University of Utrecht

Bachelor Thesis

# Game Theoretic Approach to Real Options

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# 1 Introduction

Consider an innovation race between two pharmaceutical companies for a new kind of medicine. In economical terms, two firms are facing a Research & Development investment opportunity in the new medicine. Both are in the position to invest now, but also have the option to postpone the investment. Investing now means the advantage of gaining a larger market share, obtaining a first-mover advantage. However, it also means that this firm has to make the decision without knowing what the other firms and the market will do and how they will react to his decision to invest. The firm that decides to defer investment and wait will be able to observe what the other firms will do and thereby has the advantage of having more information to base his decision on.
This simple case shows that two main elements will influence the decision the management of a firm has to make. On the one hand the management has to deal with uncertainty. It doesn't know what the market will do. Will the demand of that specific medicine increase or decrease and will production prices go up or down? Another point of uncertainty is the price of the investment since this is volatile. On the other hand, the management has to take into account the actions of other firms, in particular that of his competitors. Deferring his investment because he expects demand to fall, includes the risk that his competitors will invest first and gain a larger market share, but does give him more information. It is a decision between knowledge building and strategic positioning.
Two mathematical tools are used to weigh above-mentioned options. The theory of option pricing, introduced by Black and Scholes in 1973 (see [1]), will be used to cope with uncertainty. It gives us a way to value the option at a given moment in time. The interaction between firms will be analysed by applying a game theoretic approach. Competitive interaction is important in the valuation of real options. The theory of option games is the combination of these two successful theories, real options and game theory.
A real option is called 'real' because it usually pertains to tangible assets instead of financial instruments such as securities. The underlying asset for a real option is often illiquid and not easy to trade. In this paper the underlying asset is the present value of project's or investment's cash flows, the value of an investment to the firm. This value is driven by competition, demand and the quality of the management. The value of the underlying asset of a financial option, a share, is more easy to asses (for example on the stock market). A trader cannot influence or control the value of a financial option, while the management of a firm can control the value of the real option by managerial decisions. Their actions can affect the value of the real option.
Thus in this paper the underlying asset is a real option a firm may gain when it undertakes certain actions. For example, by investing in new machinery or R&D it can expand its business. As we have seen, a real option differs from a financial option in the sense that the underlying asset is different. However, there are more differences. For example, there doesn't exist a contract between the owner and the seller of the option. Secondly,

a real option can be held by various firms at the same time. Real options are often not exclusively for one firm. Other firms also have the possibility to invest in a certain project. Thirdly, a firm cannot trade its option. An real option is firm specific since it depends on the capabilities of the firm and the market in which the firm acts. Nevertheless, despite this differences the theory of option pricing can be used to valuate real options.
In this paper I will investigate the interaction between two firms that invest in Research & Development (R&D). Thereby I will follow the paper 'An R&D investment game under uncertainty in real option analysis' by G. Villani, 2008 (see [2]). The aim is to show the effects of moving first and acquiring a first mover advantage against the effects of deferring the investment and thereby generate information revelation.

This paper will start with an introduction in option pricing. An option is the right, but not the obligation to buy or sell a certain asset for a fixed price at a specific moment in time or before a certain date. An option is a financial derivative, which means that its payoff depends on the value of the underlying asset. The introduction will start with the simple binomial two-period model to make the idea of option pricing clear. This binomial model will be extended. In 1973 Black and Scholes introduced a way to value financial options. The limit of the binomial option pricing model is the Black-Scholes formula for pricing options. However, as we will see, the underlying assumptions of the Black-Scholes model are not all applicable to the situation of real options. We therefore have to relax some of these assumptions and introduce exchange and compound options as a way to value real options. Finally, the theory of evaluating compound exchange options will be applied to the above-mentioned problem of two firms facing an R&D-investment. We will be able to derive the final payoffs of two firms after which game theory can be applied.

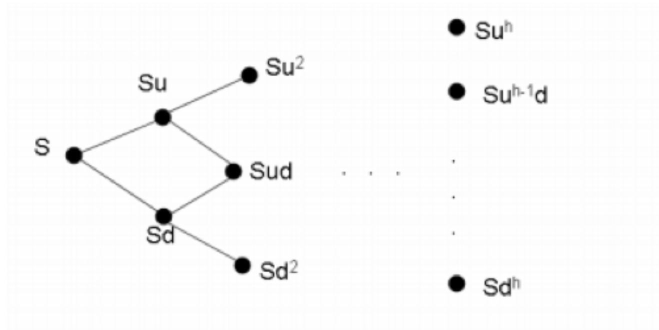
# 2 Option pricing: binomial model

As mentioned before, an option is the right, but not the obligation to buy or sell a certain asset for a fixed price at a specific date or before a certain date. (see [3], p. 5). There are different kind of options. The first distinction that can be made is that between a call and a put option. A call option is the right to buy a certain asset, for example a share, whereas a put option is the right to sell. (see [4], p.4) Another important discrimination is that between American and European options. American options give the holder of the option the right to exercise the option on any date before the expiration date $T$, while a European option can only be exercised at maturity time $T$. (see for example [5], p. 325) An American option is thus at least as valuable as the European option. For the application to real options the American call option is relevant. A firm 'owns' the right to invest in a project and can do this at any moment in time and not just at maturity time $T$.
The main difference between real options and financial options is that real options are often not exclusively for one firm. However, the valuation of a real option is analogous to that of a financial option. Instead of a stock the underlying asset is the value of the project. The value of the project will depend of future cash flows and is therefore uncertain, just like stock prices.

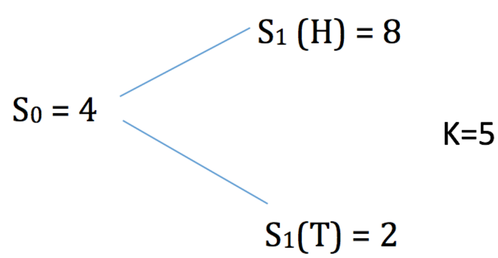
It will become clear with a simple example of an European call option, which underlying asset is a stock, in an one-period binomial model.

## 2.1 One-period binomial model

Let $S_t$ be the price per share of a certain stock at time $t$. The possible future values of $S_t$ are known. These are the values one can see in the scheme below. However, it is uncertain which of the possible future values $S_t$ will be. At time $t = 0$ we know the current value of the share but we are not sure what the future values will be. One cannot predict the future stock values. Let's assume that with probability $p$ the value of the share at time $t = 1$ will be $S_1(H) = uS_0$, where H stands for 'head' (from tossing the coin), and with probability $q = 1 - p$ that its value will become $S_1(T) = dS_0$, where T stands for 'tail'. The probabilities $p$ en $(1 - p)$ are the market probabilities. Assume that $d < 1 < u \in \mathbb{R}$. In a scheme this will look like the following:




We know the possible future stock prices. However, we don't know how the stock prices will evolve exactly. Now we will introduce an option which can protect the investor a bit against this risk of the market. At time $t = 0$ an investor can buy an option for a price $V_0$. This option gives him the right to buy the share at $t = 1$ for a fixed price K, the so-called strike price. Our target is to determine $V_0$, the value of the option at $t = 0$. An example will make things clear.
Suppose $S_0 = 4$ , $d = \frac{1}{2}$ and $u = 2$. We assume that the short-term interest $r$ rate is known and constant overtime and that it is possible to borrow and lend money at the riskless rate $r$. Additionally, the stock doesn't pay any dividend and there are no transaction costs for buying or selling shares. (see the assumptions made in [1]) Then we will get the following scheme:




Suppose further that the strike price is 5 ($K = 5$), $r = 0.25$ and that the (market)chance that the share will be worth $S_1(H)$ and $S_1(T)$ is both $\frac{1}{2}$ ($p = q$). The value of the option at $t = 1$ depends on whether the stock prices will evolve up or downward ($H$ or $T$). If the price evolves upward to 8 at time one the owner of the option will exercise the option since he can buy a share for $K = 5$ instead of 8. On the other hand, if $S_1 = 2$ it is cheaper to buy the share for 2 and therefore not use to option, the option is worthless. The value of the option will be either $V_1(H) = (S_1(H) - K)^+$ or $V_1(T) = (S_1(T) - K)^+$, thus in our example 3 or 0.[1] In general, the value of an option at $t = 1$ is: (see [5], p. 321)

$$V_1(T) = (dS_0 - K)^+ \qquad\qquad V_1(H) = (uS_0 - K)^+ \tag{1}$$

The main assumption in the theory of option pricing is that riskless profitable arbitrage is not possible. In other words, we have a perfect financial market. The option should be priced fairly such that an investor can not set up a trading strategy that guarantees him that he has zero probability of losing money and positive probability of making money. Then he would be able to make a profit without any risk of losing money. (see [6], p. 3-4)
The main implication of this assumptions is that $d < 1+r < u$. To make this clear, imagine for example that $d \geq (1 + r)$. Then the investor could borrow money at the riskless rate $r$ to buy a stock. No matter what the market does, the stock value will always increase more than the debt of the investor. He can sell the stock and still have money left after he has repaid his debt.
Under the non-arbitrage assumption it is possible to determine the (fair) value of the option. An investor composes a portfolio $\zeta$, which is a combination of cash $\Psi$ and shares $S$. The value of the portfolio at $t = 1$ is:

$$\zeta_1 = (1 + r)(C_0 - \Delta_0 S_0) + \Delta_0 S_1$$

The first part of this expression represents the amount of cash $\Psi$ the investor has. At $t = 0$ it had $C_0$. However, he might have bought some share of stock $\Delta_0$ at $t = 0$. Multiplying this term by $(1 + r)$ gives the value of his money stock at $t = 1$. The second part of the value of the portfolio is determined by the current value of his shares of stock, $\Delta_0 S_1$. The non-arbitrage assumption requires that the payoff of the option is equal to the value of the portfolio $\zeta$, thus that $\zeta_1 = V_1$ no matter if stock prices go up or down. The objective is to set up a portfolio with combinations of risk-free borrowing or lending money and buying or selling shares of the stock, or another asset, such that the same value as the option is created. (see [6], p. 5-9) One cannot create a portfolio that has always a value larger than the option since that would violate the non-arbitrage assumption. In other words, the hedge ratio $\Delta_0$ should be such that the following equalities hold: (see [7], p. 5-7)

$$\begin{aligned} \zeta_1(H) = (1 + r)(C_0 - \Delta_0 S_0) + \Delta_0 S_1(H) = V_1(H) \\ \zeta_1(T) = (1 + r)(C_0 - \Delta_0 S_0) + \Delta_0 S_1(T) = V_1(T) \end{aligned} \tag{2}$$

Substracting these equalities from each other and solving for $\Delta_0$ gives:

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}. \tag{3}$$

[1] The subscript '+' implies that you take the maximum of what is in between the brackets and zero. Thus $X = (y - 5)^+ = \max[0, y - 5]$.

