Universiteit Utrecht

Institute for Theoretical Physics

Master's Thesis

# Thermodynamics of the Ernst spacetime

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# Contents

# Introduction

Since the 1970s it has been known that black holes satisfy the laws of black hole mechanics. These laws put constraints on how the mass, charge and angular momentum can vary during classical processes. When one identifies the mass with internal energy, one quarter of the area of the event horizon with entropy, and its surface gravity over $2\pi$ with temperature, the laws of black hole mechanics are identical to the laws of thermodynamics. This suggests that thermodynamics and black holes are intrinsically related to each other. Indeed, when Hawking in 1974 showed that black holes do emit so-called Hawking radiation, this analogy became more than just an analogy: black holes are thermodynamic systems.

The laws of black hole mechanics have been proved for spacetimes that exhibit a time-translation symmetry. Such spacetimes are called stationary. Examples of stationary solutions are the Schwarzschild, Reissner-Nördstrom and Kerr black holes.

However, there also exist non-stationary black hole solutions. An example of a non-stationary black hole solution is the Ernst spacetime, which describes two oppositely charged black holes accelerated away from each other. Instead of a time-translation symmetry, this solution has a boost symmetry. The goal of this thesis is to show that the first law of black hole mechanics also holds in the Ernst spacetime. And more generally, to prove the first law of black hole mechanics for black holes in any boost symmetric spacetime.

The outline of this thesis is as follows. The subsequent sections serve as an introduction to black holes and some of their properties which are relevant in the subsequent chapters. In Chapter 1 we give a full account of the Ernst metric. In Chapter 2 we derive the first law of black hole mechanics for boost symmetric spacetimes, and we show that the Ernst metric satisfies this law.

## Uses of the Ernst metric

The Ernst metric is a solution to the Einstein-Maxwell equations. It contains two magnetically charged Reissner-Nördstrom black holes connected by an Einstein-Rosen bridge. Additionally, the spacetime is filled with a magnetic background field. By the electromagnetic duality, magnetic charges in a magnetic field behave the same as electric charges in an electric field. And because of this, the black holes are constantly accelerated. This can be seen in Figure 1. The hyperbolas represent the world lines of the inner and outer horizons of the accelerated black holes. The diagonal lines represent the acceleration horizons.

The Ernst metric has an important application in quantum gravity. It is used to describe pair production of black holes. That is, similar to Schwinger pair production of electron-positron pairs from an electric field ([2], [3]), it is possible to create pairs of oppositely magnetically charged black holes from a magnetic field [4]. Arguably even more interesting is that, as Schwinger pairs are entangled, the pair created black holes in the Ernst metric are also entangled. Recently, it has been argued that the Einstein-Rosen bridge between the two black holes is a geometric manifestation of this entanglement [5]. This goes under the name ER=EPR conjecture.

It is of interest to understand how thermodynamics works in the Ernst spacetime for at least three reasons. One is that since the black holes live in the same asymptotic region, and are connected by an Einstein-Rosen bridge, one can ask whether these separate thermodynamics systems are also thermodynamically connected in one way or another. For example, would it be possible to give an interpretation to the laws of thermodynamics in terms of entanglement entropy between the black holes? A second reason is that the black holes are immersed in a magnetic background spacetime. This background magnetic field contains energy and therefore can also be considered as a thermodynamic quantity. This was also done for stationary black holes immersed in a magnetic field [6] and [7]. It will also be interesting to see how the acceleration of the black holes affects the interplay between thermodynamic quantities.

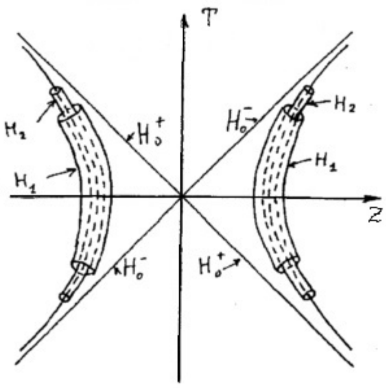



Figure 1: From [1]. A schematic representation of the Ernst metric. The hyperbolic lines represent the wordlines of the accelerated Reissner-Nördstrom black holes. The tubes represent the inner and outer horizon.

# Preliminaries

In the remaining part of this introduction we discuss some general theory and some ideas that will make it easier to understand some concepts used further on in this thesis. First, we comment on the setup and our conventions. Second, we discuss some properties of Killing horizons.

## Setup

In this thesis we denote by $(M, g)$ a manifold equipped with a metric $g$ with signature $(-,+,+,+)$. We consider the Einstein-Maxwell action given by

$$S = \frac{1}{16\pi}\int_M (-R + F \wedge *F) - \int_{\partial M} \frac{1}{8\pi} K. \qquad (1)$$

Here $F = \mathrm{d}A$ denotes the electromagnetic field strength and $A$ is the gauge field. When the manifold $M$ has a boundary, the term with the extrinsic curvature $K$ and the induced metric $h$ on its boundary is needed to render the equations of motion well defined. Varying with respect to the metric yields the Einstein equations

$$G_{ab} = 8\pi T_{ab}, \qquad (2)$$

where

$$T_{ab} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} \quad \text{and} \quad G_{ab} = R_{ab} - \frac{1}{2}R g_{ab}.$$

Varying with respect to the gauge field $A$ yields the source-free Maxwell equations in curved space

$$\partial_a(\sqrt{-g}F^{ab}) = 0. \tag{3}$$

A solution to the Einstein-Maxwell equations is a pair $(A, g)$ such that (2) and (3) are satisfied. Because (1) is conformally invariant (in four dimensions), it follows that the Ricci scalar always vanishes when the equations of motion are satisfied.

## Black holes and Killing horizons

A black hole refers to a part of spacetime from which no future directed timelike or null line can escape into asymptotic regions. Let $J^+$ denote the future asymptotic region of $(M, g)$ and $I^-(S)$ the chronological past of a spatial surface $S$. Then the black hole is defined as the region

$$\mathcal{B} := \mathcal{M} - I^-(J^+).$$

The event horizon $\mathcal{H}$ (the diagonal lines in the figure) of the black hole is the boundary of $\mathcal{B}$. This boundary is given by

$$\mathcal{H} = \overline{J^-(\mathcal{J}^+)} - J^-(\mathcal{J}^+)),$$

where $J^-(S)$ denotes the causal past of a surface $S$. The event horizon separates the interior of the black hole from its exterior. It has a couple of interesting properties.

In stationary spacetimes, the event horizon is always a Killing horizon of the time translation Killing vector. This means that the event horizon is defined by a surface on which the time translation Killing vector is null. If one chooses a convenient coordinate system, it is therefore easy to read off the event horizon from the metric. More importantly, however, the fact that the event horizon is a Killing horizon, allows us to introduce the *surface gravity* of the event horizon. This will allow us to speak of the temperature of a black hole in chapter 2.

**Definition 1.** *Let $\xi^a$ be a Killing vector. If $\xi_a\xi^a = 0$ on some hypersurface $\mathcal{H}$, then $\mathcal{H}$ is called a Killing horizon.*

According to the definition there holds that the normal vector field $n^a$ of $\mathcal{H}$ is proportional to $\xi^a$ on $\mathcal{H}$. That is,

$$\xi^a \stackrel{\mathcal{H}}{=} f n^a, \tag{4}$$

for some function $f$. It turns out that we can normalise $n_a$ in such a way that [8]

$$n^a \nabla_a n^b \stackrel{\mathcal{H}}{=} 0. \tag{5}$$

Plugging in (4), it follows that

$$\nabla_\xi \xi^a \stackrel{\mathcal{H}}{=} \kappa \xi^a,$$

where $\kappa = \xi^a \partial_a f$ is called the surface gravity.

Notice that if $\xi^a$ is a Killing vector, then $\alpha\xi^a$ ($\alpha$ constant) is also a Killing vector. And the surface gravity of this new Killing vector is $\alpha\kappa$. This tells us that the intrinsic geometry of the Killing horizon alone is not enough to unambiguously define the surface gravity. Additionally, a particular normalisation is required. For an asymptotically timelike Killing vector field $k^a$, the normalisation is chosen such that $k_a k^a = -1$ at spatial infinity.

## A boost Killing vector

In this thesis we work with a boost Killing vector, which in Minkowski spacetime is of the form

$$\xi = z\partial_T + T\partial_z.$$

In the same way that an axial Killing vector (e.g. $\eta = x\partial_y - y\partial_x$) generates rotations, the boost Killing vector generates Lorentz boosts. Whereas the orbits of an axial Killing vector field are circles, the orbits of a boost Killing vector field are hyperbolas. If a solution admits a boost symmetry, then the spacetime is invariant or "stationary" from the point of view of an accelerated observer.

Unlike a time translation Killing vector, a boost Killing vector is not asymptotically timelike. To see this, notice that in Minkowski spacetime,

$$\xi^2 = T^2 - z^2.$$

If we pick some $T$ and $z$ such that $|T| > |z|$ and let $x, y \to \infty$, then the norm of $\xi$ is positive. On the other hand, if we pick some $T$ and $z$ such that $|T| < |z|$ and again let $x, y \to \infty$, then the norm of $\xi$ is negative. Therefore it is impossible to normalise $\xi$ in such a way that $\xi^2 = -1$ at spatial infinity. To fix the definition of surface gravity for Killing horizons generated by a boost Killing vector, we need a different normalisation condition. This condition is that $\xi_a\xi^a = 1/\rho$ at the world line with proper acceleration $\rho$.

To see how in subsequent chapters a boost Killing vector arises, let us look again at Figure 1. In this picture one recognizes two hyperbolas, which correspond to the world lines of accelerated particles. The accelerated particles are Reissner-Nördstrom black holes, the inner and outer horizons of which are indicated by the inner and outer tubes. If we put ourselves onto the world line of one of these black holes, then the black hole is stationary to us. Since the two black holes are separated by an acceleration horizon, we will not be able to receive signals or messages from the other black hole. Therefore the metric is symmetric under translations in our proper time. This symmetry is generated by a boost Killing vector, which also generates the acceleration and the black hole horizons[1].

[1]This is not the case if the black holes are rotating. A form of the C-metric in which the black holes are also rotating was found in [9]. The corresponding Ernst metric (after a Harrison transformation) can be found in [10]

# Chapter 1

# The Ernst metric

The Ernst metric [11] together with the appropriate gauge field is a solution to the Einstein-Maxwell equations. It describes two oppositely magnetically charged black holes immersed in a magnetic field accelerated away from each other. Additionally, the two black holes are connected by an Einstein-Rosen bridge.

The Ernst metric is built up from two ingredients (both solutions to the Einstein-Maxwell equations). Ingredient one is the C-metric, which describes the two oppositely charged black holes accelerating away from each other in a Minkowski background spacetime [12]. Ingredient two is the magnetic background spacetime, which is called the Melvin universe [13]. These two ingredients can be "combined" by means of a Harrison transformation [14]. The goal of this chapter is to develop a working knowledge with these concepts.

The structure of this chapter is as follows. First, we successively discuss the two ingredients: the C-metric and the Melvin universe. Then we derive the Harrison transformation and apply it to the C-metric to obtain the Ernst metric. We end with a discussion on some general properties of boost symmetric spacetimes.

## 1.1 The C-metric

The C-metric is a solution to the Einstein-Maxwell equations that describes a pair of oppositely charged black holes constantly accelerated away from each other. To get to grips with the the C-metric, we will first give a "bottom-up" derivation. That is, we will start from Minkowski spacetime and manipulate it, until we end up at the C-metric. Subsequently, we start from the top (the C-metric) and formally derive more properties.

### 1.1.1 Bottom-up

The starting point is Minkowski spacetime given in cylindrical coordinates by the metric

$$\mathrm{d}s^2 = -\,\mathrm{d}T^2 + \mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2.$$

To study a constantly accelerated object, it is convenient to place ourselves in a frame in which the object is stationary to us, such that the metric components are independent of time. Therefore the first step is to write the Minkowski metric in terms of Rindler coordinates

$$t = \operatorname{arctanh}(T/z), \quad \eta = \sqrt{z^2 - T^2}. \tag{1.1}$$

Here $t$ denotes the proper time of an accelerated observer. From the inverse coordinate transformation $T = \eta \sinh(t)$ and $z = \eta \cosh(t)$, we can see that a line of constant $\eta$ corresponds the orbit of a particle with proper acceleration $A = \eta^{-1}$. In order words: the trajectory of a particle with constant proper acceleration $A$ in the $z$-direction is given by

$$T = A^{-1} \sinh(At), \quad z = A^{-1} \cosh(At).$$

The Rindler coordinates defined in (1.1) parametrise the part of Minkowski spacetime for which $|z| > |T|$ and $0 \leqslant z < \infty$. This region of Minkowski spacetime is called the right Rindler wedge, denoted by

$$\mathrm{RR} := \{(T, z) : z > 0, |z| > |T|\}.$$

The metric on this wedge is given by

$$\mathrm{d}s^2 = -\eta^2\, \mathrm{d}t^2 + \mathrm{d}\eta^2 + \mathrm{d}\rho^2 + \rho^2\, \mathrm{d}\phi^2.$$

The hypersurface at $\eta = 0$ defines the acceleration horizon.

Our final goal is to physically change the metric by placing a black hole on the world line of an accelerated observer. In order to be able to do this, we introduce another set of coordinates that allows us to think of the world line of a particle with proper acceleration $A$ as the center of a radial coordinate system. This is convenient, since black hole metrics are often written in radial coordinates because of rotational symmetry. These coordinates are defined by

$$\rho =: \frac{1}{A}\frac{\sqrt{1-x^2}}{x-y} \quad \text{and} \quad \eta =: \frac{1}{A}\frac{\sqrt{1-y^2}}{x-y}.$$

The metric in these coordinates becomes

$$\mathrm{d}s^2 = \frac{1}{A^2(x-y)^2}\left[G(y)\, \mathrm{d}t^2 - \frac{\mathrm{d}y^2}{G(y)} + \frac{\mathrm{d}x^2}{G(x)} + G(x)\, \mathrm{d}\phi^2\right], \tag{1.2}$$

where $G(x) = 1 - x^2$. For this metric to have the correct signature, $G(x)$ must be non-negative. Therefore the $x$-coordinate is allowed to take values in the range $-1 \leqslant x \leqslant 1$. And for reasons becoming clear in the next subsection, the $y$-coordinate is allowed to take values such that $y < x$. Let us now look at how the world line of an accelerated (test) particle is parametrised in these coordinates.

- If $y \longmapsto -\infty$ then $\eta \longmapsto A^{-1}$. So $y = -\infty$ corresponds to the center of the accelerating particle.
- If $y \longmapsto -1$ then $\eta = 0$. This corresponds to the acceleration horizon.

Given the domain of $x$ and $y$, we can now define a coordinate

$$r := \frac{1}{A(x-y)}, \tag{1.3}$$

which has the properties of radial coordinate. Namely, $r = 0$ ($y \to -\infty$) is the center of the accelerating particle. The coordinate $x$ can now be thought of as an azimuthal angle.

Up to this point, we still have done nothing but rewriting Minkowski spacetime in a coordinate system that satisfies some desirable properties. The next step is to actually change the metric in such a way that it includes a black hole. What do we need to add a black hole? Indeed, horizons. To create more horizons, we need to consider a function $G$ with more roots. One possibility is to let $G$ be a higher order polynomial. It turns out that choosing

$$G(x) = 1 - x^2 - 2mAx^3 - q^2A^2x^4,$$

in (1.2) still is a solution of the Einstein-Maxwell equations (see [15] and also [9] for more general solutions). Here $A$ is the proper acceleration of a particle with mass $m$ and charge $q$. The polynomial $G$ has four real zeroes for a certain range of the parameters $m, q, A$. The two additional roots (as compared to $G(x) = 1 - x^2$) correspond to the inner and outer horizon of a Reissner-Nördstrom black hole.

To see that this is indeed the case, we will now show that in the limit $A \to 0$ (vanishing acceleration) this new solution, which is called the C-metric, is indeed a Reissner-Nördstrom black hole. For this purpose it is convenient to use the advanced time coordinate

$$v := \frac{1}{A}\left(t - \int^{y} G(y')^{-1}\,\mathrm{d}y'\right).$$

Then the metric becomes

$$\mathrm{d}s^2 = -H\,\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + 2Ar^2\,\mathrm{d}v\,\mathrm{d}x + r^2\left(\frac{1}{G(x)}\,\mathrm{d}x^2 + G(x)\,\mathrm{d}\phi^2\right), \tag{1.4}$$

where $H(x,r) = -A^2r^2G(x - 1/Ar)$. As $A \to 0$, we have that $G(x) \to 1 - x^2$. So in this limit we can define $x = \cos\theta$, such that $G(x) \to \sin^2\theta$. Then $\mathrm{d}x^2 \to \sin^2\theta\,\mathrm{d}\theta^2$ and

$$H(x,r) \longrightarrow 1 - \frac{2m}{r} + \frac{q^2}{r^2}.$$

This means that (1.2) in this limit becomes

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)\mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + 2Ar^2\,\mathrm{d}v\,\mathrm{d}x + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right),$$

which is precisely the Reissner-Nördstrom metric in advanced Eddington-Finkelstein coordinates.

