Utrecht University

Institute of Theoretical Physics

Topological Superconducting Ring

Author:
C.J.J. Van Langen

Supervisor:
Prof. dr. ir. H.T.C. Stoof

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Abstract

This paper covers the theoretical analysis of the behaviour of an ion chain on a superconducting ring through which a magnetic flux passes. After the introductory first chapter, in the second chapter we shall cover the behaviour of a non-superconducting ring and show that for a finite ring a persistent current flows. Here we use a tight-binding model and the technique of second quantization to create a suitable model. Following this, in chapter three we describe the behaviour of the superconducting ring using BCS Theory. Here we will see where the topological aspect comes into play in the form of topological regimes. In this chapter we will only be accounting for the movement of the electrons and not the Cooper pairs. This gives incorrect but still interesting results for the energy and current. In the fourth chapter, we include the movement of the Cooper pairs into the analysis and derive a correct equation for the flux generated current. We discuss some of the strange characteristics of the current around the transition points of topological regimes. Finally, we show that in the thermodynamic limit, when you send a flux through a superconducting ring a persistent current will still flow.

Contents

1 Introduction

The study of superconductivity is of major interest in modern condensed matter research. This is due to the great technological and environmental potential of high temperature superconductors in the modern world. For instance, superconductors are very important in the creation of very strong magnets and quantum computers. Also, since superconductors conduct currents with zero resistance, they are very favorable to normal conductors since they do not lose useful energy to heat. Presently it is not yet practical to use superconductors to transfer energy because the currently used metal alloys have to be cooled down to very low temperatures to start superconducting which costs a lot of energy. Today, the record of highest temperature superconductor is at about 200 Kelvin (around -70 degrees Celcius) [1] and further research is being done constantly. To find a high temperature superconductor is very hard, because the physical theory for this regime of superconductors has yet to be formulated.

In the 1950's, Bardeen, Cooper and Schieffer came up with the first theory of low temperature superconductors called BCS theory. In this theory they postulated that under an arbitrarily small attractive potential, electrons form bounded pairs called Cooper pairs which are essential for superconductivity. This theory is still widely used and we use it throughout this paper.

2 Normal conducting ring

2.1 Introduction to the model

As we begin our theoretical description of the superconducting ring, it is useful to first look at the non-interacting case (normal conducting). So we consider a model of a fixed number N of non-interacting, spinless ions and chemical potential μ which form a chain on a conductor. The conductor is cooled down to $T = 0$ and acts as an electron reservoir for the chain. The particles have spacing length a such that the length of the chain is $L = Na$. Through this chain we consider a magnetic flux ϕ inducing a vector potential $\mathbf{A}$. This model requires a grand canonical approach, thus we use the grand canonical Hamiltonian of this system:

$$H = \sum_{j=1}^{N_e} \left(\frac{-\hbar^2}{2m} (\nabla_j - \frac{e}{\hbar c} A(x_j))^2 - \mu + V(x_j) \right), \tag{2.1.1}$$

where $V(x)$ is the periodic potential imposed by the chain of ions and N_e is the number of electrons in the chain. This form is essentially the "first quantizatized" form of the Hamiltonian. But for many-body problems this is a very hard approach because we have to keep track of all the positions of the particles. So we will now introduce a more convenient way to formulate the problem without having to keep track of all the positions of the electrons. This entirely equivalent way of formulating the problem is called second quantization.

2.2 Second quantization

To do the theoretical analysis, we will need to shape this Hamiltonian into a more suitable form. Here the technique of second quantization comes into play. Let $\mathcal{H}_1$ be the single particle Hilbert space. The Hilbert space of N_e identical particles (electrons in our case) is the (completion of the) tensor product of N_e single particle Hilbert spaces:

$$\mathcal{H}_{N_e} = \bigotimes_{i=1}^{N_e} \mathcal{H}_1 \quad .$$

This means that one can write a basis of the wave function of an N_e-particle system as a product of the wave functions of the single-particle states. These single-particle wave functions have to follow the symmetric/antisymmetric property of the boson/fermion wave functions. For bosons this means writing the wave function as a permanent and for fermions as a Slater determinant.

Since we work in the grand canonical ensemble with the underlying conductor as a reservoir, the amount of electrons is allowed to fluctuate. Since the amount of particles is not constant, we need to account for all possibilities of the number of electrons N_e. For this purpose, we introduce Fock space as the direct sum of the spaces $\mathcal{H}_n$ where $n = \{0, 1, 2.., \}$. So we write

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n,$$

where $n = 0$ corresponds to the vacuum states (no particles).

Now that we have the right setting, we can introduce the creation/annihilation field operators working in Fock space. Define the creation $c^\dagger(x)$ operator as the operator adding a particle (electron in our model) to the system at position x and the annihilation operator $c(x)$ as the operator removing a particle at position x.

For fermions, these operators follow the anticommutation ($\{A, B\}$ = AB + BA) relations:

$$\{c(x), c(x')\} = \{c^\dagger(x), c^\dagger(x')\} = 0 \quad , \quad \{c(x), c^\dagger(x)\} = \delta(x - x'). \tag{2.2.1}$$

So we can write a general N_e-particle state in Fock space as

$$|\Psi(t)\rangle_{N_e} = \frac{1}{\sqrt{N_e}} \left(\prod_{i=1}^{N_e} \int \mathrm{d}x_i c^\dagger(x_i) \right) \Psi(x_1, .., x_{N_e}, t) |0\rangle .$$

This gives the second quantized form of the Hamiltonian:

$$H = \int \mathrm{d}x \; c^\dagger(x) \left(\frac{-\hbar^2}{2m} (\nabla - \frac{e}{\hbar c} A(x))^2 - \mu + V(x) \right) c(x). \tag{2.2.2}$$

2.3 Tight Binding

Looking back at our model of the chain, we want to incorporate the periodicity of our chain into the model. Since we assumed a constant spacing a, we know that the potential created by the lattice is periodic in a. Under these assumptions Bloch's theorem holds so we know that the wavefunctions of the electrons in our 1D system have a specific form, i.e.,

$$\Psi_{n,k}(x) = e^{ikx} u_{n,k}(x),$$

where $u_{n,k}(x)$ is periodic in a and n, k stand for respectively the band index and the wavenumber. In a tight binding model, the potential has a minimum at the lattice points (which are stable equilibria) so the electrons will localize around the lattice points. This gives us the motivation to express our states in terms of the complete and orthogonal Wannier states $w_n(x-\alpha_j)$ where α_j is the position of the j-th lattice point. Since the lattice spacing is constant we can just choose a starting point and express the j-th lattice point as aj. This gives the wave function

$$\Psi_{n,k}(x) = \frac{1}{N}\sum_{j=1}^{N} e^{ikaj} w_n(x-aj).$$

We now have localized states so we can now look at the occupation number of a site. We denote a state in Fock space as

$$|n_1, n_2, .., n_N\rangle ,$$

where n_j is the amount of particles at site j. Now we introduce new creation and annihilation operators that create/annihilate a particle at site j in the Wannier state $w_{n,j}$. We define the new operators $c_j^\dagger$ and its Hermitian conjugate c_j. The creation operator $c_j^\dagger$ acting on a state in our Fock space adds a particle to the system at site j. The annihilation operator is the Hermitian conjugate of the creation operator c_j and removes a particle from the system at site j. Note that for fermions, due to the Pauli exclusion principle, this number can only be 0 or 1. For bosons this can be any positive number. The creation and annihilation operators work on fermionic states as follows:

$$\begin{aligned} c_j^\dagger |n_j\rangle &= \sqrt{1-n_j}\,|n_j+1\rangle \\ c_j |n_j\rangle &= \sqrt{n_j}\,|n_j-1\rangle \\ c_j^\dagger c_j |n_j\rangle &= n_j |n_j\rangle . \end{aligned}$$

Therefore, $c_j^\dagger$ annihilates all states with $n_j = 1$ and c_j annihilates all states with $n_j = 0$.

For fermions, these operators follow the anticommutation ($\{A, B\}$ = AB + BA) relations:

$$\begin{aligned} \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} &= 0 \\ \{c_i, c_j^\dagger\} &= \delta_{ij}. \end{aligned}$$

We can express the field creation/annihilation operators in terms of the localized site operators as

$$c^\dagger(x) = \sum_n w_n^*(x-aj)c_{n,j}^\dagger \quad c(x) = \sum_n w_n(x-aj)c_{n,j},$$

where $w_n(x - aj)$ is the corresponding Wannier state to the position x and we sum over the band index. Since we work at a low temperature, the kinetic energy of the electrons will be low compared to the potential bump of the lattice so we can use a single-band approximation. When we put this into (2.2.2), we get the tight binding Hamiltonian:

$$H = -\mu \sum_{j=1}^{N} c_j^\dagger c_j - \sum_{i,j=1}^{N} t_{ij} c_i^\dagger c_j, \tag{2.3.1}$$

where t_{ij} is the hopping amplitude from site i to j, defined by

$$t_{ij} = \int \mathrm{d}x \left(w^*(x - ai) \left(-\frac{\hbar^2}{2m} (\nabla - \frac{e}{\hbar c} A(x))^2 + V(x) \right) w(x - aj) \right).$$

Furthermore we make the reasonable assumption that the electrons can only hop to neighboring sites, simplifying (2.3.1) for $A = 0$ to

$$H = -\mu \sum_{j=1}^{N} c_j^\dagger c_j - t \sum_{j=1}^{N} \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right). \tag{2.3.2}$$

