Braid Floer Homology on Surfaces

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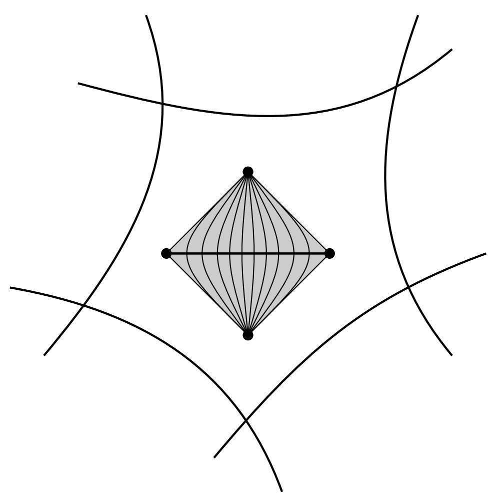

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Abstract

The Arnold Conjecture gives the existence of 1-periodic solutions of a nondegenerate Hamiltonian system on a compact symplectic manifold. This Conjecture is proved for aspherical manifolds with the use of Floer homology. After this proof, we continue with developing braid Floer homology on surfaces. We look at free and skeleton braids to define braid Floer homology. The skeleton braids are known solutions for the Hamiltonian. The free braids are unknown solutions. Braid Floer homology tells us when the skeleton forces new solutions. For a special class of skeleton and free braids on the torus, I define braid Floer homology completely.

Preface

This Thesis focusses on the Arnold Conjecture and braid Floer homology on surfaces. The first part is an extended summary of the Book [AD14]. They use Floer homolgy to prove the Arnold Conjecture in its Morse homological form. The goal of my Thesis was to use this Floer homology to develop braid Floer homology on surfaces. This is a continuation of the Article [BGVW15] about braid Floer homology on a disk.

Rob Vandervorst, one of the authors of [BGVW15], was my daily supervisor from the VU-University. Some special thanks to him about all the discussions we had about the understanding of Floer homology, proper braids and braid Floer homology on especially the torus.

This Thesis is done as a course on the VU University as partial fulfillment of the requirements of the degree *Master of Mathematical Sciences* at Utrecht University. Fabian Ziltener was my supervisor from the Utrecht University to make this collaboration possible. Also some special thanks to him, because of his advise about how to read new mathematical texts and how to organize a Thesis.

All 22 figures in my Thesis are made by me with the use of Ti*k*Z. Some of the figures are inspired from figures in [AD14] and [Wój07].

Contents

Introduction

For a symplectic manifold with a Hamiltonian system on it, we are interested in the existence of periodic solutions. This interest comes from physics, solutions of a Hamiltonian system describe the mechanics of motions. The Arnold Conjecture provides an answer. The amount of 1-periodic solutions is greater or equal to the summation of the Betti numbers, see Conjecture 1.1 of [Sal99]. To be more precise,

Arnold Conjecture. *Let (M, ω) a compact symplectic manifold. If all the 1-periodic solutions of a time dependent Hamiltonian vector field X_t are nondegenerate, then the number of such solutions is greater or equal to*

$$\sum_{i=0}^{2n} b_i(M).$$

Here denotes $b_i(M)$ the i^{th} Betti number of M.

Many mathematicians have contributed to the proof of the Arnold Conjecture. It started with Eliashberg for dimension 2 and then by Conley and Zender for the $2n$-tori. The breakthrough came with a series of papers from Floer. He developed an infinite dimensional approach to the Morse theory, the now so called Floer homology. With this homology he was able to prove the Arnold Conjecture for the monotone case. This was extended by Hofer and Salamon, by Fukaya and Ono and by Liu and Tian to the general case. I refer to the overview [Sal99] by Salamon for a more precise list of contributions and references to these works.

The first Part of my Thesis is an extensive summary of Part II of the Book [AD14]. It is about the definition of Floer homology to prove a simplified version of the Arnold Conjecture. Namely, let M be a symplectic manifold such that the second homotopy group is trivial, then the amount of 1-periodic solutions is greater or equal to the summation of the Morse homotopy groups without taking orientation, see for a precise statement Theorem 2.1. A brief outline of the proof can be found in Subsection 2.2.
As stated above, Floer homology is an important tool to show this simplified version of the Arnold Conjecture. Floer homology is made of the periodic solutions we are looking for. It turns out to be equal to the Morse homology in some special case. Therefore, the Floer homology brings the periodic solutions and the Morse homology together. This allows us to prove this simplified version of the Arnold Conjecture.

After showing the Arnold Conjecture, the next question you could ask is

Question. *If we already find some periodic solutions for a Hamiltonian system on a compact symplectic manifolds, will the topology of these solutions on the manifold forces more periodic solutions?*

The second Part of my Thesis is about developing a new kind of Floer homology to try to answer this question in the case of compact symplectic surfaces. The new kind of Floer homology is the braid Floer homology. Braid Floer homology on a disk is already developed in [BGVW15].
The idea is to view the solutions that we already know, as a closed braid on the surface. This braid is called the skeleton braid. Now, we want to show if we can find more solutions. The possible new solutions form also a braid, this is called the free braid. Braid Floer homology is made from this free braids that are restricted by the skeleton braid. This restriction goes via an isolated invariant neighbourhood, just as a technique mentioned already by Floer in [Flo89].
My thesis is a start of how to define the braid Floer homology on surfaces, but for a certain class of skeleton braids and free braids on the torus I define the braid Floer homology completely, see Theorem 10.9.

Part I
Floer homology

1 Basics of symplectic geometry

Before going into Floer homology, I start with some basics of symplectic geometry.

Definition 1.1. Let M a manifold and T_xM the tangent space of M at the point x in M. A 2-form ω on M assigns to each $x \in M$ an alternating bilinear form ω_x on the tangent space T_xM varying smoothly with x.
A 2-form is called nondegenerate if ω_x is nondegenerate for all $x \in M$.

Definition 1.2 (Symplectic manifold)**.** Let M a manifold and ω a nondegenerate closed 2-form, *i.e.* $d\omega = 0$, then the pair (M, ω) is called a symplectic manifold. Such a form ω is called a symplectic form. The notion of ω is left from the notation if there can be no confusion about the symplectic form ω we use.

Remark 1.3. Note that the dimension of a symplectic manifold (M, ω) must be even, since the corresponding alternating form ω_x on the tangent space T_xM cannot be nondegenerate if the tangent space has an odd dimension.

Lemma 1.4. *Let V a vector space of dimension $2n$. An alternating bilinear form ω is nondegenerate if and only if the n-fold exterior power is nonzero, which means:*

$$\omega^n = \omega \wedge \ldots \wedge \omega \neq 0$$

Proof. For this proof I refer to Corollary 2.5 of [MS98]. □

Corollary 1.5. *A symplectic manifold (M, ω) is orientable.*

Proof. The alternating bilinear form ω_x is nonzero in each point x of M, since $\omega_x^n \neq 0$ by Lemma 1.4. This implies that ω is a volume form, hence M is orientable. □

Corollary 1.6. *A 2-dimensional manifold is symplectic if and only if it is orientable.*

Proof. We only need to prove that a 2-dimensional orientable manifold is symplectic, since the other implication follows from Corollary 1.5.
Let M be a 2-dimensional orientable manifold. Then M has a volume form, which is a 2-form ω that is nonzero at every point x on the manifold, that is $\omega_x \neq 0$ for all x in M. We also have that ω_x is a 1-fold exterior power of the alternating bilinear form ω_x. Now apply Lemma 1.4 to ω_x, so ω_x is nondegenerate for all x in M. Hence ω is a nondegenerate form on M. Now we conclude that (M, ω) is a symplectic manifold, since every 2-form on a 2-dimensional manifold is closed. □

Remark 1.7. For a manifold to be sympletic is a quiet restrictive demand. We already saw that it must be of an even dimension and that it needs to have an orientation. In dimension 2 it is enough to have an orientation to be symplectic, but this does not hold in higher dimensions. Even some basic spaces like the spheres S^{2n} do not admit a symplectic structure for $n \geq 3$. This follows from the fact that a compact manifold without boundary such that the second cohomology space is zero cannot have a symplectic structure, see page 8 of [MS98].

Definition 1.8. Let ω be a symplectic form on a vector space V. A complex structure J compatible with ω is an endomorphism on V such that:

1) $J^2 = -\mathrm{Id}$. (J is a complex structure)

2) $\omega(Ju, Jv) = \omega(u, v)$ for all u, v in V. (J is symplectic)

3) $g(u, v) := \omega(u, Jv)$ is an inner product for all u, v in V.

Definition 1.9. Let (M, ω) a symplectic manifold. An almost complex structure on M, an endomorphism J on the tangent bundle TM such that $J^2 = -\mathrm{Id}$, is compatible with ω if the complex structure J_x in T_xM is compatible with ω_x for all x in M.

Proposition 1.10. *Every symplectic manifold (M, ω) has an almost complex structure compatible with ω.*

Proof. For this proof I refer to Proposition 4.1 Part (i) of [MS98]. □

Proposition 1.11. *The space $\mathcal{J}$ of almost complex structures on M compatible with ω is contractible.*

Proof. For this proof I refer to Proposition 5.5.6 of [AD14]. □

Definition 1.12 (Hamiltonian vector field)**.** Let (M, ω) a symplectic manifold. For a function $H : M \to \mathbb{R}$ we define the Hamiltonian vector field X_H by the relation

$$\omega_x(Y, X_H(x)) = (dH)_x(Y) \text{ for every } Y \in T_xM, \tag{1.1}$$

this can also be written as

$$i_{X_H}\omega = -dH. \tag{1.2}$$

We define a time depended Hamiltonian vector field for a function $H : M \times \mathbb{R} \to \mathbb{R}$ by $X_t = X_{H_t}$ where $H_t(x) = H(x, t)$. The corresponding Hamiltonian equation

$$\dot{x}(t) = X_t(x(t))$$

is called the Hamiltonian system.

Remark 1.13. Relation (1.1) together with the fact that ω is nondegenerate gives that the Hamiltonian vector field X_H is 0 at x if and only if x is a critical point of H, *i.e.*

$$X_H(x) = 0 \iff (dH)_x = 0.$$

Note also that since ω is alternating, that the function H is constant on the trajectories of X_H, *i.e.* $(dH)(X_H) = 0$.

Remark 1.14. We associate to a given almost complex structure J on a symplectic manifold (M, ω) and a function $H : M \to \mathbb{R}$ two vector fields. Namely, the Hamiltonian vector field X_H and the gradient vector field $\operatorname{grad} H$. The last one is defined with respect to the Riemannian metric defined by $g(X, Y) = \omega(X, JY)$ where X, Y are vector fields of M. We relate these vector fields in the following way:

$$\omega(Y, X_H)) = (dH)(Y) = g(Y, \operatorname{grad} H) = \omega(Y, J \operatorname{grad} H)$$

for all vector fields Y of M. Hence, $X_H = J \operatorname{grad} H$, since ω is nondegenerate.

Proposition 1.15. *The flow φ^t of a Hamiltonian vector field X_H preserves the symplectic form ω for all time t.*

Proof. We know that $\varphi^{t+s} = \varphi^t \varphi^s$ for all times t and s, and that

$$\left(\frac{d}{dt} (\varphi^t)^* \omega \right)_{t=0} = \mathcal{L}_{X_H} \omega = i_{X_H}(d\omega) + d(i_{X_H}\omega).$$

Now follows that

$$\begin{aligned} \frac{d}{dt}(\varphi^t)^* \omega &= (\varphi^t)^* \mathcal{L}_{X_H} \omega \\ &= (\varphi^t)^* (d(i_{X_H}\omega)) \\ &= (\varphi^t)^* (-ddH) = 0. \end{aligned}$$

This implies that $(\varphi^t)^*\omega$ is constant in t, so $(\varphi^t)^*\omega = (\varphi^0)^*\omega = \mathrm{Id}^*\omega = \omega$ for all time t. The same statement and proof holds for a time dependent Hamiltonian vector field. □

Definition 1.16. A 1-periodic solution x of a time dependent Hamiltonian vector field X_t, *i.e.* $\dot{x}(t) = X_t(x(t))$ such that $\varphi^1(x) = x$, is nondegenerate if

$$\det(\mathrm{Id} - T_{x(0)}\varphi^1) \neq 0.$$

Here is $T_{x(0)}\varphi^1$ the differential of φ^1 at the point $x(1) = x(0)$.

The next Proposition and Remark compare the definition of nondegenerate for a 1-periodic solution of a time independent Hamiltonian system $\dot{x}(t) = X_H(x(t))$ with the definition of nondegenerate for a function H. This comparison is used to compare the Floer and Morse homology.

Proposition 1.17. *If x is a nondegenerate 1-periodic solution of a time independent Hamiltonian system H, then x is nondegenerate as a critical point of the function H.*

Proof. Use local coordinates for x. Then we can define the gradient, the Jacobian and the Hessian at x, denoted by ∇_x, Jac_x and Hess_x respectively.
Note that x is nondegenerate as 1-periodic solution of the Hamiltonian system H iff

$\mathrm{Id} - \mathrm{Jac}_x\, \varphi^1$ has eigenvalue 0. Also note, x is nondegenerate as critical point of the function H iff $\mathrm{Hess}_x\, H$ has eigenvalue 0, since x is nondegenerate for the function H if the determinant of the Hessian is 0.
If we can show that $\mathrm{Id} - \mathrm{Jac}_x\, \varphi^1$ has eigenvalue 0 implies that $\mathrm{Hess}_x\, H$ has eigenvalue 0, then we are done.
In coordinates we have

$$\begin{aligned}\frac{d}{dt}(\mathrm{Jac}_x\, \varphi^t)_i &= \frac{d}{dt}\frac{\partial \varphi^t}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{d\varphi^t}{dt} = \frac{\partial}{\partial x_i}(X_H \circ \varphi^t) = \sum_j \frac{\partial X_H}{\partial x_j}\frac{\partial \varphi^t_j}{\partial x_i} = \nabla_x X_H \cdot \frac{\partial \varphi^t}{\partial x_i} \\ &= \nabla_x(-J_x \mathrm{Jac}_x\, H) \cdot \frac{\partial \varphi^t}{\partial x_i} = (-J_x \nabla_x(\mathrm{Jac}_x\, H)) \cdot \frac{\partial \varphi^t}{\partial x_i} = (-J_x \mathrm{Hess}_x\, H) \cdot \frac{\partial \varphi^t}{\partial x_i}.\end{aligned}$$

This gives

$$\frac{d}{dt}(\mathrm{Jac}_x\, \varphi^t) = (-J_x \mathrm{Hess}_x\, H) \cdot \frac{\partial \varphi^t}{\partial x},$$

this implies

$$\mathrm{Jac}_x\, \varphi^t = e^{-tJ_x \mathrm{Hess}_x\, H}.$$

Therefore,

$$\mathrm{Id} - T_{x(0)}\varphi^1 = \mathrm{Id} - \mathrm{Jac}_x\, \varphi^1 = \mathrm{Id} - e^{-J_x \mathrm{Hess}_x\, H}$$

since $x = x(0)$. From this we conclude that if $\mathrm{Hess}_x\, H$ has eigenvalue 0 then $\mathrm{Id} - \mathrm{Jac}_x\, \varphi^1$ has eigenvalue 0, this concludes the proof. □

Remark 1.18. Note that the inverse statement of Proposition 1.17 does not hold necessarily, since for any eigenvalue in $2\pi\mathbb{Z}$ of $\mathrm{Hess}_x\, H$ will give that $\mathrm{Id} - \mathrm{Jac}_x\, \varphi^1$ has eigenvalue 0. But if we choose H such that the norm of Hessian stays smaller than 2π, the two nondegeneracies are equivalent. Note in particular that if the Hessian stays smaller than 2π, the 1-periodic solutions of the Hamiltonian system are constant.

2 Arnold Conjecture

The main Theorem of the first Part of my Thesis is the Arnold Conjecture for the nonoriented Morse homology case with an extra homotopy assumption on the symplectic manifold M. This is the following Theorem.

Theorem 2.1. *Let M a compact symplectic manifold such that the homotopy group $\pi_2(M) = 0$. If all the 1-periodic solutions of a time dependent Hamiltonian vector field X_t are nondegenerate, then the number of such solutions is greater or equal to*

$$\sum_i \dim HM_i(M; \mathbb{Z}/2).$$

Here denotes $HM_i(M; \mathbb{Z}/2)$ the i^{th} Morse homology group of M without taking into account orientations.

This Theorem tells us that for such an M, there is such a 1-periodic solution if the Morse homology on M is nontrivial. The goal of this section is to discus some of the assumptions and to give an outline of the proof of the Theorem.

2.1 Assumptions

Some remarks about the assumptions of Theorem 2.1 are mentioned in this Subsection.

Remark 2.2. The assumption of $\pi_2(M) = 0$ in the Arnold Conjecture implies that every smooth map from the sphere to M, $f : S^2 \to M$, can be extended to a smooth map from the ball to M, $h : B^3 \to M$. Stoke's Theorem, see Theorem 16.11 in [Lee13], gives us now that

$$\int_{S^2} f^*\omega = \int_{B^3} h^* d\omega = 0, \tag{2.1}$$

since ω is closed. If Formula 2.1 holds for every smooth map $f : S^2 \to M$, we call the symplectic manifold (M, ω) aspherical.

Remark 2.3. The assumption $\pi_2(M) = 0$ does not only give that (M, ω) is aspherical, but it gives us also a trivialization property.
Note that for every smooth map $f : S^2 \to M$, the image $f(S^2)$ must be contractible, since $\pi_2(M) = 0$. Therefore $f(S^2)$ is homotopic to a disk D^2. Note also that for every smooth map $\psi : D^2 \to M$ the symplectic fiber bundle ψ^*TM can be trivialized, see Theorem 7.1.1 of [AD14].
This trivialization for φ together with the fact that $f(S^2)$ is homotopic to a disk D^2, gives that there exists a symplectic trivialization of the fiber bundle f^*TM for every smooth map $f : S^2 \to M$.

If we assume both Remark 2.2 and Remark 2.3 instead of the assumption $\pi_2(M)) = 0$ in Theorem 2.1, we will see that the Theorem remains to be true.

Remark 2.4. We may assume that the Hamiltonian H_t that induces the Hamiltonian vector field $X_t = X_{H_t}$ in Theorem 2.1 is periodic. If $\varphi^t : M \to M$ is a 1-periodic solution of $\dot{\varphi}^t = X_t(\varphi^t)$, then it is also a 1-periodic solution of

$$\frac{d}{dt}\left(\varphi^{\alpha(t)}\right) = \frac{d\alpha}{dt} X_{H_{\alpha(t)}}\left(\varphi^{\alpha(t)}\right) = X_{\dot{\alpha}(t)H_{\alpha(t)}}\left(\varphi^{\alpha(t)}\right),$$

where $\alpha : [0,1] \to [0,1]$ is a smooth function that is zero near zero and one near one, because $\varphi^{\alpha(1)} = \varphi^1$. Note that α is flat near zero and one, so $\dot{\alpha}(0) = \dot{\alpha}(1) = 0$, so $K_t = \dot{\alpha}(t)H_{\alpha(t)}$ can be extended as a 1-periodic function of time. Hence, we found a 1-periodic Hamiltonian K_t such that X_{K_t} has precisely the same 1-periodic solutions as X_{H_t}.

2.2 Outline of the proof

I give the outline of the proof of Theorem 2.1 here. It is also an overview of the upcoming Sections until Part II.

Proof of Theorem 2.1. The idea is to use 1-periodic solutions of a time dependent Hamiltonian vector field to define a homology which will be equal to the Morse homology. This will be the Floer homology. We denote the corresponding Hamiltonian by H. We will only use 1-periodic solutions that are contractible to define this homology. We can do this since we are looking for an inequality and not an equality in the Theorem.
To define the Floer homology we start with defining a functional, the so called action functional $\mathcal{A}_H$ (Section 3). The Action functional has this contractible 1-periodic solutions as critical points. Then we will use the trajectories of the negative gradient of the action functional to connect two critical points. These trajectories will be described as solutions of the Floer equation (Subsection 3.1). We need to choose a compatible almost complex structure J on M to define this negative gradient.
We want to connect two critical point of the action functional with these trajectories. But not every trajectory will connect two critical points in general. However, we can define an energy for these trajectories. It turns out that the finite energy ones will be exactly all the trajectories that connects two critical points. To show this, we will need that the total space $\mathcal{M}$ of finite energy solutions is compact (Subsection 3.2).
To define the Floer homology we also need an index for the critical points of the action functional. This will be the Maslov index that is denoted by μ (Section 4).
Now define $C_k(H)$ as the vector space over $\mathbb{Z}/2$ generated by critical points with Maslov index k. Furthermore, for two critical points x, y that have a Maslov index difference of one, let $n(x, y)$ denote the number modulo two of trajectories between these two critical

points. Then we define the Floer complex via the following differential (Section 6)

$$\partial : C_k(H) \longrightarrow C_{k-1}(H), \quad \partial(x) = \sum_{\mu(y)=k-1} n(x,y)y.$$

We need to prove that this differential is well defined and that $\partial \circ \partial = 0$. The differential is well defined if the total amount of critical points is finite and also if the amount of trajectories between two critical points with Maslov index difference one is finite.
That there are a finite amount of critical points is a direct consequence of the nondegeneracy assumption (Lemma 3.17). But the other two statements are a lot harder to prove. To prove the other two statements, we start with slightly adjusting the Hamiltonian H. This will be done in such a way that the spaces $\mathcal{M}(x,y)$ of solutions of the Floer equation between critical points x, y become manifolds. These manifolds will have dimension $\mu(x) - \mu(y)$, where $\mu(x), \mu(y)$ denote the Maslov index of the points x, y respectively (Section 5).
This adjustment is allowed because of a regularity property of the Floer equation. To show this regularity property we will use the Sard-Smale Theorem(5.16), an infinite dimensional analogue of Sard's theorem.
There is a natural $\mathbb{R}$ action on the spaces $\mathcal{M}(x,y)$, denoted by $\mathcal{L}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$. This action allows us to count the trajectories between two critical points with one index difference. The space $\mathcal{L}(x,y)$ turns out to be a zero dimensional compact manifold if the index difference of x and y is one (Property (1) of Theorem 6.1). Hence it consists of a finite amount of trajectories, so $n(x,y)$ is well defined.
To conclude that $\partial \circ \partial = 0$, we need to study $\mathcal{L}(x,z)$ where x and z have an index difference of two. We will see that $\overline{\mathcal{L}(x,z)}$ is a compact manifold with boundary of dimension one, such that the boundary is the union of broken trajectories (Property (2) of Theorem 6.1). Broken trajectories consists of multiple trajectories that are connected via critical points. We can conclude that $\partial \circ \partial = 0$, since every compact one dimensional manifold with boundary has an even amount of boundary points.
Now, we have a well defined chain complex depending on a Hamiltonian and on an almost complex structure J on M. The induced homology is independent of the Hamiltonian and the almost complex structure (Subsection 7.1). This homology is the Floer homology.
In a particular situation, the Hamiltonian is independent of time and sufficiently small, we can define both the Floer and Morse homology. In this case we can prove that they are equal (Subsection 7.2). Hence, the Floer and Morse homology are equal since they both do not depend on the Hamiltonian. This equality and the way how the Floer homology is constructed allows us to conclude the Theorem. $\square$

3 Action functional

The 1-periodic solutions of the Hamiltonian system $\dot{x(t)} = X_t(x(t))$ can be described as the critical points of a functional. This is the action functional, which is defined on loops on the compact symplectic manifold M. The action functional plays the same role in Floer homology as the Morse function does in the Morse homology.
We want to solve the question, how to get a chain complex between solutions of the Hamiltonian system? This question is now, how to get a chain complex between critical points of the action functional? To make a chain complex, you need a map between different critical points. This is induced from trajectories between critical points in Section 6. These trajectories travel along the negative gradient of the action functional, this is described by the Floer equation. In this Section, we show that the trajectories with finite energy that move along the negative gradient of the action functional are trajectories between critical points.

Definition 3.1 (Action functional)**.** Let (M, ω) a symplectic manifold and H_t a time dependent Hamiltonian that we assume to be periodic. To be periodic means $H_{t+1}(x) = H_t(x)$ for all time t. Define the action functional by

$$\mathcal{A}_H(x) = -\int_D u^*\omega + \int_0^1 H_t(x(t))dt \tag{3.1}$$

for a contractible loop $x : S^1 \to M$ with extension $u : D \to M$. This definition is well defined if it does not depend on the choice of an extension u. Let v be another extension. Glue u and v along their common boundary, the result of the gluing will be denoted by w, see Figure 1. Then we have by Remark 2.2 that

$$\int_D u^*\omega - \int_D v^*\omega = \int_{S^2} w^*\omega = 0.$$

Hence the action functional does not depend on the choice of the extension u of the loop x.

