Counting the number of trigonal curves of genus five over finite fields

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Abstract

The trigonal curves form a closed subscheme of M_5, the moduli space of smooth curves of genus five. The cohomological data of these spaces can be found by counting their number of points over finite fields. The trigonal curves of genus five can be represented by projective plane quintics that have one singularity that is an ordinary node or an ordinary cusp. We use a partial sieve method for plane curves to count the number of trigonal curves over any finite field. The result agrees with the findings of a computer program we have written that counts the number of trigonal curves over the finite fields of two and three elements.

We also use the partial sieve method to count the number of smooth plane quintics. This result agrees with a previous result by Gorinov where he computed the cohomology of nonsingular plane quintics using a different method.

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Notation

All our schemes are noetherian.
With a curve over an algebraically closed field k we mean a proper scheme of dimension 1 over k.
With a smooth curve we mean a curve that is smooth and irreducible.
The genus of a smooth curve C over k is given by $g(C) = \dim_k H^1(C, \mathcal{O}_C)$.
The genus of a singular curve is the genus of its normalization.
With k we will always mean a finite field with q elements unless specified otherwise. We define k_i to be the finite field extension of k that has q^i elements.
When we talk about sets of points $S \subset \mathbb{P}^2$ they will always be defined over $\bar{k}$.

1 Introduction.

Moduli spaces are spaces that parameterize objects. These objects can be many things but here we are only interested in moduli spaces of curves. By studying a moduli space of curves we can discover more about the curves themselves.

We want to know the cohomology of $\overline{M}_g$, the coarse moduli space of stable curves of genus g. The cohomology of $\overline{M}_g$ has been found for g up to four but for five it is yet unknown.

It turns out that the cohomological data can be found by counting points over finite fields. So if we know $\#\overline{M}_5(k)$ for all finite fields k then we know the cohomology of $\overline{M}_5$. Inside M_5 we have the closed subset $\mathcal{T}_5$, consisting of smooth trigonal curves of genus five. In this paper we will count $\#\mathcal{T}_5(k)$. To do this we use the fact that there is a bijection between smooth trigonal curves of genus five and projective plane quintics that have precisely one singularity that is either an ordinary node or an ordinary cusp. To count these plane quintics we use a a method for counting plane curves that we call the partial sieve method. We derive this name from the fact that we use the sieving principle up to a certain point and after that we compute the rest explicitly.

In this paper we first give a short introduction to moduli spaces. We will not actually use any theory of moduli spaces during our counting of plane curves. However since moduli spaces provide the motivation for our counting problem we still included some basic theory. After that we describe the connection with plane curves and then we develop the partial sieve method. To give some examples of how the method works we apply it to count the number of smooth plane curves of degree three, four, and five. Smooth plane quartics have been counted before (using the same partial sieve method) by Bergström in [3] and later in [5]. In [6] Gorinov has computed the cohomology corresponding to smooth plane quintics so our count gives an alternative proof.

Finally we will count the number of plane quintics that have precisely one singularity that is either an ordinary node or an ordinary cusp. Here we can reuse part of the computations for smooth plane quintics.

Theorem 10.7. *The number of smooth trigonal curves of genus five over a finite field $\mathbb{F}_q$ is given by*

$$\#\mathcal{T}_5(\mathbb{F}_q) = q^{11} + q^{10} - q^8 + 1.$$

We have written a computer program to compute $\#\mathcal{T}_5(\mathbb{F}_2)$ and $\#\mathcal{T}_5(\mathbb{F}_3)$. The results of this program support the theorem.

2 Moduli spaces.

Let k be an algebraically closed field. We want to find a scheme M_g such that there is a bijection between the k-points of M_g and the isomorphism classes of smooth genus g curves over k. Without any further conditions we could easily construct such schemes. But we want M_g to be unique and the correspondence to be natural with respect to the structure on M_g.

To achieve this we look at families of curves. A family of curves over a base scheme B is a smooth proper morphism $C \to B$ such that its geometric fibers are smooth curves of genus g.

Definition 2.1. The moduli functor $\mathcal{M}_g$ is a contravariant functor from the category of schemes to the category of sets.

$$B \mapsto \{\text{families of smooth curves of genus } g \text{ over } B \text{ up to } B\text{-isomorphism}\}$$

$$(B' \to B) \mapsto (C/B \mapsto (B' \times_B C))$$

Definition 2.2. A fine moduli space for smooth curves of genus g is a scheme M_g that represents $\mathcal{M}_g$.

This means there is an isomorphism of functors Ψ between $\mathcal{M}_g$ and h_{M_g}. Here h_{M_g} is the functor of points of M_g that sends a scheme B to (B, M_g), the set of maps from B to M_g. So every family in $\mathcal{M}_g(B)$ corresponds to a morphism $B \to M_g$.

Remark 2.3. For any field k families of curves over $\operatorname{Spec} k$ considered up to k-isomorphism, are just curves up to k-isomorphism. So if $B = \operatorname{Spec} k$ where k is an algebraically closed field then as we wanted we get a bijection between the k points of M_g and the isomorphism classes of smooth genus g curves over k.

The identity map $M_g \to M_g$ corresponds to what we call the universal family $\theta : U \to M_g$ in $\mathcal{M}_g(M_g)$. Now let us have any family $\phi : C \to B$ and $\Psi_B(\phi) = f : B \to M_g$. We look at the commutative diagram

$$\begin{tikzcd}
\mathcal{M}_g(M_g) \arrow[r, "\Psi_{M_g}"] \arrow[d, "\mathcal{M}_g(f)"] & (M_g, M_g) \arrow[d, "h_{M_g}(f)"] \\
\mathcal{M}_g(B) \arrow[r, "\Psi_B"] & (B, M_g)
\end{tikzcd}$$

and we get

$$\begin{aligned}
\phi &= \Psi_B^{-1}(f) \\
&= \Psi_B^{-1}(id_{M_g} \circ f) \\
&= (\Psi_B^{-1} \circ h_{M_g}(f) \circ \Psi_{M_g})(\theta) \\
&= (\Psi_B^{-1} \circ \Psi_B \circ \mathcal{M}_g(f))(\theta) \\
&= \mathcal{M}_g(f)(\theta) \\
&= B \times_{M_g} U
\end{aligned}$$

So every family over B is the pullback of the universal family θ via a unique map $B \to M_g$.

Remark 2.4. In a similar fashion to the above we can also define $\mathcal{M}_{g,n}$ and $M_{g,n}$ for smooth curves of genus g with n fixed points.

These fine moduli spaces are very nice but they only exist if the moduli functor is representable.

We look at a family of curves $\phi : C \to B$ such that all fibers are isomorphic to some curve C_0. If such a family exists and it is not the trivial family $C_0 \times B$ then the fine moduli space does not exist. For if a fine moduli space M_g does exist then $\Psi(\phi)$ will factor through the point in M_g corresponding to C_0. Which means that the fiber product $C_0 \times_{M_g} B$ is trivial.

Example 2.5. We will construct such a family for $g = 1$. We consider complex elliptic curves and write them in Weierstrass form $Y^2 = X^3 + ax + b$. Two elliptic curves are isomorphic if and only if they have the same j-invariant. The j-invariant is given by $j = 1728\frac{4a}{4a-9b}$ and it is zero for $a = 0, b \neq 0$. So we look at the family of curves over $\mathbb{A}^1_{\mathbb{C}} - \{0\}$ given by $Y^2 = X^3 + b$. If this family is trivial then we can extend it to a family over $\mathbb{A}^1_{\mathbb{C}}$. But its fiber over 0 would be $Y^2 = X^3$ which is not a smooth curve. So the family is not trivial and the fine moduli space M_1 does not exist.

It turns out that this problem is caused by the existence of non-trivial automorphisms. These non-trivial automorphisms enable us to create a non-trivial family such that all fibers are isomorphic. We can see a non-trivial automorphism in example 2.5 if we go from one fiber to another fiber and then back to the first fiber. The fibers over 1 and -1 are the curves $Y^2 = X^3 + 1$ and $Y^2 = X^3 - 1$. The isomorphism between these curves is given by $(x, y) \mapsto (-x, iy)$ where $i^2 = -1$. Applying it twice gives us the non-trivial automorphism $(x, y) \mapsto (x, -y)$.

There are three main solutions to this problem.

Solution 1: rigidification.
We no longer consider just the curves but we add some extra data. For example we will choose some fixed points. If an automorphism fixes enough points then it has to be the identity. (How many points is enough depends on the genus.) This way we no longer have any non-trivial automorphism. The downside of this approach is that we have to keep track of our extra data.

Solution 2: stacks.
We extend the category of schemes by considering algebraic objects called stacks. The moduli functor of smooth curves is not representable by a scheme but it is representable by a stack. This approach is very clean. However the difficulty of working with extra structure is still there, except it is now part of the theory of algebraic stacks.

Solution 3: coarse moduli spaces.
Instead of demanding a fine moduli space we lower our demands to get only a coarse moduli space.

Definition 2.6. A coarse moduli space for the moduli functor F is a scheme M and a morphism of functors $\Psi : F \to h_M$ such that:

(i) For algebraically closed fields k the map $F(k) \to M(k)$ is a bijection of sets.

(ii) Given a scheme N and a morphism of functors $\Phi : F \to h_N$, there is a unique morphism $f : M \to N$ such that $\Phi = h_f \circ \Psi$.

Coarse moduli spaces of smooth curves do exist. But these spaces may not have nice properties. For example they are not always smooth.

Remark 2.7. From now on we will act as if the fine moduli schemes M_g, $M_{g,n}$ exist. (They actually exist as stacks but since my knowledge of stacks is insufficient I pretend we are just working with a scheme or variety.)

Example 2.8. Smooth 1-pointed curves of genus 1 are elliptic curves. The j-line $\mathbb{A}^1$ is a coarse moduli space for elliptic curves. Concretely a curve $C \to B$ is sent to the map $B \to J$ that sends a point $b \in B$ to the j-invariant of the fiber C_b. The first condition is satisfied since when k is algebraically closed isomorphism classes of elliptic k-curves correspond to l-points on the j-line.

Let $m \in J(l)$ for some field l, there is a family of curves $C_m \to \operatorname{Spec} l$ such that it is the isomorphism class of curves of j-invariant m. Now for the second condition: given such N and Φ we get the morphism $f : J \to N$ by sending $m \in J$ to $\Phi(C_m)$.

We will now look at another moduli functor, namely the one for stable curves. In what follows let $g \geq 0$, $n \geq 0$ and $2g - 2 + n > 0$. Here g is the arithmetic genus.

Definition 2.9. A stable n-pointed curve of genus g is a reduced connected curve C of genus g whose only singularities are ordinary double points, coupled with a collection $P_1, \ldots, P_n$ of distinct nonsingular points, such that C has only finitely many automorphisms that fix $P_1, \ldots, P_n$. A stable curve is a stable 0-pointed curve.

Similar to the smooth case we can define moduli functors $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$ for stable curves and n-pointed stable curves. It turns out that there exist coarse moduli spaces $\overline{M}_g$, $\overline{M}_{g,n}$ and that these spaces are projective varieties.

As the notation implies the moduli space M_g is a dense open subspace of $\overline{M}_g$. We can take a look at the boundary $\Delta_g = \overline{M}_g - M_g$. Every curve in the boundary has at least one singular point so it is in the closure of some locus of curves with a singularity at that point. This means that the boundary is the union of closures of loci of curves with one singularity.

For $1 \leq i \leq \lfloor \frac{g}{2} \rfloor$ we define Δ_g^i to be the closure of the locus of curves with one singularity that are the union of two smooth curves of genera i and $g-i$. And we define Δ_g^0 to be the closure of the locus of irreducible curves with one singularity. We find that $\Delta_g = \bigcup_i \Delta_g^i$.

An irreducible curve with one ordinary double point corresponds to its normalization which is a curve with genus one lower and two points in the inverse image of the double point. This gives us a dominating map $M_{g-1,2} \to \Delta_g^0$. And we also have dominating maps $M_{i,1} \times M_{g-i,1} \to \Delta_g^i$.

3 Point counts and cohomology.

As we said in the introduction we are interested in the cohomology of $\overline{M}_g$. Let k be a finite field. We define the number of points over k of $\overline{M}_g$ to be

$$\#\overline{M}_g(k) = \sum_{C/k} \frac{1}{|\mathrm{Aut}_k(C)|},$$

where the sum is over representatives of k-isomorphism classes of stable curves of genus g over k.

Bogaart and Edixhoven have shown the following in [1]:
Let d be the dimension of $\overline{M}_g$. If there exists a polynomial $P(t) = \sum P_i t^i$, with $P_i \in \mathbb{Q}$, such that

$$\#\overline{M}_g(\mathbb{F}_{p^n}) = P(p^n) + o(p^{nd/2}) \quad (n \to \infty)$$

for all p in a set of primes of Dirichlet density 1, then the degree of $P(t)$ equals d and there exists a unique such polynomial satisfying $P_i = P_{d-i}$ for all $0 \le i \le d$. Suppose $P(t)$ satisfies this symmetry. Then it has non-negative integer coefficients and satisfies $\#\overline{M}_g(\mathbb{F}_{p^n}) = P(p^n)$ for all primes p. This polynomial also completely determines the cohomology of $\overline{M}_g$. That is, for all primes l and all $i \ge 0$ there is an isomorphism of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations

$$H^i((\overline{M}_g)_{\overline{\mathbb{Q}},\text{ét}}, \mathbb{Q}_l) \cong \begin{cases} 0 & \text{if } i \text{ is odd;} \\ \mathbb{Q}_l(-i/2)^{P_{i/2}} & \text{if } i \text{ is even.} \end{cases}$$

From Theorem F in [2, p.15] and the fact that $\overline{M}_5$ is unirational it follows that for $\overline{M}_5$ such a polynomial $P(t)$ exists.

By purity, knowing the cohomology of $\overline{M}_5$ is equivalent to knowing the Hodge Euler characteristic of $\overline{M}_5$. This Hodge Euler characteristic can be computed from that of M_5 and the $\mathbb{S}_n$-equivariant Hodge Euler characteristics of the $M_{g,n}$ occurring in the description of the boundary Δ_5. We say a function of $k = \mathbb{F}_q$ is polynomial if it is an element of $\mathbb{Q}[q]$. The occurring $\#M_{g,n}(k)$ are known and polynomial for $g \le 3$. We do not know all the occuring $\#M_{4,n}(k)$ but we do know they are polynomial from Theorem F in [2, p.15], so $\#M_5(k)$ is polynomial. By the symmetry described above it will then be sufficient to know the higher order terms of $\#M_5(k)$, from which we will be able to compute $\#M_5(k)$ and $\#\overline{M}_5(k)$. Since $M_5(k)$ has dimension 12 the higher order terms are those of order greater than or equal to 6.

We can split up $M_5(k)$ in three groups of curves:

(i) If a curve has a g^1_2 then it is hyperelliptic. We already know the number of hyperelliptic curves: $\#H_5(\mathbb{F}_q) = q^9$.

(ii) If a non-hyperelliptic curve has a g^1_3 then it is trigonal.

(iii) We call the other curves non-trigonal curves. A non-trigonal curve has a canonical embedding as a complete intersection of three quadrics in $\mathbb{P}^4$.

In this paper we count $\#\mathcal{T}_5(\mathbb{F}_q)$, the number of trigonal curves of genus 5.

Theorem 10.7. *The number of smooth trigonal curves of genus five over a finite field $\mathbb{F}_q$ is given by*

$$\#\mathcal{T}_5(\mathbb{F}_q) = q^{11} + q^{10} - q^8 + 1.$$

4 Representing trigonal curves by projective plane curves.

Proposition 4.1. *There is a bijection between the smooth trigonal curves of genus five and projective plane quintics that have precisely one singularity which is either an ordinary node or an ordinary cusp.*

Proof. Let C be a smooth trigonal curve of genus 5 and let D be a divisor in its g^1_3. We look at the linear system $|K-D|$. We see that $\deg(K-D) = 2g-2-3 = 5$. And by Riemann-Roch we have $\dim|K-D| = \dim|D| - \deg(D) + g - 1 = 2$. So we get a g^2_5.
Claim: The g^2_5 is base point free.
Proof of claim: If it has a base point P then we have $\dim|K-D-P| = \dim|K-D|$ which gives us a g^2_4. This g^2_4 is base point free, otherwise we get a g^2_3 which contradicts Clifford's theorem. It is also very ample for otherwise there are points Q, R such that $\dim|K-D-P-Q-R| = \dim|K-D-P|-1$ and we get a g^1_2, which is a contradiction with the fact that C is not hyperelliptic. So C can be embedded as a smooth plane curve of degree 4. But this means that the genus is $\frac{1}{2}(4-1)(4-2) = 3$, which is a contradiction.

Since the g^2_5 is base point free we get a corresponding morphism $\phi : C \to \mathbb{P}^2$ with $\deg(\phi)\cdot\deg(\phi(C)) = 5$. If $\deg(\phi(C)) = 1$ then $\phi(C)$ is a line, which is in contradiction with the fact that the dimension of the g^2_5 is two. So ϕ is of degree 1. The image is of degree 5 which gives us

$$5 = g = \frac{1}{2}(5-1)(5-2) - \sum_P \delta_P = 6 - \sum_P \delta_P.$$

So the image has to have precisely one singularity of delta-invariant 1. This means that it has multiplicity 2 and thus it has two tangents counting multiplicity. If these tangents are the same then we have an ordinary cusp and if they are different we have an ordinary node.

The only thing that remains is to prove that the g^1_3 (and thereby the g^2_5) is unique. If there are two different g^1_3's then they are both base point free, otherwise we would get a g^1_2, which contradicts the fact that C is not hyperelliptic. Together these two g^1_3's give a map to $\mathbb{P}^1 \times \mathbb{P}^1$. The image of this map is a curve Z of type $(3,3)$, i.e., the inverse image of a point on either $\mathbb{P}^1$ is a divisor of degree 3. A non singular curve of type (a,b) in $\mathbb{P}^1 \times \mathbb{P}^1$ has genus $(a-1)(b-1)$, so Z has genus at most 4, which is a contradiction.

Conversely, suppose that we have a curve C that has a representation as a plane quintic with one singularity that is either an ordinary node or an ordinary cusp. Then by the genus formula

$$g(C) = \frac{(5-1)(5-2)}{2} - 1 = 5.$$

Every line through the singular point intersects C in 3 other points counting multiplicity. So the pencil of lines through the singular point gives us a g_3^1. □

We want to count

$$\#\mathcal{T}_5(k) = \sum_{C/k} \frac{1}{|\mathrm{Aut}_k(C)|}$$

where the sum is over representatives of k-isomorphism classes of smooth trigonal curves of genus five over k. By the above proposition it is equivalent to let the sum go over representatives of k-isomorphism classes of plane quintics that have precisely one singularity which is either an ordinary node or an ordinary cusp.

Plane curve automorphisms can be extended to $\mathbb{P}^2$ automorphisms and the automorphism group of $\mathbb{P}^2$ is the projective general linear group PGL_3. So the automorphism group of a plane curve C is the stabilizer of the action by PGL_3 on $\mathbb{P}^2$. We get

$$\sum_{C/k} \frac{1}{|\mathrm{Aut}_k(C)|} = \sum_{C/k} \frac{1}{|\mathrm{Stab}_k(C)|}.$$

We can rewrite this as

$$\sum_{C/k} \frac{1}{|\mathrm{Stab}_k(C)|} = \sum_{C/k} \frac{|\mathrm{Orb}_k(C)|}{|\mathrm{PGL}_3(k)|} = \frac{1}{|\mathrm{PGL}_3(k)|} \sum_{C \in T(k)} 1.$$

Here T is the set of plane quintics with exactly one singularity which is either an ordinary node or an ordinary cusp.

This enables us to simply count plane curves rather than plane curves up to k-isomorphism. However before we compute $|T(k)|$ we first develop the partial sieve method for counting smooth plane curves. Later we shall adjust this method to count curves that have precisely one singularity which is either an ordinary node or an ordinary cusp.

5 Preliminaries and tools.

Definition 5.1. Let $P_1, \ldots, P_n$ be points in $\mathbb{P}^2$ and let $r_1, \ldots r_n$ be natural numbers. We define $V_k(d, r_1P_1, \ldots, r_nP_n)$ to be the set of degree d plane curves over k that have multiplicity at least r_i at P_i for $1 \leq i \leq n$.

If $S = \{P_1, \ldots, P_n\}$ is a set of points then we may write $V_k(d, rS) := V_k(d, rP_1, \ldots, rP_n)$.

The set $V_k(d, r_1P_1, \ldots, r_nP_n)$ is the result of applying linear conditions to the projective space of all degree d projective plane curves over k. When the conditions contradict each other we get the empty set and otherwise we get a linear subspace. Whether these conditions are dependent or not does not depend on the fields that the points are defined over:

Notation 5.2. When we talk about the Frobenius map $\mathcal{F}$ we mean the geometric Frobenius.

Let $\mathbb{P}^2$ be the projective plane over a field $\mathbb{F}_q$. The geometric Frobenius on $\mathbb{P}^2$ is the endomorphism $\mathbb{P}^2 \to \mathbb{P}^2$ defined by $(x : y : z) \mapsto (x^q : y^q : z^q)$.

Lemma 5.3. ([3, 2.8]) *If we have points $\{P_1, \ldots, P_n\}$ such that for every $1 \leq i \leq n$ there is a j such that $\mathcal{F}(P_i) = P_j$ and $r_i = r_j$, then $\dim_k V_k(d, r_1P_1, \ldots, r_nP_n) = \dim_{\overline{k}} V_{\overline{k}}(d, r_1P_1, \ldots, r_nP_n)$.*

Notation 5.4. We write $\lambda = [1^{\lambda_1}, \ldots, v^{\lambda_v}]$ for the partition where i appears λ_i times. This partition has weight $|\lambda| := \sum_{i=1}^{v} i \cdot \lambda_i$. We consider the empty partition [] to have weight 0. For the sake of notation we leave out the zero powers, e.g. $[1^2, 2^0, 3^0, 4^1]$ is the same as $[1^2, 4^1]$.

Given $\lambda = [1^{\lambda_1}, \ldots, v^{\lambda_v}]$ and $\mu = [1^{\mu_1}, \ldots, v^{\mu_v}]$ we use $[\lambda, \mu]$ to denote $[1^{\lambda_1+\mu_1}, \ldots, v^{\lambda_v+\mu_v}]$.

Definition 5.5. Let X be a scheme defined over k. An n-tuple $(x_1, \ldots, x_n)$ of distinct subschemes of $X_{\bar{k}}$ is called a conjugate n-tuple if $\mathcal{F}(x_i) = x_{i+1}$ for $0 \leq i < n$ and $\mathcal{F}(x_n) = x_0$, where $\mathcal{F}$ is the Frobenius map.

A $|\lambda|$-tuple $(x_1, \ldots, x_{|\lambda|})$ of distinct subschemes of $X_{\bar{k}}$ is called a λ-tuple if it consists of λ_1 conjugate 1-tuples, followed by λ_2 conjugate 2-tuples, etc.

Given a set S of distinct subschemes of $X_{\bar{k}}$, we define $\widehat{S}$ to be the partition such that S is a $\widehat{S}$-tuple.

A λ-tuple is called ordered when it is ordered as a $|\lambda|$-tuple. A λ-tuple is called unordered when for each $1 \leq i \leq v$ the λ_i conjugate i-tuples are unordered among themselves and the points within every i-tuple are unordered.

For a scheme X defined over k we write $X_{\mathrm{ord}}(\lambda)$ for the set of ordered λ-tuples of points of X and we write $X(\lambda)$ for the set of unordered λ-tuples of points of X.

We see that for a partition $\lambda = [1^{\lambda_1}, \ldots, v^{\lambda_v}]$ we get

$$|X(\lambda)| = \frac{|X_{\mathrm{ord}}(\lambda)|}{\prod_{i=1}^{v} \lambda_i! \cdot i^{\lambda_i}}$$

Lemma 5.6. ([5, 4.12]) *Let μ denote the Möbius function. For any scheme X defined over k, the number of ordered λ-tuples of points of X is equal to*

$$|X_{\mathrm{ord}}(\lambda)| = \prod_{i=1}^{v} \prod_{j=0}^{\lambda_i - 1} \left(\left(\sum_{d|i} \mu(\frac{i}{d}) \cdot |X(k_d)| \right) - i \cdot j \right).$$

Example 5.7. Since the empty product gives one we have $|X([])| = 1$.

Example 5.8. We want to know the number of ways we can pick an unordered $[2^2]$-tuple of points on a k-line L. The lemma tells us that the number is

$$\begin{aligned}
|L([2^2])| &= \frac{1}{8} \left(\left(\sum_{d|2} \mu(\frac{2}{d}) \cdot |L(k_d)| \right) - 2 \cdot 0 \right) \cdot \left(\left(\sum_{d|2} \mu(\frac{2}{d}) \cdot |L(k_d)| \right) - 2 \cdot 1 \right) \\
&= \frac{1}{8} \left(|L(k_2)| - |L(k)| \right) \cdot \left(\left(|L(k_2)| - |L(k)| \right) - 2 \right) \\
&= \frac{1}{8} \big((q^2+1) - (q+1) \big) \cdot \big(\big((q^2+1) - (q+1) \big) - 2 \big) \\
&= \frac{1}{8} (q^2 - q) \cdot ((q^2 - q) - 2) \\
&= \binom{\frac{q^2-q}{2}}{2}
\end{aligned}$$

There are $\frac{q^2-q}{2}$ conjugate 2-tuples on a line so this fits our expectations.

Notation 5.9. We say a curve C is of type $[d_1, \ldots, d_n]$ for $d_1, \ldots, d_n \in \mathbb{N}$ if it has n distinct irreducible components $C_1, \ldots, C_n$ where C_i has degree d_i.

The number of singularities on each irreducible component is restricted by

Theorem 5.10. *If C is an irreducible projective plane curve of degree n, then*

$$\sum_i \frac{m_{P_i}(m_{P_i} - 1)}{2} \leq \frac{(n-1)(n-2)}{2}$$

were m_{P_i} is the multiplicity of C at P_i.

Proof. See Fulton [8, 5.4, theorem 2] □

Here are some basic facts about lines that we will use.

Lemma 5.11. *Let $i, j \in \mathbb{N}$ be coprime and let $LCM(i, j)$ denote the least common multiple of i, j.*

(i) A conjugate i-tuple of lines and a conjugate j-tuple of lines intersect in one k-point or in a conjugate $LCM(i, j)$-tuple of points.

(ii) Through a conjugate i-tuple of points and a conjugate j-tuple of points goes one k-line or a $LCM(i, j)$-tuple of lines.

(iii) There is precisely one k-point on a conjugate 2-tuple of lines.

(iv) There is precisely one k-line through a conjugate 2-tuple of points.

(v) A conjugate 3-tuple of lines either has no k-points on it or all three lines intersect in one k-point.

(vi) A conjugate 3-tuple of points either has no k-lines through it or all three points lie on one k-line.