At $t = 0$, when he also buys the option, the investor of our example above should buy $\Delta_0 = \frac{3-0}{8-2} = \frac{1}{2}$ share of stock. We also want to know the number of units of cash $\Psi$ (risk-less bonds) the investor should hold. Solving equality (2) for $\Psi = C_0 - \Delta_0 S_0$ gives:

$$\begin{aligned}\Psi = C_0 - \Delta_0 S_0 &= \frac{V_1(H) - \Delta_0 S_1(H)}{1+r} \\ &= \frac{S_1(H)V_1(T) - S_1(T)V_1(H)}{(1+r)(S_1(H) - S_1(T))}\end{aligned}$$

To avoid arbitrage, the portfolio value and the value of the call option should also be the same at $t = 0$. (see [8], p. 18; [9], p. 1-2; [4], p. 4))

$$\begin{aligned}V_0 = \zeta_0 &= \Psi + \Delta_0 S_0 \qquad (4)\\ &= \frac{S_1(H)V_1(T) - S_1(T)V_1(H)}{(1+r)(S_1(H) - S_1(T))} + \frac{V_1(h) - V_1(T)}{S_1(H) - S_1(T)} S_0 \\ &= \frac{V_1(H) - V_1(T)}{uS_0 - dS_0} S_0 + \frac{V_1(T)uS_0 - V_1(H)dS_0}{(1+r)(uS_0 - dS_0)} \\ &= \frac{V_1(H) - V_1(T)}{u-d} + \frac{uV_1(T) - dV_1(H)}{(1+r)(u-d)} \\ &= \frac{1}{1+r}\left[\frac{1+r-d}{u-d} V_1(H) + \frac{u-1-r}{u-d} V_1(T)\right]\end{aligned}$$

One can see that the terms $\frac{r-d}{u-d}$ and $\frac{u-r}{u-d}$ add up to one which means one can see them as probabilities. We call them the risk-neutral probabilities. The time-zero price $V_0$ will be:

$$V_0 = \frac{1}{1+r} \cdot [\widehat{p}V_1(H) + \widehat{q}V_1(T)] \qquad (5)$$

$$\widehat{p} = (1+r-d)/(u-d) \qquad (6)$$

$$\widehat{q} = 1 - \widehat{p}.$$

Formula (5) , is called the risk-neutral pricing formula for the one-period binomial model, since we use risk-neutral probabilities $\widehat{p}$ and $\widehat{q}$. (see [7], p. 7). The risk-neutral probability $\check{p}$ gives the probability that the stock price becomes $uS_0$ when everyone is risk-neutral. It are probabilities such that at each $t \in [0, T]$ the stock price is equal to the discounted expectation of the future stock price,[2]

$$S_0 = \frac{1}{1+r}[\widehat{p}S_1(H) + (1-\widehat{p})S_1(T)]. = \frac{1}{1+r}\widehat{E}[S_1]. \qquad (7)$$

We divide by $1 + r$ to get the present value. In the example above $\widehat{p} = \frac{0,75}{1,5} = 0,5$ and $\widehat{q}$ is therefore also 0,5. Consequently, according to formula (5) the value of the option at $t = 0$ is $\frac{1}{1.25} \cdot (0.5 \cdot 3 + 0.5 \cdot 0) = 1.20$.

[2]In other words, the stock prices are a martingale.

### 2.2 Multi-period model

The one-period model can be extended to a multi-period model with $N$ periods. We still assume that the stock prices evolve over time according to a multiplicative binomial model over discrete periods. The stock prices follow a random walk:

$$S_{t+1} = \begin{cases} uS_t & \text{if } \omega_t = H \\ dS_t & \text{else} \end{cases} \tag{8}$$

This means that the value of an option at $t = N$ is $V_N(\omega_1, \ldots, \omega_N) = (S_N(\omega_1, \ldots, \omega_N) - K)^+$ with $\omega_t \in \{H, T\}$. The possible values of the option at expiration date are known. By going backward from the final node ($t = N$) we can calculate all the values $V_t$ until $V_0$ by applying the following formula, which is the generalization of formula (5):

$$\begin{aligned} V_t(\omega_1, \ldots, \omega_t) &= \frac{1}{1+r} \cdot [\widehat{p}V_{t+1}(\omega_1, \ldots, \omega_t H) + \widehat{q}V_{t+1}(\omega_1, \ldots, \omega_t T)] \\ &= \frac{1}{1+r}\widehat{E}[V_{t+1}] \end{aligned} \tag{9}$$

in which,

$$\begin{aligned} V_{t+1}(\omega_1, \ldots, \omega_t H) &= (uV_t(\omega_1, \ldots, \omega_t) - K)^+ \\ V_{t+1}(\omega_1, \ldots, \omega_t T) &= (dV_t(\omega_1, \ldots, \omega_t) - K)^+ \end{aligned}$$

The probabilities $\widehat{p}$ and $\widehat{q}$ won't change over time since we assumed that the interest rate is constant and we have a binomial model (which implies constant $u$ and $d$). If we use the same method as we did with the one-period binomial model to calculate $V_0$ we would have to calculate $2^N$ values which is quite undoable for a model with for example 100 periods. Instead of going backward from the final node calculating all the values $V_n$ we can use a more efficient method.
The risk-neutral pricing formula (9) can be extended such that we don't we have to calculate all the option values for every single period. Suppose we know $V_N$, then we can calculate the value of the option at any time $n$ without calculating all the values in between by the following formula:

$$V_n(\omega_1, \ldots, \omega_n) = \widehat{E}\left[\frac{V_N}{(1+r)^{N-n}}\middle| n\right](\omega_1, \ldots, \omega_n) \text{ for all } \omega_1, \ldots, \omega_n \tag{10}$$

I will show intuitionally that this is true for $V_0$. The $\widehat{E}$ is again the expected value under the risk-neutral probability measure $\widehat{P} = (\widehat{p}, 1 - \widehat{p})$. According to the formula above we can determine the value of an option at time zero without calculating all the values $V_t$

with $t \neq N$, namely by $V_0 = \widehat{E}[\frac{V_N}{(1+r)^N}]$. Consider the probability space $\Omega$ containing all the combinations of $H$ and $T$ by which $V_0$ can evolve to a value $V_N$. For the two-period model this will be $\Omega = \{HH, HT, TH, TT\}$. The probability that $V_2(HH)$ will be reached is $p^2$, $q^2$ that $V_2(TT)$ will be reached and $pq$ that it will be $V_2(HT)$ or $V_2(TH)$. In general, the probability that $V_N(\omega_1, \ldots, \omega_n)$ is reached is $p^i q^{N-i}$ in which $i$ is the number of heads and $N-i$ the number of tails (this is because the coin tosses are independent). Each path with exactly $i$ upward moves and $N-i$ downward moves gives the same stock price. The option price can be seen as the discounted average of all possible values at t=N:

$$V_0 = \frac{1}{(1+r)^N}\widehat{E}[V_N] = \frac{1}{(1+r)^N}\sum V_N(\omega_1, \ldots, \omega_N) \cdot P(\omega_1, \ldots, \omega_N) \tag{11}$$

for all elements of $\Omega$, thus for all possible paths. Since $i$ follows a binomial distribution formula (11) can be rewritten in the following way: (see [10], p. 201)

$$\begin{aligned} V_0 &= \frac{1}{(1+r)^N}\sum_{i=0}^{N}\binom{N}{i}\widehat{p}^i \cdot (1-\widehat{p})^{N-i} V_N(\omega_1, \ldots, \omega_N) \\ &= \frac{1}{(1+r)^N}\sum_{i=0}^{N}\binom{N}{i}\widehat{p}^i \cdot (1-\widehat{p})^{N-i}(u^i d^{N-i} S_0 - K, 0)^+. \end{aligned} \tag{12}$$

A lot of payoffs at the end nodes, at $t = N$, will be zero since $V_N$ will often be smaller than $K$. Let $a$ denote the positive integer that states the boundary between the values $V_T$ that are positive and that are zero. Therefore $a$ will be the smallest integer such that $u^a d^{N-a} S_0 > K$, it represents the minimum number of upward steps the stock price must take for the call to finish in-the-money. The condition $u^a d^{N-a} S_0 > K$ results in $a = \lfloor \frac{\ln[\frac{K}{S_0 d^N}]}{\ln[\frac{u}{d}]} \rfloor + 1$. Since for $i < a$ the value of the option at maturity is zero ($V_N = 0$) formula (12) can be rewritten in the following way: (see [11], p. 238)

$$V_0 = \frac{1}{(1+r)^N}\sum_{i=a}^{N}\binom{N}{i}\widehat{p}^i \cdot (1-\widehat{p})^{N-i}(u^i d^{N-i} S_0 - K)$$

One can split this expression into two components:

$$\begin{aligned} V_0 &= S_0 \sum_{i=a}^{N}\binom{N}{i}\widehat{p}^i \cdot (1-\widehat{p})^{N-i}\frac{u^i d^{N-i}}{(1+r)^N} - K\frac{1}{(1+r)^N}\sum_{i=a}^{N}\binom{N}{i}\widehat{p}^i \cdot (1-\widehat{p})^{N-i} \\ &= S_0 \Phi(\text{a, N, } \widehat{p}') - K\frac{1}{(1+r)^N}\Phi(\text{a, N, } \widehat{p}), \end{aligned} \tag{13}$$

in which

$$\widehat{p}' = \frac{up}{r} \text{ and } 1 - \widehat{p}' = \frac{d(1-p)}{r}.$$

In expression (13) $\Phi$(a, N, $\widehat{p}$') is the complementary binomial distribution which denotes the probability for having at least $a$ successes in $N$ trials. (see [12], p. 70) It represents the probability that $i < a$, the probability that the option ends up in a situation in which the strike price K will actually be paid. The first part of equation (13) is the current value of the underlying asset $S$, multiplied by a complementary binomial probability. As we saw above, this can be seen as a kind of discounted average of all possible values at $t = N$. The second part of the equation is the discounted exercise price $K$ multiplied by the complementary binomial distribution.
Conclusion, one can interpret the value of a call option as the discounted expected future values of the option in a risk-neutral world.