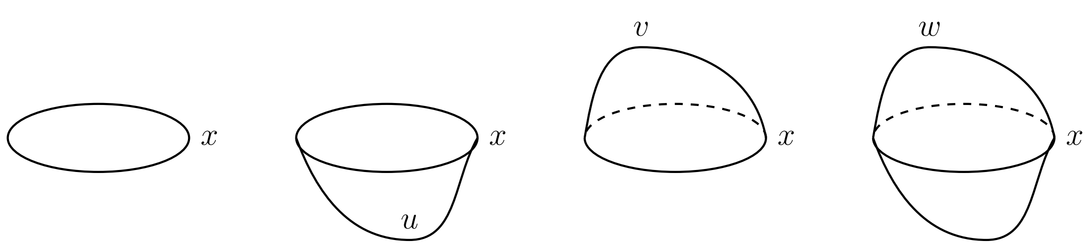


Figure 1: A contractible loop x [*left*] in M with its extensions u and v [*middle*]. The extensions u and v form together a sphere w [*right*].

Theorem 3.2. *A loop x is a critical point of the action functional $\mathcal{A}_H$ if and only if it is a periodic solution of the Hamiltonian system $\dot{x}(t) = X_t(x(t))$.*

Proof. A contractible loop x is a critical point of $\mathcal{A}_H$ if the differential of $\mathcal{A}_H$ is 0 at the point x. Let $u : z \to u(z)$ be the extension of $x : t \to x(t)$ from the definition of the action functional. Start with a computation of the differential for any vector field Y, see the proof of Theorem 6.3.3 of [AD14] for more details. Split the differential in two parts, namely

$$(d\mathcal{A}_H)_x(Y) = -\int_D u^*(\mathcal{L}_{Y(z)}\omega) + \int_0^1 (dH_t)_{x(t)}(Y(t))\, dt.$$

The first part gives

$$\begin{aligned}
-\int_D u^*(\mathcal{L}_{Y(z)}\omega) &= -\int_D u^*(di_{Y(z)}\omega) && (\omega \text{ is closed}) \\
&= -\int_{S_1} x^*(i_{Y(t)}\omega) && (\text{Stoke's Theorem}) \\
&= -\int_0^1 \omega(Y(t), \dot{x}(t))\, dt
\end{aligned}$$

and the second part gives

$$\int_0^1 (dH_t)_{x(t)}(Y(t))\, dt = \int_0^1 \omega_{x(t)}(Y(t), X_t(x(t)))\, dt \qquad (\text{by definition of } X_t).$$

Hence for any vector field Y we have

$$(d\mathcal{A}_H)_x(Y) = \int_0^1 \omega(Y(t), X_t(x(t)) - \dot{x}(t))dt, \tag{3.2}$$

so the differential of $\mathcal{A}$ is 0 if and only if $\dot{x}(t) = X_t(x(t))$, because of the nondegeneracy of ω. □

3.1 The Floer equation

In this Subsection I will introduce the negative gradient of the action functional $\mathcal{A}_H$ and the Floer equation. The solutions of the Floer equation are the trajectories of the negative gradient of the action functional.
We will study the trajectories of the negative gradient, because the trajectories are needed to define the differential of the Floer homology.

Definition 3.3 (Gradient on the loop space)**.** A compatible almost complex structure J on a symplectic manifold (M, ω) defines a metric g on M. This metric induces a metric on the space of contractible loops $\mathcal{L}M$ of M, namely for vector fields Y, Z along a contractible loop x in $\mathcal{L}M$ we can define the metric at a loop x

$$\langle Y, Z\rangle_x = \int_0^1 g_x(Y(t), Z(t))dt.$$

The gradient of a function $f : \mathcal{L}M \to \mathbb{R}$ is defined via the relation

$$\langle \operatorname{grad}_x f, Y\rangle_x = (df)_x(Y)$$

Remark 3.4. If we apply the last definition on the action functional $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$, we get for a vector field Y defined along a loop x in $\mathcal{L}M$ that

$$\begin{aligned}(d\mathcal{A}_H)_x(Y) =&\langle \operatorname{grad}_x \mathcal{A}_H, Y\rangle_x\\ =&\int_0^1 g_{x(t)}((\operatorname{grad}_{x(t)} \mathcal{A}_H)(t), Y(t))dt\\ =&\int_0^1 \omega_{x(t)}((\operatorname{grad}_{x(t)} \mathcal{A}_H)(t), JY(t))dt.\end{aligned}$$

Furthermore, Formula 3.2 gives that

$$(d\mathcal{A}_H)_x(Y) = \int_0^1 \omega_{x(t)}(J\dot{x}(t) - JX_t(x), JY)dt,$$

since J is symplectic. Hence for the negative gradient $\mathcal{X}_H$ we have

$$-\mathcal{X}_H(t) := (\operatorname{grad}_x \mathcal{A}_H)(t) = J_{x(t)}\dot{x}(t) - J_{x(t)}X_t(x(t)) = J_{x(t)}\dot{x}(t) + \operatorname{grad}_{x(t)} H_t(x(t)), \quad (3.3)$$

since ω is nondegenerate. The last identity holds, because $J^2 = -1$ and $J \operatorname{grad} H_t = X_t$.

Definition 3.5 (Floer equation)**.** Let $u : \mathbb{R} \to \mathcal{L}M : s \mapsto u(s, \cdot)$ be a path of contractible loops in M, see Figure 2.
Such a u is the trajectory of the negative gradient $\mathcal{X}_H$ of the action functional $\mathcal{A}_H$ if it is a solution of the following partial differential equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad}_u H_t(u) = 0, \quad (3.4)$$

see Equation 3.3. This differential equation is called the Floer equation.

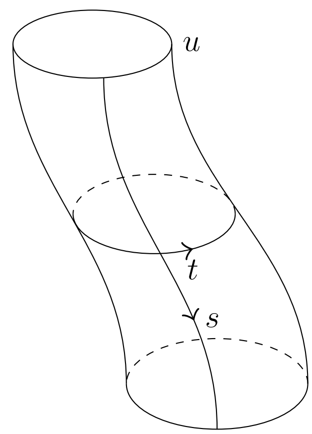


Figure 2: Schematic shape of a trajectory u.

Remark 3.6. There are two important special cases of the Floer equation. First, if the Hamiltonian H_t vanishes, then we get the Cauchy-Riemann equation

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0.$$

If this is the case, u is called J-holomorphic.
The other special case is when a solution u of the Floer equation does not depend on s. In this case the Floer equation can be rewritten as

$$\dot{u} = X_t(u).$$

Such a solution, a stationary trajectory of the negative gradient flow of the action functional $\mathcal{A}_H$, is thus a periodic solution of the Hamiltonian system of H_t. Theorem 3.2 gives now that the solution is a critical point of the action functional.

3.2 Solutions of the Floer equation

To define the differential of the Floer homology we are especially interested in smooth contractible solutions of the Floer equation with finite energy. I will show that such solutions connect two critical points of the action functional. The number of all such possible connections between critical points will be used to define the differential.

Definition 3.7. For a contractible periodic solution u of the Floer equation we define the energy as

$$E(u) = \int_{-\infty}^{+\infty} \|\mathrm{grad}_u \mathcal{A}_H\|^2 \, ds. \tag{3.5}$$

Definition 3.8. A contractible periodic solution u of the Floer equation connects two critical points x and y of the action functional if

$$\lim_{s\to-\infty} u(s,\cdot) = x \quad \text{and} \quad \lim_{s\to+\infty} u(s,\cdot) = y$$

in $\mathcal{C}^\infty(S^1; M)$. The space of such solutions between x and y is denoted by $\mathcal{M}(x, y)$. Here denotes $\mathcal{C}^\infty(S^1; M)$ the smooth maps from S^1 to $M \subset \mathbb{R}^m$ with the standard $\mathcal{C}^\infty$ topology. M is viewed as embedded in $\mathbb{R}^m$ for a large enough m, see the Whitney Embedding Theorem 6.15 of [Lee13].

Lemma 3.9. *For u a contractible periodic solution of the Floer equation, the energy has the following three properties:*

(1) The energy $E(u)$ is positive.

(2) $E(u)$ is zero if and only if u is a critical point of the action functional $\mathcal{A}_H$.

(3) If u connects two critical points x and y of $\mathcal{A}_H$, then $E(u) = \mathcal{A}_H(x) - \mathcal{A}_H(y)$. Hence the energy is finite.

Corollary 3.10. *Let x a critical point of the action functional $\mathcal{A}_H$, then $\mathcal{M}(x, x) = \{x\}$.*

Proof of Lemma 3.9. The first property follows directly from the definition of the energy. For the second property we notice first that

$$\operatorname{grad}_u \mathcal{A}_H = J(u)\frac{\partial u}{\partial t} + \operatorname{grad}_u H_t(u) = -\frac{\partial u}{\partial s},$$

since u is a solution. Hence

$$E(u) = \int_{-\infty}^{\infty} \left\|\frac{\partial u}{\partial s}\right\|^2 ds = \int_{\mathbb{R}\times S^1} \left|\frac{\partial u}{\partial s}\right|^2 ds\, dt, \tag{3.6}$$

so $E(u)$ is zero if and only if u does not depend on s. Now we are in the second special case of Remark 3.6, which says that u is a critical point of the action functional $\mathcal{A}_H$. The last property follows from the identities

$$\|\operatorname{grad}_u \mathcal{A}_H\|^2 = (d\mathcal{A}_H)_u(\operatorname{grad}_u \mathcal{A}_H) = -(d\mathcal{A}_H)_u\left(\frac{\partial u}{\partial s}\right) = -\frac{d}{ds}\mathcal{A}_H(u(s)). \tag{3.7}$$

Another formula for the energy is thus

$$E(u) = -\int_{-\infty}^{+\infty} \frac{d}{ds}\mathcal{A}_H(u(s))ds.$$

Since u connects x and y we conclude $E(u) = \mathcal{A}_H(x) - \mathcal{A}_H(y)$. ∎

Definition 3.11. Let us now define the total space of solutions of the Floer equation

$$\mathcal{M} = \{u : \mathbb{R} \times S^1 \longrightarrow M \mid u \text{ is a contractible solution of finite energy}\}.$$

View the compact symplectic manifold M as being embedded in a space $\mathbb{R}^m$ for a sufficiently large m. We use this embedding to define a topology on $\mathcal{M}$. This is the $\mathcal{C}^\infty$ uniform convergence topology on the the compact sets of $\mathbb{R} \times S^1$ for smooth maps from $\mathbb{R} \times S^1$ to $M \subset \mathbb{R}^m$. The topology is denoted by $\mathcal{C}^\infty_{\text{loc}}(\mathbb{R} \times S^1; M)$. The topology $\mathcal{C}^\infty_{\text{loc}}(\mathbb{R} \times S^1; M)$ is the topology we use on the space $\mathcal{M}$.

A crucial property to define Floer homology is that trajectories of the negative gradient of the action functional with finite energy connect critical points of the action functional. This is the following Theorem.

Theorem 3.12. *If all the periodic trajectories of X_t are nondegenerate, then for every $u \in \mathcal{M}$ there are critical points x and y of the action functional $\mathcal{A}_H$ such that u connects these two critical points, i.e.*

$$\lim_{s \to -\infty} u(s, \cdot) = x \quad \text{and} \quad \lim_{s \to +\infty} u(s, \cdot) = y$$

in $\mathcal{C}^\infty(S^1; M)$. Moreover, we have

$$\lim_{s \to \pm\infty} \frac{\partial u}{\partial s}(s, t) = 0$$

uniform in t.

Proof. For the proof see Section 3.5. □

Corollary 3.13. *Theorem 3.12 together with Item (3) of Lemma 3.9 give that*

$$\mathcal{M} = \cup_{x,y \in \text{Crit}(\mathcal{A}_H)} \mathcal{M}(x, y).$$

Before we can show Theorem 3.12, we need the following important Theorem.

Theorem 3.14 (Gromov compactness)**.** *The space $\mathcal{M}$ is compact in $\mathcal{C}^\infty_{\text{loc}}(\mathbb{R} \times S^1; M)$.*

Proof. For the proof see Section 3.4. □

3.3 Intermediate results

Before we are able to prove Theorem 3.14 and Theorem 3.12 we need some more results. From now on, I will denote $u(s, \cdot)$ with u_s.

Lemma 3.15. *Let $u \in \mathcal{M}$. There exists two critical points x and y of the action functional $\mathcal{A}_H$ such that*

$$\lim_{s \to -\infty} \mathcal{A}_H(u_s) = \mathcal{A}_H(x) \quad \text{and} \quad \lim_{s \to +\infty} \mathcal{A}_H(u_s) = \mathcal{A}_H(y).$$

Note that this Lemma does not yet contain the condition that all the periodic orbits of X_t needs to be nondegenerate. We also note that it is enough to show this Lemma only for the case $s \to +\infty$, because of symmetry.

Proof. The function $s \mapsto \mathcal{A}_H(u_s)$ is decreasing because of Formula 3.7. So it is enough to show that there is a critical point y of $\mathcal{A}_H$ and a sequence s_k going to $+\infty$ such that

$$\lim_{k\to+\infty} \mathcal{A}_H(u_{s_k}) = \mathcal{A}_H(y).$$

To show this, we split the proof in three steps.

(1) There is a sequence s_k going to $+\infty$ such that u_{s_k} goes to a limit y that is continuous, *i.e.* $y \in \mathcal{C}^0$.

(2) y is a critical point of the action functional A_H and is smooth, *i.e.* $y \in \mathcal{C}^\infty$.

(3) $\lim_{k\to+\infty} \mathcal{A}_H(u_{s_k}) = \mathcal{A}_H(y)$.

I will only sketch the proofs of these three steps. More details can be found in the proof of Proposition 6.5.7 of [AD14].
To prove the first step, create a sequence $(s_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$ in such a way that $(u_{s_k})_{k\in\mathbb{N}}$ will be an equi-continuous family of functions. To do this you need to use the finiteness of the energy of u and that M is compact. Then the compactness of M and the equi-continuous property allows us to use the Arzelà-Ascoli Theorem, see Theorem C.1.1 in [AD14]. Applying this Theorem on the sequence $(u_{s_k})_{k\in\mathbb{N}}$ gives step one.

For the second step we start with to show that $\dot{y} = X_t(y)$. To show this, we look at the difference

$$(y(t) - y(0)) - \int_0^t X_\tau(y(\tau))d\tau = \lim_{k\to\infty} \left(u_{s_k}(t) - u_{s_k}(0) - \int_0^t X_\tau(y(\tau))d\tau \right).$$

The norm of this expression can be estimated to zero, so $\dot{y} = X_t(y)$.
The relation $\dot{y} = X_t(y)$ implies now that y is differentiable, i.e. $y \in \mathcal{C}^1$, because $y \in \mathcal{C}^0$. Now we know $y \in \mathcal{C}^1$, hence $y \in \mathcal{C}^2$. In this way we conclude that $y \in \mathcal{C}^\infty$. This process is well known as bootstrapping. This completes the proof of the second step.

The difficulties of proving the third step are in the first integral of Definition 3.1, the definition of the action functional. For extensions $\tilde{u}_{s_k}$ and $\tilde{y}$ of u_{s_k} and y to the disk respectively, we want that

$$\lim_{k\to\infty} \int_D \tilde{u}_{s_k}^*\omega = \int_D \tilde{y}^*\omega.$$

If we assume that the form ω is exact, *i.e.* $\omega = d\lambda$, then follows from a short computation that uses the first two steps, that the statement is true.

The problem is that we cannot assume that ω is exact. It is even worse, a symplectic form on a compact manifold cannot be exact. This follows from Lemma 1.4 and Theorem 17.31 of [Lee13].
However, we can still use this computation. y is contractible, so pick an open U around y such that it is diffeomorphic to an open ball. Then ω is exact on U by Corrollary 17.15 (Local Exactness of Closed Forms) of [Lee13]. u_{s_k} goes to y, so for k large enough, u_{s_k} lies also in U. If we now choose the extensions $\tilde{u}_{s_k}$ and $\tilde{u}_y$ to be in U, the computation is valid. We may choose $\tilde{u}_{s_k}$ and $\tilde{u}_y$ in U, because the computation will not depend on the chosen extension, because of Remark 2.2. □

Lemma 3.16. *The set of critical points of the action functional $\mathcal{A}_H$ is compact.*

Proof. Let $(x_k)_{k\in\mathbb{N}}$ a sequence of critical points of the action functional. There exists a constant $C > 0$ such that for all k we have $|x^k| < C$. This holds because of the image of x^k lies in M and M is compact. Furthermore, $\dot{x}^k(t) = X_t(x^k(t))$, so $\dot{x}^k$ is for all k also uniformly bounded. Hence, x^k is equi-Lipschitz, and in particular equicontinuous. Now we can apply the Arzelà-Ascoli Theorem, Theorem C.1.1 in [AD14], to conclude that there exists a subsequence of $(x_k)_{k\in\mathbb{N}}$ that converges in the $\mathcal{C}^1$ sense. By differentiating the equation $\dot{x}^k(t) = X_t(x^k(t))$ multiple times, bootstrapping, we obtain the $\mathcal{C}^\infty$ convergence. □

Lemma 3.17. *If all the critical points of the action functional $\mathcal{A}_H$ are nondegenerate, then the amount of critical points is finite.*

Proof. The critical points of the action functional can be described as the intersection of two submanifolds of the compact manifold $M \times M$. Both submanifolds have the same dimension as M, which is half the dimension of $M \times M$. The two submanifolds are

(1) the diagonal $\Delta = \{(x,x) \mid x \in M\}$

(2) and the graph $\{(x, \varphi^1(x)) \mid x \in M\}$ of the flow φ^1 of X_t at time 1.

The nondegeneracy of the critical points, see Definition 1.16, gives that both submanifolds intersect transversal. Therefore, their intersection is a submanifold of dimension 0, see Theorem 6.30 of [Lee13]. Hence the amount of critical points is finite, since it is also compact by the Lemma above. □

Corollary 3.18. *There exist a constant C such that for all $u \in \mathcal{M}$ we have the following two estimates*

$$-C \leq \mathcal{A}_H(u) \leq C \quad \textit{and} \quad 0 \leq E(u) \leq C$$

Proof. Lemma 3.15 and the fact that the function $s \mapsto \mathcal{A}_H(u_s)$ is decreasing imply $\mathcal{A}_H(x) \leq \mathcal{A}_H(u) \leq \mathcal{A}_H(y)$ for x and y critical points of the action functional $\mathcal{A}_H$.
Furthermore, Lemma 3.15 and the same reasoning as in the third item of Lemma 3.9 imply $E(u) = \mathcal{A}_H(x) - \mathcal{A}_H(y)$ for x and y critical points of the action functional $\mathcal{A}_H$.
Combining these two results with Lemma 3.17, the amount of critical points is finite, proofs the Corollary. Here we must note that the first item of 3.9 gives that the energy must be positive. □

3.4 Compactness

The Gromov compactness, Theorem 3.14, is a consequence of the following Lemma. This Lemma is the core of the proof of the Theorem, but define first the following notation. Let $B(x, r)$ be the ball centered at a point x with radius r.

Lemma 3.19. *Let M be a compact symplectic manifold such that $\pi_2(M) = 0$, then there exists a constant $A > 0$ such that the norm of the gradient of u is bounded by A for all u in $\mathcal{M}$ and all (s,t) in $\mathbb{R} \times S^1$, i.e.*

$$\forall u \in \mathcal{M}, \forall (s,t) \in \mathbb{R} \times S^1, \|\operatorname{grad}_{(s,t)} u\| \leq A.$$

Proof. The proof of this Lemma contains some long computations, so I will give here only a proof sketch. All the details can be found in the proof of Lemma 6.6.2 in [AD14].
We start with assuming that we have sequences $(u_k)_{k\in\mathbb{N}}$ in $\mathcal{M}$ and $(s_k, t_k)_{k\in\mathbb{N}}$ in $\mathbb{R} \times S^1$ such that

$$\lim_{k\to\infty} \left\|\operatorname{grad}_{(s_k,t_k)} u_k\right\| = +\infty.$$

The sequence (u_k) will be used to create a sequence (v_k) that does not diverge to $+\infty$. This sequence inherits a property that will give a contradiction with our assumption $\pi_2(M) = 0$, see Remark 2.1.
Before the definition of the sequence (v_k), we make some assumptions on (u_k). There exists a sequence (ε_k) of positive numbers that converges to 0 such that

$$\lim_{k\to\infty} \varepsilon_k \left\|\operatorname{grad}_{(s_k,t_k)} u_k\right\| = +\infty \qquad \text{and}$$

$$\left\|\operatorname{grad}_{(s,t)} u_k\right\| \leq 2 \left\|\operatorname{grad}_{(s_k,t_k)} u_k\right\| \quad \text{for } (s,t) \in B((s_k,t_k), \varepsilon_k).$$

We can make this assumption because of the half maximum Lemma 6.6.3 in [AD14].
Now, define the sequence (v_k) by

$$v_k(s,t) = u_k\left(\frac{(s,t)}{R_k} + (s_k,t_k)\right) \quad \text{where } R_k = \left\|\operatorname{grad}_{(s,t)} u_k\right\|.$$

Note that $\|\operatorname{grad}_{(0,0)} v_k\| = 1$, $\varepsilon_k R_k \xrightarrow{(k\to\infty)} \infty$, and $\|\operatorname{grad}_{(s,t)} v_k\| \leq 2$ on $B(0, \varepsilon_k R_k)$.

Apply the Elliptic Regularity Lemma 12.1.1 of [AD14]. to conclude that there is a subsequence of (v_k) that converges to a v in $\mathcal{C}^2_{\text{loc}}(\mathbb{R} \times S^1; M)$ such that

(1) $\left\|\operatorname{grad}_{(0,0)} v\right\| = 1, \quad$ hence v is nonconstant,

(2) $\left\|\operatorname{grad}_{(s,t)} v\right\| \leq 2 \quad$ for all $(s,t) \in \mathbb{R}$,

(3) $\frac{\partial v}{\partial s} + J(v)\frac{\partial v}{\partial t} = 0$, hence v is J-holomorphic.

This three properties follow directly from the definition of the sequence (v_k) and the Elliptic Regularity Lemma, but we need more properties of v to run into a contradiction. An estimation of the energy of v on $B(0, \varepsilon_k R_k)$ gives

$$E(v) = \int_{B(o,\varepsilon_k R_k)} \|\text{grad } v_k\|^2 \, dt \, ds = \int_{B(o,\varepsilon_k R_k)} \left\| \frac{\partial v_k}{\partial s} \right\|^2 + \left\| \frac{\partial v_k}{\partial t} \right\|^2 dt \, ds \leq 4C,$$

where C is the constant from Corollary 3.18. The ball $B(0, \varepsilon_k R_k)$ goes in the limit to $\mathbb{R} \times S^1$, so v has a finite energy by Fatou's Lemma, see Theorem 9.11 of [Sch05]. This estimation implies also that the symplectic area of v is finite, since a computation of this area gives that

$$\int_{\mathbb{R}\times S^1} v^*\omega = \int_{\mathbb{R}\times S^1} \left\| \frac{\partial v}{\partial t} \right\|^2 dt \, ds = \int_{\mathbb{R}\times S^1} \left\| \frac{\partial v}{\partial s} \right\|^2 + \left\| \frac{\partial v}{\partial t} \right\|^2 dt \, ds \leq 4C.$$

Note also that the area of v is nonzero, since v is nonconstant.
The last ingredient we need to get a contradiction is a sequence (r_k) going to ∞ such that the length ℓ of $v(\partial B(0, r_k))$ converges to 0, *i.e.* the length of the image of the boundary of the disk $B(0, r_k))$ converges to 0. To find such a sequence, you need to use the facts that v is J-holomorphic and the symplectic area of v is bounded. For the proof of this statement see Lemma 6.6.5 of [AD14].
The length $\ell(v(\partial B(0, r_k)))$ goes to 0 and ω is a closed form, so there is a big enough k and a closed ball U such that $v(\partial B(0, r_k)) \subset U$ and $\omega = d\lambda$, see Corollary 17.15 of [Lee13]. Let D_k be a disk in U such that $\partial D_k = v(\partial B(0, r_k))$ and let S_k^2 be the sphere $D_k \cup v(B(0, r_k))$ in M. The assumption $\pi^2(M) = 0$, see Remark 2.1, gives now that

$$0 = \int_{S_k^2} \omega = \int_{D_k} \omega + \int_{v(B_k)} \omega.$$

Now we get a contradiction, since

$$\int_{v(B_k)} \omega = \int_{B_k} v^*\omega \xrightarrow{(k\to\infty)} \int_{\mathbb{R}\times S^1} v^*\omega \neq 0 \qquad \text{and}$$

$$\left| \int_{D_k} \omega \right| = \left| \int_{D_k} d\lambda \right| = \left| \int_{v(\partial B(0,r_k))} d\lambda \right| \leq \ell(v(\partial B(0, r_k))) \sup_U \|\lambda\| \xrightarrow{(k\to\infty)} 0.$$

The image $v(B(0, r_k))$ is called a bubble in this contradiction argument. Namely, the surface $v(B(0, r_k))$ increases like a bubble, since its area increases, but its boundary $v(\partial B(0, r_k))$ decreases. This phenomenon of a bubble is thus not possible under the assumption of $\pi^2(M) = 0$. □

Corollary 3.20. *Every sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$ has a subsequence that converges uniformely, as do its derivatives, to a limit that is therefore of class $\mathcal{C}^\infty$.*

Proof. Lemma 3.19 together with the Elliptic Regularity Lemma 12.1.1. of [AD14] give the result. Note that the energy on $\mathcal{M}$ is bounded, see Corollary 3.18, so the limit stays in $\mathcal{M}$. $\square$

Theorem 3.14 is a direct consequence of this corollary.

