# The Conley-Zehnder Theorem

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# Abstract

In this thesis, we prove part of the Conley-Zehnder theorem, a specific case of the Arnold conjecture, which states that every Hamiltonian symplectomorphism of the standard symplectic torus $\mathbb{T}^{2n}$ has at least $2^{2n}$ fixed points, provided that all these fixed points are non-degenerate. Our proof follows that of [MS99, Theorem 11.6], using generating functions and Conley index theory.

# Acknowledgments

Most importantly, I would like to thank Dr. Fabian Ziltener, whose teaching has introduced me to many interesting areas of mathematics and whose supervision has helped me a lot during the writing of this thesis. I would also like to thank my friends for making my time at this university even more enjoyable, and my family for supporting me throughout my whole education.

# Contents

# 1 Introduction

## 1.1 Motivation and main result

In physics, Hamiltonian mechanics was introduced by William Hamilton in 1834 as a reformulation of earlier theories in classical mechanics by Newton and Lagrange, who published their theories in 1687 and 1788, respectively. The basic idea of Hamiltonian mechanics is to derive, given a Hamiltonian, or *energy function*, a set of differential equations in terms of the generalized coordinates and momenta of a system, which, when solved, determine the path the system will undergo in so-called *phase space*. Mathematically, this space is the cotangent bundle of the *configuration space*, the space of the generalized coordinates.

From Hamiltonian mechanics originated symplectic geometry, a part of differential topology and geometry that focuses on *symplectic manifolds*. A symplectic manifold is a pair $(M, \omega)$, where $M$ is a smooth manifold and $\omega$ is a nondegenerate, closed 2-form on $M$. Symplectic geometry can be seen as a natural mathematical generalization of Hamiltonian mechanics. Indeed; for every manifold $X$ the cotangent bundle $T^*X$ can be made into a symplectic manifold in a natural way (so that symplectic geometry "works" on phase space), and one can formulate Hamilton's equation for any symplectic manifold $M$ and smooth (Hamiltonian) function on $M$.

A *symplectomorphism* between two symplectic manifolds $(M, \omega)$ and $(M', \omega')$ is a diffeomorphism $\varphi : M \to M'$ that preserves the symplectic structure: $\varphi^*\omega' = \omega$. A Hamiltonian symplectomorphism is a symplectomorphism $\varphi_H : M \to M$ that is induced by a time-dependent Hamiltonian function $H : [0,1] \times M \to \mathbb{R}$, in the sense that is is the time-1 flow of the time-dependent vector field $(X_t)_{t\in[0,1]}$ defined by $\omega(X_t, \cdot) = dH_t$. The study of fixed points of these Hamiltonian symplectomorphisms has important applications in physics, since for example these fixed points correspond to periodic solutions of Hamilton's equations. With this in mind, a natural question to ask is the following.

**Question 1.1.** *How many fixed points can we expect a given Hamiltonian symplectomorphism to have?*

The Russian mathematician Vladimir Arnold first formulated an answer to this question in 1966 in the specific case of the 2-torus. He later expanded it to compact symplectic manifolds, and formulated the following famous conjecture.

**Conjecture 1.2** (Arnold conjecture). *Let $(M, \omega)$ be a compact symplectic manifold. Then the number of fixed points of any Hamiltonian symplectomorphism is greater than or equal to the number of critical points any smooth function on $M$ must at least have. When all the fixed points are nondegenerate, this lower bound is increased to the number of critical points any Morse function on $M$ must at least have.*

As of yet, the Arnold conjecture in its strongest form as given above has not yet been proven. Yuli Rudyak and John Oprea proved the degenerate part under some extra hypotheses in [Rud97] and [RO97], but this is the best progress that has been made on the strong form. A weaker, homological version of the conjecture also exists, with milder lower bounds, which has been proven to greater extent. The nondegenerate case of this version was even proven for all compact symplectic manifolds, using Floer homology. Many mathematicians contributed to this result, and more information and references can be found for example in [MS04]. A celebrated result was the solution of the strong Arnold conjecture 1.2 for the even-dimensional tori. It was given in 1983 by Charles Conley and Eduard Zehnder in [CZ83], and it is the nondegenerate part of this theorem we will be proving in this thesis.

**Theorem 1.3** (Conley-Zehnder)**.** *Every Hamiltonian symplectomorphism on the torus* $\mathbb{T}^{2n}$ *possesses at least* $2n+1$ *fixed points. If in addition all the fixed points are nondegenerate, this number is increased to* $2^{2n}$.

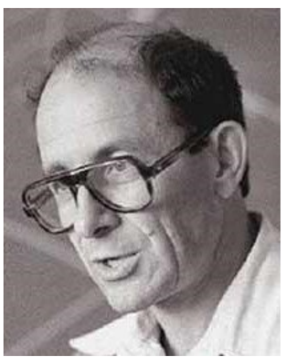
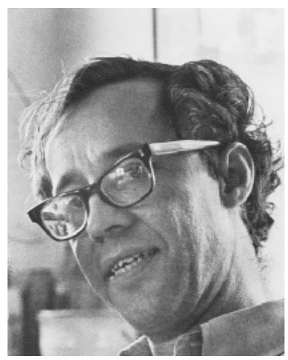


**Figure 1:** From left to right: Vladimir Arnold, Charles Conley and Eduard Zehnder.

## 1.2 Organization of this thesis

In Section 2, we discuss the basics of symplectic geometry. In particular, we define symplectic vector spaces, symplectic manifolds, (Hamiltonian) symplectomorphisms etc. We will also supply several examples, and define the necessary terminology needed to understand Theorem 1.3. Then in Section 3 and 4 we discuss the two main ingredients of the proof; generating functions of symplectomorphisms and Conley index theory. Finally, in Section 5 we use these ingredients to prove the nondegenerate part of Theorem 1.3.

## 1.3 Some remarks about terminology

Throughout this thesis we will work with the following terminology:

- By a vector space we mean a real, finite-dimensional vector space. Similarly, by a manifold we mean a real, smooth, and finite-dimensional one.

- We will use the term "iff" as an abbreviation of "if and only if".

- We will use the notation $W \subset V$ for two sets $V, W$ to indicate that $W$ is a subset of $V$, and not necessarily that also $V \neq W$. If we want to specify this, we will use the notation $W \subsetneq V$.
- We use the convention $\mathbb{N} = \{1, 2, \ldots\}$, and we will denote $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.
- When we speak of a topological manifold $X$ we assume that, in addition to being locally Euclidean, it is also Hausdorff and 2nd countable.

# 2 Basics of symplectic geometry

In this section we explain the basic notions of symplectic geometry. We start by discussing symplectic vector spaces and some of their properties, after which we will do the same for symplectic manifolds. Then, we will define Hamiltonian symplectomorphisms and state the definition of (non-)degenerate fixed points. Finally, we will look into the symplectic structure on $\mathbb{T}^{2n}$. After this, the reader should understand the statement of Theorem 1.3.

## 2.1 Symplectic vector spaces

Let us start with the definition of a symplectic vector space.

**Definition 2.1** (Symplectic vector spaces)**.** Let $V$ be a vector space. A bilinear form $\omega : V \times V \to \mathbb{R}$ is called

- *skew-symmetric* iff $\omega(v, w) = -\omega(w, v)$ for all $v, w \in V$,
- *nondegenerate* iff $\omega(v, w) = 0$ for all $w \in V$ implies that $v = 0$.

If $\omega$ is skew-symmetric and nondegenerate, it is called a *symplectic bilinear form*. The pair $(V, \omega)$ is then called a *symplectic vector space*. $\diamondsuit$

**Example 2.2.**

(i) We can equip $\mathbb{R}^{2n}$ with the bilinear form $\tilde{\omega}_0$ defined by

$$\tilde{\omega}_0 := \sum_{i=1}^{n} Q^i \wedge P_i,$$

where $Q^i$ and $P_i$ denote the canonical projections. In coordinates, denoting $v = (v_1, \dots, v_{2n})$ with respect to the standard basis of $\mathbb{R}^{2n}$, this map is given by

$$\tilde{\omega}_0(v, w) = \sum_{i=1}^{n} v_{2i-1}w_{2i} - v_{2i}w_{2i-1}.$$

Bilinearity and skew-symmetry of $\tilde{\omega}_0$ are immediate and nondegeneracy follows by considering $w = e_i$, where $\{e_i\}_{1\leq i \leq 2n}$ denotes the standard basis of $\mathbb{R}^{2n}$. Hence $(\mathbb{R}^{2n}, \tilde{\omega}_0)$ is a symplectic vector space. The map $\tilde{\omega}_0$ is called the *standard linear symplectic form* on $\mathbb{R}^{2n}$.

(ii) Consider a vector space $V$ and its dual space $V^*$. On $V \times V^*$, define the bilinear form $\omega_V$ by

$$\omega_V((v, \varphi), (v', \varphi')) := \varphi'(v) - \varphi(v').$$

A quick computation[1] shows that $\omega_V$ is symplectic, and it is often called the *canonical linear symplectic form* on $V \times V^*$.

[1] For details, see Appendix Section A.

(iii) For any two symplectic vector spaces $(V, \omega)$ and $(V', \omega')$, $(V \times V', \omega \times \omega')$ is a symplectic vector space as well. Here $(\omega \times \omega')((v, v'), (w, w')) := \omega(v, w) + \omega(v', w')$. Indeed, bilinearity and skew-symmetry are immediate, while nondegeneracy follows by considering the vectors $(w, 0), (0, w') \in V \times V'$. △

The next definition allows us to compare symplectic vector spaces in a sensible way.

**Definition 2.3** (Symplectic isomorphisms)**.** Let $(V, \omega)$ and $(V', \omega')$ be symplectic vector spaces. A linear map $\Phi : V \to V'$ is called *linear symplectic* if it preserves the symplectic structure; $\Phi^*\omega' = \omega$.[2] If $\Phi$ is also a vector space isomorphism, we call it a *symplectic isomorphism*, or simply an isomorphism. Similarly, if $\Phi$ is a vector space automorphism, we call it a symplectic automorphism, or just automorphism, in short. If an isomorphism between two symplectic vector spaces exists, we call them *isomorphic*. ♢

**Proposition 2.4.** *Let $(V_1, \omega)_1$, $(V_2, \omega_2)$ and $(V_3, \omega_3)$ be symplectic vector spaces and let $\Phi_1 : V_1 \to V_2$ and $\Phi_2 : V_2 \to V_3$ be isomorphisms. Then the following hold:*

*(i) The map $\Phi_1^{-1} : V_2 \to V_1$ is an isomorphism,*

*(ii) The map $\Phi_2 \circ \Phi_1 : V_1 \to V_3$ is an isomorphism.*

*Proof.* This follows immediately from the definitions. □

**Remark 2.5** (Symplectic group)**.** From Proposition 2.4, together with the trivial fact that the identity on any symplectic vector space is an automorphism, it follows that "being isomorphic" as symplectic vector spaces is an equivalence relation, and in particular that the set of all automorphisms on a symplectic vector space $(V, \omega)$ is a subgroup of $\mathrm{GL}(V)$. We will denote this subgroup by $\mathrm{Aut}(V, \omega)$.

The main result in this section will be the classification of symplectic vector spaces. We need the following.

**Definition 2.6** (Symplectic complement)**.** Let $(V, \omega)$ be a symplectic vector space and $W \subset V$ a linear subspace. We define the set $W^\omega$ to be the linear subspace

$$W^\omega := \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\} \subset V.$$

We call this set the *symplectic complement* of $W$ in $V$. We say the subspace $W$ is

- *symplectic* iff $W \cap W^\omega = \{0\}$,
- *isotropic* iff $W \subset W^\omega$,
- *coisotropic* iff $W^\omega \subset W$,
- *Lagrangian* iff $W = W^\omega$. ♢

[2] Recall the definition of the *pullback* by a linear map: $(\Phi^*\omega')(v, w) := \omega'(\Phi v, \Phi w)$ for $v, w \in V$.

**Remark 2.7.** Note that a subspace $W$ is symplectic iff $(W, \omega|_W)$ is a symplectic vector space, which explains the name. Here we denote $\omega|_W := \omega|_{W\times W}$. Note also that $W$ is isotropic iff $\omega|_W \equiv 0$, and that it is Langrangian iff it is both isotropic and coisotropic.

**Example 2.8.**

(i) Let $(V, \omega)$ be a symplectic vector space of dimension $2n$. Then any one-dimensional subspace is isotropic and any $(2n-1)$-dimensional subspace is coisotropic.

(ii) Recall the definition of the symplectic vector space $(V \times V^*, \omega_V)$ in Example 2.2(ii). Then the subspaces $V \times \{0\}$ and $\{0\} \times V^*$ are Lagrangian.

For details, see Appendix Section A. △

The following proposition will be of great use proving the classification theorem, and states some nice properties of the symplectic complement.

**Proposition 2.9.** *Let $(V, \omega)$ be a symplectic vector space, and $W \subset V$ a linear subspace. Then the following hold:*

*(i)* $\dim W + \dim W^\omega = \dim V$,

*(ii)* $W^{\omega\omega} := (W^\omega)^\omega = W$.

*Proof.* **(i):** Define the map $\iota_\omega : V \to V^*$ by $\iota_\omega(v) = \omega(v, \cdot)$. By bilinearity of $\omega$ this map is well-defined and linear. Since $\omega$ is nondegenerate, $\iota_\omega$ is injective. Since also $\dim V = \dim V^*$, the rank-nullity theorem implies that $\iota_\omega$ is a vector space isomorphism.[3] Now consider the map $\iota_\omega^W : V \to W^*$ defined by $\iota_\omega^W(v) = \iota_\omega(v)|_W$. This is still a linear surjective map, and its kernel is given by all vectors $v \in V$ such that $\omega(v, w) = 0$ for all $w \in W$, so $\ker \iota_\omega^W = W^\omega$. So we get, again by the rank-nullity theorem,

$$\dim V = \dim \operatorname{im} \, \iota_\omega^W + \dim \ker \iota_\omega^W = \dim W^* + \dim W^\omega = \dim W + \dim W^\omega.$$

This proves Proposition 2.9(i).
**(ii):** With Proposition 2.9(i) we get that $\dim W = \dim W^{\omega\omega}$. Together with the obvious inclusion $W \subset W^{\omega\omega}$ this implies that $W = W^{\omega\omega}$. This proves 2.9(ii) and completes the proof of Proposition 2.9. □

**Remark 2.10.** The "nice properties" we talked about are immediately clear: from Proposition 2.9(i) it follows that

(i) $\dim W \leq \frac{1}{2} \dim V$ if $W$ is isotropic,

(ii) $\dim W \geq \frac{1}{2} \dim V$ if $W$ is coisotropic,

(iii) $\dim W = \frac{1}{2} \dim V$ if $W$ is Langrangian.

[3]For a statement and proof of the rank-nullity theorem, see Appendix Section A.

Furthermore, it follows that $W \oplus W^\omega = V$ if $W$ is symplectic. From Proposition 2.9(ii) it follows in turn that $W$ is symplectic iff $W^\omega$ is symplectic, and that $W$ is isotropic iff $W^\omega$ is coisotropic, and vice versa.

Proposition 2.9 has an immediate consequence, stating that every symplectic vector space has even dimension.

**Theorem 2.11** (Dimension of symplectic vector space)**.** *Every symplectic vector space* $(V, \omega)$ *has even dimension.*

The theorem is a consequence of the following lemmata.

**Lemma 2.12.** *There is no symplectic vector space* $(V, \omega)$ *with* $\dim V = 1$*.*

**Lemma 2.13.** *Every symplectic vector space* $(V, \omega)$ *with* $\dim V \geq 2$ *has a symplectic subspace* $W \subset V$ *with* $\dim W = \dim V - 2$*.*

*Proof of Theorem 2.11.* In the case $(V, \omega) = (\{0\}, 0)$ there is nothing to prove, since the dimension of this space is even. By Lemma 2.12, we only need to check the case where $\dim V \geq 2$. Then, using Lemma 2.13 iteratively, we obtain a sequence of symplectic vector spaces $V_0 := V \supset V_1 \supset \cdots$, where $\dim V_k = \dim V - 2k \geq 2$. This sequence ends at a symplectic vector space of dimension 0 or 1, and by Lemma 2.12 we see that it has to be the former option. Hence we find that $\dim V = 2n$ for some $n \in \mathbb{N}_0$. This completes the proof of Theorem 2.11. $\square$

*Proof of Lemma 2.12.* Let $V$ be a one-dimensional vector space, and let $\omega$ be a skew-symmetric bilinear form on $V$. We show that $\omega$ can't be nondegenerate. Indeed, for any $v, w \in V$ we can write $w = \lambda v$ for some $\lambda \in \mathbb{R}$. Hence it follows from bilinearity of $\omega$ that $\omega(v, w) = \lambda \cdot \omega(v, v)$. Now, from skew-symmetry of $\omega$ it follows that $\omega(v, v) = -\omega(v, v) = 0$, so that $\omega(v, w) = 0$ for all $v, w \in V$. Hence $\omega \equiv 0$ and thus $\omega$ is certainly not nondegenerate. $\square$

*Proof of Lemma 2.13.* Since $\omega$ is nondegenerate and $V \neq \{0\}$ we can find two vectors $v, w \in V$ such that $\omega(v, w) \neq 0$. Note that just as in the proof of Lemma 2.12, it follows that $v$ and $w$ are linearly independent. Hence the linear subspace $W$ generated by $v$ and $w$ is two-dimensional. It is easily verified that $W$ is in fact a symplectic subspace of $V$: let $x = \alpha v + \beta w \in W$ such that $\omega(x, y) = 0$ for all $y \in W$. Choosing $y = v$ yields $\beta = 0$ and choosing $y = w$ yields $\alpha = 0$. Now, by Proposition 2.9 and Remark 2.10 it follows that $W^\omega$ is a symplectic subspace of $V$ of dimension $\dim V - 2$. $\square$

Theorem 2.11 already puts a pretty severe limitation on the existence of symplectic vector spaces. As we shall see now however, dimension is actually the only property distinguishing symplectic vector spaces. This result is a direct consequence of the following theorem.

**Theorem 2.14** (Symplectic basis)**.** *Let $(V,\omega)$ be a symplectic vector space and denote* $\dim V = 2n$. *Then there exists a basis $\{v_1,\dots,v_n,w_1,\dots,w_n\}$ such that $\omega(v_i,v_j) = \omega(w_i,w_j) = 0$ and $\omega(v_i,w_j) = \delta_{ij}$ for all $1 \le i,j \le n$. This is called a* symplectic basis *of $(V,\omega)$.*

*Proof.* We prove the theorem by induction over $n$. In the case that $n = 0$, the statement is certainly true. So now assume that $n \geq 1$ and that the statement is true for all $m < n$. Just as in the proof of Lemma 2.13, we can find a symplectic subspace $W$ generated by two linearly independent vectors $v, w \in V$ such that $\omega(v,w)$. Again, $(W^\omega, \omega|_{W^\omega})$ is a $2(n-1)$-dimensional symplectic subspace, and thus we can find a symplectic basis $\{v_1,\dots,v_{n-1},w_1,\dots,w_{n-1}\}$ of this space. Defining $v_n := v$ and $w_n := w/\omega(v,w)$, we claim that $\{v_1,\dots,v_n,w_1,\dots,w_n\}$ is a symplectic basis of $(V,\omega)$. It is clear that we only need to check the conditions involving $v_n$ or $w_n$. Now, since $\omega$ is bilinear we have that

$$\omega(v_n,w_n) = \frac{\omega(v,w)}{\omega(v,w)} = 1.$$

Also, since $v_n, w_n \in W$ and $\{v_1,\dots,v_{n-1},w_1,\dots,w_{n-1}\} \subset W^\omega$, we have that $\omega(v_n,v_i) = \omega(w_n,w_i) = \omega(v_n,w_i) = \omega(w_n,v_i) = 0$ for all $i \in \{1,\dots,n-1\}$. Finally, since $W$ is symplectic we have that $W \oplus W^\omega = V$, and since $\{v_1,\dots,v_{n-1},w_1,\dots,w_{n-1}\}$ is a basis of $W^\omega$ and $\{v_n,w_n\}$ one for $W$, the union is a basis of $V$. This completes the proof of Theorem 2.14. $\square$

**Example 2.15.** On $(\mathbb{R}^{2n},\omega_0)$ the "reshuffled" bases $\{e_1,e_3,\dots,e_{2n-1},e_2,e_4,\dots,e_{2n}\}$ and $\{-e_2,-e_4,\dots,-e_{2n},e_1,e_3,\dots,e_{2n-1}\}$ are symplectic bases. $\triangle$

The promised classification of vector spaces now follows immediately.

**Corollary 2.16** (Classification of symplectic vector spaces)**.** *Let $(V,\omega)$ and $(V',\omega')$ be symplectic vector spaces. Then $(V,\omega)$ is isomorphic to $(V',\omega')$ iff $\dim V = \dim V'$.*

*Proof.* First, note that the condition that $V$ and $V'$ have the dimension is obviously necessary: any symplectic isomorphism between $(V,\omega)$ and $(V',\omega')$ is also a vector space isomorphism, which requires that $\dim V = \dim V'$.

Now, assume that $\dim V = \dim V' = 2n$. We choose symplectic bases $\{v_1,\dots,w_n\}$ and $\{v_1',\dots,w_n'\}$ of $(V,\omega)$ and $(V',\omega')$, respectively. Now we define $\Phi : V \to V'$ to be the vector space isomorphism satisfying $\Phi(v_i) = v_i'$ and $\Phi(w_i) = w_i'$. To see that $\Phi$ is symplectic, note that

$$\begin{aligned}
(\Phi^*\omega')\left(\sum_{i=1}^n \alpha_i v_i + \beta_i w_i, \sum_{j=1}^n \alpha_j' v_j + \beta_j' w_j\right) &= \omega'\left(\sum_{i=1}^n \alpha_i v_i' + \beta_i w_i', \sum_{j=1}^n \alpha_j' v_j' + \beta_j' w_j'\right) \\
&= \sum_{i=1}^n \alpha_i \beta_i' - \alpha_i' \beta_i \\
&= \omega\left(\sum_{i=1}^n \alpha_i v_i + \beta_i w_i, \sum_{j=1}^n \alpha_j' v_j + \beta_j' w_j\right).
\end{aligned}$$

We see that $\Phi^*\omega' = \omega$, and this completes the proof of Corollary 2.16. $\square$

**Remark 2.17.** In particular, it follows that every symplectic vector space is isomorphic to $(\mathbb{R}^{2n}, \omega_0)$ for some $n \in \mathbb{N}_0$. Also, for every symplectic vector space $(V, \omega)$ we have a "standard form" of $\omega$: if $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ is a symplectic basis, then, denoting by $\{v^1, \ldots, v^n, w^1, \ldots, w^n\}$ the basis dual to it,

$$\omega = \sum_{i=1}^{n} v^i \wedge w^i.$$

The next result will be useful later, when we are discussing symplectic manifolds.

**Proposition 2.18** (Canonical volume form). *Let* $(V, \omega)$ *be a symplectic vector space and denote* $\dim V = 2n$*. Then* $\Omega := \frac{1}{n!}\omega^{\wedge n}$ *is a volume form.*[4] *We call it the* canonical volume form *on* $(V, \omega)$*. It is determined by* $\Omega(v_1, w_1, \ldots, v_n, w_n) = 1$*, where* $\{v_1, \ldots, w_n\}$ *is a symplectic basis of* $(V, \omega)$*.*

**Remark 2.19.** Recall that a *volume form* on a vector space of dimension $n$ is a nonzero $n$-form on $V$.

*Proof of Proposition 2.18.* We choose a symplectic basis $\{v_1, \ldots, w_n\}$ of $(V, \omega)$. We show that $\omega^{\wedge n}(v_1, w_1, \ldots, v_n, w_n) = n!$. Recall that for any set of vectors $x_1, \ldots, x_{2n}$ we have that[5]

$$\omega^{\wedge n}(x_1, \ldots, x_{2n}) = \frac{1}{2^n} \sum_{\sigma \in S_{2n}} (-1)^\sigma \omega(x_{\sigma(1)}, x_{\sigma(2)}) \cdots \omega(x_{\sigma(2n-1)}, x_{\sigma(2n)}). \tag{2.1}$$

Now in our case, the only terms of the sum on the right that will not vanish are the ones where every $v_i$ is paired with the corresponding $w_i$. In such a case, the value of the term will always be 1; indeed, changing two pairs $(v_i, w_i)$ and $(v_j, w_j)$ is an even permutation, so the sign of the permutation $\sigma$ corresponding to the term will depend on switching a pair $(v_i, w_i)$ into $(w_i, v_i)$. So $(-1)^\sigma$ in this case is $(-1)^k$, where $k$ is the number of "switched pairs". But of course, all these pairs will have value $-1$, and the rest will have value 1, so the end value of the term is $(-1)^k(-1)^k = 1$. Now, the amount of these terms possible is $2^n n!$: indeed, all of the $n$ pairs have 2 possible "internal orderings", and we can arrange the $n$ pairs in $n!$ possible ways. We conclude that indeed $\omega^{\wedge n}(v_1, w_1, \ldots, v_n, w_n) = n!$, which completes the proof of Proposition 2.18. $\square$

## 2.2 Symplectic manifolds

We are now well prepared to tackle symplectic manifolds.

**Definition 2.20** (Symplectic manifolds)**.** Let $M$ be a manifold. A differential 2-form $\omega$ on $M$ is called *symplectic* if it is closed and nondegenerate, i.e. $d\omega = 0$ and $\omega_x : T_xM \times T_xM \to \mathbb{R}$ is nondegenerate for every $x \in M$. The pair $(M, \omega)$ is called a *symplectic manifold.* $\diamondsuit$

[4] Here $\omega^{\wedge n}$ denotes the wedge product of $n$ times $\omega$.

[5] This follows from a combinatorial argument, which will be worked out in Appendix Section A.

**Remark 2.21.** From the above definition it is immediately clear that if $(M, \omega)$ is a symplectic manifold, then $(T_xM, \omega_x)$ is a symplectic vector space for every $x \in M$. So we immediately get two properties of symplectic manifolds:

(i) *Every symplectic manifold has even dimension.* This is because the dimension of a manifold is the same as the dimension of the tangent space at each point;

(ii) *Every symplectic manifold is orientable.* Denoting $\dim M = 2n$, this follows from the fact that $\omega^{\wedge n}$ is a nonvanishing $2n$-form on $M$ (indeed, $(\omega^{\wedge n})(x) = (\omega_x)^{\wedge n}$ is nonzero on $T_xM$ by Proposition 2.18), and every nonvanishing $2n$-form induces an orientation of $M$.[6]

**Example 2.22.**

(i) As we know, every vector space $V$ of dimension $n$ can be made into a manifold, with the atlas induced by one global chart, the chart map being a vector space isomorphism with $\mathbb{R}^n$. Similarly, we can make any symplectic vector space $(V, \tilde{\omega})$ into a symplectic manifold. Indeed, denoting $\dim V = 2n$, we can choose a symplectic basis $\{v_1, \ldots, w_n\}$, and by Remark 2.17 we have

$$\tilde{\omega} = \sum_{i=1}^{n} v^i \wedge w^i,$$

where $\{v^1, \ldots, w^n\}$ denotes the dual basis. Now, on the *manifold* $V$ we can define the differential 2-form

$$\omega := \sum_{i=1}^{n} dv^i \wedge dw^i.$$

This is obviously closed, and it is clear that $\omega_x = \tilde{\omega}$, for every $x \in M$.[7] Hence $(V, \omega)$ is a symplectic manifold. In particular, we have a symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$, where

$$\omega_0 = \sum_{i=1}^{n} dq^i \wedge dp_i,$$

where $q^i$ and $p^i$ denote the standard projections. This is called the *standard symplectic form* on $\mathbb{R}^{2n}$.

(ii) An interesting example on the sphere $S^2$ is given by the form

$$\omega_x(v, w) := \langle x, (v \times w) \rangle.$$

Here we view $x \in S^2$ as a vector in $\mathbb{R}^3$, and we use that

$$T_xS^2 = \{v \in \mathbb{R}^3 \mid \langle x, v \rangle = 0\} \subset \mathbb{R}^3,$$

[6] See e.g. [Lee12, Proposition 15.5].

[7] Here we canonically identify $T_xV \simeq V$.

see Figure 2. It is clear from the properties of the cross product that $\omega_x$ is skew-symmetric and nondegenerate for every $x$. Since $S^2$ has dimension 2 and $d\omega$ is a 3-form, it immediately follows that $d\omega = 0$.

(iii) If $(M, \omega)$ and $(M', \omega')$ are symplectic manifolds, then we can define a symplectic 2-form on $M \times M'$ by

$$\omega \times \omega' := \pi_1^* \omega + \pi_2^* \omega',$$

where $\pi_1$ is the projection onto $M$ and $\pi_2$ the projection onto $M'$. Explicitly, this is given by

$$(\omega \times \omega')_{(x,x')}((v, v'), (w, w')) := \omega_x(v, w) + \omega'_{x'}(v, w),$$

From these two formulas it immediately follows that $\omega \times \omega'$ is a symplectic form on $M \times M'$.

(iv) As mentioned in the introduction, the cotangent bundle $T^*X$ of any manifold $X$ carries a natural symplectic structure. Since this will take a little more work to establish, we have devoted Section 2.2.1 to it. Since it will not be relevant for any of the results in the rest of this thesis, this section can safely be skipped.

