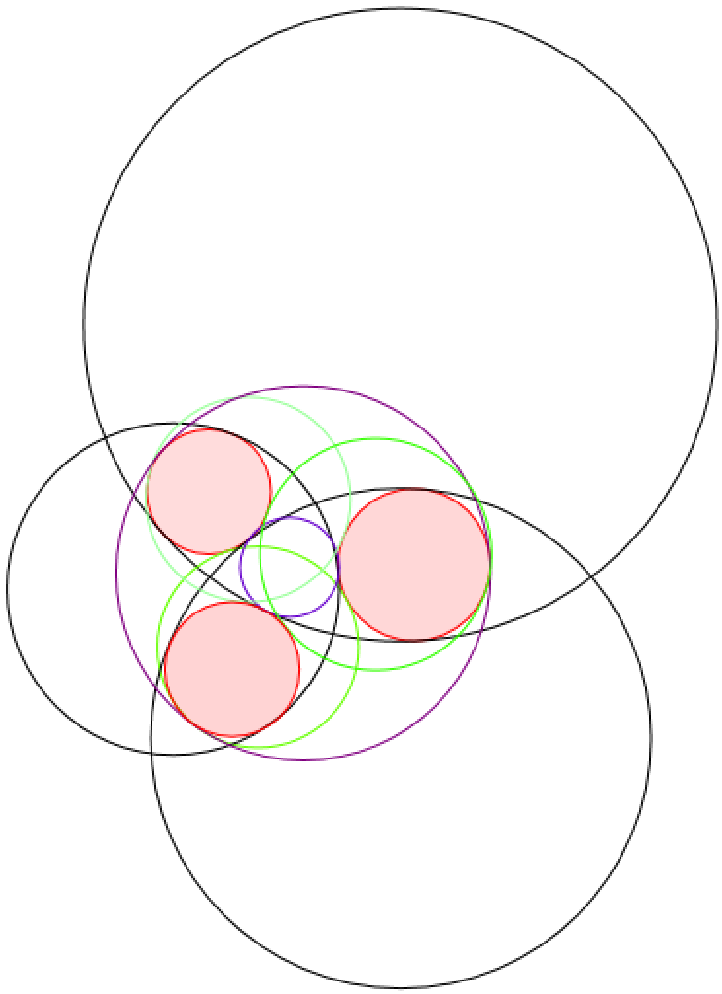
Universiteit Utrecht

Bachelor thesis

Enumerative geometry of planar conics



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Abstract

In this thesis three problems from enumerative geometry will be solved. The first problem, Apollonius circles, which states how many circles are tangent to three given circles will be solved just using some polynomials and some algebra. It turns out that there are, with reasonable assumptions, eight of those. The second problem deals with how many lines intersect with four given lines in $\mathbb{P}^3_{\mathbb{C}}$. In general, there are two of those. The last problem, how many conics are tangent to five given conics, is solved using blowups, the dual space, Buchberger's algorithm and the Chow group. It turns out that there are 3264 conics which satisfy this solution. Furthermore, it is easy to extend this solution to find how many conics are tangent to c conics, l lines and pass through p points where $p + l + c = 5$ and these points, lines and conics satisfy reasonable assumptions.

Acknowledgements

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Chapter 0

Introduction

In this thesis, we will consider some problems which have to do with enumerative geometry. Enumerative geometry is a branch of mathematics which was mostly developed during the end of the 19th century. In this branch of algebraic geometry all the problems have to do with counting how many objects there are that satisfy certain conditions. Take for example the problem of Apollonius, which was known long before the 19th century, and therefore it is one of the first problems from enumerative geometry known. The problem states that given three circles in $\mathbb{R}^2$ whose centres aren't colinear, how many circles are there that are tangent to these circles. In chapter 2, this problem will be solved. Chapter 1 consists of some definitions and propositions that are needed throughout this thesis.
Chapter 3 will solve the problem: given 4 general lines, how many lines can you find that intersect all these four lines.

The last three chapters will deal with conics. In chapter 4 we will look at what conics exactly are and how many conics pass through p points and l lines where $p + l = 5$. In chapter 6 we will look at how many conics are tangent to 5 given conics, but the theory that is needed for this will be explained in chapter 5. With these solving methods any problem that states how many conics pass through p points and are tangent to l lines and c conics where $p + l + c = 5$ can be solved.

Chapter 1

Some basic definitions

In this chapter, some definitions and propositions are stated that are used throughout this thesis.

In enumerative geometry, many of the problems are stated in the complex projective space. The complex numbers are to make sure that every equation has at least one solution and the projective space is to make sure that any two lines intersect. Recall:

Definition 1.1. The *projective space* of dimension n over a field K is the set

$$\{[k_0, \ldots, k_n] | k_i \in K\} / \sim,$$

where $\sim$ is the equivalence relation $[k_0, \ldots, k_n] \sim [\lambda k_0, \ldots, \lambda k_n]$ for $\lambda \neq 0$. This set is denoted by $\mathbb{P}^n_K$. Most of the time, it will be over $\mathbb{C}$, then it can also be denoted by $\mathbb{P}^n$.

Any element $x \in \mathbb{P}^n$, we will denote by $[X_0, \ldots, X_n]$. These coordinates are called homogeneous coordinates.

Definition 1.2. A *homogeneous polynomial* of degree d is any polynomial $F : K^n \to K$ such that $F(\lambda k_1, \ldots, \lambda k_n) = \lambda^d F(k_1, \ldots, k_n)$.

Definition 1.3. The *common zero locus* or *zero locus* of a collection of polynomials $\{f_1, \ldots, f_r\}$, where $f_i \in K[X_0, \ldots, X_n]$ for a field K for every i, is the set

$$\{p \in K^n | f_1(p) = \cdots = f_r(p) = 0\}.$$

Remark 1.4. A polynomial $F \in K[Z_0, \ldots, Z_n]$ does not define a function on $\mathbb{P}^n_K$. However, it does make sense to talk about the zero locus of homogeneous polynomials, since for X in the zero locus of F, we have $F[\lambda X_0, \ldots, \lambda X_n] = \lambda^d F[X_0, \ldots, X_n] = \lambda^d \cdot 0 = 0$, hence the zero locus is well-defined.

Definition 1.5. A *projective variety* $X \subset \mathbb{P}^n_K$ is the zero locus of a collection of homogeneous polynomials.

When you have a projective variety, it can be usefull to be able to do some operations on it. It is easier to project it first onto $\mathbb{C}^n$, and then do the operations you want. For doing this, we must define charts, atlasses and transition maps as in the course Differential Manifolds [3].

Definition 1.6. Let X be a topological space. A *chart* (U, ϕ) for X is an open subset $U \subset X$, together with a map $\phi : U \to \mathbb{R}^n$, such that for $\tilde{U} = \mathrm{Im}\phi$, $\tilde{U} \subset \mathbb{R}^n$ is open and ϕ is a homeomorphism between U and $\tilde{U}$. The dimension of the chart is n.

Definition 1.7. Let (U, ϕ) and (V, ψ) be two charts for a topological space X such that $U \cap V \neq \emptyset$. Then we define the *transition map* as $\psi \circ \phi^{-1} : \phi(U \cap V) \to \mathbb{R}^n$. Two charts are compatible if either $U \cap V = \emptyset$ or the transition map is smooth, i.e. C^∞.

Note that the transition map goes from $\mathbb{R}^n$ to $\mathbb{R}^n$ but the domain is resticted such that it is well-defined.

Definition 1.8. An *atlas* of a manifold M is a collection of charts $\{(U_i, \phi_i)\}$ such that $\{U_i\}$ forms an open cover and each of the charts are smoothly compatible with each other.

Definition 1.9. A *differential manifold* is a topological space that is locally Euclidean, Hausdoff and second countable, together with an atlas.

Besides all these definitions, we would like to know when a variety is smooth.

Definition 1.10. Let $X \subset \mathbb{P}^n$ be a projective variety, and let $F_1, \ldots, F_r \in \mathbb{P}[X_0, \ldots, X_n]$ be its homogeneous defining functions. X is called *singular* in a point p if $\frac{\partial F_i}{\partial X_j}|_p = 0$ for all $1 \leq i \leq r$ and $0 \leq j \leq n$. X is called *smooth* at p if it is not singular.

Remark 1.11. Note that we can identify $\mathbb{C}$ with $\mathbb{R}^2$. When we replace smooth (i.e. C^∞) by holomorphic, we can use the same theory using $\mathbb{C}^n$.

Now it is time for the first example:

Example 1.12. Let $X \subset \mathbb{P}^3$ be the variety defined by $X_0^3 = X_1^2 X_2$. Then X is the zero locus of the function $F : \mathbb{P}^3 \to \mathbb{C}$ $F[X_0, X_1, X_2] = X_0^3 - X_1^2 X_2$ (notice that this function is homogeneous, hence we can talk about the zero locus).

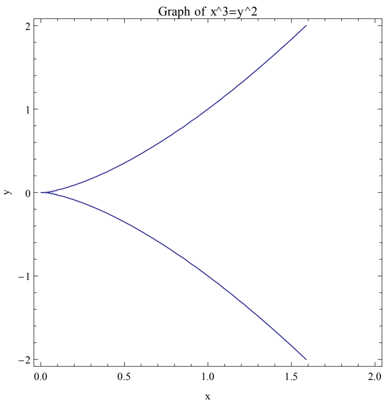
Now we look at the partial derivatives of F in $[0, 0, 1]$. We see

$$\frac{\partial F}{\partial X_0}|_{[0,0,1]} = 3X_0^3|_{[0,0,1]} = 0$$
$$\frac{\partial F}{\partial X_1}|_{[0,0,1]} = 2X_1X_2|_{[0,0,1]} = 0$$
$$\frac{\partial F}{\partial X_2}|_{[0,0,1]} = X_1^2|_{[0,0,1]} = 0$$

It follows that X has a singular point at the point $p = [0, 0, 1]$.

Note that we can also use charts to draw this curve in $\mathbb{C}^2$ (where we only draw the real part of this space to make it sit in $\mathbb{R}^2$) as follows.

Define charts $U_0 = \{[X_0, X_1, X_2] | X_0 \neq 0\}$ and $\phi_0 : U_0 \to \mathbb{C}^2$ where $\phi_0([X_0, X_1, X_2]) = (\frac{X_1}{X_0}, \frac{X_2}{X_0})$ with inverse $(x, y) \mapsto [1, x, y]$ and define analogously (U_1, ϕ_1) and (U_2, ϕ_2).
The transition map $\phi_0 \circ \phi_1^{-1} : \mathbb{C}^2 \to \mathbb{C}^2$ defined on the intersection $\{(x, y) \in \mathbb{C}^2 | x \neq 0\}$ is $(x, y) \xmapsto{\phi_1^{-1}} [x, 1, y] \xmapsto{\phi_0} (\frac{1}{x}, \frac{y}{x})$. Clearly, this function is continious for $x \neq 0$. Similar equations hold for the other transistionmaps, hence we've found an atlass.
Now we have the function $F \circ \phi_2^{-1} : \mathbb{C}^2 \to \mathbb{C}, \quad (x, y) \mapsto x^3 - y^2$.