3.5 Trajectories connect critical points

I will give a proof sketch of Theorem 3.12 in this Section. The proof starts with the following lemma.

Lemma 3.21. *Let $u \in \mathcal{M}$ and $(s_k)_{k\in\mathbb{N}}$ a sequence in $\mathbb{R}$ going to $+\infty$. Then there exists a critical point y of the action functional $\mathcal{A}_H$ and a subsequence $(s_{k'})_{k'\in\mathbb{N}}$ such that*

$$\lim_{k'\to\infty} u_{s_{k'}} = y.$$

Proof. We want to use the compactness of $\mathcal{M}$ to extract a subsequence of $(s_k)_{k\in\mathbb{N}}$ such that u_{s_k} converges to an element $v \in \mathcal{M}$. We need to be careful here, since $u_{s_k} \notin \mathcal{M}$, but $u_{s_k} \in \mathcal{L}M$.
We start with making a sequence in $\mathcal{M}$ out of the sequence $(s_k)_{k\in\mathbb{N}}$ in $\mathbb{R}$. To do this, we note that the additive group $\mathbb{R}$ acts continuously on $\mathcal{M}$ from the right in the following way $(u \cdot s)(s_0, t) = u(s + s_0, t)$. Now we define a sequence in $\mathcal{M}$ out of the sequence $(s_k)_{k\in\mathbb{N}}$ as $(u \cdot s_k)_{k\in\mathbb{N}}$. Theorem 3.14 gives that $\mathcal{M}$ is compact, so there exists a subsequence $(u \cdot s_{k'})_{k'\in\mathbb{N}}$ such that $\lim_{k'\to\infty} u \cdot s_{k'}(s,t) = v(s,t)$ for some $v \in \mathcal{M}$ and all s and t. For a fixed s_0 we have now

$$v_{s_0}(t) = \lim_{k'\to\infty} u \cdot s_{k'}(s_0, t) = \lim_{k'\to\infty} u_{s_{k'}+s_0}(t) = \lim_{k'\to\infty} u_{s_{k'}}(t).$$

Now we are in the same situation as in the first step of the proof of Lemma 3.15. The second step of that proof tells us now that v_{s_0} is a critical point y of the action functional. $\square$

Proof of Theorem 3.12. It is enough to show only the case $s \to +\infty$, because of symmetry. Lemma 3.17 gives that the amount of critical points of the action functional $\mathcal{A}_H$ is finite. Therefore, we can choose disjoint open neighbourhoods U_y in $\mathcal{L}M$ such that each of them contains one critical point y of the action functional. Define $U = \cup_{y\in\operatorname{Crit}\mathcal{A}_H} U_y$.
For all $u \in \mathcal{M}$ there exists a big enough s_K such that $u([s_K, \infty[\times S^1) \subset U$ by Lemma 3.21, hence $u([s_K, \infty[\times S^1) \cap U_y \neq \emptyset$ for some critical point y. We have even more, since $u([s_K, \infty[\times S^1)$ must be connected, then follows that $u([s_K, \infty[\times S^1) \subset U_y$. The conclusion is now $\lim_{s\to+\infty} u_s = y$.
Now we have the main statement of Theorem 3.12. The second part about $\partial u/\partial s$ follows now easily. Corollary 3.20 gives that $\lim_{s\to\infty} \partial u_s/\partial t = \dot{y}$. We also have that u is a solution of the Floer equation and y a critical point of $\mathcal{A}_H$, so

$$\lim_{s\to+\infty} \frac{\partial u}{\partial s}(s,t) = -(J(\dot{y}) + \operatorname{grad}_y H_t(y)) = -J(\dot{y} - X_t(y)) = 0. \qquad \square$$

4 The Maslov Index

To define the Floer homology, we need a chain complex. This chain complex can only be defined if we have an index for the critical points of the action functional, because these points will generate the vector spaces of the chain complex. This Section is about defining the Maslov index on critical points of the action functional. This index will be used to define the chain complex. Keep in mind that this critical points are precisely the 1-periodic contractible solutions of the Hamiltonian.
Let x a nondegenerate 1-periodic contractible solution. We will associate to x an integer in three steps.

(1) Associate x with a path of symplectic matrices in the symplectic group $\mathrm{Sp}(2n)$.

(2) Associate the path of symplectic matrices with a path in the cicle S^1.

(3) Associate such a path in S^1 with an integer.

I will discuss these steps in the rest of the Section. Each step has his own Subsection.

4.1 First step

For a nondegenerate 1-periodic contractible solution x, we will associate to it a path $t \mapsto A(t)$ of symplectic matrices such that $A(0) = \mathrm{Id}$ and $1 \notin \operatorname{Spec} A(1)$, where $\operatorname{Spec} A(1)$ denotes the spectrum of the matrix $A(1)$.
We begin with choosing a symplectic basis $Z(0) = (Z_1(0), \ldots, Z_{2n}(0))$ of $T_{x(0)}M$. Let $A(t)$ be the matrix of the linear map $T_{x(0)}\varphi^t$ in the basis $Z(0)$. These matrices are symplectic for all $t \in [0,1]$, since φ^t preserves the symplectic form ω, see Propostion 1.15. We also have that $1 \notin \operatorname{Spec} A(1)$, because x is assumed to be nondegenerate, see Definition 1.16. Now we have a path $t \mapsto A(t)$ of symplectic matrices such that $A(0) = \mathrm{Id}$ and $1 \notin \operatorname{Spec} A(1)$.
To study this path even more, we define $\mathrm{Sp}(2n)^* = \{A \in \mathrm{Sp}(2n) | \det(A - \mathrm{Id}) \neq 0\}$. $\mathrm{Sp}(2n)^*$ is the space of symplectic matrices which do not have eigenvalue 1. We are looking at paths that start at the identity and finish in the space $\mathrm{Sp}(2n)^*$. Therefore, define the space of paths $\mathcal{S} = \{A : [0,1] \longrightarrow \mathrm{Sp}(2n) \mid A(0) = \mathrm{Id},\ A(1) \in \mathrm{Sp}(2n)^*\}$.

Lemma 4.1. *The path of matrices $t \mapsto A(t)$ is unique in $\mathcal{S}$ up to homotopy in $\mathrm{Sp}(2n)$.*

Proof. See Subsection 7.1 of [AD14], Remark 2.3 is crucial in this proof. □

4.2 Second step

We will associate to a path $t \mapsto A(t) \in \mathcal{S}$ from the first step a path $\gamma : [0,1] \to S^1$ such that $\gamma(0) = 1$ and $\gamma(1) = \pm 1$. The main idea is to make the path $t \mapsto A(t)$ of symplectic matrices longer and then apply to this longer path a certain map $\rho : \mathrm{Sp}(2n) \to S^1$ to obtain the path in S^1.

The space $Sp(2n)^*$ is the complement of the hypersurface $\Sigma = \{A \in \mathrm{Sp}(2n) \mid \det(A - \mathrm{Id}) = 0\}$ in $\mathrm{Sp}(2n)$. So it is clear that $\mathrm{Sp}(2n)^*$ cannot consists of one connected component and has thus at least two components. The next propositions tells us that $\mathrm{Sp}(2n)^+ = \{A \in \mathrm{Sp}(2n) \mid \det(A - \mathrm{Id}) > 0\}$ and $\mathrm{Sp}(2n)^- = \{A \in \mathrm{Sp}(2n) \mid \det(A - \mathrm{Id}) < 0\}$ are path connected, so $Sp(2n)^*$ consists of two components.

Proposition 4.2. *The spaces $Sp(2n)^{\pm}$ are path connected.*

Proof. See Proposition 7.1.4 of [AD14]. ☐

With this Proposition we can make the paths $t \mapsto A(t) \in \mathcal{S}$ longer, such that it goes to

$$W^+ = -\mathrm{Id} \text{ or } W^- = \left(\begin{array}{cc|c} 2 & 0 & \\ 0 & 1/2 & 0 \\ \hline \multicolumn{2}{c|}{0} & -\mathrm{Id} \end{array}\right).$$

To be more precise, for a path $\alpha_0 : t \mapsto A(t) \in \mathcal{S}$ we have that $A(1) \in \mathrm{Sp}(2n)^*$, so $A(1) \in \mathrm{Sp}(2n)^+$ or $A(1) \in \mathrm{Sp}(2n)^-$. If $A(1) \in \mathrm{Sp}(2n)^+$, choose a path α_1 from $A(1)$ to $W^+ \in Sp(2n)^+$, but if $A(1) \in \mathrm{Sp}(2n)^-$, choose a path α_1 from $A(1)$ to $W^- \in \mathrm{Sp}(2n)^-$. Now, let α be the concatenation of the paths α_0 and α_1.

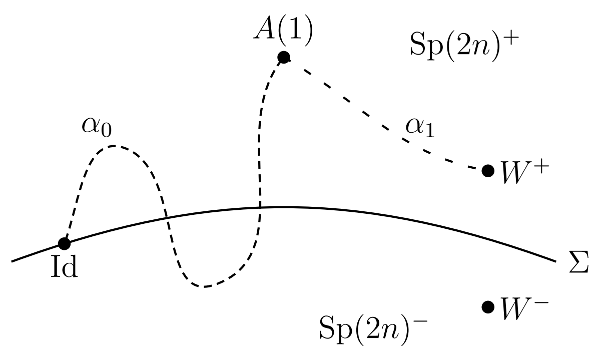


Figure 3: Path of symplectic matrices.

Theorem 4.3. *There exist a continuous map $\rho : \mathrm{Sp}(2n) \to S^1$ for every $n \in \mathbb{N}$ that satisfies the following properties:*

(1) For $A, T \in \mathrm{Sp}(2n)$ we have $\rho(TAT^{-1}) = \rho(A)$. *(Neutrality)*

(2) For $A \in \mathrm{Sp}(2n)$ and $B \in \mathrm{Sp}(2m)$ holds $\rho\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \rho(A)\rho(B)$. *(Product)*

(3) For $A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$, then $\rho(A) = \det_{\mathbb{C}}(X + iY)$. *(Determinant)*

(4) For $A \in \mathrm{Sp}(2n)$, if $\mathrm{Spec}\,(A) \subset \mathbb{R}$, then $\rho(A) = (-1)^{m/2}$, (*Normalization*)
where m denotes the total multiplicity of negative eigenvalues.

(5) For $A \in \mathrm{Sp}(2n)$, $\rho(A^t) = \rho(A^{-1}) = \overline{\rho(A)}$.

Proof. The construction of such a map ρ can be found in Subsection 7.3 of [AD14]. In the case $A \in \mathrm{Sp}(2n)$ has $2n$ distinct eigenvalues, the map will be

$$\rho(A) = (-1)^{m/2} \prod_{\substack{\lambda \in \mathrm{Spec}\,(A) \cap S^1 \\ \mathrm{Im}(\lambda) > 0}} \lambda^{\mathrm{sign}\ \mathrm{Im}\ \omega(\overline{X}, X)},$$

here denotes m the total multiplicity of real negative eigenvalues, and is X an eigenvector of the corresponding eigenvalue λ. □

If we now apply the map ρ of this Theorem to the path α, we get a path $\gamma = \rho \circ \alpha$ in S^1. Note that $\alpha(0) = \mathrm{Id}$ and $\alpha(1) = W^{\pm}$, so the fourth property of the map ρ gives that $\gamma(0) = 1$ and $\gamma(1) = \pm 1$. Now we have completed the second step of defining the Maslov Index.

4.3 Third step

To associate to the path $\gamma : [0, 1] \to S^1$ from the second step an integer, we will lift the path γ to $\mathbb{R}$. The integer will be the difference between the end points of the lifted path divided by π. This integer is called the Maslov index of the nondegenerate 1-periodic contractible solution x. Before we can lift the path γ, we need to have the following Proposition.

Proposition 4.4. *The inclusions $i^{\pm}$ of the spaces $Sp(2n)^{\pm}$ into $Sp(2n)^*$ induce zero homomorphisms on the fundamental groups.*

This Proposition can be easily proved with the following Remark and Lemma.

Remark 4.5. The third item of Theorem 4.3 can also be stated as, the map ρ is an extension of the complex determinant map on the unitary group $\mathrm{U}(n)$. This is because the intersection $\mathrm{Sp}(2n) \cap \mathrm{O}(2n)$ consists of matrices

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(2n; \mathbb{R}) \quad \text{with} \quad \begin{cases} X^t Y = Y^t X \\ X^t X + Y^t Y = \mathrm{Id} \end{cases}$$

and these conditions are equivalent to $X + iY \in \mathrm{U}(n)$. In particular, ρ induces an isomorphism on the fundamental groups

$$\rho_* : \pi_1(\mathrm{Sp}(2n)) \longrightarrow \pi_1(S^1) = \mathbb{Z}.$$

This is because $\mathrm{Sp}(2n)$ retracts on $\mathrm{U}(n)$, see Proposition 5.6.9 of [AD14], and the determinant map $\det_{\mathbb{C}} : \mathrm{U}(n) \to S^1$ induces between the fundamental groups of $\mathrm{U}(n)$ and S^1 an isomorphism, see Proposition 2.23 of [MS98].

Lemma 4.6. *There exist two continuous maps $\rho^\pm : \mathrm{Sp}(2n)^\pm \to \mathbb{R}$ such that the diagrams*

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[d, "\exp"] \\
\mathrm{Sp}(2n)^\pm \arrow[urr, "\rho^\pm"] \arrow[r, "i^\pm"'] & \mathrm{Sp}(2n) \arrow[r, "\rho"'] & S^1
\end{tikzcd}$$

commute.

Proof. This Lemma can be proven by using details of the construction of ρ. This is done in Subsection 7.3 of [AD14]. □

Proof of Proposition 4.4. The homomorphism $\exp_*$ between the fundamental groups of $\mathbb{R}$ and S^1 induced by the exponential map $\exp : \mathbb{R} \to S^1$ is the zero homomorphism. So Lemma 4.6 gives that $\rho_* \circ i^\pm_* = \exp_* \circ \rho^\pm_* = 0$. Now follows that $i^\pm_*$ are zero homomorphisms, since ρ_* is an isomorphism by Remark 4.5. □

Theorem 4.7 (Maslov index)**.** *Let x be a nondegenerate 1-periodic contractible solution. Associate via the first and second step to it a path $\gamma : [0,1] \to S^1$. Now choose a lift β of $\gamma = \rho \circ \alpha$ such that the diagram*

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[d, "\exp"] \\
{[0,1]} \arrow[urr, "\beta"] \arrow[r, "\alpha"'] & \mathrm{Sp}(2n) \arrow[r, "\rho"'] & S^1
\end{tikzcd}$$

commutes. Then

$$\mu(x) = \frac{\beta(0) - \beta(1)}{\pi}$$

is well defined and is an integer. μ is called the Maslov index.

Proof. To define the number $\mu(x)$, I made three choices. $\mu(x)$ is well defined if it does not depend on these choices.
The first choice was the symplectic basis $Z(0)$ of $T_{x(0)}M$. From this basis we defined the path $\alpha_0 : t \mapsto A(t)$ in $\mathcal{S}$. Lemma 4.1 gives that this path α_0 is unique up to homotopy in $\mathrm{Sp}(2n)$. So if $\mu(x)$ does only depend on the homotopy class of α_0, then it does not depend on our choice of symplectic basis $Z(0)$. Before I will show this, I take a look at the second choice.
I choose a path α_1 from $\alpha_0(1) = A(1)$ to W^+ or W^- in $\mathrm{Sp}(2n)^+$ or respectively $\mathrm{Sp}(2n)^-$. By Proposition 4.5 follows that this path is unique up to homotopy in $\mathrm{Sp}(2n)$. So the concatenation α of the paths α_0 and α_1 is also unique up to homotopy in $\mathrm{Sp}(2n)$. So if $\mu(x)$ does only depend on the homotopy class of α, then it does not depend on our first two choices.

Now we note that ρ_* is an isomorphism, so $\gamma = \rho \circ \alpha$ is also unique up to homotopy in S^1. Hence $\mu(x) = \beta(0) - \beta(1))/\pi$ does not depend on the first two choices and the choice of lift β. Thus $\mu(x)$ is well defined.
Furthermore, it is an integer, since $\beta(\exp(0)) = \rho(\alpha(0)) = \rho(\mathrm{Id}) = 1$ and $\beta(\exp(1)) = \rho(\alpha(1)) = \rho(W^{\pm}) = \pm 1$. $\square$

After defining the Maslov index, we need one more Proposition about the Maslov index to be able to compare the Morse and Floer homology in Subsection 7.2.

Proposition 4.8. *If S is an invertible symmetric matrix with norm $\|S\| < 2\pi$ and if $\alpha_0(t) = e^{-tJ_xS}$ with $x \in M$, then the Maslov index of this path is equal to the number of eigenvalues of S minus half the dimension of M.*

Proof. For this proof I refer to Proposition 7.2.1 of [AD14]. $\square$

5 Regularity property

The goal of this Section is to show that there is always a perturbation of the Hamiltonian H such that the space $\mathcal{M}(x,y)$, the space of solutions of the Floer equation that connects the orbits x and y, is a smooth submanifold of dimension $\mu(x)-\mu(y)$. This goal translates into the following Theorem.

Theorem 5.1. *For every nondegenerate Hamiltonian H_0 on M there exists an $h \in \mathcal{C}^\infty_\varepsilon(H_0)$ such that*

(1) $H = H_0 + h$ is nondegenerate and has the same 1-periodic solutions as H_0.

(2) for every distinct pair of critical values x and y of the action functional $\mathcal{A}_{H_0}$ the space $\mathcal{M}(x,y,H)$ is a submanifold of dimension $\mu(x)-\mu(y)$.

Such a Hamiltonian H is called regular.

The space $\mathcal{C}^\infty_\varepsilon(H_0)$ denotes the perturbations of the Hamiltonian H_0 and $\mathcal{M}(x,y,H)$ denotes the space $\mathcal{M}(x,y)$. That $\mathcal{M}(x,y)$ depends on H is important in this Section, therefore $\mathcal{M}(x,y)$ is denoted by $\mathcal{M}(x,y,H)$ in this Section.
The idea of how to show this Theorem, is to start with the space $\mathcal{Z}(x,y,H)$ of solutions that connects the orbits x and y for all perturbations around H. Then define the projection

$$\pi : \mathcal{Z}(x,y,H) \longrightarrow \mathcal{C}^\infty_\varepsilon(H),$$

so $\mathcal{M}(x,y,H) = \pi^{-1}(H)$. Now follows that $\mathcal{M}(x,y,H)$ is a submanifold if H is a regular value of π. This is precisely the case if H is regular, which needs to be shown.
The problem is that not every Hamiltonian H is regular value of π. That is why we need to look at perturbations of the Hamiltonian. The Sard-Smale Theorem will give us that there are enough Hamiltonians in $\mathcal{C}^\infty_\varepsilon(H)$ to find a Hamiltonian that is a regular value of π, which is also close to the original Hamiltonian.
The Sard-Smale Theorem works only for Banach manifolds. Therefore, the whole proof of Theorem 5.1 must be done in a Banach setting. This starts with defining a Banach manifold $\mathcal{P}(x,y,H)$ that contains $\mathcal{M}(x,y)$ and allows us to define a Floer map $\mathcal{F}$ on it. Also, the space of perturbations $\mathcal{C}^\infty_\varepsilon(H)$ must be a Banach space. Then via a transversality argument we show that $\mathcal{Z}(x,y,H)$ is a Banach submanifold of $\mathcal{P}(x,y,H)\times\mathcal{C}^\infty_\varepsilon(H)$. This allow us to use the Sard-Smale Theorem to find enough Hamiltonians that are regular values of π.
The argument that $\mathcal{M}(x,y,H)$ is a submanifold if H is a regular value of π, works in the Banach setting only if the derivative of π is a Fredholm operator. The Fredholm index of this derivative is $\mu(x)-\mu(y)$, therefore $\mathcal{M}(x,y,H)$ becomes a submanifold of dimension $\mu(x)-\mu(y)$.

5.1 Floer map on a Banach manifold

We want to define a Banach manifold $\mathcal{P}^{1,p}(x, y, H)$ that contains $\mathcal{M}(x, y, H)$ such that the Floer equation for solutions between two critical points x and y of the action functional can be described via the Floer map

$$\begin{aligned}\mathcal{F} : \mathcal{P}^{1,p}(x, y, H) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^m)\\ u &\longmapsto \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \operatorname{grad}_u(H_t).\end{aligned}$$

Here denotes $L^p(\mathbb{R} \times S^1; \mathbb{R}^m)$ the Lebesgue space with norm p. m is chosen in such a way that the manifold M can be embedded in $\mathbb{R}^m$. The manifold $\mathcal{P}^{1,p}(x, y, H)$ will be local diffeomorphic to the Sobolev space $W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^m)$.

First we note that for a smooth map $u : \mathbb{R} \times S^1 \to M : (s, t) \mapsto u(s, t)$ the variable s sits not in a compact space. This makes it unnatural to look at Sobolev spaces containing this u. Therefore we will look at maps u with the property

$$\|\frac{\partial u}{\partial s}(s, t)\| \leq Ke^{-\delta|s|} \quad \text{and} \quad \|\frac{\partial u}{\partial t}(s, t) - X_H(u)\| \leq Ke^{-\delta|s|}$$

for constants K and δ depending on u. The space of such smooth maps that also connect two critical points x and y of the action functional $\mathcal{A}_H$ is denoted by $\mathcal{C}^\infty_\searrow(x, y, H)$.
Using the fact that $\mathcal{M}(x, y, H)$ contains only elements with finite energy, the exponential decay Theorem, see Theorem 8.9.1 of [AD14], gives us that $\mathcal{M}(x, y, H) \subset \mathcal{C}^\infty_\searrow(x, y, H)$.

Definition 5.2. Define for a map $w \in \mathcal{C}^\infty_\searrow(x, y, H)$ the space $W^{1,p}(w^*TM)$. The space containing all continuous maps $Y : \mathbb{R} \times S^1 \to TM$ such that $Y(s, t) \in T_{w(s,t)}M$ for all (s, t) and the composition of Y with the projection map pr_2 on the second coordinate

$$\mathbb{R} \times S^1 \xrightarrow{Y} TM \subset T\mathbb{R}^m = \mathbb{R}^m \times \mathbb{R}^m \xrightarrow{pr_2} \mathbb{R}^m$$

is in $W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^m)$.

Remark 5.3. We need to pick $p > 2$ if we want to assume that $Y \in W^{1,p}(w^*TM)$ is continuous, see Theorem C.4.9 of [AD14].

Remark 5.4. Recall that an integral curve $\gamma : I = (-\varepsilon, \varepsilon) \to M$ of a smooth vector field V on the manifold M if

$$\dot{\gamma}(t) = V_{\gamma(t)} \quad \text{for all } t \in I.$$

An integrable curve is called complete if the domain I is $\mathbb{R}$. M is compact, so for every smooth vector field V and all points $m \in M$ there exists a unique complete integrable curve γ such that $\gamma(0) = m$, see Theorem 9.12 and Corollary 9.17 of [Lee13]. Denote evaluation of γ in time $t = 1$ by $\exp_m V$.

Definition 5.5. For $p > 2$ define $\mathcal{P}^{1,p}(x, y, H)$ as the space that contains the maps of the form

$$(s, t) \longmapsto \exp_{w(s,t)} Y(s, t) \quad \text{where } Y \in W^{1,p}(w^*TM) \text{ and } w \in \mathcal{C}^\infty_{\searrow}(x, y, H).$$

Theorem 5.6. *$\mathcal{P}^{1,p}(x, y, H)$ is a Banach manifold*

Proof. The maps defined in the previous definition are local diffeomorphims and these diffeomorphims are transition maps from $W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^m)$ to $\mathcal{P}^{1,p}(x, y, H)$, see Subsection 8.2.d of [AD14]. These transitions maps form an atlas for $\mathcal{P}^{1,p}(x, y, H)$, so it is a Banach manifold. □

Remark 5.7. The Banach manifold $\mathcal{P}^{1,p}(x, y, H)$ is defined in such a way that $\mathcal{C}^\infty_{\searrow}(x, y, H) \subset \mathcal{P}^{1,p}(x, y, H)$, therefore $\mathcal{M}(x, y, H) \subset \mathcal{P}^{1,p}(x, y, H)$.

