Comparing General Relativity with Shape Dynamics

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Master's Thesis

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Abstract

The aim of this thesis is to compare the strengths and weaknesses of the theories *shape dynamics* and *general relativity* using two well-known situations involving spherically symmetric gravitational fields; the pure vacuum solution and the gravitational collapse of a thin shell. Using the formal equivalence between the field equations in a particular spacetime foliation of general relativity we have succeeded in finding relevant solutions that permit a shape dynamics description. We subsequently provide an interpretation of these solutions in terms of the principles of shape dynamics. Although the two theories are locally equivalent at the level of the field equations, global or topological differences may yet arise in the classical theory, furthermore, the fact that the theories contain different local symmetries and corresponding constraint algebras may lead to different predictions after quantization.

Contents

1 Outline

This thesis is divided into seven chapters. Starting with chapter two we give a brief introduction to the ideas and motivation of shape dynamics (SD). In chapter three we use the Dirac/ADM Hamiltonian formulation of general relativity (GR) to find the static spherically symmetric vacuum solution to the EFE in a *maximal spacetime foliation*.[11] The maximal foliation requirement is a sufficient condition for the solution to admit a SD solution, we then give an interpretation according to the SD perspective. In chapter four we briefly show how the solution is related to the well-known Schwarzschild solution. In chapter five we give the equivalent solution using the better-known covariant formulation giving both the static exterior solution and the non-static interior solution whilst giving full expression to the fact that, in general relativity, one always has the freedom to choose coordinates at will. In chapter six we summarise the dynamics of a thin shell in the covariant formalism and present the solution for the motion of a shell made of dust with respect to both the shell's proper time and exterior time, which represents a maximal foliation and is required for a SD interpretation. Chapter seven describes the same physical situation in phase space and suggests a $(1 + 1)$ Hamiltonian that will determine the time evolution of the shell in a maximal foliation in SD. Chapter eight concludes the thesis including a critical reflection of SD.

Acknowledgements

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2 Introduction to Shape Dynamics

Before comparing the predictions of SD and GR in our chosen physical situations we need to introduce both theories. GR is covered in great detail in many good textbooks. [1–5] This chapter serves as a very brief introduction to only SD which is covered in more detail in the following papers. [6–11]

In 1898 Poincaré identified two fundamental issues relating to time: [12, 13] The definitions of *duration* and of *simultaneity at spatially separated points*. Special relativity (SR) was correctly understood once it was realised that the notion of simultaneity at spatially separated points is dependent on the motion of the observer. The generalisation of SR to incorporate gravitational phenomena led to GR and the realisation that inertial motion and gravitation were two aspects of the same phenomenon.

We find in both theories that a fundamental physical property is the *proper time*: The time measured by an *ideal* clock carried along a timelike curve between two points in spacetime. However, little importance is given to the choice of an adequate clock, moreover, they stand alone as self-sufficient entities, [14] such that their relation to other physical things in the theory is unclear.

In contrast, SD was created out of the desire to understand the nature of clocks both natural and man made. A particular goal was to highlight the manner in which duration, according to our experience, is derived from the motion of material bodies. It stresses that a crucial feature of adequate clocks is that they *march in step*. It is a physical assumption inherent in the structure of relativity that, between the same spacetime points, an ideal clock will always measure a time interval that is proportional to any other ideal clock, independent of the word line connecting the points and of the events on that world line. SD aspires to explain why this hypothesis is true in nature, so that we can have confidence that the proper time measured by ideal clocks is of physical importance. Clearly both GR and SD are concerned with the foundations of time, in the future we hope to understand how both problems of time may be solved within one coherent framework.

2.1 Motivation

As was the case for GR, an impetus for the creation of SD was the abandonment of the concept of absolute space and time, and the definition of a *relative velocity*. This was motivated by considering how motion is determined in experiment. The laws of both Newtonian mechanics (NM) and GR may be tested by observing the motion of celestial bodies against an assumed fixed background provided by the distant galaxies. In reality we acknowledge that, however small, the distant galaxies will exhibit relative motions and so the problem is then how to define displacement when nothing in the universe ever remains relatively motionless.

In NM, absolute space and time was postulated precisely to solve this problem. The equation

$$\mathbf{v} = \frac{\Delta \mathbf{x}}{\Delta t}, \tag{2.1}$$

presents no problems; because with an absolute spacetime we have a unique notion of *equilocality* and duration, that is, it is permissible to say that two spacetime points (events) are separated by a definite measure of time, and are or are not at the same

spatial location. Equation (2.1) simply states that in an absolute period of time Δt an absolute distance $\Delta \mathbf{x}$ was covered, motion is thereby definable and NM can be established via Newton's laws.
If we do not accept absolute space or time, then there are (at least) two ways to proceed: Acknowledge that the frame of reference is always relative to an observer, in terms of their coordinate system one may again define duration and displacement, and hence a relative velocity between any *two* observers, or define duration and displacement in a way that is intrinsic to, and includes *all* the observable bodies in the system, the former is implemented in GR and the latter in SD. In GR 4-velocities are compared at different spacetime points via parallel transport, the mathematics involved is well understood. In this chapter we will explain how SD defines its *holistic* relative velocity via a method called *best matching* (BM).

2.2 Best matching

The theory of SD and its method of BM begins with the concept of a instantaneous configuration of the universe. [7] Due to the finiteness of the speed of light only the part of this surface lying on the past light cone of some event can be observed, furthermore, the motion of the observer at this event will determine the particular configuration observed. We therefore consider the configurations as formal concepts that are useful in the mathematical description of SD, but recognise that they are not observable in practise.
In the field theoretic version of SD, the configuration space is obtained by quotienting larger spaces starting with the space of all possible Riemannian 3-metrics g_{ij} called *Riem*. The same 3-geometry may be represented by different 3-metrics which are related by 3-diffeomorphisms, identifying the configuration points that are related by 3-diffeomorphisms gives the quotient space called *Superspace*; the space of all distinct 3-geometries.
In contrast to GR, SD makes the radical postulate that local scale should not be considered as a physical observable, instead it may be understood as a gauge degree of freedom. This idea is motivated by the following epistemological argument; if it is assumed that all idealised measurements of length are local comparisons of the separations of bodies in the system, then both the separation representing the *ideal ruler* and the separation of the bodies being measured will be rescaled under a local scale transformation of the metric. The equivalent mathematical statement is that in SD the geometrical degrees of freedom (d.o.g) reside in the *conformal 3-geometry.* By quotienting superspace with respect to 3-conformal transformations,

$$\tilde{g}_{ij} = \omega^4 g_{ij}, \tag{2.2}$$

where $\omega = \omega(\mathbf{x})$ is a scalar function of position[1], we arrive at *conformal superspace* (CSS); the space of all distinct conformal 3-geometries. CSS is considered to be the irreducible configuration space of SD.
The method of BM then takes two distinct configurations in Riem, and transforms one configuration via 3-diffeomorphisms and conformal transformations to match as closely as possible the other configuration, thereby determining the difference in the configurations when represented in CSS. Hence, BM is a method of bringing distinct

[1] The fourth power is chosen for mathematical convenience.

configurations as close to *congruence* as possible, i.e. a principle of least incongruence. This method leads to a relative definition of displacement $\Delta\mathbf{x}$, notice that the process necessarily involves a comparison between all the bodies in the system, it is in this respect a holistic notion of displacement. We are half way to defining a holistic relative velocity; we have yet to define the duration Δt.

2.3 Duration from change

Suppose that we have best-matched a set of configurations to produce an ordered smooth curve in CSS. SD postulates that in nature the curve connecting the first and last configuration, in the best-matched order, will be an extremal curve, i.e. a *geodesic*. This is a generalisation of the geodesic principle in GR:[2] Free massive point particles traverse timelike geodesics. The difference lies in the choice of the configuration space, which in GR is a single free particle traversing a world line in spacetime, whereas in SD it is the successive shapes of a collection of observable bodies.

SD defines duration as a measure of difference between points in CSS representing different conformal 3-geometries. Notice that again this definition of Δt is holistic as it results from all observable change in the system. From this perspective we recognise that Newton's first law

$$\frac{d\mathbf{p}}{dt} = 0, \tag{2.3}$$

with $\mathbf{p} = m\mathbf{v}$ and its generalisation in GR; the geodesic equation

$$U^\alpha \nabla_\alpha U^\sigma = 0, \qquad U^\alpha = \frac{dx^\alpha}{ds}, \tag{2.4}$$

where s is an *affine* parameter, is only arrived at after consideration of all the ways in which the entire configuration of the system could change. This is the essence of *Mach's conjecture* [15], which we interpret as: The idea that local inertial motion arises through the interaction of physical fields in the universe.

What is left to do is to specify the laws by which this interaction may be described. We find that the postulates of SD laid out so far lead to equations of motion that are identical to a particular Hamiltonian formulation of GR,[3] called the *ADM formalism*, [18] and furthermore in a certain spacetime foliation called *constant mean curvature* (CMC) slicing. We can therefore appreciate that SD should admit far fewer solutions than standard GR satisfying the Einstein field equations (EFE).

2.4 ADM formalism

An early Hamiltonian formulation of GR was found by Dirac, [19] where the original 4-metric tensor with corresponding momenta were taken to be the fundamental phase space coordinates. The ADM formalism, however, by a change of variables, distinguishes between those that describe the 3-geometry and those that describe how the

[2]The Geodesic principle can be deduced from Einstein's field equations with the assumption that energy propagates along only timelike world lines, which represents a stronger form of the *dominant energy condition*.

[3]The equations of motion may also be derived from an action principle using the *BSW* action, which was first considered in the context of the *Thin-sandwich problem*. [2, 16, 17]

3-geometry changes from one spacelike surface to the next, the so called $3+1$ split,[4] on the other hand, extended spacelike surfaces are not a necessary ingredient in Dirac's formulation. A necessary requirement for the equivalence between the Hamiltonian formulations is that a canonical transformation relates the two set of variables, it has been shown that no such transformation exists, and consequently the two theories are not exactly equivalent. [20] The ADM formalism may, however, be understood as equivalent to Dirac's formalism on a reduced part of the phase space. [21]

2.5 ADM and SD phase space

The Hamiltonian formulation of SD is given on the phase space corresponding to CSS. In a particular conformal section, which means fixing the conformal factor of the 3-metric to be a particular function, it is formally equivalent to the ADM formulation with an additional constraint which, on the one hand is effecting conformal transformations in SD, but in the ADM formalism is enforcing the spatial 3-surface to have CMC. CMC foliations were first considered by Lichnerowicz, whose conformal techniques were later championed by York culminating in the solution to the *Initial value problem* (IVP) providing the inspiration for SD.[5] [22–24] York's solution to the IVP can be summarised as follows: On a CMC slice, give the conformal 3-geometry and its conjugate momenta as freely specifiable data to uniquely determine the whole generic 4-dim spacetime manifold. [2] SD turns this statement on its head and requires that the conformal 3-geometry and its conjugate momenta uniquely determine the spacetime, therefore concluding that a CMC slice must exist.
The quickest way to arrive at SD formalism is through a local symmetry trading algorithm; trading foliation invariance for spatial conformal invariance. This is achieved by enlarging the phase space of ADM to a *Linking theory* and then subsequently reducing the phase space by imposing a constraint (gauge-fixing) to arrive at SD, see Fig. 1 [8,11,25]. In this respect SD and ADM may be understood as two theories with different local symmetries and constraint algebras coexisting in the same phase space. Furthermore, they provide gauge-fixings of each other on the intersection (dictionary) of the surfaces, see Fig. 2. Finding the GR solution on the intersection allows one to find the equivalent solution in SD. This is the fastest way to arrive at a solution in SD and is the procedure followed in this thesis. In future work the solutions should be considered in a conformal section which does not lie on the intersection. According to SD such solutions are gauge-equivalent, however, they will necessarily not be a solution to the EFE since they do not lie on the ADM constraint surface.

[4]This restricts the topology of the spacetime manifold to be $\mathbb{R} \times \Sigma$, where Σ is a constant time slice, such that the spacetime is *globally hyperbolic*. This means that it can be sliced by non-intersecting space-like hypersurfaces labelled by a monotonic parameter.

[5]In particular the remark in [22] p. 1658 regarding the scalar or Hamiltonian constraint was an impetus for the search for a spatially conformally invariant theory which eventually led to SD.

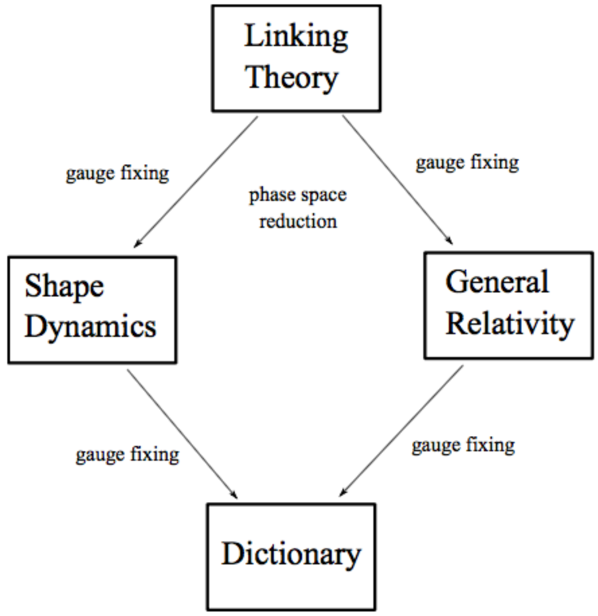

Figure 1: A schematic showing the relationship between ADM, SD and the Linking theory relating them. The dictionary corresponds to the CMC slicing of ADM and a particular conformal section in SD. This figure was taken from [11].