To end this section, we need to answer one more question. In the beginning we said that the C-metric describes a pair of black holes accelerating away from each other. Why do we only see one black hole in (1.2)? The answer to this question is that in the process of constructing this metric, we used Rindler coordinates, which describe only one wedge of Minkowski spacetime. To obtain the second (oppositely charged) black hole, one needs to do a Kruskal extension. This, among other things, will be discussed in the next section.

### 1.1.2 Top-down

The C-metric (1.2) is a solution to the Einstein-Maxwell equations if $G$ is any polynomial of second, third or fourth order [9]. A second order polynomial produces Minkowski spacetime, while a higher order polynomial yields black hole solutions. So from this we can already see that the global properties of this large class of solutions are not the same. A full topological classification of the C-metrics is given in [15].

In this thesis, we will restrict our attention to the class of C-metrics describing a pair of accelerating Reissner-Nördstrom black holes. This class is parametrised by polynomials $G$ having four non-degenerate real roots such that the fourth coefficient $a_4 = -q^2A^2 < 0$. The latter requirement is needed to provide the proper signature of the regions in-between the horizons (roots). In this section we give a full account of this class of solutions.

The structure of this section is as follows. First we present the full solution including its electromagnetic field. Then we discuss the three types of singularities of this metric: coordinate, spacetime and conical singularities. Lastly, we perform a Kruskal extension for $G$ with two roots (to warm up) and then for $G$ with four roots, and we obtain the corresponding Penrose diagrams.

### Accelerated Reissner-Nordstrom black holes

The class of C-metrics that we will consider is given by (1.2) with $G : \mathbf{R} \to \mathbf{R}$ given by

$$G(x) = a_4(x - \zeta_1)(x - \zeta_2)(x - \zeta_3)(x - \zeta_4), \tag{1.5}$$

where $a_4 \in \mathbf{R}_{<0}$ and the roots $\zeta_i$ are pairwise distinct. Together with the gauge potential given by

$$\mathcal{A} = -qx\,\mathrm{d}\phi,$$

this is a solution to the Einstein-Maxwell equations, which describes two magnetically charged black holes accelerated away from each other. In this class of solutions, $G$ assumes the form as in Figure 1.1. In our notation we will always assume the ordering

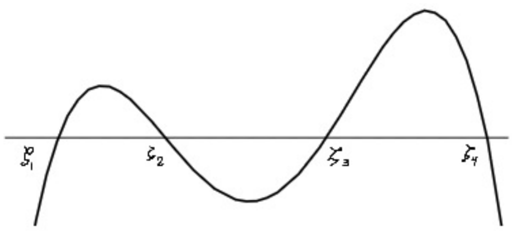


Figure 1.1: The form of the polynomial $G$ in the C-metric that we consider in this thesis.

$$\zeta_1 < \zeta_2 < \zeta_3 < \zeta_4.$$

For the C-metric to have the correct signature $(-, +, +, +)$, one must restrict the $x$-coordinate to the domain $x \in [\zeta_3, \zeta_4]$ such that $G(x)$ is non-negative. The domain of $y$ is restricted by the presence of spacetime singularities. It turns out that the Kretschmann scalar of the C-metric [16],

$$R_{abcd}R^{abcd} = \frac{8}{r^8}[6m^2r^2 + 12mq^2(2Axr - 1)r + q^4(7 - 24Axr + 24A^2x^2r^2)],$$

becomes singular at $r = 0$, where $r$ is the coordinate as defined in (1.3). So the C-metric has[1] a spacetime singularity at $r = 0$ and for this reason we require that $r > 0$, hence $x > y$. We see that the domain of the $y$-coordinate in the C-metric depends on the coordinate $x$. The coordinate ranges are thus as follows.

$$x \in [\zeta_3, \zeta_4], \quad \phi \in [0, \Delta\phi), \quad t \in \mathbf{R} \quad \text{and} \quad y < x.$$

In the next subsection we comment on the value of $\Delta\phi$. The ranges of the $(x, y)$-coordinates are shown in Figure 1.2.

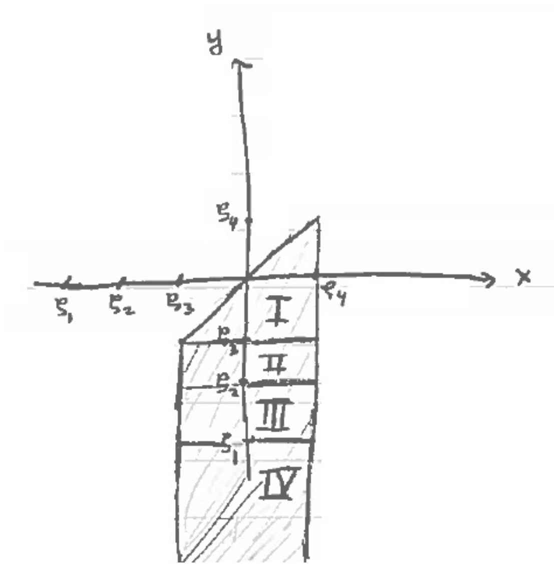



Figure 1.2: The horizontal axis denotes the $x$-coordinate, while the vertical axis denotes the $y$-coordinate. The range of the $x$- and $y$-coordinates is given by the grey area. Region I is the asymptotic region. Region II is the space between the outer black hole horizon and the acceleration horizon. Region III is the region between the inner and outer black hole horizons and region IV is the region containing the singularity at $y = -\infty$.

The roots of $G$ have a purpose. They correspond to the black hole and acceleration horizons. This is motivated by the limiting behaviour of the $C$-metric that we discussed in the previous section, but also by the fact that these horizons are Killing horizons. The $C$-metric admits two independent Killing vectors given by

$$\xi = \partial_t \quad \text{and} \quad \eta = \partial_\phi.$$

Although $\xi$ looks like a time-translation Killing vector, it is not. Namely, the coordinate $t$ is the proper time in the frame of the accelerated black hole. Hence, $\xi$ is a boost Killing vector. The hypersurfaces at which $\xi$ becomes null define the Killing horizons of $\xi$. There holds that $\xi^2 \sim G(y)$. This tells us that the horizons are the hyperplanes at constant $y = \zeta_1, \zeta_2, \zeta_3$. More specifically, $\zeta_1$ corresponds to the inner horizon, $\zeta_2$ to the out horizon and $\zeta_3$ to the acceleration horizon. Note that since $\zeta_4$ is not included in the domain of $y$, it does not define a null-hypersurface.

[1]If $m = q = 0$, the C-metric is just Minkowski space and there is no spacetime singularity at $r = 0$.

### Conical singularities

Besides a spacetime singularity at $r = 0$, the C-metric also has a conical singularity. Before we discuss this, let us first recall what a conical singularity is. Consider $\mathbf{R}^2 \setminus \{(0,0\}$ parametrised by polar coordinates $(r, \theta)$, where $0 < r < \infty$ and $0 \leqslant \theta < 2\pi$. Notice that the origin of $\mathbf{R}^2$ is not covered with these polar coordinates. The Euclidean metric on $\mathbf{R}^2 \setminus \{0,0\}$ is then given by

$$\mathrm{d}s^2 = r^2(\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2). \tag{1.6}$$

We can now introduce Euclidean coordinates $x = r\cos\theta$ and $y = r\sin\theta$ and write for the metric

$$\mathrm{d}s^2 = \,\mathrm{d}x^2 + \,\mathrm{d}y^2.$$

Since the $(x, y)$-coordinates are defined on the whole of $\mathbf{R}^2$ we have now extended the metric in (1.6) in a smooth manner to include the point $(0, 0)$.

In general, however, this is not always possible. Consider again $\mathbf{R}^2 \setminus \{0,0\}$ and suppose that we cut out the wedge $\{(r,\theta) \in \mathbf{R}_{>0} \times [0, 2\pi - \Delta\theta]\}$ for some $\Delta\theta$, and that we identify the boundaries at $\theta = 0$ and $\theta = 2\pi - \Delta\theta$. The resulting space looks like a cone, with at $r = 0$ the tip of the cone. Now it is impossible to smoothly extend the metric to the cone including the tip. We therefore say that the metric has a conical singularity[2] at $r = 0$. Spacetimes with a conical singularity are considered nonphysical.

To prevent a conical singularity, a condition must be met. Let $p$ be any point on a manifold. Let $\gamma_r$ be the curve with a constant geodesic distance $r$ from $p$ and denote by $L(r)$ the length of $\gamma_r$. Then there is no conical singularity if

$$L(r)/2\pi r \to 1 \quad \text{as } r \to 0.$$

In the above example we know that $L(r) = r\Delta\theta$. So we simply require that $\Delta\theta = 2\pi$, which indeed corresponds to flat space.

Now we turn back to the C-metric. Let us isolate a slice of constant $v$ and $r$ in (1.4). The induced metric on this slice is

$$\mathrm{d}s^2 = r^2(G^{-1}(x)\,\mathrm{d}x^2 + G(x)\,\mathrm{d}\phi^2).$$

If we apply the coordinate transformation

$$\widetilde{x} = \int_{x_3}^{x} G(x')^{-1/2}\,\mathrm{d}x',$$

then the metric on the slice of constant $v$ and $r$ becomes

$$\mathrm{d}s^2 = r^2(\,\mathrm{d}\widetilde{x}^2 + G(x)\,\mathrm{d}\phi^2).$$

The lines parametrised by the points $x = \zeta_3$ at constant $v$, describe the axis in the direction of acceleration at either side of the black hole. For a small circle with radius $\widetilde{x}$ around the half-axis $x = \zeta_3$ with $v$ and $r$ constant, its circumference is given by

$$L = \int_0^{\Delta\phi} \sqrt{G(x)}\,\mathrm{d}\phi = \Delta\phi\sqrt{G(x)}.$$

[2]Note that this singularity is purely metrical. It is possible to define a smooth manifold structure at the tip of the cone.

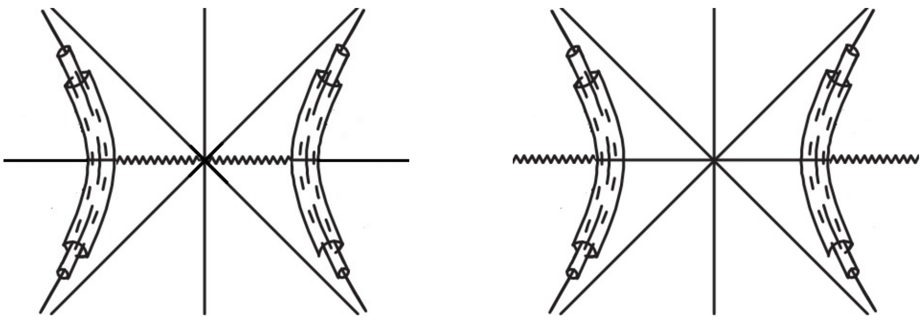
Hence, to avoid a conical singularity, we require that

$$\begin{aligned} 2\pi \equiv \lim_{\widetilde{x}\to 0}\left|\frac{\Delta\phi\sqrt{G(x)}}{\widetilde{x}}\right| &= \Delta\phi\left|\frac{\mathrm{d}}{\mathrm{d}\widetilde{x}}\sqrt{G(x)}\right|_{x=\zeta_3} \\ &= \Delta\phi\left|\frac{\partial x}{\partial\widetilde{x}}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{G(x)}\right|_{x=\zeta_3} \\ &= \Delta\phi\left|\sqrt{G(x)}\frac{1}{2\sqrt{G(x)}}G'(x)\right|_{x=\zeta_3} \\ &= \frac{\Delta\phi}{2}G'(\zeta_3). \end{aligned}$$

Following a similar procedure, at the half-axis $x = \zeta_4$, we require

$$2\pi \equiv -\frac{\Delta\phi}{2}G'(\zeta_4).$$

We can choose $\Delta\phi$ such that either at the half-axis $\zeta_3$ or at the half-axis $\zeta_4$ no conical singularity occurs. See Figure 1.3. However, the conical singularity cannot be removed



(a) (b)

Figure 1.3: The position of the conical singularities, taken from [16] and modified. (a) The conical singularity at $x = \zeta_4$. (b) The conical singularity at $x = \zeta_3$. In both figures $\Delta\phi$ is chosen such that one of the half-axes is free of conical singularities.

simultaneously at both poles. Namely, if that were the case, we would require

$$G'(\zeta_3) = -G'(\zeta_4).$$

This is a severe constraint on the coefficients of $G$. In fact, this constraint is so bad that none of the solutions in the class we consider would be allowed. Hence, any C-metric solution that we consider has a conical singularity. Fortunately, there is a way to resolve this problem. In section 1.3 we discuss how this is done.

### 1.1.3 Causal structure

We already mentioned that the C-metric in the coordinates used above does not describe the global spacetime, but only half. The "missing" black hole in the other half can be found by performing a Kruskal extension. In this section we will do this, and also find the corresponding Penrose diagrams.

The structure of this section is as follows. First, we will conformally compactify the C-metric for $m = q = 0$. Indeed, this is just Minkowski space. However, it is useful to go through this example to see how the absence of manifest spherical symmetry in the C-metric coordinates appears in these diagrams. Second, we will consider the case $m, q \neq 0$.

**The C-metric when $m = q = 0$**

Setting $m = q = 0$ implies that $G(x) = 1 - x^2$. Then the roots of $G$ are $\zeta_1 = -1$ and $\zeta_2 = 1$. From the introduction we know that the coordinates $(t, y, x, \phi)$ parametrize precisely half of Minkowski spacetime when we set $m = q = 0$. Its Kruskal extension should therefore lead to the whole Minkowski spacetime. Let us check this.

In order to find Kruskal coordinates, we must first define the tortoise coordinate by

$$y_* = \int^y G^{-1}(y)\,\mathrm{d}y = \int^y \frac{1}{1-\widetilde{y}^2}\,\mathrm{d}\widetilde{y} = \frac{1}{2}\log\left|\frac{1+y}{1-y}\right|.$$

Here we used that $G(y) = 1 - y^2$. The Eddington-Finkelstein (EF) coordinates are then defined by $u := t - y_*$ and $v := t + y_*$. Notice that

$$y_* = \frac{1}{2}(v-u).$$

Computing the differentials of the EF coordinates, we find that

$$\begin{aligned} \mathrm{d}u &= \mathrm{d}t - \mathrm{d}y_* = \mathrm{d}t - G^{-1}(y)\,\mathrm{d}y \\ \mathrm{d}v &= \mathrm{d}t + \mathrm{d}y_* = \mathrm{d}t + G^{-1}(y)\,\mathrm{d}y \end{aligned}$$

Hence, $\mathrm{d}u\,\mathrm{d}v = \mathrm{d}t^2 - G^{-2}(y)\,\mathrm{d}y^2$ and therefore the C-metric in EF coordinates becomes

$$\mathrm{d}s^2 = r^2[-G(y)\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}\Omega^2],$$

where $\mathrm{d}\Omega^2 := r^2(G^{-1}(x)\,\mathrm{d}x^2 + G(x)\,\mathrm{d}\phi^2)$. Now we are in a position to perform a Kruskal extension. Consider the part of the domain where $x < y < \zeta_2$. That is, the part where $G(y) > 0$ and define the Kruskal coordinates

$$U := -e^{-u} \quad \text{and} \quad V := e^{v}.$$

Then

$$UV = -e^{(v-u)} = -e^{2y_*} = -\frac{1+y}{1-y}.$$

The domain given by

$$\{(x, y) \in [-1, 1] \times \mathbf{R} : x < y < 1\}$$

is therefore parametrised by the coordinates $U < 0$ and $V > 0$ subject to the condition that

$$-1 \leqslant UV < -\frac{1+x}{1-x}. \tag{1.7}$$

Since there holds that

$$du\, dv = -\frac{1}{UV}\, dU\, dV$$

the metric in $U$, $V$ coordinates in this region is

$$ds^2 = r^2 \left[ \frac{G(y)}{UV}\, dU\, dV + d\Omega^2 \right].$$

However, this metric is also regular for $(U, V) \in \mathbf{R}^2$. Allowing the Kruskal coordinates to take values in this region too, defines the Kruskal extension. To see what this resulting space looks like, we will perform a conformal transformation given by

$$\widetilde{U} = \arctan U \quad \text{and} \quad \widetilde{V} = \arctan V.$$

These new coordinates are also subject to the constraint (1.7), which becomes

$$-1 \leqslant \tan \widetilde{U} \tan \widetilde{V} < \frac{1+x}{1-x}.$$

For every $x$ we can now draw the corresponding Penrose diagram. A couple of examples are given in the figure.