# 3 Convergence to The Black-Scholes model

In 1973 Fisher Black and Myron Scholes ([1]) provided a solution for pricing a call option. It was the beginning of many papers about option pricing. Black and Scholes stated the following option pricing formula:

$$V_0 = S_0 N(d_1) - Ke^{-rT} N(d_2) \tag{14}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(\frac{k}{S_0}) + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ d_2 &= \frac{\ln(\frac{k}{S_0}) + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}. \end{aligned}$$

The binomial model described in the previous chapter can be extended to a continuous time model by dividing the period till maturity time $T$ in $N$ subintervals and let $N$ approach infinity. The binomial option pricing model converges to the Black-Scholes model under certain conditions. In other words, Black-Scholes is a special limiting case of the binomial discrete situation above. (see [10], p. 205) The binomial model only allows for two different values at a particular time. If in the above situation the stock prices changed every day, making $N$ larger will make it possible to let them change every hour, every minute or even more often.
Till now $r$ represented the risk-free rate over a certain fixed period of time, for example the yearly interest rate. Now the interest rate $\check{r}$ will denote the interest rate over an interval, the compounded interest rate. However, we don't want the interest obtained after $T$ to depend on the number of intervals $N$. The total return at $t = T$ should be the same when the interest is compounded every interval ($N$) or only yearly as we saw in the binomial model. For the continuous model we need a $\check{r}$ such that the interest over the fixed time

period $T$ is always the same. (see [11], p. 247) Normally the investment opportunity grows with $(1+r)$ per year if $r$ is the annual rate: $V_t = (1+r)^t V_0$. Now we have more intervals so the interest rate is compounded more often than once a year, $V_t = V_0(1+\frac{r}{N})^{Nt}$. If we make the compounding period infinitesimally small, thus take the limit of $N$ to infinity, the continuously compounding rate can be found, $\lim_{N\to\infty}(1+\frac{\check{r}}{N})^{NT} = e^{\check{r}T}$. The Black-Scholes formula needs a continuously compounded risk-free rate. The 'new' rate in the continuous model will be $e^{\check{r}t}$ with $t \in [0,T]$. The interest rate $\check{r}$ is such that

$$e^{\check{r}T} = (1+r)^N, \tag{15}$$

in which $r$ is the risk-free interest rate. By using the continuously compounded risk-free rate we can rewrite the Cox Ross-Rubinstein Formula (13) by:

$$V_0 = S_0\Phi(\text{a, N, } \widehat{p}') - Ke^{-rT}\Phi(\text{a, N, } \widehat{p}) \tag{16}$$

This formula has the same structure as the Black-Scholes formula (14). To prove that the binomial model converges to the Black-Scholes model we have to prove that the complementary binomial distribution $\Phi$ converges to the Normal distribution $N(d_2)$. I will only prove the second term of equations (13) and (14) since the proof of the left part is equivalent.

By making $N$ approach infinity the length of the interval approaches zero. When the length of the intervals becomes smaller and smaller adjustments have to be made to the interval-dependent variables $u$, $d$ and $\widehat{p}$ such that they still have realistic values. It's not realistic that a stock price rises by for example 200% in only a second and can again rise by 200% in the following second. When prices can change every second instead of only say once a day we don't want them to have the same percentage up and down every second. They have to be dependent on the number of intervals $N$ so that we get empirically realistic results when $N$ gets larger and thus the intervals smallers. In the binomial model we saw that the stock price experienced a rate of return at each step of $u$ with probability $\widehat{p}$ and of $d$ with probability $(1-\widehat{p})$. Following Cox et al., it is easier to work with logaritmes in this case, resulting in respectively $\ln(u)$ and $\ln(d)$. These are the continuously compounded rates of return of the asset. The stock price at expiration is $S_T = S_0 u^i d^{N-i}$. Taking the logarithm on both sides gives:

$$\ln(\frac{S_T}{S_0}) = i\ln(u) + (N-i)\ln(d) \tag{17}$$

$$= i\ln(\frac{u}{d}) + N\ln d \tag{18}$$

Here, $\ln(S_T/S_0)$ is the natural logarithm of one-plus-return for holding the stock over the $N$ periods. It is equivalent to a continuously compounded return over the $N$ periods.
The number of upward movements is denoted by $i$ and is binomial distributed. The expected value and variance of $i$ will therefore be $N\widehat{p}$ and $N\widehat{p}(1-\widehat{p})$ respectively. The $\mu$ is

the stock's annual expected return and $\sigma^2$ the variance of returns. The expected value and variance of $\ln(\frac{S_t}{S_0})$ will then become:

$$\begin{aligned} E[\ln(\frac{S_t}{S_0})] &= E[i]\ln(\frac{u}{d}) + N\ln(d) \\ &= N\widehat{p}ln(\frac{u}{d}) + N\ln(d) \\ &= N\widehat{\mu} \\ Var(\ln(\frac{S_T}{S_0})) &= Var(j)\ln(\frac{u}{d})^2 + 0 \\ &= N\widehat{p}(1-\widehat{p})\ln(\frac{u}{d})^2 \\ &= N\widehat{\sigma}^2 \end{aligned}$$

We want to find appropriate expressions for $u$, $d$ and $\widehat{p}$, to solve the problem of unrealistic stock prices. As stated above, the empirical values of the mean and the variance of $ln(\frac{S_T}{S_0})$ are $\widehat{\mu}N$ and $\widehat{\sigma}^2N$ and assume that the actual ones are $\mu T$ and $\sigma^2 T$ respectively. Then we want $\widehat{\mu}N$ to converge to $\mu T$ and $\widehat{\sigma^2}N$ to $\sigma^2 T$ when $N$ approaches infinity. (see [11], p. 248-249)

$$\begin{aligned} \widehat{p}N\ln(\frac{u}{d}) + N\ln(d) &\to \mu T \\ N\widehat{p}(1-\widehat{p})\ln(\frac{d}{u})^2 &\to \sigma^2 T. \end{aligned}$$

In order to keep the binomial property let the third restriction be $d = 1/u$. Now we have three equations with three variables which can be solved, giving:

$$u = e^{\sigma\sqrt{\frac{T}{N}}} \qquad d = e^{-\sigma\sqrt{\frac{T}{N}}} \qquad \widehat{p} = \frac{1}{2} + \frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{N}} \tag{19}$$

If $u$, $d$ and $\widehat{p}$ are defined as above then $\widehat{\sigma}^2N \to \sigma^2T$ and $\widehat{\mu}N \to \mu T$ when $N \to \infty$. (see[10], p. 356) Besides, if $u$, $d$ and $\widehat{p}$ are defined like this, together with expression (17), one can show that the $\Phi(a, N, \check{p})$ converges to $N(d_2)$. We can rewrite expression (17). This will give $i = \frac{\ln(\frac{S_T}{S_0}) - N\ln(d)}{\ln(\frac{u}{d})}$. In paragraph 3.2 an expression for $a$ was derived. Rewriting this expression gives:

$$\begin{aligned} a - 1 &= \left\lfloor \frac{\ln[\frac{K}{S_0 d^N}]}{\ln[\frac{u}{d}]} \right\rfloor \\ &= \frac{\ln[\frac{K}{S_0 d^N}]}{\ln[\frac{u}{d}]} - \epsilon, \text{ with } \epsilon \in [0, 1). \end{aligned} \tag{20}$$

From the Cox Ross-Rubinstein formula (16): (see [11], p. 251)

$$
\begin{aligned}
1-\Phi(a,N,\widehat{p}) &= P(i \le a-1) \\
&= P\Big(\frac{i-N\widehat{p}}{N\widehat{p}(1-\widehat{p})} \le \frac{a-1-N\widehat{p}}{N\widehat{p}(1-\widehat{p})}\Big) \\
&= P\Big(\frac{\ln(\frac{S_T}{S_0})-N\ln(d)-N\widehat{p}\ln(\frac{u}{d})}{N\widehat{p}(1-\widehat{p})\ln(\frac{u}{d})}) \le \frac{\ln(\frac{K}{S_0})-N\ln(d)-\epsilon\ln(\frac{u}{d})-N\widehat{p}\ln(\frac{u}{d})}{\sqrt{N\widehat{p}(1-\widehat{p})}\ln(\frac{u}{d})}\Big)
\end{aligned}
\tag{21}
$$

Analogous to what we discussed before, $\ln(\frac{S_T}{S_0}) = i\ln(\frac{u}{d})+N\ln(d)$. Therefore, in the case of continuous time, the mean and variance of $\ln(\frac{S_T}{S_0})$, which is the continuously compounded rate of return of the stock, are as followed:

$$
\widehat{\mu}_{\widehat{p}} = \widehat{p}\ln(\frac{u}{d})+\ln(d) \qquad\qquad \widehat{\sigma}_{\widehat{P}} = \widehat{p}(1-\widehat{p})\ln(\frac{u}{d})^2 \tag{22}
$$

These equalities can be substituted in expression (21), resulting in:

$$
1-\Phi(a,N,\widehat{p}) = P\Big(\frac{ln(\frac{S_T}{S_0})-\widehat{\mu}_{\widehat{p}}N}{\widehat{\sigma}_{\widehat{p}}\sqrt{N}} \le \frac{\ln(\frac{K}{S_0})-\widehat{\mu}_{\widehat{p}}N-\epsilon\ln(\frac{u}{d})}{\widehat{\sigma}_{\widehat{p}}\sqrt{N}}\Big). \tag{23}
$$

Result (23) still depends on the number of intervals $N$. However, the Black-Scholes Model doesn't depend on $N$ but on $T$. By using convergence, $N \to \infty$, one can get rid of the $N$. We will evaluate how the terms in (23) will behave when N approaches infinity.

The first term to discuss is $\widehat{\sigma}_{\widehat{p}}\sqrt{N}$. From expression (22)

$$
\widehat{\sigma}_{\widehat{p}}\sqrt{N} = \sqrt{\widehat{p}(1-\widehat{p})\ln(\frac{u}{d})^2}\sqrt{N}. \tag{24}
$$

Substituting the expressions for $u$, $d$ and $\widehat{p}$ gives

$$
\widehat{\sigma}_{\widehat{p}}\sqrt{N} = \sqrt{\widehat{p}(1-\widehat{p})}\cdot 2\sigma\frac{\sqrt{T}}{\sqrt{N}}\cdot\sqrt{N} \tag{25}
$$

$$
= \sqrt{(\frac{1}{2}+\frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{N}})\cdot(\frac{1}{2}-\frac{1}{2}\frac{\mu}{\sigma}\sqrt{\frac{T}{N}})}\cdot 2\sigma\sqrt{T}. \tag{26}
$$

Now one can easily see that as $N$ approaches infinity,

$$
\widehat{\sigma}_{\widehat{p}}\sqrt{N} \to \sigma\sqrt{T}. \tag{27}
$$

**Definition 1.** *The lognormal distribution* $LN(\mu, \sigma^2)$ *is the distribution of* $e^X$ *where* $X \sim N(\mu, \sigma^2)$. *If* $Y = e^X \sim N(\mu, \sigma^2)$ *then* $E[Y] = e^{\mu + \frac{1}{2}\sigma^2}$ *and* $var(Y) = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$.