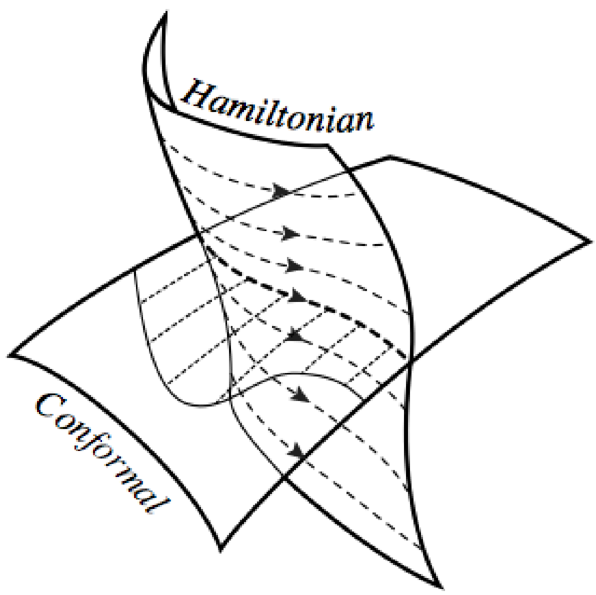

Figure 2: A schematic representation of the ADM and SD phase space. Two coexisting constraint surfaces, defined by the scalar (or Hamiltonian) constraint in ADM and the conformal constraint in SD, are good gauge-fixings of each other. Any solution on the intersection may be represented in an arbitrary conformal gauge by lifting it from the intersection to an arbitrary curve on the conformal constraint surface. This figure was taken from [7].

3 Spherically symmetric vacuum solution to Einstein's equations in the Hamiltonian formulation

SD is motivated by the desire to highlight the way in which local proper time may be derived from the dynamical gravitational degrees of freedom, which in SD are considered to be the conformally invariant spatial data, in short the shape. In addition it requires that, given the initial shape and corresponding momenta, the whole generic spacetime must be uniquely determined. In turn, this imposes the foliation of spacetime to be such that each spatial 3-space must have a spatially constant trace of extrinsic curvature.[6]

In GR proper time takes a primary physical status. In contrast, the shapes of matter in the universe are considered primary in SD. Furthermore, by assuming a choice of local scale, considered a gauge degree of freedom in SD, one would like to consider local proper time as an emergent property in SD. From the GR perspective, built upon the relativity of simultaneity, a preferred foliation seems to be a serious drawback of SD, the consolation prize might be a clearer understanding of how duration arises as measured by local proper time.

With a preferred foliation it is appropriate to present SD in its Hamiltonian formulation, if the Hamiltonian is conformally invariant then it will have especial importance in SD. As a first step in this direction, we find the spherically symmetric vacuum solution to the Einstein field equations (EFE) using the Hamiltonian formalism.[7] We make use of the ADM variables and the Hamiltonian gauge fixing procedure by imposing certain coordinate conditions dictated by shape dynamical considerations. Requiring that the coordinate conditions be preserved in time under the Hamiltonian evolution leads to the exterior solution (radii greater than the Schwarzschild radius) in *isotropic* coordinates that is static and asymptotically flat, as required by Birkhoff's theorem.

Notation: Greek indices range over the values $0, 1, 2, 3$ and latin indices over $1, 2, 3$. The coordinates t and $\mathbf{x}$ or (r, θ, ϕ) are assumed to be timelike and spacelike respectively. We use a spacetime metric $g_{\mu\nu}$ of signature $(-, +, +, +)$. Tensors such as the 3 dimensional Ricci tensor $R_{ij[g]}$ and its trace $R_{[g]}$, with the functional dependence on the relevant metric shown explicitly, are defined on a 3-dimensional spacelike hypersurface (3-space). Ordinary differentiation is denoted by a comma, covariant differentiation (g_{ij}-compatible) by a vertical line, covariant differentiation with respect to the flat metric γ_{ij} by a dot, and the time derivative by an over-dot. We employ units in which $c = 1 = 16\pi G$ (c the speed of light in vacuo and G the gravitational constant).

[6] The so called constant mean curvature slicing. SD therefore restricts the topology of the spacetime manifold to be $\mathbb{R} \times \Sigma$, where Σ is a constant time slice, such that it is *globally hyperbolic*.

[7] The time dependant and independent solutions have been found in the ADM formulation. [26, 27]

3.1 Primary Hamiltionian

The canonical theory begins with the following decomposition of the metric tensor:

$$(g_{\mu\nu}) = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ \\ N_i & g_{ij} \end{pmatrix},$$

$$(g^{\mu\nu}) = \begin{pmatrix} -N^{-2} & N^{-2}N^j \\ \\ N^{-2}N^i & h^{ij} - N^{-2}N^i N^j \end{pmatrix}, \tag{3.1}$$

with $N = (-g^{00})^{-1/2}$ the *lapse function*, $N^i = h^{ij}N_j = -g^{i0}/g^{00}$ the *shift vector field* and $g_{ij}h^{jk} = \delta_i^k$ defines the inverse metric in the 3-space.[8] The Lagrangian of GR is singular because its Hessian matrix is not invertible. In the canonical formalism the Hamiltonian function is therefore not uniquely defined, indeed any linear combination of *primary constraints* (PC), the vanishing of which define the PC surface, may be added to the Hamiltonian function to give the *primary Hamiltonian* (PH).

The PH that we start with is the ADM vacuum Hamiltonian plus the primary constraints,[9]

$$H = \int \mathcal{H}\ d^3x, \tag{3.2}$$

with

$$\mathcal{H} = N\,\mathcal{H}_\perp + N^j\,\mathcal{H}_j + \lambda_{\mu 0} p^{\mu 0}, \tag{3.3}$$

and

$$\mathcal{H}_\perp = g^{-1/2}(g_{ik}g_{jm} - \tfrac{1}{2}g_{ij}g_{km})p^{ij}p^{km} - g^{1/2}R_{[g]}, \tag{3.4}$$

$$\mathcal{H}_j = -2g_{ij}p^{ik}{}_{|k}, \tag{3.5}$$

where $\lambda_{\mu 0} = \lambda_{\mu 0}(x)$ are arbitrary functions and $p^{\mu\nu}$, $\mathcal{H}_\perp$, $\mathcal{H}_j$ and $\mathcal{H}$ are 3-densities of weight -1 under coordinate transformations in the constant time slice.

3.2 Vacuum canonical equations of motion and constraints

The fundamental Poisson brackets are

$$\left\{g_{\mu\nu}(x), p^{\alpha\beta}(x')\right\}_{t=t'} = \tfrac{1}{2}\left(\delta_\mu^\alpha\delta_\nu^\beta + \delta_\mu^\beta\delta_\nu^\alpha\right)\delta(\mathbf{x}-\mathbf{x}'). \tag{3.6}$$

Using (3.2)-(3.6), we find the following *Canonical equations of motion*, [28]

$$\dot{g}_{ij} = 2Ng^{-1/2}\left(p_{ij} - \tfrac{1}{2}pg_{ij}\right) + 2N_{(i|j)}, \tag{3.7}$$

$$\begin{aligned}\dot{p}^{ij} = &-Ng^{1/2}G^{ij} + \tfrac{1}{2}Ng^{-1/2}\left(p^{km}p_{km} - \tfrac{1}{2}p^2\right)h^{ij} - 2Ng^{-1/2}\left(p^i{}_k p^{kj} - \tfrac{1}{2}p\,p^{ij}\right) \\ &+ g^{1/2}\left(N^{|ij} - h^{ij}N_{|k}{}^k\right) + g^{1/2}\left(g^{-1/2}p^{ij}N^k\right)_{|k} - 2p^{k(i}N^{j)}{}_{|k},\end{aligned} \tag{3.8}$$

[8] The signature of the 4-metric is chosen such that $g^{00} < 0$.

[9] We postpone the discussion of spatial boundary terms until Sec. 3.6.

with $g \equiv \det g_{ij}$, $p \equiv g_{ij}p^{ij}$, $G^{ij} = R^{ij}{}_{[g]} - \frac{1}{2}R_{[g]}h^{ij}$ and $N_{(i|j)} \equiv \frac{1}{2}(N_{i|j} + N_{j|i})$. Furthermore, because the Lagrangian of GR is singular we have

$$\dot{g}_{\mu 0} = \lambda_{\mu 0}, \tag{3.9}$$

and four PC

$$p^{\mu 0} \approx 0, \tag{3.10}$$

where the ($\approx$) symbol is used to distinguish functions which vanish only on the PC surface (weakly zero), as opposed to those that vanish throughout all phase space (strongly zero).
Requiring $\dot{p}^{\mu 0} \approx 0$ leads to four *secondary constraints* (SC),

$$\mathcal{H}_\perp \approx 0, \tag{3.11}$$

$$\mathcal{H}_j \approx 0, \tag{3.12}$$

which will be referred to as the *scalar* and *vector* constraint respectively. The propagation of the SC in time give no new constraints.
The coefficients $\lambda_{\mu 0}$ in (3.9) represent an arbitrariness in the dynamical evolution because they are not determined by the EFE (3.7),(3.8),(3.11) and (3.12). In addition to this, there exists a redundancy in the initial conditions that are physically equivalent.[10] All the arbitrariness is due to the freedom in choosing a coordinate system.
The ADM formulation [18] represents a reduced part of the full phase space because it treats N, N^i as functions of g_{ij}, p^{ij} such that they are no longer independent phase space variables. In this case the primary constraints in (3.3), and equations (3.9) and (3.10) do not occur, in turn, (3.11) and (3.12) are considered as primary constraints. For the sake of generality we continue with the full phase space, nevertheless, it can be shown that the same solution may be found in the reduced ADM formalism.

3.3 Coordinate conditions

According to SD only the conformally invariant spatial degrees of freedom are considered objective and dynamical. We therefore make the following canonical transformations to separate these from the gauge degree of freedom residing in the conformal factor. [29]

$$\omega = \ln\left(\frac{g}{\gamma}\right)^{1/3}, \tag{3.13}$$

$$p = g_{ij}p^{ij}, \tag{3.14}$$

$$\tilde{g}_{ij} = \left(\frac{g}{\gamma}\right)^{-1/3} g_{ij}, \tag{3.15}$$

$$\tilde{p}^{ij} = \left(\frac{g}{\gamma}\right)^{1/3}\left(p^{ij} - \tfrac{1}{3}ph^{ij}\right), \tag{3.16}$$

where $\tilde{p}^{\,j}_j = 0$ and $\tilde{g} \equiv \det \tilde{g}_{ij} = \gamma$ are strong equations, and where for now γ_{ij} plays the role of an arbitrary reference metric. The variable ω represents the conformal

[10] More generally, physically equivalent points at a specific time. These points are related by transformations effected by the SC.

(gauge) degree of freedom and $\tilde{g}_{ij}$ the conformally invariant degrees of freedom in g_{ij} respectively. The Poisson brackets of the new variables derive from (3.6),

$$\begin{aligned}\{\omega(x), p(x')\}_{t=t'} &= \delta(\mathbf{x}-\mathbf{x}'),\\ \left\{\tilde{g}_{ij}(x), \tilde{p}^{km}(x')\right\}_{t=t'} &= \tilde{\delta}^{km}_{i\,j}\delta(\mathbf{x}-\mathbf{x}'),\\ \left\{\tilde{p}^{ij}(x), \tilde{p}^{km}(x')\right\}_{t=t'} &= \tfrac{1}{3}\left(\tilde{p}^{ij}\tilde{h}^{km}-\tilde{p}^{km}\tilde{h}^{ij}\right)\delta(\mathbf{x}-\mathbf{x}'),\end{aligned} \tag{3.17}$$

with

$$\tilde{\delta}^{km}_{i\,j} = \tfrac{1}{2}\left(\delta^k_i\delta^m_j + \delta^m_i\delta^k_j\right) - \tfrac{1}{3}\tilde{g}_{ij}\tilde{h}^{km}. \tag{3.18}$$

All other Poisson brackets are zero.
We proceed by introducing coordinate conditions which remove the redundancy in the initial conditions, allowing us to solve (3.11) and (3.12). By imposing that these conditions be propagated in time, we are able to determine the arbitrary coefficients in (3.9) and hence remove all arbitrariness in the system.
In SD we regard ω as a gauge degree of freedom, in the phase space formulation this is reflected in the (weakly) vanishing of its conjugate momentum,

$$p \approx 0, \tag{3.19}$$

which represents the additional constraint relating the phase space formulations of SD and ADM.[11] [30]
We specialise to spherically symmetric gravitational fields. In terms of our decomposition (3.1), the most general spherically symmetric 4-metric is

$$\begin{aligned} ds^2 &= -\left(N^2 - N_r N^r\right) dt^2 + 2A^4 N_r\, dr\, dt + A^4\, dr^2 + B^2\, d\Omega^2 \\ &= -N^2\, dt^2 + A^4\left(dr + N^r\, dt\right)^2 + B^2\, d\Omega^2, \end{aligned} \tag{3.20}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ and where all metric components are functions of (r,t) only. A common choice of spatial coordinates corresponds to choosing $B^2 \approx R^2$, we call these *areal* coordinates because the constant-R 2-space is intrinsically indistinguishable from an ordinary sphere of radius R and invariant area $4\pi R^2$. Yet another choice is $B^2 \approx r^2 A^4$, accordingly the spatial 3-metric is equal to the flat metric multiplied by a conformal factor

$$g_{ij} \approx A^4\gamma_{ij}, \qquad \gamma_{ij} = \mathrm{diag}(1, r^2, r^2\sin^2\theta), \tag{3.21}$$

taking the determinant we find

$$A^4 \approx \left(\frac{g}{\gamma}\right)^{1/3} = e^{\omega}, \qquad \gamma = r^4\sin^2\theta, \tag{3.22}$$

where we have used (3.13). In terms of the variables (3.13)-(3.16), the choice (3.21) takes on a particularly simple form because the conformally invariant degrees of freedom residing in (3.15) are completely fixed by (3.21)

$$\tilde{g}_{ij} \approx \gamma_{ij}, \tag{3.23}$$

[11] It is called the *maximal slicing* gauge, a subset of the CMC gauge where the trace of the extrinsic curvature of the spatial hypersurface, defined by t = constant, is zero. In the case where the 3-space is infinite, we consider the volume inside every finite domain.

leaving only the gauge degree of freedom ω, which represents the local scale. In accordance with (5.1), if spherical symmetry is required at all times we also have the following coordinate conditions that the shift vector field be radial

$$N_\theta = g_{t\theta} \approx 0, \qquad N_\phi = g_{t\phi} \approx 0, \tag{3.24}$$

which, together with (3.19) and (3.23), comprise 8 weak conditions.[12] We need to ensure that they are preserved in time, if their time derivatives are not automatically zero, they impose further constraints and the process repeats itself until no new constraints appear.