Proof. $(ii), (iv), (vi)$ are the dual versions of $(i), (iii), (v)$ so just proving the latter three is sufficient.
(i): Let L be a line in the i-tuple of lines and let R be a line in the j-tuple of lines. Let t be the minimal natural number such that $L \cap R$ is a k_t-point P. If $i \nmid t$ then $\mathcal{F}^t(L) \neq L$ and so $P \in L'$ for some conjugate L' of L. Since $\mathcal{F}^i$ fixes both L and L' and $P = L \cap L'$ we get $t|i$. So $i|t$ or $t|i$ and similarly we get $j|t$ or $t|j$. We also know that both L and R are fixed by $\mathcal{F}^{\mathrm{LCM}(i,j)}$ so $t \leq \mathrm{LCM}(i,j)$.
If $t|i$ and $t|j$ then because i, j are coprime we get $t = 1$.
If $i|t$ and $j|t$ then $t = \mathrm{LCM}(i,j)$.
If $i|t$ and $t|j$ then $i|j$ so $i = 1$ and then $t = 1$ or $t = j = \mathrm{LCM}(i,j)$. For $t|i$ and $j|t$ it is the same with i, j switched.
(iii): Let us denote one of the two lines by L. The k-points on L are the points on L that are fixed by $\mathcal{F}$, that is the points in the intersection $L \cap \mathcal{F}(L)$. And two distinct lines intersect in precisely one point.
(v): Let us denote one of the three lines by L. A k-point P is fixed by $\mathcal{F}$ so if $P \in L$ then $P \in L \cap \mathcal{F}(L) \cap \mathcal{F}^2(L)$. $\square$

6 The partial sieve method.

In this section we will develop a method to compute $|C_d(k)|$, the number of smooth degree d plane curves over k. We first count all curves of degree d and then "sieve out" the degree d curves that are singular so that we are left with the nonsingular curves.

First we count all curves (including singular curves).

$$\sum_{S \in \mathbb{P}^2([])} |V_k(d, 2S)| = |V_k(d)|$$

Every curve that has precisely one singularity has been counted one time. So we subtract every curve that has at least one singularity at a k-point. For every k-point $P \in \mathbb{P}^2$ we subtract $|V_k(d, \{P\})|$.

$$-\sum_{S \in \mathbb{P}^2([1^1])} |V_k(d, 2S)|$$

Now for the curves that have precisely two singularities we get two cases: either the singularities are both defined over k or they form a conjugate 2-tuple.

The curves that have two singularities over k have been counted once and then subtracted twice (once for each singularity). So we have to add them once.

$$+\sum_{S\in\mathbb{P}^2([1^2])}|V_k(d,2S)|$$

The curves that have a conjugate 2-tuple of singularities have been added once so we have to subtract them once.

$$-\sum_{S\in\mathbb{P}^2([2^1])}|V_k(d,2S)|$$

And we continue in this manner for curves that have at least three singularities and then curves that have at least four singularities, etc.

Definition 6.1. If $\lambda=[1^{\lambda_1},\ldots,v^{\lambda_v}]$ is a partition then by $\mu\subset\lambda$ we mean any partition $\mu=[1^{\mu_1},\ldots,v^{\mu_v}]$ such that $\mu_i\le\lambda_i$ for all $1\le i\le v$.

We define $\sigma(\lambda)\in\mathbb{Z}$ such that

$$\sigma([])=1$$

$$\sum_{\mu\subset\lambda}\sigma(\mu)\prod_{i=1}^{v}\binom{\lambda_i}{\mu_i}=0,\quad\text{for }\lambda\neq[].$$

The number of times a curve with singularities at a λ-tuple of points has to be added in the sieve method described above is given by $\sigma(\lambda)$. So if we continue our sieving we get

$$\sum_{\lambda}\Bigg(\sigma(\lambda)\cdot\sum_{S\in\mathbb{P}^2(\lambda)}|V_k(d,2S)|\Bigg).$$

We cannot keep counting indefinitely so we only count those λ for which $|\lambda|\le M$ where we define $M:=\frac{d(d-1)}{2}$. We stop at M because any degree d plane curve that has more than M singularities has a double component and thus has infinitely many singularities. So it suffices to count up to and including M and then afterwards correct the curves that have an infinite number of singularities.

However it becomes more and more difficult to compute this sum as $|\lambda|$ increases. So we will use a partial sieving method where we will only count those λ for which $|\lambda|\le N$ where N is a fixed number chosen such that $1\le N\le M$.

For the λ such that $N<|\lambda|\le M$ we will explicitly count $|C_{d,\lambda}|$, the number of degree d plane curves that have exactly a λ-tuple of singular points.

Definition 6.2. For a partition $\lambda = [1^{\lambda_1}, \dots, v^{\lambda_v}]$ we define $\tau(\lambda) \in \mathbb{Z}$ such that

$$\sum_{\substack{\mu \subset \lambda \\ |\mu| \leq N}} \sigma(\mu) \prod_{i=1}^{v} \binom{\lambda_i}{\mu_i} + \tau(\lambda) = 0$$

The number of times $|C_{d,\lambda}|$ has to be added is given by $\tau(\lambda)$. We now have

$$|C_d(k)| = s_{d,\text{sieve}} + s_{d,\text{explicit}} + s_{d,\infty}$$

where

$$s_{d,\text{sieve}} := \frac{1}{|\mathrm{PGL}_3(k)|} \sum_{|\lambda| \leq N} \Big(\sigma(\lambda) \cdot \sum_{S \in \mathbb{P}^2(\lambda)} |V_k(d, 2S)| \Big), \tag{1}$$

$$s_{d,\text{explicit}} := \frac{1}{|\mathrm{PGL}_3(k)|} \sum_{N < |\lambda| \leq M} \tau(\lambda) \cdot |C_{d,\lambda}|, \tag{2}$$

$$s_{d,\infty} := -\frac{1}{|\mathrm{PGL}_3(k)|} \sum_{C \in C_{d,\infty}} \sum_{|\lambda| \leq N} \sigma(\lambda) \cdot |C(\lambda)|. \tag{3}$$

Here $C_{d,\infty}$ is the set of degree d plane curves that have an infinite number of singularities.

We will now describe how we will compute these sums.

Lemma 6.3. *For a partition* $\lambda = [1^{\lambda_1}, \dots, v^{\lambda_v}]$ *we have*

$$\sigma(\lambda) = (-1)^{\sum_{i=1}^{v} \lambda_i},$$
$$\tau(\lambda) = \sigma(\lambda) \cdot \sum_{\substack{\mu \subset \lambda \\ |\mu| < M - N}} \sigma(\mu) \prod_j \binom{\lambda_j}{\mu_j}.$$

Proof. First we use induction to show that for a a positive integer and b a non-negative integer we have $\sigma([a^b]) = (-1)^b$. By definition it is true for $\sigma([a^0]) = \sigma([]) = 1$. If it is true for $b \leq n$ then for $b = n+1$ we get from the definition of σ

$$0 = \sum_{i=0}^{n+1} \binom{n+1}{i} \sigma([a^i]) = \sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^i + \sigma([a^{n+1}]) - (-1)^{n+1}$$

And from the properties of the binomial coefficient we get

$$\sum_{i=0}^{n+1} \binom{n+1}{i} (-1)^i = (1-1)^{n+1} = 0$$

So we find that $\sigma([a^{n+1}]) = (-1)^{n+1}$.

In a similar fashion we show that $\sigma(\lambda) = \prod_{i=1}^v \sigma([i^{\lambda_i}])$ using induction on the weight of λ. For $|\lambda| = 0$ it is clearly true and if it is true for all λ of weight up to n then we get for $|\lambda| = n+1$

$$\begin{aligned}
0 &= \sum_{a_1=0}^{\lambda_1} \cdots \sum_{a_v=0}^{\lambda_v} \sigma([1^{a_1}, \dots, v^{a_v}]) \prod_{i=1}^{v} \binom{\lambda_i}{a_i} \\
&= \sum_{a_1=0}^{\lambda_1} \cdots \sum_{a_v=0}^{\lambda_v} \prod_{i=1}^{v} \sigma([i^{a_i}]) \binom{\lambda_i}{a_i} + \sigma(\lambda) - \prod_{i=1}^{v} \sigma([i^{\lambda_i}]) \\
&= \prod_{i=1}^{v} \sum_{a_i=0}^{\lambda_i} (-1)^i \binom{\lambda_i}{a_i} + \sigma(\lambda) - \prod_{i=1}^{v} \sigma([i^{\lambda_i}]) \\
&= \prod_{i=1}^{v} (1-1)^{\lambda_i} + \sigma(\lambda) - \prod_{i=1}^{v} \sigma([i^{\lambda_i}]) \\
&= \sigma(\lambda) - \prod_{i=1}^{v} \sigma([i^{\lambda_i}]).
\end{aligned}$$

So $\sigma(\lambda) = \prod_{i=1}^v \sigma([i^{\lambda_i}])$. Now it is easy to see that $\sigma(\lambda) = (-1)^{\sum_{i=1}^v \lambda_i}$.

From the definitions of σ and τ we find that

$$\begin{aligned}
\tau(\lambda) &= \sum_{\substack{\mu \subset \lambda \\ N < |\mu| \le M}} \sigma(\mu) \prod_j \binom{\lambda_j}{\mu_j} \\
&= \sum_{\substack{\mu \subset \lambda \\ N < |\mu| \le M}} (-1)^{\sum_j \mu_j} \prod_j \binom{\lambda_j}{\mu_j} \\
&= \sum_{\substack{\mu \subset \lambda \\ |\mu| < M - N}} (-1)^{\sum_j (\lambda_j - \mu_j)} \prod_j \binom{\lambda_j}{\lambda_j - \mu_j} \\
&= \sigma(\lambda) \cdot \sum_{\substack{\mu \subset \lambda \\ |\mu| < M - N}} \sigma(\mu) \prod_j \binom{\lambda_j}{\mu_j}.
\end{aligned}$$

□

Definition 6.4. A sieving partition is a partition of $\bigcup_{|\lambda| \le N} \mathbb{P}^2(\lambda)$ into subsets $U_0, \dots, U_n$ together with numbers $w_i, u_i \in \mathbb{Z}_{\ge 0}$ for $0 \le i \le n$ such that if $S \in U_i$ then we have $|S| = w_i$ and $|V_k(d, 2S)| = u_i$. We also write $U_{i,\lambda} := \{S \in U_i | \widehat{S} = \lambda\}$.

We can use a sieving partition to compute $s_{d,\mathrm{sieve}}$.

$$s_{d,\mathrm{sieve}} = \frac{1}{|\mathrm{PGL}_3(k)|} \sum_{i=0}^{n} \left(u_i \cdot \sum_{|\lambda| = w_i} \sigma(\lambda) \cdot |U_{i,\lambda}| \right) \tag{4}$$

When we create a sieving partition we have to look at what $|V_k(d, 2S)|$ becomes for different $S \subset \mathbb{P}^2$. The set $V_k(d)$ is simply the projective space spanned by all monomials of degree d. So $V_k(d) \cong \mathbb{P}^n$ where $n = \frac{d(d+3)}{2}$. For $V_k(d, 2S)$ we add $3|S|$ linear conditions to $V_k(d)$: one condition for a point being on the curve and two for it being a singular point. So if $V_k(d, 2S)$ is not empty then its minimal dimension is $\frac{d(d+3)}{2} - 3|S|$. If there are no dependencies between the $3|S|$ conditions then this minimal dimension is the actual dimension.

If we have points $P_1, \dots, P_n$ on a line L then for any curve $C \in V_k(d, 2P_1, \dots, 2P_n)$ we get for the intersection number that $L \cdot C \geq 2n$. So if $2n > d$ then by Bézout L is a component of C. This means we can see C as a degree $d-1$ curve times L. We get $V_k(d, 2P_1, \dots, 2P_n) \cong V_k(d-1, 1P_1, \dots, 1P_n)$. The minimal dimensions are given by $\frac{d(d+3)}{2} - 3n$ for $V_k(d, 2P_1, \dots, 2P_n)$ and $\frac{(d-1)(d+2)}{2} - n$ for $V_k(d-1, 1P_1, \dots, 1P_n)$. (Note that these dimensions are the same for $2n = d+1$.)

We will create our sieve partitions as follows: We will separate $\bigcup_{|\lambda| \leq N} \mathbb{P}^2(\lambda)$ into subsets based on $|\lambda|$ and on the dependencies we can find as above using Bézout. So an example would be the subset of all sets of five points such that there are precisely four points on a line. We will move to a space of lower degree curves as often as we can if this new space has a higher minimal dimension. When we can no longer do this we have found the "highest minimal dimension". Then we only need to prove that the highest minimal dimension is the actual dimension. Of course there is no reason to think that this is always true but it will turn out to be true in all the cases that we come across.

Example 6.5. Let us have a set of points $S = \{P_1, \dots, P_6, P_7\}$ where the points in $S' = \{P_1, P_2, P_3, P_4\}$ are on a line L and the points in $S'' = \{P_5, P_6, P_7\}$ are on another line L'. We want to know the dimension of $V_k(5, 2S)$. There are four points on the line L so L is part of any curve in $V_k(5, 2S)$. So $V_k(5, 2S) \cong V_k(4, 1S', 2S'')$. Now for $d = 4$ the points in S'' determine that L' is part of any curve in $V_k(4, 1S', 2S'')$ so we find $V_k(4, 1S', 2S'') = V_k(3, 1S)$. The projective space $V_k(3, 1S)$ has minimal dimension $9 - 7 = 2$. To see that the dimension is equal to 2 we only need to prove that the seven conditions on the $\mathbb{P}^9$ of cubics are independent. We will not do that in this example but it is not very hard to do.

Remark 6.6. For our partial sieving method we generally want to choose a high N since it is less work to compute $s_{d,\text{sieve}}$ than $s_{d,\text{explicit}}$. However if we choose N too high then when we try to make a sieving partition some of the highest minimal dimensions will be negative. This is a lot harder to work with so in practice we will want to choose N as high as possible such that this problem won't occur.

Definition 6.7. For any scheme X defined over k and $w \in \mathbb{Z}_{\geq 0}$ we define

$$\pi_w(X) := \sum_{|\lambda|=w} \sigma(\lambda) \cdot |X(\lambda)|$$

This construction will occur quite often, mainly as part of (4) when we have $U_{i,\lambda} \cong X(\lambda)$ for some scheme X.

In the following table we list some values of $\pi_w(X)$ where the rows represent different X and the columns represent different w. Remember that q is the number of elements in k.

	0	1	2	3	4	5	6	7
$\mathbb{P}^1$	1	$-q-1$	q	0	0	0	0	0
$\mathbb{P}^1 - \{P\}$	1	$-q$	0	0	0	0	0	0
$\mathbb{P}^2$	1	$-q^2-q-1$	q^3+q^2+q	$-q^3$	0	0	0	0
$\mathbb{P}^2 - \{P\}$	1	$-q^2-q$	q^3	0	0	0	0	0

Lemma 6.8. *In the following cases we will have $s_{d,\infty} = 0$.*

(i) $d = 3$, $N \geq 2$.

(ii) $d = 4$, $N \geq 3$.

(iii) $d = 5$, $N \geq 5$.

Proof. If we can show that for any $C \in C_{d,\infty}$ we have $\sum_{|\lambda| \leq N} \sigma(\lambda) \cdot C(\lambda) = 0$ then we can see from (3) that $s_{d,\infty} = 0$. Since $d \leq 5$ there are the following four possible types of double components: A k-line, a k-conic, two k-lines, and a conjugate 2-tuple of lines.

Let us take a curve C that has precisely one double k-line L. We use Z to denote the set of singular points of C that are not on L. The number of points in Z is at most $\frac{(d-2)(d-3)}{2}$. When choosing a λ-tuple of singular points on C we pick part of the points from Z and the rest from L.

$$\begin{aligned}
\sum_{|\lambda| \leq N} \sigma(\lambda) \cdot C(\lambda) &= \sum_{S \subset Z} \sum_{|\lambda| \leq N - |S|} \sigma([\widehat{S}, \lambda]) \cdot |L(\lambda)| \cdot \prod_{i=1}^{v} \binom{\widehat{Z}_i}{\widehat{S}_i} \\
&= \sum_{S \subset Z} \sigma(\widehat{S}) \cdot \prod_{i=1}^{v} \binom{\widehat{Z}_i}{\widehat{S}_i} \sum_{|\lambda| \leq N - |S|} \sigma(\lambda) \cdot |L(\lambda)| \\
&= \sum_{S \subset Z} \sigma(\widehat{S}) \cdot \prod_{i=1}^{v} \binom{\widehat{Z}_i}{\widehat{S}_i} \sum_{j=0}^{N - |\widehat{S}|} \pi_j(L)
\end{aligned}$$

Since $\sum_{i=1}^{n} \pi_i(\mathbb{P}^1) = 0$ for $n \geq 2$ we see that $\sum_{|\lambda| \leq N} \sigma(\lambda) \cdot C(\lambda) = 0$ if $N \geq |\lambda| + 2 \geq \frac{(d-2)(d-3)}{2} + 2$.

For a curve C that has precisely one double k-conic we can apply the same reasoning. Except that for $d \leq 5$ there are no singularities outside the double conic. So we end up with the result that $\sum_{|\lambda| \leq N} \sigma(\lambda) \cdot C(\lambda) = 0$ if $N \geq 2$.

For a a curve C that has precisely two double k-lines L, L' there are no singularities outside the double lines and we get

$$\begin{aligned}\sum_{|\lambda| \leq N} \sigma(\lambda) \cdot C(\lambda) &= \sum_{\lambda} \sum_{|\mu| \leq N - |\lambda|} \sigma([\lambda, \mu]) \cdot |L(\lambda)| \cdot |(L - L')(\mu)| \\ &= \sum_{\lambda} \sigma(\lambda) \cdot |L(\lambda)| \sum_{|\mu| \leq N - |\lambda|} \sigma(\mu) \cdot |(L - L')(\mu)| \\ &= \sum_{i=0}^{N} \pi_i(L) \sum_{i=j}^{N-i} \pi_j(L - L').\end{aligned}$$

Since $\sum_{i=1}^{n} \pi_i(\mathbb{P}^1) = 0$ for $n \geq 2$ and $\pi_j(\mathbb{P}^1 - \{P\}) = 0$ for $j > 1$ we find that $\sum_{|\lambda| \leq N} \sigma(\lambda) \cdot C(\lambda) = 0$ if $N \geq 3$.

We are left with the curves C that have a double conjugate 2-tuple of lines L, L'. Let P denote the k-point $L \cap L'$. For a partition $\lambda = [1^{\lambda_1}, 2^{\lambda_2}, \ldots, v^{\lambda_v}]$ we define $2\lambda = [2^{\lambda_1}, 4^{\lambda_2}, \ldots, (2v)^{\lambda_v}]$. All points on $(L \cup L') - \{P\}$ are defined over fields k_i such that i is even. This means that we can work over base k_2 when choosing our points. (For example a $[1^1, 2^1]$-tuple of points over base k_2 consists of a k_2 point and conjugate 2-tuple of k_4 points.) Picking a λ-tuple of points over base k_2 on $L - \{P\}$ also determines their conjugates and is thus equivalent to picking a 2λ-tuple of points over base k on $(L \cup L') - \{P\}$. Note that $\sigma(\lambda) = \sigma(2\lambda)$. We divide the sum $\sum_{|\lambda| \leq N} \sigma(\lambda) \cdot C(\lambda)$ into two parts. First we have the sum of λ-tuples of points that do not include the point P. Over base k_2 this sum is

$$\sum_{|\lambda| \leq \lfloor \frac{N}{2} \rfloor} \sigma(2\lambda) \cdot |(L - \{P\})(\lambda)| = \sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \pi_i(L - \{P\}).$$

When we do include the point P we get over base k_2

$$\sum_{|\lambda| \leq \lfloor \frac{N-1}{2} \rfloor} \sigma([[1^1], 2\lambda]) \cdot |(L - \{P\})(\lambda)| = - \sum_{i=0}^{\lfloor \frac{N-1}{2} \rfloor} \pi_i(L - \{P\}).$$

These two sums cancel each other out if $\lfloor \frac{N}{2} \rfloor \geq 1$ and $\lfloor \frac{N-1}{2} \rfloor \geq 1$ so we find that $\sum_{|\lambda| \leq N} \sigma(\lambda) \cdot C(\lambda) = 0$ if $N \geq 3$.

The case where we have exactly a double k-line imposes the condition $N \geq \frac{(d-2)(d-3)}{2} + 2$. For $d = 3$ this means $N \geq 2$, for $d = 4$, $N \geq 3$, and for

$d = 5$, $N \geq 5$. The other three cases do not occur for $d = 3$ and for $d = 4, 5$ their strongest demand is that $N \geq 3$. $\square$

7 Smooth plane cubics.

For $d = 3$ we see that $M = \frac{d(d-1)}{2} = 3$ and we choose $N = 3$. This means that $s_{3,\text{explicit}} = 0$ and by Lemma 6.8 we have $s_{3,\infty} = 0$. So we only need to count $s_{3,\text{sieve}}$. To do this we will choose a sieving partition.

We say a tuple of points is in general position if there are no three points on a line. For every $0 \leq w \leq 3$ we have the set U_w of w-tuples of points in general position. For $S \in U_w$ we see that $V_k(3, 2S)$ is a $\mathbb{P}^9$ of cubics with $3w$ linear conditions imposed on it. Because of 5.3 we can apply a $\bar{k}$-linear transformation to map the w-tuple of points to a subset of $\{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$. This makes it easy to see that the $3w$ conditions are independent. So $u_w = |\mathbb{P}^{9-3w}|$. For $0 \leq w \leq 2$ all w-tuples are in general position.

We write U_4 for 3-tuples of points that are on a line. We get a $\mathbb{P}^5$ of conics with 3 linear conditions imposed on it. By mapping the points to $(1 : 0 : 1), (1 : 0 : 0), (0 : 0 : 1)$ we see that the three conditions are independent so $u_4 = |\mathbb{P}^2|$.

Since for $0 \leq w \leq 2$ all w-tuples of points are in general position we find that $U_{w,\lambda} = \mathbb{P}^2(\lambda)$. To get a 3-tuple of points on a line we first pick a k-line and then a 3-tuple of points on that line. So $|U_{4,\lambda}| = |\mathbb{P}^2| \cdot |\mathbb{P}^1(\lambda)|$. All 3-tuples of points that are not all on a line are in general position so $|U_{3,\lambda}| = |\mathbb{P}^2(\lambda)| - |\mathbb{P}^2| \cdot |\mathbb{P}^1(\lambda)|$.

We get

$$\begin{aligned}
s_{3,\text{sieve}} &= \frac{1}{|\mathrm{PGL}_3(k)|} \sum_{i=0}^{4} \left(u_i \cdot \sum_{|\lambda| = w_i} \sigma(\lambda) \cdot |U_{i,\lambda}| \right) \\
&= \frac{1}{|\mathrm{PGL}_3(k)|} \left(\sum_{w=0}^{2} \left(|\mathbb{P}^{9-3w}| \cdot \pi_w(\mathbb{P}^2) \right) + |\mathbb{P}^0| \cdot (\pi_3(\mathbb{P}^2) - |\mathbb{P}^2| \cdot \pi_3(\mathbb{P}^1)) + |\mathbb{P}^2| \cdot |\mathbb{P}^2| \cdot \pi_3(\mathbb{P}^1) \right) \\
&= \frac{1}{|\mathrm{PGL}_3(k)|} \left(|\mathbb{P}^9| \cdot 1 - |\mathbb{P}^6| \cdot (q^2 + q + 1) + |\mathbb{P}^3| \cdot (q^3 + q^2 + q) + |\mathbb{P}^0| \cdot (-q^3 - 0) + 0 \right) \\
&= q
\end{aligned}$$

So $|C_3(\mathbb{F}_q)| = q$.

8 Smooth plane quartics.

For $d = 4$ we see that $M = \frac{d(d-1)}{2} = 6$. As we mentioned in Remark 6.6 we want to choose N as high as we can without getting problems with negative minimal dimensions.

We say a tuple of points is in general position if there are no three points on a line and no five points on a conic. For a set of w-tuples of points in general position we get a $\mathbb{P}^{14}$ of quartics with $3w$ linear conditions imposed on it. For $1 \le w \le 4$ it is not hard to see that the $3w$ conditions are independent. But for $w = 5$ and $w = 6$ we get a negative minimal dimension.

We can evade this problem since by Theorem 5.10 we find that any curve with four or more singularities is reducible. In fact a quartic can only have exactly five singularities if it consists of two lines and a conic, in which case there will be three points on a line. And a quartic can only have exactly six singularities if it consists of four lines, in which case there would also be three points on a line. So there are no quartics that have five or six singularities in general position.

For $N = 6$ we will still run into problems but for $N = 5$ the above workaround suffices since we do not get any negative minimal dimensions in other cases. So we choose $N = 5$. By Lemma 6.8 we have $s_{3,\infty} = 0$.

In our sieving partition we get for $1 \le w \le 4$ the set of w-tuples of points in general position. We also get the set of tuples of points that have five points on a conic and multiple sets of tuples of points that have at least three points on a line. (For example the set of 5-tuples of points that have four points on a line and one point outside the line.) We do not prove here that their resulting conditions are independent but it is not hard to see. Only the sets of points in general position contribute to the count because $\pi_w(\mathbb{P}^1) = 0$ for $w \ge 3$ and $\pi_w(\mathbb{P}^1 - \{P\}) = 0$ for $w \ge 2$.

As an example we take U_i, the set of 5-tuples of points that have three points on a k-line L and three points on a different k-line L'. To count $|U_{i,\lambda}|$ we first pick the two k-lines which can be done in $\binom{q^2+q+1}{2}$ ways. The intersection P of these lines is one of our five points and we pick two more points on each line.

$$\begin{aligned}
\sum_{|\lambda|=5} \sigma(\lambda) \cdot |U_{i,\lambda}| &= \binom{q^2+q+1}{2} \sum_{|\lambda|=2} \sum_{|\mu|=2} \sigma([\lambda, \mu]) \cdot (L - \{P\})(\lambda) \cdot (L' - \{P\})(\mu) \\
&= \binom{q^2+q+1}{2} \Big(\sum_{|\lambda|=2} \sigma(\lambda) \cdot (L - \{P\})(\lambda) \Big) \Big(\sum_{|\mu|=2} \sigma(\mu) \cdot (L' - \{P\})(\mu) \Big) \\
&= \binom{q^2+q+1}{2} \cdot \pi_2(\mathbb{P}^1 - \{P\}) \cdot \pi_2(\mathbb{P}^1 - \{P\}) \\
&= 0
\end{aligned}$$

Since the points that are in general position are the only type of points that do not result in zero we can pretend all points are in general position. (We actually have to subtract the cases where the points are not in general position but since we would only subtract zero it makes no difference to the result.)

$$
\begin{aligned}
s_{4,\mathrm{sieve}} &= \frac{1}{|\mathrm{PGL}_3(k)|}\sum_{w=0}^{4} |\mathbb{P}^{14-3w}| \cdot \pi_w(\mathbb{P}^2) \\
&= \frac{1}{|\mathrm{PGL}_3(k)|}\Big(|\mathbb{P}^{14}| \cdot 1 + |\mathbb{P}^{11}| \cdot -(q^2+q+1) + |\mathbb{P}^{8}| \cdot (q^3+q^2+q) + |\mathbb{P}^{5}| \cdot -q^3 \Big) \\
&= q^6
\end{aligned}
$$

For $|\lambda| = 6$ we will explicitly count $|C_{4,\lambda}|$, the number of smooth degree 4 plane curves over k that have exactly a λ-tuple of singularities. If a plane quartic has exactly six singularities then it consists of four lines. If these four lines are a $[1^4]$-tuple then we count the number of curves as follows. We first choose two k-points.

$$
|\mathbb{P}^2([1^2])| = \binom{q^2+q+1}{2}
$$

Then through each of these points we take two k-lines that do not pass through the other point.

$$
|(\mathbb{P}^1 - \{P\})([1^2])|^2 = \binom{q}{2}^2
$$

Now we have four k-lines that intersect in six points. However for every way we pick an unordered pair of unordered pairs of lines from the four lines we get the same four lines, so we have to divide by 3.