Theorem 5.8. *The Floer map*

$$\begin{aligned} \mathcal{F} : \mathcal{P}^{1,p}(x, y, H) &\longrightarrow L^p(\mathbb{R} \times S^1; \mathbb{R}^m) \\ u &\longmapsto \frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} + \operatorname{grad}_u(H_t). \end{aligned}$$

is well defined.

Proof. The differential operators in the definition of the new Floer map are viewed as differential operators on distributions instead of operators on functions. Therefore, it is well defined if the image of $\mathcal{P}^{1,p}(x, y, H)$ lies in $L^p(\mathbb{R} \times S^1; \mathbb{R}^m)$. The crucial part to show this is that for all $u \in \mathcal{P}^{1,p}(x, y, H)$ it can be written as $u = \exp_w(Y)$ with $w \in \mathcal{C}^\infty_{\searrow}(x, y, H)$ instead of $w \in \mathcal{C}^\infty(x, y)$, see subsection 13.3 of [AD14]. That the image of $\mathcal{P}^{1,p}(x, y, H)$ lands in $L^p(\mathbb{R} \times S^1; \mathbb{R}^m) = W^{0,p}(\mathbb{R} \times S^1; \mathbb{R}^m)$ and not in $W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^m)$ is a natural consequence of the fact that $\mathcal{F}$ is a differential operator, so you expect to drop one Sobolev index. □

Definition 5.9. A continuous linear operator $L : E \to F$ between two Banach spaces E, F is called a Fredholm operator if its image is closed and both the kernel and cokernel are finite dimensional. This allows us to define the index of a Fredholm operator as the difference

$$\operatorname{Ind} L = \dim \ker L - \dim \operatorname{coker} L.$$

Theorem 5.10. *For every $u \in \mathcal{M}(x, y, H)$, $(d\mathcal{F})_u$ is a Fredholm operator of index $\mu(x) - \mu(y)$.*

Proof. For this proof I refer to Theorem 8.1.5 of [AD14]. □

5.2 Transversality

The introduction of this Section starts about finding a perturbation of the Hamiltonian H_0 such that the spaces $M(x, y, H)$ become a submanifold. Therefore, we define a space of perturbations $\mathcal{C}^\infty_\varepsilon(H)$. After this definition, we use a transversality argument to conclude that the space $\mathcal{Z}(x, y, H_0)$ of $W^{1,p}$ solutions going from x to a distinct y for all perturbations of the Hamiltonian H_0 is a Banach manifold, here are x, y critical points of the action functional A_{H_0}.
To define the space of perturbations, we use the following construction. Start with fixing a sequence $\varepsilon = (\varepsilon_n)_n$ of positive real numbers. Define a norm

$$\|h\|_\varepsilon = \sum_{k\leq 0} \varepsilon_k \sup_{(x,t)\in M\times S^1} \left|d^k h(x,t)\right|,$$

where $\left|d^k h(x,t)\right|$ denotes the maximum of $|d^\alpha h(x,t)|$ over all multi-indices α of length k. Define the space $\mathcal{C}^\infty_\varepsilon$ as the $\mathcal{C}^\infty$ functions on $M \times S^1$ with finite norm $\|\cdot\|_\varepsilon$. The space $\mathcal{C}^\infty_\varepsilon$ is a normed and complete vector space. This space has another nice property.

Lemma 5.11. *There exists a sequence $\varepsilon = (\varepsilon_n)_n$ of positive real numbers such that the space $\mathcal{C}^\infty_\varepsilon$ lies dense in $\mathcal{C}^\infty(M \times S^1)$ for the C^1 topology. Moreover, the space $\mathcal{C}^\infty_\varepsilon$ is seperable, i.e. it contains a countable dense subset.*

Proof. For this proof I refer to Proposition 8.3.1 of [AD14]. □

Definition 5.12. The space of Hamiltonian perturbations $\mathcal{C}^\infty_\varepsilon(H)$ of a Hamiltonian H_0 is defined as the subspace of $\mathcal{C}^\infty_\varepsilon$ such that for $h \in \mathcal{C}^\infty_\varepsilon(H)$ we have that $h(x, t) = 0$ in a neighbourhood of the 1-periodic solutions of H_0.

Lemma 5.13. *Let H_0 be a Hamiltonian such that all 1-periodic solutions of the induced Hamiltonian system are nondegenerate. There exist a neighbourhood U around 0 in the space $\mathcal{C}^\infty_\varepsilon(H_0)$ such that for $h \in U$ we have that $H = H_0 + h$ is nondegenerate and H has the same periodic solutions as H_0.*

Proof. Let $h \in \mathcal{C}^\infty_\varepsilon(H_0)$, then the support of h sits around the periodic solutions of H_0. So all the periodic solutions of H_0 are also periodic solutions for $H = H_0 + h$. Moreover, these periodic solutions remain to be nondegenerate.
If we take $\|h\|_\varepsilon$ small enough then there will be no other periodic solutions of H then H_0 already has, see 1.18. □

Definition 5.14. Let x, y critical points of the action functional $\mathcal{A}_{H_0}$. Define the space

$$\mathcal{Z}(x, y, H_0) = \{(u, H = H_0 + h) \mid h \in \mathcal{C}^\infty_\varepsilon(H_0) \text{ and } u \text{ a } W^{1,p} \text{ solution between } x \text{ and } y\}.$$

Theorem 5.15. *The space $\mathcal{Z}(x, y, H_0)$ of $W^{1,p}$ solutions going from x to a distinct y for all perturbations of the Hamiltonian H_0 is a Banach submanifold of $\mathcal{P}^{1,p}(x, y, H_0) \times \mathcal{C}^\infty_\varepsilon(H_0)$.*

Proof. To show that $\mathcal{Z}(x, y, H_0)$ is a manifold we use a transversality argument. Describe $\mathcal{Z}(x, y, H_0)$ as the set of zeros of a section of a fiber bundle. To do this, define for $u \in \mathcal{P}^{1,p}(x, y)$ and $h \in \mathcal{C}_\varepsilon^\infty(H_0)$ the space

$$\mathcal{E} = \{(u, h, Y) | Y \in L^p(u^*TM)\},$$

here is $L^p(u^*TM)$ defined in an analogous way as $W^{1,p}(u^*TM)$. Define now the fiber bundle as

$$\begin{aligned} \mathcal{E} &\longrightarrow \mathcal{P}^{1,p}(x, y, H_0) \times \mathcal{C}_\varepsilon^\infty(H_0) \\ (u, h, Y) &\longmapsto (u, h). \end{aligned}$$

Note that $\mathcal{C}_\varepsilon^\infty(H_0)$ is closed in $\mathcal{C}_\varepsilon^\infty(H_0)$, therefore it is a Banach space. We also have that $\mathcal{P}^{1,p}(x, y, H_0)$ is a Banach manifold by Theorem 5.6, so $\mathcal{P}^{1,p}(x, y, H_0) \times \mathcal{C}_\varepsilon^\infty(H_0)$ is a Banach manifold.
The section that has the zero set $\mathcal{Z}(x, y, H_0)$ is given by

$$\begin{aligned} \sigma : \mathcal{P}^{1,p}(x, y, H_0) \times \mathcal{C}_\varepsilon^\infty(H_0) &\longrightarrow \mathcal{E} \\ (u, h) &\longmapsto \left(u, h, \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \operatorname{grad}_u(H_0 + h)\right). \end{aligned}$$

The section is well defined because of Theorem 5.8 and the definition of $L^p(u^*TM)$. Theorem 8.1.4 of [AD14] gives now that the linearised map $(d\sigma)_{(u,h)}$ composed with the projection on the fiber is surjective when $\sigma(u, h) = 0$ and admits a continuous right inverse. To prove the surjectivity, you need that the Floer map is Fredholm. Now we say that the section σ has the transversality property. This transversality property gives that the zero set $\sigma^{-1}(0)$, which is equal to $\mathcal{Z}(x, y, H_0)$, is a Banach manifold, see Theorem 8.1.3 of [AD14]. $\square$

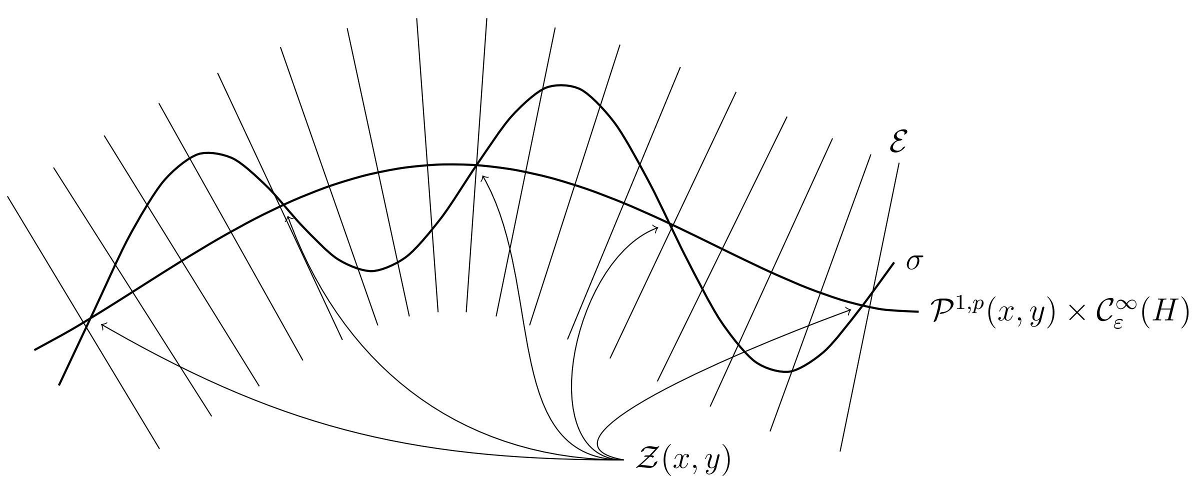


Figure 4: $\mathcal{Z}(x, y)$ as transverse intersection of the section σ and the zero section $\mathcal{P}^{1,p}(x, y) \times \mathcal{C}_\varepsilon^\infty(H)$ in the fiber bundle $\mathcal{E}$.

5.3 We may assume that $M(x, y)$ is a submanifold

Define the projection

$$\begin{aligned} \pi : \mathcal{Z}(x, y) &\longrightarrow \mathcal{C}^\infty_\varepsilon(H_0) \\ (u, H_0 + h) &\longmapsto h \end{aligned}$$

We want to conclude that there are a lot of regular values $h \in \mathcal{C}^\infty_\varepsilon(H_0)$, so we apply the Sard-Smale Theorem, see [Sma65].

Theorem 5.16 (Sard-Smale)**.** *Let E and F be two seperable Banach spaces. If for U an open set in E the map $f : U \to F$ is a smooth Fredholm map, then the set of regular values of f is a countable intersection of dense open subsets.*

This Theorem can also be used if E is a Banach manifold, since f is defined local on an open U in E. To apply this Theorem on the map π above, we need that π is a Fredholm map.

Lemma 5.17. *The map π is Fredholm.*

Proof. Use that the tangent map of the Floer map $\mathcal{F}$ is a Fredholm operator to conclude the same for π, for the proof I refer to the beginning of Subsection 8.5.c. of [AD14]. □

The last ingredient to prove the main Theorem of this Section is the following Lemma,.

Lemma 5.18. *The regular values h of π are exactly the $h \in \mathcal{C}^\infty_\varepsilon(H_0)$ such that $(d\mathcal{F})_u$ is surjective for every $u \in \mathcal{M}(x, y, H_0 + h)$.*

Proof. For this proof I refer to Lemma 8.5.9. of [AD14]. □

Now we are able to prove the main theorem of this Section.

Proof of Theorem 5.1. Apply the Sard-Smale Theorem 5.16 on $\pi = \pi_{x,y}$ for all the distinct pairs x and y of critical values of the action functional $\mathcal{A}_{H_0}$. The amount of critical values is finite because of 3.17, so we have a finite intersection of a countable intersections of dense open sets of regular values for all maps $\pi_{x,y}$. Hence this set is a countable intersection of opens, then by Baires Theorem we conclude that this set is dense in $\mathcal{C}^\infty_\varepsilon(H_0)$. Denote this set by π_{reg}.
Denote by U the open subset U defined in Lemma 5.13. Since π_{reg} is dense and U is open, we have that $U \cap \pi_{\text{reg}} \neq \emptyset$. Take $h \in U \cap \pi_{\text{reg}}$, then $H = H_0 + h$ fulfills the first property of the Theorem and h is a regular value of $\pi_{x,y}$ for all distinct pairs of critical values x and y of the action functional $\mathcal{A}_{H_0}$.
Fix now an x and an y. h is a regular value for $\pi_{x,y}$, so $\pi^{-1}_{x,y}(h)$ is a submanifold with dimension equal to its Fredholm index, see Theorem C.2.13. of [AD14]. Regular value

means that the tangent map of $\pi_{x,y}$ is surjective, so the cokernel of $(d\pi_{x,y})_{(u,H)}$ is 0. Now we obtain

$$
\begin{aligned}
\operatorname{Ind}(d\pi_{x,y})_{(u,H)} &= \dim\ker(d\pi_{x,y})_{(u,H)} \\
&= \dim\ker(d\mathcal{F})_u && \text{(Lemma 5.17)} \\
&= \operatorname{Ind}(d\mathcal{F})_u && \text{(Lemma 5.18)} \\
&= \mu(x) - \mu(y) && \text{(Theorem 5.10)}
\end{aligned}
$$

If we now show that $\pi_{x,y}^{-1}(h) = \mathcal{M}(x,y,H)$ we are done. By the Elliptic Regularity Proposition 12.1.4 of [AD14] we have $\pi_{x,y}^{-1}(h) \subset \mathcal{M}(x,y,H)$. This Proposition works only for $p > 2$, since then the space $W^{1,p}(\mathbb{R} \times S^1; M)$ consist of continuous maps, see Remark 5.3. The other direction follows by Remark 5.7, since $\mathcal{M}(x,y,H) \subset \mathcal{P}(x,y) \subset \pi_{x,y}^{-1}(h)$. This concludes the Theorem. $\square$

6 Definition of the Floer complex

To define a chain complex for the Floer homology for a regular Hamiltonian, we must look at the trajectory spaces $\mathcal{L}(x, y) = \mathcal{M}(x, y)/\mathbb{R}$. This quotient is induced by the right action explained in the proof of Lemma 3.21, denote the corresponding quotient map by π. We need two properties of the trajectory spaces $\mathcal{L}(x, y)$, namely,

Theorem 6.1 (Two properties of the trajectory spaces)**.** *Let x, y, z critical points of the action functional A_H.*

(1) If $\mu(x) - \mu(y) = 1$, then $\mathcal{L}(x, y)$ is a compact manifold of dimension 0.

(2) If $\mu(x) - \mu(z) = 2$, then $\mathcal{L}(x, z)$ is a manifold of dimension 1 such that the closure $\overline{\mathcal{L}(x, z)}$ is a compact manifold with boundary

$$\partial\mathcal{L}(x, z) = \bigcup_{\mu(x)-\mu(y)=1} \mathcal{L}(x, y) \times \mathcal{L}(y, z).$$

The precise definition of $\mathcal{L}(x, y) \times \mathcal{L}(y, z)$ will be made clear in Theorem 6.3. The definition in words is that an element of $\mathcal{L}(x, y) \times \mathcal{L}(y, z)$ consists of two trajectories such that the first one connects the critical points x and y, and the second one connects y and z. Such a space is called the space of broken trajectories between x and z via y.
The first property gives that $\mathcal{L}(x, y)$ consists only of a finite amount of trajectories. Therefore, the definition $n(x, y) = \#\mathcal{L}(x, y) \mod 2$ is well defined if $\mu(x) - \mu(y) = 1$. Since we already have shown that the total amount of critical points of the action functional is finite, Lemma 3.17, we can define the differential

$$\partial_k : C_k(H) \longrightarrow C_{k-1}(H), \quad \partial_k(x) = \sum_{\mu(y)=k-1} n(x, y)y,$$

where $C_k(H)$ denotes the vector space over $\mathbb{Z}/2$ generated by the critical points with Maslov index k. This differential induces the Floer complex. That it is really a differential needs still to be shown. It is a differential if

$$\partial \circ \partial(x) = \sum_{\substack{\mu(y)-\mu(x)=1 \\ \mu(z)-\mu(x)=2}} n(x, y)n(y, z)z = 0.$$

The second property gives that $\overline{\mathcal{L}(x, z)}$ consists of a union of a finite amount of lines and circles, the boundary of such a manifold is always even, this allows us to prove $\partial \circ \partial = 0$. See Subsection 6.2 for the details about how the second property implies $\partial \circ \partial = 0$.
The first step we must take to conclude both properties of Theorem 6.1 is to show that $\mathcal{L}(x, y)$ is a manifold.

Theorem 6.2. *Let x, y two distinct critical points of the action functional $\mathcal{A}_H$ where H is is regular. The trajectory space $\mathcal{L}(x, y)$ between x and y is then a manifold of dimension $\mu(x) - \mu(y) - 1$.*

Proof. Theorem 5.1 gives that $\mathcal{M}(x,y)$ is manifold of dimension $\mu(x)-\mu(y)$. Since $\mathcal{L}(x,y)=\mathcal{M}(x,y)/\mathbb{R}$ is a quotient space, it is a manifold of dimension $\mu(x)-\mu(y)-1$ if R acts smoothly, freely and properly on $M(x,y)$, see Theorem 21.10 of [Lee13]. To conclude that the action is indeed free, see Remark 9.1.6 of [AD14]. □

To prove the two properties of 6.1, we also need the following Theorem. It shows that the space $\overline{\mathcal{L}(x,y)}$ of broken trajectories is compact.

Theorem 6.3. *Let x,y be two critical points of the action functional $\mathcal{A}_H$. Then for every sequence (u_n) in $\mathcal{M}(x,y)$, there exists a subsequence of (u_n), also denoted by (u_n) and critical points $x=x_0,x_1,\dots,x_l,x_{l+1}=y$ such that*

$$\lim_{n\to\infty}\widetilde{u}_n\in\mathcal{L}(x_0,x_1)\times\mathcal{L}(x_1,x_2)\times\cdots\times\mathcal{L}(x_l,x_{l+1}),\ \textit{where } \widetilde{u}_n=\pi\circ u_n\in\mathcal{L}(x,y),$$

i.e. there exist sequences (s_n^k) for $0\leq k\leq l$ and $u^k\in\mathcal{M}(x_k,x_{k+1})$ such that for all $k=0,\dots,l$ we have

$$\lim_{n\to\infty}u_n\cdot s_n^k=u^k.$$

Moreover, the limits are unique.

Proof. The proof is given in Subsection 6.1. □

Corollary 6.4. *For two critical points x and y of the action functional $\mathcal{A}_H$, the space $\overline{\mathcal{L}(x,y)}$ is compact.*

Now we are able to prove Property (1) of Theorem 6.1, but we need the following Theorem to conclude also Property (2).

Theorem 6.5 (Gluing)**.** *Let x,y and z be critical points of the action functional $\mathcal{A}_H$, such that $\mu(x)=\mu(y)+1=\mu(z)+2$. We have then for $(u,v)\in\mathcal{M}(x,y)\times\mathcal{M}(y,z)$ representing trajectories $(\widetilde{u},\widetilde{v})\in\mathcal{L}(x,y)\times\mathcal{L}(y,z)$ that:*

(1) For some ρ_0 There exists a differentiable map $\psi:[\rho_0,+\infty[\to\mathcal{M}(x,z)$ such that

$$\widetilde{\psi}=\pi\circ\psi:[\rho_0,+\infty[\longrightarrow\mathcal{L}(x,z)$$

is an embedding satisfying

$$\lim_{\rho\to+\infty}\widetilde{\psi}(\rho)=(\widetilde{u},\widetilde{v})\in\overline{\mathcal{L}}(x,z).$$

(2) Every sequence (ℓ_n) in $\mathcal{L}(x,z)$ that converges to $(\widetilde{u},\widetilde{v})$ must lie in the image of $\widetilde{\psi}$ for n large enough.

Proof. For this proof I refer to Theorem 9.2.3 of [AD14]. □

Now we have all the tools to prove Theorem 6.1.

Proof of Theorem 6.1. Theorem 6.2 gives that $\mathcal{L}(x,y)$ and $\mathcal{L}(x,z)$ are a manifold and the above corollary gives that $\overline{\mathcal{L}(x,y)}$ and $\overline{\mathcal{L}(x,z)}$ are compact.
Note for Property (1) that the manifold $\mathcal{L}(x,y)$ has dimension 0. For a discrete manifold holds $\mathcal{L}(x,y) = \overline{\mathcal{L}(x,y)}$, hence $\mathcal{L}(x,y)$ is a compact manifold of dimension 0.
To show Property (2), we are only left with the statement that $\overline{\mathcal{L}(x,z)}$ is a manifold with boundary. This statement is a consequence of the so called gluing Theorem. I will not prove this Theorem, but I will give some examples that illustrate how to conclude from this second Property that $\partial \circ \partial = 0$ in Subsection 6.2. $\square$

6.1 Broken trajectories

The boundary of the space of Trajectories $\mathcal{L}(x,y)$ is the space of broken trajectories that connect x and y. These broken trajectories are unions of trajectories that connect critical points, such that it starts at x and finishes in y. Furthermore, the space of trajectories together with its broken trajectories is compact, in other words the closure of the space of trajectories is compact. These properties of the trajectory spaces are a direct consequence of Theorem 6.3, this Theorem is proved in this Subsection. The proof is a combination of the proofs of Proposition 9.1.2 and Theoerm 9.1.7 of [AD14], the first one gives the uniqueness and the second one the existence. I will only prove the existence.

Proof of Theorem 6.3. To prove the Theorem, we will view all the solutions of the Floer equation in the loop space $\mathcal{L}M$ and use the notation of the proof of Lemma 3.21. The proof to find a limit of a subsequence of (u_n) is via induction to k. The proof is illustrated in Figure 5.
Pick around each critical point of the action functional a ball with radius ε such that they are all disjoint. The solutions u_n in $\mathcal{M}(x,y)$ start at x and end up at y, so u_n exits the ball $B(x;\varepsilon)$. Denote by s_n^1 the first time that u_n exits the ball $B(x;\varepsilon)$. The space $\mathcal{M}$ is compact by Theorem 3.14, so there is a subsequence (u_n) such that $(u_n \cdot s_n^1)$ converges to a limit u^1 in $\mathcal{M}$. The time s_n^1 is chosen in such a way, that the limit u^1 must start at x and is non constant. By Corollary 3.10 follows now that $u^1 \in \mathcal{M}(x, x_1)$ for some critical point x_1 of the action functional distinct from x. If it happens to be that $x_1 = y$, then we are done, if not, then we have finished the induction basis.
To finish the proof by induction, we must show that if we found sequences $(s_n^1), \ldots, (s_n^j)$ and $u^j \in \mathcal{M}(x_{j-1}, x_j)$ such that $y = x_j$ and $\lim_{n\to\infty} u_n \cdot s_n^j = u^j$ for $1 \leq j \leq k$, then there exists another critical point x_{k+1} such that the same holds for $j = k+1$.
We have that $u^k \in \mathcal{M}(x_{k-1}, x_k)$, so there exist a time s^* such that $u^k(s)$ is in the ball $B(x_k;\varepsilon)$ for all times $s \geq s^*$. For n large enough, we have that $(u_n \cdot s_n^k)(s^*) = u_n(s_n^k + s^*)$ lies in the ball $B(x_k;\varepsilon)$, because $(u_n \cdot s_n^k)(s^*)$ converges to $(u \cdot s^k)(s^*)$. Therefore, there is a first time $s_n^{k+1} > s_n^k + s^*$ such that $u_n(s_n^k)$ exits the ball $B(x_k;\varepsilon)$, since (u_n) has to go to y. $\mathcal{M}$ is compact, so there is a subsequence (u_n) such that $u^{k+1} = \lim_{n\to\infty} u_n \cdot s_n^{k+1}$. This is the limit we are looking for, but we need to prove that u^{k+1} starts from x_k and ends in

another distinct critical point x_{k+1}.
By contradiction it can be shown that the time difference between the times $s_n^k + s^*$ and s^{k+1} tends towards infinity. Therefore, $u_n \cdot s^{k+1}(s)$ lies in the ball $B(x_k; \varepsilon)$ for n large enough and s a negative real number. Hence u^{k+1} starts at x^k. With the same argument as in the induction basis, it follows that u^{k+1} ends in another distinct critical point x_{k+1}. Now we conclude that $u^{k+1} \in \mathcal{M}(x_k, x_{k+1})$, so by induction follows that we have found a subsequence u_n that converges to a broken trajectory.
To conclude that this limit is unique, we use as mentioned above Proposition 9.1.2 of [AD14]. □

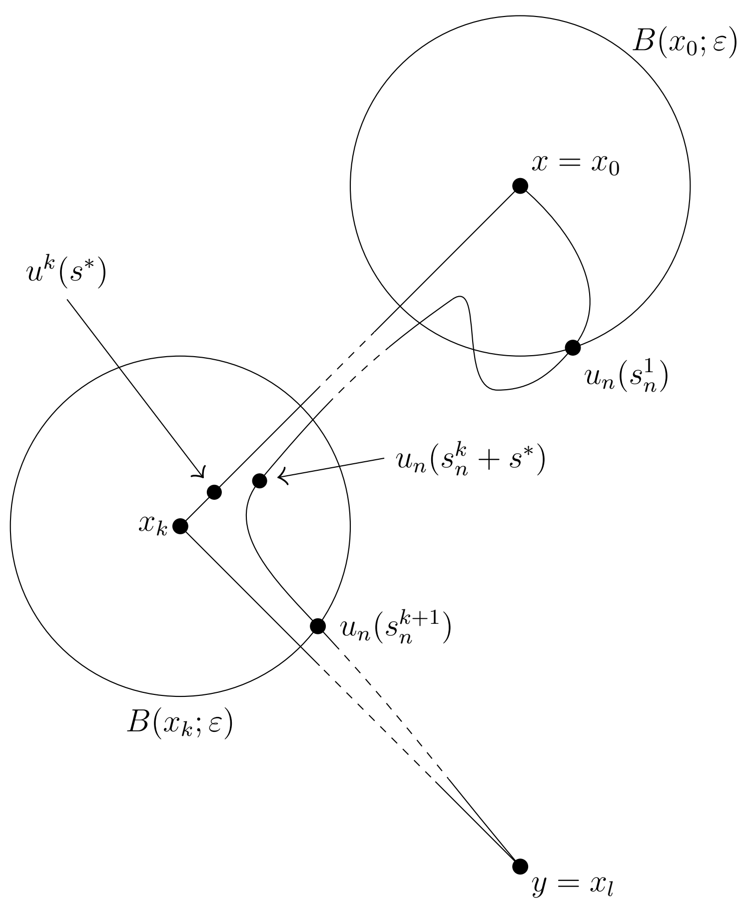


Figure 5: The illustration of how a sequence $(u_n)_n$ of trajectories becomes a broken trajectory.

