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Syntactic Characterisations in Model Theory

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Chapter 1

Introduction

The mathematician is and has always been in search of the truth. Model theory is in fact one of the specialisations in mathematics where this search has always been very important. It focuses on the relation between a formal language and its models. Out of the languages, combined with some auxiliary symbols, we can build up sentences. These sentences can help us describe things about the models. In this way we can define a truth in a model, and it is this truth definition that is the relation between the language and its models. The truth definition we call a semantic property, a property that deals with interpretation or meaning. Besides truth, another semantic property is falsity, as are the amalgamation property and the congruence extension property, which will be studied in this paper. In contrast to the semantics we have the syntax, which focuses purely on the formal structure of the language used. For example the symbols used in a sentence or the length of a sentence are syntactical properties. This gives the notion of syntactical characterisations of semantic properties, thus a equivalence between the meaning of a property and a formal description of the language that makes the property hold.

In this thesis we shall try to make clear the syntactical characterisations of some amalgamation properties as well as of the congruence extension property for certain theories. Hereby we will follow the steps of Paul Bacsich and Dafydd Rowland Hughes, who have described the characterisations in *Syntactic Characterisations of Amalgamation, Convexity and Related Properties* [2].

We shall use chapter 2 to describe our notational conventions, give some examples of syntactic characterisations and introduce the notion of generalised atomic sets of formulas. We will also prove some general lemmas which form the basis of the rest of the thesis. In chapter 3 we will focus on the syntactical characterisations of the general amalgamation property, theories for which injections are transferable and the strong amalgamation property. Finally in chapter 4, we will focus on the syntactical characterisation of the congruence extension property. Here we shall look at theories that are preserved under homomorphisms, and especially at equational theories.

Chapter 2

Preliminaries

The goal of this chapter is to give a basic background to the reader of this thesis such that this thesis is fully readable. Therefore we shall first describe some notational conventions, then some basics of model theory, some explanation about syntactical characterisations, the description of the notion of generalised atomic sets of formulas and we conclude with some important theorems in model theory.

2.1 Notational Conventions

In this thesis we presume that the reader is familiar with the general concepts of first-order logic and model theory. We shall use the notation introduced by Ieke Moerdijk and Jaap van Oosten in [4] and [5].

We will write $\mathcal{L}$ for a language in predicate logic, and Gothic letters $\mathfrak{A}, \mathfrak{B}, ...$ for $\mathcal{L}$-structures constructed from the domains $A, B, ...$

For each constant c, function symbol f and relation symbol R in $\mathcal{L}$, we call $c^{\mathfrak{A}}$, $f^{\mathfrak{A}}$ and $R^{\mathfrak{A}}$ the respective interpretations in $\mathfrak{A}$.

We shall use Greek letters $\phi, \psi, ...$ for $\mathcal{L}$-formulas and $\mathcal{L}$-sentences, and denote $\mathfrak{A} \models \phi$ for ϕ holds in $\mathfrak{A}$. We can extend the language $\mathcal{L}$ by adding a constant for each $a \in A$, this language we shall denote by $\mathcal{L}_{\mathfrak{A}}$.

We will use $s, t, ...$ for constant terms of $\mathcal{L}$ and x, y, z for variables. By $\bar{t}, \bar{x}, \bar{y}, \bar{a}, ...$ we will denote lists of constants or variables, and by variables and lists of variables which are written differently we assume them to be disjoint, unless the context proves differently. By $\bar{a} \in A$ we denote a list $\bar{a}$ whose elements are all elements of A.

We write T for an $\mathcal{L}$-theory, and use $\mathfrak{A} \models T$ if $\mathfrak{A}$ is a model of the theory T. With $T \models \phi$ we mean that the sentence ϕ holds in every model of T.

2.2 Basic Notions of Model Theory

We will now give some important definitions in Model Theory, that are used throughout this thesis. We get this definitions from Jaap van Oosten in [5].

First we look at reducts and expansions. Let $\mathcal{L} \subseteq \mathcal{L}'$ be two languages, with $\mathfrak{B}$ an $\mathcal{L}'$-structure. If we now restrict the interpretation function of $\mathfrak{B}$ to $\mathcal{L}$, we get a $\mathcal{L}$-structure $\mathfrak{A}$. We then describe $\mathfrak{B}$ as the $\mathcal{L}'$-expansion of $\mathfrak{A}$ and $\mathfrak{A}$ as the $\mathcal{L}$-reduct of $\mathfrak{B}$.

If we have the language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathfrak{A}$, we can extend the language to $\mathcal{L}_{\mathfrak{A}}$ as stated above. When we have the interpretation of $a^{\mathfrak{A}} = a$, $\forall a \in A$, then $\mathfrak{A}$ becomes an $\mathcal{L}_{\mathfrak{A}}$-structure and is called the natural expansion of $\mathfrak{A}$ to $\mathcal{L}_{\mathfrak{A}}$.

Now we shall have a look at functions in Model Theory. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures. A function $f : A \to B$ that sends every element from A to an element from B such that they commute with the interpretations of the elements in the structures $\mathfrak{A}$ and $\mathfrak{B}$, is called a homomorphism of $\mathcal{L}$-structures. To commute with the interpretation, we have the following demands:

i) For all constants c in $\mathcal{L}$, $f(c^{\mathfrak{A}}) = c^{\mathfrak{B}}$.

ii) For all function symbols g in $\mathcal{L}$ and $\bar{a} \in A$, $f(g^{\mathfrak{A}}(\bar{a})) = g^{\mathfrak{B}}(f(\bar{a}))$.

iii) For all function symbols R in $\mathcal{L}$ and $\bar{a} \in A$, if $\bar{a} \in R^{\mathfrak{A}}$ then $f(\bar{a}) \in R^{\mathfrak{B}}$.

We shall denote such a homomorphism by $f : \mathfrak{A} \to \mathfrak{B}$.

If there is a homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ and a homomorphism $g : \mathfrak{B} \to \mathfrak{A}$, such that g is the inverse of f, then f and g are called isomorphisms and the structures $\mathfrak{A}$ and $\mathfrak{B}$ are said to be isomorphic. Two isomorphic structures also satisfy the same $\mathcal{L}$-sentences. If two $\mathcal{L}$-structures $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the same $\mathcal{L}$-sentences, they are said to be elementarily equivalent, which we notate by $\mathfrak{A} \equiv \mathfrak{B}$.

If there is a homomorphism $f : \mathfrak{A} \to \mathfrak{B}$, which is injective and for every relation symbol R of $\mathcal{L}$ and every n-tuple $\bar{a} \in A$, if $f(\bar{a}) \in R^{\mathfrak{B}}$ then $\bar{a} \in R^{\mathfrak{A}}$, then f is called an embedding.

If there is an embedding $f : \mathfrak{A} \to \mathfrak{B}$, such that for every $\mathcal{L}$-formula $\phi(\bar{x})$ and every $\bar{a} \in A$, $\mathfrak{A} \models \phi(\bar{x}) \iff \mathfrak{B} \models \phi(f(\bar{x}))$ holds, then f is called an elementary embedding.

We shall now focus on different kinds of $\mathcal{L}$-formulas. If a formula contains neither connectives nor quantifiers, it is called an atomic formula. In practice this comes down to formulas of the form $t = s$ and $\bar{t} \in R$, where s and t are terms and $\bar{t}$ is a tuple of terms of $\mathcal{L}$ and R is a relation symbol of $\mathcal{L}$. Atomic and negated atomic formulas together are called basic formulas.

Positive formulas are those $\mathcal{L}$-formulas that are not negated, thus not using the negation or the implication, and open formulas are the ones not using quantifiers. When we speak of universal $\mathcal{L}$-formulas, we mean formulas of the form $\forall\bar{x}\phi$, where ϕ is an open $\mathcal{L}$-formula. The same idea holds for existential formulas, which are of the form $\exists\bar{x}\phi$.

We can also combine the definitions. Thus we have universal-existential formulas, which are formulas of the form $\forall\bar{x}\exists\bar{y}\phi$, where ϕ is again an open $\mathcal{L}$-formula, and in the same way existential-universal formulas. But also for example positive existential formulas, which are of the form $\exists\bar{x}\phi$, where ϕ is a positive open $\mathcal{L}$-formula.

Now we have seen the different kinds of formulas, we can look at theories and diagrams. Let T and T' be two $\mathcal{L}$-theories. If T' has exactly the same models as T, then T' is called a set of axioms of T.

Diagrams are sets of $\mathcal{L}$-sentences that are true in a certain $\mathcal{L}$-structure. If $\mathfrak{A}$ is an $\mathcal{L}_{\mathfrak{A}}$-structure, then the diagram of $\mathfrak{A}$ is the set of basic $\mathcal{L}_{\mathfrak{A}}$-sentences that hold in $\mathfrak{A}$. We denote this by $\Delta(\mathfrak{A})$. We also have the positive diagram, $\Delta^+(\mathfrak{A})$, the set of atomic $\mathcal{L}_{\mathfrak{A}}$-sentences that hold in $\mathfrak{A}$, and the elementary diagram, $\Delta_{el}(\mathfrak{A})$, the set of all $\mathcal{L}_{\mathfrak{A}}$-sentences that hold in $\mathfrak{A}$.

2.3 Syntactical Characterisations

Now we have made clear our notational conventions, we shall have a closer look at syntactical characterisations. At first we shall take a closer look at the classical preservation theorems, which were in fact the first syntactical characterisations. We get these from [1] and [5] and shall not prove them here, but rather use them as an explanation to syntactical characterisation.

Let T be an $\mathcal{L}$-theory. We will link the preservation of the models of T under certain operations of $\mathcal{L}$-structures to the syntactical structure of axioms of T. Here a set of axioms of T is a theory with the same models of T.

We have three general preservation theories, about *preservation under substructures, preservation under unions of chains* and *preservation under homomorphic images.* Here we say a theory T is preserved under substructures if and only if every substructure of a model of T is again a model of T. A theory T is preserved under unions of chains if and only if the union of any chain of models of T is again a model of T. And finally, a theory T is preserved under homomorphic images if and only if every homomorphic image of a model of T is a model of T.

Now we have described the semantic properties, we can give the syntactical characterisations. These are in the following proposition.

Proposition 2.3.1 *A theory T is preserved under substructures if and only if it has a set of axioms consisting of universal sentences.*

A theory T is preserved under unions of chains if and only if it has a set of axioms consisting of universal-existential sentences.

A theory T is preserved under homomorphic images if and only if it has a set of axioms consisting of positive sentences.

We can see how the syntactic characterisations link the semantic properties to some syntactical ones, such as just the use of universal sentences. The singularity of the preservation theorems is their gracefulness. They do not consist of long and complex descriptions, and all follow the same style.

The last of these theorems we should bear in mind, for we will have to use it in the last chapter about the congruence extension property.

2.4 Generalised Atomic Sets Of Formulas

To prove the syntactical characterisations in this paper, we shall use the notion of a generalised atomic set of formulas, or GA set. We get this notion from Keisler [3]. We shall first define what those sets are and how we can use them. Then we shall show how they link with diagrams and then show some general results that we can use in the following chapters.

Definition 2.4.1 *A generalised atomic set of formulas is a set F of $\mathcal{L}$-formulas such that:*

i. the set F is closed under substitution, i.e. if $\phi(x, \bar{x}) \in F$ and $x, x_1, ..., x_N$ are mutually distinct, then,

a) for all variables y, $\phi(y, \bar{x}) \in F$;

b) for all constants c, $\phi(c, \bar{x}) \in F$.

ii. the set F is closed under logical equivalence, i.e. if $\phi \in F$ and $\models \phi \leftrightarrow \psi$ then $\psi \in F$.

iii. all the formulas $x_1 = x_2 \in F$, where x_1 and x_2 are distinct variables.

iv. falsum is in F, i.e. $\perp \in F$.

Example We shall give some examples of GA sets that will be used in this paper:

1. The set $[\mathcal{L}]$ consisting of all $\mathcal{L}$-formulas.
2. The set $(\mathcal{L})$ consisting of all basic $\mathcal{L}$-formulas, i.e. all formulas equivalent to atomic and negated atomic $\mathcal{L}$-formulas.
3. The set $(\mathcal{L})^+$ consisting of all $\mathcal{L}$-formulas equivalent to atomic $\mathcal{L}$-formulas.

We can now describe some closure operations on the GA sets. We use subsets of the set $\{\exists, \forall, \wedge, \vee, \neg\}$ to describe a finitary closure operation on sets of formulas. This means:

$\exists$: if $\phi \in F$ then $\exists x \phi(x) \in F$ with x free in ϕ;

$\forall$: if $\phi \in F$ then $\forall x \phi(x) \in F$ with x free in ϕ;

$\wedge$: if $\phi, \psi \in F$ then $\phi \wedge \psi \in F$;

$\vee$: if $\phi, \psi \in F$ then $\phi \vee \psi \in F$;

$\neg$: if $\phi \in F$ then $\neg\phi \in F$.

