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Heyting-valued Models of Intuitionistic Set Theory

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0 Introduction

L.E.J. Brouwer founded intuitionism with the philosophy that mathematics is a personal endeavour that exists as mental constructions. It was an immediate result of this philosophy that the Law of Excluded Middle (LEM), a central tenant to mathematics, could not hold. For everything was a construction and therefore having $\phi \vee \neg\phi$ meant having either a construction of ϕ or a construction of $\neg\phi$ which often, as with any unproven theorem, is not the case.
Intuitionism was later formalized by Andrej Kolmogorov and Brouwer's student Arend Heyting. From their work we get the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic logic which states what is intended to be understood to be a proof for any given logical formula:

- A proof of $\phi \wedge \psi$ is a proof of ϕ and a proof of ψ.
- A proof of $\phi \vee \psi$ is a proof of either ϕ or ψ and an indication as to which is the case.
- There is no proof of $\bot$.
- A proof of $\phi \to \psi$ is an algorithm which converts a proof of ϕ into a proof of ψ.
- A proof of $\neg\phi$ is a proof of $\phi \to \bot$. In other words it is an algorithm that converts a proof of ϕ into an absurdity.
- A proof of $\exists x\phi(x)$ is an element x along with a proof of $\phi(x)$.
- A proof of $\forall x\phi(x)$ is an algorithm that converts any x into a proof of $\phi(x)$.

This, however, was not meant to be the basis of intuitionism itself. In the words of Heyting:

> Logic is not the ground upon which I stand. How could it be? It would in turn need a foundation, which would involve principles much more intricate and less direct than those of mathematics itself. A mathematical construction ought to be so immediate to the mind and its result so clear that it needs no foundation whatsoever.
> [Heyting, 1971, p. 6]

It was, nonetheless, useful as a means for studying intuitionistic mathematics. The study of logic that arises from intuitionism has since grown with, among other things, realizability as well as the logic of topoi in category theory.
This paper focuses on a version of set theory created by John Myhill in a 1971 paper as part of an attempt to "extend [the results of a modification of realizability] to an intuitionistic version of Zermelo-Fraenkel set-theory".[Myhill, 1973, p. 206] To this end he kept all the axioms of ZFC except those which implied LEM, namely regularity and choice which he replaced with the axiom he called transfinite induction (the current version also uses an alternative version of replacement). The resulting theory had models in which, among other things, all functions from the reals to the reals were continuous, the complex numbers were no longer algebraically closed, and many classical definitions of ordinals failed to coincide.
The intention of this paper is to give an idea of how Intuitionistic Zermelo Fraenkel (IZF) can be studied through the lens of Heyting-valued models, models analogous to the the Boolean-valued models used to study ZFC as shown in [Bell, 1985] and [Bell, 2014]. In doing so, however, we remain within the context of ZFC. As such we are not holding ourselves to the philosophies of intuitionism but rather approaching it as an outsider trying to recognize patterns within the system of mathematics that has developed from it. IZF is itself already a tool for such a task, like any formalization of intuitionistic logic, as is suggested by Heyting [Heyting, 1971]. We hope this approach is somewhat consistent with his ideas on the study of intuitionism through formalization.
We begin with a brief description of a formal system of logic that we use to approximate intuitionistic reasoning. We continue with a mention of the axioms in IZF and a few of its properties. Afterward we present the necessary tools from ZFC for constructing a Heyting-valued model before carrying out the construction itself. The rest of the paper is devoted to properties of the

model and proving two results about IZF, namely the independence of the Axiom of Choice from Zorn's Lemma and the consistency of there being a partial surjective function from a subset of the natural numbers object $\mathbb{N}$ to the set $\mathbb{N}^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to itself.

We assume the reader has a degree of familiarity with formal first order logic. Having some familiarity with axiomatic set theory will also be quite helpful. Both are treated in the paper but only briefly and we focus on a couple interesting or relevant aspects instead of constructing a good basis for the subject. A couple of examples touch on basic ideas in topology, specifically opens, interiors and connected spaces but knowledge of the subject is not essential to understanding the main focus.

1 Preliminaries

1.1 A Formalization for Intuionistic Logic

Let us begin with a brief description of a formal derivation system for intuitionistic first-order logic. We use a Hilbert-style calculus with the following axioms[1]

$$
\begin{aligned}
&\phi \to (\psi \to \phi) \\
&\phi \to [\psi \to (\phi \wedge \psi)] \\
&(\phi \wedge \psi) \to \phi \qquad (\phi \wedge \psi) \to \psi \\
&\phi \to (\phi \vee \psi) \qquad \psi \to (\phi \vee \psi) \\
&[\phi \to (\psi \to \chi)] \to [(\phi \to \psi) \to (\phi \to \chi)] \\
&(\phi \to \chi) \to [(\psi \to \chi) \to [(\phi \vee \psi) \to \chi]] \\
&(\phi \to \psi) \to [(\phi \to \neg\psi) \to \neg\phi] \\
&\neg\phi \to (\phi \to \psi) \\
&x = x \qquad x = y \wedge \phi(x) \to \phi(y)
\end{aligned} \tag{1}
$$

and whenever t can be substituted for x without binding any occurences of t as a variable in ϕ

$$\phi(t) \to \exists x \phi(x) \qquad \forall x \phi(x) \to \phi(t) \tag{2}$$

Definition 1.1. Given some first-order language $\mathcal{L}$ and some theory Γ define the relation $\Gamma \vdash \phi$ inductively as follows for any $\mathcal{L}$-sentence ϕ

1. If ϕ is any of the axioms stated above then $\Gamma \vdash \phi$
2. If $\phi \in \Gamma$ then $\Gamma \vdash \phi$
3. If $\Gamma \vdash \phi$ and $\Gamma \vdash \phi \to \psi$ then $\Gamma \vdash \psi$ (modus ponens)

   And for x free in ψ

4. If $\Gamma \vdash \psi \to \phi(x)$ then $\Gamma \vdash \psi \to \forall x \phi(x)$
5. If $\Gamma \vdash \phi(x) \to \psi$ then $\Gamma \vdash \exists x \phi(x) \to \psi$

A sentence ϕ is *derivable* from Γ if $\Gamma \vdash \phi$. Additionally we write $\vdash \phi$ for $\emptyset \vdash \phi$. Classical logic is obtained from intuitionistic logic by adding any one of the following (intuitionistically equivalent) axioms

Law of excluded middle (LEM)	$\phi \vee \neg\phi$
Law of double negation	$\neg\neg\phi \to \phi$
Law of contraposition	$(\neg\psi \to \neg\phi) \to (\phi \to \psi)$

None of these are derivable solely from the axioms in (1) or (2), and neither are the following[2]

De Morgan's law	$\neg(\phi \wedge \psi) \to (\neg\phi \vee \neg\psi)$
Weakened law of excluded middle	$\neg\neg\phi \vee \neg\phi$

The following, however, does hold

$$\vdash \neg\neg\neg\phi \to \neg\phi \tag{3}$$

To show this it is useful to have the following theorem [see Kleene, 1952, p. 90]

[1]For a more detailed account of the calculus see [Kleene, 1952] or [van den Berg, 2015]

[2]Chapter VI of [Kleene, 1952] addresses this along with a number of similar results for the intuitionistic calculus

Theorem 1.2 (Deduction Theorem)**.** *In the Hilbert-style proof calculus,* $\Gamma \cup \{\phi\} \vdash \psi$ *if and only if* $\Gamma \vdash \phi \to \psi$*.*

Furthermore it can be shown that $\vdash \phi \to \phi$ for all ϕ with some manipulation of the axioms and rules of deduction as in [van den Berg, 2015, p. 2]. With this in hand we proceed

Proof of (3).
First we assert that

$$\vdash \phi \to \neg\neg\phi \tag{4}$$

By the first and seventh axioms we have

$$\vdash \phi \to (\neg\phi \to \phi)$$
$$\vdash (\neg\phi \to \phi) \to \big((\neg\phi \to \neg\phi) \to \neg\neg\phi\big)$$

so by rules 1, 2 and 3 in combination with $\vdash \neg\phi \to \neg\phi$ we have

$$\{\phi\} \vdash \neg\neg\phi$$

from which the result follows via 1.2. Now, using the first axiom again we have

$$\vdash \neg\neg\neg\phi \to (\phi \to \neg\neg\neg\phi)$$

and therefore by Theorem 1.2 and (4) in combination with rule 1 we have

$$\begin{aligned} \{\neg\neg\neg\phi\} &\vdash \phi \to \neg\neg\phi \\ \{\neg\neg\neg\phi\} &\vdash \phi \to \neg\neg\neg\phi \end{aligned} \tag{5}$$

now using the seventh axiom again gives us

$$\vdash (\phi \to \neg\neg\phi) \to \big((\phi \to \neg\neg\neg\phi) \to \neg\phi\big)$$

which, using rule 3 in combination with (5) gives us

$$\{\neg\neg\neg\phi\} \vdash \neg\phi$$

from which the desired result follows via Theorem 1.2. □

We argue further in this section without explicit use of the calculus however we will think of it as the underlying notion of derivability in intuitionism and return to it later.

1.2 Intuitionistic Zermelo Fraenkel set theory

In this subsection and the next we will argue with constructive proofs, that is, without making use of the Law of Excluded Middle or its equivalents, in order to remain consistent with intuitionistic logic. The theory of IZF is constructed using the first-order language $\mathcal{L}$ with equality which has as its single binary operation $\in$. IZF has the following axioms:

1. *Extensionality*
   $\forall x, y[\forall z(z \in x \leftrightarrow z \in y) \to z = y]$

2. *Pairing*
   $\forall x, y \exists w \forall z[z \in w \leftrightarrow (z = x \vee z = y)]$
   Using this axiom singleton $\{x\}$ can be defined to be the set $\{x, x\}$ and the ordered pair $\langle x, y\rangle$ is the set $\{\{x\}, \{x, y\}\}$.

3. *Collection*
   $\forall x[(\forall y \in x \exists z \phi(y, z)) \to \exists w \forall y \in x \exists z \in w \phi(y, z)]$

4. *Powerset*
   $\forall x \exists w \forall z[z \in w \leftrightarrow \forall y \in z(y \in x)]$

5. *Separation*
   $\forall x \exists w \forall z[z \in w \leftrightarrow (z \in x \wedge \phi(z))]$

6. *Empty Set*
   $\exists x \forall z[z \in x \leftrightarrow z \neq z]$
   The set satisfying this axiom is, by extensionality, unique and we refer to it with the notation $\emptyset$.

7. *Union*
   $\forall x \exists w \forall z[z \in w \leftrightarrow \exists y \in x(z \in y)]$

8. *Infinity*
   $\exists x[\emptyset \in x \wedge \forall y \in x(y^+ \in x)]$
   Here y^+ is shorthand for the set $y \cup \{y\}$ (which is a set by union and pairing, and again, unique by extensionality).

9. *Induction*
   $\forall x[(\forall y \in x\phi(y)) \rightarrow \phi(x)] \rightarrow \forall x\phi(x)$

The variables in each axiom are understood to range over a universe of sets. It is helpful, however, to be able to talk about collections of sets, or, classes, that themselves are not necessarily sets. To this end we use $\{x \mid \phi(x)\}$ to denote a definable class, that is, the collection of all sets that satisfy some condition ϕ and try to mention explicitly when such a class is a set. In doing this we use a particular abuse of notation, for any set x and any definable class A we take $x \in A$ to mean $\phi(x)$ and also say "x is in A". When A is a set the symbol $\in$ is the usual relation symbol. The universe of all sets will be denoted by V.

Definition 1.3. A *relation* R is a class of ordered pairs and we understand xRy to mean $\langle x, y\rangle \in R$.

A function f is a special relation namely one for which $\langle x, y\rangle, \langle x, y'\rangle \in f$ implies $y = y'$. Given a function f we define $\text{dom}(f)$ to be the class $\{x \mid \exists y[\langle x, y\rangle \in f]\}$ and $\text{ran}(f)$ to be the class $\{y \mid \exists x[\langle x, y\rangle \in f]\}$.
The class $\{x \mid x = \tau \wedge \phi\}$ where τ is not free in ϕ is of particular interest and, for it, we use the shortened notation $\{\tau \mid \phi\}$. Thus defined, $\{\tau \mid \phi\}$ is a set by separation on the singleton $\{\tau\}$, furthermore $\tau \in \{\tau \mid \phi\}$ iff ϕ. A couple other notations: we use 0 interchangeably with $\emptyset$ and the shorthands 1 and 2 for the sets $\{0\}$ and $\{0, 1\}$ respectively.

In order to give a sense of how IZF differs from classical set theory we give a few examples of common notions that imply LEM. Without LEM the formula $\neg\exists x\phi(x)$ is no longer equivalent to $\forall x\neg\phi(x)$. One major consequence of this is that asserting a set x is *non-empty* or that $x \neq \emptyset$ is not the same as asserting $\exists y[y \in x]$. Any set satisfying the latter condition is *inhabited.* Furthermore call a class A is *discrete* or *decidable* if $\forall x, y \in A[x = y \vee x \neq y]$. Now we have

Proposition 1.4. *The following are equivalent to LEM*

(i) Membership is decidable: $\forall x \forall y(x \in y \vee x \notin y)$.

(ii) The universe of all sets, V, is discrete.

(iii) The powerset $P(1)$ *is equal to* 2.

(iv) 2 *is well-ordered, i.e. every inhabited subset of* 2 *has a least element.*

Proof.
LEM$\rightarrow$ (i)
This is true by definition of LEM.

$(i) \to (ii)$
For all $x, y \in V$ it follows from (i) that $x \in \{y\} \vee x \notin \{y\}$. Now $x \in \{y\} \leftrightarrow x = y$ hence the first part of the or implies $x = y$ directly, while the second leads to a contradiction and therefore $x \neq y$.

$(ii) \to (iii)$
Let $x \in P(1)$, by (ii) we have $x = 1 \vee x \neq 1$. Now $y \in x$ implies $y \in 1$ and therefore $y = 0$ so by extensionality $x = 1$. It follows that assuming $x \neq 1$ and $y \in x$ leads to a contradiction therefore $y \notin x$ for all y and so $x = 0$. Thus $x \in P(1)$ implies $x = 1 \vee x = 0$ from which (iii) follows.

$(iii) \to (iv)$
Let $x \subseteq 2$ be inhabited and let $\leq$ be the ordering defined on 2 such that $0 \leq 1$ then assuming (iii) implies $\leq$ is a well ordering. For all $y \in x$ we have $y = 0 \vee y = 1$. Now consider the set $A = \{0 \mid 0 \in x\}$. Since A is a subset of 1 it follows, by (iii), that $A = 1 \vee A \neq 1$. In the first case $0 \in x$ and therefore x has a least element, namely 0. In the second case $y = 0$ leads to a contradiction therefore $\forall y \in x[y = 1]$ and since x is inhabited $1 \in x$ therefore 1 is the least element.

(iv) implies LEM
Assuming 2 has some well ordering $\leq$ assume, without loss of generality, that $0 \leq 1$. Now consider the set $A = \{0 \mid \phi\} \cup \{1\}$, this is a subset of 2 and therefore has some least element a. Now by definition of A we have $a = 0 \vee a = 1$. In the first case $0 \in A$ and therefore ϕ must hold. In the second assuming ϕ leads to a contradiction since it follows from $0 \in A$ that $0 = 1$, so $\neg\phi$ holds. □

The following proposition is also of interest

Proposition 1.5. *The Axiom of Choice implies LEM*

Proof. Given a formula ϕ let f be the function on $2 = \{0, 1\}$ with $f(0) = \{\tau\}$ and $f(1) = \{\tau \mid \phi\}$. By the Axiom of Choice there exists a choice function g for f which, by definition, takes values in 2 and is such that $f(g(x)) = x$. Since 2 is discrete we have

$$g(\{\tau\}) \neq g(\{\tau \mid \phi\}) \vee g(\{\tau\}) = g(\{\tau \mid \phi\})$$

Now $g(\{\tau\}) = g(\{\tau \mid \phi\})$ implies

$$\{\tau\} = f(g(\{\tau\})) = f(g(\{\tau \mid \phi\})) = \{\tau \mid \phi\}$$

and therefore ϕ. On the other hand $g(\{\tau\}) \neq g(\{\tau \mid \phi\})$ implies

$$\{\tau\} \neq \{\tau \mid \phi\}$$

which contradicts ϕ therefore $\neg\phi$ follows. Thus

$$g(\{\tau\}) \neq g(\{\tau \mid \phi\}) \vee g(\{\tau\}) = g(\{\tau \mid \phi\}) \to \phi \vee \neg\phi$$

and so LEM holds. □

Let us say that a set A is infinite if it satisfies $\emptyset \in A \wedge \forall y \in A(y^+ \in A)$. Axiomatically there exists some such A in any model of IZF. We define

$$\mathbb{N} = \bigcap\{K \subset A \mid K \textit{ is an infinite set}\}$$

Since for any other infinite set A' the intersection $A \cap A'$ is both infinite and a subset of A it follows that $\mathbb{N} \subseteq A'$ so $\mathbb{N}$ is the least infinite set in IZF. From this follows immediately the principle of induction on $\mathbb{N}$ that is

$$\forall B\big[\emptyset \in B \wedge \forall b \in B[b^+ \in B] \to \mathbb{N} \subseteq B\big]$$

Hence the object $\mathbb{N}$ is analogous to the natural numbers in ZFC. We now have the following definitions

Definition 1.6.

(i) A set A is *countable* if there exists a surjection $f: \mathbb{N} \to A$

(ii) A set A is *subcountable* if there exists a partial function f on some subset of $\mathbb{N}$ that is a surjection on A

Classically these definitions are equivalent; intuitionistically, however, only $(i) \to (ii)$ holds. Assuming any partial function can be extended to a total function implies LEM and as we will we can construct models in which there exist subcountable sets that are not countable. Note that this, in turn, implies that LEM is in fact refutable in certain models of IZF.

1.3 Heyting Algebras and Frames

In this section we describe an algebra that corresponds with intuitionistic logic upon which we we will build the model. The construction is primarily derived from [Bell, 1985].

