-4)}{\epsilon} \end{aligned} \tag{324}$$

Furthermore, all of the 4 finite integrals on the right side have retained linear reducibility so that the terms in their ϵ-expansions can be integrated using HyperInt. A rule of thumb is derived from the previous example:

First rule of thumb The first step in removing overcounted divergences is to extract any positive integers out of the exponents of $\mathcal{U}^{a-\frac{d}{2}(h+1)}$ and $W^{-a+\frac{hd}{2}}$ in eq. 273. Let $\text{pow}(U)$ denote the integer power of U in the Feynman-parametrization, and similar for W. For $d = 4 - 2\epsilon$ one can distinguish the following cases:

1. If $a > 2(1+h)$ then $\text{pow}(U) > 0$, $\text{pow}(W) < 0$.
2. If $2(1+h) \geq a \geq 2h$ then $\text{pow}(U) \leq 0$, $\text{pow}(W) \leq 0$
3. If $a < 2h$ then $\text{pow}(U) < 0$, $\text{pow}(W) > 0$

Hence for cases 1 and 3 there is a positive integer power of either U or W which can/should be extracted and separated into multiple integrals. Furthermore, by looking at (permutation) symmetries of U and W the set of resulting integrals can usually be reduced to a few as in eq. 323. Two more rules of thumb are applicable to the recursive projective trick.

Second rule of thumb In step 4. of the recursive projective trick each new integral is generally considered to be a term from the product rule of the total derivative on λ. But note that one can have the following kind of scenario:

$$\frac{d}{d\lambda}\left(\alpha_1\alpha_2\alpha_3\lambda + \alpha_1\alpha_2\alpha_4\lambda + \alpha_1\alpha_4\alpha_3 + \alpha_2\alpha_4\alpha_3\right)^k\big|_{\lambda=1} = k\left(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4\right)\cdot \left(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_4\alpha_3 + \alpha_2\alpha_4\alpha_3 + \alpha_1\alpha_2\alpha_4\right)^{k-1} \quad (325)$$

Generally one can consider the term $(\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4)$ to be part of a single integral, but if there are still remaining spurious poles it might be necessary to split the term up and consider the result to be two integrals. The downside of this is that this increases the number of integrals for which the recursive projective trick is applied in the next step.

Third rule of thumb If the recursive projective trick still leads to spurious poles, or if there are simply a large number of integrals resulting from the recursive projective trick, change the order of removing the divergences.

The blowing up of the number of integrals can be illustrated with a simple example. Consider a product of three massive tadpole integrals ($I^{3,18}$.) Clearly each tadpole can be separately evaluated but suppose one is lazy and tries to automate everything in the manner discussed in this section, by going to the Feynman parametrization and evaluating by finite integral expansion. The alpha-parametrization of the three-tadpole integral is proportional (but not equal) to the integral below:

$$I_{\text{Beta}} = \prod_{i=1}^{3}\left(\int_0^\infty d\alpha_i\right)\alpha_1^{\epsilon-2}\alpha_2^{\epsilon-2}\alpha_3^{\epsilon-2}\left(\alpha_1+\alpha_2+\alpha_3\right)^{3-3\epsilon}\delta\left(1-\sum_{i=1}^{3}\alpha_i\right) \quad (326)$$

This integral can be easily exactly evaluated for $\text{Re}(\epsilon) > 1$ by using $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and recognizing eq. 431. It is written here with the additional factor so that the integral is a projective form. Taking the analytic continuation for $d = 4 - 2\epsilon \in \mathbb{C}$ gives:

$$I_{\text{Beta}} = \frac{\Gamma(\epsilon-1)^3}{\Gamma(3\epsilon-3)} \quad (327)$$

The highest order pole is seen to be of order ϵ^{-2}. (Note that a product of three massive tadpole integrals is divergent up to order ϵ^{-3}, but there are additional Γ-functions coming from the alpha-parametrization which are not included in 326.)

Let's apply the recursive projective trick to eq. 326. It turns out that if one starts removing divergent regions with a high number of parameters, the finite integral expansion consists of an enormous amount of terms. If on the other hand the divergences are removed starting from $\alpha_1 \to 0$, then $\alpha_2 \to 0$ and lastly $\alpha_3 \to 0$ this is not the case. Power counting gives:

S	$\text{div}(S)$	S	$\text{div}(S)$
$\{\alpha_1\}$	$-1+\epsilon$	$\{\alpha_1,\alpha_2\}$	$-2+2\epsilon$
$\{\alpha_2\}$	$-1+\epsilon$	$\{\alpha_1,\alpha_3\}$	$-2+2\epsilon$
$\{\alpha_3\}$	$-1+\epsilon$	$\{\alpha_2,\alpha_3\}$	$-2+2\epsilon$

If the divergence $\alpha_1 \to 0$ is removed by applying the projective trick twice one finds the following result:

$$I_{\text{Beta}} = \left(\frac{3-3\epsilon}{\epsilon-1}\right)\left(\frac{2-3\epsilon}{\epsilon}\right)\prod_{i=1}^{3}\left(\int_0^\infty d\alpha_i\right)\alpha_1^{\epsilon}\alpha_2^{\epsilon-2}\alpha_3^{\epsilon-2}\left(\alpha_1+\alpha_2+\alpha_3\right)^{1-3\epsilon} \tag{328}$$

Power counting the new integral gives the following:

S	$\text{div}(S)$	S	$\text{div}(S)$
$\{\alpha_1\}$	$1+\epsilon$	$\{\alpha_1,\alpha_2\}$	2ϵ
$\{\alpha_2\}$	$-1+\epsilon$	$\{\alpha_1,\alpha_3\}$	2ϵ
$\{\alpha_3\}$	$-1+\epsilon$	$\{\alpha_2,\alpha_3\}$	$-2+2\epsilon$

Note that $\text{div}(S)$ of the regions $S=\{\alpha_1,\alpha_2\}$ and $S=\{\alpha_1,\alpha_3\}$ has gone up, in addition to the region $\{\alpha_1\}$ being removed. Applying the same procedure for the regions $\alpha_2 \to 0$ and $\alpha_3 \to 0$ finally yields:

$$I_{\text{Beta}} = -\frac{3(-3\epsilon-2)(-3\epsilon-1)(1-3\epsilon)(2-3\epsilon)(3-3\epsilon)}{(\epsilon-1)^3\epsilon^2}\prod_{i=1}^{3}\left(\int_0^\infty d\alpha_i\right)\alpha_1^{\epsilon}\alpha_2^{\epsilon}\alpha_3^{\epsilon}\left(\alpha_1+\alpha_2+\alpha_3\right)^{-3\epsilon-3} \tag{329}$$

This result gives the finite integral expansion which consists of just a single finite integral with a prefactor with a pole of order ϵ^{-2} as is needed. Furthermore, evaluating the finite integral gives:

$$\frac{\Gamma(1+\epsilon)^3}{\Gamma(3+3\epsilon)} \tag{330}$$

So what has been found is that:

$$I_{\text{Beta}} = \frac{\Gamma(\epsilon-1)^3}{\Gamma(3\epsilon-3)} = \frac{9\left(81\epsilon^4-45\epsilon^2+4\right)}{(\epsilon-1)^2\epsilon^2}\frac{\Gamma(1+\epsilon)^3}{\Gamma(3+3\epsilon)} \tag{331}$$

Now look again at the original integral but start by removing the divergence $(\alpha_1,\alpha_2) \to (0,0)$. Applying the projective trick three times leads to:

$$I_{\text{Beta}} = \frac{3(9(\epsilon-1)\epsilon+2)}{4\epsilon(2\epsilon-1)}\prod_{i=1}^{3}\left(\int_0^\infty d\alpha_i\right)\left(\alpha_1+\alpha_2\right)^3\alpha_1^{\epsilon-2}\alpha_2^{\epsilon-2}\alpha_3^{\epsilon-2}\left(\alpha_1+\alpha_2+\alpha_3\right)^{-3\epsilon}\delta\left(1-\sum_{i=1}^{3}\alpha_i\right) \tag{332}$$

The divergences are now:

S	$\mathrm{div}(S)$	S	$\mathrm{div}(S)$
$\{\alpha_1\}$	$-1+\epsilon$	$\{\alpha_1,\alpha_2\}$	2ϵ
$\{\alpha_2\}$	$-1+\epsilon$	$\{\alpha_1,\alpha_3\}$	$-2+2\epsilon$
$\{\alpha_3\}$	$-1+\epsilon$	$\{\alpha_2,\alpha_3\}$	$-2+2\epsilon$

While $S=\{\alpha_1,\alpha_2\}$ has improved, the rest of the divergent regions are untouched unlike the previous case. Note some additional regions can be removed by expanding the term $(\alpha_1+\alpha_2)^3$ but this leads to multiple integrals. Furthermore applying the projective trick on eq. 332 for the region $\{\alpha_1,\alpha_3\}$ turns out to give multiple integrals as well. Hence in this example it is clearly seen the order of removing divergences matters for the end result. Note that in this case the problem is not the existence of spurious poles but simply the blowing up of the number of integrals after applying the projective trick multiple times.

Two simple strategies can be distinguished for the order of removing divergences:

1. From below: start removing regions S with the smallest number of alpha-parameters.
2. From above: start by removing regions with the most number of parameters.

Let's consider another example. The scalar integral in the alpha-parametrization of $I^{3,13}_{\mathrm{OS}}$ is:

$$\begin{aligned}
I^{3,13}_{\mathrm{OS,part}} &= \left(\prod_{i=1}^{4}\int_0^\infty d\alpha_i\right)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{2-3\epsilon}\left(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_4\alpha_3+\alpha_2\alpha_4\alpha_3+\alpha_1\alpha_2\alpha_4\right)^{\epsilon-2}\delta\left(1-\sum_{i=1}^{4}\alpha_i\right)\\
&= \left(\prod_{i=1}^{4}\int_0^\infty d\alpha_i\right)(4\alpha_1^2+12\alpha_2\alpha_1)(\alpha_1+\alpha_2+\alpha_3+\alpha_4)^{-3\epsilon}\cdot\\
&\quad(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_4\alpha_3+\alpha_2\alpha_4\alpha_3+\alpha_1\alpha_2\alpha_4)^{\epsilon-2}\delta\left(1-\sum_{i=1}^{4}\alpha_i\right) \qquad (333)
\end{aligned}$$

The first rule of thumb is applied in the second line. It turns out that when the divergences are removed 'from below' the finite integral expansion has two spurious poles. Using the second rule of thumb does not remove them. The first spurious pole at order ϵ^{-4} has the following integral:

$$\text{'sp. pole'}(\epsilon^{-4}) = \left(\prod_{i=1}^{4}\int_0^\infty d\alpha_i\right)\frac{240(-5\alpha_2^4\alpha_3^4\alpha_4^4\alpha_1^5+\alpha_2^5\alpha_3^3\alpha_4^4\alpha_1^5+\alpha_2^5\alpha_3^4\alpha_4^3\alpha_1^5+\alpha_2^5\alpha_3^4\alpha_4^4\alpha_1^4)}{(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_4\alpha_3+\alpha_2\alpha_4\alpha_3+\alpha_1\alpha_2\alpha_4)^7}\delta\left(1-\sum_{i=1}^{4}\alpha_i\right) \qquad (334)$$

A quick numerical calculation gives the result $-4.24\cdot10^{-11}$ which indeed indicates this integral is zero. The first nonzero coefficient which is of order ϵ^{-2} contains numerous logarithmic terms and squares of logarithmic terms. In contrast, when the divergences are removed from above, i.e. starting with the regions with the most alpha-parameters, and the second rule of thumb is applied, a finite integral expansion is derived without spurious poles. The ϵ^{-2} coefficient is:

$$\begin{aligned}
\mathrm{coef}(I^{3,13}_{\mathrm{OS,part}},\epsilon^{-2}) = \prod_{i=1}^{4}\left(\int_0^\infty d\alpha_i\right)\frac{1}{A_4^4\,S_4^3}18\alpha_1^2\alpha_2^2(\alpha_3A_4^2+\alpha_4A_4^2+\alpha_1\alpha_3\alpha_4A_4S_4+\alpha_2\alpha_3\alpha_4A_4\,S_4\\
+\alpha_3^2\alpha_4^2S_4^3)\,\delta\left(1-\sum_{i=1}^{4}\alpha_i\right)=12 \qquad (335)
\end{aligned}$$

where $A_4 \equiv \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_4\alpha_3 + \alpha_2\alpha_4\alpha_3 + \alpha_1\alpha_2\alpha_4$ and $S_4 \equiv \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ are introduced to keep the notation simple.