The cases for the other λ are computed in a similar fashion. For example when we have a $[1^1, 3^1]$-tuple of lines we first take a k-line: $q^2 + q + 1$. And then we choose a conjugate 3-tuple of lines such that these lines do not intersect each other in one k-point. To count this we first choose any conjugate 3-tuple of lines and then subtract for every k-point P the number of conjugate 3-tuples of lines through P.

$$
\frac{1}{3}(q^6+q^3-q^2-q) - (q^2+q+1)\frac{1}{3}(q^3-q) = \frac{1}{3}(q^6-q^5-q^4+q^3)
$$

We will divide $C_{4,\lambda}$ into smaller parts. Since proper subscripts would overly complicate the notation we will simply denote these parts by C for all λ. This is quite vague but since we will only use it in tables it will be clear enough what is meant.

To help see how we computed $|C|$ we make the following distinction in all our tables. We write $|(\mathbb{P}^1 - \{P\})([1^2])| = \binom{q}{2}$ and $|(\mathbb{P}^1)([2^1])| = \frac{1}{2}(q^2 - q)$.

lines	λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^4]$	$[1^6]$	1	$\frac{1}{3}\binom{q^2+q+1}{2}\binom{q}{2}^2$
$[1^2, 2^1]$	$[1^2, 2^2]$	1	$\frac{1}{2}\binom{q^2+q+1}{2}\binom{q}{2}\frac{1}{2}(q^2 - q)$
$[1^1, 3^1]$	$[3^2]$	1	$(q^2 + q + 1)\frac{1}{3}(q^6 - q^5 - q^4 + q^3)$
$[2^2]$	$[1^2, 2^2]$	1	$\binom{q^2+q+1}{2}(\frac{1}{2}(q^2 - q))^2$
$[4^1]$	$[2^1, 4^1]$	1	$\frac{1}{4}((q^8 - q^2) - (q^2 + q + 1)(q^4 - q^2))$

We get

$$\sum \frac{\tau(\lambda) \cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{24} + \frac{1}{8} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = 1$$

where the sum is over the rows in the table.
So we have proven the following theorem.

Theorem 8.1. ([3, 2.13]) *The number of smooth plane quartics over a finite field* $\mathbb{F}_q$ *is given by*

$$|C_4(\mathbb{F}_q)| = q^6 + 1.$$

9 Smooth plane quintics.

For $d = 5$ we see that $M = \frac{d(d-1)}{2} = 10$ and we choose $N = 7$. Now by Lemma 6.8 we have $s_{5,\infty} = 0$.

9.1 The sieve count.

We create a sieving partition.

9.1.1 All points in general position.

We say a tuple of points is in general position if there are no four points on a line and no six points on a conic. Note that this includes reducible conics. By Theorem 5.10 we find that any curve with seven or more singularities is reducible. So a curve with seven or more singularities either has infinitely many singularities or is of type $[1, 1, 1, 1, 1]$, $[2, 1, 1, 1]$, $[2, 2, 1]$, $[3, 1, 1]$, $[3, 2]$, or $[4, 1]$.

Lemma 9.1. *A plane quintic of type* $[1, 1, 1, 1, 1]$*,* $[2, 1, 1, 1]$*,* $[2, 2, 1]$*,* $[3, 1, 1]$*,* $[3, 2]$*, or* $[4, 1]$ *cannot have seven singularities in general position.*

Proof. If a curve of type $[1,1,1,1,1]$ has at least seven singular points then either no three lines intersect in one point or there is precisely one point where precisely three lines intersect. In either case there are four singular points on a line.

If a curve of type $[2,1,1,1]$ has at least seven singular points then it has four singular points on a line unless all three lines intersect in the same point in which case there are six singular points on the conic.

If a curve of type $[2,2,1]$ has at least seven singular points then there are six singular points on a conic unless the conics intersect in precisely three points in which case there are four singular points on the line.

If a curve of type $[3,1,1]$ has at least seven singular points then there are four singular points on a line.

If a curve of type $[3,2]$ has at least seven singular points then there are six singular points on the conic.

If a curve of type $[4,1]$ has at least seven singular points then there are four singular points on the line. $\square$

This means we can just do the dimension proof for the case where we have six points in general position, since for any lower number of points the conditions will be a subset of the conditions for some tuple of six points.

Lemma 9.2. *If $P_1, \ldots, P_6$ form a λ-tuple of points in general position where λ is any distribution of weight six, then $V_k(5, 2P_1, \ldots, 2P_6) \cong \mathbb{P}^2$.*

Proof. We get a $\mathbb{P}^{20}$ of quartics with 18 linear conditions imposed on it so we need to proof that the conditions are all independent.

Because of 5.3 we can apply a $\bar{k}$-linear transformation to map the λ-tuple of points to $(1:0:0), (0:1:0), (0:0:1), (1:1:1), (s:\alpha:\beta), (t:\gamma:\delta)$ for some $s, t, \alpha, \beta, \gamma, \delta$. If $s = t = 0$ then we have four points on a line so without loss of generality we can take $s = 1$. This leaves us with two cases: $t = 1$ and $t = 0$. We start with $t = 1$.

We look at the matrix where the columns correspond to the monomials of degree 5. For every point we get three rows: one with the values at the monomials and two with the values at the derivatives of the monomials. Here we take the derivatives to x and z.

We can remove the rows corresponding to $(1:0:0), (0:1:0), (0:0:1)$ while also removing the nine corresponding columns. Now we have a 9×12 matrix; if we can remove three columns such that we get a square matrix with nonzero determinant then the rank of the matrix is 9 and the conditions are all independent. We remove the columns corresponding to y^2z^3, xyz^3, x^2z^3 to get determinant

$$(\alpha\beta\gamma - \alpha\beta\delta - \alpha\gamma\delta + \beta\gamma\delta - \beta\gamma + \alpha\delta)^4(\alpha - \gamma)(\alpha - 1)\alpha^2(\gamma - 1)\gamma^2.$$

If $\alpha\beta\gamma - \alpha\beta\delta - \alpha\gamma\delta + \beta\gamma\delta - \beta\gamma + \alpha\delta = 0$ then there are six points on a conic.

If $\alpha = \gamma$ then we remove the columns corresponding to y^2z^3, xyz^3, y^3z^2 instead to get determinant

$$(\alpha - \beta)(\alpha - \delta)(\alpha - 1)^4\alpha^6(\beta - \delta)^5.$$

If $\alpha = 0, 1$ we have four points on a line and if $\beta = \delta$ then two of our points are the same. If $\alpha = \beta$ then we remove columns y^2z^3, xyz^3, xy^2z^2 to get

$$-(\alpha - \delta)^6(\alpha - 1)^5\alpha^7.$$

If this is zero then we have four points on a line or two points are the same. So we have proven that the conditions are independent if $\alpha = \gamma = \beta$. By symmetry we have also proven it for $\alpha = \gamma = \delta$ and thus again by symmetry for $\alpha = \gamma$ and $\beta = \delta$.

If $\alpha = 1$ then we remove columns y^2z^3, xyz^3, y^3z^2 to get

$$-(\beta\gamma - \delta)(\beta - 1)^5(\gamma - \delta)(\gamma - 1)^4\gamma^2\delta^2.$$

If $\gamma = 1$ we have four points on a line, if $\beta = 1$ then two of our points are the same, and if $\delta = 0$ then there are six points on a reducible conic. From here on we replace the rows corresponding to derivatives to x by rows corresponding to derivatives to y. If $\gamma = 0$ then we remove columns y^2z^3, y^3z^2, xy^2z^2 to get

$$-(\beta - 1)^5\delta^5.$$

If $\gamma = \delta$ then we remove columns y^2z^3, xyz^3, xy^2z^2 to get

$$-(\beta - 1)^6(\gamma - 1)^5\gamma^5.$$

If $\delta = \beta\gamma$ then we remove columns y^2z^3, xyz^3, xy^2z^2 to get

$$(\beta - 1)^6\beta^3(\gamma - 1)^5\gamma^5.$$

If $\alpha = 0$ then we remove columns y^2z^3, xyz^3, x^2yz^2 to get

$$-\beta^5(\gamma - 1)\gamma^6(\delta - 1)^4.$$

If $\gamma = 0$ we have four points on a line, if $\beta = 0$ then two of our points are the same, and if $\delta = 1$ then there are six points on a reducible conic. If $\gamma = 1$ then we have a case symmetric to $\alpha = 1, \gamma = 0$.

By symmetry we have also covered $\gamma = 0, 1$ so we are done with the case where $t = 1$.

Now for $t = 0$ we again take derivatives to x and z and we remove columns xyz^3, xy^2z^2, x^3z^2 to get determinant

$$(\alpha - \beta)^5\alpha^2\gamma^8\delta^5.$$

If $\alpha = 0$ we have four on a line, if γ or δ is zero the two points are the same, and if $\alpha = \beta$ then there are six points on a reducible conic.

We have shown that there are no dependencies. $\square$

When we descripe a subset of tuples of points in our sieving partition, we do not explicitly describe all the points. So when we say "four points on a line", we leave implicit that there are up to three other points and that these three other points are not covered by one of the other cases such as "four points on a line and three other points on another line". Also when we say "four points on a line and three other points on another line" we are referring to a total of seven different points. So the four points are different from each other and the three points are also different from each other. Furthermore our use of "other" implies that the four points are distinct from the three points.

We only show that the dimension is the expected one for the cases where we have seven points in total. Since if the conditions corresponding to seven points are independent then the conditions corresponding to a subset of the seven points are also independent. We always remove the rows and columns corresponding to $(1:0:0),(0:1:0),(0:0:1)$ from our matrices since their conditions can easily be separated from the others this way.

9.1.2 Five to seven points on a line.

The line is a double component and we are left with a $\mathbb{P}^9$ of cubics with at most six conditions from the points not on the line. It is easy to see that these conditions are independent.

9.1.3 Four points on a line and four points on another line.

Note that in cases like this where we have an equal number of points on each line, the lines could be a conjugate 2-tuple. However since we use a $\bar{k}$-linear transformation on the points this does not affect the argument.

We map the intersection point of the lines to $(0:0:1)$. The other points are mapped to $(1:0:0),(1:0:1),(\alpha:0:1)$ and $(0:1:0),(0:\beta:1),(0:\gamma:1)$ where $\alpha\neq 0,1$, $\beta,\gamma\neq 0$, and $\beta\neq\gamma$. The points determine both lines so we get a space of cubics through six points. Looking at the conditions for $(\alpha:0:1),(0:\beta:1),(0:\gamma:1)$ and columns yz^2,xz^2,y^2z,x^2z we get determinant

$$(\alpha-1)\alpha(\beta-\gamma)\beta\gamma.$$

9.1.4 Four points on a line and three other points on another line.

We map the four points on a line to $(1:0:0),(0:0:1),(\alpha:0:1),(\beta:0:1)$ with $\alpha,\beta\neq 0$ and $\alpha\neq\beta$. The two other points we map to $(0:1:0),(1:1:1),(\gamma:1:\gamma)$ such that $\gamma\neq 0,1$. The points determine both lines so we get a space of cubics through seven points. Looking at columns yz^2,xz^2,y^2z,x^2z we get determinant

$$(\alpha-\beta)\alpha\beta(\gamma-1)\gamma.$$

9.1.5 Three points on a line and three other points on another line.

We map the points on the lines to $(1:0:0), (0:0:1), (\alpha:0:1)$ and $(0:1:0), (1:1:1), (\beta:1:\beta)$ such that $\alpha \neq 0$ and $\beta \neq 0, 1$. The other point we map to $(\gamma:1:\delta)$ where $\gamma \neq \delta$. The points determine both lines so we get a space of cubics through seven points that have a singularity at $(\gamma:1:\delta)$. Taking derivatives to x and z and removing the column for xz^2 we get determinant

$$-\alpha^2(\beta-1)\beta(\gamma-\delta)^3.$$

9.1.6 Four points on a line.

We map the four points on a line to $(1:0:0), (0:0:1), (\alpha:0:1), (\beta:0:1)$ with $\alpha, \beta \neq 0$ and $\alpha \neq \beta$. The two other points we map to $(0:1:0), (1:1:1), (\gamma:1:\delta)$ such that $\gamma \neq \delta$. The points determine the line so we get a space of quartics through seven points that have singularities at $(0:1:0), (1:1:1), (\gamma:1:\delta)$. Taking derivatives to x and z and removing the columns xz^3, y^2z^2 we get determinant

$$(\alpha-\beta)\alpha^2\beta^2(\gamma-\delta)^5\gamma,$$

so $c = 0$ but then we remove xz^3, xyz^2 instead to get

$$(\alpha-\beta)\alpha^2\beta^2\delta^5,$$

so $d = 0$ but then we would have $c = d$.

9.1.7 Seven points on an irreducible conic.

The conic is a double component of the curve so we get a $\mathbb{P}^2$ of lines with no conditions.

9.1.8 Six points on an irreducible conic.

We have six points $P_1, \ldots, P_6$ on a conic C and one more point Q outside the conic. Among $P_1, \ldots, P_6$ there are no three points on a line since otherwise C would be reducible.

The points determine the conic so we get a $\mathbb{P}^9$ of cubics with 9 conditions on it. We first show that when we leave out the condition for P_6 we get 8 independent conditions.

We map $P_1, \ldots, P_5$ to $(1:0:0), (0:1:0), (0:0:1), (1:1:1), (1:\alpha:\beta)$ with $\alpha, \beta \neq 0, 1$ and $\alpha \neq \beta$. We also map Q to $(1:\gamma:\delta)$ with the condition that it is not on the conic, i.e. $\alpha\beta\gamma - \alpha\beta\delta - \alpha\gamma\delta + \beta\gamma\delta - \beta\gamma + \alpha\delta \neq 0$. We take tangent conditions dy, dz. First we remove the columns for yz^2, xz^2 to get determinant

$$-(\alpha\beta\gamma - \alpha\beta\delta - \alpha\gamma\delta + \beta\gamma\delta - \beta\gamma + \alpha\delta)(\alpha-\gamma)(\gamma-1)\gamma.$$

If $c = 0$ then we remove columns yz^2, xyz instead to get determinant

$$\alpha^2(\beta - 1)\delta^2.$$

If $c = a$ then we remove columns yz^2, xy^2 to get determinant

$$-(\beta - \delta)^2\delta(\alpha - 1)^2\alpha.$$

Now if $b = d$ then $Q = (1 : a : b)$. If $d = 0$,then we remove columns yz^2, y^2z to get determinant

$$-\beta^2(\alpha - 1)\alpha^2.$$

If $c = 1$ then we remove columns yz^2, xy^2 to get determinant

$$(\alpha - 1)^2\beta(\delta - 1)^2\delta.$$

Now if $d = 1$ then $Q = (1 : 1 : 1)$, so $d = 0$, now we remove columns yz^2, y^2z to get determinant

$$-(\alpha - 1)\beta.$$

We have shown that the determinant will not become zero and the 8 conditions are independent.

Now if we can find a cubic that passes through $P_1, \ldots, P_5$ and has a singularity at Q but does not pass through P_6 then the condition for P_6 is independent from the other 8 conditions. There is a point among $P_1, \ldots, P_5$ that is not on the line through P_6 and Q, without loss of generality we assume this point is P_5. Now taking the cubic given by the conic through $P_1, \ldots, P_4, Q$ together with the line through P_5, Q satisfies our demands.

9.1.9 Counting $s_{5,\text{sieve}}$.

Because $\pi_w(\mathbb{P}^1) = 0$ for $w \geq 4$ and $\pi_w(\mathbb{P}^1 - \{P\}) = 0$ for $w \geq 2$ all the counts become zero besides the count of the cases where all points are in general position. As we did with quartics we can pretend all points are in general position since subtracting the other cases is just subtracting zero.

$$\begin{aligned}
s_{5,\text{sieve}} &= \frac{1}{|\mathrm{PGL}_3(k)|} \sum_{w=0}^{6} |\mathbb{P}^{20-3w}| \cdot \pi_w(\mathbb{P}^2) \\
&= \frac{1}{|\mathrm{PGL}_3(k)|} \left(|\mathbb{P}^{20}| \cdot 1 + |\mathbb{P}^{17}| \cdot -(q^2 + q + 1) + |\mathbb{P}^{14}| \cdot (q^3 + q^2 + q) + |\mathbb{P}^{11}| \cdot -q^3 \right) \\
&= q^{12}
\end{aligned}$$

9.2 The explicit count.

We will count the different types of curves separately. First we note that when $|\lambda| = 8$ we get

$$\tau(\lambda) = \sigma(\lambda).$$

When $|\lambda| = 9$ we get

$$\tau(\lambda) = \sigma(\lambda)(1 - \binom{\lambda_1}{1}).$$

And when $|\lambda| = 10$ we get

$$\tau(\lambda) = \sigma(\lambda)(1 - \binom{\lambda_1}{1} - \binom{\lambda_2}{1} + \binom{\lambda_1}{2}).$$

We will not list the cases where $\tau(\lambda) = 0$ in our tables.

9.2.1 $[1,1,1,1,1]$, three lines through one point Q.

We start with the case where we have a $[1^5]$-tuple of lines, i.e. all lines are k-lines. We first take two k-points. Then we choose two k-lines through one of the k-points and three k-lines through the other such that none of these lines goes through the two chosen k-points. Together this gives us

$$\binom{q^2+q+1}{2} 2 \binom{q}{2}\binom{q}{3}.$$

The other cases are computed in a similar matter. (In all cases Q has to be a k-point.) We put the results in a table.

The values in the table should be multiplied with $\binom{q^2+q+1}{2}2$.

lines	λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^5]$	$[1^8]$	1	$\binom{q}{2}\binom{q}{3}$
$[1^3, 2^1]$, only one k-line through Q	$[1^4, 2^2]$	1	$\binom{q}{2}q\frac{1}{2}(q^2-q)$
$[1^3, 2^1]$, three k-lines through Q	$[1^2, 2^3]$	-1	$\frac{1}{2}(q^2-q)\binom{q}{3}$
$[1^1, 2^2]$	$[1^2, 2^3]$	-1	$\frac{1}{2}(q^2-q)q\frac{1}{2}(q^2-q)$
$[1^2, 3^1]$	$[1^2, 3^2]$	1	$\binom{q}{2}\frac{1}{3}(q^3-q)$
$[2^1, 3^1]$	$[1^2, 6^1]$	-1	$\frac{1}{2}(q^2-q)\frac{1}{3}(q^3-q)$

We get

$$\sum \frac{\tau(\lambda)\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{12}(q-2) + \frac{1}{4}q - \frac{1}{12}(q-2) - \frac{1}{4}q + \frac{1}{6}(q+1) - \frac{1}{6}(q+1) = 0$$

where the sum is over the rows in the table.

9.2.2 $[1,1,1,1,1]$, no three lines through one point.

For a $[1^5]$-tuple of lines we first take two k-points. Then we choose two k-lines through one of the k-points and two k-lines through the other such that none of these lines goes through the two chosen k-points. This gives $\binom{q^2+q+1}{2}\binom{q}{2}^2$. Then we pick a point P on one of the lines and a line though P but not through any of the 6 intersection points that we already have. This gives $4(q-2)(q-3)$, which we then have to divide again by 4 since we will get a chosen line four times (once for each of its intersection points with the other lines). For every way we pick an unordered pair of unordered pairs of lines from the five lines we get the same five lines, so we have to divide by 15.

We have a table with the results:

lines	λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^5]$	$[1^{10}]$	36	$\frac{1}{15}\binom{q^2+q+1}{2}\binom{q}{2}^2(q-2)(q-3)$
$[1^1, 2^2]$	$[1^2, 2^4]$	-4	$\binom{q^2+q+1}{2}(\frac{1}{2}(q^2-q))^2(q+1)(q-2)$
$[5^1]$	$[5^2]$	1	$\frac{1}{5}(q^{10}+q^5-q^2-q)-\frac{1}{5}(q^2+q+1)(q^5-q)$

We get

$$\sum \frac{\tau(\lambda)\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{3}{10}(q-2)(q-3) - \frac{1}{2}(q+1)(q-2) + \frac{1}{5}(q^2+1) = -q+3$$

where the sum is over the rows in the table.

9.2.3 Tools for conics.

Definition 9.3. A curve C in $\mathbb{P}^n$ is called strange if there is a point S which lies on all the tangent lines of C. We call S the strange point of C.

Lemma 9.4. *Let C be a plane curve and S a point not on C. The following are equivalent:*

(i) C is strange with strange point S.

(ii) Every line through S is tangent to C.

And when (i) holds S is the unique strange point of C.

Proof. $(i) \Rightarrow (ii)$: Any line L through S intersects C in at least one point P. The tangent of C at P goes through S so it is L.
$(ii) \Rightarrow (i)$: The line through S and any point on C is a tangent line. So S lies on the tangent line of C at every point of C.

If C has another strange point S' then all tangent lines pass through S and S'. But there is only one line through two points so there is only one tangent line. This means that C is a line, which contradicts the assumption that S is not on C. □

Proposition 9.5. *Given an irreducible plane $\bar{k}$-conic C.*

(i) If char(k) $\neq 2$ then for every point P that is not on the conic there are precisely two lines through P that are tangent to the conic.

(ii) If char(k) $= 2$ then the conic is strange.

Proof. Without loss of generality we let the point be $P = (0 : 0 : 1)$, we have a $\bar{k}$-conic C given by

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

where $a, b, c, d, e, f \in \bar{k}$ are the coefficients. Since P is not on C we know that $c \neq 0$. A line through P is of the form $rx - sy$. We look at the lines where $r \neq 0$ so we can take $r = 1$ and $x = sy$. This gives us

$$(as^2 + ds + b)y^2 + (es + f)yz + cz^2.$$

Now for the char(k) $\neq 2$ case the equation will have discriminant

$$(es + f)^2 - 4c(as^2 + ds + b).$$

This will have a zero in s unless a, d, e are all zero which would mean that C is not irreducible. Now we know that there is at least one tangent line and we can apply a $\bar{k}$-linear transformation such that it becomes the line y tangent at $(1 : 0 : 0)$. Then a, e are zero so we get precisely one more zero from the above equation.

For the char(k) $= 2$ case we see that we get a double zero iff $es + f = 0$. But this is only for the lines where $r \neq 0$. In general we get a double zero iff $es + rf = 0$. If $e = f = 0$ then for all lines we get a double zero and otherwise there is exactly one solution. So either P is strange or precisely one tangent line of C passes through P.

We take two different points on C. The tangent lines of C at these points are different and intersect in a point P' outside the conic. This point P' has more than one tangent line of C passing through it so it is the strange point. □

Lemma 9.6. *Let char(k) $\neq 2$, C be an irreducible k-conic and P a k-point that is not on C. We have two cases for the lines through P, there are*

1. *2 k-lines tangent to C,*
   $\frac{q-1}{2}$ k-lines that intersect C in two k-points,
   $\frac{q-1}{2}$ k-lines that intersect C in a conjugate 2-tuple of points,
   $\frac{(q-1)^2}{4}$ conjugate 2-tuples of lines that intersect C in two conjugate 2-tuples of points,
   $\frac{q^2-1}{4}$ conjugate 2-tuples of lines that intersect C in a conjugate 4-tuple of points.

2. 1 k_2-tuple of lines tangent to C,
$\frac{q+1}{2}$ k-lines that intersect C in two k-points,
$\frac{q+1}{2}$ k-lines that intersect C in a conjugate 2-tuple of points,
$\frac{(q+1)(q-3)}{4}$ conjugate 2-tuples of lines that intersect C in two conjugate 2-tuples of points,
$\frac{q^2-1}{4}$ conjugate 2-tuples of lines that intersect C in a conjugate 4-tuple of points.

When char$(k) = 2$ then the cases for the lines through P are:

1. P *is a strange point for* C.

2. 1 k-line tangent to C,
$\frac{q}{2}$ k-lines that intersect C in two k-points,
$\frac{q}{2}$ k-lines that intersect C in a 2-tuple of points,
$\frac{q(q-2)}{4}$ conjugate 2-tuples of lines that intersect C in two conjugate 2-tuples of points,
$\frac{q^2}{4}$ conjugate 2-tuples of lines that intersect C in a conjugate 4-tuple of points.

Proof. Since the proofs are all similar and simple we just do the first case for char$(k) \neq 2$: If there are two k-lines through P tangent to C, then there are $q-1$ other points on C. A point on C determines a k-line from it to P and all non-tangent lines intersect C in two points. So we get $\frac{q-1}{2}$ k-lines that intersect C in two k-points. Which leaves $q+1-2-\frac{q-1}{2} = \frac{q-1}{2}$ k-lines that intersect C in a conjugate 2-tuple of points.

So there are $\frac{q^2-1}{2}$ k_2-lines that intersect C in two k_2-points. From this we subtract the $q-1$ k-lines that intersect C in two k_2-points. This leaves us with $\frac{1}{2}(\frac{q^2-1}{2} - (q-1)) = \frac{1}{4}(q-1)^2$ conjugate 2-tuples of lines that intersect C in two k_2-points. Since a k_2-tuple of lines will not intersect C in two k points we have the answer.

There are $\frac{q^2-1}{2}$ k_2-lines that intersect C in a conjugate 4-tuple of points. □

Lemma 9.7. *Let char $\neq 2$, C be an irreducible k-conic.*
There are $\binom{q+1}{2}$ points outside C that are the intersection of two k-tangent lines of the conic.
There are $\binom{q}{2}$ points outside C that are the intersection of a conjugate 2-tuple of tangent lines of the conic.

Proof. There are $\binom{q+1}{2}$ ways to pick two different k-points of the conic and there are $\frac{1}{2}(q^2-q) = \binom{q}{2}$ k_2-tuples of points. The tangents of C at these pairs of points are pairs of tangents that intersect in a point outside C. □

Lemma 9.8. *The number of irreducible k-conics is*

$$(q^2+q+1)q^2(q-1).$$

Proof. There is a $\mathbb{P}^5$ of conics and every reducible k-conic is either a pair of different k-lines, a conjugate 2-tuple of lines, or a double k-line.

$$\frac{q^6-1}{q-1} - \binom{q^2+q+1}{2} - \frac{1}{2}(q^4-q) - (q^2+q+1) = (q^2+q+1)q^2(q-1)$$

$\square$

9.2.4 $[2,1,1,1]$, one of the lines is tangent to the conic.

The case where precisely one of the lines is tangent to the conic, no lines intersect on the conic, and the three lines do not intersect in one point.

To get a $[1^3]$-tuple of lines and a $[1^8]$-tuple of points we first pick an irreducible k-conic and a k-point Q on the conic: $(q^2+q+1)q^2(q-1)(q+1)$. The tangent line at Q is one of our three lines. We choose 4 other k-points on the conic and 2 lines through those points: $\binom{q}{4}3$. From this we subtract the cases where the resulting 3 lines intersect in one point. If $\mathrm{char}(k) \neq 2$ then through every k-point on the tangent line besides Q there are $\frac{q-1}{2}$ k-lines that intersect the conic in two k-points: $q\binom{\frac{q-1}{2}}{2}$. If $\mathrm{char}(k) = 2$ then through every non-strange k-point on the tangent line besides Q there are $\frac{q}{2}$ k-lines that intersect the conic in two k-points: $(q-1)\binom{\frac{q}{2}}{2}$.