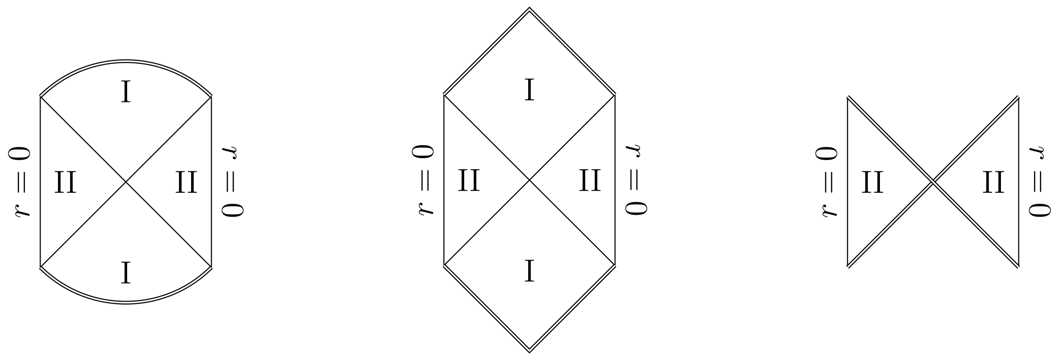



Figure 1.4: Conformal diagrams of slices defined by $x, \phi$ constant. Region I and II correspond to $y \in [-1, x)$ and $y < -1$ respectively (recall that $y \to -\infty$ corresponds to $r = 0$). In (a) $x$ is general, in (b) $x = 1$ and in (c) $x \downarrow -1$. The diagrams for different $x$ look different because of the constraint (1.7). In (c) it looks as if there is no region $I$. One must think of region $I$ being squeezed into a small neighbourhood above the horizon. The double lines represent conformal infinity. $r = 0$ are the positions of the two accelerated test particles.

### The C-metric for $m, q \neq 0$

Now we will assume that $G$ is of the form (1.5). Then the tortoise coordinate is

$$y_* = \int^y G^{-1}(y)\, dy = \sum_{i=1}^{4} \kappa(\zeta_i) \log |y - \zeta_i|,$$

where $\kappa(\zeta_i) = [G'(\zeta_i)]^{-1}$. By $\zeta_i$ we denote the roots of $G$. Eddington-Finkelstein coordinates are again given by $u = t - y_*$, $v = t + y_*$.

As we saw in the previous subsection, we need to create a conformal diagram separately for every $x$. Let us therefore fix the $x$-coordinate. Then we can consider the following three regions in the domain of the $y$-coordinate.

1. $\zeta_3 \leqslant y < x$
2. $\zeta_2 \leqslant y < \zeta_3$
3. $\zeta_1 \leqslant y < \zeta_2$

The idea is that each of these regions can be Kruskal extended separately. The overlapping parts of the Kruskal extension can then be identified to obtain the full extension. In the following, these different regions are labelled by $i$. For each region $i$ we can define Kruskal coordinates by

$$\begin{aligned} U_i :=& (-1)^{m+1} e^{-u/2\kappa(\zeta_i)} \text{sgn } \kappa(\zeta_i) \\ U_i :=& (-1)^{n+1} e^{u/2\kappa(\zeta_i)} \text{sgn } \kappa(\zeta_i) \end{aligned}$$

where $m = 1$ and $n = 0$. Later, when performing the Kruskal extension, we will allow $m$ and $n$ to take other integer values, such that $U_i$ and $V_i$ can become negative. But not yet. Note that

$$\begin{aligned} U_i V_i &= (-1)^{m+n} e^{(v-u)/2\kappa(\zeta_i)} \\ &= (-1)^{m+n} e^{y_*/\kappa(\zeta_i)} \\ &= (-1)^{m+n} \prod_{k \neq i} |y - \zeta_k|^{-\kappa(\zeta_k)/\kappa(\zeta_i)} \end{aligned} \tag{1.8}$$

In order to determine the range of the coordinates $U_i$ and $V_i$, which is constrained by the range of $x$ and $y$, similar as in the previous subsection, we need to simplify (1.8) using the following lemma.

**Lemma 1.** *Let $P_n$ be a polynomial of order n with roots $a_1, ..., a_n$. Then*

$$\frac{1}{P'(a_1)} + \frac{1}{P'(a_2)} + ... + \frac{1}{P'(a_n)} = 0.$$

Since $\kappa(\zeta_i) = 1/G'(\zeta_i)$, a direct consequence of the lemma is that

$$\sum_i \kappa(\zeta_i) = 0,$$

such that (1.8) can be reduced to

$$U_i V_i = (-1)^{m+n} \prod_{k \neq i} \left| \frac{y - \zeta_k}{y - \zeta_i} \right|^{-\kappa(\zeta_k)/\kappa(\zeta_i)} =: h_i(y).$$

For consistency, let's compare this to the previous subsection. There we had only two roots, and the corresponding $\kappa(\zeta_i)$ evaluated at those roots were $\pm 1/2$. So the new expression reduces correctly to the simpler case in the previous subsection.

We see that for $i = 3$ there is a constraint depending on $x$. It is given by

$$\min_{x' \in [\zeta_3, \zeta_4]} h(x') \leqslant U_i V_i < h(x).$$

For the other regions $i = 1, 2$, the constraint $y < x$ was already satisfied from the beginning. Next, we allow the Kruskal coordinates in each region $i$ to take on negative values too, while respecting the constraint for region $i = 3$. This means that we allow $m$ and $n$ to take other integers. For each Kruskal coordinate $U_i$ we can then draw the conformal diagram for a given $x$. This is again done by introducing conformally rescaled coordinates

$$\widetilde{U}_i = \arctan U_i \quad \text{and} \quad \widetilde{V}_i = \arctan V_i.$$

The diagrams corresponding to $i = 1$ look similar to the diagrams in the previous section for Minkowski space. In regions $i = 1$ and $i = 2$, the $y$-coordinate runs between two roots of $G$. Therefore these regions are bounded by null-surfaces. The point $r = 0$ in region $IV$ can now be found in the diagram in the Kruskal extension of $i = 3$. This is the position of the spacetime singularity, behind the two black hole horizons. See Figure 1.5.

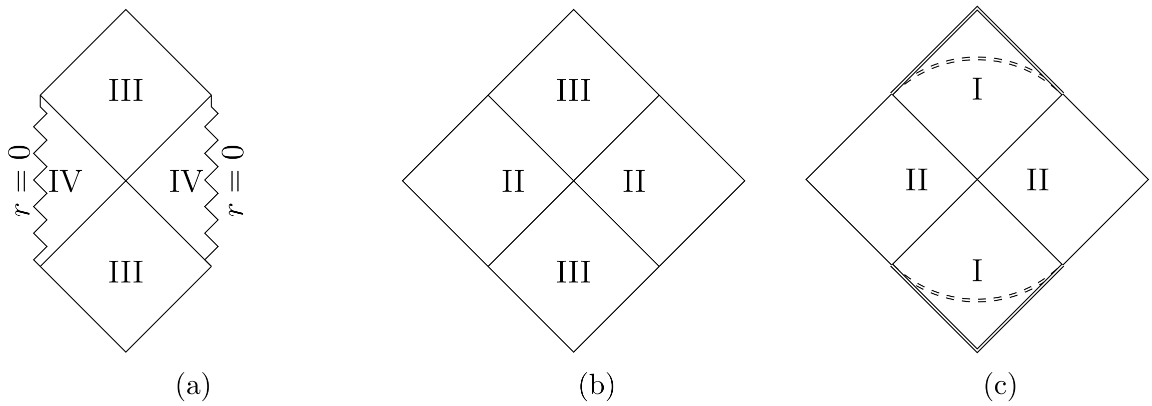
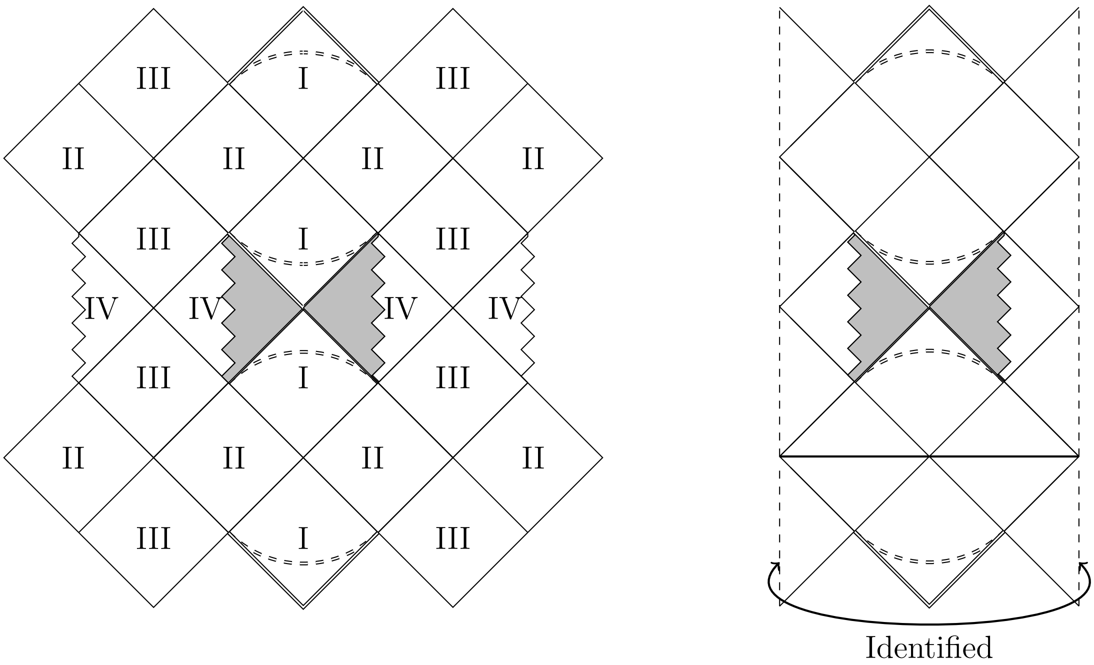



Figure 1.5: (a) The Kruskal extension of region $i = 1$. The singularity is at $r = 0$. (b) The Kruskal extension of region $i = 2$. The block labelled with II corresponds to the interior of the black hole. Region III corresponds to the region between the outer horizon and the acceleration horizon. (c) The Kruskal extension of region $i = 1$. Region I corresponds to the asymptotic region. The curved stippled lines denote the $x$-dependence of this region, as in the previous subsection.

The separate blocks are not geodesically complete spacetimes. To create a geodesically complete spacetime, the regions with the same label (I, II, III, IV) should be identified. This can be done indefinitely and so the maximally extended C-metric consists of an infinite amount of such blocks glued together. One can see the result in Figure 1.6a.

An extension of the spacetime by adding more blocks is, however, not unique. The maximally extended C-metric consists of an infinite amount of blocks glued together. But it is also possible to make identifications, without destroying causality. By doing this in the




(a) (b)

Figure 1.6: (a) If one adds diamond blocks to this diagram indefinitely and in the manner indicated, one obtains the maximally extended conformal diagram of the C-metric. The grey regions are not part of the spacetime. (b) The conformal diagram in which the interiors of the black holes are identified. The horizontal line represents the spatial slice containing the Einstein-Rosen bridge.

above diagram, one can create a space with two black holes in the same asymptotic region that are connected by an Einstein-Rosen bridge.

The lines along which the identification is made are depicted in Figure 1.6b. The bridge connecting the two separate black holes is called an Einstein-Rosen bridge. Though, the causal structure is such that the bridge cannot be used as a short-cut to go from one side of the asymptotic region to the other side. Once one is trapped behind a black hole horizon, one remains trapped.

## 1.2 The Melvin universe

In this section we will discuss the Melvin universe. This universe is similar to Minkowski spacetime, apart from the fact that it contains a magnetic field spread throughout space. In the weak-gravity limit, this is analogous to a constant magnetic field on Minkowski space.

The Melvin universe becomes important at two points in this thesis. Firstly, in the remaining part of this chapter, it is needed to construct the Ernst spacetime. Secondly, in the subsequent chapter, various results can be traced back to properties of the Melvin universe.

The Melvin universe is a solution to the Einstein-Maxwell equations given by

$$\mathrm{d}s^2 = (1 + \frac{1}{4}B^2\rho^2)^2(-\,\mathrm{d}t^2 + \mathrm{d}z^2 + \mathrm{d}\rho^2) + \frac{\rho^2}{(1 + \frac{1}{4}B^2\rho^2)^2}\,\mathrm{d}\phi^2, \tag{1.9}$$

with the field strength given by

$$F = \frac{B\rho}{(1 + \frac{1}{4}B^2\rho^2)^2}\,\mathrm{d}\rho \wedge \mathrm{d}\phi.$$

Let $S$ be a surface of constant $t$ and $z$. The flux of magnetic field lines through this surface is

$$\varPhi = \int_S F = \frac{4\pi}{B}.$$

Contrary to classical electrodynamics (ignoring gravity), we see that the flux through this infinite area is finite. Moreover, it scales with $B^{-1}$ while the usual flux is proportional to $B$. The physical explanation for this is the following.

A magnetic field contains energy. And energy curves spacetime. So intuitively, the magnetic field is held together by its own gravitational pull, in such a way that the flux becomes finite. For this reason the Melvin universe is often referred to as a flux tube. That is, the GR analogue of a tube with finite radius, and thus with a finite flux.

## 1.3 The Harrisson transformation

In this section we explain how to combine the Melvin universe and the C-metric to form the Ernst metric. This is done using the Harrison transformation. The Harrison transformation takes a solution of the Einstein-Maxwell equations and maps it into another solution. The Harrison transformation is thus a symmetry of the equations of motion. The transformed solution looks like the old one, only instead of a Minkowski background, it contains the Melvin universe as a background.

The Harrison transformations form a group $SO(1,1)$ [17]. This group, however, is a proper subgroup of a larger symmetry group $SU(2,1)$ of the equations of motion. This latter is called the Kinnersley group, which also contains scaling and gauge transformations.

In this section we follow [18] to explain how the Kinnersley group can be derived. We begin by constructing the analogous Ehlers group $SL(2,\mathbf{R})$ for the Einstein-Hilbert action. Then we extend the construction to include electromagnetic fields to find the Kinnersley group. We end with a couple of examples, the last of which continues into the next section and is the Harrison transformation applied to the C-metric.

### 1.3.1 The Ehlers group

The existence of the Ehlers symmetry group is based on the presence of at least one global Killing vector. This Killing vector generates a one-parameter group of motions[3] $G$ in the spacetime $M$. By identifying all the points that lie on each orbit, we get the

[3] A family of maps $\phi_t : \mathcal{M} \to \mathcal{M}$ such that the distance between any two points is preserved under this mapping.

three-dimensional quotient space $\Sigma = M/G$, where each point in $\Sigma$ represents a trajectory in the spacetime $M$. The canonical projection $\pi : M \to \Sigma$ defines a principal bundle with fiber $G$. And by the pull-back this projection induces a metric $\widehat{g}$ on $\Sigma$. This metric is given by the ansatz

$$g = -\sigma(\,\mathrm{d}t + a) \otimes (\,\mathrm{d}t + a) + \frac{1}{\sigma}\widehat{g}, \tag{1.10}$$

where $a = a_i\,\mathrm{d}x^i$, $\sigma$ and $\widehat{g}$ are all fields on $\Sigma$. Since we assumed[4] that $k = \partial_t$ is a Killing vector field, $\sigma$ and $a_i$ must be independent of the coordinate $t$. The field strength of $a$ is given by $f = \mathrm{d}a$, and is gauge invariant.

The Einstein-Hilbert action on $M$ can now be expressed in terms of the sigma model [18]

$$S[\widehat{g}, \sigma, a] = \int \widehat{*}(\widehat{R} - \frac{1}{2\sigma^2}\langle\,\mathrm{d}\sigma,\,\mathrm{d}\sigma\rangle + \frac{\sigma^2}{2}\langle\,\mathrm{d}a,\,\mathrm{d}a\rangle).$$

Here $\widehat{*}$ denotes the Hodge dual with respect to the three-metric $\widehat{g}$. And the inner product is defined $\langle\alpha, \alpha\rangle := \widehat{*}(\alpha \wedge \widehat{*}\alpha)$ for an arbitary $n$-form $\alpha$. The equations of motion of $\widehat{g}_{ij}$, $\sigma$ and $a$ reproduce the Einstein equations.