2.4 Influence of the magnetic field

To account for the influence of the magnetic field we use the following gauge-symmetry:

$$\mathbf{A} \to \mathbf{A} + \nabla \varphi.$$

Since the magnetic field in the superconductor is zero due to the Meissner effect, we can choose the gauge such that

$$\mathbf{A}' = \mathbf{A} + \nabla \varphi = 0 \implies \mathbf{A} = -\nabla \varphi,$$

where

$$\varphi_j = \int_0^{x_j} \mathbf{dx} \cdot \mathbf{A}.$$

We use the standard electromagnetic gauge transformation

$$c_j \to e^{i \frac{e}{\hbar c} \varphi_j} c_j.$$

For the vector potential we know that we can take the integral over the ring to obtain the flux ϕ :

$$\oint \mathrm{d}\mathbf{r} \cdot \mathbf{A} = \phi.$$

So the terms in our Hamiltonian transform as:

$$c_{j+1}^{\dagger}c_j \to e^{-ia\frac{e}{\hbar c}\phi}c_{j+1}^{\dagger}c_j$$
$$c_j^{\dagger}c_j \to c_j^{\dagger}c_j,$$

giving the Hamiltonian:

$$H = \sum_{j=1}^{N}\left(-t(e^{ia\frac{e}{\hbar c}\phi}c_j^{\dagger}c_{j+1} + e^{-ia\frac{e}{\hbar c}\phi}c_{j+1}^{\dagger}c_j) - \mu c_j^{\dagger}c_j\right).$$

One of the conditions on a wave function is that it has to be single valued. This implies that differing the phase of the wave function by integers of 2π should not change the amplitude of the wave function. When a full loop around our ring is completed, the wave function gains a phase factor $e^{\frac{ie\phi}{\hbar c}}$. For the wave function to be single valued $\frac{ie\phi}{\hbar c} = 2\pi n$ (with n an integer) has to hold. This causes the flux to be quantized, i.e, we can only add an integer flux quanta $\Phi_0 = \frac{2\pi\hbar c}{e}$ where e is the charge of the charge carriers. It was experimentally discovered (1961 [6]) that for superconductors the flux quantum is $\Phi_0 = \frac{2\pi\hbar c}{2e}$, confirming that the charge carriers are Cooper pairs. Due to the Meisnner effect, magnetic fields are expelled from the superconductor. Consequently, one can actually trap an integer amount of flux quanta inside the loop. From now on, whenever we refer to ϕ we mean a quantized $\phi = \frac{e}{\hbar c}n\Phi_0$, where we add $\frac{e}{\hbar c}$ for easy notation. This means that $\frac{e}{\hbar c}n\Phi_0 = (2)\pi n$, so in this convention one flux quantum corresponds to a flux of π in the superconductor (and 2π in the normal conductor).

2.5 Fourier Transformation of the Hamiltonian

To Fourier transform our Hamiltonian we use the relations to momentum space:

$$c_j^{\dagger} = \frac{1}{\sqrt{N}}\sum_k e^{-ikaj}c_k^{\dagger} \qquad c_j = \frac{1}{\sqrt{N}}\sum_k e^{ikaj}c_k, \tag{2.5.1}$$

where $k = \frac{2\pi m}{aN}$ with $m = \{0, 1, .., N\}$.
These operators have anti-commutation relations:

$$\{c_k^{\dagger}, c_{k'}\} = \delta_{k,k'} \quad \{c_k^{\dagger}, c_{k'}^{\dagger}\} = \{c_k, c_{k'}\} = 0. \tag{2.5.2}$$

Substituting this into our Hamiltonian gives, after a straightforward calculation,

$$H = \sum_k\left(-t(e^{ia\phi+ka} + e^{-(ia\phi+ka)})c_k^{\dagger}c_k - \mu c_k^{\dagger}c_k\right)$$

$$H = \sum_k \left(-2t\cos[a(k+\phi)] - \mu\right) c_k^\dagger c_k = \sum_k \epsilon_k c_k^\dagger c_k. \tag{2.5.3}$$

Giving the normal state dispersion $\epsilon_k = -\mu - 2t\cos[a(k+\phi)]$.

2.6 Ground state energies

Now that we have the Hamiltonian in a form which we can easily work with, we can look at the ground state energy of the system. The ground energy can be found by filling the energy band up to the Fermi level given by the chemical potential, which is the same as filling up to the Fermi momentum k_F in k-space. This gives the following for the ground state energy:

$$E_g = \sum_{|k+\phi|\leq k_F} \epsilon_k = -2t \sum_{|k+\phi|\leq k_F} \cos[a(k+\phi)] - \sum_{|k+\phi|\leq k_F} \mu, \tag{2.6.1}$$

where we let $-\pi \leq k \leq \pi$ and take steps $\mathrm{d}k = \frac{2\pi}{N}$.
The induced persistent current is related to the energy in the following way:

$$\langle J \rangle = -\frac{1}{N}\frac{\partial E}{\partial \phi} = -\frac{2t}{N} \sum_{|k|\leq k_F} \sin[a(k+\phi)]. \tag{2.6.2}$$

Note that this current is a persistent current, a current that does not vanish in time. In the following plots of the energy and current, as stated earlier, one flux quantum corresponds to a flux of 2π. For $N = 5$ and varying $\frac{\mu}{t}$, we plot the energy and current:

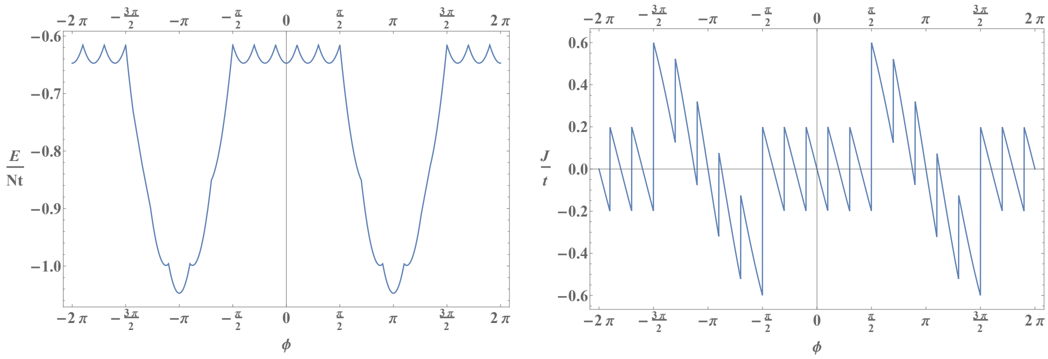

Figure 2.1: Ground state average energy and current at $\frac{\mu}{t} = 0$.

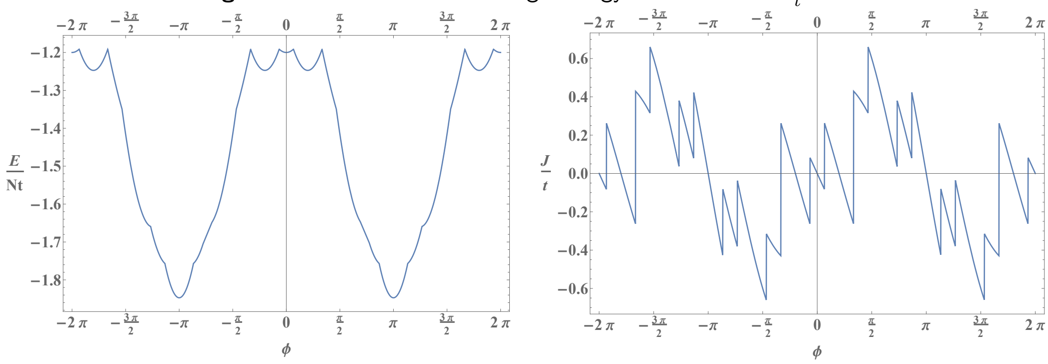

Figure 2.2: Ground state average energy and current at $\frac{\mu}{t} = 1$.

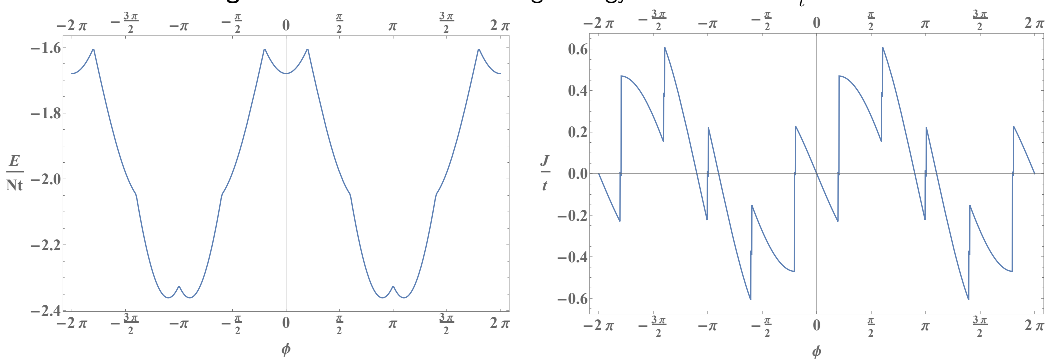

Figure 2.3: Ground state average energy and current at $\frac{\mu}{t} = 1.6$.