(v) For every $n$, the torus $\mathbb{T}^{2n}$ can be endowed with a symplectic structure. We will look extensively into this structure in Section 2.4. $\triangle$

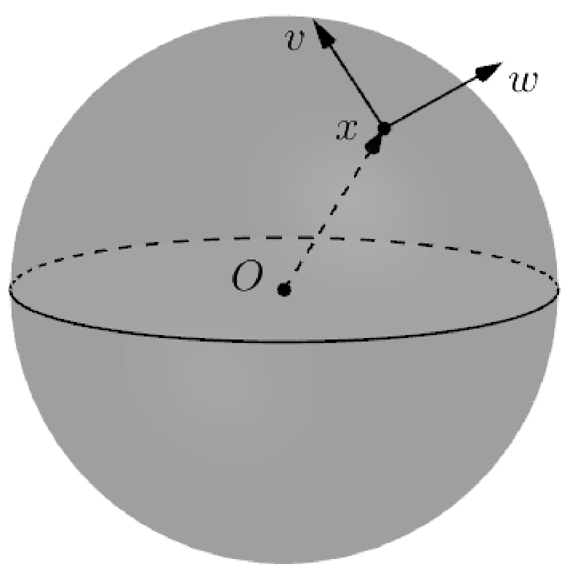


**Figure 2:** A symplectic form on $S^2$; the vectors $v$ and $w$ are perpendicular to $x$.

The following definition extends the notion of symplectic isomorphisms to symplectic manifolds.

**Definition 2.23** (Symplectomorphisms)**.** Let $(M, \omega)$ and $(M', \omega')$ be symplectic manifolds. A diffeomorphism $\varphi : M \to M'$ is called a *symplectomorphism* if

$$\varphi^*\omega' = \omega.^8 \qquad \diamondsuit$$

**Remark 2.24.** Exactly as in the linear case, inverses and compositions of symplectomorphisms are again symplectomorphisms. Again, this follows immediately from the definitions. So "being symplectomorphic" is an equivalence relation, and the set of all symplectomorphisms of a manifold $(M, \omega)$ to itself is a subgroup of $\mathrm{Diff}(M)$, the group of diffeomorphisms of $M$ to itself. We denote this subgroup by $\mathrm{Symp}(M, \omega)$.

**Example 2.25.** As we might expect, just like symplectic vector spaces of equal dimension are symplectically isomorphic, the same holds when we view them as manifolds. Indeed, if $(V, \omega)$ and $(V', \omega')$ are symplectic manifolds of dimension $2n$ induced by the symplectic vector spaces $(V, \tilde{\omega})$ and $(V', \tilde{\omega}')$, then, choosing symplectic bases $\{v_1, \dots, w_n\}$ and $\{v_1', \dots, w_n'\}$, we can define $\varphi : V \to V'$ by $v_i \mapsto v_i'$ and $w_i \mapsto w_i'$. Of course, this map is an isomorphism between $(V, \tilde{\omega})$ and $(V', \tilde{\omega}')$. It is also a diffeomorphism between the manifolds $V$ and $V'$: (pre-)composing it (or its inverse) with the chart maps for $V$ and $V'$, which are vector space isomorphisms with $\mathbb{R}^{2n}$, yields a vector space automorphism on $\mathbb{R}^{2n}$, which is smooth, since it is linear. Also, when we identify $T_xV$ with $V$ for every $x \in V$, and do the same for $V'$, the map $d\varphi(x)$ is just $\varphi$, so we also clearly have that $\varphi^*\omega' = \omega$. $\triangle$

Unlike symplectic vector spaces, there is more than dimension distinguishing symplectic manifolds. Therefore, no global classification exists; there is just the following theorem, known as Darboux' Theorem, which states that *locally* all symplectic manifolds look like $(\mathbb{R}^{2n}, \omega_0)$.

**Theorem 2.26** (Darboux)**.** *Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold and let $x \in M$. Then there exists a* Darboux chart *around $x$: an open neighbourhood $U \subset M$ of $x$ and a chart map $\varphi : U \to \mathbb{R}^{2n}$ such that*

$$\varphi^*\omega_0 = \omega.$$

Proving this theorem is somewhat involved and will not be relevant for the rest of this thesis. Therefore we will not state it here. A nice proof using Moser isotopy can be found in [MS99, Theorem 3.15].

[8] Recall that this means that for every $x \in M$, $\omega_x = d\varphi(x)^*\omega'_{\varphi(x)} = \omega'_{\varphi(x)}(d\varphi(x)\cdot, d\varphi(x)\cdot)$.

### 2.2.1 The cotangent bundle

In this section we will define the *canonical symplectic structure* on the cotangent bundle of a manifold $M$. Before we give the definition, let us first discuss some notation; in this section, $X$ is a manifold, $T^*X$ is the cotangent bundle and we denote by $\pi : T^*X \to X$ the canonical projection. Also, we will write a point $x \in T^*X$ as $x = (q, p)$, where $q \in X$ and $p \in T_q^*X$.

**Definition 2.27** (Canonical forms on the cotangent bundle)**.** For a manifold $X$, we define the *canonical 1-form* $\lambda_X^{\mathrm{can}}$ on $T^*X$ by

$$(\lambda_X^{\mathrm{can}})_x := p \circ d\pi(x) : T_x(T^*X) \to \mathbb{R}, \quad x = (q, p) \in T^*X.$$

We define the *canonical 2-form* $\omega_X^{\mathrm{can}}$ by

$$\omega_X^{\mathrm{can}} := -d\lambda_X^{\mathrm{can}}. \qquad \diamondsuit$$

The goal here is of course to show that $\omega_X^{\mathrm{can}}$ is a symplectic form for any manifold $X$. Since closedness is obvious from the definition, we are left with showing nondegeneracy. To do this, we will first consider the case that $X = V$, a vector space. In this case, there are some dramatic simplifications; first of all, we can identify the cotangent bundle with $V \times V^*$, a vector space. Hence for every $x = (v, \varphi) \in V \times V^*$ we can canonically identify $T_x(V \times V^*)$ and $V \times V^*$. Since also the projection $\pi : V \times V^* \to V$ is linear, we get that $d\pi(x) : T_x(V \times V^*) \to T_qV$ is just $\pi_1 : V \times V^* \to V$, the standard projection onto $V$. Denoting also by $\pi_2 : V \times V^* \to V^*$ the projection onto $V^*$, we get that

$$(\lambda_V^{\mathrm{can}})_x = \varphi \circ \pi_1 = \pi_2(x) \circ \pi_1.$$

To compute $\omega_V^{\mathrm{can}}$, recall the invariant formula for the exterior derivative: for any two vector fields $X_1$ and $X_2$ on $V \times V^*$, we have that

$$d\lambda_V^{\mathrm{can}}(X_1, X_2) = X_1(\lambda_V^{\mathrm{can}}(X_2)) - X_2(\lambda_V^{\mathrm{can}}(X_1)) - \lambda_V^{\mathrm{can}}([X_1, X_2]).$$

This means that for every $x \in V \times V^*$ we have that

$$(d\lambda_V^{\mathrm{can}})_x((X_1)_x, (X_2)_x) = (X_1)_x(\lambda_V^{\mathrm{can}}(X_2)) - (X_2)_x(\lambda_V^{\mathrm{can}}(X_1)) - (\lambda_V^{\mathrm{can}})_x([X_1, X_2]_x),$$

where we view $\lambda_V^{\mathrm{can}}(X_i)$ as a smooth function $x \mapsto (\lambda_V^{\mathrm{can}})_x((X_i)_x)$. Now let $x_1, x_2 \in T_x(V \times V^*) = V \times V^*$, and let $X_1$ and $X_2$ be the constant vector fields mapping to $x_1$ and $x_2$ respectively. Then $[X_1, X_2] = 0$, and using that $X_x(f) = df(x)(X_x)$ for any smooth function $f$ and any vector field $X$, we get that

$$\begin{aligned}(\omega_V^{\mathrm{can}})_x(x_1, x_2) &= (X_2)_x(\lambda_V^{\mathrm{can}}(X_1)) - (X_1)_x(\lambda_V^{\mathrm{can}}(X_2)) \\ &= d(\lambda_V^{\mathrm{can}}(X_1))(x)(x_2) - d(\lambda_V^{\mathrm{can}}(X_2))(x)(x_1).\end{aligned}$$

Now, writing $x = (v, \varphi)$, we get that

$$(\lambda_V^{\mathrm{can}}(X_i))(x) = (\lambda_V^{\mathrm{can}})_x(x_i) = \varphi(\pi_1(x_i)).$$

This map is linear, and hence its differential at any $x \in V \times V^*$ is just the map itself. Hence we get that, writing $x_i = (v_i, \varphi_i)$,

$$(\omega_V^{\mathrm{can}})_x(x_1, x_2) = \varphi_2(\pi_1(x_1)) - \varphi_1(\pi_1(x_2)) = \varphi_2(v_1) - \varphi_1(v_2).$$

This is of course the symplectic bilinear form $\omega_V$ on the vector space $V \times V^*$ defined in Example 2.2(ii). Hence we get that for every $x \in V \times V^*$,

$$(\omega_V^{\mathrm{can}})_x = \omega_V.$$

In particular, $\omega_V^{\mathrm{can}}$ is nondegenerate.

Now, to show that $\omega_X^{\mathrm{can}}$ is nondegenerate for *any* manifold $X$, we want to somehow reduce to the vector space case described above. To do this, we need the following definition.

**Definition 2.28** (Pushforward on cotangent bundles)**.** Let $X$ and $X'$ be manifolds and $\varphi : X \to X'$ a diffeomorphism. We define the *pushforward* of $\varphi$ to be the map $\Phi : T^*X \to T^*X'$ defined by $\Phi(q, p) = (\varphi(q), p \circ d\varphi(q)^{-1})$. ♢

**Remark 2.29.** Note that $\Phi$ is a smooth bundle isomorphism covering $\varphi$.

The crucial property of the pushforward map is that it intertwines the canonical forms on the cotangent bundles.

**Lemma 2.30.** *Let $X, X', \varphi$ and $\Phi$ be as in Definition 2.28. Then we have that*

$$\Phi^*\lambda_{X'}^{\mathrm{can}} = \lambda_X^{\mathrm{can}}$$

*and thus that*

$$\Phi^*\omega_{X'}^{\mathrm{can}} = \omega_X^{\mathrm{can}}.$$

*Proof.* We simply compute, for $x = (q, p) \in T^*X$:

$$\begin{aligned}(\Phi^*\lambda_{X'}^{\mathrm{can}})_x &= (\lambda_{X'}^{\mathrm{can}})_{\Phi(x)} \circ d\Phi(x) = p \circ d\varphi(q)^{-1} \circ d\pi'(\Phi(x)) \circ d\Phi(x) \\ &= p \circ d(\varphi^{-1} \circ \pi' \circ \Phi)(x) = p \circ d\pi(x) = (\lambda_X^{\mathrm{can}})_x.\end{aligned}$$

Thus, $\Phi^*\omega_{X'}^{\mathrm{can}} = \Phi^*(-d\lambda_{X'}^{\mathrm{can}}) = -d(\Phi^*\lambda_{X'}^{\mathrm{can}}) = -d\lambda_X^{\mathrm{can}} = \omega_X^{\mathrm{can}}$, and this completes the proof of Lemma 2.30. □

We are now ready to wrap up this section.

**Proposition 2.31** (Symplectic cotangent bundle)**.** *For any manifold $X$, $(T^*X, \omega_X^{\mathrm{can}})$ is a symplectic manifold.*

*Proof.* As mentioned before, all that is left is to show that $\omega_X^{\mathrm{can}}$ is nondegenerate. To see this, let $x = (q, p) \in T^*X$, and choose a chart $(U, \varphi)$ (of $X$) around $q$. Denoting $n = \dim X$, this means that $\varphi \to \mathbb{R}^n$ is a diffeomorphism, where we think of $\mathbb{R}^n$ as a manifold. Defining $\Phi : T^*U \to T^*\mathbb{R}^n$ to be the pushforward of $\varphi$, we get by Lemma 2.30 that

$$\omega_X^{\mathrm{can}}|_{T^*U} = \Phi^*\omega_{\mathbb{R}^n}^{\mathrm{can}}.$$

Since $\mathbb{R}^n$ is a vector space, we get for every $x' = (q', p') \in T^*U$ that

$$(\omega_X^{\mathrm{can}})_{x'} = \omega_{\mathbb{R}^n}(d\Phi(x)\cdot, d\Phi(x)\cdot).$$

Now, since $\Phi$ is a diffeomorphism, $d\Phi(x)$ is a vector space isomorphism. Since also $\omega_{\mathbb{R}^n}$ is nondegenerate, it follows that $(\omega_X^{\mathrm{can}})_{x'}$ is as well.

Since the above construction works for every $x \in T^*M$, it follows that $\omega_X^{\mathrm{can}}$ is nondegenerate. This completes the proof of Proposition 2.31 □

## 2.3 Hamiltonian symplectomorphisms

Now that we have established what symplectic manifolds are, we can look into what we can actually do with them. We start with several definitions.

**Definition 2.32** (Symplectic isotopies)**.** Let $(M, \omega)$ be a symplectic manifold, and let $I$ be an interval. A smooth isotopy $\varphi : I \times M \to M$ is called a *symplectic isotopy* if $\varphi_t$ is a symplectomorphism for every $t \in I$. ◇

**Remark 2.33.** Recall that a smooth isotopy on a manifold $X$ and an interval $I$ is a smooth map $\varphi : I \times X \to X$ such that $\varphi_t$ is a diffeomorphism for every $t \in I$. Hence the notion of a symplectic isotopy is a natural extension of this concept to symplectic manifolds.

**Definition 2.34** (Symplectic vector fields)**.** Let $(M, \omega)$ be a symplectic manifold. We call a vector field $X$ on $M$ *symplectic* iff the 1-form

$$\iota_X\omega := \omega(X, \cdot)$$

is closed. We denote by $\mathcal{X}(M, \omega)$ the set of all symplectic vector fields on $(M, \omega)$. ◇

The next proposition shows that if a smooth isotopy is induced by a time-dependent vector field, then the isotopy is symplectic iff the vector fields are, at any time. If the reader is not familiar with time-dependent vector fields, a quick overview and some useful facts are given in Appendix B.

**Proposition 2.35** (Characterization of symplectic isotopies)**.** *Let $(M, \omega)$ be a symplectic manifold and $\varphi : I \times M \to M$ a smooth isotopy on $M$ such that there exists $t' \in I$ such that $\varphi_{t'} \in \mathrm{Symp}(M, \omega)$. Define a smooth, time-dependent vector field $X : I \times M \to TM$ by*

$$X(t, x) := \left.\frac{d}{ds}\right|_{s=t} \varphi(s, \varphi_t^{-1}(x)).$$

*Then $\varphi_t \in \mathrm{Symp}(M, \omega)$ for every $t \in I$ iff $X_t \in \mathcal{X}(M, \omega)$ for every $t \in I$.*

*Proof.* Note that we have

$$X(t, \varphi_t(x)) = \left.\frac{d}{ds}\right|_{s=t} \varphi(s, x)$$

for every $(t, x) \in I \times M$, so that $t \mapsto \varphi_t(x)$ is a maximal integral curve for every $x \in M$. Hence we get that the flow $\psi$ of $X$ is globally defined by

$$\psi(t, t_0, x) = (\varphi_t \circ \varphi_{t_0}^{-1})(x).$$

Indeed, we get $\psi(t_0, t_0, x) = (\varphi_{t_0} \circ \varphi_{t_0}^{-1})(x) = x$ and

$$\left.\frac{d}{ds}\right|_{s=t} \psi(s, t_0, x) = \left.\frac{d}{ds}\right|_{s=t} \varphi(s, \varphi_{t_0}^{-1}(x)) = X(t, \varphi_t(\varphi_{t_0}^{-1}(x))) = X(t, \psi(t, t_0, x)).$$

Now, denoting $\psi_{t,t_0} = \psi(t, t_0, \cdot) = \varphi_t \circ \varphi_{t_0}^{-1}$, we get by Proposition B.4 that

$$\left.\frac{d}{ds}\right|_{s=t} \left(\psi_{s,t_0}^* \omega\right)_x = \left(\psi_{t,t_0}^* \left(\mathcal{L}_{X_t} \omega\right)\right)_x,$$

and from this we get

$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=t} (\varphi_s^* \omega)_x &= \left.\frac{d}{ds}\right|_{s=t} ((\psi_{s,t_0} \circ \varphi_{t_0})^* \omega)_x = \varphi_{t_0}^* \left( \left.\frac{d}{ds}\right|_{s=t} \left(\psi_{s,t_0}^* \omega\right) \right)_x \\ &= \varphi_{t_0}^* \left( \left(\psi_{t,t_0}^* \left(\mathcal{L}_{X_t} \omega\right)\right) \right)_x = \left((\psi_{t,t_0} \circ \varphi_{t_0})^* \left(\mathcal{L}_{X_t} \omega\right)\right)_x \\ &= \left(\varphi_t^* \left(\mathcal{L}_{X_t} \omega\right)\right)_x. \end{aligned} \tag{2.2}$$

Using Eq. 2.2 and Cartan's Magic Formula,

$$\mathcal{L}_{X_t} \omega = \iota_{X_t}(d\omega) + d\left(\iota_{X_t} \omega\right) = d\left(\iota_{X_t} \omega\right), \tag{2.3}$$ [9]

we can now easily prove the proposition. First, assume that $\varphi$ is a symplectic isotopy. Then for every $t \in I$

$$0 = \left.\frac{d}{ds}\right|_{s=t} \omega = \left.\frac{d}{ds}\right|_{s=t} \varphi_s^* \omega = \varphi_t^* \left(\mathcal{L}_{X_t} \omega\right).$$

[9] Here the second equality follows since $\omega$ is symplectic, and thus closed.

Since $\varphi_t$ is a diffeomorphism, it follows that $\mathcal{L}_{X_t}\omega = 0$, and by Eq. 2.3 we get that $X_t$ is symplectic.

Conversely, assume that $X_t$ is symplectic for every $t \in I$. Then by the exact same computations as above we get that

$$\left.\frac{d}{ds}\right|_{s=t} \varphi_s^*\omega = 0$$

for every $t \in I$. Fixing any $x \in M$ and $v, w \in T_xM$, this means that the map $t \mapsto (\varphi_t^*\omega)_x(v, w)$ is constant. Hence for every $t \in I$ we have that $\varphi_t^*\omega = \varphi_{t'}^*\omega = \omega$, and that $\varphi$ is symplectic. $\square$

In order to define Hamiltonian symplectomorphisms, we need to define even more specific isotopies and vector fields.

**Definition 2.36** (Hamilonian vector fields)**.** Let $(M, \omega)$ be a symplectic manifold, and let $H : M \to \mathbb{R}$ be a smooth function on $M$. We define the *Hamiltonian vector field* $X_H$ generated by $H$ by

$$\omega(X_H, \cdot) = dH,$$

i.e. for every $x \in M$ we define $X_H(x)$ by

$$\omega_x(X_H(x), \cdot) = dH(x). \qquad \diamondsuit$$

Note that since $H$ is smooth, and since $\omega$ is smooth and nondegenerate, this formula indeed gives a well-defined, smooth vector field.

**Remark 2.37.** By definition, it follows immediately that every Hamiltonian vector field is also symplectic.

**Definition 2.38** (Hamiltonian isotopies)**.** Let $I$ be an interval, $(M, \omega)$ a symplectic manifold, and $H : I \times M \to \mathbb{R}$ a smooth, time-dependent function. Denote by $X_H : I \times M \to TM$ the smooth, time-dependent vector field defined by $X_H(t, x) := X_{H_t}(x)$, where $X_{H_t}$ is as in Definition 2.36. A smooth isotopy $\varphi : I \times M \to M$ that is generated by $X_H$ in the sense that

$$\left.\frac{d}{ds}\right|_{s=t} \varphi(s, x) = X_H(t, \varphi(t, x))$$

is called a *Hamiltonian isotopy.* $\diamondsuit$

**Remark 2.39.** Since the vector fields $X_{H_t}$ are Hamiltonian, and thus also symplectic, it follows by Proposition 2.35 that every Hamiltonian isotopy is also symplectic.

**Definition 2.40** (Hamiltonian symplectomorphisms)**.** Let $(M, \omega)$ be a symplectic manifold. We call a diffeomorphism $\psi : M \to M$ a *Hamiltonian symplectomorphism* if there is a Hamiltonian isotopy $\varphi : [0, 1] \times M \to M$ from $\varphi_0 = \mathrm{Id}$ to $\varphi_1 = \psi$. $\diamondsuit$

**Remark 2.41.** Since any Hamiltonian isotopy is symplectic, and since the identity is a symplectomorphism, it follows from Proposition 2.35 that any Hamiltonian symplectomorphism is also a symplectomorphism.

**Remark 2.42.** Let us go a little bit deeper into what a Hamiltonian symplectomorphism really is. By Definition 2.38, the Hamiltonian isotopy $\varphi$ mentioned in Definition 2.40 is generated by some smooth function $H : [0,1] \times M \to \mathbb{R}$. Denoting by $X_H : [0,1] \times M \to TM$ the Hamiltonian vector field induced by $H$, we know that $X_H$ has a global flow $\Phi : I \times I \times M \to M$.[10] Since this flow is unique, we have that $\varphi_t = \Phi_{t,0}$ for all $t \in I$. Hence we can see a Hamiltonian symplectomorphism as the time-1 flow of some Hamiltonian function $H : [0,1] \times M \to \mathbb{R}$.

Note that not any time-dependent function $H : [0,1] \times M \to M$ determines a Hamiltonian symplectomorphism, since the vector field $X_H$ induced by $H$ might not have a global flow. However, if $H$ has compact support, then so does $X_H$, and hence it has a global flow. In particular, if the manifold $M$ is closed, every smooth function $[0,1] \times M \to M$ determines a Hamiltonian symplectomorphism.

We have the following (very) weak result about fixed points of Hamiltonian symplectomorphisms induced by time-*independent* Hamiltonians.

**Corollary 2.43.** *Let $(M, \omega)$ be a symplectic manifold and $\psi : M \to M$ a Hamiltonian symplectomorphism induced by a time-independent Hamiltonian $H : M \to \mathbb{R}$. Then every critical point of $H$ induces a fixed point of $\psi$. In particular, if $M$ is compact, then $\psi$ has at least two fixed points.*

*Proof.* Let $x_0 \in M$ be a critical point of $H$. Then we have

$$\omega_{x_0}(X_H(x_0), \cdot) = 0$$

and thus that $X_H(x_0) = 0$. Thus the constant map $\gamma : [0,1] \to M$ given by $\gamma(t) := x_0$ is a (maximal) integral curve of $X_H$, and we clearly have $\psi(x_0) = x_0$.

To see the second part, note that if $M$ is compact, then $H$ has a minimum and maximum on that set, at points $x_+, x_- \in M$. We show that these are critical points of $H$. Indeed, let $v \in T_{x_+}M$ and choose a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = x_+$ and $\dot\gamma(0) = v$. Then $H \circ \gamma$ has a maximum at 0 and thus we have

$$dH(x_+)(v) = dH(\gamma(0))(\dot\gamma(0)) = \left.\frac{d}{dt}\right|_{t=0} (H \circ \gamma)(t) = 0.$$

A similar argument works for $x_-$. $\square$

Similar to the groups of (linear) symplectomorphisms, we also have a group structure on the set of Hamiltonian symplectomorphisms.

[10] Just as in the proof of Proposition 2.35, this is because the flow is given by $\Phi_{t,t_0} = \varphi_t \circ \varphi_{t_0}^{-1}$.

**Definition 2.44.** For a symplectic manifold $(M, \omega)$, we denote by $\mathrm{Ham}(M, \omega)$ the set of all Hamiltonian symplectomorphisms on $(M, \omega)$. $\diamondsuit$

We will show that it is actually a normal subgroup of $\mathrm{Symp}(M, \omega)$. Unlike before, this will not simply follow from the definitions, and will require a little more work. In particular, we will use the following lemma.

**Lemma 2.45.** *Let $(M, \omega)$ be a symplectic manifold, $H : M \to \mathbb{R}$ a smooth function and $\varphi \in \mathrm{Symp}(M, \omega)$. Then*

$$X_{H\circ\varphi} = (d\varphi)^{-1} \circ X_H \circ \varphi,$$

*i.e. we have for every $x \in M$ that*

$$X_{H\circ\varphi}(x) = \left(d\varphi(x)^{-1} \circ X_H \circ \varphi\right)(x).$$

*Proof.* This follows from a straightforward computation. We need to show that, for every $x \in M$,

$$\omega_x((d\varphi(x)^{-1} \circ X_H \circ \varphi)(x), \cdot) = d(H \circ \varphi)(x).$$

Now, note that

$$d(H \circ \varphi)(x) = dH(\varphi(x)) \circ d\varphi(x) = \omega_{\varphi(x)}((X_H \circ \varphi)(x), d\varphi(x)\cdot).$$

Since $\varphi$ is a symplectomorphism it now follows that

$$d(H\circ\varphi)(x) = \omega_{\varphi(x)}((d\varphi(x)\circ d\varphi(x)^{-1}\circ X_H\circ\varphi)(x), d\varphi(x)\cdot) = \omega_x((d\varphi(x)^{-1}\circ X_H\circ\varphi)(x), \cdot).$$

$\square$

We are now ready to prove the next proposition.

**Proposition 2.46** (Hamiltonian group)**.** *Let $(M, \omega)$ be a symplectic manifold. Then* $\mathrm{Ham}(M, \omega)$ *is a normal subgroup of* $\mathrm{Symp}(M, \omega)$.

*Proof.* We can divide this proposition into three claims.

**Claim 1.** For any $\varphi, \psi \in \mathrm{Ham}(M, \omega)$, we have that $\varphi \circ \psi \in \mathrm{Ham}(M, \omega)$.

**Claim 2.** For any $\varphi \in \mathrm{Ham}(M, \omega)$, we have that $\varphi^{-1} \in \mathrm{Ham}(M, \omega)$.

**Claim 3.** For any $\varphi \in \mathrm{Ham}(M, \omega)$ and $\psi \in \mathrm{Symp}(M, \omega)$ we have that $\psi^{-1} \circ \varphi \circ \psi \in \mathrm{Ham}(M, \omega)$.

**Proof of Claim 1:** By definition, there exist smooth functions $G, H : [0, 1] \times M \to \mathbb{R}$ and Hamiltonian isotopies $\Phi, \Psi : [0, 1] \times M \to M$ such that $\Phi_0 = \Psi_0 = \mathrm{Id}$, $\Phi_1 = \varphi$, $\Psi_1 = \psi$ and

$$\left.\frac{d}{ds}\right|_{s=t} \Phi_s(x) = X_{G_t}(\Phi_t(x)),$$
$$\left.\frac{d}{ds}\right|_{s=t} \Psi_s(x) = X_{H_t}(\Psi_t(x)),$$

for every $(t,x) \in [0,1] \times M$. Then we get

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=t} (\Phi_s \circ \Psi_s)(x) &= \left.\frac{d}{ds}\right|_{s=t} (\Phi_s \circ \Psi_t)(x) + \left.\frac{d}{ds}\right|_{s=t} (\Phi_t \circ \Psi_s)(x) \\
&= X_{G_t}((\Phi_t \circ \Psi_t)(x)) + d\Phi_t(\Psi_t(x)) \left(X_{H_t}(\Psi_t(x))\right) \\
&= \left[X_{G_t} + d(\Phi_t^{-1})^{-1} \circ X_{H_t} \circ \Phi_t^{-1}\right] ((\Phi_t \circ \Psi_t)(x)) \\
&= X_{G_t + H_t \circ \Phi_t^{-1}}((\Phi_t \circ \Psi_t)(x)).
\end{aligned}$$

Here in the last equality we use Lemma 2.45, linearity of the exterior derivative and bilinearity of $\omega$.

Hence we have a Hamiltonian isotopy $\Phi' : [0,1] \times M \to M$ given by $\Phi'(t,x) = (\Phi_t \circ \Psi_t)(x)$, generated by the smooth function $(t,x) \mapsto G_t(x) + (H_t \circ \Phi_t^{-1})(x)$, with $\Phi'(0,x) = x$ and $\Phi'(1,x) = (\Phi \circ \Psi)(x)$. Hence $\Phi \circ \Psi$ is a Hamiltonian symplectomorphism. This proves Claim 1.

**Proof of Claim 2:** By definition, there exists a smooth function $H : [0,1] \times M \to \mathbb{R}$ and a Hamiltonian isotopy $\Phi : [0,1] \times M \to M$ such that $\Phi_0 = \mathrm{Id}$, $\Phi_1 = \varphi$ and

$$\left.\frac{d}{ds}\right|_{s=t} \Phi_s(x) = X_{H_t}(\Phi_t(x)),$$

for every $(t,x) \in [0,1] \times M$. In the same way as before, we have

$$\begin{aligned}
0 &= \left.\frac{d}{ds}\right|_{s=t} (\Phi_s \circ \Phi_s^{-1})(x) \\
&= X_{H_t}(x) + d\Phi_t(\Phi_t^{-1}(x)) \left( \left.\frac{d}{ds}\right|_{s=t} \Phi_s^{-1}(x) \right),
\end{aligned}$$

and thus

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=t} \Phi_s^{-1}(x) &= -d\Phi_t(\Phi_t^{-1}(x))^{-1}(X_{H_t}(x)) \\
&= -X_{H_t \circ \Phi_t}(\Phi_t^{-1}(x)) \\
&= X_{-H_t \circ \Phi_t}(\Phi_t^{-1}(x)).
\end{aligned}$$

Therefore, the smooth function $(t,x) \mapsto -(H_t \circ \Phi_t)(x)$ generates the Hamiltonian isotopy $\Phi' : [0,1] \times M \to M$ given by $\Phi'(t,x) = \Phi_t^{-1}(x)$. Since $\Phi'(0,x) = x$ and $\Phi'(1,x) = \varphi^{-1}(x)$, it follows that $\varphi^{-1}$ is a Hamiltonian symplectomorphism. This proves Claim 2.