Now let F be a GA set, then $\{\wedge, \vee\}F$ is the smallest set that contains F and is closed under finite disjunction and conjunction. We can give for example the GA set $(\mathcal{L})^+$, consisting of formulas equivalent to atomic formulas. Then we have that $\{\exists, \forall, \wedge, \vee, \neg\}(\mathcal{L})^+$ is the smallest set that contains the atomic formulas of $\mathcal{L}$ and is closed under $\exists, \forall, \wedge, \vee$ and $\neg$. This gives that $\{\exists, \forall, \wedge, \vee, \neg\}(\mathcal{L})^+ = [\mathcal{L}]$.

When the closure consists of only one element, for example $\{\vee\}F$, we shall write $\vee F$. We shall repeat the abbreviation when there is another closure of one element. We can write for example $\vee \wedge F$ for $\{\vee\}(\{\wedge\}F)$ or $\exists\{\forall, \neg\}F$ for $\{\exists\}(\{\forall, \neg\}F)$. Note that this means that, with $F = (\mathcal{L})$, we have $\forall\exists(\mathcal{L}) = \forall(\exists(\mathcal{L})) = \{\forall\}(\{\exists\}(\mathcal{L}))$. This gives that $\forall\exists(\mathcal{L}) \neq \exists\forall(\mathcal{L}) \neq \{\forall, \exists\}(\mathcal{L})$.

We shall write $\forall_0$ for the set $\{\wedge, \vee, \neg\}(\mathcal{L})^+$ of open $\mathcal{L}$-formulas, $\forall_1$ for the set $\forall(\{\wedge, \vee, \neg\}(\mathcal{L})^+)$ of universal $\mathcal{L}$-formulas, $\forall_1^+$ for the set $\forall(\{\wedge, \vee\}(\mathcal{L})^+)$ of positive universal $\mathcal{L}$-formulas, $\exists_1$ for the set $\exists(\{\wedge, \vee, \neg\}(\mathcal{L})^+)$ of existential $\mathcal{L}$-formulas and $\exists_1^+$ for the set $\exists(\{\wedge, \vee\}(\mathcal{L})^+)$ for the set of positive existential $\mathcal{L}$-formulas.

Furthermore, we shall use the notation $F(\mathfrak{A})$ for the GA-set of $\mathcal{L}_{\mathfrak{A}}$-formulas generated by F, i.e. $\{\phi(\bar{a}) : \phi(\bar{x}) \in F \text{ and } \bar{a} \in A\}$.

With $\Delta_F(\mathfrak{A})$ we denote the set of all $F(\mathfrak{A})$-sentences which hold in $\mathfrak{A}$. In the special cases of $\Delta_{(\mathcal{L})^+}(\mathfrak{A})$,$\Delta_{(\mathcal{L})}(\mathfrak{A})$ and $\Delta_{[\mathcal{L}]}(\mathfrak{A})$, we shall write respectively $\Delta^+(\mathfrak{A})$,$\Delta(\mathfrak{A})$ and $\Delta_{el}(\mathfrak{A})$, which we have seen earlier.

We will now define the morphism induced by the GA sets and show a general result of these morphisms.

Definition 2.4.2 *Let F be a GA-set of $\mathcal{L}$-formulas, let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathcal{L}$-structures and $f : \mathfrak{A} \to \mathfrak{B}$. Then f is called an F-morphism, or F-hom, if whenever $\phi \in F$, $\bar{a} \in A$ and $\mathfrak{A} \models \phi(\bar{a})$, then $\mathfrak{B} \models \phi(f(\bar{a}))$.*

Lemma 2.4.3 *Every F-morphism is an $\{\exists, \wedge, \vee\}F$-morphism.*

Proof Let f be an F-morphism. Now consider the set $G = \{\phi(\bar{x}) : \forall \bar{a} \in A$ $(\mathfrak{A} \models \phi(\bar{a})$ then $\mathfrak{B} \models \phi(f(\bar{a})))\}$. We can see by definition of the F-morphism, that $F \subseteq G$. Now we shall prove that G is closed under conjunction, disjunction and the existential quantifier.

First for conjunction, let $\phi_1(\bar{x}_1), \phi_2(\bar{x}_2) \in G$ and let, with $\bar{a}_1, \bar{a}_2 \in A$, $\mathfrak{A} \models \phi_1(\bar{a}_1) \wedge \phi_2(\bar{a}_2)$. This gives that $\mathfrak{A} \models \phi_1(\bar{a}_1)$ and $\mathfrak{A} \models \phi_2(\bar{a}_2)$. Since $\phi_1(\bar{x}_1), \phi_2(\bar{x}_2) \in G$, we have $\mathfrak{B} \models \phi_1(f(\bar{a}_1))$ and $\mathfrak{B} \models \phi_2(f(\bar{a}_2))$. Thus we see that $\mathfrak{B} \models \phi_1(f(\bar{a}_1)) \wedge \phi_2(f(\bar{a}_2))$. Hence we have that $\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \in G$.

Secondly we consider disjunction. Again let $\phi_1(\bar{x}_1), \phi_2(\bar{x}_2) \in G$, but now let, with $\bar{a}_1, \bar{a}_2 \in A$, $\mathfrak{A} \models \phi_1(\bar{a}_1) \vee \phi_2(\bar{a}_2)$. Thus we have that $\mathfrak{A} \models \phi_1(\bar{a}_1)$ or $\mathfrak{A} \models \phi_2(\bar{a}_2)$ and, with the fact $\phi_1(\bar{x}_1), \phi_2(\bar{x}_2) \in G$, $\mathfrak{B} \models \phi_1(f(\bar{a}_1))$ or $\mathfrak{B} \models \phi_2(f(\bar{a}_2))$. Thus we see that $\mathfrak{B} \models \phi_1(f(\bar{a}_1)) \vee \phi_2(f(\bar{a}_2))$ and we have $\phi_1(\bar{x}_1) \vee \phi_2(\bar{x}_2) \in G$.

Lastly we consider the existential quantifier. Let $\phi(\bar{x}) \in G$ and let $\mathfrak{A} \models \exists\bar{x}\phi(\bar{x})$. Then there is a $\bar{a} \in A$ such that $\mathfrak{A} \models \phi(\bar{a})$. With $\phi(\bar{x}) \in G$ this gives $\mathfrak{B} \models \phi(f(\bar{a}))\}$. We can see that then also $\mathfrak{B} \models \exists\bar{x}\phi(\bar{x})\}$, hence $\exists\bar{x}\phi(\bar{x}) \in G$.

We can conclude that G is closed under conjunction, disjunction and the existential quantifier. Then, since $F \subseteq G$, we can also see that $\{\wedge, \vee, \exists\}F \subseteq G$. Thus we have that f is an $\{\wedge, \vee, \exists\}F$-morphism. ∎

From lemma 2.4.3 we get with the definition of homomorphism, embedding and elementary embedding immediately the following corollary.

Corollary 2.4.4 *An $(\mathcal{L})^+$-morphism is a homomorphism, an $(\mathcal{L})$-morphism is an embedding and an $[\mathcal{L}]$-morphism is an elementary embedding.*

Now we shall have a look at the relation between the F-morphism and the diagram obtained by the set F.

Lemma 2.4.5 *The following are equivalent:*

i) There is an F-morphism $f : \mathfrak{A} \to \mathfrak{B}$;

ii) There is a $\mathcal{L}_{\mathfrak{A}}$-expansion of $\mathfrak{B}$ which is a model of $\Delta_F(\mathfrak{A})$.

Proof Firstly i) $\implies$ ii):
We assume i), so there is a $f \in F$-hom, which means $f : \mathfrak{A} \to \mathfrak{B}$ with for all $\phi \in F$ and tuples $\bar{a} \in A$, if $\mathfrak{A} \models \phi(\bar{a})$ then $\mathfrak{B} \models \phi(f(\bar{a}))$. Let now $a^{\mathfrak{B}} = f(\bar{a})$, i.e. the interpretation of the element $a \in A$ in $\mathfrak{B}$ is the element $f(a) \in B$. Now for all sentences ϕ in $\Delta_F(\mathfrak{A})$, we have $\phi \in F$ and $\mathfrak{A} \models \phi(\bar{a})$ for some $\bar{a} \in A$ (with $a^{\mathfrak{A}} = a$). This gives $\mathfrak{B} \models \phi(f(\bar{a}))$ and thus $\mathfrak{B} \models \phi(a^{\mathfrak{B}})$. This holds for all ϕ in $\Delta_F(\mathfrak{A})$ and hence $\mathfrak{B}$ with the described interpretation is a model of $\Delta_F(\mathfrak{A})$.
Now ii) $\implies$ i):
We have $\mathfrak{B} \models \phi(\bar{a}^{\mathfrak{B}})$ for all $\phi(\bar{x}) \in F$ and $\bar{a} \in A$ such that $\mathfrak{A} \models \phi(\bar{a})$. Now let $f(a) = a^{\mathfrak{B}}$, i.e. let the function f send all elements a of A to the interpretation of the element in $\mathfrak{B}$. Then we have that $\mathfrak{B} \models \phi(f(\bar{a}))$ whenever $\phi(\bar{x}) \in F$, $\bar{a} \in A$ and $\mathfrak{A} \models \phi(\bar{a})$, which gives that $f \in F$-hom. ∎

When we now consider $F = (\mathcal{L})$ and $F = [\mathcal{L}]$, this gives us the well known equivalences between homomorphisms, embeddings and expansions.

Corollary 2.4.6 *Giving a homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ is equivalent to giving an $\mathcal{L}_{\mathfrak{A}}$-expansion of $\mathfrak{B}$ which is a model of $\Delta^+(\mathfrak{A})$.*

Giving an embedding $f : \mathfrak{A} \to \mathfrak{B}$ is equivalent to giving an $\mathcal{L}_{\mathfrak{A}}$-expansion of $\mathfrak{B}$ which is a model of $\Delta(\mathfrak{A})$.

Giving an elementary embedding $f : \mathfrak{A} \to \mathfrak{B}$ is equivalent to giving an $\mathcal{L}_{\mathfrak{A}}$-expansion of $\mathfrak{B}$ which is a model of $\Delta_{el}(\mathfrak{A})$.

2.5 Basic Theorems of Model Theory

After the notion of GA sets, we shall now look at some more general model theory. In this section we shall consider some examples of methods to link models and theories. The first lemma we get from [5].

Lemma 2.5.1 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two $\mathcal{L}$-structures. Then the following are equivalent:*

i) $\mathfrak{A} \equiv \mathfrak{B}$*;*

ii) $\mathfrak{A}$ *and* $\mathfrak{B}$ *have a common elementary extension.*

Proof First we look at ii) $\implies$ i):
If $\mathfrak{A}$ and $\mathfrak{B}$ have a common elementary extension, there is a $\mathfrak{C}$ such that $\mathfrak{A} \preceq \mathfrak{C}$ and $\mathfrak{B} \preceq \mathfrak{C}$. Since $\mathfrak{A} \preceq \mathfrak{A}'$ implies $\mathfrak{A} \equiv \mathfrak{A}'$, we now have $\mathfrak{A} \equiv \mathfrak{C}$ and $\mathfrak{B} \equiv \mathfrak{C}$. Thus, $\mathfrak{A} \equiv \mathfrak{C} \equiv \mathfrak{B}$.
Now we look at i) $\implies$ ii):
To prove there is a common elementary extension, we consider the $\mathcal{L}_{\mathfrak{A}\mathfrak{B}}$-theory $T = \Delta_{el}(\mathfrak{A}) \cup \Delta_{el}(\mathfrak{B})$, with $\mathcal{L}_{\mathfrak{A}\mathfrak{B}} = \mathcal{L}_{\mathfrak{A}} \cup \mathcal{L}_{\mathfrak{B}}$. Both are $\mathcal{L}_{\mathfrak{A}\mathfrak{B}}$-theories when we take the constants from $\mathfrak{A}$ and $\mathfrak{B}$ disjoint. Now any model $\mathfrak{C}$ of T is a common elementary extension. So we have to prove a model exists. For this we shall use a proof by contradiction.
Suppose T has no model, then some finite subset of T has no model, by the Compactness Theorem. This means that there is a finite conjunction of elements of $\Delta_{el}(\mathfrak{A})$ and $\Delta_{el}(\mathfrak{B})$, which has no model. So consider $\phi(\bar{a}) \in \Delta_{el}(\mathfrak{A})$ and $\psi(\bar{b}) \in \Delta_{el}(\mathfrak{B})$ such that $\phi(\bar{a}) \wedge \psi(\bar{b})$ is inconsistent. This means that $\mathfrak{A}$ cannot be expanded with interpretations for $\bar{b}$ such that $\psi(\bar{b}^{\mathfrak{A}})$ holds. Then we know that $\mathfrak{A} \models \forall \bar{x} \neg \psi(\bar{x})$. But we know that $\mathfrak{B} \models \exists \bar{x} \psi(\bar{x})$ and $\mathfrak{A} \equiv \mathfrak{B}$, thus we have a contradiction. ∎

Lemma 2.5.2 *Let $\mathcal{L} \subseteq \mathcal{L}'$, $\mathfrak{A}$ an $\mathcal{L}$-structure and $\mathfrak{B}$ an $\mathcal{L}$'-structure. Also suppose that $\mathfrak{A}$ is elementary equivalent to the $\mathcal{L}$-reduct of $\mathfrak{B}$. Then there are an $\mathcal{L}$'-structure $\mathfrak{C}$, an elementary embedding f of $\mathcal{L}$-structures and an elementary embedding f' of $\mathcal{L}$'-structures, with the following diagram of embeddings:*

$$\begin{tikzcd}
\mathfrak{A} \arrow[dr, "f"] & \\
& \mathfrak{C} \\
\mathfrak{B} \arrow[ur, "f'"'] &
\end{tikzcd}$$

Proof We can use the second part of the proof of lemma 2.4.1 to show this result. This time we consider the $\mathcal{L}'_{\mathfrak{A}\mathfrak{B}}$-theory $T = \Delta_{el}(\mathfrak{A}) \cup \Delta_{el}(\mathfrak{B})$, with $\mathcal{L}'_{\mathfrak{A}\mathfrak{B}} = \mathcal{L}_{\mathfrak{A}} \cup \mathcal{L}'_{\mathfrak{B}}$. Again we take the constants from $\mathfrak{A}$ and $\mathfrak{B}$ disjoint. Now we get, with corollary 2.4.6, that any model $\mathfrak{C}$ of T gives indeed the required diagram.
So again for contradiction, suppose it has no model. This gives us, as above, some $\phi(\bar{a}) \in \Delta_{el}(\mathfrak{A})$ and $\psi(\bar{b}) \in \Delta_{el}(\mathfrak{B})$ such that $\phi(\bar{a}) \wedge \psi(\bar{b})$ is inconsistent.