The second term to discuss is $\widehat{\mu}_{\widehat{p}}$. Remember that we assumed that the asset prices follow a lognormal distribution, thus $\frac{S_T}{S_0}$ follows a lognormal distribution. Remember also that $\ln(\frac{S_T}{S_0})$ has mean $\mu_{\widehat{p}}N$ and variance $\sigma_{\widehat{p}}N$. According to Definition 1, $\ln(\frac{S_T}{S_0})$ follows a normal distribution with mean $\mu_{\widehat{p}}N$ and variance $\sigma_{\widehat{p}}N$ and therefore $E[\frac{S_T}{S_0}] = e^{\mu_{\widehat{p}}N + \frac{1}{2}\sigma^2_{\widehat{p}}N}$. Taking the log on both sides gives

$$\ln\left(E[\frac{S_T}{S_0}]\right) = \mu_{\widehat{p}}N + \frac{1}{2}\sigma^2_{\widehat{p}}N. \tag{28}$$

Expression (6) for the risk neutral probability $\widehat{p}$ can be rewritten,

$$1 + r = \widehat{p}u + (1 - \widehat{p})d. \tag{29}$$

Besides, remember that the stock price at time $t$ is $uS_{t-1}$ with risk-neutral probability $\widehat{p}$ and $dS_{t-1}$ with probability $(1 - \widehat{p})$. Therefore,

$$\begin{aligned} E[S_T] &= \widehat{p}uE[S_{T-\frac{T}{N}}] + (1-\widehat{p})dE[S_{T-\frac{T}{N}}] \\ &= (\widehat{p}u + (1-\widehat{p})d)E[S_{T-\frac{T}{N}}], \end{aligned}$$

in which $\frac{T}{N}$ is the length of an interval. By induction,

$$E[S_T] = (\widehat{p}u + (1-\widehat{p})d)^N \cdot E[S_0].$$

Since $N$ is the number of intervals it is also the number of times the stock price 'changes'. Using expression (29) and the expression we have found for the continuously compounded risk-free rate $e^{rT}$, (15),

$$E[\frac{S_T}{S_0}] = (1 + r)^N = e^{rT}.$$

Now take the logarithm on both sides:

$$\ln\left(E[\frac{S_T}{S_0}]\right) = rT. \tag{30}$$

Now we have two expressions for $\ln(E[\frac{S_T}{S_0}])$, namely (28) and (30). Equalising them gives

$$rT = \mu_{\widehat{p}}N + \frac{1}{2}\sigma^2_{\widehat{p}}N.$$

Finally, an expression for $\widehat{\mu}_{\widehat{p}}$ is found when N approaches infinity. Since we have seen that $\widehat{\sigma}_{\widehat{p}}\sqrt{N} \to \sigma\sqrt{T}$,

$$\widehat{\mu}_{\widehat{p}}N \to rT - \frac{1}{2}\sigma^2 T \text{ as } N \to \infty. \tag{31}$$

The third and last term of which we want to know how it behaves when $N$ approaches infinity is $\epsilon \ln(\frac{u}{d})$. That one is quite straightforward when the expressions for $u$ and $d$ (19) are substituted. Since $\epsilon \in [0, 1)$ we can conclude that

$$\epsilon \ln\big(\frac{u}{d}\big) = 2\epsilon\sigma\sqrt{\frac{T}{N}} \to 0 \text{ as } N \to \infty. \tag{32}$$

We examined the behaviour of the three terms of expression (23). We can conclude that when $N$ goes to infinity

$$\frac{\ln(\frac{K}{S_0}) - \widehat{\mu}_{\widehat{p}}N - \epsilon\ln(\frac{u}{d})}{\widehat{\sigma}_{\widehat{p}}\sqrt{N}} \to \frac{\ln(\frac{K}{S_0}) - rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.$$

By applying the Central Limit Theorem we can find that

$$1 - \Phi(a, N, \widehat{p}) \to N(z),\ z = \frac{\ln(\frac{K}{S_0}) - rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

when $N$ approaches infinity. The final step is to use the well-known property of the standard normal distribution, $1 - N(z) = N(-z)$ and that $-\ln(\frac{a}{b}) = \ln(\frac{b}{a})$. See that $-z$ is equal to $d_2$ in equation (14),

$$1 - (1 - \Phi(a, N, \widehat{p})) = \Phi(a, N, \widehat{p}) \to 1 - N(z) = N(-z) = N(d_2)$$

Therefore the proof is compleet. The binomial model of Cox-Ross and Rubinstein converges to the Black-Scholes model.

## 4 Assumptions of the Black-Scholes model

Implicitly we have made some assumptions while deriving the Black-Scholes formula. For convenience, the assumptions are shown here. The Black-Scholes model assumes that (see [1], p. 640) :

i) The option is a European option, which means that it can only be exercised at maturity time $T$.

ii) The interest rate $r$ is constant. It is possible to lend and borrow any amount of cash at the riskless interest rate $r$.

iii) The underlying asset, the stock prices, follows a random walk. The stock prices are lognormal distributed. This means that the underlying asset follows a Geometric Brownian motion (see [13], p. 115):

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{33}$$

where $W_t$ is a Wiener process and $\mu$ (the percentage drift) and $\sigma$ (the percentage volatility) are constants. A Wiener process is a stochastic process which is a limiting (continuous) case of the random walk (8) we used in the binomial model. If one takes a random walk with very small steps, one gets an approximation of the Wiener Process. Formule (33) is an example of a random walk. (see [8], p. 23)
Absolute changes in asset prices is not what interests us. We are interested in rate of returns on the stock, which can at any time $t$ be expressed by (33). In this expression $\mu dt$ is the drift term which is in this context a measure of the average rate of growth of asset prices, or the rate of return of the underlying asset. The term $\sigma dW_t$ gives the random change in asset prices due to external factors in which $dW_t$ represents a random sample from a normal distribution $N(0, 1)$.

One can also prove that if $u$ and $d$ are chosen as in (19) then the binomial distribution of $S_t$ converges to the Geometric Brownian motion (33) as the number of intervals $N$ approaches infinity. (see for a proof for example [8])

The Geometric Brownian motion (33) can also be written in the form:

$$S_t = S_0 e^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t}.$$

It is the solution to the lognormal stock price stochastic differential equation (33). (see [13], p. 116) At fixed time $t$, this expression has a log-normal distribution $LN(\ln(S_0) + \mu t, \sigma\sqrt{t})$. Now we can clearly see why we used that $S_t$ follows a log-normal distribution in the proof of the convergence of the binomial pricing model to the Black-Scholes

model.
The last comment I want to make to this item is the fact that the solution of the Black-Scholes model (14) does not contain the rate of return $\mu$ while it does contain the riskless rate $r$ which isn't a part of the Geometric Brownian motion (33). This is caused by the fact that the Black-Scholes model is based on the idea that the investor can create a riskless portfolio where the risk associated with the stochastic stock prices is eliminated. We assumed that riskless arbitrage is impossible and therefore the expected rate of return of the portfolio should be equal to the riskless rate $r$. See also assumptions ii) and viii).

iv) The underlying asset pays no dividend.

v) The variance of return $\sigma$ (or the measure of standard deviation about the mean return) on the underlying asset is constant.

vi) The exercise price $K$ is known and fixed.

vii) There exist no transaction costs.

viii) There are no riskless arbitrage opportunities.

These are quite strong assumptions and for our application to real options we need to relax some of them. First of all, an investment opportunity can usually be exercised at any moment in time and not just at maturity time $T$. Real options are more comparable to American than European options since American option can be exercised at any moment before $T$. Secondly, it is not realistic to assume that a real option doesn't pay dividend. In the case of real options, the opportunity costs that arise when deferring an investment project can be seen as dividends. As long as one doesn't exercise his option to buy shares he won't get the dividend inherent with owning shares. An analogous reasoning applies to deferring the option to invest in a project. The firm misses the potential project cash flows as long as it doesn't invest. (see [2]) Thirdly, the exercise price will not be constant over time in case of real options. The investment cost will, among other things, depend on the market share of a firm. The exercise price is particularly uncertain for R&D investments. (see [14], p. 1)
A solution for this last problem is introduced by Margrabe in 1978 ([15]). He said that the investment opportunity should not be seen as a simple call option but as an exchange option. An exchange option gives the holder the right, but not the obligation, to exchange one asset for another. He can exchange the asset he owns, the delivery asset $Y$, for the optioned asset $X$. Thus the exchange option has two underlying assets, $Y$ and $X$, whereas the simple option only one. In the context of real options $Y$ is the amount to invest in a project, the investment cost, and $X$ is the current value of the project to the firm, the present value of future cash flows. Both $X$ and $Y$ are random variables. (see [16], p. 13)

The last element that Black-Scholes doesn't take into account is the interaction between firms. An investment opportunity is normally not hold by one singular firm. Therefore, the optimal exercise strategy should take in consideration the actions of competitors also owning this option to invest. The strategic interaction will be considered by using compound options. The compound option was introduced in Geske, 1978 ([17]). A compound option is an option on an option. The underlying asset of the option is again an option. An R&D investment is often not made once and in isolation. It involves a series of investments in which exercising the first option will give the firm the right to exercise the following option as well. Latter opportunities are only available if earlier opportunities are undertaken. The R&D program is cut into smaller pieces and at each stage the firm can reconsider whether it will provide additional funding for the next stage. This might sound a bit abstract but will become clear later when we apply the theory to a real option situation. The compound option will be used to analyse the two stage R&D investment case. However, having compound options also implies that the fifth Black- Scholes assumption doesn't hold. The variance rate of the return on the stock, or project, is not constant but depends on the value of the firm.