In this picture, you can see the singularity at the point $(0, 0)$, which came from the point $[0, 0, 1]$.

A few more definitions are usefull. These definitions will be about dimensions of subspaces.

Definition 1.13. A *affine subspace* of $\mathbb{C}^n$ of dimension m is a space $\{a+v | v \in V\}$ where V is a linear subspace of $\mathbb{C}^n$ of dimension m and $a \in \mathbb{C}^n$.

Definition 1.14. The *dimension* of a linear subspace in $\mathbb{P}^n$ is defined to be the dimension of the corresponding affine subspace in $\mathbb{C}^n$.

Definition 1.15. The *codimension* of a subspace $X \subset \mathbb{P}^n$ equals $n - \dim(X)$. It is denoted by $\mathrm{codim}(X)$.

Note that if two subspaces of codimension k and l are intersected, the intersection will generically have codimension $k + l$.

Definition 1.16. Define a *hypersurface* of the space $\mathbb{P}^n$ a subspace of codimension 1.

Furthermore, we'd like to use Bézout's theorem, which is as follows ([7] page 227):

Theorem 1.17. *Let there be n homogeneous polynomials in $n + 1$ variables of degrees $d_1, \ldots, d_n$ that define n hypersurfaces in $\mathbb{P}^n$. If the number of intersectionpoints is finite, then this number is $d_1 \cdot d_2 \cdots \cdot d_n$ where the points are counted with multiplicity. If the hypersurfaces are irreducible and in general position, then every point in the intersection has multiplicity 1 and hence there are exactly $d_1 \cdot d_2 \cdots \cdot d_n$ points in the intersection.*

If the hypersurfaces satisfy the second condition, i.e. they are irreducible and in general position, they are said to intersect transverse.
This theorem won't be proved in this thesis.

Chapter 2

Problem of Apollonius

In this chapter, we will prove the problem of Apollonius.

Theorem 2.1. *Let 3 circles $C_i \subset \mathbb{R}^2$ in general position be given. Then there are at most 8 circles tangent to the given circles.*

Remark 2.2. I will not go into details about what general in this context means. It contains for example that the centers of the circles are not colinear, but it also makes sure that no circle is contained in another circle. To find out which conditions the circles exactly should satisfy, you can check all equations in the proof and put conditions on the circles such that all the terms in x_i, y_i and r_i do not cancel.

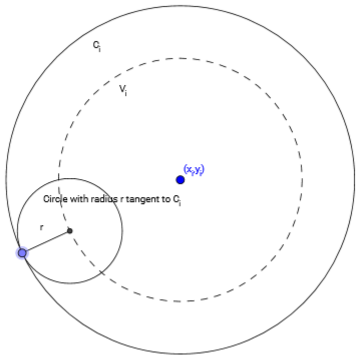
Proof. A circle in $\mathbb{R}^2$ is uniquely defined by its centre and radius. Hence we can denote the space of circles by

$$\mathcal{C} = \{(x, y, r) \in \mathbb{R}^2 \times \mathbb{R}_{>0}\}$$

Now for a given circle $C_i = (x_i, y_i, r_i)$, the set

$$V_i^{\text{in}} = \{(x, y, r) \in \mathcal{C} | (x - x_i)^2 + (y - y_i)^2 = (r - r_i)^2\}$$

is exactly the set of circles which are tangent to C_i from the inside. This can be seen by first fixing r. Then all the circles that are tangent form a circle of radius $r_i - r$ if $r < r_i$ and $r - r_i$ if $r > r_i$. This yields exactly this expression.



Similarly, the set

$$V_i^{\text{out}} = \{(x, y, r \in \mathcal{C} | (x - x_i)^2 + (y - y_i)^2 = (r + r_i)^2\}$$

is the set of circles who are tangent to C_i from the outside.
Now we can see that $\mathcal{C}$ in fact the set $\mathbb{R}^2 \times \mathbb{R}_{>0}$ is, where we have a bijection between the circles and this space.
We observe that V_i^{in} and V_i^{out} now define a cone in $\mathcal{C}$.

To show that there are exactly 8 circles, we have to show that there is exactly one point in the intersection $V_1^\alpha \cap V_2^\beta \cap V_3^\gamma$ for every choice of $\alpha, \beta, \gamma \in \{\text{in}, \text{out}\}$.
This I will not do by looking at the cones, because Bézout tells us that for every choice of α, β and γ there will be 8 possibilities, but that's with multiplicity. Note that Bézout is a theorem over $\mathbb{C}$, but we can identify $\mathbb{R}^2$ with $\mathbb{C}$ and then apply Bézout. Therefore, we will just look at the equations we have.
Let $s_1, s_2, s_3 = \pm 1$, depending on the choice of α, β and γ. Then we get:

$$\begin{aligned}(x-x_1)^2 + (y-y_1)^2 &= (r - s_1 r_1)^2 \\ (x-x_2)^2 + (y-y_2)^2 &= (r - s_2 r_2)^2 \\ (x-x_3)^2 + (y-y_3)^2 &= (r - s_3 r_3)^2 \end{aligned} \tag{2.1}$$

Writing this differently yields:

$$x^2 - 2x_1 x + x_1^2 + y^2 - 2y_1 y + y_1^2 - r^2 + 2s_1 r_1 r - r_1^2 = 0 \tag{2.2}$$
$$x^2 - 2x_2 x + x_2^2 + y^2 - 2y_2 y + y_2^2 - r^2 + 2s_2 r_2 r - r_2^2 = 0 \tag{2.3}$$
$$x^2 - 2x_3 x + x_3^2 + y^2 - 2y_3 y + y_3^2 - r^2 + 2s_3 r_3 r - r_3^2 = 0 \tag{2.4}$$

Now subtracting (2.3) from (2.2) yields a linear equation in x and y:

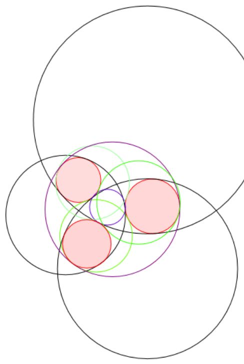
$$2(x_2 - x_1)x + x_1^2 - x_2^2 + 2(y_2 - y_1)y + y_1^2 - y_2^2 + 2(s_1 r_1 - s_2 r_2)r - r_1^2 + r_2^2 = 0$$

Using (2.4) as well, we can get seperate equations for x and y:

$$\begin{aligned} x &= f_1 + f_2 r \\ y &= f_3 + f_4 r \end{aligned}$$

where f_1, f_2, f_3 and f_4 are known functions in terms of $x_1, x_2, x_3, y_1, y_2, y_3, r_1, r_2, r_3, s_1, s_2$ and s_3.
Now we can substitute these functions back in (2.1) and solve this for r. Since these are quadratic equations, we expect to get 16 solutions in total, for every choice 2.
Suppose that (x, y, r) is a solution for some choice of α, β, γ. Then $(x, y, -r)$ is a solution for the α', β', γ' where α' is exactly the different element in $\{\text{in}, \text{out}\}$ than α (i.e. $\alpha' \in \{\text{in}, \text{out}\} - \{\alpha\}$ and similarly for β' and γ'. We see that every solution is counted at least twice, hence we can discard all the circles with radius < 0 (which isn't a circle). We have at most 8 circles left, which are the solutions to the problem.
If furthermore $r > 0$, we get exactly 8 solutions (note that this is a condition which is taken care of by the general position).

In the picture you can see that we can find at least 8 circles tangent to the given (red) circles.



$\square$

Chapter 3

Lines through given lines

In this chapter another quite intuitive problem is stated. When three lines in $\mathbb{R}^3$ are given, no two of which lie in a plane, it is quite easy to see that there are many lines that intersect all these lines. Take for example an arbitrary point on the first line. Then you can make a plane with this point and the second line. By assumption the third line doesn't lie in this plane, so it follows that, unless the line is parallel to this plane, we always have a point of intersection. However, we now give a fourth line, and the question remains whether there are lines left that intersect all these 4 given lines. To make sure that any two lines in a plane intersect (and parallel doesn't exist) we will work in $\mathbb{P}^3_{\mathbb{C}}$.

Theorem 3.1. *Let 4 lines $L_1, L_2, L_3, L_4 \subset \mathbb{P}^3_{\mathbb{C}}$ in general position be given. Then there are exactly two lines through L_1, L_2, L_3 and L_4.*

Remark 3.2. In this case *in general position* means that no two lines are in a plane, i.e. no two lines intersect.

Proof. Claim: We can do a basis transformation such that L_1 is of the form $[0, 0, p, q]$, L_2 is of the form $[p, q, 0, 0]$ and L_3 is of the form $[p, q, p, q]$, where $p, q \in \mathbb{C}$ are variables.

Proof of the claim: Notice that we can identify $L_i \subset \mathbb{P}^3$ with a plane $V_i \subset \mathbb{C}^4$ for $i = 1, 2, 3, 4$. Since the lines are in general position, it follows that V_i and V_j together span $\mathbb{C}^4$, i.e. $V_i \oplus V_j = \mathbb{C}^4$ for $i \neq j$.

Now we can choose the basis for $\mathbb{C}^4$ such that $(1, 0, 0, 0)^T$ and $(0, 1, 0, 0)^T$ are in V_1 and $(0, 0, 1, 0)^T$ and $(0, 0, 0, 1)^T$ are in V_2.

When we translate this basis back to $\mathbb{P}^3$, we see that we have met the conditions for L_1 and L_2.

Now V_3 is spanned by some vectors $v = (v_1, v_2, v_3, v_4)^T$ and $w = (w_1, w_2, w_3, w_4)^T$. Now I claim that the matrix

$$M_1 = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$$

is invertible.

But we know that V_3 and V_2 together span $\mathbb{C}^4$, hence the vectors $v, w, (0, 0, 1, 0)^T, (0, 0, 0, 1)^T$ are a basis. Therefore it follows that M_1 is invertible, with inverse A_1.

Similarly, the matrix

$$M_2 = \begin{pmatrix} v_3 & w_3 \\ v_4 & w_4 \end{pmatrix}$$

is invertible with inverse A_2. Now we define the matrix

$$A = \begin{pmatrix} a_1^{11} & a_1^{12} & 0 & 0 \\ a_1^{21} & a_1^{22} & 0 & 0 \\ 0 & 0 & a_2^{11} & a_2^{12} \\ 0 & 0 & a_2^{21} & a_2^{22} \end{pmatrix}$$

where a_1^{ij} is the element of matrix A_1 at postition (i,j) and similarly for a_2^{ij}.
Now it follows that

$$A\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \\ v_4 & w_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

hence A gives a basis transformation for v and w to the vectors we asked for in the beginning (when we translate this back to $\mathbb{P}^3$).

Note that when we do this basis transformation, we can keep the vectors $(1,0,0,0)^T$ and $(0,1,0,0)^T$ or $(0,0,1,0)^T$ and $(0,0,0,1)^T$ fixed since we only use the basisvectors v and w.