We get that every $\mathcal{L}'_{\mathfrak{B}}$-structure satisfying $\psi(\bar{b})$ cannot be expanded with interpretations for $\bar{a}$ such that it satisfies ϕ, thus it will satisfy $\forall\bar{x}\neg\phi(\bar{x})$. But this is an $\mathcal{L}$-sentence, $\mathfrak{A} \models \exists\bar{x}\phi(\bar{x})$ and $\mathfrak{A}$ is elementary equivalent to the $\mathcal{L}$-reduct of $\mathfrak{B}$, which gives a contradiction.

We take a look at the method of completing diagrams. This can be confusing because we have two kinds of diagrams, the set of sentences and the image of functions. When we speak of completing a diagram, we always focus on the second one. We will use from [2] the way of describing when a diagram can be completed. If there are for the condition Y functions $f' : \mathfrak{B} \to \mathcal{D}$ and $g' : \mathfrak{C} \to \mathcal{D}$ such that $f' \circ f = g' \circ g$, whenever there are functions $f : \mathfrak{A} \to \mathcal{B}$ and $g : \mathfrak{A} \to \mathcal{C}$ for the condition X, then we say that the diagram

$$\begin{tikzcd} & \mathfrak{B} \arrow[dr, dashed, "f'"] & \\ \mathfrak{A} \arrow[ur, "f"] \arrow[dr, "g"'] & & \mathfrak{D} \\ & \mathfrak{C} \arrow[ur, dashed, "g'"'] & \end{tikzcd}$$

X

Y

can be completed. The next lemma is an example of such a diagram.

Lemma 2.5.3 *Every diagram of elementary embeddings between $\mathcal{L}$-structures*

$$\begin{tikzcd} & \mathfrak{B} \arrow[dr, dashed, "f'"] & \\ \mathfrak{A} \arrow[ur, "f"] \arrow[dr, "g"'] & & \mathfrak{D} \\ & \mathfrak{C} \arrow[ur, dashed, "g'"'] & \end{tikzcd}$$

f,g elementary embeddings,

f',g' elementary embeddings

can be completed.

Proof Since f and g are elementary embeddings, we can use corollary 2.4.6 to see that there are $\mathcal{L}_{\mathfrak{A}}$-expansions of $\mathfrak{B}$ and $\mathfrak{C}$ that are models of $\Delta_{el}(\mathfrak{A})$. To get these expansions, we interpret the constants $a \in \mathcal{L}_{\mathfrak{A}}$ as follows, $a^{\mathfrak{B}} = f(a)$ and $a^{\mathfrak{C}} = g(a)$. Now we have that $\mathfrak{B}$ and $\mathfrak{C}$ are $\mathcal{L}_{\mathfrak{A}}$-structures and, with both models of $\Delta_{el}(\mathfrak{A})$, $\mathfrak{B}$ also elementary equivalent to the $\mathcal{L}_{\mathfrak{A}}$-reduct of $\mathfrak{C}$. Thus we can see, with lemma 2.4.2, that there are elementary embeddings f' and g' and an $\mathcal{L}_{\mathfrak{A}}$-structure $\mathfrak{D}$ as in the diagram. Since $\mathfrak{D}$ is an $\mathcal{L}_{\mathfrak{A}}$-structure, there must be interpretations of the constants $a \in \mathcal{L}_{\mathfrak{A}}$. These interpretations must be $a^{\mathfrak{D}} = f'(f(a))$ and $a^{\mathfrak{D}} = g'(g(a))$, because f' and g' are elementary embeddings and hence $\mathfrak{D}$ must be a model of $\Delta_{el}(\mathfrak{A})$. This also gives $f'(f(a)) = g'(g(a))$, thus the diagram is commutative. ∎

Now we shall look at the so-called *Diagram Lemma*, which we get from [2]. This lemma will be shown to be very useful in the rest of this paper.

Lemma 2.5.4 *Let T be an $\mathcal{L}$-theory, F a GA set of $\mathcal{L}$-formulas and $\mathfrak{A}$ an $\mathcal{L}$-structure. Then for all $\phi(\bar{x}) \in \mathcal{L}$ and $\bar{a} \in A$ the following are equivalent:*

i) $T \cup \Delta_F(\mathfrak{A}) \models \phi(\bar{a})$;

ii) $\mathfrak{A} \models \psi(\bar{a})$ *for some* $\psi(\bar{x}) \in \exists\{\wedge\}F$ *such that* $T \models \psi \to \phi$,
i.e. there is a $\psi(\bar{x}) \in \{\psi(\bar{x}) \in \exists\{\wedge\}F : T \models \psi \to \phi\}$ *with* $\mathfrak{A} \models \psi(\bar{a})$.

Proof First i) $\implies$ ii):
We assume $T \cup \Delta_F(\mathfrak{A}) \models \phi(\bar{a})$. Then by Compactness there is a finite subset $\{\chi_1, ..., \chi_n\} \subseteq \Delta_F(\mathfrak{A})$ such that $T \cup \{\chi_1, ..., \chi_n\} \models \phi(\bar{a})$. Now let χ be equivalent to $\chi_1 \wedge ... \wedge \chi_n$, and let χ' be the $\mathcal{L}$-formula, with $\bar{b} \in A - \{\bar{a}\}$, such that $\chi = \chi'(\bar{a}, \bar{b})$ and $\mathfrak{A} \models \chi'(\bar{a}, \bar{b})$. We now have $T \models \chi'(\bar{a}, \bar{b}) \to \phi(\bar{a})$, $\mathfrak{A} \models \chi'(\bar{a}, \bar{b})$ and $\chi'(\bar{x}, \bar{y}) \in \{\wedge\}F$. This gives that $T \models \exists\bar{y}\chi'(\bar{a}, \bar{y}) \to \phi(\bar{a})$, $\mathfrak{A} \models \exists\bar{y}\chi'(\bar{a}, \bar{y})$ and $\exists\bar{y}\chi'(\bar{x}, \bar{y}) \in \exists\{\wedge\}F$. Now let $\psi(\bar{x})$ be $\exists\bar{y}\chi'(\bar{x}, \bar{y})$. Thus we have $T \models \psi \to \phi$, $\mathfrak{A} \models \phi(\bar{a})$ and $\phi(\bar{x}) \in \exists\{\wedge\}F$.
Now ii) $\implies$ i):
Let there be a $\psi(\bar{x}) \in \{\psi(\bar{x}) \in \exists\{\wedge\}F : T \models \psi \to \phi\}$. There must be a $\chi(\bar{x}, \bar{y}) \in \wedge F$ such that $\psi(\bar{x})$ is $\exists\bar{y}\chi(\bar{x}, \bar{y})$. Then, with $\mathfrak{A} \models \psi(\bar{a})$, there is a $\bar{b} \in A$ such that $\mathfrak{A} \models \chi(\bar{a}, \bar{b})$. Now let $\mathfrak{B}$ be some model of $T \cup \Delta_F(\mathfrak{A})$. Since $\chi(\bar{a}, \bar{b})$ is a finite union of elements in $F(\mathfrak{A})$ and $\mathfrak{A} \models \chi(\bar{a}, \bar{b})$, now $\mathfrak{B} \models \chi(\bar{a}, \bar{b})$. This gives $\mathfrak{B} \models \exists\bar{y}\chi(\bar{a}, \bar{y}) \leftrightarrow \psi(\bar{a})$. With the fact that $T \models \psi \to \phi$, this gives $B \models \phi(\bar{a})$ and thus $T \cup \Delta_F(\mathfrak{A}) \models \phi(\bar{a})$. ∎

Finally, to conclude this chapter, we shall have a look at another lemma we get from [2]. This lemma is a method of making a lot of statements simpler and easier to read.

Lemma 2.5.5 *Let S be a set of $\mathcal{L}$-formulas and let for each $\phi \in S$, H_ϕ be a set of $\mathcal{L}$-formulas. Then the following are equivalent:*

i) $T \models \bigwedge_{\phi \in S}(\phi \to \bigvee_{\psi \in H_\phi} \psi)$;

ii) for all $\phi \in S$ there is a $\psi \in \{\vee\}H_\phi$ such that $T \models \phi \to \psi$.

Proof First i) $\implies$ ii):
We assume $T \models \bigwedge_{\phi \in S}(\phi \to \bigvee_{\psi \in H_\phi} \psi)$. Let $\phi \in S$, thus we have $T \models \phi \to \bigvee_{\psi \in H_\phi} \psi$. Now we shall use a proof by contradiction. So suppose there is no $\psi \in \{\vee\}H_\phi$ such that $T \models \phi \to \psi$. Then we have for every finite subset $\{\psi_1, ..., \psi_n\} \subset H_\phi$, $T \cup \{\phi\} \cup \{\neg\psi_1, ..., \neg\psi_n\}$ is consistent. With compactness this gives that $T \cup \{\phi\} \cup \{\neg\psi \| \psi \in H_\phi\}$ is consistent, but that contradicts with $T \models \phi \to \bigvee_{\psi \in H_\phi} \psi$. Thus we have for all $\phi \in S$ there is a $\psi \in \{\vee\}H_\phi$ such that $T \models \phi \to \psi$.
Now ii) $\implies$ i):
If for all $\phi \in S$ there is a $\psi \in \{\vee\}H_\phi$ such that $T \models \phi \to \psi$, we also have for all $\phi \in S$, $T \models \phi \to \bigvee_{\psi \in H_\phi} \psi$. This is simply equivalent to $T \models \bigwedge_{\phi \in S}(\phi \to \bigvee_{\psi \in H_\phi} \psi)$. ∎

Chapter 3

Amalgamation Properties

In this chapter we will take a look at the amalgamation properties, to give for all of the properties a syntactical characterisation. Here we will stay closely to the steps of Bacsich and Hughes in [2], so the definitions, lemmas and theorems in this chapter are all from [2]. We will start with the general amalgamation property, then have a short look at the property of Injections Transferable and afterwards we will consider the strong amalgamation property.

3.1 The Amalgamation Property

We will start this section by giving some definitions. We will not only define the amalgamation property itself, but also of the morphism property and amalgamation bases, since we need them to give a syntactical characterisation of the amalgamation property.

Definition 3.1.1 (Morphism Property) *Let T_1, T_2 and T_3 be $\mathcal{L}$-theories and let F, G be GA sets of $\mathcal{L}$-formulas. Then the triple $< T_1, T_2, T_3 >$ has the* (F,G)-Morphism Property ((F,G)-MP) *if for all $\mathfrak{A}_1 \models T_1$ and $\mathfrak{A}_2 \models T_2$ there are $\mathfrak{A}_3 \models T_3$ and $f : \mathfrak{A}_1 \to \mathfrak{A}_3$, $g : \mathfrak{A}_2 \to \mathfrak{A}_3$ with $f \in F$-hom and $g \in G$-hom.*

Definition 3.1.2 (Amalgamation Base) *Let T be an $\mathcal{L}$-theory, $\mathfrak{A} \models T$ and E, F, G and H GA sets of $\mathcal{L}$-formulas. Then $\mathfrak{A}$ is an* (E,F,G,H)-amalgamation base ((E,F,G,H)-a. base) *if any diagram $\mathfrak{M}(T)$ of the form*

$$\begin{tikzcd} & \mathfrak{B} \arrow[dr, dashed, "g"] & \\ \mathfrak{A} \arrow[ur, "e"] \arrow[dr, "f"'] & & \mathfrak{D} \\ & \mathfrak{C} \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

$e \in E$-hom, $f \in F$-hom,

$g \in G$-hom, $h \in H$-hom

can be completed.