# 5 Exchange option

An American exchange option gives the owner the right to exchange one asset for another at any moment before the expiring date $T$. Margrabe was the first to introduce a theoretical model for the valuation of an exchange option. He did that in 1978. However, he only dealt with the European exchange option, or equivalent, for American options that don't pay any dividend. Remember that the owner of an exchange option can exchange the options he owns, the delivery asset $Y$, for the optioned asset $X$. At maturity date $T$, the value of the exchange option will be given by:

$$V_T = (X_T - Y_T)^+$$

The model of the exchange options builds on the Black-Scholes model. Therefore we again assume that the the underlying assets, $X$ and $Y$, follow a geometric Brownian motion of the form (see assumption iii):

$$\frac{dX_t}{X_t} = (\mu_x - \delta_x)dt + \sigma_x Z_t^X \tag{34}$$

$$\frac{dY_t}{Y_t} = (\mu_y - \delta_y)dt + \sigma_y Z_t^Y$$

$$cov(\frac{dX_t}{X_t}, \frac{dY_t}{Y_t}) = \rho dt \tag{35}$$

In which $\mu_x$ and $\mu_y$ are, as with the Black-Scholes model for a simple European option, the expected rates of return on the assets X and Y respectively and $\sigma_x$ and $\sigma_y$ the variance rates. The $\delta_x$ en $\delta_y$ are the dividends. All these parameters are assumed to be non-negative and constant. The $Z_t$ expresses again two Wiener processes at time $t$.

McDonald and Siegel showed in 1985 ([18]) that under this condition (34) the value of the European Exchange option on an asset that pays dividend is given by:

$$V_t = X_t e^{-\delta_x(T-t)} N_1(d_1) - Y_t e^{-\delta_y(T-t)} N_1(d_2) \tag{36}$$

where

$$\begin{aligned}
N_1(d_i) &= \int_0^{d_i} \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz \text{ , the standard univariate normal distribution function} \\
d_1 &= \frac{\ln(Pe^{\delta T}) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}; \\
d_2 &= \frac{\ln(Pe^{\delta T}) - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}; \\
P &= \frac{X}{Y}; \\
\delta &= \delta_x - \delta_y; \\
\sigma^2 &= \sigma_x^2 + \sigma_y^2 - 2\rho
\end{aligned}$$

If dividends are set to zero, one gets the formula of Margrabe for European exchange options that don't pay any dividend. I argued that the model for exchange options builds on the Black-Scholes model. To see this, remember that a call option is a special kind of exchange option, where the delivery asset Y, the investment price, is constant over time. This means that $dY_t/Y_t = 0$, which in turn implies that $Y$ pays dividend at the riskless rate $r$ ($\delta_y = r$) and that the variance rate of the delivery asset $Y$ is zero ($\sigma_y^2 = 0$). Consequently, if the variance rate of $Y$ is zero then the expected rate of return of $Y$ must be the riskless rate $r$ to avoid arbitrage, $\mu_d = r$. On top of that, assumption iv) states that the underlying asset $X$ doesn't pay any dividend ($\delta_x = 0$). Substituting these parameter restrictions in formula (36) results in the Black-Scholes formula for simple call options in which $X$ is the stock price ($S$ in the Black-Scholes model) and $Y$ the strike price $K$.
The only difference in the underlying assumption of a Geometric Brownian motion when dividend is paid, is the term $\delta_x dt$ (compare (33) and (34)). Although it stands in the same position as the $\mu$ it does disappear in the final solution (36), while $\mu$ didn't appear in the basic Black-Scholes formula (33). This is because the argument about setting up a riskless portfolio doesn't hold for dividend. Dividends influence the value of the underlying asset in a positive way. Deferring investment means that one doesn't get the dividends till he invests. Lastly, one can also observe that the riskless rate $r$ disappeared in the formula for European exchange options (36). This is due to the fact that the value $V_0$ is linearly homogeneous in the asset prices $X$ and $Y$. The argument that eliminated risk in

the simple Black-Scholes model now also makes the portfolio costless. No arbitrage in the case of exchange options means a zero return on the portfolio instead of a return of $r$.

Up to this point, the focus has been on European options. However, since exercising an investment is also possible before maturity time T, we need a formula for the valuation of American exchange options on dividend-paying assets. If the underlying asset pays a positive dividend, early exercise will often be better than deferring investment till $t = T$. Imagine now that the option can be exercised at two moments in time, $T$ and $\frac{T}{2}$. It is obvious that the option will not be exercised at time $\frac{T}{2}$ if:

$$V_{\frac{T}{2}} = X_{\frac{T}{2}} e^{\delta_x \frac{T}{2}} N_1(d_1) - Y_{\frac{T}{2}} e^{\delta_y \frac{T}{2}} N_1(d_2) > X_{\frac{T}{2}} - Y_{\frac{T}{2}}. \tag{37}$$

In that case the opportunity cost of exercising the option will be more than the benefits. Carr 1988 ([16]) introduced the expression $P = X/Y$ which turns the previous expression in:

$$Pe^{-\delta_x \frac{T}{2}} N_1(d_1) - e^{-\delta_y \frac{T}{2}} N_1(d_2) > P - 1 \tag{38}$$

There is a $\check{P}$ that makes the firm indifferent at $\frac{T}{2}$ between exercising the option or not; it turns expression (38) into an equality. If $P > \check{P}$ at $t = \frac{T}{2}$ the option will be exercised. One can see this expression as an European exchange option with maturity date $T$ which pays $(X-Y)^+$. I will not go into detail in the derivation of the formula for an American exchange option. The following expression was derived by Carr ([16]) for valuing an American exchange option exercisable at time $\frac{T}{2}$ and $T$:

$$\begin{aligned} V_{carr} =& Xe^{-\delta_x T} N_2(-\check{d}_1, d_1; -\rho_1) - Ye^{-\delta_y T} N_2(-\check{d}_2, d_2; -\rho_1) \\ &+ Xe^{-\delta_x \frac{T}{2}} N_1(\check{d}_1) - Ye^{-\delta_y \frac{T}{2}} N_1(\check{d}_2) \end{aligned} \tag{39}$$

Where:

$$\delta = \delta_x - \delta_y$$

$$d_1(P,T) = \frac{\ln(p) - \delta T + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$d_2(P,T) = \frac{\ln(p) - \delta T - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

$$\check{d}_1 \equiv d_1\left(\frac{P}{\check{P}}, \frac{T}{2}\right)$$

$$\check{d}_2 \equiv d_2(\frac{P}{\check{P}}, \frac{T}{2})$$

$N_1(d)$ is the standard univariate normal distribution function

$N_2(x_1, x_2, \rho)$ is the standard bivariate normal distribution function evaluated at $x_1$ and $x_2$ with correlation $\rho$.

Armada ([19]) tried to improve the formula of Carr. He evaluated the American exchange options by using the Richardson extrapolation process on the formula of Carr which resulted in the following formula:

$$
\begin{aligned}
V_{aeo} &\simeq V_{carr} + \frac{V_{carr} - V_{eeo}}{3} \\
&= \frac{4V_{carr} - V_{eeo}}{3}
\end{aligned}
\tag{40}
$$

in which $V_{carr}$ is the formula introduced by Carr in 1988 (39) and $V_{eeo}$ the value of an European exchange option (eeo), see equation (36).

## 6 Compound exchange option

We are almost there. The last step to take before we can apply our theory to real options is to find an expression for the value of a compound American exchange option. Geske ([17]) was the first to introduce a theory for pricing options on options. A compound option is an option which underlying asset is again an option. Consequently, the variance of the rate of return on the underlying asset is not constant as the Black-Scholes model assumes (see assumption v).

A compound exchange option involves the opportunity to deliver one asset in return for an exchange option. The delivery option will be the same in each exchange. In the application to real options this will be the investment cost. It does not mean that the investment cost is constant, but that it will be cash that is exchanged for an investment at every stage. The underlying exchange option $S$ is the opportunity to make the next investment in the R&D investment stages, in our case the development stage. In short, we have a compound American exchange option which underlying asset is an exchange option.

Suppose $\beta$ is the exchange ratio, meaning that the exercise price is a fraction $\beta$ of asset $Y$, the investment cost. Thus $\beta$ is the fraction of $Y$ required for R&D. (see [20], p. 136)

The value of a compound option will be: $V_{compound} = (V_S - \beta Y)^+$, in which $V_S$ is the value of the underlying exchange option $S$. However, we want to express the value of the compound option in terms of assets $X$ and $Y$ since those values are observable. The underlying asset $S$ is an exchange option thus $V_S(X, Y, T) = (X - Y)^+$ which resulted in the McDonald & Siegel formula for an exchange option (36). This gives:

$$
\begin{aligned}
V_{compound}(X, \beta Y, T) &= (V_S - \beta Y)^+ \\
&= (Xe^{-\delta_x T} N_1(d_1(P)) - Ye^{-\delta_y T} N_1(d_2(P)) - \beta Y)^+,
\end{aligned}
\tag{41}
$$

in which $\beta Y$ is the exercise price. The compound option will be exercised if the value of the underlying exchange option $V_S$ is larger than the exercise price:

$$
Xe^{-\delta_x T} N_1(d_1(P)) - Ye^{-\delta_y T} N_1(d_2(P)) > \beta Y. \tag{42}
$$

We will follow the same line of reasoning and apply the same trick as we did above with the exchange option. Introduce $P = \frac{X}{Y}$, such that the exercise condition only depends on one random variable:

$$Pe^{-\delta_x T} N_1(d_1(P)) - e^{-\delta_y T} N_1(d_2(P)) > \beta \tag{43}$$

What is now left looks familiar. The left-hand side of this condition is the Black-Scholes formula (see formula (14)) of a call option on the price ratio $P$ with an exercise price of $K = 1$ and a risk-free rate $r$ of zero. Again there will be a $\check{P}$ such that condition (43) becomes an equality. Then the firm is indifferent between exercising the option or not. According to Villani ([21], p. 4), the value of a compound American exchange option is given by:

$$\begin{aligned} V_{caeo}(V_S(X,Y,T-t_1), \beta Y, t_1) \simeq & \frac{4V_{pcaeo}(V_{paeo}(X,Y,T-t_1), \beta Y, t_1)}{3} && (44)\\ & - \frac{V_{ceeo}(V_{eeo}(X,Y,T-t_1), \beta Y, t_1)}{3} && (45) \end{aligned}$$

Thus the value of a compound American exchange option $V_{caeo}$ is defined by this expression. $V_{pcaeo}$ is the value of a pseudo compound American exchange option. A pseudo compound American exchange option is a compound option whose underlying asset is a pseudo American exchange option. A pseudo American exchange option is an exchange option that can not, as normally with an American option, be exercised at any moment before the expiration date. However it is more flexible than an European option since it can be exercised at $\check{t} = \frac{T+t_1}{2}$ and at $t = T$ with $t_1 < T$. The price of a pseudo compound American exchange option therefore is:

$$V_{pcaeo} = e^{-rt_1} \check{E}[\max(V_{paeo}(X_{t_1}, Y_{t_1}, T) - \beta Y_{t_1}, 0)]. \tag{46}$$

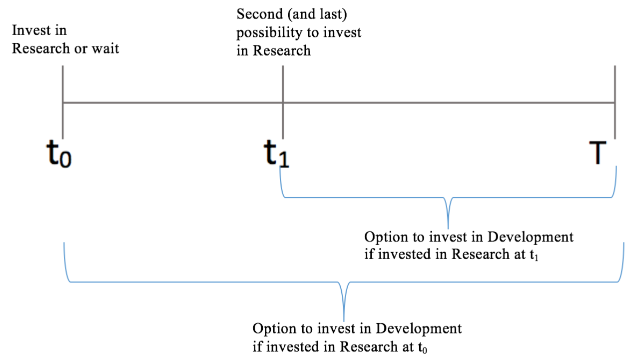
Recall that $\check{E}$ was the expected value under the risk-neutral probability measure. The value of an American exchange option that is exercisable at two moments in time is defined by the expression found by Carr, see expression (39). The expiration date of the pseudo compound American exchange option is $t_1$ and its exercise price is $\beta Y$.
$V_{ceeo}$ is the value of a compound European exchange option whose underlying asset is an European exchange option and with expiration date $t_1$. Carr (1988) showed that the value of such an option is given by:

$$\begin{aligned} V_{ceeo}(V_{eeo}(X,Y,T), \beta Y, t_1) =& Xe^{-\delta_x T} N_2(\check{d}_1, d_1, \rho) - Ye^{-\delta_y T} N_2(\check{d}_2, d_2, \rho) \\ & - \beta Y e^{-\delta_y t_1} N(\check{d}_2). \end{aligned} \tag{47}$$

It took some time but we have found an expression for the value of a compound American exchange option, namely (45). Now we have the tools to start the R&D investment game between two firms.

