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Congruences for Coefficients of Power Series Expansions of Rational Functions

Bachelor Thesis

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Contents

1 Introduction

In this thesis we study what we will call "Gaussian rational functions". A Gaussian rational function in n variables is a rational function for which the coefficients of its power series expansion around 0 satisfy certain congruences. The starting point for the definition of these congruences may be seen as the famous "Euler's Theorem", as originally formulated in 1763 by Leonhard Euler. In particular, we know that for any integer a, natural number r and prime number p,

$$a^{mp^r} \equiv a^{mp^{r-1}} \pmod{p^r} \tag{1}$$

So, for example, the coefficients a_k of the power series expansion of the rational function f given by

$$f := \frac{ax}{1-ax} = \sum_{k\geq 0} a_k x^k \tag{2}$$

satisfy the congruences

$$a_{mp^r} \equiv a_{mp^{r-1}} \pmod{p^r} \tag{3}$$

Our question will roughly be: Which rational functions satisfy the congruences (3)?

The thesis can be seen as consisting of three parts. In the first part, consisting of sections 2, 3 and 4, we will state some basic definitions, results in the theory of linear recurrent sequences, and a give a small summary of the basics of algebraic number theory that we will need along the way. In the second part, consisting of section 5, we will give a characterization of Gaussian rational functions in one variable. In the third part, consisting of section 6, we will attempt to generalize our results to give a characterization for Gaussian rational functions in n variables.

The results up to section 5 are not new. In particular, our characterization of Gaussian rational functions in one variable will be based on a result by Minton [1]. The aim of section 5 is mainly a translation of this result into a generalizable form. We also provide a slightly simpler alternative to the technique used in the proof of Minton. The results in section 6 are mostly new. Here we will provide ideas that work towards a characterization in the general case of n variables that can be applied to give a complete characterization for a special type of rational functions. This special characterization will not completely exhaust the theory we have developed, and there will be room for a possible extension of results for a wider range of applications.

2 General Definitions

This section mainly consists of definitions and notational conventions that we will use throughout the thesis. In the last subsection we introduce rational functions, the main object of our study.

2.1 Basic Definitions and Conventions

Convention 2.1 The set of natural numbers $\mathbb{N}$ is the set of positive integers $\mathbb{N} = \{1, 2, \cdots\}$. n usually denotes an arbitrary element of $\mathbb{N}$, unless stated otherwise. ∘

Convention 2.2 Given an element $\boldsymbol{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^n$ we write $\boldsymbol{k} \geq 0$ if $k_i \geq 0$ for all $1 \leq i \leq n$. We will write $\boldsymbol{k} > 0$ if $\boldsymbol{k} \geq 0$ and $k_i > 0$ for at least one i. For $d \in \mathbb{N}$, we will write $|\boldsymbol{k}| = d$ if $\boldsymbol{k} \geq 0$ and $k_1 + \cdots + k_n = d$. ∘

Definition 2.3 We call an indexed sequence of numbers $\{a_i\}_{i \in I}$ an ***n*-sequence** if $I = \mathbb{Z}^n_{\geq 0}$. A 1-sequence will simply be called a **sequence**. △

Convention 2.4 In order to avoid tedious notation, we will often abbreviate an n-sequence of numbers $\{a_{\boldsymbol{k}}\}_{\boldsymbol{k} \geq 0}$ with $\{a_{\boldsymbol{k}}\}$. ∘

Convention 2.5 When we talk about the numerator and denominator of rational numbers $q \in \mathbb{Q}$, we will usually assume that q is in reduced form. That is, we have written $q = \frac{a}{b}$ with $a \in \mathbb{Z}, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$. ∘

We would like to say that two rational numbers are congruent modulo a prime power p^r if the numerator of their difference is a multiple of p^r. This is captured by the following definition.

Definition 2.6 (Congruence for rational numbers) Let $q_1 = a_1/b_1$ and $q_2 = a_2/b_2$ be rational numbers. Let p be a prime number, let $r \in \mathbb{Z}_{\geq 0}$ and let $p_1, \cdots, p_m$ be the prime numbers dividing $b_1 b_2$. We say that q_1 is congruent to q_2 modulo p^r, written $q_1 \equiv q_2 \pmod{p^r}$, if $p \neq p_i$ for all $1 \leq i \leq m$ and $q_1 - q_2 \in p^r \mathbb{Z}[1/p_1, \cdots, 1/p_m]$. △

Definition 2.7 (v_p function) Let p be a prime number, and let $0 \neq \boldsymbol{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^n$. We define $v_p(\boldsymbol{k})$ to be the largest integer r such that $p^r \mid k_i$ for all $1 \leq i \leq n$. △

Definition 2.8 Let $m \in \mathbb{N}$ and let $a_1, \cdots, a_m \in \mathbb{Z}$. We say that the set $\{a_1, \cdots, a_m\}$ is a **complete residue system** modulo m if, for any $i, j \in \{1, \cdots, m\}$, $a_i \equiv a_j \pmod{m}$ if and only if $i = j$. △

Definition 2.9 Let $\alpha \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$. We define the generalized **binomial coefficient** as follows:

$$\binom{\alpha}{k} := \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!} \tag{4}$$

△

Lemma 2.10 *Let $p_1, \cdots, p_m$ be prime numbers and let $k \in \mathbb{Z}_{\geq 0}$. If $\alpha \in \mathbb{Z}[1/p_1, \cdots, 1/p_m]$ then also $\binom{\alpha}{k} \in \mathbb{Z}[1/p_1, \cdots, 1/p_m]$.*

Proof. Write $\alpha = a/b$ with $a \in \mathbb{Z}$ and $b = p_1^{r_1} \cdots p_m^{r_m}$. We have

$$\binom{\alpha}{k} = \frac{1}{b^k} \frac{a(a-b)\cdots(a-b(k-1))}{k!} \tag{5}$$

Let p be a prime number such that $p \neq p_i$ for all $1 \leq i \leq m$. Write $k = c_r p^r + c_{r-1} p^{r-1} + \cdots + c_0$ such that $c_0, \cdots, c_r \in \mathbb{Z}_{\geq 0}$ all strictly less than p. Now let $0 \leq j \leq r$ be an integer. Since $p \nmid b$, for any integer l the set $\{l, l-b, \cdots, l-b(p^j-1)\}$ is a complete residue system modulo p^j. It follows in particular that

$$v_p\left(\prod_{i=1}^{c_j p^j} \left(i + c_{j-1}p^{j-1} + \cdots + c_0\right)\right) \leq v_p\left(\prod_{i=1}^{c_j p^j} \left(a - b(i + c_{j-1}p^{j-1} + \cdots + c_0 - 1)\right)\right) \tag{6}$$

Since this holds for any integer $0 \leq j \leq r$ we have

$$\begin{aligned}
v_p(k!) &= v_p\left(\prod_{i=1}^{k} i\right) && (7)\\
&= v_p\left(\prod_{j=0}^{r}\prod_{i=1}^{c_j p^j} \left(i + c_{j-1}p^{j-1} + \cdots + c_0\right)\right) && (8)\\
&\leq v_p\left(\prod_{j=0}^{r}\prod_{i=1}^{c_j p^j} \left(a - b(i + c_{j-1}p^{j-1} + \cdots + c_0 - 1)\right)\right) && (9)\\
&= v_p\left(\prod_{i=1}^{k} (a - b(i-1))\right) && (10)\\
&= v_p\Big(a(a-b)\cdots(a-b(k-1))\Big) && (11)
\end{aligned}$$

Since this holds for any prime p not dividing b, we conclude using (5) that $\binom{\alpha}{k} \in \mathbb{Z}[1/p_1, \cdots, 1/p_m]$. □

Definition 2.11 (Almost all) Let S be a countable set. We will say that a condition $P(s)$ on elements of S holds for **almost every** $s \in S$, if it holds for all but finitely many $s \in S$. △

Definition 2.12 Given an integral domain I, we denote by $I[[x_1, \cdots, x_n]]$ the set of formal power series in the variables $x_1, \cdots, x_n$ with coefficients in I. △

2.2 Polynomials

Definition 2.13 Given an integral domain I and a polynomial $P = a_0 + a_1 x + \cdots + a_d x^d \in I[x]$, we define the **reciprocal polynomial** of P to be $P^* := a_0 x^d + a_1 x^{d-1} + \cdots + a_d$. △

Lemma 2.14 *Let F be a field, and let $P \in F[x]$. Assume that $P(0) \neq 0$. Then P is separable if and only if P^* is separable.*

Proof. Let $d := \deg P$. Note that $P^*(x) = x^d P\left(\frac{1}{x}\right)$. It follows that $\alpha \neq 0$ is a root of P if and only if α^{-1} is a root of P^*. Also, $P^*(0) \neq 0$ and we assumed $P(0) \neq 0$. We conclude that P has multiple zeros (in its splitting field) if and only if P^* has multiple zeros (in its splitting field). □

Definition 2.15 (θ-operator) Given a polynomial $P \in \mathbb{Z}[x_1, \cdots, x_n]$ in n variables, we define $\theta_i P$, for $1 \leq i \leq n$, to be the polynomial defined by $\theta_i P(x_1, \cdots, x_n) := x_i \frac{\partial}{\partial x_i} P(x_1, \cdots, x_n)$. That is, θ_i can be seen as the operator which acts by formally partially differentiating with respect to the i-th variable, followed by multiplication with this variable. In the case $n = 1$, we will abbreviate θ_1 by θ. △

Definition 2.16 We denote by $\mathbb{Z}[x_1, \cdots, x_n]_0$ the set of all polynomials in $\mathbb{Z}[x_1, \cdots, x_n]$ with non-zero constant term:

$$\mathbb{Z}[x_1, \cdots, x_n]_0 := \{Q \in \mathbb{Z}[x_1, \cdots, x_n] \mid Q(0) \neq 0\} \tag{12}$$

△

2.3 Rational Functions

Definition 2.17 Let $n \in \mathbb{N}$. A **rational function** $f = P/Q$ in the variables $x_1, \cdots, x_n$ is the quotient of two polynomials $P, Q \in \mathbb{Z}[x_1, \cdots, x_n]$. We will denote the set of rational functions in the variables $x_1, \cdots, x_n$ by $\mathbb{Q}(x_1, \cdots, x_n)$. △

If $Q(0) \neq 0$, i.e. the constant term of Q is non-zero, then $f = P/Q$ has a power series expansion around 0, which means that we can write

$$f(x_1, \cdots, x_n) = \sum_{\boldsymbol{k} \geq 0} a_{\boldsymbol{k}} x^{\boldsymbol{k}} := \sum_{k_1, \cdots, k_n \geq 0} a_{k_1, \cdots, k_n} x_1^{k_1} \cdots x_n^{k_n} \tag{13}$$

For coefficients $a_{\boldsymbol{k}} \in \mathbb{Z}[1/Q(0)]$ (see Proposition 2.18). The power series is uniquely determined if we demand that it converges for $x = (x_1, \cdots, x_n)$ small enough. However, the properties regarding the convergence of the power series will not be of our interest. We will often identify a rational function with its power series, so that we can regard rational functions as elements of $\mathbb{Q}[[x]]$.

Proposition 2.18 *Let $f = P/Q$ be a rational function and suppose that $Q(0) \neq 0$. Write the power series expansion of f as in* (13)*. Then $a_{\boldsymbol{k}} \in \mathbb{Z}[1/Q(0)]$ for all $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$. Hence in particular, if $Q(0) = 1$ then $a_{\boldsymbol{k}} \in \mathbb{Z}$ for all $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$.*

Proof. If Q is constant then the statement is clearly true, so assume that Q is not constant.

We can write

$$f = \frac{P}{Q(0) - Q'} = \frac{P}{Q(0)} \frac{1}{1 - \frac{Q'}{Q(0)}} \tag{14}$$

Where $Q' := Q(0) - Q$ is a non-zero polynomial with integer coefficients such that $Q'(0) = 0$. The right hand side of (14) can be expanded as a geometric series:

$$f = \frac{P}{Q(0)} \frac{1}{1 - \frac{Q'}{Q(0)}} = \frac{P}{Q(0)} \left(1 + \frac{Q'}{Q(0)} + \left(\frac{Q'}{Q(0)} \right)^2 + \cdots \right) \tag{15}$$

Because Q' has constant term equal to zero, the monomials that $\left(\frac{Q'}{Q(0)} \right)^n$ consists of all have at least degree n. Therefore the expansion in (15) well-defines a power series, which converges for all x such that $\left| \frac{Q'(x)}{Q(0)} \right| < 1$. Since $Q'(0) = 0$ and Q' is continuous at 0, we conclude by uniqueness of power series expansions that this coincides with the power series expansion of f. Now since P and Q' have integer coefficients, and $Q(0)$ is also an integer, we see immediately from (15) that the power series expansion of f has coefficients in $\mathbb{Z}[1/Q(0)]$. □

Convention 2.19 We say a rational function $P/Q \in \mathbb{Q}(x_1, \cdots, x_n)$ is written in **reduced form** if $P, Q \in \mathbb{Z}[x_1, \cdots, x_n]$ and P, Q have no common divisors in $\mathbb{Q}[x_1, \cdots, x_n]$ (except for units). ∘

3 Linear Recurrent Sequences

Definition 3.1 A **rational linear recurrent sequence** is a sequence $\{q_n\}_{n\geq 0}$ in $\mathbb{Q}$ for which there exists an $r \in \mathbb{N}$ and coefficients $c_1, \cdots, c_r \in \mathbb{Q}$ such that $q_n = c_1 q_{n-1} + c_2 q_{n-2} + \cdots + c_r q_{n-r}$ for all $n > r$. This last relation is called a **recurrence relation**. △

Remark 3.2. Note that in our definition above, for a rational linear recurrent sequence to be determined by its recurrence relation, we need initial conditions $q_1, \cdots, q_r$. Note also that there is no restriction on q_0, the value of which will not be relevant for our purposes. ◊

Definition 3.3 The **characteristic polynomial** G of a rational linear recurrent sequence with coefficients $c_1, \cdots, c_r$ is given by $G(x) = x^r - c_1 x^{d-1} - \cdots - c_r$. △

Proposition 3.4 *Let $\{u_n\}$ be a rational linear recurrent sequence with coefficients $c_1, \cdots, c_r$, and suppose that the characteristic polynomial G is separable and that $G(0) \neq 0$. Denote by $\omega_1, \cdots, \omega_r \in K$ the distinct roots of G in its splitting field K. Then there exist uniquely determined $\alpha_1, \cdots, \alpha_r \in K$ such that*

$$u_n = \sum_{i=1}^{r} \alpha_i \omega_i^n \tag{16}$$

for all $n \geq 1$.

Lemma 3.5 *Suppose that $\{u_n\}$ is as in Proposition 3.4, and that $\alpha_i \in \mathbb{Q}$ for all $1 \leq i \leq r$. If ω_i and ω_j are conjugate, then $\alpha_i = \alpha_j$.*

Proof. Let H be the Galois group of $K/\mathbb{Q}$, and suppose that ω_i and ω_j are conjugate. By a basic result in Galois theory there exists a $\sigma \in H$ be such that $\sigma(\omega_i) = \omega_j$. Note that since $u_n \in \mathbb{Q}$, it is fixed under this σ, hence

$$u_n = \sum_{i=1}^{r} \alpha_i \omega_i^n = \sigma\left(\sum_{i=1}^{r} \alpha_i \omega_i^n\right) = \sum_{i=1}^{r} \alpha_i \sigma(\omega_i)^n \tag{17}$$

Since the $\alpha_i \in K$ are uniquely determined by $\{u_n\}$ (Proposition 3.4), we conclude that $\alpha_i = \alpha_j$. □

4 Algebraic Preliminaries

This section is a summary of results and basic definitions in algebraic number theory, which will be used in the proof of Theorem 5.12. In particular the result found in Theorem 4.17 will be of great importance. Some statements are given without proof. Instead we will refer to [4] and [5].

Definition 4.1 A **number field** K is a finite field extension of $\mathbb{Q}$. △

Since any finite field extension is algebraic, a number field K consists of algebraic numbers, hence it can be seen as a subset of $\mathbb{C}$. In the remaining part of this section we will assume that K denotes a number field.