Using this basis transformation, we will construct a line through all of these lines.

Therefore, choose a point $p \in L_1$. We know that p is of the form $p = [0,0,p_2,p_3]$. Furthermore, L_2 is of the form $[x_0,x_1,0,0] = \lambda[1,0,0,0] + \mu[0,1,0,0]$, since x_0 and x_1 are variables. Now the plane through p and L_2 is of the form $V = \alpha p + \beta[x_0,x_1,0,0] = [\lambda,\mu,\alpha p_2,\alpha p_3]$, where α,λ,μ are variables.
Now we will find the point where L_3 and V cross. Therefore we see, writing $L_3 = [\gamma,\delta,\gamma,\delta]$, where γ,δ are variables:

$$[\gamma,\delta,\gamma,\delta] = [\lambda,\mu,\alpha p_2,\alpha p_3]$$

Now we first assume $p_2 \neq 0$. Then for some $l \in \mathbb{C}^*$, we have $\lambda = l\gamma$ and $\delta = l\mu$ and $\alpha = l\frac{\gamma}{p_2}$. From this it follows that $\mu = l\delta = \alpha p_3 = l\gamma\frac{p_3}{p_2}$.
Hence the point of intersection q looks like $q = [l\gamma, l\gamma\frac{p_3}{p_2}, l\gamma, l\gamma\frac{p_3}{p_2}] = [\gamma,\gamma\frac{p_3}{p_2},\gamma,\gamma\frac{p_3}{p_2}]$. Since $\gamma \neq 0$, it follows that the point of intersection looks like $q = [1,\frac{p_3}{p_2},1,\frac{p_3}{p_2}] = [p_2,p_3,p_2,p_3]$. Notice that this looks the same if we chose $p_3 \neq 0$, and this last expression therefore holds for every point $p \in L_1$ we choose.
Now we can define a line through p and q:

$$L = \lambda' p + \mu' q = [\mu' p_2, \mu' p_3, (\lambda'+\mu')p_2, (\lambda'+\mu')p_3] \tag{3.1}$$

Now we let the point p vary. This gives us a quadric Q which obeys the relation $X_0X_3 - X_1X_2 = 0$.
Now we check whether Q is smooth. Therefore we define charts $U_i \subset \mathbb{P}^3$, where $U_i = \{x \in \mathbb{P}^3 | x_i \neq 0\}$ for $i = 0,1,2,3$. Also, we define a map $\pi_i : U_i \to \mathbb{C}^3$, by (if $i = 0$) $[X_0,X_1,X_2,X_3] \mapsto (\frac{X_1}{X_0},\frac{X_2}{X_0},\frac{X_3}{X_0})$ and similarly if $i = 1,2,3$. The inverse we define by

$$\begin{aligned} \pi_0^{-1} : \mathbb{C}^3 &\to U_i \\ (x_0,x_1,x_2) &\mapsto [1,x_0,x_1,x_2] \end{aligned}$$

and similarly for $i = 1,2,3$.
For one transitionmap, we check that this is a smooth chart, the other transitionmaps are analogous. We check it for $\pi_1 \circ \pi_0^{-1}$. Let $U \subset \mathbb{C}^3$ be such that $\pi_0^{-1}(U) \subset U_1$ (i.e. $U = \{(x_1,x_2,x_3) \in \mathbb{C}^3 | x_1 \neq 0\}$).

$$\pi_1 \circ \pi_0^{-1} : U \to \mathbb{C}(x_1,x_2,x_3) \mapsto [1,x_1,x_2,x_3] \mapsto (\frac{1}{x_1},\frac{x_2}{x_1},\frac{x_3}{x_1})$$

Clearly, this is a smooth map in U, hence we've defined a smooth chart.
Now we can check whether Q is smooth. We see for the subset $V_0 = Q \cap U_0$, that

$$\pi_0(V_0) = \{(\frac{x_1}{x_0},\frac{x_2}{x_0},\frac{x_3}{x_0} | x_0x_3 - x_1x_2 = 0\} = \{(\frac{x_1}{x_0},\frac{x_2}{x_0},\frac{x_3}{x_0} | \frac{x_3}{x_0} - \frac{x_1}{x_0}\frac{x_2}{x_0} = 0\}$$

Hence in $\mathbb{C}^3$ this gives the subspace $x_3 = x_1x_2$, which is a smooth subspace. Similarly for V_1, V_2 and V_3. Hence Q is smooth.

Now we check that all the lines of the form (3.1) in Q are disjoint, and that we have all possible lines intersecting L_1, L_2 and L_3.

Suppose there is a line l intersecting all these lines, but is not in Q. Then $l \cap L_1$ gives a point p, which again gives a plane V_p when we look at the span of L_2 and p. When we do the whole construction again, we get again a line, which is in Q. Hence we have all possible lines.
Now suppose that we have 2 lines of the form (3.1) $l, m \in Q$ that intersect. Notice that every point $p \in L_1$ gave rise to a unique line of the form (3.1) in Q. When two lines therefore intersect, this can not be in the point p (otherwise the lines would be the same). It follows that L_1 is in the plane spanned by l and m. Similarly, L_3 lies also in this plane (start with a point $p' \in L_3$, do the same calculation. Then you get the same lines of the form (3.1) and hence the problem is completely symmetric). It follows that L_1 and L_3 both lie in the plane spanned by l and m and this is a contradition with the assumption.

Now we can intersect the surface Q with the line L_4. We see that Q is a two-dimensional surface and L_4 a one-dimensional line. Using Bézout, we will in general have 2 points of intersections (the product of the degrees). When we choose the line of the form (3.1) in Q where this point of intersection lies. Then this line intersects all the given lines. □

Remark 3.3. I expect that the condition of the lines being in general position already makes sure that the fourth line does not intersect Q with multiplicity 2. However, I have not checked this, but we can put this condition in the conditions of the theorem.

Chapter 4

Conics

In this chapter, there will be dealt with conics. We will mainly solve some problems where much background information is not needed yet. Also, [1] is followed in this chapter.

Definition 4.1. A *plane conic curve* or *conic* is the set of points $[X, Y, Z] \in \mathbb{P}^2_{\mathbb{C}}$ that satisfy a degree two polynomial

$$aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0, \tag{4.1}$$

where not all the coefficients are 0.

Note that a conic includes circles, hyperbola's, parabola's, any pair of lines and every other shape that you can define using quadratic formulas. For any pair of lines, we see that the formula splits into 2 factors, each factor representing one of the lines. Also, when the formula splits in two factors, we know it to be two lines (which can be the same though).

Definition 4.2. A conic is called *nondegenerate* if the polynomial is irreducible, otherwise the conic is called *degenerate*. If the conic is nondegenerate or degenerate and it consists of two different lines, the conic is called *reduced*, if the two lines are the same, the conic is called a *double line*.

Now we can ask ourselves, how many reduced conics pass through p points, are tangent to l lines and are tangent to c conics, where $p+l+c=5$ (note that we want to look at reduced conics to remove all the double lines, which will sometimes give an infinite amount of conics).
For simplicity, we first set $c = 0$.

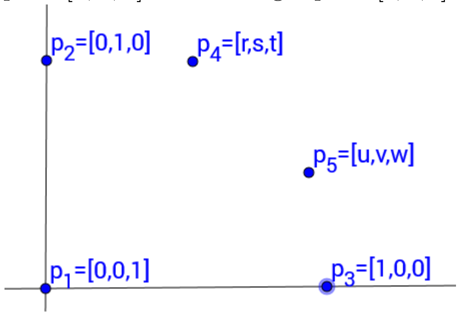
But before we're going to count these number of conics, we observe that a conic is uniquely identified by the coefficients a, b, c, d, e, f. There is only one problem, the same conic is defined by the coefficients $\lambda a, \lambda b, \ldots, \lambda f$ for $\lambda \neq 0$. The solution is that we can use homogeneous coordinates. Therefore we get a bijection between the conics and the coordinates $[a, b, \ldots, f] \in \mathbb{P}^5$.
If we impose a condition on a conic, like saying the conic should go through a point p, we get a hyperplane in $\mathbb{P}^5$.

4.1 Five points

Proposition 4.3. *Given 5 points $p_1, \ldots, p_5 \in \mathbb{P}^2$ in general position. Then there is a unique nondegenerate conic passing through these points.*

Proof. For every point p_i we get a condition on the conic, which leads to a hyperplane in $\mathbb{P}^5$ of codimension 1. Intersecting these hyperplanes, yields the conics that pass through all of these points. When the points yield linearly independent conditions, the codimension of the intersection will rise with 1 every time we intersect. Hence we remain with a linear subspace of codimension 5, hence dimension 0, which is exactly a point. So the main point to check will be whether they impose linear conditions.

Since no three points are on a line, and the question does not depend on the choice of coordinates, we can choose coordinates in $\mathbb{P}^2$ such that $p_1 = [0,0,1]$, $p_2 = [0,1,0]$ and $p_3 = [1,0,0]$. Then we get $p_4 = [r,s,t]$ and $p_5 = [u,v,w]$.



Now we see that the conic through these points should obey the following equations:

$$f = 0 \tag{4.2}$$
$$c = 0 \tag{4.3}$$
$$a = 0 \tag{4.4}$$
$$ar^2 + brs + cs^2 + drt + est + ft^2 = 0 \tag{4.5}$$
$$au^2 + buv + cv^2 + duw + evw + fw^2 = 0 \tag{4.6}$$

This can also be expressed as a matrix M, where we have $M \cdot [a,b,c,d,e,f]^T = [0,0,0,0,0]$:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ r^2 & rs & s^2 & rt & st & t^2 \\ u^2 & uv & v^2 & uw & vw & w^2 \end{pmatrix} \tag{4.7}$$

Notice that this matrix has rank < 5 if every 5x5 submatrix has determinant 0. Remember that the rank of a matrix is defined to be the number of independent rows or columns. If the rank is less than 5, there is a relation between the different rows, and hence also when you take a 5×5 submatrix. If the rank of M is actually 5, at least one 5×5 submatrix will have determinant different from 0.
Using this, we can calculate which conditions the conics through these points should obey, and also the points itself.
When we delete the 1st, 3rd or 6th colomn, we always get determinant 0. When we remove the 2nd, 4th or 5th colom, and set the determinant to be 0, we get the following conditions:

$$tw(rv - su) = 0 \tag{4.8}$$
$$su(rw - tv) = 0 \tag{4.9}$$
$$ru(sw - tv) = 0 \tag{4.10}$$

We study case by case what happens when these conditions are obeyed:

- $t = s = 0$: $p_4 = [r,s,t] = [r,0,0] = [1,0,0]$, hence p_4 and p_3 coincide.
- $t = u = 0$: The points p_2, p_3 and p_4 will be colinear, since the line through p_2 and p_3 will look like $\lambda[1,0,0] + \mu[0,1,0] = [\lambda, \mu, 0]$, and by choosing $\lambda = r$ and $\mu = s$, we see that $p_4 = [r,s,0]$ lies on this line. We assumed the points to be in general position, so it is not allowed that they are colinear.
- $t = r = 0$: p_2 and p_4 will coincide.
- $t = w = 0$: the points p_2, p_3 and p_5 will be colinear, analogous to $t = u = 0$

- $w = u = 0$: p_2 and p_5 will coincide
- $w = v = 0$: p_3 and p_5 will coincide
- $w = s = r = 0$: p_4 and p_1 will coincide

If $rv = su$, we have the following cases:

- $s = 0$, then either $r = 0$ or $v = 0$. If $r = 0$, then p_4 and p_1 will coincide
- $s = 0$ and $v = 0$, then we have $p_4 = [1, 0, 1]$, hence p_1, p_3 and p_4 will be colinear
- $u = 0$: analogous to $s = 0$
- $r = 0$, (notice that equation 2 is also satiesfied, since $sutv = rv^2t = 0$), then we have $p_4 = [0, 1, 1]$ and the points p_2, p_3 and p_4 will be colinear.
- $rw = tv$ and $sw = tv$. It follows that either $w = 0$ (p_2, p_3 and p_5 are colinear) or $s = t$, from which follows that $s = 0$ or $v = w$, from which again follows that $v = 0$ or $r = t$, and then we get $u = v$, hence p_4 and p_5 coincide

In all the cases when the matrix M failed to have maximal rank, we saw that the conditions were not satisfied. Hence the hyperplanes are linearly independent, and their intersection will be 0-dimensional. Since all the hyperplanes are linear, it follows that there is exactly 1 conic through the given 5 points.

