The McKay correspondence and resolving the Kleinian singularities

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Abstract

In this thesis we find and blow up the Kleinian singularities and show their relationship with Dynkin diagrams of type ADE. Furthermore, we construct the McKay graphs of the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ and show their connection with the ADE type Dynkin diagrams and hence the Kleinian singularities. The purpose of this thesis is to establish all of the above in great detail and to work out the specifics.

Acknowledgments

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1 Introduction

1.1 Historical note

In 1979, John McKay [McK80] outlined a remarkable correspondence between the theory of Kleinian singularities, the conjugacy classes of finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ and the Dynkin diagrams which show up in the ADE classification of simple Lie algebras. The recurring element in each of the branches of the correspondence is a relationship with the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$. These subgroups are related to the platonic solids and have been studied since antiquity.
Felix Klein characterized the structure of the quotient space $\mathbb{C}^2/\Gamma$ for each finite subgroup Γ of $\mathrm{SL}(2,\mathbb{C})$ in 1884 [Kle84]. By invariant theory, we can obtain varieties in $\mathbb{C}^3$ with an isolated singularity at the origin. These singularities have been aptly named the *Kleinian singularities*. The Kleinian singularities appear throughout the classification of surfaces and other areas of geometry. The connection between the Kleinian singularities and Dynkin diagrams was shown by Patrick Du Val in 1934 [DV34]. Du Val showed that resolving the Kleinian singularities by means of blowing up yields an exceptional divisor that can be converted to a Dynkin diagram of type ADE.
In 1979 McKay used representation theory to construct a graph called the *McKay graph* of the binary polyhedral groups. He observed the connection between these McKay graphs and the (extended) Dynkin diagrams of type ADE.
Since Du Val, further connections have been discovered by Brieskorn, Kostant and Steinberg, Grothendieck, Kronheimer and more.

1.2 Motivation and organization of this thesis

The scope of the McKay correspondence and the diversity of topics it connects make it an extremely interesting topic for a thesis. We will show the existence of the McKay correspondence and we will also mirror the work of Du Val by resolving the Kleinian singularities and construct the intersection diagrams.
In addition to obtaining the above correspondences, this thesis aims to introduce many advanced topics to bachelor students and it was written with this audience in mind. Consequently we will not introduce any theory concerning elementary algebra but we will develop all the necessary algebraic geometry, invariant theory and representation theory.
While introducing the necessary preliminary knowledge, we will state the relevant theorems and definitions without proof. If the reader is interested in those, we refer to the books where these theorems are proved. For the reader who is well versed in these topics, it should be possible to skip sections at their leisure since we have attempted to stick to conventional terminology and notation.
We have opted for a roughly chronological approach to this subject. Firstly, we characterize the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ and use invariant theory to find the Kleinian singularities. Then we will introduce the basics on algebraic geometry and introduce

blowups. This gives us enough knowledge to resolve the Kleinian singularities and obtain the intersection diagrams. This section constitutes a major part of this thesis due to the difficulties that arise when one attempts to do this in detail. However, the reader could ignore the details of the calculations and only look at the resulting diagrams to understand the correspondence. Afterwards we briefly introduce representation theory and construct the McKay graphs to complete the McKay correspondence.

1.3 Outline of the main results

Klein showed that the quotient space $\mathbb{C}^2/\Gamma$ for finite subgroups Γ of $\mathrm{SL}(2,\mathbb{C})$ is isomorphic to some surface in $\mathbb{C}^3$ defined by a single polynomial. The corresponding surfaces have (isolated) singularities at the origin. The defining polynomials are

Group	**ADE-classification**	**Defining polynomial**
Cyclic ($\mathbb{Z}_{n+1}$)	A_n	$xy - z^{n+1}$
Binary Dihedral (BD_{4n})	D_{n+2}	$x^2 + zy^2 + z^{n+1}$
Binary Tetrahedral (BT_{24})	E_6	$x^4 + y^3 + z^2$
Binary Octahedral (BO_{48})	E_7	$x^2 + y^3 + yz^3$
Binary Icosahedral (BI_{120})	E_8	$x^5 + y^3 + z^2$

Resolving these singularities involves locally blowing up the surfaces to obtain a smooth surface locally inside $\mathbb{C}^3 \times \mathbb{P}^2$. The exceptional divisor consists of projective lines that emerge from the singular point itself. These projective lines meet transversally and the graph whose vertices correspond to the irreducible components of the exceptional divisor (the projective lines), with two vertices joined if the lines intersect, is a Dynkin diagram of type ADE.

Lastly, if we find the irreducible representations of the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$, we can use the character tables to construct the McKay graph. These graphs also correspond to the Dynkin diagrams of type ADE.

2 The subgroups of $\mathrm{SL}(2,\mathbb{C})$

At the basis of this thesis lie the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$, which are conjugate to the finite subgroups of $\mathrm{SU}(2)$ [Boc, page 4]. Hence, we will start by finding the generators of these subgroups. To do so, we look at a homomorphism $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ and its kernel. We repeat the argument made in [Arm88, Chapter 19] for finding the finite subgroups of $\mathrm{SO}(3,\mathbb{R})$. When we have those, we can lift them to the finite subgroups of $\mathrm{SU}(2)$.

2.1 A homomorphism on $\mathrm{SU}(2)$

As mentioned, the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ are conjugate to the finite subgroups of $\mathrm{SU}(2)$. Recall that the special unitary group is defined as follows (A^* denotes the conjugate transpose)

$$\mathrm{SU}(2) = \{A \in \mathrm{GL}(2,\mathbb{C}) \mid AA^* = A^*A = I,\ \det(A) = 1\}$$

Suppose we have an element of the form $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. From $U^*U = I$ follows that $|a|^2 + |c|^2 = 1$, $|b|^2 + |d|^2 = 1$ and $\overline{a}b + \overline{c}d = 0$. So we only need to consider values for a, b, c, d on the unit disk in $\mathbb{C}$. From $\det(U) = 1$ we get $ad - bc = 1$. If $b = 0$, we get $|d|^2 = 1$ so $ad = 1 \leftrightarrow a = \overline{d}$. Since $ad = 1$ we get $d \neq 0$ and thus $c = 0$. So we get an element of the form $\begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}$. If $b \neq 0$, we get $a = -c\overline{d}/\overline{b}$ and $|a|^2 + |c|^2 = |c|^2 \left(\frac{|d|^2}{|b|^2} + 1\right) = 1$. Hence $|c| = |b|$ and $|a| = |d|$.

We can pick $a = e^{i\alpha}\cos(x)$ for some $x \in \mathbb{R}$ since a lies on the unit disk. Since $|a|^2 + |c|^2 = 1$, this fixes $c = e^{i\gamma}\sin(x)$. Similarly, we can pick $d = e^{i\delta}\cos(y)$ which fixes $b = e^{i\beta}\sin(y)$. Lastly, from $|a| = |d|$ follows that $x = y$. Therefore, keeping in mind that $a = -c\overline{d}/\overline{b}$, we can define the parameterizations

$$a = e^{i\alpha}\cos(x), \quad b = e^{i\beta}\sin(x), \quad c = -e^{i\gamma}\sin(x), \quad d = e^{i\delta}\cos(x)$$

From $a = -\frac{c\overline{d}}{\overline{b}}$ we get

$$e^{i\alpha}\cos(x) = \frac{e^{i\gamma}\sin(x)e^{-i\delta}\cos(x)}{e^{-i\beta}\sin(x)}$$

Hence, $e^{i(\alpha+\delta)} = e^{i(\beta+\gamma)}$. So from $\det(U) = ad - bc = 1$ we get

$$e^{i(\alpha+\delta)}\cos(x)^2 + e^{i(\beta+\gamma)}\sin(x)^2 = e^{i(\alpha+\delta)} = 1$$

We obtain that $\alpha = -\delta$ and $\beta = -\gamma$. So $a = \overline{d}$ and $b = -\overline{c}$. So we get an element of the form

$$U \in \mathrm{SU}(2), \quad U = \begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix} = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

There exists a surjective homomorphism between SU(2) and $\mathrm{SO}(3,\mathbb{R})$ (the real orthogonal 3×3 matrices). In fact, $\mathrm{SO}(3,\mathbb{R}) \cong \mathrm{SU}(2)/\mathbb{Z}_2$. This homomorphism is [Wes08]

$$\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R}),$$
$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \mapsto \begin{pmatrix} a^2-b^2-c^2+d^2 & 2ab+2cd & -2ac+2bd \\ -2ab+2cd & a^2-b^2+c^2-d^2 & 2ad+2bc \\ 2ac+2bd & -2ad+2bc & a^2+b^2-c^2-d^2 \end{pmatrix}$$

The only elements that map to the identity are I and $-I$ so the kernel of this homomorphism is $\{I,-I\}$. By the first isomorphism theorem we get $\mathrm{SU}(2)/\mathbb{Z}_2 \cong \mathrm{SO}(3,\mathbb{R})$. So for $U \in \mathrm{SU}(2)$ we get $\varphi(U) = \varphi(-U) = \widetilde{U}$ with $\widetilde{U} \in \mathrm{SO}(3,\mathbb{R})$. Suppose that G is a finite subgroup of SU(2). We have to check whether $-I \notin G$ since in that case we have $|\varphi(G)| = |G|$. Otherwise, we have $|G| = 2|\varphi(G)|$. Firstly, we need to know what the subgroups of $\mathrm{SO}(3,\mathbb{R})$ are.

2.2 Classifying the finite subgroups of $\mathrm{SO}(3,\mathbb{R})$

The classification of the finite subgroups of $\mathrm{SO}(3,\mathbb{R})$ is done in [Arm88, Chapter 19]. We repeat the argument here for the sake of completeness. We know that every element of $\mathrm{SO}(3,\mathbb{R})$ has two antipodal fixed points in the induced action on the sphere S^2 since we know that $\mathrm{SO}(3,\mathbb{R})$ acts on $\mathbb{R}^3$ by rotation. Furthermore, if it is not an identity element, it has no more fixed points. Let G be a finite subgroup of $\mathrm{SO}(3,\mathbb{R})$, then every non-identity element of G has precisely two fixed points on S^2. We define

$$F = \{p \in S^2 \mid \exists x \in G, x \neq e, x \cdot p = p\}$$

The action of G on S^2 sends F to itself since for all $q \in F$, $g \cdot q$ is held fixed by gxg^{-1} where x holds q fixed. For every $f \in F$, the stabilizer G_f contains the two antipodal points. Let R denote the number of distinct orbits of the action of G. Then by the counting theorem we get $R = \frac{1}{|G|}(2(|G|-1)+|F|)$. We know that $|F| = \sum_{i=1}^R |G(f_i)|$ for f_i representatives of the orbits ($G(f)$ denotes the orbit of f). So we obtain

$$2\left(1-\frac{1}{|G|}\right) = R - \frac{1}{|G|}\sum_{i=1}^{R} |G(f_i)| \tag{2.1}$$

By the Orbit-Stabilizer theorem we get $|G_f| = \frac{|G|}{|G(f)|}$, so we obtain

$$2\left(1-\frac{1}{|G|}\right) = R - \sum_{i=1}^{R}\frac{1}{|G_{f_i}|} = \sum_{i=1}^{R}\left(1-\frac{1}{|G_{f_i}|}\right) \tag{2.2}$$

Note that each G_{f_i} contains at leas 2 points so we get $\frac{1}{|G_{f_i}|} \leq \frac{1}{2} \Leftrightarrow 1-\frac{1}{|G_{f_i}|} \geq \frac{1}{2}$. Since $G \neq \{e\}$ we get that $1-\frac{1}{|G|} \geq \frac{1}{2}$. Obviously $1-\frac{1}{|G|} < 1$ so from the above equation we obtain that $1 \leq \sum_{i=1}^{R}(1-\frac{1}{|G_{f_i}|}) < 2$. Each term of the sum is greater than $\frac{1}{2}$ so the only possibilities for R are $R = 2$ or $R = 3$.

Case R=2
If $R = 2$, equation (2.1) above gives us $2|G| - 2 = 2|G| - (|G(f_1)| + |G(f_2)|)$. So $|G(f_1)| + |G(f_2)| = 2$. This means that there is one axis around which all the points are rotated. So G must be cyclic and isomorphic to $\mathbb{Z}_n$.

Case R=3
If $R = 3$, equation (2.2) gives us $\frac{2}{|G|} + 1 = \sum_{i=1}^{3} \frac{1}{|G_{f_i}|}$. From now on we will denote G_{f_i} as G_i. This means that $\frac{1}{G_1} + \frac{1}{G_2} + \frac{1}{G_3} > 1$. Suppose that $G_1 = 2$. If $G_2 = 2$ we get that $G_3 = n$ can be arbitrary. Suppose $G_2 = 3$, then $G_3 < 6$ which gives us the possibilities $G_3 = 3, 4, 5$ ($G_3 = 2$ is the previous case). It is easy to check that all the other combinations turn out to be permutations of these.

If $|G_1| = |G_2| = 2$ and $|G_3| = n$, we get $1 + \frac{2}{|G|} = 1 = \frac{1}{|G_3|}$. Since $|G| = 2|G_3|$ and G_3 is cyclic we conclude that G is the dihedral group of order $2|G_3| = 2n$.
If $|G_1| = 2, |G_2| = |G_3| = 3$, we get $|G| = 12$. By the orbit stabilizer theorem we know that $|G(x)| = \frac{|G|}{|G_x|}$ and thus $|G(f_3)| = 4$. Similarly the other two orbits have sizes 4 and 6. This is the symmetry group of the regular tetrahedron.
If $|G_1| = 2, |G_2| = 3, |G_3| = 4$, we get $|G| = 24$ and so the orbit sizes are 12, 8 and 6. This corresponds to the symmetry group of the regular octahedron.
If $|G_1| = 2, |G_2| = 3, |G_3| = 5$, we get $|G| = 60$ and orbit sizes 30, 20 and 12. This corresponds to the symmetry group of the dodecahedron. In total, the finite subgroups of $\mathrm{SO}(3, \mathbb{R})$ are (for more details see [Arm88, Chapter 19])

Group	**Symmetry figure**
Cyclic ($\mathbb{Z}_n$)	Oriented regular polygon
Dihedral (D_{2n})	Regular polygon
Alternating (A_4)	Regular tetrahedron
Symmetric (S_4)	Cube or regular octahedron
Alternating (A_5)	Regular dodecahedron or icosahedron

2.3 Classifying the finite subgroups of $\mathrm{SU}(2)$

We are now in a position to classify the subgroups of $\mathrm{SU}(2)$. For each of the finite subgroups of $\mathrm{SO}(3, \mathbb{R})$ we will find the generators. Using the homomorphism from section 2.1 we can obtain the generators of the corresponding subgroups of $\mathrm{SU}(2)$.

Case $\mathbb{Z}_n$ (n-gon)
We start with the cyclic group $\mathbb{Z}_n$ which has the well known generator

$$g = \begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) & 0 \\ -\sin(2\pi/n) & \cos(2\pi/n) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If we look at the homomorphism we see that $a^2 - b^2 - c^2 + d^2 = a^2 - b^2 + c^2 - d^2$ which implies $c^2 = d^2$. Furthermore, $2ab + 2cd + (-2ab) + 2cd = 4cd = \sin(2\pi/n) - \sin(2\pi/n) = 0$. Together with $c^2 = d^2$ this means that $c = d = 0$. Now we obtain that $a^2 + b^2 = 1$, $a^2 - b^2 = \cos(2\pi/n)$ and $2ab = \sin(2\pi/n)$. Note that $(a + bi)^2 = a^2 - b^2 + 2abi = \cos(2\pi/n) + i\sin(2\pi/n) = e^{2\pi i/n}$. This gives us $a + bi = e^{\pi i/n}$ which is a root of unity of order $2n$. So the generator of the cyclic subgroup of SU(2) has the form

$$U = \begin{pmatrix} e^{\pi i/n} & 0 \\ 0 & e^{-\pi i/n} \end{pmatrix} = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$$

We are not done yet since $-I \in \langle U \rangle$ and $\langle U \rangle$ is not of order n. Suppose that n is even, then we pick $m = \frac{n}{2}$ and we obtain a cyclic subgroup of order m. Suppose that n is odd, then $-U$ has order n so we choose $-U$ as generator and we obtain a cyclic subgroup of order n. So for all $n \in \mathbb{N}$ we can obtain a cyclic subgroup of SU(2) (and $\mathrm{SL}(2, \mathbb{C})$) of order n of the form $\langle U \rangle$.

Case D_{2n} (Dihedral group)
The dihedral group has the extra operation of reflection given by

$$r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

If we look at the homomorphism, the diagonal gives $a^2 = b^2$ and $2ab - 2ab + 4cd = 0$ which implies $cd = 0$. This also gives us $2ab + 2cd = 2ab = 0$. Therefore, $(a + bi)^2 = a^2 - b^2 + 2abi = 0$ so $a + bi = 0$. Furthermore, it gives us $d^2 - c^2 = 1$ and $-d^2 - c^2 = -1$. So $c = 0$ and we get the generator

$$V = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Note that $V^2 = -I$ so the finite subgroup of SU(2) generated by U and V has twice the order of the dihedral group (so $4n$). In conclusion, we get another finite subgroup of SU(2), the *Binary Dihedral group* $\mathrm{BD}_{4n} = \langle U, V \rangle$.

Case A_4 (Tetrahedron)
There are three different operations on the tetrahedron; a rotation that keeps a vertex fixed, rotation about the z-axis which joins the midpoints of two edges and rotation about the x-axis. The rotation about the y-axis is achieved by rotating about the x and z axis consecutively.

$$r = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Using the homomorphism, we get the following generators for r,x and z

$$\varphi(U) = r = -\frac{1}{2}\begin{pmatrix} 1-i & -1-i \\ 1-i & 1+i \end{pmatrix}, \quad \varphi(V) = x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varphi(W) = z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

The orders of V and W are 4 and the order of U is 6. However, $W^2 = V^2 = U^3 = -I$ so the subgroup generated by U and V has order 24. This is correct considering that it contains $-I$ and it must therefore have twice the order of the tetrahedral group (which is 12). So we get another subgroup of SU(2), the *Binary Tetrahedral group* $\mathrm{BT}_{24} = \langle U, V, W\rangle$.

Case S_4 (Octahedron and Cube)
We have three rotational operations for the octahedron,

$$a = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Note that $a^2 = z$ where z is the rotation of the tetrahedron and c is the rotation x of the tetrahedron. The rotation b is the same as r for the tetrahedron so the homomorphism yields the same element U and clearly c yields V. Using the homomorphism on a yields:

$$U = -\frac{1}{2}\begin{pmatrix} 1-i & -1-i \\ 1-i & 1+i \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varphi(W) = a = \frac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

Note that W^2 equals the generator W of the tetrahedron. So W has order 8, V has order 4 and U has order 6. We have $W^4 = U^3 = V^2 = -I$ so the subgroup generated by U, V and W has order 48. This is correct considering that it contains $-I$. So we get another subgroup of SU(2), the *Binary Octahedral group* $\mathrm{BO}_{48} = \langle U, V, W\rangle$.

Case A_5 (Icosahedron and Dodecahedron)
According to [Lit90], we have three rotational operations for the octahedron (here $\Phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio),

$$q = \frac{1}{2}\begin{pmatrix} 1 & -\Phi & \frac{1}{\Phi} \\ \Phi & \frac{1}{\Phi} & -1 \\ \frac{1}{\Phi} & 1 & \Phi \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We already know r and z from the tetrahedral case. Using the homomorphism for q yields:

$$U = -\frac{1}{2}\begin{pmatrix} 1-i & -1-i \\ 1-i & 1+i \end{pmatrix}, \quad V = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \varphi(W) = q = \frac{1}{2}\begin{pmatrix} \Phi - i & i(1-\Phi) \\ i(1-\Phi) & \Phi + i \end{pmatrix}$$

U has order 6, V has order 4 and W has order 10. We have $U^3 = V^2 = -I$ so the subgroup generated by U, V and W has order 120. This is correct considering that it contains $-I$. So we get another subgroup of SU(2), the *Binary Icosahedral group* $\mathrm{BI}_{120} = \langle U, V, W\rangle$.

3 Basics of algebraic geometry

This section consists of the definitions and theorems of introductory algebraic geometry that the reader should be familiar with. Our goal is to lay the framework for the next sections where we find the Kleinian singularities and introduce blowups. Anyone who is well-versed in these concepts should feel free to skip sections at their leisure.

3.1 Affine varieties

We will discuss varieties over a field K which is *algebraically closed*. This means that every non-constant polynomial in $K[X]$ has all its root in K. An example of such a field (and the one we will mostly consider) is $\mathbb{C}$. The problems in algebraic geometry revolve around studying objects which locally resemble zero sets to sets of polynomial equations. This usually involves studying an objects affine space.

Definition 3.1. [Har92] By the *affine space* over a field K we mean the vector space K^n without the special structure of a vector space. Alternatively, we can think of the affine space as the set of n-tuples of elements of K, $K^n = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in K\}$. We call K the *coefficient field*.

Intuitively, an affine space is the remainder of a vector space when ignoring the origin that is characterized by the identity element $\mathbf{0}$.

Definition 3.2. [Har92] An *affine variety* $X \subseteq K^n$ is the set of zeroes (zero locus) of a collection of polynomials $f_i \in K[z_1, \ldots, z_n]$. Finite unions and intersections of affine varieties are again affine varieties. Infinite intersections of affine varieties are again affine varieties but infinite unions need not be.

Definition 3.3. A variety X is irreducible if for any pair of closed subvarieties $Y, Z \subset X$ such that $Y \cup Z = X$, we have either $Y = X$ or $Z = X$.

Example: A classic example of a variety in $\mathbb{R}^2$ is defined by $f = x^2 + y^2 - 1$, which is the circle.

Example: Suppose we have the polynomial $f = (x^2 + y^2 - z^2)(z - 1) \in K[x, y, z]$. Then the zero locus of f is an affine variety that consists of a cone through 0 and a plane.

Example: Another example is the variety in $\mathbb{R}^3$ defined by $x^2 - y^2z^2 + z^3 = 0$ (see figure 1). This variety has a line of singularities where it intersects itself. What this is precisely will be discussed in section 5.