Definition 1.7. A *lattice* is a poset P such that for any two elements $x, y \in P$ the set $\{x, y\}$ has an infimum, or meet, and a supremum, or join.

For any lattice we can therefore define the binary operations $\wedge$ and $\vee$ that send a pair, x, y to their meet and join, respectively. A lattice is *bounded* if it has a maximum, or top, and minimum, or bottom, element. In this case we denote the top element by $\top$ and bottom element by $\bot$.

Examples.

(i) The two element set $\{\top, \bot\}$ with $\bot \leq \top$ is a bounded lattice.

(ii) For any set A the powerset $P(A)$, ordered by inclusion, is a bounded lattice. The meet and join operations are the intersection and union of sets respectively. The top element is A and the bottom element is $\emptyset$.

(iii) For any topological space $(X, \mathcal{T})$ the set of opens $O(X)$ ordered by inclusion is a bounded lattice. The meet and join operations are again intersection and union, and the top and bottom elements are X and $\emptyset$.

A bounded lattice can be characterized by the following equations

1. $x \vee \bot = x,\ x \wedge \top = x$
2. $x \vee x = x,\ x \wedge x = x$
3. $x \vee y = y \vee x,\ x \wedge y = y \wedge x$
4. $x \vee (y \vee z) = (x \vee y) \vee z,\ x \wedge (y \wedge z) = (x \wedge y) \wedge z$
5. $(x \vee y) \wedge y = y,\ (x \wedge y) \vee y = y$

By this the following two things are meant. First, any bounded lattice will satisfy the equations above. Second, any set P with binary operations $\wedge$ and $\vee$ along with designated elements $\top$ and $\bot$ that satisfy the above equations becomes a bounded lattice with the partial ordering $x \leq y$ iff $x \wedge y = x$.

A lattice is *distributive* if for all $x, y, z \in H$ we have

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) \quad \text{and} \quad (x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$$

The two conditions are actually equivalent. Both implications are proved similarly so only one is shown here. Assuming the first equivalence we have

$$\begin{aligned}(x \vee z) \wedge (y \vee z) &= (x \wedge (y \vee z)) \vee (z \wedge (y \vee z)) \\ &= ((x \wedge y) \vee (x \wedge z)) \vee z \\ &= (x \wedge y) \vee ((x \wedge z) \vee z) \\ &= (x \wedge y) \vee z\end{aligned}$$

Given a subset A of P we denote its supremum by $\bigvee A$ and its infimum by $\bigwedge A$ if they exist.

Definition 1.8. A lattice is *complete* if any set A has both an infimum and a supremum.

To show that poset is complete it is sufficient to show that it has all infimums since for any set, the infimum of its upper-bounds is its supremum. A similar reasoning shows that it is also sufficient to show that a poset has all supremums.

Definition 1.9. A *Heyting algebra* is a bounded lattice in which, for any two elements a and b, the set

$$\{z \mid a \wedge z \leq b\}$$

has a greatest element $a \Rightarrow b$. If, in addition, it is a complete lattice, it is called either a *complete Heyting algebra* or a *frame*.

Examples.

(i) $P(A)$ is a frame. For $B, C \in P(A)$ we have $B \Rightarrow C = \bigcup\{D \mid D \cap B \subseteq C\}$ and the operations $\bigwedge$ and $\bigvee$ are the operations $\bigcap$ and $\bigcup$ respectively.

(ii) The bounded lattice $O(X)$ of opens on some topological space is also a frame. The join of a set of opens is equal to its union and the meet equal to the interior of its intersection. In other words for $Y \subseteq O(X)$ we have

$$\bigvee Y = \bigcup Y$$

$$\bigwedge Y = \overset{\circ}{\overbrace{\bigcap Y}}$$

Furthermore for $U_1, U_2 \in O(X)$ we have

$$U_1 \Rightarrow U_2 = \bigcup\{U \mid U \cap U_1 \subseteq U_2\}$$

where U is assumed to be open.

(iii) $\{\top, \bot\}$ is a Heyting algebra however it is complete iff LEM holds. Namely, for any ϕ if join $\bigvee\{\top \mid \phi\}$ is an element of $\{\top, \bot\}$ one of $\bigvee\{\top \mid \phi\} = \top$ or $\bigvee\{\top \mid \phi\} = \bot$ must be true. The first equality implies ϕ while the second implies $\neg\phi$.

The operation $\Rightarrow$ can be characterized by the following equations

6. $x \Rightarrow x = \top$
7. $x \wedge (x \Rightarrow y) = x \wedge y$
8. $y \wedge (x \Rightarrow y) = y$
9. $x \Rightarrow (y \wedge z) = (x \Rightarrow y) \wedge (x \Rightarrow z)$

For every frame H the following holds

$$x \wedge \bigvee A = \bigvee_{a \in A} x \wedge a$$

Namely, for all $z \in H$ we have

$$\begin{aligned} x \wedge \bigvee A \leq z &\leftrightarrow \bigvee A \leq x \Rightarrow z \\ &\leftrightarrow a \leq x \Rightarrow z \text{ for all } a \in A \\ &\leftrightarrow x \wedge a \leq z \text{ for all } a \in A \\ &\leftrightarrow \bigvee_{a \in A} x \wedge a \leq z \end{aligned}$$

In particular every frame is also a distributive lattice. The reasoning above is always true if A is a two-element set, so every Heyting algebra is a distributive lattice as well.

Definition 1.10. The pseudocomplement of and element x is the element $x^* = x \Rightarrow \bot$.

An element x is *complemented* if $x \vee x^* = \top$. For such x the following holds

$$\begin{aligned} a \wedge x \leq y &\rightarrow (a \wedge x) \vee x^* \leq y \vee x^* \\ &\leftrightarrow (a \vee x) \wedge (x \vee x^*) \leq y \vee x^* \\ &\leftrightarrow (a \vee x) \leq y \vee x^* \\ &\rightarrow a \leq y \vee x^* \end{aligned}$$

Furthermore

$$x \wedge (y \vee x^*) = x \wedge y \leq y$$

And therefore $x \Rightarrow y = y \vee x^*$. In particular, $x^{**} = \bot \vee x = x$. Note that not every element is a Heyting algebra is necessarily complemented. For instance in $O(X)$ where $(X, \mathcal{T})$ is a connected topological space X and $\emptyset$ are the only complemented elements.

Definition 1.11. A *complete subalgebra* of a Heyting algebra H is a subset that is closed under the restriction of all operations in H.

1.3.1 The Frame $H_{\mathbf{C}}$

Here we will define a Heyting algebra following [Bell, 2014] that will be of particular use later. First a few definitions, in the following we assume $(P, \leq)$ to be some arbitrary poset.
For $p \in P$ let $\downarrow p = \{q \in P \mid q \leq p\}$.

Definition 1.12. A *sieve* is a set $I \subseteq P$ with the property that $\downarrow p \subseteq I$ for each $p \in I$.

A set T *sharpens* or is a *sharpening* of a set S if for each $q \in T$ there exists $p \in S$ such that $q \leq p$.

Definition 1.13. A *coverage* $\mathbf{C}$ of P is a map sending each $p \in P$ to a family of sets $\mathbf{C}(p)$ such that $C \subseteq \downarrow p$ for all $C \in \mathbf{C}(p)$ and, given $q \leq p$ and $C \in \mathbf{C}(p)$, there exists $C' \in \mathbf{C}(q)$ that sharpens C.

A set $A \subseteq P$ is ***C**-closed* if it satisfies $\exists C \in \mathbf{C}(p)[C \subseteq A] \rightarrow p \in A$

Lemma 1.14. *For any coverage $\mathbf{C}$ of a poset P the set $H_{\mathbf{C}}$ of $\mathbf{C}$-closed sieves (ordered by inclusion) is a frame.*

Proof. First note that for any non-empty set $A \subseteq H_{\mathbf{C}}$, the intersection $\bigcap A$ is itself a $\mathbf{C}$-closed sieve. Namely if $p \in \bigcap A$ and $q \leq p$ then for all $a \in A$, we have $p \in a$ and so $q \in a$ from which it follows that $q \in \bigcap A$. Similarly if there exists $S \in \mathbf{C}(p)$ such that $S \subseteq \bigcap A$ then for all $a \in A$ it holds that $S \subseteq a$ and so also $p \in a$ since each $a \in A$ is $\mathbf{C}$-closed, thus $p \in \bigcap A$. Therefore $\bigcap A$ is an element of $H_{\mathbf{C}}$ and since $H_{\mathbf{C}}$ is ordered by inclusion it is, by definition, the infimum of A. The definition for $\bigvee A$ follows immediately since $\bigvee A = \bigwedge\{b \mid \forall a \in A[a \leq b]\}$.

It is left to show that $a \Rightarrow b$ is well defined for $a, b \in H_{\mathbf{C}}$. Consider the set

$$S = \{p \mid (a \cap \downarrow p) \subseteq b\}$$

We assert that $a \Rightarrow b = S$. First, S is a sieve since for all $p \in S$ and $q \leq p$ we have $\downarrow q \subseteq \downarrow p$ and so $(a \cap \downarrow q) \subseteq (a \cap \downarrow p) \subseteq b$. Now let there be $C \in \mathbf{C}(p)$ with $C \subseteq S$, then for arbitrary $p' \in a \cap p\downarrow$ there exists $C' \in \mathbf{C}(p')$ that sharpens C since $\mathbf{C}$ is a coverage and $p' \leq p$. It follows then, because S is a sieve, that $C' \subseteq S$. Therefore for $p'' \in C'$ we have $p'' \in b$ since $\downarrow p'' \subseteq \downarrow p' = a \cap \downarrow p'$, and thus, $C' \subseteq b$. Now since b is $\mathbf{C}$-closed it follows that $p' \in b$ and because p' was an arbitrary element of $a \cap \downarrow p$ we have $a \cap \downarrow p \subseteq b$ so $p \in S$ and therefore S is $\mathbf{C}$-closed.
Now let $p \in a$, it follows from the definition of membership in S that $p \in b \cap S$ implies $p \in b$ so $a \wedge S = a \cap S \leq b$. Then for arbitrary $c \in H_{\mathbf{C}}$ such that $c \wedge a \leq b$ and $p \in c$ we have $\downarrow p \subseteq c$ since c is a sieve and therefore $a \cap \downarrow p \subseteq a \cap c \subseteq b$ and so, by definition, $p \in S$ hence $c \leq S$. Thus $S = a \Rightarrow b$. □

1.4 Classical Set Theory: ZFC

We argue from this point on in ZFC. This section follows the exercises and proofs given in [Jech, 2003] and [Moerdijk and van Oosten, 2014]. The notions of *induction* and *recursion* are essential to the construction of Heyting-valued models and they are the primary focus of this section. At the end we prove in a lemma some of the well known classical properties of ordinals that will also prove helpful. Sentences in ZFC are written, as with IZF, using the first-order language $\mathcal{L}$ with equality and the binary operation $\in$. Its axioms are all those of IZF with, instead of collection and induction the axioms

> *Replacement*
> $\forall x[(\forall y \in x \exists! z \phi(y, z)) \to \exists w \forall y \in x \exists z \in w \phi(y, z)$
>
> *Regularity*
> $\forall x \exists y \in x[y \cap x = \emptyset]$

As well as the Axiom of Choice which can be formulated

$$\forall x \exists f[\text{fun}(f) \wedge \text{dom}(f) = x \wedge \forall y \in x(y \neq \emptyset \to f(y) \in y)]$$

which corresponds to the idea that, for any collection of non-empty sets we can 'chose' an element from each. An equivalent formulation is

$$\forall f[\text{fun}(f) \to \exists s(\text{fun}(s) \wedge \text{dom}(s) = \text{ran}(f) \wedge \forall x \in \text{dom}(f)[s(f(x)) = x])]$$

which means that every surjective function $f: A \to B$ has a 'section' $s: B \to A$ such that $s(f(x)) = x$.

Both induction and recursion are dependent on the idea of a *well-founded* relation defined below.

Definition 1.15. A relation R on a class X is well-founded on X if the class $Ru = \{x \in X \mid xRu\}$ is a set for all $u \in X$ and for any set $S \subseteq X$ there exists some $s \in S$ such that $\{s' \in S \mid s'Rs\}$ is empty, that is, $s'Rs$ does not hold for any $s' \in S$.

Note that R is irreflexive for if xRx then R fails the second property for the set $\{x\}$. A set T is *R-transitive* if $Rx \subseteq T$ for every $x \in T$. Given a set S and a well-founded relation R it will be useful to be able to construct an R-transitive set containing S. To this end let $\tilde{S} = \bigcup_{s \in S} Rs$ and define the sequence $(S_n)_{n\in\mathbb{N}}$ recursively as such: $S_0 = S$ and $S_{n+1} = S_n \cup \tilde{S_n}$. Now $T(S) = \bigcup_{n\in\mathbb{N}} S_n$ has the desired property. Namely, for any $x \in T(S)$ and any arbitrary y such that yRx, there exists some n such that $x \in S_n$, and therefore $y \in S_{n+1} \subseteq T(S)$.

Lemma 1.16. *If R is a well-founded relation, any non-empty, definable class C has an R-minimal element, that is, an element e such that cRe for no c in C.*

Proof. Let $C = \{x \mid \phi(x)\}$ be non-empty. For c in C consider the set $C' = \{c' \in T(\{c\}) \mid \phi(c')\}$. Because C' is a set by separation on the set $T(\{c\})$ there exists an element $e \in C'$ such that $c'Re$ for no $c' \in C'$. Now if there exists x such that xRe and $\phi(x)$ both hold then it follows that $x \in C'$, which is contradiction. Therefore e is as desired. □

With this we can now prove

Proposition 1.17 (Principle of Induction)**.** *The following holds for any well founded relation R*

$$\forall y[\forall x\big(xRy \to \phi(x)\big) \to \phi(y)] \to \forall x\phi(x)$$

Proof. Assume $\forall y[\forall x\big(xRy \to \phi(x)\big) \to \phi(y)]$ and let $C = \{x \mid \neg\phi(x)\}$ be non-empty. By 1.16 there exists an R-minimal element e of C, so for e we have $\forall x[xRe \to \phi(x)]$. It follows then by the assumption that $\phi(e)$, a contradiction, hence C is empty from which follows $\forall x\phi(x)$. □

This will be the primary tool in a lot of the proofs we go over including the principle of recursion.

Proposition 1.18 (Principle of Recursion)**.** *For any well founded relation R on a class X and a class function F on pairs (x, g) where x is an object in X and g is a function on Rx there exists a unique function G such that $G(x) = F(x, G|_{Rx})$ for all x in X.*

Proof. First we prove uniqueness. Let G and Z satisfy the theorem for some function F and assume $\forall y[yRx \to G(y) = Z(y)]$ then

$$G(x) = F(x, G|_{Rx}) = F(x, Z|_{Rx}) = Z(x)$$

Therefore, by induction, $G(x) = Z(x)$ for all x in X so $G = Z$.
Using this we can then prove existence. Assume that for all yRx there exists G_y defined on $T(\{y\})$ such that $G_y(z) = F(z, G_y|_{Rz})$. Now let

$$G_x = \{\langle x, F(x, \bigcup_{yRx} G_y)\rangle\} \cup \bigcup_{yRx} G_y$$

That G_x is a well-defined follows from the uniqueness of the G_y's and the fact R is well-founded. First, for any elements y, y' of X we have $G_y|_{\mathrm{dom}(G_y)\cap\mathrm{dom}(G_{y'})} = G_{y'}|_{\mathrm{dom}(G_y)\cap\mathrm{dom}(G_{y'})}$ by uniqueness, it follows then that the union $G_y \cup G_y$ is a well defined function. Second $x \notin \mathrm{dom}(\bigcup_{yRx} G_y)$ since R is well-founded. To see this let $x \in T(y)$ for some yRx, there exists then, a sequence $y_1, ..., y_n$ such that y_nRy_{n-1}, y_1Ry, and xRy_n; however, as such the set $\{y, y_1, ..., y_n, x\}$ has no minimal element.
Now G_x is defined on $T(\{x\})$ and is such that $G_x(z) = F(z, G_x|_{Rz})$ thus by induction there exists such a G_x for all x. Define $G(x) = G_x(x)$ then for yRx we have $G_x(y) = G_y(y) = G(y)$ therefore $G(x) = G_x(x) = F(x, G_x|_{Rx}) = F(x, G|_{Rx})$. □

The relation $\in$ is well founded since we have $\in u = u$ and the axiom of regularity ensures that for all u there exists some $x \in u$ such that $y \notin x$ for all $y \in u$. It follows then that the principles of induction and recursion hold for $\in$. Particularly the axiom of recursion from IZF holds in ZFC as well.

Definition 1.19. An *ordinal* is an $\in$-transitive set α that is linearly ordered by $\in$, that is to say, for any $\beta, \gamma \in \alpha$ one of $\beta \in \gamma$, $\beta = \gamma$, $\gamma \in \beta$ holds.

We will usually refer to ordinals using lower-case greek letters. A few facts about ordinals are useful and are summarized in the following lemma.

Lemma 1.20.

(i) Every ordinal is itself a set of ordinals.