3.4 Propagating the constraint equations

According to (3.9), the preservation in time of (3.24) determines two of the arbitrary functions, $\lambda_{t\theta} \approx 0 \approx \lambda_{t\phi}$. Using (3.7) and (3.8) the preservation in time of the constraint $p \approx 0$ gives

$$\dot{p} \approx 2g^{1/2}\left(NR - N_{|j}{}^{j}\right) \approx 0. \tag{3.25}$$

The preservation in time of (3.23) gives

$$\dot{\tilde{g}}_{ij} - \dot{\gamma}_{ij} \approx 2\left(\frac{g}{\gamma}\right)^{-1/3}\left(Ng^{-1/2}p_{ij} + N_{(i|j)} - \tfrac{1}{3}g_{ij}h^{km}N_{(k|m)}\right) \approx 0, \tag{3.26}$$

where we have used (3.7), (3.15), (3.19) and (3.21). Therefore, we have the following relation between the spatial momenta and the shift vector

$$p_{ij} \approx N^{-1}g^{1/2}\left(N_{(i|j)} - \tfrac{1}{3}g_{ij}h^{km}N_{(k|m)}\right), \qquad N \neq 0, \tag{3.27}$$

such that our coordinate chart will cover the spacetime manifold everywhere except the surface $N = 0$. Substituting (3.27) into (3.12) yields an equation for the shift function

$$N_{i|j}{}^{j} - \tfrac{1}{3}N^{j}{}_{|ji} \approx 0. \tag{3.28}$$

In general ω, N^r and N will be time dependent, however, we choose to simplify our calculation by taking the trivial solution to (3.28) $N^r \approx 0$, we then find $\lambda_{\mu 0} \approx 0$ and $\dot{N} \approx 0$. We recognise that our simplification is equivalent to choosing a static spacetime, accordingly, from (3.27) we find that the spatial momenta must vanish weakly $p^{ij} \approx 0$. Equation (3.11) then simplifies to $R_{[g]} \approx R_{[A^4\gamma]} \approx 0$. Using (3.44) with $R_{[\gamma]} = 0$, we find that (3.11) reduces to the Laplace equation[13]

$$A_{.k}{}^{k} \approx 0. \tag{3.29}$$

Using (3.29) and (3.46), (3.25) yields

$$(AN)_{.k}{}^{k} \approx 0. \tag{3.30}$$

[12] Six conditions followed from demanding spherical symmetry at all times, and a further two due to the ambiguity associated with choosing two coordinates (r, t).

[13] Covariant differentiation with respect to γ_{ij} is denoted by a dot.

Lastly we require $\dot{p}^{ij} \approx 0$, substituting the equations $N^i \approx 0$, $p^{ij} \approx 0$, and (3.25) into (3.8) and setting to zero yields

$$N^{|ij} - NR^{ij} = N^{\cdot ij} - 2A^{-1}\left(A^{\cdot i}N^{\cdot j} + A^{\cdot j}N^{\cdot i} - A_{.k}N^{\cdot k}\gamma^{ij}\right) + 2NA^{-2}\left(AA^{\cdot ij} + A_{.k}A^{\cdot k}\gamma^{ij} - 3A^{\cdot i}A^{\cdot j}\right) \approx 0, \tag{3.31}$$

where we have used (3.43) with ${R^{ij}}_{[\gamma]} = 0$, (3.45) and (3.29) to obtain the first equality.

3.5 Solving the constraint equations

We have reduced our canonical equations of motion to $\dot{g}_{ij} = 0$ and $p^{ij} = 0$, giving a stationary spacetime with N and g_{ij} functions only of r. The most general solution satisfying our coordinate conditions (3.19), (3.23) and (3.24) may be obtained for a general function N^r. Our simplifying assumption $N^r = 0$ led to a static spacetime, in accordance with Birkhoff's theorem.
There are no new constraints to be found, we can now treat all our constraints as strong equations and subsequently solve them. The general solution to (3.29) is $A = b + a/r$ for $r \neq 0$, with a and b constant. From (3.30) we find $N = (d + c/r)/A$, where c and d are two more constants. Since we have solved two second order equations, we are left with four integration constants. However, a temporal and radial coordinate rescaling removes the constants b and d, the solutions then take the following form,

$$A = 1 + \frac{a}{r}, \qquad N = \frac{1 + c/r}{1 + a/r}. \tag{3.32}$$

Substituting (3.32) into (3.31), and taking either the $r\,r, \theta\,\theta$ or $\phi\,\phi$ component, we find that $c = -a$ yielding

$$N = \frac{1 - a/r}{1 + a/r}, \tag{3.33}$$

the requirement $N \neq 0$ gives $r \neq a$. The remaining constant, a, may be interpreted by going to the weak-field limit, which leads us to consider the spatial boundary terms as the radial coordinate tends to infinity.

3.6 On-shell Hamiltonian

When the vacuum field equations are satisfied, the value of the primary Hamiltonian (3.2) is zero due to the constraints. Only spatial boundary terms contribute to the on-shell Hamiltonian E, [31]

$$E = 2 \oint_{S(r,t)} \left(g^{-1/2}N_i p^{ij} r_j - N(k - k_0)\right) \sigma^{1/2}\, d\theta\, d\phi, \tag{3.34}$$

where $\sigma = \det \sigma_{ab}$, $a, b = 1, 2$, is the induced metric on the closed 2-surface $S(r,t)$ which forms the boundary of the 3-space. The unit normal to $S(r,t)$ is the spacelike vector r_j. k is the extrinsic curvature of $S(r,t)$ embedded in 3-space, and k_0 is the extrinsic curvature of a 2-surface of identical intrinsic geometry, but embedded in flat

3-space.[14] For stationary spacetimes $p^{ij} = 0$ therefore in the spherically symmetric case only the second term in (3.34) contributes.
The extrinsic curvature is defined as $k = r^j{}_{|j}$, and in our case the unit normal is given by $r_j = |g_{rr}|^{1/2} r_{,j} = (1 + a/r)^2\, r_{,j}$, we find

$$k = r^j{}_{|j} = g^{-1/2}\left(g^{1/2}(1+a/r)^{-2}\right)_{,r}$$
$$= \frac{2}{r}\left(1+\frac{a}{r}\right)^{-2} - \frac{4a}{r^2}\left(1+\frac{a}{r}\right)^{-3}.$$

In order to calculate k_0 we need to know how the radial coordinate in the curved space is related to that used for the flat space. By comparing coefficients of $d\Omega^2$ in the curved and flat metrics we find $r_0^2 = r^2(1 + a/r)^4$, k_0 is therefore given by

$$k_0 = g_0^{-1/2} g_0^{1/2}{}_{,r} = \frac{2}{r_0} = \frac{2}{r}\left(1+\frac{a}{r}\right)^{-2}.$$

The difference in the extrinsic curvatures is then

$$k - k_0 = -\frac{4a}{r^2}\left(1+\frac{a}{r}\right)^{-3}. \tag{3.35}$$

The induced 2-metric in our case is

$$\sigma_{ab} = \left(1+\frac{a}{r}\right)^4 \operatorname{diag}(r^2, r^2\sin^2\theta),$$

with determinant

$$\sigma = \left(1+\frac{a}{r}\right)^8 r^4 \sin^2\theta. \tag{3.36}$$

Substituting (3.33), (3.35) and (3.36) into (3.34) we find

$$E = 8a\left(1-\frac{a}{r}\right)\oint_{S(r,t)} \sin\theta\, d\theta\, d\phi$$
$$= 32\pi a\left(1-\frac{a}{r}\right) = 2a - \frac{2a^2}{r}. \tag{3.37}$$

In the last equality we are reminded that we employ units in which $16\pi G = 1$. When the Einstein field equations are satisfied, equation (3.37) represents the total energy of the system. Accordingly, in the limit $r \to \infty$ we require that it be equal to the *gravitational mass* m by choosing the constant $a = m/2$,

$$E = m - \frac{m^2}{2r}. \tag{3.38}$$

We recognise the second term as the energy contribution of the gravitational field, which has the form of (half) the Newtonian gravitational potential energy.[15] The 4-metric is then

$$ds^2 = -\left(\frac{1-m/2r}{1+m/2r}\right)^2 dt^2 + (1+m/2r)^4\left(dr^2 + r^2 d\Omega^2\right), \tag{3.39}$$

[14] The k_0 term makes the Hamiltonian finite for non-compact asymptotically flat spacetimes, furthermore for Minkowski spacetime it makes the Hamiltonian zero.

[15] It differs in two respects: The radial coordinate employed here is the isotropic coordinate, not the areal coordinate, however, in the asymptotic region the coordinate values differ by only the constant m. The mass is the gravitational mass, not the Newtonian mass, again however, in the weak field limit, these quantities will agree.

which is the *isotropic line element*. [32] Both (3.38) and (3.39) appear to superficially diverge as $r \to 0$. However, we may also notice that (3.39) is in fact invariant under the following inversion of the radial coordinate

$$r = \frac{m^2}{4r'}. \tag{3.40}$$

The region $r \to 0$ is therefore isometric to another asymptotically flat region using the coordinate r', indeed, it leaves invariant

$$R = r\left(1 + \frac{m}{2r}\right)^2, \tag{3.41}$$

where R is the Schwarzschild radial coordinate of the areal coordinate system. The transformation (3.40) also features in electrostatics using the *method of images*,[16] applied to a sphere of radius $r_s = m/2$ equal to the *Schwarzschild radius* in isotropic coordinates. [33]

The manifold described by (3.39) covers only the exterior left $r < m/2$ and right $r > m/2$ regions of the Kruskal diagram corresponding to $R > 2m$. [34, 35] Indeed, the foliation of spacetime with Schwarzschild time coordinate does not penetrate under the event horizon located at the Schwarzschild radius $R = 2m$ in areal coordinates. We restricted our solution to the exterior regions by simultaneously imposing (3.19) and $N^r \approx 0$ leading to a stationary solution valid for only the region $R > 2m$.[17] In contrast, the interior region $R < 2m$ is non-stationary and spatially homogeneous (independent of the radial coordinate); it is well known that the Killing vector field additional to the three related to spherical symmetry is timelike in the exterior and spacelike in the interior.

3.7 Shape dynamics interpretation

3.7.1 The conformal constraint

According to the standard spacetime interpretation, the gauge fixings $g_{i0} \approx 0$, $g_{ij} \approx e^{\omega}\gamma_{ij}$ are equivalent to choosing a spacetime which is diagonal and foliated by 3-spaces which are manifestly conformally flat. Owing to its contracted tensorial character we notice that our third condition, $p \approx 0$, which was required to enforce that ω be a gauge degree of freedom, does not restrict the coordinates in the 3-space, indeed, it corresponds to the requirement that the volume of every 3-space be stationary under timelike deformations. It is therefore a time coordinate condition and determines how spacetime is to be divided into space and time.

In contrast, SD interprets this last constraint as a SC which generates conformal transformations in the 3-space by transforming the variable conjugate to p, i.e. ω, considered to be a gauge degree of freedom. Equation (3.11) determines the conformal factor ω and accordingly, if SD assumes that the physical dynamics reside on the intersection

[16] Where the interaction between an electric charge at a distance r from the centre of a conducting sphere of radius $R < r$ is equivalent to that of a charge of equal magnitude but opposite sign at a distance R^2/r.

[17] There exist other maximal foliations which are related to (3.39) by a spacetime diffeomorphism that preserves (3.19), representing a different congruence of observers who are accelerated relative to the static observers. [36]. The 4-metric in [36] uses a different time coordinate, has a non zero shift, and a lapse that vanishes at $R = 3m/2$.

where both (3.11) and (3.19) hold, then (3.11) may reasonably be considered a gauge fixing of (3.19).
The scalar and conformal constraints have therefore swapped roles; the generator of a symmetry reinterpreted as a constraint which gauge fixes a different symmetry and vice versa. The two symmetries in question are spacetime refoliation invariance, assumed in GR, and 3-dim conformal invariance, assumed in SD.

3.7.2 Duration from changing shapes

Our aim was to develop a theory which highlighted the manner in which duration, as measured by proper time, emerged as a secondary construct from the underlying conformal degrees of freedom. However, for $N \neq 0$, we were able to fix the conformal degrees of freedom by choosing a conformally flat coordinate system. This is easily understandable; in SD the objective dynamical degrees of freedom are the evolving shapes of matter in the universe, by imposing spherical symmetry there is now only ever one shape allowed, the sphere.
Furthermore, according to (3.28) $N^r \approx 0$ is an admissible solution yielding a static spacetime for $R > 2m$, such that there is no evolution on the SD phase space. In this case the remaining non-zero variables are N and ω, which in GR represent physical gravitational degrees of freedom. On the other hand, in SD, ω is considered to be a gauge degree of freedom fixed by (3.11) representing a choice of local scale. This postulate was imposed mathematically in (3.19), the propagation of this led to (3.25) which, when combined with (3.23) represents a relation between N and ω.
The lapse function relates the coordinate time t to the proper time τ as measured by an *Eulerian observer*; whose 4-velocity is equal to the unit timelike vector $\mathbf{n}$ of the spacelike 3-space Σ, [37]

$$\delta\tau = N\delta t, \qquad N > 0. \tag{3.42}$$

The coordinate time also has a physical interpretation, it is equal to the proper time of an observer with $N = 1$. For our calculation, the particular coordinate time found was the Schwarzschild time; physically equivalent to the proper time of an observer with (r, θ, ϕ) constant and $r \to \infty$. Therefore, (3.42) relates the proper time measured by Eulerian observers at any r to those with $r \to \infty$.
The spherically symmetric static solution in SD is considered physically trivial due to there being no local gravitational degrees of freedom. Equation (3.30) fixes local duration in terms of the local scale, considered a gauge degree of freedom. The local scale is determined by the constant a, which in turn is given by the boundary condition that the total energy be equal to the Schwarzschild mass, see (3.38). Therefore, the boundary conditions determine the local scale and thence duration.
Asymptotic boundary conditions allow the system to be considered as an *isolated system*, but ultimately we must recognise that, when gravitational phenomena are concerned, there are no truly isolated systems, even though there are many approximately isolated systems. In a more physically realistic situation the boundary conditions may be understood as the information contained in the relationship between the system under consideration and the rest of the universe; at infinity they give rise to a coordinate system, with respect to which local scale and duration may be defined. In this respect we start to understand how local physics may be determined by the universe at large, which is the central holistic message at the heart of *Mach's principle*. [38]

3.8 Identities

Transformation identities of the Ricci tensor, Ricci scalar, second covariant derivative and its contraction under the transformation of the 3-metric $g_{ij} = A^4\gamma_{ij}$. Covariant differentiation with respect to γ_{ij} is denoted by a dot.

$$R_{ij\,[A^4\gamma]} = R_{ij\,[\gamma]} - A^{-2}(A^2{}_{.ij} + \gamma_{ij}A^2{}_{.k}{}^{k} - 8A_{.i}A_{.j}), \tag{3.43}$$

$$R_{[A^4\gamma]} = A^{-4}(R_{[\gamma]} - 8A^{-1}A_{.j}{}^{j}), \tag{3.44}$$

$$N_{|ij} = N_{.ij} - 2A^{-1}\left(A_{.i}N_{.j} + A_{.j}N_{.i} - \gamma_{ij}A_{.k}N^{.k}\right), \tag{3.45}$$

$$N_{|j}{}^{j} = A^{-5}\left(AN_{.j}{}^{j} + 2A_{.j}N^{.j}\right). \tag{3.46}$$

4 The relation between the Schwarzschild metric and the Isotropic line element

In the last chapter we found (3.39), representing the spherically symmetric solution to the vaccum EFE. Due to Birkhoff's theorem we know that this must be locally equivalent to the Schwarzschild solution, in this brief chapter we elucidate the relationship between the two solutions.