10.3. Dimensional shift relations

One can also wonder what the projective trick does to an integral considered in momentum space. For this purpose we consider the integral $I^{2,3}_{\text{OS}}$, except we the omit factor $(2\pi)^{-2d}$. For this reason we label the integral differently, to avoid confusion. Removing the only divergent region yields:

$$\begin{aligned} I^{\text{2-loop}}_{\text{OS}} &\equiv \int\int \frac{d^dk_1 d^dk_2}{(k_1^2)(k_2^2)((k_1+k_2+p_1)^2 - m^2)} \\ &= 3\pi^{4-2\epsilon} m^{2-4\epsilon} \frac{\Gamma(2\epsilon-1)(\epsilon-1)}{\epsilon} T(1,1,2-4\epsilon,1-2\epsilon,0,0,3\epsilon-4) \end{aligned} \tag{336}$$

where the notation of eq. 312 is used. It turns out that the result is related to a Feynman integral with a shifted dimension $\tilde{d} = 6 - 2\epsilon$:

$$\begin{aligned} I^{\text{2-loop}}_{\text{OS,shifted}} &\equiv \int\int \frac{d^{\tilde{d}}k_1 d^{\tilde{d}}k_2}{(k_1^2)^2(k_2^2)^2((k_1+k_2+p_1)^2 - m^2)} \\ &= -m^{2-4\epsilon}\pi^{6-2\epsilon}\Gamma(2\epsilon-1) T(1,1,2-4\epsilon,1-2\epsilon,0,0,3\epsilon-4) \end{aligned} \tag{337}$$

Note that the 6-dimensional integral has the first two propagators raised two power 2. The following relation is derived:

$$I^{\text{2-loop}}_{\text{OS}} = -\frac{3(\epsilon-1)}{\pi^2\epsilon} I^{\text{2-loop}}_{\text{OS,shifted}}(m^2) \tag{338}$$

Additionally of interest is the fact that one use integration by parts to reduce the shifted integral back into a master integral of the same form as the original one. Then one has a found a 'dimensional recurrence relation' for the integral considered. Using FIRE5 for the IBP-reduction gave the following result:

$$I^{\text{2-loop}}_{\text{OS,shifted}} = \frac{(\tilde{d}-6)(\tilde{d}-3)(3\tilde{d}-10)(3\tilde{d}-8)}{16(2\tilde{d}-9)(2\tilde{d}-7)m^4} I^{\text{2-loop},\tilde{d}}_{\text{OS}} \tag{339}$$

The notation $I^{\text{2-loop},\tilde{d}}_{\text{OS}}$ denotes the integral of eq. 336 with d replaced by $\tilde{d}$. Combining the result with equation 338 yields the dimensional recurrence relation:

$$I^{\text{2-loop}}_{\text{OS}} = -\frac{3(\epsilon-1)(2\epsilon-3)(3\epsilon-5)(3\epsilon-4)}{2\pi^2 m^4(4\epsilon-5)(4\epsilon-3)} I^{\text{2-loop},\tilde{d}}_{\text{OS}} \tag{340}$$

The analytical result of $I^{\text{2-loop}}_{\text{OS}}$ is known from the result for $I^{2,3}_{\text{OS}}$ in app. A, by using that $I^{\text{2-loop}}_{\text{OS}} = (2\pi)^{2d} I^{2,3}_{\text{OS}}$. It was checked using the relation $\Gamma(x+1) = x\Gamma(x)$ that eq. 340 is correct.

Unfortunately the integral $I^{\text{2-loop},\tilde{d}}_{\text{OS}}$ is again divergent so the singularity has been absorbed into the integral again. Indeed the proportionality factor that is derived is finite as $\epsilon \to 0$. However, the dimensional recurrence relation does expose structure of the integral which is considered. In reference [32] a method is discussed for using dimensional recurrence relation to find analytical results for master-integrals. This method is outside the scope of this thesis but possibly worth the consideration for the interested reader.

10.4. Concluding remarks

The projective trick has been introduced as a way of removing a single singular region from an alpha-parametrized Feynman integral. By repeatedly applying the projective trick in the manner of the recursive projective trick an automated method is found for rewriting divergent integrals into finite integrals.

One can encounter spurious poles in which case the rules of thumb should be used. The first rule of thumb should almost always be applied as it gets rid of most of the overcounted divergent regions. The second rule of thumb should only be applied in case the recursive projective trick gives spurious poles. Lastly, if there are still remaining spurious poles one should change the order in which the divergent regions are removed. The two strategies which were mentioned (remove 'from below' or 'from above') together with the other rules of thumb made it possible to derive finite integral expansions without spurious poles for all the (3-loop) integrals considered in this thesis.

The finite integrals can be expanded as power series in ϵ. Because the 3-loop integrals considered in this thesis are on-shell and contain a single mass scale, the mass factors out of these integrals. This makes it possible to evaluate the series coefficients of the power series numerically, which gives an automated method of evaluating the master integrals as Laurent series in ϵ up to some desired order. Exact evaluation of the series coefficients of the finite integrals is usually also possible with HyperInt.

XI. Explicit calculations of Feynman integrals

In this section a number of calculations are presented for 2- and 3-loop Feynman integrals.

11.1. 2-Loop

11.1.1 Warmup exercise: $T^{2,4}$

Consider a simple 2-loop integral $T^{2,4}(\lambda_1, \lambda_2, \lambda_3)$:

$$T^{2,4}(\lambda_1, \lambda_2, \lambda_3) = \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \frac{1}{(-k_1^2)^{\lambda_1} (-k_2^2)^{\lambda_2} (-(k_1+k_2)^2 + m^2)^{\lambda_3}} \tag{341}$$

This integral can be evaluated by successive integration. First one shifts variables to get:

$$T^{2,4}(\lambda_1, \lambda_2, \lambda_3) = \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \frac{1}{(-k_1^2)^{\lambda_1} (-(k_1+k_2)^2)^{\lambda_2} (-k_2^2 + m^2)^{\lambda_3}} \tag{342}$$

Evaluating the k_1 integral gives (the result follows from the alpha-parametrization):

$$\int \frac{d^d k_1}{(2\pi)^d} \frac{1}{(-k_1^2)^{\lambda_1} (-(k_1+k_2)^2)^{\lambda_2}} = \frac{i \left(-k_2^2\right)^{\frac{1}{2}(d-2\lambda_1-2\lambda_2)} \Gamma\left(\frac{d}{2}-\lambda_1\right) \Gamma\left(\frac{d}{2}-\lambda_2\right) \Gamma\left(-\frac{d}{2}+\lambda_1+\lambda_2\right)}{(4\pi)^{\frac{d}{2}} \Gamma(\lambda_1) \Gamma(\lambda_2) \Gamma(d-\lambda_1-\lambda_2)} \tag{343}$$

Hence one finds for $T^{2,4}(\lambda_1,\lambda_2,\lambda_3)$ that:

$$T^{2,4}(\lambda_1,\lambda_2,\lambda_3) = \frac{i\Gamma\left(\frac{d}{2}-\lambda_1\right)\Gamma\left(\frac{d}{2}-\lambda_2\right)\Gamma\left(-\frac{d}{2}+\lambda_1+\lambda_2\right)}{(4\pi)^{\frac{d}{2}}\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(d-\lambda_1-\lambda_2)}\int\frac{(2\pi)^{-d}d^dk_2}{(-k_2^2)^{-\frac{1}{2}(d-2\lambda_1-2\lambda_2)}(-(k_2)^2+m^2)^{\lambda_3}} \tag{344}$$

$$= -\frac{\Gamma\left(\frac{d}{2}-\lambda_1\right)\Gamma\left(\frac{d}{2}-\lambda_2\right)\Gamma\left(-\frac{d}{2}+\lambda_1+\lambda_2\right)\Gamma(-d+\lambda_1+\lambda_2+\lambda_3)\left(m^2\right)^{d-\lambda_1-\lambda_2-\lambda_3}}{(4\pi)^d\Gamma\left(\frac{d}{2}\right)\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)} \tag{345}$$

One can also evaluate the integral in a more brute force approach. From the alpha-parametrization one finds:

$$T^{2,4}(\lambda_1,\lambda_2,\lambda_3) = -\frac{m^{2(d-\lambda_1-\lambda_2-\lambda_3)}\Gamma(-d+\lambda_1+\lambda_2+\lambda_3)}{(4\pi)^d\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)}\left(\prod_{i=1}^3\int_0^\infty\alpha_i\right)\alpha_1^{\lambda_1-1}\alpha_2^{\lambda_2-1}\alpha_3^{d-\lambda_1-\lambda_2-1}\cdot$$
$$(\alpha_2\alpha_3+\alpha_1(\alpha_2+\alpha_3))^{-\frac{d}{2}}\,\delta\left(1-\sum_{i=1}^3\alpha_i\right) \tag{346}$$

We want to focus on the scalar integral so define:

$$T^{2,4}_{\text{part}}(\lambda_1,\lambda_2,\lambda_3) = \left(\prod_{i=1}^3\int_0^\infty\alpha_i\right)\alpha_1^{\lambda_1-1}\alpha_2^{\lambda_2-1}\alpha_3^{d-\lambda_1-\lambda_2-1}(\alpha_2\alpha_3+\alpha_1(\alpha_2+\alpha_3))^{-\frac{d}{2}}\,\delta\left(1-\sum_{i=1}^3\alpha_i\right) \tag{347}$$

Evaluate the α_3 integral (i.e. $\alpha_3 = 1-\alpha_1-\alpha_2$), and then do a change of variables:

$$\alpha_1 \to \eta\alpha_1 \tag{348}$$
$$\alpha_2 \to \eta\alpha_2 \tag{349}$$

under the contraint that $\alpha_1+\alpha_2 = 1$. This yields:

$$T^{2,4}_{\text{part}}(\lambda_1,\lambda_2,\lambda_3) = \int_0^1 d\alpha_1\int_0^{1-\alpha_1}d\alpha_2\int_0^1 d\eta\eta^{\lambda_1+\lambda_2-1}\alpha_1^{\lambda_1-1}\alpha_2^{\lambda_2-1}(1-\eta(\alpha_1+\alpha_2))^{d-1-\lambda_1-\lambda_2}$$
$$(\eta(\alpha_1+\alpha_2)+\eta^2(-\alpha_1\alpha_2-\alpha_1^2-\alpha_2^2))^{-\frac{d}{2}}\delta(1-\alpha_1-\alpha_2) \tag{350}$$

Evaluate the α_2 integral to find:

$$T^{2,4}_{\text{part}}(\lambda_1,\lambda_2,\lambda_3) = \int_0^1 d\alpha_1\int_0^1 d\eta\eta^{\lambda_1+\lambda_2-1-\frac{d}{2}}(1-\eta)^{d-1-\lambda_1-\lambda_2}\cdot$$
$$(1+\eta(-1+(1-\alpha_1)\alpha_1))^{-\frac{d}{2}}\alpha_1^{\lambda_1-1}(1-\alpha_1)^{\lambda_2-1} \tag{351}$$