Definition 3.4. [CLD15] Let $f_1, \ldots, f_s \in K[x_1, \ldots, x_n]$, then

$$\langle f_1, \ldots, f_n \rangle = \{\sum_{i=1}^{s} h_i f_i \mid h_1, \ldots, h_s \in K[x_1, \ldots, x_n]\}$$

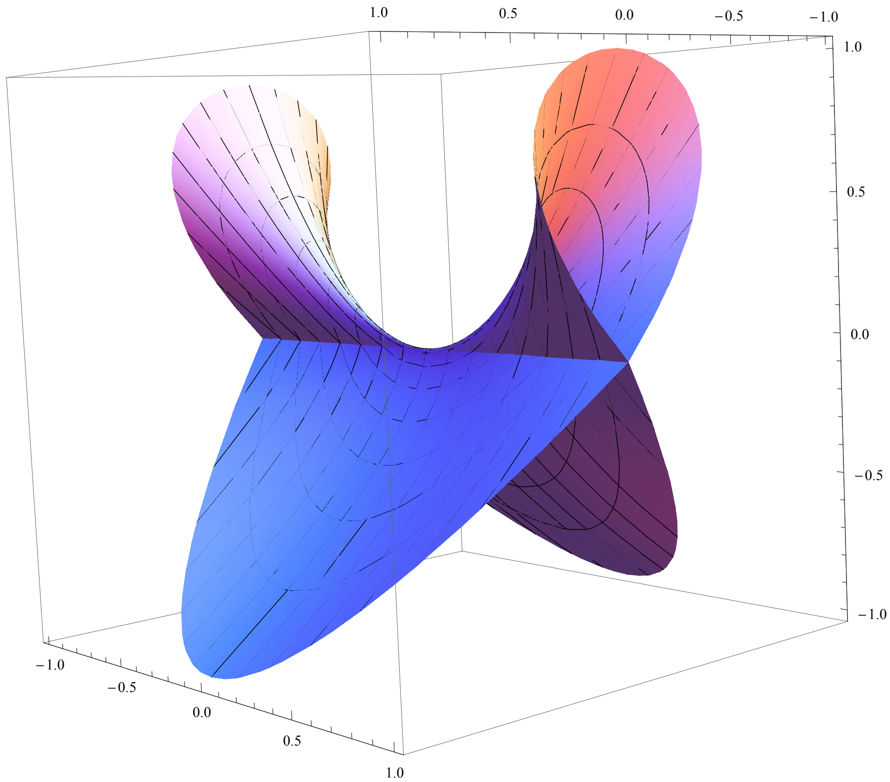

Figure 1: The variety $x^2 - y^2z^2 + z^3$.

is called the *ideal generated by* $f_1, \ldots . f_s$. Let $X \subseteq K^n$ be an affine variety. The *ideal of* X is defined as

$$I(X) = \{f \in K[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0, \; \forall (a_1, \ldots, a_n) \in X\}$$

Definition 3.5. [CLD15] Let $I \subseteq K[x_1, \ldots, x_n]$ be an ideal. The corresponding variety is defined as

$$V(I) = \{(a_1, \ldots, a_n) \in K^n \mid f(a_1, \ldots, a_n) = 0 \text{ for all } f \in I\}$$

If $I = \langle f_1, \ldots, f_s \rangle$, then $V(I) = V(f_1, \ldots, f_s)$.

3.2 Projective space

So far we have looked at varieties in affine space K^n. Now we will look at an enlargement of K^n by adding "points at infinity" to create the projective space. By the projective space $\mathbb{P}^n$ we intuitively mean the set of lines through the origin of K^{n+1}.

Definition 3.6. Let V be a vector space. The *projective space* $\mathbb{P}(V)$ of V is the set of 1-dimensional vector subspaces of V. If V has dimension n, $\mathbb{P}(V)$ has dimension $n-1$.

For a field K we will use the notation $\mathbb{P}^n = \mathbb{P}(K^{n+1})$ when it's clear what K is or if it is irrelevant. Projective spaces are in essence compactifications of affine spaces, we will formalize this intuition later. Firstly, it is useful for our intuition to consider $V = \mathbb{R}^{n+1}$, then the projective space is the set of lines through the origin. Each such line intersect the sphere $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \sum_i x_i^2 = 1\}$ so $\mathbb{P}(\mathbb{R}^{n+1})$ can be interpreted as a hemisphere of S^n with antipodal points on the equator identified.

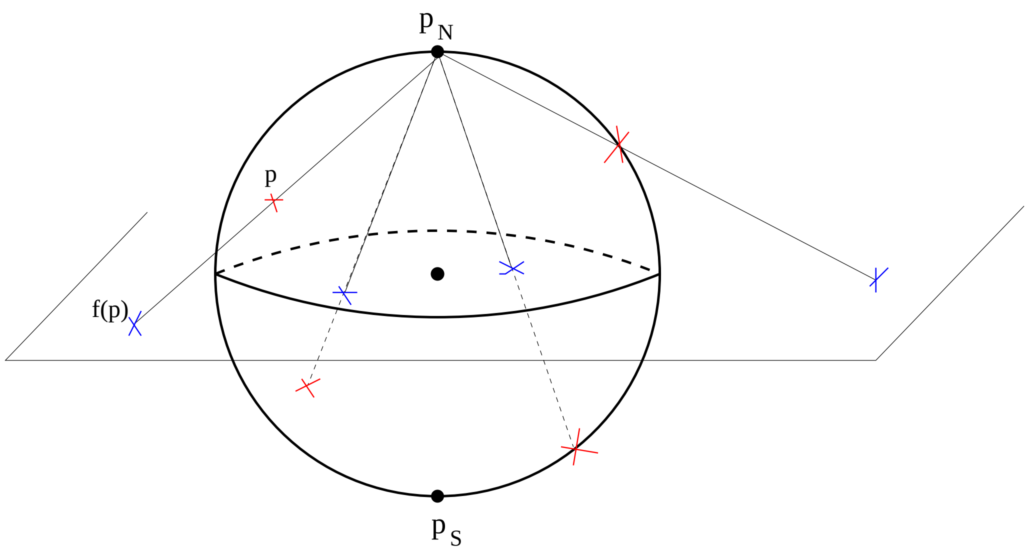

Figure 2: [Cra15, page 13] An interpretation of $\mathbb{P}^n$ in relation to the hemisphere of S^n for the case $n = 2$.

Another interpretation of the projective space involves *representative vectors* for points in $\mathbb{P}(V)$. We know that any 1-dimensional subspace of V is simply the set of linear multiples of a non-zero vector $\mathbf{v} \in V$. We then say that $\mathbf{v}$ is a representative vector for the point $[\mathbf{v}] \in \mathbb{P}(V)$. If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a basis for V we can write $\mathbf{v} = \sum_{i=1}^n x_i \mathbf{e}_i$ and $(x_1, \ldots, x_n)$ are the coordinates of $\mathbf{v}$. If $\mathbf{v} \neq \mathbf{0}$ we write $[\mathbf{v}] = [x_1 : \ldots : x_n]$ which are its *homogeneous coordinates*. Note that for $\lambda \neq 0$ we have $[\lambda x_1 : \ldots : \lambda x_n] = [x_1 : \ldots : x_n]$.

Let K be our coefficient field and $W \subseteq \mathbb{P}(V)$ such that $x_1 \neq 0$. Then we have $[x_1 : \ldots : x_n] = \left[1 : \frac{x_2}{x_1} : \ldots : \frac{x_n}{x_1}\right] = [1 : y_2 : \ldots , y_n]$. So $W \cong K^{n-1}$. If $x_1 = 0$ we are actually considering the projective space of one lower dimension.

Let $U_i \subset \mathbb{P}^n$ be the subset of points $[x_0 : \ldots : x_n]$ with $x_i \neq 0$. The map

$$[x_0 : \ldots : x_n] \mapsto \left[\frac{x_0}{x_i} : \ldots : \frac{x_{i-1}}{x_i} : 1 : \frac{x_{i+1}}{x_i} : \ldots : \frac{x_n}{x_i}\right]$$

is an isomorphism from U_i to K^n. So this map takes a line $L \subset K^{n+1}$ with $x_i \neq 0$ and returns the point of intersection with the plane $x_i = 1$. By the definition of U_i we see that $\mathbb{P}^n = U_i \cup H$ where $H = \{p \in \mathbb{P}^n \mid p = [x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n]\}$. We can identify U_i with the affine space K^n and interpret H as a *hyperplane at infinity*. It follows that there is a one-to-one correspondence between H and $\mathbb{P}^{n-1}$ so we can write $\mathbb{P}^n = K^n \cup \mathbb{P}^{n-1}$ and $\mathbb{P}^n = \cup_{i=0}^n U_i$. This makes the projective space a compactification of affine space with the sets U_i a cover of $\mathbb{P}^n$ by affine open sets.

To better illustrate that the projective space is the compactification of affine space we consider $[a_0, a_1] \in \mathbb{P}^1$. Suppose $a_0 \neq 0$, then we can write the point as $[1, a_1/a_0]$ as we have done previously and so we can see that the set of points with $a_0 \neq 0$ is isomorphic to K. If $a_0 = 0$, then $a_1 \neq 0$ and we can write this point as $[0, 1]$. Thus $\mathbb{P}^1$ is just K with one point added, the "point at infinity" you have no doubt encountered during introductory courses in topology.

3.3 Zariski topology and morphisms

Definition 3.7. [Har92] The *Zariski topology* on a variety X is the topology whose closed sets are the subvarieties of X.

So for $X \subseteq K^n$ the open sets are given by $U_f = \{p \in X \mid f(p) \neq 0\}$ for the polynomials f. From now on, if we speak of an open subset of a variety X, we mean the complement of a subvariety.

Definition 3.8. [Har92] Let $U \subset X$ be an open set and $p \in U$. A function f on U is *regular at* p if in some neighbourhood V of p it is expressible as a quotient g/h, where $g, h \in K[x_1, \ldots, x_n]$ with $h(p) \neq 0$. We say that f is *regular* on U if it is regular at every point of U.

The set of all regular functions on a variety X defines a ring with addition and multiplication defined as for polynomials. This ring is the coordinate ring $A(X)$. Each polynomial $f \in K[x_1, \ldots, x_n]$ can be seen as a function on the points of X, denoted by $\varphi \in A(X)$. Therefore, we have a homomorphism between $A(X)$ and $K[x_1, \ldots, x_n]$. The kernel of this homomorphism consists of the polynomials that are zero on X, i.e. $I(X)$. Therefore we have the following definition.

Definition 3.9. [Har92] We define the *coordinate ring* of a variety X to be the quotient

$$A(X) = K[x_1, \ldots, x_n] \big/ I(X)$$

Note that this is the set of equivalence classes modulo I. So polynomials are considered equivalent if their difference vanishes on the variety. For affine varieties we have the following.

Definition 3.10. [Wil06] For $X \subseteq K^n$ and $Y \subseteq K^m$, a map $\varphi : X \to Y$ is regular if there exists m regular functions $f_1, \ldots, f_m$ on X such that $\varphi(x) = (f_1(x), \ldots, f_m(x))$ for all $x \in X$.

If $\varphi : X \to Y$ is a regular map, then for every function g on Y we associate a function f on X by $f(x) = g(\varphi(x))$. We define $f = \varphi^*(g)$ as the *pullback* of g. So inversely, φ^* maps functions on Y to functions on X. The pullback gives us a homomorphism between coordinate rings $\varphi^* : A(Y) \to A(X)$. The kernel of φ^* is zero if and only if $\varphi(X)$ is dense in Y. When this is the case, φ^* defines an isomorphic inclusion (embedding) from $A(Y)$ to $A(X)$. Regular maps give us a notion of when affine varieties are the same. X and Y are isomorphic or *biregular* if there exist two maps $\varphi : X \to Y$ and $\psi : Y \to X$ that are inverse to one another, or equivalently $A(X) \cong A(Y)$.

Definition 3.11. [Wil06] Let $X \subset K^n$ be an irreducible affine variety. We define the *rational function field* of X as the quotient field of the coordinate ring $A(X)$. We usually denote it by $K(X)$. An element $h \in K(X)$ of the rational function field of X is called a *rational function* $h = f/g$ on X.

Note that while f and g are regular functions on X, h is not a function on X. For $X \subseteq K^n$ and $Y \subseteq K^m$, a *rational map* $\varphi : X \to Y$ is a tuple of rational functions $f_1, \ldots, f_m \in A(X)$ such that the functions f_i are regular at all points $x \in X$ and $\varphi(x) = (f_1(x), \ldots, f_m(x))$.

Definition 3.12. [Wil06] Let X be an irreducible variety and Y any variety. A *rational map* $\varphi : X \to Y$ is an equivalence class of pairs (U, γ) where $U \subset X$ is a dense Zariski open subset and $\gamma : U \to Y$ is a regular map. Two pairs (U, γ) and (V, η) are considered equivalent if $\gamma|_{U \cap V} = \eta|_{U \cap V}$.

Rational maps are important because of the connection they have with maps between function fields over varieties. A rational map is said to be birational if there exists a rational map that is its inverse. To construct a definition formally we need to define composition of rational maps. Let $\varphi : X \to Y$ and $\eta : Y \to Z$ be rational maps represented by the pairs (U, f) and (V, g) respectively. Suppose that $f^{-1}(V) \neq \emptyset$, then the composition $\eta \circ \varphi$ is defined as the equivalence class $(f^{-1}(V), g \circ f)$.

Definition 3.13. [Wil06] A rational map $\varphi : X \to Y$ is said to be *birational* if there exists a rational map $\gamma : Y \to X$ such that $\varphi \circ \gamma$ and $\gamma \circ \varphi$ are defined and equal to the identity. If there exists a birational map between two irreducible varieties we call them *birationally isomorphic* or *birational.*

Theorem 3.1. *[Wil06, Theorem 21] Two varieties X and Y are birational if and only if $R(X) \cong R(Y)$. Equivalently, they are birational if and only if there exists nonempty open subsets $U \subset X$ and $V \subset Y$ that are isomorphic.*

Example: The cusp $\mathcal{C} = V(y^2 - x^3)$ is birational to $\mathbb{A}^1$ (affine space of K^1). The map $\varphi : t \mapsto (t^2, t^3)$ has the inverse $\psi : \mathcal{C} - - \to \mathbb{A}^1, (x, y) \mapsto y/x$.

Example: $\mathbb{P}^2$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ since they both have $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ as open subset.

4 The Kleinian singularities

In this section we study the Kleinian singularities. These are surfaces in $\mathbb{C}^3$ with a singularity at the origin. Firstly, we introduce the relevant invariant theory as it is introduced in [CLD15, Chapter 7]. Secondly, we opted to include a section on Grundformen for theoretical background. The Grundformen can be used to find a single polynomial that defines a singular surface in $\mathbb{C}^3$. Some parts of this section require additional knowledge that is outside the scope of this thesis if we were to compute them explicitly. Therefore, we have opted to largely give only the results and refer to additional material for the specifics.

4.1 Invariant theory

Let Γ be a finite matrix group of $\mathrm{GL}(2,\mathbb{C})$. The subgroup Γ acts on $\mathbb{C}^2$ by means of matrix multiplication. This induces an action on $\mathbb{C}[x,y]$ by

$$(A, f(x,y)) \mapsto f(A\cdot(x,y)), \qquad \text{for } A\in\Gamma \text{ and } (x,y)\in\mathbb{C}^2$$

Definition 4.1. The *ring of* Γ*-invariant polynomials* $\mathbb{C}[x,y]^\Gamma$ consists of all polynomials $f\in\mathbb{C}[x,y]$ such that $A\cdot f(x,y) := f(A\cdot(x,y)) = f(x,y)$ for all $A\in\Gamma$.

We would like to find the generators of the ring of invariant polynomials. To do so, we define the Reynolds operator (average over G):

Definition 4.2. Let $G\subset \mathrm{GL}(n,K)$ be a finite matrix group, the *Reynolds operator* of G is the map

$$R_G : K[x_1,\dots,x_n]\to K[x_1,\dots,x_n], \quad R_G(f(\mathbf{x})) = \frac{1}{|G|}\sum_{A\in G} f(A\cdot\mathbf{x})$$

Theorem 4.1. *[CLD15, Theorem 7.3.5] Let $G\subset \mathrm{GL}(n,K)$ be a finite matrix group, let $x^{\beta_1},\dots,x^{\beta_m}$ be all monomials of total degree at most $|G|$. Then*

$$K[x_1,\dots,x_n]^G = K[R_G(x^{\beta_1}),\dots,R_G(x^{\beta_m})] = K\left[R_G(x^\beta)\;\middle|\; |\beta|\le|G|\right]$$

So checking the Reynolds operator for finitely many monomials will yield us our generators. We can define the following useful ideal.

Definition 4.3. Let $K[x_1,\dots,x_n]^G$ be generated by $f_1,\dots,f_m$ and let $F=(f_1,\dots,f_m)$ be a polynomial in $K[y_1,\dots,y_m]$. We define the *ideal of relations*

$$I_F = \{h\in K[y_1,\dots,y_m]\mid h(f_1,\dots,f_m)=0\}$$

Theorem 4.2. *[CLD15, Theorem 7.4.2] Let $K[x_1, \dots, x_n]^G$ be generated by $f_1, \dots, f_m$ and let I_F be the ideal of relations. Then there is a ring isomorphism*

$$K[x_1, \dots, x_n]^G \cong K[y_1, \dots, y_m] \big/ I_F$$

Here the quotient of $K[y_1, \dots, y_m]$ modulo I_F is the set of equivalence classes for congruence modulo I. The ideal of relations defines a variety $V(I_F) \subseteq K^m$ which is irreducible and I_F defines the ideal of all the polynomials that vanish on V_F [CLD15, Proposition 7.4.7]. Eventually we intend to obtain surfaces in $\mathbb{C}^3$. The following theorem completes the correspondence with $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$.

Theorem 4.3. *[CLD15, Theorem 7.4.10] Let $G \subset \mathrm{GL}(n, K)$ be a finite matrix group, $\mathbf{a} \in K^n$ and I_F the ideal of relations. The G-*orbit *of $\mathbf{a}$ is $G \cdot \mathbf{a} = \{A \cdot \mathbf{a} \mid A \in G\}$ and the set of all G-orbits is called the* orbit space*. The map sending the G-orbit $G \cdot \mathbf{a}$ to the point $F(\mathbf{a}) \in V_F$ induces a one-to-one correspondence.*

$$K^n \big/ G \cong V_F$$

4.2 Divisors and Grundformen

The previous section showed that the quotient space $\mathbb{C}^2/\Gamma$ defines a variety but we do not yet know how to find it explicitly. The Reynolds operator is one option and we will use it for the cyclic case. However, for the other cases the Reynolds operator turns out to be cumbersome. Therefore, we outline an alternative method here. The following section is based largely on [Dol07, section 1.2] and introduces Grundformen. Grundformen can be used to find generators for the ring of invariants.

Definition 4.4. [BT11] Let X be a connected irreducible variety of dimension d. A *prime divisor* on X is a closed irreducible subset $Z \subset X$ of dimension $d - 1$.

Definition 4.5. [BT11] A *divisor* on X is an element of the group generated by the prime divisors. This group is denoted by $Div(X)$.

We write a divisor as $\sum_Z n_Z Z$ where Z ranges over the prime divisors and n_Z are integers. Only finitely many n_Z are non-zero.

Definition 4.6. [BT11] An *effective divisor* is a divisor where each $n_Z \in \mathbb{N}$.

Let $f(x, y)$ be a homogeneous polynomial of degree d. Note that if a is a zero of f, then $f(\lambda a) = \lambda^d f(a)$ is also a zero. Therefore, a defines a line of zeroes that intersects the origin. Thus, the zero set of the polynomial in $\mathbb{C}^2$ is a set of lines through the origin. Hence, it is a set of points in $\mathbb{P}^1$. This set consists is a finite union of prime divisors (points are irreducible and have codimension 1). Hence, the set forms an effective divisor, which we denote by $V(f)$, because we can take each n_Z equal to the multiplicity of the points. Let $g \in \mathrm{SL}(2,\mathbb{C})$, we define the action of g on $\mathbb{C}[x,y]_d$ (the space of homogeneous polynomials in two variables of degree d) as [Dol07]

$$g \cdot f(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot f(x,y) = f(dx - by, ay - cx)$$

Note that

$$g \cdot f(x,y) = 0 \Leftrightarrow f(g^{-1} \cdot (x,y)) = 0 \Leftrightarrow g^{-1} \cdot (x,y) \in V(f) \Leftrightarrow (x,y) \in g(V(f))$$

So we have that $V(g \cdot f) = g(V(f))$. This leads to the following definition:

Definition 4.7. [Dol07] A homogeneous polynomial f is a *relative invariant* of G if for all $g \in G$, $g(V(f)) = V(f)$. This means that f is a relative invariant if and only if for all $g \in G$, $g \cdot f = \lambda_g f$ with $\lambda_g \in \mathbb{C}$.

The map $\chi_f : G \to \mathbb{C}, g \mapsto \lambda_g$ is called the *character* of f.

Definition 4.8. [Dol07] A polynomial Φ is called a *Grundform* if it is a relative invariant and its divisor $V(\Phi)$ is equal to an orbit with non-trivial stabilizer. So, every $g \in G$ has to send the set of points in $\mathbb{P}^1$, $V(\Phi)$, to itself while keeping some of them fixed.

The Grundformen will form a basis of the invariant polynomials. Dolgachev states the following criterion that has to be satisfied for this to be true.

Theorem 4.4. *[Dol07] If there exist two Grundformen Φ_1 and Φ_2 whose orbits have cardinalities $|G|/e_1$ and $|G|/e_2$ such that the characters satisfy*

$$\chi_{\Phi_1}^{e_1} = \chi_{\Phi_2}^{e_2}$$

Then every relative invariant is a polynomial in Grundformen.

We can now calculate the Grundformen for each finite subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ and show that the above theorem applies. This will help us when finding the generators of the ring of invariants. The full calculation can be found in [Dol07, section 1.2] but we will repeat the key points here.

Case: Cyclic A_n
The Grundformen and the corresponding characters are:

$$\Phi_1 = x, \qquad \Phi_2 = y$$

$$\chi_{\Phi_1}(u) = \epsilon_n, \qquad \chi_{\Phi_2}(u) = \epsilon_n^{-1}$$

So indeed $\chi_{\Phi_1}^n = \chi_{\Phi_2}^n$ and therefore every relative invariant is a polynomial in Grundformen.

Case: Binary Dihedral BD_{4n}
The Grundformen of BD_{4n} are:

$$\Phi_1 = x^n + y^n, \qquad \Phi_2 = x^n - y^n, \qquad \Phi_3 = xy$$

The binary dihedral group has 2 generators so we get two values for the characters:

$$\chi_{\Phi_1}(u) = -1, \qquad \chi_{\Phi_2}(u) = -1, \qquad \chi_{\Phi_3}(u) = 1$$
$$\chi_{\Phi_1}(v) = i^n, \qquad \chi_{\Phi_2}(v) = -i^n, \qquad \chi_{\Phi_3}(v) = -1$$

For Φ_1 and Φ_2 we have $e_1 = e_2 = 2$. We only need two characters for the theorem and indeed $\chi^2_{\Phi_1} = \chi^2_{\Phi_2}$ so every relative invariant is a polynomial in Grundformen.