**Proof of Claim 3:** By definition, there exists a smooth function $H : [0,1] \times M \to \mathbb{R}$ and a Hamiltonian isotopy $\Phi : [0,1] \times M \to M$ such that $\Phi_0 = \mathrm{Id}$, $\Phi_1 = \varphi$ and

$$\left.\frac{d}{ds}\right|_{s=t} \Phi_s(x) = X_{H_t}(\Phi_t(x)),$$

for every $(t,x) \in [0,1] \times M$. We get that

$$\left.\frac{d}{ds}\right|_{s=t} (\psi^{-1} \circ \Phi_s \circ \psi)(x) = d(\psi^{-1})((\Phi_t \circ \psi)(x))(X_{H_t}((\Phi_t \circ \psi)(x)))$$
$$= d\psi((\psi^{-1} \circ \Phi_t \circ \psi)(x))^{-1}(X_{H_t}((\psi \circ \psi^{-1} \circ \Phi_t \circ \psi)(x)))$$
$$= X_{H_t \circ \psi}((\psi^{-1} \circ \Phi_s \circ \psi)(x)).$$

We conclude that the smooth function $(t,x) \mapsto (H_t \circ \psi)(x)$ generates the Hamiltonian isotopy $\Phi' : [0,1] \times M \to M$ given by $\Phi'(t,x) = (\psi^{-1} \circ \Phi_t \circ \psi)(x)$. Since $\Phi'(0,x) = x$ and $\Phi'(1,x) = (\psi^{-1} \circ \varphi \circ \psi)(x)$, it follows that $\psi^{-1} \circ \varphi \circ \psi$ is a Hamiltonian symplectomorphism. This proves Claim 3 and completes the proof of Proposition 2.46. □

We will discuss one more useful lemma regarding Hamiltonian symplectomorphisms.

**Lemma 2.47** (Reparametrizing Hamiltonian isotopies)**.** *Let $(M,\omega)$ be a symplectic manifold, $I$ an interval and $\varphi_H : I \times M \to M$ a Hamiltonian isotopy induced by a smooth function $H : I \times M \to \mathbb{R}$. For any smooth map $\beta : I \to I$, the map $\widetilde{\varphi}_H : I \times M \to M$ given by $\widetilde{\varphi}^t_H(x) := \varphi_H^{\beta(t)}(x)$ is a Hamiltonian isotopy induced by the smooth function $\widetilde{H} : I \times M \to \mathbb{R}$ given by $\widetilde{H}_t(x) := \beta'(t) \cdot H_{\beta(t)}(x)$.*

*Proof.* Denote by $X_H : I \times M \to TM$ the time-dependent vector field induced by $H$. Then we see that

$$d\widetilde{H}_t = \beta'(t) \cdot dH_{\beta(t)} = \beta'(t) \cdot \omega(X_H^{\beta(t)}, \cdot) = \omega(\beta'(t) \cdot X_H^{\beta(t)}, \cdot).$$

Hence $X^t_{\widetilde{H}} = \beta'(t) \cdot X_H^{\beta(t)}$. Now, we get that

$$\left.\frac{d}{ds}\right|_{s=t} \widetilde{\varphi}^s_H(x) = \beta'(t) \cdot \left.\frac{d}{ds}\right|_{s=\beta(t)} \varphi^s_H(x) = \beta'(t) \cdot X_H^{\beta(t)}\big(\varphi_H^{\beta(t)}(x)\big)$$
$$= X^t_{\widetilde{H}}\big(\widetilde{\varphi}^t_H(x)\big).$$

This proves the lemma. □

This lemma has an immediate consequence.

**Corollary 2.48.** *Let $(M,\omega)$ be a symplectic manifold and $\varphi : [0,1] \times M \to M$ a Hamiltonian isotopy starting at the identity. Then for every $t \in [0,1]$, $\varphi^t = \varphi(t,\cdot) : M \to M$ is a Hamiltonian symplectomorphism.*

*Proof.* This follows immediately by applying Lemma 2.47 to the smooth map $\beta_t : [0,1] \to [0,1]$ give by $\beta_t(s) := st$ for any $t \in [0,1]$. □

**Remark 2.49.** Recall from Remark 2.42 that we can view a Hamiltonian symplectomorphism as the "time-1 flow" of some Hamiltonian function. Specifically, let $H : [0,1] \times M \to M$ be a smooth function and assume that its induced time-dependent vector field $X_H$ has a global flow, which we denote by $\varphi_H : [0,1] \times [0,1] M \to M$. Now, from the above corollary, it follows that not only $\varphi_H^{1,0}$ is a Hamiltonian symplectomorphism, but in fact $\varphi_H^{t_1,t_0}$ for *any* $(t_1, t_0) \in [0,1] \times [0,1]$. To see this, note that by the corrollary we have that $\varphi_H^{t_1,0}$ is a Hamiltonian symplectomorphism for any $t_1 \in [0,1]$. Now, using Proposition 2.46 and that

$$\varphi_H^{t_1,t_0} = \varphi_H^{t_1,0} \circ \varphi_H^{0,t_0} = \varphi_H^{t_1,0} \circ \left(\varphi_H^{t_0,0}\right)^{-1}$$

the result follows.

### 2.3.1 Nondegenerate fixed points

As promised, we will quickly state the definition of (non-)degenerate fixed points.

**Definition 2.50.** Let $X$ be a manifold and $\varphi : X \to X$ a diffeomorphism. A *fixed point* of $\varphi$ is a point $x \in X$ such that $\varphi(x) = x$. A fixed point $x$ of $\varphi$ is called *nondegenerate* if the differential of $\varphi$ at $x$ does not have 1 as an eigenvalue, i.e. if

$$\det(d\varphi(x) - \mathrm{Id}) \neq 0.$$

Otherwise, the fixed point is called *degenerate.* $\diamondsuit$

Closely related to this is the notion of nondegeneracy of critical points of smooth functions $X \to \mathbb{R}$. Recall the following definition.

**Definition 2.51.** Let $X$ be a manifold and $f : X \to \mathbb{R}$ a smooth function. A *critical point* of $f$ is a point $x \in X$ such that $df(x) = 0$. A critical point $x$ of $f$ is called *nondegenerate* if the Hessian $D^2 f(x)$ is invertible. Otherwise it is called *degenerate.* $\diamondsuit$

**Remark 2.52.** There are may different ways to define the Hessian at a critical point, e.g. as a linear map $D^2 f(x) : T_x X \to T_x^* X$ or a symmetric bilinear form $D^2 f(x) : T_x X \times T_x X \to \mathbb{R}$. In the latter case we say a critical point is nondegenerate if this bilinear form is nondegenerate. The important fact is that the notion of nondegeneracy of a critical point is equivalent in the different definitions of the Hessian. The most useful way of determining whether a critical point is nondegenerate is to look at the Hesse matrix, i.e. the matrix of second partial derivatives, of a coordinate representation of $f$. A critical point is nondegenerate iff this matrix is invertible.

## 2.4 The symplectic torus

In this section we will describe the torus $\mathbb{T}^{2n}$ as a quotient of $\mathbb{R}^{2n}$ under the action of $\mathbb{Z}^{2n}$ and define a symplectic form using this quotient structure. We will also show that the quotient $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is a smooth covering of the torus. For a brief overview of Lie group actions and (smooth) covering spaces, see Appendices C and D, respectively.

A natural question one might ask is under which conditions the orbit space $X/G$ of a smooth manifold $X$ under the smooth action of a Lie group $G$ has a canonical smooth structure. The answer is given by the following theorem, often referred to as the Quotient Manifold Theorem.

**Theorem 2.53** (Quotient Manifold Theorem)**.** *Let $G$ be a Lie group acting smoothly, freely and properly on a smooth manifold $X$. Then the orbit space $X/G$, endowed with the quotient topology, has a unique smooth structure such that the quotient map $\pi : X \to X/G$ is a smooth submersion. The dimension of this structure is* $\dim X - \dim G$.

We will not give any proof of this theorem here, since it is rather involved and will not be used elsewhere in this thesis. A proof can be found in e.g. [Lee12, Theorem 21.10].

This theorem only partly solves our problems, since we also want to prove that the quotient $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is a smooth covering of the torus. For this, we will use the fact that $\mathbb{Z}^{2n}$ is actually a discrete Lie group. There is a special case of the Quotient Manifold Theorem for discrete Lie groups, i.e. 0-dimensional Lie groups, that provides an answer.

**Theorem 2.54.** *Let $G$ be a discrete Lie group acting smoothly, freely and properly on a smooth manifold $X$. Then the orbit space $X/G$, endowed with the quotient topology, has a unique smooth structure such that the quotient map $\pi : X \to X/G$ is a smooth covering map.*

For the proof of this theorem, we will paraphrase [Lee12, Theorem 21.13].
We need the following lemma, of which we postpone the proof.

**Lemma 2.55.** *Let $G$ be a discrete Lie group acting continuously and freely on a topological manifold $X$. If in addition the action is proper, the following statement holds;*

(∗) *Every point $x \in X$ has an open neighbourhood $V$ with the property that for each $g \in G$ not equal $e$ we have that $(g \cdot V) \cap V = \emptyset$.*

*Proof of Theorem 2.54.* The Quotient Manifold Theorem immediately yields that $X/G$ is a topological manifold of dimension $\dim X - \dim G = \dim X$, and a unique smooth structure such that the quotient $\pi : X \to X/G$ is a smooth submersion. Since $\dim X/G = \dim X$, $\pi$ then becomes a local diffeomorphism. Hence we just need to show that $\pi$ is a topological covering map. Then it is also a smooth covering map, and since any smooth covering map is a smooth submersion, uniqueness of the structure follows from the uniqueness part of the Quotient Manifold Theorem.

Now let $x \in X$, let $V_x$ be a neighbourhood as in Proposition 2.55 and define $U_x := \pi(V_x)$. Then $U_x$ is open by Proposition C.11 and we also have that

$$\pi^{-1}(U_x) = \bigcup_{g \in G} (g \cdot V_x).$$

Since $g_1 \cdot x = g_2 \cdot x'$ for $x, x' \in V_x$ implies that $\big((g_2^{-1}g_1) \cdot V_x\big) \cap V_x \neq \emptyset$, it follows by Lemma 2.55 that then $g_1 = g_2$, and thus all sets of the form $g \cdot V_x$ are actually disjoint. Hence we are left with showing that $\pi_{g,x} := \pi|_{g \cdot V_x} : g \cdot V_x \to U_x$ is a homeomorphism. Since it is a restriction of $\pi$, it immediately follows that $\pi_{g,x}$ is open, continuous and surjective. So suppose that $\pi(y) = \pi(y')$ for $y, y' \in g \cdot v_x$. Then $h \cdot y = y'$ for some $h \in G$, and by the same reasoning as before, it follows that $h = e$, and thus that $y = y'$. So $\pi_{g,x}$ is injective, and thus a homeomorphism, since we already knew it was open, continuous and surjective. Now the cover $\{U_x\}_{x \in X}$ makes $\pi : X \to X/G$ into a topological covering space. This completes the proof of Theorem 2.54. □

*Proof of Lemma 2.55.* To show $(*)$, let $x \in X$. Since $X$ is a manifold, we can choose an open, precompact neighbourhood $V$ of $x$. By Proposition C.8 the set

$$G_{\overline{V}} = \{g \in G \mid (g \cdot \overline{V}) \cap \overline{V} \neq \emptyset\}$$

is compact, and hence finite, since $G$ is discrete. Writing $G_{\overline{V}} = \{e, g_1, \ldots, g_k\}$, we know that $g_i^{-1} \cdot x \neq x$ for all $i$, since the action is free. Since $X$ is Hausdorff, we can choose an open neighbourhood $V_i$ of $g_i^{-1} \cdot x$ such that $x \notin V_i$ for all $i$. Defining

$$W := V \setminus \bigcup_{i=1}^{k} \overline{V_i},$$

this set is also an open precompact neighbourhood of $x$, with the extra property that $g_i^{-1} \cdot x \notin \overline{W}$ for all $i$. This means that $x \notin g_i \cdot \overline{W}$ for all $i$, and thus that the set

$$U := W \setminus (g_1 \cdot \overline{W} \cup \cdots \cup g_k \cdot \overline{W})$$

is as required. This proves Lemma 2.55. □

Under suitable conditions, a differentiable $k$-form on $X$ descends to the quotient $X/G$, in the sense that there exists a unique differentiable $k$-form $\bar{\omega}$ on $X/G$ such that $\pi^*\bar{\omega} = \omega$. Since the general case requires knowledge of Lie algebras and exponential maps, we will only mention and prove the conditions for discrete Lie groups, since then the situation simplifies dramatically. We need the following definition.

**Definition 2.56.** Let $G$ be a Lie group acting smoothly on a smooth manifold $X$ and let $\omega$ be a differential $k$-form on $X$. The we say that $\omega$ is *$G$-invariant* if $\varphi_g^*\omega = \omega$ for all $g \in G$, where $\varphi_g : X \to X$ is the smooth map

$$\varphi_g(x) := g \cdot x. \qquad \diamondsuit$$

Now we have the following proposition.

**Proposition 2.57.** *Let $G$ be a discrete Lie group acting smoothly, freely and properly on a smooth manifold $X$. We endow $X/G$ with the smooth structure as in Theorem 2.54. Let $\omega$ be a $G$-invariant differential $k$-form on $X$. Then there is a unique differential $k$-form $\bar{\omega}$ on $X/G$ such that $\pi^*\bar{\omega} = \omega$.*

*Proof.* Let $\bar{x} \in X/G$ and $\bar{v}_1, \ldots, \bar{v}_k \in T_{\bar{x}}(X/G)$. We choose some $x \in \pi^{-1}(\bar{x})$ and define

$$\bar{\omega}_{\bar{x}}(\bar{v}_1, \ldots, \bar{v}_k) := \omega_x(d\pi(x)^{-1}\bar{v}_1, \ldots, d\pi(x)^{-1}\bar{v}_k).$$

To show that this is well-defined, let $x, x' \in \pi^{-1}(\bar{x})$. This means that $\varphi_g(x) = x'$ for some $g \in G$. Then, since $\pi \circ \varphi_g = \pi$, we get that $d\pi(x')d\varphi_g(x) = d\pi(x)$, and thus that

$$\begin{aligned} \omega_{x'}(d\pi(x')^{-1}\bar{v}_1, \ldots, d\pi(x')^{-1}\bar{v}_k) &= \omega_{\varphi_g(x)}(d\varphi_g(x)d\pi(x)^{-1}\bar{v}_1, \ldots, d\varphi_g(x)d\pi(x)^{-1}\bar{v}_k) \\ &= (\varphi_g^*\omega)_x(d\pi(x)^{-1}\bar{v}_1, \ldots, d\pi(x)^{-1}\bar{v}_k) \\ &= \omega_x(d\pi(x)^{-1}\bar{v}_1, \ldots, d\pi(x)^{-1}\bar{v}_k). \end{aligned}$$

Hence our definition does not depend on the choice of $x \in \pi^{-1}(\bar{x})$. Now, the fact that $\pi^*\bar{\omega} = \omega$ and uniqueness of the form follow immediately, and smoothness follows since $\pi$ is a local diffeomorphism.[11] Thus we have proved the proposition. ☐

**Remark 2.58.** One can immediately see why the case for discrete Lie groups is easier; in that case, $d\pi(x)$ is an isomorphism, and we only have to worry about independence of the choice of $x \in \pi^{-1}(\bar{x})$. In the general case, we only know that $d\pi(x)$ is surjective, and thus we have to worry about independence of the choice of $v_i \in d\pi(x)^{-1}(\bar{v}_i)$. This is where an extra condition comes in, which is void in the discrete case.

Finally, let us look at the torus. We consider the smooth manifold $\mathbb{R}^n$, with its standard smooth structure. The abelian group $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ by addition. Since $\mathbb{Z}^n$ is countable, we can endow it with the discrete topology and make it into a smooth 0-manifold. Since every map between 0-manifolds is smooth, $\mathbb{Z}^n$ then becomes a discrete Lie group. Furthermore, the action is clearly smooth, and it is free because $k + x = x$ for any $x \in \mathbb{R}^n$ of course implies $k = 0$. For properness, we use Proposition C.8(ii); if $(x_i)_{i\in\mathbb{N}}$ is a sequence in $\mathbb{R}^n$ and $(k_i)_{i\in\mathbb{N}}$ a sequence in $G$ such that $x_i \to x$ and $k_i + x_i \to x'$ for some $x, x' \in \mathbb{R}^n$, then $k_i \to x' - x$. Therefore the action is proper. Now, using Theorem 2.54 we define the $n$-dimensional torus $\mathbb{T}^n$ to be the quotient space $\mathbb{R}^n/\mathbb{Z}^n$ endowed with the unique smooth structure that makes $\pi : \mathbb{R}^n \to \mathbb{T}^n$ a smooth covering map.

**Remark 2.59.** A different, and perhaps more common definition of the $n$-dimensional torus is the Cartesian product of $n$ copies of $S^1$, endowed with the product smooth structure. Here $S^1$ is endowed with the standard smooth structure on spheres. It turns out that these two different definition of the torus are actually the same; that is, there is a diffeomorphism between the two spaces. One can show that if $X, X_1$ and $X_2$ are smooth manifolds and $\pi_1 : X \to X_1$, $\pi_2 : X \to X_2$ smooth surjective submersions that make the same identifications, then there exists a diffeomorphism $f : X_1 \to X_2$. To apply this in our current case, it is easy to show that the map $\pi' : \mathbb{R}^n \to S^1 \times \cdots \times S^1$ given by

$$\pi'(x_1, \ldots, x_n) = \Big((\cos(2\pi x_1), \sin(2\pi x_1)), \ldots, (\cos(2\pi x_n), \sin(2\pi x_n))\Big)$$

is a smooth, surjective submersion that identifies two points $x, x' \in \mathbb{R}^n$ iff $x - x' \in \mathbb{Z}^n$. Hence there is a diffeomorphism $\mathbb{T}^n \to S^1 \times \cdots \times S^1$.

[11] Indeed, locally our definition is just the push-forward of $\omega$ by $\pi$.

Finally, we can define a smooth structure on $\mathbb{T}^{2n}$, using Proposition 2.57. Let $\bar{\omega}_0$ be the unique differential 2-form on $\mathbb{T}^{2n}$ such that $\pi^*\bar{\omega}_0 = \omega_0$.

**Lemma 2.60.** *The differential 2-form $\bar{\omega}_0$ on $\mathbb{T}^{2n}$ is symplectic.*

*Proof.* This follows practically immediately from how $\bar{\omega}_0$ was constructed; for closedness, note that

$$\pi^* d\bar{\omega}_0 = d(\pi^*\bar{\omega}_0) = d\omega_0 = 0.$$

In other words, for all $x \in \mathbb{R}^{2n}$ and $v_1, v_2, v_3 \in T_x\mathbb{R}^{2n}$ we have that

$$(d\bar{\omega}_0)_{\pi(x)}(d\pi(x)v_1, d\pi(x)v_2, d\pi(x)v_3) = 0$$

Since $\pi$ is surjective, and $d\pi(x)$ is surjective for every $x \in \mathbb{R}^{2n}$, it then follows that $d\bar{\omega}_0 = 0$, hence $\bar{\omega}_0$ is closed.

Nondegeneracy follows from the fact that, for any $\bar{x} \in \mathbb{T}^{2n}$ and any $\bar{v}_1, \bar{v}_2 \in T_{\bar{x}}\mathbb{T}^{2n}$,

$$(\bar{\omega}_0)_{\bar{x}}(\bar{v}_1, \bar{v}_2) = \omega_x(d\pi(x)^{-1}\bar{v}_1, d\pi(x)^{-1}\bar{v}_2)$$

for any $x \in \pi^{-1}(\bar{x})$. If then

$$(\bar{\omega}_0)_{\bar{x}}(\bar{v}_1, \bar{v}_2) = 0$$

for all $\bar{v}_2 \in T_{\bar{x}}\mathbb{T}^{2n}$, then

$$(\omega_0)_x(d\pi(x)^{-1}\bar{v}_1, v_2)$$

for all $v_2 \in T_x\mathbb{R}^{2n}$, since $d\pi(x)^{-1}$ is surjective. Hence since $\omega_0$ is nondegenerate, it follows that $d\pi(x)^{-1}\bar{v}_1 = 0$, and thus that $\bar{v}_1 = 0$. Hence $\bar{\omega}_0$ is nondegenerate, and indeed symplectic. □

So now we have finally obtained a symplectic structure on the torus.

**Definition 2.61** (Symplectic torus)**.** On $\mathbb{T}^{2n}$, we define the *standard symplectic form* $\bar{\omega}_0$ to be the unique differential 2-form that satisfies $\pi^*\bar{\omega}_0 = \omega_0$, where $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n} = \mathbb{T}^{2n}$ is the quotient. The pair $(\mathbb{T}^{2n}, \bar{\omega}_0)$ is called the *standard symplectic torus.* ♢

From now on, when we refer to the torus $\mathbb{T}^{2n}$, we will mean the symplectic manifold $(\mathbb{T}^{2n}, \bar{\omega}_0)$.

**Remark 2.62.** The above construction of the symplectic structure on the torus is elegant, since the main results we used hold very generally for all quotients by Lie groups (provided of course that the Lie groups act freely and properly). However, for the torus specifically, we can also determine the smooth structure and differential form in a different way.

Denoting by $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ the quotient map (where we endow $\mathbb{R}^n/\mathbb{Z}^n$ with the quotient topology, but without a smooth structure), we have for any $x \in \mathbb{R}^n$ that for

$$U_x := (x_1, x_1 + 1) \times \cdots \times (x_n, x_n + 1),$$

$\pi|_{U_x} : U_x \to \pi(U_x)$ is a homeomorphism. The transition map for two of these charts is simply translation in $\mathbb{R}^n$ by a constant, and hence $\{\pi(U_x)\}_{x\in\mathbb{R}^n}$ determines a smooth structure on $\mathbb{R}^n/\mathbb{Z}^n$. In fact, it can be easily seen that this same cover makes $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ into a smooth covering map. Hence this structure is the same as the one we obtained from the above theorems.

For the symplectic structure on $\mathbb{T}^{2n}$, we use that $\mathbb{R}^{2n}/\mathbb{Z}^{2n} \simeq (\mathbb{R}/\mathbb{Z})^{2n}$, and we define the projections $q^i, p_i$ onto the $(2i-1)$-th and and $(2i)$-th component respectively. Canonically identifying $T_x(\mathbb{R}/\mathbb{Z}) \simeq \mathbb{R}$, we obtain a differential 2-form

$$\sum_{i=1}^{n} dq^i \wedge dp_i.$$

It can be computed that this is the standard form $\bar{\omega}_0$ defined above.

# 3 Generating functions

In this section we will use the theory of generating functions to take our first step towards proving Theorem 1.3. First, we will discuss the so-called *generating functions of type* $V$, after which we will use these functions to define the *discrete symplectic action*. As mentioned in Section 2.3.1, the first step of the proof will be relating the fixed points of some Hamiltonian symplectomorphism on $\mathbb{T}^{2n}$ to the critical points of this function.

## 3.1 Generating functions of type $V$

Consider the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$ and let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a symplectomorphism. Throughout this section, we will write a point in $\mathbb{R}^{2n}$ as $(x, y)$, where $x$ and $y$ are thus elements of $\mathbb{R}^n$. Taking a point $(x_0, y_0) \in \mathbb{R}^{2n}$, we can write

$$\varphi(x_0, y_0) = (u(x_0, y_0), v(x_0, y_0)) = (x_1, y_1),$$

where $u, v : \mathbb{R}^{2n} \to \mathbb{R}^n$ are the component functions of $\varphi$. Using this notation, let us move on to the definition of a generating function.

**Definition 3.1** (Generating functions)**.** Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a symplectomorphism.[12] A *generating function (of type* $V$*)* for $\varphi$ is a smooth function $V : \mathbb{R}^{2n} \to \mathbb{R}$ such that $(x_1, y_1) = \varphi(x_0, y_0)$ iff

$$x_1 - x_0 = \frac{\partial V}{\partial y}(x_1, y_0), \qquad\qquad y_1 - y_0 = -\frac{\partial V}{\partial x}(x_1, y_0). \tag{3.1}$$

Here $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ denote the standard partial derivatives of $V$. ♢

The above equations can be seen as a discrete-time version of Hamilton's equations. Our goal is to show that any symplectomorphism that is sufficiently close to the identity admits a generating function as defined above. The next proposition accomplishes this.

**Proposition 3.2** (Existence of generating function)**.** *Let* $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ *be a symplectomorphism such that*

$$\|d\varphi(z) - \mathrm{Id}\| < \frac{1}{2}$$

*for all* $z \in \mathbb{R}^{2n}$*. Then* $\varphi$ *admits a generating function of type* $V$*.*

The proof uses the following lemma.

**Lemma 3.3.** *Let* $\psi : \mathbb{R}^n \to \mathbb{R}^n$ *be a smooth map such that*

$$\|d\psi(x) - \mathrm{Id}\| < \frac{1}{2}$$

*for all* $x \in \mathbb{R}^n$*. Then* $\psi$ *is a diffeomorphism.*

[12] Whenever we call a map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ a symplectomorphism, we will mean this with respect to the standard symplectic form $\omega_0$.

We postpone the proof of this lemma.

*Proof of Proposition 3.2.* Define the map $\psi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$\psi(x_0, y_0) = (u(x_0, y_0), y_0) = (x_1, y_0).$$

For this map we have that

$$\|d\psi(z) - \mathrm{Id}\| < \frac{1}{2}.$$

Indeed, if we write $d\varphi(z) - \mathrm{Id} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then $d\psi(z) - \mathrm{Id} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$, and it is clear that

$$\|d\psi(z) - \mathrm{Id}\| \le \|d\varphi(z) - \mathrm{Id}\| < \frac{1}{2}.$$

So by Lemma 3.3 we get that $\psi$ is a diffeomorphism, and this means we can use the independent coordinates $(x_1, y_0)$. Now define maps $f, g : \mathbb{R}^{2n} \to \mathbb{R}^n$ by

$$\begin{aligned} f(x_1, y_0) &= x_0, \\ g(x_1, y_0) &= y_1. \end{aligned}$$

Next, define the differential 1-form $\alpha$ by

$$\alpha = \sum_{i=1}^{n} \left( g_i \, dx_1^i + f_i \, dy_0^i \right).$$

**Claim 1.** The 1-form $\alpha$ is closed.

We postpone the proof of this claim.
Since $\alpha$ is defined on $\mathbb{R}^{2n}$, closedness implies exactness, and thus there exists a smooth function $W : \mathbb{R}^{2n} \to \mathbb{R}$ such that $dW = \alpha$, i.e. such that

$$g(x_1, y_0) = \frac{\partial W}{\partial x}(x_1, y_0), \qquad f(x_1, y_0) = \frac{\partial W}{\partial y}(x_1, y_0).$$

Now define the (smooth) function $V : \mathbb{R}^{2n} \to \mathbb{R}$ by $V(x_1, y_0) = \langle x_1, y_0 \rangle - W(x_1, y_0)$. Then we obtain

$$\begin{aligned} \frac{\partial V}{\partial y}(x_1, y_0) &= x_1 - \frac{\partial W}{\partial y}(x_1, y_0) = x_1 - x_0, \\ -\frac{\partial V}{\partial x}(x_1, y_0) &= \frac{\partial W}{\partial x}(x_1, y_0) - y_0 = y_1 - y_0. \end{aligned}$$

**Proof of Claim 1:** First, we define $\beta := \psi^* \alpha$. Since $g \circ \psi = v$ and $f \circ \psi = \pi_1$, where $\pi_1 : (x_0, y_0) \mapsto x_0$ is the projection on the first coordinate, we get

$$\beta = \sum_{i=1}^{n} \left( v_i \, du_i + x_0^i \, dy_0^i \right).$$

Therefore,

$$d\beta = \sum_{i=1}^{n} \left( dv_i \wedge du_i + dx_0^i \wedge dy_0^i \right) = -\varphi^* \omega_0 + \omega_0 = 0,$$

since $\varphi$ is a symplectomorphism. So $\beta$ is closed, and since $\psi$ is a diffeomorphism, it follows that $\alpha$ is as well. This proves Claim 1 and completes the proof of Proposition 3.2. □

*Proof of Lemma 3.3.* Fix some $y \in \mathbb{R}^n$ and consider the map $\Psi_y : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\Psi_y(x) := y + x - \psi(x).$$

This map is obviously smooth and satisfies

$$\|d\Psi_y(x)\| = \|\mathrm{Id} - d\psi(x)\| < \frac{1}{2}$$

for all $x \in \mathbb{R}^n$. Hence it is $\frac{1}{2}$-Lipschitz continuous, and thus a contraction mapping. So by Banach's fixed-point theorem, there exists a unique fixed point $x_0 \in \mathbb{R}^n$, i.e. a unique point $x_0$ such that $y = \psi(x_0)$. Since this holds for every $y \in \mathbb{R}^n$, it follows that $\psi$ is a bijection, and we obtain an inverse $y \mapsto x_0$. Also note that $d\psi(x)$ is invertible for every $x \in \mathbb{R}^n$; indeed, for every $v \in \mathbb{R}^n$ we get that

$$\big| \|d\psi(x)v\| - \|v\| \big| \leq \|(d\psi(x) - \mathrm{Id})v\| \leq \|d\psi(x) - \mathrm{Id}\| \cdot \|v\| < \frac{1}{2}\|v\|.$$

Thus $d\psi(x)v = 0$ implies $v = 0$, and $d\psi(x)$ is invertible. Therefore, by the Inverse Function Theorem, $\psi$ is a diffeomorphism. □

## 3.2 The discrete symplectic action

Now, suppose we are given symplectomorphisms $\varphi_{N-1}, \ldots, \varphi_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, such that every $\varphi_i$ satisfies the condition of Proposition 3.2, and define $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$\varphi := \varphi_{N-1} \circ \cdots \circ \varphi_0.$$

By Remark 2.24, $\varphi$ is then also a symplectomorphism.