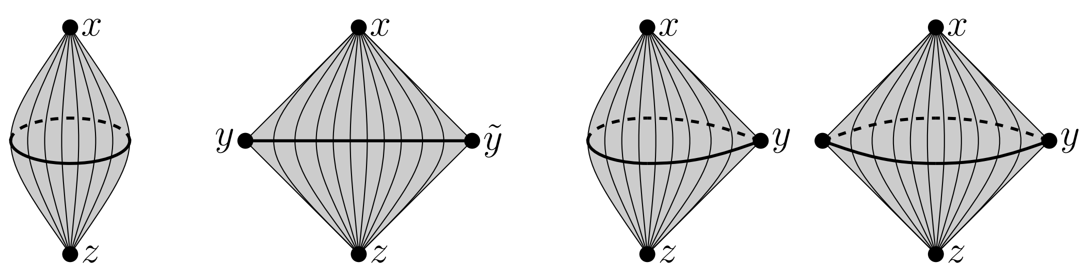
6.2 Gluing

Here I will illustrate with figures what Property (2) of the gluing Theorem 6.1 actually means, and why it gives that $\partial\circ\partial = 0$. Note that $\partial\circ\partial = 0$ iff $\sum_y n(x,y)n(y,z) = 0 \mod 2$ for all critical points x, y, z of the action functional such that $\mu(x) = \mu(y)+1 = \mu(z)+2$. The figures in Figure 6 are examples of spaces $\mathcal{M}(x,z)$ with broken trajectories as boundary. The thick lines in the middle are the corresponding trajectory spaces $\mathcal{L}(x,z) = \mathcal{M}(x,z)/\mathbb{R}$. The grey parts are the spaces $\mathcal{M}(x,z)$. The lines going from x to z are the Floer solutions from x to z. Now I will look at the four examples of Figure 6, to explain why $\partial\circ\partial = 0$.

Figure 6a shows the case where $\mathcal{L}(x,z)$ is a circle. There are no broken trajectories, so $\mathcal{L}(x,z) = \overline{\mathcal{L}(x,z)}$. $\mathcal{L}(x,z)$ is a circle, so it is a manifold, in particular it is also a manifold with boundary that has an empty boundary. This case fulfills therefore Property (2). Now we want to conclude that $\partial\circ\partial = 0$. Since there is no broken trajectory from x to z, the trajectory spaces $\mathcal{L}(x,y)$ and $\mathcal{L}(y,z)$ are all empty for all y such that $\mu(x)-\mu(y) = 1$. This means that $n(x,y) = n(y,z) = 0$ for all such y. Hence $\partial\circ\partial = 0$.

Figure 6b shows the case where $\mathcal{L}(x,z)$ is a line. There are two broken trajectories, one via y and another one via $\tilde{y}$. Therefore $\overline{\mathcal{L}(x,z)}$ is a line with two boundary points on each site of the line. We have thus that $\mathcal{L}(x,z)$ is a manifold, $\overline{\mathcal{L}(x,z)}$ is a manifold with boundary $\partial\mathcal{L}(x,z) = \mathcal{L}(x,y)\times\mathcal{L}(y,z)\cup\mathcal{L}(x,\tilde{y})\times\mathcal{L}(\tilde{y},z)$. This case fulfills therefore Property (2). Now we want to conclude again that $\partial\circ\partial = 0$. The trajectory spaces $\mathcal{L}(x,y)$, $\mathcal{L}(y,z)$, $\mathcal{L}(x,\tilde{y})$ and $\mathcal{L}(\tilde{y},z)$ consist all of one trajectory, so $n(x,y) = n(y,z) = n(x,\tilde{y}) = n(\tilde{y},z) = 1 \mod 2$. Therefore, we get $n(x,y)n(y,z) + n(x,\tilde{y})n(\tilde{y},z) = 2 \mod 2 = 0 \mod 2$, so $\partial\circ\partial = 0$.

Now we are able to conclude from Property (2) that $\partial\circ\partial = 0$. The only two connected manifolds of dimension one with boundary are the circle and the line. For both cases we have shown that $\partial\circ\partial = 0$. A manifold with boundary is thus a disjoint union of circles and lines that all add a zero amount to $\partial\circ\partial$, so it is zero.



(a) Possible. (b) Possible. (c) Not possible. (d) Not possible.

Figure 6: Example spaces $\mathcal{M}(x,z)$ in grey with there broken trajectories. The thick line in the middle shows the corresponding trajectory spaces $\mathcal{L}(x,z)$.

Figure 6c shows the case where $\mathcal{L}(x,z)$ is again a line. The problem is here that there is only one broken trajectory, since the two end points of the line are collapsed on one point. In this case $\partial \circ \partial = z$, because $n(x,y)n(y,z) = 1 \mod 2$. Thus $\partial \circ \partial \neq 0$, even though $\mathcal{L}(x,z)$ is a manifold and its boundary is a trajectory. The crucial point is that $\overline{\mathcal{L}(x,z)}$ is not a manifold with boundary. It is not a manifold with boundary because of Item (2) of Theorem 6.5.

Figure 6d shows the case where $\mathcal{L}(x,z)$ consist of two lines. Again $\mathcal{L}(x,z)$ is a manifold and its boundary are trajectories. The problem is here again that $\overline{\mathcal{L}(x,z)}$ is not a manifold with boundary as in the case above, even though $\partial \circ \partial = 0$.

7 Floer homology equals the Morse homology

The Floer complex defined in the last Section induces the Floer homology for a compatible almost complex structure J and a regular Hamiltonian H. Denote this Floer homology by $HF(H, J)$. now we want to show that the definition of $HF(H, J)$ does not depend on the chosen pair (H, J). We call a pair (H, J) regular if for the chosen compatible almost complex structure J the Hamiltonian H is regular. Denote the space of such pairs by $(\mathcal{H} \times \mathcal{J})_{\mathrm{reg}}$. The invariance Theorem is the precise statement of the independence of the pair (H, J).

Theorem 7.1 (Invariance)**.** *The Floer homology $HF(H, J)$ does not depend on the chosen regular Hamiltonian H and the compatible almost complex structure J in $(\mathcal{H} \times \mathcal{J})_{\mathrm{reg}}$.*

When we know that the Floer homology $HF(M)$ does only depend on the choice of the compact symplectic manifold (M, ω), we can choose a specific pair $(H, J) \in (\mathcal{H} \times \mathcal{J})_{\mathrm{reg}}$ to show that the Floer homology HF is equal to the Morse homology $HM(M)$, up to a shift in the indices. This is the following Theorem.

Theorem 7.2. *The Floer homology $HF_*(M)$ and Morse homology $HM_{*+n}(M)$ coincide on every compact symplectic manifold (M, ω) with homotopy group $\pi_2(M) = 0$, here is n half the dimension of M.*

7.1 Invariance

This Subsection is devoted to the proof of Theorem 7.1. Let (H^a, J^a) and (H^b, J^b) be in $(\mathcal{H} \times \mathcal{J})_{\mathrm{reg}}$. Define a smooth homotopy $\Gamma = (H, J)$ that connects (H^a, J^a) to (H^b, J^b) by

$$H : \mathbb{R} \times S^1 \times M \longrightarrow \mathbb{R} \quad \text{and} \quad J : \mathbb{R} \longrightarrow \mathcal{J} \subset \mathrm{End}(TM),$$

such that for a constant R

$$\begin{cases} H(s, \cdot, \cdot) = H^a & \text{if } s \leq -R \\ H(s, \cdot, \cdot) = H^b & \text{if } s \geq R \end{cases} \quad \text{and} \quad \begin{cases} J(s) = J^a & \text{if } s \leq -R \\ J(s) = J^b & \text{if } s \geq R. \end{cases}$$

The existence of this homotopies follows from the fact that $\mathcal{J}$ is contractible, see Theorem 1.11.
Write $\Gamma_t(s) = (H_{s,t}, J_s)$, then define the Floer equation with parameters as

$$\frac{\partial u}{\partial s} + J_s(u)\frac{\partial u}{\partial t} + \mathrm{grad}_u(H_{s,t}) = 0.$$

Similar as for the definitions of the space $\mathcal{M}$ of solutions of the Floer equation with finite energy, we can also define them for the Floer equation with parameters.

Definition 7.3. For energy defined by Formula 3.6, define analogous to $\mathcal{M}$

$$\mathcal{M}^\Gamma = \{u : \mathbb{R} \times S^1 \longrightarrow M \mid u \text{ is a contractible solution with finite energy}\}.$$

And analogous to $\mathcal{M}(x, y)$

$\mathcal{M}^\Gamma(x, y) = \{u \in \mathcal{M}^\Gamma \mid u \text{ connects a critical point } x \text{ of } \mathcal{A}_{H^a} \text{ and a critical point } y \text{ of } \mathcal{A}_{H^b}\}.$

We have analogous to Theorem 3.12 that $\mathcal{M}^\Gamma = \bigcup_{x,y} \mathcal{M}^\Gamma(x, y)$. Furthermore, analogous to Theorem 5.1, there exists an $h \in \mathcal{C}^\infty_\varepsilon(H_0)$, such that for the homotopy $h + H$ the spaces $\mathcal{M}^\Gamma(x, y)$ are submanifolds. Such a homotopy is called a regular homotopy.

Definition 7.4. Let Γ a regular homotopy connecting (H^a, J^a) and (H^b, J^b), define for every index k the morphism

$$\begin{aligned} \Phi^\Gamma_k : C_k(H^a, J^a) &\longrightarrow C_k(H^b, J^b) \\ x &\longmapsto \sum_{\substack{y \in \operatorname{Crit} \mathcal{A}_{H^b} \\ \mu(y) = k}} n^\Gamma(x, y) y, \end{aligned}$$

where $n^\Gamma = \#\mathcal{M}^\Gamma(x, y)/\mathbb{R} \mod 2$.

This morphism is well defined if $\#\mathcal{M}^\Gamma(x, y)/\mathbb{R}$ is finite for $\mu(x) = \mu(y)$ and

$$\Phi^\Gamma \circ \partial_{(H^a, J^a)} = \partial_{(H^b, J^b)} \circ \Phi^\Gamma.$$

These two properties follow from analogies of Theorem 6.3 and Theorem 6.5. For all the analogies in this Subsection I refer to Subsection 11.1 of [AD14].
The proof of Theorem 7.1 follows from the next three Propositions, which I will not prove.

Proposition 7.5. *Let $(H^a, J^a) = (H^b, J^b) \in (\mathcal{H} \times \mathcal{J})_{\text{reg}}$ and $\Gamma = \operatorname{Id}$, then Φ^Γ_k is the identity for every k.*

Proof. For this proof I refer to Proposition 11.1.14. of [AD14]. □

Proposition 7.6. *The morphisms Φ^Γ_k are for every k independent of the choice of the regular homotopy Γ between (H^a, J^a) and (H^b, J^b).*

Proof. For this proof I refer to Proposition 11.2.8. of [AD14]. □

Proposition 7.7. *Let (H^a, J^a), (H^b, J^b) and $(H^c, J^c) \in (\mathcal{H} \times \mathcal{J})_{\text{reg}}$ and let Γ' a homotopy between (H^a, J^a) and (H^b, J^b) and let Γ'' a homotopy between (H^b, J^b) and (H^c, J^c). There exists a regular homotopy Γ between (H^a, J^a) and (H^c, J^c) such that Φ^Γ_k and $\Phi^{\Gamma''}_k \circ \Phi^{\Gamma'}_k$ induce the same homomorphism for every k.*

Proof. For this proof I refer to Proposition 11.2.9. of [AD14]. □

Proof of Theorem 7.1. Let (H^a, J^a) and $(H^b, J^b) \in (\mathcal{H} \times \mathcal{J})_{\text{reg}}$. Use the same notation as the last Proposition, where we choose $(H^c, J^c) = (H^a, J^a)$ and $\Gamma = \operatorname{Id}$. Then follows by the last three Propositions

$$\Phi^{\Gamma''}_k \circ \Phi^{\Gamma'}_k = \Phi^\Gamma_k = \Phi^{\operatorname{Id}}_k = \operatorname{Id}$$

for every k, which proofs the Theorem. □

7.2 From Floer to Morse

From the last Subsection follows that the Floer homology does not depend on the pair $(H, J) \in (\mathcal{H} \times \mathcal{J})_{\text{reg}}$, Theorem 7.1. This allows us to prove that the Floer homology equals the Morse homology up to a shift in indices, Theorem 7.2. This Subsection gives the idea of the proof of this Theorem.

Before we start with the proof, we need to fix some new notations to make a clear distinction between the two homologies. Denote the vector spaces of the Floer complex corresponding to the pair $(H, J) \in (\mathcal{H} \times \mathcal{J})_{\text{reg}}$ by $CF_*(H, J)$ and denote its differential by ∂F. For H a Morse function, *i.e.* H has no nondegenerate critical points, and X a pseudo-gradient field admitting the Smale condition, see Subsection 2.2.b of [AD14], denote its Morse complex by $CM_*(H, X)$ and its differential by ∂M.

Theorem 7.2 is a direct consequence of Theorem 7.1 and the following Theorem.

Theorem 7.8. *There exists a Morse function H, such that there exist a compatible almost complex structure J and a pseudo-gradient field X admitting the Smale condition such that $HF(H, J)$ and $HM(H, X)$ both exist, furthermore $CF_*(H, J) = CM_{*+n}(H, X)$ and $\partial M_* = \partial F_{*+n}$.*

Proof. Start with a time independent nondegenerate Hamiltonian H such that the norm of its Hessian is smaller then 2π. We have that

(1) $\text{Crit}(\mathcal{A}_H) = \text{Crit}(H)$, by Remark 1.18 and Theorem 3.2. In particular, all the critical points x of the action functional are constant.

(2) $\mu(x) = \text{Ind}_{\text{Morse}}(x) - n$, where $\mu(x)$ is the Maslov index and $\text{Ind}_{\text{Morse}}(x)$ the Morse index. The Morse index is defined as the amount of negative eigenvalues of the Hessian $\text{Hess}_x(H)$. Since we already computed the path needed in the first step to define the Maslov index in Proposition 1.17 and this path has the same shape as the path in Proposition 4.8, it follows indeed that $\mu(x) = \text{Ind}_{\text{Morse}}(x) - n$.

From these two points follows that $CF_*(H, J) = CM_{*+n}(H, X)$. Now we must still define differentials ∂M_* and ∂F_{*_n}, and show that they coincide. The trajectories of the Morse homology and the Floer homology are solutions of the equations

$$\frac{\partial u}{\partial s} + X(u) = 0 \quad \text{and} \quad \frac{\partial u}{\partial s} + J(u)\left(\frac{\partial u}{\partial t} + X_H(u)\right) = 0$$

respectively, where X is a pseudo-gradient field admitting the Smale condition and J a compatible almost complex structure. The trick is now to set $X = -JX_H$ and to deform the Hamiltonian such that the solutions of the Floer equation that make the differential are constant solutions. For $X = -JX_H$ and $\frac{\partial u}{\partial t} = 0$ both equations are the same, and will induce the same differential up to an n-shift in the indices.

Subsection 10.1 of [AD14] gives that there exists a compatible almost complex structure J such that $X = -JX_H$ is a pseudo-gradient field admitting the Smale condition. From

this will then follow that then automatically holds that $(H, J) \in (\mathcal{H} \times \mathcal{J})_{\text{reg}}$. Therefore, both differentials ∂M_* and ∂F_* can be defined.
This Subsection 10.1 gives also that if we replace H by $H_k = H/k$ for a large enough k, then $\frac{\partial u}{\partial t} = 0$ for all trajectories that connect two critical points with Maslov index difference smaller or equal then two. H_k has the same critical points as H, so H_k is the Hamiltonian we are looking for, since the definition of the Floer differential uses only Maslov index differences of one and two, see Theorem 6.1. $\square$

Part II
Braid Floer homology on surfaces

8 Basics of braids

We want to develop braid Floer homology on surfaces. This is done for the closed disk $\overline{D^2}$ in [BGVW15]. This Article is the inspiration to look for a definition of braid Floer homology on other surfaces in this Thesis. Possible surfaces to define braid Floer homology on, are all the orientable compact surfaces without boundary, since they are symplecic surfaces, see Corollary 1.6.
The Floer homology that is defined in Part I works only for aspherical spaces, *i.e* the second homotopy group must be zero. The sphere is obviously a surface that is not a-spherical, so I will not look for a definition of braid Floer homology on a sphere in this Thesis.
Denote the compact orientable surfaces by M_g, where g is its genus. For $g \in \mathbb{N}_{\geq 1}$ we have that $\pi_2(M_g) = 0$, because of Proposition 4.2 in [Hat01]. So except for the sphere, all the compact orientable surfaces are aspherical. Therefore, we can use the Floer homology of Part I on these spaces.
Before we start with stating what braid Floer homology is, we need some basics about braids on these spaces. The basics are explained in this Section.

8.1 Closed braids

Let M_g be a compact orientable surface with genus g, such that $g \in \mathbb{N}_{\geq 1}$. A closed braid consists of n strands in M_g such that the begin points of the strands matches the end points of the strands. A more precise definition is given below. We will only look at closed braids in this Thesis, so with braid we refer to a closed braid.

Definition 8.1 (Closed braids)**.** A closed braid with n strands is a continuous mapping

$$\mathrm{x}(t) = (x^1(t), \cdots, x^n(t)) : [0,1] \longrightarrow M_g \times \cdots \times M_g$$

with the following two properties:

(1) $x^k(t) \neq x^{k'}(t)$ for any pair $k \neq k'$ and for all $t \in [0,1]$. This property gives that strands of a braid will never intersect each other.

(2) There exists a permutation σ in the symmetric group S_n, such that $x^k(1) = x^{\sigma(k)}(0)$ for all $k = 1, \cdots, n$. This property gives the closeness of a braid.

The space of all closed braids with n strands of the space M_g is denoted by $\Gamma F_n(M_g)$, if there is no chance of confusion I denote it by ΓF_n. For this space we use the strong metric topology of $C^0([0,1]; M_g \times \cdots \times M_g)$.

Definition 8.2 (Braid class)**.** A braid class is a path connected component of ΓF_n. The braid class of a closed braid x is denoted by $[\mathrm{x}]$.

Remark 8.3. The right action of the symmetric group on a braid is defined as

$$\mathrm{x} \cdot \sigma(1, \cdots, n) = (x^{\sigma(1)}, \cdots, x^{\sigma(n)}).$$

Note that all braids x' in a braid class $[\mathrm{x}]$ have the same unique permutation σ such that $\mathrm{x}'(0) \cdot \sigma(1, \cdots, n) = \mathrm{x}'(1)$. Therefore, the C^0-closure of the braid class $[\mathrm{x}]$ consists of mappings that satisfy Property (2) of Definition 8.1. The closure of $[\mathrm{x}]$ is denoted by $\mathrm{cl}[\mathrm{x}]$.

Remark 8.4. The union of a braid with n strands and a braid with m strands is not always a braid with $n+m$ strands. It is not guaranteed that the union of the $n+m$ strands has Property (1) of Definition 8.1. However, we can define the map

$$\Gamma F_n \times \Gamma F_m \longrightarrow \mathrm{cl}(\Gamma F_{n+m}), \quad (\mathrm{x}, \mathrm{y}) \longmapsto \mathrm{x} \;\mathrm{rel}\; \mathrm{y} = (x^1, \cdots, x^n, y^1, \cdots, y^m).$$

Definition 8.5. The space $\Gamma F_{n,m} = \{\mathrm{x} \;\mathrm{rel}\; \mathrm{y} \in \Gamma F_{n+m}\}$ is the space of relative closed braids. The braids x and y are called respectively the free braid and the skeleton braid. Furthermore, the path connected component of $\mathrm{x} \;\mathrm{rel}\; \mathrm{y}$ in $\Gamma F_{n,m}$ is denoted by $[\mathrm{x} \;\mathrm{rel}\; \mathrm{y}]$. Then for a given skeleton $\mathrm{y} \in \Gamma F_m$, the relative closed braid class fiber $[\mathrm{x}] \;\mathrm{rel}\; \mathrm{y}$ is defined as

$$[\mathrm{x}] \;\mathrm{rel}\; \mathrm{y} = \{\mathrm{x}' \in \Gamma F_n \mid \mathrm{x}' \;\mathrm{rel}\; \mathrm{y} \in [\mathrm{x} \;\mathrm{rel}\; \mathrm{y}]\}.$$

8.2 Proper braids

Definition 8.6 (Singular braids)**.** The elements in the boundary

$$\begin{aligned}&\Sigma[\mathrm{x}] = \mathrm{cl}([\mathrm{x}]) \backslash [\mathrm{x}] = \\ &\qquad \{\mathrm{x} \in \mathrm{cl}([\mathrm{x}]) \mid x^k(t_0) = x^{k'}(t_0), \text{ for at least one pair } k \neq k' \text{ and a time } t_0 \in [0,1]\}\end{aligned}$$

are called the singular braids. Collapsed singular braids are the singular braids such that at least two strands are collapsed on each other. The space of collapsed singular braids is

$$\Sigma^-[\mathrm{x}] = \{\mathrm{x} \in \Sigma[\mathrm{x}] \mid x^k(t) = x^{k'}(t), \text{ for at least one pair } k \neq k' \text{ and all time } t \in [0,1]\}.$$

Properly singular braids are braids such that there are two strands intersecting each other, but those two strands are not collapsed. The space of properly singular braids is

$$\begin{aligned}\Sigma^+[\mathrm{x}] = \{\mathrm{x} \in \Sigma[\mathrm{x}] \mid\; & x^k(t_0) = x^{k'}(t_0) \text{ and } x^k(t_1) \neq x^{k'}(t_1), \\ & \text{for at least one pair } k \neq k' \text{ and two distinct times } t_0, t_1 \in [0,1]\}.\end{aligned}$$

Note that a properly singular braid may contain collapsed strands, so the intersection between the spaces $\Sigma^-[\mathrm{x}]$ and $\Sigma^+[\mathrm{x}]$ is in general nonempty.

Definition 8.7. (Proper) A relative braid class $[\mathrm{X} \text{ rel } \mathrm{Y}]$ is defined to be proper if for all braid class fibers $[\mathrm{X}] \text{ rel } \mathrm{Y}$ we have

$$\mathrm{cl}([\mathrm{X}] \text{ rel } \mathrm{Y}) \cap \Sigma[\mathrm{X} \text{ rel } \mathrm{Y}] \subset \Sigma^{+}[\mathrm{X} \text{ rel } \mathrm{Y}],$$

otherwise $[\mathrm{X} \text{ rel } \mathrm{Y}]$ is called improper.

Remark 8.8. The above definition of a proper relative braid class $[\mathrm{X} \text{ rel } \mathrm{Y}]$ is analogous to Definition 3.2 of [BGVW15], but there is a difference. Note that M_g has no boundary, but $\overline{D^2}$ has the circle as boundary. Therefore, Definition 3.2 of [BGVW15] needs an extra boundary assumption. Namely the acylindrical assumption, *i.e.* the free strands in X cannot collapse on the boundary.