Definition 3.1.3 (Amalgamation Property) *Let E, F, G and H be GA sets* *of $\mathcal{L}$-formulas. Then an $\mathcal{L}$-theory T has the* (E, F, G, H)-Amalgamation Property ((E, F, G, H)-AP) *if every model of T is an (E, F, G, H)-a. base.*

With these definitions we can now consider some lemmas and theorems that can help obtain the syntactic characterisations. But first we have to clarify some more notation. When we consider the Joint Embedding Property (or JEP) we mean the $((\mathcal{L}), (\mathcal{L}))$-Morphism Property of the triple $< T, T, T >$. When we simply denote amalgamation base (a. base) or Amalgamation Property (AP), we consider the situation in which $E = F = G = H = (\mathcal{L})$. Lastly, the property Injections Transferable (or IT) implies the situation in which $E = H = (\mathcal{L})$ and $F = G = (\mathcal{L})^+$.

The next proposition will show how the Morphism Property is linked to the Amalgamation Property.

Proposition 3.1.4 *Let E, F, G and H be GA sets of $\mathcal{L}$-formulas, let T be an $\mathcal{L}$-theory and let $\mathfrak{A}$ be a model of T. Then the following are equivalent:*

i) $\mathfrak{A}$ is an (E, F, G, H)-amalgamation base for T;

ii) $< T \cup \Delta_E(\mathfrak{A}), T \cup \Delta_F(\mathfrak{A}), T \cup \{a = a : a \in A\} >$ has the $(G(\mathfrak{A}), H(\mathfrak{A}))$-Morphism Property.

With the properties defined, we can now start to describe how we can find a syntactic equivalence to the Amalgamation Property. We will start with the Morphism Property and build from there.

Lemma 3.1.5 *Let T_1, T_2 and T_3 be $\mathcal{L}$-theories. The following are equivalent:*

i) $< T_1, T_2, T_3 >$ has the (F, G)-MP;

ii) for all sentences $\phi \in \exists\{\vee, \wedge\}F$, $\psi \in \exists\{\vee, \wedge\}G$, if $T_1 \cup \{\phi\}$ and $T_2 \cup \{\psi\}$ are consistent then $T_3 \cup \{\phi, \psi\}$ is consistent.

Proof Firstly i) $\implies$ ii):
Suppose $T_1 \cup \{\phi\}$ and $T_2 \cup \{\psi\}$ are consistent. This gives us models $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \models T_1 \cup \{\phi\}$ and $\mathfrak{B} \models T_2 \cup \{\psi\}$. This means for some $\bar{a} \in A$, $\bar{b} \in B$ that $\mathfrak{A} \models \phi(\bar{a})$ and $\mathfrak{B} \models \psi(\bar{b})$. Now with i), there is a $\mathfrak{C} \models T_3$ with $f : \mathfrak{A} \to \mathfrak{C}$, $g : \mathfrak{B} \to \mathfrak{C}$ with $f \in F$-hom and $g \in G$-hom. We have seen in 2.4.3 that an F-morphism is equal to an $\{\exists, \vee, \wedge\}F$-morphism, hence we have $\mathfrak{C} \models \phi(f(\bar{a}))$ and $\mathfrak{C} \models \psi(g(\bar{b}))$. This gives us that $\mathfrak{C} \models T_3 \cup \{\phi, \psi\}$.
Secondly ii) $\implies$ i):
We will use contradiction, so suppose that $< T_1, T_2, T_3 >$ doesn't have the (F, G)-MP. Now let $T_1 \cup \{\phi\}$ and $T_2 \cup \{\psi\}$ be consistent, with respectively the models $\mathfrak{A}$ and $\mathfrak{B}$. We know that there cannot be a model $\mathfrak{C}$ of T_3 such that $f : \mathfrak{A} \to \mathfrak{C}$, $g : \mathfrak{B} \to \mathfrak{C}$ with $f \in F$-hom and $g \in G$-hom. Since giving such morphisms is equivalent to giving an $\mathcal{L}_{\mathfrak{A}\mathfrak{B}}$-expansion of a model such that it is a model of $\Delta_F(\mathfrak{A}) \cup \Delta_G(\mathfrak{B})$ (when we take the interpretations of the elements

of $\mathfrak{A}$ and $\mathfrak{B}$ disjoint), we now see that $T_3 \cup \Delta_F(\mathfrak{A}) \cup \Delta_G(\mathfrak{B})$ is inconsistent. Then the Compactness Theorem gives that there is a finite subtheory which is also inconsistent, thus there are some finite subsets of $\Delta_F(\mathfrak{A})$ and $\Delta_G(\mathfrak{B})$ that united with T_3 are inconsistent. This gives for some $\phi(\bar{x}) \in \wedge F$, $\psi(\bar{x}) \in \wedge G$ and lists $\bar{a} \in A$ and $\bar{b} \in B$ such that $\mathfrak{A} \models \phi(\bar{a})$ and $\mathfrak{B} \models \psi(\bar{b})$, that $T_3 \cup \{\phi(\bar{a}), \psi(\bar{b})\}$ is inconsistent. With $\phi(\bar{a}) \leftrightarrow \exists\bar{x}\phi(\bar{x})$, we now have that for all sentences $\phi \in \exists\{\vee, \wedge\}F$, $\psi \in \exists\{\vee, \wedge\}G$, $T_3 \cup \{\phi, \psi\}$ is inconsistent. Hence ii) does not hold. ∎

With this lemma we can already see the way in which the properties link between semantic properties and syntactic conditions. This lemma considers existential sentences of GA sets, but when we take the contrapositive we are able to consider universal sentences.

Corollary 3.1.6 *Let T_1, T_2 and T_3 be $\mathcal{L}$-theories. The following are equivalent:*

i) $< T_1, T_2, T_3 >$ has the (F, G)-MP

ii) for all sentences $\xi \in \forall\{\vee, \wedge\}F', \chi \in \forall\{\vee, \wedge\}G'$, if $T_3 \models \xi \vee \chi$ then $T_1 \models \xi$ or $T_2 \models \chi$, where $F' = \{\neg\phi : \phi \in F\}$ and $G' = \{\neg\psi : \psi \in G\}$.

Proof We can look at the contrapositive of ii) of lemma 3.1.5, i.e.:
for all sentences $\phi \in \exists\{\vee, \wedge\}F$, $\psi \in \exists\{\vee, \wedge\}G$, if $T_3 \cup \{\phi, \psi\}$ isn't consistent then it doesn't hold that $T_1 \cup \{\phi\}$ and $T_2 \cup \{\psi\}$ are consistent.
We know that not consistent means inconsistent and $\neg(\phi \wedge \psi) \leftrightarrow \neg\phi \vee \neg\psi$, thus:
for all sentences $\phi \in \exists\{\vee, \wedge\}F$, $\psi \in \exists\{\vee, \wedge\}G$, if $T_3 \cup \{\phi, \psi\}$ is inconsistent then $T_1 \cup \{\phi\}$ is inconsistent or $T_2 \cup \{\psi\}$ is inconsistent.
This gives, with basic conditions for theories:
for all sentences $\phi \in \exists\{\vee, \wedge\}F$, $\psi \in \exists\{\vee, \wedge\}G$, if $T_3 \models \neg\phi \vee \neg\psi$ then $T_1 \models \neg\phi$ or $T_2 \models \neg\psi$.
Now we can use $\neg\exists\bar{x}\phi(\bar{x}) \leftrightarrow \forall\bar{x}\neg\phi(\bar{x})$, which gives finally:
for all sentences $\xi \in \forall\{\vee, \wedge\}F'$, $\chi \in \forall\{\vee, \wedge\}G'$, if $T_3 \models \xi \vee \chi$ then $T_1 \models \xi$ or $T_2 \models \chi$, where $F' = \{\neg\phi : \phi \in F\}$ and $G' = \{\neg\psi : \psi \in G\}$.
It is clear now that this corollary is indeed equivalent to lemma 3.1.5. ∎

When we recall that the JEP is the $((\mathcal{L}), (\mathcal{L}))$-Morphism Property of the triple $< T, T, T >$ and that the set $\forall_1$ of all universal $\mathcal{L}$-sentences is $\forall(\{\wedge, \vee, \neg\}(\mathcal{L})^+)$, we see that the above is equivalent to the next corollary.

Corollary 3.1.7 *A $\mathcal{L}$-theory T has the JEP iff for all universal sentences ϕ, ψ of $\mathcal{L}$, if $T \models \phi \vee \psi$ then $T \models \phi$ or $T \models \psi$.*

With this equivalence we have a description of the JEP in terms of universal sentences in an $\mathcal{L}$-theory. This already gives a glimpse of how to describe a syntactic equivalence of the Amalgamation Property. But before we can do that, we have to look at the amalgamation bases first.

Theorem 3.1.8 *Let T be an $\mathcal{L}$-theory and let $\mathfrak{A}$ be a model of T. Then the following are equivalent:*

i) $\mathfrak{A}$ *is an amalgamation base for* T*;*

ii) for all $\phi_1(\bar{x}), \phi_2(\bar{x}) \in \forall_1$ *with* $T \models \phi_1 \vee \phi_2$ *there are* $\psi_1(\bar{x}), \psi_2(\bar{x}) \in \exists_1$ *such that* $T \models \psi_1 \rightarrow \phi_1$*,* $T \models \psi_2 \rightarrow \phi_2$ *and either* $\mathfrak{A} \models \forall\bar{x}\psi_1(\bar{x})$ *or* $\mathfrak{A} \models \forall\bar{x}\psi_2(\bar{x})$*.*

Proof With lemma 3.1.4 we see that $\mathfrak{A}$ is an amalgamation base for T iff $< T \cup \Delta(\mathfrak{A}), T \cup \Delta(\mathfrak{A}), T \cup \{a = a : a \in A\} >$ has the $((\mathcal{L})(\mathfrak{A}), (\mathcal{L})(\mathfrak{A}))$-morphism property. We note that $\{\neg\phi : \phi \in (\mathcal{L})\} = (\mathcal{L})$, since $(\mathcal{L})$ is closed under negation. This gives us, with corollary 3.1.6, that $< T \cup \Delta(\mathfrak{A}), T \cup \Delta(\mathfrak{A}), T \cup \{a = a : a \in A\} >$ has the $((\mathcal{L})(\mathfrak{A}), (\mathcal{L})(\mathfrak{A}))$-morphism property iff for all sentences $\phi_1(\bar{x}), \phi_2(\bar{x}) \in \forall_1$ and $\bar{a} \in A$, if $T \cup \{a = a : a \in A\} \models \phi_1(\bar{a}) \vee \phi_2(\bar{a})$ then $T \cup \Delta(\mathfrak{A}) \models \phi_1(\bar{a})$ or $T \cup \Delta(\mathfrak{A}) \models \phi_2(\bar{a})$. With lemma 2.4.3, we know that $T \cup \Delta(\mathfrak{A}) \models \phi_i(\bar{a})$ iff there is some $\psi_i(\bar{x}) \in \{\psi_i(\bar{x}) \in \exists_1 : T \models \psi_i \rightarrow \phi_i\}$ with $\mathfrak{A} \models \psi_i(\bar{a})$. Now because of the fact that $\phi_1(\bar{x}), \phi_2(\bar{x}) \in \forall_1$ and $T \cup \{a = a : a \in A\} \models \phi_1(\bar{a}) \vee \phi_2(\bar{a})$, it even holds that there is a $\psi_i\bar{x}$ such that $\mathfrak{A} \models \psi_i(\bar{a})$ for all $\bar{a} \in A$ and thus $\mathfrak{A} \models \forall\bar{x}\psi_i(\bar{x})$. We now get that for all sentences $\phi_1(\bar{x}), \phi_2(\bar{x}) \in \forall_1$ and $\bar{a} \in A$, if $T \cup \{a = a : a \in A\} \models \phi_1(\bar{a}) \vee \phi_2(\bar{a})$ then $T \cup \Delta(\mathfrak{A}) \models \phi_1(\bar{a})$ or $T \cup \Delta(\mathfrak{A}) \models \phi_2(\bar{a})$ iff for all $< \phi_1(\bar{x}), \phi_2(\bar{x}) > \in \{< \phi_1, \phi_2 >: \phi_1, \phi_2 \in \forall_1$ and $T \models \phi_1 \vee \phi_2\}$ and some $\psi_i \in \{\psi \in \exists_1 : T \models \psi_i \rightarrow \phi_i\}$, $\mathfrak{A} \models \forall\bar{x}\psi_1(\bar{x})$ or $\mathfrak{A} \models \forall\bar{x}\psi_2(\bar{x})$. Hence we have the desired equivalence. ∎

With this we almost have a syntactic characterisation of the Amalgamation Property. All that is left to do is apply the definition of the amalgamation property on theorem 3.1.8 to get the equivalence.