Case: Binary Tetrahedral BT_{24}
The Grundformen of BT_{24} are:

$$\Phi_1 = xy(x^n - y^n), \qquad \Phi_2 = x^4 + 2i\sqrt{3}x^2y^2 + y^4, \qquad \Phi_3 = x^4 - 2i\sqrt{3}x^2y^2 + y^4$$

The binary tetrahedral group has 3 generators so we get three values for the characters:

$$\chi_{\Phi_1}(u) = 1, \qquad \chi_{\Phi_2}(u) = \epsilon_3, \qquad \chi_{\Phi_3}(u) = 1$$
$$\chi_{\Phi_1}(v) = 1, \qquad \chi_{\Phi_2}(v) = \epsilon_3, \qquad \chi_{\Phi_3}(v) = 1$$
$$\chi_{\Phi_1}(w) = 1, \qquad \chi_{\Phi_2}(w) = \epsilon_3, \qquad \chi_{\Phi_3}(w) = \epsilon_3^2$$

For Φ_2 and Φ_3 we have $e_2 = e_3 = 3$. We only need two characters for the theorem and indeed $\chi^3_{\Phi_2} = \chi^3_{\Phi_3}$ so every relative invariant is a polynomial in Grundformen.

Case: Binary Octahedral BO_{48}
The Grundformen of BO_{48} are:

$$\Phi_1 = xy(x^n - y^n), \qquad \Phi_2 = x^8 + 14x^4y^4 + y^8, \qquad \Phi_3 = (x^4 + y^4)((x^4 + y^4)^2 - 36x^4y^4)$$

The binary octahedral group has 3 generators so we get three values for the characters:

$$\chi_{\Phi_1}(u) = -1, \qquad \chi_{\Phi_2}(u) = 1, \qquad \chi_{\Phi_3}(u) = -1$$
$$\chi_{\Phi_1}(v) = 1, \qquad \chi_{\Phi_2}(v) = 1, \qquad \chi_{\Phi_3}(v) = 1$$
$$\chi_{\Phi_1}(w) = 1, \qquad \chi_{\Phi_2}(w) = 1, \qquad \chi_{\Phi_3}(w) = 1$$

For Φ_1 and Φ_3 we have $e_1 = 4$ and $e_3 = 2$. We only need two characters for the theorem and indeed $\chi^4_{\Phi_1} = \chi^2_{\Phi_3}$ so every relative invariant is a polynomial in Grundformen.

Case: Binary Icosahedral BI_{120}
The Grundformen of BI_{120} are:

$$\Phi_1 = xy(x^{10} + 11x^5y^5 - y^{10}), \qquad \Phi_2 = 228(x^{15}y^5 - x^5y^{15}) - x^{20} - y^{20} - 494x^{10}y^{10}$$
$$\Phi_3 = x^{30} + y^{30} + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + x^{10}y^{20})$$

The characters are trivial, i.e. $\chi_{\Phi_1} = \chi_{\Phi_2} = \chi_{\Phi_3} = 1$. Hence, every relative invariant is a polynomial in Grundformen.

4.3 The Kleinian singularities

In this section we will determine generators for the rings of Γ-invariant polynomials for each finite subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$. Note that a polynomial $f \in \mathbb{C}[x, y]^\Gamma$ if and only if $\chi_f = 1$. We will need the following important lemma from [Dol07].

Lemma 4.5. *Suppose that an invariant polynomial F can be written as $\sum c_i \Phi_i$ with $c_i \neq 0$ and Φ_i relative invariants corresponding to different characters. Then each Φ_i is invariant.*

Now we have the prerequisite tools to find the generators for $\mathbb{C}[x, y]^\Gamma$ and construct the polynomial $g \in \mathbb{C}[x, y, z]$ that defines our surface in $\mathbb{C}^3$. For the cyclic case we do not yet need Grundformen. If we look at the generator of the cyclic subgroup (denoted by A_n)

$$U = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{pmatrix}$$

we see that on $\mathbb{C}[x, y]$ it sends $x \mapsto \epsilon_n x$ and $y \mapsto \epsilon_n^{-1} y$. Applying the Reynolds operator to monomials of the form $x^i y^j$ yields

$$R_G(x^i y^j) = \frac{1}{n} \sum_{k=1}^{n} \epsilon_k^i x^i \epsilon_k^{-j} y^j = \frac{x^i y^j}{n} \sum_{k=1}^{n} \epsilon_k^{i-j}$$

This is 0 if $i \neq j \mod n$ and it is $x^i y^j$ if $i = j \mod n$. So using Theorem 4.2 we conclude that the ring of invariants is generated by $f_1 = x^n$, $f_2 = y^n$ and $f_3 = xy$. It turns out that for each subgroup of $\mathrm{SL}(2, \mathbb{C})$ the ring of invariants has three generators.

We have the relation $f_3^n = f_1 f_2$ so if we set $x = f_1$, $y = f_2$ and $z = f_3$ we obtain the equation $xy - z^n = 0$. Klein [Kle84] proved in 1884 that the ideals of relations for the subgroups of $\mathrm{SL}(2, \mathbb{C})$ are generated by one element. In other words, $\mathbb{C}^2/\Gamma$ can be interpreted as a surface in $\mathbb{C}^3$ defined by the polynomial F. For A_n this means that

$$\mathbb{C}[x, y]^{A_n} \cong \mathbb{C}[x, y, z] \Big/ I(xy - z^n)$$

By Theorem 4.4, A_n defines a surface in $\mathbb{C}^3$ defined by $xy - z^n$. By convention, there is a shift in A_n such that A_n is defined by $xy - z^{n+1}$.

For the other four cases it is convenient to use Grundformen. The idea is that we take any invariant $F = \sum_i \Phi_1^{a_i} \Phi_2^{b_i} \Phi_3^{c_i}$ and show that it can be written as a polynomial in generators f_i. By Lemma 4.6, we know that each monomial $\Phi_1^a \Phi_2^b \Phi_3^c$ is an invariant. The next step in the procedure is to define three invariant polynomials (do this by finding monomials in Grundformen with trivial character) and show that any monomial that is not generated by those is not an invariant. The method requires a lot of calculations that we did not do ourselves. Therefore, we will show how Dolgachev computes the binary dihedral case and refer to his notes for the other cases.

For the binary dihedral case, the following generators are invariants:

$$f_1 = \Phi_1\Phi_2, \qquad f_2 = \Phi_3^2, \qquad f_3 = \Phi_3\Phi_2^2$$

If we take a monomial $\Phi_1^a\Phi_2^b\Phi_3^c$, we see that if $c \geq 2$ we can factor a power of f_2. Hence, we assume $c \leq 1$. Furthermore, if $a, b \geq 1$, we can factor f_1 until either $a = 0$ or $b = 0$. Hence, we assume that $a = 0$ or $b = 0$. We are left with the following possibilities for the monomial:

$$\{\Phi_1^k, \Phi_2^k, \Phi_1^k\Phi_3, \Phi_2^k\Phi_3, \Phi_3\}$$

Observe that the following monomials are also expressible in f_i:

$$\Phi_3\Phi_1^2 = xy((x^n + y^n)^2 - 4(xy)^n) = f_3 - 4f_2^{\frac{n+1}{2}}$$
$$\Phi_2^4 = (x^{2n} - y^{2n})^2 - 4x^ny^n(x^n - y^n)^2 = f_1^2 - 4f_3f_2^{\frac{n-1}{2}}$$
$$\Phi_1^4 = ((x^n - y^n)^2 + 4x^ny^n)^2 = f_1^2 + 4f_3f_2^{\frac{n-1}{2}} + 16f_2^n$$

Hence, we can factor any of $\Phi_1^4, \Phi_2^4, \Phi_1^2\Phi_3, \Phi_2^2\Phi_3$. This restricts the possibilities to:

$$\{\Phi_1^k, \Phi_2^k, \Phi_1\Phi_3, \Phi_2\Phi_3, \Phi_3 \mid k \leq 3\}$$

None of the elements of the set is an invariant and hence we have shown that any invariant monomial can also be written as polynomial in f_i. Hence, any invariant is generated by the f_i. Dolgachev observes the following relation:

$$f_3^2 + f_2f_1^2 + 4f_3f_2^{\frac{n+1}{2}} = (f_3 + 2f_2^{\frac{n+1}{2}})^2 - 4f_1^{n+1} - f_2f_1^2 = 0$$

If we substitute $f_3' = f_3 + 2f_2^{\frac{n+1}{2}}$ we get

$$f_3'^2 - 4f_1^{n+1} - f_2f_1^2$$

If we scale the generators properly we get

$$\mathbb{C}[x, y]^{\mathrm{BD}_{4n}} \cong \frac{\mathbb{C}[x, y, z]}{(x^2 + z(y^2 + z^n))}$$

By Theorem 4.4 and Lemma 4.6, BD_{4n} defines a surface in $\mathbb{C}^3$ defined by $x^2+z(y^2+z^n)$.

As mentioned, the rest of the calculations can be found in [Dol07, section 1,2]. Historically, Klein [Kle84] proved the following for all finite subgroups of $\mathrm{SL}(2, \mathbb{C})$.

Group	**ADE-classification**	**Defining polynomial**
Cyclic ($\mathbb{Z}_{n+1}$)	A_n	$xy - z^{n+1}$
Binary Dihedral (BD_{4n})	D_{n+2}	$x^2 + zy^2 + z^{n+1}$
Binary Tetrahedral (BT_{24})	E_6	$x^4 + y^3 + z^2$
Binary Octahedral (BO_{48})	E_7	$x^2 + y^3 + yz^3$
Binary Icosahedral (BI_{120})	E_8	$x^5 + y^3 + z^2$

The corresponding varieties are all irreducible and they all have a singularity at the origin. Therefore, they have been coined *the Kleinian Singularities*. The next pages show some graphs of the real parts of the varieties.

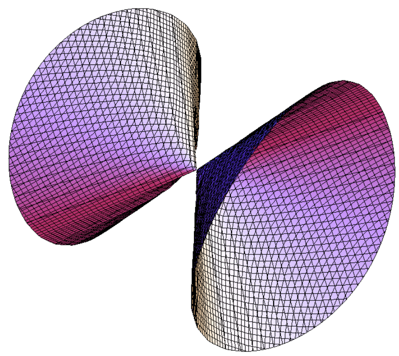

Figure 3: The graph of the real part of A_1.

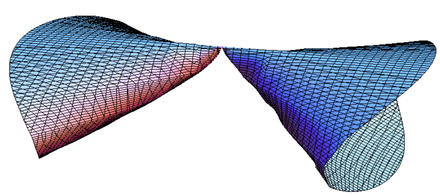

Figure 4: The graph of the real part of D_5.

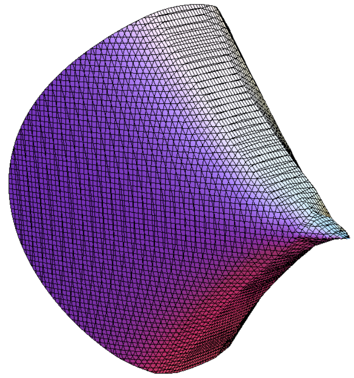

Figure 5: The graph of the real part of E_6.

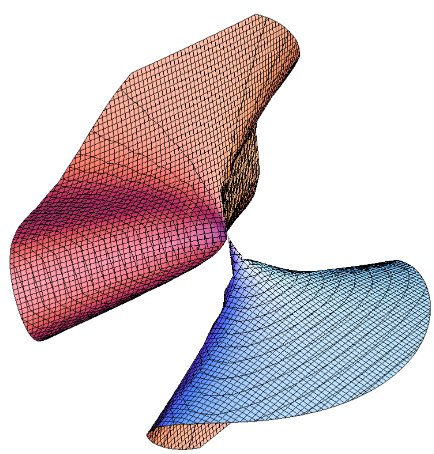

Figure 6: The graph of the real part of E_7.

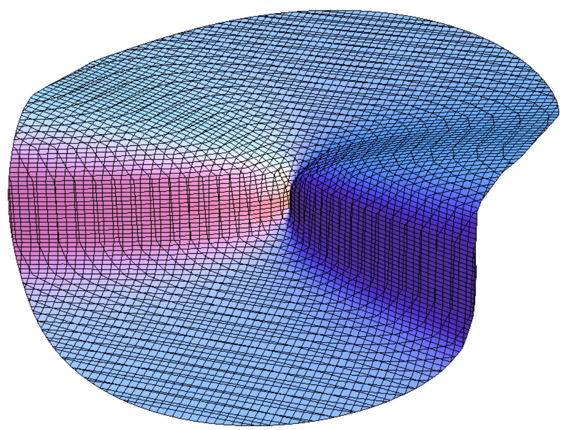

Figure 7: The graph of the real part of E_8.

5 Blowups

Many affine varieties are not entirely smooth but have singular points. Studying such an object in projective space may allow us to view it without it's singular points. We would like to find a map from a smooth variety $V \subset K^n \times \mathbb{P}^{n-1}(K)$ to an affine variety $W \subset K^n$. We would like this mapping to be such that W and V are isomorphic everywhere at perhaps the singular points. A method for removing singular points is called *blowing up*. A blowup can be thought of as a higher-dimensional parametrization of our variety such that it does not intersect itself. The blowup map is an example of a birational map that is not isomorphic. In certain cases it is an isomorphism on the regular points but not on the singular points.

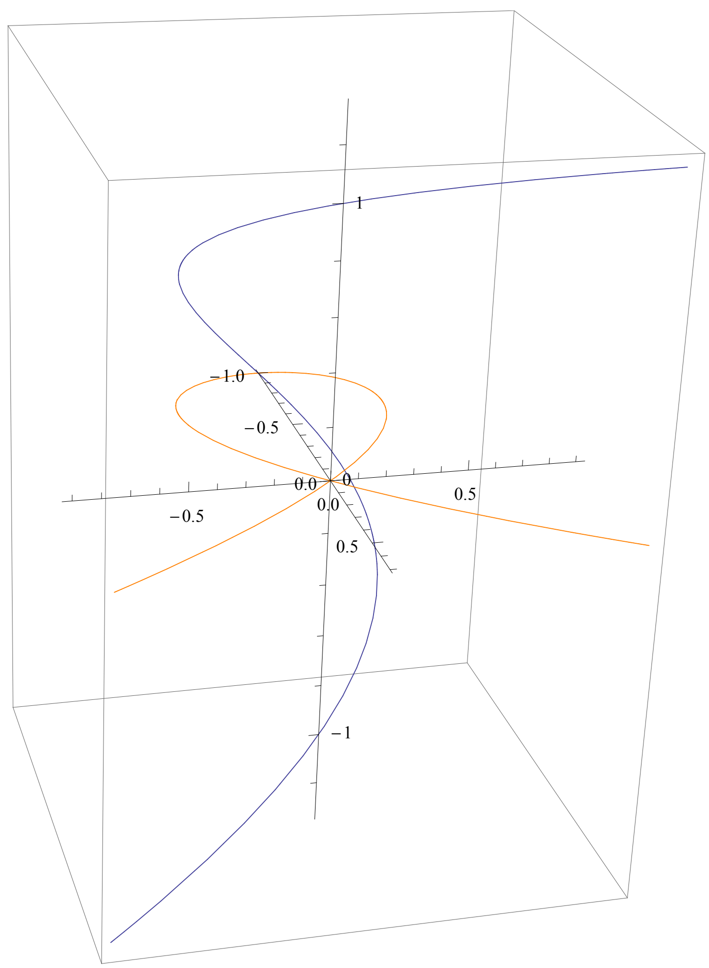

Figure 8: An example of the node $x^3 + x^2 - y^2$ and its blowup.

5.1 Blowing up at a point

Suppose we have some affine variety $V \subset K^n$ and some point $p \in V$. The idea behind blowing up at p is to leave every part of K^n unchanged except for the point p. This point we will replace with an entire copy of $\mathbb{P}^{n-1}(K)$. This means that all the lines through p can be uniquely associated with a point in $\mathbb{P}^{n-1}(K)$. This should enforce the idea of separating the lines through p. From now on we will assume p to be the origin since this could be achieved by an appropriate change of coordinates. Firstly, we will formalize the concept of a singularity.

Definition 5.1. Let $V \subset K^n$ be an affine variety described by $f_1, \ldots, f_m$. A *singularity* of V is a point $p \in V$ such that the Jacobian

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} (p)$$

has rank strictly less than $\min(m, n)$.

This requirement guarantees that the null space of the Jacobian is non-empty, i.e. we have a singular point. We can now construct a blowup surface as the set of all pairs of points $(p, q) \in K^n \times \mathbb{P}^{n-1}$ with $p \in K^n$ and $q \in \mathbb{P}^{n-1}$ the line through p and the origin.

Definition 5.2. [Har92]A *blowup surface* $Bl_O V$ of a space $V = K^n$ is defined to be:

$$Bl_O V = \{(x_1, \ldots, x_n, y_1 : \ldots : y_n) \mid x_i y_j = x_j y_i \ 1 \le i < j \le n\} \subset K^n \times \mathbb{P}^{n-1}(K)$$

Definition 5.3. The *blowup* of K^n at the origin is the blowup surface $Bl_O V$ and the projection map $\pi : Bl_O V \to K^n, (q, p) \mapsto q$.

The map π is a projection onto K^n. An important observation is that π is an isomorphism on $\mathbb{A}^n \backslash \{O\}$ so the condition that we leave $\mathbb{A}^n$ unchanged except at the origin is satisfied. Obviously the origin is where things get interesting. We can see that the preimage of the origin is $\pi^{-1}(O) = \{O\} \times \mathbb{P}^{n-1}(K) \cong \mathbb{P}^{n-1}(K)$. We call $\pi^{-1}(O)$ the *exceptional divisor* of the blowup. Let X be a variety passing through the origin. We call $\pi^{-1}(X)$ the *strict transform*. What we are interested in however is the *proper transform*.

Definition 5.4. [Wil06] Let $X \subseteq \mathbb{A}^n$ be variety through the origin. The *proper transform* of X under $\pi : Bl_O V \to K^n$ is defined as $Bl_O(X) = \overline{\pi^{-1}(X \backslash \{O\})}$ with closure defined by the Zariski topology. Whenever we talk about the blowup of a variety X in O we are referring to the proper transform.

Example : We will blow up the cusp $C = V(x^3 - y^2)$ at the origin. To do so, we first look at the blowup when we restrict $\mathbb{P}^1$ to U_0. In that case, $x_0 \neq 0$ so we get that $x_0 y = x x_1$ becomes $y = tx$. If we fill this in we get $x^3 - t^2 x^2 = x^2(x - t^2) = 0$. This means that either $x^2 = 0$ or $x = t^2$. However, $x = 0$ implies $y = 0$ which is the origin. So this is the exceptional divisor. The other option is $x = t^2$ which means that $y = t^3$.
Now we look at the other chart U_1. Then we get $x = sy$ which implies $y^2(s^3 y - 1) = 0$. Again $y^2 = 0$ is the exceptional divisor and the other solution is $s^3 y = 1$. At this point we note that the only point we missed when looking at U_0 was $[0, 1]$. This is the point where $s = 0$. This point is clearly not on $s^3 y = 1$ so the point is of no consequence.
We conclude that the exceptional divisor intersects $\pi^{-1}(C \backslash \{O\})$ at one point. This means that $\overline{\pi^{-1}(C \backslash \{O\})}$ requires the addition of one point. Note that π defines an isomorphism outside of the origin by $(x, y) \mapsto (x, y : \frac{x}{y})$ and $(x, y, : t) \mapsto (x, y)$.

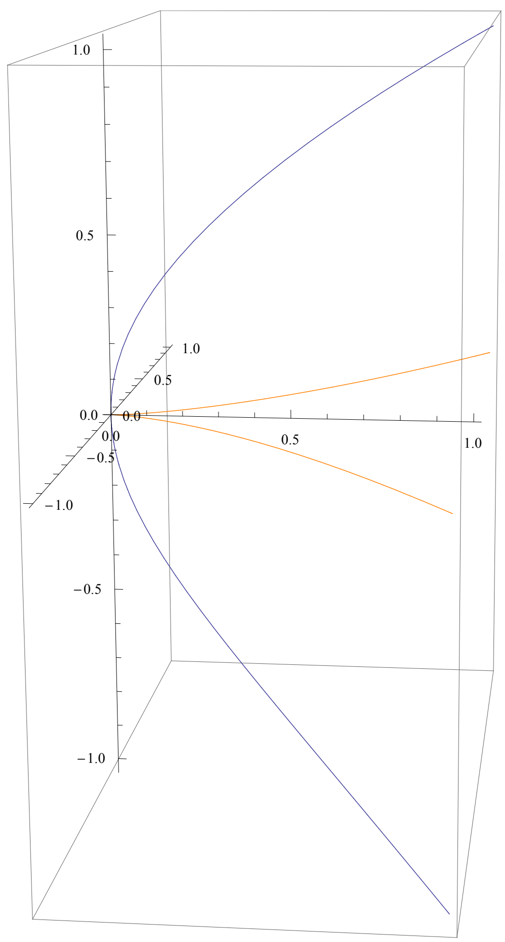

Figure 9: The blowup of the cusp $x^3 - y^2$.

The homogeneous polynomials $x_iy_j - x_jy_i$ define a variety $\Gamma \subseteq \mathbb{P}^{n-1} \times K^n$. We will illustrate why this variety corresponds with our intuition of a blowup.

Lemma 5.1. *Let $(p, q) \in \mathbb{P}^{n-1} \times K^n$ be a point and $\hat{p} \in K^n \backslash \{0\}$ the affine vector corresponding to p. Then $(p, q) \in \Gamma$ if and only if $q = t\hat{p}$ for some t.*

Proof. Let $(p, q) \in \mathbb{P}^{n-1} \times K^n$ with $p = [p_1 : \ldots : p_n]$ and not all $p_i = 0$. Suppose $(p, q) \in \Gamma$, then $p_iq_j = p_jq_i$. There exists a coordinate $p_k \neq 0$ and for this coordinate we have $p_kq_j = p_jq_k$ for all j. So $q_j = \frac{q_k}{p_k}p_j = tp_j$ and we conclude that $q = t\hat{p}$.
Suppose that $q = t\hat{p}$ for some $t \in K$. We have that $p_iq_j - p_jq_i = p_itp_j - p_jtp_i = 0$ is satisfied. Hence $(p, q) \in \Gamma$. □

So points lie on Γ if and only if they lie on some line through the origin. Now we look at what happens at the origin.

Lemma 5.2. *Let $q \in K^n$ be such that $q \neq 0$. Then for some $p \in \mathbb{P}^{n-1}$ we have $(\mathbb{P}^{n-1} \times \{q\}) \cap \Gamma = \{(p, q)\} \in \mathbb{P}^{n-1} \times K^n$. If $q = 0$, then $(\mathbb{P}^{n-1} \times \{0\}) \cap \Gamma = \mathbb{P}^{n-1} \times \{0\}$.*

Proof. Suppose $q \neq 0$, then we know that $(p, q) \in \Gamma$ if and only if $q = t\hat{p}$ for some $t \neq 0$. We have $\hat{p} = (p_1, \ldots, p_n) = \frac{q}{t}$ which is clearly unique. Hence the intersection $(\mathbb{P}^{n-1} \times \{q\}) \cap \Gamma$ consists of one point (p, q).
Suppose $q = 0$, then $(p, q) \in \Gamma$ if and only if $t\hat{p} = 0$. Because there exists $p_i \neq 0$, we require $t = 0$. So every $\hat{p} \in K^n \backslash \{0\}$ satisfies the criterion which corresponds with a copy of $\mathbb{P}^{n-1}$. So indeed $(\mathbb{P}^{n-1} \times \{0\}) \cap \Gamma = \mathbb{P}^{n-1} \times \{0\}$. □

If we take $\pi : \Gamma \to K^n$ the projection map. Then $\pi^{-1}(q)$ consists of a single point when $q \neq 0$ and when $q = 0$ it consists of a copy of $\mathbb{P}^{n-1}$. So we can regard Γ as the variety obtained by removing the origin from K^n and replacing it by a copy of $\mathbb{P}^{n-1}$.