Definition 4.2 An **algebraic integer** is an element $\omega \in \mathbb{C}$ which is a root of some monic polynomial $p \in \mathbb{Z}[x]$. We will denote by $\mathcal{A} \subseteq \mathbb{C}$ the set of all algebraic integers. △

Proposition 4.3 *$\mathcal{A}$ is a subring of $\mathbb{C}$.*

Proof. Can be found in [4] on page 68 (Proposition 6.1.5). □

Lemma 4.4 *$\mathcal{A} \cap \mathbb{Q} = \mathbb{Z}$.*

Proof. Let $q = k/l \in \mathbb{Q}$ be a rational number. We assume (Convention 2.5) that $\gcd(k, l) = 1$. Suppose that q is an algebraic integer. Then there exists an $n \in \mathbb{N}$ and $a_1, \cdots, a_n \in \mathbb{Z}$ such that

$$\left(\frac{k}{l}\right)^n + a_1 \left(\frac{k}{l}\right)^{n-1} + \cdots + a_n = 0 \tag{18}$$

It follows that $-k^n = a_1 k^{n-1} l + \cdots + a_n l^n$, hence $l \mid k^n$. We conclude that $l = 1$. Hence $\mathcal{A} \cap \mathbb{Q} \subseteq \mathbb{Z}$.

Conversely, an integer $k \in \mathbb{Z}$ is the root of the monic polynomial $x - k \in \mathbb{Z}[x]$, so $\mathbb{Z} \subseteq \mathcal{A} \cap \mathbb{Q}$. □

Definition 4.5 We denote by $\mathcal{O}_K$ the set of algebraic integers of K, i.e. $\mathcal{O}_K := \mathcal{A} \cap K$. By Proposition 4.3, $\mathcal{O}_K$ is a ring. We will call it the **ring of integers** of K. △

Proposition 4.6 *Let $\alpha \in K$. Then there exists a non-zero $d \in \mathbb{Z}$ such that $d\alpha$ is an algebraic integer.*

Proof. Since α is algebraic it is a zero of some polynomial with integer coefficients. That is, there exist $a_0, \cdots, a_n \in \mathbb{Z}$, $a_n \neq 0$ such that $a_n \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$. Multiplying this by a_n^{n-1} we find $(a_n\alpha)^n + a_{n-1}(a_n\alpha)^{n-1} + \cdots + a_n^{n-1} a_0$, hence $a_n\alpha$ is an algebraic integer. □

Definition 4.7 A **prime** P of K is a prime ideal of $\mathcal{O}_K$. △

Proposition 4.8 *For any non-zero ideal $A \subseteq \mathcal{O}_K$, $\mathcal{O}_K/A$ is finite.*

Proof. Can be found in [4] on page 176 (Proposition 12.2.3). □

Since every finite integral domain is a field, Proposition 4.8 has the following corollary:

Corollary 4.9 *Every prime of K is maximal.*

Lemma 4.10 *Suppose $A, B \subseteq \mathcal{O}_K$ are ideals such that $A \subseteq B$, then there is an ideal $C \subseteq \mathcal{O}_K$ such that $A = BC$.*

Proof. Can be found in [4] on page 179 (Proposition 12.2.7). □

Proposition 4.11 *Every non-zero proper ideal A of $\mathcal{O}_K$ can be written uniquely (up to ordering) as a (finite) product of primes of K.*

Proof. This is Proposition 12.2.8, page 180 in [4]. □

Lemma 4.12 *Let $A \subseteq \mathcal{O}_K$ be an ideal. Then $A \cap \mathbb{Z} \neq \{0\}$.*

Proof. Can be found in [4] on page 176 (Lemma 2). □

Corollary 4.13 *Let P be a prime of K. Then $P \cap \mathbb{Z} = (p)$ (ideal in $\mathcal{O}_K$ generated by p) for a unique prime number $p \in \mathbb{N}$. We will say that P is a **prime above** p.*

Lemma 4.14 *For every prime number p, there exist finitely many primes P above p.*

Proof. By Proposition 4.11, we can uniquely factorize $(p) = P_1^{e_1} \cdots P_g^{e_g}$ as a product of primes of K. It follows by Lemma 4.10 that $P_1, \cdots, P_g$ are precisely the primes above p. In particular there are finitely many such primes. □

Definition 4.15 Let P be a prime of K. We call $\mathcal{O}_K/P$ the **residue field** of P. △

Lemma 4.16 *Let p be a prime number. For every prime P above p, the residue field of P is a finite field of characteristic p.*

Proof. It follows directly from Proposition 4.8 and Corollary 4.9 that $\mathcal{O}_K/P$ is a finite field. Since $p \in P$ its characteristic is p. □

Theorem 4.17 *Let K be a number field and let $\alpha \in \mathcal{O}_K$. Suppose that, for almost every prime P of K, the residue $\bar{\alpha}$ of α in the residue field κ of P satisfies $\bar{\alpha}^p = \bar{\alpha}$. Then $\alpha \in \mathbb{Z}$.*

Proof. This follows from the Frobenius Density Theorem. A proof can be found in [5] on p.134 (Thm 5.2). □

5 One Variable Case

5.1 Definitions and statement of result

In this section we will give a characterization for the so-called Gaussian rational functions in one variable. Part of our result is based on a proof for rational linear recurrent sequences by Minton, see [1]. In one variable the approach using rational linear recurrent sequences and our approach by rational functions are almost equivalent. We however choose the viewpoint of rational functions, because then there is a straightforward generalization to the multivariable case. We begin with the definition of Gauss sequences in one variable.

Definition 5.1 (Gauss sequences) Let $\{q_n\}_{n\geq 0}$ be a sequence of rational numbers. Let p be a prime number. We say that $\{q_n\}$ is **Gauss at p** if for all $m, r \in \mathbb{N}$, we have (see Definition 2.6)

$$q_{mp^r} \equiv q_{mp^{r-1}} \pmod{p^r} \tag{19}$$

Furthermore, we call the sequence $\{q_n\}$ **Gauss** if it is Gauss at almost every prime p, and **strictly Gauss** if it is integral and Gauss at every prime p.

We will call a power series $f = a_0 + a_1 x + \cdots \in \mathbb{Q}[[x]]$ (strictly) Gauss if the corresponding sequence of coefficients $\{a_n\}_{n\geq 0}$ is (strictly) Gauss. △

Remark 5.2. Using the v_p-function as defined in 2.7, we can rewrite condition (19) as $q_k \equiv q_{k/p} \pmod{p^{v_p(k)}}$ for all $k \in p\mathbb{Z}$. ◊

Example 5.3 Let the power series $f \in \mathbb{Z}[[x]]$ be given by

$$f = \frac{2-x}{1-x-x^2} = 2 + x + 3x^2 + 4x^3 + 7x^4 + 11x^5 + 18x^6 + 29x^7 \cdots \tag{20}$$

Notice how each term in the sequence of power series coefficients is the sum of the previous two terms. This Fibonacci-like sequence (also known as the Lucas sequence) will, among other sequences of the same type, turn out to be strictly Gauss. ☆

Definition 5.4 We call a rational function $f = P/Q$, $P, Q \in \mathbb{Z}[x]$, $Q(0) \neq 0$ in one variable of type L if $f(x) = 1$ or $P = \theta Q$ (Definition 2.15). △

Remark 5.5. The set of non-constant functions of type L is closed under addition, since if $Q_1, Q_2 \in \mathbb{Z}[x]$ then

$$\frac{\theta Q_1}{Q_1} + \frac{\theta Q_2}{Q_2} = \frac{x\left(\frac{d}{dx}Q_1\right)Q_2 + Q_1 x\left(\frac{d}{dx}Q_2\right)}{Q_1 Q_2} = \frac{\theta(Q_1 Q_2)}{Q_1 Q_2} \tag{21}$$

◊

The following Theorem is the main result of this section.

Theorem 5.6 *Let $P, Q \in \mathbb{Z}[x]$ such that $Q(0) \neq 0$. Let $f \in \mathbb{Q}[[x]]$ be the power series expansion of P/Q. Suppose that Q is separable. Then f is Gauss if and only if it is a $\mathbb{Q}$-linear sum of functions of type L.*

Remark 5.7. Note that, for a rational function of type L, the degree of its numerator is always equal to the degree of its denominator. Therefore, Theorem 5.6 implies in particular that if a rational function P/Q in one variable is Gauss, then $\deg P \leq \deg Q$. $\Diamond$

The implication " $\Longleftarrow$ " of Theorem 5.6 will be shown in more general form in section 6. The implication " $\Longrightarrow$ " will be of our concern for the rest of this section. Our proof will be an alternative and simpler version of the strategy used in [1].

Example 5.8 Consider f as in Example 5.3. Note that:

$$f = \frac{2-x}{1-x-x^2} = 2 - \frac{-x-2x^2}{1-x-x^2} = 2 - \frac{\theta(1-x-x^2)}{1-x-x^2} \tag{22}$$

Which is indeed a $\mathbb{Q}$-linear sum of rational functions of type L. $\star$

5.2 Proof of the first part of Theorem 5.6

As stated earlier, the result in [1] regards rational linear recurrent sequences, but we are interested in a result for rational functions. The relation between the two is shown by Lemma 5.9.

Lemma 5.9 *Let $P, Q \in \mathbb{Z}[x]$, $Q(0) \neq 0$ such that $\deg P \leq \deg Q$. Let $f = c_0 + c_1 x + \cdots \in \mathbb{Q}[[x]]$ be the power series expansion of P/Q. Then the corresponding sequence of coefficients $\{c_n\}_{n\geq 0}$ is a rational linear recurrent sequence, with characteristic polynomial $G = Q^*/Q(0)$ (see Definition 2.13).*

Proof. It follows from Proposition 2.18 that $c_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$.

Write $Q = b_0 + b_1 x + \cdots + b_d x^d$. Note that:

$$\begin{aligned} P &= Qf && (23)\\ &= b_0 \sum_{k\geq 0} c_k x^k + \cdots + b_d \sum_{k\geq 0} c_k x^{k+d} && (24)\\ &= \sum_{k\geq 0} b_0 c_k x^k + \cdots + \sum_{k\geq d} b_d c_{k-d} x^k && (25)\\ &= \sum_{k>d} (b_0 c_k + b_1 c_{k-1} + \cdots + b_d c_{k-d}) x^k + Q_2(x) && (26) \end{aligned}$$

where $Q_2(x)$ is a polynomial of degree $\leq d$.

Since $\deg P \leq \deg Q = d$ we find that for all $k > d$:

$$c_k = -\frac{1}{b_0}\left(b_1 c_{k-1} + \cdots + b_d c_{k-d}\right) \tag{27}$$

We conclude that $\{c_n\}$ is indeed a rational linear recurrent sequence, with characteristic polynomial

$$G = x^d + \frac{b_1}{b_0} x^{d-1} + \cdots + \frac{b_d}{b_0} = \frac{Q^*}{Q(0)} \tag{28}$$

$\square$

In Lemma 5.9 we assumed that $\deg P \leq \deg Q$. In order to be able to extend our result to the case that $\deg P > \deg Q$ we will need the following definition.

Definition 5.10 We call a sequence of rational numbers $\{v_n\}$ **vanishing** if there exists a $k \in \mathbb{N}$ such that $v_n = 0$ for all $n > k$. The smallest such $k \in \mathbb{N}$ is called the **support** of $\{v_n\}$. △

Lemma 5.11 *Let P, Q, f and $\{c_n\}_{n\geq 0}$ be as in Lemma 5.9, but without the restriction that* $\deg P \leq \deg Q$. *Then we can write $\{c_n\} = \{u_n\} + \{v_n\}$ for a rational linear recurrent sequence $\{u_n\}$ with characteristic polynomial $Q^*/Q(0)$ and a vanishing sequence $\{v_n\}$.*

Proof. Let $n := \deg P$ and $d := \deg Q$. By Euclidean division on polynomials we can write $P = QV + R$ for polynomials $V, R \in \mathbb{Q}[x]$ such that $\deg R < \deg Q$. Now we see that

$$f = \frac{P}{Q} = \frac{QV + R}{Q} = V + \frac{R}{Q} \tag{29}$$

By Lemma 5.9 the sequence of power series coefficients $\{u_n\}$ of R/Q is a rational linear recurrent sequence with characteristic polynomials $Q^*/Q(0)$. Since V is a polynomial, its sequence of power series coefficients $\{v_n\}$ is vanishing. The desired result follows from the fact that $\{c_n\} = \{u_n\} + \{v_n\}$. □

The main mechanism of our proof of Theorem 5.6 is the following result:

Theorem 5.12 *Suppose $\{u_n\}_{n\geq 0}$ is a rational linear recurrent sequence with separable characteristic polynomial G, $G(0) \neq 0$. Let $\{v_n\}$ be a vanishing sequence and let $\{c_n\} = \{u_n\} + \{v_n\}$. Let $\omega_1, \cdots, \omega_r$ be the roots of G in its splitting field K and write u_n as in Proposition 3.4 Equation* (16). *If $\{c_n\}$ is Gauss then $\alpha_i \in \mathbb{Q}$ for all $1 \leq i \leq r$.*

Proof. Since $\omega_1, \cdots, \omega_r, \alpha_1, \cdots, \alpha_r \in K$, it follows from Proposition 4.6 that there exist non-zero integers $d_{\omega_1}, \cdots, d_{\omega_r}, d_{\alpha_1}, \cdots, d_{\alpha_r} \in \mathbb{Z}$ such that $\{d_{\omega_1}\omega_1, \cdots, d_{\omega_r}\omega_r, d_{\alpha_1}\alpha_1, \cdots, d_{\alpha_r}\alpha_r\} \subseteq \mathcal{O}_K$. Now define

$$d_\omega := \prod_{i=1}^{r} d_{\omega_i}, \qquad d_\alpha := \prod_{i=1}^{r} d_{\alpha_i}, \qquad \nu_i := d_\omega \omega_i, \qquad \gamma_i := d_\alpha \alpha_i \tag{30}$$

Note that the ν_i and γ_i are all algebraic integers.

Now consider the following matrix:

$$\Omega := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \nu_1 & \nu_2 & \cdots & \nu_r \\ \nu_1^2 & \nu_2^2 & \cdots & \nu_r^2 \\ \vdots & \vdots & \ddots & \vdots \\ \nu_1^{r-1} & \nu_2^{r-1} & \cdots & \nu_r^{r-1} \end{pmatrix} \tag{31}$$

By the Vandermonde determinant formula we have

$$\det(\Omega) = \prod_{1\leq i<j\leq n} (\nu_i - \nu_j) \tag{32}$$

Now suppose H is the Galois group of the field extension $K/\mathbb{Q}$, and let $\sigma \in H$. Since σ permutes the roots $\omega_1, \cdots, \omega_n$ of G, it also induces a permutation on the ν_i's. Thus we see, either directly from (32), or by noting that σ permutes the columns of Ω, that σ fixes $\det(\Omega)^2 = \prod_{1 \leq i < j \leq n} (\nu_i - \nu_j)^2$. Since this holds for any $\sigma \in H$, $\det(\Omega)^2 \in \mathbb{Q}$. Since $\det(\Omega)^2$ is also an algebraic integer, we conclude by Lemma 4.4 that $\det(\Omega)^2$ is in fact an integer.

Now let s be the support of $\{v_n\}$. Let p be a prime number at which $\{c_n\}$ is Gauss, and such that $p \nmid G(0) d_\omega \det(\Omega)^2$.

Note that since G is separable we have $\det(\Omega) \neq 0$. We also assumed $G(0) \neq 0$ and constructed d_ω such that it is non-zero. Therefore $p \nmid G(0) d_\omega \det(\Omega)$ holds for almost every prime number p. Since $\{c_n\}$ is Gauss at almost every prime p, we conclude that the assumptions we make on p above hold for almost all prime numbers p.