Corollary 3.1.9 *Let* T *be a* $\mathcal{L}$*-theory. The following are equivalent:*

i) T *has the Amalgamation Property;*

ii) for all $\phi_1(\bar{x}), \phi_2(\bar{x}) \in \forall_1$ *with* $T \models \phi_1 \vee \phi_2$ *there are* $\psi_1(\bar{x}), \psi_2(\bar{x}) \in \exists_1$ *such that* $T \models \psi_1 \rightarrow \phi_1$*,* $T \models \psi_2 \rightarrow \phi_2$ *and* $T \models \psi_1 \vee \psi_2$*.*

Proof We know that T has the AP if and only if every model of T is an amalgamation base. Thus with theorem 3.1.8 we have that every model $\mathfrak{A}$ of T, $\mathfrak{A} \models \forall\bar{x}\psi_1(\bar{x})$ or $\mathfrak{A} \models \forall\bar{x}\psi_2(\bar{x})$ for all $< \phi_1(\bar{x}), \phi_2(\bar{x}) > \in \{< \phi_1, \phi_2 >: \phi_1, \phi_2 \in \forall_1$ and $T \models \phi_1 \vee \phi_2\}$ and some $\psi_i \in \{\psi \in \exists_1 : T \models \psi_i \rightarrow \phi_i\}$. This gives that for all $\phi_1, \phi_2 \in \forall_1$ and $T \models \phi_1 \vee \phi_2$ we have some $\psi_1, \psi_2 \in \{\psi \in \exists_1 : T \models \psi_i \rightarrow \phi_i\}$ such that $T \cup \{\psi_1\}$ or $T \cup \{\psi_2\}$ is consistent, which gives the characterisation. ∎

3.2 Injections Transferable

In this section we shall have a look at theories for which injections are transferable. We recall that a theory T has IT if T has the $((\mathcal{L}), (\mathcal{L})^+, (\mathcal{L})^+, (\mathcal{L}))$-amalgamation property. Since we can see that this is close to the general amalgamation property, we can use the same steps to get the characterisation. So we will start to look at a base for Injections Transferable.

Lemma 3.2.1 *Let T be a $\mathcal{L}$-theory and $\mathfrak{A} \models T$. The following are equivalent:*

i) *$\mathfrak{A}$ is a $((\mathcal{L}), (\mathcal{L})^+, (\mathcal{L})^+, (\mathcal{L}))$-amalgamation base, thus a base for injections transferable;*

ii) *for all formulas $\phi_1 \in \exists_1^+$ and $\phi_2 \in \exists_1$, such that $T \cup \{\phi_1, \phi_2\}$ is inconsistent, there exist formulas $\psi_1 \in \exists_1$ and $\psi_2 \in \exists_1^+$, such that $T \cup \{\phi_1, \psi_1\}$ and $T \cup \{\phi_2, \psi_2\}$ are inconsistent and either $\mathfrak{A} \models \forall \bar{x} \psi_1(\bar{x})$ or $\mathfrak{A} \models \forall \bar{x} \psi_2(\bar{x})$.*

Proof We know that $\mathfrak{A}$ is a $((\mathcal{L}), (\mathcal{L})^+, (\mathcal{L})^+, (\mathcal{L}))$-amalgamation base if and only if $< T \cup \Delta(\mathfrak{A}), T \cup \Delta(\mathfrak{A}), T \cup \{a = a : a \in A\} >$ has the $((\mathcal{L}_{\mathfrak{A}})^+, (\mathcal{L}_{\mathfrak{A}})$-morphism property. We use lemma 3.1.5 and see that this holds if and only if for all $\phi_1(\bar{x}) \in \exists_1^+, \phi_2(\bar{x}) \in \exists_1$, if $T \cup \Delta(\mathfrak{A}) \cup \{\phi_1\}$ and $T \cup \Delta(\mathfrak{A}) \cup \{\phi_2\}$ are consistent then $T \cup \{a = a : a \in A\} \cup \{\phi_1, \phi_2\}$ is consistent. This is equivalent to if $T \cup \{a = a : a \in A\} \not\models \phi_1(\bar{a}) \wedge \phi_2(\bar{a})$ then $T \cup \Delta(\mathfrak{A}) \not\models \phi_1(\bar{a})$ or $T \cup \Delta(\mathfrak{A}) \not\models \phi_2(\bar{a})$. We can replace the last part by saying $T \cup \Delta(\mathfrak{A}) \models \neg\phi_1(\bar{a})$ or $T \cup \Delta(\mathfrak{A}) \models \neg\phi_2(\bar{a})$. With lemma 2.4.3 this is equivalent to for all $\phi_1(\bar{x}) \in \exists_1^+, \phi_2(\bar{x}) \in \exists_1$, if $T \cup \{a = a : a \in A\} \not\models \phi_1(\bar{a}) \wedge \phi_2(\bar{a})$ then there are some $\psi_1(\bar{x}) \in \exists_1$ and $\psi_2(\bar{x}) \in \exists_1^+$ such that $T \models \psi_1 \to \neg\phi_1$, $T \models \psi_2 \to \neg\phi_2$ and $\mathfrak{A} \models \psi_1(\bar{a}) \vee \psi_2(\bar{a})$ for all $\bar{a} \in A$. Now by noticing that $T \cup \{a = a : a \in A\} \not\models \phi_1(\bar{a}) \wedge \phi_2(\bar{a})$ is equivalent to $T \cup \{\phi_1, \phi_2\}$ is inconsistent and $T \models \psi_i \to \neg\phi_i$ equivalent to $T \cup \{\phi_i, \psi_i\}$ is inconsistent, we have the characterisation. ∎

Now with the same proof as in 3.1.9, we can get from the characterisation of a base for Injections Transferable to the syntactical characterisation of Injections Transferable. Hence we have the following corollary.

Corollary 3.2.2 *Let T be a $\mathcal{L}$-theory. The following are equivalent:*

i) *Injections are transferable in T;*

ii) *for all $\phi_1(\bar{x}) \in \exists_1^+, \phi_2(\bar{x}) \in \exists_1$ with $T \cup \{\phi_1(\bar{x}), \phi_2(\bar{x})\}$ is inconsistent, there are $\psi_1(\bar{x}) \in \exists_1$ and $\psi_2(\bar{x}) \in \exists_1^+$ such that $T \cup \{\phi_1(\bar{x}), \psi_1(\bar{x})\}$ and $T \cup \{\phi_2(\bar{x}), \psi_2(\bar{x})\}$ are inconsistent and $T \models \psi_1(\bar{x}) \vee \psi_2(\bar{x})$.*

3.3 The Strong Amalgamation Property

The last section of this chapter consists of getting a syntactic characterisation of the strong amalgamation theorem. Our steps will be almost the same as in section 3.1, with the general amalgamation property. The difference is that we will not look at a 'Strong Joint Embedding Property", but rather at a 'Strong Embedding Property'. Once again we shall start with the definitions necessary in this section.

Definition 3.3.1 (Strong Amalgamation Base) *Let T be an $\mathcal{L}$-theory and $\mathfrak{A}$ be a model of T. Then $\mathfrak{A}$ is a* strong amalgamation base (s.a. base) *for T if every diagram of the form*

$$\begin{tikzcd} & \mathfrak{B} \arrow[dr, dashed, "f'"] & \\ \mathfrak{A} \arrow[ur, "f"] \arrow[dr, "g"'] & & \mathfrak{D} \\ & \mathfrak{C} \arrow[ur, dashed, "g'"'] & \end{tikzcd}$$

f, g, f' and g' embeddings

f'f($\mathfrak{A}$) = f'($\mathfrak{B}$)$\cap$ g'($\mathfrak{C}$)

can be completed.

Definition 3.3.2 (Strong Amalgamation Property) *An $\mathcal{L}$-theory T has the* Strong Amalgamation Property (SAP) *if every model of T is a strong amalgamation base.*

Definition 3.3.3 (Strong Embedding Property) *Let T_1, T_2 and T_3 be $\mathcal{L}$ theories. Then the triple $< T_1, T_2, T_3 >$ has the* Strong Embedding Property (SEP) *if given $\mathfrak{A}_1 \models T_1$ and $\mathfrak{A}_2 \models T_2$ there is $\mathfrak{A}_3 \models T_3$ and embeddings $f_1 : \mathfrak{A}_1 \to \mathfrak{A}_3$ and $f_2 : \mathfrak{A}_2 \to \mathfrak{A}_3$, such that $f_1(\mathfrak{A}_1) \cap f_2(\mathfrak{A}_2) = D(\mathfrak{A}_3)$. Here is for any langage $\mathcal{L}$, $D(\mathfrak{A})$ the substructure of $\mathfrak{A}$ built on the constant terms of $\mathcal{L}$.*

We can now see the difference between the JEP and the SEP. The next proposition shows that our SEP is well chosen, and it shows, quite similar to proposition 3.1.4, the link between the strong embedding property and the strong amalgamation property.

Proposition 3.3.4 *Let T be an $\mathcal{L}$-theory and $\mathfrak{A}$ be a model of T. Then the following are equivalent:*

i) $\mathfrak{A}$ is a strong amalgamation base for T;

ii) $< T \cup \Delta(\mathfrak{A}), T \cup \Delta(\mathfrak{A}), T \cup \{a = a : a \in A\} >$ has the SEP.

Now we shall look at the syntactical characterisation of the strong embedding property, which already gives a glimp of the characterisation of the strong amalgamtion property itself.

Theorem 3.3.5 *Let T_1, T_2 and T_3 be $\mathcal{L}$ theories. Then the following are equivalent:*

i) $< T_1, T_2, T_3 >$ has the SEP;

ii) for all $\phi(\bar{x}), \psi(\bar{y}) \in \wedge(\mathcal{L})$, if $T_3 \models \phi(\bar{x}) \wedge \psi(\bar{y}) \to \bar{x} \cap \bar{y} \neq \emptyset$ then there is a list $\bar{t}$ of constant terms of $\mathcal{L}$ such that either $T_1 \models \phi(\bar{x}) \to \bar{x} \cap \bar{t} \neq \emptyset$ or $T_2 \models \psi(\bar{y}) \to \bar{y} \cap \bar{t} \neq \emptyset$.

Proof Firstly i) $\implies$ ii):
At first we assume that $< T_1, T_2, T_3 >$ has the SEP. Now suppose that $T_3 \models \phi(\bar{x}) \wedge \psi(\bar{y}) \to \bar{x} \cap \bar{y} \neq \emptyset$. If $T_1 \cup \{\phi(\bar{x})\}$ is inconsistent, then $T_1 \models \phi(\bar{x}) \to \perp \to \bar{x} \cap \bar{t} \neq \emptyset$ thus $T_1 \models \phi(\bar{x}) \to \bar{x} \cap \bar{t} \neq \emptyset$ holds. The same idea if $T_2 \cup \{\psi(\bar{x})\}$ is inconsistent.

So we will now consider the case where both $T_1 \cup \{\phi(\bar{x})\}$ and $T_2 \cup \{\psi(\bar{x})\}$ are consistent. So let $\mathfrak{A}_1 \models T_1 \cup \{\phi(\bar{a})\}$ and $\mathfrak{A}_2 \models T_2 \cup \{\psi(\bar{b})\}$ be some arbitrary models. Since $< T_1, T_2, T_3 >$ has the SEP, there is a $\mathfrak{A}_3 \models T_3$ and embeddings $f_1 : \mathfrak{A}_1 \to \mathfrak{A}_3$, $f_2 : \mathfrak{A}_2 \to \mathfrak{A}_3$ such that $f_1(\mathfrak{A}_1) \cap f_2(\mathfrak{A}_2) = D(\mathfrak{A}_3)$.

Since f_1 and f_2 are embeddings and $\phi(\bar{x}), \psi(\bar{y}) \in \wedge(\mathcal{L})$, we know with lemma 2.4.3 that $\mathfrak{A}_3 \models \phi(f_1(\bar{a})) \wedge \psi(f_2(\bar{b}))$. Now by the fact that $T_3 \models \phi(\bar{x}) \wedge \psi(\bar{y}) \to \bar{x} \cap \bar{y} \neq \emptyset$, there must be some $a \in \bar{a}$ and some $b \in \bar{b}$ such that $f_1(a) = f_2(b)$. Then by $f_1(\mathfrak{A}_1) \cap f_2(\mathfrak{A}_2) = D(\mathfrak{A}_3)$, $f_1(a) = f_2(b) \in D(\mathfrak{A}_3)$. Since $D(\mathfrak{A})$ is the substructure of $\mathfrak{A}$ built on the constant terms of $\mathcal{L}$, now $a \in D(\mathfrak{A}_1)$ and $b \in D(\mathfrak{A}_2)$ must hold. Hence we see that there must be some list $\bar{t}$ of constant terms of $\mathcal{L}$ such that $\mathfrak{A}_1 \models \phi(\bar{a}) \to \bar{a} \cap \bar{t} \neq \emptyset$ and $\mathfrak{A}_2 \models \psi(\bar{b}) \to \bar{b} \cap \bar{t} \neq \emptyset$. Since $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are some arbitrary models of respectively $T_1 \cup \{\phi(\bar{x})\}$ and $T_2 \cup \{\psi(\bar{x})\}$, this must hold for all models. Thus we have $T_1 \models \phi(\bar{x}) \to \bar{x} \cap \bar{t} \neq \emptyset$ and $T_2 \models \psi(\bar{y}) \to \bar{y} \cap \bar{t} \neq \emptyset$, which certainly implies at least one of them holds.
Secondly ii) $\implies$ i):
We know that for $< T_1, T_2, T_3 >$ to have the SEP, there must be for all $\mathfrak{A}_1 \models T_1$, $\mathfrak{A}_2 \models T_2$ a model $\mathfrak{A}_3 \models T_3$ such that also $\mathfrak{A}_3 \models \Delta(\mathfrak{A}_1)$, $\mathfrak{A}_3 \models \Delta(\mathfrak{A}_2)$ and $\mathfrak{A}_3 \models \{a \neq b : a \in A - D(\mathfrak{A}_1), b \in B - D(\mathfrak{A}_2)\}$, i.e. $T_3 \cup \Delta(\mathfrak{A}_1) \cup \Delta(\mathfrak{A}_2) \cup \{a \neq b : a \in A_1 - D(\mathfrak{A}_1), b \in A_2 - D(\mathfrak{A}_2)\}$ is consistent. Here we can consider $\Delta(\mathfrak{A})$ in the language $\mathcal{L}_{\mathfrak{A} - D(\mathfrak{A})}$, since $D(\mathfrak{A})$ consists of constant terms of $\mathcal{L}$.