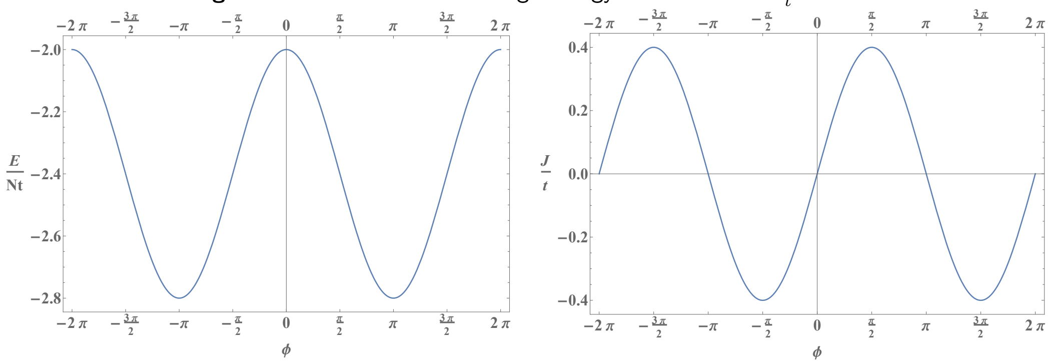

Figure 2.4: Ground state average energy and current at $\frac{\mu}{t} = 2$.

We can see from these plots that the energy and current show a bumpy behaviour that goes toward a smooth Cosine/Sine as it comes closer to $\frac{\mu}{t} = 2$ and onward.

For negative $\frac{\mu}{t}$ the behaviour is slightly different:

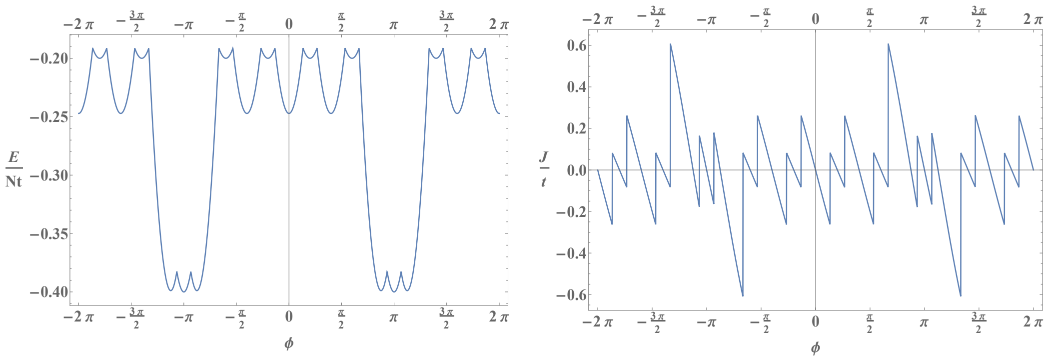

Figure 2.5: Ground state average energy and current at $\frac{\mu}{t} = -1$.

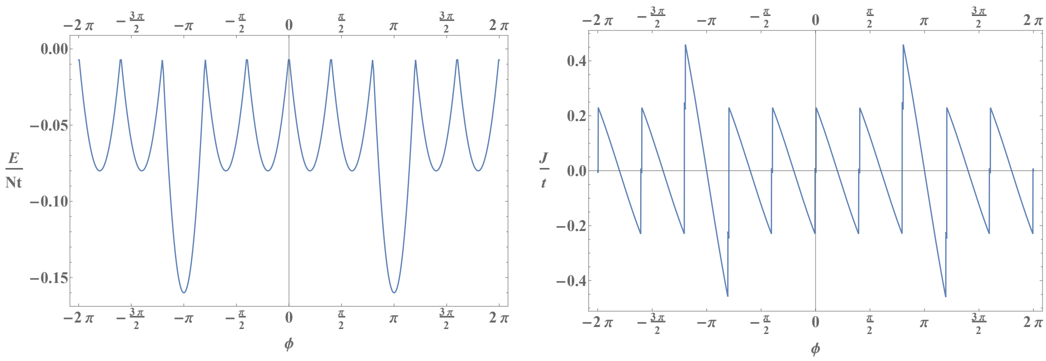

Figure 2.6: Ground state average energy and current at $\frac{\mu}{t} = -1.6$.

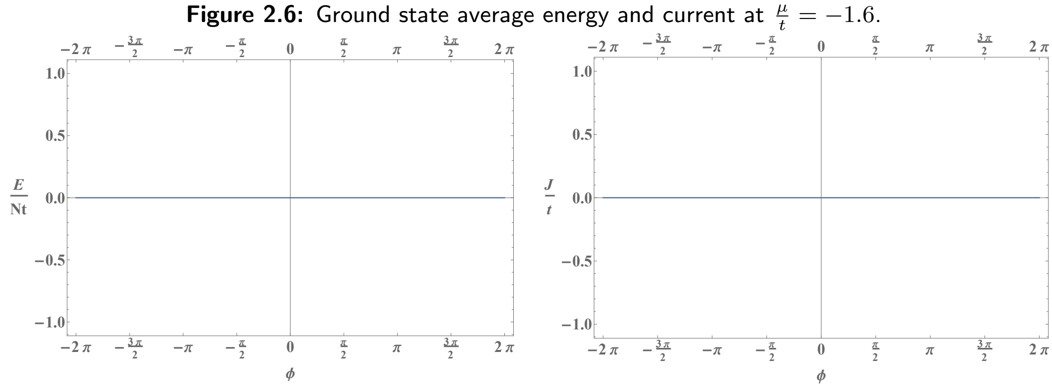

Figure 2.7: Ground state average energy and current at $\frac{\mu}{t} = -2$.

For negative $\frac{\mu}{t}$ we see even more bumpy behaviour and the energy and current going to zero at $\frac{\mu}{t} = -2$. This is because the hopping energy compared to the chemical potential is now too low to cause hopping from site to site. It is good to see that we have a period of 2π in the energy and current, due to the flux quantum being 2π in the normal conducting case.

By applying a flux, we are shifting the energy of the points in momentum space where we sum over. This can cause a point to contribute a relative large amount of energy at one value of the flux and none at a different value of the flux. This is because suddenly $\frac{\mu}{t}$ can be bigger than $-2\cos[a(k+\phi)]$. That means at this point in momentum space it will cost energy to put an electron into the system and thus it will not be added. We can illustrate this with the following picture in which we plot the energies of points in momentum space which we sum over together with the green line μ:

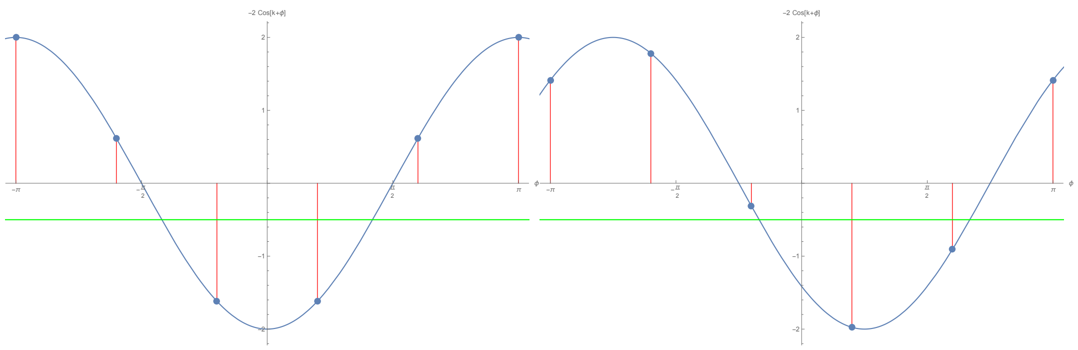

Figure 2.8: Energy contributions at $\phi = 0, -\frac{\pi}{4}$.

In Figure 2.8, we see that in the second plot the left point just falls out and the energy becomes significantly lower.

When we increase the particle number N, the bumps become smaller relative to the amplitude so they become less apparent. This is because when we increase N, we take more points in momentum space on which we measure the energy as seen in Figure 2.8, so one point falling out of range does not cause a big shift in energy. However, the current does fluctuate fast on a small scale. We see the current decreasing when we increase the particle number N, which we would expect since the ring is getting bigger.

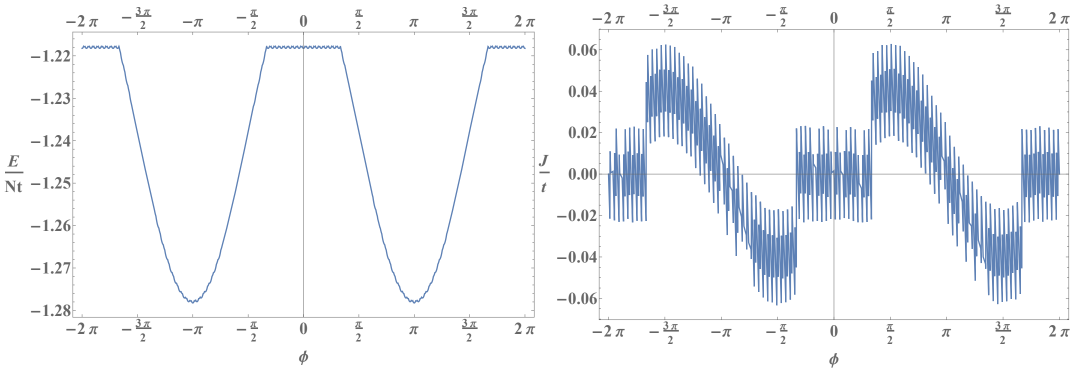

Figure 2.9: Ground energy and current at $\mu = 1$ and $n = 50$.