Example 8.9. Take a free braid X that consists of one strand and pick a skeleton braid Y, in such a way that $[\mathrm{X} \text{ rel } \mathrm{Y}]$ is proper. In this case there is no possibility that two different free strands may intersect on the boundary $\Sigma[\mathrm{X} \text{ rel } \mathrm{Y}]$, since there is just only one free strand. Therefore, $\Sigma[\mathrm{X} \text{ rel } \mathrm{Y}]$ is proper if the braid strand X does not collapse on a strand of the skeleton Y. For this simplified case there are two easy examples of proper braid classes $[\mathrm{X} \text{ rel } \mathrm{Y}]$.
For the first example, take a skeleton braid $\mathrm{Y} \in \Gamma F_m$ in such a way that $y^l(1) \neq y^l(0)$ for all $1 \leq l \leq m$. This is the same as saying that the permutation $\sigma(l)$ is non trivial for all $1 \leq l \leq m$, since $y^l(1) = y^{\sigma(l)}(0)$. Now holds for any free braid $\mathrm{X} \in \Gamma F_1$ such that $\mathrm{X} \text{ rel } \mathrm{Y}$ is a braid, that $[\mathrm{X} \text{ rel } \mathrm{Y}]$ is proper. This holds, because $x_1(1) = x_1(0)$ and $y^l(1) \neq y^l(0)$ for all $1 \leq l \leq m$, therefore the free strand x_1 of $\mathrm{X} = (x_1)$ cannot collapse on a strand of the skeleton Y.
For the other example, take a skeleton braid Y in such a way that all the strands of the skeleton have a nontrivial homotopy in the space M_g. Pick then a free braid $\mathrm{X} \in \Gamma F_1$ with trivial homotopy in M_g such that $\mathrm{X} \text{ rel } \mathrm{Y}$ is a braid. We have again that X cannot collapse on a strand of the skeleton Y, so $[\mathrm{X} \text{ rel } \mathrm{Y}]$ is proper.

8.3 Braids as solutions of the Hamilton equation

This Subsection is about the first link between braids and Floer homology.

Definition 8.10 (Critical, Stationary)**.** Let H be a Hamiltonian on M_g. A closed braid $\mathrm{X} \in \Gamma F_n(M_g)$ satisfies the Hamilton equation if its strands x^k satisfy the Hamilton equation

$$\dot{x}^k(t) = X_H(t, x^k(t))$$

for all $k = 1, \cdots, n$ and if the boundary conditions $x^k(t+1) = x^{\sigma(k)}(t)$ are satisfied for all t. Such a braid is called critical or stationary.

Remark 8.11. The terms critical and stationary are used in different situations, even though they mean the same. Look at a relative braid $\mathrm{X} \text{ rel } \mathrm{Y}$. If the skeleton Y satisfied the Hamiltonian equation, we say that Y is a stationary skeleton. And if both the free braid X and the skeleton Y satisfies the Hamiltonian equation, we say that $\mathrm{X}\,\mathrm{rel}\,\mathrm{Y}$ is a critical relative braid.

Definition 8.12. Fix a stationary skeleton Y and a fiber $[\mathrm{X}]\,\mathrm{rel}\,\mathrm{Y}$. The set of all free braids $\mathrm{X}' \in [\mathrm{X}]\,\mathrm{rel}\,\mathrm{Y}$ such that the relative braids $\mathrm{X}'\,\mathrm{rel}\,\mathrm{Y}$ are critical is denoted by $\mathrm{Crit}_H([\mathrm{X}]\,\mathrm{rel}\,\mathrm{Y})$.

Definition 8.13. Let $\mathrm{U} = (u_1, \dots, u_n) : \mathbb{R} \times [0,1] \to M_g^n$ be a smooth map with the periodicity condition

$$u_k(s,1) = u_{\sigma(k)}(s,0) \quad \text{for all } s, \text{ for all } k = 1, \dots, n \text{ and for some permutation } \sigma \in S_n.$$

Define analogous to Formula 3.6 its energy as

$$E(\mathrm{U}) = \sum_{k=1}^{n} \int_{\mathbb{R}\times[0,1]} \left| \frac{\partial u_k}{\partial s} \right|^2 ds\, dt.$$

Remark 8.14. Let U as in the above Definition. Note that $\mathrm{U}(s,\cdot) \in \mathrm{cl}(\mathcal{L}F_n(M_g^n))$ for all $s \in \mathbb{R}$, where $\mathcal{L}F_n(M_g^n)$ denotes the loop space of n-braids on M_g. Also note that the space $\mathrm{U}(\mathbb{R}\times[0,1])$ is a union of cylindrical shapes. Every cycle of the permutation gives a cylindrical shape with the length in the s direction and its circle shapes in the t direction. The time t it takes to go around a circle shape is equal to the length of the corresponding cycle. See figure 2 for the case of a braid with one strand.

Definition 8.15. Define analogous to $\mathcal{M}$ the space $\mathcal{M}(\mathrm{cl}(\mathcal{L}F_n))$. Let U as above. Denote the space of such U with finite energy and contractible cylindrical shapes such that u_k is a solution of the Floer equation for every $k = 1, \dots, n$ by $\mathcal{M}(\mathrm{cl}(\mathcal{L}F_n))$.

Instead of defining a space analogous to $\mathcal{M}$ for a map U such that $\mathrm{U}(s,\cdot)$ is a braid for every $s \in \mathbb{R}$, we can also define it such that $\mathrm{U}(s,\cdot)$ is in a relative braid class fiber.

Definition 8.16. For a stationary skeleton Y and a free braid X such that $[\mathrm{X}]\,\mathrm{rel}\,\mathrm{Y}$ is a relative braid class fiber, define the space

$$\mathcal{M}([\mathrm{X}]\,\mathrm{rel}\,\mathrm{Y}) = \{\mathrm{U} \in \mathcal{M}(\mathrm{cl}(\mathcal{L}F_n)) \mid \mathrm{U}(s,\cdot) \in [\mathrm{X}]\,\mathrm{rel}\,\mathrm{Y} \text{ for all } s \in \mathbb{R}\}.$$

The last definition in this Subsection is analogous to the definition of $\mathcal{M}(x,y)$. We do this for a relative braid class fiber.

Definition 8.17. Let again Y a stationary skeleton and X a free braid such that $\mathrm{X}\,\mathrm{rel}\,\mathrm{Y}$ is a relative braid, define

$$\mathcal{M}(\mathrm{U}_-, \mathrm{U}_+, [\mathrm{X}]\,\mathrm{rel}\,\mathrm{Y}) = \{\mathrm{U} \in \mathcal{M}(\mathrm{cl}(\mathcal{L}F_n)) | \lim_{s\to\pm\infty} \mathrm{U}(s,\cdot) = \mathrm{U}_\pm \in [\mathrm{X}]\,\mathrm{rel}\,\mathrm{Y}\}$$

Remark 8.18. Note that in the last Definition we did not assume that U lies in $\mathcal{M}([\mathrm{X}]\,\mathrm{rel}\,\mathrm{Y})$ but that it lies in $\mathcal{M}(\mathrm{cl}(\mathcal{L}F_n))$ to reach a limit in $\mathcal{M}([\mathrm{X}]\,\mathrm{rel}\,\mathrm{Y})$.

9 Braid Floer homology

The Arnold Conjecture tells us, that if the Morse homology is nontrivial, then for every nondegenerate Hamiltonian vector field on a compact symplectic manifold there exists at least one 1-period solutions. The next question to ask is, if we know the existence of solutions, does they force new solutions? This is the main motivation of developing braid Floer homology.
Look at a relative braid class fiber $[\mathrm{x}]\,\mathrm{rel}\,\mathrm{y}$ on M_g, for $g \in \mathbb{N}_{\geq 1}$. For such a fiber we want to define the braid Floer homology. The skeleton y are the known solutions, and we want to define for the free braid x the braid Floer homology. Then again, as in the Arnold Conjecture, if the braid Floer homology is nonempty, then there is a free braid x' in the relative braid class fiber $[\mathrm{x}]\,\mathrm{rel}\,\mathrm{y}$ that is a solution. Before we start with defining the braid Floer homology we need three assumptions.
The first assumption we need to define the braid Floer homology is that $[\mathrm{x}]\,\mathrm{rel}\,\mathrm{y}$ lies in a proper relative braid class $[\mathrm{x}\,\mathrm{rel}\,\mathrm{y}]$. This is just as in the case of the closed disk, see [BGVW15]. This properness assumption gives an isolation property, that allows us to define braid Floer homology. Isolation is already used by Floer in [Flo89] to define an earlier version of Floer homology.
The next assumption is that the free braid x can only consists of strands that forms contractible loops in M_g. This is not an issue on the disk, since there are all the loops contractible. It is an issue on M_g, since the Floer homology defined in Part I works only for contractible loops, see the definition of the action functional, Definition 3.1.
The last assumption is that the free braid x consists of only one strand. This last assumption is probably not needed, since this assumption is not needed in the case of a disk, but makes everything a lot easier.
To define the braid Floer homology for this case, we will use the Floer homology from Part I. A crucial Principle to be able to use Floer homology to define braid Floer homology is the Monotonicity Principle.

Monotonicity Principle. *Let* $\mathrm{u} \in \mathcal{M}(\mathrm{cl}(\mathcal{L}F_n))$ *and* $\mathrm{y} \in \Gamma F_m$*. Let* Cross *a braid invariant. If for an* $s_0 \in \mathbb{R}$ *we have that* $\mathrm{u}(s_0, \cdot)\,\mathrm{rel}\,\mathrm{y} \in \Sigma$*, then either* $\mathrm{u}(s_0, \cdot)\,\mathrm{rel}\,\mathrm{y} \in \Sigma^-$ *or there exists an* $\varepsilon_0 > 0$ *such that* $\mathrm{u}(s_0 \pm \varepsilon, \cdot)\,\mathrm{rel}\,\mathrm{y} \in \mathcal{L}F_n$ *for all* $0 < \varepsilon < \varepsilon_0$ *and*

$$\mathrm{Cross}(\mathrm{u}(s_0 - \varepsilon, \cdot)) > \mathrm{Cross}(\mathrm{u}(s_0 + \varepsilon, \cdot)).$$

The problem of this Principle is that it needs a braid invariant. This braid invariant is the braid crossing number in the case of a closed disk, that is why I used Cross to denote a braid invariant in the Principle.
The definition of the braid Floer homology will be proven for the case $[\mathrm{x}\,\mathrm{rel}\,\mathrm{y}]$ is proper and x consists of one strand, except for the Monotonicity Principle in my Thesis. I will only show the Monotonicity Principle with two extra restrictive assumptions. The first assumption is that $g = 2$, so we are looking only on a torus $M_g = \mathbb{T}^2$. The second assumption is that all the solutions in the skeleton y are contractible in $\mathbb{T}^2$.

The Monotonicity Principle will be proven for these restrictive assumptions in the next Section, the rest of the definition of the Floer homology will be shown for the less restrictive assumptions in this Section.

9.1 Floer solutions between critical braids

From this moment on, we will look at free braid $\mathrm{x} \in \Gamma F_1(M_g)$ with only one contractible strand, to define braid Floer homology for a relative fiber $[\mathrm{x}] \operatorname{rel} \mathrm{y}$ of a proper braid class $[\mathrm{x} \operatorname{rel} \mathrm{y}]$. Before we can define the braid Floer homology, we need to take a closer look at the spaces $\mathcal{M}(\mathrm{u}_-, \mathrm{u}_+, [\mathrm{x}] \operatorname{rel} \mathrm{y})$, for $\mathrm{u}_-, \mathrm{u}_+ \in \operatorname{Crit}_H([\mathrm{x}] \operatorname{rel} \mathrm{y})$. I will use the Monotonicity Principle without proving it.

Lemma 9.1. *Let $[\mathrm{x} \operatorname{rel} \mathrm{y}]$ be a relative braid class where y is a stationary skeleton and x is a braid that consists only of one contractible strand, then the set $\operatorname{Crit}_H([\mathrm{x}] \operatorname{rel} \mathrm{y})$ is finite.*

Proof. For $\mathrm{x}' = (x_1') \in \operatorname{Crit}_H([\mathrm{x}] \operatorname{rel} \mathrm{y})$, note $x_1' \in \operatorname{Crit} \mathcal{A}_H$, so $\operatorname{Crit}_H([\mathrm{x}] \operatorname{rel} \mathrm{y}) \subset \operatorname{Crit} \mathcal{A}_H$. Since Lemma 3.17 gives that $\operatorname{Crit} \mathcal{A}_H$ is finite, it follows that $\operatorname{Crit}_H([\mathrm{x}] \operatorname{rel} \mathrm{y})$ is finite. $\square$

The next Theorem is analogous to the Gromov compactness of $\mathcal{M}$, see Theorem 3.14. We do not actually need this Theorem to define the braid Floer homology, since we already have the Gromov compactness Theorem. This Theorem is also analogues to Proposition 6.2 for the case of a disk in [BGVW15]. Since they wrote a self containing braid Floer homology, that does not use the normal Floer homology, they needed this Theorem to define the braid Floer homology on a disk. In particular, it shows an isolation property in the case of a disk. Therefore, I will state and prove the next Theorem, since it is also insightful of how the properness assumption brings isolation, which will be Theorem 9.3.

Theorem 9.2. *Let $[\mathrm{x} \operatorname{rel} \mathrm{y}]$ a proper relative braid class, where x consists of one contractible strand. For any fiber $[\mathrm{x}] \operatorname{rel} \mathrm{y}$ with y a stationary skeleton the set $\mathcal{M}([\mathrm{x}] \operatorname{rel} \mathrm{y})$ is compact.*

Proof. Since x consists of one closed strand and $\mathcal{M}(\operatorname{cl}(\mathcal{L}F_1)) = \mathcal{M}$, it follows that $\mathcal{M}([\mathrm{x}] \operatorname{rel} \mathrm{y}) \subset \mathcal{M}$. $\mathcal{M}$ is compact, so for a sequence $\{\mathrm{u}_i\} \subset \mathcal{M}([\mathrm{x}] \operatorname{rel} \mathrm{y})$ there exists a subsequence that converges to a limit u in $\mathcal{M}$. To prove that $\mathcal{M}([\mathrm{x}] \operatorname{rel} \mathrm{y})$ is compact, we must show that $\mathrm{u} \in \mathcal{M}([\mathrm{x}] \operatorname{rel} \mathrm{y})$. This means that we must show $\mathrm{u}(s, \cdot) \operatorname{rel} \mathrm{y} \in [\mathrm{x}] \operatorname{rel} \mathrm{y}$ for all $s \in \mathbb{R}$.
Before we show this, note that the definition of proper for a relative braid class $[\mathrm{x} \operatorname{rel} \mathrm{y}]$ where the free braid consists of one strand can replaced be by

$$\operatorname{cl}([\mathrm{x}] \operatorname{rel} \mathrm{y}) \cap \Sigma[\mathrm{x} \operatorname{rel} \mathrm{y}] \subset \Sigma[\mathrm{x} \operatorname{rel} \mathrm{y}] \setminus \Sigma^-[\mathrm{x} \operatorname{rel} \mathrm{y}],$$

see Example 8.9.
Now we will show that $\mathrm{u}(s, \cdot) \operatorname{rel} \mathrm{y} \in [\mathrm{x}] \operatorname{rel} \mathrm{y}$ for all $s \in \mathbb{R}$ to conclude the proof. For

$\text{U} = (u)$, if $u(s_0, t_0) \notin [\text{X}]\text{rel}\,\text{Y}$ for some $s_0 \in \mathbb{R}$ and $t_0 \in [0,1]$, then $u(s_0, t_0)\text{rel}\,\text{Y} \in \Sigma[\text{X}\,\text{rel}\,\text{Y}]$. $u(s_0, t_0)$ rel Y $\in \Sigma[\text{X rel Y}]$ means that $u(s_0, t_0) = y_k(t_0)$ for a strand y_k of the skeleton Y. Then by the Monotonicity Principle, either U rel Y $\in \Sigma[\text{X rel Y}]^- = \Sigma[\text{X rel Y}] \setminus \Sigma^+[\text{X rel Y}]$ or there exists an $\varepsilon_0 > 0$ such that $\text{U}(s_0 \pm \varepsilon, \cdot) \in \mathcal{L}F_n$ rel Y and $\text{Cross}(\text{U}(s_0 - \varepsilon, \cdot)) > \text{Cross}(\text{U}(s_0 + \varepsilon, \cdot))$, for all $0 < \varepsilon < \varepsilon_0$. The first case contradicts with the fact that [X rel Y] is a proper class and the second case contradicts with the fact that Cross is a braid class invariant, so $\text{Cross}(\text{U}(s_0 - \varepsilon, \cdot)) = \text{Cross}(\text{U}(s_0 + \varepsilon, \cdot))$. Therefore, $u(s_0, t_0) \in$ [X] rel Y for all $s_0 \in \mathbb{R}$ and all $t_0 \in [0,1]$, hence $\mathcal{M}([\text{X}]\text{ rel Y})$ is compact. □

The following Theorem gives an isolation property. It is analogous to the statement $\mathcal{M} = \cup_{x,y \in \text{Crit}(\mathcal{A}_H)} \mathcal{M}(x,y)$, see Corollary 3.13.

Theorem 9.3 (Isolation property)**.** *Let* [X rel Y] *be a proper relative braid class, with* X *be a free braid that consists of one strand. Then we have for every braid class fiber of this relative braid class that*

$$\mathcal{M}([\text{X}]\text{ rel Y}) = \bigcup_{\substack{\text{U}_-,\text{U}_+ \in \\ \text{Crit}_H([\text{X}]\text{rel}\text{Y})}} \mathcal{M}(\text{U}_-, \text{U}_+, [\text{X}]\text{ rel Y}).$$

Proof. Start with the proof of the inclusion

$$\mathcal{M}([\text{X}]\text{ rel Y}) \subset \bigcup_{\substack{\text{U}_-,\text{U}_+ \in \\ \text{Crit}_H([\text{X}]\text{rel}\text{Y})}} \mathcal{M}(\text{U}_-, \text{U}_+, [\text{X}]\text{ rel Y}). \tag{9.1}$$

We have that $\mathcal{M}([\text{X}]\text{ rel Y}) \subset \mathcal{M} = \cup_{x,y \in \text{Crit}(\mathcal{A}_H)} \mathcal{M}(x,y)$, so for $\text{U} \in \mathcal{M}([\text{X}]\text{ rel Y})$ there exist $\text{U}_-, \text{U}_+ \in \text{Crit}(\mathcal{A}_H)$ such that $\text{U} \in \mathcal{M}(\text{U}_-, \text{U}_+)$. If we now show that $\text{U}_-, \text{U}_+ \in [\text{X}]\text{rel}\text{Y}$, then will follow that $\text{U}_-, \text{U}_+ \in \text{Crit}_H([\text{X}]\text{ rel Y})$, hence the inclusion holds.
To show that $\text{U}_-, \text{U}_+ \in$ [X] rel Y, note first that $\text{U}_\pm = \lim_{s \to \pm\infty} \text{U}(s, \cdot)$. We also know that $\text{U}(s, \cdot) \in [x]$ rel Y for all $s \in \mathbb{R}$ by definition of $\mathcal{M}([\text{X}]\text{ rel Y})$, so $\text{U}_-, \text{U}_+ \in \text{cl}([\text{X}]\text{ rel Y})$. If we assume that U_- rel Y or U_+ rel Y lies on the boundary $\Sigma[\text{X rel Y}]$, we want to find a contradiction with our properness assumption. For U_- rel Y or U_+ rel Y on the boundary $\Sigma[\text{X rel Y}]$ means that the strand of U_+ or U_- intersects a stand of the skeleton Y. Both are solutions of the Hamiltonian equation, hence by uniqueness they are the same. This implies that U_- or U_+ is collapsed on a strand of the skeleton. Now we obtain a contradiction with the properness assumption.
The other inclusion

$$\mathcal{M}([\text{X}]\text{ rel Y}) \supset \bigcup_{\substack{\text{U}_-,\text{U}_+ \in \\ \text{Crit}_H([\text{X}]\text{rel}\text{Y})}} \mathcal{M}(\text{U}_-, \text{U}_+, [\text{X}]\text{ rel Y}) \tag{9.2}$$

will be proven in the same way as Corollary 6.3 of [BGVW15]. Let $\text{U} \in \mathcal{M}(\text{U}_-, \text{U}_+, [\text{X}]\text{rel}\text{Y})$ for some $\text{U}_-, \text{U}_+ \in \text{Crit}_H([\text{X}]\text{ rel Y}])$. To show the inclusion, we must show that $\text{U}(s, \cdot) \in$ [X] rel Y for all $s \in \mathbb{R}$. First note [X] rel Y is open, so by continuity of U there exists an

$S > 0$ such that $\text{u}(s,\cdot) \in [\text{x}] \text{ rel } \text{y}$ for all $|s| > S$. Let Cross be the braid class invariant mentioned in the Monotonicity Principle, then we have $\text{Cross}(\text{u}(s,\cdot)) = \text{Cross}(\text{u}_\pm)$ for all $|s| > S$.
If we assume that there exists an s_0 such that $\text{u}(s_0,\cdot) \notin [\text{x}] \text{ rel } \text{y}$, then by continuity of u there exists an s_* such that $\text{u}(s_*,\cdot) \text{ rel } \text{y} \in \Sigma[\text{x} \text{ rel } \text{y}]$. By the Monotonicity Principle follows that $\text{u}(s_*,\cdot) \text{ rel } \text{y} \in \Sigma^-[\text{x} \text{ rel } \text{y}]$ or

$$\text{Cross}(\text{u}_-) \geq \text{Cross}(\text{u}(s_* - \varepsilon,\cdot)) > \text{Cross}(\text{u}(s_* + \varepsilon,\cdot)) \geq \text{Cross}(\text{u}_+).$$

Both possibilities give a contradiction, the first one with $\text{u}(s,\cdot) \in [\text{x}] \text{ rel } \text{y}$ for all $|s| > S$ and the second one with the fact that $\text{Cross}(\text{u}_-) = \text{Cross}(\text{u}_+)$. $\square$

Corollary 9.4. *Let* $[\text{x}] \text{ rel } \text{y}$ *be the fiber class as in the Theorem before, then for* $\text{u}_-\text{u}_+ \in \text{Crit}([\text{x}] \text{ rel } \text{y})$ *we have*

$$\mathcal{M}(\text{u}_-, \text{u}_+, [\text{x}] \text{ rel } \text{y}) = \mathcal{M}(\text{u}_-, \text{u}_+).$$

9.2 Definition braid Floer homology

Now, we have all the tools to define the braid Floer homology for a fiber class $[\text{x}] \text{ rel } \text{y}$ of a proper relative braid class $[\text{x} \text{ rel } \text{y}]$, where the free braid x consists only of one contractible strand. I will define it without proving the Monotonicity Principle. This homology will depend on the chosen regular Hamiltonian H, see Theorem 5.1, and the chosen compatible almost complex structure J and the chosen braid fiber class $[\text{x}] \text{ rel } \text{y}$.
Recall the trajectory spaces $\mathcal{L}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$, for $x, y \in \text{Crit}(\mathcal{A}_H)$. Since we have the equality $\mathcal{M}(\text{u}_-, \text{u}_+, [\text{x}] \text{ rel } \text{y}) = \mathcal{M}(\text{u}_-, \text{u}_+)$ by Corollary 9.4, we have $\mathcal{L}(\text{u}_-, \text{u}_+) = \mathcal{M}(\text{u}_-, \text{u}_+)/\mathbb{R}$ for all $\text{u}_-, \text{u}_+ \in \text{Crit}([\text{x}] \text{rel} \text{y})$. This implies that if $\mu(\text{u}_-) - \mu(\text{u}_+) = 1$, then $\mathcal{L}(\text{u}_-, \text{u}_+)$ is a compact submanifold of dimension 0, see Property (1) of Theorem 6.1.
Recall the definition $n(\text{u}_-, \text{u}_+) = \#\mathcal{L}(\text{u}_-, \text{u}_+) \mod 2$ and that $\text{Crit}_H([\text{x}] \text{ rel } \text{y})$ is finite, see Lemma 9.1. Now we are able to define the differential of the braid Floer homology for every k

$$\partial_k : C_k(H, [\text{x}] \text{ rel } \text{y}) \longrightarrow C_{k-1}(H, [\text{x}] \text{ rel } \text{y}), \quad \partial_k(\text{x}) = \sum_{\substack{\text{x}' \in \text{Crit}([\text{x}]\text{rel}\text{y}) \\ \mu(\text{x}') = k-1}} n(\text{x}, \text{x}')\text{x}',$$

where $C_k(H, [\text{x}] \text{ rel } \text{y})$ denotes the vector space over $\mathbb{Z}/2$ generated by the critical points in $\text{Crit}_H([\text{x}] \text{ rel } \text{y})$ with Maslov index k. I use the same notation for this differential as the differential in Floer homology, because it is just a restriction. The complex induced by this new differential is therefore a chain subcomplex of the Floer complex.
Now we still need to show that $\partial \circ \partial = 0$, to conclude that ∂ is a differential. To show this, we want to use the results that are obtained in Section 6. If we can show that Theorem 6.3 holds also in this case, then we can directly use Theorem 6.5 and a property analogous to Property (2) of Theorem 6.1 to conclude $\partial \circ \partial = 0$.