In flat space, r has the standard interpretation of distance, however, generalising to curved spaces that are spherically symmetric it is best to define r via the equation $A = 4\pi r^2$, where A is the surface area of a sphere that may be measured by observers on its surface. Hence we refer to (t, r, θ, ϕ) as *areal* coordinates, where (t, r) constant describes a 2-space with invariant area A, or alternatively as *curvature* coordinates since the 2-space has an intrinsic Gaussian curvature $2/r^2$. r is related to the isotropic radial coordinate ρ by the following transformation

$$r = \rho\left(1 + \frac{m}{2\rho}\right)^2, \tag{4.1}$$

with m the *gravitational mass*. The transformation (4.1) is not injective because for every $r > 2m$, outside the horizon, there are two distinct values of ρ that are mapped to it,

$$\rho_\pm = \frac{r-m}{2} \pm \frac{r}{2}\sqrt{1 - \frac{2m}{r}}, \tag{4.2}$$

from which we find

$$\rho_+ = \frac{m^2}{4\rho_-}. \tag{4.3}$$

This inversion maps r to itself, it is an isometry of the *isotropic line element*,

$$ds^2 = -\left(\frac{1 - m/2\rho}{1 + m/2\rho}\right)^2 dt^2 + (1 + m/2\rho)^4 \left(d\rho^2 + \rho^2 d\Omega^2\right), \tag{4.4}$$

which is the spherically symmetric vacuum solution of the Einstein field equations using isotropic radial coordinate ρ. It is therefore locally isometric to the Schwarzschild solution.

The coordinates $\rho_\pm$ cover two geometrically identical exterior regions where $r > 2m$,[18] but neither one covers the interior region where $r < 2m$, hence (4.1) is also not surjective, see Figs.3 and 4.

Because (4.1) is not a bijection, the coordinate charts defined using r and ρ seem to give rise to manifolds that are not everywhere diffeomorphic to each other. However, it is well known that they cover different regions of the *maximally extended* Schwarzschild manifold which may be globally covered, for example by the *Kruskal-Szekeres* coordinates. [34, 35]

[18] These regions are often referred to as *mirror* images of each other, but this is not correct as *parity* transformations involve the angular coordinates θ, ϕ and therefore are not related to (4.3). [39]

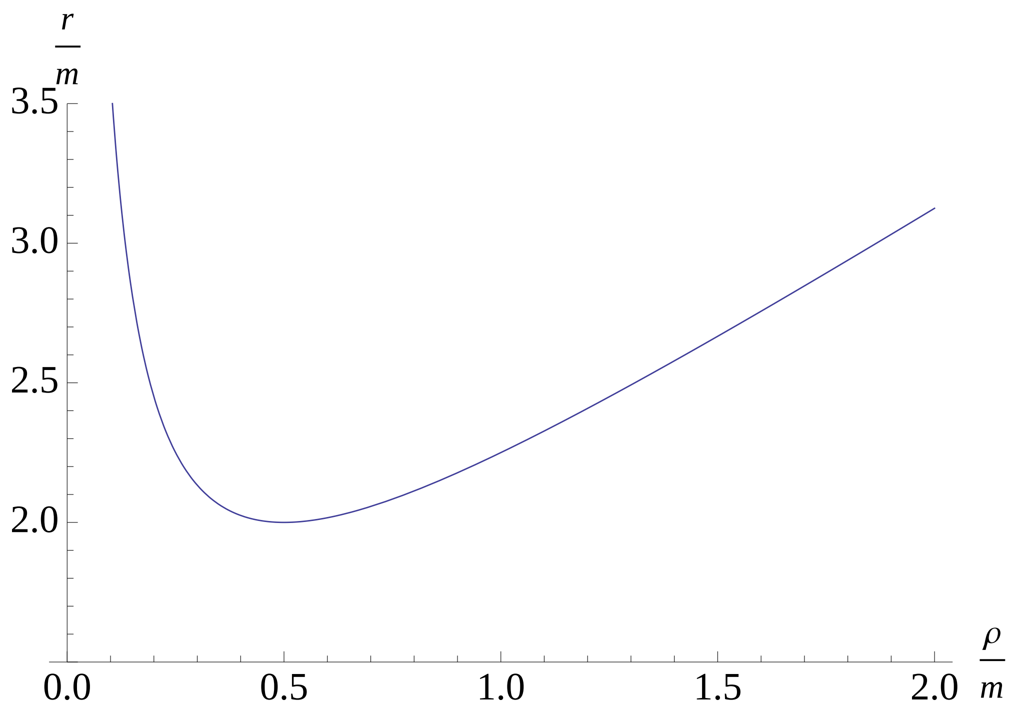

Figure 3: The coordinate mapping from the isotropic radial coordinate ρ to the Schwarzschild radial coordinate r, with m the gravitational mass.

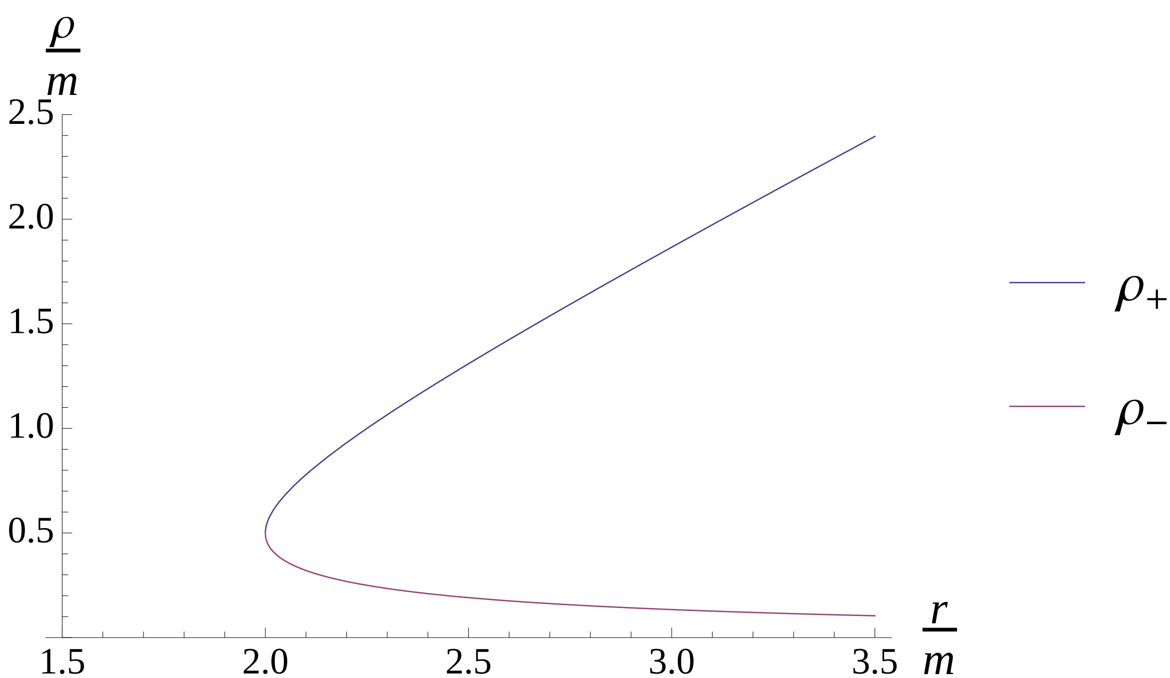

Figure 4: The inverse coordinate mapping from the Schwarzschild radial coordinate r to the the isotropic radial coordinates $\rho_{\pm}$, with m the gravitational mass.

5 Spherically symmetric vacuum solution to Einstein's equations in the covariant formulation

In the last two chapters we derived the Isotropic line element (3.39), showed precisely how it is related to the Schwarzschild solution and explained which part of the maximally extended manifold it covers. In this chapter we solve the same problem in the covariant formalism to make clear the relation between the exterior static solution and the interior solution. We also show explicitly that the EFE only fix the coordinate independent spacetime geometry, as is well known. This is the reason why one is free to choose either the isotropic radial coordinate or the Schwarzschild areal coordinate. The time-independent (exterior) solution to the EFE for a spherically symmetric gravitational field was first found by Schwarzschild and then subsequently by Droste, Hilbert and Weyl. [32,40–43] Later on it was generalised to the time dependent case by [44–47], now known as Birkhoff's theorem: The Schwarzschild metric is the unique vacuum solution with spherical symmetry.
Notation: We assume t to be timelike coordinate and r, θ, ϕ to be spacelike coordinates. We denote ordinary differentiation by a comma and covariant differentiation ($g_{\mu\nu}$-compatible) by a vertical line. In this chapter we deliberately do not set the constants c and G to unity.

5.1 The spherically symmetric EFE in vacuo

The most general spherically symmetric 4-metric of signature $(-, +, +, +)$ is

$$ds^2 = -e^{\gamma}\, dt^2 + e^{\alpha}\, dr^2 + e^{\beta}\, d\Omega^2, \qquad d\tau = -c\, ds, \tag{5.1}$$

where τ is the proper time and $d\Omega^2$ is the metric on a unit two-sphere,

$$d\Omega^2 = d\theta^2 + \sin^2\, d\phi^2. \tag{5.2}$$

The ranges of the coordinates are assumed to be

$$-\infty < t < \infty, \qquad 0 \leqslant r < \infty, \qquad 0 \leqslant \theta \leqslant \pi, \qquad 0 \leqslant \phi < 2\pi. \tag{5.3}$$

We assume the EFE with zero cosmological constant $\Lambda = 0$,

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R^{\sigma}_{\sigma} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{5.4}$$

where $G_{\mu\nu}$ is the Einstein tensor. In vacuo $T_{\mu\nu} = 0$, hence we start by setting the five non-zero components of the $(1,1)$ Einstein tensor G^{μ}_{ν} to zero.[19]

$$G^t_t = e^{-\gamma}\left(\frac{\beta_{,t}^{\,2}}{4} + \frac{\alpha_{,t}\beta_{,t}}{2}\right) + e^{-\alpha}\left(-\beta_{,rr} - \frac{3\beta_{,r}^{\,2}}{4} + \frac{\alpha_{,r}\beta_{,r}}{2}\right) + e^{-\beta} = 0, \tag{5.5}$$

$$G^r_r = e^{-\alpha}\left(-\frac{\beta_{,r}^{\,2}}{4} - \frac{\beta_{,r}\gamma_{,r}}{2}\right) + e^{-\gamma}\left(\beta_{,tt} + \frac{3\beta_{,t}^{\,2}}{4} - \frac{\beta_{,t}\gamma_{,t}}{2}\right) + e^{-\beta} = 0, \tag{5.6}$$

$$\begin{aligned} G^\theta_\theta = G^\phi_\phi = e^{-\alpha}&\left(-\frac{\beta_{,rr}}{2} - \frac{\beta_{,r}^{\,2}}{4} - \frac{\gamma_{,rr}}{2} - \frac{\gamma_{,r}^{\,2}}{4} - \frac{\beta_{,r}\gamma_{,r}}{4} + \frac{\alpha_{,r}\beta_{,r}}{4} + \frac{\alpha_{,r}\gamma_{,r}}{4}\right) \\ &+ e^{-\gamma}\left(\frac{\beta_{,tt}}{2} + \frac{\beta_{,t}^{\,2}}{4} + \frac{\alpha_{,44}}{2} + \frac{\alpha_{,t}^{\,2}}{4} + \frac{\beta_{,t}\alpha_{,t}}{4} - \frac{\gamma_{,t}\beta_{,t}}{4} - \frac{\alpha_{,t}\gamma_{,t}}{4}\right) = 0, \end{aligned} \tag{5.7}$$

$$e^{\alpha}G^r_t = -e^{-\gamma}G^t_r = \beta_{,rt} + \frac{\beta_{,r}\beta_{,t}}{2} - \frac{\beta_{,r}\alpha_{,t}}{2} - \frac{\beta_{,t}\gamma_{,r}}{2} = 0. \tag{5.8}$$

The local solution to eqs. (5.5) to (5.8) depends on whether the hyper surfaces $\beta =$ constant are timelike or spacelike, the lightlike case being the limiting case where the two solutions coincide.[20]

5.2 Timelike case $\beta = \beta(r)$

First we consider the timelike case, $\beta = \beta(r)$, (5.8) then gives $\alpha_{,t} = 0$. Changing variables to $R = e^{\beta/2}$ the remaining EFE become

$$G^t_t = e^{-\alpha}\left(-\frac{2R_{,rr}}{R} - \frac{R_{,r}^{\,2}}{R^2} + \frac{\alpha_{,r}R_{,r}}{R}\right) + \frac{1}{R^2} = 0, \tag{5.9}$$

$$G^r_r = e^{-\alpha}\left(-\frac{R_{,r}^{\,2}}{R^2} - \frac{\gamma_{,r}R_{,r}}{R}\right) + \frac{1}{R^2} = 0, \tag{5.10}$$

$$G^\theta_\theta = G^\phi_\phi = e^{-\alpha}\left(-\frac{R_{,rr}}{R} - \frac{\gamma_{,rr}}{2} - \frac{\gamma_{,r}^{\,2}}{4} + \frac{R_{,r}}{2R}(\alpha_{,r} - \gamma_{,r}) + \frac{\alpha_{,r}\gamma_{,r}}{4}\right) = 0. \tag{5.11}$$