2.7 Thermodynamic and continuum limit

Now that we know the behaviour for a discrete finite ring, we would like to find out what the behaviour is in the limiting cases. First up is the thermodynamic limit. We let $N \to \infty$ and keep the chain spacing constant. For simplicity, let $a = 1$. The sum over the discrete k now becomes an integral. We take the average energy per particle:

$$\frac{E_g}{N} = \int\limits_{|k+\phi|\leq k_F} (-\mu - 2t\cos(k+\phi))\,\mathrm{d}k = -2\mu k_F - 4t\sin(k_F).$$

The energy goes to a constant in the thermodynamic limit, meaning no persistent current can flow. The reason for this is that there can no longer be any global wave functions covering the whole ring due to the statistical damping of the electron wave functions. If such wave functions do not exist, no flux quanta can be added to the system so no persistent current will flow. We will see that this is not the case when we have a superconductor instead of a normal conductor. For normal conducting systems that are smaller in scale, the wave functions are still able to wrap around the ring and cause a persistent current. For very small systems this current has been experimentally measured [2].

In the continuum limit, we let $N \to \infty$ but we now also let the lattice spacing $a \to 0$. Physically, we now have a different system as there is no periodic potential anymore due to the periodicity of the lattice disappearing. So the periodic energy band completely disappears and the model is now effectively described by the free electron model with new dispersion:

$$\epsilon_k = \frac{\hbar^2(k+\phi)^2}{2m}, \tag{2.7.1}$$

where m is the effective mass. To calculate the energy, we have to use a density of states approach. A straightforward calculation gives the density of states

$$D(\epsilon) = N\sqrt{\frac{2m}{\pi^2\hbar^2\epsilon}}.$$

From which we can calculate the energy:

$$E = \int_0^{\epsilon_F} \epsilon D(\epsilon)\,\mathrm{d}\epsilon = \frac{2N}{3\pi}\sqrt{\frac{2m}{\hbar^2}}(\epsilon_F)^{\frac{3}{2}},$$

where ϵ_F is the Fermi energy. Since ϵ_F is independent of ϕ, the current is zero in this limit. So physically, we need a discrete chain of ions and not a continuum to generate a current by a flux.

3 Superconducting ring with stationary Cooper pairs

3.1 Introducing the new model

We now consider a model where we have an interaction between the electrons, causing the formation of Cooper pairs. We follow the model for a superconducting chain as introduced by Kitaev [3]:

$$H = \sum_{j=1}^{N} \left(-t(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu c_j^\dagger c_j + \Delta^* c_j c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger \right) = H_0 + H_{int}, \quad (3.1.1)$$

where Δ is the superconducting gap. One can physically interpret this as the gap electrons from the underlying superconductor have to overcome to tunnel to the chain and form a Cooper pair. Similarly for a Cooper pair from the chain tunneling back into the superconductor. In this model we assume the value of Δ to be a constant intrinsic value of the superconductor.

As before, we have to make the Hamiltonian gauge-invariant. H_{int} is already gauge-invariant because of Δ, so we only have to make H_0 gauge-invariant the same way we did in chapter 2. When we Fourier transform H_{int}, we get

$$H_{int} = \sum_k \left(\Delta^* e^{iak} c_{-k} c_k + \Delta e^{-iak} c_k^\dagger c_{-k}^\dagger \right). \quad (3.1.2)$$

Note that the phase of Δ is not accounted for here. This is because when we first tried to understand this system, we did not take account of the movement of the Cooper pairs due to the flux. As we later found out, this was a mistake as the Cooper pair superfluid will definitely start moving with the electrons. We will expand on this in chapter four. Yet it is still informative to look at the difference in results when we do and when we do not account for the phase of Δ. As we will explain later, these results do hold when we look at the system with flux zero. For the rest of this thesis, we assume $\Delta = t$ since this is the interesting case and allows us to analytically solve the system in most cases.

The full Hamiltonian is given by

$$H = \sum_k \epsilon_k c_k^\dagger c_k + \sum_k \left(\Delta^* e^{iak} c_{-k} c_k + \Delta e^{-iak} c_k^\dagger c_{-k}^\dagger \right), \quad (3.1.3)$$

where $\epsilon_k = -\mu - 2t\cos[a(k+\phi)]$ is the dispersion of the non-interacting system. Using the commutation relations, we can write this Hamiltonian in matrix form:

$$H = \frac{1}{2}\sum_k \begin{pmatrix} c_k^\dagger & c_{-k} \end{pmatrix} \begin{pmatrix} \epsilon_k & \Delta_k \\ \Delta_k^* & -\epsilon_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} + \frac{1}{2}\sum_k \epsilon_k, \tag{3.1.4}$$

where $\Delta_k = -2i\Delta\sin(ak)$. To find the new dispersion, we will need to diagonalize 3.1.4.

3.2 Boguliubov Transformations

To diagonalize (3.1.4) we will use a unitary transformation on the system named after the Russian physicist Boguliubov called the Boguliubov transformation. He came up with the idea to take a linear combination of the creation/annihilation operators to define new operators that create or annihilate a quasi-particle in the system. Solving for the right parameter functions will make the Hamiltonian diagonal and one can see this as describing the system with new excitations which do not necessarily correspond to the creation or annihilation of physical particles. Boguliubov defined the transformation as

$$\begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k^* & -v_k \\ v_k^* & u_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}, \tag{3.2.1}$$

where γ is the new operator called the Boguliubov operator. For this transformation to make sense physically we need it to be unitary, so we pose the following condition:

$$|u_k|^2 + |v_k|^2 = 1.$$

Furthermore, the operators still need to be fermionic so the fermion commutation relations need to hold:

$$\{\gamma_k^\dagger, \gamma_k\} = |u_k|^2 + |v_k|^2 = 1, \tag{3.2.2}$$

$$\{\gamma_k, \gamma_{-k}\} = u_k v_{-k} - u_{-k} v_k = 0, \tag{3.2.3}$$

giving the extra condition

$$u_k = u_{-k} \quad , \quad v_k = v_{-k} \quad .$$

The inverse transformation of 3.2.1 is given by

$$\begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ -v_k^* & u_k^* \end{pmatrix} \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix}. \tag{3.2.4}$$

Since we want do diagonalize the Hamiltonian, we require that all cross terms with $\gamma_k\gamma_k$ and $\gamma_k^\dagger\gamma_{-k}^\dagger$ are zero. Inserting 3.2.4 into the Hamiltonian and setting the cross terms to zero poses the following condition on u_k and v_k:

$$2\epsilon_k u_k v_k + \Delta_k(u_k^2 - v_k^2) = 0. \tag{3.2.5}$$

Solving this together with the constraints gives us

$$u_k = \sqrt{\frac{1}{2}(1 + \frac{\epsilon_k}{E(k)})} \qquad v_k = \sqrt{\frac{1}{2}(1 - \frac{\epsilon_k}{E(k)})},$$

where

$$E(k) = \sqrt{\epsilon_k^2 + |\Delta_k|^2} = \sqrt{(-\mu - 2t\cos(k+\phi))^2 + 4\Delta^2 \sin(k)^2} \tag{3.2.6}$$

is the dispersion of the quasi-particles.

This gives the Hamiltonian

$$H = \frac{1}{2}\sum_k \epsilon_k + \frac{1}{2}\sum_k \begin{pmatrix} \gamma_k^\dagger & \gamma_{-k} \end{pmatrix} \begin{pmatrix} E(k) & 0 \\ 0 & -E(k) \end{pmatrix} \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix}.$$

The dispersion has a gap when both terms in the square root are zero. This gives the condition that the dispersion has a gap at $\frac{\mu}{t} = \pm 2\cos(\phi)$.

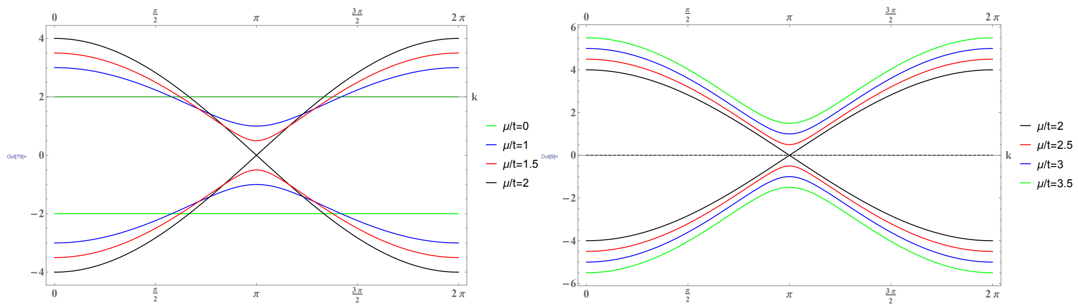

Figure 3.1: Dispersion of the quasi-particles for $\Delta = t$ and $\phi = 0$. The first figure shows the regime $|\mu/t| < 2$, the topological phase of the superconductor. The second figure shows the regime $|\mu/t| > 2$, the topologically trivial phase of the superconductor.

3.3 Topological superconductivity

The superconductor in this model has two distinct phases: the topological phase $|\mu/t| < 2|\cos(\phi)|$ and the topologically trivial phase $|\mu/t| > 2|\cos(\phi)|$. The Hamiltonian defines a map $h : S^1 \to C$ from the Brillouin Zone living on S^1 to the wave function which lives on the circle (ring) C. As we let k go from $-\pi$ to π, the topologically trivial phase maps the Brillouin Zone to a point on the circle and then back to the starting point, without going around the circle.