Now let $d \in \mathbb{N}$ such that $d > s$. Since $c_n = u_n$ for all $n > s$, we have

$$u_{dp} \equiv u_d \pmod{p} \tag{33}$$

Furthermore, by Fermat's Little Theorem we have

$$u_d^p \equiv u_d \pmod{p} \tag{34}$$

Subtracting equations (33) and (34) and filling in $u_n = \sum_{i=1}^r \alpha_i \omega_i^n$ we find

$$\sum_{i=1}^r \alpha_i \omega_i^{dp} - \left(\sum_{i=1}^r \alpha_i \omega_i^d \right)^p \equiv 0 \pmod{p} \tag{35}$$

If we consider this congruence modulo $p\mathcal{O}_K$ (ideal in $\mathcal{O}_K$ generated by p) then it still holds (in particular $p\mathbb{Z} \subseteq p\mathcal{O}_K$), hence

$$\sum_{i=1}^r \alpha_i \omega_i^{dp} - \left(\sum_{i=1}^r \alpha_i \omega_i^d \right)^p \equiv 0 \pmod{p\mathcal{O}_K} \tag{36}$$

Now multiply this equation by $d_\omega^{dp} d_\alpha^p$. We obtain

$$\sum_{i=1}^r d_\alpha^p \alpha_i \left(d_\omega \omega_i\right)^{dp} - \left(\sum_{i=1}^r d_\alpha \alpha_i \left(d_\omega \omega_i\right)^d \right)^p \equiv 0 \pmod{p\mathcal{O}_K} \tag{37}$$

By Fermat's Little Theorem, $d_\alpha^p \equiv d_\alpha \pmod{p}$, so in combination with (30) this reduces to

$$\sum_{i=1}^r \gamma_i \nu_i^{dp} - \left(\sum_{i=1}^r \gamma_i \nu_i^d \right)^p \equiv 0 \pmod{p\mathcal{O}_K} \tag{38}$$

Therefore

$$\sum_{i=1}^r \left(\gamma_i - \gamma_i^p\right) \nu_i^{dp} \equiv 0 \pmod{p\mathcal{O}_K} \tag{39}$$

Now let P be a prime above p in K. Denote by $\bar{\alpha}$ the residue of α in $\mathcal{O}_K/P$. It follows from (39) that

$$\sum_{i=1}^r \left(\bar{\gamma}_i - \bar{\gamma}_i^p\right) \bar{\nu}_i^{dp} = 0 \tag{40}$$

Since this holds for every $d > s$ we find that

$$(\bar{\gamma}_1 - \bar{\gamma}_1^p)\begin{pmatrix}\bar{\nu}_1^{(s+1)p}\\ \vdots \\ \bar{\nu}_1^{(s+r)p}\end{pmatrix} + \cdots + (\bar{\gamma}_r - \bar{\gamma}_r^p)\begin{pmatrix}\bar{\nu}_r^{(s+1)p}\\ \vdots \\ \bar{\nu}_r^{(s+r)p}\end{pmatrix} = \begin{pmatrix}0\\ \vdots \\ 0\end{pmatrix} \tag{41}$$

If we now define the matrix

$$\Gamma := \begin{pmatrix}\bar{\nu}_1^{(s+1)p} & \cdots & \bar{\nu}_r^{(s+1)p}\\ \bar{\nu}_1^{(s+2)p} & \cdots & \bar{\nu}_r^{(s+2)p}\\ \vdots & & \vdots \\ \bar{\nu}_1^{(s+r)p} & \cdots & \bar{\nu}_r^{(s+r)p}\end{pmatrix} \tag{42}$$

Then we have

$$\det(\Gamma) = \begin{vmatrix}1 & \cdots & 1\\ \bar{\nu}_1^p & \cdots & \bar{\nu}_r^p\\ \vdots & & \vdots \\ \bar{\nu}_1^{(r-1)p} & \cdots & \bar{\nu}_r^{(r-1)p}\end{vmatrix} \prod_{k=1}^{r} \bar{\nu}_k^{(s+1)p} = \prod_{1\le i<j\le r} (\bar{\nu}_i^p - \bar{\nu}_j^p) \prod_{k=1}^{r} \bar{\nu}_k^{(s+1)p} = \overline{\det(\Omega)}^p \prod_{i=1}^{r} \bar{\nu}_i^{(s+1)p} \tag{43}$$

We will now show that $\det(\Gamma) \neq 0$.

Assume first to the contrary that $\det(\Omega) \in P$. Then also $\det(\Omega)^2 \in P$. But since $\det(\Omega)^2$ is an integer, we have $\det(\Omega)^2 \equiv 0 \pmod{p}$. This contradicts $p \nmid \det(\Omega)^2$. We conclude that $\det(\Omega)^p \notin P$.

Suppose now that $\bar{\nu}_i = 0$ for some i $(1 \le i \le r)$. We know that $\bar{\nu}_i = d_\omega \bar{\omega}_i$, $p \nmid d_\omega$ and that P is a prime ideal, thus $\bar{\omega}_i = 0$. But then $0 = G(\bar{\omega}_i) = \overline{G(0)}$. This contradicts our assumption that $p \nmid G(0)$. We conclude that $\bar{\nu}_i \neq 0$ for all i.

Using once again that P is a prime ideal, it now follows that indeed $\det(\Gamma) \neq 0$. Now let $1 \le i \le r$. Combined with equation (41) we find $\bar{\gamma}_i = \bar{\gamma}_i^p$.

This holds for every prime P above almost every prime number p. So by Proposition 4.14, the residue $\bar{\gamma}_i$ of γ_i in P equals $\bar{\gamma}_i^p$ for almost every prime P of $\mathcal{O}_K$. Hence, by Theorem 4.17, $\gamma_i \in \mathbb{Z}$. We conclude by (30) that $\alpha_i \in \mathbb{Q}$ for all $1 \le i \le r$. □

We would now like to translate the result of Theorem 5.12 into a result for rational functions, which will form the promised proof of the implication " $\Longrightarrow$ " in Theorem 5.6. The translation will be established by combining Lemma 5.11 with Lemma 3.5.

Proof of " $\Longrightarrow$ " in Theorem 5.6. Let $P, Q \in \mathbb{Z}[x]$, $Q(0) \neq 0$, $f := P/Q$, Q separable, and assume that f is Gauss. Let $\{c_n\}_{n\ge 0}$ be the sequence of power series coefficients of f and write $\{c_n\} = \{u_n\} + \{v_n\}$ as in Lemma 5.11. The characteristic polynomial of $\{u_n\}$ is $G := Q^*/Q(0)$. Since Q is separable, it follows from Lemma 2.14 that G is separable. It should also be clear that $G(0) \neq 0$, since we assume $Q(0) \neq 0$. If we again write u_n as in Proposition 3.4, Theorem 5.12 tells us that $\alpha_i \in \mathbb{Q}$ for all $1 \le i \le r$. It follows from Lemma 3.5 that after relabelling the roots ω_i of G we can write

$$u_n = \beta_1 \left(\omega_1^n + \cdots + \omega_{k_1}^n\right) + \cdots + \beta_l \left(\omega_{k_{l-1}+1}^n + \cdots + \omega_{k_l}^n\right) \tag{44}$$

For rational numbers $\beta_1, \cdots, \beta_l$ and $k_0 = 0, k_1, \cdots, k_{l-1}, k_l = r \in \mathbb{Z}_{\geq 0}$, such that for every $1 \leq i \leq l$, $\{\omega_{k_{i-1}+1}, \cdots, \omega_{k_i}\}$ is a complete set of conjugate algebraic numbers, which are precisely the roots of the minimal polynomial

$$m_i := \prod_{j=k_{i-1}+1}^{k_i} (x - \omega_j) \in \mathbb{Q}[x]$$

Now define $S_i(n) := \omega_{k_{i-1}+1}^n + \cdots + \omega_{k_i}^n$. Using this abbreviation, (44) reduces to

$$u_n = \sum_{i=1}^{l} \beta_i S_i(n) \tag{45}$$

Define $f_i(x) := \sum_{n\geq 0} S_i(n) x^n$ to be the generating function of the sequence $\{S_i(n)\}_{n\geq 0}$. Note that

$$f_i(x) = \sum_{n\geq 0} \sum_{j=k_{i-1}+1}^{k_i} \omega_j^n x^n = \sum_{j=k_{i-1}+1}^{k_i} \frac{1}{1-\omega_j x} = \sum_{j=k_{i-1}+1}^{k_i} \frac{1-\omega_j x + \omega_j x}{1-\omega_j x} = k_i - k_{i-1} + \sum_{j=k_{i-1}+1}^{k_i} \frac{\omega_j x}{1-\omega_j x} \tag{46}$$

Now if we define $g_j := 1 - \omega_j x$, and $G_i := \prod_{j=k_{i-1}+1}^{k_i} g_j$, (46) transforms into (see Remark 5.5)

$$f_i = k_i - k_{i-1} - \sum_{j=k_{i-1}+1}^{k_i} \frac{\theta g_j}{g_j} = k_i - k_{i-1} - \frac{\theta G_i}{G_i} \tag{47}$$

It follows that the generating function U of $\{u_n\}$ is given by

$$U = \sum_{i=1}^{l} \beta_i \left(k_i - k_{i-1} - \frac{\theta G_i}{G_i} \right) \tag{48}$$

Now note that $G_i = m_i^* \in \mathbb{Q}[x]$, hence (after rescaling G_i to have integer coefficients) we see that the generating function of $\{u_n\}$ is a $\mathbb{Q}$-linear sum of functions of type L.

If we were able to prove that $v_m = 0$ for all $m \in \mathbb{N}$, then it follows that the generating functions of $\{u_n\}$ and $\{c_n\}$ are the same up to a possible constant. This implies the desired result, namely that f is a $\mathbb{Q}$-linear sum of functions of type L. In order to show that this actually holds, we will need the " $\Longleftarrow$ "-part of Theorem 5.6 (which we will prove in section 6), namely that every $\mathbb{Q}$-linear combination of functions of type L is Gauss.

Assuming this result, we find that U as in (48) is Gauss, hence so is the sequence $\{u_n\}$. Now let $m \in \mathbb{N}$ and let p be a prime number greater than the support of $\{v_n\}$ such that $\{u_n\}$ is Gauss at p. Since we assume $\{c_n\} = \{u_n\} + \{v_n\}$ to be Gauss it follows that

$$u_m \equiv u_{mp} \pmod{p} \qquad \text{and} \qquad u_m + v_m \equiv u_{mp} \pmod{p} \tag{49}$$

From which it follows that $v_m \equiv 0 \pmod{p}$. Since this holds for infinitely many primes p, we conclude that $v_m = 0$. $\square$

6 Multiple Variables

In section 5 we gave a characterization of Gaussian rational functions in one variable with separable denominator. In this section we work towards giving a similar characterization for rational functions in multiple variables. For now, this number of variables will be denoted by an arbitrary but fixed $n \in \mathbb{N}$.

The section consists of three parts. In the first part we will show sufficient conditions for a rational function in n variables to be Gauss, in the second part we will be concerned with finding necessary conditions, and in the third part we will apply our results to give a complete characterization for a special type of rational functions.

6.1 Definitions and sufficiency

Definition 6.1 (General Gauss sequences) Let $\{q_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^n_{\geq 0}}$ be an n-sequence of rational numbers. Let p be a prime number. We say that $\{q_{\boldsymbol{k}}\}$ is **Gauss at $\boldsymbol{p}$** if for all $\boldsymbol{m} \in \mathbb{Z}^n_{\geq 0}$ and $r \in \mathbb{N}$ we have

$$q_{\boldsymbol{m}p^r} \equiv q_{\boldsymbol{m}p^{r-1}} \pmod{p^r} \tag{50}$$

We call the n-sequence $\{q_{\boldsymbol{k}}\}$ **Gauss** if it is Gauss at almost every prime p. We call it **strictly Gauss** if it is integral and Gauss at every prime p.

We will call a power series $f = \sum_{\boldsymbol{k}\geq 0} a_{\boldsymbol{k}} x^{\boldsymbol{k}} \in \mathbb{Q}[[x_1, \cdots, x_n]]$ (strictly) Gauss if the corresponding n-sequence of coefficients $\{a_{\boldsymbol{k}}\}$ is (strictly) Gauss. △

Remark 6.2. Alternatively, we can write condition (50) as $q_{\boldsymbol{k}} \equiv q_{\boldsymbol{k}/p} \pmod{p^{v_p(\boldsymbol{k})}}$ for all $\boldsymbol{k} \in p\mathbb{Z}^n_{\geq 0}$. ♢

Definition 6.3 Let $\{a_{\boldsymbol{k}}\}, \{b_{\boldsymbol{k}}\}$ be n-sequences of rational numbers, and let $q \in \mathbb{Q}$. We define sum of and scalar multiplication with such sequences in the straightforward way: $\{a_{\boldsymbol{k}}\} + \{b_{\boldsymbol{k}}\} := \{a_{\boldsymbol{k}} + b_{\boldsymbol{k}}\}$ and $q\{a_{\boldsymbol{k}}\} := \{qa_{\boldsymbol{k}}\}$. △

Proposition 6.4 *The set $\mathcal{G}$ of Gauss n-sequences, together with the operations in Definition 6.3, is a $\mathbb{Q}$-vector space.*

Proof. Suppose $\{a_{\boldsymbol{k}}\}, \{b_{\boldsymbol{k}}\} \in \mathcal{G}$ are Gauss n-sequences, and let $q \in \mathbb{Q}$. Let P_1, P_2 be the finite set of prime numbers for which condition (50) does not hold for $\{a_{\boldsymbol{k}}\}$ and $\{b_{\boldsymbol{k}}\}$ respectively, and let P_3 be the set of primes p such that p divides the denominator of q. Then $\{a_{\boldsymbol{k}}\} + \{b_{\boldsymbol{k}}\}$ satisfies condition (50) for all primes p such that $p \notin P_1 \cup P_2$, and $q\{a_{\boldsymbol{k}}\}$ satisfies (50) for all primes p such that $p \notin P_1 \cup P_3$. Since $P_1 \cup P_2$ and $P_1 \cup P_3$ are both finite sets, we conclude that $\{a_{\boldsymbol{k}}\} + \{b_{\boldsymbol{k}}\} \in \mathcal{G}$ and $q\{a_{\boldsymbol{k}}\} \in \mathcal{G}$. □

Proposition 6.5 *Let $p_1, \cdots, p_m$ be prime numbers, $\mathcal{Z} := \mathbb{Z}[1/p_1, \cdots, 1/p_m]$. Let $f \in \mathcal{Z}[[x_1, \cdots, x_n]]$ such that $f(0) = 1$. Then there exist $a_{k_1,\cdots,k_n} = a_{\boldsymbol{k}} \in \mathcal{Z}$ such that*

$$f = \prod_{\boldsymbol{k}>0} \left(1 - x^{\boldsymbol{k}}\right)^{-a_{\boldsymbol{k}}} \tag{51}$$

We will first illustrate the method of obtaining the $a_{\boldsymbol{k}}$ in Proposition 6.5 by considering the case $n = 1$. Then we can write $f = 1 + f_1 x + f_2 x^2 + \cdots$ for $f_i \in \mathcal{Z}$. Note that

$$f(1-x)^{f_1} = f \sum_{l \geq 0} \binom{f_1}{l} (-x)^l = (1 + f_1 x + f_2 x^2 + \cdots)(1 - f_1 x + \cdots) = 1 + g_2 x^2 + g_3 x^3 + \cdots \tag{52}$$

For coefficients $g_i \in \mathcal{Z}$ (see Lemma 2.10). Using the same argument we find

$$f(1-x)^{f_1}(1-x^2)^{g_2} = (1 + g_2 x^2 + g_3 x^3 + \cdots) \sum_{l \geq 0} \binom{g_2}{l} (-x^2)^l = 1 + h_4 x^4 + h_5 x^5 + \cdots \tag{53}$$

For coefficients $h_i \in \mathcal{Z}$. Continuing this process we find $a_k \in \mathcal{Z}$ (note that $a_k = 0$ is allowed) such that $f \prod_{k>0} (1-x^k)^{a_k} = 1$. For the case of general n we can use a similar argument, by "factoring out" the terms of a given degree in each step.

Convention 6.6 Let $d \in \mathbb{Z}_{\geq 0}$. If a power series $\sum_{\boldsymbol{k} \geq 0} a_{\boldsymbol{k}} x^{\boldsymbol{k}}$ satisfies $a_{\boldsymbol{k}} = 0$ for all $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$ such that $|\boldsymbol{k}| < d$, then we will say that it is $\mathcal{O}(\boldsymbol{x}^d)$. We may abbreviate such a power series with $\mathcal{O}(\boldsymbol{x}^d)$. ∘

Proof of Proposition 6.5. Let $d \in \mathbb{N}$. We say that a power series $g \in \mathcal{Z}[[x_1, \cdots, x_n]]$ satisfies condition P_d if $g(0) = 1$, and $g - 1$ only contains terms of degree at least d (i.e. is $\mathcal{O}(\boldsymbol{x}^d)$).