We will show that only two of these equations are independent, the third being a consequence of the other two. Subtracting (5.10) from (5.9) we find

$$\alpha_{,r} + \gamma_{,r} = \frac{2R_{,rr}}{R_{,r}}. \tag{5.12}$$

Taking the time derivative of (5.12) gives

$$\gamma_{,rt} = 0, \qquad \gamma(r,t) = f(r) + g(t), \tag{5.13}$$

such that (5.12) becomes

$$\alpha_{,r} + f_{,r} = \frac{2R_{,rr}}{R_{,r}}. \tag{5.14}$$

[19]See [1] p. 272.
[20]See [48] p. 370

Substituting (5.14) into (5.11) yields

$$f_{,rr} + f_{,r}^2 = f_{,r}\left(\frac{R_{,rr}}{R_{,r}} - \frac{2R_{,r}}{R}\right). \tag{5.15}$$

Substituting (5.12) into (5.10) gives

$$\begin{aligned} R_{,r} &= e^{-\alpha}\left(-\alpha_{,r} R R_{,r}^2 + R_{,r}^3 + 2RR_{,r}R_{,rr}\right) \\ &= \left(e^{-\alpha} R R_{,r}^2\right)_{,r}, \end{aligned} \tag{5.16}$$

which we may integrate to give

$$e^{\alpha} = R_{,r}^2 \left(1 - \frac{c_1}{R}\right)^{-1}, \tag{5.17}$$

for $R > |c_1|$ where c_1 is a constant in spacetime. By integrating (5.14) we find

$$e^{\alpha+f} = c_2 R_{,r}^2, \tag{5.18}$$

where $c_2 > 0$ is another constant in spacetime. Substituting (5.49) into (5.18) yields

$$e^{f} = c_2\left(1 - \frac{c_1}{R}\right), \tag{5.19}$$

valid for $R > |c_1|$. Let us take stock of what we have found so far. Our three equations are now given by (5.15), representing a differential equation between γ and R, (5.49), a differential equation between α and R, and (5.19) an algebraic relation between f and R. We may differentiate (5.19) once and twice yielding

$$f_{,r} e^{f} = \frac{c_1 c_2 R_{,r}}{R^2}, \tag{5.20}$$

$$e^{f}\left(f_{,rr} + f_{,r}^2\right) = c_1 c_2 \left(\frac{R_{,rr}}{R^2} - \frac{2R_{,r}^2}{R^3}\right). \tag{5.21}$$

Dividing (5.21) by (5.20) we find

$$f_{,rr} + f_{,r}^2 = f_{,r}\left(\frac{R_{,rr}}{R_{,r}} - \frac{2R_{,r}}{R}\right), \tag{5.22}$$

which is precisely (5.15), showing that it is not an independent equation, but is a consequence of (5.19). This, in turn, is because the Einstein tensor satisfies the twice-contracted *Bianchi identities*,

$$G^{\mu\nu}{}_{|\nu} = 0, \tag{5.23}$$

representing four differential identities which reduce the number of independent EFE to only six. Using (5.13) we have

$$e^{\gamma} = c_2\left(1 - \frac{c_1}{R}\right) e^{g}, \tag{5.24}$$

hence the general solution that we have found is

$$ds^2 = -c_2\left(1 - \frac{c_1}{R}\right) e^{g}\, dt^2 + \left(1 - \frac{c_1}{R}\right)^{-1} R_{,r}^2\, dr^2 + R^2\, d\Omega^2, \tag{5.25}$$

for $R > |c_1|$. This solution was shown independently by Combridge and Jannes [49–51] for the time independent case $g_{,t} = 0$. Since $R = R(r)$ and $g = g(t)$ represent two undetermined functions, the spherically symmetric vacuum EFE represent an undetermined system. This is appropriate since the EFE fix only the coordinate-independent degrees of freedom, whereas the function $R(r)$ and $g(t)$ must be determined by the choice of radial coordinate r and time coordinate t. A particularly simple choice for r is the function R itself. Using the chain rule of differentiation we have

$$R_{,r}\, dr = \frac{dR}{dr} dr = dR, \tag{5.26}$$

and we may define a new time coordinate t'

$$t' = \int e^{g/2}\, dt, \tag{5.27}$$

relabelling $t' \to t$ and $R \to r$ (5.25) becomes

$$ds^2 = -c_2 \left(1 - \frac{c_1}{r}\right) dt^2 + \left(1 - \frac{c_1}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2, \tag{5.28}$$

where $|c_1| < r < \infty$.

5.2.1 Boundary conditions

Since we have solved a system of differential equations, the highest order of which was second order, we have in our general solution (5.28) two integration constants. The first, c_1, has dimensions of length, the second, c_2, has dimension velocity squared. In order to determine the constants we require two boundary conditions. First consider the radial lightlike paths, hence we set $d\theta = 0$, $d\phi = 0$ and $ds^2 = 0$ in (5.28),

$$- c_2 \left(1 - \frac{c_1}{r}\right) dt^2 + \left(1 - \frac{c_1}{r}\right)^{-1} dr^2 = 0; \tag{5.29}$$

which gives

$$\left(\frac{dr}{dt}\right)^2 = c_2 \left(1 - \frac{c_1}{r}\right)^2. \tag{5.30}$$

Our first boundary condition is to impose that in the limit

$$\frac{c_1}{r} \to 0, \qquad \left(\frac{dr}{dt}\right)^2 \to c^2, \tag{5.31}$$

from which we conclude that

$$c_2 = c^2, \tag{5.32}$$

allowing us to interpret the coordinate t as the proper time of an observer in the limit (5.31) and (r, θ, ϕ) constant.

The second boundary condition may be provided by by requiring agreement with Newtonian theory in the weak gravitational field limit.

$$c_1 = \frac{2Gm}{c^2} \equiv R_s, \tag{5.33}$$

with m the *active gravitational mass* and R_s the *Schwarzschild radius.* With our integration constants determined, we have the following particular solution

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) c^2\, dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2, \tag{5.34}$$

which is the Schwarzschild metric for $r > R_s$, hence Birkhoff's theorem has been proven.

5.3 Spacelike case $\beta = \beta(t)$

In the case where $\beta = \beta(t)$, (5.8) then gives $\gamma_{,r} = 0$. Changing variables to $T = e^{\beta/2}$ the EFE become

$$G^t_t = e^{-\gamma}\left(\frac{T_{,t}^2}{T^2} + \frac{\alpha_{,t} T_{,t}}{T}\right) + \frac{1}{T^2} = 0, \tag{5.35}$$

$$G^r_r = e^{-\gamma}\left(-\frac{2T_{,tt}}{T} + \frac{T_{,t}^2}{T^2} - \frac{\gamma_{,t} T_{,t}}{T}\right) + \frac{1}{T^2} = 0, \tag{5.36}$$

$$G^\theta_\theta = G^\phi_\phi = e^{-\gamma}\left(\frac{T_{,tt}}{T} + \frac{\alpha_{,tt}}{2} + \frac{\alpha_{,t}^2}{4} + \frac{T_{,t}}{2T}(\alpha_{,t} - \gamma_{,t}) - \frac{\alpha_{,t}\gamma_{,t}}{4}\right) = 0. \tag{5.37}$$

As before we will show that only two of these equations are independent, the third being a consequence of the other two. Subtracting (5.36) from (5.35) we find

$$\alpha_{,t} + \gamma_{,t} = \frac{2T_{,tt}}{T_{,t}}. \tag{5.38}$$

Taking the derivative of (5.38) with respect to r gives

$$\alpha_{,rt} = 0, \qquad \alpha(r,t) = f(r) + g(t), \tag{5.39}$$

such that (5.38) becomes

$$g_{,t} + \gamma_{,t} = \frac{2T_{,tt}}{T_{,t}}. \tag{5.40}$$

Substituting (5.42) into (5.37) yields

$$g_{,tt} + g_{,t}^2 = g_{,t}\left(\frac{T_{,tt}}{T_{,t}} - \frac{2T_{,t}}{T}\right). \tag{5.41}$$

Substituting (5.38) into (5.35) gives

$$\begin{aligned} T_{,t} &= e^{-\gamma}\left(\gamma_{,t} T T_{,t}^2 - T_{,t}^3 - 2T T_{,t} T_{,tt}\right) \\ &= \left(e^{-\gamma} T T_{,t}^2\right)_{,t}, \end{aligned} \tag{5.42}$$

which we may integrate to give

$$e^\gamma = -T_{,t}^2\left(1 - \frac{c_1}{T}\right)^{-1}, \tag{5.43}$$

for $T < |c_1|$ where c_1 is a spacetime constant. By integrating (5.42) we find

$$e^{g+\gamma} = c_2 T_{,t}^{\,2}, \tag{5.44}$$

where $c_2 > 0$ is another spacetime constant. Substituting (5.43) into (5.44) yields

$$e^{g} = -c_2 \left(1 - \frac{c_1}{T}\right), \tag{5.45}$$

valid for $T < |c_1|$. Let us take stock of what we have found so far. Our three equations are now given by (5.41), representing a differential equation between g and T, (5.43), a differential equation between γ and T, and (5.45) an algebraic relation between g and T. We may differentiate (5.45) once and twice yielding

$$g_{,t} e^{g} = \frac{c_1 c_2 T_{,t}}{T^2}, \tag{5.46}$$

$$e^{g}\left(g_{,tt} + g_{,t}^{\,2}\right) = c_1 c_2 \left(\frac{T_{,tt}}{T^2} - \frac{2T_{,t}^{\,2}}{T^3}\right). \tag{5.47}$$

Dividing (5.47) by (5.46) we find

$$g_{,tt} + g_{,t}^{\,2} = g_{,t}\left(\frac{T_{,tt}}{T_{,t}} - \frac{2T_{,t}}{T}\right), \tag{5.48}$$

which is precisely (5.41), showing that it is not an independent equation, but is a consequence of (5.45). Using (5.39) we have

$$e^{\alpha} = -c_2 \left(1 - \frac{c_1}{T}\right) e^{f}, \tag{5.49}$$

hence the general solution that we have found is

$$d\tau^2 = \left(1 - \frac{c_1}{T}\right)^{-1} T_{,t}^{\,2}\, dt^2 - c_2 \left(1 - \frac{c_1}{T}\right) e^{f}\, dr^2 + T^2\, d\Omega^2, \tag{5.50}$$

for $T < |c_1|$. Using the chain rule of differentiation we have

$$T_{,t}\, dt = \frac{dT}{dt} dt = dT, \tag{5.51}$$

and we may define a new radial coordinate r'

$$r' = \int e^{f/2}\, dr, \tag{5.52}$$

relabelling $r' \to r$ and $T \to t$ (5.50) becomes

$$d\tau^2 = \left(1 - \frac{c_1}{t}\right)^{-1} dt^2 - c_2 \left(1 - \frac{c_1}{t}\right) dr^2 + t^2\, d\Omega^2, \tag{5.53}$$

where $-\infty < t < |c_1|$. Matching this local solution to the previous one as $t \to |c1|$ determines the constants c_1 and c_2 giving the particular solution

$$d\tau^2 = \left(1 - \frac{R_s}{t}\right)^{-1} dt^2 - \left(1 - \frac{R_s}{t}\right) c^2\, dr^2 + t^2\, d\Omega^2, \tag{5.54}$$

which is the Schwarzschild metric for $r < R_s$ [48].

5.4 Discussion

In 1960 Kruskal and Szekeres [34, 35] gave a coordinate system that was in a one-to-one correspondence with all spacetime points of the maximally extended manifold. The two local solutions considered here can be matched at the event horizon $r = R_s$ to form part of this maximal manifold. The number of solutions to (5.4) are infinite on account of the four arbitrary coordinates that may be chosen. The solution we have found describes three qualitatively distinct physical scenarios which, due to spherical symmetry, are distinguished by only the ratio R/R_s, where R is the radius of the gravitating body, they are

$$R < R_s, \qquad R > R_s, \qquad R = R_s. \tag{5.55}$$

Systems that may be described by these solutions include: For the first, a non-rotating spherical black or white hole, the second could describe a non-rotating (but possibility radially pulsating) spherical star and the third is the point where the two solutions meet, which could describe the instantaneous state of a star collapsing to a black hole. In order to understand this last situation in greater detail, the next section analyses the simplest such case where a star is modelled by a thin spherical shell collapsing under its own gravitational attraction.

6 Thin shell dynamics

Having successfully found the static maximal foliations of the spherically symmetric vacuum spacetime and recognising that there exist non-static slices that also admit a SD solution,[17] we move on to the non-vacuum case by considering a thin spherical shell in vacuo. In the context of general relativity the dynamics of a thin shell has been studied by many and is understood fairly well. [28, 52–58] In this chapter we give a review of the subject by deriving the equations of motion for a thin spherical shell embedded in a spherically symmetric spacetime using the static maximal foliation that will necessarily admit a SD solution. We then solve the equations of motion for the simplest case of a single shell, composed of *dust*, in vacuo. We employ units in which $c = 1 = 16\pi G$.