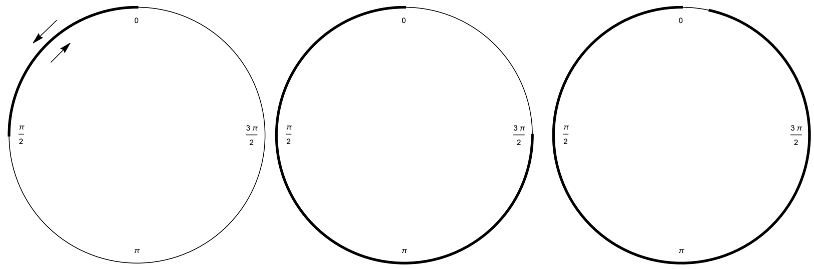

Figure 3.2: The map h as $|\mu/t|$ gets closer to the topological region. At $|\mu/t| = 2|\cos(\phi)|$, the dispersion is gapped and the topology of the map changes so a phase transition occurs.

The Hamiltonians with $|\mu/t| > 2|\cos(\phi)|$ belong to the same homotopy group (and $|\mu/t| < 2|\cos(\phi)|$ Hamiltonians also have their own group), so one can continuously deform the Hamiltonian from one to another without closing the gap, which means that they are topologically the same. At the transition $|\mu/t| = 2|\cos(\phi)|$, h maps the Brillouin Zone exactly one time around the circle, which is topologically different than before. This can be generalized to higher dimensional superconductors [4]. To get a closer look at what happens at the transition, we look at the ground state wave function of the system around this point.

Following [5], we can examine the ground state of the quasi-particles $|\varphi_g\rangle$ and find

$$|\varphi_g\rangle \propto \prod_{0<k<\pi} \left(1 + \varphi_{cp}(k) c^\dagger_{-k} c^\dagger_k\right) |0\rangle \,,$$
$$|\varphi_{cp}(k)| = \left|\frac{v_k}{u_k}\right| = \sqrt{\frac{E(k) - \epsilon_k}{E(k) + \epsilon_k}},$$

where one can loosely interpret φ_{cp} as the wavefunction of a Cooper pair formed between fermions with opposing momentum. In the topological regime $|\mu/t| < 2|\cos(\phi)|$, the Cooper pair size is infinite because $E(k) + \epsilon_k$ has a zero. This is why this is often called the "weak pairing" regime. For $|\mu/t| > 2|\cos(\phi)|$, the Cooper pairs are bound closer together and so this is called the "strong pairing" regime. We can loosely interpret the transition between regimes as the Cooper pair wave function transitioning from a locally bound state on the chain to being able to "loop around" the chain and touch its tail.

3.4 Superconducting ground state energy and current

We apply the same method as before, we write the matrix in diagonal form and use the commutation relations:

$$
\begin{aligned}
H &= \frac{1}{2}\sum_k \epsilon_k + \frac{1}{2}\sum_k \begin{pmatrix} \gamma_k^\dagger & \gamma_{-k} \end{pmatrix} \begin{pmatrix} E(k) & 0 \\ 0 & -E(k) \end{pmatrix} \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix} \\
&= \frac{1}{2}\sum_k (\epsilon_k - E(k)) + \sum_k E(k)\gamma_k^\dagger \gamma_k .
\end{aligned}
$$

In this form we can clearly see a ground energy (first sum) and excitations (second sum) in the Hamiltonian. In the following plots, as stated earlier, one flux quantum now corresponds to a flux of π due to the Cooper Pairs carrying the charge. For $N = 50$ we plot the average ground-state energy as a function of ϕ:

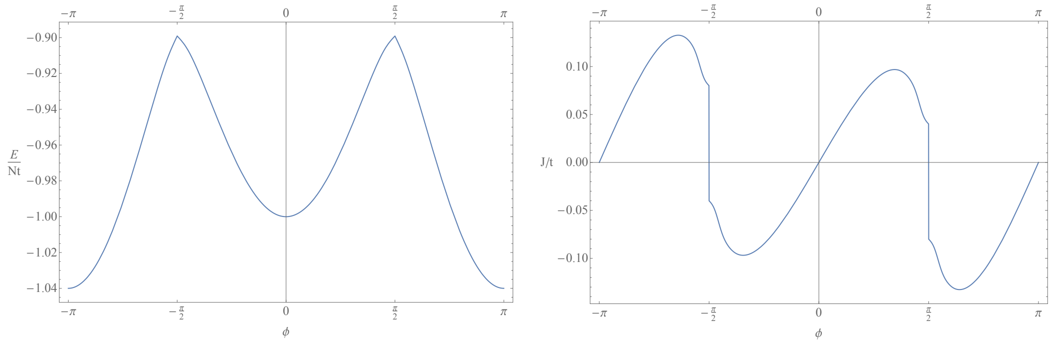

Figure 3.3: Superconducting average ground state energy and current at $\frac{\mu}{t} = 0$. The dispersion closes the gap at $\phi = \pm\frac{\pi}{2}$.

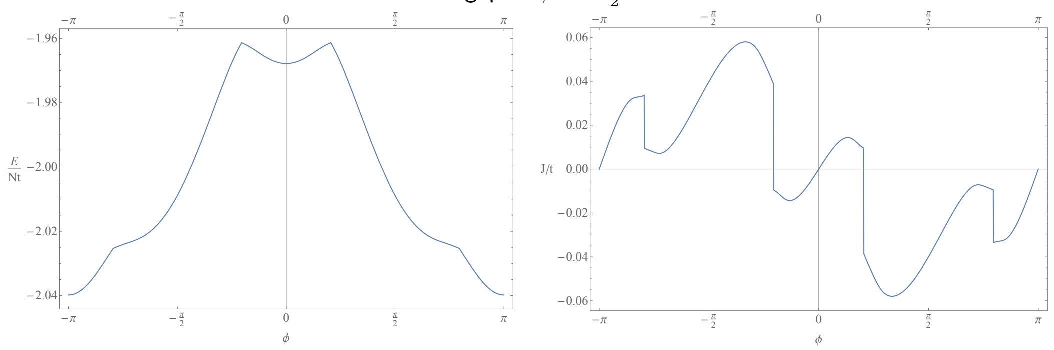

Figure 3.4: Superconducting average ground state energy and current at $\frac{\mu}{t} = 1.6$. The dispersion closes the gap around $\phi = \pm\frac{\pi}{5} + \pi n$.

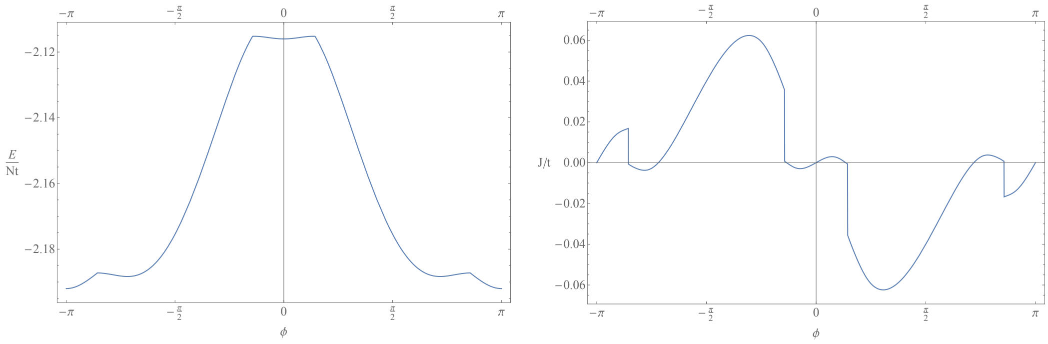

Figure 3.5: Superconducting average ground state energy and current at $\frac{\mu}{t} = 1.8$. The dispersion closes the gap around $\phi = \pm\frac{\pi}{7} + \pi n$.

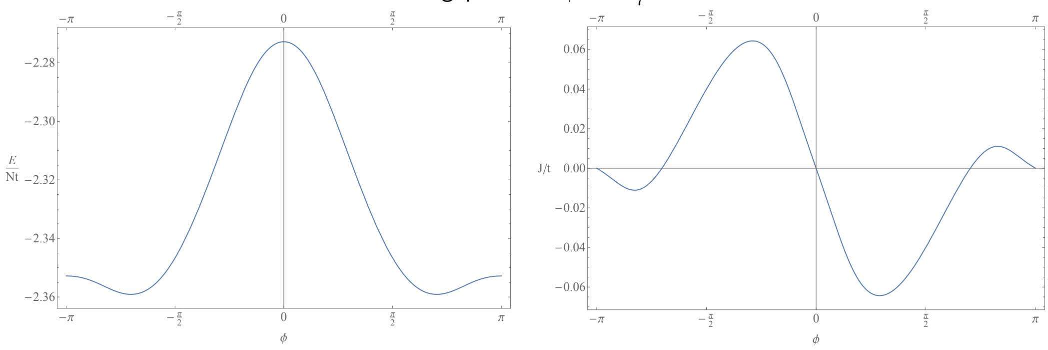

Figure 3.6: Superconducting average ground state energy and current at $\frac{\mu}{t} = 2$. The dispersion closes the gap at $\phi = \pi n$

The behaviour around negative $\frac{\mu}{t}$ is similar in form. In the topological regime ($\left|\frac{\mu}{t}\right| < 2$) the behaviour of the energy and current is fairly erratic. This is because for values $\left|\frac{\mu}{t}\right| < 2$ the dispersion can have a gap for certain values of the flux. At those values of the flux, the current has a big shift. For values $\left|\frac{\mu}{t}\right| > 2$ (topologically trivial), the dispersion can no longer have a gap so the energy and current behave smoothly.
Furthermore, we see that flux zero is only a local minimum of the energy in the topological region, in the topologically trivial region flux zero is a local maximum of the energy. This means that the ground state is always a state with flux nonzero. In the transition region between topological regimes, we see the current shifting in direction (sign). This is a sign of the phase transition occurring. We will expand on this in chapter 4.