**Example 3.4.** The model example for this, and the way we will be using the theory laid out below, is the case that $\varphi^{t_1,t_0} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a Hamiltonian flow. Defining then

$$\varphi := \varphi^{1,0}, \qquad\qquad \varphi_i := \varphi^{(i+1)/N, i/N},$$

we obtain $\varphi = \varphi_{N-1} \circ \cdots \circ \varphi_0$. Furthermore, when $N$ is sufficiently large, every $\varphi_i$ will be close to the identity, and by Proposition 3.2 will thus admit a generating function of type $V$. △

Back to the general situation, we denote by $V_i : \mathbb{R}^{2n} \to \mathbb{R}$ the generating function for $\varphi_i$, and thus for every $i$ we get that $(x_{i+1}, y_{i+1}) = \varphi_i(x_i, y_i)$ iff

$$x_{i+1} - x_i = \frac{\partial V_i}{\partial y}(x_{i+1}, y_i), \qquad\qquad y_{i+1} - y_i = -\frac{\partial V_i}{\partial x}(x_{i+1}, y_i). \tag{3.2}$$

These equations are often called the *Hamiltonian difference equations.*
Next, we define

$$X_{N,n} \simeq \mathbb{R}^{2nN}$$

to be the space of all $N$-periodic sequences in $\mathbb{R}^{2n}$, i.e. of sequences $\{z_j\}_{j\in\mathbb{Z}} = \{(x_j, y_j)\}_{j\in\mathbb{Z}}$ such that $z_j = z_{j+N}$ for all $j \in \mathbb{Z}$. As an element of $\mathbb{R}^{2nN}$ we write such a sequence as $(x_0, y_0, \dots, x_{N-1}, y_{N-1})$. We are now ready to define the discrete symplectic action.

**Definition 3.5** (Discrete symplectic action)**.** Let $\varphi_{N-1}, \dots, \varphi_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be symplectomorphisms that all admit a generating function of type $V$. In the notation given above, we define the *discrete symplectic action* $\Phi : X_{N,n} \to \mathbb{R}$ by

$$\Phi\Big(x_0, y_0, \dots, x_{N-1}, y_{N-1}\Big) := \sum_{i=0}^{N-1} \Big(\langle y_i, x_{i+1} - x_i\rangle - V_i(x_{i+1}, y_i)\Big). \tag{3.3}$$

Here we use the convention $x_N = x_0$. ♢

The following proposition establishes the correspondence between fixed points of $\varphi$ and critical points of $\Phi$.

**Proposition 3.6.** *Let $\varphi_{N-1}, \dots, \varphi_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be symplectomorphisms that all satisfy the condition of Proposition 3.2, and let $\Phi : X_{N,n} \to \mathbb{R}$ be the discrete symplectic action induced by their generating functions. Defining $\varphi := \varphi_{N-1} \circ \cdots \circ \varphi_0$, there is a bijection between the fixed points of $\varphi$ and the critical points of $\Phi$. The same holds when we just consider nondegenerate fixed points and critical points.*

**Remark 3.7.** In the proof of this proposition, we will abbreviate partial derivatives like so

$$\frac{\partial \Phi}{\partial x_i} = \partial_{x_i}\Phi, \qquad\qquad \frac{\partial^2 \Phi}{\partial x_i \partial y_j} = \partial_{x_i y_j}\Phi, \qquad\qquad \text{etc.}$$

to simplify notation.

*Proof.* The proposition will follow from three claims.

**Claim 1.** Let $(x_0, y_0, \dots, x_{N-1}, y_{N-1})$ be a critical point of $\Phi$. Then $(x_0, y_0)$ is a fixed point of $\varphi$.

**Claim 2.** Let $(x_0, y_0)$ be a fixed point of $\varphi$, and inductively define the periodic sequence $(x_0, y_0, \ldots, x_{N-1}, y_{N-1})$ by $(x_{i+1}, y_{i+1}) = \varphi_i(x_i, y_i)$.[13] Then this sequence is a critical point of $\Phi$.

**Claim 3.** If $(x_0, y_0)$ is a fixed point of $\varphi$ and $(x_0, y_0, \ldots, x_{N-1}, y_{N-1})$ the corresponding critical point of $\Phi$, then

(i) for any sequence $(u_0, v_0, \ldots, u_{N-1}, v_{N-1}) \in \ker D^2\Phi(x_0, y_0, \ldots, x_{N-1}, y_{N-1})$ we have that $(u_0, v_0) \in \ker(d\varphi(x_0, y_0) - \mathrm{Id})$,

(ii) for $(u_0, v_0) \in \ker(d\varphi(x_0, y_0) - \mathrm{Id})$, defining a sequence $(u_0, v_0, \ldots, u_{N-1}, v_{N-1})$ inductively by $(u_{i+1}, v_{i+1}) = d\varphi_i(x_i, y_i)(u_i, v_i)$, this sequence is in the kernel of $D^2\Phi(x_0, y_0, \ldots, x_{N-1}, y_{N-1})$.

**Proof of Claim 1 and Claim 2:** These claims follow from a simple calculation of the partial derivatives of $\Phi$. Indeed, for $\{z_j\} = (x_0, y_0, \ldots, x_{N-1}, y_{N-1})$ we get

$$\begin{aligned} \partial_{x_i}\Phi(\{z_j\}) &= -y_i + y_{i-1} - \partial_x V_{i-1}(x_i, y_{i-1}), \\ \partial_{y_i}\Phi(\{z_j\}) &= x_{i+1} - x_i - \partial_y V_i(x_{i+1}, y_i). \end{aligned} \tag{3.4}$$

Comparing these equations to Eq. 3.2 immediately yields what we want; for Claim 1, the partial derivatives to $x_{i+1}$ and $y_i$ being zero shows that $(x_{i+1}, y_{i+1}) = \varphi_i(x_i, y_i)$, and thus the definition of $\varphi$, together with the periodicity, shows that

$$\varphi(x_0, y_0) = (\varphi_{N-1} \circ \cdots \circ \varphi_0)(x_0, y_0) = \varphi_{N-1}(x_{N-1}, y_{N-1}) = (x_N, y_N) = (x_0, y_0).$$

For Claim 2, the very definition of $(x_{i+1}, y_{i+1})$ shows that the partial derivatives to $x_{i+1}$ and $y_i$ are zero, and hence the defined sequence is a critical point. This proves Claim 1 and Claim 2.

**Proof of Claim 3:** First, let us compute the Hessian $D^2\Phi(\{z_j\})$ of $\Phi$. Differentiating Eq. 3.4 yields first of all that

$$\partial_{x_i x_j}\Phi(\{z_j\}) = \partial_{y_i y_j}\Phi(\{z_j\}) = 0$$

for $i \neq j$ and that

$$\partial_{x_i x_i}\Phi(\{z_j\}) = -\partial_{xx} V_{i-1}(x_i, y_{i-1}), \qquad \partial_{y_i y_i}\Phi(\{z_j\}) = -\partial_{yy} V_i(x_{i+1}, y_i).$$

Furthermore,

$$\partial_{x_i y_j}\Phi(\{z_j\}) = 0$$

except when $i = j$ or $i - j = 1$. In these cases,

$$\partial_{x_i y_i}\Phi(\{z_j\}) = -\mathrm{Id}, \qquad \partial_{x_i y_{i-1}}\Phi(\{z_j\}) = \mathrm{Id} - \partial_{xy} V_{i-1}(x_i, y_{i-1}).$$

[13] Note that since $(x_0, y_0) = \varphi(x_0, y_0) = \varphi_{N-1}(x_{N-1}, y_{N-1}) = (x_N, y_N)$, this does indeed define a periodic sequence.

So, writing

$$(u'_0, v'_0, \dots, u'_{N-1}, v'_{N-1}) = D^2\Phi(\{z_j\})(u_0, v_0, \dots, u_{N-1}, v_{N-1}),$$

we get

$$\begin{aligned}
u'_i &= \partial_{x_i y_{i-1}}\Phi(\{z_j\})v_{i-1} + \partial_{x_i x_i}\Phi(\{z_j\})u_i + \partial_{x_i y_i}\Phi(\{z_j\})v_i \\
&= [\mathrm{Id} - \partial_{xy}V_{i-1}(x_i, y_{i-1})]v_{i-1} - \partial_{xx}V_{i-1}(x_i, y_{i-1})u_i - v_i, \\
v'_i &= \partial_{x_i y_i}\Phi(\{z_j\})u_i + \partial_{y_i y_i}\Phi(\{z_j\})v_i + \partial_{x_{i+1} y_i}\Phi(\{z_j\})u_{i+1} \\
&= -u_i - \partial_{yy}V_i(x_{i+1}, y_i)v_i + [\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]u_{i+1}.
\end{aligned}$$

Hence a sequence $(u_0, v_0, \dots, u_{N-1}, v_{N-1})$ is in the kernel of $D^2\Phi(\{z_j\})$ iff

$$\begin{aligned}
u_{i+1} &= [\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]^{-1}(u_i + \partial_{yy}V_i(x_{i+1}, y_i)v_i), \\
v_{i+1} &= -\partial_{xx}V_i(x_{i+1}, y_i)u_{i+1} + [\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]v_i.
\end{aligned}$$

Here we are using the following claim, of which we postpone the proof.

**Claim 4.** The linear map $\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)$ is invertible for every $i$.

Next, let us determine $d\varphi_i(x_i, y_i)$. Writing the components of $\varphi$ as $(f_i, g_i) = \varphi$, we denote

$$d\varphi_i(x_i, y_i) = \begin{pmatrix} \frac{\partial f_i}{\partial x}(x_i, y_i) & \frac{\partial f_i}{\partial y}(x_i, y_i) \\ \frac{\partial g_i}{\partial x}(x_i, y_i) & \frac{\partial g_i}{\partial y}(x_i, y_i) \end{pmatrix} = \begin{pmatrix} \partial_x f_i(x_i, y_i) & \partial_y f_i(x_i, y_i) \\ \partial_x g_i(x_i, y_i) & \partial_y g_i(x_i, y_i) \end{pmatrix}.$$

Differentiating Eq. 3.2, we now obtain

$$\begin{aligned}
\partial_x f_i(x_i, y_i) &= \mathrm{Id} + \partial_{xy}V_i(x_{i+1}, y_i)\partial_x f_i(x_i, y_i), \\
\partial_y f_i(x_i, y_i) &= \partial_{yy}V_i(x_{i+1}, y_i) + \mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)\partial_y f_i(x_i, y_i), \\
\partial_x g_i(x_i, y_i) &= -\partial_{xx}V_i(x_{i+1}, y_i)\partial_x f_i(x_i, y_i), \\
\partial_y g_i(x_i, y_i) &= \mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i) - \partial_{xx}V_i(x_{i+1}, y_i)\partial_y f_i(x_i, y_i).
\end{aligned}$$

Hence we get

$$\begin{aligned}
\partial_x f_i(x_i, y_i) &= [\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]^{-1}, \\
\partial_y f_i(x_i, y_i) &= [\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]^{-1}\partial_{yy}V_i(x_{i+1}, y_i), \\
\partial_x g_i(x_i, y_i) &= -\partial_{xx}V_i(x_{i+1}, y_i)[\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]^{-1}, \\
\partial_y g_i(x_i, y_i) &= \mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i) - \partial_{xx}V_i(x_{i+1}, y_i)[\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]^{-1}\partial_{yy}V_i(x_{i+1}, y_i).
\end{aligned}$$

So we see that $(u_{i+1}, v_{i+1}) = d\varphi_i(x_i, y_i)(u_i, v_i)$ iff

$$\begin{aligned}
u_{i+1} &= \partial_x f_i(x_i, y_i)u_i + \partial_y f_i(x_i, y_i)v_i \\
&= [\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]^{-1}u_i + [\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]^{-1}\partial_{yy}V_i(x_{i+1}, y_i)v_i \\
v_{i+1} &= \partial_x g_i(x_i, y_i)u_i + \partial_y g_i(x_i, y_i)v_i \\
&= -\partial_{xx}V_i(x_{i+1}, y_i)[\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]^{-1}u_i + \\
&\quad [\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]v_i - \partial_{xx}V_i(x_{i+1}, y_i)[\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]^{-1}\partial_{yy}V_i(x_{i+1}, y_i)v_i \\
&= -\partial_{xx}V_i(x_{i+1}, y_i)u_{i+1} + [\mathrm{Id} - \partial_{xy}V_i(x_{i+1}, y_i)]v_i.
\end{aligned}$$

So by the above computations, we see that a sequence $(u_0, v_0, \dots, u_{N-1}, v_{N-1}) \in \ker D^2\Phi(x_0, y_0, \dots, x_{N-1}, y_{N-1})$ being in the kernel of $D^2\Phi(\{z_j\})$ is equivalent to that $(u_{i+1}, v_{i+1}) = d\varphi_i(x_i, y_i)(u_i, v_i)$ for every $i$. Hence Claim 3 follows.

**Proof of Claim 4:** Recall that we have

$$\partial_x f_i(x_i, y_i) = \mathrm{Id} + \partial_{xy} V_i(x_{i+1}, y_i)\partial_x f_i(x_i, y_i).$$

Also, since $\|d\varphi_i(x_i, y_i) - \mathrm{Id}\| < \frac{1}{2}$, we also have that $\|\partial_x f_i(x_i, y_i) - \mathrm{Id}\| < \frac{1}{2}$, and thus $\partial_x f_i(x_i, y_i)$ is invertible. Then

$$\|\partial_x f_i(x_i, y_i)^{-1}\| \leq \frac{1}{1 - \|\mathrm{Id} - \partial_x f_i(x_i, y_i)\|} < 2.$$

Thus we get

$$\|\partial_{xy} V_i(x_{i+1}, y_i)\| = \|(\partial_x f_i(x_i, y_i) - \mathrm{Id})\partial_x f_i(x_i, y_i)^{-1}\| \leq \|\partial_x f_i(x_i, y_i) - \mathrm{Id}\| \cdot \|\partial_x f_i(x_i, y_i)^{-1}\| < 1.$$

Hence, by the Neumann Series, $\mathrm{Id} - \partial_{xy} V_i(x_{i+1}, y_i)$ is invertible. Since the above argument works for every $i$, this proves Claim 4 and completes the proof of Proposition 3.6. $\square$

# 4 Conley index theory

In this section we will discuss so-called *Conley index theory.* In general, this theory works for flows on locally compact metric spaces,[14] but we will apply it to the pseudo-gradient flow of the discrete symplectic action defined in the previous section. We will also discuss the Morse inequalities; these are the crucial ingredient in the proof of the nondegenerate Conley-Zehnder Theorem.

## 4.1 The Conley index

For the remainder of this section, let $M$ be a locally compact metric space and $\varphi : \mathbb{R} \times M \to M$ a flow on $M$, i.e. a continuous collection of maps $\varphi^t : M \to M$ such that

$$\varphi^0 = \mathrm{Id}, \qquad \varphi^t \circ \varphi^s = \varphi^{t+s}.$$

**Definition 4.1.** An *invariant set* is a subset $S \subset M$ such that

$$S = \varphi(\mathbb{R} \times S).$$

Given any subset $N \subset M$, the *maximal invariant subset* of $N$ is given by

$$I(N) := \{x \in N \mid \varphi(\mathbb{R} \times \{x\}) \subset N\} = \bigcap_{t \in \mathbb{R}} \varphi^t(N). \qquad \diamondsuit$$

**Remark 4.2.** To see that $I(N)$ as defined above is in fact the largest invariant set contained in $N$, note that for any set $N$, the set $\varphi(\mathbb{R} \times N)$ is the set consisting of all the orbits that go through $N$. Hence a set is invariant precisely if it is a union of orbits. Since $I(N)$ is defined to be the union of all orbits contained in $N$, it is clear that $I(N)$ is invariant. Similarly, any invariant set $N'$ contained in $N$ is then obviously contained in $I(N)$.

**Definition 4.3** (Isolated invariant sets)**.** A set $S \subset M$ is called an *isolated invariant set* if it is a compact invariant set such that there exists an *isolating neighbourhood* $N$ of $S$, i.e. a compact neighbourhood $N$ such that $S = I(N)$. $\diamondsuit$

Our goal is to define the *Conley index* for these isolated invariant sets. To do this, we need to define index pairs.

**Definition 4.4** (Index pairs)**.** Let $S \subset M$ be an isolated invariant set. An *index pair* for $S$ is pair $(N, L)$ of compact sets $L \subset N \subset M$ such that

(i) $S \subset \mathrm{int}(N \setminus L)$ and $S = I(\mathrm{cl}(N \setminus L))$,

[14]In fact, it works for locals flows on Hausdorff topological spaces, but since we will apply it to a locally compact metric space, we will not consider this more general case.

(ii) if $x \in L, t \geq 0$ are such that $\varphi([0,t] \times \{x\}) \subset N$, then $\varphi^t(x) \in L$,

(iii) if $x \in N$ is such that $\varphi([0,\infty) \times \{x\}) \not\subset N$, then there exists some $t \geq 0$ such that $\varphi([0,t] \times \{x\}) \subset N$ and $\varphi^t(x) \in L$. ♢

**Remark 4.5.** Since the conditions are a bit abstract let us look into what they actually mean. Condition (i) simply says that $N \setminus L$ is a neighbourhood of $S$ and that $\mathrm{cl}(N \setminus L)$ is an isolating neighbourhood of $S$. Condition (ii) is actually a more general concept: when $N$ is *any* compact set $N \subset M$ and $L$ *any* set $L \subset N$, $L$ is said to be *positively invariant in* $N$ if (ii) holds. This can be interpreted as saying that any point in $L$ that does not get pushed out of $N$ by the flow, also stays in $L$. Finally, condition (iii) says that any point that leaves $N$ by the flow, has to go through $L$ in doing so. The set $L$ is sometimes called the *exit set* of $N$ in this case.

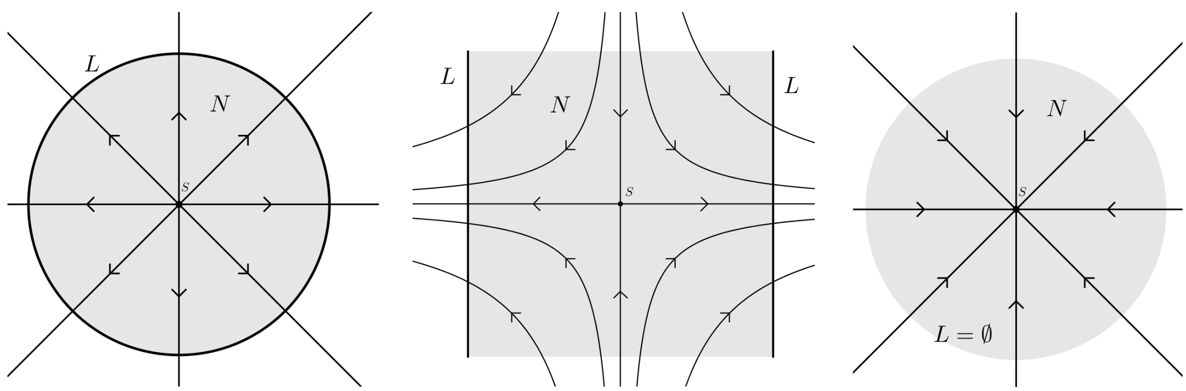


**Figure 3:** Some examples of index pairs.

Now that we have defined index pairs, we can also define the Conley index.

**Definition 4.6** (Conley index)**.** Let $S \subset M$ be an isolated invariant set. The Conley index $h(S)$ of $S$ is the homotopy type of the pointed space $N/L$, where $(N, L)$ is an index pair for $S$. ♢

Obviously, this definition hardly makes sense in our current position. There are two obvious problems:

(i) How do we know an index pair exists for an arbitrary isolated invariant set?

(ii) How dow we know the Conley index is well-defined, i.e. how do we know that $N/L$ and $N'/L'$ are homotopy equivalent for any two index pairs $(N, L)$, $(N', L')$ of the same isolated invariant set?

Thankfully, both these problems can be solved, and this is the content of the following proposition.

**Proposition 4.7** (Well-definedness of Conley index)**.** *Let $S \subset M$ be an isolated invariant set. Then*

*(i) there exists an index pair $(N, L)$ for $S$, and*

*(ii) for any two index pairs $(N, L)$ and $(N', L')$ for $S$ we have that $(N, L)$ and $(N', L')$ are homotopy equivalent.*

We will not prove this proposition here, since the proof is rather lengthy and not very insightful. A proof can be found in e.g. [Sal85, Chapter 4].

It is convenient to define the following special class of index pairs.

**Definition 4.8** (Regular index pairs)**.** Let $S \subset M$ be an isolated invariant set. An index pair $(N, L)$ of $S$ is called *regular* if $L$ is a neighbourhood deformation retract in $N$, that is, if there is some neighbourhood of $L$ in $N$ that deformation retracts onto $L$. $\diamondsuit$

The reason regular index pairs are useful is that since $L$ is closed in $N$, the homology of $N/L$ agrees with the homology of the pair $(N, L)$.[15] When the homology is finite-dimensional, we can define the characteristic polynomial of an isolated invariant set.

**Definition 4.9** (Characteristic polynomial)**.** Let $S \subset M$ be an isolated invariant set and $(N, L)$ a regular index pair for $S$. If the homology of the pair $(N, L)$ is finite-dimensional, then the *characteristic polynomial* $p_S$ of $S$ is defined as

$$p_S(s) := \sum_{k \in \mathbb{N}} \dim H_k(N, L)\ s^k. \qquad \diamondsuit$$

It can be shown, see e.g. [RS88], that when $M$ is a manifold, the homology of all index pairs are finite-dimensional.

## 4.2 The Morse inequalities

In this section we will state and prove the Morse inequalities. These inequalities provide a lower bound for the number of critical points of a Morse function in a given isolated invariant set. They will be vital in the proof of the Conley-Zehnder theorem, as we will use them to determine a lower bound for the number of critical points of the discrete symplectic action. Before we state the inequalities, we need some preparation.

**Definition 4.10** (Morse functions)**.** Let $X$ be a manifold and $f : X \to \mathbb{R}$ a smooth function. Then $f$ is called a *Morse function* if all of its critical points are nondegenerate. $\diamondsuit$

The following theorem is known as the Morse lemma, and it gives a very explicit image of what a smooth function looks like near a nondegenerate critical point.

[15] See e.g. [Hat10, Proposition 2.22].

**Theorem 4.11** (Morse lemma)**.** *Let $X$ be an $n$-dimensional manifold, $f : X \to \mathbb{R}$ a smooth function and $x_0 \in X$ a nondegenerate critical point of $f$. Then there is an $i \in \{0, \dots, n\}$ and a chart $\varphi : U \to V \subset \mathbb{R}^n$ centered at $x_0$ such that*

$$(f \circ \varphi^{-1})(x_1, \dots, x_n) = f(x_0) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

A proof of this theorem can be found in e.g. [ADE14, Theorem 1.3.1]. A chart as in Morse's lemma is called a *Morse chart.* The integer $i$ is called the *index* of $x_0$, and is often denoted as $\operatorname{ind}(x_0)$. It can be shown that this number depends only on the critical point, and not on the Morse chart used.

**Corollary 4.12.** *Let $X$ be an $n$-dimensional manifold and $f : X \to \mathbb{R}$ a Morse function. Then the critical points of $f$ are isolated.*

*Proof.* Let $x_0 \in X$ be a critical point of $f$. Then by Morse's lemma, we find a Morse chart $\varphi : U \to V \subset \mathbb{R}^n$ around $x_0$. Now, for any critical point $x' \in U$, $\varphi(x')$ is a critical point of $f \circ \varphi^{-1}$, since by the chain rule

$$d(f \circ \varphi^{-1})(\varphi(x')) = df(x') \circ d(\varphi^{-1})(\varphi(x')) = 0.$$

However, the only critical point of $f \circ \varphi$ is $0$, and hence the only critical point in $U$ is $x_0$. □

Before we move on to the Morse inequalities, we need one more definition.

**Definition 4.13** (Pseudo-gradients)**.** Let $X$ be an $n$-dimensional manifold and $f : X \to \mathbb{R}$ a Morse function. Then a *pseudo-gradient field* adapted to $f$ is a smooth vector field $V : X \to TX$ with the following properties.

(i) For every $x \in X$ we have that $df(x)(V(x)) \leq 0$, where equality holds iff $x$ is a critical point of $f$.

(ii) For any Morse chart $\varphi : U \to V \subset \mathbb{R}^n$ around a critical point $x_0$, the pushforward $\varphi_* V$ is just the negative gradient on $\mathbb{R}^n$. ♢

**Remark 4.14.** Recall that if $X$ and $Y$ are smooth manifolds, $F : X \to Y$ is a diffeomorphism and $V : X \to TX$ a smooth vector field on $X$, then the pushforward $F_* V : Y \to TY$ is defined as

$$(F_* V)_y := dF(F^{-1}(y))(V(F^{-1}(y))).$$

A pseudo-gradient vector field has the nice property that $f$ decreases along its flow lines, just like the standard negative gradient field. Furthermore, it has a "standard" form under a Morse chart.

We are now ready to state and prove the Morse inequalities. Let $X$ be an $n$-dimensional manifold and $f : X \to \mathbb{R}$ a Morse function. Let $V : X \to TX$ be a pseudo-gradient field adapted to $f$ and assume that it has a global flow $\varphi : \mathbb{R} \times X \to X$. Let $S \subset X$ be an isolated invariant set for $\varphi$ and let $(N, L)$ be a regular index pair for $S$. We write

$$c_k(S) := \#\{x \in S \mid df(x) = 0, \operatorname{ind}(x) = k\}$$

for the number of critical points of $f$ in $S$ with index $k$. We also define the *Conley-Betti numbers*

$$b_k(S) := \dim H_k(N, L).$$

**Theorem 4.15** (Morse inequalities). *For any $k \in \{0, \dots, n\}$ we have that*

$$b_k(S) - b_{k-1}(S) + \cdots \pm b_0(S) \leq c_k(S) - c_{k-1}(S) + \cdots \pm c_0(S)$$

*where equality holds for $k = n$. Also,*

$$\sum_{k=0}^{n} b_k \leq \sum_{k=0}^{n} c_k.$$

*Proof.* Let us first show that the second assertion follows from the first. Since for every $k \in \{1, \dots, n\}$ we have that both

$$b_k(S) - b_{k-1}(S) + \cdots \pm b_0(S) \leq c_k(S) - c_{k-1}(S) + \cdots \pm c_0(S)$$

and

$$b_{k-1}(S) - b_{k-2}(S) + \cdots \mp b_0(S) \leq c_{k-1}(S) - c_{k-2}(S) + \cdots \mp c_0(S),$$

it follows by adding these two inequalities that $b_k(S) \leq c_k(S)$. Together with the obvious inequality $b_0(S) \leq c_0(S)$, the second inequality follows immediately.

Now we prove the first assertion. First, let us introduce some notation. For any regular value $a \in \mathbb{R}$ of $f|_N$ we define

$$N_a := \{x \in N \mid f(x) \leq a\} \cup L.$$

For any critical value $c \in \mathbb{R}$ of $f|_N$ we define

$$S_c := \{x \in S \mid df(x) = 0, f(x) = c\}.$$

**Claim 1.** For any critical value $c \in \mathbb{R}$ the set $S_c$ is an isolated invariant set, and a regular index pair is given by $(N_b, N_a)$, where $a$ and $b$ are regular values $a < c < b$ such that $c$ is the only critical value of $f|_N$ in $[a, b]$. The characteristic polynomial of $S_c$ is given by

$$p_{S_c}(s) = \sum_{k=0}^{n} \dim H_k(N_b, N_a) s^k = \sum_{df(x)=0, f(x)=c} s^{\operatorname{ind}(x)}.$$

We postpone the proof of this claim. We introduce some more notation, namely we define the numbers

$$b_k^a(S) := \dim H_k(N_a, L), \qquad c_k^a(S) := \#\{x \in S \cap N_a \mid df(x) = 0, \mathrm{ind}(x) = k\}$$

for a regular value $a$ of $f|_N$. With this new terminology, the result of Claim 1 can be written as

$$\dim H_k(N_b, N_a) = c_k^b(S) - c_k^a(S).$$

Indeed; the claim says that $\dim H_k(N_b, N_a)$ is equal to the number of critical points in $N$ of index $k$ with critical value $c$. Since $c$ is the only critical value between $a$ and $b$, the above equality follows.

Now, recall the long exact sequence of a triple $(X, A, B)$, where $B \subset A \subset X$;

$$\cdots \to H_{k+1}(X, A) \to H_k(A, B) \to H_k(X, B) \to H_k(X, A) \to H_{k-1}(A, B) \to \cdots$$

Applied to the triple $(N_b, N_a, L)$, this becomes

$$\cdots \to H_{k+1}(N_b, N_a) \to H_k(N_a, L) \to H_k(N_b, L) \to H_k(N_b, N_a) \to H_{k-1}(N_a, L) \to \cdots$$

Defining

$$d_k^{ab}(S) := \mathrm{rank}\big(H_{k+1}(N_b, N_a) \to H_k(N_a, L)\big),$$

we get, iteratively,

$$\begin{aligned} \mathrm{rank}\big(H_k(N_a, L) \to H_k(N_b, L)\big) &= b_k^a(S) - d_k^{ab}(S), \\ \mathrm{rank}\big(H_k(N_b, L) \to H_k(N_b, N_a)\big) &= b_k^b(S) - b_k^a(S) + d_k^{ab}(S), \\ d_{k-1}^{ab}(S) = \mathrm{rank}\big(H_k(N_b, N_a) \to H_{k-1}(N_a, L)\big) &= c_k^b(S) - c_k^a(S) - b_k^b(S) + b_k^a(S) - d_k^{ab}(S). \end{aligned}$$

Hence we obtain

$$d_{k-1}^{ab}(S) + d_k^{ab}(S) = c_k^b(S) - c_k^a(S) - b_k^b(S) + b_k^a(S).$$

Defining the polynomials

$$p_S^a(s) := \sum_{k=0}^n b_k^a(S) s^k, \qquad p_{\mathrm{crit}}^a(s) := \sum_{k=0}^n c_k^a(S) s^k, \qquad p^{ab}(s) := \sum_{k=0}^n d_k^{ab}(S) s^k,$$

this can be rephrased as

$$p_{\mathrm{crit}}^b(s) - p_S^b(s) = p_{\mathrm{crit}}^a(s) - p_S^a(s) + (1+s)p^{ab}(s).$$

Now note that since every $d_k^{ab}(S)$ is nonnegative by definition, the polynomial $p^{ab}(s)$ has nonnegative coefficients.