Lemma 5.3. *Let L be a line through the origin of K^n parametrized by tv with $v \in K^n \backslash \{O\}$. Then L defines a curve in Γ that intersects $\mathbb{P}^{n-1} \times \{0\}$ in a distinct point.*

Proof. Let L be parametrized by tv with $v \in K^n \backslash \{0\}$ and w the point in $\mathbb{P}^{n-1}$ defined by L. We have for (w, tv) that $w_itv_j - w_jtv_i = tv_iv_j - tv_jv_i = 0$ so $(w, tv) \in \Gamma$. L intersects the origin when $t = 0$. This corresponds to the unique point $(w, 0) \in \mathbb{P}^{n-1} \times \{0\}$. Hence the curve defined by L intersects the origin in a distinct point in $\mathbb{P}^{n-1} \times \{0\}$. □

The lemma illustrates that there is a one-to-one correspondence between lines through the origin in K^n and points in $\pi^{-1}(0)$. Thus, Γ separates tangent directions at the origin. This should reinforce your intuition on what a blowup does.

5.2 Blowing up along a variety

Let $X \subseteq K^n$ be an affine variety and $Y \subseteq X$ a subvariety. For blowing up along a subvariety we will define a regular birational map $\pi : BL_O(X) \to X$ associated to Y that is an isomorphism away from Y.

Definition 5.5. Let the ideal of Y be generated by the functions $f_0, \ldots, f_n$ (we can do this locally and the set does not have to be minimal). We define $\varphi : X \to \mathbb{P}^n, p \mapsto [f_0(p), \ldots, f_n(p)]$. Note that φ is not defined on Y. For the graph of φ we have $\mathrm{graph}(\varphi) = \{(p, \varphi(p)) \mid p \in X \backslash Y\} \subset X \times \mathbb{P}^n$. We define the blowup as $Bl_Y(X) = \overline{\mathrm{graph}(\varphi)}$.

Example : We look at the case where $Y = (0,0)$ is the origin in affine space $\mathbb{A}^2$. The ideal is $I(Y) = (x, y)$ and the graph is $graph(\varphi) = \{(x, y : x : y) \mid (x, y) \neq (0,0)\}$. We define $F = xx_1 - yx_0$ where x_1, x_0 are homogeneous coordinates and graph(φ) lies in the zero set of F. By the definition of closed sets in the Zariski topology, this zero set is closed. Furthermore, since graph(φ) is 2D, F is irreducible and $2D$ we conclude that the zero set is the smallest set containing graph(φ). So our new definition corresponds to our earlier definition of blowing up at a point.

Example : Now we will blow up along the circle $x^2 + y^2 - 1 = 0, z = 0$. The ideal is $I(S^1) = \langle z, x^2+y^2-1\rangle$ and the graph is $\{(x, y, z, z : x^2+y^2-1) \mid (z, x^2+y^2-1) \neq (0,0)\}$. We define $F = x_0(x^2 + y^2 - 1) - zx_1$ and obtain our blowup.

6 Blowing up the Kleinian singularities

In this section we blow up the Kleinian singularities. Du Val obtained the following description of a resolution of a Kleinian singularity [Bur83, page 111]. The preimage of a singularity s is a connected union of projective lines that intersect transversally

$$\pi^{-1}(s) = C_1 \cup \cdots \cup C_m, \quad C_i \cong \mathbb{P}^1$$

We associate a vertex with each curve and two vertices are connected by an edge if the corresponding curves intersect. The resulting graph is called a *intersection diagram.* To blow up the Kleinian singularities, we find a sufficiently large set of polynomial equations that our surfaces must satisfy and use irreducibility to show that these equations must constitute the proper transform (the Zariski closure of $\pi^{-1}(X/\{O\})$). We check irreducibility in every chart but to guarantee irreducibility in our entire space we need the following theorem.

Theorem 6.1. *Let X be a connected variety and suppose we have a finite collection of non-empty open sets $\{U_i\}$ that cover X. Suppose that $X \cap U_i$ is irreducible for all i, then X is irreducible.*

Proof. Suppose that X is reducible, then X has a minimal decomposition into irreducible varieties $X = C_1 \cup \cdots \cup C_m$ [Har92, Theorem 5.7]. We consider $Y = C_a \cup C_b$ such that Y is connected, i.e. $C_a \cap C_b \neq \emptyset$. Since $\{U_i\}$ is a cover of X, we choose a U_i that contains a point $p \in C_a \cap C_b$ such that $U_i \cap C_a \not\subseteq U_i \cap C_b$ and $U_i \cap C_b \not\subseteq U_i \cap C_a$. Hence $Y \cap U_i = (U_i \cap C_a) \cup (U_i \cap C_b)$ is the union of two non-empty irreducible varieties. So $Y \cap U_i$ is reducible and hence there exists an i such that $X \cap U_i$ is reducible. $\square$

The Kleinian surfaces are connected and a connected variety remains connected after blowing up. Therefore, using the theorem it is sufficient for us to check for irreducibility in each of the charts U_i. One could check this using Eisenstein's criterion but doing so would add a significant number of pages to this thesis. Hence we checked for irreducibility using the computer.

6.1 Resolution of Kleinian singularities: A_n

In this section we resolve the A_{k-1} singularity $xy - z^k$ and construct the corresponding intersection diagram. We will blow up $xy - z^k$ at the singularity $Y = (0, 0, 0)$. We take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0y - X_2z^{k-1}, \quad X_0X_1 - X_2^2z^{k-2}$$

- $U_0 \cap Bl_Y X$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations:

  $$xu - y, \quad xv - z, \quad yv - zu, \quad y - vz^{k-1}, \quad u - v^2z^{k-2}$$

  Note that $xv - z = xvu - zu = yv - zu$ if $u \neq 0$. Hence $yv - zu$ is redundant. Similarly, $u - v^2z^{k-2} = xu - xv^2z^{k-2} = y - vz^{k-1}$ if $x \neq 0$ so $y - vz^{k-1}$ is redundant. We are left with three irreducible, independent equations.

  $$xu - y, \quad xv - z, \quad u - v^2z^{k-2}$$

  Our current space is isomorphic with $\mathbb{C}^5_{x,y,z,u,v}$ so the variety defined by these equations has dimension 2. Hence, it is defines the proper transform of our blowup. Using the first two equations, we can embed our surface into $\mathbb{C}^3_{x,u,v}$ with the remaining equation $u - x^{k-2}v^k$.
  Note: From now on, we will give the three relevant equation without argument to make these sections less cumbersome.

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$yu - x, \quad yv - z, \quad u - v^2z^{k-2}$$

  This is isomorphic to $u - y^{k-2}v^k$ in $\mathbb{C}^3_{u,y,v}$.

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$zu - x, \quad zv - y, \quad uv - z^{k-2}$$

  This is isomorphic to $uv - z^{k-2}$ in $\mathbb{C}^3_{u,v,z}$.

Case $k = 2$

- $U_0 \cap Bl_Y X$: After blowing up, we obtain the equation $u - v^2$ which has Jacobian

  $$(0, \;\; 1, \;\; -2v),$$

  which is smooth. *No singularity.*

- $U_1 \cap Bl_Y X$: After blowing up, we obtain the equation $u - v^2$ which has Jacobian

  $$(0, \;\; 1, \;\; -2v),$$

  which is smooth. *No singularity.*

- $U_2 \cap Bl_Y X$: After blowing up, we obtain the equation $uv - 1$ which has Jacobian

$$(v, \ u, \ 0),$$

which is smooth. *No singularity.*

Going back to the equations, the exceptional divisor $\pi^{-1}(O)$ gives $X_0X_1 - X_2^2$. This is a smooth quadric which is isomorphic to $\mathbb{P}^1$. So the intersection diagram is a single vertex. The following picture illustrates that $\pi^{-1}(O)$ contracts the circle ($\mathbb{P}^1$) to a point.

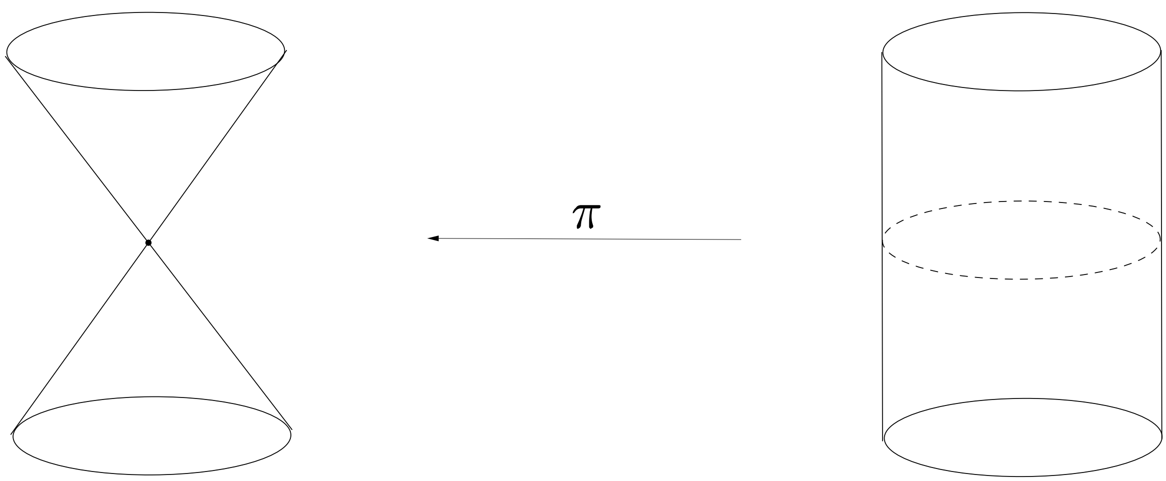

Figure 10: The resolution of A_1 type singularity.

Case $k = 3$

- $U_0 \cap Bl_Y X$: After blowing up, we obtain the equation $u - xv^3$ which has Jacobian

$$(-v^3, \ 1, \ -3xv^2),$$

which is smooth. *No singularity.*

- $U_1 \cap Bl_Y X$: After blowing up, we obtain the equation $u - yv^3$ which has Jacobian

$$(-v^3, \ 1, \ -3yv^2),$$

which is smooth. *No singularity.*

- $U_2 \cap Bl_Y X$: After blowing up, we obtain the equation $uv - z$ which has Jacobian

$$(v, \ u, \ -1),$$

which is smooth. *No singularity.*

The exceptional divisor $\pi^{-1}(O)$ gives $X_0X_1 = 0$. This constitutes two intersecting projective lines.

Case $k \geq 4$

- $U_0 \cap Bl_Y X$: After blowing up, we obtain the equation $u - x^{k-2}v^k$ which has Jacobian
$$(-(k-2)x^{k-3}v^k, \;\; 1, \;\; -kx^{k-2}v^{k-1}),$$
which is smooth. *No singularity.*

- $U_1 \cap Bl_Y X$: After blowing up, we obtain the equation $u - y^{k-2}v^k$ which has Jacobian
$$(-(k-2)y^{k-3}v^k, \;\; 1, \;\; -ky^{k-2}v^{k-1}),$$
which is smooth. *No singularity.*

- $U_2 \cap Bl_Y X$: After blowing up, we obtain the equation $uv - z^{k-2}$ which has Jacobian
$$(v, \;\; u, \;\; -(k-2)z^{k-3}),$$
which has a *singularity at* $(0,0,0)$. Note that this is exactly the equation for A_{k-3}.

The exceptional divisor $\pi^{-1}(O)$ gives $X_0X_1 = 0$. This constitutes two intersecting projective lines. The singularity found in the U_2 chart lies at the intersection of these two lines since $u = v = 0$ implies $X_0 = X_1 = 0$. If we blow up again, we get another two projective lines. This continues until we reach A_1 or A_2 which can be resolved by one last blowup. Hence we have: Using the procedure for finding the intersection

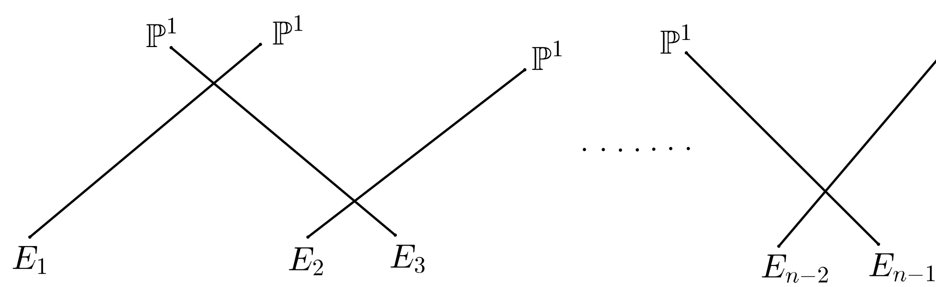

Figure 11: The exceptional divisor of A_{n-1} type singularity.

diagram described above, we get the following graph for A_n (there are n nodes):

$$A_n : \quad \overset{n}{\circ - \circ \cdots \circ - \circ}$$

6.2 Resolution of Kleinian singularities: D_n

In this section we resolve the D_{n+2} singularity $x^2 + y^2z + z^{n+1}$ and construct the corresponding intersection diagram. We will blow up $x^2+y^2z+z^{n+1}$ at the singularity $Y = (0,0,0)$. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2 + zX_1^2 + X_2^2 z^{n-1}$$

- $U_0 \cap Bl_Y X$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:
$$xu - y, \quad xv - z, \quad 1 + zu^2 + v^2z^{n-1}$$
This is isomorphic to $1 + xvu^2 + x^{n-1}v^{n+1}$ in $\mathbb{C}^3_{x,u,v}$.

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:
$$yu - x, \quad yv - z, \quad u^2 + z + v^2z^{n-1}$$
This is isomorphic to $u^2 + yv + y^{n-1}v^{n+1}$ in $\mathbb{C}^3_{u,y,v}$.

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:
$$zu - x, \quad zv - y, \quad u^2 + v^2z + z^{n-1}$$
This is isomorphic to $u^2 + v^2z + z^{n-1}$ in $\mathbb{C}^3_{u,v,z}$.

Case $n = 2$

- $U_0 \cap Bl_Y X$: After blowing up, we obtain the equation $1 + xvu^2 + xv^3$ which has Jacobian
$$(v(v^2+u^2), \;\; 2xuv, \;\; x(u^2+3v^2)),$$
which is smooth. *No singularity.*

- $U_1 \cap Bl_Y X$: After blowing up, we obtain the equation $u^2 + yv + yv^3$ which has Jacobian
$$(2u, \;\; v(1+v^2), \;\; y(1+3v^2)),$$
which has *three singularities* at $(0,0,0), (0,0,i)$ and $(0,0,-i)$.

- $U_2 \cap Bl_Y X$: After blowing up, we obtain the equation $u^2 + v^2z + z$ which has Jacobian
$$(2u, \;\; 2vz, \;\; 1+v^2),$$
which has *two singularities* at $(0,i,0)$ and $(0,-i,0)$.

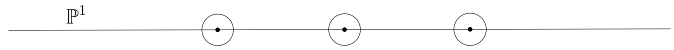

Figure 12: The exceptional divisor of D_4 at this moment.

However, note that $v = \pm i$ in the U_2 chart implies $X_1 = \pm i X_2$. Hence, the two singularities found in the U_2 chart are the same as the singularities $(0,0,i)$ and $(0,0,-i)$ found in the U_1 chart. Going back to the equations, the exceptional divisor $\pi^{-1}(O)$ gives X_0^2. This is a single projective line $\mathbb{P}^1$. Note that in the U_1 chart, $u = 0$ for all singularities so all three lie on the exceptional divisor.

To further resolve the D_4 singularity we have to blow up at the three singularities consecutively. If we translate our blowup surface by $\epsilon = v, v - i, v + i$ we can do this easily. This is however quite cumbersome so we will do the center singularity $(0,0,0)$ here and leave the other two to the reader if they so desire.

Convention: To avoid a plethora of variables, we resort back to x, y, z coordinates after each blowup.

So we will blow up $x^2 + yz + yz^3$ at the singularity $Y = (0,0,0)$. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2 + X_1X_2 + X_1X_2z^2$$

- $U_0 \cap Bl_Y X$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$xu - y, \quad xv - z, \quad 1 + uv + uvz^2$$

  This is isomorphic to $1 + uv + ux^2v^3$ in $\mathbb{C}^3_{x,u,v}$. The Jacobian is

  $$(2uxv^3, \;\; v(1 + x^2z^2), \;\; u(1 + 3x^2v^2)),$$

  which is smooth. *No singularity.*

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$yu - x, \quad yv - z, \quad u^2 + v + vz^2$$

  f This is isomorphic to $u^2 + v + y^2v^3$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

  $$(2u, \;\; 2yv^3, \;\; 1 + 3y^2v^2),$$

  which is smooth. *No singularity.*

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$zu - x, \quad zv - y, \quad u^2 + v + vz^2$$

  This is isomorphic to $u^2 + v + vz^2$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is

$$(2u, \ \ 1 + z^2, \ \ 2vz),$$

  which has *two singularities* at $(0, 0, i)$ and $(0, 0, -i)$.

The two singularities at $(0, 0, i)$ and $(0, 0, -i)$ are the previous two we have not yet resolved. After resolving these two we obtain a smooth surface. Going back to the equations, the exceptional divisor $\pi^{-1}(O)$ gives $X_0^2 + X_1X_2$. This is a smooth quadric which is isomorphic to $\mathbb{P}^1$. Hence we have:

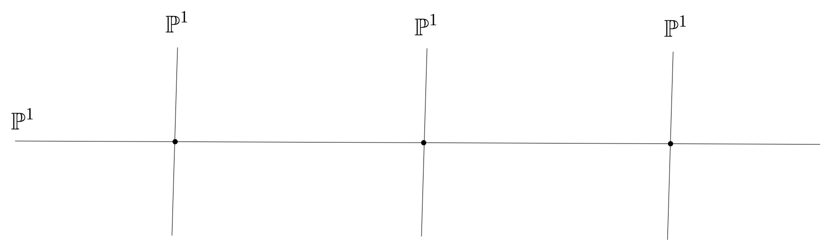

Figure 13: The exceptional divisor of D_4 at this moment.

The corresponding intersection diagram is

$$D_4 : \quad \begin{array}{ccccc} \circ & - & \circ & - & \circ \\ & & | & & \\ & & \circ & & \end{array}$$

The following picture illustrates how $\pi^{-1}(O)$ contracts the circles ($\mathbb{P}^1$) to a point.

Case $n \geq 3$

- $U_0 \cap Bl_Y X$: After blowing up, we obtain the equation $1 + xvu^2 + x^{n-1}v^{n+1}$ which has Jacobian

$$(v(u^2 + (n-1)x^{n-2}v^n), \ \ 2xvu, \ \ x(u^2 + (n+1)x^{n-2}v^n)),$$

  which is smooth. *No singularity.*

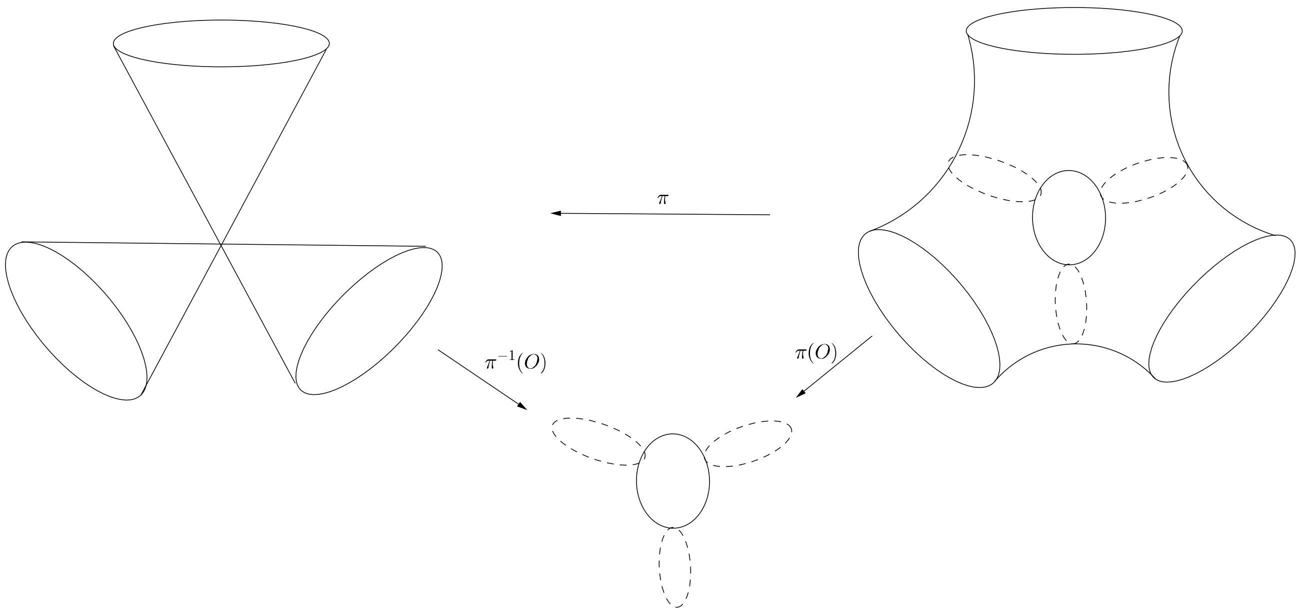

Figure 14: [Bur83, page 112] The blow up of D_4 and the exceptional divisor.

- $U_1 \cap Bl_Y X$: After blowing up, we obtain the equation $u^2+yv+y^{n-1}v^{n+1}$ which has Jacobian

$$(2u, \;\; v(1+(n-1)y^{n-2}v^n), \;\; y(1+(n+1)y^{n-2}v^n)),$$

  which has a *singularity at* $(0,0,0)$.

- $U_2 \cap Bl_Y X$: After blowing up, we obtain the equation $u^2+v^2z+z^{n-1}$ which has Jacobian

$$(2u, \;\; 2vz, \;\; v^2+(n-1)z^{n-2}),$$

  which has a *singularity at* $(0,0,0)$. Note that this is exactly the equation for D_{n-2}.

The exceptional divisor $\pi^{-1}(O)$ gives $X_0^2 = 0$. This is one projective line. Both the singularity found in the U_2 chart and the singularity found in the U_1 chart lie on this projective line.