(ii) For ordinals $\alpha \subseteq \beta$ we have $\alpha = \beta \vee \alpha \in \beta$

(iii) The class of all ordinals is linearly ordered by $\in$

(iv) Any non-empty class of ordinals has a least element

(v) The union of a set of ordinals is itself an ordinal

Proof.
(i)
For any ordinal α let β be such that $\beta \in \alpha$. We wish to show that β is $\in$-transitive and linearly order by $\in$. To this end consider arbitrary ξ, γ such that $\xi \in \gamma \in \beta$. First, it follows from transitivity of α, that $\gamma \in \alpha$ and therefore $\xi \in \alpha$. Since $\xi = \beta$ and $\beta \in \xi$ both contradict the fact that $\in$ is well-founded (consider the set $\{\xi, \gamma, \beta\}$) we must have $\xi \in \beta$. Second, for all $\beta \in \alpha$ and $\gamma, \gamma' \in \beta$ we have $\gamma, \gamma' \in \alpha$ therefore one of $\gamma' \in \gamma$, $\gamma' = \gamma$, $\gamma \in \gamma'$ must hold.
(ii)
Assume $\alpha \subsetneq \beta$ then $\beta - \alpha$ has a minimal element, call it γ. Now if $\xi \in \gamma$, then ξ must be in β but not in $\beta - \alpha$ so $\xi \in \alpha$ and therefore $\gamma \subseteq \alpha$. Furthermore for $\zeta \in \alpha$, since β is linearly ordered by $\in$, one of $\gamma \in \zeta$, $\gamma = \zeta$, $\zeta \in \gamma$ must hold, the first two imply $\gamma \in \alpha$ which contradicts $\gamma \in \beta - \alpha$ so $\zeta \in \gamma$ and therefore $\alpha \subseteq \gamma$ and so $\gamma = \alpha$.
(iii)
Given any two ordinals α, β assume $\alpha \neq \beta$. If $\beta - \alpha$ is empty then $\beta \subsetneq \alpha$ and so by *(ii)* $\beta \in \alpha$. Otherwise, let γ be a minimal element of $\beta - \alpha$ then $\gamma \subseteq \alpha$ as shown above, and since $\gamma \subsetneq \alpha$ would imply $\gamma \in \alpha$ which is a contradiction, we must have $\gamma = \alpha$ and so $\alpha \in \beta$.
(iv)
Given a class of ordinals A we can find a minimal element ξ by 1.16 which is then also a least element by *(iii)*.
(v)
Let A be a set of ordinals. For all $\beta \in \bigcup A$ and $\gamma \in \beta$ we have $\beta \in \alpha$ for some $\alpha \in A$ and therefore $\gamma \in \alpha$ since α is transitive, thus $\gamma \in \bigcup A$ and therefore $\bigcup A$ is transitive. Now for any $\beta, \beta' \in \bigcup A$ we have $\beta \in \alpha$ and $\beta' \in \alpha'$ for some ordinals α, α' in A which means by *(i)* that β and β' are also ordinals and therefore by *(iii)* one of $\beta \in \beta'$, $\beta = \beta'$, $\beta' \in \beta$ must hold, so $\bigcup A$ is linearly ordered by $\in$. □

Note that for any ordinal α the successor $\alpha^+ = \alpha \cup \{\alpha\}$ is an ordinal as well as this is a specific instance of (v).

2 Frame Valued Models

We now proceed to construct the model following the procedure in [Bell, 1985] and [Bell, 2014]. It is common to associate x with a characteristic function χ_x for which $x \subseteq \mathrm{dom}(\chi_x)$. We can think of a characteristic function as holding all necessary information about the set that it represents. Commonly, a characteristic function will take the values $\chi_x(z) = 1$ if $z \in x$ and $\chi_x(z) = 0$ otherwise. As such, it takes values on $\{0, 1\}$ can therefore be seen as a mapping to some algebra of truth values (think of $\{0, 1\}$ as the two valued Boolean algebra $\{\top, \bot\}$). We generalize this notion by examining functions that maps potential elements of a set to any Heyting algebra. This

is motivation for the following construction.
Given some Heyting algebra H we wish to construct a universe $V^{(H)}$ of so called *H-valued functions*, that is functions with range in H, that will come to act as the universe of all sets. As characteristic functions are mappings of sets to truth values we would also like this to be the case for $u \in V^{(H)}$ and so we consider only functions that are *homogenous* that is, functions whose domains are themselves sets of homogenous H-valued functions.
To this end we define for each ordinal α the set

$$V_\alpha^{(H)} = \{u \mid \mathrm{fun}(u) \wedge \mathrm{ran}(u) \subseteq H \wedge \exists \xi \in \alpha[\mathrm{dom}(u) \subseteq V_\xi^{(H)}]\}$$

The universe of homogenous H-valued functions is then the class

$$V^{(H)} = \{u \mid \exists \alpha[u \in V_\alpha^{(H)}]\}$$

As such, $V^{(H)}$ is precisely the class of all sets u that satisfy

$$\mathrm{fun}(u) \wedge \mathrm{ran}(u) \subseteq H \wedge \mathrm{dom}(u) \subseteq V^{(H)}$$

To see this, first, given $v \in V^{(H)}$, let $\mathrm{rank}(v)$ denote the least ordinal α such that $v \in V_\alpha^{(H)}$. Now for some function u with domain in $V^{(H)}$ that takes values in H consider then the ordinal $\alpha = \bigcup\{\mathrm{rank}(v) \mid v \in \mathrm{dom}(u)\}$. For all $v \in \mathrm{dom}(u)$ we then have $\mathrm{rank}(v) \in \alpha^+$ and therefore $u \in V_{\alpha^+}^{(H)}$.

Lemma 2.1. *The relation $v \in dom(u)$ is a well-founded relation on $V^{(H)}$.*

Proof. Since $\mathrm{dom}(u)$ is a set it suffices to show that there exists $v' \in \mathrm{dom}(u)$ such that $w \in \mathrm{dom}(u)$ for no $w \in \mathrm{dom}(v')$. The set of ordinals $\{\mathrm{rank}(v) \mid v \in \mathrm{dom}(u)\}$ has a least element α, let $v' \in \mathrm{dom}(u)$ be such that $\mathrm{rank}(v') = \alpha$. Now for arbitrary $w \in \mathrm{dom}(v')$ there exists $\xi \in \alpha$ such that $w \in V_\xi^{(H)}$ and therefore either $\mathrm{rank}(w) \in \xi$ or $\mathrm{rank}(w) = \xi$. It follows then by transitivity of α that $\mathrm{rank}(w) \in \alpha$. Therefore we must have $w \notin \mathrm{dom}(u)$ since α is minimal with regard to $\in$ in $\{\mathrm{rank}(v) \mid v \in \mathrm{dom}(u)\}$. □

It follows immediately from 2.1 that we can perform induction on $y \in \mathrm{dom}(x)$.

The structure of $V^{(H)}$ allows for a mapping of logical sentences ϕ to truth values $[\![\phi]\!]$ in H. Such sentences are written in the first order language $\mathcal{L}^H$ with equality that has the relation $\in$ and a constant for every element $u \in V^{(H)}$. Given definitions for $[\![\phi]\!]$ and $[\![\psi]\!]$ we define the mapping $[\![\cdot]\!]$ as follows:

$$[\![\phi \wedge \psi]\!] = [\![\phi]\!] \wedge [\![\psi]\!]$$
$$[\![\phi \vee \psi]\!] = [\![\phi]\!] \vee [\![\psi]\!]$$
$$[\![\phi \to \psi]\!] = [\![\phi]\!] \Rightarrow [\![\psi]\!]$$
$$[\![\neg\phi]\!] = [\![\phi]\!]^*$$
$$[\![\exists x[\phi(x)]]\!] = \bigvee_{u \in V^{(H)}} [\![\phi(u)]\!]$$
$$[\![\forall x[\phi(x)]]\!] = \bigwedge_{u \in V^{(H)}} [\![\phi(u)]\!].$$

Now all that is left is to determine values for the atomic sentences $[\![u = v]\!]$ and $[\![u \in v]\!]$. The guiding intuition in doing this will be as follows: We, again, wish to have each function in $V^{(H)}$ act as a characteristic function and so it should carry the structure of the set it represents. Hence, given some u, v in $V^{(H)}$ we want

$$[\![u \in v]\!] = \bigvee_{y \in \mathrm{dom}(v)} \big[v(y) \wedge [\![u = y]\!]\big].$$

In other words we wish u to be equal to some y which v indicates is within the set it represents. The notion of equality is guided by the axioms of extensionality, namely

$$[\![u = v]\!] = \bigvee_{y \in \mathrm{dom}(v)} \big[v(y) \Rightarrow [\![y \in u]\!]\big] \wedge \bigvee_{x \in \mathrm{dom}(u)} \big[u(x) \Rightarrow [\![x \in v]\!]\big].$$

Which can be thought to mean "u and v are equal if whenever u indicates x is within the set it represents then x is in the set represented by v and vis versa".
The desired definitions can be achieved by simultaneous recursion. We start by defining a new relation R on the class of ordered pairs $\{\langle u, v\rangle \mid u, v \in V^{(H)}\}$. Let $\langle u', v'\rangle R\langle u, v\rangle$ iff $u' \in \mathrm{dom}(u)$ and $v' = v$ or $v' \in \mathrm{dom}(v)$ and $u' = u$.

Lemma 2.2. *The relation R is well founded.*

Proof. First $R\langle u, v\rangle$ is a set by replacement on $\mathrm{dom}(u) \times \mathrm{dom}(v)$. Now, given a set A of ordered pairs, the sets $\{u \mid \exists v[\langle u, v\rangle \in A]\}$ and $\{v \mid \exists u[\langle u, v\rangle \in A]\}$ have, respectively, minimal elements u' and v' with regard to the relation $x \in \mathrm{dom}(y)$. Therefore, $\langle u', v'\rangle$ is a minimal element of A, for if there existed $\langle u'', v''\rangle R\langle u', v'\rangle$ it would contradict the minimality of either u' or v' by definition of R. □

Now define a class function F as follows, given $\langle u, v\rangle \in V^{(H)} \times V^{(H)}$ and a function $g\colon R\langle u, v\rangle \to H^3$ with components g_i, let $F(\langle u, v\rangle, g) = \langle a, b, c\rangle$ where

$$\begin{aligned} a &= \bigvee_{y \in \mathrm{dom}(v)} [v(y) \wedge g_3(\langle u, y\rangle)] \\ b &= \bigvee_{x \in \mathrm{dom}(u)} [u(x) \wedge g_3(\langle x, v\rangle)] \\ c &= \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow g_1(\langle x, v\rangle)] \wedge \bigwedge_{y \in \mathrm{dom}(v)} [v(y) \Rightarrow g_2(\langle u, y\rangle)] \end{aligned}$$

Now by Proposition 1.18 there exists a unique function G that satisfies $G(x) = F(x, G|_{Rx})$ for all $x \in V^{(H)} \times V^{(H)}$. We then define $[\![u = v]\!]$ and $[\![u \in v]\!]$ by saying $\langle [\![u \in v]\!], [\![v \in u]\!], [\![u = v]\!]\rangle = G(\langle u, v\rangle)$. The following lemma shows that, as such, $[\![u \in v]\!]$ is well defined.

Lemma 2.3. *We have the equalities $G_1(\langle u, v\rangle) = G_2(\langle v, u\rangle)$ and $G_3(\langle u, v\rangle) = G_3(\langle v, u\rangle)$*

Proof. By induction on R: let the statement be true for $R\langle u, v\rangle$ then we have

$$G_1(\langle u, v\rangle) = \bigvee_{y \in \mathrm{dom}(v)} [v(y) \wedge G_3(\langle u, y\rangle)] = \bigvee_{y \in \mathrm{dom}(v)} [v(y) \wedge G_3(\langle y, u\rangle)] = G_2(\langle v, u\rangle)$$

and

$$\begin{aligned} G_3(\langle u, v\rangle) = \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow G_1(\langle x, v\rangle)] \wedge \bigwedge_{y \in \mathrm{dom}(v)} [v(y) \Rightarrow G_2(\langle u, y\rangle)] = \\ \bigwedge_{y \in \mathrm{dom}(v)} [v(y) \Rightarrow G_1(\langle y, u\rangle)] \wedge \bigwedge_{x \in \mathrm{dom}(u)} [u(x) \Rightarrow G_2(\langle v, x\rangle)] = G_3(\langle v, u\rangle) \end{aligned} \tag{6}$$

□

We have then as an immediate consequence, the following corollary

Corollary 2.3.1. *For all $u, v \in V^{(H)}$ the equality $[\![u = v]\!] = [\![v = u]\!]$ holds.*

Note that $V^{(H)} \models [\![\phi \to \psi]\!]$ iff $[\![\phi]\!] \leq [\![\psi]\!]$ since exactly then is $[\![\phi]\!] \Rightarrow [\![\psi]\!] = \top$, this fact will frequently be used without mention.

2.1 Consistency with Intuitionistic logic

We say a sentence ϕ is *true in* $V^{(H)}$ (notation $V^{(H)} \models \phi$) if $[\![\phi]\!] = \top$. All the axioms of the Hilbert-style calculus for intuitionistic first-order logic are true in $V^{(H)}$ and the rules of inference hold in the following sense: If Γ is a theory such that ϕ is true in $V^{(H)}$ for all $\phi \in \Gamma$ then $\Gamma \vdash \psi$ implies $V^{(H)} \models \psi$. This result follows fairly quickly from the definition of $[\![\cdot]\!]$ and the nature of the operations $\wedge, \vee, \Rightarrow$ in any Heyting algebra for most of the axioms. We will treat a specific few cases relating to the language $\mathcal{L}^H$ here. An important consequence of this is that for any theory Γ that is true in $V^{(H)}$ showing that $V^{(H)} \models \phi$ implies $\neg\phi$ is not a consequence of Γ. If it were we would have $\top = [\![\phi]\!] \wedge [\![\neg\phi]\!] = [\![\phi]\!] \wedge [\![\phi]\!]^* = \bot$, a contradiction.

Lemma 2.4. *The following is true for arbitrary u, v, and w:*

(i) $V^{(H)} \models u = u$, *additionally* $u(x) \leq [\![x \in u]\!]$ *for all* $x \in dom(u)$.

(ii) $V^{(H)} \models u = v \wedge u = w \to v = w$

(iii) $V^{(H)} \models u = v \wedge v \in w \to u \in w$

(iv) $V^{(H)} \models u = v \wedge w \in v \to w \in u$

(v) $V^{(H)} \models u = v \wedge \phi(v) \to \phi(u)$

Proof.

(i) We proceed by induction, assume $[\![x = x]\!]$ for all $x \in \mathrm{dom}(u)$ then for $x \in \mathrm{dom}(u)$ we have

$$[\![x \in u]\!] = \bigvee_{x' \in \mathrm{dom}(u)} \big[u(x') \wedge [\![x' = x]\!]\big] \geq u(x) \wedge [\![x = x]\!] = u(x) \tag{7}$$

and so

$$[\![u = u]\!] = \bigwedge_{x \in \mathrm{dom}(u)} \big[u(x) \Rightarrow [\![x \in u]\!]\big] \geq \bigwedge_{x \in \mathrm{dom}(u)} \big[u(x) \Rightarrow u(x)\big] = \top$$

It is an immediate consequence of this that $u(x) \leq [\![x \in u]\!]$ for all $u \in V^{(H)}$ and $x \in \mathrm{dom}(u)$ since *(i)* implies that (7) is universally true.

(ii) We, again, use the induction principle for $V^{(H)}$. Assume that for $x \in \mathrm{dom}(u)$ we have $\forall v, w \in V^{(H)} \big[[\![x = v]\!] \wedge [\![x = w]\!] \leq [\![v = w]\!]\big]$. Note that by definition of $[\![\cdot = \cdot]\!]$

$$v(y) \wedge [\![u = v]\!] \leq v(y) \wedge \bigwedge_{y \in v} \big[v(y) \Rightarrow [\![y \in u]\!]\big] \leq [\![y \in u]\!]$$

Using this twice we get

$$\begin{aligned}
[\![u = w]\!] \wedge [\![u = v]\!] \wedge v(y) &\leq [\![u = w]\!] \wedge [\![y \in u]\!] \\
&= \bigvee_{x \in \mathrm{dom}(u)} \big[u(x) \wedge [\![u = w]\!] \wedge [\![y = x]\!]\big] \\
&\leq \bigvee_{x \in \mathrm{dom}(u)} \big[[\![x \in w]\!] \wedge [\![y = x]\!]\big]
\end{aligned}$$

Now, by the induction hypothesis we have

$$\begin{aligned}
[\![x \in w]\!] \wedge [\![y = x]\!] &= \bigvee_{z \in \mathrm{dom}(w)} \big[w(z) \wedge [\![z = x]\!] \wedge [\![y = x]\!]\big] \\
&\leq \bigvee_{z \in \mathrm{dom}(w)} \big[w(z) \wedge [\![z = y]\!]\big] \qquad (8) \\
&= [\![y \in w]\!]
\end{aligned}$$

thus

$$[\![u = w]\!] \wedge [\![u = v]\!] \wedge v(y) \leq [\![y \in w]\!]$$

and therefore $[\![u = w]\!] \wedge [\![u = v]\!] \leq v(y) \Rightarrow [\![y \in w]\!]$ for all $y \in \mathrm{dom}(v)$. By symmetry we then also have $[\![u = w]\!] \wedge [\![u = v]\!] \leq w(z) \Rightarrow [\![z \in v]\!]$ for all $z \in \mathrm{dom}(w)$ so

$$[\![u = w]\!] \wedge [\![u = v]\!] \leq \bigwedge_{y \in \mathrm{dom}(v)} \left[v(y) \Rightarrow [\![y \in w]\!]\right] \wedge \bigwedge_{z \in \mathrm{dom}(w)} \left[w(z) \Rightarrow [\![z \in v]\!]\right] = [\![v = w]\!]$$

as desired.

(iii) Given *(ii)*, the equation (2) provides the desired result.