To prove that the conic is nondegenerate, suppose it is not. Then the conic will consist of two lines. But the points are in general position, hence no three points will be on a line. Therefore we will need at least three lines to pass through all the points, but that is a contradiction. Hence the conic is nondegenerate. □

4.2 Four points and a line

Now we will show that there are exactly 2 conics that are tangent to one line and intersect 4 given points in general position. We will use Bézout's theorem.
The first claim is that the space of conics as a subspace of $\mathbb{P}^5$ tangent to a given line is a four-dimensional hypersurface of degree 2. Take for example the line $y = 0$. Then all the conics that intersect this line are given by (by substituting $y = 0$)

$$aX^2 + dXZ + fZ^2 = 0$$

for some X, Z.

Since this is a quadratic equation in X, we will in general have 2 solutions. Only when the discriminant $d^2 - 4af = 0$, the conic is tangent to the line $y = 0$. Hence all the conics that are tangent to $y = 0$ form the hypersurface $\{[a, b, c, d, e, f]|d^2 - af = 0\}$, which is indeed of degree 2 and 4 dimensional. The same calculation can be done for a general line. Let $L = \{[X, Y, Z]|AX + BY + CZ = 0\}$. Assume $A \neq 0$. Then we have $AX = -(BY + CZ)$ for points on the line. When we intersect it with a conic we get:

$$a(BY + CZ)^2 - b(BY + CZ)AY + cA^2Y^2 - d(BY + CZ)AZ + eA^2YZ + fA^2Z^2 = 0$$
$$(aB^2 - bAB + cA^2)Y^2 + (2aBC - bAC - dAB + eA^2)YZ + (aC^2 - dAC + fA^2)Z^2 = 0$$

The discriminant of this equation looks like

$$
\begin{aligned}
&(2aBC - bAC - dAB + eA^2)^2 - 4(aB^2 - bAB + cA^2)(aC^2 - dAC + fA^2)\\
&= 4a^2B^2C^2 + b^2A^2C^2 + d^2A^2B^2 + e^2A^4 - 4abABC^2 - 4adAB^2C + 4aeA^2BC\\
&\quad - 2beA^3C + 2bdA^2BC - 2deA^3B - 4a^2B^2C^2 + 4adAB^2C\\
&\quad - 4afA^2B^2 + 4abABC^2 - 4bdA^2BC + 4bfA^3B - 4acA^2C^2 + 4cdA^3C - 4cfA^4\\
&= b^2A^2C^2 + d^2A^2B^2 + e^2A^4 + 4aeA^2BC - 2beA^3C - 2bdA^2BC - 2deA^3B\\
&\quad - 4afA^2B^2 + 4bfA^3B - 4acA^2C^2 + 4cdA^3C - 4cfA^4\\
&=A^2\big((e^2 - 4cf)A^2 + (d^2 - 4af)B^2\\
&\quad +(b^2 - 4ac)C^2 + (4ae - 2bd)BC + (4bf - 2de)AB + (4cd - 2be)AC\big)
\end{aligned}
$$

For a line to be tangent to the conic, the discriminant should be 0. Since we assumed $A \neq 0$, it follows that we have the following relation for A, B and C:

$$
\begin{aligned}
(e^2 - 4cf)A^2 + (d^2 - 4af)B^2 + (b^2 - 4ac)C^2 + (4ae - 2bd)BC+\\
(4bf - 2de)AB + (4cd - 2be)AC = 0
\end{aligned}
\tag{4.11}
$$

The same result follows if $B \neq 0$ or $C \neq 0$.
Hence all the conics that are tangent to a given line form a degree 2 hypersurface and is fourdimensional.

Now applying Bézout, we see that there are exactly 2 conics that satisfy the conditions.

Actually, we do have to check that when the points and lines are in general position the multiplicity is 1 (and the intersection is transverse). This I will not do in this thesis.

4.3 Three points and two lines

Again using Bézout we get 4 conics that are tangent to two given lines and pass through 3 given points. Again we should check whether the intersection is transverse, but this will not happen in this thesis.

4.4 Two points and 3 lines

For the number of conics through two points and tangent to three lines, we can't use Bézout directly anymore. This is because the double line through the two points is tangent to the 3 lines as well, hence the hyperplanes will not intersect transversally anymore. To solve this problem, we will go to the *dual space* of $\mathbb{P}^2$.

Consider a point $p \in \mathbb{P}^2$. In coordinates, $p = [p_0, p_1, p_2]$. Note that we can uniquely identify this point with the line $L = \{[X, Y, Z] | p_0X + p_1Y + p_2Z = 0\}$. Define the dual of a point to be this line.

Similarly, consider a line $L = \{[X, Y, Z]|aX + bY + cZ = 0\}$. Clearly, we can denote this line by the point $[a, b, c]$, since the line does not change if we multiply with a scalar. Hence we define the dual of a line to be this corresponding point.

The dual space, we denote by $\check{\mathbb{P}}$, whereas we denote the dual of the line L by $\check{L}$ and the dual of a point p by $\check{p}$.

It would be nice if taking the dual of something respects inclusing. Take therefore a point $p \in L$ for a line L. Denote the line L by $\{[X, Y, Z]|aX + bY + cZ = 0\}$. Since $p \in L$, we know that $ap_0 + bp_1 + cp_2 = 0$. Then $\check{p} = \{[X, Y, Z]|p_0X + p_1Y + p_2Z = 0\}$ and $\check{L} = [a, b, c]$. It follows that $\check{L} \in \check{p}$ and duality indeed respects inclusion.

An interesting point is what happens to conics when looking at them in the dual projective space. We define the dual of a conic Q to be the set

$$\check{Q} = \{\check{L}|L \text{ is tangent to } Q\}.$$

It follows that if $\check{Q}$ contains a point $\check{L}$, then $L \in \mathbb{P}^2$ is tangent to Q.
Let $AX + BY + CZ = 0$ be a line and $Q : aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0$ represent a conic. In (4.11) we found the following relation when L is tangent to Q:

$$(e^2 - 4cf)A^2 + (d^2 - 4af)B^2 + (b^2 - 4ac)C^2 + (4ae - 2bd)BC \tag{4.12}$$
$$+(4bf - 2de)AB + (4cd - 2be)AC = 0 \tag{4.13}$$

This is again a conic. Hence the dual of a conic is a conic.

Proposition 4.4. *The dual of a nondegenerate conic is again nondegenerate, the dual of a pair of disjoint lines is a double line and the dual of a double line is the whole space* $\check{\mathbb{P}}^5$

Proof. For the first part, suppose that the dual of a nondegenerate conic is degenerate. Then it has the form

$$(AX + BY + CZ)(DX + EY + FZ) = 0$$
$$ADX^2 + (BD + AE)XY + BEY^2 + (AF + CD)XZ + (BF + CE)YZ + CFZ^2 = 0$$

for some $A, B, C, D, E, F \in \mathbb{C}$.

The original conic had the form $aX^2+bXY+cY^2+dXZ+eYZ+fZ^2 = 0$. Combining this with (4.13), we get the following relations:

$$AD = l(e^2 - 4cf) \tag{4.14}$$
$$BD + AE = l(d^2 - 4af) \tag{4.15}$$
$$BE = l(b^2 - 4ac) \tag{4.16}$$
$$AF + CD = l(4ae - 2bd) \tag{4.17}$$
$$BF + CE = l(4b - 2de) \tag{4.18}$$
$$CF = l(4cd - 2be) \tag{4.19}$$

for some $l \in \mathbb{C}^*$. Note that this l is neccecary because the coefficients are an element of $\mathbb{P}^5$ and then you have this equivalence relation which states that the points can differ a scalar.

Now (4.14), (4.16) and (4.19) reduce to

$$A = l\frac{e^2 - 4cf}{D} \tag{4.20}$$
$$B = l\frac{b^2 - 4ac}{E} \tag{4.21}$$
$$C = l\frac{4cd - 2be}{F}. \tag{4.22}$$

Now we can fill (4.20),(4.21) and (4.22) in the equations (4.15),(4.17) and (4.18) to get the following relations:

$$(b^2 - 4ac)\frac{D}{E} + (e^2 - 4cf)\frac{E}{D} = d^2 - 4af$$
$$(e^2 - 4cf)\frac{F}{D} + (4cd - 2be)\frac{D}{F} = 4ae - 2be$$
$$(b^2 - 4ac)\frac{F}{E} + (4cd - 2be)\frac{E}{F} = 4bf - 2de,$$

or similarly

$$(b^2 - 4ac)D^2 + (e^2 - 4cf)E^2 - (d^2 - 4af)ED = 0 \tag{4.23}$$
$$(e^2 - 4cf)\frac{F}{D} + (4cd - 2be)\frac{D}{F} = 4ae - 2be \tag{4.24}$$
$$(b^2 - 4ac)F^2 + (4cd - 2be)E^2 - (4bf - 2de)EF. \tag{4.25}$$

The equations (4.23) and (4.24) give the following relations for D and F:

$$D = E \cdot \frac{(d^2 - 4af) \pm \sqrt{(d^2-4af)^2 - 4(b^2-4ac)(e^2-4cf)}}{2(b^2-4ac)} \tag{4.26}$$

$$F = E \cdot \frac{(4bf - 2de) \pm' \sqrt{(4bf-2de)^2 - 4(b^2-4ac)(4cd-2be)}}{2(4bf-2de)} \tag{4.27}$$

From this we can get an expression for $\frac{F}{D}$ and $\frac{D}{F}$ in terms of a, b, c, d, e and f. When we fill this in in (4.24), we get quite a nasty expression, but it turns out that this leads to a contradiction for every choice of $\pm$ and $\pm'$. Hence the dual of a nondegenerate conic is nondegenerate

The second part, let the conic be given by $(aX + bY + cZ)(dX + eY + fZ) = 0$. Then the dual conic looks like (first expanding this equation, then filling it in in the expression for the dual conic):

$$\begin{aligned}
&(b^2f^2 + c^2e^2 - 2bcef)A^2\\
&+ (2bcdf + 2acef - 2abf^2 - 2c^2de)AB\\
&+ (a^2f^2 + c^2d^2 - 2acdf)B^2\\
&+ (2abef + 2bcde - 2b^2df - 2ace^2)AC\\
&+ (2abdf + 2acde - 2bcd^2 - 2a^2ef)BC\\
&+ (b^2d^2 + a^2e^2 - 2abde)C^2\\
&= (bf-ce)^2A^2 + 2(bf-ce)(cd-af)AB + (cd-af)^2B^2\\
&+ 2(bf-ce)(ae-bd)AC + 2(cd-af)(ae-bd)BC + (bd-ae)^2C^2\\
&= ((bf-ce)A + (cd-af)B + (ae-bd)C)^2 = 0
\end{aligned}$$

This is exactly the form of a double line, hence the dual of a pair of disjoint lines is a double conic.