Actually, to define ∂ we already used implicitly Theorem 6.3, since Property (1) of Theorem 6.1 is a consequence of it. This consequence follows from the case in Theorem 6.3 where $\mu(\text{U}_-) - \mu(\text{U}_+) = 1$, but this gives no trouble yet. Theorem 6.3 does not hold any more if $\mu(\text{U}_-) - \mu(\text{U}_+) > 1$. Then we are looking at all the critical points of the action functional $\mathrm{Crit}(\mathcal{A}_H)$, but that is not allowed, since we are now only working with the subset $\mathrm{Crit}_H([\text{X}] \,\mathrm{rel}\, \text{Y}) \subset \mathrm{Crit}(\mathcal{A}_H)$. The next Theorem will replace Theorem 6.3 to show that the space of broken trajectories between U_- and U_+ viewed in $\mathrm{Crit}_H([\text{X}] \,\mathrm{rel}\, \text{Y})$ is the same as the space of broken trajectories $\overline{\mathcal{L}(\text{U}_-, \text{U}_+)}$ where U_- and U_+ are viewed as critical points of the action functional.

Theorem 9.5. *Let* $\text{U}_-, \text{U}_+ \in \mathrm{Crit}_H([\text{X}]\mathrm{rel}\text{Y})$ *for a proper relative braid class* $[\text{X}\mathrm{rel}\text{Y}]$*. Then for every sequence* (V_n) *in* $\mathcal{M}(\text{U}_-, \text{U}_+)$*, there exists a subsequence of* (V_n)*, also denoted by* (V_n) *and critical points* $\text{U}_- = \text{U}_0, \text{U}_1, \ldots, \text{U}_l, \text{U}_{l+1} = \text{U}_+$ *in* $\mathrm{Crit}_H([\text{X}] \,\mathrm{rel}\, \text{Y})$ *such that*

$$\lim_{n\to\infty} \widetilde{\text{V}}_n \in \mathcal{L}(\text{U}_0, \text{U}_1) \times \mathcal{L}(\text{U}_1, \text{U}_2) \times \cdots \times \mathcal{L}(\text{U}_l, \text{U}_{l+1}),\ \textit{where}\ \widetilde{\text{V}}_n = \pi \circ \text{V}_n \in \mathcal{L}(\text{U}_-, \text{U}_+),$$

i.e. there exist sequences (s_n^k) *for* $0 \leq k \leq l$ *and* $\text{V}^k \in \mathcal{M}(\text{U}_k, \text{U}_{k+1})$ *such that for all* $k = 0, \ldots, l$ *we have*

$$\lim_{n\to\infty} \text{V}_n \cdot s_n^k = \text{V}^k.$$

Moreover, the limits are unique.

Proof. We already have Theorem 6.3, so we only need to show that the critical points $\text{U}_1, \ldots, \text{U}_l$ of the action functional lie in $\mathrm{Crit}_H([\text{X}] \,\mathrm{rel}\, \text{Y})$.
To exclude that $\text{U}_i \in [\text{X}] \,\mathrm{rel}\, \text{Y}$ with $1 \leq i \leq l$, assume $\text{U}_i \notin [\text{X}] \,\mathrm{rel}\, \text{Y}$. Divide the problem in two cases. The first case is $\text{U}_i \,\mathrm{rel}\, \text{Y} \notin \Sigma$ and the second is $\text{U}_i \,\mathrm{rel}\, \text{Y} \in \Sigma$.
If $\text{U}_i \,\mathrm{rel}\, \text{Y} \notin [\text{X}] \,\mathrm{rel}\, \text{Y}$ and $\text{U}_i \,\mathrm{rel}\, \text{Y} \notin \Sigma$, then for a big enough n, there exists an $s \in \mathbb{R}$ such that $\text{V}_n(s, \cdot)$ is close enough to U_i to conclude that $\text{V}_n(s, \cdot) \notin [\text{X}] \,\mathrm{rel}\, \text{Y}$. This contradicts Inclusion 9.2, this finishes the first case.
If $\text{U}_i \,\mathrm{rel}\, \text{Y} \in \Sigma$, then the strand of U_i intersects a strand of the skeleton Y. Both are solutions of the Hamiltonian equation, hence by uniqueness they are the same. This implies that U_i is collapsed on a strand of the skeleton. Now we obtain a contradiction with the properness assumption. □

Theorem 9.6. $\partial \circ \partial = 0$ *holds for a proper relative braid class fiber* $[\text{X}] \,\mathrm{rel}\, \text{Y}$.

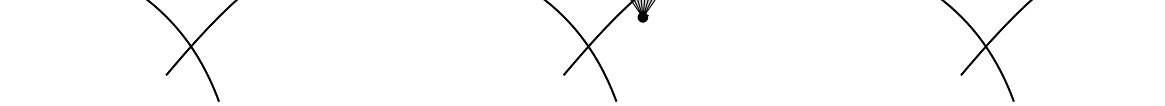
Proof. The last Theorem shows that the broken trajectory spaces in the braid Floer homology are equal to broken trajectory spaces in the Floer homology case. Therefore, we can use the gluing Theorem 6.5 to conclude Property (2) of Theorem 6.1 for only picking all $y \in \mathrm{Crit}_H([\text{X}] \,\mathrm{rel}\, \text{Y})$ instead of picking y from all the critical points of the action functional, *i.e.* we conclude:

Let $\text{U}_-, \text{U}_+ \in \text{Crit}_H([\text{X}] \,\text{rel}\, \text{Y})$ such that $\mu(\text{U}_-) - \mu(\text{U}_+) = 2$, then $\mathcal{L}(\text{U}_-, \text{U}_+)$ is a manifold of dimension 1 such that the closure $\overline{\mathcal{L}(\text{U}_-, \text{U}_+)}$ is a compact manifold with boundary

$$\partial\mathcal{L}(\text{U}_-, \text{U}_+) = \bigcup_{\substack{\mu(\text{U}_-)-\mu(y)=1 \\ y\in\text{Crit}_H([\text{X}]\text{rel}\text{Y})}} \mathcal{L}(\text{U}_-, y) \times \mathcal{L}(y, \text{U}_+).$$

Now follows that $\partial \circ \partial = 0$ in the same way as it did in Section 6. $\square$

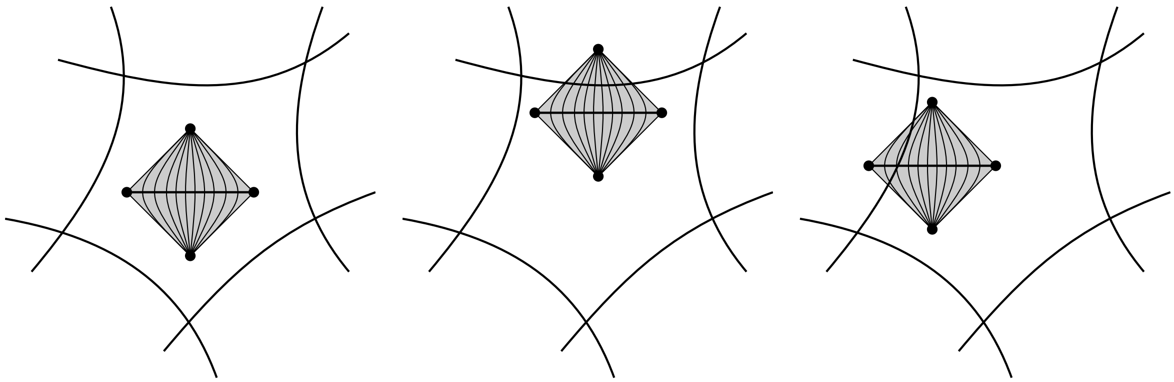
We have defined the braid Floer complex in the case that $[\text{X} \,\text{rel}\, \text{Y}]$ is proper and the free braid X consist of only one contractible strand, without proving the Monotonicity Principle. Denote the braid Floer homology by $BFH(H, J, [\text{X}] \,\text{rel}\, \text{Y})$, since it depends on H, J and $[\text{X}] \,\text{rel}\, \text{Y}$.
To get a better intuition of why $\partial \circ \partial = 0$, see the next Figures. The five lines on the outside, are boundaries between braid classes. The braid class that is surrounded by this lines is the proper braid class $[\text{X} \,\text{rel}\, \text{Y}]$. The grey figures are examples of the space $\mathcal{M}(\text{U}_-, \text{U}_+)$ with U_-, U_+ critical points of the action functional, just as in Figure 6.



(a) Possible. (b) Possible. (c) Not possible.

Figure 7: Proper braid class $[\text{X} \,\text{rel}\, \text{Y}]$ with examples of spaces $\mathcal{M}(\text{U}_-, \text{U}_+)$.

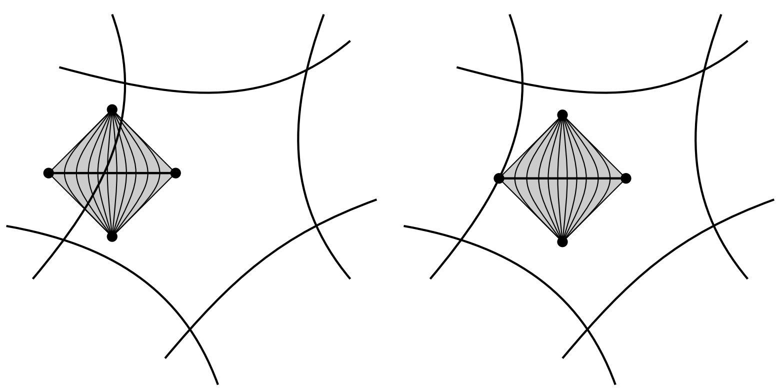
Figure 7a, both U_-, U_+ lie $\text{Crit}_H([\text{X}] \,\text{rel}\, \text{Y})$, so $\mathcal{M}(\text{U}_-, \text{U}_+)$ belongs to the Floer homology and the braid Floer homology. Note $\partial \circ \partial = 0$ as in Figure 6a.
Figure 7b, the bottom point U_+ lies not in $\text{Crit}_H([\text{X}] \,\text{rel}\, \text{Y})$, so $\mathcal{M}(\text{U}_-, \text{U}_+)$ belongs only to the Floer homology and not to the braid Floer homology.
Figure 7b, this is not possible by Theorem 9.3.



(a) Possible. (b) Possible. (c) Not possible.

Figure 8: Proper braid class $[\mathrm{X} \,\mathrm{rel}\, \mathrm{Y}]$ with more examples of spaces $\mathcal{M}(\mathrm{U}_-, \mathrm{U}_+)$.

Figure 8a, both $\mathrm{U}_-, \mathrm{U}_+$ lie $\mathrm{Crit}_H([\mathrm{X}] \,\mathrm{rel}\, \mathrm{Y})$, so $\mathcal{M}(\mathrm{U}_-, \mathrm{U}_+)$ belongs to the Floer homology and the braid Floer homology. Note $\partial \circ \partial = 0$ as in Figure 6b
Figure 8b, the top point U_- lies not in $\mathrm{Crit}_H([\mathrm{X}] \,\mathrm{rel}\, \mathrm{Y})$, so $\mathcal{M}(\mathrm{U}_-, \mathrm{U}_+)$ belongs only to the Floer homology and not to the braid Floer homology.
Figure 8c, this is not possible by Theorem 9.3.



(a) Possible. (b) Not possible.

Figure 9: Proper braid class $[\mathrm{X} \,\mathrm{rel}\, \mathrm{Y}]$ with even more examples of spaces $\mathcal{M}(\mathrm{U}_-, \mathrm{U}_+)$.

Figure 9a, the top point U_- lies not in $\mathrm{Crit}_H([\mathrm{X}] \,\mathrm{rel}\, \mathrm{Y})$, so $\mathcal{M}(\mathrm{U}_-, \mathrm{U}_+)$ belongs only to the Floer homology and not to the braid Floer homology.
Figure 9b, this is not possible by Theorem 9.5.

10 Monotonicity Principle

The last Section gives a definition for braid Floer homology, without proving the Monotonicity Principle. This Section is about proving the Monotonicity Principle for a restrictive case. The proof is for the case that we have a relative braid class fiber $[\mathrm{X}]$ rel $\mathrm{Y} \in \Gamma F_{1,n}(\mathbb{T}^2)$ on the torus, such that the free braid X consists of one contractible braid. Also assume that the loops formed by the strands of the skeleton Y are contractible on the torus.

The Monotonicity Principle needs two main ingredients. The first one is a local property of the Floer equation that works for all braids on a space M_g. The first Subsection shows this property.

The second ingredient is a braid invariant. We get a braid crossing number on the torus from the braid crossing number on a plane. This can only be done within the restrictive case mentioned above.

10.1 Floer equation implies a local property

Start with local solutions $u, u' : U \subset \mathbb{R}^2 \to M_g$ of the Floer equation, such that there exists a chart $\varphi : (V, J) \to (\mathbb{R}^2, J')$ with the properties $J \circ \varphi = J'$ and $u, u' : U \to V \subset M$. Define $v = \varphi \circ u$, $v' = \varphi \circ u'$ and $w = v - v'$.

Theorem 10.1. *Let $u, u' : U \subset \mathbb{R}^2 \to M$ two local solutions of the Floer equation as above. If $u(s_0, t_0) = u'(s_0, t_0)$ for some $(s_0, t_0) \in G = [a, a'] \times [b, b'] \subset U$ and $u(z) \neq u'(z)$ for all points $z \in \partial G$, then (s_0, t_0) is an isolated point such that $u(s_0, t_0) = u'(s_0, t_0)$ and we have for the degree $\deg(w, G, 0) < 0$.*

Proof. To prove this Lemma, I use the same approach as Lemma 5.1 of [BGVW15], but since the statement to prove is more general, the computation is longer. First, we need to take into account that in this case we are working on a surface not equal to the unit disk in $\mathbb{R}^2$. Secondly, the almost complex structure J is not constant any more.

The Floer equation for u is

$$\frac{\partial u}{\partial s} + J(s, u)\frac{\partial u}{\partial t} + \operatorname{grad}_{g(s,u)} H_t(u) = 0.$$

Rewriting the Floer equation in u into v gives

$$\left(\sum_{i=1,2} \frac{\partial \varphi^{-1}}{\partial x_i}\bigg|_{v_i} \right) \frac{\partial v}{\partial s} + J'(s, v) \left(\sum_{i=1,2} \frac{\partial \varphi^{-1}}{\partial x_i}\bigg|_{v_i} \right) \frac{\partial v}{\partial t} + \operatorname{grad}_{g'(s,v)} H_t(\varphi^{-1} \circ v) = 0,$$

where x_1 and x_2 are the local coordinates.

The chart φ is a diffeomorphism, so we can rewrite the equation as

$$\frac{\partial v}{\partial s} + J'(s, v)\frac{\partial v}{\partial t} + \left(\sum_{i=1,2} \frac{\partial \varphi^{-1}}{\partial x_i}\bigg|_{v_i} \right)^{-1} \operatorname{grad}_{g'(s,v)} H_t(\varphi^{-1} \circ v) = 0.$$

This equation holds also for v'. Now we are interested in the difference of the two equations of v and v'. To write this difference as an equation in w, we need to Taylor expand multiple parts of the equation. Start with the first order Taylor expansion of

$$T(s,v) = \left(\sum_{i=1,2} \frac{\partial \varphi^{-1}}{\partial x_i}\bigg|_{v_i} \right)^{-1} \operatorname{grad}_{g'(s,v)} H_t(\varphi^{-1} \circ v)$$

to v. This gives that there exists a smooth function A such that

$$T(s,v) - T(s,v') = A(s,t)(v - v') = A(s,t)w. \tag{10.1}$$

The first order Taylor expansion of

$$J'(s,v)$$

gives that there exists a smooth function B such that

$$J'(s,v) - J'(s,v') = B(s,t)w$$

So

$$\begin{aligned} J'(s,v)\frac{\partial v}{\partial t} - J'(s,v')\frac{\partial v'}{\partial t} &= J'(s,v)\frac{\partial v}{\partial t} - J'(s,v')\frac{\partial v}{\partial t} + J'(s,v')\frac{\partial v}{\partial t} - J'(s,v')\frac{\partial v'}{\partial t} \\ &= B(s,t)\frac{\partial v}{\partial t}w + J'(s,v')\frac{\partial w}{\partial t} \end{aligned} \tag{10.2}$$

So by using Equalities 10.1 and 10.2 we get

$$\begin{aligned} 0 = 0 - 0 =& \frac{\partial v}{\partial s} + J'(s,v)\frac{\partial v}{\partial t} + T(s,v) - \left(\frac{\partial v'}{\partial s} + J'(s,v')\frac{\partial v'}{\partial t} + T(s,v') \right) \\ =& \frac{\partial w}{\partial s} + J'(s,v')\frac{\partial w}{\partial t} + \left(B(s,t)\frac{\partial v}{\partial t} + A(s,t) \right) w. \end{aligned}$$

Note that v' depends on (s,t) and $C(s,t) = B(s,t)\frac{\partial v}{\partial t} + A(s,t)$ depends also on (s,t). So if we define the complex coordinates $z = s - s_0 + i(t - t_0)$, the above Equation gets the following shape

$$\frac{\partial w}{\partial s} + J'(z)\frac{\partial w}{\partial t} + C(z)w = 0,$$

with initial condition $w(0) = 0$ and where $(J')^2 = -\mathrm{Id}$.
Identify now the target space $\mathbb{R}^2$ of w with $\mathbb{C}$. Then by applying Theorem 12 of Appendix A.6 of [HZ94], there exists a $\delta > 0$ such that on $D_\delta = \{z| \, |z| \leq \delta\} \subset G$ there exist a holomorphic map $h : D_\delta \to \mathbb{C}$ and a continuous map $\Phi : D_\delta \to \mathrm{GL}_{\mathbb{R}}(\mathbb{R}^2)$ such that for all $z \in D_\delta$ we have

$$\det \Phi(z) > 0, \qquad J(z)\Phi(z) = \Phi(z)i, \qquad w(z) = \Phi(z)\bar{h}(z).$$

Since $\Phi(z)$ is invertible for all $z \in D_\delta$, the last property could also be written as $\bar{h}(z) = w(z)\Phi^{-1}(z)$. For $z = 0$ we get that $0 = w(0)\Phi^{-1}(0) = \bar{h}(0) = h(0)$. The map h is holomorphic, so $z = 0$ is an isolated zero in D_δ, or h is constant to zero in D_δ. If h is constant to zero, then $\bar{h}$ too, hence w is zero on D_δ. This implies that $u(z) = u'(z)$ for all $z \in D_\delta$. Via an analytical continuation argument by repeating this argument we get that $u(z) = u'(z)$ for all $z \in G$, this contradicts the assumption that $u(z) \neq u'(z)$ for all points $z \in \partial G$. Therefore, $z = 0$ is an isolated zero in D_δ of the map h, so also for w. This implies that (s_0, t_0) is an isolated point such that $u(s_0, t_0) = u'(s_0, t_0)$.
Now we are left with the proof that $\deg(w, G, 0) < 0$. Since we have chosen G to be compact, there are only a finite amount of points z_i in $G \subset U$ such that $u(z_i) = u'(z_i)$. For sufficiently small $\varepsilon_i > 0$ holds

$$\deg(w, G, 0) = \sum_i \deg(w, B(z_i; \varepsilon_i), 0) = \sum_i \deg(\bar{h}, B(z_i; \varepsilon_i), 0) = -\sum_i \deg(h, B(z_i; \varepsilon_i), 0)$$

since $\det \Phi(z) > 0$. The map h is holomorphic and has an isolated zero z_i, this implies $\deg(h, B(z_i; \varepsilon_i), 0) \geq 1$. Hence $\deg(w, G, 0) < 0$.

□

10.2 Crossing number for braids on the plane

The braid crossing number that is used to define the braid Floer homology on the disk in [BGVW15] is defined via the winding number around the origin.

Definition 10.2. Let $\gamma : I \to \mathbb{R}^2 \backslash \{(0,0)\}$ be a curve in the plane $\mathbb{R}^2$ that does not go through the origin for I a bounded interval. The winding number $\mathcal{W}(\gamma, 0)$ around the origin is then defined as

$$\mathcal{W}(\gamma, 0) = \frac{1}{2\pi} \int_\gamma \alpha,$$

where $\alpha = (-ydx + xdy)/(x^2 + y^2)$ for coordinates $(x, y) \in \mathbb{R}^2$.

Definition 10.3. For a closed braid $\mathrm{z} = (z^1, \ldots, z^n)$ defined on the plane $\mathbb{R}^2$ with n strands. This is defined in the same way as a closed braid on M_g for $g > 2$. Define the braid crossing number for z as

$$\mathrm{Cross}(\mathrm{z}) = \sum_{k \neq k'} \mathcal{W}(z^k - z^{k'}, 0).$$

Lemma 10.4. *The braid crossing number* $\mathrm{Cross}(\mathrm{z})$ *for a braid* z *on the plane is an integer and a braid invariant, i.e.* $\mathrm{Cross}(\mathrm{z}) = \mathrm{Cross}(\mathrm{z}')$ *for all* $\mathrm{z}' \in [\mathrm{z}]$.

Proof. The difference of two strands cannot be zero, because we are working with braids. Therefore, the braid crossing number is well defined.
The winding number for a loop around the origin is an integer. We are only working with closed braids, so we have that the braid z consists of one or multiple loops. The braid crossing number is the sum of winding numbers of the difference of these loops, so it is an integer.
To conclude that it is an braid invariant, I refer to Lemma 5.4 of [BGVW15]. $\square$

We want to relate this braid crossing number with the local property of the last Subsection. Therefore we need the following notation.

Definition 10.5. Let $G = [a, a'] \times [b, b']$ for real numbers a, a', b, b' and Let $w : G \to \mathbb{R}^2$ such that $w|_{\partial G} \neq 0$. Then denote by $\mathcal{W}_a^{[b,b']}(w)$, $\mathcal{W}_{a'}^{[b,b']}(w)$, $\mathcal{W}_{[a,a']}^{b}(w)$ and $\mathcal{W}_{[a,a']}^{b'}(w)$ the winding numbers $\mathcal{W}(w(a, \cdot), 0)$, $\mathcal{W}(w(a', \cdot), 0)$, $\mathcal{W}(w(\cdot, b), 0)$ and $\mathcal{W}(w(\cdot, b'), 0)$ respectively.

Lemma 10.6. *Let G and w as in the above Definition, then*

$$\left(\mathcal{W}_{a'}^{[b,b']}(w) - \mathcal{W}_{a}^{[b,b']}(w)\right) - \left(\mathcal{W}_{[a,a']}^{b'}(w) - \mathcal{W}_{[a,a']}^{b}(w)\right) = \deg(w, G, 0).$$

Proof. A short computation gives

$$\begin{aligned}
\deg(w, G, 0) = \mathcal{W}(w|_{\partial G}) &= \frac{1}{2\pi}\int_{w|_{\partial G}} \alpha \\
&= \frac{1}{2\pi}\int_{w|_{\{a'\}\times[b,b']}} \alpha - \frac{1}{2\pi}\int_{w|_{\{a\}\times[b,b']}} \alpha - \frac{1}{2\pi}\int_{w|_{[a,a']\times\{b'\}}} \alpha + \frac{1}{2\pi}\int_{w|_{[a,a']\times\{b\}}} \alpha \\
&= \left(\mathcal{W}_{a'}^{[b,b']}(w) - \mathcal{W}_{a}^{[b,b']}(w)\right) - \left(\mathcal{W}_{[a,a']}^{b'}(w) - \mathcal{W}_{[a,a']}^{b}(w)\right)
\end{aligned}$$

The first equality follows from the definition of the degree, see Subsection 1.6 of [Van14]. $\square$

Corollary 10.7. *Let $G = [a, a'] \times [b, b']$ and $w : G \to \mathbb{R}^2$ as in Theorem 10.1. Then for every zero (s_0, t_0) of w in the interior of G, there exists an $\varepsilon_0 > 0$ such that*

$$\mathcal{W}_{s_0+\varepsilon}^{[b,b']}(w) - \mathcal{W}_{s_0-\varepsilon}^{[b,b']}(w) < \mathcal{W}_{[s_0-\varepsilon,s_0+\varepsilon]}^{b'}(w) - \mathcal{W}_{[s_0-\varepsilon,s_0+\varepsilon]}^{b}(w)$$

for all $0 < \varepsilon \leq \varepsilon_0$.