We will blow up the singularity in the U_1 chart first. So we blow up $x^2+yz+y^{n-1}z^{n+1}$ at the singularity $Y = (0,0,0)$. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2 + X_1X_2 + X_2^2(yz)^{n-1}$$

- $U_0 \cap Bl_Y X$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$xu - y, \quad xv - z, \quad 1 + uv + v^2(yz)^{n-1}$$

This is isomorphic to $1 + uv + u^{n-1}v^{n+1}x^{2(n-1)}$ in $\mathbb{C}^3_{x,u,v}$. The Jacobian is

$$(2(n-1)u^{n-1}v^{n+1}x^{2n-3},\ \ v(1+(n-1)u^{n-2}v^n x^{2(n-1)}),\ \ u(1+(n+1)u^{n-2}v^n x^{2(n-1)})),$$

which is smooth. *No singularity.*

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:
$$yu - x, \quad yv - z, \quad u^2 + v + v^2(yz)^{n-1}$$
This is isomorphic to $u^2 + v + v^{n+1}y^{2(n-1)}$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is
$$(2u,\ \ 2(n-1)v^{n+1}y^{2n-3},\ \ 1 + (n+1)v^n y^{2(n-1)}),$$
which is smooth. *No singularity.*
- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:
$$zu - x, \quad zv - y, \quad u^2 + v + (yz)^{n-1}$$
This is isomorphic to $u^2 + v + v^{n-1}z^{2(n-1)}$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is
$$(2u,\ \ 1 + (n-1)v^{n-2}z^{2(n-1)},\ \ 2(n-1)v^{n-1}z^{2n-3}),$$
which is smooth. *No singularity.*

The exceptional divisor $\pi^{-1}(O)$ gives $X_0^2 + X_1X_2$. This is a smooth quadric which is isomorphic to $\mathbb{P}^1$. So blowing up D_n gives two intersecting projective lines with a D_{n-2} type singularity on one of them. Resolving it will eventually lead to the following:
Using the procedure for finding the intersection diagram, we get the following graph

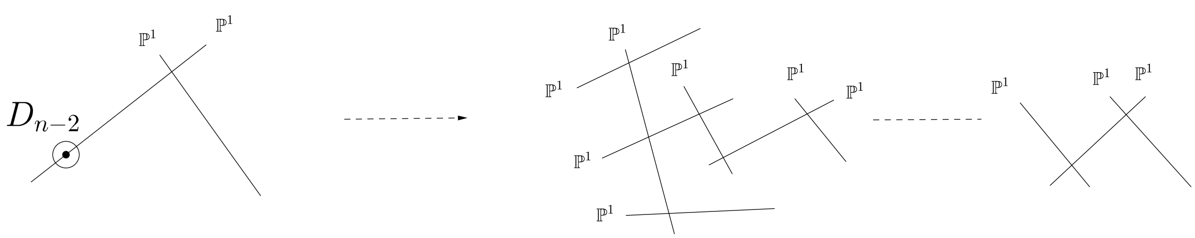

Figure 15: The exceptional divisor of D_n type singularity.

for D_n for $n \geq 4$ (there are n vertices in total):

$$\begin{array}{cccccccc} & \circ & & & n & & & \\ & | & & & & & & \\ D_n : & \circ & - & \circ & \cdots & \circ & - & \circ \\ & | & & & & & & \\ & \circ & & & & & & \end{array}$$

6.3 Resolution of Kleinian singularities: E_6

In this section we resolve the E_6 singularity $x^4+y^3+z^2$ and construct the corresponding intersection diagram. We will blow up $x^4+y^3+z^2$ at the singularity $Y=(0,0,0)$. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2x^2 + X_1^2y + X_2^2$$

- $U_0 \cap Bl_YX$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$xu - y, \quad xv - z, \quad x^2 + u^2y + v^2$$

  This is isomorphic to $x^2 + xu^3 + v^2$ in $\mathbb{C}^3_{x,u,v}$. The Jacobian is

  $$(2x + u^3, \;\; 3xu^2, \;\; 2v),$$

  which has a *singularity at* $(0,0,0)$.

- $U_1 \cap Bl_YX$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$yu - x, \quad yv - z, \quad u^2x^2 + y + v^2$$

  This is isomorphic to $y^2u^4 + y + v^2$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

  $$(4y^2u^3, \;\; 1 + 2yu^4, \;\; 2v),$$

  which is smooth. *No singularity.*

- $U_2 \cap Bl_YX$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$zu - x, \quad zv - y, \quad 1 + u^2x^2 + v^2y$$

  This is isomorphic to $1 + zv^3 + z^2u^4$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is

  $$(4z^3u^3, \;\; 3zv^2, \;\; v^3 + 2zu^4),$$

  which is smooth. *No singularity.*

The exceptional divisor $\pi^{-1}(O)$ gives $X_2^2 = x = y = z = 0$. In U_0 this is $x = 0 \wedge v = 0$. This is one projective line with a singularity on it. We will blow up $x^2 + xy^3 + z^2$ at the singularity $Y = (0,0,0)$. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2 + X_1^2xy + X_2^2$$

- $U_0 \cap Bl_YX$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$xu - y, \quad xv - z, \quad 1 + u^2xy + v^2$$

  This is isomorphic to $1 + x^2u^3 + v^2$ in $\mathbb{C}^3_{x,u,v}$. The Jacobian is

  $$(2xu^3, \;\; 3x^2u^2, \;\; 2v),$$

  which is smooth. *No singularity.*

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:
$$yu - x, \quad yv - z, \quad u^2 + xy + v^2$$
This is isomorphic to $u^2 + uy^2 + v^2$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is
$$(2u + y^2, \;\; 2uy, \;\; 2v),$$
which has a *singularity at* $(0, 0, 0)$.

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:
$$zu - x, \quad zv - y, \quad 1 + u^2 + v^2 xy$$
This is isomorphic to $1 + u^2 + uz^2v^3$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is
$$(2u + z^2v^3, \;\; 3uz^2v^2, \;\; 2uzv^3),$$
which is smooth. *No singularity.*

The exceptional divisor of the previous blowup gave us $x = 0 \wedge v = 0$ which, after the change in coordinates, is $x = 0 \wedge z = 0$ in this blowup. In $\mathbb{C}^5_{x,y,z,u,v}$ this means that $yu = 0 \wedge yv = 0$ which gives us two cases:

- $y = 0$ This gives us $u^2 + v^2 = 0$ which constitutes two intersecting projective lines $(u, 0, iu)$ and $(u, 0, -iu)$ that intersect at $(0, 0, 0)$.
- $u = 0 \wedge v = 0$ This gives us another projective line $(0, y, 0)$ which intersects the other at the origin.

After the second blowup we have an exceptional divisor which consists of three projective lines that intersect in the origin. The origin is also the position of the singularity.

We will blow up $x^2 + xy^2 + z^2$ at the singularity $Y = (0, 0, 0)$. we take a sufficiently large collection of equations:
$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2 + X_1^2 x + X_2^2$$

- $U_0 \cap Bl_Y X$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:
$$xu - y, \quad xv - z, \quad 1 + xu^2 + v^2$$
This is isomorphic to $1 + xu^2 + v^2$ in $\mathbb{C}^3_{x,u,v}$. The Jacobian is
$$(u^2, \;\; 2xu, \;\; 2v),$$
which is smooth. *No singularity.*

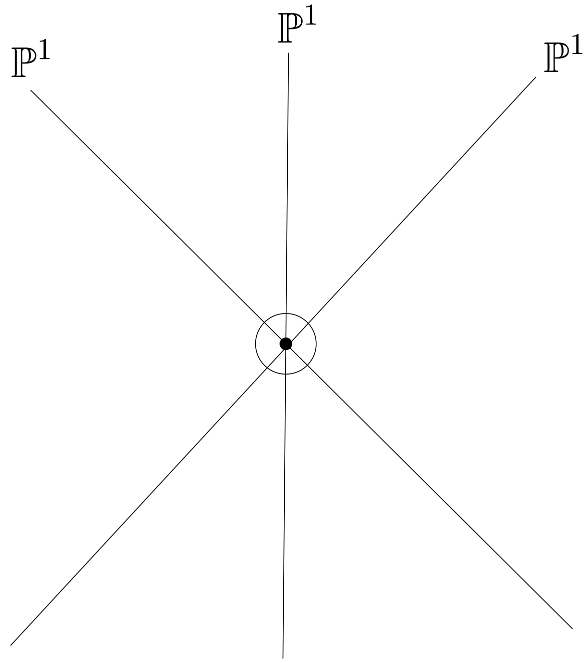

Figure 16: The exceptional divisor of E_6 after the second blowup.

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$yu - x, \quad yv - z, \quad u^2 + x + v^2$$

  This is isomorphic to $u^2 + yu + v^2$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

$$(2u + y, \quad , u \quad 2v),$$

  which has a *singularity at* $(0, 0, 0)$.

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$zu - x, \quad zv - y, \quad 1 + u^2 + v^2 x$$

  This is isomorphic to $1 + u^2 + zuv^2$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is

$$(2u + zv^2, \;\; 2zuv, \;\; uv^2),$$

  which is smooth. *No singularity.*

The exceptional divisor of the previous blowup gave us three cases in $\mathbb{C}^5_{x,y,z,u,v}$. Finding the entire exceptional divisor is not as straightforward this time since we have to check all the charts. We do U_1 first. To keep track of the various lines we designate each $\mathbb{P}^1$ by some E_i during this procedure,

- $y = 0 \wedge x^2 + z^2 = 0$ This translates to $y = 0 \wedge y^2(u^2 + v^2) = 0$ and it has to satisfy $u^2 + yu + v^2$. Thus we get $y = 0 \wedge u^2 + v^2 = 0$ which is still a cross of projective lines that intersect at $(0, 0, 0)$. The two components of the cross are E_1 and E_2.

- $z = 0 \wedge x = 0$ This translates to $yv = 0 \wedge yu = 0$ which gives us two options. $y = 0$ gives us the same cross as the previous case. The case $u = 0 \wedge v = 0$ gives us a new projective line (E_3) that intersects the cross at the origin.

At this point the exceptional divisor has not significantly changed with respect to the second blowup. Now we check the U_0 chart.

- $y = 0 \wedge x^2 + z^2 = 0$ This translates to $xu = 0 \wedge x^2(1 + v^2) = 0$, which gives us two cases:
  If $x = 0$, we get $v = \pm i$ which constitute two projective lines that never intersect (E_1' and E_2').
  If $u = 0$, we get $v = \pm i$ which are also two non-intersecting lines (E_3' and E_4'). Note that E_3' intersects E_1' at $(0, 0, i)$ and E_4' intersects E_2' at $(0, 0, -i)$.

- $z = 0 \wedge x = 0$ This translates to $xv = 0 \wedge x = 0$ which gives us E_1' and E_2'.

We do not have seven different projective lines but rather perceive portions of the same ones in the different charts. We can compare the various lines by keeping in mind that on $U_0 \cap U_1$ we have $u_1 \mapsto \frac{1}{u_0}$ and $v_1 \mapsto \frac{v_0}{u_0}$. Firstly, note that E_3' and E_4' do not lie in U_1 since $u_0 = 0$ implies $X_1 = 0$. Note that $u_1^2 + v_1^2 \mapsto \frac{1}{u_0^2}(1 + v_0^2)$ which is indeed satisfied by $v_0 = \pm i$. So E_1' and E_2' correspond to E_1 and E_2.
To sum up, we have the three lines that intersect in the origin in U_1 but two of its components (E_1 and E_2) intersect with two lines that are not visible in U_1. The situation is illustrated in the following diagram.

Obviously it is necessary to check the U_2 chart as well. However, it yields no changes to the exceptional divisor and, considering the already substantial length of this section, we refrain from doing it here.

We will blow up $x^2 + xy + z^2$ at the singularity $Y = (0, 0, 0)$. Although we continue in the U_1 chart, the two projective lines E_3' and E_4' are still present but we can no longer perceive them. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2 + X_0X_1 + X_2^2$$

- $U_0 \cap Bl_Y X$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$xu - y, \quad xv - z, \quad 1 + u + v^2$$

  This is isomorphic to $1 + u + v^2$ in $\mathbb{C}^3_{x,u,v}$. The Jacobian is

  $$(0, \ 1, \ 2v),$$

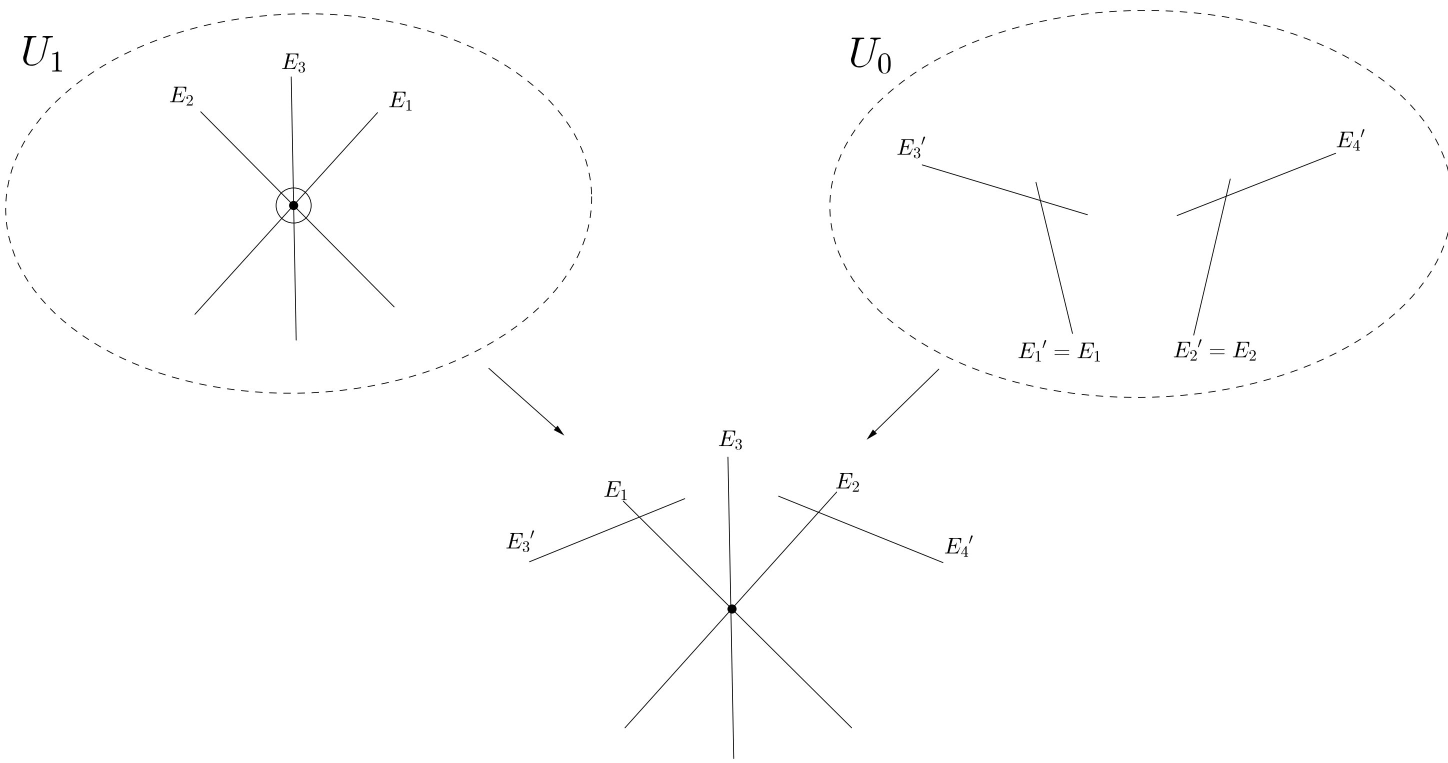

Figure 17: The exceptional divisor of E_6 after the third blowup.

which is smooth. *No singularity.*

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$yu - x, \quad yv - z, \quad u^2 + u + v^2$$

This is isomorphic to $u^2 + u + v^2$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

$$(2u + 1, \quad ,0 \quad 2v),$$

which is smooth. *No singularity.*

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$zu - x, \quad zv - y, \quad 1 + uv + u^2$$

This is isomorphic to $1 + uv + u^2$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is

$$(2u + v, \quad u, \quad 0),$$

which is smooth. *No singularity.*

The E_6 singularity has been resolved. The exceptional divisor of the previous blowup gave us three cases in $\mathbb{C}^5_{x,y,z,u,v}$. Finding the entire exceptional divisor requires only checking U_0 and U_1 since the other chart yields no changes (the reader can check this). For U_0:

- $y = 0 \wedge x^2 + z^2 = 0$ This translates to $xu = 0 \wedge x^2(1 + v^2) = 0$, which gives us two cases:
  If $x = 0$, we get $1 + u + v^2 = 0$ which is one projective line. This is a new projective line we will denote by E_4.
  If $u = 0$, we get $v = \pm i$ which are two non-intersecting lines (E_1 and E_2). Note that they intersect E_4 in $(0, 0, i)$ and $(0, 0, -i)$ respectively.

- $z = 0 \wedge x = 0$ This translates to $xv = 0 \wedge x = 0$ which gives us E_4 again.

Now we check the U_1 chart.

- $y = 0 \wedge x^2 + z^2 = 0$ This translates to $y = 0 \wedge y^2(u^2 + v^2) = 0$, which means that $y = 0 \wedge u^2 + u + v^2 = 0$. A simple change in coordinates on $U_0 \cap U_1$ will show that this the portion of E_4 that lies in U_1.

- $x = 0 \wedge z = 0$ This translates to $yu = 0 \wedge yv = 0$ which gives us two cases. If $y = 0$ we obtain the previous case. If $u = 0 \wedge v = 0$ we obtain a new projective line. Note that in $\mathbb{C}^5_{x,y,z,u,v}$ this implies $x = 0 \wedge z = 0$ so this is E_3 from the previous blowups.

The fourth blowup separated the cross into three non-intersecting lines who each intersect a fourth. If we recall that E_1 and E_2 intersected with E_3' and E_4' we get the following illustration. Using the procedure for finding the intersection diagram, we get

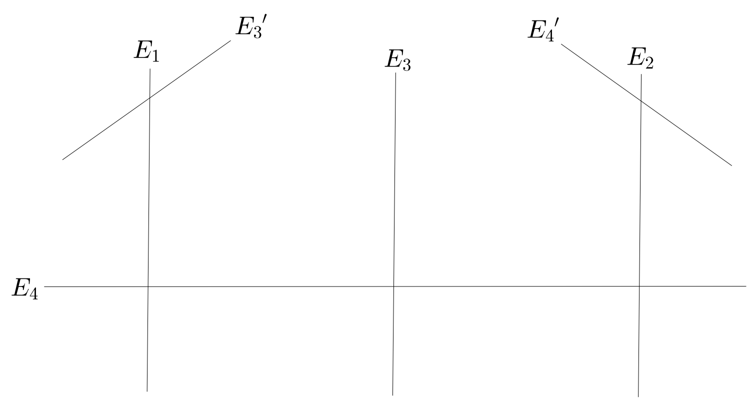

Figure 18: The exceptional divisor of E_6 after the fourth and final blowup.

the following graph for E_6:

$$
E_6 : \quad \begin{array}{ccccccccc}
\circ & - & \circ & - & \circ & - & \circ & - & \circ \\
 & & & & | & & & & \\
 & & & & \circ & & & &
\end{array}
$$

6.4 Resolution of Kleinian singularities: E_7

In this section we resolve the E_7 singularity $x^2+y^3+yz^3$ and construct the corresponding intersection diagram. This procedure is very elaborate and we will therefore refrain from discussing any charts that do not contribute anything. The reader can check for them self that these charts do not contain singularities or any additional projective lines.

We will blow up $x^2+y^3+yz^3$ at the singularity $Y=(0,0,0)$. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2 + yX_1^2 + yzX_2^2$$

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$zu - x, \quad zv - y, \quad u^2 + yv^2 + yz$$

  This is isomorphic to $u^2 + zv^3 + z^2 v$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is

  $$(2u, \;\; z(3v^2+z), \;\; v(v^2+2z)),$$

  which has a *singularity at* $(0,0,0)$.

The exceptional divisor $\pi^{-1}(O)$ gives $X_0^2 = x = y = z = 0$. In U_2 this is $u=0 \wedge z=0$. This is one projective line with a singularity on it. We will blow up $x^2+zy^3+z^2y$ at the singularity $Y=(0,0,0)$. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2 + yzX_1^2 + yX_2^2$$

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$yu - x, \quad yv - z, \quad u^2 + yz + yv^2$$

  This is isomorphic to $u^2 + vy^2 + yv^2$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

  $$(2u, \;\; v(v+2y), \;\; y(y+2v)),$$

  which has a *singularity at* $(0,0,0)$.

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$zu - x, \quad zv - y, \quad u^2 + yzv^2 + y$$

  This is isomorphic to $u^2 + zv + z^2v^3$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is

  $$(2u, \;\; z(1+3zv^2), \;\; v(1+2zv^2)),$$

  which has a *singularity at* $(0,0,0)$.

We have a situation where each chart has one singularity which is not visible from the other chart. Since blowing up is a local procedure, we can blow up the singularities consecutively provided we keep proper track of all the projective lines. The exceptional divisor of the previous blowup gave us $u = 0 \wedge z = 0$ which, after the change in coordinates, is $x = 0 \wedge z = 0$ in this blowup. We consider each chart separately.

Chart U_1 We have $x = 0 \wedge z = 0$ which means that $yu = 0 \wedge yv = 0$ which gives us two cases:

- $y = 0$ This gives us $u^2 = 0$ which is one projective line $(0, 0, v)$ through the origin.
- $u = 0 \wedge v = 0$ This gives us one projective line that intersects at the origin.

Chart U_2 We have $x = 0 \wedge z = 0$ which means that $zu = 0 \wedge z = 0$ which gives us two cases:

- $z = 0$ This gives us $u^2 = 0$ which is one projective line $(0, v, 0)$ through the origin.

Obviously the line $u = 0 \cap v = 0$ in U_1 does not exist in U_2. The line in U_2 is a segment of the other one in U_1. Hence we get the following situation:

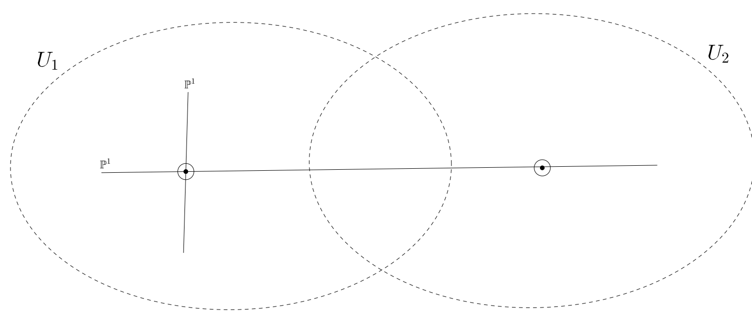

Figure 19: The exceptional divisor of E_7 after the second blowup.