**The Ernst potential**

The above form of the action is not yet the desired form. In order to be able read off the Ehlers group actions, we need to rewrite this action in terms of the Ernst potential. We will do this now. First note that the equation of motion of $a$ is

$$\mathrm{d}\widehat{*}(\sigma^2\,\mathrm{d}a) = 0.$$

This implies that locally there exists a potential $Y$ such that

$$\mathrm{d}Y = -\widehat{*}(\sigma^2\,\mathrm{d}a). \tag{1.11}$$

$Y$ is called the twist potential. To get a feeling for what this is, we will find another representation for the right-hand side of (1.11). For this purpose, we define a quantity associated to the Killing vector $k$ given by

$$\begin{aligned} 2\omega &:= *\,(k^\flat \wedge \,\mathrm{d}k^\flat) \\ &= *\,k^\flat(\sigma^{-1}\,\mathrm{d}\sigma \wedge k^\flat + \sigma\,\mathrm{d}a) \\ &= -\,\sigma^2\widehat{*}\,\mathrm{d}a, \end{aligned}$$

where $k^\flat = -\sigma(\,\mathrm{d}t + a)$. It immediately follows that

$$\mathrm{d}Y = 2\omega.$$

$\omega$ is the twist form associated to $k$ and it measures the extent to which the Killing vector $k$ fails to be orthogonal to the family of 3-surfaces $\Sigma$. That is, a non-zero twist indicates the presence of a rotation in the spacetime geometry. Since $Y$ is a potential for $\omega$, $Y$ is called the twist potential. As we will not encounter such twisted geometries, we continue without further ado.

[4]One may replace $k$ by any other Killing vector field.

To write the above action in terms of this twist potential $Y$, we simply add a Lagrange multiplier constraint $\mathrm{d}(\sigma^2 \widehat{*}\, \mathrm{d}Y) = 0$ to the Lagrangian. Solving for $a$ then yields [18]

$$S[\widehat{g}, \sigma, Y] = \int \widehat{*}\Big(\widehat{R} - \frac{\langle\, \mathrm{d}\sigma,\, \mathrm{d}\sigma\rangle + \langle\, \mathrm{d}Y,\, \mathrm{d}Y\rangle}{2\sigma^2}\Big),$$

The last step is to define the complex potential $\mathcal{E} := \sigma + iY$ such that the action becomes

$$S[\widehat{g}, \mathcal{E}] = \int \widehat{*}\Big(\widehat{R} - 2\frac{\langle\, \mathrm{d}\mathcal{E},\, \mathrm{d}\overline{\mathcal{E}}\rangle}{(\mathcal{E} + \overline{\mathcal{E}})^2}\Big). \tag{1.12}$$

$\mathcal{E}$ is called the Ernst potential. We are now in a position to read off the actions on $\mathcal{E}$ that leaves this action invariant.

**The Ehlers group actions**

In this subsection we discuss the symmetries of (1.12). First, it is obvious that

$$\mathcal{E} \longmapsto \mathcal{E} + ib, \quad b \in \mathbf{R},$$

leaves the action invariant. Furthermore, we have the rescaling symmetry

$$\mathcal{E} \longmapsto \alpha\overline{\alpha}\mathcal{E}, \quad \alpha \in \mathbf{C}.$$

And finally we have a non-linear symmetry given by

$$\mathcal{E} \longmapsto \mathcal{E}(1 + ic\mathcal{E})^{-1}, \quad c \in \mathbf{R}.$$

The latter is the most complicated of the three. Showing that this is indeed a symmetry is straightforward, but tedious. The three vector fields generating these symmetries form an $sl(2, \mathbf{R})$ algebra under the Lie bracket of vector fields.

## 1.3.2 The Kinnersley group

In this subsection we generalise the above action and potentials to include electromagnetic fields. Besides the decomposition of the metric, we now also have to decompose the gauge potential. The gauge potential $A$ can be decomposed into parallel and orthogonal components to $\Sigma$. Let $\widehat{A}$ be the projection of $A$ on $\Sigma$ by the pull back. Then we may write

$$A = \varphi(\, \mathrm{d}t + a) + \widehat{A}, \tag{1.13}$$

where $\varphi$ is a scalar field on $\Sigma$ and $a$ is the same one-form as in the metric ansatz. In terms of this decomposition, the field strength becomes

$$F = \mathrm{d}A = \mathrm{d}\varphi \wedge (\, \mathrm{d}t + a) + \widehat{F} + \varphi f,$$

where $\widehat{F} = \mathrm{d}\widehat{A}$ and $f = \mathrm{d}a$. Then the matter term in the Lagrangian becomes

$$F \wedge *F = \mathrm{d}t \wedge \big[\sigma(\widehat{F} + \varphi f) \wedge \widehat{*}(\widehat{F} + \varphi f) - \sigma^{-1}\, \mathrm{d}\varphi \wedge \widehat{*}\, \mathrm{d}\varphi\big].$$

Substituting this into the action, together with the results from the previous subsection, we find

$$S[\widehat{g},\sigma,a] = \int \widehat{*}\Big(\widehat{R} - \frac{1}{2\sigma^2}\langle\,\mathrm{d}\sigma,\,\mathrm{d}\sigma\rangle + \frac{\sigma^2}{2}\langle\,\mathrm{d}a,\,\mathrm{d}a\rangle + \frac{2}{\sigma}\langle\,\mathrm{d}\varphi\rangle^2 - 2\sigma\langle\,\mathrm{d}\widehat{A}+\varphi\,\mathrm{d}a\rangle^2\Big),$$

Again, since the symmetries of the action are not yet manifest, we must now find potentials that we can use to simplify this expression. This is a little bit more messy than in the Ehlers case in the previous subsection.

### The electromagnetic Ernst potentials

To find these Ernst potentials, we first need to look at the equation of motion for $\widehat{A}$. It is given by

$$\mathrm{d}\widehat{*}[\sigma(\widehat{F}+\varphi f)] = 0.$$

This implies that locally there exists a potential $\psi$ such that

$$\mathrm{d}\psi := \sigma\widehat{*}(\widehat{F}+\varphi f). \tag{1.14}$$

Second, the equation of motion for $a$ is

$$\mathrm{d}\widehat{*}[\sigma^2 f - 4\sigma\varphi(\widehat{F}+\varphi f)] = 0. \tag{1.15}$$

Before we define the corresponding potential, notice that

$$\begin{aligned} 2\,\mathrm{d}(\omega+\varphi\,\mathrm{d}\psi-\psi\,\mathrm{d}\varphi) &= \mathrm{d}\widehat{*}(-\sigma^2\widehat{*}f + 2\varphi\,\mathrm{d}\psi - 2\psi\,\mathrm{d}\varphi) \\ &= \mathrm{d}\widehat{*}[\sigma^2 f - 4\sigma\varphi(\widehat{F}+\varphi f)] \\ &= 0. \end{aligned}$$

So instead of defining $Y$ through (1.15) as we did in the previous subsection, we can define a potential $Y$ such that

$$\mathrm{d}Y := 2[\omega+\varphi\,\mathrm{d}\psi-\psi\,\mathrm{d}\varphi]. \tag{1.16}$$

This expression is much cleaner. But more importantly, it is already expressed in terms of the other potential $\psi$. In the previous subsection we called $Y$ a twist potential. We see that this name still makes sense. Namely, if $k$ is a time translation Killing vector, then $\mathrm{d}\psi = H$ and $\mathrm{d}\varphi = E$, where $H$ and $E$ are the magnetic[5] and electric field respectively, the exterior derivative of $\varphi\,\mathrm{d}\psi - \psi\,\mathrm{d}\varphi$ is proportional to the Poynting vector $E\wedge H$, which measures the twists in electromagnetic fields. The action in terms of the potentials $Y$ and $\psi$ becomes [18]

$$S = \int \widehat{*}\Big(\widehat{R} - 2\frac{\langle\,\mathrm{d}\varphi\rangle^2 + \langle\,\mathrm{d}\psi\rangle^2}{\sigma} - \frac{\langle\,\mathrm{d}\sigma\rangle^2 + \langle\,\mathrm{d}Y - 2\varphi\,\mathrm{d}\psi + 2\psi\,\mathrm{d}\varphi\rangle^2}{2\sigma^2}\Big).$$

Next we define the Ernst potentials

$$\mathcal{E} = -\sigma - (\varphi^2+\psi^2) + iY \quad\text{and}\quad \varPhi = -\varphi + i\psi.$$

[5] Notice that both $H$ and $E$ are one-forms.

Here $\varphi$ and $\psi$ are called the electric and magnetic potentials respectively. According to the decomposition (1.13) and the definition (1.14), there holds that

$$\mathrm{d}\varphi = -\iota_k F \quad \text{and} \quad \mathrm{d}\psi = \iota_k * F.$$

In terms of these potentials, the action becomes

$$S = \int \widehat{*}\left(\widehat{R} - \frac{2}{\sigma}|\,\mathrm{d}\Phi|^2 - \frac{1}{2\sigma^2}|\,\mathrm{d}\mathcal{E} + 2\overline{\Phi}\,\mathrm{d}\Phi|^2\right).$$

From this action one can now read off the Kinnersley symmetries.

**The Kinnersley group actions**

We worked hard to rewrite the action in such a way that symmetries become manifest. The question now is: what are those symmetries? One can check that there are five distinct transformations that leave the action invariant. They are given by [19]

$$\begin{aligned} \mathcal{E}' &= \alpha\overline{\alpha}\mathcal{E}, & \Phi' &= \alpha\Phi \\ \mathcal{E}' &= \mathcal{E} + ib, & \Phi' &= \Phi \\ \mathcal{E}' &= \mathcal{E}(1 + ic\mathcal{E})^{-1}, & \Phi' &= \Phi(1 + ic\mathcal{E})^{-1} \\ \mathcal{E}' &= \mathcal{E} - 2\overline{\beta}\Phi - \beta\overline{\beta}, & \Phi' &= \Phi + \beta \\ \mathcal{E}' &= \mathcal{E}(1 - 2\overline{\gamma}\Phi - \gamma\overline{\gamma}\mathcal{E})^{-1}, & \Phi' &= (\Phi + \gamma\mathcal{E})(1 - 2\overline{\gamma}\Phi - \gamma\overline{\gamma}\mathcal{E})^{-1}. \end{aligned} \tag{1.17}$$

The first three we recognise as the Ehlers symmetry group from the previous subsection. And together with the other two, they form the Kinnersley group, which is $SU(2,1)$. The most complicated ones, (1.17), are known as the Harrison transformations. They form a subgroup $SO(1,1)$. The remaining part of this section consists of a couple of examples in which we apply the Harrison transformation.

### 1.3.3 Examples

In order to see how the metric transforms under a Harrison transformation, we need to know how $\sigma$ transforms. And to see this, we need to express $\sigma$ in terms of the Ernst potentials. This is not difficult, since by definition we have

$$\sigma = -\mathrm{Re}\ \mathcal{E} - |\Phi|^2.$$

Consider the Harrison transformation given by

$$\mathcal{E}' = \mathcal{E}\Lambda^{-1}, \quad \text{and} \quad \Phi' = (\Phi + \gamma\mathcal{E})\Lambda^{-1},$$

where $\varLambda = 1 - 2\overline{\gamma}\varPhi - \gamma\overline{\gamma}\mathcal{E}$. Under this transformation, $\sigma$ transforms as

$$\begin{aligned}
\sigma' &= -\mathrm{Re}\big(\varLambda^{-1}\mathcal{E}\big) - |\varLambda^{-1}(\varPhi + \gamma\mathcal{E})|^2 \\
&= -\frac{1}{2}\left(\frac{\mathcal{E}}{\varLambda} + \frac{\overline{\mathcal{E}}}{\overline{\varLambda}}\right) - \frac{1}{\varLambda\overline{\varLambda}}(\varPhi + \gamma\mathcal{E})(\overline{\varPhi} + \overline{\gamma}\overline{\mathcal{E}}) \\
&= -\frac{1}{2|\varLambda|^2}\left[\varLambda\overline{\mathcal{E}} + \overline{\varLambda}\mathcal{E} + 2(\overline{\gamma}\varPhi\overline{\mathcal{E}} + \gamma\overline{\varPhi}\mathcal{E}) + 2\gamma\overline{\gamma}\mathcal{E}\overline{\mathcal{E}}\right] - \frac{|\varPhi|^2}{|\varLambda|^2} \\
&= -\frac{1}{|\varLambda|^2}(\mathrm{Re}\ \mathcal{E} + |\varPhi|^2) \\
&= \frac{1}{|\varLambda|^2}\sigma
\end{aligned}$$

So, the Harrison transformation comes down to substituting

$$\sigma \longmapsto \frac{1}{|\varLambda|^2}\sigma, \tag{1.18}$$

in the metric. We will now apply this substitution to two instructive examples: Minkowski spacetime and the Schwarzschild black hole. In the next section we do the same for the C-metric to find the Ernst metric.

**Example 1: the Minkowski metric**

Minkowski space is

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \,\mathrm{d}z^2 + \,\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2$$

with $A_\mu = 0$. Taking the Killing vector $\xi = \partial_\phi$, the Minkowski metric in cylindrical coordinates is for the form (1.10) with $\sigma = -\rho^2$ and $a = 0$. Furthermore the complex electromagnetic potential vanishes: $\varphi = \psi = 0$, such that the Ernst potentials become

$$\mathcal{E} = \rho^2 \quad \text{and} \quad \varPhi = 0.$$

According to (1.18) when setting $\gamma = iB/2$, the transformed metric becomes

$$\mathrm{d}s^2 = \varLambda^2\big(-\,\mathrm{d}t^2 + \,\mathrm{d}z^2 + \,\mathrm{d}\rho^2\big) + \frac{\rho^2}{\varLambda^2}\,\mathrm{d}\phi^2,$$

where $\varLambda = 1 + \frac{1}{4}B^2\rho^2$. The complex electromagnetic potential transforms to

$$\varPhi' = \frac{i}{2\varLambda}B\rho^2.$$

Since $\mathrm{d}\psi = \iota_{\partial_\phi} * F$, we find that in the transformed solution, the gauge potential becomes

$$A = \frac{1}{2\varLambda}B\rho^2\,\mathrm{d}\phi,$$

where we used that $\varPhi = i\psi$. The field strength then becomes

$$\begin{aligned}
F &= (\varLambda^{-1}\rho B - \varLambda^{-2}\frac{1}{4}B^3\rho^3)\,\mathrm{d}\rho \wedge\,\mathrm{d}\phi \\
&= \varLambda^{-2}(\rho B\varLambda - \frac{1}{4}B^3\rho^3)\,\mathrm{d}\rho \wedge\,\mathrm{d}\phi \\
&= \varLambda^{-2}\rho B\,\mathrm{d}\rho \wedge\,\mathrm{d}\phi.
\end{aligned}$$

Comparing this to (1.9), we see that this is the magnetic Melvin universe.

**Example 2: the Schwarzschild black hole**

The Schwarzschild black hole is given by the metric

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2(\theta)\,\mathrm{d}\phi^2. \tag{1.19}$$

where $f(r) = 1 - 2M/r$. The gauge potential vanishes. This solution also has an axial Killing vector $\partial_\phi$. Hence, (1.19) is of the form (1.10) when we choose $\sigma = -r^2\sin^2(\theta)$ and $a = 0$, hence $\omega = 0$. Then the Ernst potentials corresponding to the Killing vector $\partial_\phi$ are

$$\mathcal{E} = r^2\sin^2\theta \quad \text{and} \quad \Phi = 0.$$

Therefore, the transformed metric becomes

$$\mathrm{d}s^2 = \Lambda^2(-f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2) + \frac{r^2\sin^2(\theta)}{\Lambda^2}\,\mathrm{d}\phi^2,$$

where $\Lambda = 1 - iB\Phi + \frac{1}{4}B^2r^2\sin^2(\theta)$. The complex electromagnetic potential becomes

$$\Phi' = \frac{i}{2\Lambda}Br^2\sin^2\theta.$$

Hence, the gauge potential becomes

$$A = \frac{1}{2}\Lambda^{-1}Br^2\sin^2\theta\,\mathrm{d}\phi.$$

And hence the field strength

$$F = \Lambda^{-2}Br\sin\theta(\sin\theta\,\mathrm{d}r + r\cos\theta\,\mathrm{d}\theta)\wedge\,\mathrm{d}\phi.$$

The resulting solution is a Schwarzschild black hole immersed in a magnetic field. Other black hole solutions such as the Reissner-Nördstrom and Kerr black holes can also be immersed in a magnetic field. This, however, requires more work, since the electric field of the black hole, together with the magnetic field of the Melvin universe contribute to a non-zero twist potential defined in (1.16). The Harrison transformed RN and Kerr metrics can be found in [7].