Lastly we look at the dual of a double line. The line looks like $A^2X^2 + 2ABXY + B^2Y^2 + 2ACXZ + 2BCYZ + C^2Z^2$. When we now fill these values in in equation (4.11), we see:

$$\begin{aligned}
e^2 - 4cf &= (2CB)^2 - 4B^2C^2 = 0\\
4bf - 2de &= 8ABC^2 - 2ACBC = 0\\
d^2 - 4af &= 4A^2C^2 - 4A^2C^2 = 0\\
4cd - 2be &= 8B^2AC - 8AB^2C = 0\\
4ae - 2bd &= 8A^2BC - 8A^2BC = 0\\
b^2 - 4ac &= 4A^2B^2 - 4A^2B^2 = 0
\end{aligned}$$

Hence all lines possible obey (4.11), and hence every line is tangent to the double line. It follows that for a double line Q, we have $\check{Q} = \check{\mathbb{P}}^5$. □

Now we know that if a line L is tangent to a nondegenerate conic Q, then by definition $\check{L} \in \check{Q}$. We would like to know whether it is the other way around as well, hence for a point $p \in Q$, does it hold that $\check{p}$ is tangent to $\check{Q}$?
To show that this holds, let L_1 be a line tangent to Q in p. Then $\check{L}_1 \in \check{Q}$ and the point $\check{L}_1$ lies on the line $\check{p}$. When we want to check whether $\check{p}$ is tangent to $\check{Q}$, it is sufficient to find a different intersectionpoint and show that these points coincide (since that's how tangency is defined, a line is tangent in a point, exactly if it intersects the conic with double multiplicity). Since a conic is quadratic, we will have 2 points of intersection, call them $\check{L}_1$ and $\check{L}_2$, both of which are a point in $\check{p}$. Now we observe that both lines L_1 and L_2 are tangent to Q in the point p. When the conic is nondegenerate, it follows that

these lines coincide, hence $\check{L}_1 = \check{L}_2$ and hence $\check{p}$ is tangent to $\check{Q}$ in the point $\check{L}_1$.
So tangency is conserved when taking the dual.

Now we can again look at the problem how many conics are tangent to three given lines and pass through two given points. When we look at the dual of these points and lines, we see that there are exactly 4 (dual) conics tangent to the dual points and dual lines. Since we now have reduced conics, we know that their was originally only one that gave this conic, hence there are exactly 4 conics tangent to three given lines that pass through two given points.

4.5 Remaining point line problems

For the two remaining problems, namely the number of conics tangent to four given lines and that pass through a given point and the number of conics tangent to 5 given lines, we look at the dual of these problems, and we see that there are respectively 2 and 1 conics that satisfy these conditions.

Chapter 5

Background

5.1 Blow ups

In 6 we will solve the problem how many conics are tangent to five given conics. In this chapter, we will develop some tools to be able to solve this. The first tool will be a blow up.
Let $X \subset \mathbb{P}^n$ be a variety and $Y \subset X$ a closed subvariety. Let $I = (f_1, \dots, f_r)$ be the ideal that locally describes Y in X (i.e. all the functions by which Y is defined, such that $f_i(y) = 0$ for $y \in Y$). Clearly, this is an ideal, because when you multiply f_i by an arbitrary function g, the product will still be zero for points in X.
Now define $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^r$

$$p \mapsto [f_1(p), \dots, f_r(p)]$$

Notice that is map isn't defined on Y, hence the dashed arrow. Now define $\Gamma = \text{graph}\phi \subset X \times \mathbb{P}^r$.

Definition 5.1. Define the *blow up* of a closed subvariety $Y \subset X$ to be $\overline{\Gamma}$. It is denoted by $Bl_Y(X)$.

We have two natural projections from $Bl_Y(X)$ to X and $\mathbb{P}^r$, namely:

$$\pi : \overline{\Gamma} \to X \quad (p, q) \mapsto p$$
$$pr : \overline{\Gamma} \to \mathbb{P}^r \quad (p, q) \mapsto q$$

where $p \in X$ and $q \in \mathbb{P}^r$. Outside Y, π gives an isomorphism between $X - Y$ and $Bl_Y(X)$ (since we can uniquely identify p with (p, q)).

Definition 5.2. $\pi^{-1}(X)$ is called the *exeptional divisor*, where X and π are as before.

Here is an example of a blow ups.

Example 5.3. Let $X = \mathbb{C}^2$ and $Y = (0, 0)$. Then the ideal is generated by $I = (x, y)$. The map ϕ is given by

$$\phi : \mathbb{C}^2 \dashrightarrow \mathbb{P}^1$$
$$(x, y) \mapsto [x, y]$$

The graph of ϕ is given by $\text{graph}\phi = \{(x, y, [x, y]) | (x, y) \neq (0, 0)\} \subset \mathbb{C}^2 \times \mathbb{P}^1$
Now we find some polynomials in which graphϕ is contained. We choose coordinates $(x, y, X_0, X_1) \in \mathbb{C}^2 \times \mathbb{P}^1$. These polynomials should be homogeneous in the coordinates X_0 and X_1. We therefore see that $xX_1 = yX_0$ is one polynomial. Furthermore, graphϕ is completely contained in it, and since this polynomial is irreducible, this is the only polynomial. Hence $\overline{\Gamma} = \{(x, y, X_0, X_1) \in \mathbb{C}^2 \times \mathbb{P}^1 | xX_1 = yX_0\}$ is the blowup of Y.

Intuitively, what happens when you find the blow up of a point or an ideal is that you replace the point by a copy of $\mathbb{P}^n$ for some n. Outside the point nothing really happens, since there, you have an isomorphism between the blow up and the original points.

5.2 Syzygies and Buchbergers Algorithm

Let $K[x_1, \ldots, x_n]$ be a ring. To be able to understand syzyzies and this algorithm, we need first some more definitions about an ordening on monomials as it is done in [4].

Definition 5.4. A *monomial* is a polynomial of the form $k \cdot x_1^{m_1} \cdot \cdots \cdot x_n^{m_n}$ where $m_1, \ldots, m_n \geq 0$ and $k \in K$.

In other words, a monomial is just a polynomial which just contains one term.

Definition 5.5. A *monomial ordening* over a ring $K[x_1, \ldots, x_n]$ is an ordening $>$ over the monomials, which obeys if $p > q$, then $p \cdot r > q \cdot r$ for $r \in K[x_1, \ldots, x_n]$ a monomial and $p > 1$ for all monomials $p \neq 1$.

As an example, take the lexicographic ordening, which first orders the monomials to the degree of the first variable, if they are the same, it orders it to the degree of the second variable and so on, i.e. $x_1^5 x_2 > x_1^4 x_2^9 > x_1^4$.
For this ordening the given condition holds, i.e. $x_1^2 > x_1 x_2$ and multiplying with $x_1 x_2$ yields $x_1^3 x_2 > x_1^2 x_2^2$, which is true.
Now it is possible to define the leading term of a polynomial:

Definition 5.6. Let $K[x_1, \ldots, x_n]$ be a ring with a monomial ordening. Let $p \in K[x_1, \ldots, x_n]$ be a polynomial and let $p_1, \ldots, p_q$ be monomials such that $p = p_1 + \cdots + p_q$, where $p_1 > p_2 > \cdots > p_q$. Then define the *leading term* of p to be p_1, denoted by $\text{in}(p)$ (initial term).

Now let $I \subset K[x_1, \ldots, x_n]$ be an ideal and let

$$\{g_1, \ldots, g_t | g_i \in I \ \forall 1 \leq i \leq t, \text{in}(g_1) > \cdots > \text{in}(g_t)\}$$

be a subset that generates this ideal I.

Definition 5.7. The elements $g_1, \ldots, g_t$ form a *Gröbner basis* if $\{\text{in}(g_1), \ldots, \text{in}(g_n)\}$ generate the ideal I as well.

For example, let $I = (xy^2 - y^3, xy) \subset \mathbb{R}[x, y]$ (i.e. the ideal generated by these polynomials) with respect to lexografic ordening. Then the leading terms are xy^2 and xy, and together they generate the ideal (xy). Notice that $y^3 \notin (xy)$, but since $xy^2 - y^3 - x(xy) = y^3 \in I$, it follows that this is not a Gröbner basis. We need to find some more polynomials that will eventually form a Gröbner basis. However, this is quite hard work, because it's hard to be sure that you've checked all possible polynomials. For this, we have an algorithm, called Buchberger's Algorithm, which will give some special relations between the generators (called syzygies), which will lead to a subset that is a Gröbnerbasis

Definition 5.8. Let $K[x_1, \ldots, x_n]$ again be a ring with a monomial ordening $<$, and let I be an ideal. Let $f_1, f_2 \in I$. Then the *S-polynomial* of (f_1, f_2) on $<$ is

$$S(f_1, f_2) = \frac{\text{in}(f_2)}{GCD(\text{in}(f_1), \text{in}(f_2))} f_1 - \frac{\text{in}(f_1)}{GCD(\text{in}(f_1), \text{in}(f_2))} f_2,$$

where GCD of two monomials is defined to be $GDC(x_1^{a_1} \ldots x_n^{a_n}, x_1^{b_1} \ldots x_n^{b_n}) = x_1^{min(a_1,b_1)} \ldots x_n^{min(a_n,b_n)}$.

The following two theorems are stated from [4], page 332.

Theorem 5.9. *(Buchberger's criterion) Let $I \subset K[x_1, \ldots, x_n]$ be an ideal and let*

$$\{g_1, \ldots, g_t | g_i \in I \quad \forall 1 \leq i \leq t, in(g_1) > \cdots > in(g_t)\}$$

be a subset that generates this ideal I. Then $g_1, \ldots, g_t$ form a Gröbner basis if and only if $S(g_i, g_j) = 0$ for all pairs $1 \leq i, j \leq t$.