Firstly, we resolve the singularity $(0, 0, 0)$ in the U_2 chart. We will blow up $x^2 + yz + y^3z^2$ at the singularity $Y = (0, 0, 0)$. At this point it is important to note that the projective line that contains both the singularities is $x = 0 \wedge z = 0$ in the U_2 chart at this point. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2 + X_1X_2 + X_2^2y^3$$

- $U_1 \cap Bl_YX$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$yu - x, \quad yv - z, \quad u^2 + v + v^2y^3$$

This is isomorphic to $u^2 + v + v^2y^3$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

$$(2u, \;\; 3y^2v^2, \;\; 1 + 2vy^3),$$

which is smooth. *No singularity.*

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$zu - x, \quad zv - y, \quad u^2 + v + y^3$$

  This is isomorphic to $u^2 + v + v^3z^3$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is

$$(2u, \;\; 1 + 3v^2z^2, \;\; 3v^3z^2),$$

  which is smooth. *No singularity.*

The exceptional divisor of the previous blowup gave us one case in $\mathbb{C}^5_{x,y,z,u,v}$. We only need to check U_1.

- $x = 0 \wedge z = 0$ This translates to $yu = 0 \wedge yv = 0$ which gives us two options. $y = 0$ gives us $y = 0 \wedge u^2 + v = 0$ which is a projective line through the origin. The case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one.

We need to figure out which one connects to the other singularity. We can do this by determining what each line "folds down to". Recall that the line we are looking for corresponded to $x = 0 \wedge z = 0$.

The line $z = 0 \wedge u^2 + v = 0$ folds down to $x = 0$, $y = 0$ and $z = 0$. Hence it contracts to the singularity. The line $u = 0 \wedge v = 0$ in U_1 corresponds to $x = 0$ and $z = 0$ when folded down. Hence this last projective line connects to the other singularity. This gives us the following situation:

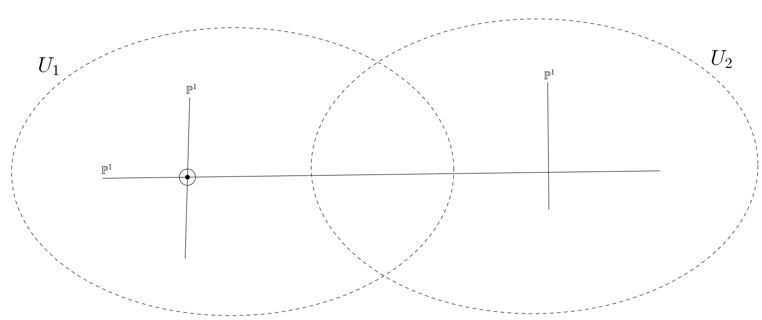

Figure 20: The exceptional divisor of E_7 after resolving the first of two singularities during the third blowup.

Now we resolve the singularity $(0,0,0)$ in the U_1 chart. We will blow up $x^2+zy^2+yz^2$ at the singularity $Y=(0,0,0)$. At this point it is important to note that the projective line that connected the two singularities (although one is now resolved) is $x=0 \wedge y=0$ in the U_1 chart. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2 + X_1^2 z + X_2^2 y$$

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$yu - x, \quad yv - z, \quad u^2 + z + v^2 y$$

  This is isomorphic to $u^2 + vy + v^2 y$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

  $$(2u, \ \ v(1+v), \ \ y(1+2v)),$$

  which has two *singularities at* $(0,0,0)$ and $(0,0,-1)$.

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$zu - x, \quad zv - y, \quad u^2 + v^2 z + y$$

  This is isomorphic to $u^2 + v^2 z + vz$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is

  $$(2u, \ \ z(1+2v) \ \ v(1+v)),$$

  which has two *singularities at* $(0,0,0)$ and $(0,-1,0)$.

The singularities $(0,0,-1)$ in U_1 and $(0,-1,0)$ in U_2 are the same singularity. We have the curious situation of both charts containing two singularities of which one is shared. The exceptional divisor of the previous blowup gave us two cases in $\mathbb{C}^5_{x,y,z,u,v}$. We do each chart separately.

Chart U_1

- $x=0 \wedge y=0$ This translates to $yu=0 \wedge y=0$, which gives us $y=0 \wedge u=0$ which is a projective line through both singularities.

- $x=0 \wedge z=0$ This translates to $yu=0 \wedge yv=0$ which gives us two options. $y=0$ gives the other projective line and the case $u=0 \wedge v=0$ gives us a new projective line that intersects the other one in the $(0,0,0)$ singularity.

Chart U_2

- $x=0 \wedge z=0$ This translates to $zu=0 \wedge z=0$, which gives us $z=0 \wedge u=0$ which is a projective line through both singularities.

- $x=0 \wedge y=0$ This translates to $zu=0 \wedge zv=0$ which gives us two options. $z=0$ gives the other projective line and the case $u=0 \wedge v=0$ gives us a new projective line that intersects the other one in the $(0,0,0)$ singularity.

At this point the reader should be familiar with mapping the u_i and v_i to one another. We obtain that $y = 0 \wedge u = 0$ in U_1 is a segment of the same line as $z = 0 \wedge u = 0$ in U_2. We are left with three distinct projective lines, one of which connects tot the tail section we have already found. We check which one by folding down:

- $u_1 = 0 \wedge y = 0$ in U_1 This collapses to $x = y = z = 0$ which is the singularity.
- $u_1 = 0 \wedge v_1 = 0$ in U_1 This collapses to $x = z = 0$ which is not the correct projective line.
- $u_2 = 0 \wedge v_2 = 0$ in U_2 This collapses to $x = y = 0$ which is the correct line.

We conclude that the line $u_2 = 0 \wedge v_2 = 0$ in U_2 connects to the tail section we already had. This gives us the following situation:

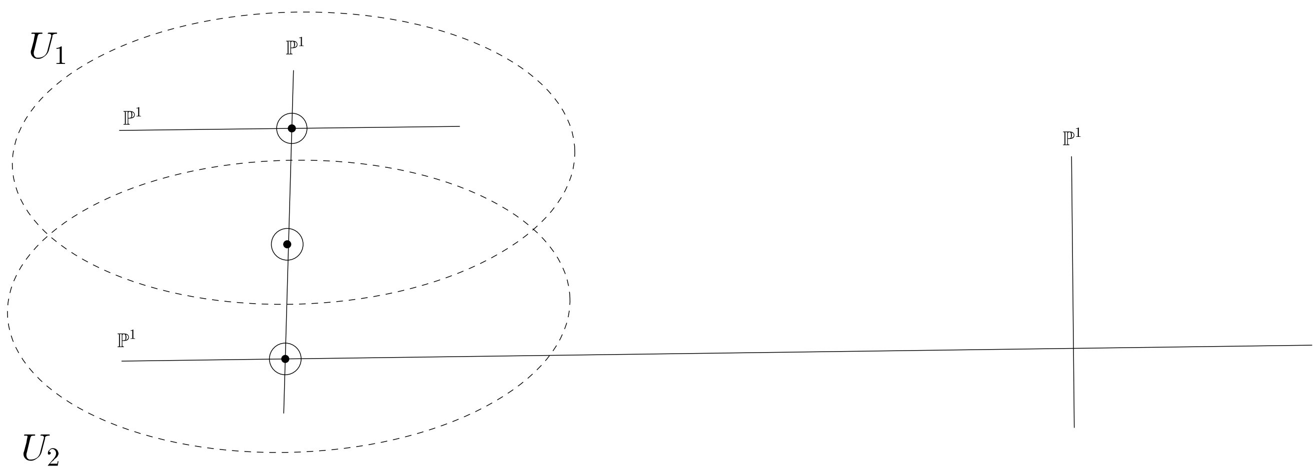

Figure 21: The exceptional divisor of E_7 after resolving both singularities of the second blowup.

We have the situation of both charts containing two singularities of which one is shared. We blow up $x^2 + yz + y^2z$ at $(0, -1, 0)$ in the U_2 chart first. If $y = y' - 1$ we obtain $x^2 + y'^2z - y'z$ and blow up in $(0, 0, 0)$. we take a sufficiently large collection of equations:

$$xX_1 - y'X_0, \quad xX_2 - zX_0, \quad y'X_2 - zX_1, \quad X_0^2 - X_1X_2 + X_1^2z$$

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y',z,u,v}$:

  $$y'u - x, \quad y'v - z, \quad u^2 + z - v$$

  This is isomorphic to $u^2 - v + y'v$ in $\mathbb{C}^3_{u,y',v}$. The Jacobian is

  $$(2u, \ \ v, \ \ y' - 1),$$

  which has a *singularities at* $(0, 1, 0)$.

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y',z,u,v}$:

$$zu - x, \quad zv - y', \quad u^2 + v^2 z - v$$

This is isomorphic to $u^2 + v^2 z - v$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is

$$(2u, \;\; 2vz - 1 \;\; v^2),$$

which is smooth. *No singularity.*

The singularity $(0, 1, 0)$ in U_1 is the old singularity that is visible in the U_2 chart. The exceptional divisor of the previous blowup gave us two cases in $\mathbb{C}^5_{x,y',z,u,v}$. We only need to check U_1.

Chart U_1

- $x = 0 \wedge z = 0$ This translates to $y'u = 0 \wedge y'v = 0$ which gives us two options. $y' = 0$ gives us $y' = 0 \wedge v - u^2 = 0$ which is a projective line that does not intersect the singularity. The case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one and also crosses the singularity.

- $x = 0 \wedge y' = 1$ This translates to $y'u = 0 \wedge y' = 1$. This gives us $u = 0 \wedge y' = 1$ which intersects $u = 0 \wedge v = 0$ in the singularity $(0, 1, 0)$.

We are left with three distinct projective lines. Note that $u = 0 \wedge y' = 1$ in U_1 collapses to $x = 0 \wedge y = 0$ so it connects to our tail section. This gives us the following situation:

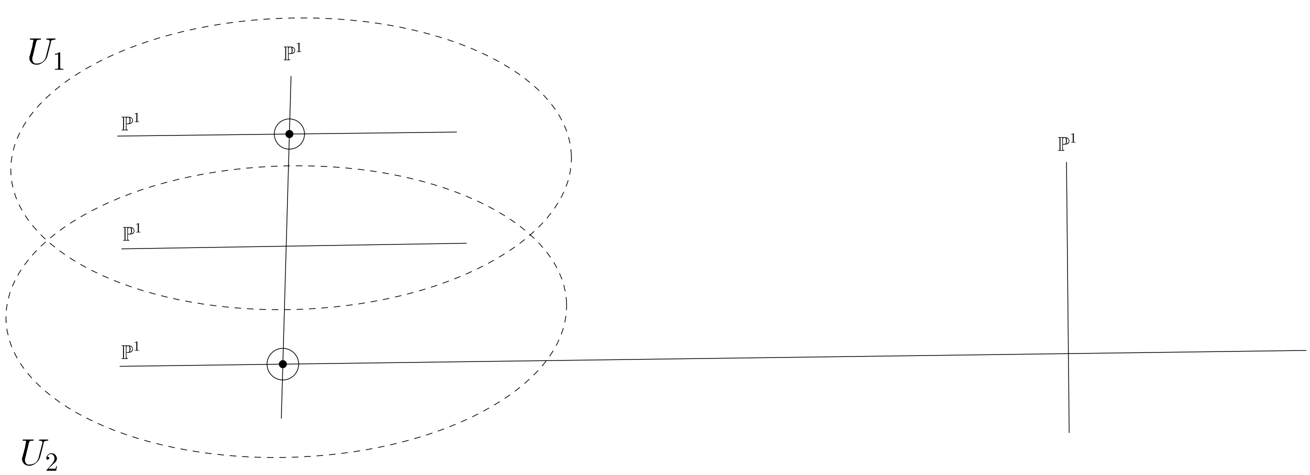

Figure 22: The exceptional divisor of E_7 after resolving one of the singularities during the fourth blowup.

Now we want to blow up $x^2 + y'z - z$ at $(0, 1, 0)$ in the U_1 chart. However, suppose that $y' = 1 - y$, then we obtain $x^2 - yz$ which is a singularity of type A_1. Furthermore, blowing up at $(0, 0, -1)$ in U_1 is symmetrical to the U_2 case since $x^2 + yz^2 + yz$ leads to $x^2 + y^2z + yz$ by exchanging y and z. Therefore, the other singularity is also of type A_1.
We only blow up the U_2 case, i.e. we blow up $x^2 - yz$ in $(0, 0, 0)$ (after the coordinate change). Recall that the two relevant exceptional divisors are $x = 0 \wedge y = 0$ which connects to the tail section and $x = 0 \wedge z = 0$ which also intersects the second singularity in U_1. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2 - X_1X_2$$

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$yu - x, \quad yv - z, \quad u^2 - v$$

  This is isomorphic to $u^2 - v$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

  $$(2u, \ \ 0, \ \ -1),$$

  which is smooth. *No singularity.*

- $U_2 \cap Bl_Y X$: We define $u = \frac{X_0}{X_2}$ and $v = \frac{X_1}{X_2}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$zu - x, \quad zv - y, \quad u^2 - v$$

  This is isomorphic to $u^2 - v$ in $\mathbb{C}^3_{u,v,z}$. The Jacobian is

  $$(2u, \ \ -1, \ \ 0),$$

  which is smooth. *No singularity.*

The exceptional divisor of the previous blowup gave us two cases in $\mathbb{C}^5_{x,y,z,u,v}$. We do each chart separately.

Chart U_1

- $x = 0 \wedge y = 0$ This translates to $yu = 0 \wedge y = 0$, which gives us $y = 0 \wedge u^2 - v = 0$ which is a projective line through the origin.
- $x = 0 \wedge z = 0$ This translates to $yu = 0 \wedge yv = 0$ which gives us two options. $y = 0$ gives the other projective line and the case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one in $(0, 0, 0)$.

Chart U_2

- $x = 0 \wedge z = 0$ This translates to $zu = 0 \wedge z = 0$, which gives us $z = 0 \wedge u^2 - v = 0$ which is a projective line through the origin.

- $x = 0 \wedge y = 0$ This translates to $zu = 0 \wedge zv = 0$ which gives us two options. $z = 0$ gives the other projective line and the case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one in $(0, 0, 0)$.

The two lines $u_i^2 - v_i$ are segments of the same exceptional divisor. Furthermore, the projective line $u = 0 \wedge v = 0$ in U_2 connects to our tail section while $u = 0 \wedge v = 0$ in U_1 connects to the other singularity in the previous U_1 chart. As mentioned, both of the remaining singularities were of the type A_1 and both resolved similarly. This leads to the following: Using the procedure for finding the intersection diagram, we get the

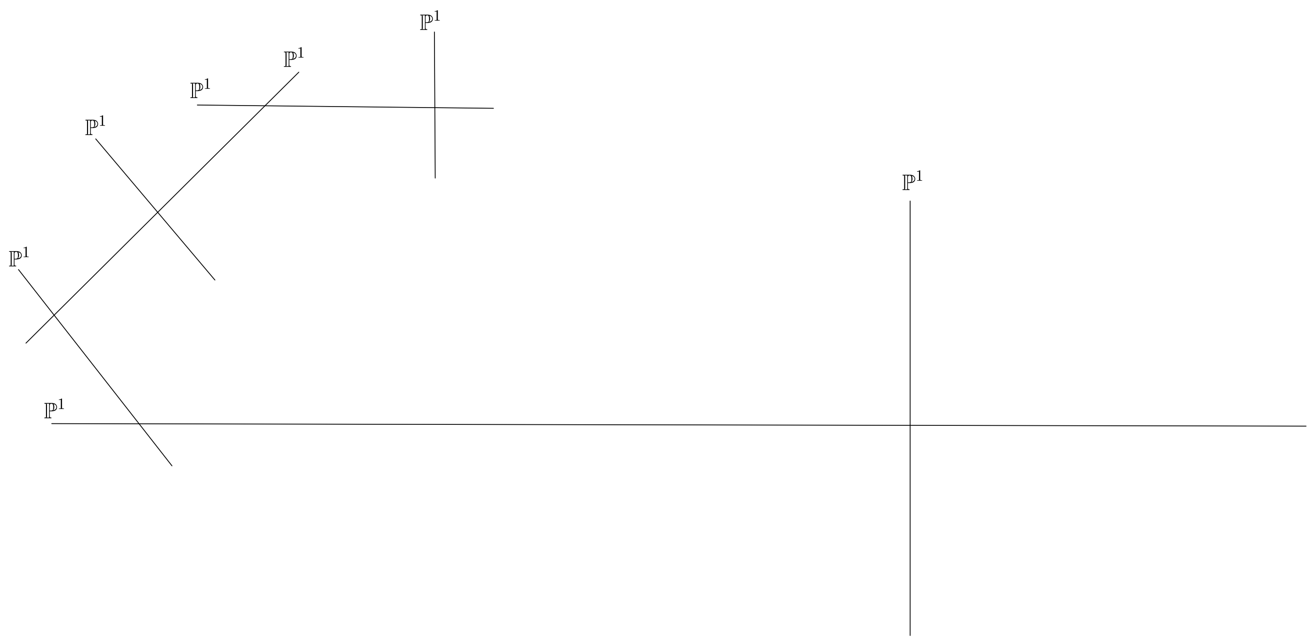

Figure 23: The exceptional divisor of E_7 after resolving all the singularities.

following graph for E_7:

$$E_7 : \begin{array}{ccccccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & & & | & & & & & & \\ & & & & \circ & & & & & & \end{array}$$

6.5 Resolution of Kleinian singularities: E_8

In this section we resolve the E_8 singularity $x^5+y^3+z^2$ and construct the corresponding intersection diagram. This procedure is very elaborate and we will therefore refrain from discussing any charts that do not contribute anything. The reader can check for them self that these charts do not contain singularities or any additional projective lines.

We will blow up $x^5+y^3+z^2$ at the singularity $Y = (0,0,0)$. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2x^3 + X_1^2y + X_2^2$$

- $U_0 \cap Bl_Y X$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$xu - y, \quad xv - z, \quad x^3 + u^2y + v^2$$

  This is isomorphic to $x^3 + xu^3 + v^2$ in $\mathbb{C}^3_{x,u,v}$. The Jacobian is

  $$(3x^2 + u^3, \;\; 3u^2x, \;\; 2v),$$

  which has a *singularity at* $(0,0,0)$.

The exceptional divisor $\pi^{-1}(O)$ gives $X_2^2 = 0$ (and $x = y = z = 0$). This is one projective line with the singularity on it. We will blow up $x^3+xy^3+z^2$ at the singularity $Y = (0,0,0)$. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2x + X_1^2yx + X_2^2$$

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$yu - x, \quad yv - z, \quad u^2x + yx + v^2$$

  This is isomorphic to $u^3y + uy^2 + v^2$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

  $$(y(y + 3u^2), \;\; u(u^2 + 2y), \;\; 2v),$$

  which has a *singularity at* $(0,0,0)$.

The exceptional divisor of the previous blowup gave us $v = 0 \wedge x = 0$ which, after the change in coordinates, is $x = 0 \wedge z = 0$ in this blowup. So we get $yu = 0 \wedge yv = 0$. If $y = 0$, then $v = 0$ so we have one projective line $y = 0 \wedge v = 0$ that crosses the singularity. The other case is $u = 0 \wedge v = 0$ which intersects the previous projective line in the origin.

We will blow up $x^3y + xy^2 + z^2$ at the singularity $Y = (0,0,0)$. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2xy + X_1^2x + X_2^2$$

- $U_0 \cap Bl_Y X$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$xu - y, \quad xv - z, \quad xy + u^2x + v^2$$

  This is isomorphic to $ux^2 + xu^2 + v^2$ in $\mathbb{C}^3_{x,u,v}$. The Jacobian is

$$(u(u+2x), \ \ x(x+2u), \ \ 2v),$$

  which has a *singularity at* $(0,0,0)$.

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$yu - x, \quad yv - z, \quad xyu^2 + x + v^2$$

  This is isomorphic to $uy + v^2 + y^2u^3$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

$$(3y^2u^2, \ \ u(1+2yu^2), \ \ 2v),$$

  which has a *singularity at* $(0,0,0)$.

We have one singularity in each chart. The exceptional divisor of the previous blowup gave us two cases in $\mathbb{C}^5_{x,y,z,u,v}$. We do each chart separately.

Chart U_0

- $x = 0 \wedge z = 0$ This translates to $x = 0 \wedge xv = 0$, which gives us $x = 0 \wedge v = 0$ which is a projective line through the singularity. This line folds down to a point if we reverse the blowup.

- $y = 0 \wedge z = 0$ This translates to $xu = 0 \wedge xv = 0$ which gives us two options. $x = 0$ gives the other projective line and the case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one in the $(0,0,0)$ singularity. This line is $y = 0 \wedge z = 0$ from before the blowup.

Chart U_1

- $y = 0 \wedge z = 0$ This translates to $y = 0 \wedge yv = 0$, which gives us $y = 0 \wedge v = 0$ which is a projective line through the singularity. This line is a segment of the $x = 0 \wedge v = 0$ line in U_0.

- $x = 0 \wedge z = 0$ This translates to $yu = 0 \wedge yv = 0$ which gives us two options. $y = 0$ gives the other projective line and the case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one in the $(0,0,0)$ singularity. This line is $x = 0 \wedge z = 0$ from before the blowup.

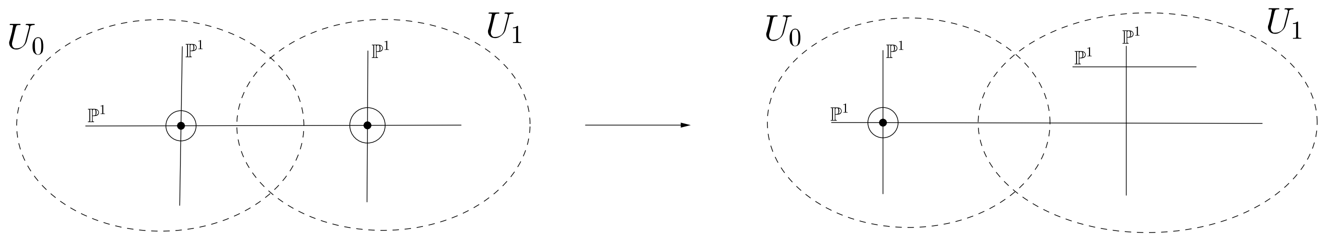

Figure 24: The exceptional divisor of E_8 after the third blowup.

If we look at the U_1 chart we obtain the equation $xy + x^3y^2 + z^2$. This is the same equation as the third blowup of E_7 if we exchange the variables $x \mapsto y$, $y \mapsto z$ and $z \mapsto x$. So we can immediately obtain: We will blow up $x^2y+xy^2+z^2$ at the singularity $Y = (0,0,0)$. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0^2y + X_1^2x + X_2^2$$

- $U_0 \cap Bl_Y X$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:
$$xu - y, \quad xv - z, \quad y + u^2x + v^2$$
This is isomorphic to $ux + u^2x + v^2$ in $\mathbb{C}^3_{x,u,v}$. The Jacobian is
$$(u(1+u), \;\; x(1+2u), \;\; 2v),$$
which has *two singularities at* $(0,0,0)$ and $(0,-1,0)$.
- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:
$$yu - x, \quad yv - z, \quad x + u^2y + v^2$$
This is isomorphic to $uy + u^2y + v^2$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is
$$(y(1+2u), \;\; u(1+u), \;\; 2v),$$
which has *two singularities at* $(0,0,0)$ and $(-1,0,0)$.