Here we take also the opportunity to mention that the thermodynamics of magnetised Kerr black holes was recently worked out in [7].

## 1.4 The Ernst metric

Continuing with the previous subsection, our third and last example will be the C-metric. The C-metric is axially symmetric and the corresponding Killing vector is given by $\partial_\phi$ in the coordinates of (1.2). This means that we can write also the C-metric in the form (1.10). The Ernst potentials are

$$\mathcal{E} = -\left(\frac{G(x)}{A^2(x-y)^2} + \widehat{q}^2x^2\right) \quad \text{and} \quad \Phi = -i\widehat{q}x.$$

Hence, the Harrison transformed potentials are

$$\mathcal{E}' = \Lambda^{-1}\mathcal{E} \quad \text{and} \quad \Phi' = \Lambda^{-1}(\Phi + \frac{i}{2}\widehat{B}\mathcal{E}),$$

where

$$\Lambda(x,y) = (1 + \frac{1}{2}\widehat{q}\widehat{B}x)^2 + \frac{\widehat{B}G(x)}{4A^2(x-y)^2}.$$

Note that we now write $\widehat{B}$ instead of $B$. Namely, we have reserved the letter $B$ for the asymptotic value of the magnetic field of the Melvin universe. It turns out that as opposed to in the previous examples, $\widehat{B}$ is not equal to $B$. The transformed metric is

$$\mathrm{d}s^2 = \frac{\Lambda^2}{A^2(x-y)^2}\left(G(y)\,\mathrm{d}t^2 - \frac{1}{G(y)}\,\mathrm{d}y^2 + \frac{1}{G(x)}\,\mathrm{d}x^2\right) + \frac{G(x)}{A^2\Lambda^2(x-y)^2}\,\mathrm{d}\phi^2. \tag{1.20}$$

The complex potential transforms to

$$\begin{aligned}
\Phi' &= \frac{1}{\Lambda}(\Phi + \frac{i}{2}\widehat{B}\mathcal{E}) \\
&= \frac{i}{\Lambda}\left[ - \widehat{q}x - \frac{1}{2}\widehat{B}\left(\frac{G(x)}{A^2(x-y)^2} + \widehat{q}^2x^2\right)\right] \\
&= \frac{i}{\Lambda}\left[ - \widehat{q}x - \frac{1}{\widehat{B}}\left(2\Lambda - 2(1+\frac{1}{2}\widehat{q}\widehat{B}x)^2\right) - \frac{1}{2}\widehat{q}^2\widehat{B}x^2\right] \\
&= \frac{2i}{\Lambda\widehat{B}}\left[ - \frac{1}{2}\widehat{q}\widehat{B}x - \Lambda + (1+\frac{1}{2}\widehat{q}\widehat{B}x)^2 - \frac{1}{4}\widehat{q}^2\widehat{B}^2x^2\right] \\
&= \frac{2i}{\Lambda\widehat{B}}\left[1 + \frac{1}{2}\widehat{q}\widehat{B}x\right] + \text{constant}
\end{aligned}$$

Hence, the connection one form is given by

$$A = \frac{2}{\Lambda\widehat{B}}(1 + \frac{1}{2}\widehat{q}\widehat{B}x)\,\mathrm{d}\phi$$

The metric in (1.20) is called the Ernst metric. It is similar to the C-metric, apart from the fact that the Ernst metric is asymptotic to the Melvin universe. So almost all the properties of the C-metric that we discussed in the previous sections also hold for the Ernst metric. In the following subsections we highlight some properties of the Ernst metric that we will need in chapter 2.

### Removal of the conical singularity

Recall from section 1.1.2 the C-metric had a conical singularity at $x \in \{\zeta_3, \zeta_4\}$. We were able to remove this singularity at one of the two poles by scaling the length of the $\phi$ coordinate. However, the singularity at the other pole always remained.

In the Ernst metric, we have an additional parameter $\widehat{B}$ which we can choose such that the conical singularity at both poles vanishes. To see this, notice that the condition for the conical singularity to be absent at both poles is that

$$1 \equiv= \frac{\Delta\phi}{4\pi\Lambda(\zeta_i,y)^2}|G'(\zeta_i)|, \tag{1.21}$$

for $i = 3, 4$. This follows from a procedure similar to that employed in section 1.1.2. Since $\Lambda$ is a function of $x$ and $y$, we require (1.21) to hold for all $y$. Fortunately this is the case, since $G(\zeta_i) = 0$ and therefore $\Lambda(\zeta_i, y)$ is independent of $y$. Henceforth we can thus write $\Lambda(\zeta_i, y) =: \Lambda(\zeta_i)$. Then (1.21) are the constraints

$$\frac{G'(\zeta_4)}{\Lambda(\zeta_4)^2} = -\frac{G'(\zeta_3)}{\Lambda(\zeta_3)^2} \quad \text{and} \quad \Delta\phi = \frac{4\pi\Lambda(\zeta_3)^2}{G'(\zeta_3)}. \tag{1.22}$$

The Ernst solution is thus a solution with three free parameters. Namely, $m, \widehat{q}, A$ and $\widehat{B}$ minus one due to the first constraint. In the Ernst solution without conical singularity, $\Delta\phi$ must be adjusted so as to satisfy the second constraint in (1.22).

**The form of Hong and Teo**

In the next chapter we need to know explicitly the expressions for the roots $\zeta_i$ of $G$. With $G$ written in the form (1.5), the expressions for $\zeta_i$ are very complicated. Too complicated for us to work with. Fortunately, Hong and Teo in [20] showed by performing a coordinate transformation and by simultaneously rescaling $A$, $m$ and $\widehat{q}$ in a particular way, that it is possible to write

$$G(x) = (1 - x^2)(1 + r_- Ax)(1 + r_+ Ax), \tag{1.23}$$

where

$$r_\pm = m \pm \sqrt{m^2 - \widehat{q}^2},$$

such that the roots of $G$ are given by

$$\zeta_1 = -\frac{1}{r_- A}, \quad \zeta_2 = -\frac{1}{r_+ A}, \quad \zeta_3 = -1 \quad \text{and} \quad \zeta_4 = 1. \tag{1.24}$$

These roots are manageable. Therefore, in the next chapter the form of Hong and Teo is used, instead of (1.5). Notice that in these expressions, the parameters $A$, $m$, $\widehat{q}$ and $\widehat{B}$ are rescaled versions of the original ones. The precise relation between the new and old parameters is given by [20].

### 1.4.1 Symmetries

Since the metric components are independent of the coordinates $t$ and $\phi$, it can be deduced that the Ernst metric admits two independent Killing vectors $\partial_t$ and $\partial_\phi$. Since $t$ denotes the Rindler time, $\partial_t$ is a boost Killing vector, rather than a time translation Killing vector. The Killing vector $\partial_\phi$ generates axial symmetry.

Let us say a bit more about the boost Killing vector $\partial_t$. To see that this indeed corresponds to the usual boost symmetry of Minkowski spacetime, let us introduce the following definition of a boost symmetric spacetime. A spacetime is said to be boost symmetric if there exist coordinates such that the invariant line element can be written in the form [21]

$$\mathrm{d}s^2 = -\frac{e^\mu}{z^2 - T^2}\left[(z\,\mathrm{d}T - T\,\mathrm{d}z) + A(z^2 - T^2)\,\mathrm{d}\phi\right]^2 + e^\lambda\left[\frac{(z\,\mathrm{d}z - T\,\mathrm{d}T)^2}{z^2 - T^2} + \mathrm{d}\rho^2\right] + e^{-\mu}\rho^2\frac{\gamma^2}{\beta^2}\,\mathrm{d}\phi^2. \tag{1.25}$$

Here $\mu$, $\lambda$ and $A$ are specific functions of $\rho$ and $z^2 - T^2$ such that $\partial_T$ is asymptotically timelike. $\beta$ and $\gamma$ are constants. In these coordinates, one can see that

$$\xi = z\partial_T + T\partial_z, \tag{1.26}$$

is a Killing vector. Since $\partial_T$ is asymptotically timelike, we can think of $T$ as Minkowski time. And hence $\xi$ generates a boost symmetry. It can be shown that the C-metric and the Ernst metric are indeed boost symmetric spacetimes by finding coordinates such that the metric assumes the form of (1.25). This is done in [21] and [22] .

Another important feature of spacetimes as in (1.25) is that they are invariant under $z \mapsto -z$ and $T \mapsto -T$.

# Chapter 2

# Thermodynamics of the Ernst spacetime

It is known that stationary black holes satisfy the four laws of black hole thermodynamics [23]. The black holes in the Ernst spacetime, however, are not stationary. So it is not immediately clear that these are also subject to the laws of thermodynamics.

In this chapter we propose that the black holes in the Ernst spacetime obey the first law of thermodynamics. And more generally, we argue that this law is satisfied in any boost symmetric spacetime. According to our knowledge of the literature, this is a new result. A different version of the first law in boost symmetric spacetimes was already derived in [24]. It would be interesting to see how their results are related to ours.

The structure of this chapter is as follows. First, we given an elementary introduction to the mechanics of black holes. Second, we present and derive the first law in boost symmetric spacetimes. And third, we verify that the first law does indeed hold in the Ernst spacetime.

## 2.1 Overview of black hole thermodynamics

Black holes began to play an important role in the 1970s, when it was realised that black holes are thermodynamic objects. Bardeen, Carter and Hawking ([23] 1973) showed that if one identifies (up to a constant) surface gravity with temperature and the area of the horizon with entropy, black holes obey the four laws of thermodynamics.

**The zeroth law** states that the surface gravity along the event horizon is constant. This law is analogous to the zeroth law of thermodynamics stating that the temperature throughout a body at thermal equilibrium is constant.

**The first law** is a law of conservation of energy and states that when an amount of matter is thrown into the black hole, the variation of the mass $M$ is given by

$$\mathrm{d}M = \frac{1}{8\pi}\kappa\,\mathrm{d}A_{\mathcal{H}} + \Omega\,\mathrm{d}J + \Phi\,\mathrm{d}Q.$$

where $\kappa$ is the surface gravity and $A_{\mathcal{H}}$ is the area of the black hole horizon. This law corresponds to the first law of thermodynamics.

**The second law** states that the area of the black hole horizon cannot decrease during classical processes. I.e., $\mathrm{d}A \geqslant 0$. This is analogous to the second law of thermodynamics, which states that the entropy cannot decrease.

**The third law** states that it is impossible to reduce the surface gravity to zero by a finite number of processes. This is analogous to the third law of thermodynamics, stating that the temperature cannot become zero.

The first law was first proved for stationary black holes living in an asymptotically flat spacetime. The proofs have been generalised to include also asymptotically (anti)-de Sitter spacetimes ([25],[26]). Furthermore, in [6] it was shown that Kerr-Newman-Melvin black holes (stationary, electrically charged and rotating black holes immersed in a magnetic field), also satisfy the first law.

The derivation of the first law is based on a satisfactory notion of mass of the black hole. In order to derive the first law for boost symmetric spacetimes, we should thus first agree on a notion of mass.

## 2.2 Mass in GR

If the spacetime is asymptotically flat and admits a time translation Killing vector $k^a$, then it is possible to define the mass of an area enclosed by a 2-surface $S$ by

$$m[S] := -\frac{1}{4\pi}\int_S \nabla^a k^b \, \mathrm{d}\Sigma_{ab}. \tag{2.1}$$

If the spacetime does not admit a global time translation Killing vector, but only a vector field that is asymptotically Killing and timelike, then it is only possible to define the total mass of the spacetime by

$$m_\infty := m[S_\infty], \tag{2.2}$$

where $S_\infty$ is a closed 2-surface at spatial infinity. The integral in (2.1) is called a Komar integral. And the quantity $m[S]$ is called the Komar mass. In stationary asymptotically flat spacetimes, the Komar mass agrees with the ADM mass.

The Ernst metric admits a vector field that is asymptotically Killing and timelike. So we can compute $m_\infty$ according to the above definition. However, since the Ernst metric is asymptotically Melvin, and the Melvin universe contains an infinite amount of energy, the integral defining $m_\infty$ first has to be regularised. In [27] it is explained how this can be done. It turns out that the regularised mass $m_\infty$ vanishes for the Ernst metric. In order to understand why this is the case, it is useful to recall that the Ernst metric can be used to describe pair production of black holes. In the process that a pair of black holes is created from the energy in the magnetic field of the Melvin universe, the energy is conserved. So if we compute the energy of the final state (the Ernst metric) and regularise it by subtracting the energy of the initial state (the Melvin universe), then we end up at zero energy. So $m_\infty = 0$ just tells us that the energy is conserved during the pair creation process.

The result that the Komar mass vanishes (after subtracting a background) is generally true for boost and rotation symmetric spacetimes [28]. Namely, the Komar mass is the zeroth component of the 4-momentum of the spacetime, which must be invariant under the action of any Killing vector fields. Boost and rotation symmetric spacetimes contain a boost and an axial Killing vector. The action of the boost Killing vector mixes the components $\partial_t$ and $\partial_z$, while the axial Killing vector mixes $\partial_x$ and $\partial_y$. The only vector that is left invariant under these actions is the zero vector.

The conclusion is that the Komar mass (2.2) is not a useful quantity when doing thermodynamics of boost and rotation symmetric spacetimes. In order to be able do thermodynamics, we have to introduce the boost mass. Before we give the definition, let us introduce some notation.

**Notation 1.** *$\Sigma$ denotes a hypersurface at constant Rindler time*[1]. *We define*

$$L := LR \cap \Sigma \quad \text{and} \quad R := RR \cap \Sigma.$$

*Here $LR$ and $RR$ are the left and right Rindler wedge respectively. We denote by $\mathcal{B}$ the interior of a black hole, and by $\mathcal{H}$ its horizon. If the black hole is contained in the left Rindler wedge, then*

$$LR \cap \mathcal{B} = \mathcal{B}.$$

*That is, in our notation $LR$ includes the black hole region.*

*$\mathcal{H}$ and $\mathcal{A}$ denote the black hole and acceleration horizon respectively. We will often write $\mathcal{H}$ and $\mathcal{A}$ and $\mathcal{B}$ while we actually mean $\mathcal{H} \cap \Sigma$, $\mathcal{A} \cap \Sigma$ and $\mathcal{B} \cap \Sigma$ respectively. This should be clear from the context. By $S_\infty$ we denote the $S^2$ boundary at spatial infinity.*

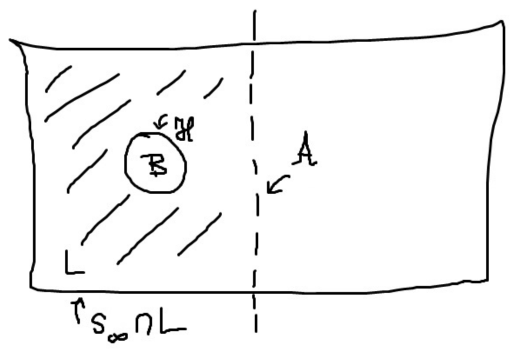



Figure 2.1: A spatial slice $\Sigma$ at constant Rindler time divided up into two regions intersecting the left and right Rindler wedges. These regions are separated by the acceleration horizon $\mathcal{A}$. The $S^2$ boundary at spatial infinity is denoted by $S_\infty$.

**Definition 2** (Boost mass)**.** *Let $\xi$ be a boost Killing vector field as in* (1.26)*. Then the boost mass of an area enclosed by a surface $S$ is*

$$M[S] := -\frac{1}{8\pi} \int_S \nabla^a \xi^b \, \mathrm{d}\Sigma_{ab}.$$

*We define the same quantity for surfaces $S$ that are not closed.*

[1]We refer to Rindler time whenever we talk about the proper time of an accelerated object in some spacetime.

In defining the boost mass, compared to the usual mass, we replaced the time translation Killing vector by a boost Killing vector. Since the boost Killing vector is global, this means that the boost mass is defined for any closed surface $S$ in $\Sigma$. This will allow us to define a non-zero and finite mass of a black hole.

The definition of boost mass is similar to the definition of Komar angular momentum. If $\eta^a$ is an axial Killing vector, then the latter is given by

$$J[S] := -\frac{1}{8\pi}\int_S \nabla^a \eta^b \,\mathrm{d}\Sigma_{ab}.$$

This is no surprise, since together with the angular momentum, the boost mass forms the antisymmetric tensor $\mathcal{J}^{ab}$. The charges corresponding to boost symmetries are the $\mathcal{J}^{0i}$ components, while the $\mathcal{J}^{ij}$ components contain the angular momentum. The angular momentum of a black hole may become negative if the black hole is rotating in the "opposite" direction. Similarly, the boost mass may become negative if the black hole is boosted in the "opposite" direction. This implies that if we place a black hole with positive boost mass in one Rindler wedge, then the same black hole in the other Rindler wedge has a negative and opposite boost mass. Namely, the Killing time is reversed in the opposite wedge.