Since $N$ is compact, the number of critical points of $f|_N$ is finite, and thus so is the number of critical values. Denote the set of critical values of $f|_N$ by $\{c_1, \dots, c_r\}$ and choose regular values of $f|_N$

$$a_1 < c_1 < a_2 < \cdots < a_r < c_r < a_{r+1}$$

such that

$$a_1 < \inf_N f, \qquad \sup_N f < a_{r+1}.$$

In this case, we have $N^{a_1} = L$, and thus that $b_k^{a_1}(S) = c_k^a(S) = 0$ for all $k$ ($c_k^a(S)$ is zero since $S \cap L = \emptyset$, by the definition of an index pair). Hence we get that

$$p_{\text{crit}}^{a_2}(s) - p_S^{a_2}(s) = p_{\text{crit}}^{a_1}(s) - p_S^{a_1}(s) + (1+s)p^{a_1 a_2}(s) = (1+s)p^{a_1 a_2}(s).$$

Repeating this argument by induction, we get, for every $k \in \{1, \dots, r+1\}$ that

$$p_{\text{crit}}^{a_j}(s) - p_S^{a_j}(s) = (1+s)p^{a_j}(s)$$

for some polynomial $p^{a_j}(s)$ with nonnegative coefficients. Indeed, the result holds for $j = 1, 2$, and if it holds for $a_j$, then

$$p_{\text{crit}}^{a_{j+1}}(s) - p_S^{a_{j+1}}(s) = p_{\text{crit}}^{a_j}(s) - p_S^{a_j}(s) + (1+s)p^{a_j a_{j+1}}(s) = (1+s)\Big(p^{a_j}(s) + p^{a_j a_{j+1}}(s)\Big).$$

In particular, it holds for $a_{r+1}$. However, since we chose $a_{r+1}$ larger than the supremum of $f$ on $N$, it actually follows that $N^{a_{r+1}} = N$, and thus that $b_k^{a_{r+1}}(S) = b_k(S)$ and $c_k^{a_{r+1}}(S) = c_k(S)$, since $S \cap N^{a_{r+1}} = S \cap N = S$. So defining

$$p_{\text{crit}}(s) := \sum_{k=0}^{n} c_k(S)s^k,$$

we get that

$$p_{\text{crit}}(s) - p_S(s) = (1+s)p(s)$$

for some polynomial $p(s)$ with nonnegative coefficients. If we write

$$p(s) = \sum_{k=0}^{n} d_k s^k,$$

then we get by induction that

$$d_k = \sum_{j=0}^{k} (-1)^j (c_{k-j}(S) - b_{k-j}(S)). \tag{4.1}$$

Indeed, we get that

$$d_0 = c_0 - b_0,$$

and for any $k \in \{0, n-1\}$ we have that

$$d_{k+1} + d_k = c_{k+1}(S) - b_{k+1}(S).$$

Hence if the equality holds for $k$, then

$$d_{k+1} = c_{k+1}(S) - b_{k+1}(S) - \sum_{j=0}^{k} (-1)^j (c_{k-j}(S) - b_{k-j}(S)) = \sum_{j=0}^{k+1} (-1)^j (c_{k+1-j}(S) - b_{k+1-j}(S)).$$

Therefore, we indeed obtain Eq. 4.1. Now, for every $k$ we have that $d_k \geq 0$, and we have $d_n = 0$, since $p_{\text{crit}}(s) - p_S(s)$ is of degree $n$. But these are exactly the Morse inequalities.

**Proof of Claim 1:** First, let us show that $S_c$ is an isolated invariant set with regular index pair $(N_b, N_a)$. It is clear that $S_c$ is invariant, since the pseudogradient $V$ is zero at critical points. Hence every critical point is an orbit of $\varphi$. To see that $S_c$ is compact, note that since $N$ is compact and nondegenerate critical points are isolated, there are only finitely many points in $S_c$. Hence it is a finite unions of singletons, all of which are of course compact. Hence $S_c$ is compact as well. To see that it is isolated, we show that

$$S_c = I(N_{ab}) = I(\text{cl}(N_b \setminus N_a))$$

where we write

$$N_{ab} := \{x \in N \mid a \leq f(x) \leq b\}.$$

We have to show that the only orbits contained in $N_{ab}$ are the critical points, $S_c$. To see this, note that for any $x \in N_{ab} \setminus S_c$, the flow $\varphi$ pushes $x$ *down* along $f$. Hence any orbit that passes through $N_{ab}$ (and is not a critical point) has to exit $N_{ab}$ through the level set $\{x \in N \mid f(x) = a\}$ and/or enter it through the level set $\{x \in N \mid f(x) = b\}$. Hence the only orbits contained in $N_{ab}$ are the critical points, $S_c$. To show that $(N_b, N_a)$ is an index pair for $S_c$ we still need to show part (ii) and (iii) of Definition 4.4. So let $x \in N_a$ and $t \geq 0$ be such that $\varphi([0,t] \times \{x\}) \subset N_b$. If $x \in L$, then since $\varphi([0,t] \times \{x\}) \subset N_b \subset N$, it follows since $(N, L)$ is an index pair that $\varphi^t(x) \in L \subset N_a$. If $x \notin L$, then $x \in \{x \in N \mid f(x) \leq a\}$. The fact that $\varphi([0,t] \times \{x\}) \subset N_b$ then means, in particular, that $\varphi([0,t] \times \{x\}) \subset N$. This, combined with the fact that $f$ decreases along $\varphi$, implies that $\varphi^t(x) \in N_a$; it is in $N$, and since $f(x) \leq a$, it follows that $f(\varphi^t(x)) \leq f(x) \leq a$.

To show (iii), let $x \in N_b$ be such that $\varphi([0,\infty) \times \{x\}) \not\subset N_b$. The crucial point here is that this also implies that $\varphi([0,\infty) \times \{x\}) \not\subset N$; indeed, if $x$ flows out of $N_b$, but not out of $N$, it has to flow into the region $\{x \in N \mid b < f(x)\}$. But again, this is not possible, since $f$ decreases along $\varphi$. Hence we actually get that $\varphi([0,\infty) \times \{x\}) \not\subset N$, and since $x \in N_b \subset N$ it follows since $(N, L)$ is an index pair that there is some $t \geq 0$ such that $\varphi([0,t] \times \{x\}) \subset N$ and $\varphi^t(x) \in L$. Since $f(x) \leq b$, it follows that for every

$t' \geq 0$ we have that $f(\varphi^{t'})(x) \leq b$, and thus the above implies that $\varphi([0,t] \times \{x\}) \subset N_b$ and $\varphi^t(x) \in L \subset N_a$. Hence (iii) holds, and thus $(N_b, N_a)$ is an index pair for $S_c$.

To see that $(N_b, N_a)$ is regular, note that we can use the flow of the pseudo-gradient to deformation retract a neighbourhood of $N_a$ onto $N_a$. Indeed, take some small neighbourhood $U$ of $L$ in $N$, such that $U$ does not contain a critical value. This is possible since $(N, L)$ is a regular pair. We now write $U_b := U \cap N_b$. Now choose some $a < a' < c$. Then $N_{a'} \cup U_b$ is a neighbourhood of $N_a$ in $N_b$. Then since $f$ decreases along the flow, and since $L$ is an exit set for the flow in $N_b$, the flow (after rescaling if necessary) deformation retracts $N_{a'} \cup U_b$ onto $N_a$.

Now all that is left to prove is that

$$\sum_{k=0}^{n} \dim H_k(N_b, N_a) s^k = \sum_{df(x)=0, f(x)=c} s^{\mathrm{ind}(x)}.$$

We will achieve this by moving to a different index pair for $S_c$, of which we can easily compute the relative homology groups, and applying Proposition 4.7 (ii).

For now, consider a single critical point $x_0 \in S_c$. Since $V$ is a pseudo-gradient, there is a Morse chart $\Phi : U_0 \to V_0 \subset \mathbb{R}^n$ around $x_0$, such that

$$(f \circ \Phi^{-1})(y) = c - \sum_{j=1}^{\mathrm{ind}(x_0)} y_j^2 + \sum_{j=\mathrm{ind}(x_0)+1}^{n} y_j^2$$

and

$$d\Phi(\Phi^{-1}(y))(V(\Phi^{-1}(y))) = -(\mathrm{grad}(f \circ \Phi^{-1}))(y).$$

We claim that the flow $\psi^t$ of $-(\mathrm{grad}(f \circ \Phi^{-1}))$, where it is defined, is given by

$$\psi^t(y) = (\Phi \circ \varphi^t \circ \Phi^{-1})(y).$$

Indeed,

$$\psi^0(y) = (\Phi \circ \mathrm{Id} \circ \Phi^{-1})(y) = y$$

and

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=t} \psi^s(y) &= d\Phi((\varphi^t \circ \Phi^{-1})(y)) \left( \left.\frac{d}{ds}\right|_{s=t} \varphi^s(\Phi^{-1}(y)) \right) \\
&= d\Phi((\varphi^t \circ \Phi^{-1})(y)) \big( V((\varphi^t \circ \Phi^{-1})(y)) \big) \\
&= d\Phi(\Phi^{-1}(\psi^t(y))) \big( V(\Phi^{-1}(\psi^t(y))) \big) \\
&= -(\mathrm{grad}(f \circ \Phi^{-1}))(\psi^t(y)).
\end{aligned}$$

Now, since the gradient is given by

$$-(\operatorname{grad}(f \circ \Phi^{-1}))(y) = \begin{pmatrix} 2y_1 \\ \vdots \\ 2y_{\operatorname{ind}(x_0)} \\ -2y_{\operatorname{ind}(x_0)+1} \\ \vdots \\ -2y_n \end{pmatrix},$$

it is clear that an index pair for the isolated invariant set $\{0\} \subset \mathbb{R}^n$ is given by

$$\begin{aligned} N_0 &:= [-\epsilon, \epsilon]^n, \\ L_0 &:= \{-\epsilon\} \times [-\epsilon, \epsilon]^{n-1} \cup \{\epsilon\} \times [-\epsilon, \epsilon]^{n-1} \cup \dots \cup [-\epsilon, \epsilon]^{\operatorname{ind}(x_0)-1} \times \{\epsilon\} \times [-\epsilon, \epsilon]^{n-\operatorname{ind}(x_0)}, \end{aligned}$$

where $\epsilon > 0$ is small enough such that $N_0 \subset V_0$. That $(N_0, L_0)$ is in fact an index pair for $\{0\}$ is clear; the only bounded orbit (and thus the only orbit contained in $N_0$) is $\{0\}$, and since the first $\operatorname{ind}(x_0)$ coordinates increase of the flow and the others decrease, it is clear that $L_0$ is invariant in $N_0$ and that it is an exit set for $N_0$. It is also clearly regular.

When $\operatorname{ind}(x_0) > 1$, $L_0$ has the homotopy type of a sphere $S^{\operatorname{ind}(x_0)-1}$. Hence the exact sequence of a pair $(X, A)$, where $A \subset X$,

$$\cdots \to H_k(X) \to H_k(X, A) \to H_{k-1}(A) \to H_{k-1}(X) \to \cdots$$

becomes in this case

$$\cdots \to 0 \to H_{\operatorname{ind}(x_0)}(N_0, L_0) \to \mathbb{Z} \to 0 \to \cdots$$

and

$$\cdots \to 0 \to H_k(N_0, L_0) \to 0 \to 0 \to \cdots$$

for $k > \operatorname{ind}(x_0)$ and $1 < k < \operatorname{ind}(x_0)$. Finally, we have the exact sequence

$$\cdots \to 0 \to H_1(N_0, L_0) \to \mathbb{Z} \to \mathbb{Z} \to 0 \to \cdots$$

In the case $\operatorname{ind}(x_0) = 1$ we simply have

$$\cdots \to 0 \to H_1(N_0, L_0) \to \mathbb{Z}^2 \to \mathbb{Z} \to 0 \to \cdots$$

From the above sequences it immediately follows that, for $\operatorname{ind}(x_0) \geq 1$,

$$H_k(N_0, L_0) = \begin{cases} \mathbb{Z} & \text{if } k = \operatorname{ind}(x_0), \\ 0 & \text{else.} \end{cases}$$

Since the pair $(N_0, L_0)$ is regular, it follows that, for $\operatorname{ind}(x_0) \geq 1$,

$$H_k(N_0/L_0) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \operatorname{ind}(x_0), \\ 0 & \text{else.} \end{cases}$$

In the case that $\mathrm{ind}(x_0) = 0$, $(N_0, L_0)$ is not regular, since then $L_0 = \emptyset$. However, now $N_0/L_0$ is the disjoint union of two contractible spaces, namely $N_0$ and the isolated point corresponding to the empty set $L_0$. Hence

$$H_k(N_0/L_0) = \begin{cases} \mathbb{Z}^2 & \text{if } k = 0, \\ 0 & \text{else.} \end{cases}$$

Now that we have established the above, we wish to show that $(N_0', L_0') := (\Phi^{-1}(N_0), \Phi^{-1}(L_0))$ is an index pair for the isolated invariant set $\{x_0\}$. First of all, $\{x_0\}$ is clearly compact and invariant, since it is a critical point. Now,

$$\begin{aligned} I(\mathrm{cl}(N_0' \setminus L_0')) &= \bigcap_{t\in\mathbb{R}} \varphi^t(\mathrm{cl}(\Phi^{-1}(N_0) \setminus \Phi^{-1}(L_0))) = \bigcap_{t\in\mathbb{R}} \varphi^t(\mathrm{cl}(\Phi^{-1}(N_0 \setminus L_0))) \\ &= \bigcap_{t\in\mathbb{R}} (\Phi^{-1} \circ \Phi \circ \varphi^t \circ \Phi^{-1})(\mathrm{cl}(N_0 \setminus L_0)) \\ &= \Phi^{-1}\left(\bigcap_{t\in\mathbb{R}} \psi^t(\mathrm{cl}(N_0 \setminus L_0))\right) \\ &= \Phi^{-1}(0) = x_0. \end{aligned}$$

To see that $L_0'$ is invariant in $N_0'$, let $x \in L_0'$ and $t \geq 0$ be such that $\varphi([0,t]\times\{x\}) \subset N_0'$. Then $\Phi(x) \in L_0$ and

$$\psi([0,t], \{\Phi(x)\}) = \Phi(\varphi([0,t] \times \{x\})) \subset \Phi(N') = N_0.$$

Therefore, $\Phi(x) \in L_0$, since $(N_0, L_0)$ is an index pair, and thus $x \in L_0'$.

To see that $L_0'$ is an exit set for $N_0'$, let $x \in N_0'$ be such that $\varphi([0,\infty) \times \{x\}) \not\subset N_0'$. Then $\Phi(x) \in N_0$ and

$$\psi([0,\infty), \{\Phi(x)\}) = \Phi(\varphi([0,\infty) \times \{x\})) \not\subset \Phi(N') = N_0.$$

Therefore, there exists some $t \geq 0$ such that $\psi([0,t] \times \{\Phi(x)\}) \subset N_0$ and $\psi^t(\Phi(x)) \in L_0$. But then we have that

$$\varphi([0,t] \times \{x\}) = \Phi^{-1}\left(\psi([0,t] \times \{\Phi(x)\})\right) \subset \Phi^{-1}(N_0) = N_0'$$

and

$$\varphi^t(x) = \Phi^{-1}(\psi^t(\Phi(x))) \in \Phi^{-1}(L_0) = L_0'.$$

So $(N_0', L_0')$ is in fact an index pair for $x_0$. It is also regular, since $(N_0, L_0)$ is. Since $\Phi$ induces a homeomorphism $N_0'/L_0' \to N_0/L_0$, we have that

$$H_k(N_0'/L_0') = \begin{cases} \mathbb{Z} & \text{if } k = 0, \mathrm{ind}(x_0), \\ 0 & \text{else.} \end{cases}$$

if $\operatorname{ind}(x_0) \geq 1$ and

$$H_k(N'_0/L'_0) = \begin{cases} \mathbb{Z}^2 & \text{if } k = 0, \\ 0 & \text{else.} \end{cases}$$

if $\operatorname{ind}(x_0) = 0$.

We can proceed as above for *every* element of $S_c$. Since $N$ is compact, we know that $S_c$ is finite, and we can write $S_c = \{x_0, \ldots, x_l\}$ for some integer $l \in \mathbb{Z}$. Then for $i \in \{0, \ldots, l\}$ we obtain index pairs $(N'_i, L'_i)$ for $x_i$ with homology groups as described above. Now comes the crucial point; we can choose these index pairs such that they are actually disjoint; choosing small neighbourhoods of the $x_i$ such that these neighbourhoods are all disjoint, we can choose the $\epsilon$ in the construction of the index pairs small enough such that $[-\epsilon, \epsilon]^n$ is contained in the image of the small neighbourhood under the Morse chart. Now, it is easy to see that

$$(N', L') := (N'_0 \cup \cdots \cup N'_l, L'_0 \cup \cdots \cup L'_l)$$

is an index pair of $S_c$. The very useful property of this index pair is that we know its homology groups; indeed, we have that

$$N'/L' = \bigwedge_{i=0}^{l} N'_i/L'_i$$

and since all pairs $(N'_i, L'_i)$ are regular, it follows by e.g. [Hat10, Corollary 2.25] that the (reduced) homology groups of $N'/L'$ are the direct sum of the (reduced) homology groups of all the $N'_i/L'_i$; for $k > 0$ we have

$$H_k(N'/L') = \mathbb{Z}^{n_k},$$

where $n_k$ is the number of critical points in $S_c$ with index $k$. For $k = 0$ we have

$$H_0(N'/L') = \mathbb{Z}^{n_0+1}.$$

Now, by Proposition 4.7 (ii) $N'/L'$ and $N_b/N_a$ are homotopy equivalent. Then, since $(N_b, N_a)$ is regular, we obtain

$$H_k(N_b, N_a) = \mathbb{Z}^{n_k}$$

for *all* $k$ (in the case for $k = 0$ the extra generator cancels since $H_k(N_b, N_a)$ is actually isomorphic to the *reduced* homology group of $N_b/N_a$), and thus we finally obtain that

$$\sum_{k=0}^{n} \dim H_k(N_b, N_a) s^k = \sum_{df(x)=0, f(x)=c} s^{\operatorname{ind}(x)}.$$

This proves Claim 1 and completes the proof of Theorem 4.15. $\square$

# 5 Proof of Theorem 1.3 (Conley-Zehnder, nondegenerate case)

Finally, with the preparations made in the preceding sections, we are ready to prove the nondegenerate Conley-Zehnder Theorem. We will do this in two steps. First, we will tweak the theory of Section 3 to fit Hamiltonian symplectomorphisms of the torus $\mathbb{T}^{2n}$ instead of $\mathbb{R}^{2n}$. This step will establish that there are at least as many fixed points of the Hamiltonian symplectomorphism on $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ as critical points of the discrete symplectic action $\Phi$ on $\mathbb{R}^{2nN}/\mathbb{Z}^{2n}$. Then in step two, we will apply Conley index theory to $\Phi : \mathbb{R}^{2nN}/\mathbb{Z}^{2n} \to \mathbb{R}$ and, in particular, use the Morse inequalities to complete the proof.

## Step 1: a nicer discrete symplectic action

The general idea for this step is to look at lifts of symplectomorphisms of $\mathbb{T}^{2n}$ to the universal covering space $\mathbb{R}^{2n}$. The first thing we will do is refine Proposition 3.2 to obtain a better suited generating function, namely one that is invariant under the action of $\mathbb{Z}^{2n}$ on $\mathbb{R}^{2n}$. First, we need some preparation.

Let $H : [0,1] \times \mathbb{T}^{2n} \to \mathbb{R}$ be a smooth function, $\varphi_H : [0,1] \times [0,1] \times \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ the Hamiltonian flow induced by $H$ and $\psi := \varphi_H^{1,0}$ the corresponding Hamiltonian symplectomorphism. Our goal is to lift this flow to $\mathbb{R}^{2n}$. Define the smooth function $\widetilde{H} := H \circ (\mathrm{Id} \times \pi) : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$, where $\pi : \mathbb{R}^{2n} \to \mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is the quotient map. This function determines a Hamiltonian vector field $X_{\widetilde{H}}$.

**Lemma 5.1.** *The time-dependent vector field $X_{\widetilde{H}}$ has a global flow $\varphi_{\widetilde{H}} : [0,1] \times [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\varphi_{\widetilde{H}}^{t,t_0}$ is a lift of $\varphi_H^{t,t_0}$ for every $t, t_0$.*[16]

*Proof.* First, recall that $\pi^*\bar{\omega}_0 = \omega_0$, and that $d\pi(x)$ is invertible for every $x \in \mathbb{R}^{2n}$, since it is a local diffeomorphism. Then, just as in the proof of Lemma 2.45,

$$\begin{aligned} d\widetilde{H}_t(x) &= dH_t(\pi(x)) \circ d\pi(x) = (\bar{\omega}_0)_{\pi(x)}((X_H^t \circ \pi)(x), d\pi(x)\cdot) \\ &= (\bar{\omega}_0)_{\pi(x)}((d\pi(x) \circ d\pi(x)^{-1} \circ X_H^t \circ \pi)(x), d\pi(x)\cdot) \\ &= (\omega_0)_x((d\pi(x)^{-1} \circ X_H^t \circ \pi)(x), \cdot). \end{aligned}$$

Hence for every $x \in \mathbb{R}^{2n}$ and $t \in [0,1]$, $X_{\widetilde{H}}^t(x) = (d\pi(x)^{-1} \circ X_H^t \circ \pi)(x)$. Next, we lift $\varphi_H : [0,1] \times [0,1] \times \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ to a smooth map $\tilde{\varphi}_H : [0,1] \times [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with initial contition $\tilde{\varphi}_H^{0,0} = \mathrm{Id}$.

**Claim 1.** For every $t \in [0,1]$, we have that $\tilde{\varphi}_H^{t,t} = \mathrm{Id}$.

[16] Recall from Appendix D that a lift $\tilde{f} : \widetilde{X} \to \widetilde{X}$ of a map $f : X \to X$ to the universal covering space $\widetilde{X}$ of $X$ satisfies $p \circ \tilde{f} = f \circ p$, where $p : \widetilde{X} \to X$ is the covering map.

We will prove this claim below.
Note also that

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=t_1}(\pi\circ\tilde{\varphi}_H^{t,t_0})(x) &= d\pi(\tilde{\varphi}_H^{t_1,t_0}(x))\left(\left.\frac{d}{dt}\right|_{t=t_1}\tilde{\varphi}_H^{t,t_0}(x)\right)\\ &= \left.\frac{d}{dt}\right|_{t=t_1}(\varphi_H^{t,t_0}\circ\pi)(x) = \left(X_H^{t_1}\circ\varphi_H^{t_1,t_0}\circ\pi\right)(x).\end{aligned}$$

Hence

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=t_1}\tilde{\varphi}_H^{t,t_0}(x) &= \left(d\pi(\tilde{\varphi}_H^{t_1,t_0}(x))^{-1}\circ X_H^{t_1}\circ\varphi_H^{t_1,t_0}\circ\pi\right)(x)\\ &= \left(d\pi(\tilde{\varphi}_H^{t_1,t_0}(x))^{-1}\circ X_H^{t_1}\circ\pi\right)(\tilde{\varphi}_H^{t_1,t_0}(x))\\ &= X_{\widetilde{H}}^{t_1}(\tilde{\varphi}_H^{t_1,t_0}(x)).\end{aligned}$$

This means that $\tilde{\varphi}_H$ is the (global) flow of $X_{\widetilde{H}}$.

**Proof of Claim 1:** First note that for every $x\in\mathbb{R}^{2n}$ and $t\in[0,1]$ we have that

$$(\pi\circ\tilde{\varphi}_H^{t,t})(x) = (\varphi_H^{t,t}\circ\pi)(x) = \pi(x).$$

Hence $\tilde{\varphi}_H^{t,t}(x) = x+k(t,x)$ for some $k(t,x)\in\mathbb{Z}^{2n}$. We want to show that $k(t,x)=0$ for all $x\in\mathbb{R}^{2n}$ and $t\in[0,1]$. To see this, let $x\in\mathbb{R}^{2n}$ and consider the map $\Phi_x:[0,1]\to\mathbb{R}^{2n}$ given by $\Phi_x(t)=\tilde{\varphi}_H^{t,t}(x)$. Then $\Phi_x$ is smooth and $\Phi_x(0)=x$. Now assume that $\Phi_x(t)\neq x$ for some $t\in[0,1]$, and define

$$t_0 := \inf\{t\in[0,1]\mid \Phi_x(t)\neq x\}.$$

Now there are two cases: either $\Phi_x(t_0)=x$ or not. In the first case, we easily reach a contradiction; for every $\epsilon>0$ there is some $t_\epsilon$ such that $|t_\epsilon - t_0|<\epsilon$ and $\|\Phi_x(t_\epsilon)-\Phi_x(t_0)\|\geq 1$. This of course contradicts continuity of $\Phi_x$.

The second case is not much different; in this case we know that $t_0>0$ and that $\Phi_x(t)=x$ for all $t<t_0$. Then we can argue in the same way as in the first case, except now approaching from below. So we reach a contradiction, and this proves Claim 1 and completes the proof of Lemma 5.1. □

Now, fixing some $N\in\mathbb{N}$ and denoting $\widetilde{\psi}:=\varphi_{\widetilde{H}}^{1,0}, \widetilde{\psi}_i := \varphi_{\widetilde{H}}^{(i+1)/N,i/N}$, we get that

$$\widetilde{\psi} = \widetilde{\psi}_{N-1}\circ\cdots\circ\widetilde{\psi}_0$$

and we wish to apply the theory of generating functions as in Example 3.4. However, since the $\widetilde{\psi}_i$ defined above are not just any symplectomorphisms, we can derive extra properties of their generating functions, namely that they descend to the torus.

The next proposition shows that lifts of Hamiltonian symplectomorphisms have "nicer" generating functions.