We have two singularities in each chart, one of which is shared. The exceptional divisor of the previous blowup gave us two cases in $\mathbb{C}^5_{x,y,z,u,v}$. Note that before the blowup, the line that connected to the tail section was $x = 0 \wedge z = 0$. We do each chart separately.

Chart U_0

- $x = 0 \wedge z = 0$ This translates to $x = 0 \wedge xv = 0$, which gives us $x = 0 \wedge v = 0$ which is a projective line through both singularities. This line folds down to a point if we reverse the blowup.

- $y = 0 \wedge z = 0$ This translates to $xu = 0 \wedge xv = 0$ which gives us two options. $x = 0$ gives the other projective line and the case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one in the $(0,0,0)$ singularity. This line is $y = 0 \wedge z = 0$ from before the blowup.

Chart U_1

- $y = 0 \wedge z = 0$ This translates to $y = 0 \wedge yv = 0$, which gives us $y = 0 \wedge v = 0$ which is a projective line through both singularities. This line is a segment of the $x = 0 \wedge v = 0$ line in U_0.
- $x = 0 \wedge z = 0$ This translates to $yu = 0 \wedge yv = 0$ which gives us two options. $y = 0$ gives the other projective line and the case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one in the $(0,0,0)$ singularity. This line is $x = 0 \wedge z = 0$ from before the blowup.

We have the following:

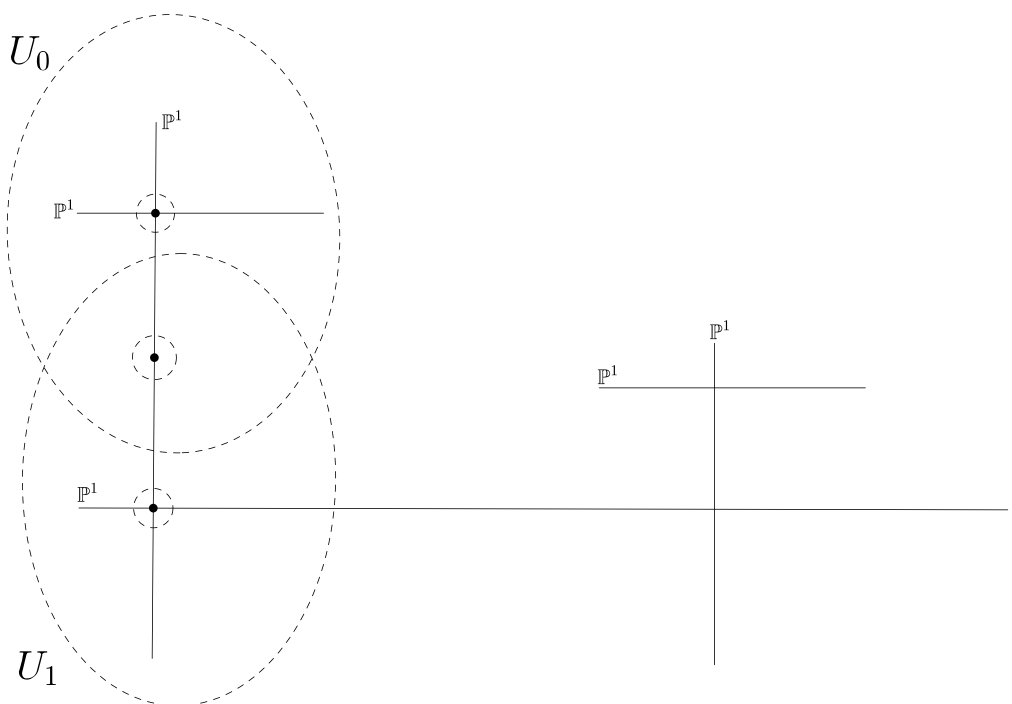

Figure 25: The exceptional divisor of E_8 after the fourth blowup.

We will blow up $xy + x^2y + z^2$ at the singularity $Y = (0,0,0)$ in the U_1 chart. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_0X_1 + X_0^2y + X_2^2$$

- $U_0 \cap Bl_Y X$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$xu - y, \quad xv - z, \quad u + y + v^2$$

  This is isomorphic to $u + xu + v^2$ in $\mathbb{C}^3_{x,u,v}$. The Jacobian is

$$(u, \ \ 1 + x, \ \ 2v),$$

  which has a *singularity at* $(-1, 0, 0)$.

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

$$yu - x, \quad yv - z, \quad u + u^2 y + v^2$$

  This is isomorphic to $u + u^2 y + v^2$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

$$(1 + 2uy, \ \ u^2, \ \ 2v),$$

  which is smooth. *No singularity.*

We have one singularity in the U_0 chart which is the one from the previous blowup. The exceptional divisor of the previous blowup gave us two cases in $\mathbb{C}^5_{x,y,z,u,v}$. Note that before the blowup, the line that connected to the tail section was $x = 0 \wedge z = 0$. We do each chart separately.

Chart U_0

- $x = 0 \wedge z = 0$ This translates to $x = 0 \wedge xv = 0$, which gives us $x = 0 \wedge u + v^2 = 0$ which is a projective line that folds down to a point if we reverse the blowup.

- $y = 0 \wedge z = 0$ This translates to $xu = 0 \wedge xv = 0$ which gives us two options. $x = 0$ gives the other projective line and the case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one in $(0, 0, 0)$ and it also crosses the singularity. This line is $y = 0 \wedge z = 0$ from before the blowup.

Chart U_1

- $y = 0 \wedge z = 0$ This translates to $y = 0 \wedge yv = 0$, which gives us $y = 0 \wedge u + v^2 = 0$ which is a segment of the $x = 0 \wedge u + v^2 = 0$ line in U_0.

- $x = 0 \wedge z = 0$ This translates to $yu = 0 \wedge yv = 0$ which gives us two options. $y = 0$ gives the other projective line and the case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one. This line is $x = 0 \wedge z = 0$ from before the blowup.

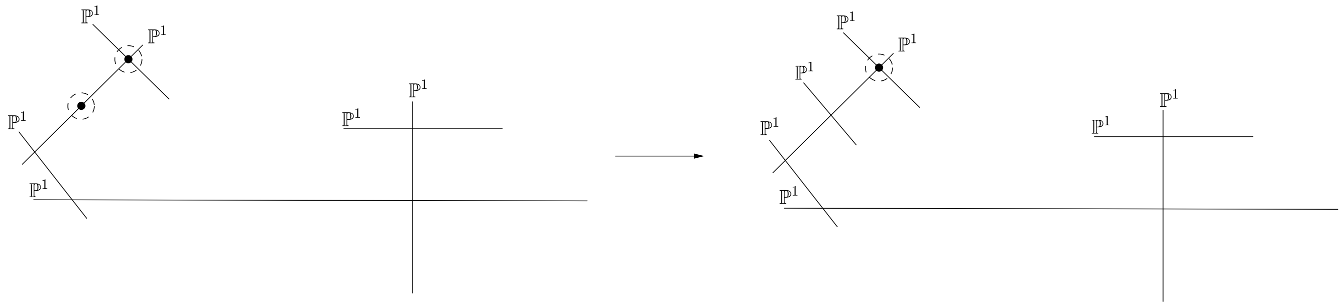

Figure 26: The exceptional divisor of E_8 after the fifth and sixth blowup.

So the line $u = 0 \wedge v = 0$ in U_1 connects to the tail section. Moreover, note that in U_0 we have the singularity $(-1, 0, 0)$ and the equation $y + xy + z^2$. If we substitute $x = -x' - 1$ we get $z^2 - x'y$ which is a singularity of type A_1. Hence, we have the following:

The reader can compute the blowup of $(0, -1, 0)$ in the U_0 chart to see that, after a suitable change of coordinates, we end up with $z^2 - xy$. This obviously has a singularity at $(0, 0, 0)$ and we also have three projective lines: $x = 0 \wedge z = 0$, $y = 0 \wedge z = 0$ and $y = 1 \wedge x = z^2$. The latter does not intersect with the singularity and can therefore be ignored. Finally, We will blow up $z^2 - xy$ at the singularity $Y = (0, 0, 0)$ in the U_0 chart. we take a sufficiently large collection of equations:

$$xX_1 - yX_0, \quad xX_2 - zX_0, \quad yX_2 - zX_1, \quad X_2^2 - X_0X_1$$

- $U_0 \cap Bl_Y X$: We define $u = \frac{X_1}{X_0}$ and $v = \frac{X_2}{X_0}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$xu - y, \quad xv - z, \quad v^2 - u$$

  This is isomorphic to $v^2 - u$ in $\mathbb{C}^3_{x,u,v}$. The Jacobian is

  $$(0, \ -1, \ 2v),$$

  which is smooth. *No singularity.*

- $U_1 \cap Bl_Y X$: We define $u = \frac{X_0}{X_1}$ and $v = \frac{X_2}{X_1}$ and obtain for our equations in $\mathbb{C}^5_{x,y,z,u,v}$:

  $$yu - x, \quad yv - z, \quad v^2 - u$$

  This is isomorphic to $v^2 - u$ in $\mathbb{C}^3_{u,y,v}$. The Jacobian is

  $$(-1, \ 0, \ 2v),$$

  which is smooth. *No singularity.*

The exceptional divisor of the previous blowup gave us two cases in $\mathbb{C}^5_{x,y,z,u,v}$. We do each chart separately.

Chart U_0

- $x = 0 \wedge z = 0$ This translates to $x = 0 \wedge xv = 0$, which gives us $x = 0 \wedge v^2 - u = 0$ which is a projective line that folds down to a point if we reverse the blowup.
- $y = 0 \wedge z = 0$ This translates to $xu = 0 \wedge xv = 0$ which gives us two options. $x = 0$ gives the other projective line and the case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one. This line is $y = 0 \wedge z = 0$ from before the blowup.

Chart U_1

- $y = 0 \wedge z = 0$ This translates to $y = 0 \wedge yv = 0$, which gives us $y = 0 \wedge v^2 - u = 0$ which is a segment of the $x = 0 \wedge v^2 - u = 0$ line in U_0.
- $x = 0 \wedge z = 0$ This translates to $yu = 0 \wedge yv = 0$ which gives us two options. $y = 0$ gives the other projective line and the case $u = 0 \wedge v = 0$ gives us a new projective line that intersects the other one. This line is $x = 0 \wedge z = 0$ from before the blowup.

We have resolved the E_8 singularity and obtained: Using the procedure for finding the

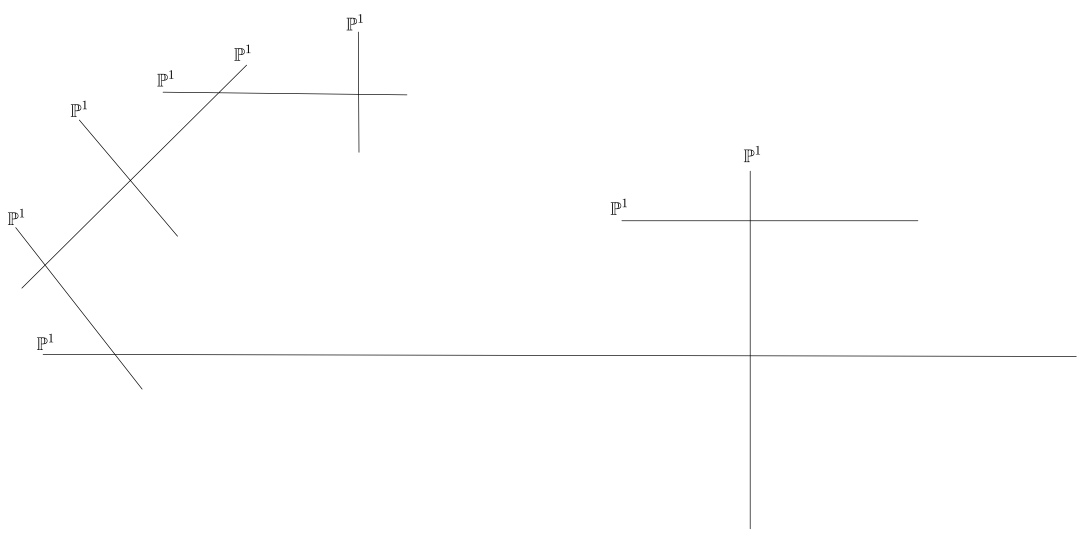

Figure 27: The exceptional divisor of E_8 after resolving all the singularities.

intersection diagram, we get the following graph for E_8:

$$E_8 : \quad \begin{array}{ccccccccccccc} \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & & & | & & & & & & & & \\ & & & & \circ & & & & & & & & \end{array}$$

7 The McKay correspondence

In this section we establish the McKay correspondence. Firstly, we introduce the relevant representation theory as it is introduced in [JL12] and [FH91]. Then we use it to find the McKay graphs of the finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ and show their relationship to the Dynkin diagrams of type ADE.

7.1 Representation theory

Definition 7.1. [JL12] Let G be a group and let F be $\mathbb{R}$ or $\mathbb{C}$. A *representation* of G over F is a homomorphism ρ from G to $\mathrm{GL}(n, F)$, for some n. The *degree* of ρ is the integer n.

The representation gives the vector space F^n the so called structure of an FG-module which has the following formal definition.

Definition 7.2. [JL12] Let V be a finite dimensional vector space over F and let G be a group. Then V is an *FG-module* if a multiplication vg is defined satisfying:

- $vg \in V$
- $v(gh) = (vg)h$
- $ve = e$
- $(\lambda v)g = \lambda(vg)$
- $(u + v)g = ug + vg$

A subset W of V is an *FG-submodule* if it a subspace that is itself an FG-module, i.e. $wg \in W$ for all $w \in W$, $g \in G$.

Definition 7.3. [JL12] An FG-module V is *irreducible* if it is non-zero and its only FG-submodules are $\{0\}$ and V.

We will adhere to the convention of denoting a representation of V by V itself when there is little ambiguity about the map ρ. Whenever we write gv (with $v \in V$ and $g \in G$) we mean $\rho(g)v$. There is a correspondence from the FG-modules to representations by the map $g \mapsto [g]_B$ where $[g]_B$ is the matrix of $v \mapsto gv$ relative to the basis B. Similarly, a representation yields an FG-module by setting $V = F^n$ and defining multiplication as $gv = \rho(g)v$. We say that a representation is irreducible if its corresponding FG-module is irreducible.

Both the direct sum and tensor product of two representations are again representations. Recall that a sum $V = U_1 + \cdots + U_r = \{u_1 + \cdots + u_r \mid u_i \in U_i\}$ is a *direct sum* if every element of the sum can be written in a unique way as $u_1 + \cdots + u_r$ for $u_i \in U_i$. In this case we write $V = U_1 \oplus \cdots \oplus U_r$.

Definition 7.4. [FH91] Let V and W be vector spaces over $\mathbb{C}$ with bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$. The *tensor product* space $V \otimes W$ is an mn-dimensional vector space over $\mathbb{C}$ with a basis given by

$$\{v_i \otimes w_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$

So $V \otimes W$ consists of all expressions of the form

$$\sum_{i,j} \lambda_{ij}(v_i \otimes v_j), \quad \lambda_{ij} \in \mathbb{C}$$

We define the tensor product of two representations V and W such that $g(v \otimes w) = gv \otimes gw$. The nth power tensor product $V^{\otimes n}$ simply means $V^{\otimes n} = V \otimes \cdots \otimes V$ (n times). Similarly, $V^{\oplus n}$ means $V \oplus \cdots \oplus V$ (n times).

Theorem 7.1. *[JL12, Theorem 8.1] (Maschke's Theorem) Let G be a finite group and V an FG-module. If U is an FG-submodule of V, then there is an FG-submodule W of V such that $V = U \oplus W$.*

A consequence of Maschke's is that every non-zero FG-module is decomposable into irreducible components. Formally, an FG-module V is said to be *completely reducible* if $V = U_1 \oplus \cdots \oplus U_r$ where each U_i is irreducible.

Theorem 7.2. *[JL12, Theorem 8.7] If G is a finite group, then every non-zero FG-module is completely reducible.*

Theorem 7.3. *[JL12, Theorem 9.1] (Schur's Lemma) Let V and W be irreducible $\mathbb{C}G$-modules.*

- *If $f : V \to W$ is a homomorphism, then either f is an isomorphism or $f(v) = 0$ for all $v \in V$.*
- *If $f : V \to V$ is an isomorphism, then f is a scalar multiple of the identity map.*

The above theorems and definitions neatly summarize into the following theorem:

Theorem 7.4. *[FH91, Theorem 1.8] For any representation V of a finite group G there is a unique decomposition (up to reordering)*

$$V \cong V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$$

Where V_i are distinct irreducible representations.

One of our remaining major objectives is determining this irreducible decomposition. For finite abelian groups it is fairly straightforward.

Theorem 7.5. *[JL12, Theorem 9.8] Let G be the abelian group $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$. We define the representation $\rho = \rho_{\lambda_1 \cdots \lambda_r}$ as $\rho(g) = \rho(g_1^{i_1} \cdots g_r^{i_r}) = \lambda_1^{i_1} \cdots \lambda_r^{i_r}$ (here λ_i is an n_ith root of unity). The representation ρ is irreducible and has degree 1. There are $|G|$ of these representations, and every irreducible representation of G is equivalent to precisely one of them.*

Another important theorem that we will use is

Theorem 7.6. *[JL12, Theorem 11.12] Let $V_1, \ldots, V_k$ form a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules. Then*

$$\sum_{i=1}^{k} (\dim V_i)^2 = |G|$$

7.2 Characters

Definition 7.5. [JL12] Let $\rho : G \to \mathrm{GL}(V)$ be a representation. For any $g \in G$, we define the *character* of ρ as $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$ where Tr is the trace. The character of an irreducible representation is called an *irreducible character*.

Theorem 7.7. *[JL12, Theorem 13.5] The number of irreducible characters of G is equal to the number of conjugacy classes of G.*

We have that $\chi_{V \oplus W} = \chi_V + \chi_w$ and $\chi_{V \otimes W} = \chi_V \chi_W$ and $\chi_V(h^{-1}gh) = \chi_V(g)$. So the character is really a function on the conjugacy classes of G which motivates the definition of a *character table*. Here we list the conjugacy classes of G at the top in the form of some representative and denote the number of elements in the class above it. The irreducible representations U_i of G are listed in the leftmost column and the value of $\chi_V(g)$ is shown in the appropriate position.

We define an inner product for the characters as

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_1(g)} \chi_2(g)$$

And we obtain that a representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$ [JL12, Theorem 14.20].

7.3 The McKay graphs

We now have sufficient machinery to define and determine the *McKay graph*. Let $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ be a finite subgroup. We already know the representation $V : \Gamma \to \mathrm{GL}(\mathbb{C}^2)$. The representation is decomposable into irreducible representations $V_0, \ldots, V_k$ (V_0 is trivial). Next, we determine

$$V_i \otimes V \cong V_0^{\oplus a_{i0}} \oplus \cdots \oplus V_k^{\oplus a_{ik}}$$

to obtain a_{ij}. Using what we know from characters we know that $a_{ij} = \langle \chi_{V_i \otimes V}, \chi_{V_j} \rangle = \langle \chi_{V_i}\chi_V, \chi_{V_j} \rangle$. We now construct the graph with vertices V_i and the number of edges between V_i and V_j is a_{ij}. Lastly, we remove the trivial vertex V_0 and its accompanying edges to obtain the McKay graph.

Case cyclic $\mathbb{Z}_n$

The cyclic group $\mathbb{Z}_n$ is abelian and so it has n 1-dimensional representations. We denote the elements of $\mathbb{Z}_n$ as ϵ^k where ϵ is an nth root of unity. The irreducible representations are given by $V_i(\epsilon^k) = \epsilon^{ik}$. We take as our representation $V : \mathbb{Z}_n \to \mathrm{GL}(\mathbb{C}^2)$ the representation given in section 2.3, which is

$$V : \mathbb{Z}_n \to \mathrm{GL}(\mathbb{C}^2), \;\; \epsilon^k \mapsto \begin{pmatrix} \epsilon^k & 0 \\ 0 & \epsilon^{-k} \end{pmatrix}$$

If we calculate the characters we see that $\chi_V(\epsilon^k) = \epsilon^k + \epsilon^{-k}$ and obviously $\chi_{V_i}(\epsilon^k) = \epsilon^{ik}$. So we get for a_{ij}

$$\begin{aligned} a_{ij} = \langle \chi_{V_i}\chi_V, \chi_{V_j} \rangle &= \frac{1}{n}\sum_{k=1}^{n} \overline{\chi_{V_i}(\epsilon^k)\chi_V(\epsilon^k)}\chi_{V_j}(\epsilon^k) \\ &= \frac{1}{n}\sum_{k=1}^{n} \overline{\epsilon^{k(i+1)} + \epsilon^{k(i-1)}}\epsilon^{jk} = \frac{1}{n}\sum_{k=1}^{n} \epsilon^{k(j-i-1)} + \epsilon^{k(j-i+1)} \end{aligned}$$

Note that $\frac{1}{n}\sum_{k=1}^{n} \epsilon^{ak}$ is 0 if $a \neq 0$ and 1 if $a = 0$. So $a_{ij} = 0$ if $j \neq i+1$ $(mod\ n)$ and $j \neq i-1$ $(mod\ n)$ and $a_{ij} = 0$ if $j = i+1$ $(mod\ n)$ or $j = i-1$ $(mod\ n)$. So in the McKay graph, each vertex V_i is only connected to V_{j-1} and V_{j+1} (cyclically so V_0 is connected to V_{n-1} and V_1). Hence, our graph is currently a cycle. If we remove the vertex V_0 we end up with a Dynkin diagram of type A_{n-1}.

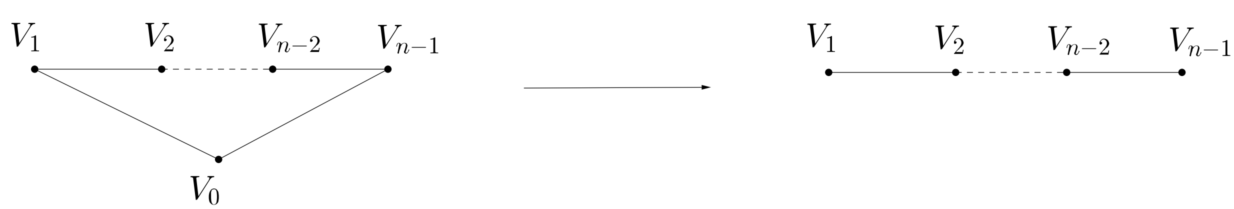

Figure 28: The McKay graph of the cyclic subgroup of order n.