(iv) Again using definition of $[\![\cdot = \cdot]\!]$ in combination with *(iii)* we get

$$[\![u = v]\!] \wedge [\![w \in v]\!] = \bigvee_{y \in \mathrm{dom}(v)} \left[v(y) \wedge [\![y = w]\!] \wedge [\![u = v]\!]\right] \leq \bigvee_{y \in \mathrm{dom}(v)} \left[[\![y = w]\!] \wedge [\![y \in u]\!]\right] \leq [\![w \in u]\!]$$

(v) We proceed by induction on the structure of ϕ. The atomic cases are handled in *(ii)-(iv)*. The following equations cover induction for $\wedge, \vee, \exists$ and $\forall$ respectively:

$$[\![\phi(v) \wedge \psi(v)]\!] \wedge [\![u = v]\!] = [\![\phi(v)]\!] \wedge [\![u = v]\!] \wedge [\![\psi(v)]\!] \wedge [\![u = v]\!] \leq [\![\phi(u)]\!] \wedge [\![\psi(u)]\!]$$

$$[\![\phi(v) \vee \psi(v)]\!] \wedge [\![u = v]\!] = ([\![\phi(v)]\!] \wedge [\![u = v]\!]) \vee ([\![\psi(v)]\!] \wedge [\![u = v]\!]) \leq [\![\phi(u)]\!] \vee [\![\psi(u)]\!]$$

$$[\![v = u]\!] \wedge [\![\exists \phi(w, v)]\!] = \bigvee_{w \in V^{(H)}} \left[[\![v = u]\!] \wedge [\![\phi(w, v)]\!]\right] \leq \bigvee_{w \in V^{(H)}} [\![\phi(w, u)]\!]$$

$$[\![v = u]\!] \wedge [\![\forall \phi(w, v)]\!] = \bigwedge_{w \in V^{(H)}} \left[[\![v = u]\!] \wedge [\![\phi(w, v)]\!]\right] \leq \bigwedge_{w \in V^{(H)}} [\![\phi(w, u)]\!]$$

For the implication conjunction we have then

$$\begin{aligned}[\![\phi(u)]\!] \wedge [\![v = u]\!] \wedge ([\![\phi(v)]\!] \Rightarrow [\![\psi(v)]\!]) &\leq [\![v = u]\!] \wedge [\![\phi(v)]\!] \wedge ([\![\phi(v)]\!] \Rightarrow [\![\psi(v)]\!]) \\ &\leq [\![v = u]\!] \wedge [\![\psi(v)]\!] \\ &\leq [\![\psi(u)]\!]\end{aligned}$$

Therefore

$$[\![v = u]\!] \wedge ([\![\phi(v)]\!] \Rightarrow [\![\psi(v)]\!]) \leq [\![\phi(u)]\!] \Rightarrow [\![\psi(u)]\!].$$

The case for $\neg\phi$ follows directly since $[\![\neg\phi]\!] = [\![\phi]\!]^* = [\![\phi]\!] \Rightarrow \bot$.

□

The following lemma is another consequence of 2.4 and will be used often in calculating values for $[\![\cdot]\!]$.

Lemma 2.5. *The following equalities hold:*

(i) $[\![\exists v \in u \phi(v)]\!] = \bigvee_{x \in u} \left[u(x) \wedge [\![\phi(x)]\!]\right]$

(ii) $[\![\forall v \in u \phi(v)]\!] = \bigwedge_{x \in u} \left[u(x) \Rightarrow [\![\phi(x)]\!]\right]$

Proof.

(i) To begin, by definition of the mapping $[\![\cdot]\!]$ we have

$$\begin{aligned}[\![\exists v \in u \phi(v)]\!] &= [\![\exists v(v \in u \wedge \phi(v))]\!] \\ &= \bigvee_{v \in V^{(H)}} \left[[\![v \in u]\!] \wedge [\![\phi(v)]\!]\right] \\ &= \bigvee_{v \in V^{(H)}} \bigvee_{x \in \mathrm{dom}(u)} \left[u(x) \wedge [\![v = x]\!] \wedge [\![\phi(v)]\!]\right] \\ &= \bigvee_{x \in \mathrm{dom}(u)} \left[u(x) \wedge \bigvee_{v \in V^{(H)}} [\![v = x \wedge \phi(v)]\!]\right]\end{aligned}$$

Now by Lemma 2.4 we have $[\![v = x \wedge \phi(v)]\!] \leq [\![\phi(x)]\!]$ and by Corollary 2.3.1 we have $[\![x = x \wedge \phi(x)]\!] = [\![\phi(x)]\!]$ therefore

$$\bigvee_{x \in \mathrm{dom}(u)} \left[u(x) \wedge \bigvee_{v \in V^{(H)}} [\![v = x \wedge \phi(v)]\!]\right] = \bigvee_{x \in u} \left[u(x) \wedge [\![\phi(x)]\!]\right]$$

which gives the desired equality.

(ii) First

$$[\![\forall v \in u \phi(v)]\!] = [\![\forall v(v \in u \to \phi(v))]\!] = \bigwedge_{v \in V^{(H)}} \left[[\![v \in u]\!] \Rightarrow [\![\phi(v)]\!]\right] \tag{9}$$

Now given $v \in V^{(H)}$ we have

$$\begin{aligned}\bigwedge_{x \in u} \left[u(x) \Rightarrow [\![\phi(x)]\!]\right] \wedge [\![v \in u]\!] &= \bigwedge_{x \in u} \left[u(x) \Rightarrow [\![\phi(x)]\!]\right] \wedge \bigvee_{x \in \mathrm{dom}(u)} \left(u(x) \wedge [\![v = x]\!]\right) \\ &= \bigvee_{x \in \mathrm{dom}(u)} \left[\bigwedge_{x \in u} \left[u(x) \Rightarrow [\![\phi(x)]\!]\right] \wedge u(x) \wedge [\![v = x]\!]\right] \\ &\leq \bigvee_{x \in \mathrm{dom}(u)} \left[[\![\phi(x)]\!] \wedge [\![x = v]\!]\right] \\ &\leq [\![\phi(v)]\!]\end{aligned}$$

Therefore for all $v \in V^{(H)}$ we have

$$\bigwedge_{x \in \mathrm{dom}(u)} \left[u(x) \Rightarrow [\![\phi(x)]\!]\right] \leq [\![v \in u]\!] \Rightarrow [\![\phi(v)]\!]$$

Furthermore

$$\bigwedge_{v \in V^{(H)}} \left[[\![v \in u]\!] \Rightarrow [\![\phi(v)]\!]\right] \leq \bigwedge_{v \in \mathrm{dom}(u)} \left[[\![v \in u]\!] \Rightarrow [\![\phi(v)]\!]\right] \leq \bigwedge_{v \in \mathrm{dom}(u)} \left[u(v) \Rightarrow [\![\phi(v)]\!]\right]$$

so (9) is equivalent to $\bigwedge_{x \in \mathrm{dom}(u)} \left[u(x) \Rightarrow [\![\phi(x)]\!]\right]$ as desired.

□

Finally we have

Proposition 2.6. *$V^{(H)} \models$ LEM iff H is a Boolean algebra.*

Proof. Assume $V^{(H)} \models$ LEM then given $a \in H$ let u be the function $\{\langle \emptyset, a \rangle\}$ then $[\![\emptyset \in u]\!] = a$. It then follows from $V^{(H)} \models \emptyset \in u \vee \neg(\emptyset \in u)$ that

$$a \vee a^* = [\![\emptyset \in u]\!] \vee [\![\neg(\emptyset \in u)]\!] = \top$$

so a is complimented. Since a is an arbitrary element of H we can conclude that all elements of H are complemented and that it is therefore a Boolean algebra.
Conversely, if H is a Boolean algebra we have $[\![\phi \vee \neg\phi]\!] = [\![\phi]\!] \vee [\![\phi]\!]^* = \top$ and so $V^{(H)} \models$ LEM. □

2.2 Existence in $V^{(H)}$

The fact that a sentence of the form $\exists x\phi(x)$ is true in $V^{(H)}$ need not imply that we can actually find an element u such that $V^{(H)} \models \phi(u)$. There are, however, certain conditions that ensure this is the case and we present two in this section.

Proposition 2.7 (Unique Existence Principle)**.** *If* $V^{(H)} \models \exists! x\phi(x)$ *then there is a* $u \in V^{(H)}$ *such that* $V^{(H)} \models \phi(u)$

Proof. Since $\bigvee_{x \in V^{(H)}} [\![\phi(x)]\!] = \top$ we can find, using what is called a *collection argument*, an ordinal α such that $\bigvee_{x \in V_\alpha^{(H)}} [\![\phi(x)]\!] = \top$. Let $\tilde{H} = \{h \in H \mid \exists x \in V^{(H)}(h = [\![\phi(x)]\!])\}$ we can then find a set X (be it be AC and replacement or collection) such that $\forall h \in \tilde{H} \exists x \in X[[\![\phi(x)]\!] = h]$, now let $\alpha = \bigcup_{x \in X} \text{rank}(x)$ then $X \subseteq V_\alpha^{(H)}$ and therefore

$$\bigvee_{x \in V_\alpha^{(H)}} [\![\phi(x)]\!] = \bigvee_{x \in V^{(H)}} [\![\phi(x)]\!] = \top$$

Let $\text{dom}(u) = V_\alpha^{(H)}$ and $u(v) = [\![\exists w[\phi(w) \wedge v \in w]]\!]$. Now unique existence implies that $[\![\phi(a)]\!] \wedge [\![\phi(b)]\!] \leq [\![a = b]\!]$. It follows that

$$\begin{aligned}
[\![\phi(x)]\!] \wedge [\![\exists w[\phi(w) \wedge v \in w]]\!] &= \bigvee_{w \in V^{(H)}} \left[[\![\phi(x)]\!] \wedge [\![\phi(w)]\!] \wedge [\![v \in w]\!]\right] \\
&\leq \bigvee_{w \in V^{(H)}} \left[[\![x = w]\!] \wedge [\![v \in w]\!]\right] \\
&\leq [\![v \in x]\!]
\end{aligned}$$

and so $[\![\phi(x)]\!] \leq [\![\exists w[\phi(w) \wedge v \in w]\!] \Rightarrow [\![v \in x]\!]$. Additionally the fact that $[\![\phi(x)]\!] \wedge x(y) \leq [\![y \in x \wedge \phi(x)]\!]$ along with the definition of $u(y)$ gives us, for arbitrary $x \in V^{(H)}$

$$x(y) \Rightarrow [\![y \in u]\!] \geq x(y) \Rightarrow u(y) \geq x(y) \Rightarrow [\![y \in x \wedge \phi(x)]\!] \geq [\![\phi(x)]\!]$$

therefore

$$[\![u = x]\!] = \bigwedge_{y \in \text{dom}(x)} \left[x(y) \Rightarrow [\![y \in u]\!]\right] \wedge \bigwedge_{v \in V_\alpha^{(H)}} \left[[\![\exists w[\phi(w) \wedge v \in w]]\!] \Rightarrow [\![v \in x]\!]\right] \geq [\![\phi(x)]\!]$$

Thus it can be concluded that $[\![\phi(u)]\!] = [\![\exists x[\phi(x) \wedge x = u]]\!] = [\![\exists x[\phi(x)]]\!] = \top$. ∎

Before we move onto the next condition we will need a couple definitions and an important lemma.

Definition 2.8. Given a Heyting algebra H an *anti-chain* is a subset A of H such that $a_1 \wedge a_2 = \bot$ for all $a_1, a_2 \in A$ with $a_1 \neq a_2$.

Definition 2.9. Given some collection of sets $\{u_i \mid i \in I\}$ and $\{a_i \mid i \in I\} \subseteq H$ the *mixture* $\sum_{i \in I} a_i \cdot u_i$ is the function u with $dom(u) = \bigcup_{i \in I} dom(u_i)$ and $u(x) = \bigvee_{i \in I} a_i \wedge [\![x \in u_i]\!]$

Lemma 2.10 (Mixing Lemma)**.** *Let* u *be the mixture* $\sum_{i \in I} a_i \cdot u_i$. *If*

$$a_i \wedge a_j \leq [\![u_i = u_j]\!]$$

for all $i, j \in I$ *then*

$$a_i \leq [\![u_i = u]\!].$$

In particular the result holds when $\{a_i \mid i \in I\}$ *is an anti-chain.*

Proof.
First, we have for $x \in$dom(u)

$$\begin{aligned} a_i \wedge \bigvee_{j\in J} a_j \wedge [\![x \in u_j]\!] &= \bigvee_{j\in J} a_i \wedge a_j \wedge [\![x \in u_j]\!] \\ &\leq \bigvee_{j\in J} [\![u_i = u_j]\!] \wedge [\![x \in u_j]\!] \\ &\leq [\![x \in u_i]\!] \end{aligned}$$

Furthermore for y in dom(u_i)

$$a_i \wedge u_i(y) \leq a_i \wedge [\![y \in u_i]\!] \leq u(y) \leq [\![y \in u]\!]$$

Therefore $a_i \leq u(x) \Rightarrow [\![x \in u_i]\!]$ for all $x \in$dom(u) and $a_i \leq u_i(y) \Rightarrow [\![y \in u]\!]$ for all y in dom(u_i). Hence $a_i \leq [\![u_i = u]\!]$. □

A set B *refines* a set A if for all $b \in B$ there exists some $a \in A$ such that $b \leq a$. A Heyting algebra H is *refinable* if for every subset $A \subseteq H$ there exists some anti-chain B in H that refines A and has the same join, that is, $\bigvee A = \bigvee B$.

Proposition 2.11 (Refinable Existence Principle)**.** *If H is refinable then $V^{(H)}$ satisfies the existence principle, that is, if $V^{(H)} \models \exists x \phi(x)$ we can find $u \in V^{(H)}$ such that $V^{(H)} \models \phi(u)$.*

Proof. Assuming $V^{(H)} \models \exists x \phi(x)$ we can, by a collection argument, find some ordinal α such that

$$\bigvee_{v \in V_\alpha^{(H)}} [\![\phi(v)]\!] = \top$$

Now let $\{a_i \mid i \in I\}$ be a refinement of the set

$$\{[\![\phi(v)]\!] \mid v \in V_\alpha^{(H)}\}$$

Such that $\bigvee_{i\in I} a_i = \bigvee_{v \in V_\alpha^{(H)}} [\![\phi(v)]\!]$. Then, using the Axiom of Choice, we can choose for every a_i some $v_i \in V^{(H)}$ such that $a_i \leq [\![\phi(v_i)]\!]$. Now let u be the mixture $\sum_{i\in I} a_i \cdot v_i$. By the Mixing Lemma we have

$$\begin{aligned} \top &= \bigvee_{i\in I} a_i \\ &= \bigvee_{i\in I} a_i \wedge [\![v_i = u]\!] \\ &\leq \bigvee_{i\in I} [\![\phi(v_i)]\!] \wedge [\![v_i = u]\!] \\ &\leq [\![\phi(u)]\!] \end{aligned}$$

Therefore $V^{(H)} \models \phi(u)$. □

2.3 The natural mapping $\widehat{(\cdot)}\colon V \to V^{(H)}$

Given a complete subalgebra H' of H we can think of the associated model $V^{(H')}$ as a submodel of $V^{(H)}$, that is to say $V^{(H')} \subseteq V^{(H)}$ and the mappings $[\![\cdot]\!]^{H'}$ and $[\![\cdot]\!]^{H}$ are consistent. The first assertion can be seen quickly since any H'-valued function is also an H-valued function and so it follows by induction on $x \in$ dom(u). For the second assertion we have the following lemma.

Definition 2.12. A formula ϕ is *restricted* if all its quantifiers are of the form $\exists y \in x$ or $\forall y \in x$

Given this definition we have the following lemma

Lemma 2.13. *For any complete subalgebra H' of H and any restricted formula $\phi(x_1,...,x_n)$ with variables in $V^{(H')}$ the equality $[\![\phi(x_1,...,x_n)]\!]^H = [\![\phi(x_1,...,x_n)]\!]^{H'}$ holds.*

Proof. We proceed by induction on the complexity of ϕ. First, for the atomic formulas $[\![u = v]\!]$ and $[\![u \in v]\!]$: given $\langle u, v\rangle \in V^{(H')} \times V^{(H')}$ let the following

$$[\![x = y]\!]^H = [\![x = y]\!]^{H'}$$

$$[\![x \in y]\!]^H = [\![x \in y]\!]^{H'}$$

$$[\![y \in x]\!]^H = [\![y \in x]\!]^{H'}$$

be true for all $\langle x, y\rangle$ such that either $x = u$ and $y \in \mathrm{dom}(v)$ or $x \in \mathrm{dom}(u)$ and $y = v$ (this is a well founded relation as shown in Lemma 2.2). Then

$$[\![u \in v]\!]^{H'} = \bigvee_{y\in v}^{H'} \left[v(x) \wedge [\![u = y]\!]^{H'}\right] = \bigvee_{y\in v}^{H'} \left[v(x) \wedge [\![u = y]\!]^{H}\right] = [\![u \in v]\!]^H$$

and

$$[\![u = v]\!]^{H'} = \bigwedge_{y\in v}^{H'} \left[v(y) \Rightarrow [\![y \in u]\!]^{H'}\right] \wedge \bigwedge_{x\in u}^{H'} \left[u(x) \Rightarrow [\![x \in v]\!]^{H'}\right]$$

$$= \bigwedge_{y\in v}^{H} \left[v(y) \Rightarrow [\![y \in u]\!]^{H}\right] \wedge \bigwedge_{x\in u}^{H} \left[u(x) \Rightarrow [\![x \in v]\!]^{H}\right] = [\![u = v]\!]^H$$

We have then by induction that the atomic cases are true for all pairs $u, v \in V^{(H')}$.
Now given ϕ, ψ such that $[\![\phi]\!]^{H'} = [\![\phi]\!]^H$ and $[\![\psi]\!]^{H'} = [\![\psi]\!]^H$ we also have the following by completeness of H':

$$[\![\phi \wedge \psi]\!]^{H'} = [\![\phi]\!]^{H'} \wedge^{H'} [\![\psi]\!]^{H'} = [\![\phi]\!]^H \wedge^H [\![\psi]\!]^H = [\![\phi \wedge \psi]\!]^H$$

$$[\![\phi \vee \psi]\!]^{H'} = [\![\phi]\!]^{H'} \vee^{H'} [\![\psi]\!]^{H'} = [\![\phi]\!]^H \vee^H [\![\psi]\!]^H = [\![\phi \vee \psi]\!]^H$$

$$[\![\phi \to \psi]\!]^{H'} = [\![\phi]\!]^{H'} \Rightarrow^{H'} [\![\psi]\!]^{H'} = [\![\phi]\!]^H \Rightarrow^H [\![\psi]\!]^H = [\![\phi \to \psi]\!]^H$$

$$[\![\neg\phi]\!]^{H'} = ([\![\phi]\!]^{H'})^* = ([\![\phi]\!]^H)^* = [\![\neg\phi]\!]^H$$

The final step is by induction on the well founded relation $x \in \mathrm{dom}(u)$, let $[\![\phi(x)]\!]^{H'} = [\![\phi(x)]\!]^H$ for all such x then

$$[\![\exists x \in u\phi(v)]\!]^{H'} = \bigvee_{x\in\mathrm{dom}(u)}^{H'} \left[u(x) \wedge^{H'} [\![\phi(x)]\!]^{H'}\right] = \bigvee_{x\in\mathrm{dom}(u)}^{H} \left[u(x) \wedge^{H} [\![\phi(x)]\!]^{H}\right] = [\![\exists v \in u\phi(v)]\!]^H$$

and

$$[\![\forall x \in u\phi(v)]\!]^{H'} = \bigwedge_{x\in\mathrm{dom}(u)}^{H'} \left[u(x) \Rightarrow^{H'} [\![\phi(x)]\!]^{H'}\right] = \bigwedge_{x\in\mathrm{dom}(u)}^{H} \left[u(x) \Rightarrow^{H} [\![\phi(x)]\!]^{H}\right] = [\![\exists v \in u\phi(v)]\!]^H$$

□

There is a natural mapping $\widehat{(\cdot)}\colon V \to V^{(\{\top,\bot\})}$ defined by $\widehat{u} = \{\langle \widehat{v}, \top\rangle \mid v \in u\}$ (this is well defined by recursion on $v \in \mathrm{dom}(u)$). Since $\top, \bot$ is a complete subalgebra of any Heyting algebra H we can also think of $\widehat{(\cdot)}$ as being a mapping from V to $V^{(H)}$. We have the following properties for $\widehat{(\cdot)}$

Lemma 2.14.