# 7 The game: real options

Two firms, firm A and B, are facing an R&D investment. Following Villani ([2]), we simplify the game by assuming that it is a two-stage investment. First a firm has to invest in 'Research' and then in 'Development'. I will clarify the game by using a timeline:




The expiration date of the compound option is $t_1$. Both company A and B have two options at $t_0$. They can either invest in R&D at $t_0$ or wait and defer their investment decision till $t_1$. If a firm decides to invest while the other decides to wait the first firm will gain a first mover advantage. By moving first it will be able to gain a larger market share than the firm that decided to postpone investment. He will gain a market share opportunity $\alpha \in (\frac{1}{2}, 1]$ which implies that the other firm (the follower) will gain a market share of $(1-\alpha) \in [0, \frac{1}{2}]$. However, waiting can also be an advantage since that will give that firm more information about what the other firm will do and what the effect of the first investment is. For example, suppose the first investment (the research) stage is a first drilling. If oil is found the firm will have a first-mover advantage. However, the other firm also knows whether there is oil in that area or not and can therefore make a better, more accurate, decision in the future since it has more information to base its decision on.
Assume that the probabilities of having a successful research investment are $q$ and $p$ for firm A and B respectively, which means that the investment success is Bernoulli distributed. When the first investment of A is a success, $X = 1$, and else $X = 0$. Same holds for B, $Y = 1$ means a successful investment, while $Y = 0$ means a failure. If the investment of A is successful, for example if oil is found during the first drilling, the success probability $p$ of firm B will change to $p^+$. He has more information so if he decides to invest in a later stage this decision will be based on more information so the probability of a successful will increase. On the other hand, if A's investment fails then B's success probability $p$ changes to $p^-$. We want to find explicit expressions for this $p^+$ and $p^-$. We will start with $p^+$.

$$p^+ = P(X = 1|Y = 1) = \frac{P(X = 1 \bigcap Y = 1)}{P(Y = 1)} \tag{48}$$

There are four combinations possible for $X$ and $Y$, depending on whether both, none or one of them is successful: (0, 0), (1, 0), (0, 1) and (1, 1). The probabilities that these combinations occur are respectively $p_{00}$, $p_{10}$, $p_{01}$ and $p_{11}$, the so called bivariate bernoulli distribution. We need to find an expression for $p_{11}$. The variables $X$ and $Y$ are Bernoulli distributed thus the expected values are $E[X] = q$ and $E[Y] = p$ and the variances $q(1-q)$ and $p(1-p)$ respectively. Then the correlation between $X$ and $Y$ is:

$$\begin{aligned} corr(X,Y) &= \frac{cov(X,Y)}{\sqrt{var(X)}\sqrt{var(Y)}} \\ &= \frac{E[XY] - E[X]E[Y]}{\sqrt{q(1-q)}\sqrt{p(1-p)}} \\ &= \frac{(1 \cdot p_{11} + 0 \cdot p_{10} + 0 \cdot p_{01} + 0 \cdot p_{00}) - pq}{\sqrt{q(1-q)}\sqrt{p(1-p)}} \\ &= \frac{p_{11} - pq}{\sqrt{q(1-q)}\sqrt{p(1-p)}} \end{aligned} \tag{49}$$

Rewriting this expression gives:

$$p_{11} = corr(X,Y)\sqrt{q(1-q)}\sqrt{p(1-p)} + pq.$$

Therefore $p^+$ is:

$$\begin{aligned} p^+ &= \frac{p_{11}}{q} \\ &= \frac{corr(X,Y)\sqrt{q(1-q)}\sqrt{p(1-p)} + pq}{q} \\ &= p + \sqrt{\frac{1-q}{q}} \cdot \sqrt{p(1-p)} \cdot \rho(X,Y). \end{aligned}$$

Analogous to described above we can derive expressions for the other probabilities:

$$\begin{aligned} p^+ &= p + \sqrt{\frac{1-q}{q}} \cdot \sqrt{p(1-p)} \cdot \rho(X,Y) \\ p^- &= p - \sqrt{\frac{q}{1-q}} \cdot \sqrt{p(1-p)} \cdot \rho(X,Y) \\ q^+ &= q + \sqrt{\frac{1-p}{p}} \cdot \sqrt{q(1-q)} \cdot \rho(Y,X) \\ q^- &= q - \sqrt{\frac{p}{1-p}} \cdot \sqrt{q(1-q)} \cdot \rho(Y,X). \end{aligned}$$

The correlation ($\rho$) between $X$ and $Y$ (or $Y$ and $X$) is a measure of information revelation from $Y$ to $X$ (and vice versa). When both firms decide to invest at $t_0$ or defer their

investment till $t_1$ no information will be revealed to one of the firms, thereby $\rho$ will be 0. There are three situation possible. Both firms can decide to invest at $t_0$ or decide simultaneously to defer investment. If both firms decide to invest at the same time they will both gain equal market share, namely $\alpha = \frac{1}{2}$. The third situation is that one of the two firms moves first, while the others waits. Suppose the total market share that can be gained by the R&D investment is $X$ (thus $X$ is the current value of the project to the firm) and $Y$ is again the investment cost, but this time the investment cost for the total investment. That means that the investment cost is proportional to the market share. If investing will make firm $A$ gain a market share of $\alpha$, the investment cost will be $\alpha Y$. Let $R$ denote the first-stage investment in research and $R = \beta Y$, a fraction of the total investment cost $Y$. Another important assumption is that the firms can exercise their investment opportunity (the second stage) at anytime before maturity time $T$. This reflects the flexibility that is normal with investment opportunities. The management of a firm can decide whether it wants to invest and when. The investment opportunity can therefore be considered as an American option. Assuming that the dynamics of the value of the investment to the firm ($X$) and the investment cost ($Y$) are reasonably approximated by a geometric Brownian motion, the option pricing theory can be used to value the investment option.
We will now go through the three possible situations. In all these situation we will consider the payoff of the players at $t_0$.

### 7.1 Firm A moves first, B waits

Without loss of generality, suppose that A decides to invest in research at $t_0$ while B decides to postpone investment. As a result, A will gain a market share $\alpha \in (\frac{1}{2}, 1]$ and B a share of $1 - \alpha$. If the investment of A turns out to be successful then B's success probability will become $p^+$. Firm B still owns the (development) investment option which will cost him $(1 - \alpha)Y$ and he can execute it at any moment between $t_1$ and $T$. Investing will give him a market share $(1 - \alpha)X$. However, if B wants to undertake this investment he first has to do the first investment $R$. The option to this investment $R$ expires at $t_1$. It is a compound American exchange option with maturity date $t_1$ and exercise price $R = \beta Y$. The underlying option is the development option which can be executed at anytime between $t_1$ and $T$ and with exercise price $(1 - \alpha)Y$. The underlying option is an exchange option since one asset, the investment cost $(1-\alpha)Y$, can be exchanged for another asset, in this case a share of the total market, $(1 - \alpha)X$. In terms of the theory above, the underlying exchange option is $p^+V_S((1 - \alpha)X, (1 - \alpha)Y, T - t_1)$ and the compound option $V_{caeo}(p^+V_S((1-\alpha)X, (1-\alpha)Y, T-t_1), \beta Y, t_1)$, in which $p^+$ in the probability that the R&D investment will be successful. Firm B will only exercise the compound option if the value of the underlying option $V_S$ is larger than the exercise price $R$:

$$V_{caeo}(p^+V_S, \beta Y, t_1) = (p^+V_S((1 - \alpha)X, (1 - \alpha)Y, T - t_1)) - R)^+ \tag{50}$$

Using equation (45), the value of the compound option of firm B when he postpones his investment till $t_1$ will be

$$\begin{aligned} V_{caeo}(p^+) \simeq & \frac{4V_{pcaeo}(p^+V_{paeo}((1-\alpha)X,)1-\alpha)Y,T-t_1),\beta Y,t_1)}{3} \\ & -\frac{V_{ceeo}(p^+V_{eeo}((1-\alpha)X,(1-\alpha)Y,T),\beta Y,t_1)}{3} \end{aligned} \tag{51}$$

in case of a successful R&D investment of firm A, and

$$\begin{aligned} V_{caeo}(p^-) \simeq & \frac{4V_{pcaeo}(p^-V_{paeo}((1-\alpha)X,)1-\alpha)Y,T-t_1),\beta Y,t_1)}{3} \\ & -\frac{V_{ceeo}(p^-V_{eeo}((1-\alpha)X,(1-\alpha)Y,T),\beta Y,t_1)}{3} \end{aligned} \tag{52}$$

when the investment of A will turn out to be a failure. Since the chance of a failure of the R&D investment of firm A is (1-q) and that of a success $q$ the expected value of the option of B at $t_0$ will be:

$$\Pi_B = qV_{caeo}(p^+) + (1-q)V_{caeo}(p^-). \tag{53}$$

The payoff of firm A when he moves first will be different. He invests an amount $R$ at $t_0$ and then owns the American exchange option to undertake the development investment, $\alpha Y$. He can execute this option at anytime between $t_0$ and $T$. The payoff will therefore be

$$\begin{aligned} L_A &= -R + qV_{aeo} \\ &= -R + q\left(\frac{4V_{carr}(\alpha X, \alpha Y, T) - V_{eeo}(\alpha X, \alpha Y, T)}{3}\right). \end{aligned} \tag{54}$$

In which the expression of Armada, (40), is used to value the American exchange option $V_{aeo}$.
Now we will turn over to the second situation.