For char$(k) \neq 2$.
The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)(q+1)$.

lines	λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3]$	$[1^8]$	1	$\binom{q}{4}3 - q\binom{\frac{q-1}{2}}{2}$
	$[1^6,2^1]$	-1	$\binom{q}{2}\frac{1}{2}(q^2-q) - q\left(\frac{q-1}{2}\right)^2$
	$[1^4,2^2]$	1	$\binom{\frac{1}{2}(q^2-q)}{2} - q\binom{\frac{q-1}{2}}{2}$
$[1^1,2^1]$	$[1^2,2^3]$	-1	$\binom{\frac{1}{2}(q^2-q)}{2}2 - q\frac{(q-1)^2}{4}$
	$[1^2,2^1,4^1]$	1	$\frac{1}{4}(q^4-q^2) - q\frac{q^2-1}{4}$

We get

$$\sum \frac{\tau(\lambda)\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{8}\Big((q-3)^2 - 2(q-1)^2 + (q-1)^2 - 2(q^2-2q-1) + 2(q+1)(q-1)\Big) = 1$$

where the sum is over the rows in the table.

For char$(k) \neq 2$.

The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)(q+1)$.

lines	λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3]$	$[1^8]$	-	$\binom{q}{4}3-(q-1)\binom{\frac{q}{2}}{2}$
	$[1^6,2^1]$	+	$\binom{q}{2}\frac{1}{2}(q^2-q)-(q-1)\left(\frac{q}{2}\right)^2$
	$[1^4,2^2]$	-	$\binom{\frac{1}{2}(q^2-q)}{2}-(q-1)\binom{\frac{q}{2}}{2}$
$[1^1,2^1]$	$[1^2,2^3]$	+	$\binom{\frac{1}{2}(q^2-q)}{2}2-(q-1)\frac{q(q-2)}{4}$
	$[1^2,2^1,4^1]$	-	$\frac{1}{4}(q^4-q^2)-(q-1)\frac{q^2}{4}$

We get

$$\sum \frac{\tau(\lambda)\cdot\lvert C\rvert}{\lvert \mathrm{PGL}_3(k)\rvert}=\frac{1}{8}\Big((q-2)(q-4)-2q(q-2)+q(q-2)-2(q^2-2q)+2q^2\Big)=1$$

where the sum is over the rows in the table.

9.2.5 $[2,1,1,1]$, nine singularities.

We get nine singularities when none of the lines is tangent to the conic, no lines intersect on the conic, and the three lines do not intersect in one point.

We will have to subtract the case where we have three lines through one point so we count that first.

To get a $[1^3]$-tuple of lines and a $[1^7]$-tuple of points we pick an irreducible k-conic and choose a k-point Q outside the conic that is the intersection of two k-tangents of the conic. By Lemma 9.7 this gives $(q^2+q+1)q^2(q-1)\binom{q+1}{2}$. Then we pick three lines through Q that intersect the conic in two k-points: $\binom{\frac{q-1}{2}}{3}$. We can also choose a k-point Q outside the conic that is the intersection of a k_2-tuple of tangents of the conic: $\binom{q}{2}$. And picking three lines through Q that intersect the conic in two k-points gives $\binom{\frac{q+1}{2}}{3}$.

For char$(k)\neq 2$.

The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)$.

lines	λ points	$\lvert C\rvert$
$[1^3]$	$[1^7]$	$\binom{q+1}{2}\binom{\frac{q-1}{2}}{3}+\binom{q}{2}\binom{\frac{q+1}{2}}{3}$
	$[1^5,2^1]$	$\binom{q+1}{2}\binom{\frac{q-1}{2}}{2}\frac{q-1}{2}+\binom{q}{2}\binom{\frac{q+1}{2}}{2}\frac{q+1}{2}$
	$[1^3,2^2]$	$\binom{q+1}{2}\frac{q-1}{2}\binom{\frac{q-1}{2}}{2}+\binom{q}{2}\frac{q+1}{2}\binom{\frac{q+1}{2}}{2}$
	$[1^1,2^3]$	$\binom{q+1}{2}\binom{\frac{q-1}{2}}{3}+\binom{q}{2}\binom{\frac{q+1}{2}}{3}$
$[1^1,2^1]$	$[1^3,2^2]$	$\binom{q+1}{2}\frac{q-1}{2}\frac{(q-1)^2}{4}+\binom{q}{2}\frac{q+1}{2}\frac{(q+1)(q-3)}{4}$
	$[1^3,4^1]$	$\binom{q+1}{2}\frac{q-1}{2}\frac{q^2-1}{4}+\binom{q}{2}\frac{q+1}{2}\frac{q^2-1}{4}$
$[3^1]$	$[1^1,3^2]$	$q^2\frac{1}{6}(q^3-q)$
	$[1^1,6^1]$	$q^2\frac{1}{6}(q^3-q)$

For char$(k)=2$.
The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)(q^2-1)$.

lines	λ points	$\lvert C\rvert$
$[1^3]$	$[1^7]$	$\binom{\frac{q}{2}}{3}$
	$[1^5,2^1]$	$\binom{\frac{q}{2}}{2}\frac{q}{2}$
	$[1^3,2^2]$	$\frac{q}{2}\binom{\frac{q}{2}}{2}$
	$[1^1,2^3]$	$\binom{\frac{q}{2}}{3}$
$[1^1,2^1]$	$[1^3,2^2]$	$\frac{q}{2}\frac{q(q-2)}{4}$
	$[1^3,4^1]$	$\frac{q}{2}\frac{q^2}{4}$
$[3^1]$	$[1^1,3^2]$	$\frac{1}{6}(q^3-q)$
	$[1^1,6^1]$	$\frac{1}{6}(q^3-q)$

Now back to the case where there are no three lines through one point. For a $[1^3]$-tuple of lines and a $[1^9]$-tuple of points we pick an irreducible k-conic and choose six k-points on the conic: $(q^2+q+1)q^2(q-1)\binom{q+1}{6}$. We then choose three lines through these six points: 15.

The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)$.

lines	λ points	$\tau(\lambda)$	$\|C\|$
$[1^3]$	$[1^9]$	8	$\binom{q+1}{6}15$
	$[1^7,2^1]$	-6	$\binom{q+1}{4}\frac{1}{2}(q^2-q)3$
	$[1^5,2^2]$	4	$\binom{q+1}{2}\binom{\frac{1}{2}(q^2-q)}{2}$
	$[1^3,2^3]$	-2	$\binom{\frac{1}{2}(q^2-q)}{3}$
$[1^1,2^1]$	$[1^3,2^3]$	-2	$\binom{q+1}{2}\binom{\frac{1}{2}(q^2-q)}{2}2$
	$[1^3,2^1,4^1]$	2	$\binom{q+1}{2}\frac{1}{4}(q^4-q^2)$
$[3^1]$	$[3^3]$	-1	$\binom{\frac{1}{3}(q^3-q)}{2}3$
	$[3^1,6^1]$	1	$\frac{1}{6}(q^6-q^3-q^2+q)$

For char$(k)\neq 2$.

We get

$$\sum\frac{\tau(\lambda)\cdot|C|}{|\mathrm{PGL}_3(k)|}=\frac{1}{6}(q-2)(q-3)(q-4)-\frac{3}{8}q(q-1)(q-2)+\frac{1}{4}(q+1)q(q-2)$$
$$-\frac{1}{24}(q-2)(q^2-q-4)-\frac{1}{4}(q+1)q(q-2)+\frac{1}{4}(q+1)q^2-\frac{1}{6}(q^3-q-3)+\frac{1}{6}(q^3+q-1)$$
$$-\left(\frac{1}{6}(q-3)^2-\frac{3}{8}(q-1)^2+\frac{1}{4}(q-1)^2-\frac{1}{24}(q-3)^2-\frac{1}{4}(q^2-2q-1)+\frac{1}{4}(q+1)(q-1)-\frac{1}{6}q^2+\frac{1}{6}q^2\right)=$$
$$4(q-1)-1=4q-5$$

where the sum is over the rows in the table minus the cases from the table where there are three lines through a point.

For char$(k)=2$.
The contribution of the case where we have three lines through one point is given by

$$\frac{1}{6}q(q-2)(q-4)-\frac{3}{8}q^2(q-2)+\frac{1}{4}q^2(q-2)-\frac{1}{24}q(q-2)(q-4)$$
$$-\frac{1}{4}q^2(q-2)+\frac{1}{4}q^3-\frac{1}{6}(q+1)q(q-1)+\frac{1}{6}(q+1)q(q-1)=1$$

so the result is the same.

9.2.6 Tools for $[2,2,1]$.

Let there be a λ-tuple of points P_1,P_2,P_3,P_4 such that no three points are on a line. We write $\mathcal{P}$ for the pencil of conics through P_1,P_2,P_3,P_4. This pencil of conics is a $\mathbb{P}^1$. Let L be a k-line not through any of P_1,P_2,P_3,P_4.

For char $\neq 2$ we look at the separable finite morphism $\mathbb{P}^1 \to \mathbb{P}^1$ where a point Q on L gets sent to the conic in $\mathcal{P}$ through Q. This map has degree 2 so using Hurwitz's theorem we see that there are two branch points. So L is the tangent of two conics.

For char $= 2$ we choose coordinates such that P_1, P_2, P_3, P_4 are $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1)$. From a few simple computations it follows that that the line $x + y + z$ is tangent to all conics in $\mathcal{P}$ and that all other lines that do not pass through the four points are the tangent of precisely one conic in $\mathcal{P}$.

We define for L and P_1, P_2, P_3, P_4:

- A_k, the number of pairs of k-lines in $\mathcal{P}(k)$ such that their intersection is not on L.
- B_k, the number of conics in $\mathcal{P}(k)$ that L is tangent to.
- X_k, the number of reducible conics in $\mathcal{P}(k)$.
- Y_k, the number of irreducible conics in $\mathcal{P}(k)$ that L is tangent to.

We also define the following:

- C_k, the number of irreducible conics in $\mathcal{P}(k)$ that intersect L in two k-points.
- C_{k_2}, the number of irreducible conics in $\mathcal{P}(k)$ that intersect L in a conjugate 2-tuple of points.
- D_{k_2}, the number of conjugate 2-tuples of irreducible conics in $\mathcal{P}(k_2)$ that intersect L two conjugate 2-tuples of points.
- D_{k_4}, the number of conjugate 2-tuples of irreducible conics in $\mathcal{P}(k_2)$ that intersect L a conjugate 4-tuple of points.

For C_k we note that any k-point on L gives a k-conic in $\mathcal{P}(k)$. We do not want the conic to be reducible so we subtract $2A_k$ and we do not want a conic that has L as tangent so we subtract B_k. Every conic that is not tangent to L intersects L in two points so we have to divide by two.

$$C_k = \frac{1}{2}(q + 1 - 2A_k - B_k)$$

The irreducible k-conics in $\mathcal{P}(k)$ intersect L in one k-point, two k-points, or a conjugate 2-tuple of points. So we get

$$\begin{aligned} C_{k_2} &= q + 1 - X_k - Y_k - C_k(L) \\ &= \frac{1}{2}(q + 1 + 2A_k + B_k - 2X_k - 2Y_k). \end{aligned}$$

If we take the k_2-conics that intersect L in two k_2-points and subtract the cases where we get a k-conic, then we get the conjugate 2-tuples of points that give conjugate 2-tuples of conics.

$$\begin{aligned} D_{k_2} &= \frac{1}{4}(q^2 + 1 - 2A_{k_2} - B_{k_2} - 2C_k - 2C_{k_2}) \\ &= \frac{1}{4}(q^2 - 2q - 1 - 2A_{k_2} - B_{k_2} + 2X_k + 2Y_k) \end{aligned}$$

And lastly we get

$$\begin{aligned} D_{k_4} &= \frac{1}{2}(q^2 - q - (X_{k_2} - X_k) - (Y_{k_2} - Y_k) - 2D_{k_2}) \\ &= \frac{1}{4}(q^2 + 1 + 2A_{k_2} + B_{k_2} - 2X_{k_2} - 2Y_{k_2}). \end{aligned}$$

When λ is $[1^4]$,$[1^2, 2^1]$ or $[2^2]$ all the reducible conics in $\mathcal{P}(k)$ are pairs of conjugate 2-tuples of lines. This means that we have $X_{k_2} = 3$ and $A_{k_2} + B_{k_2} - X_{k_2} - Y_{k_2} = 0$

For char$(k) \neq 2$.
We have $B_{k_2} = 2$ and we use this to get

$$\begin{aligned} D_{k_2} &= \frac{1}{4}(q^2 - 2q - 1 + B_{k_2} + 2X_k + 2Y_k - 2X_{k_2} - 2Y_{k_2}) \\ &= \frac{1}{4}(q^2 - 2q - 5 + 2X_k + 2Y_k - 2Y_{k_2}) \end{aligned}$$

and

$$D_{k_4} = \frac{1}{4}(q^2 + 1 - B_{k_2}) = \frac{1}{4}(q^2 - 1).$$

For $\lambda = [1^4]$ there are three pairs of k-lines through P_1, P_2, P_3, P_4. We denote their singular points by B_1, B_2, B_3. If B_1, B_2, B_3 are on a line then that line would be the tangent of least three conics which is a contradiction.
So there are 3 k-lines through two of B_1, B_2, B_3.
There are $3(q-3)$ k-lines through precisely one of B_1, B_2, B_3 and not through any of P_1, P_2, P_3, P_4.
There are $(q-3)^2$ k-lines not through any of $B_1, B_2, B_3, P_1, P_2, P_3, P_4$.
There are $q-2$ irreducible conics in $\mathcal{P}(k)$ that each have $q-3$ non-intersection k-points. This gives us $(q-3)(q-2)$ k-tangents to irreducible conics in $\mathcal{P}(k)$. From this we subtract $3(q-3)$ for the lines that have only one k-conic tangent to it. This leaves us with $(q-3)(q-5)$ tangent points and $\frac{1}{2}(q-3)(q-5)$ lines that have two k-conics tangent to it. And we get $(q-3)^2 - \frac{1}{2}(q-3)(q-5) = \frac{1}{2}(q-3)(q-1)$ lines that have two k_2-conics tangent to it.

When we look at the types of lines and their properties we get the following table:

For $\lambda = [1^4]$ we have $X_k = 3$.

number	A_k	B_k	Y_k	Y_{k_2}	$2C_k$	$2C_{k_2}$	$4D_{k_2}$
3	1	2	0	0	$q-3$	$q-1$	q^2-2q+1
$3(q-3)$	2	2	1	1	$q-5$	$q-1$	q^2-2q+1
$\frac{1}{2}(q-3)(q-5)$	3	2	2	2	$q-7$	$q-1$	q^2-2q+1
$\frac{1}{2}(q-1)(q-3)$	3	0	0	2	$q-5$	$q+1$	q^2-2q-3

Similarly we find for other distributions λ:
For $\lambda = [1^2, 2^1]$ we have $X_k = 1$.

number	A_k	B_k	Y_k	Y_{k_2}	$2C_k$	$2C_{k_2}$	$4D_{k_2}$
1	1	0	0	0	$q-1$	$q+1$	q^2-2q-3
$q-1$	0	2	1	1	$q-1$	$q-1$	q^2-2q-3
$\frac{1}{2}(q-1)^2$	1	2	2	2	$q-3$	$q-1$	q^2-2q-3
$\frac{1}{2}(q+1)(q-3)$	1	0	0	2	$q-1$	$q+1$	q^2-2q-7

For $\lambda = [2^2]$ we have $X_k = 3$.

number	A_k	B_k	Y_k	Y_{k_2}	$2C_k$	$2C_{k_2}$	$4D_{k_2}$
1	1	2	0	0	$q-3$	$q-1$	q^2-2q+1
2	0	2	0	0	$q-1$	$q-3$	q^2-2q+1
$2(q-1)$	1	2	1	1	$q-3$	$q-3$	q^2-2q+1
$q-3$	0	2	1	1	$q-1$	$q-5$	q^2-2q+1
$\frac{1}{2}(q-1)(q-3)$	1	2	2	2	$q-3$	$q-5$	q^2-2q+1
$\frac{1}{2}(q+1)(q-1)$	1	0	0	2	$q-1$	$q-3$	q^2-2q-3

For $\lambda = [1^1, 3^1]$ we have $A_k = A_{k_2} = X_k = X_{k_2} = 0$.

number	B_k	Y_k	Y_{k_2}	$2C_k$	$2C_{k_2}$	$4D_{k_2}$	$4D_{k_4}$
$\frac{1}{2}q(q+1)$	2	2	2	$q-1$	$q-1$	q^2-2q+1	q^2-1
$\frac{1}{2}q(q-1)$	0	0	2	$q+1$	$q+1$	q^2-2q-3	q^2-1

For $\lambda = [4^1]$ we have $A_k = 0$, $X_k = 1$, $X_{k_2} = 3$.

number	A_{k_2}	B_k	Y_k	Y_{k_2}	$2C_k$	$2C_{k_2}$	$4D_{k_2}$	$4D_{k_4}$
1	1	0	0	0	$q+1$	$q-1$	q^2-2q-3	q^2-1
$q+1$	0	2	1	1	$q-1$	$q-1$	q^2-2q+1	q^2-5
$\frac{1}{2}(q+1)(q-1)$	1	2	2	1	$q-1$	$q-3$	q^2-2q+1	q^2-5
$\frac{1}{2}(q+1)(q-1)$	1	0	0	2	$q+1$	$q-1$	q^2-2q-3	q^2-5

For char$(k) = 2$.
We do not consider the line that is tangent to all concis in $\mathbb{P}$. We have $B_k = B_{k_2} = 1$ and $Y_k = Y_{k_2}$. When λ is $[1^4]$,$[1^2, 2^1]$ or $[2^2]$ we use $X_{k_2} = 3$ and $A_{k_2} + B_{k_2} - X_{k_2} - Y_{k_2} = 0$ to get

$$D_{k_2} = \frac{1}{4}(q^2 - 2q - 6 + 2X_k), \qquad \text{and} \qquad D_{k_4} = \frac{1}{4}q^2.$$

For $\lambda = [1^4]$ there are three pairs of k-lines through P_1, P_2, P_3, P_4. As above we denote their singular points by B_1, B_2, B_3. The points B_1, B_2, B_3 all lie on the line $x + y + z$. There are $3(q-2)$ k-lines through one of B_1, B_2, B_3 and not through any of P_1, P_2, P_3, P_4. There are $(q-2)(q-4)$ k-lines not through any of $B_1, B_2, B_3, P_1, P_2, P_3, P_4$.

For $\lambda = [1^4]$ we have $X_k = 3$, so $D_{k_2} = \frac{1}{4}(q^2 - 2q)$.

number	A_k	Y_k	$2C_k$	$2C_{k_2}$
$3(q-2)$	2	0	$q-4$	q
$(q-2)(q-4)$	3	1	$q-6$	q

For $\lambda = [1^2, 2^1]$ we have $X_k = 1$, so $D_{k_2} = \frac{1}{4}(q^2 - 2q - 4)$.

number	A_k	Y_k	$2C_k$	$2C_{k_2}$
$q-2$	0	0	q	q
$q(q-2)$	1	1	$q-2$	q

For $\lambda = [2^2]$ we have $X_k = 3$, so $D_{k_2} = \frac{1}{4}(q^2 - 2q)$.

number	A_k	Y_k	$2C_k$	$2C_{k_2}$
$q-2$	0	0	q	$q-4$
$2q$	1	0	$q-2$	$q-2$
$q(q-2)$	1	1	$q-2$	$q-4$

For $\lambda = [1^1, 3^1]$ we have $A_k = A_{k_2} = X_k = X_{k_2} = 0$, $Y_k = 1$.

number	$2C_k$	$2C_{k_2}$	$4D_{k_2}$	$4D_{k_4}$
$(q+1)(q-1)$	q	q	q^2-2q	q^2

For $\lambda = [4^1]$ we have $A_k = 0$, $X_k = 1$, $X_{k_2} = 3$.

number	A_{k_2}	Y_k	$2C_k$	$2C_{k_2}$	$4D_{k_2}$	$4D_{k_4}$
q	0	0	q	q	q^2-2q	q^2-4
q^2	1	1	q	$q-2$	q^2-2q	q^2-4

When counting the number of ways we can choose a λ-tuple of four points it turns out that it is given by

$$\nu(\lambda) \cdot |\mathrm{PGL}_3(k)|.$$

Where $\nu(\lambda)$ is given by

λ	$[1^4]$	$[1^2, 2^1]$	$[1^1, 3^1]$	$[2^2]$	$[4^1]$
$\nu(\lambda)$	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{1}{4}$

9.2.7 $[2, 2, 1]$, eight singularities and the conics are definined over k.

We get eight singularities when the line intersects each conic in 2 points and the conics intersect in 4 other points.
For a $[1^8]$-tuple of points we pick four k-points $P_1, \ldots, P_4$ such that there are no three on a line.

$$\nu([1^8]) \cdot |\mathrm{PGL}_3(k)| = \frac{1}{24}(q+1)q^3(q^2+q+1)(q^2-2q+1)$$

We then use the tables above to choose a line and choose two conics through P_1, P_2, P_3, P_4 such that each conic intersects the line in two k-points.

For char$(k) \neq 2$.
The values in the table should be multiplied with $(q+1)q^3(q-1)^2(q^2+q+1)$.

4 points	λ points	$\tau(\lambda)\nu(\lambda)$	$\lvert C\rvert$
$[1^4]$	$[1^8]$	$\frac{1}{24}$	$3\binom{\frac{1}{2}(q-3)}{2}+3(q-3)\binom{\frac{1}{2}(q-5)}{2}$ $+\frac{1}{2}(q-3)(q-5)\binom{\frac{1}{2}(q-7)}{2}+\frac{1}{2}(q-1)(q-3)\binom{\frac{1}{2}(q-5)}{2}$
	$[1^6, 2^1]$	$-\frac{1}{24}$	$3\frac{1}{2}(q-3)\frac{1}{2}(q-1)+3(q-3)\frac{1}{2}(q-5)\frac{1}{2}(q-1)$ $+\frac{1}{2}(q-3)(q-5)\frac{1}{2}(q-7)\frac{1}{2}(q-1)$ $+\frac{1}{2}(q-1)(q-3)\frac{1}{2}(q-5)\frac{1}{2}(q+1)$
	$[1^4, 2^2]$	$\frac{1}{24}$	$3\binom{\frac{1}{2}(q-1)}{2}+3(q-3)\binom{\frac{1}{2}(q-1)}{2}$ $+\frac{1}{2}(q-3)(q-5)\binom{\frac{1}{2}(q-1)}{2}+\frac{1}{2}(q-1)(q-3)\binom{\frac{1}{2}(q+1)}{2}$
$[1^2, 2^1]$	$[1^6, 2^1]$	$-\frac{1}{4}$	$\binom{\frac{1}{2}(q-1)}{2}+(q-1)\binom{\frac{1}{2}(q-1)}{2}$ $+\frac{1}{2}(q-1)^2\binom{\frac{1}{2}(q-3)}{2}+\frac{1}{2}(q+1)(q-3)\binom{\frac{1}{2}(q-1)}{2}$
	$[1^4, 2^2]$	$\frac{1}{4}$	$\frac{1}{2}(q-1)\frac{1}{2}(q+1)+(q-1)\frac{1}{2}(q-1)\frac{1}{2}(q-1)$ $+\frac{1}{2}(q-1)^2\frac{1}{2}(q-3)\frac{1}{2}(q-1)$ $+\frac{1}{2}(q+1)(q-3)\frac{1}{2}(q-1)\frac{1}{2}(q+1)$
	$[1^2, 2^3]$	$-\frac{1}{4}$	$\binom{\frac{1}{2}(q+1)}{2}+(q-1)\binom{\frac{1}{2}(q-1)}{2}$ $+\frac{1}{2}(q-1)^2\binom{\frac{1}{2}(q-1)}{2}+\frac{1}{2}(q+1)(q-3)\binom{\frac{1}{2}(q+1)}{2}$
$[1^1, 3^1]$	$[1^5, 3^1]$	$\frac{1}{3}$	$\frac{1}{2}q(q+1)\binom{\frac{1}{2}(q-1)}{2}+\frac{1}{2}q(q-1)\binom{\frac{1}{2}(q+1)}{2}$
	$[1^3, 2^1, 3^1]$	$-\frac{1}{3}$	$\frac{1}{2}q(q+1)\frac{1}{2}(q-1)\frac{1}{2}(q-1)+\frac{1}{2}q(q-1)\frac{1}{2}(q+1)\frac{1}{2}(q+1)$
	$[1^1, 2^2, 3^1]$	$\frac{1}{3}$	$\frac{1}{2}q(q+1)\binom{\frac{1}{2}(q-1)}{2}+\frac{1}{2}q(q-1)\binom{\frac{1}{2}(q+1)}{2}$
$[2^2]$	$[1^4, 2^2]$	$\frac{1}{8}$	$\binom{\frac{1}{2}(q-3)}{2}+2\binom{\frac{1}{2}(q-1)}{2}+2(q-1)\binom{\frac{1}{2}(q-3)}{2}+(q-3)\binom{\frac{1}{2}(q-1)}{2}$ $+\frac{1}{2}(q-1)(q-3)\binom{\frac{1}{2}(q-3)}{2}+\frac{1}{2}(q+1)(q-1)\binom{\frac{1}{2}(q-1)}{2}$
	$[1^2, 2^3]$	$-\frac{1}{8}$	$\frac{1}{2}(q-3)\frac{1}{2}(q-1)+2\frac{1}{2}(q-1)\frac{1}{2}(q-3)$ $+2(q-1)\frac{1}{2}(q-3)\frac{1}{2}(q-3)+(q-3)\frac{1}{2}(q-1)\frac{1}{2}(q-5)$ $+\frac{1}{2}(q-1)(q-3)\frac{1}{2}(q-3)\frac{1}{2}(q-5)$ $+\frac{1}{2}(q+1)(q-1)\frac{1}{2}(q-1)\frac{1}{2}(q-3)$
	$[2^4]$	$\frac{1}{8}$	$\binom{\frac{1}{2}(q-1)}{2}+2\binom{\frac{1}{2}(q-3)}{2}+2(q-1)\binom{\frac{1}{2}(q-3)}{2}+(q-3)\binom{\frac{1}{2}(q-5)}{2}$ $+\frac{1}{2}(q-1)(q-3)\binom{\frac{1}{2}(q-5)}{2}+\frac{1}{2}(q+1)(q-1)\binom{\frac{1}{2}(q-3)}{2}$
$[4^1]$	$[1^4, 4^1]$	$-\frac{1}{4}$	$\binom{\frac{1}{2}(q+1)}{2}+(q+1)\binom{\frac{1}{2}(q-1)}{2}$ $+\frac{1}{2}(q+1)(q-1)\binom{\frac{1}{2}(q-1)}{2}+\frac{1}{2}(q+1)(q-1)\binom{\frac{1}{2}(q+1)}{2}$
	$[1^2, 2^1, 4^1]$	$\frac{1}{4}$	$\frac{1}{2}(q+1)\frac{1}{2}(q-1)+(q+1)\frac{1}{2}(q-1)\frac{1}{2}(q-1)$ $+\frac{1}{2}(q+1)(q-1)\frac{1}{2}(q-1)\frac{1}{2}(q-3)$ $+\frac{1}{2}(q+1)(q-1)\frac{1}{2}(q+1)\frac{1}{2}(q-1)$
	$[2^2, 4^1]$	$-\frac{1}{4}$	$\binom{\frac{1}{2}(q-1)}{2}+(q+1)\binom{\frac{1}{2}(q-1)}{2}$ $+\frac{1}{2}(q+1)(q-1)\binom{\frac{1}{2}(q-3)}{2}+\frac{1}{2}(q+1)(q-1)\binom{\frac{1}{2}(q-1)}{2}$

We get

$$\sum \frac{\tau(\lambda)\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{192}\Big((q^2-9q+17)(q-3)(q-5)-2(q^2-5q+3)(q-1)(q-3)+(q^2-q+1)(q-1)(q-3)$$
$$-6(q^2-3q+1)(q-1)(q-3)+12(q^3-2q^2-1)(q-1)-6(q^2-3q+1)(q+1)(q-1)+8(q+1)q(q-1)(q-2)$$
$$-16q^2(q+1)(q-1)+8(q+1)q(q-1)(q-2)+3(q^3-2q^2-2q-1)(q-3)-6(q^2-q-1)(q-1)(q-3)$$
$$+3(q^2-q-1)(q+1)(q-1)-6(q^2-q-1)(q+1)(q-1)+12(q^2-q+1)(q+1)(q-1)-6(q^2-q-1)(q-1)(q-3)\Big) =$$
$$-\frac{1}{2}(q-1)$$

where the sum is over the rows in the table.