Recognize the Euler integral of a hypergeometric function, (see eq. 447.) Evaluating the η integral yields:

$$\int_0^1 d\eta\eta^{\lambda_1+\lambda_2-1-\frac{d}{2}}(1-\eta)^{d-1-\lambda_1-\lambda_2}(1-\eta\gamma)^{-\frac{d}{2}} =$$
$$\frac{\Gamma(-\frac{d}{2}+\lambda_1+\lambda_2)\Gamma(d-\lambda_1-\lambda_2)}{\Gamma(\frac{d}{2})}{}_2F_1\left(\frac{d}{2},-\frac{d}{2}+\lambda_1+\lambda_2;\frac{d}{2}|\gamma\right) \tag{352}$$

where $\gamma = 1-(1-\alpha_1)\alpha_1$. The HGF that is found simplifies because $a = c$, and using eqs. 442, 443 this yields:

$$ {}_2F_1\left(\frac{d}{2}, -\frac{d}{2}+\lambda_1+\lambda_2; \frac{d}{2}|\gamma\right) = \sum_{k\geq 0}\frac{(-\frac{d}{2}+\lambda_1+\lambda_2)_k}{k!}\gamma^k = \sum_{k\geq 0}\binom{\frac{d}{2}-\lambda_1-\lambda_2}{k}(-\gamma)^k = (1-\gamma)^{\frac{d}{2}-\lambda_1-\lambda_2} \tag{353} $$

Plugging in $\gamma = 1-(1-\alpha_1)\alpha_1$ gives $1-\gamma = (1-\alpha_1)\alpha_1$ and combining terms leads to:

$$ \begin{aligned} T^{2,4}_{\text{part}}(\lambda_1,\lambda_2,\lambda_3) &= \frac{\Gamma(-\frac{d}{2}+\lambda_1+\lambda_2)\Gamma(d-\lambda_1-\lambda_2)}{\Gamma(\frac{d}{2})}\int_0^1 d\alpha_1 \alpha_1^{\frac{d}{2}-1-\lambda_2}(1-\alpha_1)^{\frac{d}{2}-1-\lambda_1} \\ &= \frac{\Gamma(-\frac{d}{2}+\lambda_1+\lambda_2)\Gamma(d-\lambda_1-\lambda_2)}{\Gamma(\frac{d}{2})}\frac{\Gamma\left(\frac{d}{2}-\lambda_1\right)\Gamma\left(\frac{d}{2}-\lambda_2\right)}{\Gamma\left(d-\lambda_1-\lambda_2\right)} \end{aligned} \tag{354} $$

This gives the final result, which agrees with the result from successive integration in momentum space:

$$ T^{2,4}(\lambda_1,\lambda_2,\lambda_3) = -\frac{m^{2(d-\lambda_1-\lambda_2-\lambda_3)}}{(4\pi)^d}\frac{\Gamma(\lambda_1+\lambda_2+\lambda_3-d)\Gamma\left(-\frac{d}{2}+\lambda_1+\lambda_2\right)\Gamma\left(\frac{d}{2}-\lambda_1\right)\Gamma\left(\frac{d}{2}-\lambda_2\right)}{\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)\Gamma\left(\frac{d}{2}\right)} \tag{355} $$

11.1.2 A topology from the NNLO renormalization: $T^{2,2}$

Consider the following integral:

$$ T^{2,2}(\lambda_1,\lambda_2,\lambda_3) \equiv \int\frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d}\frac{1}{(-k_1^2)^{\lambda_1}(-(k_1+k_2)^2)^{\lambda_2}(-(k_2+p_1)^2+m^2)^{\lambda_3}} \tag{356} $$

The master integral $I^{2,2}_{\text{OS}}$ from the 2-loop on-shell calculation is a special case. This integral can again be evaluated by successive integration in momentum space. The k_1 integral is the same as for $T^{2,4}$. Hence one finds:

$$ T^{2,2}(\lambda_1,\lambda_2,\lambda_3) = \frac{i\Gamma\left(\frac{d}{2}-\lambda_1\right)\Gamma\left(\frac{d}{2}-\lambda_2\right)\Gamma\left(-\frac{d}{2}+\lambda_1+\lambda_2\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\lambda_1\right)\Gamma\left(\lambda_2\right)\Gamma\left(d-\lambda_1-\lambda_2\right)}\int\frac{(2\pi)^{-d}d^dk_2}{(-k_2^2)^{-\frac{1}{2}(d-2\lambda_1-2\lambda_2)}(-(k_2+p_1)^2+m^2)^{\lambda_3}} \tag{357} $$

Using the result for $T^{1,2}$ in app. A and assuming $|p_1^2/m^2| < 1$ one finds:

$$ \begin{aligned} T^{2,2}(\lambda_1,\lambda_2,\lambda_3) = &-\frac{\Gamma\left(\frac{d}{2}-\lambda_1\right)\Gamma\left(\frac{d}{2}-\lambda_2\right)\Gamma\left(-\frac{d}{2}+\lambda_1+\lambda_2\right)\Gamma\left(-d+\lambda_1+\lambda_2+\lambda_3\right)\left(m^2\right)^{d-\lambda_1-\lambda_2-\lambda_3}}{(4\pi)^d\Gamma\left(\frac{d}{2}\right)\Gamma\left(\lambda_1\right)\Gamma\left(\lambda_2\right)\Gamma\left(\lambda_3\right)}\cdot \\ &{}_2F_1\left(-\frac{d}{2}+\lambda_1+\lambda_2, -d+\lambda_1+\lambda_2+\lambda_3; \frac{d}{2}\middle|\frac{p_1^2}{m^2}\right) \end{aligned} \tag{358} $$

A brute force approach is again possible too. From the alpha-parametrization one has:

$$ T^{2,2}(\lambda_1,\lambda_2,\lambda_3) = -\frac{\Gamma(\lambda_1+\lambda_2+\lambda_3-d)}{(4\pi)^d\Gamma(\lambda_1)\Gamma(\lambda_2)\Gamma(\lambda_3)}T^{2,2}_{\text{part}}(\lambda_1,\lambda_2,\lambda_3) \tag{359} $$

where $T^{2,4}_{\text{part}}(\lambda_1,\lambda_2,\lambda_3)$ is equal to:

$$T^{2,2}_{\text{part}}(\lambda_1,\lambda_2,\lambda_3)=\int_0^\infty\int_0^\infty\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \alpha_1^{\lambda_1-1}\alpha_2^{\lambda_2-1}\alpha_3^{\lambda_3-1}(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{\lambda_1+\lambda_2+\lambda_3-\frac{3d}{2}}$$
$$(-p_1^2\alpha_1\alpha_2\alpha_3+m^2\alpha_3(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3))^{d-\lambda_1+\lambda_2-\lambda_3}\delta(1-\alpha_1-\alpha_2-\alpha_3) \quad (360)$$

The alpha-parametrization looks like that of $T^{2,4}$, but there is now a composite term with the external momentum p_1 in it. It is possible to get rid of this term using a binomial expansion, assuming that $|p_1^2/m^2|<1$.

$$(-p_1^2\alpha_1\alpha_2\alpha_3+m^2\alpha_3(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3))^{d-\lambda_1+\lambda_2-\lambda_3}=\sum_{k\geq 0}\binom{d-\lambda_1-\lambda_2-\lambda_3}{k}\left(-\frac{p_1^2}{m^2}\right)^k\cdot$$
$$(m^2)^{d-\lambda_1-\lambda_2-\lambda_3}(\alpha_3(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3))^{d-\lambda_1-\lambda_2-\lambda_3-k}(\alpha_1\alpha_2\alpha_3)^k \quad (361)$$

Combining terms together gives:

$$T^{2,2}_{\text{part}}(\lambda_1,\lambda_2,\lambda_3)=\sum_{k\geq 0}\binom{d-\lambda_1-\lambda_2-\lambda_3}{k}\left(-\frac{p_1^2}{m^2}\right)^k(m^2)^{d-\lambda_1-\lambda_2-\lambda_3}\int_0^\infty\int_0^\infty\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3$$
$$\alpha_1^{k+\lambda_1-1}\alpha_2^{k+\lambda_2-1}\alpha_3^{d-\lambda_1-\lambda_2-1}\left(\alpha_2\alpha_3+\alpha_1\left(\alpha_2+\alpha_3\right)\right)^{-\frac{d}{2}-k}\delta\left(1-\sum_{i=1}^3\alpha_i\right) \quad (362)$$

Let $(\%\alpha\%)$ denote the scalar integral in $T^{2,2}_{\text{part}}(\lambda_1,\lambda_2,\lambda_3)$ without the prefactor:

$$(\%\alpha\%)=\int_0^\infty\int_0^\infty\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3\alpha_1^{\lambda_1-1+k}\alpha_2^{\lambda_2-1+k}\alpha_3^{d-1-\lambda_1-\lambda_2}(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{-\frac{d}{2}-k}\delta\left(1-\sum_{i=1}^3\alpha_i\right) \quad (363)$$

Similar to before one can now do consecutively:

- Evaluate the α_3 integral so that $\alpha_3\to 1-\alpha_1-\alpha_2$.
- Change variables to: $\alpha_1\to\eta\alpha_1$, $\alpha_2\to\eta(1-\alpha_1)$.

The result is:

$$(\%\alpha\%)=\int_0^1\int_0^1 d\alpha_1 d\eta\alpha_1^{\lambda_1-1+k}(1-\alpha_1)^{\lambda_2-1+k}\eta^{\lambda_1+\lambda_2-1+k-\frac{d}{2}}(1-\eta)^{d-1-\lambda_1-\lambda_2}\cdot$$
$$(1+\eta(-1+\alpha_1-\alpha_1^2))^{-\frac{d}{2}-k} \quad (364)$$

Recognizing the Euler integral over η, and doing the integration, one finds:

$$\int_0^1 d\eta\eta^{-1+\lambda_1+\lambda_2+k-\frac{d}{2}}(1-\eta)^{-1+d-\lambda_1-\lambda_2}(1+\eta(-1+\alpha_1-\alpha_1^2))^{-\frac{d}{2}-k}=$$
$$\frac{\Gamma(d-\lambda_1-\lambda_2)\Gamma(-\frac{d}{2}+k+\lambda_1+\lambda_2)((1-\alpha_1)\alpha_1)^{\frac{d}{2}-k-\lambda_1-\lambda_2}}{\Gamma(\frac{d}{2}+k)} \quad (365)$$

So that:

$$(\%\alpha\%)=\frac{\Gamma(d-\lambda_1-\lambda_2)\Gamma(-\frac{d}{2}+k+\lambda_+\lambda_2)}{\Gamma(\frac{d}{2}+k)}\int_0^1 d\alpha_1\,\alpha_1^{\frac{d}{2}-\lambda_2-1}(1-\alpha_1)^{\frac{d}{2}-\lambda_1-1} \quad (366)$$

The remaining integral is clearly a beta function so we find that:

$$(\%\alpha\%) = \frac{\Gamma(\frac{d}{2}-\lambda_2)\Gamma(\frac{d}{2}-\lambda_1)\Gamma(-\frac{d}{2}+k+\lambda_1+\lambda_2)}{\Gamma(\frac{d}{2}+k)} \tag{367}$$

Putting everything together the following is found:

$$T^{2,2}(\lambda_1,\lambda_2,\lambda_3) = -(m^2)^{d-\lambda_1-\lambda_2-\lambda_3}\frac{\Gamma(-d+\lambda_1+\lambda_2+\lambda_3)\Gamma(\frac{d}{d}-\lambda_2)\Gamma(\frac{d}{2}-\lambda_1)}{(4\pi)^d\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma(\lambda_3)}\cdot \tag{368}$$

$$\cdot\sum_{k\geq 0}\left(-\frac{p_1^2}{m^2}\right)^k\binom{d-\lambda_1-\lambda_2-\lambda_3}{k}\frac{\Gamma(-\frac{d}{2}+k+\lambda_1+\lambda_2)}{\Gamma(\frac{d}{2}+k)} \tag{369}$$

In the latest expression one can recognize a hypergeometric function, after rewriting the binomial coefficient using 442 and the final result becomes:

$$T^{2,2}(\lambda_1,\lambda_2,\lambda_3) = -(m^2)^{d-\lambda_1-\lambda_2-\lambda_3}\frac{\Gamma(-\frac{d}{2}+\lambda_1+\lambda_2)\Gamma(-d+\lambda_1+\lambda_2+\lambda_3)\Gamma(\frac{d}{d}-\lambda_2)\Gamma(\frac{d}{2}-\lambda_1)}{(4\pi)^d\Gamma(\frac{d}{2})\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma(\lambda_3)}\cdot$$
$$ {}_2F_1\left(-\frac{d}{2}+\lambda_1+\lambda_2,-d+\lambda_1+\lambda_2+\lambda_3;\frac{d}{2}\middle|\frac{p_1^2}{m^2}\right) \tag{370}$$

This is the same as what was obtained using successive integration in momentum space.