## 7.2 Firm A and B invest in R&D

In this situation both firms invest simultaneously in R&D at $t_0$. More precisely, they invest in research $R$ at $t_0$ and between $t_0$ and $T$ they can both decide to also undertake the second-stage investment. However, they can only take the development investment if the research investment was successful. As a result, both firm A and B hold the development option with respectively probability $q$ and $p$. We have already seen that in this case information revelation will not take place, thus $\rho(X,Y) = \rho(Y,X) = 0$ and therefore $p = p^+ = p^-$ and $q = q^+ = q^-$, and both firms will gain an equal share of the market, $\alpha = \frac{1}{2}$. At $t_0$ both firms have already invested $R$ and own the exchange option to the development investment (see the formula of Armada , (40)). Consequently, their payoffs will look like this:

$$
\begin{aligned}
Z_A &= -R + qV_S(\frac{1}{2}X, \frac{1}{2}Y, T) \\
&\simeq -R + q\left(\frac{4V_{carr}(\frac{1}{2}X, \frac{1}{2}Y, T) - V_{eeo}(\frac{1}{2}X, \frac{1}{2}Y, T)}{3}\right) && (55) \\
Z_B &= -R + pV_S(\frac{1}{2}X, \frac{1}{2}Y, T) \\
&\simeq -R + p\left(\frac{4V_{carr}(\frac{1}{2}X, \frac{1}{2}Y, T) - V_{eeo}(\frac{1}{2}X, \frac{1}{2}Y, T)}{3}\right) && (56)
\end{aligned}
$$

Now the final situation will be discussed.

### 7.3 Both firm A and B wait

The last possible situation is the one in which both firms decide to not invest $R$ at $t_0$ and postpone their investment decision till $t_1$. Since again both firms will invest simultaneously the market share $\alpha$ will be $\frac{1}{2}$ and $\rho(X,Y) = \rho(Y,X) = 0$. The value of the compound option at $t_0$ will therefore be:

$$W_A = V_{caeo}\big(qV_S(\frac{1}{2}X, \frac{1}{2}Y, T), \beta Y, t_1\big) \tag{57}$$

$$W_B = V_{caeo}\big(pV_S(\frac{1}{2}X, \frac{1}{2}Y, T), \beta Y, t_1\big) \tag{58}$$

### 7.4 In a matrix

The three situation with their corresponding payoffs can be put in a matrix. Both firms have two possible strategies at $t_0$, namely waiting or investing. The best strategy of a firm depends on what the other firm does and the amount of information revelation. I will discuss this in chapter 9. The matrix belonging to this game is:

Payoff Matrix Investment Game

| | | Firm B | |
|---|---|---|---|
| | | Invest | Wait |
| Firm A | Invest | $Z_A$, $Z_B$ | $L_A$, $\Pi_B$ |
| | Wait | $\Pi_A$, $L_B$ | $W_A$, $W_B$ |

Decisions of the firms are made simultaneously. The corresponding game is therefore a strategic form game. The firms are not informed about which action the other firm has taken.

# 8 The strategies

Game theory is the study of mathematical models of conflicts and cooperation between rational decision-makers. (see [24], p.1) The main idea in game theory is that rational players will choose the strategies that belong to a Nash equilibrium. If there is only one Nash equilibrium the players know what strategy to follow. However, in many cases there are multiple equilibria. In such a case, it's not immediately clear what the players should do. Then the Nash equilibria have to be observed more closely to see what is the best strategy for players. One can do this for example by introducing extra rationality criteria. In the matrix above it is not immediately clear what the best strategies for firm A and B are since the payoffs in the different situations depend on various parameters. Let's try to find the Nash equilibria of this game.

**Definition 2.** *Let A en B denote the set of possible strategies of firm A and B respectively. A pair of strategies* $(a^*, b^*)$*,* $a \in A$ *and* $b \in B$*, is a Nash equilibrium if* $a^*$ *is the best reply to* $b^*$ *and vice versa. In other words* $U_{player1}(a^*, b^*) \geq U_{player1}(a, b^*)$ *for every* $a \in A$ *and* $U_{player2}(a*, b*) \geq U_{player2}(a^*, b)$ *for every* $b \in B$*. The* $U$ *denotes the utility (or payoff) the players obtain from playing that strategy.*

In this game the sets of strategies are clear, $S_A = S_B =$\{Invest, Wait\}. The main assumption in game theory is that players are rational. (see [22], p. 4) Both firms will try to maximise the profit they can obtain given what the other firm will do. They will solve the problem $\max_{a \in S_A} U(g(a))$ and $\max_{b \in S_B} U(g(a))$ in which $g(j)$ is a function that associates a payoff (utility) with each action.

## 8.1 The influence of the project value $X$

Assume that firm B chooses to invest. Then firm A will wait if $\Pi_A > Z_A$ because of the rationality assumption. There will be a situation in which all parameters are such that $\Pi_j^* = Z_j^*$ for $j \in \{A, B\}$. In words, if one of the firms chooses to invest, the other firm will be indifferent between investing and waiting. Let $X_{za}^*$ and $X_{zb}^*$ be the corresponding value of the project to firm A en B respectively in this case. It depends on the amount of the future cashflows the firm expects to receive due to the investment. If demand of the product seems to be in a rising trent or prices go up because of scarcity then the firm may expect the future cashflows to increase. Therefore $X$ will increase.
Denoting $X_z^* = \max(X_{za}^*, X_{zb}^*)$, if $X > X_z^*$ both firms will prefer investment at $t_0$ above waiting when the other firm invests since $\Pi_j < Z_j$. In the situation of decreasing demand or prices, $X$ will be smaller than $X_k^* = \min(X_{za}^*, X_{zb}^*)$, both firms will prefer waiting, since in that case $\Pi_j > Z_j$. Thus in case $X > X_z^*$ there will be at least one Nash equilibrium, namely (Invest, Invest), since none of the firms will then have an incentive to change his strategy.
Following the same line of reasoning there will exist $X_{wa}^*$ and $X_{wb}^*$ for which both firms are indifferent between investing and waiting if the other firm decides to wait, $L_j = W_j$ for $j \in \{A, B\}$. Let the maximum of the two project values be denoted by $X_r^* = \max(X_{wa}^*, X_{wb}^*)$ and $X_q^* = \min(X_{wa}^*, X_{wb}^*)$. If $X > X_r^*$ it will be rational for both players to invest if the other firm decides to wait since $L_j > W_j$.
Now suppose $X > \max(X_r^*, X_z^*)$. Then there will be one Nash equilibria:

| | | Firm B: Invest | Firm B: Wait |
|---|---|---|---|
| Firm A | Invest | $Z_A^*$, $Z_B^*$ | $L_A^*$, $\Pi_B$ |
| Firm A | Wait | $\Pi_A$, $L_B^*$ | $W_A$, $W_B$ |

The stars (*) show the rational reactions of the firms to the possible actions of the other firm. Only in the case of (Invest, Invest) the actions are the best replies to the action of the other firm and consequently it is a Nash equilibrium, see Definition 2. In the situation that $X < \min(X_k^*, X_q^*)$, $L_j < W_j$ and $\Pi_j > Z_j$. We will get the following matrix:

| | | Firm B | |
|---|---|---|---|
| | | Invest | Wait |
| Firm A | Invest | $Z_A$, $Z_B$ | $L_A$, $\Pi_B^*$ |
| | Wait | $\Pi_A^*$, $L_B$ | $W_A^*$, $W_B^*$ |

In this case there is again one Nash equilibrium, namely (Wait, Wait).
Now assume that $X_r^* < X_k^*$, implying $X_{wj} < X_{zj}$. If the value of the project $X$ is smaller than $X_q^*$ then $\Pi_j > Z_j$ and $W_j > L_j$. This is the situation described above. The corresponding Nash equilibrium is (Wait, Wait). If $X \in (X_q^*, X_r^*)$ then $\Pi_j > Z_j$ for all $j$, and $W_j < L_j$ in which $j$ is the firm that has the lowest critical value $X_{wj}^*$ and $W_l > L_l$ with $l$ being the other firm. Then the Nash equilibrium will also be (Wait, Invest) if $X_{wa}^* > X_{wb}^*$ and (Invest, Wait) if $X_{wa}^* < X_{wb}$.
The third possibility is that $X \in (X_r^*, X_k^*)$. Then investing is the best reaction for both firms if the other firm waits ($\Pi_j > Z_j$). However, if one firm decides to invest, the best reply for the other firm is to wait since $W_j < L_j$. Thus when the value of the project falls within this range there will be two pure strategy Nash equilibria, (Wait, Invest) and (Invest, Wait). In this situation there exists, apart from the two pure Nash equilibria (Wait, Invest) and (Invest, Wait), a third Nash equilibrium. It's Nash equilibrium in mixed strategies. A mixed strategy is an assignment of a probability to each pure strategy. Imagine that player A invests if the coin lands heads and waits if the coin lands tails. Then player A is considered to play a mixed strategy. A mixed strategy Nash equilibrium is an equilibrium where at least one player is playing a mixed strategy. Consider the strategy (q, 1-q) of firm B. He will invest with probability $q$ and wait with probability $1 - q$. The best reply of firm A to this strategy is invest respectively wait if:

$$0 \cdot q + 7(1-q) > 2q + 6(1-q)$$
$$0 \cdot q + 7(1-q) < 2q + 6(1-q).$$

Thus 'invest' is best reply if $q < \frac{1}{3}$ and 'wait' is best reply if $q > \frac{1}{3}$. They are both best replies when $q = \frac{1}{3}$. The same holds for firm B. Consider the strategy (p, 1-p) of firm A. Then waiting is the best reply for firm B if $q > \frac{1}{3}$ and investing if $q < \frac{1}{3}$. Again both replies are best replies in the case $q$ is equal to $\frac{1}{3}$. Therefore, there is a mixed strategy Nash equilibrium where both firms invest by probability $\frac{1}{3}$. We won't consider the case of mixed strategies further here since it is not very relevant in the case of real options. It's unlikely that a firm will use probabilities in defining its strategy in investments. The management of a firm will not make its investment decisions dependent on throwing a coin. (see [23],

p. 7)

If $X$ increases even more, one of the equilibria (Invest, Wait) and (Wait, Invest) will disappear since waiting is in that case not always the best reply for the firms when the other firm invests. The only equilibrium left then is (Wait, Invest) or (Invest, Wait), again depending on whether firm A or firm B has the highest critical value $X^*_{wj}$. When $X$ rises above the last critical value $X^*_z$ it will be bigger than all the critical values. For both firms investing will therefore be the best reply when the other firm invest but also if the other firm decides to wait. Investing is a dominant strategy for both firms. The Nash equilibrium in this case is (Invest, Invest).