It could thus happen that the total angular momentum and total boost mass of a spacetime vanish, while the Komar integral is finite and non-zero when integrated over e.g. half of a spatial slice ($L$ or $R$). In this thesis, we only consider spacetimes with this property.

## 2.3 The first law

In this section we derive the first law for boost symmetric spacetimes. The derivation is analogous to the derivation given in [23] for stationary spacetimes. However, it differs in places due to the different definition of mass. We begin by discussing how to define the boost mass of a black hole and then derive a Smarr relation. Then we vary the Smarr relation to obtain the first law.

### 2.3.1 A Smarr formula

The Smarr formula relates the *total mass* of the black hole to its actual mass behind the horizon, its angular momentum and the energy of its electromagnetic field in the exterior. In a stationary spacetime, the electromagnetic field of the black hole extends throughout the whole space. And therefore the mass formula is found by integrating the Komar mass (2.1) over a surface $S_\infty$. In the boost symmetric case, however, we integrate over just half of the spacetime. In order to see why, note that from the point of view of a left Rindler observer, the electromagnetic field of a black hole in the left Rindler wedge LR will never leave LR. And any field inside the right Rindler wedge RR, will never enter LR. Hence, only fields on LR contributee to the mass of the black hole. So we define

$$M := M[(S_\infty \cup \mathcal{A}) \cap L], \tag{2.3}$$

as the mass of a black hole in the left Rindler wedge.

The mass of a black hole is related to its charge, temperature, angular momentum and entropy through the Smarr formula. The Smarr relation for boost symmetric spacetimes is given by

$$\boxed{2M = 2\int_{L\backslash\mathcal{B}} (T_{ab} - \frac{1}{2}Tg_{ab})\xi^a \,\mathrm{d}\Sigma^b + 2\Omega J + \frac{\kappa_{\mathcal{H}}}{4\pi}A_{\mathcal{H}}} \tag{2.4}$$

Here $\xi$ is the boost Killing vector, which generates the acceleration horizon. The black hole horizon is generated by $l^a = \xi^a + \Omega\eta^a$ for some $\Omega$, where $\eta^a$ is an axial Killing vector. And the angular momentum is defined by

$$J := J[\mathcal{H}].$$

**Derivation of** (2.4)

For any Killing vector $X^a$, the Ricci identity

$$\nabla_a\nabla^b X^a = R^b_a X^a,$$

holds. Using Gauss' law of divergence, we can thus write

$$2M = \frac{1}{4\pi}\int_{L\backslash\mathcal{B}} R^a_b\xi^b \,\mathrm{d}\Sigma_a + 2M[\mathcal{H}] \tag{2.5}$$

Upon substituting $\xi^a = l^a - \Omega\eta^a$, the second term can be written

$$\begin{aligned} 2M[\mathcal{H}] &= -\frac{1}{4\pi}\int_{\mathcal{H}} \nabla^a l^b \,\mathrm{d}\Sigma_{ab} + 2\Omega J \\ &= \frac{1}{4\pi}\kappa_{\mathcal{H}}A_{\mathcal{H}} + 2\Omega J. \end{aligned}$$

In the last step we used that the volume element $\mathrm{d}\Sigma_{ab}$ can be expressed as

$$\mathrm{d}\Sigma_{ab} = \xi_{[a}n_{b]}\,\mathrm{d}\Sigma,$$

where $n_a$ is the second null vector orthogonal to $\mathcal{H}$ normalised such that $\xi^a n_a = -1$ on $\mathcal{H}$. Namely, then the surface gravity, which is constant on the horizon, is given by $\kappa_{\mathcal{H}} = -n_a l_b \nabla^b l^a$ on $\mathcal{H}$. Using Einstein's equations, the first term in (2.5) can be written as

$$\frac{1}{4\pi}\int_{L\backslash\mathcal{B}} R^a_b\xi^b \,\mathrm{d}\Sigma_a = 2\int_{\Sigma} (T_{ab} - \frac{1}{2}Tg_{ab})\xi^b \,\mathrm{d}\Sigma^a.$$

This concludes the derivation.

### 2.3.2 The total boost mass vanishes

We consider spacetimes in which the total boost mass vanishes. The Ernst metric is an example of such a spacetime [27]. Such spacetimes exhibit an even stronger property. Namely, that the Komar integral over half of spatial infinity also vanishes.

To see why this is the case, recall that according to the definition of a boost symmetric spacetime (1.25), the metric is symmetric under $z \mapsto -z$ and $T \mapsto -T$ (here $T$ is associated to the usual Minkowski time). One can thus map $L$ onto $R$ while preserving the metric and its Killing vectors. This implies that

$$M[S_\infty \cap L] = \frac{1}{2} M[S_\infty] = 0. \tag{2.6}$$

We can thus use (2.6) in (2.3) to write

$$M = M[\mathcal{A}].$$

And by the definition of surface gravity (5) we can evaluate this integral to

$$M = \frac{1}{8\pi} \kappa_{\mathcal{A}} A_{\mathcal{A}}, \tag{2.7}$$

The boost mass of a black hole thus allows for a simple representation in terms of the area of the acceleration horizon $A_{\mathcal{A}}$ and its surface gravity $\kappa_{\mathcal{A}}$. However, since the area of the acceleration horizon $A_{\mathcal{A}}$ is infinite, the boost mass diverges. In [27] it is explained how the area can be regularised. The regularised acceleration horizon area shall be denoted by $\varDelta A_{\mathcal{A}}$. And from now on we mean by $M$ the regularised mass

$$M = \frac{1}{8\pi} \kappa_{\mathcal{A}} \varDelta A_{\mathcal{A}}. \tag{2.8}$$

As opposed to the usual Komar mass, the regularised boost mass may be non-vanishing.

**Remark 1.** *The assumption that the total boost mass of the spacetime vanishes is required in our derivation of the first law. If this statement turns out to be true in any boost symmetric spacetime, then this assumption can be dropped. We conjecture that this is the case, based on the classical analogy that $J^{ab} = 2X^{[a}P^{b]}$, where $X^a$ and $P^a$ are the position and momentum four-vectors respectively. We already know that $P^a = 0$ in boost symmetric spacetimes, so it would follow that $J^{ab} = 0$, too. A possible proof of, or counterexample to this claim is left for future work.*

### 2.3.3 Varying the mass formula

In this section we shall use the Smarr formula to derive an expression for the difference $\delta M$ between the boost masses of two slightly different boost symmetric black holes.

The first law of black hole mechanics in a boost symmetric spacetime is given by

$$\boxed{\delta M = \frac{\kappa_{\mathcal{H}}}{8\pi} \delta A_{\mathcal{H}} + \varOmega \delta J + \frac{\varDelta A_{\mathcal{A}}}{8\pi} \delta \kappa_{\mathcal{A}} + \int_{L \setminus \mathcal{B}} (\delta T^{ab}) g_{ab} \xi^c \, \mathrm{d}\varSigma_c.} \tag{2.9}$$

Here $\varDelta A_{\mathcal{A}}$ is the regularised area of the acceleration horizon and $\kappa_{\mathcal{A}}$ is its surface gravity. The following lemma is needed in the derivation of the first law.

**Lemma 2.** *Let $X^a$ and $Y^a$ be two Killing vectors such that a Killing horizon $\mathcal{K}$ is generated by $Z^a = X^a + \omega Y^a$ for some $\omega$. Let $n^a$ be orthogonal to $\mathcal{K}$, normalised such that $n^a l_a = -1$ on $\mathcal{K}$. Consider $(M, g)$ and a slightly perturbed spacetime $(M', g + \delta g)$. Write $h_{ab} = \delta g_{ab}$*

*and* $h = g^{ab}h_{ab}$*. Also assume that the surfaces* $S_\infty$*,* $\mathcal{H}$ *and* $\mathcal{A}$ *as well as the Killing vectors* $\xi$ *and* $\eta$ *are the same in both solutions. I.e.,*

$$\delta\xi^a = \delta\eta^a = 0$$

*and*

$$\delta\xi_a = h_{ab}\xi^b, \quad \delta\eta_a = h_{ab}\eta^b.$$

*Then*

$$\delta\kappa_{\mathcal{K}} = -\frac{1}{2}Z^b\nabla^a h_{ab} + \delta\omega(\nabla_n\eta^a)Z_a.$$

*Proof.* See [23] or [18]. □

Since the black hole horizon is generated by $l^a = \xi^a + \varOmega\eta^a$, a direct consequence of the lemma is that

$$\delta\kappa_{\mathcal{H}} = -\frac{1}{2}l^b\nabla^a h_{ab} + \delta\varOmega(\nabla_n\eta^a)l_a. \tag{2.10}$$

Here $\kappa_{\mathcal{H}} = -n_a l_b \nabla^b l^a$. And $n^a$ is normalised such that $l^a n_a = -1$ on $\mathcal{H}$. Similarly, since the acceleration horizon is generated by $\xi^a$, we also have

$$\delta\kappa_{\mathcal{A}} = -\frac{1}{2}\xi^b\nabla^a h_{ab}, \tag{2.11}$$

where $\kappa_{\mathcal{A}} = -\widetilde{n}_a\xi_b\nabla^b\xi^a$. Here $\widetilde{n}^a$ is normalised such that $\xi^a\widetilde{n}_a = -1$ on $\mathcal{A}$. Notice that since the two Killing horizons have a different generator, in the definitions of $\kappa_{\mathcal{H}}$ and $\kappa_{\mathcal{A}}$ two different Killing vectors are used.

**Derivation of** (2.9)

Consider again a perturbation of $(M, g)$ such as in the lemma. Varying the Smarr-like formula gives

$$2\delta M = \delta\left[\int_{L\setminus\mathcal{B}}(2T_{ab} + \frac{1}{8\pi}Rg_{ab})\xi^a\,\mathrm{d}\varSigma^b\right] + \frac{1}{4\pi}(\kappa\delta A_{\mathcal{H}} + A_{\mathcal{H}}\delta\kappa) + 2(J\delta\varOmega + \varOmega\delta J) \tag{2.12}$$

Here we used that in four dimensions,

$$8\pi T = R^a{}_a - \frac{1}{2}g^a_a R = -R.$$

We shall now work out the integral term in (2.12). The variation of the term involving $R$ gives [29]

$$\delta\left(\frac{1}{8\pi}\int_{L\setminus\mathcal{B}} Rg_{ab}\xi^a\,\mathrm{d}\varSigma^b\right) = -\frac{1}{8\pi}\int_{L\setminus\mathcal{B}}\left[\left(R_{ab} - \frac{1}{2}g_{ab}R\right)h^{ab} + \nabla^a\nabla_b h^b_a - \nabla^a\nabla_a h\right]\xi^c\,\mathrm{d}\varSigma_c \tag{2.13}$$

Furthermore, the variation of the term involving $T^{ab}$ gives

$$\delta\left(\int_{L\setminus\mathcal{B}} 2T_{ab}\xi^a\,\mathrm{d}\varSigma^b\right) = \int_{L\setminus\mathcal{B}} T_{ab}h^{ab}\xi^c\,\mathrm{d}\varSigma_c + \int_L(\delta T^{ab})g_{ab}\xi^c\,\mathrm{d}\varSigma_c, \tag{2.14}$$

where for the first term we used that $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{ab}h_{ab}$. Adding (2.13) and (2.14) and using $R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}$, we are left with

$$-\int_{L\backslash\mathcal{B}} (\nabla^a\nabla_b h_a^b - \nabla^a\nabla_a h)\xi^c \, \mathrm{d}\Sigma_c + \int_{L\backslash\mathcal{B}} (\delta T^{ab}) g_{ab}\xi^c \, \mathrm{d}\Sigma_c.$$

Using Gauss' law, we can write the first term as a boundary term

$$-\left(\int_{S_\infty\cap L} + \int_{\mathcal{A}} - \int_{\mathcal{H}}\right)(\nabla_b h^{ab} - \nabla^a h)\xi^c \, \mathrm{d}\Sigma_{ca}. \tag{2.15}$$

Using (2.10), it follows that [23]

$$\int_{\mathcal{H}} (\nabla_b h^{ab} - \nabla^a h)\xi^c \, \mathrm{d}\Sigma_{ca} = -\frac{1}{4\pi} A_{\mathcal{H}}\delta\kappa_{\mathcal{H}} - 2J\delta\Omega$$

Similarly, using (2.11), it follows that

$$\int_{\mathcal{A}} (\nabla_b h^{ab} - \nabla^a h)\xi^c \, \mathrm{d}\Sigma_{ca} = -\frac{1}{4\pi} A_{\mathcal{A}}\delta\kappa_{\mathcal{A}}$$

Now it remains to show that the integral over $S_\infty \cap L$ vanishes. Following the steps towards (2.15) in reversed order, we find that

$$\int_{S_\infty\cap L} (\nabla_b h^{ab} - \nabla^a h)\xi^c \, \mathrm{d}\Sigma_{ca} = \delta M[S_\infty \cap L] = 0$$

where we used (2.6) (so this is the place where we need the assumption that the total boost mass of the spacetime vanishes). Wrapping up, we find that

$$\delta\left[\int_{L\backslash\mathcal{B}} (2T_{ab} + \frac{1}{8\pi}Rg_{ab})\xi^a \, \mathrm{d}\Sigma^b\right] = -\frac{1}{4\pi}A_{\mathcal{H}}\delta\kappa_{\mathcal{H}} - 2J\delta\Omega + \frac{1}{4\pi}A_{\mathcal{A}}\delta\kappa_{\mathcal{A}} + \int_{L\backslash\mathcal{B}} (\delta T^{ab}) g_{ab}\xi^c \, \mathrm{d}\Sigma_c.$$

Substituting this back into (2.12) yields the first law.

### 2.3.4 Interpretation of the first law

Compared to the usual first law, the first law in boost symmetric spacetimes differs in the $\Delta A_{\mathcal{A}}\delta\kappa_{\mathcal{A}}/4$ term. The surface gravity $\kappa_{\mathcal{A}}$ can be interpreted as being proportional to the Unruh temperature: the temperature of the Rindler ground state. The Rindler ground state has an energy that is different from the energy of the Minkowski vacuum. The first law thus tells us that when varying the boost mass, changes of the acceleration and thus the energy in the Rindler ground state have to be taken into account.

Upon identifying$\kappa_{\mathcal{A}}$ with the Unruh temperature, it also makes sense to try to interpret $\Delta A_{\mathcal{A}}$ in terms of entropy. An interpretation of this quantity as the entanglement entropy between the left and right Rindler wedges is given in the last section of this chapter.

### 2.3.5 Including electromagnetism

When including electromagnetic fields, the term involving the Ricci tensor in the Smarr relation can be evaluated as follows.

$$\frac{1}{4\pi}\int_{L\setminus\mathcal{B}} R_{ab}\xi^a \,\mathrm{d}\Sigma^b = \int_{\partial(L\setminus\mathcal{B})} (\phi * F + \psi F).$$

It turns out that the electric and magnetic potential are constant on the Killing horizons [18]. Since in boost symmetric spacetimes there also holds that

$$\int_{S_\infty} F = \int_{S_\infty} *F = 0,$$

we can thus write

$$\int_{\mathcal{H}\cup\mathcal{A}\cup(L\cap S_\infty)} (\phi * F + \psi F) = (\phi_\mathcal{H} - \phi_\mathcal{A})q + (\psi_\mathcal{H} - \psi_\mathcal{A})g + \int_{L\cap S_\infty} (\psi F + \phi * F),$$

where we used that the electric and magnetic charges are given by

$$q = \int_{S^2} *F \quad \text{and} \quad g = \int_{S^2} F.$$

Here $S^2$ is any surface enclosing the black hole. This result tells us that in the grand canonical ensemble the chemical potential is determined by the potential difference between the black hole and acceleration horizons, instead of the black hole horizon and infinity.