6.1 Junction conditions

Let the time-like 3-space Σ be the history of a thin spherical shell of matter which divides spacetime into two distinct four-dimensional manifolds $V^{\pm}$ mapped by independent coordinate charts $x^{\mu}_{\pm}$ and with metrics[21]

$$ds^2_{\pm} = g_{\mu\nu}dx^{\mu}dx^{\nu}|_{\pm}. \tag{6.1}$$

Their common boundary Σ is described by the following intrinsic 3-metric

$$ds^2_{\Sigma} = h_{ij}d\zeta^i d\zeta^j, \tag{6.2}$$

where ζ^i are the intrinsic coordinates of Σ. The intrinsic 3-metric is induced on Σ by each of the 4-geometries via

$$h_{ij} = g_{\mu\nu}e^{\mu}_i e^{\nu}_j, \qquad e^{\mu}_i = \frac{\partial x^{\mu}}{\partial \zeta^i}, \tag{6.3}$$

where e^{μ}_i are the basis vectors[22] tangent to Σ.
Our first junction condition requires (6.3) to be equal on either side of Σ,

$$[h_{ij}] = h^+_{ij} - h^-_{ij} \equiv h_{ij}(V^+)|_{\Sigma} - h_{ij}(V^-)|_{\Sigma} = 0. \tag{6.4}$$

This is certainly required if Σ is to have a well-defined geometry.
In general the *extrinsic curvature* K_{ij} will differ in V^+ and V^- in the presence of a shell of matter. Our second junction condition therefore relates the jump in K_{ij} across Σ to the *surface stress-energy tensor* S_{ij} of the shell,[23]

$$-8\pi S_{ij} = [K_{ij}] - h_{ij}[K], \tag{6.5}$$

where $K = h^{ij}K_{ij}$. We notice that (6.5) is the surface analogue of the Einstein field equations (EFE),

$$8\pi T_{\mu\nu} = G_{\mu\nu}, \tag{6.6}$$

[21] Greek indices range over $0, 1, 2, 3$ and Latin indices $0, 2, 3$.
[22] More correctly, the components of the basis vectors.
[23] Geometric unit system $G = 1 = c$.

which holds in the surrounding continuous medium, where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R^{\sigma}_{\sigma}g_{\mu\nu}$ is the Einstein tensor. Indeed, both can be derived from the *principle of stationary action* using

$$S = S_G + S_M,$$

as the total action with gravitational part[24]

$$16\pi S_G = \int_{V-\Sigma} R^{\sigma}_{\sigma}\sqrt{g}\, d^4x - 2\int_{\Sigma}[K]\sqrt{h}\, d^3y, \tag{6.7}$$

where $V \equiv V^+ \cup\ V^-$, $g = |\det g_{\mu\nu}|$ and $h = |\det h_{ij}|$. The thin spherical shell of matter is represented by a timelike boundary Σ with common *unit normal vector* n^{μ}, directed from V^- to V^+. Performing a variation of (6.7) with respect to the inverse metric $g^{\mu\nu}$ leads to

$$16\pi\delta S_G = \int_{V^{\pm}-\Sigma} G_{\mu\nu}\delta g^{\mu\nu}\sqrt{g}\, d^4x - \int_{\Sigma}[K_{ij} - h_{ij}K]\delta h^{ij}\sqrt{h}\, d^3y, \tag{6.8}$$

Variation of the material part S_M gives

$$S_M = \tfrac{1}{2}\int_{V^{\pm}-\Sigma} T_{\mu\nu}\delta g^{\mu\nu}\sqrt{g}\, d^4x - \tfrac{1}{2}\int_{\Sigma} S_{ij}\delta h^{ij}\sqrt{h}\, d^3y, \tag{6.9}$$

where $T_{\mu\nu}$ corresponds to the surrounding continuum and S_{ij} to the shell. In the standard derivation of (6.6) by a variational principle, the second term in (6.8) vanishes because Σ is considered to be the enclosing boundary of the entire spacetime on which one requires $\delta h^{ij} = 0$. In our case we allow non-zero variations of the metric on Σ, requiring then that $\delta S = 0$ gives both (6.5) and (6.6).
By contracting the *Gauss-Codazzi equations* we find the following relations between the extrinsic curvatures $K^{\pm}_{ij}$ and the normal components of the Einstein tensor on Σ,

$$-2G_{\mu\nu}n^{\mu}n^{\nu}|^{\pm} = {}^{3}R + K^{ij}K_{ij} - K^2|^{\pm}, \tag{6.10}$$

$$G_{\mu\nu}e^{\mu}_{j}n^{\nu}|^{\pm} = K^{i}{}_{j|i} - K_{,j}|^{\pm}, \tag{6.11}$$

where the vertical bar denotes h_{ij}-compatible covariant differentiation and ${}^{3}R$ is the intrinsic curvature invariant of Σ. Using (6.5) and (6.6), we can write (6.10) and (6.11) in the following form,

$$S^{ij}\left(K^{+}_{ij} + K^{-}_{ij}\right) = 2\left[T_{\mu\nu}n^{\mu}n^{\nu}\right], \tag{6.12}$$

$$S_{i}{}^{j}{}_{|j} = -2\left[T_{\mu}{}^{\nu}e^{\mu}_{i}n_{\nu}\right]. \tag{6.13}$$

It has be shown that equation (6.12) expresses the normal force acting on the respective sides of the shell as a consequence of the gravitational self-attraction. Equation (7.7) is an energy conservation law, describing the response of the shell to energy fluxes and stresses from the surrounding continuum.

[24] Equation (6.7) assumes that the entire spatial 3-volume is a closed manifold. Also assumed is the existence of coordinates near Σ such that a discontinuity occurs only in the first derivative of $g_{\mu\nu}$ at Σ. [59]

6.2 Equation of motion

We are concerned with the specific case of a spherical shell moving in a spacetime with interior and exterior 4-metric given by

$$ds_\pm^2 = -f_\pm(r_\pm)dt_\pm^2 + f_\pm^{-1}(r_\pm)dr_\pm^2 + r_\pm^2\, d\Omega_\pm^2, \tag{6.14}$$

where $t_\pm$ = constant are static maximal slices and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric on a unit two-sphere. The history of the shell represents a time-like hypersurface Σ which we parameterise as follows

$$\begin{aligned} t &= T(\tau), \\ x &= R(\tau)\sin\theta\cos\phi, \\ y &= R(\tau)\sin\theta\sin\phi, \\ z &= R(\tau)\cos\theta, \end{aligned} \tag{6.15}$$

or in spherical coordinates

$$r = \sqrt{x^2+y^2+z^2} = R(\tau), \tag{6.16}$$

where $R = R(\tau)$ is the shell's radius and τ is the *proper time* for observers with coordinates $\zeta^i = (\tau, \theta, \phi)$ *comoving* with the shell. In terms of these coordinates the intrinsic metric (6.2) is given by

$$ds_\Sigma^2 = -d\tau^2 + R^2 d\Omega^2. \tag{6.17}$$

Using (6.3) and (6.4) we find the following relations

$$f_\pm(R)\dot{t}_\pm = \lambda\sqrt{f_\pm(R) + \dot{R}^2}, \tag{6.18}$$

where $\lambda = \mathrm{sgn}(n^u\partial_\mu r) = \pm 1$ determines whether r increases or decreases in the direction of n^μ from V^- to V^+. Equation (6.18) may be integrated to find $t_\pm(\tau)$. We find that (6.4) also requires $r_\pm = r$ and $\theta_\pm = \theta$ and $\phi_\pm = \phi$. The *unit tangent* v^i to Σ is defined as the unit vector in the timelike eigen-direction of the surface stress-energy tensor,

$$S_{ij}v^j = -\sigma v_i, \quad v^i v_i = -1, \tag{6.19}$$

where $\sigma = \sigma(R(\tau))$ is the *proper surface energy*, i.e. the surface energy in the rest frame of the shell. From (6.5) and (6.19) it follows that

$$-8\pi\sigma = [K_{ij}]v^i v^j + [K]. \tag{6.20}$$

In terms of the coordinates of Σ we have

$$v^i \equiv \mathrm{d}\zeta^i/\mathrm{d}\tau = (\dot{\tau}, \dot{\theta}, \dot{\phi}) = (1, 0, 0). \tag{6.21}$$

Using (6.21) and (6.17), (6.20) simplifies to

$$[K_{\theta\theta}] = -4\pi R^2\sigma \equiv -M(\tau), \tag{6.22}$$

where M is the total *proper mass*, equal to the sum of the masses of the constituent particles of the shell M_0, when infinitely dispersed and at rest, plus the internal thermal energy of the shell U, i.e. $M = M_0 + U$. The thermal energy of the shell is due to the random transversal kinetic motion of the particles in the shell. The *four-velocity* $u^\mu \equiv dx^\mu/d\tau$ is as follows

$$u^\mu_\pm = (\dot{t}_\pm, \dot{R}, 0, 0). \tag{6.23}$$

Substituting (6.23) into the identities $u^\mu u_\mu = -1$, $n^\mu n_\mu = 1$ and $u^\mu n_\mu = 0$ we find

$$n^\pm_\mu = \lambda(-\dot{R}, \dot{t}_\pm, 0, 0), \tag{6.24}$$

The extrinsic curvature tensor K_{ij} is defined as the projection of the covariant derivative of the unit normal vector n_μ onto Σ,

$$K_{ij} \equiv n_{\mu;\nu} e^\mu_i e^\nu_j, \tag{6.25}$$

where the semicolon denotes $g_{\mu\nu}$-compatible covariant differentiation. Substituting (6.24) into (6.25) and using (6.18) we find

$$K^\pm_{\theta\theta} = \lambda R \sqrt{f_\pm(R) + \dot{R}^2}. \tag{6.26}$$

Substituting (6.26) into (6.22) yields the equation of motion of the shell,

$$\lambda\sqrt{f_-(R) + \dot{R}^2} - \lambda\sqrt{f_+(R) + \dot{R}^2} = \frac{M}{R}, \tag{6.27}$$

which follows directly from (6.5) and the requirement of spherical symmetry, in particular, no explicit form of the surface energy was assumed. Upon squaring equation (6.27) we find

$$f_-(R) - f_+(R) = \frac{2\lambda M}{R}\sqrt{f_\pm(R) + \dot{R}^2} \pm \frac{M^2}{R^2}. \tag{6.28}$$

Specialising to the case of an electrically neutral shell of total gravitational mass, the Birkhoff theorem dictates that the regions $V^\pm$ are given by the Schwarzschild manifold giving

$$m_\Sigma = \lambda M\sqrt{f_\pm(R) + \dot{R}^2} \pm \frac{M^2}{2R} \tag{6.29}$$

where $m_\Sigma = m_+ - m_-$, and $f_\pm = 1 - 2m_\pm/r$ in the regions $V^\pm$.[25] Equation (6.29) shows that all forms of energy contribute to the total gravitational mass of the shell, however, although it permits a splitting of the inertial and gravitational terms, we must remember that in the general theory of relativity this split is arbitrary, that is, it depends on the frame of reference in accordance with the *equivalence principle*. Indeed, we may multiply (6.27) by $M/2$, upon substituting the result into (6.29) we find

$$m_\Sigma = \lambda\frac{M}{2}\left(\sqrt{f_+(R) + \dot{R}^2} + \sqrt{f_-(R) + \dot{R}^2}\right), \tag{6.30}$$

where the distinction between inertial and gravitational terms has disappeared.

6.3 A single shell in vacuo

By assuming the interior region V^- to be the Minkowski manifold $m_- = 0$, [26] we further specialise to the case of a single shell of total gravitational mass $m_+ = m$ in empty space.[27] Both equations given in (6.29) are equally valid and are a consequence

[25] Interchanging $\pm$ requires also changing the sign of λ.

[26] This can be understood by taking the limit $r \to 0$, the ratio of the circumference of a small circle to its radius should be 2π, this enforces $m_- = 0$, hence the interior metric must be flat.

[27] We choose r to increase in the direction n^μ, i.e. from V^- to V^+, thus $\lambda = +1$.

of (6.27), however, in the literature the negative sign equation appears more frequently because its terms are recognisable from special relativity and Newtonian theory,

$$m = M\sqrt{1+\dot{R}^2} - \frac{M^2}{2R}. \tag{6.31}$$

The first term on the right-hand side is the shell's relativistic inertial mass-energy, which in the classical limit $dR/dt_\pm \ll 1$ comprises exactly the rest mass and kinetic energy of the shell.
The second term is a potential energy term due to the binding of the shell; it is the negative of the work done by the gravitation field in moving the shell to a radius R from infinity. Notice that it is exactly half the Newtonian potential energy, reflecting the fact that the shell responds to the average field on either side of the hypersurface Σ.

6.4 Conservation equations

The EFE (6.6) require all geometries with metrics of the form (7.3) to have $T_t{}^t = T_r{}^r$. Substituting (6.24) and $T_\mu{}^\nu$ into (7.7) and using $e^\mu_i n_\mu = 0$ we find

$$S^{ij}{}_{|j} = 0. \tag{6.32}$$

Thus for spherically symmetric spacetimes, any pressures due to the surrounding continuum do no work, as a result the total internal energy of the shell is locally conserved.

6.5 Pressureless fluid (dust)

Let us first consider the simplest of all continua namely *dust*, which, on Σ, is defined by the surface stressless energy tensor

$$S_{ij} = \sigma v_i v_j. \tag{6.33}$$

From the normalisation (6.19) we have

$$v_i v^i{}_{|j} = 0. \tag{6.34}$$

To investigate the streamlines of dust, we substitute (6.33) into (7.9) and obtain

$$(\sigma v^j)_{|j} v^i + \sigma v^j v^i{}_{|j} = 0. \tag{6.35}$$

After multiplying by v_i and using (6.34) we find that

$$(\sigma v^j)_{|j} = 0, \tag{6.36}$$

and so by (6.35)

$$v^j v^i{}_{|j} = 0. \tag{6.37}$$

Equation (6.37) expresses that for dust the streamlines are *geodesics*. Using (6.17) we find

$$(\sigma v^i)_{|i} = \frac{1}{\sqrt{h}}(\sqrt{h}\sigma v^i)_{,i} = \dot{\sigma} + 2\sigma\frac{\dot{R}}{R} = 0, \tag{6.38}$$

where $h = R^4 \sin^2\theta$. The solution to equation (6.38) is $\sigma(\tau) = \sigma_0/R^2(\tau)$ with σ_0 a constant. Substituting this into (6.22) reveals that the proper mass of the shell is a conserved quantity, $M(\tau) = 4\pi\sigma_0 \equiv M_0$; equal to the mass of each free particle of dust at rest, multiplied by the number of particles in the shell.

6.6 Perfect fluid

Next in order of simplicity comes the *perfect fluid* which, on Σ, is defined by two quantities, σ and an isotropic rest frame surface pressure p. Due to spherical symmetry, the two eigenvalues of the stress-energy tensor corresponding to spacelike eigenvectors must be equal, therefore the material comprising a thin spherically symmetric shell is necessarily a 2-dim perfect fluid. Its surface stress-energy tensor is

$$S_{ij} = (\sigma + p)v_i v_j + p h_{ij}. \tag{6.39}$$

Comparison of (7.8) with (6.33) shows that a perfect fluid degenerates into dust when p tends to zero. To investigate its motion, we substitute (7.8) into (7.9) and obtain

$$[(\sigma + p)v^j]_{|j} v_i + [(\sigma + p)v^j] v_{i|j} + p_{|i} = 0. \tag{6.40}$$

On multiplying by v^i and using (6.34) we find

$$(\sigma v^i)_{|i} = -p v^i{}_{|i}, \tag{6.41}$$

which states that the rate of increase of surface energy is equal to minus the rate of work done by the pressure in expanding the shell. Using (6.17) we find

$$\dot{\sigma} = -2\frac{\dot{R}}{R}(\sigma + p). \tag{6.42}$$

Using (6.22) we can rewrite (6.42) as

$$dM = dU = -p\, d(4\pi R^2). \tag{6.43}$$

From which we note that in contrast to dust, the proper mass of an element of a fluid under surface pressure changes with time.
Due to (6.22), we see that equations (6.42) and (6.31) contain three unknown functions R, σ and p. Therefore, they become a closed set once a further *equation of state* of the form $p = p(\sigma)$ has been given.[28] The motion of the shell is then determined, with M_0 and m the constants of motion.