**Proposition 5.2.** *Let $\widetilde{\varphi} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the lift of a Hamiltonian symplectomorphism $\varphi$ of $\mathbb{T}^{2n}$ that is sufficiently close to the identity. Then $\widetilde{\varphi}$ admits a generating function $\widetilde{V} : \mathbb{R}^{2n} \to \mathbb{R}$ as in Definition 3.1 with the extra property that*

$$\widetilde{V}(z+k) = \widetilde{V}(z)$$

*for all $z \in \mathbb{R}^{2n}$ and $k \in \mathbb{Z}^{2n}$.*

*Proof.* Consider the space $\mathbb{T}^{2n} \times \mathbb{T}^{2n}$, and denote the projection map by $\pi : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{T}^{2n} \times \mathbb{T}^{2n}$. Writing an element of this space as $(\bar{x}_0, \bar{y}_0, \bar{x}_1, \bar{y}_1)$, we denote by $\bar{x}_0^i : \mathbb{T}^{2n} \times \mathbb{T}^{2n} \to \mathbb{T}^1$ the projection onto the $i$-th component, and use similar notation for the other projections. As discussed in Remark 2.62, the differentials $d\bar{x}_0^i$ and similar can be seen as a differential 1-form on $\mathbb{T}^{2n} \times \mathbb{T}^{2n}$ by canonically identifying $T_x\mathbb{T}^1 \simeq \mathbb{R}$. Now, define an open neighbourhood of the diagonal

$$U := (\mathbb{T}^{2n} \times \mathbb{T}^{2n}) \setminus \{(\bar{x}_0, \bar{y}_0, \bar{x}_1, \bar{y}_1) \mid x_0^i = x_1^i + k + \frac{1}{2} \text{ or } y_0^i = y_1^i + k + \frac{1}{2} \text{ for some } i \in \{1, \dots, n\}\},$$

i.e. the points $(\bar{x}_0, \bar{y}_0, \bar{x}_1, \bar{y}_1)$ in $\mathbb{T}^{2n} \times \mathbb{T}^{2n}$ such that for any two points in the pre-image of $\pi$ of this point, all the coordinates do not differ by an integer plus one half. The reason we define this neighbourhood is that we wish is that we wish to define functions "$y_0^i - y_1^i$" and "$x_1^i - x_0^i$" that measure the difference between coordinates. Of course, this is not quite possible on the torus, but on $U$ we can actually make these functions well defined, by requiring that this difference lies in the interval $(-\frac{1}{2}, \frac{1}{2})$. Specifically, we define the function $Y_i : U \to \mathbb{R}$ by $Y_i(\bar{x}_0, \bar{y}_0, \bar{x}_1, \bar{y}_1) := t$, where $t$ is the number $t \in (-\frac{1}{2}, \frac{1}{2})$ such that $\overline{y_0^i} = \overline{y_1^i + t}$. This is clearly unique, since $(\bar{x}_0, \bar{y}_0, \bar{x}_1, \bar{y}_1) \in U$, so the difference between the coordinates is not equal to an integer plus one half, and hence $\overline{y_0^i} = \overline{y_1^i + t'}$ for some $t' \in \mathbb{R} \setminus \{k + \frac{1}{2} \mid k \in \mathbb{Z}\}$. Since $\overline{y_1^i + k} = \overline{y_1^i}$ for any $k \in \mathbb{Z}$, it follows that there is a unique $t \in (-\frac{1}{2}, \frac{1}{2})$ such that $\overline{y_0^i} = \overline{y_1^i + t}$. Now, to see that $Y_i$ is smooth, note that the map $Y_i \circ \pi$ is locally just the map $(x_0, y_0, x_1, y_1) \mapsto y_0^i - y_1^i$ on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$, which is clearly smooth. Hence $Y_i$ is also locally smooth around every point in $U$, and thus $Y_i$ is smooth. We can similarly define $X_i : U \to \mathbb{R}$ by $X_i(\bar{x}_0, \bar{y}_0, \bar{x}_1, \bar{y}_1) := t$, where $t$ is the number $t \in (-\frac{1}{2}, \frac{1}{2})$ such that $\overline{x_1^i} = \overline{x_0^i + t}$. Now we define a differential 1-from $\alpha$ on $U$ by

$$\alpha := \sum_{i=1}^{n} Y_i \; d\bar{x}_1^i + X_i \; d\bar{y}_0^i.$$

It is clear from the definitions that we have $dY_i = d\bar{y}_0^i - d\bar{y}_1^i$ and $dX_i = d\bar{x}_1^i - d\bar{x}_0^i$. Hence we get

$$\begin{aligned} d\alpha &= \sum_{i=1}^{n} \left[ d\bar{y}_0^i \wedge d\bar{x}_1^i - d\bar{y}_1^i \wedge d\bar{x}_1^i + d\bar{x}_1^i \wedge d\bar{y}_0^i - d\bar{x}_0^i \wedge d\bar{y}_0^i \right] \\ &= \sum_{i=1}^{n} \left[ d\bar{x}_1^i \wedge d\bar{y}_1^i - d\bar{x}_0^i \wedge d\bar{y}_0^i \right] \\ &= (-\bar{\omega}_0) \oplus \bar{\omega}_0. \end{aligned}$$

This means that when $(v_1, v_2, v_1', v_2') \in (T_{(\bar{x}_0,\bar{y}_0)}\mathbb{T}^{2n})\times(T_{(\bar{x}_1,\bar{y}_1)}\mathbb{T}^{2n}) = T_{(\bar{x}_0,\bar{y}_0,\bar{x}_1,\bar{y}_1)}(\mathbb{T}^{2n}\times\mathbb{T}^{2n})$ we have that

$$d\alpha_{(\bar{x}_0,\bar{y}_0,\bar{x}_1,\bar{y}_1)}(v_1, v_2, v_1', v_2') = -(\bar{\omega}_0)_{(\bar{x}_0,\bar{y}_0)}(v_1, v_2) + (\bar{\omega}_0)_{(\bar{x}_1,\bar{y}_1)}(v_1', v_2').$$

Now, denote by $H : [0,1]\times\mathbb{T}^{2n} \to \mathbb{T}^{2n}$ the smooth function that induces $\varphi$. Denote also by $X_H$ the induced Hamiltonian vector field and by $\varphi_H$ the induced Hamiltonian isotopy. When $\varphi$ is sufficiently close to the identity, for every $t \in [0,1]$, the image of $\mathrm{gr}\ \varphi_H^t$ is contained in $U$, and we can pull by $\alpha$ by this map. We then define

$$\beta_t := (\mathrm{gr}\ \varphi_H^t)^*\alpha.$$

Now, note that

$$\left.\frac{d}{ds}\right|_{s=t}(\mathrm{gr}\ \varphi_H^s)(\bar{x}_0,\bar{y}_0) = (0, X_H(t, \varphi_H^t(\bar{x}_0,\bar{y}_0))) \in (T_{(\bar{x}_0,\bar{y}_0)}\mathbb{T}^{2n})\times(T_{(\bar{x}_1,\bar{y}_1)}\mathbb{T}^{2n}).$$

We will now use the following generalized Cartan formula, which we will not prove here

**Claim 1.** Let $X_1, X_2$ be smooth manifolds, $f : I\times X_1 \to X_2$ a smooth isotopy and $\omega$ a differential $k$-form ($k \geq 1$) on $X_2$. Defining the differential $(k-1)$- and $k$-forms, respectively,

$$\begin{aligned}
(\omega_t^1)_x(v_1,\ldots,v_{k-1}) &:= \omega_{f_t(x)}\left(\left.\frac{d}{ds}\right|_{s=t} f_s(x), df_t(x)v_1, \ldots, df_t(x)v_{k-1}\right),\\
(\omega_t^2)_x(v_1,\ldots,v_k) &:= (d\omega)_{f_t(x)}\left(\left.\frac{d}{ds}\right|_{s=t} f_s(x), df_t(x)v_1, \ldots, df_t(x)v_k\right),
\end{aligned}$$

we have

$$\left.\frac{d}{ds}\right|_{s=t} f_t^*\omega = \omega_t^2 + d\omega_t^1.$$

Applying this claim to what we have we get in this case

$$\begin{aligned}
(\omega_t^2)_{(\bar{x}_0,\bar{y}_0)} &= (\bar{\omega}_0)_{\varphi_H^t(\bar{x}_0,\bar{y}_0)}\big(X_H(t,\varphi_H^t(\bar{x}_0,\bar{y}_0)), d\varphi_H^t(x)\,\cdot\,\big)\\
&= ((\varphi_H^t)^*\iota_{X_H^t}\bar{\omega}_0)_{(\bar{x}_0,\bar{y}_0)}\\
&= ((\varphi_H^t)^* dH_t)_{(\bar{x}_0,\bar{y}_0)}.
\end{aligned}$$

Hence we get

$$\left.\frac{d}{ds}\right|_{s=t}\beta_t = (\varphi_H^t)^* d\big(H_t + \omega_1^t\big) = d\left[(\varphi_H^t)^*\big(H_t + \omega_1^t\big)\right].$$

So by the fundamental theorem of calculus we get

$$(\mathrm{gr}\varphi)^*\alpha = \int_0^1 d\left[(\varphi_H^t)^*\big(H_t+\omega_1^t\big)\right]dt = d\left[\int_0^1 (\varphi_H^t)^*\big(H_t+\omega_1^t\big)dt\right]$$

So we see that $(\mathrm{gr}\varphi)^*\alpha$ is exact. To shorten notation, we write $(\mathrm{gr}\varphi)^*\alpha = dV$, where $V : \mathbb{T}^{2n} \to \mathbb{R}$ is a smooth function. Now, defining $\widetilde{V} = V \circ \pi$, we get

$$(\mathrm{gr}\varphi \circ \pi)^*\alpha = \pi^*(\mathrm{gr}\varphi)^*\alpha = d\widetilde{V}.$$

Now, writing a point in $\mathbb{R}^{2n}$ as $(x_0, y_0)$, we write, just as in Section 3.1,

$$\widetilde{\varphi}(x_0, y_0) = (u(x_0, y_0), v(x_0, y_0)) = (x_1, y_1).$$

It is easy to see from the definitions that

$$(\mathrm{gr}\varphi \circ \pi)^*\alpha = \sum_{i=1}^{n} (y_0^i - v_i)du_i + (u_i - x_0^i)dy_0^i.$$

Here we use that, by the assumption that $\widetilde{\varphi}$ is sufficiently small, $\widetilde{\varphi}(z) - z \in (-\frac{1}{2}, \frac{1}{2})$ for every $z \in \mathbb{R}^{2n}$. We also assume that $\|d\widetilde{\varphi}(z) - \mathrm{Id}\| < \frac{1}{2}$ for every $z \in \mathbb{R}^{2n}$, and thus, as in the proof of Proposition 3.2, we can use the coordinates $(x_1, x_0)$ on $\mathbb{R}^{2n}$. In these coordinates we get that

$$(\mathrm{gr}\varphi \circ \pi)^*\alpha = \sum_{i=1}^{n} (y_0^i - y_1^i)dx_1^i + (x_1^i - x_0^i)dy_0^i = d\widetilde{V}.$$

This means exactly that $\widetilde{V}$ is a generating function for $\widetilde{\varphi}$, since $\widetilde{\varphi}(x_0, y_0) = (x_1, y_1)$ iff

$$x_1 - x_0 = \frac{\partial V}{\partial y}(x_1, y_0), \qquad y_0 - y_1 = -\frac{\partial V}{\partial x}(x_1, y_0).$$

Of coures, by construction $\widetilde{V}$ is a lift of a function on $\mathbb{T}^{2n}$, and thus clearly satisfies

$$\widetilde{V}(z + k) = \widetilde{V}(z)$$

for all $z \in \mathbb{R}^{2n}$ and $k \in \mathbb{Z}^{2n}$. $\square$

**Remark 5.3.** The hypothesis that $\widetilde{\varphi}$ is "sufficiently small" may seem quite strict, since, as becomes clear in the proof, we require that $\widetilde{\varphi}(z) - z \in (-\frac{1}{2}, \frac{1}{2})$ for every $z \in \mathbb{R}^{2n}$. However, this is not unreasonable; of course, the way we want to apply this proposition to $\varphi_{\widetilde{H}}^{(i+1)/N, i/N}$, which is a lift of $\varphi_H^{(i+1)/N, i/N}$. Now, $\varphi_H^{(i+1)/N, i/N}$ is Hamiltonian for every $i$ by Remark 2.49, and it can be shown using the flux homomorphism[17] that for any lift $\varphi' : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ of $\varphi_H^{(i+1)/N, i/N}$ we have that $\varphi'(z + k) = \varphi'(z) + k$ for any $z \in \mathbb{R}^{2n}$ and $k \in \mathbb{Z}^{2n}$. In particular, $\varphi' - \mathrm{Id}$ is periodic. Thus the condition $\widetilde{\varphi}(z) - z \in (-\frac{1}{2}, \frac{1}{2})$ only needs to hold on the compact set $[0, 1]^{2n}$, and this is clearly doable by letting $N$ get big enough, since $\varphi_{\widetilde{H}}^{t,t} = \mathrm{Id}$ for every $t$.

[17] See e.g. [MS99, Chapter 10].

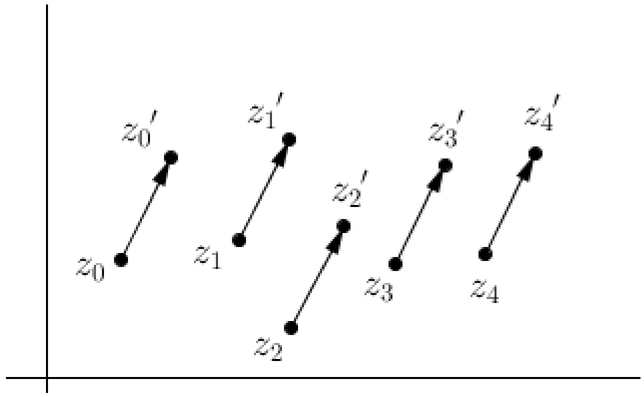


**Figure 4:** The action of $\mathbb{Z}^{2n}$ on $\mathbb{R}^{2n(N-1)}$ for $n = 1$ and $N = 5$; $(1, 2) \cdot \{z_0, \ldots, z_4\} = \{z_0', \ldots, z_4'\}$.

As before, denote by $V_i : \mathbb{R}^{2n} \to \mathbb{R}$ the generating function for $\widetilde{\psi}_i$ as in Proposition 5.2 and define the discrete symplectic action $\Phi : X_{N,n} \to \mathbb{R}$ by

$$\Phi\Big(x_0, y_0, \ldots, x_{N-1}, y_{N-1}\Big) := \sum_{i=0}^{N-1} \Big(\langle y_i, x_{i+1} - x_i\rangle - V_i(x_{i+1}, y_i)\Big). \tag{5.1}$$

Recall that $X_{N,n} \simeq \mathbb{R}^{2nN}$ is the space of $N$-periodic sequences in $\mathbb{R}^{2n}$. As before, due to Proposition 3.6, (nondegenerate) fixed points $\widetilde{\psi}$ correspond to (nondegenerate) critical points of $\Phi$. However, this is not quite enough for our purposes; two distinct fixed points of $\widetilde{\psi}$ might just come down to *one* fixed point of $\psi$ if they are related by the action of $\mathbb{Z}^{2n}$ on $\mathbb{R}^{2n}$. So to overcome this problem, we have to use the additional properties of the generating functions $V_i$. First, we define the action of $\mathbb{Z}^{2n}$ on $X_{N,n}$ by

$$(k, l) \cdot \{(x_i, y_i)\}_{i\in\mathbb{Z}} = \{(x_i + k, y_i + l)\}_{i\in\mathbb{Z}}$$

for $(k, l) \in \mathbb{Z}^{2n}$ and $\{(x_i, y_i)\}_{i\in\mathbb{Z}} \in X_{N,n}$. This can be seen as translating the whole sequence $\{z_0, \ldots, z_{N-1}\}$ by $(k, l)$, see Figure 4. This action is clearly smooth, free and proper, and thus by Theorem 2.54, $X_{N,n}/\mathbb{Z}^{2n}$ has a unique smooth structure such that the quotient map $\pi' : X_{N,n} \to X_{N,n}/\mathbb{Z}^{2n}$ is a smooth covering map. Now note that

$$\begin{aligned}\Phi\Big((k, l) \cdot \{(x_i, y_i)\}_{i\in\mathbb{Z}}\Big) &= \sum_{i=0}^{N-1} \Big(\langle y_i + l, x_{i+1} + k - x_i - k\rangle - V_i(x_{i+1} + k, y_i + l)\Big) \\ &= \sum_{i=0}^{N-1} \Big(\langle y_i, x_{i+1} - x_i\rangle - V_i(x_{i+1}, y_i)\Big) + \sum_{i=0}^{N-1} \langle l, x_{i+1} - x_i\rangle \\ &= \Phi\Big(x_0, y_0, \ldots, x_{N-1}, y_{N-1}\Big).\end{aligned}$$

Here we use that $V_i$ is invariant under the action of $\mathbb{Z}^{2n}$ for every $i$ and that the sequence $\{x_i\}_{i\in\mathbb{Z}}$ is periodic. Hence we get that $\Phi$ descends to the quotient $\mathbb{R}^{2nN}/\mathbb{Z}^{2n}$, since it is invariant under the action of $\mathbb{Z}^{2n}$. Now we are ready to formulate the adapted version of Proposition 3.6.

**Proposition 5.4.** *Let $\varphi_{N-1}, \dots, \varphi_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be symplectomorphisms that all satisfy the condition of Proposition 5.2, and let $\Phi : X_{N,n} \to \mathbb{R}$ be the discrete symplectic action induced by their generating functions. Writing $\varphi = \varphi_{N-1} \circ \dots \circ \varphi_0$, every critical point of $\Phi$ on $X_{N,n}/\mathbb{Z}^{2n}$ induces exactly one fixed point of $\varphi$ on $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$. Furthermore, a critical point of $\Phi$ on $X_{N,n}/\mathbb{Z}^{2n}$ is nondegenerate iff the corresponding fixed point is nondegenerate.*

Before we start the proof, let us establish some notation. We denote by $\bar{\varphi}_i : \mathbb{R}^{2n}/\mathbb{Z}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ and $\bar{\Phi} : \mathbb{R}^{2nN}/\mathbb{Z}^{2n} \to \mathbb{R}$ the maps that satisfy $\pi \circ \varphi_i = \bar{\varphi}_i \circ \pi$ and $\bar{\Phi} \circ \pi' = \Phi$, where $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ and $\pi' : \mathbb{R}^{2nN} \to \mathbb{R}^{2nN}/\mathbb{Z}^{2n}$ are the quotient maps. Furthermore, for $(x_0, y_0) \in \mathbb{R}^{2n}$ we denote $(\bar{x}_0, \bar{y}_0) = \pi(x_0, y_0)$ and for $\{z_j\} = \{(x_i, y_i)\}_{i\in\mathbb{Z}}$ we denote $\overline{\{z_j\}} = \pi'(\{z_j\})$.

*Proof of Proposition 5.4.* The crucial fact that we will use in this proof is that as in Remark 2.62, there is a chart around any point $\overline{\{z_j\}} \in \mathbb{R}^{2nN}/\mathbb{Z}^{2n}$ that is given by $\overline{\{z_j\}} \mapsto \{z_j\}$. This follows again from the fact that for sufficiently small neighbourhoods of a point $\{z_j\} \in \mathbb{R}^{2nN}$, $\pi'$ is injective in this neighbourhood, and we can use the (local) inverse of $\pi'$ as a chart. A similar case of course holds for points $(\bar{x}_0, \bar{y}_0) \in \mathbb{R}^{2n}/\mathbb{Z}^{2n}$.

Now let $\overline{\{z_j\}} \in \mathbb{R}^{2nN}/\mathbb{Z}^{2n}$ be a critical point of $\bar{\Phi}$. Using a chart as described as above, this implies that all the partial derivatives of $\Phi$ are zero, at any point in the pre-image $\pi'^{-1}(\overline{\{z_j\}})$. Then by Proposition 3.6, every point in the pre-image induces a fixed point of $\varphi$, namely writing a point in the pre-image as $(x_0, y_0, \dots, x_{N-1}, y_{N-1})$, $(x_0, y_0)$ is a fixed point of $\varphi$. Now, all the other points in the pre-image $\pi'^{-1}(\overline{\{z_j\}})$ can be written as $(x_0 + k, y_0 + l, \dots, x_{N-1} + k, y_{N-1} + l)$ for some $(k, l) \in \mathbb{Z}^{2n}$, and hence all the induced fixed points are identified by the action of $\mathbb{Z}^{2n}$: indeed, the set of fixed points of $\varphi$ induced by the critical point of $\bar{\Phi}$ is given by

$$\{(x_0 + k, y_0 + l) \mid (k, l) \in \mathbb{Z}^{2n}\}.$$

Now every fixed point $(x_0, y_0)$ of $\varphi$ induces a fixed point of $\bar{\varphi}$, since $\bar{\varphi}(\bar{x}_0, \bar{y}_0) = \pi(\varphi(x_0, y_0)) = (\bar{x}_0, \bar{y}_0)$. Also, two fixed points of $\varphi$ that are identified by the action of $\mathbb{Z}^{2n}$ induce the same fixed point of $\bar{\varphi}$.

In the end, we have that every critical point of $\bar{\Phi}$ induces a set

$$\{(x_0 + k, y_0 + l) \mid (k, l) \in \mathbb{Z}^{2n}\}$$

of fixed points of $\varphi$, and since this whole set gets mapped to *one* fixed point of $\bar{\varphi}$, we indeed get that every critical point of $\bar{\Phi}$ induces exactly one fixed point of $\bar{\varphi}$, and we have proved the first part of the proposition.

For the second part, note first that the Hessian of a critical point $\overline{\{z_j\}} \in \mathbb{R}^{2nN}/\mathbb{Z}^{2n}$ in a chart described at the beginning of the proof if just the Hessian matrix of $\Phi$ at any point in $\pi'^{-1}(\overline{\{z_j\}})$. Hence a critical point $\overline{\{z_j\}} \in \mathbb{R}^{2nN}/\mathbb{Z}^{2n}$ is nondegenerate iff the points in $\pi'^{-1}(\overline{\{z_j\}})$ are nondegenerate as critical points of $\Phi$.

Now let $(\bar{x}_0, \bar{y}_0) \in \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be a fixed point of $\bar{\varphi}$ that is induced by a critical point of $\bar{\Phi}$, that is, such that any $(x_0, y_0) \in \pi^{-1}(\bar{x}_0, \bar{y}_0)$ is a fixed point of $\varphi$.[18] Then in a chart as described at the beginning of the proof, the differential $d\bar{\varphi}(\bar{x}_0, \bar{y}_0)$ is just the Jacobian matrix of $\varphi$ at any point in the pre-image $\pi^{-1}(\bar{x}_0, \bar{y}_0)$. Hence the fixed point $(\bar{x}_0, \bar{y}_0)$ is nondegenerate iff all corresponding fixed points of $\varphi$ are nondegenerate.

Combining the above two results with the fact that by Proposition 3.6 the nondegenerate critical points of $\Phi$ and nondegenerate fixed points of $\varphi$ are in one-to-one correspondence, we have indeed proven the second part of the proposition. □

For the sake of clarity, let us quickly explain how the results in this section complete step 1. Recall our definitions at the start of this step; let $H : [0,1] \times \mathbb{T}^{2n} \to \mathbb{R}$ be a smooth function, $\varphi_H : [0,1] \times [0,1] \times \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ the Hamiltonian flow induced by $H$ and $\psi := \varphi_H^{1,0}$ the corresponding Hamiltonian symplectomorphism. Define the smooth function $\widetilde{H} := H \circ (\mathrm{Id} \times \pi) : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$. Define the maps

$$\widetilde{\psi} := \varphi_{\widetilde{H}}^{1,0}, \qquad \widetilde{\psi}_i := \varphi_{\widetilde{H}}^{(i+1)/N, i/N},$$

where $\varphi_{\widetilde{H}}$ is the flow of $\widetilde{H}$. Then by Lemma 5.1 every $\widetilde{\psi}_i$ is a lift of $\varphi_H^{(i+1)/N, i/N}$, and is thus a lift of a Hamiltonian symplectomorphism of the torus. Also, since $\varphi_{\widetilde{H}}^{t,t} = \mathrm{Id}$ for every $t$ it follows that for sufficiently large $N$, every $\widetilde{\psi}_i$ is sufficiently close to the identity, and thus admits a generating function as in Proposition 5.2. Since $\widetilde{\psi}$ is a lift of the Hamiltonian symplectomorphism $\psi$, it follows then by Proposition 5.4 that $\psi$ has at least as many fixed points as the discrete symplectic action $\Phi : \mathbb{R}^{2nN}/\mathbb{Z}^{2n} \to \mathbb{R}$ associated with it. Furthermore, when we assume that $\psi$ has only nondegenerate fixed points, it follows that $\Phi : \mathbb{R}^{2nN}/\mathbb{Z}^{2n} \to \mathbb{R}$ is in fact a Morse function, and thus we can use Conley index theory and the Morse inequalities to establish a lower bound for the number of critical points of $\Phi$, and thus also for the number of fixed points of $\psi$. This completes step 1.

## Step 2: Conley index theory for the discrete symplectic action

In this step, we will define a pseudo-gradient flow adapted to $\Phi$ on $\mathbb{R}^{2nN}/\mathbb{Z}^{2n}$, find a regular index pair for it and compute its Conley-Betti numbers. The sum of these Conley-Betti numbers will then be a lower bound for the number of critical points of $\Phi$, by Theorem 4.15 (Morse inequalities). Combining this with step 1, the sum of these Conley-Betti

[18]Note that this not necessarily hold for any fixed point of $\bar{\varphi}$; indeed, this is actually the reason that this proposition does not establish a one-to-one correspondence as before. If $(\bar{x}_0, \bar{y}_0) \in \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ is a fixed point of $\bar{\varphi}$, all we know is that $\varphi(x_0, y_0) = (x_0 + k, y_0 + l))$ for some $(k,l) \in \mathbb{Z}^{2n}$. Only when the fixed point is induced by a critical point of $\bar{\Phi}$ do we know that a "lift" of $(\bar{x}_0, \bar{y}_0)$ is a fixed point of $\varphi$.

numbers will then be a lower bound for the number of fixed points of $\psi$.

First, we will identify $\mathbb{R}^{2nN}/\mathbb{Z}^{2n}$ with a slightly nicer space, namely $\mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}$. Indeed, the smooth[19] map $F : \mathbb{R}^{2nN} \to \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}$ defined by

$$F(z_0, \ldots, z_{N-1}) := (\bar{z}_0, z_1 - z_0, \ldots, z_{N-1} - z_{N-2})$$

is invariant under the action of $\mathbb{Z}^{2n}$ on $\mathbb{R}^{2nN}$, and thus descends to a smooth map

$$\bar{F} : \mathbb{R}^{2nN}/\mathbb{Z}^{2n} \to \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}, \;\; \overline{(z_0, \ldots, z_{N-1})} \mapsto (\bar{z}_0, z_1 - z_0, \ldots, z_{N-1} - z_{N-2}).$$

Similarly, the map $F' : \mathbb{R}^{2nN} \to \mathbb{R}^{2nN}/\mathbb{Z}^{2n}$ defined by

$$F'(z_0, \ldots, z_{N-1}) := \overline{(z_0, z_1 + z_0, \ldots, z_{N-1} + \cdots + z_0)}.$$

also descends to a smooth map

$$\bar{F}' : \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)} \to \mathbb{R}^{2nN}/\mathbb{Z}^{2n}, \;\; (\bar{z}_0, \ldots, z_{N-1}) \mapsto \overline{(z_0, z_1 + z_0, \ldots, z_{N-1} + \cdots + z_0)}.$$

Now, $\bar{F}$ and $\bar{F}'$ are clearly inverses of each other, and thus $\bar{F} : \mathbb{R}^{2nN}/\mathbb{Z}^{2n} \to \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}$ is a diffeomorphism. Now, the discrete symplectic action $\Phi : \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)} \to \mathbb{R}$ is given by[20]

$$\begin{aligned}
\Phi(\bar{z}_0, \ldots, z_{N-1}) &= \sum_{i=0}^{N-1} \Big( \langle y_i + \cdots + y_0, x_{i+1} \rangle - \langle y_{N-1} + \cdots + y_0, x_{i+1} \rangle \\
&\quad - V_i(x_{i+1} + \cdots + x_0, y_i + \cdots + y_0) \Big) \\
&= \sum_{i=0}^{N-1} \Big( - \langle y_{N-1} + \cdots + y_{i+1}, x_{i+1} \rangle - V_i(x_{i+1} + \cdots + x_0, y_i + \cdots + y_0) \Big),
\end{aligned}$$

where we write $z_j = (x_j, y_j)$ as before.

Now, we define the function $W : \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)} \to \mathbb{R}$ by

$$W(\bar{z}_0, \ldots, z_{N-1}) = - \sum_{i=0}^{N-1} V_i(x_{i+1} + \cdots + x_0, y_i + \cdots + y_0).$$

[19] It is smooth since it is the composition of the clearly smooth map $(z_0, \ldots, z_{N-1}) \mapsto (z_0, z_1 - z_0, \ldots, z_{N-1} - z_{N-2})$ from $\mathbb{R}^{2nN}$ to itself and the quotient map $\pi \times \mathrm{Id} : \mathbb{R}^{2nN} \to \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}$.

[20] Technically, this is a different map $\Phi' : \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)} \to \mathbb{R}$ defined by $\Phi' := \Phi \circ \bar{F}'$, but since it is still essentially the same map, we will refer to it by the same name. Since $\bar{F}'$ is a diffeomorphism, this "new" $\Phi$ is still a Morse function, with the same number of critical points as before.

Then for any $(k_1, l_1, \dots, k_{N-1}, l_{N-1}) \in \mathbb{Z}^{2n(N-1)}$ we have that

$$\begin{aligned}
W(\bar{z}_0, \dots, x_{N-1} + k_{N-1}, y_{N-1} + l_{N-1}) &= -\sum_{i=0}^{N-1} V_i(x_{i+1} + \cdots + x_0 + k'_{i+1}, y_i + \cdots + y_0 + l'_i) \\
&= -\sum_{i=0}^{N-1} V_i(x_{i+1} + \cdots + x_0, y_i + \cdots + y_0) \\
&= W(\bar{z}_0, \dots, x_{N-1}, y_{N-1}),
\end{aligned}$$

since $k'_i = k_1 + \cdots + k_i \in \mathbb{Z}$, $l'_i = l_1 + \cdots + l_i \in \mathbb{Z}$ and $V_i$ is periodic for any $i$. Hence $W$ is completely determined by its values on $\mathbb{T}^{2n} \times [0,1]^{2n(N-1)}$, and in particular, $W$ is bounded.