The integral over infinity gives a contribution of the electromagnetic fields in the background. A similar contribution appears in the study of thermodynamics of electric Kerr-Newman-Melvin black holes [6]. There this term is shown to be equal to $\mu B$, where $B$ is the asymptotic value of the magnetic field of the Melvin universe, and $\mu$ has the interpretation of being the magnetic moment of the system. In the next section, we will find that for the magnetically charged black holes in the Ernst metric, this term evaluates to $PB$, where we shall refer to $P$ as the *magnetic polarisation* of the system. In general we thus expect that the integral over infinity reduces to

$$\int_{L\cap S_\infty} \psi F = (P + \mu)B.$$

However, because it is not clear how to regularise this integral, we are unable to give intrinsic definitions of $\mu$ and $P$. If all of the above contributions are taken into account, we expect that the first law reads

$$\mathrm{d}M = \frac{\kappa_\mathcal{H}}{8\pi}\,\mathrm{d}A_\mathcal{H} + \frac{\Delta A_\mathcal{A}}{8\pi}\,\mathrm{d}\kappa_\mathcal{A} + \phi\,\mathrm{d}g + \psi\,\mathrm{d}q - (\mu + P)\,\mathrm{d}B. \tag{2.16}$$

In the next section we will show that this law is indeed satisfied for the Ernst spacetime, where $J = \mu = g = 0$.

## 2.4 The first law for the Ernst metric

In this section we show that (2.16) holds in the Ernst spacetime. First, we will do this for variations with the magnetic field $B$ held constant. Second, we will allow also $B$ to vary.

**Free variables**

The Ernst solution has four parameters, three of which can be varied independently. We choose $\widehat{q}$, $A$ and $\widehat{B}$ to be our free parameters in the form of Hong and Teo (see section 1.4). The parameter $m$ can then be solved for in terms of the free parameters.

Evaluating the derivative of (1.23) at the roots $\zeta_3$ and $\zeta_4$ gives

$$\frac{1}{2}G'(\zeta_3) = 1 - 2Am - A^2\widehat{q}^2 \quad \text{and} \quad \frac{1}{2}G'(\zeta_4) = 1 + 2Am + A^2\widehat{q}^2.$$

Define

$$u := \frac{1 + \frac{1}{2}\widehat{q}\widehat{B}\zeta_3}{1 + \frac{1}{2}\widehat{q}\widehat{B}\zeta_4}, \tag{2.17}$$

where $\zeta_3 = -1$ and $\zeta_4 = 1$ (such that $u$ is independent of $m$ and $A$). The constraint in (1.22) then becomes

$$\frac{1 - 2Am - A^2\widehat{q}^2}{1 + 2Am + A^2\widehat{q}^2} = u^4. \tag{2.18}$$

Solving for $m$ gives

$$m = \frac{1}{2}\frac{(1+u^4)(1+A^2\widehat{q}^2)}{A(1-u^4)}.$$

As will be discussed in section 2.6, the domain of $u$ is $(0,1) \cup (0,1)$. Hence, $m$ is positive[2].

### 2.4.1 The magnetic potential

In the electric case, when one has a time translation symmetry given by the Killing vector field $\partial_t$, one can define a potential $\phi$ such that the electric field is given by $E = \mathrm{d}\phi$. By the electromagnetic duality, one can do the same for the magnetic field. I.e., define a potential $\psi$ such that $H = \mathrm{d}\psi$, where $H = \widehat{*}B$ in the notation from the previous chapter. We compute $\psi$ for the Ernst metric.

The gauge field of the Ernst solution is given by

$$A = \frac{2}{\widehat{B}\Lambda}(1 + \frac{1}{2}\widehat{B}\widehat{q}x)\,\mathrm{d}\phi.$$

So the field strength $F = \mathrm{d}A$ is

$$F = \partial_x A_\phi \,\mathrm{d}x \wedge \mathrm{d}\phi + \partial_y A_\phi \,\mathrm{d}y \wedge \mathrm{d}\phi.$$

The dual field strength is ${}^*F_{ab} = \frac{1}{2}\sqrt{-g}\varepsilon_{abcd}F^{cd}$, where $\varepsilon_{abcd}$ is the Levi-civita symbol. We have

$$F^{\phi x} = F_{\phi x}g^{\phi\phi}g^{xx} = A^4(x-y)^4\partial_x A_\phi,$$
$$F^{\phi y} = F_{\phi y}g^{\phi\phi}g^{yy} = -A^4(x-y)^4\frac{G(y)}{G(x)}\partial_y A_\phi.$$

[2] There is also a negative solution for $m$, but we assumed $m$ to be positive.

And

$$\sqrt{-g} = \frac{\Lambda^2}{A^4(x-y)^4}.$$

Hence, the dual field strength becomes

$$\begin{aligned} *F &= \Lambda^2\left(\partial_x A_\phi \, \mathrm{d}t \wedge \, \mathrm{d}y - \frac{G(y)}{G(x)}\partial_y A_\phi \, \mathrm{d}t \wedge \, \mathrm{d}x\right) \\ &= -\frac{2}{\widehat{B}}\left[\partial_x \Lambda\left(1+\frac{1}{2}\widehat{q}\widehat{B}x\right)+\frac{1}{2}\widehat{q}\widehat{B}\Lambda\right]\mathrm{d}t \wedge \, \mathrm{d}y + \left[\frac{\widehat{B}G(y)}{(x-y)^3}\left(1+\frac{1}{2}\widehat{q}\widehat{B}x\right)\right]\mathrm{d}t \wedge \, \mathrm{d}x \end{aligned}.$$

Up to a constant, the potential $\psi$ is now defined through the following conditions.

$$\partial_x\psi = {}^*F_{tx} \quad \text{and} \quad \partial_y\psi = {}^*F_{ty}.$$

One can verify that

$$\psi(x,y) = -\frac{\widehat{B}G(y)}{2(x-y)^2}\left[\left(1+\frac{1}{2}\widehat{q}\widehat{B}x\right)+\frac{\widehat{q}\widehat{B}}{x-y}\right]+\widetilde{\psi}(y),$$

where

$$\widetilde{\psi}(y) := \frac{y}{4A^2}\left[A^2\widehat{B}^2\widehat{q}^3y^2+(2A\widehat{B}^2m\widehat{q}-2A\widehat{B}\widehat{q}^2+4\widehat{q}A^2)y+\widehat{q}\widehat{B}^2+4\widehat{q}A^2-4\widehat{B}Am-A^2\widehat{B}^2\widehat{q}^3\right]$$

satisfies these equations. Since $G(\zeta_i) = 0$, it follows that

$$\psi(x,\zeta_2) = \widetilde{\psi}(\zeta_2) \quad \text{and} \quad \psi(x,\zeta_3) = \widetilde{\psi}(\zeta_3).$$

Hence, the potential is constant on the black hole horizons. The difference between the potential on the black hole and acceleration horizons shall be used as the thermodynamic variable conjugate to the charge $q$. I.e.,

$$\psi = \psi(x,\zeta_2) - \psi(x,\zeta_3).$$

This turns out to be equivalent to

$$\psi = -\widehat{q}(\zeta_2-\zeta_3)(1+\frac{1}{2}\widehat{q}\widehat{B}\zeta_4)(1+\frac{1}{2}\widehat{q}\widehat{B}\zeta_1).$$

### 2.4.2 Other thermodynamic quantities

Our next task is to discuss all other thermodynamic quantities that appear in the first law. These are the charge $q$, the magnetic field $B$, the area of the black hole $A_{\mathcal{H}}$ horizon and acceleration horizon $\Delta A_{\mathcal{A}}$, and their surface gravities. (Recall that the boost mass is given by (2.8)). Since these quantities were already computed in e.g. [4], we just cite the results.

### The charge, magnetic field and areas

The parameter $\widehat{q}$ that appears in $G$ is the charge of the black hole in the limit where the acceleration goes to zero. I.e., in the limit where the Ernst metric reduces to the Reissner-Nördstrom metric. However, when working outside of this limit, which we are going to do, the physical charge $q$ is different from $\widehat{q}$. Namely, the magnetic charge is given by [4]

$$q = \frac{\Delta\phi}{4\pi}\frac{2}{\widehat{B}}\left[\left(1+\frac{1}{2}\widehat{q}\widehat{B}\zeta_3\right)^{-1} - \left(1+\frac{1}{2}\widehat{q}\widehat{B}\zeta_4\right)^{-1}\right]. \tag{2.19}$$

Here

$$\Delta\phi = \frac{4\pi\Lambda(\zeta_3)^2}{G'(\zeta_3)},$$

as in (1.22). Something similar is the case for the magnetic field the Ernst solution is asymptotic to. In the limit $m, \widehat{q} \to 0$, $\widehat{B}$ approaches the magnetic field of the Melvin universe. Outside of this limit, however, the physical asymptotic magnetic field is given by [4]

$$B = \widehat{B}\left(1+\frac{1}{2}\widehat{q}\widehat{B}\zeta_3\right)^{-3}\frac{1}{2}G'(\zeta_3).$$

The area of the black hole horizon is

$$A_{\mathcal{H}} = \int_0^{\Delta\phi} \mathrm{d}\phi \int_{\zeta_3}^{\zeta_4} \sqrt{g_{xx}g_{\phi\phi}}\,\mathrm{d}x = \frac{\Delta\phi(\zeta_4-\zeta_3)}{A^2(\zeta_3-\zeta_2)(\zeta_4-\zeta_2)}.$$

As mentioned earlier, the area of the acceleration horizon of the Ernst spacetime is infinite. It can be made finite by subtracting the area of the acceleration horizon of the Melvin universe. In [27] it is explained how this can be done. The result is that the regularised area of the acceleration horizon of the Ernst metric, denoted by $\Delta A_{\mathcal{A}}$, is

$$\Delta A_{\mathcal{A}} = -\frac{\Delta\phi}{A^2}\left(\frac{1}{\zeta_3-\zeta_2}+\frac{1}{\zeta_3-\zeta_1}\right).$$

### Surface gravity

The boost Killing vector $\xi^a$ generates $\mathcal{H}$ as well as $\mathcal{A}$. Hence, the respective surface gravities are determined by

$$\xi^a\nabla_a\xi^b \overset{\mathcal{H}}{=} \kappa_{\mathcal{H}}\xi^b \quad \text{and} \quad \xi^a\nabla_a\xi^b \overset{\mathcal{A}}{=} \kappa_{\mathcal{A}}\xi^b,$$

which are understood to be evaluated on the horizon. In computing these quantities, we must take care of choosing a coordinate system that is not singular on the horizons. We take the metric in advanced Eddingtong-Finkelstein coordinates as in (1.4), modified by factors of $\Lambda$ after the Harrison transformation. The boost Killing vector (with the correct normalisation) in these coordinates is given by $\xi^a = (1,0,0,0)$. That is $\xi^v = 1$ and the other components vanish. One can derive that

$$\Gamma^v_{vv} = -G(x-(Ar)^{-1})A^2\left[\frac{\partial_r\Lambda(x,r)}{\Lambda(x,r)}r^2+r\right] - \frac{1}{2}G'(x-(Ar)^{-1}).$$

Hence,

$$\xi^a \nabla_a \xi^v \stackrel{\mathcal{H}}{=} \xi^v \Gamma^v_{vv} \xi^v \stackrel{\mathcal{H}}{=} -\frac{1}{2}\frac{1}{G'(\zeta_2)},$$

and

$$\xi^a \nabla_a \xi^v \stackrel{\mathcal{A}}{=} \xi^v \Gamma^v_{vv} \xi^v \stackrel{\mathcal{A}}{=} -\frac{1}{2}\frac{1}{G'(\zeta_3)},$$

where we used that $x - \frac{1}{Ar} = y$ and the horizons are at $y = \zeta_i$ for $i = 1, 2$ (recall $G(\zeta_i) = 0$). Hence

$$\kappa_{\mathcal{H}} = -\frac{1}{2}\frac{1}{G'(\zeta_2)} \quad \text{and} \quad \kappa_{\mathcal{A}} = -\frac{1}{2}\frac{1}{G'(\zeta_3)} \tag{2.20}$$

The surface gravities of $\mathcal{H}$ and $\mathcal{A}$ have opposite sign. I.e., $\kappa_{\mathcal{H}} > 0$ and $\kappa_{\mathcal{A}} < 0$. The temperatures of the horizons are associated with the absolute values of the surface gravities.

In order to check that we chose the right normalisation of the boost Killing vector, one can compute the temperature of either horizon by requiring that the analytic continuation of the metric expanded about either horizon is free of conical singularities. The length of the imaginary time axis $\beta$ then corresponds to the inverse temperature through $\beta = 2\pi T^{-1}$. The identification $\kappa/2\pi = T$ tells us what normalisation we had to choose. Using this method for the black hole horizon, it turns out that $\beta = -4\pi/G'(\zeta_2)$ and thus

$$T = -\frac{1}{4\pi}G'(\zeta_2).$$

Since this result agrees with (2.20), we chose the right normalisation.

### 2.4.3 Verifying the first law

Now we are in a position to verify the first law. First we keep the magnetic field $B = B_0$ for some constant $B_0$. Then the number of free parameters is reduced by one. Choosing $\widehat{q}$ and $\widehat{B}$ as free parameters, we find that

$$A = \frac{1}{4\widehat{B}(1 + \frac{1}{2}\widehat{q}\widehat{B}\zeta_3)}\sqrt{\widehat{B}(1 + \frac{1}{2}\widehat{q}\widehat{B}\zeta_3)[B_0(\widehat{q}^4\widehat{B}^4 + 24\widehat{q}^2\widehat{B}^2 + 16) - 16\widehat{B}(1 + \frac{1}{2}\widehat{q}\widehat{B}\zeta_3)]}.$$

Substituting this expression for $A$ into the thermodynamic variables, one indeed finds that

$$\frac{1}{8\pi}\big(\kappa_{\mathcal{A}}\partial_w \Delta A_{\mathcal{A}} - \kappa_{\mathcal{H}}\partial_w A_{\mathcal{H}}\big) = \psi \partial_w q$$

is satisfied for $w = \widehat{q}$ and $w = \widehat{B}$. Hence, the first law is satisfied.

Second, we allow $B$ to change. Varying with respect to $\widehat{q}$, $\widehat{B}$ and $A$ we find that

$$\frac{1}{8\pi}\big(\kappa_{\mathcal{A}}\partial_w \Delta A_{\mathcal{A}} - \kappa_{\mathcal{H}}\partial_w A_{\mathcal{H}}\big) = \psi \partial_w q - P\partial_w B$$

has to be satisfied for $w = \widehat{q}$ and $w = \widehat{B}$ and $w = A$. Since we have only one unknown $P$, and three parameters for which the above equation must hold, the solution for $P$, if it exists, is non-trivial. It turns out that this solution exists and is given by

$$P = \frac{\Delta\phi}{A^2}\frac{\widehat{q}\Lambda(\zeta_3)}{4\pi}.$$

Furthermore, with this definition of $P$ it can also be verified that the Smarr relation

$$2M = \frac{1}{4\pi}\kappa_{\mathcal{H}} A_{\mathcal{H}} + \psi q + PB$$

holds.

## 2.5 Thermal equilibrium

In this section we look at a subclass of Ernst metrics for which the black hole is at equilibrium with the Unruh radiation. These solutions have the property that $G'(\zeta_2) = -G'(\zeta_3)$ such that the polynomial $G$ is symmetric. The advantage of systems at equilibrium is that one can define a partition function $Z$ and identify the free energy $F$ with the Euclidean action $I_E$. They are related by

$$Z = e^{-\beta F} = e^{-I_E}.$$

Henceforth, we denote the equilibrium temperature by $T = \beta^{-1}$. Let us also denote the entropy of the black hole by $S = A_{\mathcal{H}}/4$. Then the free energy is given by

$$F = M - TS. \tag{2.21}$$

For the Ernst metric, one can compute the instanton action $I_E$ that is also used to describe pair production of black holes. The action denoted by $I_E$ in [4] and [27] is half of the total Euclidean action. Since we describe only one of the two black holes in the Ernst metric at the time, $I_E$ is indeed the right quantity in this context. In [27] it is shown this action has a representation in terms of the horizon areas:

$$I_E = -\frac{1}{4}(A_{\mathcal{H}} + \Delta A_{\mathcal{A}}).$$

Comparing this to (2.21) we see that $M = -\frac{1}{4}T\Delta A_{\mathcal{A}} = (8\pi)^{-1}\kappa_{\mathcal{A}}\Delta A$. This supports our definition of boost mass in (2.8).

The first law of thermodynamics can now be expressed

$$T\,\mathrm{d}I_E = \psi\,\mathrm{d}q - P\,\mathrm{d}B. \tag{2.22}$$

From the appendix in [30] we can deduce that

$$T = \frac{\sqrt{2}(1-Bq)^4}{2\pi\sqrt{1+(1-Bq)^8}} \quad \text{and} \quad I_E = \frac{4\pi(1-Bq)^2 q^2}{1-(1-Bq)^4}.$$

The chemical potential is given by

$$\psi = K(q,B)\Big(-2 + 6Bq - 4B^2q^2 + B^3q^3\Big),$$

And the magnetic polarisation is

$$P = K(q,B)\frac{q}{B}\Big(-2 + 4Bq - 6B^2q^2 + 4B^3q^3 + B^4q^4\Big)$$

where

$$K(q,B) := \frac{4\sqrt{2}(Bq-1)^5}{B(Bq-2)^2(B^2q^2+2-2Bq)^2\sqrt{1+(1-Bq)^8}}$$

One can verify that

$$(T\partial_B I_E + P)\,\mathrm{d}B - (T\partial_q I_E - \psi)\,\mathrm{d}q = 0,$$

which is another way of saying that the first law is indeed satisfied.