The master integral $I^{2,2}_{\text{OS}}$ from the 2-loop calculation follows from putting $\lambda_{1,2,3}=1$ and $p_1^2=m^2$. Plugging in these values gives:

$$\sum_{k\geq 0}\left(-\frac{p_1^2}{m^2}\right)^k\frac{\Gamma(d-\lambda_1-\lambda_2-\lambda_3+1)}{\Gamma(d-\lambda_1-\lambda_2-\lambda_3-k+1)k!}\frac{\Gamma\left(-\frac{d}{2}+k+\lambda_1+\lambda_2\right)}{\Gamma\left(\frac{d}{2}+k\right)} = \sum_{k\geq 0}\frac{(-1)^k\Gamma(d-2)\Gamma\left(2-\frac{d}{2}+k\right)}{\Gamma(d-2-k)\Gamma\left(\frac{d}{2}+k\right)k!}$$
$$= \frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma(2d-5)}{\Gamma(d-2)\Gamma\left(\frac{3}{2}(d-2)\right)} \tag{371}$$

This gives the specific result:

$$I^{2,2}_{\text{OS}} \equiv \int\frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d}\frac{1}{(-k_1^2)(-k_2^2)(-(k_1+k_2+p_1)^2+m^2)}$$
$$= \frac{-(m^2)^{d-3}\Gamma(3-d)\Gamma\left(\frac{d}{2}-1\right)^2\Gamma\left(2-\frac{d}{2}\right)\Gamma(2d-5)}{(4\pi)^d\Gamma(d-2)\Gamma\left(\frac{3}{2}(d-2)\right)} \tag{372}$$

11.1.3 Master integral from the NNLO renormalization: $I^{2,3}_{\text{OS}}$

The integral $I^{2,3}_{\text{OS}}$ comes from the 2-loop on-shell renormalization and turns out to be significantly harder to calculate than $I^{2,2}_{\text{OS}}$.

$$I^{2,3}_{\text{OS}} \equiv \int\frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d}\frac{1}{(-k_1^2+m^2)(-k_2^2+m^2)(-(k_1+k_2-p_1)^2+m^2)} \tag{373}$$

$$= -(4\pi)^{-d} m^{2d-6} \Gamma(3-d) \left(\prod_{i=1}^{3} \int_0^\infty d\alpha_i \right) (\alpha_1+\alpha_2)^{d-3} (\alpha_1+\alpha_3)^{d-3} (\alpha_2+\alpha_3)^{d-3}$$
$$(\alpha_2\alpha_3 + \alpha_1(\alpha_2+\alpha_3))^{3-\frac{3d}{2}} \delta\left(1 - \sum_{i=1}^{3} \alpha_i\right) \tag{374}$$

It is not possible to use successive integration to calculate this integral due to the fact that every propagator is massive. For example one has:

$$\int \frac{d^d k_1}{(2\pi)^d} \frac{1}{(-k_1^2+m^2)(-(k_1+k_2)^2+m^2)} = i(4\pi)^{-\frac{d}{2}} \Gamma\left(2-\frac{d}{2}\right) \int_0^1 d\alpha_1 \left((\alpha_1-1)\alpha_1 k_2^2 + m^2\right)^{\frac{d-4}{2}} \tag{375}$$

This integral yields a non-trivial function of k_2 and it is not possible to do the remaining k_2 integral using the usual methods. Furthermore, a satisfactory brute force approach as was found for $T^{2,2}$ was not obtained either. In fact quite some time was spend on finding a method to evaluate the alpha-parametrized integral as an analytic function in d, but the effort did not lead to a result. For this reason the alternative became to look at the ϵ-expansion of $I^{2,3}_{\text{OS}}$. The finite integral expansion is given by:

$$\begin{aligned} I^{2,3}_{\text{OS}} = &-\frac{m^{2-4\epsilon}}{(4\pi)^{4-2\epsilon}} \Gamma(2\epsilon-1) \left(\prod_{i=1}^{3} \int_0^\infty d\alpha_i \right) \Big(\\ &- \frac{18(\epsilon-1)}{\epsilon} \alpha_1^2\alpha_2^2\alpha_3 (\alpha_1+\alpha_2)^{-2\epsilon} (\alpha_1+\alpha_3)^{-2\epsilon} (\alpha_2+\alpha_3)^{-2\epsilon} (\alpha_2\alpha_3+\alpha_1(\alpha_2+\alpha_3))^{3\epsilon-4} + \\ &12\alpha_1^2\alpha_2\alpha_3 (\alpha_1+\alpha_2)^{-2\epsilon} (\alpha_1+\alpha_3)^{-2\epsilon-1} (\alpha_2+\alpha_3)^{-2\epsilon} (\alpha_2\alpha_3+\alpha_1(\alpha_2+\alpha_3))^{3(\epsilon-1)} + \\ &12\alpha_1^2\alpha_2^2 (\alpha_1+\alpha_2)^{-2\epsilon-1} (\alpha_1+\alpha_3)^{-2\epsilon} (\alpha_2+\alpha_3)^{-2\epsilon} (\alpha_2\alpha_3+\alpha_1(\alpha_2+\alpha_3))^{3(\epsilon-1)} + \\ &2\alpha_1\alpha_2\alpha_3 (\alpha_1+\alpha_2)^{-2\epsilon} (\alpha_1+\alpha_3)^{-2\epsilon} (\alpha_2+\alpha_3)^{-2\epsilon} (\alpha_2\alpha_3+\alpha_1(\alpha_2+\alpha_3))^{3(\epsilon-1)} \\ &\Big) \delta\left(1-\sum_{i=1}^{3}\alpha_i\right) \end{aligned} \tag{376}$$

Expanding the finite integrals in ϵ and evaluating the series coefficients using HyperInt yields the result stated in app. A.

11.2. 3-Loop

There are a number of 3-loop integrals reported in app. A, which were derived by finding finite integral expansions and integrating the coefficients using HyperInt whenever possible. For the integral $I^{3,8}_{\text{OS}}$ a linearly reducible integration order was not found.

11.2.1 $I^{3,8}_{\text{OS}}$

The integral $I^{3,8}_{\text{OS}}$ is represented diagrammatically in fig. 7, which follows from contracting to a point the propagators raised to power zero in fig. 8.

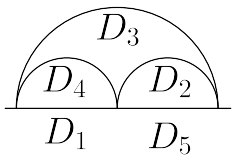


Figure 7: *Diagrammatic representation of $I^{3,8}$.*

The integral could not be performed using HyperInt because a linearly reducible integration order could not be found. For this reason a numerical computation was performed on the finite integral expansion using Mathematica's 'NIntegrate' which gave the following result:

$$I^{3,8}_{\text{OS}} \approx -i\pi^{6-3\epsilon} m^{2(1-3\epsilon)} \Gamma(3\epsilon-1) \left(\frac{3}{\epsilon^2} + \frac{7}{\epsilon} - 14.8043 \; - 17.9616\epsilon + 110.071\epsilon^2 + \mathcal{O}(\epsilon^3) \right) \tag{377}$$

Precise errors are not given on the coefficients as this calculation is presented as a proof of concept. The first two coefficients were reported to be very close to 3 and 7 with an error of 10^{-4}. The ϵ^2 coefficient was reported with an error of order 0.1. It is seen in app. A that all the 3-loop integrals considered have rational numbers for the first two coefficients in their ϵ-expansions. Therefore it seems very likely that the first two coefficients are indeed exactly 3 and 7.

The integral can also be stated with a prefactor according to the conventions of [24], leading to:

$$I^{3,8}_{\text{OS}} \approx -i\pi^{6-3\epsilon} m^{2-6\epsilon} \Gamma(\epsilon+1)^3 \left(-\frac{1}{\epsilon^3} - \frac{16}{3\epsilon^2} - \frac{16.0000}{\epsilon} - 43.9109 - 154.914\epsilon + \mathcal{O}\left(\epsilon^2\right) \right) \tag{378}$$

The exact result taken from [24] is stated below. (More coefficients are given in the paper.)

$$\begin{aligned} I^{3,8}_{\text{OS}} = -i\pi^{6-3\epsilon} m^{2-6\epsilon} \Gamma(\epsilon+1)^3 \Bigg(& -\frac{1}{\epsilon^3} - \frac{16}{3\epsilon^2} - \frac{16}{\epsilon} + \left(2\zeta(3) - \frac{8\pi^2}{3} - 20 \right) + \\ & \epsilon \left(-\frac{200\zeta(3)}{3} - \frac{3\pi^4}{10} - 28\pi^2 + \frac{364}{3} + 16\pi^2 \log(2) \right) + \mathcal{O}\left(\epsilon^2\right) \Bigg) \end{aligned} \tag{379}$$

Taking the absolute value of the difference between the numerical and exact calculation of the ϵ^0 and ϵ^1 coefficients gives:

$$|(\text{exact - numerical})(\epsilon^0)| \approx 0.004 \tag{380}$$

$$|(\text{exact - numerical})(\epsilon^1)| \approx 0.004 \tag{381}$$

This gives confidence the finite integral expansion and subsequent numerical integration can be done with sufficient accuracy.

XII. Conclusion

The renormalization of the top quark was studied which is the heaviest quark and is of significant phenomenological interest. For this purpose an explicit treatment of multiloop renormalization was first given, using diagrammatic formulas. At multiloop order there are diagrams with counterterm insertions which are not seen at 1-loop order. It was shown in addition that renormalization can be done by rescaling calculations from the bare theory with the renormalization constants and expanding the renormalization constants in the coupling constant afterwards. This makes it possible to renormalize a theory without explicitely looking at counterterms. As an introduction the 1-loop renormalization constants were derived in the MS, $\overline{\text{MS}}$ and on-shell schemes in a manual calculation.

The phenomenological context of top quark renormalization was then touched upon, which included a short treatment of top quark measurements in colliders, asymptotic series and the renormalon ambiguity in the quark pole mass. Furthermore, the electroweak vacuum instability was presented which implicates that a precise determination of the top quark mass could shed light on the ultimate fate of the universe.

The building blocks of a computational setup for doing automated calculations of Feynman diagrams were discussed next. Using an implementation developed for this thesis, a number of 1, 2, and 3-loop results were presented. In particular the full 2-loop renormalization of the top quark mass and field was done in the on-shell scheme. These results relied on finding explicit results for master integrals, usually as a Laurent series in ϵ.

An explicit treatment was given of methods for the derivation of master integrals. We started by deriving the alpha-parametrization. A new method called the 'projective trick' was introduced for deriving finite integral expansions, and rules of thumb were developed to circumvent the creation of spurious poles for the 2- and 3-loop on-shell integrals considered. The presented approach allows one to use existing general purpose numerical integration software to derive numerical results for on-shell Feynman integrals with one mass scale up to finite order or higher in ϵ in an automated manner. Furthermore, using the Maple package HyperInt we succeeded in deriving exact results for the series coefficients of most integrals.

Appendices

A. Results for Feynman integrals

The following is a list of integrals that were needed in this thesis. The subscript OS is used for integrals where $p_1^2 = m^2$. The exponents λ_i of the propagators in the topologies are assumed to be nonzero numbers in $\mathbb{C}$. A factor $(2\pi)^{-d}$ is included for each internal momentum. Note that the integrals are given with a minus in front of the momenta in the propagators.