Suppose g satisfies condition P_d. We can then write ($g_{\boldsymbol{k}} \in \mathcal{Z}$)

$$g = 1 + \sum_{|\boldsymbol{k}| = d} g_{\boldsymbol{k}} x^{\boldsymbol{k}} + \mathcal{O}(\boldsymbol{x}^{d+1}) \tag{54}$$

Define $h := g \prod_{|\boldsymbol{k}|=d} (1 - x^{\boldsymbol{k}})^{g_{\boldsymbol{k}}}$. Then (we implicitly use Lemma 2.10 for the second equality)

$$\begin{aligned} h &= \left(1 + \sum_{|\boldsymbol{k}|=d} g_{\boldsymbol{k}} x^{\boldsymbol{k}}\right) \prod_{|\boldsymbol{k}'|=d} (1 - x^{\boldsymbol{k}'})^{g_{\boldsymbol{k}'}} + \mathcal{O}(\boldsymbol{x}^{d+1}) && (55) \\ &= \left(1 + \sum_{|\boldsymbol{k}|=d} g_{\boldsymbol{k}} x^{\boldsymbol{k}}\right) \prod_{|\boldsymbol{k}'|=d} \left(1 - g_{\boldsymbol{k}'} x^{\boldsymbol{k}'}\right) + \mathcal{O}(\boldsymbol{x}^{d+1}) && (56) \\ &= 1 + \mathcal{O}(\boldsymbol{x}^{d+1}) && (57) \end{aligned}$$

So h satisfies condition P_{d+1}. Applying this inductively on f we obtain coefficients $a_{\boldsymbol{k}} \in \mathcal{Z}$ such that

$$f \prod_{\boldsymbol{k} > 0} \left(1 - x^{\boldsymbol{k}}\right)^{a_{\boldsymbol{k}}} = 1 \tag{58}$$

□

For the rest of this section we will assume that $\mathcal{Z}$ denotes a ring of the form $\mathbb{Z}[1/p_1, \cdots, 1/p_m]$ for prime numbers $p_1, \cdots, p_m$ ($m \in \mathbb{Z}_{\geq 0}$).

Theorem 6.7 *Let $l \leq n$ be a positive integer, and let $f_1, \cdots, f_l \in \mathcal{Z}[[x_1, \cdots, x_n]]$ be power series such that $f_i(0) = 1$ for all $1 \leq i \leq l$. Let $\frac{\partial(f_1,\cdots,f_l)}{\partial(x_1,\cdots,x_l)}$ be the Jacobian matrix. Then the power series $f \in \mathcal{Z}[[x_1, \cdots, x_n]]$ defined by*

$$f := \frac{x_1 \cdots x_l}{f_1 \cdots f_l} \left| \frac{\partial(f_1, \cdots, f_l)}{\partial(x_1, \cdots, x_l)} \right| \tag{59}$$

is Gauss. Furthermore, if $\mathcal{Z} = \mathbb{Z}$ then f is strictly Gauss.

Proof. First consider the special case $f_i = 1 - x^{\boldsymbol{k}_i}$ $(1 \leq i \leq l)$ with $\boldsymbol{k}_i = (k_{i1}, \cdots, k_{in}) \in \mathbb{Z}^n_{\geq 0}$. Then we have, for any $1 \leq i, j \leq l$,

$$x_j \frac{\partial f_i}{\partial x_j} = -k_{ij} x^{\boldsymbol{k}_i} \tag{60}$$

So if we define K_l to be the $l \times l$ matrix (k_{ij}) we find:

$$f = \frac{x_1 \cdots x_l}{f_1 \cdots f_l} \left| \frac{\partial(f_1, \cdots, f_l)}{\partial(x_1, \cdots, x_l)} \right| = (-1)^l \det(K_l) \frac{x^{\boldsymbol{k}_1} \cdots x^{\boldsymbol{k}_l}}{f_1 \cdots f_l} = (-1)^l \det(K_l) \sum_{m_1, \cdots, m_l \geq 1} x^{m_1 \boldsymbol{k}_1 + \cdots + m_l \boldsymbol{k}_l} \tag{61}$$

If $\det(K_l) = 0$ then we see immediately that f is strictly Gauss, so assume that $\det(K_l) \neq 0$. Then we have, for any rational numbers $r_1, \cdots, r_l, s_1, \cdots, s_l$:

$$r_1 \boldsymbol{k}_1 + \cdots + r_l \boldsymbol{k}_l = s_1 \boldsymbol{k}_1 + \cdots + s_l \boldsymbol{k}_l \quad \Longrightarrow \quad (r_1, \cdots, r_l) = (s_1, \cdots, s_l) \tag{62}$$

Now denote by $a_{\boldsymbol{k}}$ the power series coefficients of f, (61) and (62) tell us in particular that, given $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$, $a_{\boldsymbol{k}}$ is either equal to 0 or equal to $(-1)^l \det(K_l)$, the latter holding precisely when $\boldsymbol{k} = m_1 \boldsymbol{k}_1 + \cdots + m_l \boldsymbol{k}_l$ for some $m_1, \cdots, m_l \in \mathbb{N}$.

Let p be a prime number and let $\boldsymbol{k} \in p\mathbb{Z}^n$. If there do not exist $m_1, \cdots, m_l \in \mathbb{N}$ such that $m_1 \boldsymbol{k}_1 + \cdots + m_l \boldsymbol{k}_l = \boldsymbol{k}$, then there certainly do not exist $m_1, \cdots, m_l \in \mathbb{N}$ such that $m_1 \boldsymbol{k}_1 + \cdots + m_l \boldsymbol{k}_l = \boldsymbol{k}/p$. It follows that $a_{\boldsymbol{k}} = a_{\boldsymbol{k}/p} = 0$, so in particular $a_{\boldsymbol{k}} \equiv a_{\boldsymbol{k}/p} \pmod{p^{v_p(\boldsymbol{k})}}$.

Now suppose there do exist $m_1, \cdots, m_l \in \mathbb{N}$ such that $m_1 \boldsymbol{k}_1 + \cdots + m_l \boldsymbol{k}_l = \boldsymbol{k}$. If $p \mid m_i$ for all i then $a_{\boldsymbol{k}} = a_{\boldsymbol{k}/p} = (-1)^l \det(K_l)$, so again $a_{\boldsymbol{k}} \equiv a_{\boldsymbol{k}/p} \pmod{p^{v_p(\boldsymbol{k})}}$. The only case left to consider is $p \nmid m_i$ for some i. Assume without loss of generality that $p \nmid m_1$. Then we have the following:

$$\begin{aligned} v_p\left(\det(K_l)\right) &= v_p\left(\det(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_l)\right) = v_p\left(m_1 \det(\boldsymbol{k}_1, \cdots, \boldsymbol{k}_l)\right) = v_p\left(\det(m_1 \boldsymbol{k}_1, \cdots, \boldsymbol{k}_l)\right) & (63) \\ &= v_p\left(\det(m_1 \boldsymbol{k}_1 + \cdots + m_l \boldsymbol{k}_l, \cdots, \boldsymbol{k}_l)\right) \geq v_p(m_1 \boldsymbol{k}_1 + \cdots + m_l \boldsymbol{k}_l) & (64) \end{aligned}$$

So $a_{\boldsymbol{k}} = (-1)^l \det(K_l) \equiv 0 \equiv a_{\boldsymbol{k}/p} \pmod{p^{v_p(\boldsymbol{k})}}$.[1] We conclude that f is indeed strictly Gauss.

Now consider f as in the general case, i.e. f is given by (59) with $f_1, \cdots, f_l \in \mathcal{Z}[[x_1, \cdots, x_n]]$, $f_i(0) = 1$ for all i. Consider the following differential form Ω:

$$\Omega := x_1 \cdots x_l \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_l}{f_l} = \frac{x_1 \cdots x_l}{f_1 \cdots f_l} \left| \frac{\partial(f_1, \cdots, f_l)}{\partial(x_1, \cdots, x_l)} \right| dx_1 \wedge \cdots \wedge dx_l = f \, dx_1 \wedge \cdots \wedge dx_l \tag{65}$$

Using Proposition 6.5 we know that we can write $f_i = \prod_{\boldsymbol{k}_i > 0} \left(1 - x^{\boldsymbol{k}_i}\right)^{-a_{\boldsymbol{k}_i}(i)}$ for coefficients $a_{\boldsymbol{k}_i}(i) \in \mathcal{Z}$ for all i. Using this we find

$$\frac{df_i}{f_i} = \sum_{\boldsymbol{k}_i > 0} a_{\boldsymbol{k}_i}(i) \frac{dx^{\boldsymbol{k}_i}}{1 - x^{\boldsymbol{k}_i}} \tag{66}$$

[1] It follows in fact from (62) that $a_{\boldsymbol{k}/p} = 0$, but whether it is equal to $(-1)^l \det(K_l)$ or equal to 0 is irrelevant in this case.

Now for any $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$, define the power series $g^{(\boldsymbol{k})}$ by $g^{(\boldsymbol{k})} := 1 - x^{\boldsymbol{k}}$. Combining this with (66) we can rewrite Ω as:

$$\begin{aligned}
\Omega &= x_1 \cdots x_l \, \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_l}{f_l} && (67)\\
&= x_1 \cdots x_l \sum_{\boldsymbol{k}_1 > 0, \cdots, \boldsymbol{k}_l > 0} \prod_{i=1}^{l} a_{\boldsymbol{k}_i}(i) \frac{dx^{\boldsymbol{k}_1}}{1 - x^{\boldsymbol{k}_1}} \wedge \cdots \wedge \frac{dx^{\boldsymbol{k}_l}}{1 - x^{\boldsymbol{k}_l}} && (68)\\
&= (-1)^l \sum_{\boldsymbol{k}_1 > 0, \cdots, \boldsymbol{k}_l > 0} \prod_{i=1}^{l} a_{\boldsymbol{k}_i}(i) \; x_1 \cdots x_l \, \frac{dg^{(\boldsymbol{k_1})}}{g^{(\boldsymbol{k_1})}} \wedge \cdots \wedge \frac{dg^{(\boldsymbol{k_l})}}{g^{(\boldsymbol{k_l})}} && (69)\\
&= (-1)^l \sum_{\boldsymbol{k}_1 > 0, \cdots, \boldsymbol{k}_l > 0} \prod_{i=1}^{l} a_{\boldsymbol{k}_i}(i) \; \frac{x_1 \cdots x_l}{g^{(\boldsymbol{k_1})} \cdots g^{(\boldsymbol{k_l})}} \left| \frac{\partial \left(g^{(\boldsymbol{k_1})}, \cdots, g^{(\boldsymbol{k_1})} \right)}{\partial (x_1, \cdots, x_l)} \right| dx_1 \wedge \cdots \wedge dx_l && (70)
\end{aligned}$$

Note that, in view of equation (61), the sum in (70) well-defines a power series: The term of the sum corresponding to $\boldsymbol{k}_1, \cdots, \boldsymbol{k}_l$ is $\mathcal{O}(\boldsymbol{x}^{|\boldsymbol{k}_1| + \cdots + |\boldsymbol{k}_l|})$, hence each of the coefficients of the power series that the sum defines can be computed as a finite sum of elements of $\mathcal{Z}$.

Now the special case above tells us that the power series defined by $\dfrac{x_1 \cdots x_l}{g^{(\boldsymbol{k_1})} \cdots g^{(\boldsymbol{k_l})}} \left| \dfrac{\partial \left(g^{(\boldsymbol{k_1})}, \cdots, g^{(\boldsymbol{k_1})} \right)}{\partial (x_1, \cdots, x_l)} \right|$ is strictly Gauss for any $\boldsymbol{k}_1, \cdots, \boldsymbol{k}_l \in \mathbb{Z}^n_{\geq 0}$. Multiplying this by $\prod_{i=1}^{l} a_{\boldsymbol{k}_i}(i)$ we obtain a power series that is Gauss, or even strictly Gauss if $\mathcal{Z} = \mathbb{Z}$. The same will hold after we sum over all $\boldsymbol{k}_1, \cdots, \boldsymbol{k}_l$. Combined with (65) we conclude that f is Gauss, and strictly Gauss if $\mathcal{Z} = \mathbb{Z}$. □

Definition 6.8 We say a rational function $f = P/Q$, $P, Q \in \mathbb{Z}[x_1, \cdots, x_n]$, $Q(0) \neq 0$ is of type L if there is some subset of the variables $X = \{x_{i_1}, \cdots, x_{i_l}\} \subseteq \{x_1, \cdots, x_n\}$ and there are polynomials $f_1, \cdots, f_l \in \mathbb{Z}[x_1, \cdots, x_n]_0$ such that

$$f = \frac{x_{i_1} \cdots x_{i_l}}{f_1 \cdots f_l} \left| \frac{\partial (f_1, \cdots, f_l)}{\partial (x_{i_1}, \cdots, x_{i_l})} \right| \tag{71}$$

Here we use the convention that if $X = \emptyset$, then $f = 1$. △

Remark 6.9. Note that the more general Definition 6.8 for rational functions of type L in n variables coincides with Definition 5.4 for $n = 1$. ♢

The following Proposition says that every $\mathbb{Q}$-linear sum of functions of type L is Gauss, which has as a Corollary the promised implication " $\Longleftarrow$ " in Theorem 5.6.

Proposition 6.10 *Suppose that $h_1, \cdots, h_m$ are rational functions in n variables of type L and let $q_1, \cdots, q_m \in \mathbb{Q}$. Then*

$$f := q_1 h_1 + \cdots + q_m h_m \tag{72}$$

is Gauss.

Proof. By Proposition 6.4, it suffices to prove that every function h of type L is Gauss. If $h = 1$ then clearly h is Gauss. Now let $X = \{x_{i_1}, \cdots, x_{i_l}\}$ be a non-empty subset of $\{x_1, \cdots, x_n\}$ and let

$f_1, \cdots, f_l \in \mathbb{Z}[x_1, \cdots, x_n]_0$. Applying Theorem 6.7 to $g_i := f_i/f_i(0)$ after a suitable relabelling of our variables we find that

$$\frac{x_{i_1} \cdots x_{i_l}}{f_1 \cdots f_l} \left| \frac{\partial(f_1, \cdots, f_l)}{\partial(x_{i_1}, \cdots, x_{i_l})} \right| = \frac{x_{i_1} \cdots x_{i_l}}{g_1 \cdots g_l} \left| \frac{\partial(g_1, \cdots, g_l)}{\partial(x_{i_1}, \cdots, x_{i_l})} \right| \tag{73}$$

is Gauss. □

Proof of "$\Longleftarrow$" in Theorem 5.6. This follows directly from Proposition 6.10 with $n = 1$. □

The following example illustrates an application of Proposition 6.10 for a two-variable rational function.

Example 6.11 Define $f_1 := 1 + xy^2 + x^2y - x^3y^3$ and $f_2 := 1 + xy^2$. Check that

$$3f_2 = 3f_1f_2 + f_1(\theta_x f_2 - 2\theta_y f_2) + f_2(\theta_y f_1 - 2\theta_x f_1) + (\theta_x f_1)(\theta_y f_2) - (\theta_x f_2)(\theta_y f_1) \tag{74}$$

From which it follows that

$$\begin{aligned}
\frac{1}{f_1} &= \frac{3f_1f_2 + f_1(\theta_x f_2 - 2\theta_y f_2) + f_2(\theta_y f_1 - 2\theta_x f_1) + (\theta_x f_1)(\theta_y f_2) - (\theta_x f_2)(\theta_y f_1)}{3f_1f_2} && (75)\\
&= 1 + \frac{1}{3}\left(\frac{\theta_y f_1 - 2\theta_x f_1}{f_1} + \frac{\theta_x f_2 - 2\theta_y f_2}{f_2} + \frac{(\theta_x f_1)(\theta_y f_2) - (\theta_x f_2)(\theta_y f_1)}{f_1 f_2} \right) && (76)\\
&= 1 + \frac{1}{3}\left(\frac{y\frac{\partial f_1}{\partial y} - 2x\frac{\partial f_1}{\partial x}}{f_1} + \frac{x\frac{\partial f_2}{\partial x} - 2y\frac{\partial f_2}{\partial y}}{f_2} + \frac{xy}{f_1 f_2} \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} \right) && (77)
\end{aligned}$$

Hence by Proposition 6.10,

$$\frac{1}{f_1} = \frac{1}{1 + xy^2 + x^2y - x^3y^3} \tag{78}$$

is Gauss. ☆

Example 6.11 involves some tedious computational work, and it is probably not clear how (74) was obtained. Luckily enough, the result that $1/f_1$ as given by (78) is Gauss, can be obtained in a much more elegant way, that avoids the use of "magic". The key to this is the observation that, defining $X := x^2y$ and $Y := xy^2$, f_1 reduces to $f_1 = 1 + X + Y - XY$. Before we can show that this indeed implies that $1/f_1$ is Gauss, we will need to do some work. The starting point for this is Theorem 6.12. This Theorem roughly says that if a variable x_i appears only as a single power in the polynomial Q, i.e. we can write $Q = P_1 + x_i^m R_1$ where both P_1 and R_1 do not depend on x_i, then P_1/Q is Gauss, and that we can obtain more Gaussian rational functions by "repeatedly removing variables" in this way. That is, if we were able to write $P_1 = P_2 + x_j^{m_2} R_2$ for some other variable x_j such that P_2 and R_2 do not depend on x_j, then P_2/Q is also Gauss, etcetera for a possible P_3.