For char$(k)=2$.
The values in the table should be multiplied with $(q+1)q^3(q-1)^2(q^2+q+1)$.

4 points	λ points	$\tau(\lambda)\nu(\lambda)$	$\lvert C\rvert$
$[1^4]$	$[1^8]$	$\frac{1}{24}$	$3(q-2)\binom{\frac{1}{2}(q-4)}{2}+(q-2)(q-4)\binom{\frac{1}{2}(q-6)}{2}$
	$[1^6,2^1]$	$-\frac{1}{24}$	$3(q-2)\frac{1}{2}(q-4)\frac{1}{2}q+(q-2)(q-4)\frac{1}{2}(q-6)\frac{1}{2}q$
	$[1^4,2^2]$	$\frac{1}{24}$	$3(q-2)\binom{\frac{1}{2}q}{2}+(q-2)(q-4)\binom{\frac{1}{2}q}{2}$
$[1^2,2^1]$	$[1^6,2^1]$	$-\frac{1}{4}$	$(q-2)\binom{\frac{1}{2}q}{2}+q(q-2)\binom{\frac{1}{2}(q-2)}{2}$
	$[1^4,2^2]$	$\frac{1}{4}$	$(q-2)\frac{1}{2}q\frac{1}{2}q+q(q-2)\frac{1}{2}(q-2)\frac{1}{2}q$
	$[1^2,2^3]$	$-\frac{1}{4}$	$(q-2)\binom{\frac{1}{2}q}{2}+q(q-2)\binom{\frac{1}{2}q}{2}$
$[1^1,3^1]$	$[1^5,3^1]$	$\frac{1}{3}$	$(q+1)(q-1)\binom{\frac{1}{2}q}{2}$
	$[1^3,2^1,3^1]$	$-\frac{1}{3}$	$(q+1)(q-1)\frac{1}{2}q\frac{1}{2}q$
	$[1^1,2^2,3^1]$	$\frac{1}{3}$	$(q+1)(q-1)\binom{\frac{1}{2}q}{2}$
$[2^2]$	$[1^4,2^2]$	$\frac{1}{8}$	$(q-2)\binom{\frac{1}{2}q}{2}+2q\binom{\frac{1}{2}(q-2)}{2}+q(q-2)\binom{\frac{1}{2}(q-2)}{2}$
	$[1^2,2^3]$	$-\frac{1}{8}$	$(q-2)\frac{1}{2}q\frac{1}{2}(q-4)+2q\frac{1}{2}(q-2)\frac{1}{2}(q-2)$ $+q(q-2)\frac{1}{2}(q-2)\frac{1}{2}(q-4)$
	$[2^4]$	$\frac{1}{8}$	$(q-2)\binom{\frac{1}{2}(q-4)}{2}+2q\binom{\frac{1}{2}(q-2)}{2}+q(q-2)\binom{\frac{1}{2}(q-4)}{2}$
$[4^1]$	$[1^4,4^1]$	$-\frac{1}{4}$	$q\binom{\frac{1}{2}q}{2}+q^2\binom{\frac{1}{2}q}{2}$
	$[1^2,2^1,4^1]$	$\frac{1}{4}$	$q\frac{1}{2}q\frac{1}{2}q+q^2\frac{1}{2}q\frac{1}{2}(q-2)$
	$[2^2,4^1]$	$-\frac{1}{4}$	$q\binom{\frac{1}{2}q}{2}+q^2\binom{\frac{1}{2}(q-2)}{2}$

We get

$$\sum \frac{\tau(\lambda)\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{192}\Big(+(q-2)(q-4)(q-5)(q-6)-2q(q-2)(q-3)(q-4)+q(q-1)(q-2)^2$$
$$-6q(q-2)^2(q-3)+12q^2(q-1)(q-2)-6q(q+1)(q-2)^2+8q(q+1)(q-1)(q-2)-16q^2(q+1)(q-1)$$
$$+8q(q+1)(q-1)(q-2)+3q(q^2-3q-2)(q-2)-6q^2(q-2)(q-3)+3(q^2-3q-6)(q-2)(q-4)$$
$$-6q^2(q+1)(q-2)+12q^3(q-1)-6q^2(q-2)(q-3)\Big) =$$
$$-\frac{1}{2}(q-1)$$

where the sum is over the rows in the table.

9.2.8 $[2,2,1]$, eight singularities with a conjugate 2-tuple of conics.

For char$(k) \neq 2$.
The values in the table should be multiplied with $\frac{1}{4}(q+1)q^3(q-1)^2(q^2+q+1)$.

4 points	λ points	$\tau(\lambda)\nu(\lambda)$	$\lvert C\rvert$
$[1^4]$	$[1^4,2^2]$	$\frac{1}{24}$	$\frac{1}{2}(q^2-2q+3)(q^2-2q+1)$ $+\frac{1}{2}(q-1)(q-3)(q^2-2q-3)$
	$[1^4,4^1]$	$-\frac{1}{24}$	$(q^2-3q+3)(q^2-1)$
$[1^2,2^1]$	$[1^2,2^3]$	$-\frac{1}{4}$	$\frac{1}{2}(q^2+1)(q^2-2q-3)$ $+\frac{1}{2}(q+1)(q-3)(q^2-2q-7)$
	$[1^2,2^1,4^1]$	$\frac{1}{4}$	$(q^2-q-1)(q^2-1)$
$[1^1,3^1]$	$[1^1,3^1,2^2]$	$\frac{1}{3}$	$\frac{1}{2}(q^2+q)(q^2-2q+1)$ $+\frac{1}{2}(q^2-q)(q^2-2q-3)$
	$[1^1,3^1,4^1]$	$-\frac{1}{3}$	$q^2(q^2-1)$
$[2^2]$	$[2^4]$	$\frac{1}{8}$	$\frac{1}{2}(q^2+2q-1)(q^2-2q+1)$ $+\frac{1}{2}(q+1)(q-1)(q^2-2q-3)$
	$[2^2,4^1]$	$-\frac{1}{8}$	$(q^2+q-1)(q^2-1)$
$[4^1]$	$[2^2,4^1]$	$-\frac{1}{4}$	$\frac{1}{2}(q+1)^2(q^2-2q+1)$ $+\frac{1}{2}(q^2+1)(q^2-2q-3)$
	$[4^2]$	$\frac{1}{4}$	$(q^2-1)+(q^2+q)(q^2-5)$

We get

$$\sum \frac{\tau(\lambda)\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{96}\Big(+(q^3-4q^2+4q+3)(q-1)-(q^2-3q+3)(q+1)(q-1)$$
$$-6(q^2-q-3)(q+1)(q-3)+6(q^2-q-1)(q+1)(q-1)+8(q+1)q(q-1)(q-2)-8(q+1)q^2(q-1)$$
$$+3(q^3-4q-1)(q-1)-3(q^2+q-1)(q+1)(q-1)-6(q^3-2q^2-1)(q+1)+6(q^3-4q-1)(q+1)\Big) =$$
$$-\frac{1}{2}(q+1)$$

where the sum is over the rows in the table.

For char$(k)=2$.
The values in the table should be multiplied with $\frac{1}{4}(q+1)q^3(q-1)^2(q^2+q+1)$.

4 points	λ points	$\tau(\lambda)\nu(\lambda)$	$\lvert C\rvert$
$[1^4]$	$[1^4,2^2]$	$\frac{1}{24}$	$(q-1)(q-2)(q^2-2q)$
	$[1^4,4^1]$	$-\frac{1}{24}$	$(q-1)(q-2)q^2$
$[1^2,2^1]$	$[1^2,2^3]$	$-\frac{1}{4}$	$(q+1)(q-2)(q^2-2q-4)$
	$[1^2,2^1,4^1]$	$\frac{1}{4}$	$(q+1)(q-2)q^2$
$[1^1,3^1]$	$[1^1,3^1,2^2]$	$\frac{1}{3}$	$(q+1)(q-1)(q^2-2q)$
	$[1^1,3^1,4^1]$	$-\frac{1}{3}$	$(q+1)(q-1)q^2$
$[2^2]$	$[2^4]$	$\frac{1}{8}$	$(q+2)(q-1)(q^2-2q)$
	$[2^2,4^1]$	$-\frac{1}{8}$	$(q+2)(q-1)q^2$
$[4^1]$	$[2^2,4^1]$	$-\frac{1}{4}$	$(q^2+q)(q^2-2q)$
	$[4^2]$	$\frac{1}{4}$	$(q^2+q)(q^2-4)$

We get

$$\sum \frac{\tau(\lambda)\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{96}\Big(+(q-1)(q-2)(q^2-2q)-(q-1)q^2(q-2)$$
$$-6(q+1)(q-2)(q^2-2q-4)+6(q+1)q^2(q-2)+8(q+1)(q-1)(q^2-2q)-8q^2(q+1)(q-1)$$
$$+3(q+2)q(q-1)(q-2)-3(q+2)q^2(q-1)-6(q+1)q^2(q-2)+6(q+2)(q+1)q(q-2)\Big) =$$
$$-\frac{1}{2}(q+1)$$

where the sum is over the rows in the table.

9.2.9 $[3,1,1]$, eight singularities.

We get eight singularities when the lines each intersect the cubic in three points and intersect each other outside the conic.

Lemma 9.9. *If we have two different lines L, L' and three point on either line that do not include the intersection of L and L'. Then for a point P outside the lines there is exactly one cubic that goes through the six points on the lines and has a singularity at P. This cubic is reducible if and only if there is a line through P and two of the six points on the line.*

Proof. There is a $\mathbb{P}^9$ of cubics and we have six conditions for the conics plus three for the singularity at P. It is easy to see that these conditions are all independent.

If there is a line J through P and two of the six points on the line then the intersection of the cubic and J is four. So by Bézout the line is part of the cubic.

Now for the other way around: If L (or L') is part of the cubic then we are left with a conic that intersects L' (or L) in three points so both L and L' are part of the cubic and then P can no longer be singular. So L and L' are not part of the cubic. If the cubic is reducible then it consists of a line J and a (possibly reducible) conic. This conic intersects L and L' in four of the six points. So J intersects L and L' in the other two points. □

For a $[1^2]$-tuple of lines and a $[1^8]$-tuple of points we choose two k-lines and a k-point Q outside those lines. This gives

$$\binom{q^2+q+1}{2}(q^2-q) = \frac{1}{2}(q^2+q+1)(q+1)q^2(q-1).$$

We then choose three k-points on each line such that there are no two points on a line through Q: $\binom{q}{3}\binom{q-3}{3}$. By Lemma 9.9 this gives us an irreducible conic through the six points on the lines and with a singularity at Q.

The values in the table should be multiplied with $\frac{1}{2}(q^2+q+1)(q+1)q^2(q-1)$.

lines	λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^2]$	$[1^8]$	$+1$	$\binom{q}{3}\binom{q-3}{3}$
	$[1^6, 2^1]$	-1	$2\binom{q}{3}(q-3)\frac{1}{2}(q^2-q)$
	$[1^4, 2^2]$	$+1$	$q\frac{1}{2}(q^2-q)(q-1)\frac{1}{2}(q^2-q-2)$
	$[1^5, 3^1]$	$+1$	$2\binom{q}{3}\frac{1}{3}(q^3-q)$
	$[1^2, 3^2]$	$+1$	$\frac{1}{3}(q^3-q)\frac{1}{3}(q^3-q-3)$
	$[1^3, 2^1, 3^1]$	-1	$2\frac{1}{3}(q^3-q)\frac{1}{2}(q^2-q)q$
$[2^1]$	$[1^2, 2^3]$	-1	$\frac{1}{6}(q^2-q)(q^2-q-2)(q^2-q-4)$
	$[1^2, 2^1, 4^1]$	$+1$	$(q^2-q)\frac{1}{2}(q^4-q^2)$
	$[1^2, 6^1]$	-1	$\frac{1}{3}(q^6-q^2)-\frac{1}{3}(q^3-q)$

We get

$$\sum \frac{\tau(\lambda)\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{72}\Big(+(q-2)(q-3)(q-4)(q-5) - 6q(q-1)(q-2)(q-3)$$
$$+9(q+1)q(q-1)(q-2) + 4(q+1)q(q-1)(q-2) + 4(q^3-q-3)(q+1)$$
$$-12(q+1)q^2(q-1) - 6(q^2-q-4)(q+1)(q-2) + 18(q+1)q^2(q-1) - 12(q^3+q-1)(q+1)\Big) =$$
$$-2q+1$$

where the sum is over the rows in the table.

9.3 Results.

We add everything from the explicit counting together to get

$$s_{5,\text{explicit}} = 0 - q + 3 + 1 + 4q - 5 - \frac{1}{2}(q-1) - \frac{1}{2}(q+1) - 2q + 1 = 0.$$

Now we just add $s_{5,\text{explicit}}$ and $s_{5,\text{sieve}}$ together which gives us the end result.

Theorem 9.10. *The number of smooth plane quintics over $C_5(\mathbb{F}_q)$ is given by*

$$|C_5(\mathbb{F}_q)| = q^{12}.$$

10 Counting plane quintics with an ordinary singularity.

We continue where we left of at the end of section 4. We want to count $\frac{|T(k)|}{|\mathrm{PGL}_3(k)|}$, where we remember that T is the set of plane conics with exactly one singularity which is either an ordinary node or an ordinary cusp. We can divide $T(k)$ into three subsets:

1. $T_{\text{split}}(k)$, the set of plane quintics over k that have precisely one singularity that is a split node (i.e. the tangents are defined over k).

2. $T_{\text{non-split}}(k)$, the set of plane quintics over k that have precisely one singularity that is a non-split node (i.e. the tangents are a conjugate 2-tuple).

3. $T_{\text{cusp}}(k)$, the set of plane quintics over k that have precisely one singularity that is a cusp.

We have

$$|T(k)| = |T_{\text{split}}(k)| + |T_{\text{non-split}}(k)| + |T_{\text{cusp}}(k)|.$$

Let C be a curve in T and let P be its singular point. The Frobenius map sends a singularity to a singularity, so P is a k-point as it is the only singularity of C. We can apply a k-linear coordinate change such that P

gets mapped to $(0:0:1)$ and the tangent lines at P become certain fixed lines L, L'. What these fixed lines are depends on whether we are dealing with a split node, a non-split node, or a cusp.

Definition 10.1. Let $P = (0:0:1)$ and let L, L' be two lines through P. We define $T_{P;L;L'}$ to be the set of plane quintics that have P as their only singularity, such that P has multiplicity 2 and the tangents at P are given by the lines L, L'. We define $G_{P;L;L'}$ to be the subgroup of $\mathrm{PGL}_3(k)$ that fixes P and fixes $\{L, L'\}$.

Note that $G_{P;L;L'}$ is the group that fixes $T_{P;L;L'}$.

Any curve in $T_{\text{split}}(k)$ has a singularity at a k-point with two distinct k-tangents. This means we can apply a k-linear coordinate change such that the singularity is the point $P = (0:0:1)$ and the tangents at P are given by the lines x, y. And any curve in $T_{P;L;L'}$ is part of $T_{\text{split}}(k)$ so we get

$$\frac{|T_{\text{split}}(k)|}{|\mathrm{PGL}_3(k)|} = \frac{|T_{P;x;y}|}{|G_{P;x;y}|}.$$

The matrices that fix P, x and y have the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & d & 1 \end{pmatrix}$$

where $ab \neq 0$. There are $q^2 \cdot (q-1)^2 = q^4 - 2q^3 + q^2$ such matrices. We can also permute x and y which adds a factor two so $G_{P;x;y}$ contains $2(q^4 - 2q^3 + q^2)$ matrices.

We do the equivalent thing for non-split nodes. Just like with a split node we fix the point $P = (0:0:1)$ and two tangents at the point. Let α be an element in k_2 that is not in k. We take as tangents $x + \alpha y, x + \mathcal{F}(\alpha) y$ and we get

$$\frac{|T_{\text{non-split}}(k)|}{|\mathrm{PGL}_3(k)|} = \frac{|T_{P;x+\alpha y;x+\mathcal{F}(\alpha)y}|}{|G_{P;x+\alpha y;x+\mathcal{F}(\alpha)y}|}.$$

If $\mathrm{char}(k) \neq 2$ then the field k has a quadratic nonresidue r and we can take α such that $\alpha^2 = r$. The matrices that fix P, $x + \alpha y$ and $x + \mathcal{F}(\alpha) y$ have the form

$$\begin{pmatrix} a & rb & 0 \\ b & a & 0 \\ c & d & 1 \end{pmatrix}$$

where $a \neq 0$ or $b \neq 0$.
If $\mathrm{char}(k) = 2$ then we get matrices

$$\begin{pmatrix} a & \frac{\alpha\cdot\mathcal{F}(\alpha)\cdot(a+b)}{\alpha+\mathcal{F}(\alpha)} & 0 \\ \frac{a+b}{\alpha+\sigma(\alpha)} & b & 0 \\ c & d & 1 \end{pmatrix}$$

where $a \neq 0$ or $b \neq 0$.
In either case we get $q^2 \cdot (q^2 - 1) = q^4 - q^2$ matrices. We can also permute the two tangents so $G_{P;x+\alpha y;x+\mathcal{F}(\alpha)y}$ contains $2(q^4 - q^2)$ matrices.

Now for cusps we can fix the point P and the double tangent y at P. However this is not enough since a curve in $T_{P,y,y}$ can have a cusp that is not ordinary. In fact the cusp is ordinary if and only if the coefficient of x^3z^2 is nonzero.

Definition 10.2. We use $C_{x^3z^2\neq 0}$ to denote the set of plane quintics that have a nonzero coefficient for x^3z^2 in their defining polynomial, and $C_{x^3z^2=0}$ to denote the set of plane quintics that have a zero coefficient for x^3z^2 in their defining polynomial.

It is easy to see that $G_{P,y,y}$ fixes not only $T_{P,y,y}$ but also $T_{P,y,y} \cap C_{x^3z^2\neq 0}$. This gives us

$$\frac{|T_{\mathrm{cusp}}(k)|}{|\mathrm{PGL}_3(k)|} = \frac{|T_{P,y,y} \cap C_{x^3z^2\neq 0}|}{|G_{P,y,y}|}.$$

The matrices that fix P and y have the form

$$\begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ d & e & 1 \end{pmatrix}$$

where $ac \neq 0$. There are $q^3 \cdot (q-1)^2 = q^5 - 2q^4 + q^3$ such matrices. Here permuting the tangents has no effect so $G_{P,y,y}$ contains $q^5 - 2q^4 + q^3$ matrices.

Now we will introduce some notation so we can count $|T_{P;L;L'}|$ using a modification of the partial sieve method.

Definition 10.3. Let $P_1, \ldots, P_n$ be points in $\mathbb{P}^2 - \{P\}$ and let $r_1, \ldots r_n$ be natural numbers. Also let $P = (0 : 0 : 1)$ and let L, L' be two lines through P. We define $C_{P;L;L'}$ to be the set of plane quintics that have a singularity at P of multiplicity two, such that the tangents at P are given by the lines L, L'. We define $V_k(d, 2P; L; L', r_1P_1, \ldots, r_nP_n)$ to be the union of $V_k(d, 3P, r_1P_1, \ldots, r_nP_n)$ and of the subset of $V_k(d, 2P, r_1P_1, \ldots, r_nP_n)$ that consists of curves that have multiplicity 2 at P and tangents L, L' at P.

The set $V_k(d, 2P; L; L', r_1P_1, \ldots, r_nP_n)$ is the result of applying two linear conditions to $V_k(d, 2P, r_1P_1, \ldots, r_nP_n)$. And taking the intersection with $C_{x^3z^2=0}$ is equivalent to applying yet another linear condition. Similar to lemma 5.3 we get the following lemma.

Lemma 10.4. *If we have points $\{P_1, \ldots, P_n\}$ such that for every $1 \leq i \leq n$ there is a j such that $\mathcal{F}(P_i) = P_j$ and $r_i = r_j$, and the Frobenius map also fixes $\{L, L'\}$, then*

$$\dim_k V_k(d, 2P; L; L', r_1P_1, \ldots, r_nP_n) = \dim_{\overline{k}} V_{\overline{k}}(d, 2P; L; L', r_1P_1, \ldots, r_nP_n),$$
$$\dim_k(V_k(5, \ldots) \cap C_{x^3z^2=0}) = \dim_{\overline{k}}(V_{\overline{k}}(5, \ldots) \cap C_{x^3z^2=0}).$$

We will now describe the sieving for split nodes. We first count all curves in $C_{P;x;y}$.

$$\sum_{S \in (\mathbb{P}^2 - \{P\})([])} |V_k(5, 2S) \cap C_{P;x;y}| = |C_{P;x;y}|$$

Then we subtract all curves in $C_{P;x;y}$ that have a singularity outside P.

$$\sum_{S \in (\mathbb{P}^2 - \{P\})([1^1])} |V_k(5, 2S) \cap C_{P;x;y}|$$

We can go on like this as before to get our sieving sum. We get

$$s_{\text{sieve}}^{\text{s}} := \frac{1}{|G_{P;x;y}|} \sum_{|\lambda| \leq N} \left(\sigma(\lambda) \cdot \sum_{S \in (\mathbb{P}^2 - \{P\})(\lambda)} |Y^{\text{s}}(S)| \right),$$
$$s_{\text{sieve}}^{\text{ns}} := \frac{1}{|G_{P;x+\alpha y;x+\mathcal{F}(\alpha)y}|} \sum_{|\lambda| \leq N} \left(\sigma(\lambda) \cdot \sum_{S \in (\mathbb{P}^2 - \{P\})(\lambda)} |Y^{\text{ns}}(S)| \right),$$
$$s_{\text{sieve}}^{\text{c}} := \frac{1}{|G_{P,y,y}|} \sum_{|\lambda| \leq N} \left(\sigma(\lambda) \cdot \sum_{S \in (\mathbb{P}^2 - \{P\})(\lambda)} |Y^{\text{c}}(S)| \right).$$

where

$$Y^{\text{s}}(S) = V_k(5, 2P; x; y, 2S) - V_k(5, 3P, 2S),$$
$$Y^{\text{ns}}(S) = V_k(5, 2P; x + \alpha y; x + \mathcal{F}(\alpha)y, 2S) - V_k(5, 3P, 2S),$$
$$Y^{\text{c}}(S) = (V_k(5, 2P; y; y, 2S) - V_k(5, 3P, 2S)) \cap C_{x^3z^2 \neq 0}.$$

The intersection with $C_{x^3z^2\neq 0}$ poses no problem because we have

$$V_k(5, \ldots) \cap C_{x^3z^2 \neq 0} = V_k(5, \ldots) - V_k(5, \ldots) \cap C_{x^3z^2=0}.$$

As before we can compute these sums using sieve partitions.

Definition 10.5. A sieving partition for split nodes is a partition of $\bigcup_{|\lambda| \leq N} (\mathbb{P}^2 - \{P\})(\lambda)$ into subsets $U_0, \ldots, U_n$ together with numbers $w_i, u_i \in \mathbb{Z}_{\geq 0}$ for $0 \leq i \leq n$ such that if $S \in U_i$ then we have $|\widehat{S}| = w_i$ and $|Y^{\text{s}}| = u_i$. We also write $U_{i,\lambda} := \{S \in U_i | \widehat{S} = \lambda\}$.

We get a similar definition for non-split nodes and cusps.

For split nodes we get the explicit count

$$s^{\mathrm{s}}_{\mathrm{explicit}} := \frac{1}{|G_{P;x;y}|} \sum_{N<|\lambda|\leq M} \tau(\lambda)|C_{5,P;x;y,\lambda}|$$

where $C_{P;x;y,\lambda} \subset C_{P;x;y}$ is the set of curves that have exactly a λ-tuple of singularities besides P. Since $G_{P;x;y}$ is the group that fixes $C_{P;x;y,\lambda}$ we can instead pick a curve that does not have a fixed point at P and then choose a split node on it. This means we get

$$s^{\mathrm{s}}_{\mathrm{explicit}} := \frac{1}{|\mathrm{PGL}_3(k)|} \sum_{N<|\lambda|\leq M} \tau(\lambda) \sum_{C\in C_{5,[[1^1],\lambda]}} \#\mathrm{s}(C)$$

where $\#\mathrm{s}(C)$ is the number of ordinary split nodes on C. We get equivalent definitions for $s^{\mathrm{ns}}_{\mathrm{explicit}}$ and $s^{\mathrm{c}}_{\mathrm{explicit}}$ using $\#\mathrm{ns}(C)$ and $\#\mathrm{c}(C)$.

Note that $M = \frac{5(5-1)}{2} - 1 = 9$ is one lower than before since we already have the singularity at P.

Let $s^{\mathrm{s}}_{\infty}, s^{\mathrm{ns}}_{\infty}, s^{\mathrm{c}}_{\infty}$ be the corrections for curves with an infinite number of singularities. Similar to lemma 6.8 we get

Lemma 10.6. *For $N \geq 4$ the numbers $s^{s}_{\infty}, s^{ns}_{\infty}, s^{c}_{\infty}$ are all zero.*

For our sieving we choose $N = 5$, which means we can ignore the curves with an infinite number of singularities.

10.1 The sieve count for nodes.

We first make a sieving partition that works for both split nodes and non-split nodes. We write L, L' for the tangent lines at P. For split nodes these are x, y and for non-split nodes they are $x + \alpha y, x + \sigma(\alpha)y$.

Both $Y^{\mathrm{s}}(S)$ and $Y^{\mathrm{ns}}(S)$ are given by $V_k(5, 2P; L; L', 2S) - V_k(5, 3P, 2S)$. The two linear conditions for L, L' will always be conditions on the coefficients of x^2z^3, xyz^3, y^2z^3. So if we can do our dimension proofs for P and the points in S as before but without using the columns corresponding to the coefficients of x^2z^3, xyz^3, y^2z^3 then the tangent conditions are independent of the other conditions. It is sufficient to prove the cases where we have P and five other points because when we have less points it can be realized as a subset of these cases.

10.1.1 P and five other points in general position.

We say the six points are in general position if there are no four points on a line, no six on an irreducible conic, and no three on a line through P. Because of 10.4 we can apply a $\bar{k}$-linear transformation to map the five points to $(1 : 0 : 0), (0 : 1 : 0), (1 : 1 : 1), (1 : \alpha : \beta), (1 : \gamma : \delta)$ while leaving P fixed, for some $\alpha, \beta, \gamma, \delta$ where $\alpha, \gamma \neq 0, 1$ and $\alpha \neq \gamma$. We also have $\alpha\beta\gamma - \alpha\beta\delta - \alpha\gamma\delta + \beta\gamma\delta - \beta\gamma + \alpha\delta \neq 0$ for otherwise there would be six on a conic. The lines L, L' will also get transformed but this is fine since it does not change the fact that their conditions on the coefficients of y^2z^3, xyz^3, x^2z^3.