(i) $[\![u \in \widehat{v}]\!] = \bigvee_{x \in v} [\![u = \widehat{x}]\!]$ *for all* $v \in V$ *and* $u \in V^{(H)}$

(ii) $u \in v \leftrightarrow V^{(H)} \models \widehat{u} \in \widehat{v}$ *and* $u = v \leftrightarrow V^{(H)} \models \widehat{u} = \widehat{v}$

(iii) For all $x \in V^{(\{\top,\bot\})}$ *there exists a unique* $v \in V$ *such that* $V^{(\{\top,\bot\})} \models x = \widehat{v}$

(iv) For any formula ϕ

$$\phi(x_1, ..., x_n) \leftrightarrow V^{(\{\top,\bot\})} \models \phi(\widehat{x}_1, ..., \widehat{x}_n)$$

moreover for any restricted formula ψ

$$\psi(x_1, ..., x_n) \leftrightarrow V^{(H)} \models \psi(\widehat{x}_1, ..., \widehat{x}_n)$$

Proof.
(i)
$[\![u \in \widehat{v}]\!] = \bigvee_{y \in \mathrm{dom}(\widehat{v})} \widehat{v}(y) \wedge [\![u = y]\!] = \bigvee_{x \in v} \top \wedge [\![u = \widehat{x}]\!] = \bigvee_{x \in v} [\![u = \widehat{x}]\!]$
(ii)
Assume $u \in v$ then by (i)

$$[\![\widehat{u} \in \widehat{v}]\!] = \bigvee_{y \in v} [\![\widehat{u} = \widehat{y}]\!] \geq [\![\widehat{u} = \widehat{u}]\!] = \top$$

It follows then that, assuming $u = v$

$$\begin{aligned}[\![\widehat{u} = \widehat{v}]\!] &= \bigwedge_{x \in \widehat{u}} \left[\widehat{u}(x) \Rightarrow [\![x \in \widehat{v}]\!]\right] \wedge \bigwedge_{y \in \widehat{v}} \left[\widehat{v}(y) \Rightarrow [\![y \in \widehat{u}]\!]\right] \\ &= \bigwedge_{x \in u} [\![\widehat{x} \in \widehat{v}]\!] \wedge \bigwedge_{y \in v} [\![\widehat{y} \in \widehat{u}]\!] \\ &= \top\end{aligned}$$

since $x \in u \leftrightarrow x \in v$.
For the converse implication we proceed by induction. Let $[\![\widehat{x} \in \widehat{y}]\!] = \top \to x \in y$, $[\![\widehat{y} \in \widehat{x}]\!] = \top \to y \in x$, and $[\![\widehat{x} = \widehat{y}]\!] = \top \to x = y$ be true for all $\langle x, y\rangle$ such that one of $x \in u$ and $y = v$, $x = u$ and $y \in v$ is true. We have then

$$[\![\widehat{v} \in \widehat{u}]\!] = \bigvee_{x \in u} [\![\widehat{x} = \widehat{v}]\!]$$

by *(i)*. Note then that since the mapping $\widehat{(\cdot)}\colon V \to V^{(H)}$ is actually a composition of the mapping $\widehat{\cdot}\colon V \to V^{(\{\top,\bot\})}$ and the inclusion of $V^{(\{\top,\bot\})}$ into $V^{(H)}$ we have $[\![\widehat{x} = \widehat{v}]\!]^H = [\![\widehat{x} = \widehat{v}]\!]^{\{\top,\bot\}}$ by 2.13. Therefore $[\![\widehat{x} = \widehat{v}]\!] = \top$ or $[\![\widehat{x} = \widehat{v}]\!] = \bot$ for all $x \in u$. It follows then, if we assume $[\![\widehat{v} \in \widehat{u}]\!] = \top$, there must exist some $x' \in u$ such that $[\![\widehat{x'} = \widehat{v}]\!] = \top$. By the induction hypothesis this means $x' = v$ and therefore $v \in u$. The case for $u \in v$ can be shown in the same way using symmetry of $[\![\cdot = \cdot]\!]$
Similarly $[\![\widehat{v} = \widehat{u}]\!] = \bigwedge_{x \in u} [\![\widehat{x} \in \widehat{v}]\!] \wedge \bigwedge_{y \in v} [\![\widehat{y} \in \widehat{u}]\!]$ and so $[\![\widehat{x} \in \widehat{v}]\!] = [\![\widehat{y} \in \widehat{u}]\!] = \top$ for all $x \in u$ and $y \in v$ therefore by the induction hypothesis $z \in u \leftrightarrow z \in v$ and so by extensionality $u = v$. Thus the hypothesis is true for all pairs $\langle u, v\rangle \in V \times V$.
(iii)
Uniqueness follows from *(ii)* since $[\![x = \widehat{v}]\!] \wedge [\![x = \widehat{u}]\!] \leq [\![\widehat{v} = \widehat{u}]\!]$. For existence we proceed once again by induction. Let the hypothesis be true for all $y \in \mathrm{dom}(x)$ then by the axioms of replacement and separation we have a set $v = \{z \mid \exists y_z \in \mathrm{dom}(x)[x(y_z) = \top \wedge [\![\widehat{z} = y_z]\!] = \top]\}$. As such, for all $z \in v$ we have

$$[\![\widehat{z} \in x]\!] = \bigvee_{y \in \mathrm{dom}(x)} \left[x(y) \wedge [\![\widehat{z} = y]\!]\right] \geq x(y_z) \wedge [\![y_z = \widehat{z}]\!] = \top$$

And for $y \in \mathrm{dom}(x)$ either $x(y) = \bot$ in which case $x(y) \Rightarrow [\![y \in \widehat{v}]\!] = \top$ or $y = y_{z'}$ for some $z' \in v$ from which it follows that $[\![y \in \widehat{v}]\!] = \bigvee_{z \in v} [\![y_{z'} = z]\!] \geq [\![y_{z'} = z']\!] = \top$ Therefore we have

$$[\![x = \widehat{v}]\!] = \bigwedge_{y \in \mathrm{dom}(x)} \left[x(y) \Rightarrow [\![y \in \widehat{v}]\!]\right] \wedge \bigwedge_{z \in v} [\![\widehat{z} \in x]\!] \geq \top$$

(iv)
The first part is proved by induction on the structure of ϕ. The second follows immediately by 2.13. The atomic cases for the first part are proven in (*iii*). For the connectives note that since $[\![\cdot]\!]$ maps to $\{\top, \bot\}$ we have

$$[\![\phi \wedge \psi]\!] = [\![\phi]\!] \wedge [\![\psi]\!] = \top \text{ iff } [\![\phi]\!] = \top \text{ and } [\![\psi]\!] = \top \text{ iff } \phi \wedge \psi$$

$$[\![\phi \vee \psi]\!] = [\![\phi]\!] \vee [\![\psi]\!] = \top \text{ iff } [\![\phi]\!] = \top \text{ or } [\![\psi]\!] = \top \text{ iff } \phi \vee \psi$$

Furthermore

$$[\![\neg\phi]\!] = [\![\phi]\!]^* = \top \text{ iff } [\![\phi]\!] = \bot$$

which by the induction hypothesis contradicts ϕ since $\phi \leftrightarrow [\![\phi]\!] = \top$ therefore $[\![\neg\phi]\!] = \top \rightarrow \neg\phi$. Then, assuming $\neg\phi$ we have

$$[\![\neg\phi]\!] = [\![\phi]\!]^* = \top^* = \bot$$

so $\neg\phi \rightarrow [\![\neg\phi]\!] = \bot$ and thus

$$\begin{aligned}[\![\phi \rightarrow \psi]\!] = [\![\phi]\!] \Rightarrow [\![\psi]\!] = \top \ &\text{iff } [\![\phi]\!] = \bot \text{ or } [\![\psi]\!] = \top \\ &\text{iff } [\![\phi]\!]^* = \top \text{ or } [\![\psi]\!] = \top \\ &\text{iff } \neg\phi \vee \psi \\ &\text{iff } \phi \rightarrow \psi\end{aligned}$$

Finally, for a sentence of the form $\exists x\phi(x)$ if $[\![\exists x\phi(x)]\!] = \bigvee_{x \in V^{\{\top,\bot\}}} [\![\phi(x)]\!] = \top$ then there must exist some $x \in V^{\{\top,\bot\}}$ such that $[\![\phi(x)]\!] = \top$ and we can find, by *(iii)*, $v \in V$ such that $[\![\widehat{v} = x]\!] = \top$ and since $\top = [\![\widehat{v} = x]\!] \wedge [\![\phi(x)]\!] \leq [\![\phi(\widehat{v})]\!]$ it follows that $\phi(v)$ holds and therefore also $\exists x\phi(x)$.
Assume then that $\exists x\phi(x)$ holds, we can find some $v \in V$ such that $\phi(v)$ holds and it follows that $[\![\exists x\phi(x)]\!] = \bigvee_{x \in V^{\{\top,\bot\}}} [\![\phi(x)]\!] \geq [\![\phi(\widehat{v})]\!] = \top$.

This also proves the case for $\forall x\phi(x)$ since in $\{\top, \bot\}$ it holds that $[\![\neg\exists x\neg\phi(x)]\!] = \left(\bigvee_{x \in V^{\{\top,\bot\}}} [\![\phi(x)]\!]^*\right)^* = \bigwedge_{x \in V^{\{\top,\bot\}}} [\![\phi(x)]\!]^{**} = \bigwedge_{x \in V^{\{\top,\bot\}}} [\![\phi(x)]\!] = [\![\forall x\phi(x)]\!]$

□

Note that in particular (*iv*) implies that for ϕ satisfying the necessary conditions we have

$$\neg\phi \leftrightarrow [\![\phi]\!]^* = \top \leftrightarrow [\![\phi]\!] = \bot$$

2.4 The Axioms of IZF in $V^{(H)}$

We have now built up enough theory that we can prove all the axioms of IZF to be true in $V^{(H)}$.

Extensionality

Proof. We have, given $x, y \in V^{(H)}$

$$\begin{aligned}
[\![\forall z[z \in x \leftrightarrow z \in y]\!] &= [\![\forall z[(z \in x \to z \in y) \wedge (z \in y \to z \in x)]]\!] \\
&= \bigwedge_{z \in V^{(H)}} \big[[\![z \in x]\!] \Rightarrow [\![z \in y]\!]\big] \wedge \bigwedge_{z \in V^{(H)}} \big[[\![z \in y]\!] \Rightarrow [\![z \in x]\!]\big] \\
&\leq \bigwedge_{z \in \mathrm{dom}(x)} \big[[\![z \in x]\!] \Rightarrow [\![z \in y]\!]\big] \wedge \bigwedge_{z \in \mathrm{dom}(y)} \big[[\![z \in y]\!] \Rightarrow [\![z \in x]\!]\big] \\
&\leq \bigwedge_{z \in \mathrm{dom}(x)} \big[x(z) \Rightarrow [\![z \in y]\!]\big] \wedge \bigwedge_{z \in \mathrm{dom}(y)} \big[y(z) \Rightarrow [\![z \in x]\!]\big] \\
&= [\![x = y]\!]
\end{aligned}$$

Which gives us

$$[\![\forall x, y[\forall z(z \in x \leftrightarrow z \in y) \to z = y]]\!] = \top$$

Furthermore for arbitrary $z \in V^{(H)}$ we have $[\![x = y]\!] \wedge [\![z \in x]\!] \leq [\![z \in y]\!]$ so $[\![x = y]\!] \leq [\![z \in x]\!] \Rightarrow [\![z \in y]\!]$ and thus

$$[\![\forall x, y[x = y \to \forall z(z \in x \leftrightarrow z \in y)]]\!] = \top$$

□

Given a sets $u, v \in V^{(H)}$ the existence of their pairing as well as their powersets, unions and any set by separation on u or v is unique by extensionality. Therefore, for any of the axioms just named, we can find, by the Unique Existence Principle, some $w \in V^{(H)}$ that satisfies that axiom for u and v. In the following such sets will be denoted in the usual way so for instance $P(a)$ will denote an element of $V^{(H)}$ satisfying $\forall x[x \in P(a) \leftrightarrow \forall y \in x[y \in a]]$.
Lemma 2.4 has the consequence that, given $u \in V^{(H)}$ such that $V^{(H)} \models \forall x[x \in u \leftrightarrow \phi(x)]$, we have for any $v \in V^{(H)}$

$$[\![\forall y(y \in v \leftrightarrow \phi(y))]\!] = [\![v = u]\!]$$

Since the two sentences are equivalent by extensionality. From here on we will use this fact without mention.

Pairing

Proof. Given $u, v \in V^{(H)}$ consider the function $w = \{\langle u, \top\rangle, \langle v, \top\rangle\}$. For w it holds that

$$[\![z \in w]\!] = \big(w(u) \wedge [\![z = u]\!]\big) \vee \big(w(v) \wedge [\![z = v]\!]\big) = [\![z = u]\!] \vee [\![z = v]\!] = [\![z = u \vee z = v]\!]$$

□

Note in particular that $x \in z \wedge y \in z \wedge \forall w \in z[w = x \vee w = y]$ is a restricted formula in variables x, y, z so by 2.14 $\widehat{\{x, y\}}$ satisfies the axiom of pairing for $\widehat{x}, \widehat{y}$ in $V^{(H)}$. In particular $\langle \widehat{x}, \widehat{y}\rangle = \widehat{\langle x, y\rangle}$ is true in $V^{(H)}$.

Powerset

Proof. Given $u \in V^{(H)}$ let w be a function with domain $H^{\mathrm{dom}(u)}$ (the set of all functions from $\mathrm{dom}(u)$ to H) and let $w(x) = [\![\forall y \in x(y \in u)]\!]$ then by 2.4

$$[\![v \in w]\!] = \bigvee_{x \in \mathrm{dom}(w)} \big[[\![\forall y \in x(y \in u)]\!] \wedge [\![x = v]\!]\big] \leq [\![\forall y \in v(y \in u)]\!]$$

Now given v consider the function a with $\mathrm{dom}(a) = \mathrm{dom}(u)$ and $a(z) = [\![z \in u]\!] \wedge [\![z \in v]\!]$. It follows that $a(z) \Rightarrow [\![z \in v]\!] = \top$ for all $z \in \mathrm{dom}(a)$ since $a(z) \leq [\![z \in v]\!]$ therefore

$$\begin{aligned}
[\![\forall y \in v[y \in u]]\!] &= \bigwedge_{y \in \mathrm{dom}(v)} \big[v(y) \Rightarrow [\![y \in u]\!]\big] \\
&= \bigwedge_{y \in \mathrm{dom}(v)} \big[v(y) \Rightarrow \big([\![y \in u]\!] \wedge v(y)\big)\big] \\
&\leq \bigwedge_{y \in \mathrm{dom}(v)} \big[v(y) \Rightarrow a(y)\big] \\
&\leq \bigwedge_{y \in \mathrm{dom}(v)} \big[v(y) \Rightarrow [\![y \in a]\!]\big] \wedge \bigwedge_{z \in \mathrm{dom}(a)} \big[a(z) \Rightarrow [\![z \in v]\!]\big] \\
&= [\![v = a]\!]
\end{aligned}$$

Now by construction we have $a \in \mathrm{dom}(w)$. Additionally $[\![\forall y \in a(y \in u)]\!] = \top$ since $a(y) \leq [\![y \in u]\!]$ for all $y \in \mathrm{dom}(a)$. Thus

$$[\![\forall y \in v[y \in u]]\!] \leq [\![\forall y \in a(y \in u)]\!] \wedge [\![v = a]\!] = w(a) \wedge [\![v = a]\!] \leq [\![v \in w]\!]$$

□

Union

Proof. Given $u \in V^{(H)}$ let w be the function with $\mathrm{dom}(w) = \bigcup_{v \in \mathrm{dom}(u)} \mathrm{dom}(v)$ and $w(x) = \bigvee_{v \in A_x} v(x)$ where $A_x = \{v \in \mathrm{dom}(u) \mid x \in \mathrm{dom}(v)\}$. Then

$$\begin{aligned}
[\![y \in w]\!] = \bigvee_{x \in \mathrm{dom}(w)} \Big[[\![x = y]\!] \wedge \bigvee_{v \in A_x} v(x)\Big] &= \bigvee_{x \in \mathrm{dom}(w)} \bigvee_{v \in A_x} \big[v(x) \wedge [\![x = y]\!]\big] \\
&= \bigvee_{v \in \mathrm{dom}(u)} \bigvee_{x \in \mathrm{dom}(v)} \big[v(x) \wedge [\![x = y]\!]\big] = [\![\exists v \in u(y \in v]\!]
\end{aligned}$$

□

Separation

Proof. Given $u \in V^{(H)}$ let $\mathrm{dom}(w) = \mathrm{dom}(u)$ and $w(x) = [\![x \in u]\!] \wedge [\![\phi(x)]\!]$ then

$$[\![z \in w]\!] = \bigvee_{y \in \mathrm{dom}(w)} [\![y \in u]\!] \wedge [\![\phi(y)]\!] \wedge [\![y = z]\!] \leq \bigvee_{y \in \mathrm{dom}(w)} [\![\phi(z)]\!] \wedge [\![z \in u]\!] = [\![\phi(z) \wedge z \in u]\!]$$

and

$$\begin{aligned}
[\![\phi(z) \wedge z \in u]\!] = \bigvee_{y \in \mathrm{dom}(u)} u(y) \wedge [\![z = y]\!] \wedge [\![\phi(z)]\!] &\leq \bigvee_{y \in \mathrm{dom}(u)} [\![y \in u]\!] \wedge [\![z = y]\!] \wedge [\![\phi(y)]\!] \\
&= \bigvee_{y \in \mathrm{dom}(u)} w(y) \wedge [\![z = y]\!] = [\![z \in w]\!]
\end{aligned}$$

□

Empty Set

Note that since $[\![u = u]\!] = \top$ for all $u \in V^{(H)}$ it follows that $[\![u \neq u]\!] = [\![u = u]\!]^* = \bot$. Therefore, for any function $w \in V^{(H)}$ with $\mathrm{ran}(w) \subseteq \{\bot\}$ we have

$$[\![u \in w]\!] = \bigvee_{x \in \mathrm{dom}(w)} \big[w(x) \wedge [\![u = x]\!]\big] = \bot = [\![u \neq u]\!]$$

and so w satisfies the empty set axiom.