Proof. This is a direct consequence of Theorem 10.1 and the above Lemma. $\square$

10.3 Monotonicity Principle on the torus

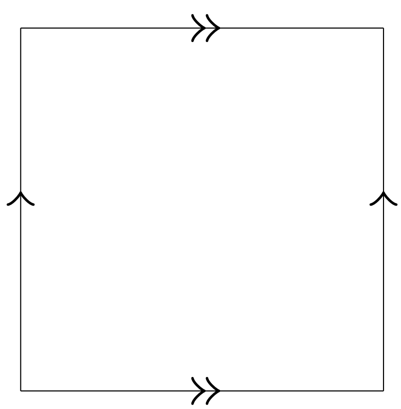
The Monotonicity Principle brings the local change described in Theorem 10.1 and a braid invariant, to conclude that such a local change gives a global change, *i.e.* a change in this braid invariant. The Mononocity Principle depends highly on this braid invariant that gets changed by the local crossing of two strands. The braid crossing number is used in the case of a disk, see [BGVW15].

As we see in the definition of the braid crossing number in Definition 10.3, we need to take the difference between two strands of a braid. This is the point wise difference between two points of the plane for every time t. The problem is that for the spaces M_g with $g \geq 1$, they do not have a difference operation on them. To avoid this problem, I will use the braid crossing number on a plane to find a braid invariant for certain braids on the torus $\mathbb{T}^2$.

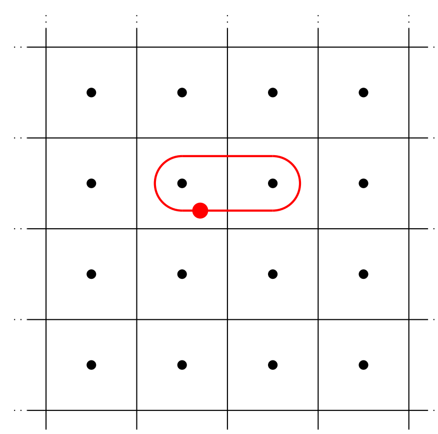
Let $[\mathrm{X}]\,\mathrm{rel}\,\mathrm{Y} \in \Gamma F_{1,n}(\mathbb{T}^2)$ be relative braid class fiber on the torus, such that the free braid X consists of one contractible braid. Also assume that the loops formed by the strands of the skeleton Y are contractible in the torus, *i.e.* they have trivial homotopy. I define a braid invariant for such braids. Note that we do not assume that $[\mathrm{X}\ \mathrm{rel}\ \mathrm{Y}]$ is a proper class, this is only needed in the theorems of Section 9.

The torus can be viewed as a square such that its sides are identified as is indicated by the arrows in Figure 10a. The representation of the torus by a square with its sides identified, makes it possible to visualize a braid on a torus, see for example Figure 11a. Two strands are going from the top to the bottom of the cube in this Figure. The sides of the cube are identified with each other just as the square on the top indicates.

A closed braid can be made periodic, by running the time t in $\mathbb{R}$ instead of $[0,1]$ and then repeat the braid over and over. The top and bottom faces of the cube are in this case also identified in Figure 11a. Therefore, the cube represents the $\mathbb{T}^3$ torus of dimension 3.



(a) Torus as square.

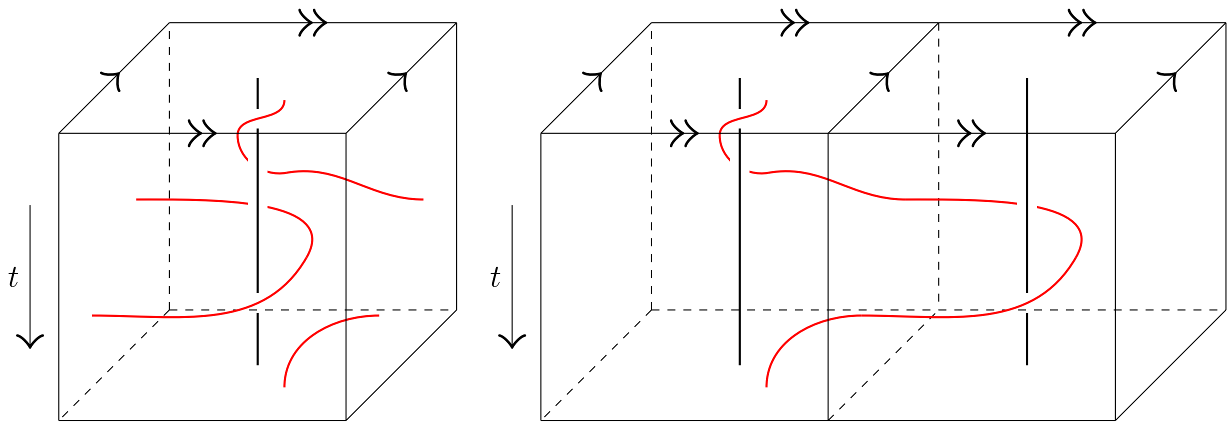


(b) Projection from above.

Figure 10: Torus.

Another way to describe the torus is $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, see Figure 10b. The plane $\mathbb{R}^2$ is the universal cover of the torus, see [Row16]. Use the map induced by the quotient $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, to define lifts from the torus to the plane. The black dots in Figure 10b are all the lifts from the black strand of Figure 11a viewed as a projection from above. The red loop is one particular lift of the red strand of Figure 11a, also viewed as a projection from above. The red dot indicates the starting and finishing point of the red strand. The lift of the red strand together with the lifts of the black strand are a closed braid in the plane with an infinite amount of strands. The part of this braid that contains the red lift is illustrated in Figure 11b.

As mentioned above, note that all the lifts of the black strand together with this one specific lift of the red strand form a closed braid in the plane with infinite strands. If we define the crossing number for such a braid in the same way as in Definition 10.3, it is in general not well defined. Then we would get an infinite sum that might diverges. Luckily, in our case of black strands and a red strand, there are only three that contribute to the crossing number, the rest has no winding with each other. This three strands are the three drawn in Figure 11b. Viewed as projections in Figure 10b, this are the red loop and the two black dots that lie in this red loop.



(a) A braid on the torus.

(b) A braid on the torus as braid on the plane.

Figure 11: Lifting braids from the torus to the plane.

The problem is of course that not every lift via this procedure forms a closed braid with infinite strands on the plane. And if it forms a braid with infinite strands, there are maybe an infinite amount of strands that contribute to the crossing number in the plane. To prevent such trouble, look at a relative braid $\mathrm{X}\,\mathrm{rel}\,\mathrm{Y}$ with a free strand X that consists of one contractible strand, and that loops formed by the strands of the skeleton Y are contractible.

Use the projection from above to look at loop formed by the strand of the free braid. This loop is contractible by assumption, so there is a homotopy that brings it to a point. If we also transform the skeleton by the same homotopy, the relative braid class $[\mathrm{X} \operatorname{rel} \mathrm{Y}]$ is also represented by a braid such that the strand of the free braid goes as a straight line. Since we are looking for a braid invariant, we may assume that the strand of the free braid is straight. Therefore, let the strand of the free braid be the black strand from the above construction.

The skeleton Y plays the same roll as the red strand from the above construction. Lift the skeleton Y on the torus $\mathbb{T}^2/\mathbb{Z}^2$ just once to the braid $\widetilde{\mathrm{Y}}$ on the plane. The braid $\widetilde{\mathrm{Y}}$ is closed, because loops formed by the strands of the skeleton Y are contractible. We also have that $\widetilde{\mathrm{Y}}$ cannot be on an infinite amount of 1 by 1 tiles of the plane. Therefore, there are only a finite amount of strands that contribute to the crossing number in the plane. Hence we found a crossing number for our relative braid X rel Y. Denote this crossing number by $\mathrm{Cross}_{\mathbb{T}^2}(\mathrm{X} \operatorname{rel} \mathrm{Y})$.

Note that this crossing number is a braid invariant for the lifted braid on the plane. Therefore, this number can only change after two strands of this braid on the plane go through each other. If two strands of the lift go through each other, then this also happen for two strands on the torus. Hence $\mathrm{Cross}_{\mathbb{T}^2}(\mathrm{X} \operatorname{rel} \mathrm{Y})$ is a braid invariant for the relative braid class $[\mathrm{X} \operatorname{rel} \mathrm{Y}]$.

It is not always the other way around too, since if two strands of the skeleton braid Y on the torus go through each other, it does not always changes the crossing number $\mathrm{Cross}_{\mathbb{T}^2}(\mathrm{X} \operatorname{rel} \mathrm{Y})$. However, if the strand of the free braid X goes through a strand of the skeleton braid Y, it does change the crossing number $\mathrm{Cross}_{\mathbb{T}^2}(\mathrm{X} \operatorname{rel} \mathrm{Y})$. This will be really important in the next Theorem.

Theorem 10.8. *Let $[\mathrm{X}]\operatorname{rel}\mathrm{Y}$ be a relative braid class on the torus $\mathbb{T}^2$. Also assume that the free braid X consists of only one contractible strand and that loops formed by the strands of the skeleton Y are contractible.*
Take $\mathrm{U} \in \mathcal{M}([\mathrm{X}] \operatorname{rel} \mathrm{Y})$, if for an $s_0 \in \mathbb{R}$ we have that $\mathrm{U}(s_0, \cdot) \operatorname{rel} \mathrm{Y} \in \Sigma$, then either $\mathrm{U}(s_0, \cdot) \operatorname{rel} \mathrm{Y} \in \Sigma^-$ or there exists an $\varepsilon_0 > 0$ such that $\mathrm{U}(s_0 \pm \varepsilon, \cdot) \in \mathcal{L}F_1 \operatorname{rel} \mathrm{Y}$ for all $0 < \varepsilon < \varepsilon_0$ and

$$\mathrm{Cross}_{\mathbb{T}^2}(\mathrm{U}(s_0 - \varepsilon, \cdot) \operatorname{rel} \mathrm{Y}) > \mathrm{Cross}_{\mathbb{T}^2}(\mathrm{U}(s_0 + \varepsilon, \cdot) \operatorname{rel} \mathrm{Y}).$$

Proof. Denote $\mathrm{U} = (u)$ and $\mathrm{Y} = (y^1, \cdots, y^m)$. Then $\mathrm{U}(s_0, \cdot) \operatorname{rel} \mathrm{Y} \in \Sigma$ gives that there is a $t_0 \in [0, 1]$ such that $u = y^k$ for some $1 \leq k \leq m$. Theorem 10.1 gives now that either $u = y^k$ via analytical continuation, hence $\mathrm{U}(s_0, \cdot) \operatorname{rel} \mathrm{Y} \in \Sigma^-$ or that (s_0, t_0) is an isolated zero. We are left with the case that (s_0, t_0) is an isolated zero.

Pick the two lifts $\widetilde{u}(s_0, \cdot)$ and $\widetilde{y}^k(s_0, \cdot)$ of the strands $u(s_0, \cdot)$ and $y^k(s_0, \cdot)$ such that $\widetilde{u}(s_0, t_0) = \widetilde{y}^k(s_0, t_0)$. Now define $\widetilde{w} = \widetilde{u} - \widetilde{y}^k$. There is a smallest positive integer l such that $\widetilde{w}(s, t) = \widetilde{w}(s, t + l)$ for all $(s, t) \in \mathbb{R}^2$. Now take all the times $(t_0, \ldots, t_n)$ of the interval $[0, l]$ such that (s_0, t_i) for $i = 1, \ldots, n$ are isolated zeros of $\widetilde{w}$. Note that a point is isolated for the braid on the plane then it is isolated for the braid on the torus.

If we pick ε_0 sufficiently small, then (s_0, t_i) is the only zero in $[s_0-\varepsilon, s_0+\varepsilon]\times[t_i-\varepsilon, t_i+\varepsilon]$, for all $i = 1, \ldots, n$ and all $0 < \varepsilon \leq \varepsilon_0$. Furthermore, by isolation we can choose ε_0 sufficiently small to ensure that $\mathrm{U}(s, \cdot)$ rel $\mathrm{Y} \notin \Sigma$ for all $0 < |s - s_0| \leq \varepsilon_0$.
Now we start with computing $\mathrm{Cross}_{\mathbb{T}^2}(\mathrm{U}(s, \cdot) \text{ rel } \mathrm{Y})$ for $s = s_0 - \varepsilon$ and $s = s_0 + \varepsilon$ for all $0 < \varepsilon \leq \varepsilon_0$, see also figure 12. Denote with $t_{n+1} = t_0 + l$.

$$
\begin{aligned}
&\mathcal{W}^{[t_0-\varepsilon,t_0-\varepsilon+l]}_{s_0+\varepsilon}(\widetilde{w}) - \mathcal{W}^{[t_0-\varepsilon,t_0-\varepsilon+l]}_{s_0-\varepsilon}(\widetilde{w}) = \\
&\sum_{i=0}^{n}\Big(\mathcal{W}^{[t_i-\varepsilon,t_i+\varepsilon]}_{s_0+\varepsilon}(\widetilde{w}) - \mathcal{W}^{[t_i-\varepsilon,t_i+\varepsilon]}_{s_0-\varepsilon}(\widetilde{w}) + \mathcal{W}^{[t_i+\varepsilon,t_{i+1}-\varepsilon]}_{s_0+\varepsilon}(\widetilde{w}) - \mathcal{W}^{[t_i+\varepsilon,t_{i+1}-\varepsilon]}_{s_0-\varepsilon}(\widetilde{w})\Big) - 0 - 0 = \\
&\sum_{i=0}^{n}\Big(\mathcal{W}^{[t_i-\varepsilon,t_i+\varepsilon]}_{s_0+\varepsilon}(\widetilde{w}) - \mathcal{W}^{[t_i-\varepsilon,t_i+\varepsilon]}_{s_0-\varepsilon}(\widetilde{w}) + \mathcal{W}^{[t_i+\varepsilon,t_{i+1}-\varepsilon]}_{s_0+\varepsilon}(\widetilde{w}) - \mathcal{W}^{[t_i+\varepsilon,t_{i+1}-\varepsilon]}_{s_0-\varepsilon}(\widetilde{w})\Big) - \\
&\sum_{i=0}^{n}\Big(\mathcal{W}^{t_i+\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w}) - \mathcal{W}^{t_{i+1}+\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w})\Big) - \sum_{i=0}^{n}\Big(\mathcal{W}^{t_{i+1}-\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w}) - \mathcal{W}^{t_i-\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w})\Big) = \\
&\sum_{i=0}^{n}\Big(\Big(\mathcal{W}^{[t_i-\varepsilon,t_i+\varepsilon]}_{s_0+\varepsilon}(\widetilde{w}) - \mathcal{W}^{[t_i-\varepsilon,t_i+\varepsilon]}_{s_0-\varepsilon}(\widetilde{w})\Big) - \Big(\mathcal{W}^{t_i+\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w}) - \mathcal{W}^{t_i-\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w})\Big)\Big) + \\
&\sum_{i=0}^{n}\Big(\Big(\mathcal{W}^{[t_i+\varepsilon,t_{i+1}-\varepsilon]}_{s_0+\varepsilon}(\widetilde{w}) - \mathcal{W}^{[t_i+\varepsilon,t_{i+1}-\varepsilon]}_{s_0-\varepsilon}(\widetilde{w})\Big) - \Big(\mathcal{W}^{t_{i+1}-\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w}) - \mathcal{W}^{t_{i+1}+\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w})\Big)\Big) < 0
\end{aligned}
$$

The inequality follows from the following two arguments. First note that locally the winding number of $\widetilde{w}$ on the plane is the same as on the torus, so

$$
\sum_{i=0}^{n}\Big(\Big(\mathcal{W}^{[t_i-\varepsilon,t_i+\varepsilon]}_{s_0+\varepsilon}(\widetilde{w}) - \mathcal{W}^{[t_i-\varepsilon,t_i+\varepsilon]}_{s_0-\varepsilon}(\widetilde{w})\Big) - \Big(\mathcal{W}^{t_i+\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w}) - \mathcal{W}^{t_i-\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w})\Big)\Big) < 0,
$$

because of Corollary 10.7. Secondly, $\widetilde{w}$ has no zeros in $[s_0-\varepsilon, s_0+\varepsilon]\times[t_i+\varepsilon, t_{i+1}-\varepsilon]$, so

$$
\begin{aligned}
&\sum_{i=0}^{n}\Big(\Big(\mathcal{W}^{[t_i+\varepsilon,t_{i+1}-\varepsilon]}_{s_0+\varepsilon}(\widetilde{w}) - \mathcal{W}^{[t_i+\varepsilon,t_{i+1}-\varepsilon]}_{s_0-\varepsilon}(\widetilde{w})\Big) - \Big(\mathcal{W}^{t_{i+1}-\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w}) - \mathcal{W}^{t_{i+1}+\varepsilon}_{[s_0-\varepsilon,s_0+\varepsilon]}(\widetilde{w})\Big)\Big) = \\
&\qquad\qquad\sum_{i=0}^{n}\deg(\widetilde{w}, [s_0-\varepsilon, s_0+\varepsilon]\times[t_i+\varepsilon, t_{i+1}-\varepsilon], 0) = 0.
\end{aligned}
$$

Note that $\mathrm{Cross}(\widetilde{\mathrm{U}}(s, \cdot) \text{ rel } \widetilde{\mathrm{Y}}) = \mathrm{Cross}_{\mathbb{T}^2}(\mathrm{U}(s, \cdot) \text{ rel } \mathrm{Y})$ for $0 < |s - s_0| \leq \varepsilon$. We also have

$$
\mathcal{W}^{[t_0-\varepsilon,t_0-\varepsilon+l]}_{s_0-\varepsilon}(\widetilde{w}) < \mathcal{W}^{[t_0-\varepsilon,t_0-\varepsilon+l]}_{s_0+\varepsilon}(\widetilde{w}).
$$

The braid crossing number on the plane is made of such terms, so

$$
\mathrm{Cross}_{\mathbb{T}^2}(\mathrm{U}(s_0-\varepsilon, \cdot) \text{ rel } \mathrm{Y}) > \mathrm{Cross}_{\mathbb{T}^2}(\mathrm{U}(s_0+\varepsilon, \cdot) \text{ rel } \mathrm{Y}).
$$

$\square$

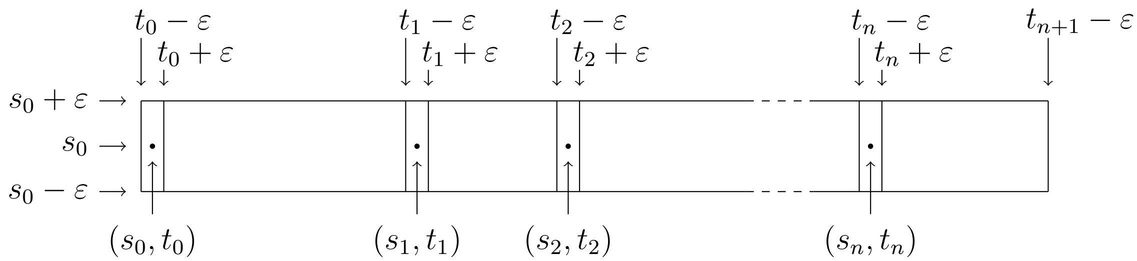


Figure 12: To compute $\mathcal{W}_{s_0+\varepsilon}^{[t_0-\varepsilon,t_0-\varepsilon+l]}(\widetilde{w}) - \mathcal{W}_{s_0-\varepsilon}^{[t_0-\varepsilon,t_0-\varepsilon+l]}(\widetilde{w})$, we use winding numbers over the time intervals indicated in this Figure.

10.4 Conclusion

This Subsection goes back to the question of the Introduction. Namely

Question. *If we already find some periodic solutions for a Hamiltonian system on a compact symplectic manifolds, will the topology of these solutions on the manifold forces more periodic solutions?*

If we summarise the results into a Theorem, we get close to an answer.

Theorem 10.9. *Let* $[\mathrm{X} \,\mathrm{rel}\, \mathrm{Y}]$ *be a proper relative closed braid class on the torus such that*

(1) The free braid X *consists of one contractible strand.*

(2) The strands of the skeleton Y *form contractible loops.*

Then we can define for a relative braid class fiber $[\mathrm{X}'] \,\mathrm{rel}\, \mathrm{Y} \subset [\mathrm{X} \,\mathrm{rel}\, \mathrm{Y}]$ *and a regular pair* (H, J) *its braid Floer homology* $\mathit{BFH}(H, J, [\mathrm{X}'] \,\mathrm{rel}\, \mathrm{Y})$*.*

Proof. Braid Floer homology is defined from a subcomplex of the Floer complex. Therefore, the first thing we need is that the Floer homology from Part I is well defined. The torus is a symplectic manifold and (H, J) is a regular pair, then by Theorem 7.1 the Floer homology is defined.

The assumptions, $[\mathrm{X} \mathrm{rel} \mathrm{Y}]$ is proper and the free braid X consists of one contractible strand give that the braid Floer homology $\mathit{BFH}(H, J, [\mathrm{X}'] \,\mathrm{rel}\, \mathrm{Y})$ is well defined if $[\mathrm{X} \,\mathrm{rel}\, \mathrm{Y}]$ admits the Monotonicity Principle, see Section 9.

For the extra assumptions that we are working on a torus and that the strands of the skeleton Y form contractible loops, Theorem 10.8 gives that the Monotonicity Principle holds. □

The conclusion is now that if we are in the situation of the above Theorem and the braid Floer homology is in that case nontrivial, then we can conclude that the known solutions represented by the skeleton Y forces new solutions X.

The above Theorem has a lot of restrictive assumptions. The next goal will be to remove some of this assumptions, but some cannot be removed. We are using a Floer homology that is made from contractible solutions, so we must assume that the strands of the free braid X form contractible loops. Also the properness assumption is crucial.
On the other hand, the following assumptions are possibly not necessary:

(1) We are only looking for braids on a torus.

(2) The free braid X consist of just one strand.

(3) The strands of the skeleton Y form contractible loops.

The only Theorem that depends on the assumption of the torus instead of an orientable compact surface M_g for $g \geq 1$ is Theorem 10.8. The property we used from the torus was that its universal cover is the plane. On the plane we defined then a crossing number for the torus. If $g \geq 2$, the universal cover of M_g is the disk, see [Row16]. This gives maybe also a possibility to define a crossing number for braids on M_g for $g \geq 2$. The case of a sphere $g = 0$ is even worse, which is already explained in the introduction of Section 8. Then you would need a Floer homology that also works for the sphere $M_0 = S^2$.
The second assumption is not needed for braid Floer homology on a disk in [BGVW15]. They do not assume that the free braid X consits of just one strand. To remove this assumption in our case, you should probaply use the same tricks as in case of a disk. For a free braid X with n stands, you can rescale the time by dividing it with $n!$. Then the loops formed with strands of a critical relative braid X rel Y are 1-periodic solutions for the Hamiltonian. You must also define a permuted Maslov index, just like the permuted Conley-Zehnder index in the case of a disk, see Subsection 7.2 of [BGVW15]. The only real problem will be finding a braid invariant that helps you to prove the Monotonicity Principle.
The last assumption is a tricky one. The first thing to note is that I used a braid invariant that changes if two strands go through each other. The problem on the torus is that there is an example of a braid class that stays in the same class after an intersection of two strands. This is the example where the skeleton Y consist of one strand that is one of the homotopy generators on the torus and the free braid X that consist of one constant strand. If you now move the free braid along the other homotopy generator of the torus, it will pass the skeleton braid without changing the relative braid class fiber $[x]$ rel Y.

References

[AD14] Michèle Audin and Mihai Damian. *Morse theory and Floer homology*. Universitext. Springer, London; EDP Sciences, Les Ulis, 2014. Translated from the 2010 French original by Reinie Erné.

[BGVW15] J.B. van den Berg, R. Ghrist, R.C. Vandervorst, and W. Wójcik. Braid Floer homology. *J. Differential Equations*, 259(5):1663–1721, 2015.

[Flo89] Andreas Floer. Symplectic fixed points and holomorphic spheres. *Comm. Math. Phys.*, 120(4):575–611, 1989.

[Hat01] Allen Hatcher. *Algebraic topology*. Cambridge University Press, New York, 2001.

[HZ94] Helmut Hofer and Eduard Zehnder. *Symplectic invariants and Hamiltonian dynamics*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 1994.

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[MS98] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1998.

[Row16] Todd Rowland. "Universal cover". From *MathWorld*, created by Eric W. Weisstein, 2016. http://mathworld.wolfram.com/UniversalCover.html.

[Sal99] Dietmar Salamon. Lectures on Floer homology. In *Symplectic geometry and topology (Park City, UT, 1997)*, volume 7 of *IAS/Park City Math. Ser.*, pages 143–229. Amer. Math. Soc., Providence, RI, 1999.

[Sch05] René L. Schilling. *Measures, integrals and martingales*. Cambridge University Press, New York, 2005.

[Sma65] S. Smale. An infinite dimensional version of Sard's theorem. *Amer. J. Math.*, 87:861–866, 1965.

[Van14] R.C.A.M. Vandervorst. Topological methods for nonlinear differntial equations. VU Course Notes, Novemeber 2014.

[Wój07] Wojciech Wójcik. *Briads, Floer homology and forcing in two and three dimensional dynamics*. PhD thesis, Vrije Universiteit, 2007.

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List of Symbols