1.1. Topologies

1.1.1 1-Loop

$$T^{1,1}(\lambda) \equiv \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{(-k_1^2+m^2)^\lambda} = \frac{i\,\Gamma\left(\lambda_1 - \frac{d}{2}\right)\left(m^2\right)^{\frac{1}{2}(d-2\lambda_1)}}{(4\pi)^{\frac{d}{2}}\Gamma(\lambda_1)} \tag{382}$$

$$\begin{aligned} T^{1,2}(\lambda_1,\lambda_2) &\equiv \int d^d k_1 \frac{1}{(-k_1^2+m^2)^{\lambda_1}(-(k_1-p_1)^2)^{\lambda_2}} \qquad (383)\\ &= \frac{i\,\Gamma\left(-\frac{d}{2}+\lambda_1+\lambda_2\right)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\lambda_1\right)\Gamma\left(\lambda_2\right)} \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2\, \alpha_1^{\lambda_1-1}\alpha_2^{\lambda_2-1}\left(\alpha_1\left(m^2-\alpha_2 p_1^2\right)\right)^{\frac{1}{2}(d-2\lambda_1-2\lambda_2)}\delta(1-\sum_{i=1}^{2}\alpha_i)\\ &= \frac{i\Gamma\left(-\frac{d}{2}+\lambda_1+\lambda_2\right)^2\Gamma\left(\frac{d}{2}-\lambda_2\right)\left(m^2\right)^{\frac{1}{2}(d-2\lambda_1-2\lambda_2)}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)\Gamma\left(\lambda_1\right)} {}_2F_1\left(\lambda_2, -\frac{d}{2}+\lambda_1+\lambda_2; \frac{d}{2}\middle|\frac{p_1^2}{m^2}\right)\\ &\quad (\text{Assuming } |p_1^2/m^2|<1) \end{aligned}$$

$$T^{1,2}_{\text{OS}}(\lambda_1,\lambda_2) = \frac{i\,\Gamma\left(d-\lambda_1-2\lambda_2\right)\Gamma\left(-\frac{d}{2}+\lambda_1+\lambda_2\right)\left(m^2\right)^{\frac{1}{2}(d-2\lambda_1-2\lambda_2)}}{(4\pi)^{\frac{d}{2}}\Gamma\left(\lambda_1\right)\Gamma\left(d-\lambda_1-\lambda_2\right)} \tag{384}$$

1.1.2 2-Loop

$$\begin{aligned} T^{2,2}(\lambda_1,\lambda_2,\lambda_3) &\equiv \int \frac{d^d k_1}{(2\pi)^d}\int \frac{d^d k_2}{(2\pi)^d} \frac{1}{(-k_1^2)^{\lambda_1}(-k_2^2)^{\lambda_2}(-(k_1+k_2+p_1)^2+m^2)^{\lambda_3}} \qquad (385)\\ &= -(m^2)^{d-\lambda_1-\lambda_2-\lambda_3}\frac{\Gamma(-\frac{d}{2}+\lambda_1+\lambda_2)\Gamma(-d+\lambda_1+\lambda_2+\lambda_3)\Gamma(\frac{d}{d}-\lambda_2)\Gamma(\frac{d}{2}-\lambda_1)}{(4\pi)^d\Gamma(\frac{d}{2})\Gamma(\lambda_1)\Gamma(\lambda_3)\Gamma(\lambda_3)}\cdot\\ &\quad {}_2F_1\left(-\frac{d}{2}+\lambda_1+\lambda_2, -d+\lambda_1+\lambda_2+\lambda_3; \frac{d}{2}\middle|\frac{p_1^2}{m^2}\right)\ (\text{Assuming } |p_1^2/m^2|<1) \end{aligned}$$

$$\begin{aligned} T^{2,4}(\lambda_1,\lambda_2,\lambda_3) &\equiv \int \frac{d^d k_1}{(2\pi)^d}\int \frac{d^d k_2}{(2\pi)^d} \frac{1}{(-k_1^2)^{\lambda_1}(-k_2^2)^{\lambda_2}(-(k_1+k_2)^2+m^2)^{\lambda_3}} \qquad (386)\\ &= -\frac{\Gamma\left(\frac{d}{2}-\lambda_1\right)\Gamma\left(\frac{d}{2}-\lambda_2\right)\Gamma\left(-\frac{d}{2}+\lambda_1+\lambda_2\right)\Gamma\left(-d+\lambda_1+\lambda_2+\lambda_3\right)\left(m^2\right)^{d-\lambda_1-\lambda_2-\lambda_3}}{(4\pi)^d\Gamma\left(\frac{d}{2}\right)\Gamma\left(\lambda_1\right)\Gamma\left(\lambda_2\right)\Gamma\left(\lambda_3\right)} \end{aligned}$$

1.2. Integrals

1.2.1 1-Loop

$$I^{1,1} \equiv \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{(-k_1^2+m^2)} = \frac{i\,\Gamma\left(1-\frac{d}{2}\right)\left(m^2\right)^{\frac{1}{2}(d-2)}}{(4\pi)^{\frac{d}{2}}} \tag{387}$$

$$I^{1,2} \equiv \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{(-k_1^2)(-(k_1-p_1)^2)} = \frac{i\,\left(-p_1^2\right)^{\frac{d-4}{2}}\Gamma\left(2-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}-1\right)^2}{(4\pi)^{\frac{d}{2}}\Gamma(d-2)} \tag{388}$$

$$I^{1,3} \equiv \int \frac{d^d k_1}{(2\pi)^d} \frac{1}{(-k_1^2+m^2)(-(k_1-p_1)^2+m^2)} = i(4\pi)^{-\frac{d}{2}}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 d\alpha_1 \left(m^2-p_1^2(1-\alpha_1)\alpha_1\right)^{\frac{1}{2}(d-4)} \tag{389}$$

1.2.2 2-Loop

$$I^{2,1} \equiv \int \frac{d^d k_1}{(2\pi)^d}\int \frac{d^d k_2}{(2\pi)^d} \frac{1}{(-k_1^2+m^2)(-k_2^2+m^2)} \tag{390}$$
$$= -\frac{1}{(4\pi)^d}\Gamma\left(1-\frac{d}{2}\right)^2 (m^2)^{d-2}$$

$$I^{2,2}_{\text{OS}} \equiv \int \frac{d^d k_1}{(2\pi)^d}\int \frac{d^d k_2}{(2\pi)^d} \frac{1}{(-k_1^2)(-k_2^2)(-(k_1+k_2+p_1)^2+m^2)} \tag{391}$$
$$= -\frac{(m^2)^{d-3}}{(4\pi)^d}\frac{\Gamma(3-d)\Gamma\left(\frac{d}{2}-1\right)^2\Gamma(2-\frac{d}{2})\Gamma(2d-5)}{\Gamma(d-2)\Gamma\left(\frac{3}{2}(d-2)\right)}$$

$$I^{2,3}_{\text{OS}} \equiv \int \frac{d^d k_1}{(2\pi)^d}\int \frac{d^d k_2}{(2\pi)^d} \frac{1}{(-k_1^2+m^2)(-k_2^2+m^2)(-(k_1+k_2-p_1)^2+m^2)} \tag{392}$$
$$= -\frac{m^{2-4\epsilon}}{(4\pi)^{4-2\epsilon}}\Gamma(2\epsilon-1)\left(\frac{5}{2}+\frac{3}{\epsilon}+\left(-\frac{9}{4}-\frac{\pi^2}{2}\right)\epsilon+\epsilon^2\left(-\frac{171}{8}+\frac{9\pi^2}{4}+6\zeta_3\right)+\epsilon^3\Bigg(-\frac{1377}{16}+$$
$$\frac{99\pi^2}{8}-\frac{3\pi^4}{40}-16\pi^2\log(2)+61\zeta_3\Bigg)+\epsilon^4\Bigg(-\frac{9315}{32}+\frac{729\pi^2}{16}+\frac{59\pi^4}{80}-72\pi^2\log(2)+48\pi^2\log^2(2)+$$
$$192\zeta_{1,-3}+\frac{495\zeta(3)}{2}-\pi^2\zeta_3-336\log(2)\zeta_3+18\zeta_5\Bigg)+\epsilon^5\Bigg(-\frac{59049}{64}+\frac{4779\pi^2}{32}+\frac{117\pi^4}{32}-$$
$$\frac{79\pi^6}{5040}-252\pi^2\log(2)-\frac{24}{5}\pi^4\log(2)+216\pi^2\log^2(2)-96\pi^2\log^3(2)+864\zeta_{1,-3}-1152\log(2)\zeta_{1,-3}-$$
$$1152\zeta_{1,1,-3}+\frac{3357\zeta_3}{4}+\frac{129\pi^2\zeta_3}{2}-1512\log(2)\zeta_3+1008\log^2(2)\zeta_3+6\zeta_3^2-$$
$$657\zeta_5+\mathcal{O}(\epsilon^6)\Bigg)\Bigg)$$

1.2.3 3-Loop

The integrals below were found by deriving finite integral expansions and integrating the series coefficients in ϵ using HyperInt, except for $I^{3,8}_{\text{OS}}$ for which a linearly reducible integration order was

not found. A numerical integration was done for the first 5 coefficients in eq. 377 and compared to [24]. The numbering of the integrals is done consistent with [24]. The integral $I_{\text{OS}}^{3,19}$ is not in [24], and is therefore labeled number 19. All the other integrals can be expressed in the following topology:

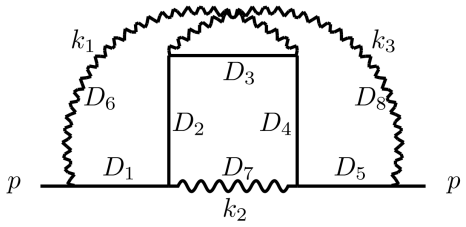


Figure 8: *3-Loop topology that includes the master integrals from the N^{3}LO calculation. (Figure taken from [24].)*

The denominator factors are given by:

$$\begin{aligned}
D_1 &= -(p-k_1)^2 + m^2 - i\epsilon & D_2 &= -(p-k_1-k_2)^2 + m^2 - i\epsilon \\
D_3 &= -(p-k_1-k_2-k_3)^2 + m^2 - i\epsilon & D_4 &= -(p-k_2-k_3)^2 + m^2 - i\epsilon \\
D_5 &= -(p-k_3)^2 + m^2 - i\epsilon & D_6 &= -k_1^2 - i\epsilon \\
D_7 &= -k_2^2 - i\epsilon & D_8 &= -k_3^2 - i\epsilon
\end{aligned} \tag{393}$$

The computation of the integrals in HyperInt was usually terminated after the last coefficient took about an hour to evaluate. This means some integrals are reported with less precision than in [24], although given a few days or a better computer the results from [24] could be improved upon. As a proof of concept $I_{\text{OS}}^{3,14}$ is stated with the ϵ^3 coefficient, one more than in [24].