Theorem 6.12 *Let $0 \leq k \leq n$ and suppose that $Q \in \mathbb{Z}[x_1, \cdots, x_n]_0$ satisfies the following condition: Defining $P_n := Q$, there exist polynomials $P_k, \cdots, P_{n-1}, R_k, \cdots, R_{n-1} \in \mathbb{Z}[x_1, \cdots, x_n]$ and natural numbers $m_{k+1}, \cdots, m_n \in \mathbb{N}$ such that for every $k \leq l < n$, we have that $P_l, R_l \in \mathbb{Z}[x_1, \cdots, x_l]$ and $P_{l+1} = P_l + x_{l+1}^{m_{l+1}} R_l$. Then P_k/Q is Gauss.*

Proof. Let everything as above. Note that for every $k \leq l < n$, we have $P_l = \left(1 - \frac{\theta_{l+1}}{m_{l+1}}\right) P_{l+1}$. Therefore

$$\frac{P_k}{Q} = \frac{P_{n-1}}{P_n}\frac{P_{n-2}}{P_{n-1}} \cdots \frac{P_k}{P_{k+1}} = \frac{\left(1 - \frac{\theta_n}{m_n}\right) P_n}{P_n} \cdots \frac{\left(1 - \frac{\theta_{k+1}}{m_{k+1}}\right) P_{k+1}}{P_{k+1}} \tag{79}$$

Now let $i_1, \cdots, i_r \in \mathbb{N}$ such that $k < i_1 < \cdots < i_r \leq n$. Since P_{i_b} does not depend on x_{i_a} for $a < b$, we see that

$$\frac{x_{i_1} \cdots x_{i_r}}{P_{i_1} \cdots P_{i_r}} \left|\frac{\partial(P_{i_1}, \cdots, P_{i_r})}{\partial(x_{i_1}, \cdots, x_{i_r})}\right| = \prod_{j=1}^{r} \frac{\theta_{i_j} P_{i_j}}{P_{i_j}} \tag{80}$$

Hence, looking back at (79), we see that P_k/Q is a $\mathbb{Q}$-linear sum of rational functions of type L. We conclude, by Proposition 6.10, that P_k/Q is Gauss. □

Example 6.13 Define $Q := x^4 + 1 + y^9(4 + 5z - z^7) + z^3$. Then, by applying Theorem 6.12 (for a suitably chosen ordering of the variables), P_i/Q is Gauss for each of the following polynomials:

$$P_1 := 1 + x^4 + z^3, \quad P_2 := 1 + x^4, \quad P_3 := 1 + z^3, \quad P_4 := 1, \quad P_5 := 1 + y^9(4 + 5z - z^7) + z^3 \tag{81}$$

☆

It looks like, given a polynomial Q, we can sometimes "filter out" some terms T of Q using the approach in Theorem 6.12 to find that T/Q is Gauss. This gives rise to the following interesting Corollary:

Corollary 6.14 *Suppose that $Q = \sum_{\boldsymbol{k}} a_{\boldsymbol{k}} x^{\boldsymbol{k}} \in \mathbb{Z}[x_1, \cdots, x_n]_0$ is linear in each variable, i.e. has at most degree 1 in x_i for all $1 \leq i \leq n$. Then for each term $T = a_{\boldsymbol{k}} x^{\boldsymbol{k}}$ appearing in Q, T/Q is Gauss.*

Proof. Define the ring of operators $\Theta := \mathbb{Q}[1 - \theta_1, \cdots, 1 - \theta_n] = \mathbb{Q}[\theta_1, \cdots, \theta_n]$. We will first show that for any $D \in \Theta$, DQ/Q is Gauss.

Define $P_n := Q$, and inductively for all $0 \leq l < n$:

$$P_l := (1 - \theta_{l+1})\, P_{l+1}, \qquad R_l := \frac{1}{x_{l+1}} \theta_{l+1} P_{l+1} \tag{82}$$

Then the polynomials P_l, R_l satisfy the conditions in Theorem 6.12 for $k = 0$ and $m_1 = \cdots = m_n = 1$. It follows by the Theorem that $\left((1 - \theta_l)(1 - \theta_{l+1}) \cdots (1 - \theta_n) Q\right)/Q$ is Gauss for any $1 \leq l \leq n$.

Now note that for any $1 \leq i \leq n$ and $0 \leq l \leq n$, $(1 - \theta_i)^2 P_l = 0$, because P_l is linear in x_i. We can run the argument above for any ordering of our variables x_i, so for any $1 \leq i_1, \cdots, i_r \leq n$, we have that $\left((1 - \theta_{i_1}) \cdots (1 - \theta_{i_r}) Q\right)/Q$ is Gauss. We conclude (Proposition 6.4) that DQ/Q is Gauss for any $D \in \Theta$.

It remains to prove the following statement: For any term T occurring in Q, there exists a $D \in \Theta$ such that $DQ = T$. We will show this by induction on the number of variables n in Q.

If $n = 0$ then the statement is clearly true. Suppose that the statement holds in the case of $n - 1$ variables. Since Q is linear in each variable, we can write any term as $T = a x_1^{k_1} \cdots x_n^{k_n}$, where $k_1, \cdots, k_n \in \{0, 1\}$.

If $k_n = 0$, then T appears in P_{n-1}. By our induction hypothesis, there exists a $D_0 \in \mathbb{Q}[\theta_1, \cdots, \theta_{n-1}] \subseteq \Theta$ such that $D_0 P_{n-1} = a x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}$, so choose $D := D_0(1 - \theta_n)$.

If $k_n = 1$, then T appears in $x_n Q_{n-1}$. In this case we find $D_1 \in \Theta$ such that $D_1 Q_{n-1} = a x_1^{k_1} \cdots x_{n-1}^{k_{n-1}}$, so choose $D := D_1 \theta_n$.

In both cases, we end up with $T = DQ$. □

Example 6.15 Suppose that Q is given by $Q = 1 + x + y - xy$. Then Corollary 6.14 tells us that for any polynomial $P \in \mathbb{Z}[x, y]$ such that P is linear in both x and y (i.e. $P = a + bx + cy + dxy$ for $a, b, c, d \in \mathbb{Z}$), P/Q is Gauss. In fact, if one runs the arguments in Theorem 6.12 and 6.14 carefully, one can even obtain that P/Q is strictly Gauss. We will not work this out here, because our main focus will be on rational functions that are not necessarily strictly Gauss. ☆

To return to our problem of determining why $1/f_1$ as given by (78) is Gauss in an elegant way, note that Example 6.15 tells us in particular that $1/(1 + X + Y - XY)$ is Gauss. We would therefore like that the "Gaussness" is not destroyed after substituting $X = x^2 y$ and $Y = xy^2$. This gives rise to Proposition 6.16.

Proposition 6.16 *Let $f = \sum_{\boldsymbol{k} \geq 0} a_{(k_1, \cdots, k_n)} x_1^{k_1} \cdots x_n^{k_n} \in \mathbb{Q}[[x_1, \cdots, x_n]]$ be a power series. Let $K = (k_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix consisting of non-negative integer entries, and suppose that $\det(K) \neq 0$. Then f is Gauss if and only if $g := f\left(t_1^{k_{11}} \cdots t_n^{k_{n1}}, \cdots, t_1^{k_{1n}} \cdots t_n^{k_{nn}}\right)$ is Gauss (seen as a power series in $t_1, \cdots, t_n$).*[2]

Proof. The power series expansion of g looks as follows:

$$g := \sum_{\boldsymbol{k} \geq 0} b_{\boldsymbol{k}} t^{\boldsymbol{k}} = \sum_{\boldsymbol{m} \geq 0} a_{(m_1, \cdots, m_n)} t_1^{m_1 k_{11} + \cdots + m_n k_{1n}} \cdots t_n^{m_1 k_{n1} + \cdots + m_n k_{nn}} = \sum_{\boldsymbol{m} \geq 0} a_{\boldsymbol{m}} t^{K\boldsymbol{m}} \tag{83}$$

" $\Longrightarrow$ " Let p be a prime number such that f is Gauss at p, and such that $p \nmid \det(K)$. Note that, since $\det(K) \neq 0$, almost all prime numbers are of this form, so if we are able to prove that g is Gauss at p then we are done. Let $\boldsymbol{k} \in p\mathbb{Z}_{\geq 0}^n$. If there does not exist an $\boldsymbol{m} \in \mathbb{Z}_{\geq 0}^n$ such that $\boldsymbol{k} = K\boldsymbol{m}$, then $b_{\boldsymbol{k}} = 0$. But then there certainly does not exist an $\boldsymbol{m} \in \mathbb{Z}_{\geq 0}$ such that $\boldsymbol{k}/p = K\boldsymbol{m}$. So we find $b_{\boldsymbol{k}} \equiv 0 \equiv b_{\boldsymbol{k}/p} \pmod{p^{v_p(\boldsymbol{k})}}$.

Now suppose that there does exist an $\boldsymbol{m} \in \mathbb{Z}_{\geq 0}^n$ such that $\boldsymbol{k} = K\boldsymbol{m}$. Denoting by $\operatorname{adj}(K)$ the adjugate of K we obtain:

$$\boldsymbol{m} = \frac{1}{\det(K)} \operatorname{adj}(K)\, \boldsymbol{k} \tag{84}$$

Since $p \nmid \det(K)$, it follows that $v_p(\boldsymbol{k}) \leq v_p(\boldsymbol{m})$. In particular we find $\boldsymbol{m} \in p\mathbb{Z}_{\geq 0}^n$. Since f is Gauss at p, we have $a_{\boldsymbol{m}} \equiv a_{\boldsymbol{m}/p} \pmod{p^{v_p(\boldsymbol{m})}}$. Since $b_{\boldsymbol{k}} = a_{\boldsymbol{m}}$ and $b_{\boldsymbol{k}/p} = a_{\boldsymbol{m}/p}$, we also have $b_{\boldsymbol{k}} \equiv b_{\boldsymbol{k}/p} \pmod{p^{v_p(\boldsymbol{k})}}$. We conclude that g is Gauss at p.

" $\Longleftarrow$ " Since K has integer entries, it follows immediately that for any $\boldsymbol{m} \in \mathbb{Z}_{\geq 0}^n$, $v_p(\boldsymbol{m}) \leq v_p(K\boldsymbol{m})$. So letting p be a prime number at which g is Gauss, $\boldsymbol{m} \in p\mathbb{Z}_{\geq 0}$, $\boldsymbol{k} := K\boldsymbol{m}$, we find that $b_{\boldsymbol{k}} \equiv b_{\boldsymbol{k}/p} \pmod{p^{v_p(\boldsymbol{m})}}$. Since $b_{\boldsymbol{k}} = a_{\boldsymbol{m}}$ and $b_{\boldsymbol{k}/p} = a_{\boldsymbol{m}/p}$, we conclude that f is Gauss at p.

□

[2] To avoid ambiguity: g is obtained from f by substituting $x_i \mapsto t_1^{k_{1i}} \cdots t_n^{k_{ni}}$ for all $1 \leq i \leq n$.

Example 6.17 Define $Q := 1 + X + Y - XY$, and let $a, b, c, d \in \mathbb{Z}_{\geq 0}$ be such that $ad - bc \neq 0$. We saw in Example 6.15 that $1/Q$ is Gauss in the variables X, Y. It follows by Proposition 6.16 that $1/(1 + x^a y^c + x^b y^d - x^{a+b} y^{c+d})$ is Gauss in the variables x, y. In particular, in the case that $a = d = 2$ and $b = c = 1$, we see that $1/f_1$ as given by (78) is Gauss. ☆

This is a good point to look back at our original problem, which was that of giving a characterization of Gaussian rational functions in n variables. We saw that in the case of $n = 1$, under the condition of separable denominator, the rational functions given by Proposition 6.10, i.e. the $\mathbb{Q}$-linear sums of functions of type L, are precisely the ones are Gauss. One might hope that something similar holds in the case of n variables. All the results in this section up to Proposition 6.16 that "produce" Gaussian rational functions, followed from Proposition 6.10. Indeed, all the rational functions that we proved to be Gauss are $\mathbb{Q}$-linear sums of functions of type L. However, Proposition 6.16 gives us a seemingly new method for finding Gaussian rational functions, and the natural question arises: Are these functions actually new? That is, given Proposition 6.10, does Proposition 6.16 "produce" Gaussian rational functions that are not $\mathbb{Q}$-linear sums of functions of type L? The answer to this question is no. We can prove this using Lemma 6.19. Before stating the Lemma, we will need a definition.

Definition 6.18 Let $M = (a_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix with integer entries $a_{i,j} \in \mathbb{Z}$ and let $m \in \{1, \cdots, n\}$. Let $A, B \subseteq \{1, \cdots, n\}$ such that $|A| = |B| = m$. Write $A = \{i_1, \cdots, i_m\}$, $B = \{j_1, \cdots, j_m\}$ such that $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$. We define $M_{(A,B)}$ to be the $m \times m$ submatrix $(a_{i_k, j_l})_{1 \leq k,l \leq m}$ of M. Now define $N := \binom{n}{m}$ and let $\mathcal{A} = \{A_1, \cdots, A_N\}$ be an enumeration of the subsets of $\{1, \cdots, n\}$ that have cardinality m. We define $M_{[\mathcal{A}]} := \left(\det\left(M_{(A_i, A_j)}\right)\right)_{1 \leq i,j \leq N}$ to be the $N \times N$ matrix whose entries are the determinants of all $m \times m$ submatrices of M under the ordering of $\mathcal{A}$. △

Lemma 6.19 *Let M be an $n \times n$ matrix, let $m \in \{1, \cdots, n\}$, let $N := \binom{n}{m}$ and let $\mathcal{A} = \{A_1, \cdots, A_N\}$ be an enumeration of the subsets of $\{1, \cdots, n\}$ of cardinality m. If $\det(M) \neq 0$ then $\det(M_{[\mathcal{A}]}) \neq 0$.*

Proof. Let $\Lambda^m \mathbb{R}^n := \{v_1 \wedge \cdots \wedge v_m \mid v_1, \cdots, v_m \in \mathbb{R}^n\} \subseteq \otimes^m \mathbb{R}^n$ be the m-th exterior power of $\mathbb{R}^n$. Denoting by $\{e_1, \cdots, e_n\}$ the standard basis for $\mathbb{R}^n$, a basis for Λ^m is given by $\mathcal{B} := \{e_{i_1} \wedge \cdots \wedge e_{i_m} \mid 1 \leq i_1 < \cdots < i_m \leq n\}$. We can label the elements $b_1, \cdots, b_N$ of $\mathcal{B}$ with respect to $\mathcal{A}$ in the following way: For $1 \leq j \leq N$, write $A_j = \{i_1, \cdots, i_m\}$ such that $i_1 < \cdots < i_m$, and then define $b_j := e_{i_1} \wedge \cdots \wedge e_{i_m}$. Now let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map corresponding to the matrix A (with respect to the standard basis $\{e_1, \cdots, e_n\}$). Since $\det(M) \neq 0$, T has an inverse T^{-1}. T induces a linear map $\Lambda^m T : \Lambda^m \mathbb{R}^n \to \Lambda^m \mathbb{R}^n$ by setting $\Lambda^m T(v_1 \wedge \cdots \wedge v_m) := Av_1 \wedge \cdots \wedge Av_m$. Now the matrix of $\Lambda^m T$ with respect to the basis $\{b_1, \cdots, b_N\}$ is precisely $M_{[\mathcal{A}]}$. Denoting by $I_N : \Lambda^m \mathbb{R}^n \to \Lambda^m \mathbb{R}^n$ the identity map, we have $I_N = \Lambda^m(TT^{-1}) = \Lambda^m(T)\Lambda^m(T^{-1})$. We conclude that $\Lambda^m T$ is invertible, hence $\det(M_{[\mathcal{A}]}) \neq 0$. □

Proposition 6.20 *Suppose that f is a rational function. Let $K = (k_{i,j})_{1\leq i,j\leq n}$ be an $n \times n$ matrix with non-negative integer entries $k_{i,j} \in \mathbb{Z}_{\geq 0}$ such that $\det(K) \neq 0$. Then f is $\mathbb{Q}$-linear sum of rational functions of type L (in the variables $x_1, \cdots, x_n$) if and only if $g := f\left(t_1^{k_{11}} \cdots t_n^{k_{n1}}, \cdots, t_1^{k_{1n}} \cdots t_n^{k_{nn}}\right)$ is a $\mathbb{Q}$-linear sum of rational functions of type L (in the variables $t_1, \cdots, t_n$).*