### 8.2 The effect of information revelation

As I mentioned before, more information revelation (higher $\rho$) will lead to a higher success probability $p^+$ and thus to higher payoffs if the R&D investment of the first mover was successful. However, in case of failure a higher level of information revelation will lead to a lower success probability $p^-$ for the firm that moves second. To see this, remember the success probabilities were:

$$
\begin{aligned}
p^+ &= p + \sqrt{\frac{1-q}{q}} \cdot \sqrt{p(1-p)} \cdot \rho(X,Y) \\
p^- &= p - \sqrt{\frac{q}{1-q}} \cdot \sqrt{p(1-p)} \cdot \rho(X,Y) \\
q^+ &= q + \sqrt{\frac{1-p}{p}} \cdot \sqrt{q(1-q)} \cdot \rho(Y,X) \\
q^- &= q - \sqrt{\frac{p}{1-p}} \cdot \sqrt{q(1-q)} \cdot \rho(Y,X)
\end{aligned}
$$

where $q$ and $p$ are the success probabilities of firm $A$ and $B$ respectively. Remember, the correlation between $X$ and $Y$, $\rho(X,Y)$, is a measure of information revelation from $Y$ to $X$. For simplicity, assume that the information revelation from the investment of firm A to B is the same as to the investment of firm B to firm A, meaning $\rho(X,Y) = \rho(Y,X)$. There are some restrictions on $\rho(X,Y)$. Since $p^+$ and $p^-$ are probabilities $0 \leq p^- \leq 1$ and $0 \leq p^+ \leq 1$. That implies that $\rho(X,Y)$ must be such that:

$$
\begin{aligned}
p^+ &= p + \frac{\sqrt{1-q}}{\sqrt{q}} \sqrt{p(1-p)} \cdot \rho(X,Y) \leq 1 \\
p^- &= p + \frac{\sqrt{q}}{\sqrt{1-q}} \sqrt{p(1-p)} \cdot \rho(X,Y) \geq 0.
\end{aligned}
$$

These two restrictions imply that

$$\rho(X,Y) = \min\left[\frac{\sqrt{p(1-q)}}{\sqrt{q(1-p)}}, \frac{\sqrt{q(1-p)}}{\sqrt{p(1-q)}}\right]. \tag{59}$$

The correlation between $X$ and $Y$ represents the information revelation. Consequently, information revelation will therefore only have an effect on the Nash equilibria if one firm decides to invest at $t_0$ while the other decides to postpone its investment. This means that in the case above information revelation will only affect $\Pi_i$ with $i \in \{A, B\}$, the payoff of the firm moving second. An increase in information revelation means an increase in $\Pi_i$ and as a result an increase in $X^*_{za}$ and $X^*_{zb}$, which means an increase in both $X^*_z$ and $X^*_k$. This in turn will enlarge the ranges $(X^*_r, X^*_k)$ and $(X^*_k, X^*_z)$, which have two and one Nash equilibrium respectively as we have seen before. The reason that also the latter interval enlarges is due to the fact that $X^*_z$ will increase faster compared to $X^*_k$ as $\rho(X,Y)$ increases. An increase in $X^*_z$ implies that the value of the project to the firms $X$ has to be higher in order to achieve the Nash equilibrium (invest, invest). This makes sense, since an increase in information revelation makes waiting more attractive compared to investing at $t_0$. If the first-stage investment R of one firm reveals a lot of useful information for the follower the success probability of the second firm will increase.

## 8.3 The effect of first mover advantage

In the previous section one could see that an increase in $\rho(X,Y)$ makes waiting more attractive for both firms. The effect of an increase in $\alpha$ will have the opposite effect. Nevertheless, $\alpha$ effects all the payoffs. As a result, not only $X^*_k$ and $X^*_z$ will decrease when $\alpha$ becomes larger, but also $X^*_q$ and $X^*_r$ will be affected negatively. Overall, investing will become more attractive for both firms. The payoff to the leader will increase and the payoff to the follower will decrease.

# 9 Conclusion

Despite the differences in nature of financial and real option, the theory of option pricing is found to be useful for evaluating the value of an R&D investment option. By considering exchange options the value of flexibility of time is taken into account, since the value of an investment will not be constant over time. By considering compound options, we take into account the different stages in R&D. Moreover, game theory can be used by firms to evaluate the competitive interactions. The decisions taken by the players of the R&D investment game is a decisions between strategic positioning and knowledge building. Making decisions fast means that you are a step ahead of your competitors resulting in a larger market share. On the other hand, patience can also be worth a lot. An increase in the information revelation between the first and second mover increases the success probability of the second firm, making waiting a better action. However, this positive effect only appears in the case of a successful research investment by the firm going first. We have also seen that an increase in the market share that can be gained, indeed makes investing first a better action.
The critical values $X_q^*$, $X_r^*$, $X_k^*$ and $X_z^*$ can help the firm evaluating its best strategy. These values can be used to determine the ranges. When $X < X_q^*$ we found one Nash equilibrium, namely (Wait, Wait). When $X \in (X_q^*, X_r^*)$ the optimal Nash strategy is waiting for the firm with the highest critical value of the project $X_{zj}^*$ with $j \in \{a, b\}$. That means that the firm with a higher success probability invests earlier in R&D then the other firm. In this interval an increase in information revelation won't have any effect since in this interval the value of the project is below the critical values $X_{wa}^*$ and $X_{wb}^*$ anyways; firms already prefer waiting above investing when the other firm invests.
When $X \in (X_r^*, X_k^*)$ investing is best for the firm if the other firm waits and vice versa. An increase in information revelation will increase this interval. The critical market values go up. When $X \in (X_k^*, X_z^*)$ there is again one Nash equilibrium. The firm with a higher success probability invests earlier in R&D then the other firm. However, now information revelation does have an effect. An increase in information revelation will enlarge this range. It increases the critical project values for investing of both firm since information revelation makes waiting more attractive. If the information revelation is higher for one firm this might increase the critical value such that it exceeds the critical value of the other firm, making waiting more attractive. The Nash equilibrium strategy will switch from investing to waiting for that firm and from waiting to investing for the other firm. If the success probability of firm A is higher than B for example, then investing will be the Nash policy for firm A and waiting for firm B.
The effect of the size of the market share is opposite to that of information revelation. An increase in the market share decreases all the critical values of both firms. A growth in the first mover's advantages results in an increase in the value of the firm going first and a decrease in the value of the firm going second. Evidently, in the case of $\alpha = 1$, the value of

the option to the second firm is zero. Instead, an increase in information revelation results in an increase in the value of the follower, but it doesn't affect the value of the firm going first.

Firms can use the option game theory to help making their managerial decisions. Monte Carlo simulations can be used to evaluate an R&D compound exchange option. In further studies, it might be interesting to investigate the advantages to both firms of cooperating in join ventures. They can cooperate in research to save costs and benefit both from it. Information revelation will then not be of any relevance. Another interesting point to look at is the effect of the height of the 'dividends' of the project. The higher the 'dividends' of the project, the higher the opportunity cost of deferring investment.

## References


[1] F. Black, M. Scholes, The pricing of options and corporate liabilities, Journal of Political Economy, Vol. 81, No. 3 (May - Jun 1973), p. 637-654.

[2] G. Villani, An R&D investment game under uncertainty in real option analysis, Computational economics, Vol. 32, 2008, p. 199-219.

[3] J. van der Hoek, R.J. Elliot, Binomial models in Finance, Springer: New York, 2006.

[4] J. Franke, W.K. Härdle, C.M. Hafner, Statistics of Financial Markets, Springer: New York, 2015.

[5] V. Capasso, D. Bakstein, An introduction to continuous-time stochastic processes, Springer: New York, 2015, Imprint: Birkhäuser.

[6] F. Delbaen, W. Schachermayer, Mathematics of arbitrage, Springer: Berlin, 2008.

[7] S. E. Shreve, Stochastic Calculus for Finance I, Springer: New York, 2004.

[8] T.K.K. Kamanu, Continuous Time Limit of the Binomial Model, African Institute for Mathematical Sciences 2003/04.

[9] R.J. Elliot, Binomial Models in Finance, Springer: New York, 2006.

[10] T. Copeland, V. Antikarov, Real options, New York: Thomson Texere, 2003.

[11] J.C. Cox, S.A. Ross, M. Rubinstein, Option pricing: A simplified approach, Journal of Financial Economics, Vol. 7, September 1979, p. 229-263.

[12] Y. Kwok, Mathematical model of Financial Derivatives, Springer: Berlin, 2008.

[13] C. Chiarella, X. He, C.S. Nikitopoulos, Derivative security pricing, Springer: New York, 2015.

[14] R.S. Pindyck, Investment of uncertain cost, Journal of Financial Economics, Vol. 34, No.1, August 1993, p. 53-76.

[15] W. Margrabe, The Value of an Option to Exchange One Asset for Another, The Journal of Finance, Vol 33 (June) No.1, p. 177-186.

[16] P. Carr, The valuation of American exchange options with application to real options. In L. Trigeorgis (ed.), Real Options in Capital Investment, Westport, CT: Praeger, p. 109-120.

[17] R. Geske, The valuation of compound options, Journal of Financial Economics, Vol. 7, No.1, March 1979, p. 63-81.

[18] R. L. McDonald, D.R. Siegel, Investment and the Valuation of Firms when there is an Option to Shut Down, International Economic Review, Vol. 26 (June), p. 331-349.

[19] M.R. Armada, L. Kryzanowski, P.J. Pereira, A modified finite-lived American exchange option methodology applied to real options valuation, Global Finance Journal, Vol. 17, No.3, March 2007, p. 419-438.

[20] D.A. Paxson, Real R&D options, Butterworth Heinemann: Oxford, 2003.

[21] G. Villani, A Strategic R&D investment with flexible development time in real option game analysis,CESifo working paper no. 2728, Venice, 2009.

[22] M.J. Osborne, A. Rubinstein, A course in game theory, Cambridge: The Mit Press, 1996.

[23] H. Peters, Speltheorie voor economen, University of Maastricht: 2007.

[24] R. B. Myerson, Game Theory: Analysis of Conflict, Harvard University Press, 1991.