Now suppose it is inconsistent. Then by the Compactness Theorem there must be a finite subtheory that is inconsistent, hence we have some $\phi(\bar{x}), \psi(\bar{y}) \in \wedge(\mathcal{L})$ and lists $\bar{a} \in A_1 - D(\mathfrak{A}_1), \bar{b} \in A_2 - D(\mathfrak{A}_2)$ such that $T_3 \cup \{\phi(\bar{a}), \psi(\bar{b}), \bar{a} \cap \bar{b} = \emptyset\}$ is inconsistent, $\mathfrak{A}_1 \models \phi(\bar{a})$ and $\mathfrak{A}_2 \models \psi(\bar{b})$. This holds for all $\mathfrak{A}_1 \models T_1$, $\mathfrak{A}_2 \models T_2$ and thus we see that $T_3 \models \phi(\bar{x}) \wedge \psi(\bar{y}) \to \bar{x} \cap \bar{y} \neq \emptyset$. But since $\bar{a} \in A_1 - D(\mathfrak{A}_1)$ and $\mathfrak{A}_1 \models \phi(\bar{a})$, we see that for all lists $\bar{t}$ of constant terms of $\mathcal{L}$ we have $\mathfrak{A}_1 \models \phi(\bar{a}) \to \bar{a} \cap \bar{t} = \emptyset$. For the same reason $\mathfrak{A}_2 \models \psi(\bar{b}) \to \bar{b} \cap \bar{t} = \emptyset$. This gives that $T_1 \cup \{\phi(\bar{x})\} \cup \{x \neq t : x \in \bar{x}$, t constant term of $\mathcal{L}\}$ and $T_2 \cup \{\psi(\bar{y})\} \cup \{y \neq t : y \in \bar{y}$, t constant term of $\mathcal{L}\}$ are consistent and never $T_1 \models \phi(\bar{x}) \to \bar{x} \cap \bar{t} \neq \emptyset$ or $T_2 \models \psi(\bar{y}) \to \bar{y} \cap \bar{t} \neq \emptyset$. ∎

Since the path to the syntactical characterisation of the amalgamation property is quite similar to the one to the characterisation of the general amalgamation property, we shall now look at the characterisation for a strong amalgamation base. Since we already noted that we won't consider a strong joint embedding property, there will be no bypass and we can continue straight away.

Theorem 3.3.6 *Let T be an $\mathcal{L}$-theory and $\mathfrak{A}$ a model of T. Then the following are equivalent:*

i) $\mathfrak{A}$ is a strong amalgamation base for T;

ii) for all $\phi_1(\bar{x}_1, \bar{z}), \phi_2(\bar{x}_2, \bar{z}) \in \wedge(\mathcal{L})$ with $T \models \phi_1(\bar{x}_1, \bar{z}) \wedge \phi_2(\bar{x}_2, \bar{z}) \to \bar{x}_1 \cap \bar{x}_2 \neq \emptyset$, there are $\psi_1(\bar{w}_n, \bar{z}), \psi_2(\bar{w}_n, \bar{z}) \in \{\psi(\bar{w}_n, \bar{z}) \in \vee \wedge (\mathcal{L}) : T \models \phi(\bar{x}, \bar{z}) \wedge \psi(\bar{w}_n, \bar{z}) \to \bar{x} \cap \bar{w}_n \neq \emptyset\}$, where $n \in \mathbb{N}$ and $\bar{w}_n$ is $w_0, w_1, ..., w_n$, such that $\mathfrak{A} \models \forall \bar{z}(\exists \bar{w}_n \psi_1(\bar{w}_n, \bar{z}) \vee \exists \bar{w}_n \psi_2(\bar{w}_n, \bar{z}))$.

Proof Firstly i) $\implies$ ii):
So we assume $\mathfrak{A}$ is a stong amalgamation base for T. This gives with lemma

3.3.4 that $< T \cup \Delta(\mathfrak{A}), T \cup \Delta(\mathfrak{A}), T \cup \{a = a : a \in A\} >$ has the SEP. Now we can use theorem 3.3.5 to see that for all $\phi_1(\bar{x}_1, \bar{z}), \phi_2(\bar{x}_2, \bar{z}) \in \wedge(\mathcal{L})$ and all lists $\bar{a} \in A$, if $T \cup \{a = a : a \in A\} \models \phi_1(\bar{x}_1, \bar{a}) \wedge \phi_2(\bar{x}_2, \bar{a}) \rightarrow \bar{x}_1 \cap \bar{x}_2 \neq \emptyset$ there is a list $\bar{b} \in A - \{\bar{a}\}$ such that $T \cup \Delta(\mathfrak{A}) \models \phi_1(\bar{x}_1, \bar{a}) \rightarrow \bar{x}_1 \cap (\bar{b} \cup \bar{a}) \neq \emptyset$ or $T \cup \Delta(\mathfrak{A}) \models \phi_2(\bar{x}_2, \bar{a}) \rightarrow \bar{x}_2 \cap (\bar{b} \cup \bar{a}) \neq \emptyset$. Here we must use $(\bar{b} \cup \bar{a})$ because the interpretations of the list $\bar{t}$ of constant terms of $\mathcal{L}$ can already (partially) be in $\bar{a}$ or is (partially) in $A - \{\bar{a}\}$, thus the list $\bar{b}$ consists of the interpretations of the missing constant terms.

We know that $T \cup \Delta(\mathfrak{A})$ is consistent and thus every subtheory must be consistent. Since $\Delta(\mathfrak{A}$ consists of all open sentences that hold in $\mathfrak{A}$, this gives that there must be an $n \in \mathbb{N}$ and some $\psi_1(\bar{w}_n, \bar{z}), \psi_2(\bar{w}_n, \bar{z}) \in \vee \wedge (\mathcal{L})$ with $\bar{w}_n = (w_0, ..., w_m)$, such that we can extend the list $\bar{b}$ from above to a list $\bar{b} = (b_0, ..., b_n) \in A - \{\bar{a}\}$ to get $T \cup \{\psi_1(\bar{b}, \bar{a})\} \models \phi_1(\bar{x}_1, \bar{a}) \rightarrow \bar{x}_1 \cap (\bar{b} \cup \bar{a}) \neq \emptyset$, $T \cup \{\psi_2(\bar{b}, \bar{a})\} \models \phi_2(\bar{x}_2, \bar{a}) \rightarrow \bar{x}_2 \cap (\bar{b} \cup \bar{a}) \neq \emptyset$ and $\mathfrak{A} \models \psi_1(\bar{b}, \bar{a}) \vee \psi_2(\bar{b}, \bar{a})$. We know that $\mathfrak{A} \models \psi_1(\bar{b}, \bar{a}) \vee \psi_2(\bar{b}, \bar{a})$ is logically equivalent with $\mathfrak{A} \models \forall \bar{z}(\exists \bar{w}_n \psi_1(\bar{w}_n, \bar{z}) \vee \exists \bar{w}_n \psi_2(\bar{w}_n, \bar{z}))$, which is what we wanted.

Secondly ii) $\implies$ i):

We assume that for all $\phi_1(\bar{x}_1, \bar{z}), \phi_2(\bar{x}_2, \bar{z}) \in \wedge(\mathcal{L})$ with $T \models \phi_1(\bar{x}_1, \bar{z}) \wedge \phi_2(\bar{x}_2, \bar{z}) \rightarrow \bar{x}_1 \cap \bar{x}_2 \neq \emptyset$, there are $\psi_1(\bar{w}_n, \bar{z}), \psi_2(\bar{w}_n, \bar{z}) \in \{\psi(\bar{w}_n, \bar{z}) \in \vee \wedge (\mathcal{L}) : T \models \phi(\bar{x}, \bar{z}) \wedge \psi(\bar{w}_n, \bar{z}) \rightarrow \bar{x} \cap \bar{w}_n \neq \emptyset\}$, where $n \in \mathbb{N}$ and $\bar{w}_n$ is $w_0, w_1, ..., w_n$, such that $\mathfrak{A} \models \forall \bar{z}(\exists \bar{w}_n \psi_1(\bar{w}_n, \bar{z}) \vee \exists \bar{w}_n \psi_2(\bar{w}_n, \bar{z}))$. Now we can extend the notion $T \models \phi(\bar{x}, \bar{z}) \wedge \psi(\bar{w}_n, \bar{z}) \rightarrow \bar{x} \cap \bar{w}_n \neq \emptyset$ to $T \models \phi(\bar{x}, \bar{z}) \wedge \psi(\bar{w}_n, \bar{z}) \rightarrow \bar{x} \cap (\bar{w}_n \cup \bar{z}) \neq \emptyset$, since if the intersection of $\bar{x}$ with the $\bar{w}_n$ isn't empty, it still won't be empty when we unite the $\bar{z}$ with the $\bar{w}_n$. Now we can substitute some z's for w's such that we get some ψ' with $T \models \phi(\bar{x}, \bar{z}) \wedge \psi'(\bar{w}_m, \bar{z}) \rightarrow \bar{x} \cap (\bar{w}_m \cup \bar{z}) \neq \emptyset$ and $\bar{w}_m \cap \bar{z} = \emptyset$. Then we get for all $\bar{a} \in A$ there is a $\bar{b} \in A - \{\bar{a}\}$ such that $\mathfrak{A} \models \psi'_1(\bar{b}, \bar{a}) \vee \psi'_2(\bar{b}, \bar{a})$ and $T \models \psi'_1(\bar{w}_m, \bar{z}) \wedge \phi_1(\bar{x}, \bar{z}) \rightarrow \bar{x} \cap (\bar{w}_m \cup \bar{z}) \neq \emptyset$ and $T \models \psi'_2(\bar{w}_m, \bar{z}) \wedge \phi_2(\bar{x}, \bar{z}) \rightarrow \bar{x} \cap (\bar{w}_m \cup \bar{z}) \neq \emptyset$. Since $\mathfrak{A} \models \psi'_1(\bar{b}, \bar{a}) \vee \psi'_2(\bar{b}, \bar{a})$ gives that ψ'_1 or ψ'_2 in $\Delta(\mathfrak{A})$, we get that $T \cup \Delta(\mathfrak{A}) \models \phi_1(\bar{x}, \bar{z}) \rightarrow \bar{x} \cap (\bar{b} \cup \bar{a}) \neq \emptyset$ or $T \cup \Delta(\mathfrak{A}) \models \phi_2(\bar{x}, \bar{z}) \rightarrow \bar{x} \cap (\bar{b} \cup \bar{a}) \neq \emptyset$. We know that $T \models \phi_1(\bar{x}_1, \bar{z}) \wedge \phi_2(\bar{x}_2, \bar{z}) \rightarrow \bar{x}_1 \cap \bar{x}_2 \neq \emptyset$ implies $T \cup \{a = a : a \in A\} \models \phi_1(\bar{x}_1, \bar{a}) \wedge \phi_2(\bar{x}_2, \bar{a}) \rightarrow \bar{x}_1 \cap \bar{x}_2 \neq \emptyset$. Now we have that for all $\phi_1(\bar{x}_1, \bar{z}), \phi_2(\bar{x}_2, \bar{z}) \in \wedge(\mathcal{L})$ with $T \cup \{a = a : a \in A\} \models \phi_1(\bar{x}_1, \bar{a}) \wedge \phi_2(\bar{x}_2, \bar{a}) \rightarrow \bar{x}_1 \cap \bar{x}_2 \neq \emptyset$, $T \cup \Delta(\mathfrak{A}) \models \phi_1(\bar{x}, \bar{z}) \rightarrow \bar{x} \cap (\bar{b} \cup \bar{a}) \neq \emptyset$ or $T \cup \Delta(\mathfrak{A}) \models \phi_2(\bar{x}, \bar{z}) \rightarrow \bar{x} \cap (\bar{b} \cup \bar{a}) \neq \emptyset$. With theorem 3.3.5 this is equivalent to $< T \cup \Delta(\mathfrak{A}), T \cup \Delta(\mathfrak{A}), T \cup \{a = a : a \in A\} >$ has the SEP, which gives with proposition 3.3.4 that $\mathfrak{A}$ is a strong amalgamation base for T. ∎

As we have seen in the sections 3.1 and 3.2, the characterisation for a base is the most important step towards the characterisation of the whole property. Thus we can use a same proof as in 3.1.9 to get the syntactical characterisation of the strong amalgamation property.