Proof. Let $1 \leq m \leq n$, $Q_1, \cdots, Q_n \in \mathbb{Z}[x_1, \cdots, x_n]_0$, let $N := \binom{n}{m}$ and let $\mathcal{A} = \{A_1, \cdots, A_N\}$ be an enumeration of the subsets of $\{1, \cdots, n\}$ of cardinality m. For $1 \leq r \leq N$, write $A_r = \{j_1, \cdots, j_m\}$ such that $j_1 < \cdots < j_m$. We define $J_r^{(x)}$ and $J_r^{(t)}$ by

$$J_r^{(x)} := \frac{x_{j_1} \cdots x_{j_m}}{Q_{j_1} \cdots Q_{j_m}} \left|\frac{\partial(Q_{j_1}, \cdots, Q_{j_m})}{\partial(x_{j_1}, \cdots, x_{j_m})}\right|, \qquad J_r^{(t)} := \frac{t_{j_1} \cdots t_{j_m}}{Q_{j_1} \cdots Q_{j_m}} \left|\frac{\partial(Q_{j_1}, \cdots, Q_{j_m})}{\partial(t_{j_1}, \cdots, t_{j_m})}\right| \tag{85}$$

Note that, by the chain rule, for any $1 \leq r \leq N$ we have

$$J_r^{(t)} = \frac{t_{j_1} \cdots t_{j_m}}{Q_{j_1} \cdots Q_{j_m}} \left|\frac{\partial(Q_{j_1}, \cdots, Q_{j_m})}{\partial(t_{j_1}, \cdots, t_{j_m})}\right| = \frac{t_{j_1} \cdots t_{j_m}}{Q_{j_1} \cdots Q_{j_m}} \left| \begin{pmatrix} \frac{\partial Q_{j_1}}{\partial x_1} & \cdots & \frac{\partial Q_{j_1}}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial Q_{j_m}}{\partial x_1} & \cdots & \frac{\partial Q_{j_m}}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial t_{j_1}} & \cdots & \frac{\partial x_1}{\partial t_{j_m}} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t_{j_1}} & \cdots & \frac{\partial x_n}{\partial t_{j_m}} \end{pmatrix} \right| \tag{86}$$

Using the Cauchy-Binet formula we can work out the determinant of the product of the $m \times n$ and $n \times m$ matrices on the right hand side of (86) (see Definition 6.18). We obtain

$$J_r^{(t)} = \sum_{l=1}^{N} \det\left(K_{(A_r, A_l)}\right) J_l^{(x)} \tag{87}$$

From which it follows that

$$\begin{pmatrix} J_1^{(t)} \\ \vdots \\ J_N^{(t)} \end{pmatrix} = K_{[\mathcal{A}]} \begin{pmatrix} J_1^{(x)} \\ \vdots \\ J_N^{(x)} \end{pmatrix} \tag{88}$$

By Lemma 6.19, $\det\left(K_{[\mathcal{A}]}\right) \neq 0$, so we obtain

$$\begin{pmatrix} J_1^{(x)} \\ \vdots \\ J_N^{(x)} \end{pmatrix} = \frac{1}{\det\left(K_{[\mathcal{A}]}\right)} \mathrm{adj}\left(K_{[\mathcal{A}]}\right) \begin{pmatrix} J_1^{(t)} \\ \vdots \\ J_N^{(t)} \end{pmatrix} \tag{89}$$

We see from (88) and (89) that every $J_r^{(t)}$ can be written as a $\mathbb{Q}$-linear (in fact $\mathbb{Z}$-linear) sum of $J_1^{(x)}, \cdots, J_N^{(x)}$ and every $J_r^{(x)}$ can be written as a $\mathbb{Q}$-linear (in fact $\mathbb{Z}[1/\det(K_{[\mathcal{A}]})]$-linear) sum of $J_1^{(t)}, \cdots, J_N^{(t)}$. The desired result follows. □

6.2 Restricting to subcones and necessity

At this point the only Gaussian rational functions that we know of are given by Proposition 6.10. In an attempt to characterize Gaussian rational functions in n variables, one might try to reduce the problem of determining whether a rational function in n variables is Gauss to a simpler problem; for instance that of determining whether a rational function in $n-1$ variables is Gauss. The motivation behind this approach is, of course, that we have a characterization for Gaussian rational functions in one variable. The following Proposition illustrates a special case of functions for which this approach works.

Proposition 6.21 *Let $P, Q, R \in \mathbb{Z}[x]$ such that Q is separable and $Q(0) \neq 0$. Then $f := P/(Q - yR)$ is Gauss (in the variables x, y) if and only if P/Q is Gauss (in the variable x).*

Proof. " $\Longrightarrow$ "

Suppose that f is Gauss with power series coefficients $a_{\boldsymbol{k}} \in \mathbb{Z}[1/Q(0)]$. Then in particular, for every $m \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{N}$ and almost every p prime

$$a_{(mp^r,0)} \equiv a_{(mp^{r-1},0)} \pmod{p^r} \tag{90}$$

We can see (90) as the congruence conditions "restricted" to the coefficients of our power series corresponding to pure powers of x. Now note that we can expand f as follows:

$$f = \frac{P}{Q - yR} = \frac{P}{Q}\frac{1}{1 - y\frac{R}{Q}} = \frac{P}{Q}\left(1 + y\frac{R}{Q} + y^2\left(\frac{R}{Q}\right)^2 + \cdots\right) \tag{91}$$

In particular, we see that the part of the power series expansion of f containing the pure powers of x is precisely the power series expansion of P/Q. We conclude that P/Q must be Gauss.

" $\Longleftarrow$ "

Suppose that P/Q is Gauss. By Theorem 5.6, we can write

$$\frac{P}{Q} = \left(q_0 + q_1\frac{\theta_x Q_1}{Q_1} + \cdots + q_l\frac{\theta_x Q_l}{Q_l}\right) \tag{92}$$

for rational numbers $q_0, \cdots, q_l \in \mathbb{Q}$ and $Q_1, \cdots, Q_l \in \mathbb{Z}[x]_0$. Define $g := Q - yR$. Note that:

$$f = \frac{P}{g} = \frac{Q}{g}\frac{P}{Q} = \frac{(1-\theta_y)g}{g}\left(q_0 + q_1\frac{\theta_x Q_1}{Q_1} + \cdots + q_l\frac{\theta_x Q_l}{Q_l}\right) \tag{93}$$

It follows from application of Theorem 6.7 that

$$\frac{(\theta_y g)(\theta_x Q_i)}{gQ_i} = \frac{yx}{gQ_i}\left|\frac{\partial(g, Q_i)}{\partial(y, x)}\right| \tag{94}$$

is Gauss for every i. We conclude that every term in the expansion of the right hand side of (93) is Gauss, hence so is f. □

Later on we will be able to prove a stronger version of Proposition 6.21. This will be in the form of Theorem 6.48.

The key to the implication " $\Longrightarrow$ " in Proposition 6.21, was examining the congruences restricted to the "subspace" of $\mathbb{Z}^2_{\geq 0}$ generated by $(1,0)$, corresponding to the pure powers of x in the power series. We were able to "split" the power series into two parts, with the first part containing all the terms with powers of x, and the second part the rest. The first part turned out to be the power series expansion of a rational function (namely P/Q), which we could conclude to be Gauss. The rest of this section will be an attempt to generalize this method.

Definition 6.22 We define the **support** $\operatorname{supp}(f)$ of a power series $f = \sum_{\boldsymbol{k} \geq 0} a_{\boldsymbol{k}} x^{\boldsymbol{k}} \in \mathbb{C}[[x_1, \cdots, x_n]]$ as the powers $\boldsymbol{k}$ of x such that $x^{\boldsymbol{k}}$ has non-zero coefficient:

$$\operatorname{supp}(f) := \{\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0} \mid a_{\boldsymbol{k}} \neq 0\} \tag{95}$$

For a collection of power series $f_1, \cdots, f_l$ we define:

$$\operatorname{supp}(f_1, \cdots, f_l) := \bigcup_{i=1}^{l} \operatorname{supp}(f_i) \tag{96}$$

△

Definition 6.23 We call a subset $X \subseteq \mathbb{R}^n_{\geq 0}$ a **cone** if it is closed under $\mathbb{R}_{\geq 0}$-linear sums. That is, for any $x_1, x_2 \in X$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ we have $\lambda_1 x_1 + \lambda_2 x_2 \in X$. △

Definition 6.24 For a subset $S \subseteq \mathbb{R}^n_{\geq 0}$, we define $\operatorname{Cone}(S)$ to be the cone **generated by** S. That is,

$$\operatorname{Cone}(S) := \bigcup_{m \in \mathbb{N}} \{\lambda_1 a_1 + \cdots + \lambda_m a_m \mid \lambda_1, \cdots, \lambda_m \in \mathbb{R}_{\geq 0}, a_1, \cdots, a_m \in S\} \tag{97}$$

△

Remark 6.25. We can see $\operatorname{Cone}(S)$ as the "smallest cone containing S". In fact, $\operatorname{Cone}(S)$ is the intersection of all cones containing S. ♢

Definition 6.26 We define the **cone of a collection of power series** $f_1, \cdots, f_l \in \mathbb{C}[[x_1, \cdots, x_n]]$, which we will denote by $\operatorname{Cone}(f_1, \cdots, f_l)$, to be the cone generated by $\operatorname{supp}(f_1, \cdots, f_l)$. △

Definition 6.27 We say a subset $L \subseteq \mathbb{R}^n_{\geq 0}$ is **positive-linearly dependent** if there is a $k \in \mathbb{N}$ for which there exist pairwise distinct $a_0, \cdots, a_k \in L$ and (not necessarily distinct) $\lambda_1, \cdots, \lambda_k \in \mathbb{R}_{\geq 0}$ such that

$$a_0 = \lambda_1 a_1 + \cdots + \lambda_k a_k \tag{98}$$

If this is not the case we say that L is **positive-linearly independent**. △

Definition 6.28 Let X be a cone. We say a set $\mathcal{B} \subseteq \mathbb{R}^n_{\geq 0}$ is a **basis** for X if $\mathcal{B}$ generates X (i.e. $X = \operatorname{Cone}(\mathcal{B})$) and $\mathcal{B}$ is positive-linearly independent. △

Lemma 6.29 *Let $S \subseteq \mathbb{R}^n_{\geq 0}$ be a finite subset and let $X := \operatorname{Cone}(S)$. Then there is a basis $\mathcal{B}$ for X such that $\mathcal{B} \subseteq S$.*

Proof. If $S = \emptyset$ then we can take $\mathcal{B} = \emptyset$. So assume that we can write $S = \{a_1, \cdots, a_m\}$ for some $m \in \mathbb{N}$. Now define the set T_i inductively for $0 \leq i \leq m$ in the following way:

$$T_0 := S \tag{99}$$

$$T_i := \begin{cases} T_{i-1} \setminus \{a_i\} & \text{if } a_i \in \text{Cone}\left(T_{i-1} \setminus \{a_i\}\right) \\ T_{i-1} & \text{otherwise} \end{cases} \tag{100}$$

We will show that $\mathcal{B} := T_m$ is a basis for X.

We start by showing that T_m is positive-linearly independent. First of all note that $T_i \subseteq T_{i-1}$ for all $1 \leq i \leq m$, hence $T_m \subseteq T_i$ for all $0 \leq i \leq m$. Now suppose to the contrary that there exist pairwise distinct $t_0, \cdots t_k \in T_m$ and $\lambda_1, \cdots, \lambda_k \in \mathbb{R}_{\geq 0}$ such that $t_0 = \lambda_1 t_1 + \cdots + \lambda_r t_r$. Since $T_m \subseteq T_0 = S$ we have that $t_0 = a_i$ for some $1 \leq i \leq m$. Since $T_m \subseteq T_{i-1}$ and $t_0, \cdots, t_r$ are pairwise distinct, $t_1, \cdots, t_r \subseteq T_{i-1} \setminus \{a_i\}$. It follows that $a_i \in \text{Cone}\left(T_{i-1} \setminus \{a_i\}\right)$, hence $a_i \notin T_i$. Since $T_m \subseteq T_i$ we also have $a_i \notin T_m$. This contradicts our assumption $a_i = t_0 \in T_m$. We conclude that T_m is positive-linearly independent.

Now note that it follows almost directly from the definition that $\text{Cone}(T_i) = \text{Cone}(T_{i-1})$ for all $1 \leq i \leq m$. Namely, if $T_i = T_{i-1}$ then the result is immediate, and if $T_i = T_{i-1} \setminus \{a_i\}$ then $a_i \in \text{Cone}(T_i)$, hence $\text{Cone}(T_{i-1}) \subseteq \text{Cone}(T_i) \subseteq \text{Cone}(T_{i-1})$. Since $\text{Cone}(T_0) = X$, it follows that $X = \text{Cone}(T_m) = \text{Cone}(\mathcal{B})$. We conclude that $\mathcal{B}$ is indeed a basis for X. □

Lemma 6.30 *Let $P, Q \in \mathbb{Z}[x_1, \cdots, x_n]$, $Q(0) \neq 0$. Then* $\text{Cone}(P/Q) \subseteq \text{Cone}(P, Q)$.

Proof. This follows from the standard geometric expansion of $1/Q$. Write $Q = Q(0) - R$ for $R \in \mathbb{Z}[x_1, \cdots, x_n]$. Then

$$\frac{P}{Q} = \frac{P}{Q(0)}\left(1 + \frac{R}{Q(0)} + \left(\frac{R}{Q(0)}\right)^2 + \cdots\right) \tag{101}$$

Since $\text{supp}(R) \subseteq \text{supp}(Q)$ and $Q(0) \neq 0$, we have $\text{Cone}\left(\sum_{k \geq 0} \left(R/Q(0)\right)^k\right) \subseteq \text{Cone}(Q)$. It follows that $\text{supp}(P/Q) \subseteq \text{Cone}(P, Q)$. We conclude that $\text{Cone}(P/Q) \subseteq \text{Cone}(P, Q)$. □

Definition 6.31 Let $X \subseteq \mathbb{R}^n_{\geq 0}$ be a cone. We say that a subset $Y \subseteq X$ is a **subcone** of X if Y is itself a cone. △

Definition 6.32 Let $X \subseteq \mathbb{R}^n_{\geq 0}$ be a cone and let $Y \subseteq X$ be a subcone. We say Y is a **face** of X if it satisfies the following condition: If $x_1, x_2 \in X$ are such that $x_1 + x_2 \in Y$, then $x_1 \in Y$ and $x_2 \in Y$. △

Lemma 6.33 *Let $f \in \mathbb{Q}[[x_1, \cdots, x_n]]$. Let $X \subseteq \mathbb{R}^n_{\geq 0}$ be a cone. Suppose that we can write $f = g + h$ for power series $g, h \in \mathbb{Q}[[x_1, \cdots, x_n]]$ such that* $\text{supp}(g) \subseteq X$ *and* $\text{supp}(h) \cap X = \emptyset$. *Then f is Gauss if and only if both g and h are Gauss.*

Proof. " $\Longleftarrow$ " follows directly from Proposition 6.4.