Case binary dihedral BD_{4n}

The binary dihedral group generated by u and v has the form $\langle u, v \mid u^{2n} = v^4 = 1,\ v^2 = u^n,\ uv = vu^{-1} \rangle$. Looking at the 1-dimensional case, we can map $u \mapsto \epsilon$ where ϵ is $2n$th root of unity. We know that $(u^n)^2 = 1$ so $u^n = \pm 1 = v^2$. Therefore, $v = 1, -1, i, -i$. We know $uv = vu^{-1}$ so $u = u^{-1}$ and therefore $u = \pm 1$. So we have the representations:

$$\begin{aligned}
\rho_1 &: \mathrm{BD}_{4n} \to \mathrm{GL}(\mathbb{C}), \quad u \mapsto 1, \quad v \mapsto 1 \\
\rho_2 &: \mathrm{BD}_{4n} \to \mathrm{GL}(\mathbb{C}), \quad u \mapsto 1, \quad v \mapsto -1 \\
\rho_3 &: \mathrm{BD}_{4n} \to \mathrm{GL}(\mathbb{C}), \quad u \mapsto -1, \quad v \mapsto i \\
\rho_4 &: \mathrm{BD}_{4n} \to \mathrm{GL}(\mathbb{C}), \quad u \mapsto -1, \quad v \mapsto -i
\end{aligned}$$

Since $v^2 = u^n$, we can easily construct the character table

	u^k	vu^k
ρ_1	1	1
ρ_2	1	-1
ρ_3	$(-1)^k$	$(-1)^k i$
ρ_4	$(-1)^k$	$(-1)^{k+1} i$

For the 2D representations, recall that we obtained

$$\sigma : \mathrm{BD}_{4n} \to \mathrm{GL}(\mathbb{C}^2), \quad u \mapsto \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad v \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

In general, we can define additional representations for $0 < j < n$ as

$$\sigma_j : \mathrm{BD}_{4n} \to \mathrm{GL}(\mathbb{C}^2), \quad u \mapsto \begin{pmatrix} \epsilon^j & 0 \\ 0 & \epsilon^{-j} \end{pmatrix}, \quad v \mapsto \begin{pmatrix} 0 & i^j \\ i^j & 0 \end{pmatrix}$$

Looking at the characters gives us $\chi_{\sigma_j}(u^k) = \epsilon^{jk} + \epsilon^{-jk}$ and $\chi_{\sigma_j}(vu^k) = 0$. The latter means that we only have to take into account the elements u^k when calculating the inner products (in particular a_{ij}) since otherwise one of the characters in the inner product is 0. We see that

$$\langle \chi_{\sigma_j}, \chi_{\sigma_j} \rangle = \frac{1}{4n} \sum_{k=1}^{2n} \overline{\chi_{\sigma_j}(u^k)} \chi_{\sigma_j}(u^k) = 1$$

Therefore, we know that each σ_j is irreducible. Note that $\sigma = \sigma_1$. By Theorem 7.6 we have $4 * 1^2 + (n-1) * 2^2 = 4n$ so these are all the representations. Furthermore, this means that ρ_1 and ρ_2 link to the same vertices, as do ρ_3 and ρ_4.

The reader can easily check that $a_{\rho_i \rho_j} = 0$ for all i and j. Now we look at connections between ρ_1 and σ_j.

$$\begin{aligned}
a_{\rho_1 \sigma_j} = \langle \chi_{\rho_1} \chi_\sigma, \chi_{\sigma_j} \rangle &= \frac{1}{4n} \sum_{k=1}^{2n} \overline{\chi_{\rho_1}(\epsilon^k) \chi_\sigma(\epsilon^k)} \chi_{\sigma_j}(\epsilon^k) = \frac{1}{4n} \sum_{k=1}^{2n} (\epsilon^k + \epsilon^{-k})(\epsilon^{jk} + \epsilon^{-jk}) \\
&= \frac{1}{4n} \sum_{k=1}^{2n} \epsilon^{k(j+1)} + \epsilon^{-k(j+1)} + \frac{1}{4n} \sum_{k=1}^{2n} \epsilon^{k(j-1)} + \epsilon^{-k(j-1)}
\end{aligned}$$

For convenience we introduce $S_a = \frac{1}{4n}\sum_{k=1}^{2n} \epsilon^{ak} + \epsilon^{-ak}$ for $0 \leq a \leq n$ which is 1 if $a = 0$ and 0 otherwise. Hence, $a_{\rho_1\sigma_j} = 1$ if $j = 1$ and 0 otherwise (this also holds for ρ_2). Now we look at connections between ρ_3 and σ_j.

$$a_{\rho_3\sigma_j} = \frac{1}{4n}\sum_{k=1}^{2n}(-1)^k(\epsilon^{k(j+1)} + \epsilon^{-k(j+1)}) + \frac{1}{4n}\sum_{k=1}^{2n}(-1)^k(\epsilon^{k(j-1)} + \epsilon^{-k(j-1)})$$

This means that $a_{\rho_3\sigma_j}$ is 1 if $j = n-1$ and 0 otherwise. Lastly, we check the connections between σ_i and σ_j.

$$a_{\sigma_i\sigma_j} = S_{j+i+1} + S_{j+i-1} + S_{j-i+1} + S_{j-i-1}$$

This means that for $1 < i < n-1$ the vertex σ_i has an edge with σ_{i-1} and with σ_{i+1}. If $i = 1$, it only has a connection with σ_2 and σ_{n-1} is connected to σ_{n-2}. Finally, we remove ρ_1 since it is the trivial representation to obtain our McKay graph. We end up with a Dynkin diagram of type D_{n+2}.

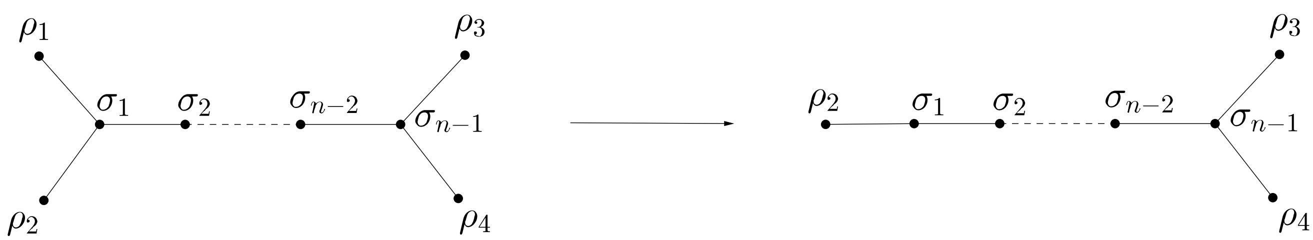

Figure 29: The McKay graph of the binary dihedral subgroup of order $4n$.

Case binary tetrahedral BT_{24}

The binary tetrahedral group is $\langle X, Y, Z \mid X^3 = Y^3 = Z^2 = XYZ = T, T^2 = 1\rangle$. We look at the 1-dimensional representations first. If $T = -1$, we get $Z = \pm i$ but then $X^3 = Y^3 = \pm iXY = -1$ which can never be satisfied. If $T = 1$ and $Z = -1$ we get $Z^3 = Y^3 = -XY = 1$ which can also never be satisfied. Lastly, if $T = 1$ and $Z = 1$ we get three options: $X = Y = 1$ or $X = \epsilon$ and $Y = \epsilon^2$ or $X = \epsilon^2$ and $Y = \epsilon$ (here ϵ is a 3th root of unity). So we have three 1-dimensional representations which we will denote by $\sigma_0, \sigma_1, \sigma_2$ respectively.

By theorem 7.6, we know that we have to form 21 by a sum of squares that does not involve 1^2. The only option is $3^2 + 3 \cdot 2^2 = 21$. Hence, we have an additional three 2 dimensional representation ρ_0, ρ_1, ρ_2 and another 3-dimensional representation τ.

For the binary tetrahedral case, the character table is [Boc]:

σ_0	1	1	1	1	1	1	1
σ_1	1	ϵ	ϵ^2	ϵ	1	ϵ^2	1
σ_2	1	ϵ^2	ϵ	ϵ^2	1	ϵ	1
ρ_0	2	-1	-1	1	-2	1	0
ρ_1	2	$-\epsilon^2$	$-\epsilon$	ϵ^2	-2	ϵ	0
ρ_2	2	$-\epsilon$	$-\epsilon^2$	ϵ	-2	ϵ^2	0
τ	3	0	0	0	3	0	-1

We denote by ρ the original representation we derived in section 2. Bocklandt denotes the original representation as $\rho_0 = \sigma_0 \otimes \rho$ since σ_0 is the identity. Recall that we put an edge between ρ_i and ρ_j if ρ_j appears in the direct summand of $\rho_0 \otimes \rho_i$. We know that $\chi_{V\oplus W} = \chi_V + \chi_w$ and $\chi_{V\otimes W} = \chi_V \chi_w$ so for each representation and each representative of a conjugacy class, we look at $\chi_{\rho_0}\chi_{\rho_i}$ and write it as a sum of other characters. If this is possible, we immediately know that there is an edge between ρ_i and the representations present in the sum. For convenience, we write the character function as a vector $(x_1, \ldots, x_k)$ where x_i is the value for each of the conjugacy classes.

Firstly, for τ we get $\chi_{\rho_0}\chi_\tau = (6,0,0,0,-6,0,0)$. This is equal to $\chi_{\rho_0} + \chi_{\rho_1} + \chi_{\rho_2}$ so there are edges between τ and ρ_0, ρ_1, ρ_2. We already know that $\rho_0 = \rho \otimes \sigma_0$ and so the trivial representation σ_0 only has an edge with ρ_0. For ρ_0 we get $\chi_{\rho_0}\chi_{\rho_0} = (4,1,1,1,4,1,0) = \chi_{\sigma_0} + \chi_\tau$ so ρ_0 has no other edges.
For ρ_1 we get $\chi_{\rho_0}\chi_{\rho_1} = (4,\epsilon^2,\epsilon,\epsilon^2,4,\epsilon,0) = \chi_\tau + \chi_{\sigma_2}$. Similarly, $\chi_{\rho_0}\chi_{\rho_2} = (4,\epsilon,\epsilon^2,\epsilon,4,\epsilon^2,0) = \chi_\tau + \chi_{\sigma_1}$. Lastly, $\chi_{\rho_0}\chi_{\sigma_1} = (2,-\epsilon,-\epsilon^2,\epsilon,-2,\epsilon^2,0) = \chi_{\rho_2}$ and $\chi_{\rho_0}\chi_{\sigma_2} = (2,-\epsilon^2,-\epsilon,\epsilon^2,-2,\epsilon,0) = \chi_{\rho_1}$ so there are no further edges.
Now that we have checked every case we can construct our McKay graph. We see that it is a Dynkin diagram of type E_6.

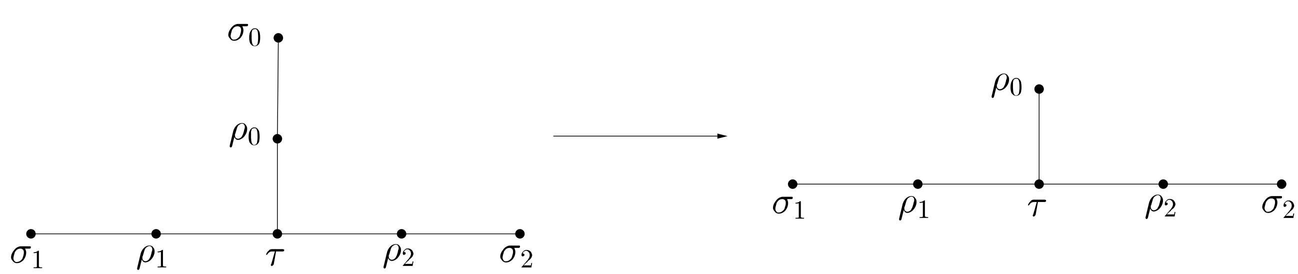

Figure 30: The McKay graph of the binary tetrahedral subgroup.

Case binary octahedral BO_{48}
The binary octahedral group is $\langle X,Y,Z \mid X^3 = Y^4 = Z^2 = XYZ = T, T^2 = 1\rangle$. We look at the 1-dimensional representations first. If $T = -1$, we get $Z = \pm i$ but then $X^3 = Y^4 = \pm iXY = -1$ which can never be satisfied. If $T = 1$ and $Z = 1$ we have $X^3 = Y^4 = XY = 1$. This means that $X = 1$ and $Y = 1$ which is the trivial representation σ_0. If $T = 1$ and $Z = -1$ we once again require $X = 1$. This gives $Y^4 = -Y = 1$ which gives $Y = -1$ and another representation σ_1. So we have two 1-dimensional representations. This means that we have to solve $a2^2+b3^2+c4^2+d5^2+e6^2 = 46$. We obtain three possibilities:

$$48 = 5^2 + 3^2 + 3\cdot 2^2 + 2\cdot 1^2 = 4^2 + 2\cdot 3^2 + 3\cdot 2^2 + 2\cdot 1^2 = 2\cdot 3^2 + 7\cdot 2^2 + 2\cdot 1^2$$

The symmetries of the octahedron give us a 3-dimensional representation τ which gives us two others $\tau_i = \tau \otimes \sigma_i$. Hence the first option is no longer possible. According to Bocklandt there is a four dimensional representation. Hence, we have the case $4^2 + 2\cdot 3^2 + 3\cdot 2^2 + 2\cdot 1^2$.

For the binary octahedral case, the character table is [Boc]

σ_0	1	1	1	1	1	1	1	1
σ_1	1	1	1	1	-1	-1	-1	1
μ	2	-1	-1	2	0	0	0	2
ρ_0	2	-1	1	-2	0	$\sqrt{2}$	$-\sqrt{2}$	0
ρ_1	2	-1	1	-2	0	$-\sqrt{2}$	$\sqrt{2}$	0
τ_0	3	0	0	3	1	-1	-1	-1
τ_1	3	0	0	3	-1	1	1	-1
ν	4	1	-1	4	0	0	0	0

This time, we go about calculating the McKay graph more systematically. We already know that σ_0 is only connected to ρ_0 (since $\rho_0 = \sigma_0 \otimes \rho$). We then construct the chain by looking at the next representation and construct the whole chain.

$$\begin{aligned}
\chi_{\rho_0}\chi_{\rho_0} &= (4,1,1,4,0,2,2,0) = \chi_{\sigma_0} + (3,0,0,3,-1,1,1,-1) = \chi_{\sigma_0} + \chi_{\tau_1}\\
\chi_{\rho_0}\chi_{\tau_1} &= (6,0,0,-6,0,\sqrt{2},-\sqrt{2},0) = \chi_{\rho_0} + (4,1,-1,-4,0,0,0,0) = \chi_{\rho_0} + \chi_{\nu}\\
\chi_{\rho_0}\chi_{\nu} &= (8,-1,-1,8,0,0,0,0) = \chi_{\tau_1} + (5,-1,-1,5,1,-1,-1,1) = \chi_{\tau_1} + \chi_{\tau_0} + \chi_{\mu}\\
\chi_{\rho_0}\chi_{\mu} &= (4,1,-1,-4,0,0,0,0) = \chi_{\nu}\\
\chi_{\rho_0}\chi_{\tau_0} &= (6,0,0,-6,0,-\sqrt{2},\sqrt{2},0) = \chi_{\nu} + (2,-1,1,-2,0,-\sqrt{2},\sqrt{2},0) = \chi_{\nu} + \chi_{\rho_1}\\
\chi_{\rho_0}\chi_{\rho_1} &= (4,1,1,4,0,-2,-2,0) = \chi_{\tau_0} + (1,1,1,1,-1,-1,-1,1) = \chi_{\tau_0} + \chi_{\sigma_1}\\
\chi_{\rho_0}\chi_{\sigma_1} &= (2,-1,1,-2,0,-\sqrt{2},\sqrt{2},0) = \chi_{\rho_1}
\end{aligned}$$

If we remove the trivial representation σ_0 we obtain the McKay graph. We see that it is a Dynkin diagram of type E_7.

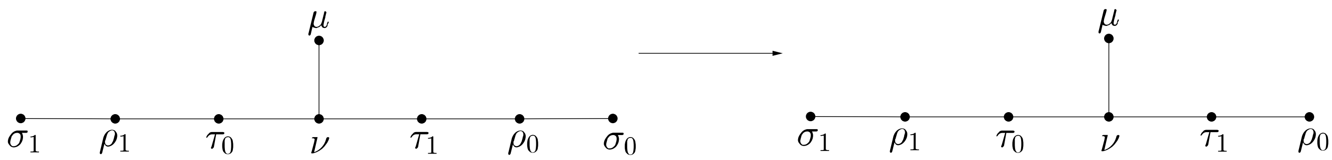

Figure 31: The McKay graph of the binary octahedral subgroup.

Case binary icosahedral BI_{120}
The binary icosahedral group is $\langle X, Y, Z \mid X^3 = Y^5 = Z^2 = XYZ = T, T^2 = 1\rangle$. We look at the 1-dimensional representations first. The only solution yields $T = 1, X = 1, Y = 1, Z = 1$ which is the trivial representation σ. Unfortunately, using Theorem 7.6 yields 69 possibilities. We refer to [Boc] for the classification of the representations.

For the binary icosahedral case, the character table is [Boc] ($\Phi = \frac{1+\sqrt{5}}{2}$, the golden ratio)

σ	1	1	1	1	1	1	1	1	1
ρ_1	2	-1	1	-2	0	Φ	$-\Phi$	$\Phi - 1$	$1 - \Phi$
ρ_2	2	-1	1	-2	0	$1 - \Phi$	$\Phi - 1$	$-\Phi$	Φ
τ_1	3	0	0	3	-1	$1 - \Phi$	$1 - \Phi$	Φ	Φ
τ_2	3	0	0	3	-1	Φ	Φ	$1 - \Phi$	$1 - \Phi$
ν_1	4	1	1	4	0	-1	-1	-1	-1
ν_1	4	1	-1	-4	0	1	-1	-1	1
μ	5	-1	-1	5	1	0	0	0	0
ζ	6	0	0	-6	0	-1	1	1	-1

The representation $\rho_1 = \sigma \otimes \rho$ is our original representation. We know that the trivial representation σ is only connected to ρ_1. We construct the chain by starting with ρ_1.

$$
\begin{aligned}
\chi_{\rho_1}\chi_{\rho_1} &= (4,1,1,4,0,1+\Phi,1+\Phi,2-\Phi,2-\Phi)\\
&= \chi_\sigma + (3,0,0,3,-1,\Phi,\Phi,1-\Phi,1-\Phi) = \chi_\sigma + \chi_{\tau_2}\\
\chi_{\rho_1}\chi_{\tau_2} &= (6,0,0,-6,0,1+\Phi,-1-\Phi,\Phi-2,2-\Phi)\\
&= \chi_{\rho_1} + (4,1,-1,-4,0,1,-1,-1,1) = \chi_{\rho_1} + \chi_{\nu_2}\\
\chi_{\rho_1}\chi_{\nu_2} &= (8,-1,-1,8,0,\Phi,\Phi,1-\Phi,1-\Phi)\\
&= \chi_{\tau_2} + (5,-1,-1,5,1,0,0,0,0) = \chi_{\tau_2} + \chi_\mu\\
\chi_{\rho_1}\chi_\mu &= (10,1,-1,-10,0,0,0,0,0) = \chi_{\nu_2} + (6,0,0,-6,0,-1,1,1,-1) = \chi_{\nu_2} + \chi_\zeta\\
\chi_{\rho_1}\chi_\zeta &= (12,0,0,12,0,-\Phi,-\Phi,\Phi-1,\Phi-1)\\
&= \chi_\mu + (7,1,1,7,-1,-\Phi,-\Phi,\Phi-1,\Phi-1) = \chi_\mu + \chi_{\nu_1} + \chi_{\tau_1}\\
\chi_{\rho_1}\chi_{\tau_1} &= (6,0,0,-6,0,-1,1,1,-1) = \chi_\zeta\\
\chi_{\rho_1}\chi_{\nu_1} &= (8,-1,1,-8,0,-\Phi,\Phi,1-\Phi,1-\Phi)\\
&= \chi_\zeta + (2,-1,1,-2,0,1-\Phi,\Phi-1,-\Phi,\Phi) = \chi_\zeta + \chi_{\rho_2}\\
\chi_{\rho_1}\chi_{\rho_2} &= (4,1,1,4,0,-1,-1,-1,-1) = \chi_{\nu_1}
\end{aligned}
$$

If we remove the trivial representation σ we obtain the McKay graph. We see that it is a Dynkin diagram of type E_8.

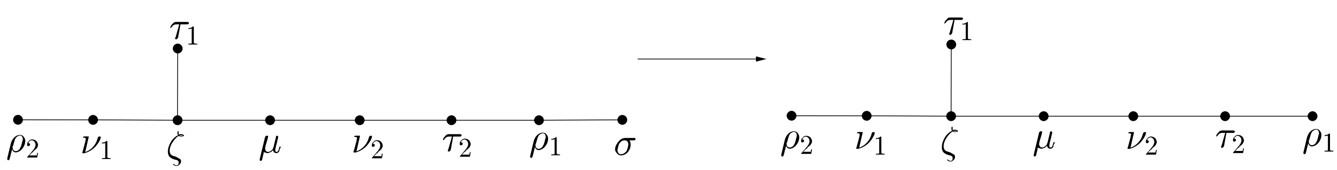

Figure 32: The McKay graph of the binary icosahedral subgroup.

McKay's observation can be formulated in the following way [McK80]:

Theorem 7.8. *(McKay correspondence) Let* Γ *be a finite subgroup of* $\mathrm{SL}(2,\mathbb{C})$*. One attaches to* Γ *a graph by associating to each irreducible representation a vertex and connecting the ith and jth by arrows in the manner described in the previous section. The resulting McKay graph is one of the extended Dynkin diagrams of type* A_n, D_n, E_6, E_7, E_8*, which occur respectively for cyclic, binary dihedral, binary tetrahedral, binary octahedral, binary icosahedral groups.*

8 Conclusion

Let Γ be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. We have shown how to obtain these subgroups and their generators. As Klein did in 1884, we used invariant theory to associate to Γ a surface in $\mathbb{C}^3$ with an isolated singularity at the origin. We followed in Du Val's footsteps and resolved these singularities in great detail to obtain the intersection diagrams of the exceptional divisors. These intersection diagrams turned out to be the Dynkin diagrams of type ADE. Lastly, we used characters to construct the McKay graph of Γ which also turned out to be Dynkin diagrams of type ADE. This peculiar correspondence is often called the *classical McKay correspondence*.

We have neglected to discuss what Dynkin diagrams are and the reader may wonder whether there is a more intricate relation between the irreducible representations of Γ and the resolution of the Kleinian singularities. Indeed, all of what we have shown here is related to Lie groups which is where the Dynkin diagrams come from. Since Klein, much more of this correspondence has been discovered by for example Kronheimer and Grothendieck. A more fundamental connection concerning K theory has been discovered in 1999 by Kapranov and Vasserot [KE99]. A further generalization was obtained in 2001 by Bridgeland, King and Reid [BKR01].

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