3.5 Thermodynamic limit

Similar to the second chapter, we will now look at the thermodynamic limit for the system. In the normal conducting case we saw that the current in this case became zero. In the superconducting case however, we expect that there will still be a persistent current due to the nature of the Cooper pair wave function. The average ground-state energy in the thermodynamic limit is

$$\frac{E_g}{Nt} = \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{d}k \left(-\mu - 2\cos(k+\phi) - \sqrt{(-\mu - 2\cos(k+\phi))^2 + 4\sin^2(k)}\right).$$

This integral is not analytically solvable, however we can solve it numerically for different values of $\frac{\mu}{t}$. The current is given by

$$\frac{J}{t} = \frac{1}{2}\int_{-\pi}^{\pi} \mathrm{d}k \left(2\sin(k+\phi) - \frac{2\sin(k+\phi)(2-2\cos(k+\phi))}{\sqrt{(2-2\cos(k+\phi))^2 + 4\sin^2(k)}}\right),$$

which is also not analytically solvable. For varying values of $\frac{\mu}{t}$ we plot the ground-state energy and current in the thermodynamic limit.

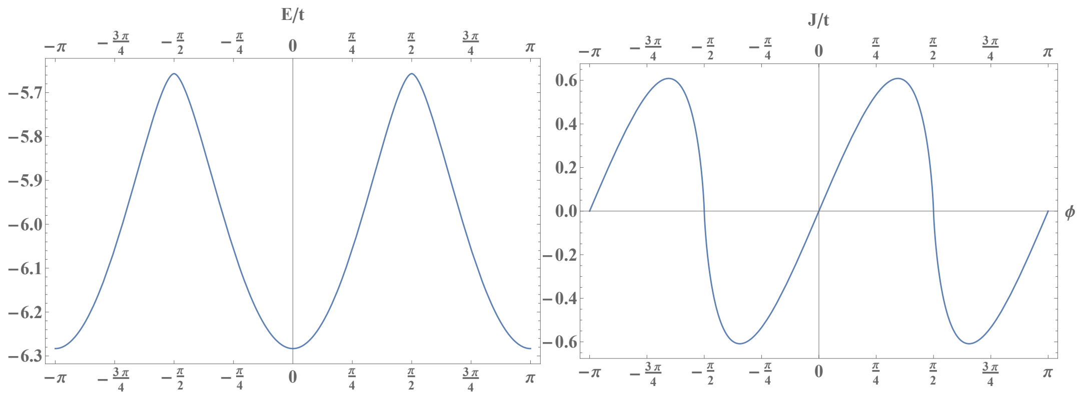

Figure 3.7: Superconducting ground state energy and current at $\frac{\mu}{t} = 0$ in the thermodynamic limit. The dispersion closes the gap at $\phi = \pm\frac{\pi}{2}$.

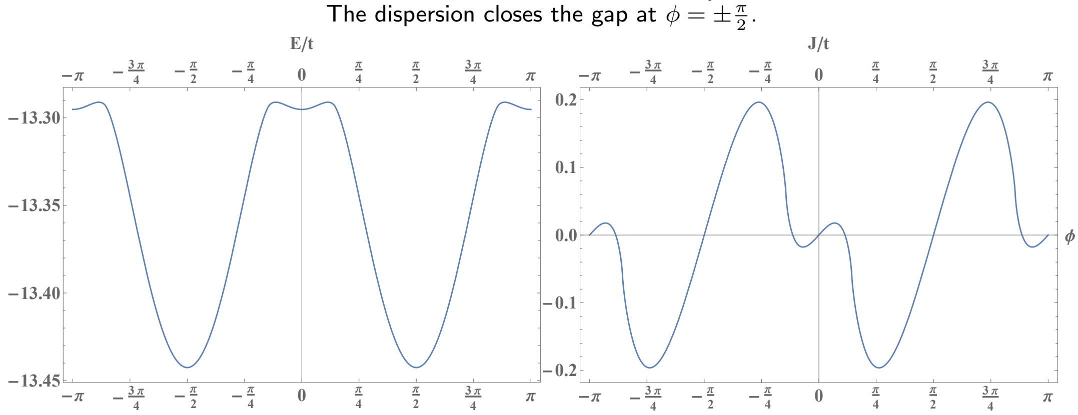

Figure 3.8: Superconducting ground state energy and current at $\frac{\mu}{t} = 1.8$ in the thermodynamic limit. The dispersion closes the gap around $\phi = \pm\frac{\pi}{8} + \pi n$.

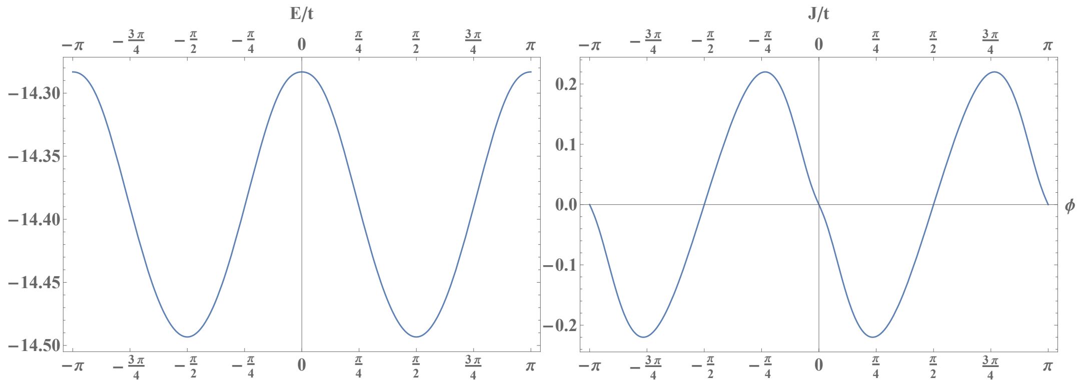

Figure 3.9: Superconducting ground state energy and current at $\frac{\mu}{t} = 2$ in the thermodynamic limit. The dispersion closes the gap at $\phi = \pi n$.

The behaviour around negative $\frac{\mu}{t}$ is the same. Looking back at the plots for a finite ring with $N = 50$, we can see resemblance. The plots have similar form but the erratic behaviour around the gap points is now smoother. The difference between the topological regime and the topologically trivial regime is also clearly visible. The behaviour from $\left|\frac{\mu}{t}\right| > 2$ onward is the same in form and behaves smoothly whereas the behaviour in the topological regime is irregular in form. The reason the current is nonzero in this limit is because the Cooper pairs form a Bose-Einstein condensate, more on this in chapter 4.

4 Superconducting ring with moving Cooper pairs

4.1 Accounting for the phase of Δ

To include the effect of the phase of Δ we have to go back to the beginning where we formed the gauge-invariant Hamiltonian. The phase of Δ is e^{iaqx} where we label $q = 2\phi = 2n\Phi_0$, so one Cooper pair adds 2 times the flux quantum since Cooper pairs can be seen as two coupled electrons. When we add the phase of Δ, the interacting part of the Hamiltonian becomes

$$H_{int} = \sum_{j=1}^{N} \left(\Delta^* e^{-iaqj} c_j c_{j+1} + \Delta e^{iaqj} c_{j+1}^\dagger c_j^\dagger \right). \tag{4.1.1}$$

We use 2.5.1 again to Fourier Transform the Hamiltonian. For the non-interacting part this stays the same. The interacting part gives

$$\begin{aligned}
H_{int} &= \frac{1}{N} \sum_{j=1}^{N} \sum_{k,k'} \left(\Delta^* e^{ia(k'j+k(j+1)-qj)} c_{k'} c_k + \Delta e^{-ia(k(j+1)+k'j-qj)} c_k^\dagger c_{k'}^\dagger \right) \\
&= \frac{1}{N} \sum_{j=1}^{N} \sum_{k,k'} \left(\Delta^* e^{iaj(k+k'-q)} e^{iak} c_{k'} c_k + \Delta e^{-iaj(k+k'-q)} e^{-iak} c_k^\dagger c_{k'}^\dagger \right) \\
&= \sum_{k,k'} \left(\Delta^* \boldsymbol{\delta}_{k,-k'+q} e^{iak} c_{k'} c_k + \Delta \boldsymbol{\delta}_{k,-k'+q} e^{-iak} c_k^\dagger c_{k'}^\dagger \right) \\
&= \sum_{k} \left(\Delta^* e^{iak)} c_{-k+q} c_k + \Delta e^{-iak} c_k^\dagger c_{-k+q}^\dagger \right).
\end{aligned}$$

So the full Hamiltonian is given by

$$H = \sum_{k} \left(\epsilon_k c_k^\dagger c_k + \Delta^* e^{iak} c_{-k+q} c_k + \Delta e^{-iak} c_k^\dagger c_{-k+q}^\dagger \right). \tag{4.1.2}$$

Physically, the change $-k \to -k+q$ means that now the Cooper pairs are not formed by electrons of opposite momentum, but with momentum k and $-k+q$. And oppositely, from a Cooper pair two electrons can form with momentum k and $-k+q$. This is due to the extra momentum the Cooper pairs receive from the flux. This is exactly the effect that we did not incorporate in our calculations last chapter. 4.1.2 in matrixform is

$$H = \frac{1}{2} \sum_{k} \begin{pmatrix} c_k^\dagger & c_{-k+q} \end{pmatrix} \begin{pmatrix} \epsilon_k & \Delta_k \\ \Delta_k^* & -\epsilon_{-k+q} \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k+q}^\dagger \end{pmatrix} + \sum_{k>0} \epsilon_k.$$