Theorem 5.10. *(Buchberger's Algorithm) Let $I, g_1, \ldots, g_t$ be as in the previous theorem. Compute $S(g_i, g_j)$ for every i, j. If $S(g_i, g_j) = 0$ for every i, j, then we have a Gröbner basis. If for a pair (i, j) $h_{ij} = S(g_i, g_j) \neq 0$, repeat the proces with $g_1, \ldots, g_t, h_{ij}$. Since the ideal generated by the leading terms of this latter set, is strictly bigger than the previous ideal generated by the leading terms, this process must terminate, and hence we can find a Gröbner basis for every ideal.*

Note that when we have a Gröbner basis, the relations $S(g_i, g_j)$ are called syzygies.

Proof. Read in [4].

Note that, although this algorithm gives a way of calculating the Gröbner basis, by hand it stays quite difficult. The length of the algorithm grows exponentially with the number of equations and the number of variables, so most of the time, you will need to make use of a computer program.

5.3 Chow group

The Chow group of a variety is defined analogous to homology for topological spaces. The elements from this group are defined as formal combinations of cycles (where, in the case of a algebraic variety, a cycle is a closed irreducible subvariety of codimension k). However, some cycles are equivalent to eachother, hence we will get an equivalence relation as well. The definition is:

Definition 5.11. Let X be a smooth variety over $\mathbb{C}$ of dimension n. Define the *Chow group* of X to be

$$A^k(X) = \mathbb{Z}\,[\text{closed irreducible subvarieties of codimension } k]\,/\sim,$$

with $\sim$ some equivalence relation.

For $k = 1$, we will investigate this relation, called rational equivalence more clearly. However, to define the equivalence relation, we first need the notion of a scheme. I will not define it very rigorously, but just give some intuition for it.

A *scheme* is an algebraic variety which keeps track of multiplicity. As an example, $x = 0$ and $x^2 = 0$ define the same variety (namely $x = 0$), but as a scheme they differ by their multiplicity in $x = 0$, which is 2 for the variety $x^2 = 0$.

Now we can have define when two varieties are equivalent ([6]).

Definition 5.12. Let Γ_1, Γ_2 be two closed irreducible subvarieties of X of codimension 1. Then $\Gamma_1 \sim_{rat} \Gamma_2$ if and only if there exists a scheme $\mathcal{D} \subset \mathbb{P}^1 \times X$ and a surjective funtion (which is the projectionfunction on the first coordinate) $f : \mathcal{D} \to \mathbb{P}^1$ such that $\mathcal{D}|_{[0,1]} = \Gamma_1$ and $\mathcal{D}|_{[1,0]} = \Gamma_2$.

Remark 5.13. Note that it is equivalent to say that $\Gamma_1 \sim_{rat} \Gamma_2$ or $\Gamma_1 - \Gamma_2 \sim_{rat} 0$.

With this notion, we can define when when a cycle is equivalent to 0 (from which follows when two cycles are equivalent).

Definition 5.14. An arbitrary cycles $\alpha = \sum_{i=1}^{k} n_i \Gamma_i$ is equivalent to 0 if and only if there exists schemes $\mathcal{D}_1, \ldots, \mathcal{D}_t$ such that $\alpha = \sum_{i=1}^{t} \mathcal{D}_i|_{[0,1]} - \mathcal{D}_i|_{[1,0]}$

Remark 5.15. This is indeed an equivalence relation.

Proof. Let Γ_1, Γ_2 and Γ_3 be closed irreducible subvarieties of codimension 1 of X. Then define $\mathcal{D} = \{([s,t], x) \in \mathbb{P}^1 \times X | x \in \Gamma_1\}$ and $f : \mathcal{D} \to \mathbb{P}^1 \quad ([s,t], x) \mapsto [s,t]$. Then $\mathcal{D}|_{[0,1]} = \Gamma_1 = \mathcal{D}|_{[1,0]}$. Hence $\Gamma_1 \sim_{rat} \Gamma_1$.
Now suppose $\Gamma_1 \sim_{rat} \Gamma_2$. Then there exists a $\mathcal{D} \subset \mathbb{P} \times X$ with a surjective function $f : \mathcal{D} \to \mathbb{P}^1$. Now define $g : \mathcal{D} \to \mathbb{P}^1 \times X$, where $([s,t], x) \mapsto ([t,s], x)$. Call the image $\mathcal{D}'$ (notice that it is still a scheme). Using the same projectionfunction f, it follows that $\Gamma_2 \sim_{rat} \Gamma_1$.
Let $\Gamma_1 \sim_{rat} \Gamma_2$ and $\Gamma_2 \sim_{rat} \Gamma_3$. Then let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the corresponding schemes and define

$$\begin{aligned} \mathcal{D} = \{([s,t], x) \in \mathbb{P}^1 \times X | ([s,t], x) &= \mathcal{D}_1|_{[|\frac{s}{t}|, 1-|\frac{s}{t}|]} \text{ if } t \neq 0 \text{ and } \left|\frac{s}{t}\right| \leq 1 \\ ([s,t], x) &= \mathcal{D}_2|_{[1-|\frac{t}{s}|, |\frac{t}{s}|]} \text{ if } s \neq 0 \text{ and } \left|\frac{t}{s}\right| \leq 1\} \end{aligned}$$

Note that this is well defined, since for $|s| \neq |t|$, they are defined differently and if $|s| = |t|$ we have $([s,t], x) = \mathcal{D}_1|_{[1,0]} = \Gamma_2 = \mathcal{D}_2|_{[0,1]}$.
Furthermore $\mathcal{D}|_{[1,0]} = \Gamma_3$ and $\mathcal{D}|_{[0,1]} = \Gamma_1$, and hence it follows that $\Gamma_1 \sim_{rat} \Gamma_3$ and $\sim_{rat}$ is indeed an equivalence relation. $\square$

Proposition 5.16. *All lines as a subset of $\mathbb{P}^2$ are rational equivalent.*

Proof. Let $L_1 = \{[X,Y,Z] | AX+BY+CZ = 0\}$ and $L_2 = \{[X,Y,Z] | DX+EY+FZ = 0\}$ be two arbitrary lines. Define

$$\mathcal{D} = \{([s,t],[X,Y,Z]) | (sA+tD)X + (sB+tE)Y + (sC+tF)Z = 0\}$$

Note that the function that defines $\mathcal{D}$ is homogeneous in both s,t and in X,Y,Z, and hence it is welldefined.
Furthermore $\mathcal{D}|_{[0,1]} = L_2$ and $\mathcal{D}|_{[1,0]} = L_1$ and hence $L_1 \sim_{rat} L_2$. $\square$

In the Chow group we have by definition the operation addition. It is also possible to define a product on $A(X) \times A(X) \to A(X)$, where $A(X) = \oplus_{k=0}^{\dim(X)} A^k(X)$. This product satisfies some conditions, like when you restict the direct sum to single groups we get $A^k(X) \times A^l(X) \to A^{k+l}(X)$. Furthermore, on varieties that intersect transversally, this product corresponds to taking the intersection of the varieties. The product is welldefined modulo rational equivalence. These facts about the product, I will just assume.

Example 5.17. Let $C = \{x \in \mathbb{P}^2 | f(x) = 0\}$ be a variety where f is a polynomial of degree n. Then $C \sim_{rat} nL$, where L is a line.

Proof. Write $f(x) = \sum_{i,j,i+j \leq n} a_{ij} X^i Y^j Z^{n-i-j}$. Define $\mathcal{D} = \{([s,t],[X,Y,Z] \in \mathbb{P}^1 \times \mathbb{P}^2 | \sum_{i,j,i+j \leq n} s a_{ij} X^i Y^j Z^{n-i-j} + tX^n\}$. Then this is a scheme (since it is an algebraic variety and it remembers the multiplicity) and $\mathcal{D}_{[1,0]} = C$ and $\mathcal{D}|_{[0,1]} = \{[X,Y,Z] | X^n\} = 0$, which is just a line with multiplicity n, hence $\mathcal{D}|_{[0,1]} = nL$. Hence $C \sim_{rat} nL$. $\square$

Remark 5.18. Using this, we can give a short proof of Bézout's theorem. Let C, D be two varieties of degree m and n. Then their intersection can be written as $C \cdot D \sim_{rat} nLmL \sim_{rat} mnLL'$, with L and L' two different lines. The intersection of those lines is just a point, and hence the intersection of C and D contains exactly mn points (where points with multiplicity $p > 1$ are counted p times).

Similar, we can prove the same results with hypersurfaces in $\mathbb{P}^n$.

Proposition 5.19. *All hyperplanes as a subset of $\mathbb{P}^n$ are rational equivalent.*

Proof. Let $H_1 = \{[X_0, \ldots, X_n] | A_0 X_0 + \cdots + A_n X_n = 0\}$ and $H_2 = \{[X_0, \cdot, X_n] | B_0 X_0 + \cdots + B_n X_n = 0\}$ be two arbitrary hypersurfaces. Define

$$\mathcal{D} = \{([s,t],[X_0, \ldots, X_n]) | (sA_0 + tB_0)X_0 + \cdots + (sA_n + tB_n)X_n = 0\}$$

Note that the function that defines $\mathcal{D}$ is homogeneous in both s,t and in $X_0, \ldots, X_n$, and hence it is welldefined. Furthermore $\mathcal{D}|_{[0,1]} = H_2$ and $\mathcal{D}|_{[1,0]} = H_1$ and hence $H_1 \sim_{rat} H_2$. $\square$

The same result also holds with hypersurfaces of degree d (i.e. a hypersurface $Z(f)$ with f a polynomial of degree d is equivalent to dH, with H a hyperplane), where the prove goes analogous, only you work in $n+1$ coordinates and you have hyperplanes instead of lines.

Chapter 6

Conics continued

6.1 Veronese surface

With the theory presented in the previous chapter, we are now ready to look at the problem of how many reduced conics are tangent to 5 given conics. We have the following theorem.

Theorem 6.1. *Let 5 conics in general position be given. Then there are exactly 3264 reduced conics tangent to the given conics.*

In this chapter the proof of this theorem will be presented.

Since there are many double lines that intersect all the given conics with multiplicity 2, we can't just use Bézout to calculate the number of conics. Therefore, we will make the blowup for the set of double lines in $\mathbb{P}^5$ and then intersect the hypersurfaces of conics tangent to a given conic. In this chapter [1] is followed.

Remark 6.2. I will not discuss the condition about the conics being in general position in detail, but in the end, there will be a short remark about it.