" $\Longrightarrow$ " Let $a_{\boldsymbol{k}}, b_{\boldsymbol{k}} \in \mathbb{Q}$, $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$ be the power series coefficients of f and g respectively and let $Z := \mathbb{Z}^n_{\geq 0} \cap X$. Since $\text{supp}(g) \cap Z = \emptyset$ and since X is a cone we have for every $\boldsymbol{m} \in Z$, p prime and $r \in \mathbb{Z}_{\geq 0}$, that $a_{\boldsymbol{m}p^r} = b_{\boldsymbol{m}p^r}$. Now if f is Gauss then in particular, for every $\boldsymbol{m} \in Z$, almost every p prime and $r \in \mathbb{N}$,

$a_{\boldsymbol{m}p^r} \equiv a_{\boldsymbol{m}p^{r-1}} \pmod{p^r}$. Since $\mathrm{supp}(g) \subseteq Z$ we conclude that g must be Gauss, hence the same holds for $h = f - g$. □

Theorem 6.34 *Let $P, Q \in \mathbb{Z}[x_1, \cdots, x_n]$, $Q(0) \neq 0$. Let $X \subseteq \mathrm{Cone}(P, Q)$ be a face. Write $P = P_1 + P_2$, $Q = Q_1 - Q_2$ for polynomials $P_1, Q_1, P_2, Q_2 \in \mathbb{Z}[x_1, \cdots, x_n]$ such that $\mathrm{supp}(P_1), \mathrm{supp}(Q_1) \subseteq X$ and $\mathrm{supp}(P_2) \cap X = \mathrm{supp}(Q_2) \cap X = \emptyset$. If P/Q is Gauss then P_1/Q_1 is Gauss.*

Proof. Note that $0 \in X$, hence $Q_1(0) = Q(0) \neq 0$ and $Q_2(0) = 0$. We can expand:

$$\begin{aligned} \frac{P}{Q} &= \frac{P_1 + P_2}{Q_1} \frac{1}{1 - \frac{Q_2}{Q_1}} = \frac{P_1 + P_2}{Q_1} \left[1 + \frac{Q_2}{Q_1} + \left(\frac{Q_2}{Q_1} \right)^2 + \cdots \right] && (102) \\ &= \frac{P_1}{Q_1} \left[1 + \frac{Q_2}{Q_1} + \left(\frac{Q_2}{Q_1} \right)^2 + \cdots \right] + \frac{P_2}{Q_1} \left[1 + \frac{Q_2}{Q_1} + \left(\frac{Q_2}{Q_1} \right)^2 + \cdots \right] && (103) \\ &= \frac{P_1}{Q_1} + (P_2 + Q_2) \left[\frac{1}{Q_1} + \frac{Q_2}{Q_1^2} + \frac{Q_2^2}{Q_1^3} + \cdots \right] && (104) \end{aligned}$$

Now note that $\mathrm{Cone}(Q) = \mathrm{Cone}(Q_1, Q_2)$ and $\mathrm{Cone}(P) = \mathrm{Cone}(P_1, P_2)$. It follows, by Lemma 6.30, that

$$\mathrm{Cone}\left[\frac{1}{Q_1} \sum_{k \geq 0} (Q_2/Q_1)^k \right] \subseteq \mathrm{Cone}(Q_1, Q_2) \subseteq \mathrm{Cone}(P, Q), \quad \mathrm{Cone}(P_2 + Q_2) \subseteq \mathrm{Cone}(P, Q) \tag{105}$$

Therefore, since $\mathrm{supp}(P_2) \cap X = \mathrm{supp}(Q_2) \cap X = \emptyset$ and X is a face, it follows that

$$\mathrm{supp}\left((P_2 + Q_2) \left[\frac{1}{Q_1} \sum_{k \geq 0} (Q_2/Q_1)^k \right] \right) \cap X = \emptyset \tag{106}$$

Also, using Lemma 6.30 again, we have $\mathrm{supp}(P_1/Q_1) \subseteq \mathrm{Cone}(P_1, Q_1) \subseteq X$. We conclude, by Lemma 6.33, that P_1/Q_1 is Gauss. □

Lemma 6.35 *Let $P \in \mathbb{Q}[x_1, \cdots, x_n]$. If P is Gauss then P is constant.*

Proof. Suppose that P is not constant, and write $P = \sum_{\boldsymbol{k} \geq 0} a_{\boldsymbol{k}} x^{\boldsymbol{k}}$ ($a_{\boldsymbol{k}} \in \mathbb{Q}$). Since P is not constant, there exists an $\boldsymbol{m} > 0$ such that $a_{\boldsymbol{m}} \neq 0$. Since P is a polynomial, there exists an $s \in \mathbb{N}$ such that $a_{\boldsymbol{k}} = 0$ for all $|\boldsymbol{k}| \geq s$. Now if p is a prime number greater than the maximum of the numerator of $a_{\boldsymbol{m}}$ and s, then $a_{\boldsymbol{m}} \not\equiv 0 \pmod{p}$ but $a_{\boldsymbol{m}p} \equiv 0 \pmod{p}$. Since this holds for infinitely many primes p, we conclude that P is not Gauss. □

Proposition 6.36 *Let $P, Q \in \mathbb{Z}[x_1, \cdots, x_n]$, $Q(0) \neq 0$. If P/Q is Gauss, then $\mathrm{Cone}(P) \subseteq \mathrm{Cone}(Q)$.*

Proof. By Lemma 6.29, we can choose a basis $\mathcal{B} \subseteq \mathrm{supp}(P, Q)$ for $\mathrm{Cone}(P, Q)$.

Now suppose that $\mathrm{Cone}(P) \not\subseteq \mathrm{Cone}(Q)$. Then $\mathrm{Cone}(Q) \subsetneq \mathrm{Cone}(P, Q)$. It follows that there is a $\boldsymbol{b} \in \mathcal{B}$ such that $\boldsymbol{b} \notin \mathrm{supp}(Q)$. Since $\mathcal{B}$ is positive-linearly independent, $X := \{\lambda \boldsymbol{b} \mid \lambda \in \mathbb{R}_{\geq 0}\}$ is a face of $\mathrm{Cone}(P, Q)$. Writing $P = P_1 + P_2$ and $Q = Q_1 - Q_2$ as in the statement of Theorem 6.34, we have $Q_1 = Q(0)$ and P_1 not constant. According to Lemma 6.35, P_1/Q_1 is not Gauss. Hence by Theorem 6.34, P/Q is also not Gauss. □

Example 6.37 Define $P := xy - 5x^2y^2 + 7x^3y - 2x^2y^4$ and $Q := 1 + xy^2 - x^3y$ and consider the figure below.

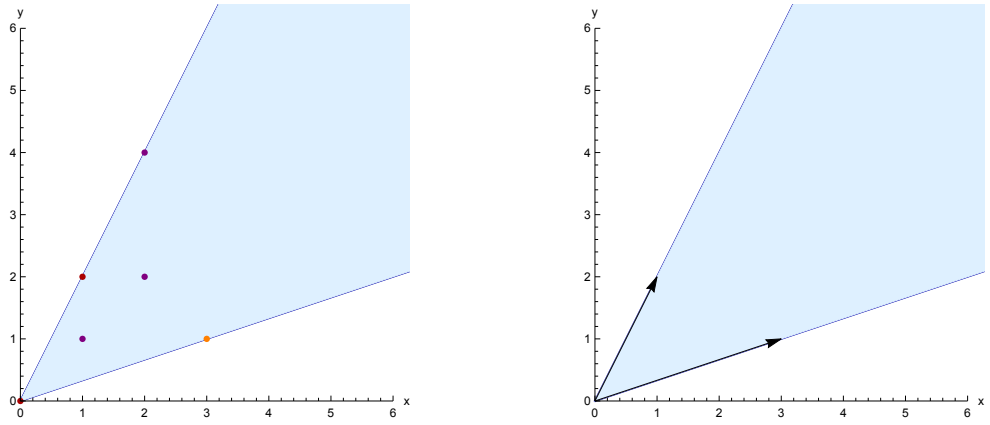

Figure 1: Visual representation of a cone in $\mathbb{R}^2_{\geq 0}$

The area in blue represents $X := \text{Cone}(P, Q)$. We see that a basis for X is given by $\{(3, 1), (1, 2)\}$. Now suppose that P/Q is Gauss. By using Theorem 6.34 for the faces generated by $(3, 1)$ and $(1, 2)$ respectively, we find that f_1 and f_2 as given by

$$f_1 := \frac{7x^3y}{1 - x^3y}, \qquad f_2 := \frac{-2x^2y^4}{1 + xy^2} \tag{107}$$

must both be Gauss. That f_1 is Gauss follows directly from Proposition 6.10 by noting that $\theta_y(1 - x^3y) = x^3y$. However, f_2 is not Gauss, which follows from Proposition 6.16 with the substitution $X = xy^2$, $Y = y$ combined with Remark 5.7. We conclude that P/Q is not Gauss.

☆

6.3 Characterization for a special type of rational functions

In this subsection we will apply our results to characterize Gaussian rational functions in n variables in a special case, the main result being Theorem 6.48. However, the theory have developed and will develop is useful in its own right and will possibly have wider applications than is illustrated by Theorem 6.48. We will start by giving a useful condition for determining whether a rational function is Gauss.

Definition 6.38 For $m \in \mathbb{N}$, we denote by $\zeta_m := e^{2\pi i/m}$ the standard m-th root of unity. △

Lemma 6.39 *Let $m \in \mathbb{N}$ and $l \in \mathbb{Z}$. Then*

$$\sum_{k=0}^{m-1} \zeta_m^{kl} = \begin{cases} m & \text{if} \quad m \mid l \\ 0 & \text{if} \quad m \nmid l \end{cases} \tag{108}$$

Proof. Suppose first that $m \mid l$. Then $\zeta_m^l = 1$, hence the sum indeed equals m.

If $m \nmid l$, then $\zeta_m^l \neq 1$. Now note that

$$\left(\zeta_m^l - 1\right) \sum_{k=0}^{m-1} \zeta_m^{kl} = \left(\zeta_m^l - 1\right) \left(1 + \zeta_m^l + \cdots + \zeta_m^{l(m-1)}\right) = \zeta_m^{ml} - 1 = 0 \tag{109}$$

So in this case the desired sum indeed equals 0. □

Definition 6.40 Let p be a prime number and let $r \in \mathbb{N}$.

We say that a rational function $f = P/Q$ (written in reduced form such that $P, Q \in \mathbb{Z}[x_1, \cdots, x_n]$) is congruent to 0 modulo p^r, written $f \equiv 0 \pmod{p^r}$, if there exists an $R \in \mathbb{Z}[x_1, \cdots, x_n]$ such that $P = p^r R$.

We say that a power series $g = \sum_{\boldsymbol{k} \geq 0} a_{\boldsymbol{k}} x^{\boldsymbol{k}} \in \mathbb{Q}[x_1, \cdots, x_n]$ is congruent to 0 modulo p^r (written $g \equiv 0 \pmod{p^r}$) if $a_{\boldsymbol{k}} \equiv 0 \pmod{p^r}$ for every $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$. △

The following Lemma shows that Definition 6.40 makes sense, i.e. the definitions for congruence modulo a prime power for rational functions and power series (almost) coincide.

Lemma 6.41 *Let $f = P/Q$ be a rational function in n variables, $Q(0) \neq 0$. Write the power series expansion of f as $\sum_{\boldsymbol{k} \geq 0} a_{\boldsymbol{k}} x^{\boldsymbol{k}}$, $a_{\boldsymbol{k}} \in \mathbb{Z}[1/Q(0)]$. Let p be a prime number such that $p \nmid Q(0)$ and let $r \in \mathbb{N}$. Then $f \equiv 0 \pmod{p^r}$ (seen as a rational function) if and only if $a_{\boldsymbol{k}} \equiv 0 \pmod{p^r}$ for all $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$.*

Proof. If $f \equiv 0 \pmod{p^r}$ then there exists a $P_2 \in \mathbb{Z}[x_1, \cdots, x_n]$ such that $p^r P_2 = P$. Now P_2/Q has a power series expansion $\sum_{\boldsymbol{k} \geq 0} b_{\boldsymbol{k}} x^{\boldsymbol{k}}$, with coefficients $b_{\boldsymbol{k}} \in \mathbb{Z}[1/Q(0)]$. We have $a_{\boldsymbol{k}} = p^r b_{\boldsymbol{k}}$ for all $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$, hence $a_{\boldsymbol{k}} \equiv 0 \pmod{p^r}$ for all $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$.

Conversely, if $a_{\boldsymbol{k}} \equiv 0 \pmod{p^r}$ for all $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$, then $b_{\boldsymbol{k}} := a_{\boldsymbol{k}}/p^r \in \mathbb{Z}[1/Q(0)]$ for all $\boldsymbol{k} \in \mathbb{Z}^n_{\geq 0}$. We now have

$$P = p^r Q \sum_{\boldsymbol{k} \geq 0} b_{\boldsymbol{k}} x^{\boldsymbol{k}} \tag{110}$$

It follows that

$$P/p^r = Q\sum_{\boldsymbol{k}\geq 0} b_{\boldsymbol{k}}x^{\boldsymbol{k}} \in \mathbb{Z}[1/Q(0)][[x_1,\cdots,x_n]] \cap \mathbb{Z}[1/p][x_1,\cdots,x_n] = \mathbb{Z}[x_1,\cdots,z_n] \tag{111}$$

with the last equality by our assumption $p \nmid Q(0)$. □

Definition 6.42 Let p be a prime number and $r \in \mathbb{Z}_{\geq 0}$. We define the operator $\mathcal{A}_{p^r} : \mathbb{C}[[x_1,\cdots,x_n]] \to \mathbb{C}[[x_1,\cdots,x_n]]$ by

$$\mathcal{A}_{p^r} f(x_1,\cdots,x_n) := \frac{1}{p^{nr}} \sum_{l_1=0}^{p^r-1} \cdots \sum_{l_n=0}^{p^r-1} f(\zeta_{p^r}^{l_1}x_1,\cdots,\zeta_{p^r}^{l_n}x_n) \tag{112}$$

The operator $\mathcal{H}_p : \mathbb{C}[[x_1,\cdots,x_n]] \to \mathbb{C}[[x_1,\cdots,x_n]]$ by

$$\mathcal{H}_p f(x_1,\cdots,x_n) := f(x_1^p,\cdots,x_n^p) \tag{113}$$

And $T_{p^r} : \mathbb{C}[[x_1,\cdots,x_n]] \to \mathbb{C}[[x_1,\cdots,x_n]]$ by:

$$T_{p^r} := \mathcal{A}_{p^r} - \mathcal{H}_p\mathcal{A}_{p^{r-1}} \tag{114}$$

△

Lemma 6.43 *Let $f = P/Q \in \mathbb{Q}(x_1,\cdots,x_n)$ be a rational function, p a prime number and $r \in \mathbb{N}$. Then $\mathcal{A}_{p^r} f \in \mathbb{Q}(x_1,\cdots,x_n)$, $\mathcal{H}_p f \in \mathbb{Q}(x_1,\cdots,x_n)$ and $T_{p^r} f \in \mathbb{Q}(x_1,\cdots,x_n)$.*

Proof. Let everything as above.

It follows directly from (113) that $\mathcal{H}_p f \in \mathbb{Q}(x_1,\cdots,x_n)$.

Define $M := p^{nr}$. We can write (see (112))

$$\mathcal{A}_{p^r} f = \frac{1}{M}\sum_{i=1}^{M} \frac{P_i}{Q_i} = \frac{1}{M}\frac{\sum_{i=1}^{M}\prod_{j\neq i} P_iQ_j}{\prod_{i=1}^{M} Q_i} \tag{115}$$

With $P_i = P(\zeta_{p^r}^{l(i)_1}x_1,\cdots,\zeta_{p^r}^{l(i)_n}x_n)$ and $Q_i = Q(\zeta_{p^r}^{l(i)_1}x_1,\cdots,\zeta_{p^r}^{l(i)_n}x_n)$ for some enumeration (i.e. bijection) $\boldsymbol{l} : \{1,\cdots,M\} \to \{0,\cdots,p^r-1\}^n$. Note that the P_i and Q_i are elements of $\mathbb{Z}[\zeta_{p^r}][x_1,\cdots,x_n]$, hence the same holds for $g := \sum_{i=1}^{M}\prod_{j\neq i} P_iQ_j$ and $h := M\prod_{i=1}^{M} Q_i$. Now let G be the Galois group of the field extension $\mathbb{Q}(\zeta_{p^r})/\mathbb{Q}$ and let $\sigma \in G$. σ induces a permutation on $\{\zeta_{p^r},\cdots,\zeta_{p^r}^{p^r-1}\}$, which in turn induces a permutation on the P_i and Q_i ($P_i \mapsto P_{\sigma(i)}$, $Q_i \mapsto Q_{\sigma(i)}$). It follows that σ fixes both g and h. Therefore $g \in \mathbb{Z}[x_1,\cdots,x_n]$ and $h \in \mathbb{Z}[x_1,\cdots,x_n]$. We conclude that $\mathcal{A}_{p^r} f = g/h$ is indeed an element of $\mathbb{Q}(x_1,\cdots,x_n)$.

That $T_{p^r} f \in \mathbb{Q}(x_1,\cdots,x_n)$ now follows from the results for $\mathcal{A}_{p^r}$ and $\mathcal{H}_p$ shown above.[3] □

Proposition 6.44 *Let $P, Q \in \mathbb{Z}[x_1,\cdots,x_n]$, $Q(0) \neq 0$, $f := P/Q$. Then f is Gauss if and only if for almost every p prime and every $r \in \mathbb{N}$, $T_{p^r} f \equiv 0 \pmod{p^r}$ (seen as a rational function).*

[3]Strictly speaking, we have not yet proven the result for T_{p^r} if $r = 1$, since this involves $\mathcal{A}_{p^0}$. However, $\mathcal{A}_{p^0}$ is simply the identity operator, so the desired result still holds in this case.

Proof. Note that it follows from Lemma 6.43 that $T_{p^r} f \in \mathbb{Q}(x_1, \cdots, x_n)$.