Infinity

Proof. The set $\widehat{\emptyset}$ is the empty function and therefore satisfies the empty set axiom in $V^{(H)}$. So, because

$$\widehat{\emptyset} \in x \wedge \forall y \in x(y^{+} \in x)$$

is a restricted formula it follows from 2.14 that $\widehat{\mathbb{N}}$ satisfies the axiom of infinity since $\mathbb{N}$ satisfies the axiom in V. □

It is true in $V^{(H)}$ that $\widehat{\mathbb{N}}$ is the smallest set satisfying the axiom of infinity. Let A be such that $V^{(H)} \models \emptyset \in A \wedge \forall a \in A[a^{+} \in A]$. We immediately have $\widehat{\emptyset} \in A$, and since $\widehat{n}^{+} = \widehat{n^{+}}$ it follows by induction on $\mathbb{N}$ that

$$[\![\widehat{\mathbb{N}} \subseteq A]\!] = \bigwedge_{n \in \mathbb{N}} [\![\widehat{n} \in A]\!] = \top$$

Collection

Proof. Given $u \in V^{(H)}$ and $x \in \mathrm{dom}(u)$ there exists by a collection argument some ordinal α_x such that $\bigvee_{y \in V^{(H)}} [\![\phi(x,y)]\!] = \bigvee_{y \in V^{(H)}_{\alpha_x}} [\![\phi(x,y)]\!]$. For $\alpha = \{\alpha_x \mid x \in \mathrm{dom}(u)\}$ and v, the function with domain $V^{(H)}_{\alpha}$ and range $\{\top\}$, we have

$$[\![\forall x \in u \exists y \phi(x,y)]\!] = \bigwedge_{x \in \mathrm{dom}(u)} \Big[u(x) \Rightarrow \bigvee_{y \in V^{(H)}} [\![\phi(x,y)]\!]\Big] = \bigwedge_{x \in \mathrm{dom}(u)} \Big[u(x) \Rightarrow \bigvee_{y \in V^{(H)}_{\alpha}} [\![\phi(x,y)]\!]\Big]$$

$$= \bigwedge_{x \in \mathrm{dom}(u)} \Big[u(x) \Rightarrow [\![\exists y \in v \phi(x,y)]\!]\Big] = [\![\forall x \in u \exists y \in v \phi(x,y)]\!] \leq [\![\exists w \forall x \in u \exists y \in w \phi(x,y)]\!].$$

□

Induction

Proof. We proceed by induction on the well-founded relation $y \in \mathrm{dom}(x)$, let

$$a = [\![(\forall y \in x \phi(y)) \to \phi(x)]\!] = \bigwedge_{y \in \mathrm{dom}(x)} \Big[x(y) \Rightarrow [\![\phi(y)]\!]\Big] \Rightarrow [\![\phi(x)]\!]$$

assuming $a \leq [\![\phi(y)]\!]$ for all $y \in \mathrm{dom}(x)$ it follows that $a \leq b \Rightarrow [\![\phi(y)]\!]$ for any b so we have in particular $a \leq \bigwedge_{y \in \mathrm{dom}(x)} \Big[x(y) \Rightarrow [\![\phi(y)]\!]\Big]$. Therefore, by definition of a we have

$$a = a \wedge \bigwedge_{y \in \mathrm{dom}(x)} \Big[x(y) \Rightarrow [\![\phi(y)]\!]\Big] \leq [\![\phi(x)]\!]$$

So by induction we have for all $x, v \in V^{(H)}$

$$v \leq \bigwedge_{u \in V^{(H)}} [\![(\forall y \in u \phi(y)) \to \phi(u)]\!] \leq [\![(\forall y \in x \phi(y)) \to \phi(x)]\!] \leq [\![\phi(x)]\!].$$

Therefore $\bigwedge_{u \in V^{(H)}} [\![(\forall y \in u \phi(y)) \to \phi(u)]\!] \leq \bigwedge_{u \in V^{(H)}} [\![\phi(u)]\!]$. □

With the above axioms we now have the equipment to define the notion of a function in $V^{(H)}$. We begin by constructing the object $A \times B$ for $A, B \in V^{(H)}$. Let C be the set of all singletons of A constructed using separation on $P(A)$ and C' the set

$$\{x \in P(A \cup B) \mid \exists a \in A, b \in B[x = \{a, b\}]\}$$

(using pairing with again powerset and separation). Now $A \times B$ is the set

$$\{x \in P(C \cup C') \mid \exists a \in A, b \in B[x = \langle a, b\rangle]\}$$

Using this we can define the formula $\text{fun}(f)$ as

$$\exists A, B[x \in f \to x \in A \times B] \wedge \forall a, b, b'[(\langle a, b\rangle \in f \wedge \langle a, b'\rangle \in f) \to b = b']$$

Given f such that $V^{(H)} \models \text{fun}(f)$ we write $\text{dom}(f)$ for the set $\{a \mid \exists b[\langle a, b\rangle \in f]\}$ and $\text{ran}(f)$ for $\{b \mid \exists a[\langle a, b\rangle \in f]\}$ the existence of both follows from separation on $A \times B$. Furthermore we will occasionally use the notation $[\![f(x) = y]\!] = [\![\langle x, y\rangle \in f]\!]$.

3 Implications in IZF

3.1 Zorn's Lemma and the Axiom of Choice

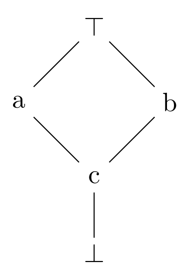
In this section we give a proof that, by assuming Zorn's lemma when constructing the $V^{(H)}$ it ends up always being in the model. We let Zorn's lemma take the form:

If X is a poset such that every chain in X has a supremum in X, then it has a maximal element.

This is slightly different than the usual statement which only requires that chains in X have upper bounds in X. However, though the existence of suprema is a stronger requirement on X this version is, classically, still strong enough to prove the Axiom of Choice which, in turn, proves the usual statement of Zorn's lemma.[3]

Definition 3.1. A set $u \in V^{(H)}$ is *inhabited* if it has a *definite element*, that is if there exists some $v \in V^{(H)}$ such that $[\![v \in u]\!] = \top$.

Note that $V^{(H)} \models \exists x(x \in u)$ does not necessarily imply that u has a definite element. For instance take $\tilde{H}$ to be the Heyting algebra $\{\top, \bot, a, b, c\}$ with $c \leq a$, $c \leq b$ as well as $a, b, c \leq \top$ and $\bot \leq a, b, c$ as suggested below



We have

$$[\![v = \emptyset]\!] = \bigwedge_{y \in \text{dom}(v)} v(y)^*$$

Note that $h^* = \bot$ for all $h \in \tilde{H} - \{\bot\}$. Therefore $[\![v = \emptyset]\!] = \top$ iff $\text{ran}(v) \subseteq \{\bot\}$ otherwise $[\![v = \emptyset]\!] = \bot$. Furthermore for v with $\text{ran}(v) \subseteq \{\bot\}$ we have

$$[\![v = \widehat{1}]\!]^{\tilde{H}} \leq [\![\emptyset \in v]\!]^{\tilde{H}} = \bot$$

Let u be the function $\{\langle \emptyset, a\rangle, \langle \widehat{1}, b\rangle\}$. Then

$$[\![x \in u]\!]^{\tilde{H}} = (a \wedge [\![x = \emptyset]\!]^{\tilde{H}}) \vee (b \wedge [\![x = \widehat{1}]\!]^{\tilde{H}}) < \top$$

[3]See for instance the proof of Proposition 1.4.3 from Moerdijk and van Oosten [2014] which works with either formulation

However

$$[\![\exists x(x \in u)]\!]^{\tilde{H}} = \bigvee_{x \in V^{(\tilde{H})}} [\![x \in u]\!]^{\tilde{H}} \leq [\![\emptyset \in u]\!]^{\tilde{H}} \vee [\![\widehat{1} \in u]\!]^{\tilde{H}} = a \vee b = \top$$

So $V^{(\tilde{H})} \models \exists x(x \in u)$ while u has no definite element. This motivates the following definition and lemma

Definition 3.2. Given $u \in V^{(H)}$ a *core* C of u is a subset $C \subseteq V^{(H)}$ such that $[\![c \in u]\!] = \top$ for all $c \in C$ and for all $v \in V^{(H)}$ that satisfy $[\![v \in u]\!] = \top$ there exists some $c \in C$ for which $[\![v = c]\!] = \top$.

The following two proofs follow those in [Bell, 2014].

Lemma 3.3. *Every $u \in V^{(H)}$ has a core.*

Proof. Given $u \in V^{(H)}$ let

$$a_x = \{\langle z, u(z) \wedge [\![z = x]\!]\rangle \mid z \in \mathrm{dom}(u)\}.$$

By collection on the set

$$\{f \in H^{\mathrm{dom}(u)} \mid \exists y \in V^{(H)}[f = a_y]\}$$

there is a set $W \subseteq V^{(H)}$ such that for all $y \in V^{(H)}$ there exists $x \in W$ such that $a_x = a_y$. Now let

$$C = \{x \in W \mid [\![x \in u]\!] = \top\}.$$

For any y if x is such that $a_x = a_y$ then

$$u(z) \wedge [\![z = x]\!] = u(z) \wedge [\![z = y]\!]$$

for all $z \in$dom(u). Therefore $[\![y \in u]\!] = \top$ implies there exists $x \in W$ for which

$$\top = \bigvee_{z \in \mathrm{dom}(u)} \big[u(z) \wedge [\![z = y]\!]\big] = \bigvee_{z \in \mathrm{dom}(u)} \big[u(z) \wedge [\![z = y]\!] \wedge [\![z = x]\!]\big] \leq [\![x = y]\!]$$

It also follows then that $[\![x \in u]\!] = \top$, so $x \in C$. Hence C is a core for u □

Proposition 3.4. *Zorn's Lemma is true in $V^{(H)}$.*

Proof. Let $(X, \leq_X)$ be such that

$$V^{(H)} \models (X, \leq_X) \textit{ is an inhabited poset and every chain in } X \textit{ has a supremum}$$

More formally, X and $\leq_X$ are elements of $V^{(H)}$ and for $\leq_X$ the following holds

$$\begin{aligned} V^{(H)} \models \forall z \in \leq_X [z \in X \times X] \wedge \forall x, y[(\langle x, y\rangle \in \leq_X \wedge \langle y, x\rangle \in \leq_X) \rightarrow x = y] \\ \wedge \forall x \in X[\langle x, x\rangle \in \leq_X] \wedge \forall x, y[(\langle x, y\rangle \in \leq_X \wedge \langle y, z\rangle \in \leq_X) \rightarrow \langle x, z\rangle \in \leq_X]. \quad (10) \end{aligned}$$

Which states that $\leq_X$ is a poset on X. Furthermore, using $v \subseteq X$ as shorthand for $\forall u \in v[u \in X]$ and $x \leq_X y$ as shorthand for $\langle x, y\rangle \in \leq_X$ we have

$$\begin{aligned} V^{(H)} \models \forall v\big[[v \subseteq X \wedge \forall x, y \in v(x \leq_X y \vee y \leq_X x)] \\ \rightarrow \exists z \in X[\forall x \in v(x \leq_X z) \wedge \forall w \in X(\forall x \in v(x \leq_X w) \rightarrow z \leq_X w]\big] \quad (11) \end{aligned}$$

Or in other words, every chain in X has a supremum. By taking $v = \emptyset$ we see that X must be inhabited. Now let C be a core for X, we define a new relation $\leq_C$ by saying that $x \leq_C y$ iff $[\![x \leq_X y]\!] = \top$ for any $x, y \in C$. That $(C, \leq_C)$ is a poset follows immediately from (10). Moreover, since suprema are unique, there exists, by the Unique Exsitence Principle, some $z \in V^{(H)}$ such that $V^{(H)} \models z \in X$ (again taking the supremum of the empty set). Therefore, since C is a core for X, C must also be inhabited. Now given a chain K in C the set

$$k = \{\langle x, \top\rangle \mid x \in K\}$$

is a chain in X. To see this note that

$$[\![y \in k]\!] = \bigvee_{x\in K} [\![y = x]\!] \leq [\![y \in X]\!].$$

Furthermore for all $x, y \in K$ we have either $x \leq_C y$ or $y \leq_C x$ and therefore $[\![x \leq_X y \vee y \leq_X x]\!] = \top$. It then follows that

$$\begin{aligned}
[\![u \in k]\!] \wedge [\![v \in k]\!] &= \bigvee_{x\in K} [\![u = x]\!] \wedge \bigvee_{y\in K} [\![v = y]\!] \\
&= \bigvee_{x\in K} \bigvee_{y\in K} \left[[\![v = y]\!] \wedge [\![u = x]\!]\right] \\
&= \bigvee_{x\in K} \bigvee_{y\in K} \left[[\![v = y]\!] \wedge [\![u = x]\!] \wedge [\![x \leq_X y \vee y \leq_X x]\!]\right] \\
&\leq [\![u \leq_X v \vee v \leq_X u]\!].
\end{aligned}$$

Because k is a chain in X we have

$$V^{(H)} \models \exists z \in X[\forall x \in k(x \leq_X z) \wedge \forall w \in X(\forall x \in k(x \leq_X z) \to w \leq_X z]]$$

and again, since suprema are unique, there exists by the Unique Existence Principle $s \in V^{(H)}$ for which *s is in X and is the supremum of k* is true in $V^{(H)}$. Now let $s' \in C$ be such that $[\![s' = s]\!] = \top$ then s' is the supremum of K in C. Thus every chain in C has a supremum and it follows from Zorn's lemma (in ZFC) that C has a maximal element c. We claim

$$V^{(H)} \models c \textit{ is a maximal element of } X.$$

Given $a \in V^{(H)}$ define Z by dom(Z) =dom(X) and

$$Z(x) = [\![x = a \wedge x \in X \wedge c \leq_X x]\!] \vee [\![x = c]\!].$$

As such we have

$$\begin{aligned}
[\![a \in X \wedge c \leq_X a]\!] &= \bigvee_{x\in \mathrm{dom}(X)} \left[X(x) \wedge [\![a = x]\!] \wedge [\![c \leq_X a]\!]\right] \\
&\leq \bigvee_{x\in \mathrm{dom}(X)} \left[[\![x \in X]\!] \wedge [\![a = x]\!] \wedge [\![c \leq_X x]\!]\right] \\
&\leq \bigvee_{x\in \mathrm{dom}(X)} \left[Z(x) \wedge [\![x = a]\!]\right] \\
&= [\![a \in Z]\!].
\end{aligned} \tag{12}$$

Now we consider arbitrary $u, v \in V^{(H)}$. It follows quickly from $[\![c \in X]\!] = \top$ that

$$[\![u \in Z]\!] \leq [\![u \in X]\!].$$

Furthermore, we have

$$\begin{aligned}
[\![u \in Z \wedge v \in Z]\!] &\leq \big([\![u = a \wedge u \in X \wedge c \leq_X u]\!] \vee [\![u = c]\!]\big) \\
&\qquad \wedge \big([\![v = a \wedge v \in X \wedge c \leq_X v]\!] \vee [\![v = c]\!]\big) \\
&\leq \big([\![c \leq_X u]\!] \wedge [\![v = c]\!]\big) \vee \big([\![c \leq_X v]\!] \wedge [\![u = c]\!]\big) \\
&\qquad \vee \big([\![v = c]\!] \wedge [\![u = c]\!]\big) \vee \big([\![u = a]\!] \wedge [\![v = a]\!]\big) \\
&\leq [\![v \leq_X u]\!] \vee [\![u \leq_X v]\!] \vee [\![u = v]\!]
\end{aligned}$$

Therefore Z is a chain in X and we can again find a definite element w of X that is the supremum of Z. Furthermore

$$[\![c \in Z]\!] = \bigvee_{x \in \mathrm{dom}(X)} [\![x = c]\!] \geq [\![c \in X]\!] = \top$$

so $[\![c \leq_X w]\!] = \top$. Now for $w' \in C$ such that $[\![w = w']\!] = \top$ we have $c \leq_C w'$ and therefore $w' = c$ since c is maximal in C. Combining this with 12 gives us

$$[\![a \in X \wedge c \leq_x a]\!] \leq [\![a = c]\!].$$

And therefore it is true in $V^{(H)}$ that c is maximal in X. □

By Proposition 1.5 the Axiom of Choice implies LEM and therefore, by 2.6 it can only hold in $V^{(H)}$ if H is a Boolean algebra. Proposition 3.4 shows that, in contrast, Zorn's lemma is always true in $V^{(H)}$ and therefore that it, intuitionistically, does not imply AC.