### Smarr relation

We end this section by verifying that also the Smarr relation is satisfied. It is given by

$$2M = 2TS + \psi q + PB. \tag{2.23}$$

We already checked the Smarr relation directly for the more general case in the previous section. However, the simplification made in this section allows us to perform a neat trick such that we can avoid tedious computations. In order to see how this works, we need the following lemma.

**Lemma 3.** *Let $f(x,y) = c_1(xy)$ and similarly let $g(x,y) = x^2 c_2(xy)$, where $c_{1,2}$ are arbitrary functions depending only on the product $xy$. Furthermore, define the differential operator*

$$\Delta := x\partial_x - y\partial_y.$$

*Then*

$$\Delta f = 0 \quad \text{and} \quad \Delta g = 2g.$$

**Lemma 4.** *The Smarr relation* (2.23) *is satisfied whenever* (2.22) *holds.*

*Proof.* The temperature is a function $T(q,B) = c_1(qB)$ and the Euclidean action is of the form $I_E(q,B) = q^2 c_2(qB)$, where $c_1$ and $c_2$ are determined by the previous section. Identifying $x = q$ and $y = B$ in the previous lemma, it follows that

$$\Delta T = 0 \quad \text{and} \quad \Delta I_E = 2I_E.$$

Hence,

$$\begin{aligned} 2(M - TS) &= 2TI_E \\ &= T\Delta(I_E) \\ &= qT\partial_q I_E - BT\partial_B I_E \\ &= \psi q + PB \end{aligned}$$

This is equivalent to the Smarr relation given in (2.23). □

## 2.6 Plots of the thermal equilibrium solution

It will be interesting to see how the various thermodynamic quantities behave as we vary the physical charge and magnetic field. Before we can do this in a meaningful way, we must first determine the range of $q$ and $B$. We will do this now.

The class of solutions that we study in this section is such that $G'(\zeta_2) = -G'(\zeta_3)$. This allows us to solve for $m$ in terms of $\widehat{q}$ and $A$. In the form of Hong and Teo, we find that

$$m = \sqrt{1 + A^2\widehat{q}^2}\widehat{q}.$$

After plugging this value for $m$ into the roots $\zeta_i$ in (1.24), the ordering $\zeta_1 < \zeta_2 < \zeta_3$ tells us that

$$0 < \widehat{q}A < \frac{1}{\sqrt{3}}. \tag{2.24}$$

We can determine $\widehat{q}A$ by the constraint in (2.18). This gives

$$\widehat{q}A = \frac{(1-u^4)}{\sqrt{(3+u^4)(3u^4+1)}},$$

where $u$ is defined as in (2.17). Plugging this into (2.24), we require that

$$0 < \frac{(1-u^4)}{\sqrt{(3+u^4)(3u^4+1)}} < \frac{1}{\sqrt{3}}.$$

And this tells us that

$$u \in (-1, 0) \cup (0, 1) \quad \text{or equivalently} \quad \widehat{q}\widehat{B} \in (0, 2) \cup (2, \infty).$$

Now we are able to determine the range of the product $qB$. Using that $qB = 1 - u$ from [4], we find that

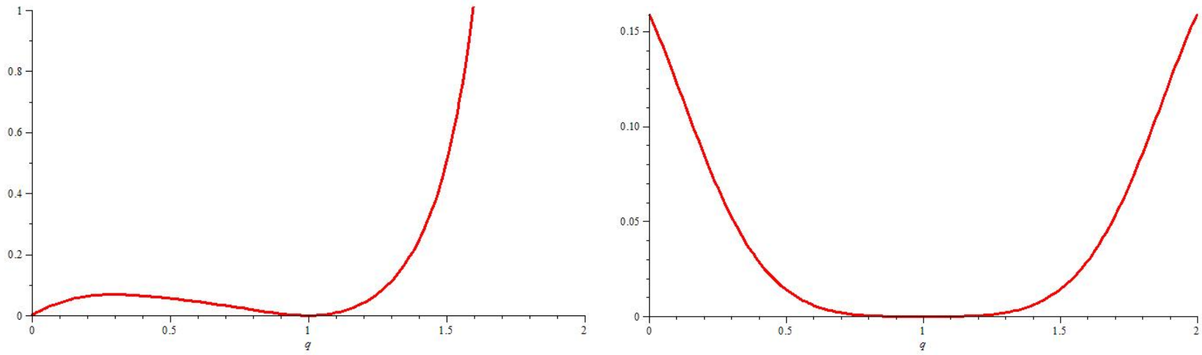
$$qB \in (0, 1) \cup (1, 2)$$

We will now fix the asymptotic magnetic field of the Melvin background at $B = 1$ such that we are able to plot $M$, $T$, $\psi$ and $P$ as function of $q$. The results are presented in Figure 2.3.

There are two branches of solutions, denoted by I and II given by $qB \in (0, 1)$ and $qB \in (1, 2)$ respectively. The two branches are separated by the critical point $qB = 1$, at which the acceleration and black hole horizon coincide. A black hole in either branch can thus never evolve to a black hole in the other branch by varying $q$ continuously. This also follows from the fact that $\partial q/\partial\widehat{q} \to 0$ as $qB \to 1$, which can be deduced from (2.19).

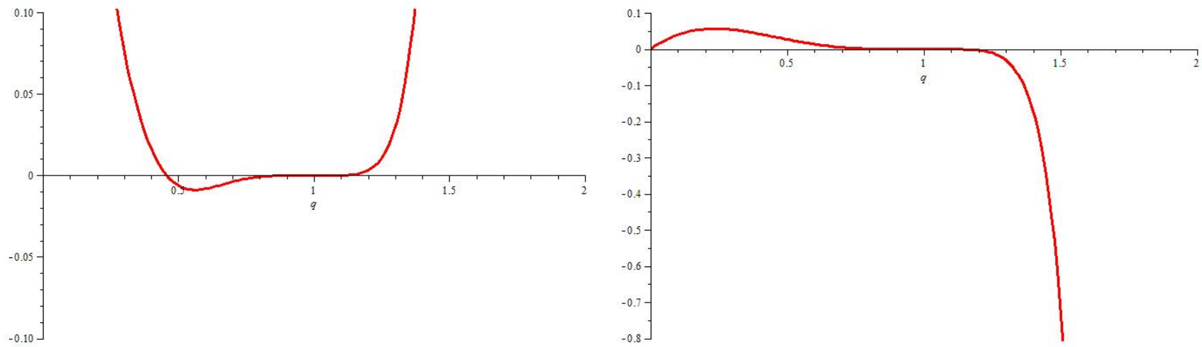
The temperature is symmetric about the critical point. Hence, for every allowed temperature, there exist two solutions: one in each branch. We end with some comments about the chemical potential.

- In branch I, the chemical potential $\psi$ has a sign flip. The chemical potential is the energy required to overcome the potential difference between the acceleration horizon and the black hole horizon. If it is positive, then one requires external energy to add a (positively charged) particle to the black hole. However, if it is negative, energy is released from the system upon adding a particle. In this latter case, the energy is extracted from the background magnetic field.
- In branch II, there is no sign flip.




(a) (b)

Figure 2.2: In both plots $B = 1$ and the point $q = 1$ is not part of the domain. (a) The boost mass $M$ as a function of $q$. (b) The temperature $T$ of the black hole, which is equal to the Unruh temperature.




(a) (b)

Figure 2.3: In both plots $B = 1$ and the point $q = 1$ is not part of the domain. (a) The chemical potential $\psi$ as a function of $q$. This is the potential difference between the black hole and acceleration horizons. (b) The magnetic polarisation $P$ as a function of $q$.

## 2.7 Entanglement interpretation

The boost mass has a representation in terms of the area of the acceleration horizon. Given that we identified the entropy of a black hole with the area of its horizon, a natural question is whether the area of the acceleration horizon also has an interpretation in terms of entropy. In this section we argue (following [31]) that this is the case: the area of the acceleration horizon is proportional to the entanglement entropy between the left and right Rindler wedges. This allows us to develop some intuition behind the boost mass formula in (2.7).

The structure of this section is as follows. First, we review some basic concepts including the density matrix formalism and entanglement entropy in quantum theory. Second, we prove a lemma from [31] relating the boost mass to entanglement entropy. Third, we apply this lemma to the Ernst spacetime.

## The density matrix formalism

In quantum mechanics, a pure state is represented by a state vector in a Hilbert space. For a pure state $|\Psi\rangle$, the density matrix is given by

$$\rho = |\Psi\rangle\langle\Psi|.$$

The expectation value of an operator $\mathcal{O}$ can then be written

$$\langle\mathcal{O}\rangle = \mathrm{Tr}(\rho\mathcal{O}).$$

This explains why $\rho$ is called the density matrix.

Generic quantum systems need to be pure. There also exist mixed states. A state is called a mixed state if it is a probablistic combination of pure states. The density matrix of a mixed state is given by

$$\rho = \sum_n p_n|\Psi_n\rangle\langle\Psi_n|,$$

where $p_m$ are amplitudes and $|\Psi_n\rangle$ are pure states. An important example of a mixed state is given by the thermal density matrix for the canonical ensemble

$$\rho = \frac{e^{-\beta\widehat{H}}}{\mathrm{Tr}(e^{-\beta\widehat{H}})},$$

where $\widehat{H}$ is the Hamiltonian operator.

Suppose that we have a pure density matrix $\rho$ and consider the decomposition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $A$ and $B$ denote two complementary regions of spacetime. Then one can define the reduced density matrices

$$\rho_A = \mathrm{Tr}_B\rho \quad \text{and} \quad \rho_B = \mathrm{Tr}_A\rho.$$

These density matrices are in general mixed. The entanglement entropy between regions $A$ and $B$ is defined

$$S_B := -\mathrm{Tr}_B(\rho_B \log\rho_B) = -\mathrm{Tr}_A(\rho_A \log\rho_A) =: S_A.$$

This is usually referred to as von Neumann entropy. It measures the information that went "missing" by tracing out a part of the Hilbert space.

### A nod to black hole entropy

A black hole is a region of space that an outside observer does not have access to. Consider again a decomposition of the Hilbert space by $\mathcal{H} = \mathcal{H}_\mathrm{A} \otimes \mathcal{H}_\mathrm{B}$, where $A$ and $B$ now denote the exterior and interior regions of the black hole. Then it makes sense to talk about entanglement entropy $S_A$ as the information "lost" behind the black hole horizon. Indeed, there have been made attempts to argue that the entropy of a black hole finds its origin in entanglement with its exterior (see [32], [33] and references therein). In fact, in the subsequent section we will show that this is true perturbatively for any bifurcation horizon.

### A first law of entanglement entropy

**Lemma 5.** *Let $L$ and $R$ be as before. Let $S$ be the entanglement entropy between $L$ and $R$ and let $\widehat{H}$ be the Rindler Hamiltonian (boost energy) of $R$. Then a change in $H = \langle \widehat{H} \rangle$ is given by*

$$\delta H = T \delta S. \tag{2.25}$$

*Proof.* Based on [33]. Consider a Minkowski vacuum $|0\rangle$. For the right Rindler observer, the reduced density matrix is given by

$$\rho_0 = \mathrm{Tr}_L |0\rangle\langle 0| = \frac{e^{-\beta \widehat{H}}}{Z}. \tag{2.26}$$

The entanglement entropy between $L$ and $R$ is then given by

$$S = -\mathrm{Tr}(\rho_0 \log \rho_0).$$

Now consider adding a particle by perturbing the density matrix by

$$\rho = \rho_0 + \delta\rho.$$

The change in entropy is

$$\delta S = -\mathrm{Tr}\big[\rho \log \rho - \rho_0 \log \rho_0\big].$$

Since we assume $\delta\rho$ small, we can approximate this change by a Taylor expansion around $\rho_0$. Then we find

$$\delta S \approx -\mathrm{Tr}\left[\frac{\delta}{\delta\rho}(\rho \log \rho)\delta\rho\right]_{\rho=\rho_0} = -\mathrm{Tr}\big[(1 + \log \rho_0)\delta\rho\big] = -\mathrm{Tr}\big[\delta\rho \log \rho_0\big],$$

where we used[3] $\mathrm{Tr}(\delta\rho) = 0$. Plugging in (2.26), we find

$$\begin{aligned} \delta S &\approx -\mathrm{Tr}\big[\delta\rho \log(e^{-\beta\widehat{H}}/Z)\big] \\ &= \mathrm{Tr}\big(\beta \widehat{H} \delta\rho\big) + \mathrm{Tr}\big(\delta\rho\big) \log Z \\ &= \beta \mathrm{Tr}\big(\widehat{H} \delta\rho\big) \\ &= \beta \delta H, \end{aligned}$$

where we used again $\mathrm{Tr}(\delta\rho) = 0$ to get rid of the term with $Z$. □

Equation (2.25) is a first law of entanglement entropy between the left and right Rindler wedges.

[3]This follows from the fact that both density matrices are of the form $\rho = \exp(-\beta\widehat{H})/Z$, where $Z = \mathrm{Tr}(\exp(-\beta\widehat{H}))$.

### 2.7.1 Application to the Ernst spacetime

A variation in the boost mass (2.7) while keeping the Unruh temperature $T$ constant gives

$$\delta M = T\frac{1}{4}\delta(\Delta A_{\mathcal{A}}).$$

Upon comparing this to (2.25), it follows that we can interpret $\Delta A_{\mathcal{A}}/4$ as the difference in entanglement entropy between $L$ and $R$ before and after adding a black hole. So, (2.7) implies a first law of the acceleration horizon,

$$\delta M = T\delta S_{\text{ent}}.$$

Note that, as opposed to the claim in the lemma, this result is also true non-perturbatively, as the tunneling process of black holes out of a magnetic field is non-perturbative.

A property of the entanglement entropy between the left and right Rindler wedges is that it is unchanged under unitary evolution in the Killing (Rindler) time. Since the black hole pair production process (the transition from the Melvin universe to the Ernst spacetime) is a unitary process, how can it then be that the entanglement entropy between $L$ and $R$ changes in the Ernst spacetime? The answer to this question is the following. In the Melvin universe there holds that $L$ is complementary to $R$ in $\Sigma$. In the Ernst spacetime, however, we have $L \cap R = \mathcal{B}$. So $L$ is not complementary to $R$. Indeed, we must compute the entanglement entropy between $L$ and $\Sigma \setminus L$. The difference between $R$ in the Melvin universe and $\Sigma \setminus L$ in the Ernst spacetime is precisely one black hole interior. This change of entanglement entropy give rise to a non-zero boost mass in the Ernst spacetime.

# Conclusions and outlook

The main result of this thesis is that a first law of black hole mechanics is satisfied in boost symmetric spacetimes. It differs from the usual first law in an extra contribution coming from a change in the energy of the ground state of an accelerated observer. The computation of thermodynamic variables is also different. For instance, in stationary spacetimes we usually take the potential difference between the black hole horizon and spatial infinity as the chemical potential conjugate to the charge. However, in the boost symmetric case, this chemical potential is defined by the potential difference between the black hole horizon and the acceleration horizon.

We also explicitly verified the first law for the Ernst metric. And we looked at a special class of Ernst metrics for which the black holes are at equilibrium with the Unruh radiation. The main result of this analysis is that the chemical potential may flip in sign as the charge varies. This is due to the background magnetic field, which for particular setups renders the potential difference between the black hole horizon and acceleration horizon negative.

Lastly, we briefly looked at an interpretation of the boost mass in terms of the difference of entanglement entropy between the left and right Rindler wedges before and after the black hole pair is created. We suggested that the different topology between the Melvin and Ernst spacetime plays a role in defining this difference. However, this suggestion is to be made more precise in future work.

The ideas presented in this thesis also apply to black holes in asymptotically de Sitter spacetimes, where instead of an acceleration horizon one has a cosmological horizon. A comment on this may be presented elsewhere.

For future work it would be interesting to study the thermodynamics of accelerated (and possibly rotating) black holes with an electric as well as magnetic charge. This may help in finding an intrinsic definition of the magnetic dipole moment and the magnetic polarisation. It would also be of interest to verify the thermodynamics of other boost symmetric solutions.

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