We switch the indexes of ϵ_k to make the matrix more symmetric:

$$H = \frac{1}{2} \sum_{k} \begin{pmatrix} c_k^\dagger & c_{-k+q} \end{pmatrix} \begin{pmatrix} \epsilon_{k+\frac{q}{2}} & \Delta_k \\ \Delta_k^* & -\epsilon_{-k+\frac{q}{2}} \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k+q}^\dagger \end{pmatrix} + \sum_{k>0} \epsilon_{-k+\frac{q}{2}}.$$

We can rewrite the Hamiltonian to take the diagonal part out. This part is not affected by the Boguliubov transformation:

$$\begin{pmatrix} \epsilon_{k+\frac{q}{2}} & \Delta_k \\ \Delta_k^* & -\epsilon_{-k+\frac{q}{2}} \end{pmatrix} = \frac{1}{2}(\epsilon_{k+\frac{q}{2}} - \epsilon_{-k+\frac{q}{2}}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}(\epsilon_{k+\frac{q}{2}} + \epsilon_{-k+\frac{q}{2}}) & \Delta_k \\ \Delta_k^* & -\frac{1}{2}(\epsilon_{k+\frac{q}{2}} + \epsilon_{-k+\frac{q}{2}}) \end{pmatrix}.$$

Now to apply the Boguliubov Transformations on the non-diagonal part, we have to slightly change the earlier definition of the Boguliubov Operators:

$$\begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k^* & -v_k \\ v_k^* & u_k \end{pmatrix} \begin{pmatrix} c_k \\ c_{-k+q}^\dagger \end{pmatrix}.$$

Similar to the last chapter, when we apply this transformation and equate the cross terms to zero, we get the following condition on the parameters:

$$(\epsilon_{k+\frac{q}{2}} + \epsilon_{-k+\frac{q}{2}})u_k v_k + \Delta_k^2(u_k^2 - v_k^2) = 0. \tag{4.1.3}$$

Equation 4.1.3 has a similar solution to last chapter, but now the terms with ϵ_k are slightly different:

$$u_k = \sqrt{\frac{1}{2}\left(1 + \frac{\frac{1}{2}(\epsilon_{k+\frac{q}{2}} + \epsilon_{-k+\frac{q}{2}})}{E(k)}\right)} \quad v_k = \sqrt{\frac{1}{2}\left(1 - \frac{\frac{1}{2}(\epsilon_{k+\frac{q}{2}} + \epsilon_{-k+\frac{q}{2}})}{E(k)}\right)},$$

where the new dispersion is $E(k) = \sqrt{(\frac{1}{2}(\epsilon_{k+\frac{q}{2}} + \epsilon_{-k+\frac{q}{2}}))^2 + \Delta_k^2}$. Using $q = 2\phi$, we write this out to see the difference with the dispersion in chapter three:

$$\begin{aligned} E(k) &= \sqrt{\left(\frac{1}{2}(-\mu - 2t\cos\left(k + \phi + \frac{q}{2}\right) - \mu - 2t\cos\left(-k - \phi + \frac{q}{2}\right))\right)^2 + 4\Delta^2\sin^2(k)} \\ &= \sqrt{(-\mu - 2t\cos(k+\phi)\cos(\phi))^2 + 4\Delta^2\sin^2(k)}. \end{aligned}$$

Comparing with (3.2.6), the new dispersion has an extra factor $\cos(\phi)$ in the first term. Here we see why the results from chapter three do hold for $\phi = 0$.

The gap is now closed at $\left|\frac{\mu}{t}\right| = 2\left|\cos^2(\phi)\right|$. So the new topological regime is $-2\cos^2(\phi) \leq \frac{\mu}{t} \leq 2\cos^2(\phi)$. Values $\phi = \frac{\pi}{2} + n\pi$ even completely destroy the topological regime.

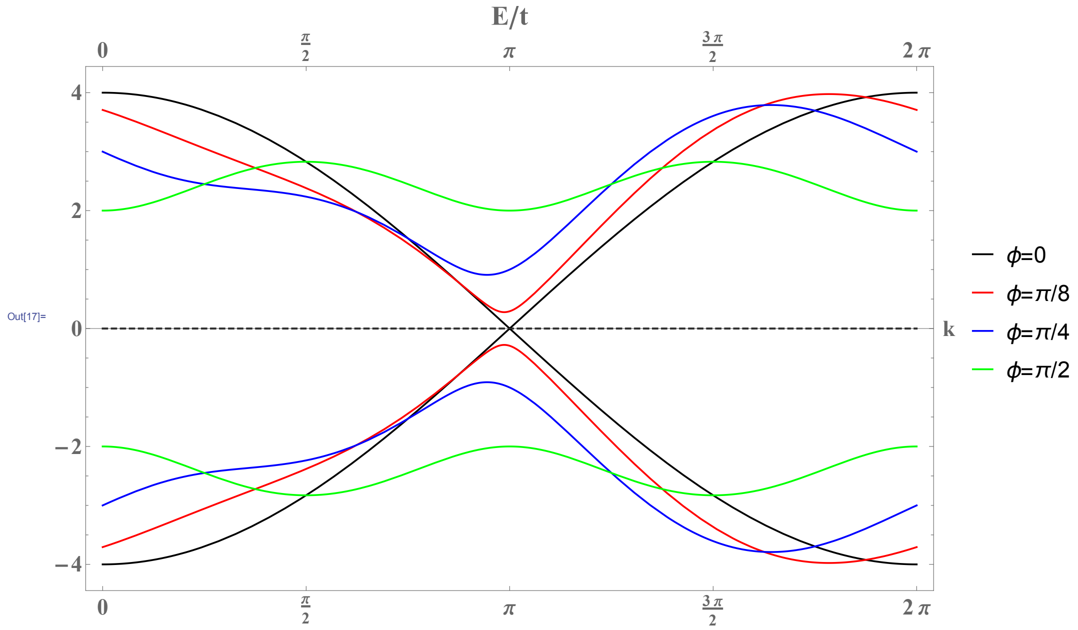

Figure 4.1: Dispersion plotted for different flux, μ/t fixed at $\mu/t = 2$.

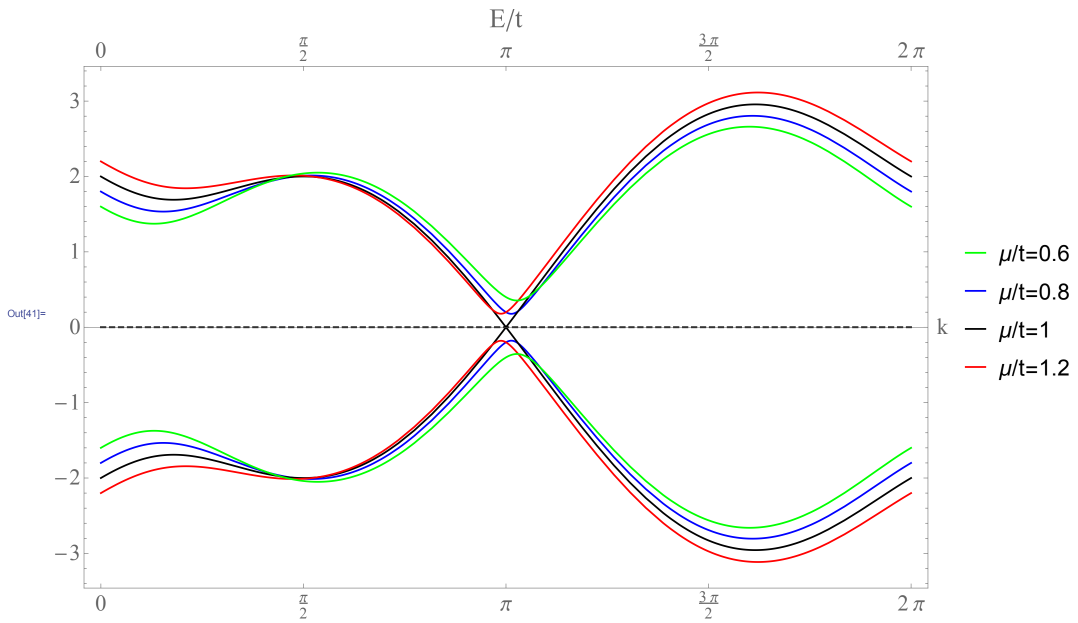

Figure 4.2: Dispersion plotted for different μ, ϕ fixed at $\phi = \frac{\pi}{4}$.

Consequently, at a fixed $\frac{\mu}{t}$, a small change in the flux will destroy the peaked behaviour around the gap. So for a fixed $\frac{\mu}{t}$ we expect that in the energy there will be strange behaviour around the value of ϕ that eliminates the gap. We confirm this by finding the ground state energy and current.