6.7 The motion of a shell composed of dust

Let us postpone the investigation of the perfect fluid for future work, and consider now the simpler case where the shell is made of dust, then $p = 0$ and the proper mass of the shell $M = M_0$ is a constant of motion. Let us also assume that both m and M_0 are strictly positive, hence so is their ratio $a = m/M_0$. Equation (6.31) can be rewritten as follows,

$$1 + \dot{R}^2 = \left(a + \frac{m}{2aR}\right)^2. \tag{6.44}$$

We distinguish three cases $a < 1$, $a > 1$ and $a = 1$ corresponding respectively to a shell with positive, negative and zero *binding energy* $E_{bin} = M_0 - m$.

[28] A fluid whose pressure is a function of only density is called a *barotropic fluid*.

6.8 Positive binding energy $m < M_0$

For $a < 1$, (6.44) shows that there exists a non-trivial solution ($m \neq 0$) where the velocity is permitted to vanish. The stationary point will occur at a maximum radius given by

$$R_{\max} = \frac{m}{2a(1-a)}. \tag{6.45}$$

Differentiating (6.45) w.r.t a and setting to zero, we find that at $a = 1/2$, R_{max} has the extremal value $R_s = 2m$ equal to the Schwarzschild radius. The second derivative at $a = 1/2$ is positive, therefore it is a minimum; this is understandable since a stationary value $\dot{R} = 0$ for $R < R_s$ would require the shell to have a spacelike 4-velocity, prohibited for a massive shell $M_0 > 0$.[29] In view of (6.45) we choose to parameterise the radius in the following way,

$$R(x) = R_{\max}\frac{x+a}{1+a}, \qquad -a \leqslant x \leqslant 1. \tag{6.46}$$

The integration of (6.44) yields $\tau(x)$,

$$\tau(x) = \pm\frac{m}{2a(1-a^2)^{3/2}}\left(a\sin^{-1}x - \sqrt{1-x^2}\right) + \tau_0, \tag{6.47}$$

where τ_0 is a constant of integration. We plot in parametric form the motion of the shell with respect to proper time $R^* = R/R_s$ for $a = 0.9$, see Fig. 6. For $a < 1$ we may calculate the amount of proper time taken for the shell to collapse from R_{max} to zero,

$$\tau_{col}(a) = \frac{m}{2a(1-a^2)^{3/2}}\left(a\sin^{-1}(1) + a\sin^{-1}(a) + \sqrt{1-a^2}\right), \tag{6.48}$$

which we have plotted as a function of a, see Fig. 5.[30]

6.9 Negative binding energy $m > M_0$

For $a > 1$, (6.44) shows that the velocity is not permitted to vanish for non-trivial solutions, therefore, we choose to parameterise the radius in the following way,

$$R(x) = R_{\max}\frac{a-x}{1+a}, \qquad a \leqslant x < \infty. \tag{6.49}$$

The integration of (6.44) yields $\tau(x)$,

$$\tau(x) = \pm\frac{m}{2a(a^2-1)^{3/2}}\left(a\ln|x+\sqrt{x^2-1}| - \sqrt{x^2-1}\right) + \tau_0, \tag{6.50}$$

where τ_0 is a constant of integration. We plot in parametric form the motion of the shell with respect to proper time $R^* = R/R_s$ for $a = 1.1$, see Fig. 6.

[29] We may also see this from (6.27), since for $\dot{R} = 0$ and $R < 2m$, $f < 0$ hence we obtain a negative number inside the square root.

[30] Both R_{max} and $\tau_{col} \to \infty$ as $a \to 0, 1$.

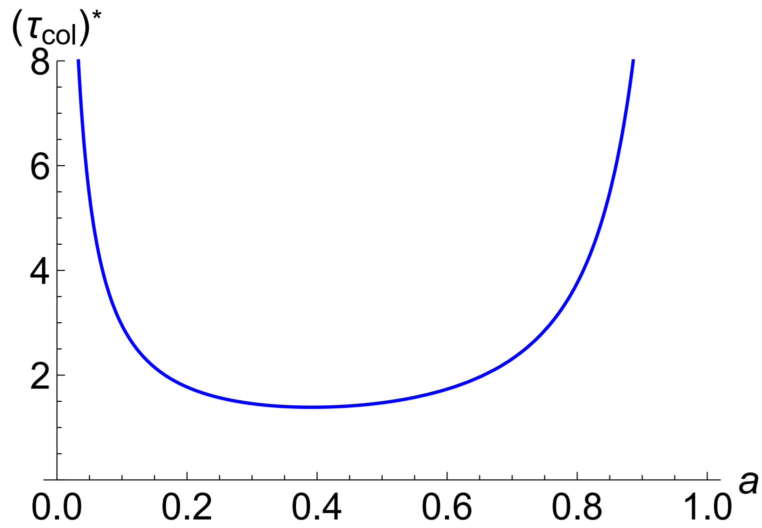

Figure 5: Normalised proper time $\tau^*_{col} = \tau_{col}/R_s$ taken for the shell to collapse from R_{max} to zero as a function of $a = m/M_0$.

6.10 Zero binding energy $m = M_0$

For $a = 1$, (6.44) shows that the velocity is permitted to vanish only as $R \to \infty$, it is obviously the limit where the previous two solutions coincide. In this case we parameterise the radius as follows,

$$R(x) = \frac{m}{4}x, \qquad 0 \leqslant x < \infty. \tag{6.51}$$

The integration of (6.44) yields $\tau(x)$,

$$\tau(x) = \pm\frac{m}{12}(x-2)\sqrt{x+1} + \tau_0, \tag{6.52}$$

where τ_0 is a constant of integration. We plot in parametric form the motion of the shell with respect to proper time $R^* = R/R_s$ for $a = 1.0$, see Fig. 6.

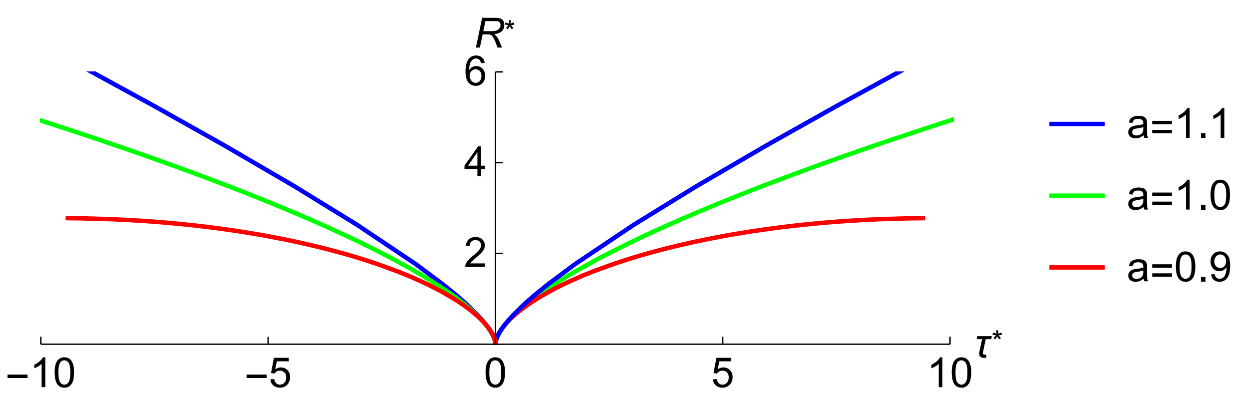

Figure 6: Normalised shell radius $R^* = R/R_s$ as a function of $\tau^* = \tau/R_s$ with τ proper time, $a = 0.9, 1.0$ and 1.1, $R(0) = 0$ and $R_s = 2m$.

6.11 Discussion

Figure 6 clearly shows the similarity between the results of the motion of a dust shell and the spatially homogenous and isotropic FLRW dust models, where there the parameter analogous to a is the constant spatial curvature of the constant time slices.[31] For positive and negative binding energies, the radius may also be parameterised using a parameter ξ related to x by

$$x = -\cos\xi, \qquad a < 1, \tag{6.53}$$
$$x = -\cosh\xi, \qquad a > 1. \tag{6.54}$$

In this parameterisation, (6.46), (6.47) and (6.49), (6.50) are given respectively by

$$R = R_0(a - \cos\xi), \qquad \tau = R_0(\xi - \sin\xi) + \tau_0, \qquad a < 1, \tag{6.55}$$
$$R = R_0(a - \cosh\xi), \qquad \tau = R_0(\xi - \sinh\xi) + \tau_0, \qquad a > 1, \tag{6.56}$$

where R_0 is a constant function of a and m, as before τ_0 is a constant of integration.[32] Equations (6.55) and (6.56) show that the solution for $a > 1$ can be obtained from $a < 1$ by taking $\xi \to i\xi$ and $\tau \to i\tau$.

6.12 Shape dynamics interpretation

Using (6.55) and (6.56) the solution for $t_\pm(\xi)$ can be obtained from (6.18) with $f_- = 1$ and $f_+ = 1 - 2m/R$. A noteworthy example is $R(t_+)$, because the spatial slices defined by $t_+ =$ constant are maximal and therefore permit a SD solution. We have plotted $R(t_+)$ for $a = 0.88$, see Fig. 7.[33] From (7.3) we know that t_+ corresponds physically to the proper time of an observer moving at constant (r, θ, ϕ) as $r \to \infty$, therefore in GR the interpretation is straightforward; an observer at infinity never actually sees the shell enter its Schwarzschild radius, indeed, it is well known that the coordinate t_+ becomes singular on the Schwarzschild sphere $r = R_s$.
In SD the interpretation could be different if the coordinate t_+ is given a different physical interpretation. The problem seems to be that the evolution of the shell with respect to its proper time can only be considered in SD if the proper time defines a CMC foliation. The question of how to interpret, in SD, the evolution of the shell after it reaches its Schwarzschild sphere is currently under investigation. [61]

6.13 Null shell limit

Lastly, there exists a non-trivial solution for $M \to 0$, $a \to \infty$ and $aM \to m$; it is the limit of a null-shell. Equation (6.18) does not apply in this case since the history of the shell is no longer time-like, it is a null hypersurface, hence $\tau = 0$. However, setting (7.3) to zero and subsequently integrating, we find the following equations for

[31] See [60] p.722.

[32] Incidentally (6.55) is the equation of a cycloid; the curve traced out by a point on the circumference of a circle rolling in a horizontal straight line, see Fig. 6, $a = 0.9$.

[33] Compare with [61] where the same solution was found by solving the equations of motion in the $3+1$ ADM decomposition and using a *twin shell* model to select the particular static maximal foliation.

a spherically symmetric shell of photons,

$$R = |t_-| + t_{-0}, \tag{6.57}$$

$$R + 2m\ln(R - 2m) = |t_+| + t_{+0}, \tag{6.58}$$

where $t_{\pm 0}$ are constants of integration. In (6.58) we again notice the singular nature of the exterior coordinate t_+ at $R = 2m$.

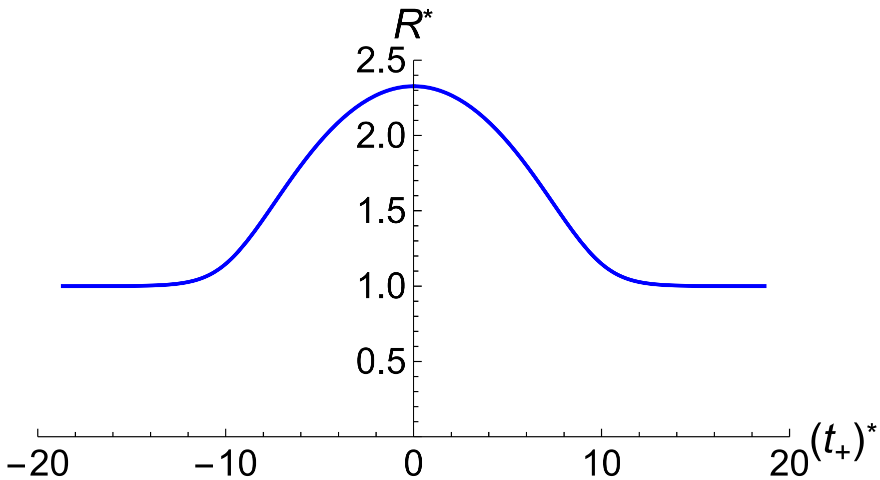

Figure 7: Normalised shell radius $R^* = R/R_s$ as a function of $t_+^* = t_+/R_s$ with t_+ exterior time, $a = 0.9$, $R(0)/R_s = R_{\max}$ and $R_s = 2m$.