Every double line is of the form

$$\begin{aligned} (AX + BY + CZ)^2 &= 0 \\ AX^2 + 2ABXY + B^2Y^2 + 2ACXZ + 2BCYZ + C^2Z^2 &= 0 \end{aligned}$$

The double lines form a surface, and this surface, seen as a subset of $\mathbb{P}^5$ is called the *Veronese surface*. Since we want to find the blow up for this set, we first will find the ideal that generates this surface. Notice that points in the Veronese surface all obey the following relations (when you choose coordinates $[a, b, c, d, e, f] \in \mathbb{P}^5$):

$$\begin{aligned} b^2 - 4ac &= 0 \\ d^2 - 4af &= 0 \\ e^2 - 4bf &= 0 \\ bd - 2ae &= 0 \\ de - 2bf &= 0 \\ be - 2cd &= 0 \end{aligned} \tag{6.1}$$

We can show that (6.1) defines an irreducible variety of dimension 2. We show this by showing that this variety is irreducible of dimension 2 in every chart.

Choose $a \neq 0$. Then the equations from (6.1) reduce to (by dividing through a, using the chart U_a, and defining $b' = \frac{b}{a}, \dots, f' = \frac{f}{a}$):

$$\begin{aligned} b'^2 - 4f' &= 0 \\ d'^2 - 4f' &= 0 \\ 2e' - b'd' &= 0 \\ e'^2 - 4b'f' &= 0 \\ d'e' - 2b'f' &= 0 \\ b'e' - 2c'd' &= 0 \end{aligned}$$

Notice that from the first three equations we can derive the other three. Hence in this chart, the Veronese surface is an irreducible variety of dimension 2 (a surface). Similar equations arise when we choose a different coordinate to be nonzero (but then we need different equations to start with from which we can derive the other equations). However, we see that in every case, we have defined the surface with these equations, and hence the Veronese surface is a irreducible variety of dimension 2. Note that in (6.1) we do need all these 6 equations, since they are linearly independent.

6.2 Blowing up the Veronese surface

Define the map

$$\begin{aligned} &\phi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^5 \\ &[a, b, c, d, e, f] \mapsto [b^2 - 4ac, d^2 - 4af, e^2 - 4bf, bd - 2ae, de - 2bf, be - 2cd] \end{aligned}$$

This map isn't defined exactly in the Veronese surface, containing all the double lines. Hence we can find the blowup of this surface.

$$\begin{aligned} \Gamma = \text{graph}\phi = \{([a, b, c, d, e, f], [r, s, t, u, v, w]) | r = e^2 - 4cf, s = 4bf - 2de, t = d^2 - 4af, \\ u = 4cd - 2be, v = 4ae - 2bd, w = b^2 - 4ac, [a, b, c, d, e, f] \text{ not on the Veronese}\} \end{aligned}$$

Now the closure of this graph obeys all the relations $x_i y_j = y_i x_j$ for all pairs $(x_i, y_i), (x_j, y_j) \in \{(a, r), (b, s), (c, t), (d, u), (e, v), (f, w)\}$. However, this are not all the relations we can find.

We can use Buchberger's algorithm (5.2) to find a Gröbner basis of the Veronese surface. For every syzygy we add to the set of generating functions ((6.1)), we can find a relation which points in the blowup satisfy. We therefore find the following relations (note that I will not do it by hand, since it becomes a very large calculation):

$$\begin{aligned} bu + 2ew + 2cv &= 0 \\ eu + 2br + 2cs &= 0 \\ ds + et + 2fv &= 0 \\ es + 2dr + 2fu &= 0 \\ bv + 2dw + 2au &= 0 \\ dv + 2bt + 2as &= 0 \\ 4ar - 4ct + du - ev &= 0 \\ 4ct - 4fw + bs - du &= 0 \end{aligned}$$

For a quick check, we can check whether points from the Veronese do obey these relation. In every equation, we can replace r, s, t, u, v and w with their corresponding equations, and check whether the relations are satisfied. Take for example the first equation:

$$\begin{aligned} bu + 2ew + 2cv &= b(4cd - 2be) + 2e(b^2 - 4ac) + 2c(4ae - 2bd) \\ &= 4bcd - 2b^2 + 2b^2e - 8ace + 8ace - 4bcd = 0 \end{aligned}$$

Since it equals 0, the relation is satiesfied.
For all the other equation it holds as well.
Even with these relations, outside the Veronese surface, we still have an isomorphism (the first 15 equations completely define r, s, t, u, v and w). For points in the Veronese, we can see that we've replaced every point with a copy of $\mathbb{P}^2$. Take for example the point which came from the line $(X + Y + Z)^2 = 0$ (i.e. $A = B = C = 1$). Then the point is given by $[1, 2, 1, 2, 2, 1]$ in the Veronese surface. The first 15 equations all reduce to $0 = 0$, since we know that we're on the Veronese, hence $x_i = 0$ for all i. The other 8 equations reduce to the following equations (every equation is gotten twice and some equations were just a linear combination of these equations)

$$\begin{aligned} u + v + 2w &= 0 \\ u + s + 2r &= 0 \\ s + t + v &= 0 \end{aligned}$$

Now we've got 3 equations and 6 unknown, hence in the end, we can choose three of these free and the other will follow. Hence we've replaced every point with a copy of $\mathbb{P}^2$.
We call the exceptional divisor E (the preimage of the Veronese surface).
To solve the problem of how many conics are tangent to the given conics, we will intersect the following sets:

Definition 6.3. For a hypersurface $Y \subset \mathbb{P}^5$, the *proper transform* is $\overline{\pi^{-1}(Y - V)}$, where π the projection function from $\mathbb{P}^5 \times \mathbb{P}^5$ on the first $\mathbb{P}^5$ and V the Veronese surface. Denote the proper transform by $\tilde{Y}$.

Notice that if Y doesn't contain the Veronese, it is still an isomorphism. If Y did contain (part of) the Veronese, it will in the blow up intersect E, but it will not entirely contain it. This will make it possible to make use of the theory of the Chow group, which we will calculate of the blowup.
Notice that in $\mathbb{P}^5$, any two hyperplanes are rationally equivalent, as proved in section 5.3. Call the class of hyperplanes $[H]$, and any hypersurface of degree d is rationally equivalent to $d[H]$.
Now take a hypersurface Y of degree d. If it doesn't contain (part of) the Veronese, we know by the isomorphism that $\tilde{Y} \sim_{rat} d[H]$.

The exeptional divisor behaves totally different, since it is not part of $\mathbb{P}^5$ and therefore it doesn't behave like any hypersurface. Hence the exeptional divisor gives rise to a new element in the Chow group. Call this new element of the group $[E]$. We know that we have all the generators now, because outside the exceptional divisor we have the isomorphism.
We get for a hypersurface Y that $d[\tilde{H}] = [Y] = \pi^{-1}(Y) = m[\tilde{Y}] + n[E]$ and hence $[\tilde{Y}] = d[\tilde{H}] - n[E]$.

If $n = 0$, m denotes the degree of the surface in H. Similar, n denotes the degree of the hypersurface in E (by definition of sums in the Chow ring). Since it is quite hard to check the degree, you can also check to which order the formal derivatives on the set on which we made the blowup don't dissapear anymore.

The following definitions come from [4], page 105.

Definition 6.4. A function F *vanishes to order* n along a variety X if F and all its partial derivatives to order n dissapear on X.

Now let $k[x_1, \ldots, x_n]$ be a polynomial ring over an algebraically closed field k of characteristic 0. Let $P \subset k[x_1, \ldots, x_n]$ be a prime ideal, so that P is the set of all polynomials vanishing on X (i.e. P is the ideal that defines X). Let for $n \geq 1$

$$P^{\langle n \rangle} = \{f \in S | f \text{ vanishes to order} \geq n \text{ at every point of } X\}.$$

Furthermore, we need the following definitions as well:

Definition 6.5. In a ring R, an ideal Q is called *primary* if for every $xy \in Q$ either $x \in Q$ or $y^n \in Q$.
The radical of Q (that is, $\{r \in R | r^n \in Q$ for some $n \in \mathbb{N}\}$) is a prime-ideal P, and Q is called *$P-$primary.*

Definition 6.6. Let P be a maximal ideal defining a variety X. The *n-th symbolic power* of P is the smallest ideal Q such that P^n is contained in it, where

$$P^n = \{p_1 \cdot p_2 \cdot \dots \cdot p_n | p_i \in P \forall 1 \leq i \leq n, \text{ not nececarily different}\}$$

Denote it by $P^{(n)}$.

Now we have the following theorem, which I will not prove (see [4], page 106).

Theorem 6.7. *Suppose that k is an algebraically closed field and S is a polynomial ring over k. If P is a prime ideal of S defining X, then $P^{\langle n \rangle} = P^{(n)}$*

Now denote the ideal of functions that dissapear on the Veronese surface by $\mathbb{I}(V)$ (notice that this set is generated by (6.1)).

Now we can give a precise definition for this n. Let P_Y be a polynomial that describes a hypersurface $H \subset \mathbb{P}^5$ (i.e. $P_Y(x) = 0$ exactly when $x \in H$. Then the inverse image $\pi^{-1}(H)$ is defined by the set on which the equation $P_Y \circ \pi = 0$ holds. Now n is the order to which $P_Y \circ \pi$ vanishes on E.
By theorem 6.2 this is the same as the order of P_Y which vanishes on V (the prime ideal associated with E is exactly V). Hence when we'd like to calculate the corresponding element of the chowgroup, all we need to do is calculate the degree (for m) and the order to which it vanishes on the Veronese surface V.

6.3 Elements in the Chow group for points, lines and conics

Now we will calculate the elements for the hyperplanes that are formed by all the conics through a point, all the conics tangent to a line and all the conics tangent to a given conic.

For a point, all the conics that go through this point form a linear hyperplane. Therefore, the degree of the hypersurface is 1. Furthermore, the hypersurface does not contain the full Veronese surface. Hence the polynomial that defines the hypersurface doesn't vanish on the whole Veronese surface and $n = 0$. Denote the hyperplane by H_p. Then we have $[\tilde{H}_p] = [\tilde{H}]$.

For a line, we've seen in (4.11) that the degree of the hypersurface is 2.
Also, the hypersurface for the line $\{[X, Y, Z] | AX + BY + CZ = 0\}$ is defined by the following equation (4.11).

$$(e^2 - 4cf)A^2 + (d^2 - 4af)B^2 + (b^2 - 4ac)C^2 + (4ae - 2bd)BC + (4bf - 2de)AB + (4cd - 2be)AC = 0$$

Clearly, the polynomial on the left vanishes on V. Now, the first partial derivative (where we derive to a) looks like:

$$-4fB^2 + 4eBC - 4cC^2$$

The point $[0, 0, 0, 0, 0, 1]$ lies on the Veronese, but the polyomial doesn't dissapear in this point. Hence $n = 1$ for the hyperplane of conics tangent by a given line. Denote this hyperplane by H_l. Then we have $[\tilde{H}_l] = 2[\tilde{H}] - [E]$.