The derived results are completely in agreement with [24] up to the number of coefficients that were derived in this thesis. To observe this eq. 244 has to be used to rewrite $\zeta_{1,-3}$.

$$I_{\text{OS}}^{3,8} \equiv \int \left(\prod_{i=1}^{3} \frac{d^d k_i}{(2\pi)^d}\right) \frac{1}{D_1 D_2 D_3 D_4 D_5} \tag{394}$$

$$I_{\text{OS}}^{3,9} \equiv \int \left(\prod_{i=1}^{3} \frac{d^d k_i}{(2\pi)^d}\right) \frac{1}{D_2 D_3 D_5 D_6 D_7} \tag{395}$$

$$\begin{aligned}
&= -i(4\pi)^{-\frac{3}{2}(4-2\epsilon)} m^{2-6\epsilon} \Gamma(\epsilon+1)^3 \Bigg(-\frac{2}{3\epsilon^3} - \frac{10}{3\epsilon^2} + \frac{-26-\pi^2}{3\epsilon} + \frac{1}{3}\left(-16\zeta_3 - 11\pi^2 - 6\right) + \\
&\left(\pi^2\left(16\log(2) - \frac{73}{3}\right) - \frac{13\pi^4}{45} - \frac{248\zeta_3}{3} + \frac{398}{3}\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\Bigg)
\end{aligned}$$

$$\begin{aligned}
I_{\text{OS}}^{3,13} &\equiv \int \left(\prod_{i=1}^{3} \frac{d^d k_i}{(2\pi)^d}\right) \frac{1}{D_1 D_2 D_4 D_5} \\
&= -i(4\pi)^{-\frac{3}{2}(4-2\epsilon)} m^{4-6\epsilon} \Gamma(\epsilon+1)^3 \left(\frac{2}{\epsilon^3} + \frac{23}{3\epsilon^2} + \frac{35}{2\epsilon} + \frac{275}{12} + \left(\frac{112\zeta_3}{3} - \frac{189}{8}\right)\epsilon + \mathcal{O}\left(\epsilon^2\right)\right)
\end{aligned} \tag{396}$$

$$I_{\text{OS}}^{3,14} \equiv \int \left(\prod_{i=1}^{3} \frac{d^d k_i}{(2\pi)^d}\right) \frac{1}{D_3 D_5 D_6 D_7} \tag{397}$$

$$= -i(4\pi)^{-\frac{3}{2}(4-2\epsilon)} m^{4-6\epsilon} \Gamma(\epsilon+1)^3 \left(\frac{1}{3\epsilon^3} + \frac{7}{6\epsilon^2} + \frac{25}{12\epsilon} + \left(\frac{8\zeta_3}{3} - \frac{5}{24} \right) + \left(\frac{28\zeta_3}{3} - \frac{2\pi^4}{15} - \frac{959}{48} \right) \epsilon + \left(\frac{50\zeta_3}{3} - \frac{7\pi^4}{15} + 48\zeta_5 - \frac{10493}{96} \right) \epsilon^2 + \left(\frac{1}{3}\zeta_3(32\zeta_3 - 5) + 168\zeta_5 - \frac{5\pi^4}{6} - \frac{4\pi^6}{21} - \frac{85175}{192} \right) \epsilon^3 + \mathcal{O}\left(\epsilon^4\right) \right)$$

$$I_{\text{OS}}^{3,15} \equiv \int \left(\prod_{i=1}^{3} \frac{d^d k_i}{(2\pi)^d} \right) \frac{1}{D_2 D_3 D_4 D_5} \tag{398}$$

$$= -i(4\pi)^{-\frac{3}{2}(4-2\epsilon)} m^{4-6\epsilon} \Gamma(\epsilon+1)^3 \left(\frac{3}{2\epsilon^3} + \frac{23}{4\epsilon^2} + \frac{105}{8\epsilon} + \left(\frac{275}{16} + \frac{4\pi^2}{3} \right) + \epsilon \left(\pi^2(10 - 8\log(2)) + 28\zeta_3 - \frac{567}{32} \right) + \epsilon^2 \left(96\zeta_{1,-3} + 210\zeta_3 - 168\zeta_3 \log(2) + \frac{28\pi^4}{45} + \frac{145\pi^2}{3} - \frac{14917}{64} + 24\pi^2 \log^2(2) - 60\pi^2 \log(2) \right) + \mathcal{O}\left(\epsilon^3\right) \right)$$

$$I_{\text{OS}}^{3,16} \equiv \int \left(\prod_{i=1}^{3} \frac{d^d k_i}{(2\pi)^d} \right) \frac{1}{D_3 D_4 D_7 D_8} \tag{399}$$

$$= -i(4\pi)^{-\frac{3}{2}(4-2\epsilon)} m^{4-6\epsilon} \Gamma(\epsilon+1)^3 \left(\frac{1}{2\epsilon^3} + \frac{7}{4\epsilon^2} + \frac{\frac{25}{8} + \frac{\pi^2}{3}}{\epsilon} + \left(4\zeta_3 + \frac{7\pi^2}{6} - \frac{5}{16} \right) + \left(14\zeta_3 + \frac{16\pi^4}{45} + \frac{25\pi^2}{12} - \frac{959}{32} \right) \epsilon + \left(25\zeta_3 + \frac{56\pi^4}{45} + \frac{1}{24}\pi^2(64\zeta_3 - 5) + 72\zeta_5 - \frac{10493}{64} \right) \epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \right)$$

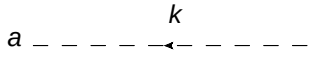
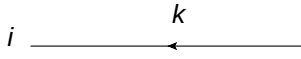
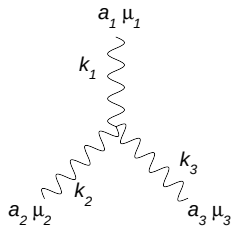
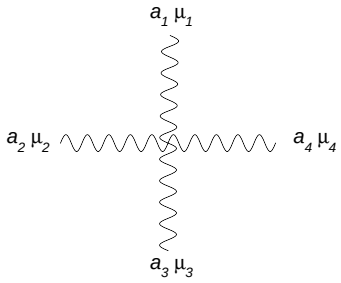
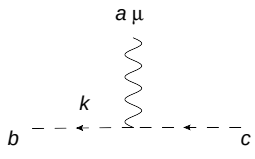
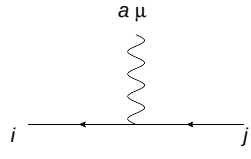
$$I^{3,18} \equiv \int \left(\prod_{i=1}^{3} \frac{d^d k_i}{(2\pi)^d} \right) \frac{1}{D_1 D_4 D_5} = -i(4\pi)^{-\frac{3}{2}(4-2\epsilon)} m^{6-6\epsilon} \Gamma(\epsilon-1)^3$$

$$I_{\text{OS}}^{3,19} \equiv \int \left(\prod_{i=1}^{3} \frac{d^d k_i}{(2\pi)^d} \right) \frac{1}{k_1^2 \left(-k_1 - k_2\right)^2 \left(k_1 + k_3 - p_1\right)^2 \left(-k_1 - k_2 - k_3 + p_1\right)^2 \left(m^2 - \left(k_2 + p_1\right)^2\right)} \tag{400}$$

$$= -i(4\pi)^{-\frac{3}{2}(4-2\epsilon)} m^{2-6\epsilon} \Gamma(\epsilon+1)^3 \left(-\frac{1}{3\epsilon^3} - \frac{4}{3\epsilon^2} - \frac{2\left(2+\pi^2\right)}{3\epsilon} - \frac{2}{3}\left(19\zeta_3 + 4\pi^2 - 24\right) + \left(-\frac{152\zeta_3}{3} - \frac{151\pi^4}{90} - \frac{8\pi^2}{3} + 144 \right) \epsilon + \left(-\frac{76}{3}\left(2+\pi^2\right)\zeta_3 - \frac{302\pi^4}{45} + 32\pi^2 - 430\zeta_5 + 832 \right) \epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \right)$$

B. Feynman rules

The Feynman rules that are used in this thesis are summarized in the table below.

Propagator	**Diagrammatic Representation**	**Mathematical Expression**
gluon A		$-i\delta_{ab}\frac{d_{\mu\nu}(k)}{k^2}$
ghost ξ		$i\delta_{ab}\frac{1}{k^2}$
quark ψ		$i\delta_{ij}\frac{1}{\not{k}-m}$
Vertex	**Diagrammatic Representation**	**Mathematical Expression**
3-gluon		$-gf^{a_1a_2a_3}V_{\mu_1\mu_2\mu_3}(k_1,k_2,k_3)$
4-gluon		$ig^2W^{a_1a_2a_3a_4}_{\mu_1\mu_2\mu_3\mu_4}$
gluon-ghost		$-gf^{abc}k_\mu$
gluon-quark		$-ig\gamma_\mu T^a_{ij}$

The following definitions apply:

$$d_{\mu\nu}(k) = \left(\eta^{\mu\nu} - (1-\alpha)\frac{k^\mu k^\nu}{k^2}\right) \tag{401}$$

$$V^{\mu_1\mu_2\mu_3}(k_1,k_2,k_3) = (k_1-k_2)_{\mu_3}g_{\mu_1\mu_2} + (k_2-k_3)_{\mu_1}g_{\mu_2\mu_3} + (k_3-k_1)_{\mu_2}g_{\mu_3\mu_1} \tag{402}$$

$$W^{a_1\ldots a_4}_{\mu_1\ldots\mu_4} = (f^{13,24} - f^{14,32})g_{\mu_1\mu_2}g_{\mu_3\mu_4} + (f^{12,34} - f^{14,23})g_{\mu_1\mu_3}g_{\mu_2\mu_4} + (f^{13,42} - f^{12,34})g_{\mu_1\mu_4}g_{\mu_3\mu_2} \tag{403}$$

Where in the last line the following combination was used:

$$f^{ij,kl} \equiv f^{a_i a_j a} f^{a_k a_l a} \tag{404}$$

- Additionally, internal momenta should be integrated over, and each such integral comes with a factor $(2\pi)^{-d}$ as part of the dimensional regularization prescription. (If dimensional regularization were not used each integral would come with a factor $(2\pi)^{-4}$.)
- Furthermore, as usual, each fermion loop gives a minus sign. This means ghost and quark loops come with a minus sign.
- Lastly every diagram should be multiplied by its combinatorial symmetry factor so there is no overcounting.

C. Derivation of a Gaussian integral

The aim in this section is to derive a specific Gaussian integral stated in eq. 416, which is needed to derive the alpha-parametrization of Feynman integrals. While most readers will be familiar with Gaussian integral identities from quantum field theory, the full derivation is provided below for completeness. First note the following result, for $A \in \mathbb{R}_{>0}$:

$$\int_0^\infty e^{-Ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{A}} \tag{405}$$

This is seen from squaring the integral and integrating using polar coordinates:

$$\left(\int_0^\infty e^{-Ax^2}dx\right)^2 = \int_0^\infty dx \int_0^\infty dy\, e^{-A(x^2+y^2)} = \int_0^{\frac{\pi}{2}} d\theta \int_0^\infty dr\, re^{-Ar^2} = \frac{\pi}{2}\cdot\left[\frac{-1}{2A}e^{-Ar^2}\right]_0^\infty = \frac{\pi}{4A} \tag{406}$$

Next consider a similar integral with a factor i in the exponent, and again with $A \in \mathbb{R}_{>0}$. The following results hold:

$$\int_{-\infty}^\infty dk\, e^{iAk^2} = e^{\frac{i\pi}{4}}\sqrt{\frac{\pi}{A}} \tag{407}$$

To derive this we can focus on the positive real axis:

$$\int_{-\infty}^\infty dk\, e^{iAk^2} = 2\int_0^\infty dk\, e^{iAk^2} \tag{408}$$

Next consider the contour $\gamma = \gamma_1\gamma_R\gamma_2$, where the product denotes the composition of curves, drawn in fig. 9. Using Cauchy's residue theorem and the fact that the integrand has no poles in

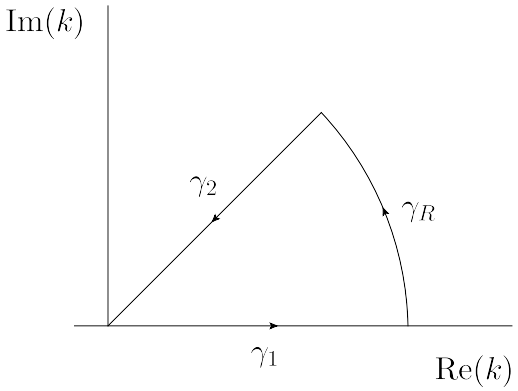


Figure 9

the interior of γ, the following integral is zero:

$$0 = \int_{\gamma} dk\, e^{iAk^2} = \int_{\gamma_1} dk\, e^{iAk^2} + \int_{\gamma_R} dk\, e^{iAk^2} + \int_{\gamma_2} dk\, e^{iAk^2} \tag{409}$$