Corollary 3.3.7 *Let T be an $\mathcal{L}$-theory. Then the following are equivalent:*

i) T has the SAP;

ii) for all $\phi_1(\bar{x}, \bar{z}), \phi_2(\bar{y}, \bar{z}) \in \wedge(\mathcal{L})$ there exist $\psi_1(\bar{u}, \bar{z}), \psi_2(\bar{v}, \bar{z}) \in \vee\wedge(\mathcal{L})$ such *that if* $T \models \phi_1(\bar{x}, \bar{z}) \wedge \phi_2(\bar{y}, \bar{z}) \to \bar{x} \cap \bar{y} \neq \emptyset$ *then* $T \models \psi_1(\bar{u}, \bar{z}) \vee \psi_2(\bar{v}, \bar{z})$, $T \models \phi_1(\bar{x}, \bar{z}) \wedge \psi_1(\bar{u}, \bar{z}) \to \bar{x} \cap \bar{u} \neq \emptyset$ *and* $T \models \phi_2(\bar{y}, \bar{z}) \wedge \psi_2(\bar{v}, \bar{z}) \to \bar{y} \cap \bar{v} \neq \emptyset$.

Chapter 4

The Congruence Extension Property

In this chapter we look at the congruence extension property. To obtain a syntactical characterisation we will once more follow the steps of Bacsich and Hughes in [2]. We shall not give a general syntactical characterisation, as we did in the last chapter, but solely focus on theories preserved under homomorphic images. When we find a characterisation, we shall be even more specific and look at equational theories. To give the characterisations we shall not, as in the last chapter, define other notions to make small intermediate steps. Therefore the proofs are quite long, but still they are understandable. However, before we go to these characterisations, we shall look at the definition of the congruence extension property.

Definition 4.0.1 (Congruence Extension Property) *An $\mathcal{L}$-theory T has the* Congruence Extension Property (CEP) *if every diagram of the form*

$$\begin{tikzcd} & \mathfrak{B} \arrow[dr, dashed, "g"] & \\ \mathfrak{A} \arrow[ur, "e"] \arrow[dr, "f"'] & & \mathfrak{D} \\ & \mathfrak{C} \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

e and h embeddings

f and g surjections

can be completed.

4.1 Preserved Under Homomorphisms

In this section we shall give a syntactical characterisation of the congruence extension property for theories preserved under homomorphic images. We consider only those theories, because it gives us the possibility to change the requirement of a surjection $g : \mathfrak{B} \to \mathfrak{D}$ to a simple homomorphism. It is possible, since $g(\mathfrak{B})$ is a model of T by the fact that T is preserved under homomorphic images and since $g : \mathfrak{B} \to g(\mathfrak{B})$ is a surjection. What we gain by this transformation is the opportunity to use, like we did before, the method of diagrams, since for the homomorphism we only have to find a model of $\Delta^+(\mathfrak{B})$, as we have seen in corollary 2.4.6.

By concluding this, we can now give the syntactical characterisation.

Theorem 4.1.1 *Let T be an $\mathcal{L}$-theory preserved under homomorphic images. Then the following are equivalent:*

i) T has the CEP;

ii) for all $\phi_1(\bar{x}) \in \exists \wedge (\mathcal{L})^+$ and $\phi_2(\bar{x}) \in \wedge(\mathcal{L})$ with $T \cup \{\phi_1(\bar{x}), \phi_2(\bar{x})\}$ is inconsistent, there are $\psi_1(\bar{x}) \in \exists_1$ and $\psi_2(\bar{x}) \in \exists_1^+$ such that $T \models \psi_1(\bar{x}) \vee \psi_2(\bar{x})$ and $T \cup \{\phi_1(\bar{x}), \psi_1(\bar{x})\}$ and $T \cup \{\phi_2(\bar{x}), \psi_2(\bar{x})\}$ are inconsistent.

Proof We recall that T has the CEP if the diagram of the form

$$\begin{tikzcd} & \mathfrak{B} \arrow[dr, dashed, "g"] & \\ \mathfrak{A} \arrow[ur, "e"] \arrow[dr, "f"'] & & \mathfrak{D} \\ & \mathfrak{C} \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

e and h embeddings

f and g surjections

can be completed. We have seen that we can weaken the demand for g of a surjection to an homomorphism, since T is preserved under homomorphisms. Thus we have that T has the CEP if and only if for every $\mathfrak{A} \models T$, there are $\mathfrak{B}, \mathfrak{C} \models T$ such that there is an embedding e from $\mathfrak{A}$ in $\mathfrak{B}$, a surjection f form $\mathfrak{A}$ to $\mathfrak{C}$ and $T \cup \Delta^+(\mathfrak{B}) \cup \Delta(\mathfrak{C})$ is consistent.

With the help of the surjection f, we can denote the diagram of $\mathfrak{C}$ with constants from $\mathfrak{A}$. Thus we denote $\Delta(f(\mathfrak{A})) = \{\chi(a_1, ..., a_n) \in (\mathcal{L}_{\mathfrak{A}}) : \mathfrak{C} \models \chi(f(a_1), ..., f(a_n))\}$. This gives that T has the CEP if and only if for every $\mathfrak{A} \models T$ the following holds: if there are $\mathfrak{B}, \mathfrak{C} \models T$ such that there is an embedding e from $\mathfrak{A}$ in $\mathfrak{B}$ and a surjection f from $\mathfrak{A}$ to $\mathfrak{C}$, then $T \cup \Delta^+(\mathfrak{B}) \cup \Delta(f(\mathfrak{A}))$ is consistent. We shall call this expression (1).

Expression (2) states: for some $\bar{a} \in A$ and for all $\phi_1(\bar{x}) \in \exists \wedge (\mathcal{L})^+$ and $\phi_2(\bar{x}) \in \wedge(\mathcal{L})$ with $T \cup \{\phi_1(\bar{x}), \phi_2(\bar{x})\}$ is inconsistent, $T \cup \Delta(\mathfrak{A}) \cup \{\phi_1(\bar{a})\}$ or $T \cup \Delta^+(\mathfrak{A}) \cup \{\phi_2(\bar{a})\}$ is inconsistent.

Now we claim that (1) and (2) are equivalent. We shall prove both ways by contradiction. Firstly (1) implies (2):

Suppose (2) fails, thus there is a $\bar{a} \in A$ such that $T \cup \Delta(\mathfrak{A}) \cup \{\phi_1(\bar{a})\}$ and $T \cup \Delta^+(\mathfrak{A}) \cup \{\phi_2(\bar{a})\}$ are consistent, for all $\phi_1(\bar{x}) \in \exists \wedge (\mathcal{L})^+$ and $\phi_2(\bar{x}) \in \wedge(\mathcal{L})$ with $T \cup \{\phi_1(\bar{x}), \phi_2(\bar{x})\}$ is inconsistent. This gives a $\mathfrak{B} \models T \cup \Delta(\mathfrak{A}) \cup \{\phi_1(\bar{a})\}$, which implies there is an embedding from $\mathfrak{A}$ in $\mathfrak{B}$, and a $\mathfrak{C}' \models T \cup \Delta^+(\mathfrak{A}) \cup \{\phi_2(\bar{a})\}$, which implies there is an homomorphism $\mathfrak{A}$ to $\mathfrak{C}'$. Since T is preserved under homomorphisms, the last one also gives a substructure $\mathfrak{C}$ of $\mathfrak{C}'$ which is a model of $T \cup \Delta^+(\mathfrak{A}) \cup \{\phi_2(\bar{a})\}$ and which is the image of the homomorphism from $\mathfrak{A}$ to $\mathfrak{C}'$. Now we have found $\mathfrak{B}, \mathfrak{C} \models T$ such that there is an embedding e from $\mathfrak{A}$ in $\mathfrak{B}$, a surjection f from $\mathfrak{A}$ to $\mathfrak{C}$ and $\mathfrak{B} \models \{\phi_1(\bar{a})\}$ and $\mathfrak{C} \models \{\phi_2(\bar{a})\}$. With $T \cup \{\phi_1(\bar{x}), \phi_2(\bar{x})\}$ is inconsistent, we see that $T \cup \Delta^+(\mathfrak{B}) \cup \Delta(f(\mathfrak{A}))$ cannot be consistent and thus (1) fails.

Now secondly (2) implies (1). We suppose that (1) fails. This gives some $\mathfrak{B}, \mathfrak{C} \models T$ such that there is an embedding e from $\mathfrak{A}$ in $\mathfrak{B}$, a surjection f from $\mathfrak{A}$ to $\mathfrak{C}$ and $T \cup \Delta^+(\mathfrak{B}) \cup \Delta(f(\mathfrak{A}))$ is inconsistent. This gives with the Compactness Theorem that there must be a finite subset of $\Delta^+(\mathfrak{B})$ and a finite subset of $\Delta(f(\mathfrak{A}))$ such that the union with T is inconsistent. Hence there are $\phi_1^{'}(\bar{x}, \bar{y}) \in \wedge(\mathcal{L})^+, \phi_2(\bar{x}) \in \wedge(\mathcal{L}), \bar{a} \in A$ and $\bar{b} \in B - A$ such that $T \cup \{\phi_1^{'}(\bar{x}, \bar{y}), \phi_2(\bar{x})\}$ is inconsistent, $\mathfrak{B} \models T \cup \Delta(\mathfrak{A}) \cup \{\phi_1^{'}(\bar{a}, \bar{b})\}$ and $\mathfrak{C} \models T \cup \Delta^+(\mathfrak{A}) \cup \{\phi_2(\bar{a})\}$. Now we can choose $\phi_1(\bar{x})$ to be $\exists \bar{y} \phi_1^{'}(\bar{x}, \bar{y})$. We see that we have for some $\phi_1(\bar{x}) \in \exists \wedge (\mathcal{L})^+$ and $\phi_2(\bar{x}) \in \wedge(\mathcal{L})$ with $T \cup \{\phi_1(\bar{x}), \phi_2(\bar{x})\}$ inconsistent, a $\bar{a} \in A$ such that $\mathfrak{B} \models T \cup \Delta(\mathfrak{A}) \cup \{\phi_1(\bar{a})\}$ and $\mathfrak{C} \models T \cup \Delta^+(\mathfrak{A}) \cup \{\phi_2(\bar{a})\}$. Thus (2) fails. We see that (1) and (2) are equivalent and thus that T has the CEP if (2) holds for all $\mathfrak{A} \models T$.

Now we can rewrite in (2) '$T \cup \Delta(\mathfrak{A}) \cup \{\phi_1(\bar{a})\}$ or $T \cup \Delta^+(\mathfrak{A}) \cup \{\phi_2(\bar{a})\}$ is inconsistent' by '$T \cup \Delta(\mathfrak{A}) \cup \{\phi_1(\bar{a})\} \models \perp$ or $T \cup \Delta^+(\mathfrak{A}) \cup \{\phi_2(\bar{a})\} \models \perp$'. This we can rewrite to '$T \cup \Delta(\mathfrak{A}) \models \phi_1(\bar{a}) \rightarrow \perp$ or $T \cup \Delta^+(\mathfrak{A}) \models \phi_2(\bar{a}) \rightarrow \perp$'. With lemma 2.4.3 this gives us that (2) is equivalent to for some $\bar{a} \in A$ and for all $\phi_1(\bar{x}) \in \exists \wedge (\mathcal{L})^+$ and $\phi_2(\bar{x}) \in \wedge(\mathcal{L})$ with $T \cup \{\phi_1(\bar{x}), \phi_2(\bar{x})\}$ is inconsistent, there is a $\psi_1(\bar{x}) \in \{\psi_1(\bar{x}) \in \exists_1 : T \models \psi_1(\bar{x}) \rightarrow (\phi_1(\bar{x}) \rightarrow \perp)\}$ and a $\psi_2(\bar{x}) \in \{\psi_2(\bar{x}) \in \exists_1^+ : T \models \psi_2(\bar{x}) \rightarrow (\phi_2(\bar{x}) \rightarrow \perp)\}$ such that $\mathfrak{A} \models \psi_1(\bar{a}) \vee \psi_2(\bar{a})$. If this must hold for all $\mathfrak{A} \models T$, we have that $T \models \psi_1(\bar{x}) \vee \psi_2(\bar{x})$. Now we notice that $T \models \psi_1(\bar{x}) \rightarrow (\phi_1(\bar{x}) \rightarrow \perp)$ is equivalent to $T \cup \{\psi_1(\bar{x})\} \models \phi_1(\bar{a}) \rightarrow \perp$, which is again equivalent to $T \cup \{\phi_1(\bar{x}), \psi_1(\bar{x})\} \models \perp$. The last is the same as saying $T \cup \{\phi_1(\bar{x}), \psi_1(\bar{x})\}$ is inconsistent. The same holds for $T \models \psi_2(\bar{x}) \rightarrow (\phi_2(\bar{x}) \rightarrow \perp)$ and $T \cup \{\phi_2(\bar{x}), \psi_2(\bar{x})\}$ is inconsistent. Thus we have T has the CEP if and only if for all $\phi_1(\bar{x}) \in \exists \wedge (\mathcal{L})^+$ and $\phi_2(\bar{x}) \in \wedge(\mathcal{L})$ with $T \cup \{\phi_1(\bar{x}), \phi_2(\bar{x})\}$ is inconsistent, there is a $\psi_1(\bar{x}) \in \exists_1$ such that $T \cup \{\phi_1(\bar{x}), \psi_1(\bar{x})\}$ is inconsistent and a $\psi_2(\bar{x}) \in \exists_1^+$ such that $T \cup \{\phi_2(\bar{x}), \psi_2(\bar{x})\}$ is inconsistent, together such that $T \models \psi_1(\bar{x}) \vee \psi_2(\bar{x})$. Hence we have the desired characterisation. ∎

4.2 Equational theories

In this section we will consider a special case of theories preserved under homomorphic images, namely the equational theories. These are theories of the language $\mathcal{L}$, which is without relation symbols, with a set of axioms based on equations, i.e. an equality $s = t$ with s and t terms of the language $\mathcal{L}$. This clearly gives that the set of axioms of T consists of positive sentences only. Thus, when we recall the preservation theorems of proposition 2.2.1, we see that T is preserved under homomorphisms. So we can use theorem 4.1.1 to give a characterisation.