We take derivatives to x and z and we remove the columns corresponding to y^2z^3, xyz^3, x^2z^3 to get determinant

$$(\alpha\beta\gamma - \alpha\beta\delta - \alpha\gamma\delta + \beta\gamma\delta - \beta\gamma + \alpha\delta)^4(\alpha - \gamma)(\alpha - 1)\alpha^2(\gamma - 1)\gamma^2.$$

So the conditions are independent.

10.1.2 P and at least four more points on a line.

The line is a double component which contradicts the fact that P is an ordinary node. So $V_k(5, 2P; L; L', 2S) - V_k(5, 3P, 2S) = \emptyset$.

10.1.3 Five points on a line and P outside the line.

We map the four points on the line to $(0 : 1 : 0), (1 : 0 : 0), (1 : \alpha : 0), (1 : \beta : 0), (1 : \gamma : 0)$ with α, β, γ all different and nonzero. The points determine the line so we get a space of quartics. Here the fact that we don't use x^2z^3, y^2z^3, xyz^3 corresponds to not using x^2z^2, y^2z^2, xyz^2. Taking derivatives to x and z and taking the columns for xy^3, x^2y^2, x^3y, x^4 we get determinant

$$-(\alpha - \beta)(\alpha - \gamma)\alpha(\beta - \gamma)\beta\gamma.$$

10.1.4 Four points on a line and P and another point on another line.

We map the four points on the line to $(1 : 0 : 0), (0 : 1 : 0), (1 : 1 : 0), (\alpha : 1 : 0)$ with $\alpha \neq 0, 1$. The two other point we map to $(1 : 1 : 1)$. The points determine the two lines so we get a space of cubics. We take columns x^2z, xy^2, x^2y to get determinant

$$-(\alpha - 1)\alpha.$$

10.1.5 Four points on a line and P and another point outside the line.

We map the four points on the line to $(1 : 0 : 0), (0 : 1 : 0), (\alpha : 1 : 0), (\beta : 1 : 0)$ with $\alpha, \beta \neq 0, 1$ and $\alpha \neq \beta$. The other point we map to $(1 : 1 : 1)$. The points determine the line so we get a space of quartics. Taking derivatives to x and z and taking the columns for $xy^2z, x^2yz, xy^3, x^2y^2, x^3y$ we get determinant

$$(\alpha - \beta)(\alpha - 1)\alpha(\beta - 1)\beta.$$

10.1.6 P and three points on one line and P and two points on another line.

We map the three other points on the line to $(1:0:0), (\alpha:0:1), (\beta:0:1)$ with $\alpha, \beta \neq 0$ and $\alpha \neq \beta$. The two other points we map to $(0:1:0), (0:1:1)$. The points determine the two lines so we get a space of cubics. We take columns xz^2, y^2z, x^2z to get determinant

$$(\alpha-\beta)\alpha\beta.$$

10.1.7 P and three more points on a line.

We map the three other points on the line to $(1:0:0), (\alpha:0:1), (\beta:0:1)$ with $\alpha, \beta \neq 0$ and $\alpha \neq \beta$. The two other points we map to $(0:1:0), (1:1:1)$. The points determine the line so we get a space of quartics. Taking derivatives to x and z and taking the columns for $x^2z^2, x^2yz, x^3z, x^2y^2, x^3y$ we get determinant

$$(\alpha-\beta)\alpha^2\beta^2.$$

10.1.8 P and two more points on a line and the three other points on a line.

We map the three other points on the line to $(1:0:0), (\alpha:0:1)$ with $\alpha \neq 0$. Two other points we map to $(0:1:0), (1:1:0), (1:\beta:0)$ where $\beta \neq 0, 1$. The points determine the two lines so we get a space of cubics. We take columns x^2z, xy^2, x^2y to get determinant

$$\alpha^2(\beta-1)\beta.$$

10.1.9 P and two points on one line and P and two points on another line.

We map the three other points on the line to $(1:0:0), (\alpha:0:1)$ with $\alpha, \beta \neq 0$ and $\alpha \neq \beta$. Two other points we map to $(0:1:0), (0:1:1)$. Then we got one more point $(1:\beta:\gamma)$ where $\beta \neq 0$. The points determine the two lines so we get a space of cubics. Taking derivatives to x and z and taking the columns for $y^2z, xyz, x^2z, xy^2, x^2y$ we get determinant

$$-\alpha^2\beta^4.$$

10.1.10 P and two more points on a line.

We map the three other points on the line to $(1:0:0), (\alpha:0:1)$ with $\alpha \neq 0$. Two other points we map to $(0:1:0), (1:1:1), (1:\beta:\gamma)$ where $\beta \neq 0, 1$ and $\gamma \neq 1$. The points determine the line so we get a space of quartics. Taking derivatives to x and z and taking the columns for $y^2z^2, xyz^2, x^2z^2, xy^2z, x^2yz, x^2y^2, x^3y$ we get determinant

$$\alpha^2(\beta-1)\beta^4(\gamma-1)^4.$$

10.1.11 Six points on a conic.

We first map four points to $(1:0:0), (0:1:0), (1:1:1), (1:\alpha:\beta)$ with α, β different and nonzero while leaving P fixed. Through these five points we then have the conic $(\beta-\alpha\beta)xy+(\alpha\beta-\alpha)xz+(\alpha-\beta)yz$. This means that the sixth point is of the form $(\alpha\beta+\alpha\gamma-\beta\gamma-\alpha : \gamma(\alpha\beta+\alpha\gamma-\beta\gamma-\alpha) : \alpha\beta\gamma-\beta\gamma)$ where $\gamma \neq \alpha, \beta, 0, 1$ and $\alpha\beta+\alpha\gamma-\beta\gamma-\alpha \neq= 0$. We take columns x^2z, xy^2, x^2y to get determinant

$$(\alpha\beta+\alpha\gamma-\beta\gamma-\alpha)^2(\alpha-\beta)^2(\alpha-\gamma)(\gamma-1)\gamma.$$

10.1.12 The result of the sieving.

Because $\pi_w(\mathbb{P}^1)=0$ for $w \geq 4$ and $\pi_w(\mathbb{P}^1-\{P\})=0$ for $w \geq 2$ all the counts become zero besides the count of the cases where all points are in general position. As we did before we can pretend all points are in general position since subtracting the other cases is just subtracting zero. For P and a set S of $3w$ points that are in general position with P we have

$$V_k(5,2P;L;L',2S)-V_k(5,3P,2S)=\mathbb{P}^{15-3w}-\mathbb{P}^{15-3w-1}=\mathbb{A}^{15-3w}.$$

So for split nodes we have

$$\begin{aligned} s^{\mathrm{s}}_{\mathrm{sieve}} &= \frac{1}{|G_{P;x;y}|}\sum_{w=0}^{5}|\mathbb{A}^{15-3w}|\cdot\pi_w(\mathbb{P}^2-\{P\}) \\ &= \frac{1}{|G_{P;x;y}|}(|\mathbb{A}^{15}|\cdot 1-|\mathbb{A}^{12}|\cdot(q^2+q)+|\mathbb{A}^9|\cdot q^3) \\ &= \frac{q^{15}-q^{14}-q^{13}+q^{12}}{2(q^4-2q^3+q^2)} \\ &= \frac{1}{2}(q^{11}+q^{10}) \end{aligned}$$

And in the same way we get

$$\begin{aligned} s^{\mathrm{ns}}_{\mathrm{sieve}} &= \frac{1}{|G_{P;x+\alpha y;x+\mathcal{F}(\alpha)y}|}\sum_{w=0}^{5}|\mathbb{A}^{15-3w}|\cdot\pi_w(\mathbb{P}^2-\{P\}) \\ &= \frac{q^{15}-q^{14}-q^{13}+q^{12}}{2(q^4-q^2)} \\ &= \frac{1}{2}(q^{11}-q^{10}) \end{aligned}$$

10.2 The sieve count for cusps.

We have

$$Y^{\mathrm{c}}(S) := (V_k(5, 2P; y; y, 2S) - V_k(5, 3P, 2S)) \cap C_{x^3z^2 \neq 0}.$$

So if we can do our dimension proofs for P and the points in S without using the columns corresponding to the coefficients of $x^2z^3, xyz^3, y^2z^3, x^3z^2$ then all the conditions are independent. Since we want to use the condition that x^3z^2 is nonzero we will keep the double tangent y fixed during linear transformations. This means we can no longer send a point to $(1 : 0 : 0)$.

We cannot have P and two other points on a line or P and four other points on a conic that has tangent y at P. Since then the points would determine that line or conic which makes it impossible for P to be an ordinary cusp.

10.2.1 P and five other points in general position.

We say the six points are in general position if there are no four points on a line, no six on an irreducible conic, no three on a line through P, and no five on a conic through P that has tangent y at P. However if there are six singular points on an irreducible plane quintic then they all have to be of delta-invariant 1. So if P is a non-ordinary cusp then that means a curve through the six points has to be of type $[1, 1, 1, 1, 1]$, $[2, 1, 1, 1]$, $[2, 2, 1]$, $[3, 1, 1]$, $[3, 2]$, or $[4, 1]$. And these types cannot have six singularities of which one is a non-ordinary cusp such that the six points are in general position. So for P and a set S of five other points such that they are in general position with P we have

$$(V_k(5, 2P; y; y, 2S) - V_k(5, 3P, 2S)) \cap C_{x^3z^2 \neq 0} = V_k(5, 2P; y; y, 2S) - V_k(5, 3P, 2S).$$

This means that we do not need the condition on x^3z^2 and we can use its corresponding column. So we can use the same proof as the one we used for nodes in section 10.1.1.

Note that the above argument does not work when we have P and less than five points in general position.

10.2.2 P and four other points in general position.

We map the four points to $(0 : 1 : 0), (1 : 1 : 1), (1 : \alpha : \beta), (1 : \gamma : \delta)$ where $\alpha, \gamma \neq 1$ and $\alpha \neq \gamma$. We also have $\alpha\beta\gamma - \alpha\gamma\delta - \alpha\beta + \gamma\delta + \alpha - \gamma \neq 0$ for otherwise there would be five points on a conic that has tangent y at P. Taking derivatives to y and z and taking the columns for $y^3z^2, xy^3z, x^2y^2z, x^3yz, x^4z, x^2y^3, x^3y^2, x^4y, x^5$ we get determinant

$$-(\alpha\beta\gamma - \alpha\gamma\delta - \alpha\beta + \gamma\delta + \alpha - \gamma)^3(\alpha - \gamma)^2(\alpha - 1)^2(\gamma - 1)^2.$$

10.2.3 Five points on a line and P outside the line.

We map the four points on the line to $(0:1:0),(1:\alpha:0),(1:\beta:0),(1:\gamma:0),(1:\delta:0)$ with $\alpha,\beta,\gamma,\delta$ all different. The points determine the line so we get a space of quartics. Taking derivatives to x and z and taking the columns for xy^3,x^2y^2,x^3y,x^4 we get determinant

$$-(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta).$$

10.2.4 Four points on a line and P and another point outside the line.

We map the four points on the line to $(0:1:0),(1:\alpha:0),(1:\beta:0),(1:\gamma:0)$ with $\alpha\neq\beta\neq\gamma\neq\alpha$ and $\alpha,\beta,\gamma\neq 1$. The other point we map to $(1:1:1)$. The points determine the line so we get a space of quartics. Taking derivatives to x and z and taking the columns for $xy^2z,x^2yz,xy^3,x^2y^2,x^3y,x^4$ we get determinant

$$(\alpha-\beta)(\alpha-\gamma)(\alpha-1)(\beta-\gamma)(\beta-1)(\gamma-1).$$

10.2.5 The result of the sieving.

Because $\pi_w(\mathbb{P}^1)=0$ for $w\geq 4$ and $\pi_w(\mathbb{P}^1-\{P\})=0$ for $w\geq 2$ all the counts become zero besides the count of the cases where all points are in general position. Again we pretend all points are in general position. For P and a set S of $3w$ points that are in general position with P we have

$$V_k(5,2P;y;y,2S)\cap C_{x^3z^2\neq 0}-V_k(5,3P,2S)\cap C_{x^3z^2\neq 0}=\mathbb{A}^{15-3w}-\mathbb{A}^{15-3w-1}.$$

So we get

$$\begin{aligned}
s^{c}_{\text{sieve}} &= \frac{1}{|G_{P,y,y}|}\sum_{w=0}^{5}(|\mathbb{A}^{15-3w}|-|\mathbb{A}^{14-3w}|)\cdot\pi_w(\mathbb{P}^2-\{P\})\\
&= \frac{1}{|G_{P,y,y}|}(((|\mathbb{A}^{15}|-|\mathbb{A}^{14}|)\cdot 1-(|\mathbb{A}^{12}|-|\mathbb{A}^{11}|)\cdot(q^2+q)+(|\mathbb{A}^{9}|-|\mathbb{A}^{8}|)\cdot q^3)\\
&= \frac{q^{15}-2q^{14}+2q^{12}-q^{11}}{q^5-2q^4+q^3}\\
&= q^{10}-q^8.
\end{aligned}$$

10.3 The explicit count.

We note that when we have P and a λ-tuple of points we get for $|\lambda|=7$

$$\tau(\lambda)=\sigma(\lambda)(1-\binom{\lambda_1}{1}).$$

For $|\lambda| = 8$ we get

$$\tau(\lambda) = \sigma(\lambda)(1 - \binom{\lambda_1}{1} - \binom{\lambda_2}{1} + \binom{\lambda_1}{2}).$$

For $|\lambda| = 9$ we get

$$\tau(\lambda) = \sigma(\lambda)(1 - \binom{\lambda_1}{1} - \binom{\lambda_2}{1} - \binom{\lambda_3}{1} + \binom{\lambda_1}{2} - \binom{\lambda_1}{3} + \binom{\lambda_1}{1}\binom{\lambda_2}{1}).$$

Remember that for the explicit count we no longer fix a point P but instead have

$$s^{\mathrm{s}}_{\mathrm{explicit}} := \frac{1}{|\mathrm{PGL}_3(k)|} \sum_{N<|\lambda|\leq M} \tau(\lambda) \sum_{C\in C_{5,[[1^1],\lambda]}} \#\mathrm{s}(C).$$

And similar for $s^{\mathrm{ns}}_{\mathrm{explicit}}, s^{\mathrm{c}}_{\mathrm{explicit}}$.

By Theorem 5.10 curves of type [5] have at most six singularities so we do not need to consider them. For curves C of type $[1,1,1,1,1]$, $[2,1,1,1]$, $[2,2,1]$ we will have $\#\mathrm{c}(C) = 0$ so we can ignore the counts for cusps in those cases.

10.3.1 $[1,1,1,1,1]$, three lines through one point Q.

We can reuse our findings from section 9.2.1 with different $\tau(\lambda)$. And instead of an 8-tuple of points we instead write it as a k-point P and a 7-tuple of points. However keep in mind that we do not fix P here, the difference between the tables is one of notation. Besides the fact that we leave out the cases where $\tau(\lambda) = 0$ and the cases where $\#\mathrm{s}(C) = \#\mathrm{ns}(C) = \#\mathrm{c}(C) = 0$.

The values in the table should be multiplied with $\binom{q^2+q+1}{2}2$.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^5]$	$[1^8]$	6	$\binom{q}{2}\binom{q}{3}$
$[1^3, 2^1]$, both k_2-lines through Q	$[1^4, 2^2]$	2	$\binom{q}{2}q\frac{1}{2}(q^2-q)$

For $[1^5]$ we find that $\#\mathrm{s}(C) = 8$ and for $[1^3, 2^1]$ we have $\#\mathrm{s}(C) = 3$. We get for split nodes

$$\sum \frac{\tau(\lambda)\cdot\#\mathrm{s}(C)\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{7}{2}(q-2) + \frac{3}{2}q = 5q - 7$$

where the sum is over the rows in the table. There are no non-split nodes to be counted here.

10.3.2 $[1,1,1,1,1]$, no three lines through one point.

We reuse our findings from section 9.2.2.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^5]$	$[1^{10}]$	56	$\frac{1}{15}\binom{q^2+q+1}{2}\binom{q}{2}^2(q-2)(q-3)$
$[1^3, 2^1]$	$[1^4, 2^3]$	6	$\frac{1}{3}\binom{q^2+q+1}{2}2\binom{q}{2}\frac{1}{2}(q^2-q)q(q-1)$
$[1^2, 3^1]$	$[1^1, 3^3]$	2	$(q^2+q+1)\binom{q+1}{2}\frac{1}{3}(q+1)q^3(q-1)^2$

For split nodes we get

$\sum \frac{\tau(\lambda)\cdot\#\mathrm{s}(C)\cdot\lvert C\rvert}{\lvert\mathrm{PGL}_3(k)\rvert} = \frac{14}{3}(q-2)(q-3) + \frac{3}{2}q(q-1) + \frac{1}{3}q(q+1) = \frac{1}{2}(13q^2 - 49q + 56)$

where the sum is over the rows in the table. And for non-split nodes we get

$$\sum \frac{\tau(\lambda)\cdot\#\mathrm{ns}(C)\cdot\lvert C\rvert}{\lvert\mathrm{PGL}_3(k)\rvert} = \frac{1}{2}q(q-1)$$

where the sum is over the rows in the table.

10.3.3 $[2,1,1,1]$, two of the lines are tangent to the conic.

Precisely two of the lines are tangent to the conic, no lines intersect on the conic, and the three lines do not intersect in one point.

For a $[1^3]$-tuple of lines and a $[1^7]$-tuple of points we pick an irreducible k-conic: $(q^2+q+1)q^2(q-1)$. We pick two k-points on the conic and take the tangent lines at those points, they meet in a point Q: $\binom{q+1}{2}$. We take two more k-points on the conic and take the line through them. We then have to subtract the case where the resulting line goes through Q. If $\mathrm{char}(k) \neq 2$ then by 9.6 this gives $\binom{q-1}{2} - \frac{q-1}{2}$. If $\mathrm{char}(k) = 2$ then Q is the strange point so we get $\binom{q-1}{2}$.

For $\mathrm{char}(k) \neq 2$.
The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)$.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3]$	$[1^7]$	1	$\binom{q+1}{2}\left(\binom{q-1}{2} - \frac{q-1}{2}\right)$
	$[1^5, 2^1]$	-1	$\binom{q+1}{2}\left(\frac{1}{2}(q^2-q) - \frac{q-1}{2}\right)$
$[1^1, 2^1]$	$[1^3, 2^2]$	1	$\frac{1}{2}(q^2-q)\left(\binom{q+1}{2} - \frac{q+1}{2}\right)$
	$[1^1, 2^3]$	-1	$\frac{1}{2}(q^2-q)\left(\frac{1}{2}(q^2-q-2) - \frac{q+1}{2}\right)$

For split nodes we get

$$\sum \frac{\tau(\lambda) \cdot \#\mathrm{s}(C) \cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{5}{4}(q-3) - \frac{3}{4}(q-1) + \frac{1}{2}(q-1) = \frac{1}{2}(2q-7)$$

where the sum is over the rows in the table. And for non-split nodes we get

$$\sum \frac{\tau(\lambda) \cdot \#\mathrm{ns}(C) \cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{4}(q-1) - \frac{1}{4}(q-3) = \frac{1}{2}$$

where the sum is over the rows in the table.

For char$(k) = 2$.
The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)$.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3]$	$[1^7]$	1	$\binom{q+1}{2}\binom{q-1}{2}$
	$[1^5, 2^1]$	-1	$\binom{q+1}{2}\frac{1}{2}(q^2-q)$
$[1^1, 2^1]$	$[1^3, 2^2]$	1	$\frac{1}{2}(q^2-q)\binom{q+1}{2}$
	$[1^1, 2^3]$	-1	$\frac{1}{2}(q^2-q)\frac{1}{2}(q^2-q-2)$

For split nodes we get

$$\sum \frac{\tau(\lambda) \cdot \#\mathrm{s}(C) \cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{5}{4}(q-2) - \frac{3}{4}q + \frac{1}{2}q = \frac{1}{2}(2q-5)$$

where the sum is over the rows in the table. And for non-split nodes we get

$$\sum \frac{\tau(\lambda) \cdot \#\mathrm{ns}(C) \cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{4}q - \frac{1}{4}(q-2) = \frac{1}{2}$$

10.3.4 $[2,1,1,1]$, one of the lines is tangent to the conic.

Precisely one of the lines is tangent to the conic, the intersections of the lines are not on the conic, and the three lines do not intersect in one point. We reuse our findings from section 9.2.4.

For char$(k) \neq 2$.
The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)(q+1)$.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3]$	$[1^8]$	6	$\binom{q}{4}3 - q\binom{\frac{q-1}{2}}{2}$
	$[1^6, 2^1]$	-4	$\binom{q}{2}\frac{1}{2}(q^2-q) - q\left(\frac{q-1}{2}\right)^2$
	$[1^4, 2^2]$	2	$\binom{\frac{1}{2}(q^2-q)}{2} - q\binom{\frac{q-1}{2}}{2}$

For split nodes we get
$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{s}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{21}{4}(q-3)^2 - 5(q-1)^2 + \frac{3}{4}(q-1)^2 = q^2 - 23q + 43$$
where the sum is over the rows in the table. There are no non-split nodes to be counted here.

For char$(k) = 2$.
The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)(q+1)$.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3]$	$[1^8]$	6	$\binom{q}{4}3 - (q-1)\binom{\frac{q}{2}}{2}$
	$[1^6, 2^1]$	-4	$\binom{q}{2}\frac{1}{2}(q^2-q) - (q-1)\left(\frac{q}{2}\right)^2$
	$[1^4, 2^2]$	2	$\binom{\frac{1}{2}(q^2-q)}{2} - (q-1)\binom{\frac{q}{2}}{2}$

For split nodes we get
$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{s}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{21}{4}(q-2)(q-4) - 5q(q-2) + \frac{3}{4}q(q-2) = q^2 - 23q + 42$$
where the sum is over the rows in the table.

10.3.5 $[2,1,1,1]$, the lines intersect on the conic.

None of the lines is tangent to the conic, the intersection of two of the lines is on the conic, and the other intersections of the lines are not on the conic.

For a $[1^3]$-tuple of lines and a $[1^7]$-tuple of points we pick an irreducible k-conic and choose a k-point Q on the conic: $(q^2+q+1)q^2(q-1)(q+1)$. Then we pick two k-lines through Q that are not tangent to the conic: $\binom{q}{2}$. And then we choose two k-points on the conic that are not on the two lines. We take the line through these points: $\binom{q-2}{2}$.

If we have a $[1^1, 2^1]$-tuple of lines such that the k_2-tuple of lines intersects on a point outside the conic. Then the intersection of the k-line with one of the k_2-lines is a k_2-point Q on the conic. The conjugate of this point is on the k-line, on the other k_2-line, and on the conic. So two of the intersection points of the lines are on the conic which is a contradiction. This means that we do not get any non-split nodes.

The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)(q+1)$.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3]$	$[1^7]$	1	$\binom{q}{2}\binom{q-2}{2}$
	$[1^5, 2^1]$	-1	$\binom{q}{2}\frac{1}{2}(q^2-q)$
$[1^1, 2^1]$	$[1^3, 2^2]$	1	$\frac{1}{2}(q^2-q)\binom{q}{2}$

For split nodes we get

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{s}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{3}{2}(q-2)(q-3) - q(q-1) + \frac{1}{2}q(q-1) = q^2 - 7q + 9$$

where the sum is over the rows in the table.

10.3.6 $[2,1,1,1]$, three lines through one point.

None of the lines is tangent to the conic and the three lines intersect in one point outside the conic. We reuse our findings from section 9.2.5.

For char$(k) \neq 2$.
The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)$.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3]$	$[1^7]$	1	$\binom{q+1}{2}\binom{\frac{q-1}{2}}{3} + \binom{q}{2}\binom{\frac{q+1}{2}}{3}$
	$[1^5, 2^1]$	-1	$\binom{q+1}{2}\binom{\frac{q-1}{2}}{2}\frac{q-1}{2} + \binom{q}{2}\binom{\frac{q+1}{2}}{2}\frac{q+1}{2}$
	$[1^3, 2^2]$	1	$\binom{q+1}{2}\frac{q-1}{2}\binom{\frac{q-1}{2}}{2} + \binom{q}{2}\frac{q+1}{2}\binom{\frac{q+1}{2}}{2}$
	$[1^1, 2^3]$	-1	$\binom{q+1}{2}\binom{\frac{q-1}{2}}{3} + \binom{q}{2}\binom{\frac{q+1}{2}}{3}$
$[1^1, 2^1]$	$[1^3, 2^2]$	1	$\binom{q+1}{2}\frac{q-1}{2}\frac{(q-1)^2}{4} + \binom{q}{2}\frac{q+1}{2}\frac{(q+1)(q-3)}{4}$
	$[1^1, 2^3]$	-1	$\binom{q+1}{2}\frac{q-1}{2}\frac{(q-1)^2}{4} + \binom{q}{2}\frac{q+1}{2}\frac{(q+1)(q-3)}{4}$
	$[1^3, 4^1]$	-1	$\binom{q+1}{2}\frac{q-1}{2}\frac{q^2-1}{4} + \binom{q}{2}\frac{q+1}{2}\frac{q^2-1}{4}$
	$[1^1, 2^1, 4^1]$	1	$\binom{q+1}{2}\frac{q-1}{2}\frac{q^2-1}{4} + \binom{q}{2}\frac{q+1}{2}\frac{q^2-1}{4}$

For split nodes we get

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{s}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{8}(q-3)^2 - \frac{1}{4}(q-1)^2 + \frac{1}{8}(q-1)^2 + \frac{1}{4}(q^2-2q-1) - \frac{1}{4}(q+1)(q-1) = -(q-1)$$

where the sum is over the rows in the table. There are no non-split nodes to be counted here.

For char$(k) = 2$.

The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)(q^2-1)$.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3]$	$[1^7]$	1	$\binom{\frac{q}{2}}{3}$
	$[1^5, 2^1]$	-1	$\binom{\frac{q}{2}}{2}\frac{q}{2}$
	$[1^3, 2^2]$	1	$\frac{q}{2}\binom{\frac{q}{2}}{2}$
	$[1^1, 2^3]$	-1	$\binom{\frac{q}{2}}{3}$
$[1^1, 2^1]$	$[1^3, 2^2]$	1	$\frac{q}{2}\frac{q(q-2)}{4}$
	$[1^1, 2^3]$	-1	$\frac{q}{2}\frac{q(q-2)}{4}$
	$[1^3, 4^1]$	-1	$\frac{q}{2}\frac{q^2}{4}$
	$[1^1, 2^1, 4^1]$	1	$\frac{q}{2}\frac{q^2}{4}$

For split nodes we get

$$\sum \frac{\tau(\lambda)\cdot\#s(C)\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{8}(q-2)(q-4) - \frac{1}{4}q(q-2) + \frac{1}{8}q(q-2) + \frac{1}{4}q(q-2) - \frac{1}{4}q^2 = -(q-1)$$

where the sum is over the rows in the table.

10.3.7 $[2, 1, 1, 1]$, nine singularities.

None of the lines is tangent to the conic, no lines intersect on the conic, and the three lines do not intersect in one point. We reuse our findings from section 9.2.5.