The integral over the arc drops out in the limit $R \to \infty$. We see this from explicitely parametrizing the arc by $\gamma_R(t) = R(\cos(\frac{\pi}{4}t) + i\sin(\frac{\pi}{4}t))$:

$$\lim_{R\to\infty} \int_{\gamma_R} dk\, e^{iAk^2} = \lim_{R\to\infty} \left| \int_0^1 e^{iAR^2(\cos(\frac{\pi}{2}t)+i\sin(\frac{\pi}{2}t))} \frac{i\pi}{4} R e^{\frac{i\pi t}{4}} dt \right| \leq \lim_{R\to\infty} \frac{\pi R}{4} \int_0^1 e^{-AR^2\sin(\frac{\pi}{2}t)} dt = 0 \tag{410}$$

It can be concluded that:

$$\int_0^{\infty} dk\, e^{iAk^2} = \lim_{R\to\infty} \int_{\gamma_1} dk\, e^{iAk^2} = -\lim_{R\to\infty} \int_{\gamma_2} dk\, e^{iAk^2} = e^{\frac{i\pi}{4}} \int_0^{\infty} dk\, e^{-Ak^2} = \frac{1}{2} e^{\frac{i\pi}{4}} \sqrt{\frac{\pi}{A}} = \frac{1}{2}\sqrt{\frac{i\pi}{A}} \tag{411}$$

It was assumed in the beginning that $A \in \mathbb{R}_{>0}$. Now suppose $A \in \mathbb{R}_{<0}$. This case can be treated analogously, but now the contour γ (the same notation is reused) of fig. 10 is considered. Similar

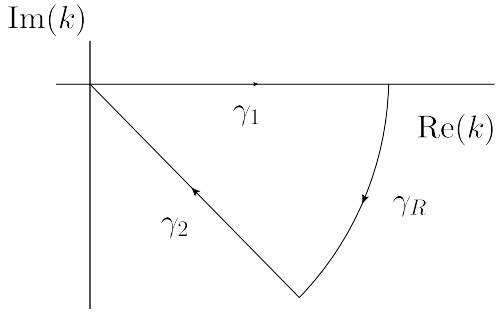


Figure 10

to before the integral over the arc goes to zero so that one finds:

$$\int_0^\infty dk\, e^{iAk^2} = \lim_{R\to\infty}\int_{\gamma_1} dk\, e^{iAk^2} = -\lim_{R\to\infty}\int_{\gamma_2} dk\, e^{iAk^2} = e^{\frac{-i\pi}{4}}\int_0^\infty dk\, e^{Ak^2} = \frac{1}{2}e^{\frac{-i\pi}{4}}\sqrt{\frac{\pi}{-A}} = \frac{1}{2}\sqrt{\frac{i\pi}{A}} \tag{412}$$

Thus one finds that in general for $A \neq 0, A \in \mathbb{R}$,

$$\int_{-\infty}^\infty dk\, e^{iAk^2} = \sqrt{\frac{i\pi}{A}} \tag{413}$$

Next we generalize the previous result by adding a linear term. Consider $A \neq 0, A \in \mathbb{R}$ and $B \in \mathbb{R}$. It holds that:

$$\int_{-\infty}^\infty e^{i(Ak^2+2Bk)} = e^{-i\frac{B^2}{A}}\int_{-\infty}^\infty e^{i\left[(k+\frac{B}{A})^2 A\right]}dk = e^{-i\frac{B^2}{A}}\sqrt{\frac{i\pi}{A}} \tag{414}$$

which follows from the identity $(k+\frac{B}{A})^2 A - \frac{B^2}{A} = Ak^2 + 2Bk$, shifting the integration parameter and using eq. 413. Now that we have the formula for a complex Gaussian integral we generalize to the case of a multidimensional Lorentzian integral, by picking $B \in \mathbb{R}^d$, and the Lorentzian dot product $B \cdot k = B^0k^0 - \ldots - B^dk^d$:

$$\begin{aligned}
\int_{-\infty}^\infty d^dk\, e^{i(Ak^2+2B\cdot k)} &= \int_{-\infty}^\infty dk^0 \ldots \int_{-\infty}^\infty dk^d \exp\left(i(A((k^0)^2 - \ldots - (k^d)^2) + 2B^0k^0 - \ldots - 2B^dk^d)\right) \\
&= \left(\int_{-\infty}^\infty dk^0 e^{i(A(k^0)^2+2B^0K^0)}\right)\ldots\left(\int_{-\infty}^\infty dk^d e^{-i(A(k^d)^2+2B^dK^d)}\right) \\
&= \left(\frac{i\pi}{A}\right)^{\frac{1}{2}}\cdot\left(\frac{i\pi}{-A}\right)^{\frac{d-1}{2}} e^{\frac{i}{A}\left(-(B^0)^2+\cdots+(B^d)^2\right)} \\
&= \left(\frac{\pi}{A}\right)^{\frac{d}{2}} e^{\frac{i\pi}{2}(1-\frac{d}{2})} e^{\frac{-iB^2}{A}}
\end{aligned} \tag{415}$$

Lastly the result will be completely generalized by considering multiple momenta, numbered by an index on the subscript. We let A be a symmetric matrix with real entries. That means in particular A is diagonalizable by an orthogonal matrix Q so that $A' = Q^TAQ$ is diagonal. A summation will be implied for repeated indices as usual: $A_{ij}k_i \cdot k_j = \sum_{i,j} A_{ij}k_i \cdot k_j$. The following formula will be proven:

$$\int d^dk_1 \ldots d^dk_h \exp\left(i(A_{ij}k_i \cdot k_j + 2q_i \cdot k_i)\right) = e^{\frac{i\pi}{2}(1-\frac{d}{2})}\pi^{h\frac{d}{2}}(\det A)^{-\frac{d}{2}}\exp\left(-iA^{-1}_{ij}q_i \cdot q_j\right) \tag{416}$$

where A^{-1} denotes the inverse of A so that $A_{ij}A^{-1}_{jk} = \delta_{ik}$, and the q_i^μ are space-time vectors which are not integrated over. Equation 416 can be derived by doing the transformation $k_i \to Q_{ij}k_j$, which is volume preserving due to orthogonality of Q. For the proof we define $q_jQ_{ji} \equiv \tilde{q}_i$. Performing the change of variables leads to:

$$\begin{aligned}
\int d^dk_1 \ldots d^dk_h \exp\left(i(A_{ij}k_i \cdot k_j + 2q_i \cdot k_i)\right) &= \int d^dk_1 \ldots d^dk_h \exp\left(i\left(\sum_i A'_{ii}k_i^2 + 2Q_{ij}\, q_i \cdot k_j\right)\right) \\
&= \prod_i\left(\int d^dk_i \exp\left(i(A'_{ii}k_i^2 + 2\tilde{q}_i \cdot k_i)\right)\right)
\end{aligned}$$

$$= e^{\frac{i\pi}{2}h(1-\frac{d}{2})} \prod_i \left(\left(\frac{\pi}{A'_{ii}} \right)^{\frac{d}{2}} e^{\frac{-i\tilde{q}_i^2}{A_{ii}}} \right)$$
$$= e^{\frac{i\pi}{2}h(1-\frac{d}{2})} \prod_i \left(\frac{\pi^h}{A'_{11} \cdot \ldots \cdot A'_{hh}} \right) e^{-i\sum_i \frac{\tilde{q}_i^2}{A_{ii}}} \tag{417}$$

Note that: $\sum_i \tilde{q}_i^2 / A'_{ii} = \vec{q}^T Q A' Q^T \vec{q} = \vec{q}^T A \vec{q}$, and also that $A'_{11} \cdot \ldots \cdot A'_{hh} = \det(A)$. Using this eq. 416 follows.

D. Mathematical definitions

4.1. Euler gamma and beta function

Euler gamma function The Euler gamma function is the unique analytic generalization of the factorial function, and is defined for $t \in \mathbb{C}$, $\mathrm{Re}(t) > 0$ by:

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx \tag{418}$$

with the integral converging absolutely in the chosen domain. From integration by parts one finds:

$$\Gamma(t+1) = \int_0^\infty x^t e^{-x} dx = \left[-x^t e^{-x} \right]_{x=0}^{x=\infty} + t \int_0^\infty x^{t-1} e^{-x} dx = t \cdot \Gamma(t) \tag{419}$$

Using that $\Gamma(1) = 1$ this proofs for $n \in \mathcal{N}_{\geq 0}$ that

$$\Gamma(n+1) = n! \tag{420}$$

Furthermore, the gamma function can be analytically continued to a meromorphic function on $\mathbb{C}$ with poles for the negative integers using eq. 419. For example:

$$\Gamma(-1.5) = \frac{1}{1.5} \cdot \Gamma(-0.5) = \frac{1}{1.5 \cdot 0.5} \cdot \Gamma(0.5) = \ldots \tag{421}$$

This analytic continuation can also be obtained in a more formal manner. One splits up the integral representation of the Γ function in the following way:

$$\Gamma(t) = \int_0^1 x^{t-1} e^{-x} dx + \int_1^\infty x^{t-1} e^{-x} dx \tag{422}$$

For $\mathrm{Re}(t) \leq 0$ the integration on the domain $[0, 1]$ will diverge, whereas the integral on the domain $[1, \infty)$ converges everywhere. Within the domain of convergence one can expand the exponent in the first integral and switch summation and integration to find:

$$\sum_{k \geq 0} \frac{(-1)^k \int_0^1 x^{t+k-1} dx}{k!} = \sum_{k \geq 0} \frac{(-1)^k}{k!(t+k)} \tag{423}$$

Then it is found that:

$$\Gamma(t) = \sum_{k \geq 0} \frac{(-1)^k}{k!(t+k)} + \int_1^\infty x^{t-1} e^{-x} dx \tag{424}$$

It turns out that the sum converges for all complex $k \notin \mathcal{N}_{\leq 0}$. Hence eq. 424 gives the analytic continuation of the gamma function.

Euler beta function The Euler beta function, denoted B, is seen many times in the evaluation of Feynman integrals. It admits the following formula:

$$\int_0^1 t^{p-1}(1-t)^{q-1}dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \equiv B(p,q) \text{ for } \mathrm{Re}(p) > 0, \mathrm{Re}(q) > 0 \tag{425}$$

The relation of $B(p,q)$ in terms of gamma functions is derived below. First note that:

$$\Gamma(p)\Gamma(q) = \int_0^\infty dx \int_0^\infty dy\, x^{p-1}y^{q-1}e^{-(p+q)} \tag{426}$$

Then change variables in the following way:

$$x = zt, \;\; y = z(1-t) \tag{427}$$

$$dxdy = \left| \det \begin{pmatrix} t & z \\ 1-t & -z \end{pmatrix} \right| dzdt = z\, dzdt \tag{428}$$

This shows that:

$$\begin{aligned} \Gamma(p)\Gamma(q) &= \int_0^\infty dz \int_0^\infty dt\, z(zt)^{p-1}(z(1-t))^{q-1}e^{-z} \\ &= \int_0^\infty dz\, z^{p+q-1}e^{-z} \int_0^\infty dt\, t^{p-1}(1-t)^{q-1} \\ &= \Gamma(p+q)B(p,q) \end{aligned} \tag{429}$$

Note that the beta function can be analytically continued by taking the analytic continuation of the Γ-functions. Another integral represention of the beta function follows from changing variables to $t = \frac{u^2}{u^2+1}$, so that $dt = \frac{2u}{(u^2+1)^2}du$. This leads to:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt = 2\int_0^\infty \frac{(u^2)^{p-1/2}}{(1+u^2)^{p+q}}du \tag{430}$$

Generalized Euler beta function The generalized Euler beta function is given by:

$$B(a_1,\ldots,a_n) = \left(\prod_{i=1}^n \int_0^\infty dt_i\right)\left(\prod_{i=1}^n t_i^{a_i-1}\right)\delta\left(1-\sum_{i=1}^n t_i\right) \tag{431}$$