3.2 Subcountability of $\mathbb{N}^{\mathbb{N}}$

In this section we construct a model in which $\mathbb{N}^{\mathbb{N}}$ is subcountable as done in [Bell, 2014]. To clarify: $\mathbb{N}$ is the natural numbers object as defined at the end of section 1.2. In section 2.4 we showed that $\widehat{\mathbb{N}}$ satisfied this condition. So we seek to construct a model $V^{(H)}$ that satisfies

$$V^{(H)} \models \textit{There exists a partial surjection } f\colon \widehat{\mathbb{N}} \to \widehat{\mathbb{N}}^{\widehat{\mathbb{N}}}$$

It happens that a more general case is true. We can construct, for any arbitrary sets A and J^I with A infinite, a model in which the above holds for $\widehat{A}$ and $\widehat{J}^{\widehat{I}}$. We we show this in two parts. First we give a number of conditions on H from which the desired result follows. We then show that a specific instance of the Heyting algebra $H_{\mathbf{C}}$ (see 1.3.1) meets those conditions.

3.2.1 Subquotients and Preserving Exponentials

Let us begin with a condition for the existence of some partial function $f\colon \widehat{A} \to \widehat{B}$ for arbitrary sets A, B.

Proposition 3.5. *Given sets A and B, the following are equivalent:*

- *(i) There exists a subset $\{u_{ab} \mid a \in A, b \in B\}$ of H such that $\{u_{ab} \mid b \in B\}$ is an anti-chain for each $a \in A$ and $\bigvee_{a \in A} u_{ab} = \top$ for all $b \in B$.*
- *(ii) There exists f such that $V^{(H)} \models fun(f) \wedge dom(f) \subseteq \widehat{A} \wedge ran(f) = \widehat{B}$*

Proof. (i)→(ii)
Define $f \in V^{(H)}$ as follows: $\mathrm{dom}(f) = \{\langle \widehat{a}, \widehat{b} \rangle \mid a \in A, b \in B\}$ and $f(\langle \widehat{a}, \widehat{b} \rangle) = u_{ab}$. First,

$$[\![x \in f]\!] = \bigvee_{\langle a,b \rangle \in A \times B} u_{ab} \wedge [\![x = \langle \widehat{a}, \widehat{b} \rangle]\!] \leq \bigvee_{\langle a,b \rangle \in A \times B} [\![x = \langle \widehat{a}, \widehat{b} \rangle]\!] = [\![x \in \widehat{A} \times \widehat{B}]\!]$$

Furthermore note that

$$[\![\langle \widehat{a}, \widehat{b} \rangle \in f]\!] = \bigvee_{\langle a',b' \rangle \in A \times B} u_{a'b'} \wedge [\![\langle \widehat{a'}, \widehat{b'} \rangle = \langle \widehat{a}, \widehat{b} \rangle]\!] = u_{ab}$$

since $[\![\langle \widehat{a'}, \widehat{b'} \rangle = \langle \widehat{a}, \widehat{b} \rangle]\!] = \bot$ by Lemma 2.14 for $\langle a', b' \rangle \neq \langle a, b \rangle$. Hence we have

$$[\![\langle \widehat{a}, \widehat{b} \rangle \in f]\!] \wedge [\![\langle \widehat{a}, \widehat{b'} \rangle \in f]\!] = u_{ab} \wedge u_{ab'} \leq [\![\widehat{b} = \widehat{b'}]\!],$$

because $\{u_{ab} \mid a \in A, b \in B\}$ is an anti-chain. Thus, $V^{(H)} \models \text{fun}(f) \wedge \text{dom}(f) \subseteq \widehat{A}$. Additionally

$$[\![\forall b \in \widehat{B} \exists a \in \widehat{A}[\langle \widehat{a}, \widehat{b} \rangle \in f]]\!] = \bigwedge_{b \in B} \bigvee_{a \in A} [\![\langle \widehat{a}, \widehat{b} \rangle \in f]\!] = \bigwedge_{b \in B} \bigvee_{a \in A} u_{ab} = \top$$

so $V^H \models \text{ran}(f) = \widehat{B}$.

(ii) → (i)
Given f that satisfies *(ii)* let $u_{ab} = [\![\langle \widehat{a}, \widehat{b} \rangle \in f]\!]$. For $b \neq b'$ it follows from $V^{(H)} \models \text{fun}(f)$ that

$$u_{ab} \wedge u_{ab'} = [\![\langle \widehat{a}, \widehat{b} \rangle \in f]\!] \wedge [\![\langle \widehat{a}, \widehat{b'} \rangle \in f]\!] \leq [\![\widehat{b} = \widehat{b'}]\!] = \bot$$

so $\{u_{ab} \mid b \in B\}$ is an anti-chain. Furthermore

$$\bigwedge_{b \in B} \bigvee_{a \in A} [\![\langle \widehat{a}, \widehat{b} \rangle \in f]\!] = [\![\forall b \in \widehat{B} \exists a \in \widehat{A}[f(a) = b]]\!] = [\![\text{ran}(f) = \widehat{B}]\!] = \top$$

So, $\bigvee_{a \in A} u_{ab} = \top$ for all $b \in B$. □

We say B is a *subquotient* of A in $V^{(H)}$ when A and B satisfy the conditions in Proposition 3.5. The idea now is, given some A, I, J, to have this be true for A and J^I. This, however, is not yet the result we desire. We want there to be a partial surjection onto the H-set of all functions $\widehat{I}$ to $\widehat{J}$. However it may be true in $V^{(H)}$ that g is a function from $\widehat{I}$ to $\widehat{J}$ without g being an element of $\widehat{J^I}$. The following condition on H eliminates this possibility.

Definition 3.6. Given sets I, J, a Heyting alegebra is said to be $\bot - (I, J)$ distributive if, for all $\{a_{ij} \mid i \in I, j \in J\}$ such that $\{a_{ij} \mid j \in J\}$ is an anti-chain for each $i \in I$ the following holds

$$\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{f \in J^I} \bigwedge_{i \in I} a_{if(i)}$$

It is *completely* $\bot$-distributive if it is $\bot - (I, J)$ distributive for all I, J.

Now, we define the set $J^I \in V^{(H)}$ of all functions from I to J more precisely using the axioms of IZF as the set

$$\{f \in P(I \times J) \mid \text{fun}(f) \wedge \text{dom}(f) = I\}$$

We then have

Proposition 3.7. *The following are equivalent*

(i) H is completely $\bot$-distributive

(ii) $V^{(H)}$ preserves exponentials, that is, $V^{(H)} \models (\widehat{J})^{\widehat{I}} = \widehat{(J^I)}$ for all sets I, J

Proof. (i)→(ii)
First note

$$\begin{aligned} [\![g \in \widehat{J}^{\widehat{I}}]\!] &\leq [\![\text{fun}(g) \wedge \text{dom}(g) = \widehat{I} \wedge \text{ran}(g) \subseteq \widehat{J}]\!] \\ &= [\![\text{fun}(g)]\!] \wedge [\![\forall i \in \widehat{I} \exists j \in \widehat{J}[\langle i, j \rangle \in g]]\!] \\ &= \bigwedge_{i \in I} \bigvee_{j \in J} \Big[[\![\text{fun}(g)]\!] \wedge [\![\langle \widehat{i}, \widehat{j} \rangle \in g]\!] \Big] \end{aligned} \tag{13}$$

Now $[\![\text{fun}(g)]\!] \wedge [\![\langle \widehat{i}, \widehat{j} \rangle \in g]\!] \wedge [\![\langle \widehat{i}, \widehat{j'} \rangle \in g]\!] \leq [\![\widehat{j} = \widehat{j'}]\!] = \bot$ for $j \neq j'$ so $\{[\![\text{fun}(g)]\!] \wedge [\![\langle \widehat{i}, \widehat{j} \rangle \in g]\!] \mid j \in J\}$ is an anti-chain and it follows from H being $\bot$-distributive that (13) is equal to

$$\bigvee_{f\in J^I}\bigwedge_{i\in I}\Big[\llbracket \mathrm{fun}(g)\rrbracket \wedge \llbracket \langle \widehat{i}, \widehat{f(i)}\rangle \in g\rrbracket\Big]$$

Now for arbitrary $f \in J^I$ we have

$$\begin{aligned}
\llbracket g \in \widehat{J}^{\widehat{I}}\rrbracket \wedge \llbracket v \in g\rrbracket \wedge \bigwedge_{i\in I}\llbracket \langle \widehat{i}, \widehat{f(i)}\rangle \in g\rrbracket &= \llbracket g \in \widehat{J}^{\widehat{I}}\rrbracket \wedge \llbracket \exists i \in \widehat{I} \exists j \in \widehat{J}[v = \langle \widehat{i}, \widehat{j}\rangle \wedge \langle \widehat{i}, \widehat{j}\rangle \in g]\rrbracket \wedge \bigwedge_{i\in I}\llbracket \langle \widehat{i}, \widehat{f(i)}\rangle \in g\rrbracket \\
&= \bigvee_{i\in I}\bigvee_{j\in J}\Big[\llbracket v = \langle \widehat{i}, \widehat{j}\rangle \wedge \langle \widehat{i}, \widehat{j}\rangle \in g\rrbracket \wedge \llbracket g \in \widehat{J}^{\widehat{I}}\rrbracket \wedge \bigwedge_{i'\in I}\llbracket \langle \widehat{i'}, \widehat{f(i')}\rangle \in g\rrbracket\Big] \\
&\leq \bigvee_{i\in I}\bigvee_{j\in J}\Big[\llbracket v = \langle \widehat{i}, \widehat{j}\rangle \wedge \langle \widehat{i}, \widehat{j}\rangle \in g\rrbracket \wedge \llbracket g \in \widehat{J}^{\widehat{I}}\rrbracket \wedge \llbracket \langle \widehat{i}, \widehat{f(i)}\rangle \in g\rrbracket\Big] \\
&\leq \bigvee_{i\in I}\bigvee_{j\in J}\llbracket v = \langle \widehat{i}, f(\widehat{i})\rangle\rrbracket \\
&= \bigvee_{i\in I}\llbracket v = \langle \widehat{i}, f(\widehat{i})\rangle\rrbracket \\
&= \llbracket v \in \widehat{f}\rrbracket
\end{aligned}$$

And additionally

$$\begin{aligned}
\llbracket v \in \widehat{f}\rrbracket \wedge \bigwedge_{i\in I}\llbracket \langle \widehat{i}, \widehat{f(i)}\rangle \in g\rrbracket &= \bigvee_{i\in I}\Big[\llbracket v = \langle \widehat{i}, \widehat{f(i)}\rangle\rrbracket \wedge \bigwedge_{i'\in I}\llbracket \langle \widehat{i'}, \widehat{f(i')}\rangle \in g\rrbracket\Big] \\
&\leq \bigvee_{i\in I}\Big[\llbracket v = \langle \widehat{i}, \widehat{f(i)}\rangle\rrbracket \wedge \llbracket \langle \widehat{i}, \widehat{f(i)}\rangle \in g\rrbracket\Big] \leq \bigvee_{i\in I}\llbracket v \in g\rrbracket = \llbracket v \in g\rrbracket
\end{aligned}$$

Therefore $\llbracket g \in \widehat{J}^{\widehat{I}}\rrbracket \wedge \bigwedge_{i\in I}\llbracket \langle \widehat{i}, \widehat{f(i)}\rangle \in g\rrbracket \leq \llbracket g = \widehat{f}\rrbracket$, combined this produces

$$\llbracket g \in \widehat{J}^{\widehat{I}}\rrbracket = \bigwedge_{i\in I}\bigvee_{j\in J}\Big[\llbracket g \in \widehat{J}^{\widehat{I}}\rrbracket \wedge \llbracket \langle \widehat{i}, \widehat{j}\rangle \in g\rrbracket\Big] = \bigvee_{f\in J^I}\bigwedge_{i\in I}\Big[\llbracket g \in \widehat{J}^{\widehat{I}}\rrbracket \wedge \llbracket \langle \widehat{i}, \widehat{f(i)}\rangle \in g\rrbracket\Big] \leq \bigvee_{f\in J^I}\llbracket g = \widehat{f}\rrbracket = \llbracket g \in \widehat{J^I}\rrbracket$$

Hence $\widehat{J}^{\widehat{I}} \subseteq \widehat{(J^I)}$. To show the other direction we begin by noting that for all $f \in J^I$ we have

$$\begin{aligned}
\llbracket \forall x \in \widehat{f} \exists i \in \widehat{I} \exists j \in \widehat{J}[x = \langle i, j\rangle]\rrbracket &= \bigwedge_{\langle i, f(i)\rangle \in f}\llbracket \exists i \in \widehat{I} \exists j \in \widehat{J}[\widehat{\langle i, f(i)\rangle} = \langle i, j\rangle]\rrbracket \\
&\geq \bigwedge_{\langle i, f(i)\rangle \in f}\llbracket \widehat{\langle i, f(i)\rangle} = \langle \widehat{i}, \widehat{f(i)}\rangle\rrbracket = \top.
\end{aligned}$$

Additionally,

$$\llbracket \langle \widehat{i}, \widehat{j}\rangle \in \widehat{f}\rrbracket = \bigvee_{\langle i', f(i')\rangle \in f}\llbracket \langle \widehat{i}, \widehat{j}\rangle = \langle \widehat{i'}, \widehat{f(i')}\rangle\rrbracket = \llbracket \langle \widehat{i}, \widehat{j}\rangle = \langle \widehat{i}, \widehat{f(i)}\rangle\rrbracket$$

since $\llbracket \widehat{u} = \widehat{v}\rrbracket = \bot$ for $u \neq v$. Therefore,

$$\llbracket \langle \widehat{i}, \widehat{j}\rangle \in \widehat{f}\rrbracket \wedge \llbracket \langle \widehat{i}, \widehat{j'}\rangle \in \widehat{f}\rrbracket = \llbracket \langle \widehat{i}, \widehat{j}\rangle = \langle \widehat{i}, \widehat{f(i)}\rangle\rrbracket \wedge \llbracket \langle \widehat{i}, \widehat{j'}\rangle = \langle \widehat{i}, \widehat{f(i)}\rangle\rrbracket \leq \llbracket \widehat{j} = \widehat{j'}\rrbracket$$

and it follows that $V^{(H)} \models \mathrm{fun}(\widehat{f}) \wedge \mathrm{dom}(\widehat{f}) \subseteq \widehat{J}$. Lastly

$$\llbracket \forall i \in \widehat{I} \exists j \in \widehat{J}[\langle i, j\rangle \in \widehat{f}]\rrbracket = \bigwedge_{i\in I}\bigvee_{j\in J}\llbracket \langle \widehat{i}, \widehat{j}\rangle \in \widehat{f}\rrbracket \geq \bigwedge_{i\in I}\llbracket \langle \widehat{i}, \widehat{f(i)}\rangle \in \widehat{f}\rrbracket = \top.$$

From there it can be concluded that

$$[\![\forall f \in \widehat{(J^I)}[f \in \widehat{J}^{\widehat{I}}]]\!] \geq \bigwedge_{f \in J^I} [\![\mathrm{fun}(\widehat{f}) \wedge \mathrm{dom}(\widehat{f}) = \widehat{I} \wedge \mathrm{ran}(\widehat{f}) \subseteq \widehat{J}]\!] = \top,$$

so $\widehat{(J^I)} \subseteq \widehat{J}^{\widehat{I}}$.