The values in the table should be multiplied with $(q^2 + q + 1)q^2(q - 1)$.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3]$	$[1^9]$	28	$\binom{q+1}{6}15$
	$[1^7, 2^1]$	-9	$\binom{q+1}{4}\frac{1}{2}(q^2-q)3$
	$[1^5, 2^2]$	1	$\binom{q+1}{2}\binom{\frac{1}{2}(q^2-q)}{2}$
	$[1^3, 2^3]$	3	$\binom{\frac{1}{2}(q^2-q)}{3}$
$[1^1, 2^1]$	$[1^3, 2^3]$	3	$\binom{q+1}{2}\binom{\frac{1}{2}(q^2-q)}{2}2$
	$[1^1, 2^4]$	-3	$\binom{\frac{1}{2}(q^2-q)}{3}3\cdot 2$
	$[1^3, 2^1, 4^1]$	-1	$\binom{q+1}{2}\frac{1}{4}(q^4-q^2)$
	$[1^1, 2^2, 4^1]$	1	$\frac{1}{2}(q^2-q)\frac{1}{4}(q^4-q^2)$

For char$(k) \neq 2$.
We subtract the case where we have three lines through one point to get for split nodes

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{s}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{21}{4}(q-2)(q-3)(q-4) - \frac{63}{16}q(q-1)(q-2) + \frac{5}{16}(q+1)q(q-2)$$
$$+\frac{3}{16}(q-2)(q^2-q-4) + \frac{3}{4}(q+1)q(q-2) - \frac{1}{4}(q+1)q^2 - \Big(\frac{21}{4}(q-3)^2 - \frac{63}{16}(q-1)^2$$
$$+\frac{5}{16}(q-1)^2 + \frac{3}{16}(q-3)^2 + \frac{3}{4}(q^2-2q-1) - \frac{1}{4}(q+1)(q-1)\Big) =$$
$$\frac{1}{2}(2q^3 - 51q^2 + 184q - 186) - (q^2 - 19q + 33) =$$
$$\frac{1}{2}(2q^3 - 53q^2 + 222q - 252)$$

For non-split nodes:

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{ns}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{3}{8}(q+1)q(q-2) - \frac{3}{8}(q-2)(q^2-q-4) - \frac{1}{8}(q+1)q^2$$
$$+\frac{1}{8}q^2(q-1) - \Big(\frac{3}{8}(q^2-2q-1) - \frac{3}{8}(q^2-2q-1) - \frac{1}{8}(q+1)(q-1) + \frac{1}{8}(q+1)(q-1)\Big) =$$
$$\frac{1}{2}(q^2-6) - 0 = \frac{1}{2}(q^2-6)$$

For char$(k) = 2$.
Subtracting three lines through one point results in $-(q^2 - 19q + 33)$ and -0 just as for char$(k) \neq 2$. So the end result is the same.

10.3.8 $[2,2,1]$, the line is tangent to one of the conics.

The line is tangent to precisely one of the conics and intersects the other conic in 2 points. The conics intersect in 4 other points.

If we have a conjugate 2-tuple of conics then the k-line has to intersect both conics in the same number of points so this will not happen.

For char$(k) \neq 2$.
For a $[1^2]$-tuple of conics and a $[1^7]$-tuple of points we pick four k-points P_1, P_2, P_3, P_4 such that there are no three on a line.

$$\frac{1}{24}(q^2+q+1)(q+1)q^3(q-1)^2$$

Then we use the tables in section 9.2.6 to pick a k-line L that is tangent to precisely one irreducible conic through P_1, P_2, P_3, P_4: $3(q-3)$. We take this conic and choose another conic through two k-points on L: $C_k = \frac{q-5}{2}$. We can also pick a k-line L that is tangent to precisely two irreducible k-conics through P_1, P_2, P_3, P_4: $\frac{1}{2}(q-3)(q-5)$. We pick one of the two irreducible conics and a conic through two k-points on L: $2C_k = q-7$.

The values in the table should be multiplied with $(q^2+q+1)(q+1)q^3(q-1)^2$.

4 points	P, λ points	$\tau(\lambda)\nu(\lambda)$	$\lvert C\rvert$
$[1^4]$	$[1^7]$	$\frac{1}{24}$	$3(q-3)\frac{q-5}{2}+\frac{1}{2}(q-3)(q-5)(q-7)$
	$[1^5,2^1]$	$-\frac{1}{24}$	$3(q-3)\frac{q-1}{2}+\frac{1}{2}(q-3)(q-5)(q-1)$
$[1^2,2^1]$	$[1^5,2^1]$	$-\frac{1}{4}$	$(q-1)\frac{q-1}{2}+\frac{1}{2}(q-1)^2(q-3)$
	$[1^3,2^2]$	$\frac{1}{4}$	$(q-1)\frac{q-1}{2}+\frac{1}{2}(q-1)^2(q-1)$
$[1^1,3^1]$	$[1^4,3^1]$	$\frac{1}{3}$	$\frac{1}{2}(q^2+q)(q-1)$
	$[1^2,2^1,3^1]$	$-\frac{1}{3}$	$\frac{1}{2}(q^2+q)(q-1)$
$[2^2]$	$[1^3,2^2]$	$\frac{1}{8}$	$(q-3)\frac{q-1}{2}+2(q-1)\frac{q-3}{2}+\frac{1}{2}(q-1)(q-3)(q-3)$
$[4^1]$	$[1^3,4^1]$	$-\frac{1}{8}$	$(q+1)\frac{q-1}{2}+\frac{1}{2}(q^2-1)(q-1)$

For split nodes we get

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{s}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{8}(q-3)(q-4)(q-5) - \cdot\frac{1}{12}(q-1)(q-2)(q-3) - \cdot\frac{1}{2}(q-1)^2(q-2)$$
$$+\cdot\frac{1}{4}q(q-1)^2 + \cdot\frac{1}{2}(q+1)q(q-1) - \cdot\frac{1}{6}(q+1)q(q-1) + \cdot\frac{1}{8}q(q-1)(q-3) - \cdot\frac{1}{4}(q+1)q(q-1) = 3(q-2)$$

where the sum is over the rows in the table. There are no non-split nodes to be counted here.

For char$(k) = 2$.
The values in the table should be multiplied with $(q^2+q+1)(q+1)q^3(q-1)^2$.

4 points	P, λ points	$\tau(\lambda)\nu(\lambda)$	$\lvert C\rvert$
$[1^4]$	$[1^7]$	$\frac{1}{24}$	$(q-2)(q-4)\frac{q-6}{2}$
	$[1^5,2^1]$	$-\frac{1}{24}$	$(q-2)(q-4)\frac{q}{2}$
$[1^2,2^1]$	$[1^5,2^1]$	$-\frac{1}{4}$	$q(q-2)\frac{q-2}{2}$
	$[1^3,2^2]$	$\frac{1}{4}$	$q(q-2)\frac{q}{2}$
$[1^1,3^1]$	$[1^4,3^1]$	$\frac{1}{3}$	$(q+1)(q-1)\frac{q}{2}$
	$[1^2,2^1,3^1]$	$-\frac{1}{3}$	$(q+1)(q-1)\frac{q}{2}$
$[2^2]$	$[1^3,2^2]$	$+\frac{1}{8}$	$q(q-2)\frac{q-2}{2}$
$[4^1]$	$[1^3,4^1]$	$-\frac{1}{4}$	$q^2\frac{q}{2}$

For split nodes we get

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{s}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{8}(q-2)(q-4)(q-6) - \cdot\frac{1}{12}q(q-2)(q-4) - \cdot\frac{1}{2}(q-2)^2$$
$$+\cdot\frac{1}{4}q^2(q-2) + \cdot\frac{1}{2}(q+1)q(q-1) - \cdot\frac{1}{6}(q+1)q(q-1) + \cdot\frac{1}{8}q(q-2)^2 - \cdot\frac{1}{4}6q^3 =$$
$$3(q-2)$$

where the sum is over the rows in the table.

10.3.9 $[2,2,1]$, eight singularities and the conics are definined over k.

The line is intersects each conic in 2 points. The conics intersect in 4 other points. We reuse our findings from section 9.2.7.

For char$(k) \neq 2$.
The values in the table should be multiplied with $(q^2+q+1)(q+1)q^3(q-1)^2$.

4 points	P, λ points	$\tau(\lambda)\nu(\lambda)$	$\lvert C\rvert$
$[1^4]$	$[1^8]$	$\frac{6}{24}$	$3\binom{\frac{1}{2}(q-3)}{2} + 3(q-3)\binom{\frac{1}{2}(q-5)}{2}$ $+\frac{1}{2}(q-3)(q-5)\binom{\frac{1}{2}(q-7)}{2} + \frac{1}{2}(q-1)(q-3)\binom{\frac{1}{2}(q-5)}{2}$
	$[1^6,2^1]$	$-\frac{4}{24}$	$3\frac{1}{2}(q-3)\frac{1}{2}(q-1) + 3(q-3)\frac{1}{2}(q-5)\frac{1}{2}(q-1)$ $+\frac{1}{2}(q-3)(q-5)\frac{1}{2}(q-7)\frac{1}{2}(q-1)$ $+\frac{1}{2}(q-1)(q-3)\frac{1}{2}(q-5)\frac{1}{2}(q+1)$
	$[1^4,2^2]$	$\frac{2}{24}$	$3\binom{\frac{1}{2}(q-1)}{2} + 3(q-3)\binom{\frac{1}{2}(q-1)}{2}$ $+\frac{1}{2}(q-3)(q-5)\binom{\frac{1}{2}(q-1)}{2} + \frac{1}{2}(q-1)(q-3)\binom{\frac{1}{2}(q+1)}{2}$
$[1^2,2^1]$	$[1^6,2^1]$	$-\frac{4}{4}$	$\binom{\frac{1}{2}(q-1)}{2} + (q-1)\binom{\frac{1}{2}(q-1)}{2}$ $+\frac{1}{2}(q-1)^2\binom{\frac{1}{2}(q-3)}{2} + \frac{1}{2}(q+1)(q-3)\binom{\frac{1}{2}(q-1)}{2}$
	$[1^4,2^2]$	$\frac{2}{4}$	$\frac{1}{2}(q-1)\frac{1}{2}(q+1) + (q-1)\frac{1}{2}(q-1)\frac{1}{2}(q-1)$ $+\frac{1}{2}(q-1)^2\frac{1}{2}(q-3)\frac{1}{2}(q-1)$ $+\frac{1}{2}(q+1)(q-3)\frac{1}{2}(q-1)\frac{1}{2}(q+1)$
$[1^1,3^1]$	$[1^5,3^1]$	$\frac{3}{3}$	$\frac{1}{2}q(q+1)\binom{\frac{1}{2}(q-1)}{2} + \frac{1}{2}q(q-1)\binom{\frac{1}{2}(q+1)}{2}$
	$[1^3,2^1,3^1]$	$-\frac{1}{3}$	$\frac{1}{2}q(q+1)\frac{1}{2}(q-1)\frac{1}{2}(q-1) + \frac{1}{2}q(q-1)\frac{1}{2}(q+1)\frac{1}{2}(q+1)$
	$[1^1,2^2,3^1]$	$-\frac{1}{3}$	$\frac{1}{2}q(q+1)\binom{\frac{1}{2}(q-1)}{2} + \frac{1}{2}q(q-1)\binom{\frac{1}{2}(q+1)}{2}$
$[2^2]$	$[1^4,2^2]$	$\frac{2}{8}$	$\binom{\frac{1}{2}(q-3)}{2} + 2\binom{\frac{1}{2}(q-1)}{2} + 2(q-1)\binom{\frac{1}{2}(q-3)}{2} + (q-3)\binom{\frac{1}{2}(q-1)}{2}$ $+\frac{1}{2}(q-1)(q-3)\binom{\frac{1}{2}(q-3)}{2} + \frac{1}{2}(q+1)(q-1)\binom{\frac{1}{2}(q-1)}{2}$
$[4^1]$	$[1^4,4^1]$	$-\frac{2}{4}$	$\binom{\frac{1}{2}(q+1)}{2} + (q+1)\binom{\frac{1}{2}(q-1)}{2}$ $+\frac{1}{2}(q+1)(q-1)\binom{\frac{1}{2}(q-1)}{2} + \frac{1}{2}(q+1)(q-1)\binom{\frac{1}{2}(q+1)}{2}$

For split nodes we get

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{s}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{24}\Big(6(q^2-9q+17)(q-3)(q-5)-6(q^2-5q+3)(q-1)(q-3)$$
$$+(q^2-q+1)(q-1)(q-3)-18(q^2-3q+1)(q-1)(q-3)+12(q^3-2q^2-1)(q-1)$$
$$+15(q+1)q(q-1)(q-2)-6q^2(q+1)(q-1)-(q+1)q(q-1)(q-2)$$
$$+3(q^3-2q^2-2q-1)(q-3)-6(q^2-q-1)(q+1)(q-1)\Big) =$$
$$\frac{1}{2}(19q^2-101q+120)$$

where the sum is over the rows in the table. There are no non-split nodes to be counted here.

For char$(k) = 2$.
The values in the table should be multiplied with $(q^2+q+1)(q+1)q^3(q-1)^2$.

4 points	P, λ points	$\tau(\lambda)\nu(\lambda)$	$\|C\|$
$[1^4]$	$[1^8]$	$\frac{6}{24}$	$3(q-2)\binom{\frac{1}{2}(q-4)}{2}+(q-2)(q-4)\binom{\frac{1}{2}(q-6)}{2}$
	$[1^6,2^1]$	$-\frac{4}{24}$	$3(q-2)\frac{1}{2}(q-4)\frac{1}{2}q+(q-2)(q-4)\frac{1}{2}(q-6)\frac{1}{2}q$
	$[1^4,2^2]$	$\frac{2}{24}$	$3(q-2)\binom{\frac{1}{2}q}{2}+(q-2)(q-4)\binom{\frac{1}{2}q}{2}$
$[1^2,2^1]$	$[1^6,2^1]$	$-\frac{4}{4}$	$(q-2)\binom{\frac{1}{2}q}{2}+q(q-2)\binom{\frac{1}{2}(q-2)}{2}$
	$[1^4,2^2]$	$\frac{2}{4}$	$(q-2)\frac{1}{2}q\frac{1}{2}q+q(q-2)\frac{1}{2}(q-2)\frac{1}{2}q$
$[1^1,3^1]$	$[1^5,3^1]$	$\frac{3}{3}$	$(q+1)(q-1)\binom{\frac{1}{2}q}{2}$
	$[1^3,2^1,3^1]$	$-\frac{1}{3}$	$(q+1)(q-1)\frac{1}{2}q\frac{1}{2}q$
	$[1^1,2^2,3^1]$	$-\frac{1}{3}$	$(q+1)(q-1)\binom{\frac{1}{2}q}{2}$
$[2^2]$	$[1^4,2^2]$	$\frac{2}{8}$	$(q-2)\binom{\frac{1}{2}q}{2}+2q\binom{\frac{1}{2}(q-2)}{2}+q(q-2)\binom{\frac{1}{2}(q-2)}{2}$
$[4^1]$	$[1^4,4^1]$	$-\frac{2}{4}$	$q\binom{\frac{1}{2}q}{2}+q^2\binom{\frac{1}{2}q}{2}$

For split nodes we get

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{s}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{24}\Big(6(q-2)(q-4)(q-5)(q-6)-6q(q-2)(q-3)(q-4)$$
$$+q(q-1)(q-2)^2-18q(q-2)^2(q-3)+12q^2(q-1)(q-2)+15q(q+1)(q-1)(q-2)$$
$$-6q^2(q+1)(q-1)-q(q+1)(q-1)(q-2)+q(q^2-3q-2)(q-2)-6q^2(q+1)(q-2)\Big) =$$
$$\frac{1}{2}(19q^2-101q+120)$$

where the sum is over the rows in the table.

10.3.10 $[2,2,1]$, eight singularities with a conjugate 2-tuple of conics.

We reuse our findings from section 9.2.8.

For char$(k) \neq 2$.
The values in the table should be multiplied with $\frac{1}{4}(q^2+q+1)(q+1)q^3(q-1)^2$.

4 points	P, λ points	$\tau(\lambda)\nu(\lambda)$	$\lvert C\rvert$
$[1^4]$	$[1^4, 2^2]$	$\frac{2}{24}$	$\frac{1}{2}(q^2-2q+3)(q^2-2q+1)$ $+\frac{1}{2}(q-1)(q-3)(q^2-2q-3)$
	$[1^4, 4^1]$	$-\frac{2}{24}$	$(q^2-3q+3)(q^2-1)$
$[1^1, 3^1]$	$[1^1, 2^2, 3^1]$	$-\frac{1}{3}$	$\frac{1}{2}(q^2+q)(q^2-2q+1)$ $+\frac{1}{2}(q^2-q)(q^2-2q-3)$
	$[1^1, 3^1, 4^1]$	$\frac{1}{3}$	$q^2(q^2-1)$

For non-split nodes we get

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{ns}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{12}\Big((q^3-4q^2+4q+3)(q-1)$$
$$-(q^2-3q+3)(q+1)(q-1)-(q+1)q(q-1)(q-2)+(q+1)q^2(q-1)\Big) =$$
$$\frac{1}{2}q(q-1)$$

where the sum is over the rows in the table. There are no split nodes to be counted here.

For char$(k) = 2$.
The values in the table should be multiplied with $\frac{1}{4}(q+1)q^3(q-1)^2(q^2+q+1)$.

4 points	P, λ points	$\tau(\lambda)\nu(\lambda)$	$\lvert C\rvert$
$[1^4]$	$[1^4, 2^2]$	$\frac{2}{24}$	$(q-1)(q-2)(q^2-2q)$
	$[1^4, 4^1]$	$-\frac{2}{24}$	$(q-1)(q-2)q^2$
$[1^1, 3^1]$	$[1^1, 2^2, 3^1]$	$-\frac{1}{3}$	$(q+1)(q-1)(q^2-2q)$
	$[1^1, 3^1, 4^1]$	$+1\frac{1}{3}$	$(q+1)(q-1)q^2$

For non-split nodes we get

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{ns}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{12}\Big((q-1)(q-2)(q^2-2q)$$
$$-(q-1)q^2(q-2)-(q+1)(q-1)(q^2-2q)+8q^2(q+1)(q-1)\Big) = \frac{1}{2}q(q-1)$$

where the sum is over the rows in the table.

10.3.11 Tools for two conics that intersect in three points.

Let there be a line T, a point Q on T and a λ-tuple of points P_1, P_2 not on T such that Q, P_1, P_2 are not on a line.

We write $\mathcal{P}$ for the pencil of conics trough Q, P_1, P_2 that have tangent T at Q. Let L be a k-line not through any of Q, P_1, P_2. We can define A_k, B_k, C_k, etc. similar to how we did before in section 9.2.6.

There are two reducible conics in $\mathcal{P}$ and they are both k-conics so we have $X_k = X_{k_2} = 2$.

For char$(k) \neq 2$.
We get $B_{k_2} = 2$ and $A_{k_2} = Y_{k_2}$ so $D_{k_4} = \frac{1}{4}(q^2-1)$.
For $\lambda = [1^2]$ we have $A_k = A_{k_2}$.

number	A_k	B_k	Y_k	$2C_k$	$2C_{k_2}$	$4D_{k_2}$
$q-1$	1	2	1	$q-3$	$q-1$	q^2-2q+1
$\frac{1}{2}(q-1)(q-3)$	2	2	2	$q-5$	$q-1$	q^2-2q+1
$\frac{1}{2}(q-1)^2$	2	0	0	$q-3$	$q+1$	q^2-2q-3

For $\lambda = [1^2]$ we get

number	A_k	A_{k_2}	B_k	Y_k	$2C_k$	$2C_{k_2}$	$4D_{k_2}$
$q-1$	0	1	2	1	$q-1$	$q-3$	q^2-2q+1
$\frac{1}{2}(q-1)^2$	1	2	2	2	$q-3$	$q-3$	q^2-2q+1
$\frac{1}{2}(q+1)(q-1)$	1	2	0	0	$q-1$	$q-1$	q^2-2q-3

For char$(k) = 2$.
We get $B_k = B_{k_2} = 1$, $Y_k = Y_{k_2}$ and $A_{k_2} = Y_{k_2} + 1$ so $4D_{k_2} = \frac{1}{4}(q^2-2q)$ and $D_{k_4} = \frac{1}{4}q^2$.

For $\lambda = [1^2]$ we have $A_k = A_{k_2}$.

number	A_k	Y_k	$2C_k$	$2C_{k_2}$
$q-1$	1	0	$q-2$	q
$(q-1)(q-2)$	2	1	$q-4$	q

10.3.12 $[2,2,1]$, two k-conics intersect in 3 points.

The line intersects each conic in 2 points. The conics intersect in 3 other points.

For char$(k) \neq 2$.
For a $[1^2]$-tuple of conics and a $[1^7]$-tuple of points we pick a line T, a point Q on T and two k-points P_1, P_2 not on T such that Q, P_1, P_2 are not on a line.

$$(q^2+q+1)(q+1)\binom{q^2}{2} - \binom{q^2+q+1}{2}2\binom{q}{2} = \frac{1}{2}(q^2+q+1)(q+1)q^3(q-1)$$

If instead we choose P_1, P_2 to be a k_2-tuple of points then we get

$$(q^2+q+1)(q+1)\tfrac{1}{2}(q^4-q^2) - \tbinom{q^2+q+1}{2}2\tfrac{1}{2}(q^2-q) = \tfrac{1}{2}(q^2+q+1)(q+1)q^3(q-1)$$

The values in the table should be multiplied with $\frac{1}{2}(q^2+q+1)(q+1)q^3(q-1)$.

2 points	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^2]$	$[1^7]$	1	$(q-1)\binom{\frac{1}{2}(q-3)}{2} + \frac{1}{2}(q-1)(q-3)\binom{\frac{1}{2}(q-5)}{2} + \frac{1}{2}(q-1)^2\binom{\frac{1}{2}(q-3)}{2}$
	$[1^5, 2^1]$	-1	$(q-1)\frac{1}{2}(q-3)\frac{1}{2}(q-1) + \frac{1}{2}(q-1)(q-3)\frac{1}{2}(q-5)\frac{1}{2}(q-1) + \frac{1}{2}(q-1)^2\frac{1}{2}(q-3)\frac{1}{2}(q+1)$
	$[1^3, 2^2]$	1	$(q-1)\binom{\frac{1}{2}(q-1)}{2} + \frac{1}{2}(q-1)(q-3)\binom{\frac{1}{2}(q-1)}{2} + \frac{1}{2}(q-1)^2\binom{\frac{1}{2}(q+1)}{2}$
$[2^1]$	$[1^5, 2^1]$	-1	$(q-1)\binom{\frac{1}{2}(q-1)}{2} + \frac{1}{2}(q-1)^2\binom{\frac{1}{2}(q-3)}{2} + \frac{1}{2}(q+1)(q-1)\binom{\frac{1}{2}(q-1)}{2}$
	$[1^3, 2^2]$	1	$(q-1)\frac{1}{2}(q-1)\frac{1}{2}(q-3) + \frac{1}{2}(q-1)^2\frac{1}{2}(q-3)\frac{1}{2}(q-3) + \frac{1}{2}(q+1)(q-1)\frac{1}{2}(q-1)\frac{1}{2}(q-1)$

For split nodes we get

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{s}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{8}\Big(3(q-3)^2(q-5)$$
$$-4(q-1)^2(q-3)+(q-1)^3-2(q-1)^2(q-3)+2(q-1)^3\Big) =$$
$$-\frac{3}{2}(q-2)(q-5)$$

where the sum is over the rows in the table. There are no non-split nodes to be counted here.

For char$(k)=2$**.**
The values in the table should be multiplied with $\frac{1}{2}(q^2+q+1)(q+1)q^3(q-1)$.

2 points	P,λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^2]$	$[1^7]$	1	$(q-1)\binom{\frac{1}{2}(q-2)}{2}$ $+(q-1)(q-2)\binom{\frac{1}{2}(q-4)}{2}$
	$[1^5,2^1]$	-1	$(q-1)\frac{1}{2}(q-2)\frac{1}{2}q$ $+(q-1)(q-2)\frac{1}{2}(q-4)\frac{1}{2}q$
	$[1^3,2^2]$	1	$(q-1)\binom{\frac{1}{2}q}{2}$ $+(q-1)(q-2)\binom{\frac{1}{2}q}{2}$
$[2^1]$	$[1^5,2^1]$	-1	$(q-1)\binom{\frac{1}{2}q}{2}$ $+q(q-1)\binom{\frac{1}{2}(q-2)}{2}$
	$[1^3,2^2]$	1	$(q-1)\frac{1}{2}q\frac{1}{2}(q-2)$ $+q(q-1)\frac{1}{2}(q-2)\frac{1}{2}(q-2)$

For split nodes we get

$$\sum \frac{\tau(\lambda)\cdot \#\mathrm{s}(C)\cdot |C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{8}(q-2)\Big(3(q-4)(q-5)$$
$$-4(q-3)q+q(q-1)-2q(q-3)+2q(q-1)\Big) =$$
$$-\frac{3}{2}(q-2)(q-5)$$

where the sum is over the rows in the table.

10.3.13 $[2,2,1]$, a conjugate 2-tuple of conics intersects in 3 points.

For $\mathbf{char}(k) \neq 2$**.**
The values in the table should be multiplied with $\frac{1}{2}(q^2+q+1)(q+1)q^3(q-1)$.

P,λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3,2^2]$	1	$\frac{1}{2}(q-1)^2\frac{1}{4}(q^2-2q+1)+\frac{1}{2}(q-1)^2\frac{1}{4}(q^2-2q-3)$
$[1^3,4^1]$	-1	$(q-1)^2\frac{1}{4}(q^2-1)$

For non-split nodes we get

$$\sum\frac{\tau(\lambda)\cdot\#\mathrm{ns}(C)\cdot|C|}{|\mathrm{PGL}_3(k)|}=\frac{1}{4}(q-1)\Big((q^2-2q-1)-(q^2-1)\Big)=-\frac{1}{2}q(q-1)$$

where the sum is over the rows in the table. There are no split nodes to be counted here.

For $\mathbf{char}(k) = 2$**.**
The values in the table should be multiplied with $\frac{1}{2}(q+1)q^3(q-1)(q^2+q+1)$.

P,λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^3,2^2]$	1	$(q-1)^2\frac{1}{4}(q^2-2q)$
$[1^3,4^1]$	-1	$(q-1)^2\frac{1}{4}q^2$

For non-split nodes we get

$$\sum\frac{\tau(\lambda)\cdot\#\mathrm{ns}(C)\cdot|C|}{|\mathrm{PGL}_3(k)|}=\frac{1}{4}(q-1)\Big((q^2-2q)-q^2\Big)=-\frac{1}{2}q(q-1)$$

where the sum is over the rows in the table.

10.3.14 $[3,1,1]$, eight singularities.

We get eight singularities when the lines each intersect the cubic in three points and intersect each other outside the conic. We reuse our findings from section 9.2.9.

The values in the table should be multiplied with $\frac{1}{2}(q^2+q+1)(q+1)q^2(q-1)$.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^2]$	$[1^8]$	6	$\binom{q}{3}\binom{q-3}{3}$
	$[1^6, 2^1]$	-4	$2\binom{q}{3}(q-3)\frac{1}{2}(q^2-q)$
	$[1^4, 2^2]$	2	$q\frac{1}{2}(q^2-q)(q-1)\frac{1}{2}(q^2-q-2)$
	$[1^5, 3^1]$	3	$2\binom{q}{3}\frac{1}{3}(q^3-q)$
	$[1^3, 2^1, 3^1]$	-1	$2\frac{1}{3}(q^3-q)\frac{1}{2}(q^2-q)q$

We do not know whether the singular point of te cubic is a split node, a non-split node, or a cusp. This means we can no longer compute the number of split nodes, non-split nodes, and cusps separately. However we can still compute their sum. We get

$$\sum \frac{\tau(\lambda)\cdot(\#\mathrm{s}(C)+\#\mathrm{ns}(C)+\#\mathrm{c}(C))\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{1}{6}\Big(4(q-2)(q-3)(q-4)(q-5)$$
$$-12q(q-1)(q-2)(q-3)+6(q+1)q(q-1)(q-2)+5(q+1)q(q-1)(q-2)-3(q+1)q^2(q-1)\Big) =$$
$$-(q^3-24q^2+87q-80)$$

where the sum is over the rows in the table.