Now, define the $n(N-1) \times n(N-1)$ matrix $B$ by

$$B := \begin{pmatrix} \mathrm{Id}_n & \mathrm{Id}_n & \cdots & \mathrm{Id}_n \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathrm{Id}_n \\ 0 & \cdots & 0 & \mathrm{Id}_n \end{pmatrix},$$

where $\mathrm{Id}_n$ denotes the $n \times n$ identity matrix. Now we define the $2n(N-1) \times 2n(N-1)$ matrix $P$ by

$$P := \begin{pmatrix} 0 & -B \\ -B^T & 0 \end{pmatrix}.$$

Now, writing

$$\mathbf{z} = (\mathbf{x}, \mathbf{y}), \qquad \mathbf{x} = (x_1, \dots, x_{N-1}), \qquad \mathbf{y} = (y_1, \dots, y_{N-1}),$$

we get that

$$P\mathbf{z} = -\begin{pmatrix} B\mathbf{y} \\ B^T\mathbf{x} \end{pmatrix} = -\begin{pmatrix} y_{N-1} + \cdots + y_1 \\ y_{N-1} + \cdots + y_2 \\ \vdots \\ y_{N-1} \\ x_1 \\ x_1 + x_2 \\ \vdots \\ x_1 + \cdots + x_{N-1} \end{pmatrix}.$$

This means that

$$\langle \mathbf{z}, P\mathbf{z} \rangle = -2\langle x_1, y_{N-1} + \cdots + y_1 \rangle - \cdots - 2\langle x_{N-1}, y_{N-1} \rangle.$$

Thus we find that

$$\Phi(\bar{z}_0, \mathbf{z}) = \frac{1}{2}\langle \mathbf{z}, P\mathbf{z} \rangle + W(\bar{z}_0, \mathbf{z}).$$

We already know that $W$ is bounded, so let us find out some more information about $P$. First of all, note that $B$ is invertible; if $(x_1, \dots, x_{N-1}) \in \ker B$, then this means that

$$B \begin{pmatrix} x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} = \begin{pmatrix} x_{N-1} + \dots + x_1 \\ x_{N-1} + \dots + x_2 \\ \cdots \\ x_{N-1} \end{pmatrix} = 0.$$

This then means that $x_{N-1} = 0$, $x_{N-2} = -x_{N-1} = 0$, etc. Hence the kernel of $B$ is trivial, and $B$ is invertible. But then so is $B^T$, and also $P$. Now let $(v_1, v_2) \in \mathbb{R}^{2n(N-1)}$ be an eigenvector of $P$, with eigenvalue $\lambda$. This means that

$$P \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -Bv_2 \\ -B^T v_1 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

But then

$$P \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix} = \begin{pmatrix} Bv_2 \\ -B^T v_1 \end{pmatrix} = -\lambda \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}.$$

This, combined with the fact that $P$ only has nonzero eigenvalues, means that the map $T : \mathbb{R}^{2n(N-1)} \to \mathbb{R}^{2n(N-1)}$ defined by

$$T(v_1, v_2) := (v_1, -v_2)$$

satisfies

$$T(E_P^+) \subset E_P^-, \qquad\qquad T(E_P^-) \subset E_P^+,$$

where $E_P^+$ is the positive eigenspace and $E_P^-$ the negative eigenspace of $P$. Since $T$ is invertible with inverse $T$, this implies that $T$ is a linear bijection between $E_P^+$ and $E_P^-$, which then implies that

$$\dim E_P^+ = \dim E_P^- = n(N-1) =: m.$$

So there is a splitting

$$\mathbb{R}^{2n(N-1)} = E_P^+ \oplus E_P^-$$

into the positive and negative eigenspace of $P$.

Now, we will define a pseudo-gradient flow on $\mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}$ adapted to $\Phi$. The most important property we will use is that $W$ is bounded, and thus that far away from the origin, $\Phi$ basically becomes the quadratic form

$$\mathbf{z} \mapsto \frac{1}{2} \langle \mathbf{z}, P\mathbf{z} \rangle$$

on $\mathbb{R}^{2n(N-1)}$. First, note that this means that all critical points of $\Phi$ are contained in some large compact set; for some sufficiently large $R > 0$ we have for all $\mathbf{z} \in \mathbb{R}^{2n(N-1)}$

with $\|\mathbf{z}\| > R$ that $P\mathbf{z}$ becomes so big compared to $W$ that the differential is never zero anymore. Indeed, defining $\widetilde{W} := W \circ (\pi \times \mathrm{Id})$, the differential becomes, in local coordinates,

$$d\Phi(\bar{z}_0, \mathbf{z}) = (P\mathbf{z})^T + (\text{grad } \widetilde{W})(z_0, \mathbf{z})^T,$$

where we view it as an operator on $\mathbb{R}^{2nN}$, and where we view $P\mathbf{z}$ and $(\text{grad } \widetilde{W})(z_0, \mathbf{z})$ as vectors in $\mathbb{R}^{2nN}$ ($z_0$ is the local coordinate for $\bar{z}_0$). Letting this act on the vector

$$v = \begin{pmatrix} 0 \\ P\mathbf{z} \end{pmatrix}$$

we get

$$d\Phi(\bar{z}_0, \mathbf{z})v = \|P\mathbf{z}\|^2 + \langle \partial_{\mathbf{z}} \widetilde{W}(z_0, \mathbf{z}), P\mathbf{z} \rangle$$

where

$$\partial_{\mathbf{z}} \widetilde{W}(z_0, \mathbf{z}) = \begin{pmatrix} \frac{\partial \widetilde{W}}{\partial z_{1,1}}(z_0, \mathbf{z}) \\ \vdots \\ \frac{\partial \widetilde{W}}{\partial z_{N-1,2n}}(z_0, \mathbf{z}) \end{pmatrix}$$

is just the gradient in the last $\mathbf{z}$ coordinates. Now since $W$ is bounded, so is $\partial_{\mathbf{z}} \widetilde{W}$, and we can choose $R > 0$ so that for all $\|\mathbf{z}\| > R$ we have that

$$\|P\mathbf{z}\| > \sup_{(z_0, \mathbf{z}') \in \mathbb{R}^{2nN}} \|\partial_{\mathbf{z}} \widetilde{W}(z_0, \mathbf{z}')\|.$$

Returning to the above case for $\|\mathbf{z}\| > R$ we then get that

$$\|P\mathbf{z}\|^2 > \|\partial_{\mathbf{z}} \widetilde{W}(z_0, \mathbf{z})\| \cdot \|P\mathbf{z}\| \geq |\langle \partial_{\mathbf{z}} \widetilde{W}(z_0, \mathbf{z}), P\mathbf{z} \rangle|$$

and thus

$$d\Phi(\bar{z}_0, \mathbf{z})v \neq 0.$$

So we know that all critical points are contained in the compact set $X_R := \mathbb{T}^{2n} \times \{\mathbf{z} \in \mathbb{R}^{2n(N-1)} \mid \|\mathbf{z}\| \leq R\}$. In particular, this means that there are only finitely many. Denote the set of critical points by $\{x_0, \ldots, x_r\}$ and choose a Morse chart $\varphi_i : U_i \to V_i \subset \mathbb{R}^{2nN}$ around every critical point $x_i$. Adding extra open charts $\varphi_j : U_j \to V_j \subset \mathbb{R}^{2nN}$ we obtain a finite open cover $\{U_1, \ldots, U_r, \ldots, U_k\}$ of $X_R$. Without loss of generality, we can assume that every critical point $x_j$ is only contained in the open $U_j$, not in any of the others. Now on every $U_j$, define a vector field $X_j$ by pulling pack the negative gradient on $\mathbb{R}^{2nN}$;

$$X_j(\bar{z}_0, \mathbf{z}) := -d(\varphi_j^{-1})(\varphi_j(\bar{z}_0, \mathbf{z}))\Big((\text{grad } \Phi \circ \varphi_j^{-1})(\varphi_j(\bar{z}_0, \mathbf{z}))\Big)$$

Note that we then have

$$d\Phi(\bar{z}_0, \mathbf{z})\big(X_j(\bar{z}_0, \mathbf{z})\big) = -d(\Phi \circ \varphi_j^{-1})(\varphi_j(\bar{z}_0, \mathbf{z}))\Big((\text{grad } \Phi \circ \varphi_j^{-1})(\varphi_j(\bar{z}_0, \mathbf{z}))\Big) < 0,$$

when $(\bar{z}_0, \mathbf{z})$ is not a critical point, since the differential on $\mathbb{R}^{2nN}$ is essentially the transpose of the gradient, and that thus

$$d\Phi(\bar{z}_0, \mathbf{z})\big(X_j(\bar{z}_0, \mathbf{z})\big) = -\|(\text{grad } \Phi \circ \varphi_j^{-1})(\varphi_j(\bar{z}_0, \mathbf{z}))\|^2.$$

Now, we complete the cover by adding the set $U_{k+1} := \mathbb{T}^{2n} \times \{\mathbf{z} \in \mathbb{R}^{2n(N-1)} \mid \|\mathbf{z}\| > R\}$. On $U_{k+1}$, we define the vector field

$$X_{k+1}(\bar{z}_0, \mathbf{z}) := \begin{pmatrix} 0 \\ -P\mathbf{z} \end{pmatrix},$$

where we view the tangent space at $(\bar{z}_0, \mathbf{z})$ as

$$T_{(\bar{z}_0,\mathbf{z})}\big(\mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}\big) = T_{\bar{z}_0}\mathbb{T}^{2n} \oplus T_{\mathbf{z}}\mathbb{R}^{2n(N-1)} = T_{\bar{z}_0}\mathbb{T}^{2n} \oplus \mathbb{R}^{2n(N-1)}.$$

By the same computations we did to prove that there were no critical points in $U_{k+1}$ we get that

$$d\Phi(\bar{z}_0, \mathbf{z})\big(X_{k+1}(\bar{z}_0, \mathbf{z})\big) = -\|P\mathbf{z}\|^2 - \langle \partial_{\mathbf{z}}\widetilde{W}(z_0, \mathbf{z}), P\mathbf{z}\rangle < 0.$$

Now, we need to patch these vector fields together. Choose a smooth partition of unity $\{\rho_1, \dots, \rho_{k+1}\}$ subordinate to $\{U_1, \dots, U_{k+1}\}$ and define the (global) smooth vector fields

$$\widetilde{X}_j := \begin{cases} \rho_j(\bar{z}, \mathbf{z})X_j(\bar{z}, \mathbf{z}) & \text{if } (\bar{z}, \mathbf{z}) \in U_j, \\ 0 & \text{else.} \end{cases}$$

Then, we define

$$X(\bar{z}, \mathbf{z}) := \sum_{i=1}^{k+1} \widetilde{X}_i(\bar{z}, \mathbf{z}).$$

We claim that this is a pseudo-gradient field adapted to $\Phi$. First, note that when $(\bar{z}, \mathbf{z})$ is not a critical point, we get that

$$d\Phi(\bar{z}_0, \mathbf{z})\big(X(\bar{z}_0, \mathbf{z})\big) = \sum_{i=1}^{k+1} d\Phi(\bar{z}_0, \mathbf{z})\big(\widetilde{X}_i(\bar{z}, \mathbf{z})\big) < 0,$$

since $\rho_j \geq 0$ for every $j$ and we have $\rho_i(\bar{z}_0, \mathbf{z}) \neq 0$ for at least one $i$. To see the second condition, remember that by construction a critical point $x_i$ is only contained in the open $U_i$. Hence, we can take a very small open $U_i'$ that is disjoint with all the other open sets in the covering. Then we must have $\rho_i|_{U_i'} \equiv 1$ and thus $X$ under the Morse chart $(U_i', \varphi_i|_{U_i'})$ is the negative the gradient on $\mathbb{R}^{2nN}$. Thus $X$ is indeed a pseudo-gradient.

To see that $X$ has a global flow, note that outside a large compact set the flow of $X$ is just the flow of $X_{k+1}$. Now, $X_{k+1}$ actually has a global flow, since the flow of this vector field is just the flow of linear, autonomous differential equation, which is globally defined. But this implies that $X$ also has a global flow; if $X$ does not have a global flow, an integral curve has to flow out of every compact subset of $\mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}$ in finite time. In particular, it needs to escape very large compact sets, where the flow is just given by the flow of $X_{k+1}$. But since this does not flow out of every compact set in finite time, the original integral curve does not either, so $X$ indeed has a global flow, which we call $\varphi : \mathbb{R} \times (\mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}) \to \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}$.

Now all that is left is to find an isolated invariant set containing all the critical points of $\Phi$, and to find a regular index pair for this set. We define $S \subset \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}$ to be the union of all bounded orbits of $\varphi$, i.e. the points $x \in \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}$ such that $\varphi^t(\mathbb{R} \times \{x\}) \subset \mathbb{T}^{2n} \times K$, where $K$ is some bounded subset of $\mathbb{R}^{2n(N-1)}$. By definition, $S$ is invariant and contains all the critical points of $\Phi$, since a critical point *is* a bounded orbit of $\varphi$. Now note that $S$ is contained in a compact set; by construction, there is some $R' > 0$ such that for every $(\bar{z}_0, \mathbf{z}) \in \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}$ with $\|\mathbf{z}\| > R'$ the vector field $X$ is given by

$$X(\bar{z}_0, \mathbf{z}) = \begin{pmatrix} 0 \\ -P\mathbf{z} \end{pmatrix}.$$

Then it follows that the flow here is (at least locally) given by $(t, \bar{z}_0, \mathbf{z}) \mapsto (\bar{z}_0, e^{-tP}\mathbf{z})$, where

$$e^{-tP} := \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} A^k$$

is the exponential of the matrix $-tP$. In particular, denoting by $\lambda_1^+, \dots, \lambda_m^+$ the positive and by $\lambda_1^-, \dots, \lambda_m^-$ the negative eigenvalues of $P$, and by $\{v_1^+, \dots, v_m^-\}$ an orthonormal basis of eigenvectors, this is given by

$$\left(t, \sum_{k=1}^{m} (a_k^+ v_k^+ + a_k^- v_k^-)\right) \mapsto \sum_{k=1}^{m} (a_k^+ e^{-t\lambda_k^+} v_k^+ + a_k^- e^{-t\lambda_k^-} v_k^-).$$

In particular, we see that as $t \to \infty$, the coefficients of the "negative" eigenvectors will diverge to either infinity or minus infinity, depending on whether they were positive or negative originally (or will stay zero if they were originally). A similar situation holds for the "positive" eigenvectors for $t \to -\infty$. Since obviously at least one of the coefficients has to be zero, it follows that an orbit through a point $(\bar{z}_0, \mathbf{z}) \in \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)}$ with $\|\mathbf{z}\| > R'$ will escape every bounded set, for either negative or positive time (or both).

So now we know that $S \subset \mathbb{T}^{2n} \times \{\mathbf{z} \in \mathbb{R}^{2n(N-1)} \mid \|\mathbf{z}\| \leq R'\} =: X_{R'}$, a compact set. So to prove that $S$ is compact, we only need to show that it is closed. So let $(x_i)_{i \in \mathbb{N}} \subset S$ be a sequence that converges to some $x \in X_{R'}$. Then for every $t \in \mathbb{R}$, the sequence $(\varphi^t(x_i))_{i \in \mathbb{N}}$ is contained in $X_{R'}$ (since the orbit of every $x_i$ is contained in $X_{R'}$) and converges $\varphi^t(x)$. Since $X_{R'}$ is compact, it then follows that $\varphi^t(x) \in X_{R'}$. Since this holds for every $t \in \mathbb{R}$, it follows that the oribt of $x$ is contained in $X_{R'}$, and thus that $x \in S$.

That $S$ is isolated is easy; recall that $S$ is isolated if there is some neighbourhood $N$ of $S$ such that $S$ is the set of all orbits contained in $N$. So then it immediately follows that $X_{R'}$ (and in fact, any set that contains $X_{R'}$) is such a neighbourhood for $S$.

Now, we need to find an index pair for $S$. We will once again exploit the explicit structure the flow has outside $X_{R'}$. Indeed, for some sufficiently large $R'' > 0$ we define

$$
\begin{aligned}
N &:= \{(\bar{z}_0, \mathbf{z}^+ + \mathbf{z}^-) \in \mathbb{T}^{2n} \times \mathbb{R}^{2n(N-1)} \mid \|\mathbf{z}^+\|, \|\mathbf{z}^-\| \leq R''\}, \\
L &:= \{(\bar{z}_0, \mathbf{z}^+ + \mathbf{z}^-) \in N \mid \|\mathbf{z}^-\| = R''\}.
\end{aligned}
$$

This is clearly a regular index pair for $S$; Under the (positive time) flow, $\|\mathbf{z}^-\|$ only gets bigger, and thus $L$ is clearly invariant in $N$. By similar reasoning, $L$ is also an exit set of $N$; under the (positive time) flow $\|\mathbf{z}^+\|$ can only become smaller., and thus if a point in $N$ flows out of $N$, it has to do this by having $\|\mathbf{z}^-\|$ become bigger than $R''$, meaning that the orbit leaves $N$ through $L$. The index pair is also regular; indeed, the map $F : [0,1] \times \{(\bar{z}_0, \mathbf{z}^+ + \mathbf{z}^-) \in N \mid \|\mathbf{z}^-\| \neq 0\} \to \{(\bar{z}_0, \mathbf{z}^+ + \mathbf{z}^-) \in N \mid \|\mathbf{z}^-\| \neq 0\}$ given by

$$
(t, \bar{z}_0, \mathbf{z}^+ + \mathbf{z}^-) \mapsto (\bar{z}_0, \mathbf{z}^+ + \frac{tR'' + (1-t)\|\mathbf{z}^-\|}{\|\mathbf{z}^-\|}\mathbf{z}^-)
$$

is a deformation retract onto $L$.

Now, by Theorem 4.15, the number of critical points of $\Phi$ is bounded below by the sum of the Conley-Betti numbers $b_k(S) := \dim H_k(N, L)$. Now, note that the homology of $N$ is just the homology of the torus $\mathbb{T}^{2n}$, and that the homology of $L$ is just the homology of $\mathbb{T}^{2n} \times S^{m-1}$ (recall that the negative eigenspace has dimension $m$). So all that is left to do is to compute the homology groups of these spaces and then use the exact sequence of a pair. We need the following lemma, which can be seen as a special case of a Künneth formula.

**Lemma 5.5.** *Let $X$ be a topological space and $n \in \mathbb{N}$. Then for all $k$,*

$$
H_k(X \times S^n) \simeq H_k(X) \oplus H_{k-n}(X).
$$

*Here we use the convention $H_k = 0$ for $k < 0$.*

*Proof.* The proof follows from two claims.

**Claim 1.** For $x_0 \in S^n$ we have that

$$
H_i(X \times S^n) \simeq H_k(X) \oplus H_k(X \times S^n, X \times \{x_0\}).
$$

**Claim 2.** We have that

$$
H_k(X \times S^n, X \times \{x_0\}) \simeq H_{k-1}(X \times S^{n-1}, X \times \{x_0\}).
$$

We postpone the proofs of these claims. Now, by applying Claim 1 and then Claim 2 $n$ times, we get that

$$
H_k(X \times S^n) \simeq H_k(X) \oplus H_{k-n}(X \times S^0, X \times \{x_0\})
$$

for $x_0 \in S^0$. However, since $S^0$ is just two points, this last group is just $H_{k-n}(X)$, and we are done.
**Proof of Claim 1:** Note that for $x_0 \in S^n$, we have a retract $r : X \times S^n \to X \times \{x_0\}$ given by

$$r(x, x') = (x, x_0).$$

Denoting by $\iota : X \times \{x_0\} \to X \times S^n$ the inclusion map, this means that $r \circ \iota = \mathrm{Id}$, and thus that the induced map $\iota_* : H_k(X \times \{x_0\}) \to H_k(X \times S^n)$ is injective, since also $r_* \circ \iota_* = \mathrm{Id}$. But then the long exact sequence of the pair $(X \times S^n, X \times \{x_0\})$ breaks up into short exact sequences

$$0 \to H_k(X \times \{x_0\}) \to H_k(X \times S^n) \to H_k(X \times S^n, X \times \{x_0\}) \to 0.$$

Then it follows from the Splitting Lemma (see e.g. [Hat10, p. 147]) that

$$H_k(X \times S^n) \simeq H_k(X \times \{x_0\}) \oplus H_k(X \times S^n, X \times \{x_0\}).$$

However, since clearly $H_k(X \times \{x_0\}) \simeq H_k(X)$, this proves Claim 1.
**Proof of Claim 2:** Writing $S^n$ as a subset of $\mathbb{R}^{n+1}$

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \dots + x_{n+1}^2 = 1\},$$

we define the sets

$$U_+ := \{(x_1, \dots, x_{n+1}) \in S^n \mid x_1 > -\epsilon\}, \tag{5.2}$$
$$U_- := \{(x_1, \dots, x_{n+1}) \in S^n \mid x_1 < \epsilon\}, \tag{5.3}$$

for some small $\epsilon > 0$. Choosing $x_0 \in S^n$ such that its first coordinate is zero, the interiors of the sets $X \times U_+$ and $X \times U_-$ cover $X \times S^n$, and, choosing some small contractible neighbourhood $U_0 \subset U_+ \cap U_+$ of $x_0$, the interior of $X \times U_0$ covers $X \times \{x_0\}$. Now, $X \times U_\pm$ is contractible and $(X \times U_+) \cap (X \times U_-)$ is homotopy equivalent to

$$X \times \{(x_1, \dots, x_{n+1}) \in S^n \mid x_1 = 0\} \simeq X \times S^{n-1}.$$

Hence the relative Mayer-Vietoris sequence yields

$$0 \to H_k(X \times S^n, X \times \{x_0\}) \to H_{k-1}(X \times S^{n-1}, X \times \{x_0\}) \to 0.$$

This proves Claim 2 and completes the proof of Lemma 5.5. $\square$

With this lemma, we can directly compute the homology groups of the torus $\mathbb{T}^n$, since we can view it as the product of $n$ copies of $S^1$. We claim that

$$H_k(\mathbb{T}^n) = \begin{cases} \mathbb{Z}^{\binom{n}{k}} & \text{if } 0 \leq k \leq n, \\ 0 & \text{else.} \end{cases}$$

Indeed, for $n = 1$, we just have $\mathbb{T}^1 = S^1$ and the above clearly holds. Now if the above holds for $n$, then by Lemma 5.5 we get that

$$H_k(\mathbb{T}^{n+1}) \simeq H_k(\mathbb{T}^n \times S^1) \simeq H_k(\mathbb{T}^n) \oplus H_{k-1}(\mathbb{T}^n).$$

Hence the dimension of $H_k(\mathbb{T}^{n+1})$ is given by

$$\begin{aligned}\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{(n+1-k)\cdot n!}{k!(n+1-k)!} + \frac{k \cdot n!}{k!(n+1-k)!} \\ &= \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}.\end{aligned}$$

So indeed, we get the result by induction. So now we get

$$H_k(N) = \begin{cases} \mathbb{Z}^{\binom{2n}{k}} & \text{if } 0 \leq k \leq 2n, \\ 0 & \text{else.} \end{cases}$$

and

$$H_k(L) = \begin{cases} \mathbb{Z}^{\binom{2n}{k}} & \text{if } 0 \leq k \leq 2n, \\ \mathbb{Z}^{\binom{2n}{k-m+1}} & \text{if } m-1 \leq k \leq 2n+m-1, \\ 0 & \text{else.} \end{cases}$$

Here we use that, without loss of generality, we can assume that $m - 1 > 2n$. Now we can use the exact sequence of the pair $(N, L)$,

$$\cdots \to H_k(L) \to H_k(N) \to H_k(N, L) \to H_{k-1}(L) \to H_{k-1}(N) \to \cdots$$

to compute the Conley-Betti numbers. Indeed, for $k > 2n + m$ we get

$$\cdots \to 0 \to H_k(N, L) \to 0 \to 0 \to \cdots$$

which implies $H_k(N, L) = 0$. Now, for $m \leq k \leq 2n + m$ we have

$$\cdots \to 0 \to H_k(N, L) \to \mathbb{Z}^{\binom{2n}{k-m}} \to 0 \to \cdots$$

This implies that for these $k$,

$$\dim H_k(N, L) = \binom{2n}{k-m}.$$

Next, for $2n + 1 < k < m$ we have

$$\cdots \to 0 \to H_k(N, L) \to 0 \to 0 \to \cdots$$

which again implies $H_k(N, L) = 0$. Finally, we have the exact sequence

$$0 \to H_{2n+1}(N, L) \to H_{2n}(L) \to H_{2n}(N) \to \cdots \to H_0(N, L) \to 0.$$

Now, since all the maps $H_i(L) \to H_i(N)$ in this sequence are isomorphisms, it follows also here that $H_i(N, L) = 0$ for $0 \leq i \leq 2n + 1$.

In conclusion, we find that the sum of the Conley-Betti numbers is equal to

$$\sum_{k=m}^{2n+m} \binom{2n}{k-m} = \sum_{k=0}^{2n} \binom{2n}{k} = 2^{2n}.$$

Thus by Theorem 4.15 and step 1 we find that the number of fixed points of a Hamiltonian symplectomorphism on the torus $\mathbb{T}^{2n}$, whose fixed points are all nondegenerate, is bounded below by $2^{2n}$. This completes the proof of the nondegenerate part of Theorem 1.3.

# A Some details omitted from Section 2

**Theorem A.1** (Rank-nullity theorem, used in the proof of Proposition 2.9)**.** *Let $V, W$ be vector spaces and $T : V \to W$ a linear map. Then we have that*

$$\dim \ker T + \dim \mathrm{im}\ T = \dim V.$$

*Proof.* We denote $n := \dim V$ and $m := \dim \ker T$. We choose a basis $\{v_1, \dots, v_m\}$ of $\ker T$, and extend it to a basis $\{v_1, \dots, v_m, w_1, \dots, w_{n-m}\}$ of $V$. The theorem is a consequence of the two following claims.

**Claim 1.** The set $\{Tw_1, \dots, Tw_{n-m}\}$ is linearly independent.

**Claim 2.** The set $\{Tw_1, \dots, Tw_{n-m}\}$ spans im $T$.

We postpone the proofs of these claims.
By Claim 1 and Claim 2, we get that $\{Tw_1, \dots, Tw_{n-m}\}$ is a basis of im $T$. Hence $\dim \mathrm{im}\ T = n - m = \dim V - \dim \ker T$, which is what we wanted.
**Proof of Claim 1:** To the contrary, assume there exists real numbers $\alpha_1, \dots, \alpha_{n-m}$ such that

$$\sum_{i=1}^{n-m} \alpha_i \cdot Tw_i = 0.$$

Linearity of $T$ then implies that

$$T\left(\sum_{i=1}^{n-m} \alpha_i \cdot w_i\right) = 0,$$

so that $\sum_{i=1}^{n-m} \alpha_i \cdot w_i \in \ker T$. This is a contradiction, since we chose $\{w_1, \dots, w_{n-m}\}$ complementary to $\ker T$. This proves Claim 1.
**Proof of Claim 2:** Let $w \in \mathrm{im}\ T$. Choose $v \in V$ such that $Tv = w$. Writing

$$v = \sum_{i=1}^{m} \alpha_i v_i + \sum_{j=1}^{n-m} \beta_j w_j,$$

we get that

$$w = \sum_{j=1}^{n-m} \beta_j \cdot Tw_j,$$

since $\sum_{i=1}^{m} \alpha_i v_i \in \ker T$ and $T$ is linear. This proves Claim 2 and completes the proof of Theorem A.1. □

*Proof of Example 2.2(ii).* Since addition on $V^*$ is defined pointwise and since every map in $V^*$ is linear by definition, it is clear that $\omega_V$ is in fact bilinear. Skew-symmetry is also obvious. To prove that $\omega_V$ is nondegenerate, let $(v, \varphi) \in V \times V^*$ such that

$$\omega_V((v, \varphi), (v', \varphi')) = \varphi'(v) - \varphi(v') = 0$$

for all $(v', \varphi') \in V \times V^*$. Choosing in particular $v' = 0$ shows that $\varphi'(v) = 0$ for all $\varphi' \in V^*$, implying that $v = 0$ (indeed, taking $\varphi'$ to be the projection onto the $i$-th coordinate shows that all coordinates of $v$ are zero). Similarly, choosing $\varphi' \equiv 0$ shows that $\varphi(v') = 0$ for all $v' \in V$, which means that $\varphi \equiv 0$. Hence $\omega_V$ is nondegenerate. □

*Proof of Example 2.8.* **(i):** First, consider a one-dimensional subspace $W$, and let $w \in W$. For any $w' \in W$, write $w' = \lambda w$. Thus, we find that $\omega(w, w') = \lambda \cdot \omega(w, w) = 0$, so $w \in W^\omega$. Hence $W \subset W^\omega$ and $W$ is isotropic.
Now, let $U$ be a $(2n-1)$-dimensional subspace. Then by Proposition 2.9(i) we have that $U^\omega$ is a one-dimensional subspace, which is isotropic by the above. Hence by Remark 2.10 we have that $U$ is coisotropic.
**(ii):** First we consider the subspace $V \times \{0\}$. Let $(v, \varphi) \in (V \times \{0\})^{\omega_V}$, so that

$$\omega_V((v, \varphi), (v', 0)) = -\varphi(v') = 0$$

for all $(v', 0) \in V \times \{0\}$. It is clear that this is satisfied precisely if $\varphi \equiv 0$, with no restrictions on $v$. Hence $V \times \{0\} = (V \times \{0\})^{\omega_V}$.
Next, we consider $\{0\} \times V^*$. Let $(v, \varphi) \in (\{0\} \times V^*)^{\omega_V}$, so that

$$\omega_V((v, \varphi), (0, \varphi')) = \varphi'(v) = 0$$

for all $(0, \varphi') \in \{0\} \times V^*$. It is clear that this is satisfied precisely if $v = 0$, with no restrictions on $\varphi$. Hence $\{0\} \times V^* = (\{0\} \times V^*)^{\omega_V}$. □

*Proof of Proposition 2.18, equation (2.1).* The equation is only interesting when $n \geq 2$, so we will proceed by induction over $n$, starting at $n = 2$. In this case, we have directly by the definition of wedge products that

$$\begin{aligned}(\omega \wedge \omega)(x_1, x_2, x_3, x_4) &= \frac{1}{4!} \cdot \frac{4!}{2! \cdot 2!} \sum_{\sigma \in S_4} \omega(x_{\sigma(1)}, x_{\sigma(2)}) \cdot \omega(x_{\sigma(3)}, x_{\sigma(4)}) \\ &= \frac{1}{2^2} \sum_{\sigma \in S_4} \omega(x_{\sigma(1)}, x_{\sigma(2)}) \cdot \omega(x_{\sigma(3)}, x_{\sigma(4)}).\end{aligned}$$

Now, assume that Equation (2.1) holds for $n = k$. We have again, by definition,

$$\begin{aligned}&(\omega^{\wedge k+1})(x_1, \ldots, x_{2k+2}) = (\omega^{\wedge k} \wedge \omega)(x_1, \ldots, x_{2k+2}) \\ &= \frac{1}{(2k+2)!} \cdot \frac{(2k+2)!}{(2k)! \cdot 2!} \sum_{\sigma \in S_{2k+2}} (-1)^\sigma \omega^{\wedge k}(x_{\sigma(1)}, \ldots, x_{\sigma(2k)}) \cdot \omega(x_{\sigma(2k+1)}, x_{\sigma(2k+2)}) \\ &= \frac{1}{2^{k+1} \cdot (2k)!} \sum_{\sigma \in S_{2k+2}} \sum_{\tau \in S_{2k}} (-1)^\sigma (-1)^\tau \omega(x_{\tau(\sigma(1))}, x_{\tau(\sigma(2))}) \cdots \omega(x_{\sigma(2k+1)}, x_{\sigma(2k+2)}).\end{aligned}$$

Here, officially, for a fixed $\sigma \in S_{2k+2}$ we consider $S_{2k}$ as the group of bijections of $\{\sigma(1), \ldots, \sigma(2k)\}$.