(ii)→(i)
Let $V^{(H)}$ preserve exponentials and let $\{a_{ij} \mid i \in I, j \in J\}$ be such that $\{a_{ij} \mid j \in J\}$ is an anti-chain for all $i \in I$. Take then $f \in V^{(H)}$ with domain $\{\langle \widehat{i}, \widehat{j}\rangle \mid i \in I, j \in J\}$ and $f(\langle \widehat{i}, \widehat{j}\rangle) = a_{ij}$ then $V^{(H)} \models \mathrm{fun}(f) \wedge \mathrm{ran}(f) \subseteq \widehat{J} \wedge \mathrm{dom}(f) \subseteq \widehat{I}$. This can be seen as in Proposition 3.5. First,

$$[\![x \in f]\!] = \bigvee_{i \in I, j \in J} a_{ij} \wedge [\![x = \langle \widehat{i}, \widehat{j}\rangle]\!] \leq [\![\exists i \in \widehat{I} \exists j \in \widehat{J}[x = \langle i, j\rangle]]\!]$$

Additionally $[\![\langle \widehat{i}, \widehat{j}\rangle \in f]\!] = a_{ij}$ since, again, $[\![\langle \widehat{i}, \widehat{j}\rangle = \langle \widehat{i}, \widehat{j'}\rangle]\!]$ is equal to $\top$ if $j = j'$ and equal to $\bot$ otherwise. So $[\![\langle \widehat{i}, \widehat{j}\rangle \in f]\!] \wedge [\![\langle \widehat{i}, \widehat{j'}\rangle \in f]\!] = a_{ij} \wedge a_{ij'} \leq [\![\widehat{j} = \widehat{j'}]\!]$ since $\{a_{ij} \mid j \in J\}$ is an anti-chain. Which gives the desired result. Now

$$\begin{aligned}\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} &= \bigwedge_{i \in I} \bigvee_{j \in J} [\![\langle \widehat{i}, \widehat{j}\rangle \in f]\!] = [\![\forall i \in \widehat{I} \exists j \in \widehat{J}[\langle i, j\rangle \in f]\!] = [\![\mathrm{dom}(f) = I]\!] = [\![f \in \widehat{J}^{\widehat{I}}]\!] \\ &= [\![f \in \widehat{(J^I)}]\!] = \bigvee_{g \in J^I} [\![f = \widehat{g}]\!] = \bigvee_{g \in J^I} \Big[\bigwedge_{i \in I, j \in J} a_{ij} \Rightarrow [\![\langle \widehat{i}, \widehat{j}\rangle \in \widehat{g}]\!] \wedge \bigwedge_{i \in I} [\![\langle \widehat{i}, \widehat{g(i)}\rangle \in f]\!] \Big] \quad (14)\end{aligned}$$

Note that $[\![\langle \widehat{i}, \widehat{j}\rangle \in \widehat{g}]\!] = \bot$ for $j \neq g(i)$ and so $a_{ij} \Rightarrow [\![\langle \widehat{i}, \widehat{j}\rangle \in \widehat{g}]\!] = a_{ij}^* \geq a_{ig(i)}$ since $a_{ij} \wedge a_{ig(i)} = \bot$ as well for such j. Furthermore when $j = g(i)$ then $[\![\langle \widehat{i}, \widehat{j}\rangle \in \widehat{g}]\!] = \top = a_{ij} \Rightarrow [\![\langle \widehat{i}, \widehat{j}\rangle \in \widehat{g}]\!]$. Therefore since $[\![\langle \widehat{i}, \widehat{g(i)}\rangle \in f]\!] = a_{ig(i)}$ we see that (2) is equal to $\bigvee_{g \in J^I} \bigwedge_{i \in I} a_{ig(i)}$. □

An equivalent condition is that H be *locally connected*. An element $c \in H$ is *connected* if, whenever A is an anti-chain, $\bigvee A = c$ implies $c = a$ for some $a \in A$. A Heyting algebra H is *connected* if its top element $\top$ is connected. A Heyting algebra is locally connected if all its elements are the join of a set of connected elements.[4] We now have

Proposition 3.8. *The following are equivalent*

(i) H is completely $\bot$ distributive

(ii) H is locally connected

Proof. (i)→(ii)
Call an element b is a-complemented if there exists c such that $b \wedge c = \bot$ and $b \vee c = a$. Note that the a-complement of b is unique for if c and c' are a-complements of b then $c = c \vee (c' \wedge b) = (c \vee c') \wedge a = c \vee c'$ so $c \geq c'$ and a similar argument shows that $c' \geq c$. Let $a - b$ denote the a-complement of b if it exists. Now given $a \in H$ let I be the set of all a-complemented elements of H and define $\{b_{ij} \mid i \in I, j \in \{0,1\}\}$ as $b_{i0} = i$ and $b_{i1} = a - i$. Then, $b_{i0} \wedge b_{i1} = \bot$ for all i. Furthermore, given some $g \in \{0,1\}^I$ let $F_g = \{0,1\}^I - \{g\}$. Since H is completely $\bot$ distributive it is $\bot - (I, \{0,1\})$ distributive and so we have

$$a = \bigwedge_{i \in I} b_{i0} \vee b_{i1} = \bigvee_{f \in \{0,1\}^I} \bigwedge_{i \in I} b_{if(i)} = \bigwedge_{i \in I} b_{ig(i)} \vee \bigvee_{f \in F_g} \bigwedge_{i \in I} b_{if(i)}$$

It will be shown that all $\bigwedge_{i \in I} b_{ig(i)}$ are connected and therefore that a is the join of a set of connected elements. For $f \neq g$ there exists some i such that $f(i) \neq g(i)$ and so $b_{if(i)} \wedge b_{ig(i)} = \bot$. Therefore

[4]Note if we take H to be the Heyting algebra of opens on some topology these two conditions are equivalent to it being connected or locally connected in the topological sense.

$\bigwedge\limits_{i\in I} b_{ig(i)} \wedge \bigvee\limits_{f\in F_g} \bigwedge\limits_{i\in I} b_{if(i)} = \bot$ so $\bigwedge\limits_{i\in I} b_{ig(i)}$ is a-complemented. Now let $d \wedge d' = \bot$ and $d \vee d' = \bigwedge\limits_{i\in I} b_{ig(i)}$ then d has a-complement $d' \vee \bigvee\limits_{f\in F_g} \bigwedge\limits_{i\in I} b_{if(i)}$ so for some i, the element $b_{ig(i)}$ equals either d or its a-compliment. In the first case $\bigwedge\limits_{i\in I} b_{ig(i)} = d$ and in the second case $\bigwedge\limits_{i\in I} b_{ig(i)} \leq d' \vee \bigvee\limits_{f\in F_g} \bigwedge\limits_{i\in I} b_{if(i)}$ so

$$\bigwedge_{i\in I} b_{ig(i)} = \bigwedge_{i\in I} b_{ig(i)} \wedge \Big(d' \vee \bigvee_{f\in F_g} \bigwedge_{i\in I} b_{if(i)}\Big) = d' \vee \bot = d'$$

(ii)→(i)
Let $\{a_{ij} \mid i \in I, j \in J\}$ be a set in H such that $\{a_{ij} \mid j \in J\}$ is an anti-chain for each $i \in I$. Let c be connected, since H is locally connected it suffices to show that $c \leq \bigvee\limits_{f\in J^I} \bigwedge\limits_{i\in I} a_{if(i)} \leftrightarrow c \leq \bigwedge\limits_{i\in I} \bigvee\limits_{j\in J} a_{ij}$ since each is the join of all connected elements less than or equal to them. First if $c \leq \bigvee\limits_{f\in J^I} \bigwedge\limits_{i\in I} a_{if(i)}$ then for fixed i, we have $c \leq \bigvee\limits_{f\in J^I} a_{if(i)} = \bigvee\limits_{j\in J} a_{ij}$ so $c \leq \bigwedge\limits_{i\in I} \bigvee\limits_{j\in J} a_{ij}$. Next let $c \leq \bigwedge\limits_{i\in I} \bigvee\limits_{j\in J} a_{ij}$ then for fixed i, we have $c \leq \bigvee\limits_{j\in J} a_{ij}$ so $c = c \wedge \bigvee\limits_{j\in J} a_{ij} = \bigvee\limits_{j\in J} c \wedge a_{ij}$. Note now that since $\{a_{ij} \mid j \in J\}$ is an anti-chain so is $\{c \wedge a_{ij} \mid j \in J\}$; therefore it follows from c being connected that $c = c \wedge a_{ij}$ for some $j_i \in J$. Now if $c \neq \bot$ then $c \leq a_{ij}$ and $c \leq a_{ij'}$ implies $j = j'$ since $a_{ij} \wedge a_{ij'} = \bot$ otherwise, hence j_i is unique. Let $g(i) = j_i$ for each i then $c \leq \bigwedge\limits_{i\in I} a_{ig(i)} \leq \bigvee\limits_{f\in J^I} \bigwedge\limits_{i\in I} a_{if(i)}$ (if $c = \bot$ this result is trivial). □

3.2.2 Constructing $V^{(H_{\mathbf{C}})}$

We will now fix A and B for the rest of this section and assume A is infinite. For $H_{\mathbf{C}}$ we use the underlying poset P of all finite partial functions from A to B with the ordering $q \leq p$ iff $p \subseteq q$. Consider then the mapping $\mathbf{C}$ defined by

$$C \in \mathbf{C}(p) \leftrightarrow \exists b \in B[C = \{q \mid b \in \text{range}(q) \wedge q \leq p\}]$$

When an element $C \in \mathbf{C}(p)$ is defined as above we will refer to it as the cover of p *determined* by its corresponding element b.
In order to check that $\mathbf{C}$ is indeed a coverage of P as defined in Definition 1.13 we need to show that for all $C \in \mathbf{C}(p)$ it holds that $C \subseteq p\downarrow$ and for $p' \leq p$ and there exists some $C' \in \mathbf{C}(p')$ that sharpens C. The first condition follows directly from the definition of $\mathbf{C}$. For the second condition note that if b determines $C \in \mathbf{C}(p)$ then the element $C' \in \mathbf{C}(p')$ also determined by b is a subset of C since

$$b \in \text{range}(q) \wedge q \leq p' \rightarrow b \in \text{range}(q) \wedge q \leq p$$

and hence also a sharpening thereof since for all $q \in C'$ we have $q \in C$ and $q \leq q$.
Before we continue we provide a useful rule for calculating joins in $H_{\mathbf{C}}$ with the help of a small lemma.

Lemma 3.9. *For all $S \in \mathbf{C}(p)$ and $q_1, q_2 \in S$ there exist $q_3, q_4, q_5 \in S$ such that $q_1 \leq q_3$, $q_2 \leq q_4$ and $q_5 \leq q_3, q_4$.*

Proof. Let b be the element that determines S. If $b \in p$ then $q_3 = q_4 = q_5 = p$ satisfy the lemma. Assume then that $b \notin p$. There exist some a_1, a_2 such that $q_1(a_1) = q_2(a_2) = b$ so that $q_3 = p \cup \{\langle a_1, b\rangle\} \subseteq q_1$ and $q_4 = p \cup \{\langle a_2, b\rangle\} \subseteq q_2$ are well defined functions such that $q_1 \leq q_3$ and $q_2 \leq q_4$. Finally $q_5 = q_3 \cup q_4$ is well defined as well and $q_5 \leq q_3, q_4$. □

Now we have

Lemma 3.10. *If $\{U_i \mid i \in I\}$ be an anti-chain of $\mathbf{C}$-closed sieves then $\bigcup\limits_{i\in I} U_i$ is $\mathbf{C}$-closed.*

Proof. To start $\emptyset$ is trivially $\mathbf{C}$-closed and so it is the bottom element in $H_{\mathbf{C}}$. In light of this $\{U_i \mid i \in I\}$ being an anti-chain is equivalent with it being a set of pairwise disjoint sieves. Let then $S \subseteq \bigcup_{i\in I} U_i$ and let q_1, q_2 be arbitrary elements of S. Since $S \subseteq \bigcup_{i\in I} U_i$ and $\{U_i \mid i \in I\}$ is pairwise disjoint there exists some unique $i_1, i_2 \in I$ such that $q_1 \in U_{i_1}$ and $q_2 \in U_{i_2}$. Now construct q_3, q_4, q_5 as in Lemma 3.9. There exists a unique $i_3 \in I$ such that $q_3 \in U_{i_3}$. However, since U_{i_3} is a sieve and $q_1 \leq q_3$ it must also be true that $q_1 \in U_{i_3}$ and so $i_3 = i_1$ and $q_3 \in U_{i_1}$. By the same logic $q_4 \in U_{i_2}$. Now $q_5 \in U_{i_1}$ since $q_5 \leq q_3$ and $q_5 \in U_{i_2}$ since $q_5 \leq q_4$ so $i_1 = i_2$. Since q_1, q_2 were arbitrary it follows that $S \subseteq U_i$ for some $i \in I$ and therefore $p \in U_i \subseteq \bigcup_{i\in I} U_i$. □

The following corollary is an immediate consequence of Lemma 3.10 and the fact that $H_{\mathbf{C}}$ is ordered by inclusion.

Corollary 3.10.1. *For any anti-chain $\{U_i \mid i \in I\}$ in $H_{\mathbf{C}}$ the join $\bigvee_{i\in I} U_i = \bigcup_{i\in I} U_i$.*

Using this we can show that $H_{\mathbf{C}}$ as constructed satisfies the conditions of the previous section in the next two propositions.

Proposition 3.11. *$H_{\mathbf{C}}$ is completely $\perp$-distributive and therefore preserves exponentials.*

Proof. Let $\{U_{ij} \mid i \in I, j \in J\}$ be such that $\{U_{ij} \mid j \in J\}$ is an anti-chain for all $i \in I$ then in light of Corollary 3.10.1 it suffices to show that $\bigcap_{i\in I} \bigcup_{j\in J} U_{ij} = \bigcup_{f\in J^I} \bigcap_{i\in I} U_{if(i)}$ since $\bigvee A = \bigcup A$ for any anti-chain $A \subseteq H_{\mathbf{C}}$.
First for arbitrary $\tilde{f} \in J^I$ and $\tilde{i} \in I$ we have

$$\bigcap_{i\in I} U_{i\tilde{f}(i)} \subseteq U_{\tilde{i}\tilde{f}(\tilde{i})} \subseteq \bigcup_{j\in J} U_{\tilde{i}j}$$

and so $\bigcup_{f\in J^I} \bigcap_{i\in I} U_{if(i)} \subseteq \bigcap_{i\in I} \bigcup_{j\in J} U_{ij}$. Now let $p \in \bigcap_{i\in I} \bigcup_{j\in J} U_{ij}$ then given $\tilde{i}$, $p \in \bigcup_{j\in J} U_{\tilde{i}j}$ and since each $\{U_{\tilde{i}j} \mid j \in J\}$ is pairwise disjoint there exists a unique $\tilde{j}$ such that $p \in U_{\tilde{i},\tilde{j}}$. Let $g \in J^I$ be the function for which $g(\tilde{i}) = \tilde{j}$ for all $\tilde{i} \in I$ then $p \in \bigcap_{i\in I} U_{ig(i)}$ and so $\bigcap_{i\in I} \bigcup_{j\in J} U_{ij} \subseteq \bigcup_{f\in J^I} \bigcap_{i\in I} U_{if(i)}$. Hence $H_{\mathbf{C}}$ is completely $\perp$-distributive and so by Proposition 3.7 it also preserves exponentials. □

Proposition 3.12. $V^{(H_C)} \models \widehat{B}$ *is a subquotient of* $\widehat{A}$

Proof. By Proposition 3.5 it suffices to show that there exists a set $\{U_{ab} \mid a \in A, b \in B\}$ of $\mathbf{C}$-closed sieves such that $\{U_{ab} \mid b \in B\}$ is an anti-chain for each $a \in A$ and $\bigvee_{a\in A} U_{ab} = P = \top$ for each $b \in B$. Let $U_{ab} = \{p \in P \mid \langle a, b\rangle \in p\}$. As defined, U_{ab} is $\mathbf{C}$-closed for let $S \subseteq U_{ab}$ for some $S \in \mathbf{C}(p)$ determined by b'. Since p is finite and A infinite we can find $a_1, a_2 \in A - \mathrm{dom}(p)$ such that $a_1 \neq a_2$. Now $g_1 = p \cup \{\langle a_1, b\rangle\}$ and $g_2 = p \cup \{\langle a_2, b\rangle\}$ are both elements of S and $g_1 \cap g_2 = p$. Since $g_1, g_2 \in U_{ab}$, we have $\langle a, b\rangle \in g_1 \cap g_2$, and so $\langle a, b\rangle \in p$ from which it follows that $p \in U_{ab}$. Additionally for $b \neq b'$ the join $U_{ab} \cap U_{ab'}$ is empty since $p \in U_{ab} \cap U_{ab'}$ implies p is not well defined. Finally given $b \in B$ let W be a $\mathbf{C}$-closed sieve such that $U_{ab} \subseteq W$ for all $a \in A$. Then for $p \in P$, the C-cover of p determined by b is a subset of W, and so $p \in W$, hence $W = P$. □

Thus if we let $A = \mathbb{N}$ and $B = \mathbb{N}^{\mathbb{N}}$ then by Proposition 3.12 we have

$$V^{(H_{\mathbf{C}})} \models \widehat{\mathbb{N}^{\mathbb{N}}} \textit{ is a subquotient of } \widehat{\mathbb{N}}$$

and by Proposition3.11

$$V^{(H_{\mathbf{C}})} \models \widehat{\mathbb{N}^{\mathbb{N}}} = \widehat{\mathbb{N}}^{\widehat{\mathbb{N}}}$$

and therefore the object $\mathbb{N}^{\mathbb{N}}$ is subcountable in $V^{(H_{\mathbf{C}})}$.

Proposition 3.13. *LEM is refutable in $V^{(H_C)}$, specifically, not all partial functions can be extended to total functions.*

Proof. Assume LEM holds in $V^{(H_\mathbf{C})}$ and Fix $g \in \mathbb{N}^\mathbb{N}$ and let $\tilde{f}$ be an element of $V^{(H_\mathbf{C})}$ that satisfies

$$V^{(H_\mathbf{C})} \models \tilde{f} = f \cup \{\langle \hat{n}, \hat{g} \rangle \mid \hat{n} \notin \mathrm{dom}(f)\}$$

Then $\mathrm{fun}(\tilde{f})$ and $\mathrm{ran}(\tilde{f}) = \widehat{\mathbb{N}}^{\widehat{\mathbb{N}}}$ are both true in $V^{(H_\mathbf{C})}$. Since LEM holds $\hat{n} \in \mathrm{dom}(f) \vee \hat{n} \notin \mathrm{dom}(f)$ must also be true in $V^{(H_\mathbf{C})}$ for all $n \in \mathbb{N}$ and it follows that

$$V^{(H_\mathbf{C})} \models \tilde{f} \textit{ is a surjection from } \widehat{\mathbb{N}} \textit{ to } \widehat{\mathbb{N}}^{\widehat{\mathbb{N}}} \tag{15}$$

Now let

$$h = \{\langle\langle \hat{n}, \tilde{f}(\hat{n})(\hat{n})^+ \rangle, \top\rangle \mid n \in \mathbb{N}\}$$

Then $h \in \widehat{\mathbb{N}}^{\widehat{\mathbb{N}}}$ is true in $V^{(H_\mathbf{C})}$. Additionally, for all $m \in \mathbb{N}$ we have

$$[\![\langle \hat{m}, h \rangle \in \tilde{f}]\!] \leq [\![\tilde{f}(\hat{m})(\hat{m}) = \tilde{f}(\hat{m})(\hat{m})^+]\!] = \bot$$

Therefore

$$[\![\exists m \in \hat{\mathbb{N}}(\langle m, h \rangle \in \tilde{f})]\!] = \bigvee_{m \in \mathbb{N}} [\![\langle \hat{m}, h \rangle \in \tilde{f}]\!] = \bot$$

Which contradicts (15) □

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