Let $\sum_{\boldsymbol{k} \geq 0} a_{\boldsymbol{k}} x^{\boldsymbol{k}}$ be the power series expansion of f. Let p be a prime number and let $r \in \mathbb{N}$. We will first show how $\mathcal{A}_{p^r}$ acts on the power series expansion of f. Let $\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^n$ be given. Then, by Lemma 6.39,

$$\frac{1}{p^{nr}} \sum_{l_1=0}^{p^r-1} \cdots \sum_{l_n=0}^{p^r-1} a_{\boldsymbol{k}} \left(\zeta_{p^r}^{l_1} x_1\right)^{k_1} \cdots \left(\zeta_{p^r}^{l_n} x_n\right)^{k_n} = \begin{cases} a_{\boldsymbol{k}} x^{\boldsymbol{k}} & \text{if } p^r \mid \boldsymbol{k} \\ 0 & \text{if } p^r \nmid \boldsymbol{k} \end{cases} \tag{116}$$

Therefore

$$\mathcal{A}_{p^r} f = \sum_{\boldsymbol{k} \in p^r \mathbb{Z}_{\geq 0}^n} a_{\boldsymbol{k}} x^{\boldsymbol{k}} \qquad \text{and} \qquad \mathcal{H}_p \mathcal{A}_{p^{r-1}} f = \sum_{\boldsymbol{k} \in p^{r-1} \mathbb{Z}_{\geq 0}^n} a_{\boldsymbol{k}} x^{p\boldsymbol{k}} = \sum_{\boldsymbol{k} \in p^r \mathbb{Z}_{\geq 0}^n} a_{\boldsymbol{k}/p} x^{\boldsymbol{k}} \tag{117}$$

Hence we have

$$T_{p^r} f = \sum_{\boldsymbol{k} \in p^r \mathbb{Z}_{\geq 0}^n} \left(a_{\boldsymbol{k}} - a_{\boldsymbol{k}/p}\right) x^{\boldsymbol{k}} \tag{118}$$

So we see that f is Gauss at p if and only if for every $r \in \mathbb{N}$, $T_{p^r} f \equiv 0 \pmod{p^r}$ (seen as a power series). Note also that $T_{p^r} f \in \mathbb{Z}[1/Q(0)][[x_1, \cdots, x_n]]$, since $a_{\boldsymbol{k}} \in \mathbb{Z}[1/Q(0)]$ for all $\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^n$. The desired result now follows from Lemma 6.41. □

Definition 6.45 Let $P \in \mathbb{Z}[x_1, \cdots, x_n]$ and let $1 \leq i \leq n$. Let $d := \deg_{x_i} P$ be the degree of P in the variable x_i. We define $P^{x_i *} := x_i^d P(x_1, \cdots, x_{i-1}, \frac{1}{x_i}, x_{i+1}, \cdots, x_n)$. That is, if we view P as a polynomial in x_i with coefficients in $\mathbb{Z}[x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n]$, then $P^{x_i *} = P^*$. △

Lemma 6.46 *Let $f \in \mathbb{Q}(x_1, \cdots, x_n)$, let $1 \leq i \leq n$, let p be a prime number and let $r \in \mathbb{N}$. Then $f \equiv 0 \pmod{p^r}$ if and only if $g := f(x_1, \cdots, x_{i-1}, 1/x_i, x_{i+1}, \cdots, x_n) \equiv 0 \pmod{p^r}$.*

Proof. Write $f = P/Q$ in reduced form. Then $g = x^k P^{x_i *}/Q^{x_i *}$ for some $k \in \mathbb{Z}$ ($k = \deg_{x_i} Q - \deg_{x_i} P$). Note that it follows almost directly from Definition 6.45 that $p^r \mid P$ if and only if $p^r \mid P^{x_i *}$ (and similarly for Q). Therefore

$$f \equiv 0 \pmod{p^r} \iff \frac{P^{x_i *}}{Q^{x_i *}} \equiv 0 \pmod{p^r} \iff g \equiv 0 \pmod{p^r} \tag{119}$$

□

Proposition 6.47 *Let $P, Q \in \mathbb{Z}[x_1, \cdots, x_n]$, $Q(0) \neq 0$, $f := P/Q$, $1 \leq i \leq n$. Define the rational function $g := f(x_1, \cdots, x_{i-1}, 1/x_i, x_{i+1}, \cdots, x_n)$ and suppose that g has a power series expansion (around 0). Then f is Gauss if and only if g is Gauss.*

Proof. Let p be a prime number and $r \in \mathbb{N}$. By Lemma 6.46, $T_{p^r} f \equiv 0 \pmod{p^r}$ if and only if

$$T_{p^r} g = (T_{p^r} f)\left(x_1, \cdots, x_{i-1}, \frac{1}{x_i}, x_{i+1}, \cdots, x_n\right) \equiv 0 \pmod{p^r} \tag{120}$$

In the (first) equality we implicitly used that the substitution $x_i \mapsto 1/x_i$ commutes with T_{p^r}, which can be seen directly from (112) and (113). We conclude, by Proposition 6.44, that f is Gauss if and only if g is Gauss. □

Theorem 6.48 *Let $m_1, \cdots, m_n \in \mathbb{N}$ and let $P_0, Q_0, \cdots, P_n, Q_n \in \mathbb{Z}[x_0]$ such that Q_i is separable and $Q_i(0) \neq 0$ for all $0 \leq i \leq n$. Define $P := P_0 + x_1^{m_1} P_1 + \cdots + x_n^{m_n} P_n$, $Q := Q_0 + x_1^{m_1} Q_1 + \cdots + x_n^{m_n} Q_n$. Then $f := P/Q$ is Gauss (in the variables $x_0, \cdots, x_n$) if and only if for every $0 \leq i \leq n$, P_i/Q_i is Gauss.*

Proof. " $\Longrightarrow$ " Suppose that f is Gauss. Applying Theorem 6.34 to f with the face

$$X_0 := \{(\lambda, 0, \cdots, 0) \in \mathbb{R}_{\geq 0}^{n+1} \mid \lambda \in \mathbb{R}_{\geq 0}\} \cap \mathrm{Cone}(P, Q) \tag{121}$$

of $\mathrm{Cone}(P, Q)$, we find that P_0/Q_0 must be Gauss.

Now define, for every $1 \leq i \leq n$,

$$\begin{aligned} g_i &:= f(x_0, x_1, \cdots, x_{i-1}, 1/x_i, x_{i+1}, \cdots, x_n) && (122) \\ &= \frac{x_i^{m_i} P_0 + \cdots + x_i^{m_i} x_{i-1}^{m_{i-1}} P_{i-1} + P_i + x_i^{m_i} x_{i+1}^{m_{i+1}} P_{i+1} + \cdots + x_i^{m_i} x_n^{m_n} P_n}{x_i^{m_i} Q_0 + \cdots + x_i^{m_i} x_{i-1}^{m_{i-1}} Q_{i-1} + Q_i + x_i^{m_i} x_{i+1}^{m_{i+1}} Q_{i+1} + \cdots + x_i^{m_i} x_n^{m_n} Q_n} && (123) \end{aligned}$$

Then by Proposition 6.47, g_i is Gauss for every $1 \leq i \leq n$.

By applying Theorem 6.34 to g_i with the face

$$X_i := \{(\lambda, 0, \cdots, 0) \in \mathbb{R}_{\geq 0}^{n+1} \mid \lambda \in \mathbb{R}_{\geq 0}\} \cap \mathrm{Cone}(P_i, Q_i) \tag{124}$$

of $\mathrm{Cone}(P_i, Q_i)$, we find that P_i/Q_i is Gauss for every $1 \leq i \leq n$.

" $\Longleftarrow$ " Suppose that for every $0 \leq i \leq n$, P_i/Q_i is Gauss. Define $h := Q_0 + x_1^{m_1} Q_1 + \cdots + x_n^{m_n} Q_n$. In order to show that f is Gauss, we will show that P_0/h is Gauss and that $x_i^{m_i} P_i/h$ is Gauss for every $1 \leq i \leq n$.

By Theorem 5.6, we can write

$$\frac{P_i}{Q_i} = q_0^{(i)} + q_1^{(i)} \frac{\theta_0 Q_1^{(i)}}{Q_1^{(i)}} + \cdots + q_l^{(i)} \frac{\theta_0 Q_l^{(i)}}{Q_l^{(i)}} \tag{125}$$

for $q_j^{(i)} \in \mathbb{Q}$, $Q_j^{(i)} \in \mathbb{Z}[x_0]$ and sufficiently large $l \in \mathbb{N}$.

Now note that

$$\frac{P_0}{h} = \frac{\left(1 - \frac{\theta_1}{m_1}\right) \cdots \left(1 - \frac{\theta_n}{m_n}\right) h}{h} \left(q_0^{(0)} + q_1^{(0)} \frac{\theta_0 Q_1^{(0)}}{Q_1^{(0)}} + \cdots + q_l^{(0)} \frac{\theta_0 Q_l^{(0)}}{Q_l^{(0)}} \right) \tag{126}$$

By application of Theorem 6.12, and by noting that $Q_j^{(0)}$ does not depend on $x_1, \cdots, x_n$, it follows that each term of the right hand side of (126) is Gauss. Hence the same holds for their sum P_0/h.

Now note that, for every $1 \leq i \leq n$,

$$\frac{x_i^{m_1} P_i}{h} = \frac{Q_i}{h} \frac{P_i}{Q_i} = \frac{\frac{\theta_i}{m_i} h}{h} \left(q_0^{(i)} + q_1^{(i)} \frac{\theta_0 Q_1^{(i)}}{Q_1^{(i)}} + \cdots + q_l^{(i)} \frac{\theta_0 Q_l^{(i)}}{Q_l^{(i)}} \right) \tag{127}$$

Which is Gauss by the same reasoning as for P_0/h. □

Definition 6.49 We say an operator $S : \mathbb{Q}(x_1, \cdots, x_n) \to \mathbb{Q}(x_1, \cdots, x_n)$ is **respectful** if for almost every prime number p, for every $r \in \mathbb{N}$ and for every $f \in \mathbb{Q}(x_1, \cdots, x_n)$

$$T_{p^r} S f = S T_{p^r} f, \qquad \text{and} \qquad f \equiv 0 \pmod{p^r} \iff Sf \equiv 0 \pmod{p^r} \tag{128}$$

△

Proposition 6.50 *Let $f \in \mathbb{Q}(x_1, \cdots, x_n)$ and let $S : \mathbb{Q}(x_1, \cdots, x_n) \to \mathbb{Q}(x_1, \cdots, x_n)$ be a respectful operator. Suppose that both f and Sf have a power series expansion around 0 (i.e. the constant term of the numerator of both f and Sf is non-zero). Then f is Gauss if and only if Sf is Gauss.*

Proof. Since S is respectful, for almost every prime number p and every $r \in \mathbb{N}$ we have that

$$T_{p^r} f \equiv 0 \pmod{p^r} \iff S T_{p^r} f \equiv 0 \pmod{p^r} \iff T_{p^r} S f \equiv 0 \pmod{p^r} \tag{129}$$

The desired result now follows from Proposition 6.44. □

The following Proposition is a stronger version of Proposition 6.16 and, using Proposition 6.50, also has Proposition 6.47 as an immediate Corollary.

Proposition 6.51 *Let $K = (k_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix with (not necessarily non-negative) integer entries, and suppose that $\det(K) \neq 0$. Then the operator $S_K : \mathbb{Q}(x_1, \cdots, x_n) \to \mathbb{Q}(x_1, \cdots, x_n)$, defined by $S_K f(x_1, \cdots, x_n) := f(x_1^{k_{11}} \cdots x_n^{k_{n1}}, \cdots, x_1^{k_{1n}} \cdots x_n^{k_{nn}})$, is respectful.*

Proof. We will first show that S_K commutes with T_{p^r} (for all $r \in \mathbb{N}$) for all prime numbers p such that $p \nmid \det(K)$. Since we assume $\det(K) \neq 0$, this is satisfied by almost every prime number.

Let p be a prime number such that $p \nmid \det(K)$ and let $r \in \mathbb{Z}_{\geq 0}$. It is immediate that S_K commutes with $\mathcal{H}_p$, so it remains to show that S_K commutes with $\mathcal{A}_{p^r}$. Let $f \in \mathbb{Q}(x_1, \cdots, x_n)$. Define $g_1 := \mathcal{A}_{p^r} S_K f(x_1, \cdots, x_n)$ and $g_2 := S_K \mathcal{A}_{p^r} f(x_1, \cdots, x_n)$. We see that

$$\begin{aligned} g_1 &= \frac{1}{p^{nr}} \sum_{l_1=0}^{p^r-1} \cdots \sum_{l_n=0}^{p^r-1} f\left(\left(\zeta_{p^r}^{l_1} x_1\right)^{k_{11}} \cdots \left(\zeta_{p^r}^{l_n} x_n\right)^{k_{n1}}, \cdots, \left(\zeta_{p^r}^{l_1} x_1\right)^{k_{n1}} \cdots \left(\zeta_{p^r}^{l_n} x_n\right)^{k_{nn}} \right) && (130) \\ &= \frac{1}{p^{nr}} \sum_{l_1=0}^{p^r-1} \cdots \sum_{l_n=0}^{p^r-1} f\left(\zeta_{p^r}^{l_1 k_{11} + \cdots + l_n k_{n1}} x_1^{k_{11}} \cdots x_n^{k_{n1}}, \cdots, \zeta_{p^r}^{l_1 k_{1n} + \cdots + l_n k_{nn}} x_1^{k_{1n}} \cdots x_n^{k_{nn}} \right) && (131) \end{aligned}$$

Now note that, for any $\boldsymbol{k}_1, \boldsymbol{k}_2 \in \mathbb{Z}_{\geq 0}^n$ such that $K^T \boldsymbol{k}_1 = \boldsymbol{k}_2$,

$$\boldsymbol{k}_1 = \frac{1}{\det(K^T)} \operatorname{adj}\left(K^T\right) \boldsymbol{k}_2 = \frac{1}{\det(K)} \operatorname{adj}\left(K^T\right) \boldsymbol{k}_2 \tag{132}$$

Since $p \nmid \det(K)$, we find $v_p(\boldsymbol{k}_1) \geq v_p(\boldsymbol{k}_2)$. It follows in particular for any $\boldsymbol{l} = (l_1, \cdots, l_n) \in \mathbb{Z}_{\geq 0}^n$ that

$$K^T \boldsymbol{l} \equiv 0 \pmod{p^r} \implies \boldsymbol{l} \equiv 0 \pmod{p^r} \tag{133}$$

Therefore, looking back at (131):

$$g_1 = \frac{1}{p^{nr}} \sum_{l_1=0}^{p^r-1} \cdots \sum_{l_n=0}^{p^r-1} f\left(\zeta_{p^r}^{l_1} x_1^{k_{11}} \cdots x_n^{k_{n1}}, \cdots, \zeta_{p^r}^{l_n} x_1^{k_{1n}} \cdots x_n^{k_{nn}} \right) = g_2 \tag{134}$$

Hence S_K indeed commutes with A_{p^r}.

Now let p be an arbitrary prime number, $r \in \mathbb{N}$ and again $f \in \mathbb{Q}(x_1, \cdots, x_n)$. Write $f = P/Q$ for $P, Q \in \mathbb{Z}[x_1, \cdots, x_n]$ in reduced form. There exist $\boldsymbol{k} = \{k_1, \cdots, k_n) \in \mathbb{Z}^n_{\geq 0}$ and $\boldsymbol{l} = \{l_1, \cdots, l_n) \in \mathbb{Z}^n_{\geq 0}$ such that $P' := x_1^{k_1} \cdots x_n^{k_n} S_K P \in \mathbb{Z}[x_1, \cdots, x_n]$ and $Q' := x_1^{l_1} \cdots x_n^{l_n} S_K Q \in \mathbb{Z}[x_1, \cdots, x_n]$. Since $\det(K) \neq 0$, the elements of $\mathrm{supp}(P)$ and $\mathrm{supp}(P')$ are in one-to-one correspondence. Therefore $p^r \mid P$ if and only if $p^r \mid P'$. By the same argument, $p^r \mid Q$ if and only if $p^r \mid Q'$. It follows that

$$S_K f = \frac{P'}{Q'} \frac{x_1^{l_1} \cdots x_n^{l_n}}{x_1^{k_1} \cdots x_n^{k_n}} \equiv 0 \pmod{p^r} \iff f = \frac{P}{Q} \equiv 0 \pmod{p^r} \tag{135}$$

We conclude that S_K is respectful. □

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