Now, we want to switch the sums, i.e. let $\sigma$ go through $S_{2k+2}$ for some fixed $\tau$. However, this is not quite possible in the current definition, since every $\tau$ depends on a certain choice of $\sigma$. So now, we will consider any $\tau \in S_{2k}$ as a permutation of the first $2k$ entries in the sequence $\{x_{\sigma(1)}, \ldots, x_{\sigma(2k+2)}\}$.[21] While this not quite agree with the formal definition of $S_{2k}$ (namely as the group of bijections of a fixed set of cardinality $2k$) it is clear that with this new interpretation of $\tau$ it is now possible to consider a fixed $\tau \in S_{2k}$, and then letting $\sigma$ move through $S_{2k+2}$. Also, this new definition of $\tau$ clearly does not change $(-1)^\tau$ from the formal definition we used previously.

With this new definition, for any fixed $\tau \in S_{2k}$, it is clear that also for any $\sigma \in S_{2k+2}$, the "composition" $\tau\sigma$, obtained by first applying $\sigma$ to the indices, and then swapping the first $2k$ entries with $\tau$, can be regarded as an element of $S_{2k+2}$. In this case, we clearly have $(-1)^{\tau\sigma} = (-1)^\sigma(-1)^\tau$, and letting $\sigma$ go through $S_{2k+2}$, this "composition" $\tau\sigma$ goes through $S_{2k+2}$ as well. Since $|S_{2k}| = (2k)!$, we obtain

$$\begin{aligned}
&\frac{1}{2^{k+1}\cdot(2k)!}\sum_{\sigma\in S_{2k+2}}\sum_{\tau\in S_{2k}}(-1)^\sigma(-1)^\tau\omega\big(x_{\tau(\sigma(1))}, x_{\tau(\sigma(2))}\big)\cdots\omega\big(x_{\sigma(2k+1)}, x_{\sigma(2k+2)}\big)\\
&= \frac{1}{2^{k+1}\cdot(2k)!}\cdot(2k)!\sum_{\tau\sigma\in S_{2k+2}}(-1)^{\tau\sigma}\omega\big(x_{\tau\sigma(1)}, x_{\tau\sigma(2)}\big)\cdots\omega\big(x_{\tau\sigma(2k+1)}, x_{\tau\sigma(2k+2)}\big)\\
&= \frac{1}{2^{k+1}}\sum_{\sigma\in S_{2k+2}}(-1)^\sigma\omega\big(x_{\sigma(1)}, x_{\sigma(2)}\big)\cdots\omega\big(x_{\sigma(2k+1)}, x_{\sigma(2k+2)}\big).
\end{aligned}$$

This completes the proof of Proposition 2.18, Eq. (2.1). □

# B Time-dependent vector fields

**Definition B.1** (Time-dependent vector fields)**.** Let $X$ be a manifold and $I$ an interval. A *smooth time-dependent vector field* on $X$ is a smooth map $V : I \times X \to TX$ such that $V(t,x) \in T_xX$ for every $(t,x) \in I \times X$. An *integral curve* of $V$ is a smooth curve $\gamma : I' \to X$ such that $\dot\gamma(t) = V(t,\gamma(t))$ for all $t \in I'$, where $I'$ is an interval contained in $I$. ♢

**Remark B.2.** It follows immediately that for any $t \in I$, the map $V_t := V(t,\cdot)$ is a smooth (time-independent) vector field. However, note that an integral curve for $V$ is in general *not* an integral curve for $V_t$, and vice versa.

Similarly to the time-independent case, there is a fundamental theorem on flows of time-dependent vector fields. We will not prove this theorem here, but a proof can be found in e.g. [Lee12, Theorem 9.48].

[21] This sequence is of course the order in which we put the $x_i$'s in the product of $\omega$'s.

**Theorem B.3** (Fundamental theorem on time-dependent flows)**.** *Let $X$ be a manifold and $V : I \times X \to TX$ a smooth, time-dependent vector field on $X$. Then there exists an open set $\mathcal{D} \subset I \times I \times X$ and a smooth map $\varphi : \mathcal{D} \to X$, called the* time-dependent flow *of $V$, such that the following hold:*

*(i) For any $t_0 \in I$ and any $x \in X$, the set $\mathcal{D}_{t_0,x} := \{t \in I \mid (t, t_0, x) \in \mathcal{D}\}$ is an interval, open in $I$ and containing $t_0$, and the smooth curve $\varphi_{t_0,x} : \mathcal{D}_{t_0,x} \to X,\ t \mapsto \varphi(t, t_0, x)$ is the unique maximal integral curve of $V$ such that $\varphi_{t_0,x}(t_0) = x$.*

*(ii) If $t_1 \in \mathcal{D}_{t_0,x}$ and $x' = \varphi_{t_0,x}(t_1)$, then $\mathcal{D}_{t_1,x'} = \mathcal{D}_{t_0,x}$ and $\varphi_{t_1,x'} = \varphi_{t_0,x}$.*

*(iii) For any $(t_1, t_0) \in I \times I$, the set $X_{t_1,t_0} := \{x \in X \mid (t_1, t_0, x) \in \mathcal{D}\}$ is open in $X$ and the map $\varphi_{t_1,t_0} : X_{t_1,t_0} \to X,\ x \mapsto \varphi(t_1, t_0, x)$ is a diffeomorphism between $X_{t_1,t_0}$ and $X_{t_0,t_1}$ with inverse given by $\varphi_{t_0,t_1}$.*

*(iv) If $x \in X_{t_1,t_0}$ and $\varphi_{t_1,t_0}(x) \in X_{t_2,t_0}$, then $x \in X_{t_2,t_0}$ and $(\varphi_{t_2,t_1} \circ \varphi_{t_1,t_0})(p) = \varphi_{t_2,t_0}(p)$.*

Just as in the time-independent case, when the manifold $X$ is compact, we have that $\mathcal{D} = I \times I \times X$, and hence Theorem B.3 implies that for any $(t_1, t_0) \in I \times I$, the map $\varphi_{t_1,t_0}$ is a diffeomorphism of $X$, and that $\varphi_{t_2,t_1} \circ \varphi_{t_1,t_0} = \varphi_{t_2,t_0}$ on the whole of $X$, for any $t_2 \in I$.

The next proposition is used in the proof of Proposition 2.35.

**Proposition B.4.** *Let $X$ be a manifold, $V : I \times X \to X$ a time-dependent vector field on $X$ and $\varphi : \mathcal{D} \to X$ its flow. Then for any differential form $\omega$ and any $(t_1, t_0, x) \in \mathcal{D}$ we have that*

$$\left.\frac{d}{dt}\right|_{t=t_1} \left(\varphi_{t,t_0}^* \omega\right)_x = \left(\varphi_{t_1,t_0}^* \left(\mathcal{L}_{V_{t_1}} \omega\right)\right)_x .$$

*Proof.* This proposition is a consequence of the following claim, of which we postpone the proof.

**Claim 1.** We have that

$$\left.\frac{d}{dt}\right|_{t=t_0} \left(\varphi_{t,t_0}^* \omega\right)_x = \left(\mathcal{L}_{V_{t_0}} \omega\right)_x .$$

Now, using Claim 1 it simply follows that

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=t_1} \left(\varphi_{t,t_0}^* \omega\right)_x &= \left.\frac{d}{dt}\right|_{t=t_1} \left((\varphi_{t,t_1} \circ \varphi_{t_1,t_0})^* \omega\right)_x = \left.\frac{d}{dt}\right|_{t=t_1} \left(\varphi_{t_1,t_0}^* \varphi_{t,t_1}^* \omega\right)_x \\ &= \left(\varphi_{t_1,t_0}^* \left(\left.\frac{d}{dt}\right|_{t=t_1} \left(\varphi_{t,t_1}^* \omega\right)\right)\right)_x = \left(\varphi_{t_1,t_0}^* \left(\mathcal{L}_{V_{t_1}} \omega\right)\right)_x .\end{aligned}$$

**Proof of Claim 1:** First, consider a 0-form $f$, which is of course just a smooth function. In this case, we simply have $\varphi_{t,t_0}^* f = f \circ \varphi_{t,t_0}$, and thus it follows that

$$\left.\frac{d}{dt}\right|_{t=t_0} \left(\varphi_{t,t_0}^* f\right)_x = \left.\frac{d}{dt}\right|_{t=t_0} \left(f \circ \varphi_{t,t_0}\right)_x = df(\varphi_{t_0,t_0}(x))(\dot{\varphi}_{t_0,x}(t_0))$$
$$= df(\varphi_{t_0,t_0}(x))(V(t_0, \varphi_{t_0,t_0}(x))) = df(x)(V(t_0,x)) = \left(V_{t_0} f\right)_x .$$

Now, denote by $\psi : \mathcal{D}' \to X$ the (time-independent) flow of $V_{t_0}$. Then we get that

$$\left(\mathcal{L}_{V_{t_0}} f\right)_x = \left.\frac{d}{dt}\right|_{t=0} \left(\psi_t^* f\right)_x = \left.\frac{d}{dt}\right|_{t=0} \left(f \circ \psi_t\right)_x = df(\psi_0(x))(\dot{\psi}_x(0))$$
$$= df(\psi_0(x))(V_{t_0}(\psi_0(x))) = df(x)(V_{t_0}(x)) = \left(V_{t_0} f\right)_x .$$

Hence the equation holds for any 0-form on $X$. Next, consider an exact 1-from $df$. Since the exterior derivative commutes with pullbacks, the Lie derivative and with $\frac{d}{dt}$, we get that

$$\left.\frac{d}{dt}\right|_{t=t_0} \left(\varphi_{t,t_0}^* df\right)_x = \left.\frac{d}{dt}\right|_{t=t_0} d\left(\varphi_{t,t_0}^* f\right)_x = d\left(\left.\frac{d}{dt}\right|_{t=t_0} \left(\varphi_{t,t_0}^* f\right)\right)_x$$
$$= d\left(\mathcal{L}_{V_{t_0}} f\right)_x = \left(\mathcal{L}_{V_{t_0}} df\right)_x$$

Next, suppose the claim holds for two differential forms $\omega_1$ and $\omega_2$. We show that it also holds for $\omega_1 \wedge \omega_2$. This follows easily from the fact that both the left and right hand side of the equation satisfy the same product rule; on the one hand we have

$$\left(\mathcal{L}_{V_{t_0}}(\omega_1 \wedge \omega_2)\right)_x = \left(\mathcal{L}_{V_{t_0}} \omega_1\right)_x \wedge (\omega_2)_x + (\omega_1)_x \wedge \left(\mathcal{L}_{V_{t_0}} \omega_2\right)_x .$$

Similarly,

$$\left.\frac{d}{dt}\right|_{t=t_0} \left(\varphi_{t,t_0}^*(\omega_1 \wedge \omega_2)\right)_x = \left.\frac{d}{dt}\right|_{t=t_0} \left((\varphi_{t,t_0}^* \omega_1) \wedge (\varphi_{t,t_0}^* \omega_2)\right)_x$$
$$= \left.\frac{d}{dt}\right|_{t=t_0} \left(\varphi_{t,t_0}^* \omega_1\right)_x \wedge (\varphi_{t_0,t_0}^* \omega_2)_x + (\varphi_{t_0,t_0}^* \omega_1)_x \wedge \left.\frac{d}{dt}\right|_{t=t_0} \left(\varphi_{t,t_0}^* \omega_2\right)_x$$
$$= \left.\frac{d}{dt}\right|_{t=t_0} \left(\varphi_{t,t_0}^* \omega_1\right)_x \wedge (\omega_2)_x + (\omega_1)_x \wedge \left.\frac{d}{dt}\right|_{t=t_0} \left(\varphi_{t,t_0}^* \omega_2\right)_x .$$

Since the equation holds for $\omega_1$ and $\omega_2$, it also holds for $\omega_1 \wedge \omega_2$. Now, consider an arbitrary differential $k$-form $\omega$. In local coordinates $(x_i)$, it can be written as

$$\omega = \sum_I \omega_I \, dx_I,$$

where $\omega_I$ are smooth functions and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for $I = \{i_1, \ldots, i_k\}$. Hence $\omega$ can locally be written as a wedge product of smooth functions and exact 1-forms. By the three "steps" we proved above, it follows that

$$\frac{d}{dt}\bigg|_{t=t_0} \left(\varphi_{t,t_0}^* \omega\right)_x = \left(\mathcal{L}_{V_{t_0}} \omega\right)_x .$$

This proves Claim 1 and completes the proof of Proposition B.4. $\square$

# C Lie group actions

Let us start with several definitions.

**Definition C.1** (Topological and Lie groups)**.** A *topological group* is a pair $(G, m)$ of a topological space $G$ and an operation $m : G \times G \to G$ such that $(G, m)$ is a group and such that the multiplication map $m$ and the inversion map $g \mapsto g^{-1}$ are both continuous with respect to the topology of $G$ and the induced product topology of $G \times G$. A *Lie group* is a topological group $(G, m)$ such that $G$ is in fact a smooth manifold, and the multiplication and inversion map are smooth with respect to this smooth structure. $\diamondsuit$

**Remark C.2.** We will often denote multiplication operator by $m(g, h) = gh$, to significantly shorten notation. Sometimes, when $G$ is a vector space structure, we also denote $m(g, h) = g + h$. We will also often refer to a group by just specifying the set $G$, with it being understood that there is an underlying operation.

**Definition C.3** (Group actions)**.** Let $G$ be a group and $X$ a set. Then a *(left) group action* of $G$ on $X$ is a map $G \times X \to X$, written as $(g, x) \mapsto g \cdot x$, such that

(i) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$,

(ii) $e \cdot x = x$ for all $x \in X$, where $e$ denotes the identity element of $G$.

When $G$ is in fact a topological group, $X$ a topological space and the action continuous, we call it a *continuous action*. We also say that the topological group $G$ *acts continuously* on $X$. We use similar terminology when $G$ is a Lie group, $X$ a smooth manifold and the action smooth. $\diamondsuit$

We will use the following terminology regarding group actions.

**Definition C.4.** Let $G$ be a group, $X$ a set and $(g, x) \mapsto g \cdot x$ a left group action of $G$ on $X$. For $x \in X$, the *orbit* of $x$ is the set

$$G \cdot x := \{g \cdot x \mid g \in G\}.$$

We denote the space of all orbits by $X/G$. We say that the action is *free* if the identity $e$ is the only element of $G$ that fixes any $x \in X$. $\diamondsuit$

**Remark C.5** (Orbit space)**.** When $X$ is a topological space, we can endow the *orbit space* $X/G$ with the quotient topology, i.e. the largest topology such that the quotient map $\pi : X \to X/G$ is continuous. In the remainder of this section, when we mention the orbit space $X/G$ of a topological space $X$, this space is always assumed to be endowed with the quotient topology.

**Definition C.6** (Proper group actions)**.** Let $G$ be a topological group, $X$ a topological space and $(g, x) \mapsto g \cdot x$ a continuous group action of $G$ on $X$. Then we say that the action is *proper* if the map $G \times X \to X \times X$ defined by $(g, x) \mapsto (g \cdot x, x)$ is a proper map. $\diamondsuit$

**Remark C.7.** Recall that a map $f : X \to Y$ between topological spaces is called *proper* if $f^{-1}(K)$ is compact in $X$ for every compact set $K \subset Y$.

The definition of an proper action is somewhat abstract, and the following proposition is very useful for determining whether a given action is proper. We paraphrase the proof given in [Lee12, Proposition 21.5].

**Proposition C.8** (Characterization of proper group actions)**.** *Let $G$ be a Lie group, $X$ a topological manifold and $(g, x) \mapsto g \cdot x$ a continuous group action of $G$ on $X$. Then the following statements are equivalent.*

*(i) The action is proper;*

*(ii) For any two sequences $(x_i)_{i\in\mathbb{N}}$ in $X$ and $(g_i)_{i\in\mathbb{N}}$ in $G$ such that $(x_i)_{i\in\mathbb{N}}$ and $(g_i \cdot x_i)_{i\in\mathbb{N}}$ both converge, $(g_i)_{i\in\mathbb{N}}$ has a convergent subsequence;*

*(iii) For any compact set $K \subset M$, the set $G_K := \{g \in G \mid (g \cdot K) \cap K \neq \emptyset\} \subset G$ is compact.*

*Proof.* We will denote by $\Theta : G \times X \to X \times X$ the map $\Theta(g, x) = (g \cdot x, x)$. In the proof, we will use that for subsets of $G$, $X$ or $G \times X$, compactness is equivalent to sequential compactness, since all these spaces are topological manifolds, and hence metrizable.
**(i) $\implies$ (ii):** Let $(x_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ be two sequences in $X$ and $G$ respectively such that both $(x_i)_{i\in\mathbb{N}}$ and $(g_i \cdot x_i)_{i\in\mathbb{N}}$ converge in $X$. Denote

$$x := \lim_{i\to\infty} x_i \qquad\qquad x' := \lim_{i\to\infty} g_i \cdot x_i.$$

Since $X$ is a manifold, we can choose compact neighbourhoods $K$ of $x$ and $K'$ of $x'$. Then, since $x_i \to x$ and $g_i \cdot x_i \to x'$, we know that there is some $N \in \mathbb{N}$ such that for all $i \geq N$, $(g_i \cdot x_i, x_i) \in K \times K'$. This also means that for all $i \geq N$, $(g_i, x_i) \in \Theta^{-1}(K \times K')$. Since $K \times K'$ is compact and $\Theta$ is proper, this means that $(g_i, x_i)_{i\geq N}$ is a sequence in a compact set. Hence a subsequence of $(g_i, x_i)_{i\geq N}$ converges, and thus also a subsequence of $(g_i)_{i\in\mathbb{N}}$ converges.
**(ii) $\implies$ (iii):** Let $K \subset X$ be compact. We will show that $G_K$ is sequentially compact,

and thus compact. So let $(g_i)_{i\in\mathbb{N}}$ be a sequence in $G_K$. Then for every $i \in \mathbb{N}$ there exists some $x_i \in (g_i \cdot K) \cap K$. Hence the sequence $(x_i)_{i\in\mathbb{N}}$ lies in $K$, and since $K$ is compact, it has a convergent subsequence $(x_{i_j})_{j\in\mathbb{N}}$. Now, the sequence $(g_{i_j}^{-1} \cdot x_{i_j})_{j\in\mathbb{N}}$ also lies in $K$, and thus has a convergent subsequence $(g_{i_{j_k}}^{-1} \cdot x_{i_{j_k}})_{k\in\mathbb{N}}$. Since clearly $(x_{i_{j_k}})_{k\in\mathbb{N}}$ also converges, we get that $(g_{i_{j_k}}^{-1})_{k\in\mathbb{N}}$ has a convergent subsequence $(g_{i_{j_{k_l}}}^{-1})_{l\in\mathbb{N}}$. Then, since the inversion map is continuous, it also follows that $(g_{i_{j_{k_l}}})_{l\in\mathbb{N}}$ converges. Hence $(g_i)_{i\in\mathbb{N}}$ has a convergent subsequence.
**(iii) $\Longrightarrow$ (i):** Let $K \subset X \times X$ be compact and define $K' := \pi_1(K) \cup \pi_2(K)$, where $\pi_i$ is the projection onto the $i$-th factor of $X \times X$. Since these projections are continuous, it follows that $K'$ is a finite union of compact sets, and hence compact. Now, we get

$$\Theta^{-1}(K) \subset \{(g, x) \mid g \cdot x \in K' \text{ and } x \in K'\} \subset G_{K'} \times K'.$$

Now, since $X$ is Hausdorff, $K$ is closed, and thus $\Theta^{-1}(K)$ is as well. Then since $G_{K'} \times K'$ is compact, it follows that $\Theta^{-1}(K)$ is compact as well. Hence $\Theta$ is a proper map. □

We will prove one more useful proposition, regarding Hausdorffness of the orbit space. Again, we paraphrase the proof given in [Lee12, Proposition 21.4].

**Proposition C.9.** *Let $G$ be a Lie group, $X$ a topological manifold and $G \times X \to X$ a continuous, proper action. Then the orbit space $X/G$ is Hausdorff.*

We need the following three lemmata, of which we postpone the proofs.

**Lemma C.10.** *Let $X$ be a topological space, $Y$ a locally compact, Hausdorff space and $f : X \to Y$ a continuous, proper map. Then $f$ is closed.*

**Lemma C.11.** *Let $G$ be a topological group, $X$ a topological space and $G \times X \to X$ a continuous action. Then the quotient map $\pi : X \to X/G$ is open.*

**Lemma C.12.** *Let $X$ and $Y$ be topological spaces and $\pi : X \to Y$ an open quotient map. Then $Y$ is Hausdorff iff*

$$R := \{(x_1, x_2) \in X \times X \mid \pi(x_1) = \pi(x_2)\}$$

*is closed in $X \times X$.*

*Proof of Proposition C.9.* Denote again by $\Theta : G \times X \to X \times X$ the proper map $\Theta(g, x) = (g \cdot x, x)$ and by $\pi : X \to X/G$ the quotient map. Define

$$R := \{(x_1, x_2) \in X \times X \mid \pi(x_1) = \pi(x_2)\}.$$

Now, since two points in $X$ are identified by $\pi$ exactly when they are related by some $g \in G$ (in the sense that $g \cdot x_1 = x_2$ for some $g \in G$), it follows that $R = \Theta(G \times X)$. Since $X \times X$ is a topological manifold and $\Theta$ is continuous and proper, it follows by Lemma C.10 that $R$ is closed in $X \times X$. Now by Lemmata C.11 and C.12 it follows that $X/G$ is Hausdorff. □

*Proof of Lemma C.10.* Let $K \subset X$ be closed, and let $y$ be a limit point of $f(K)$, i.e. a point such that all of its neighbourhoods have nonempty intersection with $f(K)$. We need to show that $y \in f(K)$. Choose a compact neighbourhood $V$ of $y$. Then $y$ is also a limit point of $f(K) \cap V$. By properness of $f$, $f^{-1}(V)$ is compact, and hence $K \cap f^{-1}(V)$ is as well. Since $f$ is continuous, $f(K \cap f^{-1}(V)) = f(K) \cap V$ is compact as well, and therefore closed, since $Y$ is Hausdorff. Therefore, $f(K) \cap V$ contains all its limit points, and in particular $y$. But this means that $y \in f(K)$, and thus $f(K)$ is closed. $\square$

*Proof of Lemma C.11.* First, note that for any $g \in G$ the map $L_g : X \to X$ defined by

$$L_g(x) = g \cdot x$$

is a homeomorphism; indeed, continuity follows since the action is continuous, and an inverse of $L_g$ is given by $L_{g^{-1}}$. Now let $U \subset X$ be an open set. Then we have

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} L_g(U),$$

and since every $L_g(U)$ is open, it follows that $\pi^{-1}(\pi(U))$ is open. Now, since $X/G$ is endowed with the quotient topology, it follows that $\pi(U)$ is open. $\square$

*Proof of Lemma C.12.* First, suppose that $R$ is closed. Let $y_1, y_2 \in Y$ be distinct and choose $x_1 \in \pi^{-1}(y_1), x_2 \in \pi^{-1}(y_2)$. Then $(x_1, x_2) \in (X \times X) \setminus R$, and thus there exist open neighbourhoods $U_1$ and $U_2$ of $x_1$ and $x_2$ respectively, such that $U_1 \times U_2 \subset (X \times X) \setminus R$. Now, $\pi(U_1)$ and $\pi(U_2)$ are open neighbourhoods of $y_1$ and $y_2$ respectively, and since $(U_1 \times U_2) \cap R = \emptyset$ we also have that they are distinct.[22] Hence $Y$ is Hausdorff.

Now assume that $Y$ is Hausdorff. Then for any $y_1 \neq y_2 \in Y$ we can choose distinct open neighbourhoods $U_{y_1}, U_{y_2}$. Now, we have that

$$(X \times X) \setminus R = \bigcup_{y_1 \neq y_2 \in Y} \pi^{-1}(U_{y_1}) \times \pi^{-1}(U_{y_2}),$$

and hence $R$ is closed in $X \times X$. $\square$

# D (Smooth) covering spaces

Let us start with topological covering spaces.

**Definition D.1** (Covering spaces)**.** Let $X$ be topological space. Then a *covering space* of $X$ is a topological space $\widetilde{X}$ and a map $p : \widetilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}_{\alpha \in A}$ of $X$ with the property that for each $\alpha \in A$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped homeomorphically to $U_\alpha$ by $p$. $\diamondsuit$

[22] Since $R$ contains the diagonal, it follows that $U_1$ and $U_2$ are distinct. So the only way that $\pi(U_1)$ and $\pi(U_2)$ are not distinct is if the action of $G$ identifies a point in $U_1$ with one in $U_2$. But again, this is not possible, since $(U_1 \times U_2) \cap R = \emptyset$.

**Remark D.2.** Sometimes it is conventient to require the space $X$ to be path-connected and locally path-connected, the covering space $\widetilde{X}$ to be path-connected and the covering map $p : \widetilde{X} \to X$ to be surjective. However, we will use a more general definition.

We can easily derive some properties of covering maps; in particular, they are always open maps and local homeomorphisms. When a covering map is surjective, it is also a quotient map.

One of the most important applications of covering spaces is the lifting of maps.

**Definition D.3** (Lifts to a covering space)**.** Let $X$ and $Y$ be a topological spaces, $p : \widetilde{X} \to X$ a covering space and $f : Y \to X$ a continuous map. Then a *lift* of $f$ is a continuous map $\tilde{f} : Y \to \widetilde{X}$ such that $p \circ \tilde{f} = f$. $\diamondsuit$

We have the following propositions about uniqueness and existence of lifts of maps. Proofs of these propositions can be found in e.g. [Hat10, Chapter 1.3].

**Remark D.4.** In the following proposition we use the notion of maps between pointed spaces: for topological spaces $X$ and $Y$ and specified points $x_0 \in X$ and $y_0 \in Y$, a map $f : (Y, y_0) \to (X, x_0)$ is a map $Y \to X$ with the property that $f(y_0) = x_0$. For a map such as this, we write $f_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ for the induced map on the fundamental group.

**Proposition D.5** (Lifting criterion)**.** *Let $X$ be a topological space, $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ a covering space, $Y$ a path-connected and locally path-connected space and $f : (Y, y_0) \to (X, x_0)$ a continuous map. Then a lift $\tilde{f} : (Y, y_0) \to (\widetilde{X}, \tilde{x}_0)$ of $f$ exists iff*

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \tilde{x}_0)).$$

**Proposition D.6** (Unique lifting property)**.** *Let $X$ and $Y$ be topological spaces, $p : \widetilde{X} \to X$ a covering space and $f : Y \to X$ a continuous map. Then if $Y$ is connected, any two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \widetilde{X}$ that are the same at one point in $Y$, are the same on all of $Y$.*

Finally, we have the well known *homotopy lifting property*.

**Proposition D.7** (Homotopy lifting property)**.** *Let $X$ and $Y$ be topological spaces, $p : \widetilde{X} \to X$ a covering space and a homotopy $f : [0, 1] \times Y \to X$. Given a lift $\tilde{f}_0 : Y \to \widetilde{X}$ of $f_0$, there exists a unique homotopy $\tilde{f} : [0, 1] \times Y \to \widetilde{X}$ starting at $\tilde{f}_0$ that lifts $f$.*

Before moving to smooth covering spaces, we will mention a specific type of covering space; for a topological space $X$, a simply connected covering space $p : \widetilde{X} \to X$ is called a *universal covering space* of $X$. An interesting property of the universal covering space is that we can always lift a continuous map $f : X \to X$ to a map $\tilde{f} : \widetilde{X} \to \widetilde{X}$, in the sense that $p \circ \tilde{f} = f \circ p$. Indeed, the map $f \circ p : \widetilde{X} \to X$ satisfies the conditions of Proposition D.5, since $\widetilde{X}$ is simply connected.

Now, let us move on to smooth covering spaces. The definitions themselves are almost identical to the ones before, except that we move to smoothness instead of continuity.

**Definition D.8** (Smooth covering spaces)**.** Let $X$ be a smooth manifold. Then a *smooth covering space* of $X$ is a smooth manifold $\widetilde{X}$ together with a smooth map $p : \widetilde{X} \to X$ such that there is an open cover $\{U_\alpha\}_{\alpha \in A}$ of $X$ with the property that for each $\alpha \in A$, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped diffeomorphically to $U_\alpha$ by $p$. $\diamondsuit$

**Remark D.9.** Just as before, it is often convenient to require the smooth manifolds $\widetilde{X}$ and $X$ to be connected[23] and the map $p : \widetilde{X} \to X$ to be surjective.

Since every smooth covering is in particular a topological one, the properties we mentioned before also hold for smooth covering maps. Of course, every smooth covering map is in fact a local diffeomorphism. One useful, easily verified fact is that a topological covering map between smooth manifolds is a smooth covering map iff it is a local diffeomorphism.

All the definitions and propositions we mentioned for topological covering spaces transfer over to smooth covering spaces; indeed, a lift of a smooth map is a smooth map satisfying the same conditions as in the topological case, and the propositions regarding existence and uniqueness of lifts are the same, except the need to change continuous maps to their smooth counterparts.

[23] Note that since they are manifolds, this is equivalent to the extra requirements one might put on topological covering spaces.

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