Now we look at the hypersurface H_Q of conics tangent to the conic Q. First we calculate H for a specific conic, given by $XZ = Y^2$. Then we get (for $Z \neq 0$):

$$aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0$$

Now using $Z \neq 0$, we can use $X = Y^2/Z$ and multiply the whole equation with Z^2 to get

$$aY^4 + bY^3Z + (c+d)Y^2Z^2 + eYZ^3 + fZ^4 = 0$$

This is a degree 4 polynomial, and the discriminant then looks like:

$$\begin{aligned} e^2(c+d)^2b^2 - 4e^3b^3 - 4e^2(c+d)^3b + 18e^3(c+d)ba - 27e^4a^2 + 256f^3a^4 \\ +f(-4(c+d)^3b^2 + 18e(c+d)b^3 + 16(c+d)^4a - 80e(c+d)^2ba - 6e^2b^2a + 144e^2(c+d)a^2) \\ +f^2(-27b^4 + 144(c+d)b^2a - 128(c+d)^2a^2 - 192eba^2 \end{aligned}$$

Setting this to 0 yields tangency, and this is a degree 6 polynomial. Hence $m = 6$.
Notice that all the hyperplanes of conics tangent to a given conic are now degree 6, since when you first start with a circle, you can get by a coordinate transform an ellips. From this you can get a parabola and hyperbola, and taking again a limit (getting the right coordinate 0), you can get two lines. From these you can again make a double line, and it follows that all the hyperplanes can be gotten by some coordinate transform. These transforms are all linear, hence the degree of the hypersurface stays the same.
For the number n, it can be checked that this polynomial is 0 on the Veronese (which you would expect, since every double line is tangent to every conic) and all the partial derivatives vanish as well. However, the second partial derivatives don't vanish all anymore, hence $n = 2$. Again, only using linear transforms, the fact whether the polynomial vanishes on the Veronese stays the same, hence for every conic Q it follows that $n = 2$.
For the hyperplane H_q we therefore get $[\tilde{H}_q] = 6[\tilde{H}] - 2[E]$.

6.4 Getting to 3264

Now we have the following equations:

$$\begin{aligned} [\tilde{H}_p] &= [\tilde{H}] \\ [\tilde{H}_l] &= 2[\tilde{H}] - [E] \\ [\tilde{H}_q] &= 6[\tilde{H}] - 2[E] \end{aligned}$$

from which follows that $[\tilde{H}_q] = 2[\tilde{H}_l] + 2[\tilde{H}_p]$.
Furthermore, in section 4, we've seen that

$$\begin{aligned} [\tilde{H}_p]^5 = [\tilde{H}_l]^5 &= 1 \\ [\tilde{H}_p][\tilde{H}_l]^4 = [\tilde{H}_p]^4[\tilde{H}_l] &= 2 \\ [\tilde{H}_p]^2[\tilde{H}_l]^3 = [\tilde{H}_p]^3[\tilde{H}_l]^2 &= 4 \end{aligned}$$

Now to calculate how many conics are tangent to 5 given conics we see that we need to calculate $[H_Q]^5$. We therefore get

$$\begin{aligned} [\tilde{H}_q]^5 = (2[\tilde{H}_l] + 2[\tilde{H}_p])^5 = 2^5 \left([\tilde{H}_l]^5 + 5[\tilde{H}_l]^4[\tilde{H}_p]^1 + 10[\tilde{H}_l]^3[\tilde{H}_p]^2 + 10[\tilde{H}_l]^2[\tilde{H}_p]^3 + 5[\tilde{H}_l]^1[\tilde{H}_p]^4 + [\tilde{H}_p]^5\right) \\ = 2^5(1 + 5\cdot 2 + 10\cdot 4 + 10\cdot 4 + 5\cdot 2 + 1 = 2^5\cdot 102 = 3264 \end{aligned}$$

With this we have proved that there are at most 3264 reduced conics tangent to 5 given conics.
Similar, we can get numbers for how many degenerate conics are tangent to p given points, l given lines and $5 - p - l$ given conics, for example for 2 lines, 1 point and 2 conics we get

$$[\tilde{H}_q]^2[\tilde{H}_l]^2[\tilde{H}_p] = 4([\tilde{H}_p] + [\tilde{H}_l])^2[\tilde{H}_l]^2[\tilde{H}_p] = 4[\tilde{H}_l]^2[\tilde{H}_p]^3 + 8[\tilde{H}_l]^3[\tilde{H}_p]^2 + 4[\tilde{H}_l]^4[\tilde{H}_p] = 56$$

6.5 Proof that this is the correct answer

In this section, we will prove that the way of calculating the number of reduced conics tangent to 5 given conics is correct. To prove this, we will use the Zariski topology:

Definition 6.8. Let $A \subset \mathbb{P}^n$ for some n. Then A is closed in the *Zariski topology* if it is the set of common solutions to a collection of polynomial equations. A is open if its complement is closed.

Proposition 6.9. *The Zariski topology is indeed a topology*

Proof. Let $A, B \subset \mathbb{P}^n$ be closed in the Zariski topology. Then we can write $A = \{x \in \mathbb{P}^n | f_1(x) = \cdots = f_m(x) = 0\}$ and $B = \{x \in \mathbb{P}^n | g_1(x) = \cdots = g_k(x) = 0\}$. Then the union is given by

$$A \cup B = \{x \in \mathbb{P}^n | f_i(x) g_j(x) = 0 \text{ for all } 1 \leq i \leq m \text{ and } 1 \leq j \leq k\}$$

This set is indeed closed, hence $A \cup B$ is closed.
Now let $\{B_\alpha\}_{\alpha \in A}$ where A is some indexset, be a set of closed sets. Then we write $B_\alpha = \{x \in \mathbb{P}^n | f_{\alpha,1}(x) = \cdots = f_{\alpha, m_\alpha}(x) = 0\}$ for every set. Then the intersection is given by

$$\bigcap_{\alpha \in A} B_\alpha = \{x \in \mathbb{P}^n | f_{\alpha,i}(x) = 0 \text{ for all } \alpha \in A \text{ and } 1 \leq i \leq m_\alpha\}$$

This set is again closed.
Furthermore we have for the function $h_1(x) = 0$ for all $x \in \mathbb{P}^n$ that $A = \{x \in \mathbb{P}^n | h_1(x) = 0\} = \mathbb{P}^n$ and for the function $h_2(x) = 1$ for all $x \in \mathbb{P}^n$ that $B = \{x \in \mathbb{P}^n | h_2(x) = 0\} = \emptyset$. Hence the Zariski topology is indeed a topology. □

To get some feeling for the Zariski topology, we check some property of the Zarisky topology:

Proposition 6.10. *The Zariski topology on $\mathbb{P}^1$ is not Hausdorff.*

Proof. Suppose that the Zariski topology is Hausdorff. Then we have for x, y some opens U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. However, by definition of the open sets, it follows that U^C and V^C are some solutions of polynomial equations, and hence finite. Hence $U^C \cup V^C = (U \cap V)^C$ is finite. Since we have infinite many points, it follows that $U \cap V$ is nonempty and hence we have a contradiction. □

Now we can prove that we've got the right answer with the following theorem:

Theorem 6.11. *For conics $Q_i \in \mathbb{P}^5$, the set of conics$(Q_1, \ldots, Q_5) \in (\mathbb{P}^5)^5$ such that the $\tilde{H}_{Q_i}$ intersect transversally in points corresponding to reduced conics is open in the Zariski topology.*

Proof. We will prove that its complement is closed.
Let for a conic Q_i its defining equation be given by

$$a_i X^2 + b_i XY + c_i Y^2 + d_i XZ + e_i YZ + f_i Z^2 = 0.$$

Choose $p \in \tilde{H}_{Q_1} \cap \tilde{H}_{Q_2} \cap \tilde{H}_{Q_3} \cap \tilde{H}_{Q_4} \cap \tilde{H}_{Q_5}$.
Now we will look at the tangent space of each of the hypersurfaces.
Note that $Bl_V(\mathbb{P}^5) \subset \mathbb{P}^5 \times \mathbb{P}^5$, hence we can look at the tangent space at p of $Bl_V(\mathbb{P}^5)$ as a subspace of $\mathbb{P}^5 \times \mathbb{P}^5$. This is a linear subspace (where the point p lies in the origin of this linear subspace). Denote the tangent space by $\mathbb{T}_{Bl_V(\mathbb{P}^5)}$.
The tangent space of p of $\tilde{H}_{Q_i}$ is now a linear hyperplane in $\mathbb{T}_{Bl_V(\mathbb{P}^5)} \subset \mathbb{P}^5 \times \mathbb{P}^5$.

Now we can choose charts on $\mathbb{P}^5 \times \mathbb{P}^5$ such that we have a map to $\mathbb{C}^{10}$ (in total we will need 36 charts). When we restrict those charts to Bl_V, we get polynomials

$g_i(a_i, \ldots, f_i, x_0, \ldots, x_9)$ which define the image of the hyperplanes of the tangent spaces of $\tilde{H}_{Q_i}$ at p, where $x_0, \ldots, x_9$ denote the point p.

These hypersurfaces intersect transversally, exactly when the jacobian $J = \left(\frac{\partial g_i}{\partial x_j}\right)$ has full rank. Note that this is a 5×10 matrix. It fails to have full rank exactly when all the 5×5 submatrices vanish. Since that become just polynomials, we have that the sets $\tilde{H}_{Q_1}, \ldots, \tilde{H}_{Q_5}$ don't intersect transversally exactly when some polynomials all become 0. Hence the set

$$S = \{(Q_1, \ldots, Q_5, p) | p \in \tilde{H}_{Q_1} \cap \cdots \cap \tilde{H}_{Q_5} \text{ and the intersection is not transverse}\} \subset (\mathbb{P}^5)^5 \times Bl_V(\mathbb{P}^5)$$

is closed in the Zariski topology.
Note that also the Veronese surface is closed in $\mathbb{P}^5$ (it is the set of common zeros of some polynomials) and since π is continous, we have $E = \pi^{-1}(V)$ is closed.

Now we will look at some sets in $(\mathbb{P}^5)^5 \times Bl_V(\mathbb{P}^5)$. Note that $(\mathbb{P}^5)^5 \times E$ is closed and S is closed as well. Hence $S' = \big((\mathbb{P}^5)^5 \times E\big) \cup S$ is closed in the Zariski topology. Now we can take the projectionfunction to $(\mathbb{P}^5)^5$, which is a continious function that maps S' to some set in $(\mathbb{P}^5)^5$ and it follows that the set conics $(Q_1, Q_2, Q_3, Q_4, Q_5)$ whose proper transforms fail to intersect transversally is closed in the Zariski topology. $\square$

When this set is non-empty, we have shown that for most configurations (we have excluded a very small set) the proper transforms intersect transversally, and this shows that we were able to use the theory we've seen before. Once we have one example for which we can find 3264 unique conics that are tangent, we're done. This example can be found in [1]. Hence for most configurations there are exactly 3264 conics that are tangent to the 5 given conics.

Remark 6.12. The condition of the conics being in general position is satisfied if the given conics are such that the corresponding hyperplanes H_{Q_i} intersect transversally. All the 5-tuples of conics for which this is not the case, we exclude using the general position.

A similar prove can be given for the other problems which involve p points, l lines and c conics where $p + l + c = 5$.

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