7 Effective thin shell dynamics

In the previous chapter we were able to simplify the study of thin shell dynamics by imposing spherical symmetry. By doing so we reduced the dynamics to a problem involving only one degree of freedom; the radius of the shell $R(\tau)$. This warrants a search for a $(1+1)$ *effective action* whose corresponding Hamiltonian generates the same evolution as its $(3+1)$ counterpart.[34]

7.1 Total action

We begin with the following total $(3+1)$ action

$$S = S_G + S_M,$$

with gravitational part

$$S_G = \int_{V-\Sigma} R^\mu_\mu \sqrt{g}\, d^4x - 2\int_\Sigma [K]\sqrt{h}\, d^3y, \tag{7.1}$$

and matter part

$$S_M = -\int_{V-\Sigma} \rho\sqrt{g}\, d^4x - \int_\Sigma \sigma\sqrt{h}\, d^3y, \tag{7.2}$$

where $V \equiv V^+ \cup V^-$ represents our spacetime with $g = |\det g_{\mu\nu}|$ and $h = |\det h_{ij}|$. ρ is the energy density in $V - \Sigma$ and σ the surface energy density of the spherical shell, represented by a timelike boundary Σ with common *unit normal vector* n^μ, directed from V^- to V^+, and *extrinsic curvature* K_{ij}, with trace $K = h^{ij}K_{ij}$ where

$$[K] = K_+ - K_- \equiv K(V^+)|_\Sigma - K(V^-)|_\Sigma,$$

represents the jump discontinuity across Σ. Once again, we are concerned with the specific case of a spherical shell moving in a spacetime with metric

$$ds^2_\pm = -f_\pm dt^2_\pm + f^{-1}_\pm dr^2 + r^2\, d\Omega^2, \tag{7.3}$$

where $t_\pm$ = constant are static maximal time slices and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ is the metric on a unit two-sphere, $f_\pm = 1 - 2m_\pm/r$, with $m_\pm$ the *Schwarzschild mass* corresponding to $V_\pm$. We have the following two junction conditions

$$\mathcal{E}_\pm \equiv f_\pm(R)t_{\pm\,\tau} = \lambda\sqrt{f_\pm(R) + R^2_\tau}, \tag{7.4}$$

$$[\mathcal{E}] = -\frac{M}{R}, \tag{7.5}$$

where $\lambda = \mathrm{sgn}(n^u\partial_\mu r) = \pm 1$ determines whether r increases or decreases in the direction of n^μ from V^- to V^+, the subscript τ indicates $d/d\tau$ with τ the *proper time*, $M = 4\pi R^2\sigma$ is the *proper mass* of the shell, and $\mathcal{E}$ may be interpreted as the energy per unit rest mass of a particle in the shell.

Equation (7.4) is obtained by requiring the induced metric h_{ij} to be continuous across

[34] The effective action idea came from [62] Appendix: Review of shell dynamics. Many of the equations given there are reused in this chapter.

Σ. Equation (7.5) relates the discontinuity of the extrinsic curvature across Σ to the proper mass of the shell, since

$$K^\theta_\theta = \frac{\mathcal{E}}{R}.$$

By squaring (7.5), we can write it in either of the two forms ($\pm$),

$$\frac{M}{R}\left(2\mathcal{E}_\pm \pm \frac{M}{R}\right) = f_- - f_+,$$

addition gives

$$\frac{2M}{R}\bar{\mathcal{E}} = f_- - f_+, \tag{7.6}$$

where $\bar{\mathcal{E}} \equiv \frac{1}{2}\left(\mathcal{E}_+ + \mathcal{E}_-\right)$, subtraction gives again (7.5).
As well as the junction conditions (7.4) and (7.5), we have the following conservation equation for metrics of the form (7.3),

$$S_i{}^j{}_{|j} = -2\left[T_\mu{}^\nu e^\mu_i n_\nu\right] = 0, \tag{7.7}$$

where $T_{\mu\nu}$ is the *stress-energy tensor* of V, S_{ij} is the *surface stress-energy tensor* of Σ and e^μ_i are the tangent basis vectors of Σ. A spherically symmetric thin shell is necessarily a 2-dim *perfect fluid* with *isotropic proper surface pressure* p. Its surface stress-energy tensor is

$$S_{ij} = (\sigma + p)v_i v_j + p h_{ij}, \tag{7.8}$$

where v^i is the *unit tangent* to Σ. Substituting (7.8) into (7.7) we find that the proper mass obeys the following conservation equation

$$dM = -p\, d(4\pi R^2). \tag{7.9}$$

7.2 Effective action

We obtain an effective action for a shell in vacuo, i.e. $\rho = 0$ and $R^\mu_\mu = 0$ in $V - \Sigma$, by retaining only the non-geometrical parts residing in (7.2),

$$I = -\int M\, d\tau, \tag{7.10}$$

which we recognise as the action of a relativistic particle with variable proper mass. The standard Euler-Lagrange variation for the relativistic particle action starts by substituting for the infinitesimal separation $d\tau$,

$$d\tau = \left(-g_{\mu\nu}\frac{dx^\mu}{d\xi}\frac{dx^\nu}{d\xi}\right)^{1/2} d\xi$$

where ξ is some arbitrary parameter. Since the system is reparameterisation invariant, the *canonical Hamiltonian* will be zero, the *primary Hamiltonian* is given by the primary constraint,

$$M^2 + g_{\mu\nu}p^\mu p^\nu = 0,$$

which we recognise as the *mass shell condition*. In order to fix the Lagrangian we require a time coordinate which is invariant under variation of the shell's worldline;

for this reason the proper time is inadmissible. We choose instead to parameterise the worldline of the shell with a static observer's time split evenly between $t_\pm$:

$$I = \int \overline{L\,dt}, \qquad L = -M\frac{f}{\mathcal{E}}, \tag{7.11}$$

where $\bar{t} \equiv (t_+ + t_-)/2$ and we have made use of (7.4). Since $t_\pm$ = constant are both maximal foliations the foliation $\bar{t}$ = constant will also be maximal and will therefore permit a SD solution.
The corresponding conjugate radial momentum and canonical Hamiltonian are given respectively by

$$P = \frac{\partial L}{\partial R_t} = \frac{M}{f}R_\tau, \tag{7.12}$$

$$H = PR_t - L = M\mathcal{E}. \tag{7.13}$$

Hamilton's equation of motion for the momentum,

$$\bar{P}_{\bar{t}} = \{\bar{P}, \bar{H}\},$$

yields an identical equation to the total time derivative of (7.6). Hence the equations of motion, obtained using the effective action (7.11), ensure the propagation in time of the junction condition (7.5). Substituting (7.6) into (7.13) we find that $\bar{H}$ is equal to the total energy of the shell,

$$\bar{H} = m_+ - m_- = m_\Sigma, \tag{7.14}$$

which is conserved by Hamilton's equations for a stationary (time-independent) spacetime metric. Using (7.5) we find

$$\begin{aligned} \bar{H} &= M\bar{\mathcal{E}} \\ &= M\mathcal{E}_\pm \pm \frac{M^2}{2R}. \end{aligned} \tag{7.15}$$

The first line is understood from the point of view that each particle comprising the shell responds in equal share to the geometry on either side of Σ. According to the *equivalence principle* the second line is more suprising, since it permits a separation of the inertial mass-energy and gravitational energy, as before we must not put any fundamental significance in this fact, the separation is merely useful in relating the terms to concepts in predecessor theories, namely special relativity and Newtonian theory.
Interestingly both terms in (7.15) feature the proper mass M, where in the latter it stands in for the Newtonian gravitational mass. This means that, in the particular decomposition (7.15) of the inertial and gravitational terms, the gravitational interaction is determined, not by the total mass m or even the inertial mass $M\sqrt{f_\pm + R_\tau^2}$, but by the proper mass M. The proper mass includes the rest mass, and for non-zero pressure, the internal thermal energy due to the random motion of the particles in the shell. Substituting $f_\pm = 1 - 2m_\pm/r$ into (7.15), squaring and solving for R_τ we find

$$R_\tau^2 = a^2 - 1 + \frac{m_+ + m_-}{R} + \left(\frac{m_\Sigma}{2aR}\right)^2, \tag{7.16}$$

where $a = m_\Sigma/M$.

7.3 Phase space for a single shell in vacuo

For a single shell in empty space, $m_- = 0$, $m_+ = m_\Sigma = m$,

$$R_\tau^2 = \left(a + \frac{m}{2aR}\right)^2 - 1. \tag{7.17}$$

From (7.12) we have the following expression for the shell momentum,

$$\begin{aligned}\bar{P} &= \frac{m}{a}\overline{\left(\frac{1}{f}\right)}R_\tau \\ &= \pm\frac{m}{a}\left(\frac{R-m}{R-2m}\right)\left[\left(a + \frac{m}{2aR}\right)^2 - 1\right]^{1/2},\end{aligned} \tag{7.18}$$

where we have substituted (7.17). Using (7.18) we can make a phase space diagram for the motion of the shell with normalised shell momentum $\bar{P}^* = \bar{P}/R_s$ and radius $R^* = R/R_s$ where $R_s = 2m$ is the *Schwarzschild radius*, see Fig. 8.[35]

We may notice that, although the proper velocity R_τ is regular at $R = R_s$, the momentum tends to infinity $\bar{P} \to \infty$ due to $f_+ \to 0$. Indeed, from (7.4) we notice that $d\tau/dt_+$ has a zero eigenvalue at $R = R_s$ since $f(R_s) = 0$, i.e. the exterior coordinate t_+ is singular on the Schwarzschild sphere $R = R_s$. Events recorded using the proper time τ are not in a one-to-one mapping with those using t_+. The physical interpretation is well understood; an observer at infinity, with proper time $\tau = t_+$, does not actually observe the sphere collapse through its Schwarzschild sphere. The exterior coordinate t_+ gives a static maximal foliation and will therefore permit a SD solution. The challenge then is to understand the singular nature of t_+ in terms the framework of SD.

Equation (7.18) is not limited to a shell of dust; it is valid also for a shell with non-zero pressure. This is because the phase space representation does not tell you how the motion of the shell takes place in time. Only when requiring the solution for the radius of the shell as a function of time (proper, exterior, interior or our average time) is it necessary to know the pressure of the fluid composing the shell so that the change in proper mass can be calculated using (7.9). Therefore, Fig. 8 represents the most general phase space solution for the motion of a spherically symmetric shell in vacuo with $R > R_s$ and Hamiltonian given by (7.15).

[35] Compare with [61].

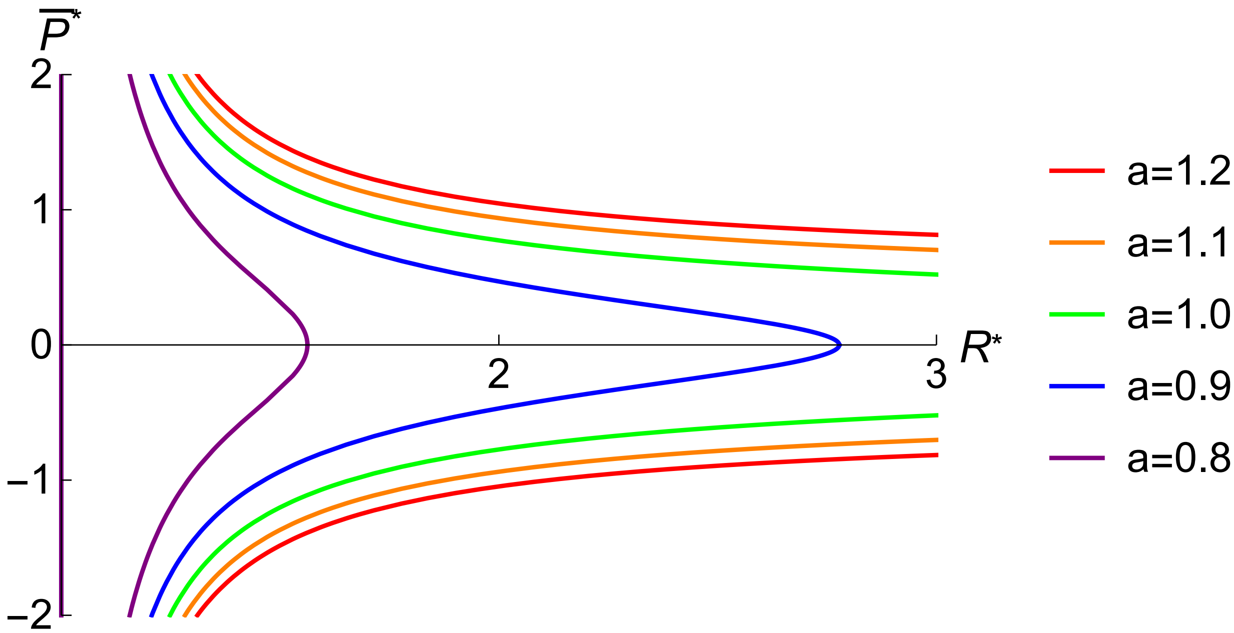

Figure 8: Phase space with normalised shell momentum $\bar{P}^* = \bar{P}/R_s$ and radius $R^* = R/R_s$ where $R_s = 2m$, $a = 0.8, 0.9, 1.0, 1.1$ and 1.2.

8 Conclusion

In this thesis we have made extensive use of the formal equivalence of the equations of motion of SD and GR in a particular spacetime foliation known as the maximal foliation (a subset of the CMC foliation). Using two physical situations involving spherically symmetric gravitational fields: The pure vacuum case and the collapse of a thin shell we have successfully found the maximal foliations that permit a SD solution and, when possible, have offered an interpretation in terms of the principles of SD. In future work the SD solutions that lie off the intersection with the ADM constraint surface,[36] i.e. in a different conformal section, should be investigated.

8.1 Critique of shape dynamics

A common misconception of SD is that it represents merely a choice of time gauge in GR. This interpretation is understandable since when spacetime permits a CMC foliation there must necessarily be a corresponding SD solution. However, this naive interpretation cannot be correct since SD has a conformal gauge symmetry that GR does not possess, furthermore, there exist SD solutions that will not be solutions to the EFE. However, according to SD, these solutions are considered physically equivalent because they must be related by a local conformal transformation.

SD originated from Machian ideas, most notably it aspires to close the gap between the immediate data that is observed in experiment and the basic concepts of the theory aiming to explain that data. The most serious criticism of SD then comes from considering what the immediate data actually is in astronomy. What one is in fact observing when one looks out at the stars is the celestial sphere; the rays of light on the past light cone of a point in spacetime, clearly this is also true of any experimental detector.[37] In practise then, the immediate data actually has no obvious link with spacelike distances or angles on a spacelike hypersurface. Although nothing prevents us from imagining the universe at an instant with a certain shape configuration, these data are not directly observable, therefore, from a positivistic point of view, they should not be used as the fundamental input of the theory.

8.2 What can we learn from shape dynamics?

In the case where the CMC preference turns out to by physically meaningful, it should be considered as a selection principle for physically viable theories. GR admits many solutions which are ruled out as models of the physical universe on account of the energy-momentum distributions that they give rise to. The statement that the spacetime solution for the physical universe should admit a CMC foliation could play a similar role. One example of a spacetime that cannot be CMC foliated is a spacetime containing closed timelike curves, such scenarios are indeed considered unphysical by most researchers.

[36] See Fig. 2.

[37] The celestial sphere does in fact have a conformal structure, where any shape perceived by one observer is mapped to a different observer, at the same spacetime point but moving at a relative velocity, by a conformal transformation. However, the celestial sphere has the structure of the Riemann sphere which is 2-dimensional. [60] p.429.

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