10.3.15 $[3, 1, 1]$, a smooth cubic.

The lines each intersect the smooth cubic in three points and intersect each other outside the conic.

For a $[1^2]$-tuple of lines and a $[1^7]$-tuple of points we choose two k-lines L, L' and three k-points on each line that are not the intersection of the lines: $\binom{q^2+q+1}{2}\binom{q}{3}^2$.

There is a $\mathbb{P}^3$ of cubics through the 6 points. From this we subtract the reducible cubics through the 6 points. We get a reducible conic when we take the lines L, L and any other line more line, this gives a $\mathbb{P}^2$. Another way to get a reducible conic is by taking a k-line that is not L, L', through two of the points. And then there are q k-conics, that are not LL', through the four other points. So we get $|\mathbb{P}^3(k)| - |\mathbb{P}^2(k)| - 9q$ for the number of irreducible cubics. However the case where we have three k-lines, that are not L, L', through the six points has been subtracted once for every line. So to compensate we have to add it twice. There are 6 ways to choose three lines through the six points so we add 12.

After this we also have to subtract the cases where we get an irreducible singular cubic. We have already computed these cases above.

lines	P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^2]$	$[1^7]$	1	$\binom{q^2+q+1}{2}\binom{q}{3}^2(q^3-9q+12)$
	$[1^5,2^1]$	-1	$\binom{q^2+q+1}{2}2\binom{q}{3}q\frac{1}{2}(q^2-q)(q^3-3q)$
	$[1^3,2^2]$	1	$\binom{q^2+q+1}{2}q^2(\frac{1}{2}(q^2-q))^2(q^3-q)$
	$[1^4,3^1]$	1	$\binom{q^2+q+1}{2}2\binom{q}{3}\frac{1}{3}(q^3-q)q^3$
	$[1^1,3^2]$	1	$\binom{q^2+q+1}{2}(\frac{1}{3}(q^3-q))^2(q^3-3)$
	$[1^2,2^1,3^1]$	-1	$\binom{q^2+q+1}{2}2q\frac{1}{2}(q^2-q)\frac{1}{3}(q^3-q)q^3$
$[2^1]$	$[1^1,2^3]$	-1	$\frac{1}{2}(q^4-q)\binom{q^2}{3}(q^3-3q+2)$
	$[1^1,2^1,4^1]$	1	$\frac{1}{2}(q^4-q)q^2\frac{1}{2}(q^4-q^2)(q^3-q)$
	$[1^1,6^1]$	-1	$\frac{1}{2}(q^4-q)\frac{1}{3}(q^6-q^2)(q^3-1)$

We get

$$\sum\frac{\tau(\lambda)\cdot(\#\mathrm{s}(C)+\#\mathrm{ns}(C)+\#\mathrm{c}(C))\cdot|C|}{|\mathrm{PGL}_3(k)|}=\frac{1}{72}\Bigg(7(q^3-9q+12)(q-2)^2-30(q^2-3)(q-2)q^2$$
$$+27(q+1)(q-1)q^3+16(q+1)(q-2)q^3+4(q^3-3)(q+1)^2-24(q+1)q^4-6(q^2-2)(q+2)(q-1)^2$$
$$+18(q+1)(q-1)q^3-12(q^2+q+1)(q^2+1)(q-1)-\Big(7\cdot(q-2)(q-3)(q-4)(q-5)$$
$$-30q(q-1)(q-2)(q-3)+27(q+1)q(q-1)(q-2)+16(q+1)q(q-1)(q-2)+4(q^3-q-3)(q+1)$$
$$-24(q+1)q^2(q-1)-6(q^2-q-4)(q+1)(q-2)+18(q+1)q^2(q-1)-12(q^3+q-1)(q+1)\Big)\Bigg)=$$
$$(2q^2-9q+5)-(2q^2-12q+11)=3(q-2)$$

where the sum is over the rows in the tables.

10.3.16 $[3,1,1]$, one of the lines is tangent to the cubic.

One of the lines intersects the cubic in three points and the other line intersects the cubic in two points. The lines intersect each other outside the conic. And the cubic has a singular point.

The lines have to be a $[1^2]$-tuple. For a $[1^7]$-tuple of points we choose two k-lines and a k-point Q outside those lines: $\frac{1}{2}(q^2+q+1)(q+1)q^2(q-1)$. We then choose one of the two lines and three k-points on the line: $2\binom{q}{3}$. Then we pick two points on the other line L such that there are no three points on a line through Q. And we finally choose one of these two points R: $\binom{q-3}{2}2$.

Similar to lemma 9.9 there is exactly one irreducible cubic through Q and the five points on the lines with tangent L at R and with a singularity at Q.

The values in the table should be multiplied with $2(q^2+q+1)(q+1)q^2(q-1)$.

P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^7]$	$+1$	$\binom{q}{3}\binom{q-3}{2}$
$[1^5, 2^1]$	-1	$q\frac{1}{2}(q^2-q)\binom{q-1}{2}$
$[1^4, 3^1]$	$+1$	$\frac{1}{3}(q^3-q)\binom{q}{2}$

We get

$$\sum \frac{\tau(\lambda)\cdot(\#\mathrm{s}(C)+\#\mathrm{ns}(C)+\#\mathrm{c}(C))\cdot|C|}{|\mathrm{PGL}_3(k)|} = (q-2)(q-3)(q-4)-2q(q-1)(q-2)+(q+1)q(q-1) = -3(q^2-7q+8)$$

where the sum is over the rows in the tables.

10.3.17 $[3, 2]$.

The cubic and the conic intersect in six points and the cubic has one singular point.

For char$(k) \neq 2$.
For a $[1^7]$-tuple of points we pick an irreducible k-conic C: $(q^2+q+1)q^2(q-1)$. We use 9.7 pick a k-point Q that is the intersection of two k-tangents of C: $\binom{q+1}{2}$. We then pick six points on C such that the tangent at none of them goes through Q and there are no two on a line through Q. There are two k-points on C such that their tangents go through Q. If we pick both these points and four others then we get

$$\frac{1}{4!}(q-1)(q-3)(q-5)(q-7).$$

If we pick one of these points and five others we get

$$2\frac{1}{5!}(q-1)(q-3)(q-5)(q-7)(q-9).$$

If we pick six other points we get

$$\frac{1}{6!}(q-1)(q-3)(q-5)(q-7)(q-9)(q-11).$$

So in total we have

$$(q-1)(q-3)(q-5)(q-7)(\frac{1}{4!}+2\frac{1}{5!}(q-9)+\frac{1}{6!}(q-9)(q-11)).$$

Similar to lemma 9.9 there is exactly one irreducible cubic through Q and the six points on C with a singularity at Q.

We can also pick the k-point Q to be the intersection of a k_2-tuple of tangents of C. ($\binom{q}{2}$) In this case we get

$$\frac{1}{6!}(q+1)(q-1)(q-3)(q-5)(q-7)(q-9).$$

For a $[1^1, 6^1]$-tuple of points there are $\frac{1}{3}(q^3-q)$ k_3-tuples of lines through Q. For two k_3-tuples of points on C there is one k_3-tuple of lines through Q. So there are

$$\frac{1}{3}(q^3-q) - \frac{1}{2}\frac{1}{3}(q^3-q) = \frac{1}{6}(q^3-q)$$

k_3-tuples of lines through Q that intersect C in a k_6-tuple of points.

The values in the table should be multiplied with $(q^2+q+1)q^2(q-1)$.

P, λ points	$\tau(\lambda)$	$\lvert C\rvert$
$[1^7]$	1	$\binom{q+1}{2}(q-1)(q-3)(q-5)(q-7)$ $(\frac{1}{4!}+2\frac{1}{5!}(q-9)+\frac{1}{6!}(q-9)(q-11)$ $+\binom{q}{2}\frac{1}{6!}(q+1)(q-1)(q-3)(q-5)(q-7)(q-9)$
$[1^5, 2^1]$	-1	$\binom{q+1}{2}(\frac{1}{2}(q^2-q)-\frac{q-1}{2})(q-1)(q-3)$ $\frac{1}{2!}+2\frac{1}{3!}(q-5)+\frac{1}{4!}(q-5)(q-7)$ $+\binom{q}{2}(\frac{1}{2}(q^2-q)-\frac{q+1}{2})\frac{1}{4!}(q+1)(q-1)(q-3)(q-5)$
$[1^3, 2^2]$	1	$\binom{q+1}{2}\frac{1}{2!}(\frac{1}{2}(q^2-q)-\frac{q-1}{2})(\frac{1}{2}(q^2-q)-\frac{q-1}{2}-2)$ $(1+2(q-1)+\frac{1}{2!}(q-1)(q-3))+\binom{q}{2}(\frac{1}{2}(q^2-q-2)-\frac{q+1}{2})$ $(1+\frac{1}{2!}(\frac{1}{2}(q^2-q-2)-\frac{q+1}{2}-2))\frac{1}{2!}(q+1)(q-1)$
$[1^1, 2^3]$	-1	$\binom{q+1}{2}\frac{1}{3!}(\frac{1}{2}(q^2-q)-\frac{q-1}{2})(\frac{1}{2}(q^2-q)-\frac{q-1}{2}-2)(\frac{1}{2}(q^2-q)-\frac{q-1}{2}-4)$ $+\binom{q}{2}(\frac{1}{2}(q^2-q-2)-\frac{q+1}{2})(\frac{1}{2}(q^2-q-2)-\frac{q+1}{2}-2)$ $(\frac{1}{2!}+\frac{1}{3!}(\frac{1}{2}(q^2-q-2)-\frac{q+1}{2}-4)$
$[1^4, 3^1]$	1	$\binom{q+1}{2}\frac{1}{3}(q^3-q)(q-1)(1+2\frac{1}{2!}(q-3)+\frac{1}{3!}(q-3)(q-5))$ $+\binom{q}{2}\frac{1}{3}(q^3-q)\frac{1}{3!}(q+1)(q-1)(q-3)$
$[1^1, 3^2]$	1	$q^2\frac{1}{2!}\frac{1}{3}(q^3-q)\frac{1}{3}(q^3-q-6)$
$[1^2, 2^1, 3^1]$	-1	$\binom{q+1}{2}\frac{1}{3}(q^3-q)(\frac{1}{2}(q^2-q)-\frac{q-1}{2})(q+1)$ $+\binom{q}{2}\frac{1}{3}(q^3-q)(\frac{1}{2}(q^2-q)-\frac{q+1}{2})(q+1)$
$[1^3, 4^1]$	-1	$(\frac{1}{4}(q^4-q^2)-\frac{q^2-1}{4})(\binom{q+1}{2}(1+2(q-1)+\frac{1}{2!}(q-1)(q-3))$ $+\binom{q}{2}\frac{1}{2!}(q+1)(q-1))$
$[1^1, 2^1, 4^1]$	1	$(\frac{1}{4}(q^4-q^2)-\frac{q^2-1}{4})(\binom{q+1}{2}(\frac{1}{2}(q^2-q)-\frac{q-1}{2})+\binom{q}{2}(\frac{1}{2}(q^2-q)-\frac{q+1}{2}))$
$[1^2, 5^1]$	1	$q^2\frac{1}{5}(q^5-q)(q+1)$
$[1^1, 6^1]$	-1	$q^2(\frac{1}{6}(q^6-q^3-q^2+q)-\frac{1}{6}(q^3-q))$

We get

$$\sum \frac{\tau(\lambda)\cdot(\#\mathrm{s}(C)+\#\mathrm{ns}(C)+\#\mathrm{c}(C))\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{7}{720}(q^2-9q+15)(q-3)(q-5)(q-7)$$
$$-\frac{5}{48}(q^3-6q^2+10q-1)(q-1)(q-3)+\frac{3}{16}(q^2-q-1)(q+1)(q-1)(q-3)$$
$$-\frac{1}{48}(q^2-q-3)(q^2-2q-7)(q-3)+\frac{2}{9}(q^2-3q+3)(q+1)q(q-1)+\frac{1}{18}(q^2+2q+3)q^2(q-2)$$
$$-\frac{1}{3}(q^2-q-1)(q+1)q(q-1)-\frac{3}{8}(q^2-q+1)(q+1)^2(q-1)+\frac{1}{8}(q^2-q-1)(q+1)(q-1)^2$$
$$+\frac{2}{5}(q^2+1)q^2(q+1)-\frac{1}{6}(q^2+q+2)q^2(q-1) =$$
$$-(2q^2-15q+14)$$

where the sum is over the rows in the tables.

For char$(k)=2$.
For a $[1^7]$-tuple of points we pick an irreducible k-conic C and a k-point Q outside the conic that is not the strange point of C: $(q^2+q+1)q^2(q-1)(q^2-1)$. We then pick six points on C such that the tangent at none of them goes through Q and there are no two on a line through Q.

$$q(q-2)(q-4)(q-6)(q-8)\left(\frac{1}{5!}+\frac{1}{6!}(q-10)\right).$$

The values in the table should be multiplied with $(q^2+q+1)(q+1)q^2(q-1)^2$.

P,λ points	$\tau(\lambda)$	$\|C\|$
$[1^7]$	1	$q(q-2)(q-4)(q-6)(q-8)(\frac{1}{5!}+\frac{1}{6!}(q-10))$
$[1^5,2^1]$	-1	$(\frac{1}{2}(q^2-q)-\frac{q}{2})q(q-2)(q-4)(\frac{1}{3!}+\frac{1}{4!}(q-6))$
$[1^3,2^2]$	1	$\frac{1}{2!}(\frac{1}{2}(q^2-q)-\frac{q}{2})(\frac{1}{2}(q^2-q)-\frac{q}{2}-2)q(1+\frac{1}{2!}(q-2))$
$[1^1,2^3]$	-1	$\frac{1}{3!}(\frac{1}{2}(q^2-q)-\frac{q}{2})(\frac{1}{2}(q^2-q)-\frac{q}{2}-2)(\frac{1}{2}(q^2-q)-\frac{q}{2}-4)$
$[1^4,3^1]$	1	$\frac{1}{3}(q^3-q)q(q-2)(\frac{1}{2!}+\frac{1}{3!}(q-4))$
$[1^1,3^2]$	1	$\frac{1}{2!}\frac{1}{3}(q^3-q)\frac{1}{3}(q^3-q-6)$
$[1^2,2^1,3^1]$	-1	$\frac{1}{3}(q^3-q)(\frac{1}{2}(q^2-q)-\frac{q}{2})(q+1)$
$[1^3,4^1]$	-1	$(\frac{1}{4}(q^4-q^2)-\frac{q^2}{4})q(1+\frac{1}{2!}(q-2))$
$[1^1,2^1,4^1]$	1	$(\frac{1}{4}(q^4-q^2)-\frac{q^2}{4})(\frac{1}{2}(q^2-q)-\frac{q}{2})$
$[1^2,5^1]$	1	$\frac{1}{5}(q^5-q)(q+1)$
$[1^1,6^1]$	-1	$\frac{1}{6}(q^6-q^3-q^2+q)-\frac{1}{6}(q^3-q)$

We get

$$\sum \frac{\tau(\lambda)\cdot(\#\mathrm{s}(C)+\#\mathrm{ns}(C)+\#\mathrm{c}(C))\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{7}{720}(q-2)(q-4)^2(q-6)(q-8)$$
$$-\frac{5}{48}(q-2)^3q(q-4)+\frac{3}{16}(q^2-2q-4)q^2(q-2)-\frac{1}{48}(q^2-2q-4)(q+2)(q-2)(q-4)$$
$$+\frac{2}{9}(2q-5)(q+1)q(q-1)(q-2)+\frac{1}{18}(q+1)q(q-1)^2(q-2)-\frac{1}{3}(q+1)^2q(q-1)(q-2)$$
$$-\frac{3}{8}(q^2-2)q^3+\frac{1}{8}(q^2-2)q^2(q-2)+\frac{2}{5}(q^2+1)(q+1)^2(q-1)-\frac{1}{6}(q^2+q+2)(q+1)(q-1)^2 =$$
$$-(2q^2-15q+14)$$
where the sum is over the rows in the tables.

10.3.18 [4, 1].

The line and the quartic intersect in four points and the quartic has three singular points.

For a $[1^7]$-tuple of points we pick three k-points Q_1, Q_2, Q_3 that are not on the same line.

$$\binom{q^2+q+1}{3} - (q^2+q+1)\binom{q+1}{3} = \frac{1}{6}(q^2+q+1)(q+1)q^3$$

We then pick a k-line L not through any of the three points: $(q-1)^2$. Then we pick four k-points on L such that none of them is on a line through two of Q_1, Q_2, Q_3: $\binom{q-2}{4}$. Now we want to pick a quartic through Q_1, Q_2, Q_3 and the four points on L with singular points at Q_1, Q_2, Q_3. There is a $\mathbb{P}^1$ of such quartics. The quartic is reducible if it consists of L and the three lines through Q_1, Q_2, Q_3, or if it consists of two conics through Q_1, Q_2, Q_3 where each conic goes through two different points on L. There are 3 such pairs of conics. All the other quartics are irreducible so we get $q-3$ irreducible quartics.

For picking a $[1^1, 2^1]$-tuple of points and a line not through any of them we get

$$((q^2+q+1)\tfrac{1}{2}(q^4-q) - (q^2+q+1)(q+1)\tfrac{1}{2}(q^2-q))(q^2-1) = \tfrac{1}{2}(q^2+q+1)(q+1)q^3(q-1)^2.$$

For a $[3^1]$-tuple we get

$$(\tfrac{1}{3}(q^6+q^3-q^2-q) - (q^2+q+1)\tfrac{1}{3}(q^3-q))(q^2+q+1) = \tfrac{1}{3}(q^2+q+1)(q+1)q^3(q-1)^2.$$

The values in the table should be multiplied with $(q^2+q+1)(q+1)q^3(q-1)^2$.

3 points	P, λ points	$\tau(\lambda)$	$\|C\|$
$[1^3]$	$[1^7]$	1	$\frac{1}{6}\binom{q-2}{4}(q-3)$
	$[1^5, 2^1]$	-1	$\frac{1}{6}\binom{q-2}{2}\frac{1}{2}(q^2-q)(q-1)$
	$[1^3, 2^2]$	1	$\frac{1}{6}\binom{\frac{1}{2}(q^2-q)}{2}(q-3)$
	$[1^4, 3^1]$	1	$\frac{1}{6}(q-2)\frac{1}{3}(q^3-q)q$
	$[1^3, 4^1]$	-1	$\frac{1}{6}\frac{1}{4}(q^4-q^2)(q-1)$
$[1^1, 2^1]$	$[1^5, 2^1]$	-1	$\frac{1}{2}\binom{q}{4}(q-3)$
	$[1^3, 2^2]$	1	$\frac{1}{2}\binom{q}{2}\frac{1}{2}(q^2-q-2)(q-1)$
	$[1^1, 2^3]$	-1	$\frac{1}{2}\binom{\frac{1}{2}(q^2-q-2)}{2}(q-3)$
	$[1^2, 2^1, 3^1]$	-1	$\frac{1}{2}q\frac{1}{3}(q^3-q)q$
	$[1^1, 2^1, 4^1]$	1	$\frac{1}{2}\frac{1}{4}(q^4-q^2)(q-1)$
$[3^1]$	$[1^4, 3^1]$	1	$\frac{1}{3}\binom{q+1}{4}(q-3)$
	$[1^2, 2^1, 3^1]$	-1	$\frac{1}{3}\binom{q+1}{2}\frac{1}{2}(q^2-q)(q-1)$
	$[1^1, 3^2]$	1	$\frac{1}{3}(q+1)\frac{1}{3}(q^3-q-3)q$

We get

$$\sum \frac{\tau(\lambda)\cdot(\#\mathrm{s}(C)+\#\mathrm{ns}(C)+\#\mathrm{c}(C))\cdot|C|}{|\mathrm{PGL}_3(k)|} = \frac{7}{144}(q-2)(q-3)^2(q-4)(q-5)$$
$$-\frac{5}{24}q(q-1)^2(q-2)(q-3)+\frac{1}{16}(q+1)q(q-1)(q-2)(q-3)+\frac{2}{9}(q+1)q^2(q-1)(q-2)$$
$$-\frac{1}{8}(q+1)q^2(q-1)^2-\frac{5}{48}q(q-1)(q-2)(q-3)^2+\frac{3}{8}(q+1)q(q-1)^2(q-2)$$
$$-\frac{1}{16}(q^2-q-4)(q+1)(q-2)(q-3)-\frac{1}{3}(q+1)q^3(q-1)+\frac{1}{8}(q+1)q^2(q-1)^2$$
$$+\frac{1}{18}(q+1)q(q-1)(q-2)(q-3)-\frac{1}{6}(q+1)q^2(q-1)^2+\frac{1}{9}(q^3-q-3)(q+1)q =$$
$$-2(5q^2-12q+8)$$

where the sum is over the rows in the tables.

10.4 Results.

We have added the results together and put them in the tables below. In a column for a certain weight w we have entered the sum of the contributions for the cases where we consider curves with P and a w-tuple of points. We have also added rows for some computer computations that we did for $\mathbb{F}_2$ and $\mathbb{F}_3$. The program we created to do the computations is written in c code and can be found at https://github.com/Wennink/countingtrigonalcurves. Before that we also computed $T_{\text{split}}(\mathbb{F}_2)$, $T_{\text{non-split}}(\mathbb{F}_2)$, and $T_{\text{cusp}}(\mathbb{F}_2)$ using a simple sage program. This sage program produced the same results as the program in c code but it was a lot slower which made it more difficult to scale to $\mathbb{F}_3$. It was also impractical to use the sage program for more detailed computations of the in-between results.

In the c computer program we go through all plane quintics over k that have an ordinary split node/non-split node/cusp with fixed tangents at P. For each curve we test for all points besides P whether they are singular or not. This way we find out how many curves there are for every distribution of singular points. We then use this information to compute all the in-between results for different weights. We can also directly grasp the end results $T_{\text{split}}(k)$, $T_{\text{non-split}}(k)$, and $T_{\text{cusp}}(k)$ by looking at how many curves the program counted that have no singularities besides P.

For the explicit cases where we have a singularity on an irreducible cubic or quartic we do not know the contribution to $s^{\text{s}}_{\text{explicit}}$, $s^{\text{ns}}_{\text{explicit}}$ or $s^{\text{c}}_{\text{explicit}}$ so there will be some question marks in the tables.
For split nodes we have

weight	q	2	3
0	$\frac{q^{13}}{2(q-1)^2}$	4096	$\frac{1594323}{8}$
1	$-\frac{q^{12}+q^{11}}{2(q-1)^2}$	-3072	$-\frac{177147}{2}$
2	$\frac{q^{10}}{2(q-1)^2}$	512	$\frac{59049}{8}$
3	0	0	0
4	0	0	0
5	0	0	0
6	?	1	0
7	?	-5	-9
8	$\frac{1}{2}(2q^3-53q^2+222q-252)$	-2	$-\frac{9}{2}$
9	$\frac{1}{2}(13q^2-49q+56)$	5	13
sum	?	1535	$\frac{236195}{2}$

For non-split nodes we have

weight	q	2	3
0	$\frac{q^{13}}{2(q^2-1)}$	$\frac{4096}{3}$	$\frac{1594323}{16}$
1	$-\frac{q^{11}}{2(q-1)}$	-1024	$-\frac{177147}{4}$
2	$\frac{q^{10}}{2(q^2-1)}$	$\frac{512}{3}$	$\frac{59049}{16}$
3	0	0	0
4	0	0	0
5	0	0	0
6	?	3	-4
7	?	1	5
8	$\frac{1}{2}(q^2-6)$	-1	$\frac{3}{2}$
9	$\frac{1}{2}q(q-1)$	1	3
sum	?	516	$\frac{118109}{2}$

For cusps we have

weight	q	2	3
0	$\frac{q^{11}}{q-1}$	2048	$\frac{177147}{2}$
1	$-\frac{q^{10}+q^9}{q-1}$	-1536	-39366
2	$\frac{q^8}{q-1}$	256	$\frac{6561}{2}$
3	0	0	0
4	0	0	0
5	0	0	0
6	?	-1	-4
7	?	-1	0
8	0	0	0
9	0	0	0
sum	?	766	52484

For the sum of the split nodes, non-split nodes, and cusps we get

weight	q when $\mathrm{char}(k) = 2$	2	q when $\mathrm{char}(k) \neq 2$	3
0	$\frac{q^{11}(q^3+q^2-1)}{(q-1)^2(q+1)}$	$\frac{22528}{3}$	$\frac{q^{11}(q^3+q^2-1)}{(q-1)^2(q+1)}$	$\frac{6200145}{16}$
1	$-\frac{q^9(q^3+q^2-1)}{(q-1)^2}$	-5632	$-\frac{q^9(q^3+q^2-1)}{(q-1)^2}$	$-\frac{688905}{4}$
2	$\frac{q^8(q^3+q^2-1)}{(q-1)^2(q+1)}$	$\frac{2816}{3}$	$\frac{q^8(q^3+q^2-1)}{(q-1)^2(q+1)}$	$\frac{229635}{16}$
3	0	0	0	0
4	0	0	0	0
5	0	0	0	0
6	$-16q^2 + 70q - 73$	3	$-16q^2 + 70q - 74$	-8
7	$-q^3 + 35q^2 - 156q + 175$	-5	$-q^3 + 35q^2 - 156q + 176$	-4
8	$q^3 - 26q^2 + 111q - 129$	-3	$q^3 - 26q^2 + 111q - 129$	-3
9	$7q^2 - 25q + 28$	6	$7q^2 - 25q + 28$	16
sum	$q^{11} + q^{10} - q^8 + 1$	2817	$q^{11} + q^{10} - q^8 + 1$	229636

So we have computed $\#\mathcal{T}_5(k)$ and the computer results over $\mathbb{F}_2$ and $\mathbb{F}_3$ are consistent with our findings.

Theorem 10.7. *The number of smooth trigonal curves of genus five over a finite field $\mathbb{F}_q$ is given by*

$$\#\mathcal{T}_5(\mathbb{F}_q) = q^{11} + q^{10} - q^8 + 1.$$

References

[1] T. van den Bogaart and B. Edixhoven, *Algebraic stacks whose number of points over finite fields is a polynomial*, Number Fields and Function Fields- Two Parallel Worlds, Progress in Mathematics, Vol. 239, Birkhäuser Boston, Boston, 2005.

[2] Gaëtan Chenevier and Jean Lannes, *Formes automorphes et voisins de Kneser des réseaux de Niemeier*, http://arxiv.org/abs/1409.7616, 2014.

[3] J. Bergström, *Curves of genus three over finite fields*, master thesis at KTH, Stockholm, Sweden.

[4] J. Bergström and O. Tommasi, *The rational cohomology of* $\overline{\mathcal{M}}_4$, Math. Ann., 338(1):207239, 2007.

[5] J. Bergström, *Cohomology of moduli spaces of curves of genus three via point counts*, J. Reine Angew. Math., 622:155187, 2008.

[6] A. Gorinov, *eal cohomology groups of the space of nonsingular curves of degree 5 in* $\mathbb{C}P^2$, Annales de la Facult des Sciences de Toulouse, 14(3):395-434, 2005.

[7] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[8] W. Fulton, *Algebraic curves*, Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1989.