Now before we shall give the characterisation, we shall first prove the following lemma, in order to keep the proof comprehensible.

Lemma 4.2.1 *Let T be an equational $\mathcal{L}$-theory. Then the following are equivalent:*

i) T has the CEP;

ii) for all $\mathfrak{B} \models T$, embeddings from $\mathfrak{A}$ in $\mathfrak{B}$ and $a, b \in A$, we have $\Delta^+(\mathfrak{B}) \cup \Delta^+(\mathfrak{A}) \cup \{a = b\} \cup \{c \neq d : \Delta^+(\mathfrak{A}) \cup \{a = b\} \not\models c = d\}$ is consistent.

Proof Since T is an $\mathcal{L}$-theory preserved under homomorphic images, we can use the same as in theorem 4.0.9, to show that T has the CEP if and only if for all $\mathfrak{A} \models T$ holds if there are $\mathfrak{B}, \mathfrak{C} \models T$ such that there is an embedding e from $\mathfrak{A}$ in $\mathfrak{B}$ and a surjection f from $\mathfrak{A}$ to $\mathfrak{C}$, then $T \cup \Delta^+(\mathfrak{B}) \cup \Delta(f(\mathfrak{A}))$ is consistent. Now as $\mathfrak{B} \models T$ it must hold that all equations in T are true in $\mathfrak{B}$. But then all these equations also hold in $\Delta^+(\mathfrak{B})$, so it is the same to say $\Delta^+(\mathfrak{B}) \cup \Delta(f(\mathfrak{A}))$ is consistent.

Now we shall have a closer look at $\mathfrak{C}$, the homomorphic image of $\mathfrak{A}$. We recall that giving a homomophism from $\mathfrak{A}$ to $\mathfrak{B}$ is the same as giving an $\mathcal{L}_{\mathfrak{A}}$-expansion of $\mathfrak{B}$ which is a model of $\Delta^+(\mathfrak{A})$. Considering our homomorphism also has to be a surjection, we get that these homomorphic images are the quotients of $\mathfrak{A}$ by a set of equations. So let X be a set of equations $a = b$ with $a, b \in A$. The quotient of $\mathfrak{A}$ modulo X is now the model $\mathfrak{A}/X$ and the surjective homomorphism $f : \mathfrak{A} \to \mathfrak{A}/X$. We can see straightaway that $\mathfrak{A}/X$ must be a model of $\Delta^+(\mathfrak{A}) \cup X \cup \{c \neq d : \Delta^+(\mathfrak{A}) \cup X \not\models c = d\}$. Thus we can see that T has the CEP if and only if for all $\mathfrak{A} \models T$ holds if there is a $\mathfrak{B} \models T$ such that there is an embedding e from $\mathfrak{A}$ in $\mathfrak{B}$ and there is a set of equations X (as described above), then $\Delta^+(\mathfrak{B}) \cup \Delta^+(\mathfrak{A}) \cup X \cup \{c \neq d : \Delta^+(\mathfrak{A}) \cup X \not\models c = d\}$ is consistent.

To show that the case $X = \{a = b\}$ is enough to prove T has the CEP, have to make two remarks. At first we notice, with the Compactness Theorem, that for consistency we only have to look at finite sets X. Secondly it is simple to prove that if $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$, then giving a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}/X$ is equal to giving a homomorphism from $\mathfrak{A}$ to $\mathfrak{A}/X_1$ and afterwards giving a homomorphism from $\mathfrak{A}/X_1$ to $(\mathfrak{A}/X_1)X_2$. We shall prove this by showing that $\mathfrak{A}/X$ and $(\mathfrak{A}/X_1)X_2$ are logically equivalent. We know that $\mathfrak{A}/X \models \Delta^+(\mathfrak{A}) \cup X \cup \{c \neq d : \Delta^+(\mathfrak{A}) \cup X \not\models c = d\}$. This gives that $\mathfrak{A}/X_1 \models \Delta^+(\mathfrak{A}) \cup X_1 \cup \{c \neq d : \Delta^+(\mathfrak{A}) \cup X_1 \not\models c = d\}$, which thereupon gives that $(\mathfrak{A}/X_1)X_2 \models \Delta^+(\mathfrak{A}/X_1) \cup X_2 \cup \{c \neq d : \Delta^+(\mathfrak{A}/X_1) \cup X_2 \not\models c = d\}$. We note that $\Delta^+(\mathfrak{A}/X_1) = \Delta^+(\mathfrak{A}) \cup X_1$, because $\Delta^+(\mathfrak{A}) \cup X_1$ consists of all equations, and thus all positive sentences, in $\mathfrak{A}/X_1$. Hence we have $(\mathfrak{A}/X_1)X_2 \models \Delta^+(\mathfrak{A}) \cup X_1 \cup X_2 \cup \{c \neq d : \Delta^+(\mathfrak{A}) \cup X_1 \cup X_2 \not\models c = d\}$ and with $X = X_1 \cup X_2$ this is equal to $(\mathfrak{A}/X_1)X_2 \models \Delta^+(\mathfrak{A}) \cup X \cup \{c \neq d : \Delta^+(\mathfrak{A}) \cup X \not\models c = d\}$, thus $\mathfrak{A}/X$ and $(\mathfrak{A}/X_1)X_2$ are logically equivalent.

At last we can use induction to show that the case $X = \{a = b\}$ for some $a, b \in A$ proofs T has the CEP. At first we will look at the diagram in the case $\Delta^+(\mathfrak{B}) \cup \Delta^+(\mathfrak{A}) \cup \{a = b\} \cup \{c \neq d : \Delta^+(\mathfrak{A}) \cup \{a = b\} \not\models c = d\}$ is consistent. Let $\mathfrak{D}_1$ be a model. We get a diagram of the form

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "e"] \arrow[d, "f"'] & \mathfrak{B} \arrow[d, dashed, "g"] \\ \mathfrak{A}/\{a = b\} \arrow[r, dashed, "h"'] & \mathfrak{D}_1 \end{tikzcd}$$

e and h embeddings

f and g surjections

and we see that in this case the diagram can be completed. Now our induction hypothesis is that for some set $X_n = \{a_1 = b_1, ..., a_n = b_n\}$, for some n-tuples $\bar{a}, \bar{b} \in A$, the diagram can be completed. We consider the set $X_{n+1} = \{a_1 = b_1, ..., a_{n+1} = b_{n+1}\}$. This gives that $X_{n+1} = X_n \cup \{a_{n+1} = b_{n+1}\}$. Our induction hypothesis gives us a model $\mathfrak{D}'$ such that

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "e"] \arrow[d, "f'"'] & \mathfrak{B} \arrow[d, dashed, "g'"] \\ \mathfrak{A}/X_n \arrow[r, dashed, "h'"'] & \mathfrak{D}' \end{tikzcd}$$

e and h embeddings

f and g surjections

can be completed. But now we have a $\mathfrak{D}' \models T$, an embedding from $\mathfrak{A}/X_n$ into $\mathfrak{D}'$ and $a_{n+1}, b_{n+1} \in \mathfrak{A}/X_n$. We see that $\Delta^+(\mathfrak{D}') \cup \Delta^+(\mathfrak{A}/X_n) \cup \{a_{n+1} = b_{n+1}\} \cup \{c \neq d : \Delta^+(\mathfrak{A}/X_n) \cup \{a_{n+1} = b_{n+1}\} \not\models c = d\}$ is consistent, thus it gives a $\mathfrak{D}$ and the diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "e"] \arrow[d, "f'"'] & \mathfrak{B} \arrow[d, "g'"] \\ \mathfrak{A}/X_n \arrow[r, "h'"'] \arrow[d, "f''"'] & \mathfrak{D}' \arrow[d, "g''"] \\ \mathfrak{A}/X_{n+1} \arrow[r, "h"'] & \mathfrak{D} \end{tikzcd}$$

e, h and h' embeddings

f, f', g and g' surjections

where $f" \circ f'$ is a surjection from $\mathfrak{A}$ to $\mathfrak{A}/X_{n+1}$ and $g" \circ g'$ a surjection from $\mathfrak{B}$ to $\mathfrak{D}$, which completes the diagram. ∎

With the help of this lemma, we can now give the syntactical characterisation of the congruence extension property for equational theories. Here we use the set S, consisting of all the formulas $\phi(s, t, x, y, \bar{z}) \in \{\wedge\}(\mathcal{L})^+$ such that $T \models \phi(t, t, x, y, \bar{z}) \to x = y$.

Theorem 4.2.2 *Let T be an equational $\mathcal{L}$-theory. Then the following are equivalent:*

i) T has the CEP;

ii) for all $\phi(s, t, x, y, \bar{z}) \in S$ there is a $\psi(s, t, x, y) \in S$ such that $T \models \phi(s, t, x, y, \bar{z}) \to \psi(s, t, x, y)$.

Proof With lemma 4.2.1 we see that T has the CEP if and only if for all $\mathfrak{B} \models T$, embeddings from $\mathfrak{A}$ in $\mathfrak{B}$ and $a, b \in A$, $\Delta^+(\mathfrak{B}) \cup \Delta^+(\mathfrak{A}) \cup \{a = b\} \cup \{c \neq d : \Delta^+(\mathfrak{A}) \cup \{a = b\} \not\models c = d\}$ is consistent. Since there is an embedding from $\mathfrak{A}$ in $\mathfrak{B}$, all atomic formulas that hold in $\mathfrak{A}$ must also hold in $\mathfrak{B}$, thus $\Delta^+(\mathfrak{A}) \subseteq \Delta^+(\mathfrak{B})$. This gives $\Delta^+(\mathfrak{B}) \cup \Delta^+(\mathfrak{A}) \cup \{a = b\} \cup \{c \neq d : \Delta^+(\mathfrak{A}) \cup \{a = b\} \not\models c = d\}$ is consistent if and only if for all $c, d \in A$ such that $\Delta^+(\mathfrak{A}) \cup \{a = b\} \not\models c = d$, $\Delta^+(\mathfrak{B}) \cup \{a = b, c \neq d\}$ is consistent. We can rewrite the last as for all $c, d \in A$, if $\Delta^+(\mathfrak{B}) \cup \{a = b, c \neq d\}$ is inconsistent (thus if $\Delta^+(\mathfrak{B}) \cup \{a = b\} \models c = d$), then $\Delta^+(\mathfrak{A}) \cup \{a = b\} \models c = d$. Here we notice that the only assumptions in

$\Delta^+(\mathfrak{A})$ relevant for this statement are about c or d. So we are only interested in the substructure of $\mathfrak{B}$ that is generated by the set $\{a, b, c, d\}$ and we can replace in the statement $\Delta^+(\mathfrak{A})$ by $\Delta^+(\{a, b, c, d\})$, the positive diagram of the substructure generated by $\{a, b, c, d\}$. Thus we get that T has the CEP if and only if for all $\mathfrak{B} \models T$ and $a, b, c, d \in B$, if $\Delta^+(\mathfrak{B}) \cup \{a = b\} \models c = d$ then $\Delta^+(\{a, b, c, d\}) \cup \{a = b\} \models c = d$. This gives that if there is a tuple $\bar{e} \in B$ and a sentence $\phi'(a, b, c, d, \bar{e}) \in \Delta^+(\mathfrak{B})$ such that $\phi'(a, b, c, d, \bar{e}) \wedge a = b \to c = d$, then there is a sentence $\psi'(a, b, c, d) \in \Delta^+(\{a, b, c, d\})$ such that $\psi'(a, b, c, d) \wedge a = b \to c = d$. Since this must hold for all $\mathfrak{B} \in T$ and all $a, b, c, d \in B$, we get that there are $\phi(s, t, x, y, \bar{z}), \psi(s, t, x, y) \in S$ such that $T \models \forall s, t, x, y(\exists \bar{z} \phi(s, t, x, y, \bar{z}) \to \psi(s, t, x, y))$. ∎

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