This integral can be evaluated in an iterative manner. First integrate out t_n, which gets rid of the delta function, and then rescale the remaining parameters in the following way:

$$t_i = \eta t_i' \quad \text{for } i = 1,\ldots,n-1 \tag{432}$$

$$\text{with } \sum_{i=1}^{n-1} t_i' = 1 \text{ enforced by introducing a new delta function} \tag{433}$$

The Jacobian of this transformation is η^{n-2}, which is one factor η less than is expected from a naive rescaling of the differentials dt_i. The resulting expression gives:

$$\begin{aligned} B(a_1,\ldots,a_n) &= \left(\prod_{i=1}^{n-1} \int_0^\infty dt_i\right)\left(\prod_{i=1}^{n-1} t_i^{a_i-1}\right)\delta\left(1-\sum_{i=1}^{n-1} t_i\right)\int_0^1 d\eta\eta^{a_n-1}(1-\eta)^{\sum_{i=1}^{n-1}a_i-1} \\ &= B(a_1,\ldots,a_{n-1})B\left(\sum_{i=1}^{n-1}a_i, a_n\right) = B(a_1,\ldots,a_{n-1})\frac{\Gamma(\sum_{i=1}^{n-1}a_i)\Gamma(a_n)}{\Gamma(\sum_{i=1}^n a_i)} \end{aligned} \tag{434}$$

Repeating the previous procedure leads to:

$$B(a_1,\ldots,a_n) = B(a_1,a_2)\cdot B(a_1+a_2,a_3)\cdot\ldots\cdot B\left(\sum_{i=1}^{n-1} a_i, a_n\right)$$
$$= \frac{\Gamma(a_1)\cdot\ldots\cdot\Gamma(a_n)}{\Gamma(a_1+\ldots+a_n)} \tag{435}$$

4.2. Pochhammer symbols and the binomial theorem

The rising and falling Pochhammer symbols denoted $(a)_n$ and $a^{(n)}$ respectively, with $a \in \mathbb{C}, n \in \mathbb{C}$ are defined by:

$$(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)}, \quad a^{(n)} \equiv \frac{\Gamma(a+1)}{\Gamma(a-n+1)} \tag{436}$$

For $n \in \mathcal{N}_{\geq 0}$ they can also be written as:

$$(a)_n = a\cdot(a+1)\cdot\ldots\cdot(a+n-1) \tag{437}$$
$$a^{(n)} = a\cdot(a-1)\cdot\ldots\cdot(a-n+1) \tag{438}$$

In some texts a different notation is used where the subscript and superscript are switched. In the theory of hypergeometric functions a subscript is always used for the rising factorial and here we stick to that notation. The following identities hold for $n \in \mathbb{Z}_{\geq 0}$:

$$(a)_n = (-1)^n(-a)^{(n)} \tag{439}$$
$$a^{(n)} = (-1)^n(-a)_n \tag{440}$$

Note that for integer $a \geq 0, a \geq n \geq 0$ it follows from the definition that:

$$\binom{a}{n} = \frac{a!}{n!(a-n)!} = \frac{\Gamma(a+1)}{n!\Gamma(a-n+1)} = \frac{a^{(n)}}{n!} \tag{441}$$

Generalizing to $a \in \mathbb{C}, n \in \mathbb{Z}_{\geq 0}$ one can therefore define the generalized binomial coefficient by:

$$\binom{a}{n} \equiv \frac{a^{(n)}}{\Gamma(n+1)}, \quad \binom{-a}{n} = (-1)^{(n)}\frac{(a)_n}{\Gamma(n+1)} \tag{442}$$

Using this definition the binomial theorem can be generalized for $\lambda \in \mathbb{C}$ and $|x| > |y|$:

$$(x+y)^\lambda = \sum_{k=0}^{\infty}\binom{\lambda}{k}x^{\lambda-k}y^k \tag{443}$$

This is of course nothing else but the Taylor series of $(x+y)^\lambda$ in y around 0.

4.3. Hypergeometric functions

The generalized hypergeometric function $F(\alpha,\beta|z)$ for $\alpha = (\alpha_1,\ldots,\alpha_k) \in \mathbb{C}^n$ (the numerator parameters) and $\beta = (\beta_1,\ldots,\beta_k) \in \mathbb{C}^n, \beta_i \notin \mathcal{Z}_{\leq 0}$ (the denominator parameters) can be introduced in a number of ways. It admits a power series expansion around 0 given by:

$$F(\alpha,\beta|z) = \sum_{n\geq 0}\frac{(\alpha_1)_n\ldots(\alpha_k)_n}{(\beta_1)_k\ldots(\beta_k)_n}z^k \text{ for } |z|<1 \tag{444}$$

It can also be introduced as the local solution around 0 of the generalized hypergeometric differential equation with parameters α and β given by:

$$[z(\theta+\alpha_1)\dots(\theta+\alpha_n)-(\theta+\beta_1-1)\dots(\theta+\beta_n-1)]\,f(z)=0 \tag{445}$$

A special case of the hypergeometric equation is ${}_2F_1(a,b;c|z)$, also called the Clausen-Thomae HGF, defined by:

$${}_2F_1(a,b;c|z)=\sum_{n\geq 0}\frac{(a)_n(b)_n}{(c)_n k!}z^k \text{ for } |z|<1 \tag{446}$$

which comes from using $\alpha=(a,b)$ and $\beta=(c,1)$ in eq. 444. This function can also be represented using an Euler integral:

$${}_2F_1(a,b;c|z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \tag{447}$$

Provided that $\mathrm{Re}(c)>\mathrm{Re}(b)>0$ and $|z|<1$. This can be proven by doing a binomial expansion on the term $(1-tz)^{-a}$:

$$(1-tz)^{-a}=\sum_{k\geq 0}\binom{-a}{k}(-tz)^k=\sum_{k\geq 0}\frac{(a)_k}{k!}(tz)^k \tag{448}$$

This leads to:

$$\begin{aligned}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt &= \sum_{k\geq 0}\frac{(a)_k}{k!}z^k\int_0^1 t^{k+b-1}(1-t)^{c-b-1}dt\\ &= \sum_{k\geq 0}\frac{(a)_k}{k!}\frac{\Gamma(b+k)\Gamma(c-b)}{\Gamma(c+k)}z^k\\ &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\sum_{k\geq 0}\frac{(a)_k(b)_k}{(c)_k\,k!}z^k\end{aligned} \tag{449}$$

which proofs eq. 447.

4.4. Multiple zeta values

The Riemann zeta function $\zeta(s)$ is the analytical continuation of the infinite series

$$\zeta(s)\equiv\sum_{n=1}^{\infty}\frac{1}{n^s} \tag{450}$$

which converges for $\mathrm{Re}(s)>1$. Zeta values with $s\in\mathbb{Z}_{\geq 1}$ show up in the power series expansions of Feynman integrals, in which case we put s in the subscript: $\zeta(s)=\zeta_s$. At high powers in ϵ so-called multiple zeta values show up. Using the notation from [1] these are defined by:

$$\zeta_{n_1,\dots,n_r}\equiv \mathrm{Li}_{|n_1|,\dots,|n_r|}\left(\frac{n_1}{|n_1|},\dots,\frac{n_r}{|n_r|}\right) \tag{451}$$

with indices $n_1,\dots,n_r\in\mathbb{Z}\setminus\{0\}$, $n_r\neq 1$. The multiple polylogarithms are defined by:

$$\mathrm{Li}_{n_1,\dots,n_r}(z_1,\dots,z_r)\equiv\sum_{0<k_1<\dots<k_r}\frac{z_1^{k_1}\dots z_r^{k_r}}{k_1^{n_1}\dots k_r^{n_r}} \tag{452}$$

References

[1] E. Panzer, "Algorithms for the symbolic integration of hyperlogarithms with applications to feynman integrals," arXiv:1403.3385.

[2] The ATLAS and CDF and CMS and D0 Collaborations, "First combination of tevatron and lhc measurements of the top-quark mass," arXiv:1403.4427.

[3] T. Muta, *Foundations of Quantum Chromodynamics: An Introduction to Perturbative Methods in Gauge Theories*. World Scientific, 2010.

[4] T. van Ritbergen, A. N. Schellekens, and J. A. M. Vermaseren, "Group theory factors for feynman diagrams," arXiv:hep-ph/9802376.

[5] A. Zee, *Quantum field theory in a nutshell*. Princeton university press, 2010.

[6] G. 't Hooft and M. Veltman, "Regularization and renormalization of gauge fields," *Nuclear Physics B* **44** no. 1, (1972) 189 – 213.

[7] V. A. Smirnov, *Evaluating feynman integrals*. Springer, 2005.

[8] K. Melnikov and T. van Ritbergen, "The three-loop relation between the ms and the pole quark masses," *Physics Letters B* **482** no. 1, (2000) 99–108.

[9] A. Grozin, "Lectures on qed and qcd," arXiv:hep-ph/0508242.

[10] G. Sterman, *An Introduction to Quantum Field Theory*. Cambridge University Press, 1993.

[11] D. H. Perkins, *Introduction to high energy physics*. Cambridge University Press, 2000.

[12] F. Blekam, "Top quark pair production in proton antiproton collisions," tech. rep., 2005.

[13] I. Brock and T. Schörner-Sadenius, *Physics at the Terascale*. John Wiley & Sons, 2011.

[14] CMS Collaboration, "Measurement of the top quark mass using charged particles in pp collisions at sqrt(s) = 8 tev," arXiv:1603.06536.

[15] ATLAS Collaboration, "Measurement of the top quark mass with the template method in the top antitop -> lepton + jets channel using atlas data," arXiv:1203.5755.

[16] F. Gournaris, *A study of the top quark production threshold at a future electron-positron linear collider*. PhD thesis, UCL (University College London), 2010.

[17] C. M. Bender and S. A. Orszag, *Advanced mathematical methods for scientists and engineers I*. Springer Science & Business Media, 1999.

[18] M. Beneke and V. M. Braun, "Heavy quark effective theory beyond perturbation theory: Renormalons, the pole mass and the residual mass term," arXiv:hep-ph/9402364.

[19] I. Stewart, "Mit ocw 8.851 - effective field theory - video lecture 11 (renormalons)," Spring 2013. http://ocw.mit.edu/courses/physics/8-851-effective-field-theory-spring-2013/.

[20] J. R. Espinosa, "Implications of the top (and higgs) mass for vacuum stability," arXiv:1512.01222.

[21] P. Nogueira, "Automatic feynman graph generation," *Journal of Computational Physics* **105** no. 2, (1993) 279–289.

[22] S. Laporta, "High-precision calculation of multi-loop feynman integrals by difference equations," arXiv:hep-ph/0102033.

[23] A. V. Smirnov, "Fire5: a c++ implementation of feynman integral reduction," arXiv:1408.2372.

[24] S. Laporta and E. Remiddi, "The analytical value of the electron (g-2) at order α^3 in qed," arXiv:hep-ph/9602417.

[25] J. S. Hoff, *Methods for multiloop calculations and Higgs boson production at the LHC*. PhD thesis, Karlsruher Institut für Technologie (KIT), 2015.

[26] G. 't Hooft and M. Veltman, "Scalar one-loop integrals," *Nuclear Physics B* **153** (1979) 365 – 401.

[27] H. Cheng and T. T. Wu, *Expanding protons: Scattering at high energies*. The MIT Press, 1987.

[28] E. Panzer, "On hyperlogarithms and feynman integrals with divergences and many scales," arXiv:1401.4361.

[29] E. Panzer, "Feynman integrals and hyperlogarithms," arXiv:1506.07243.

[30] E. Remiddi and J. A. M. Vermaseren, "Harmonic polylogarithms," arXiv:hep-ph/9905237.

[31] J. A. M. Vermaseren, "Harmonic sums, mellin transforms and integrals," arXiv:hep-ph/9806280.

[32] R. N. Lee, "Space-time dimensionality d as complex variable: calculating loop integrals using dimensional recurrence relation and analytical properties with respect to d," arXiv:0911.0252.