4.2 Superconducting ground state energy and current

The diagonalized Hamiltonian is

$$H = \frac{1}{2}\sum_k \begin{pmatrix} \gamma_k^\dagger & \gamma_{-k} \end{pmatrix} \left(\frac{1}{2}(\epsilon_{k+\frac{q}{2}} - \epsilon_{-k+\frac{q}{2}}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} E(k) & 0 \\ 0 & -E(k) \end{pmatrix} \right) \begin{pmatrix} \gamma_k \\ \gamma_{-k}^\dagger \end{pmatrix} + \sum_{k>0} \epsilon_{-k+\frac{q}{2}}.$$

Similar to chapter three, we use the commutation relations to rewrite the Hamiltonian so we can see a ground state energy and excited states:

$$H = \frac{1}{2}\sum_k \left(\frac{1}{2}(\epsilon_{k+\frac{q}{2}} + \epsilon_{-k+\frac{q}{2}}) - E(k) \right) + \sum_k (E(k) + \frac{1}{2}(\epsilon_{k+\frac{q}{2}} - \epsilon_{-k+\frac{q}{2}}))\gamma_k^\dagger \gamma_k.$$

So the ground state energy is

$$\frac{1}{2}\sum_k \left(-\mu - 2t\cos(k+\phi)\cos(\phi) - \sqrt{(-\mu - 2t\cos(k+\phi)\cos(\phi))^2 + 4\Delta^2 \sin(k)} \right). \tag{4.2.1}$$

For $N = 50$ and varying $\frac{\mu}{t}$ we plot the average energy and current, where one flux quantum corresponds to a flux of π.

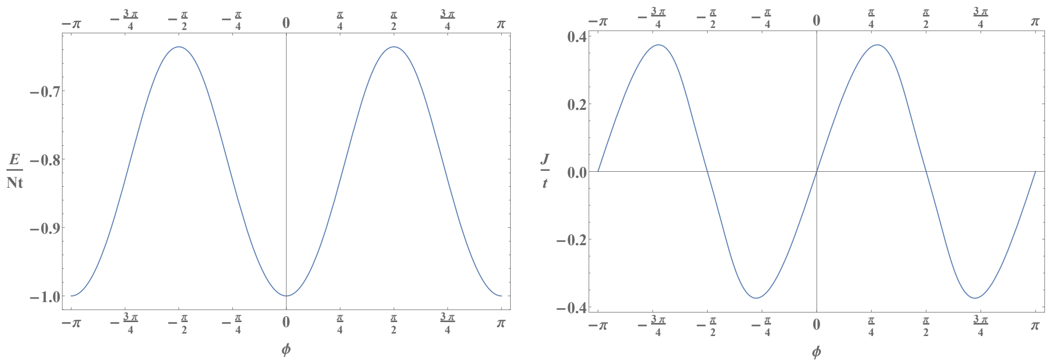

Figure 4.3: Superconducting average ground state energy and current at $\frac{\mu}{t} = 0$. The gap in the dispersion closes at $\phi = \pm\frac{1}{2}\pi + \pi n$.

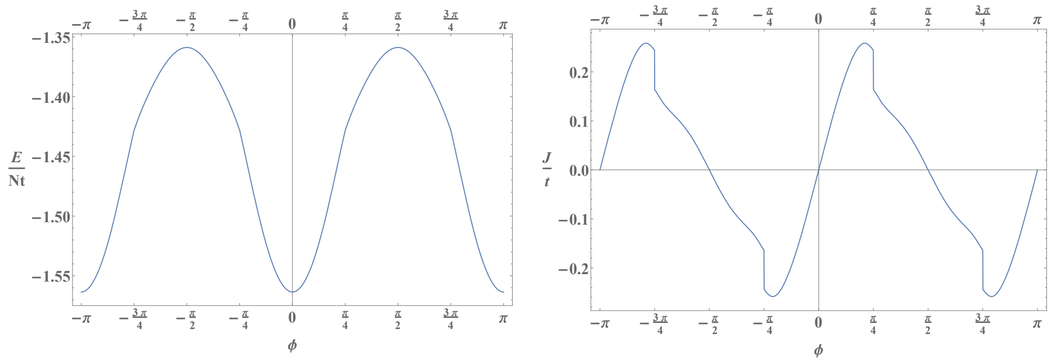

Figure 4.4: Superconducting average ground state energy and current at $\frac{\mu}{t} = 1$. The gap in the dispersion closes at $\phi = \pm\frac{1}{4}\pi + \pi n$.

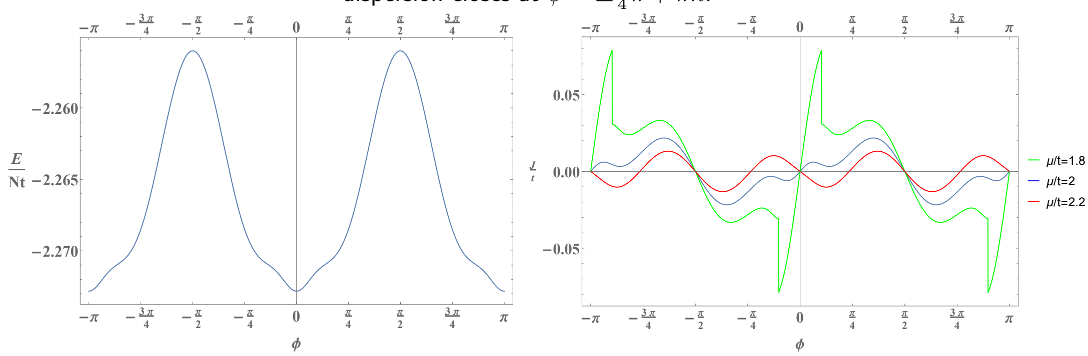

Figure 4.5: Superconducting average ground state energy at $\frac{\mu}{t} = 2$. The current is plotted around the transition point. The dispersion at $\frac{\mu}{t} = 2$ has no gap at $\phi = \pi n$

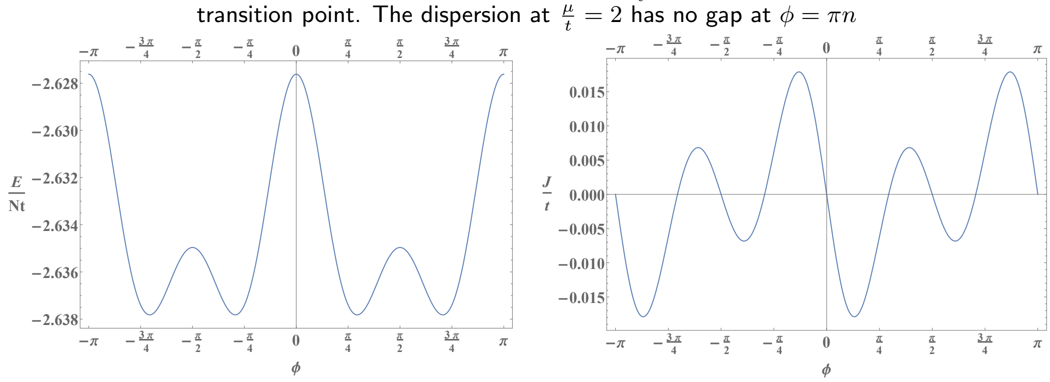

Figure 4.6: Superconducting average ground state energy and current at $\frac{\mu}{t} = 2.4$. In the topologically trivial regime the dispersion is gapped for all ϕ. The energy now has a maximum at $\phi = 0$ and the current is shifted in sign in certain regions compared to $\frac{\mu}{t} < 2$.

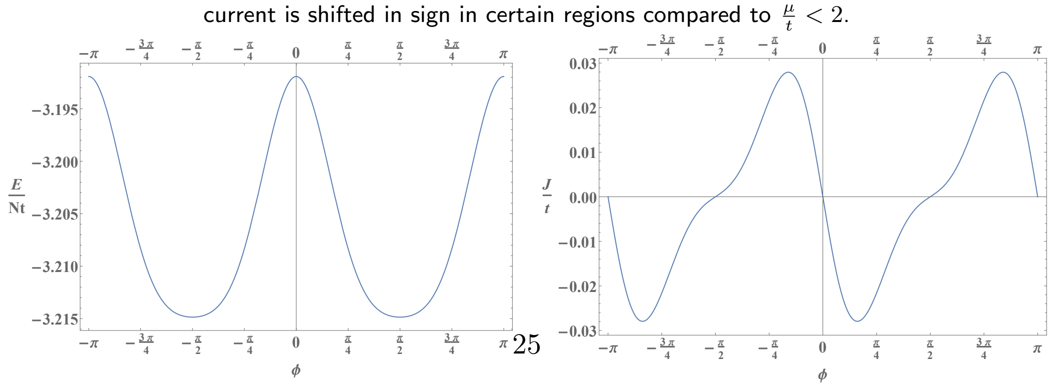

Figure 4.7: Superconducting average ground state energy and current at $\frac{\mu}{t} = 3$. The current has now completely shifted in direction compared to the topological regime.

The period is again π, indicating that the Cooper Pairs carry the charge. We see that the energy is symmetric in ϕ, which we would expect since changing the sign of the flux simply means switching the direction the flux is threading through the ring. The current is anti-symmetric in ϕ since switching the direction of the flux would make the current also switch direction, giving a negative current. The behaviour for negative $\frac{\mu}{t}$ is similar in form due to particle-hole symmetry.

There is a distinct jump in current on the points where the dispersion closes the gap, indicating a phase transition. When $\left|\frac{\mu}{t}\right| > 2$, the dispersion will have a gap for all values of the flux and we are fully in the topologically trivial regime. Consequently, for no value of the flux is there a jump in the current. Furthermore, for values $\left|\frac{\mu}{t}\right| > 2.4$ the energy no longer has a minimum in flux 0, but a maximum. This causes a shift in direction of the current since positive flux now gives a negative current.

To better visualize what happens at the transition point, we plot the average energy and current as a function of μ for various fixed values of ϕ.

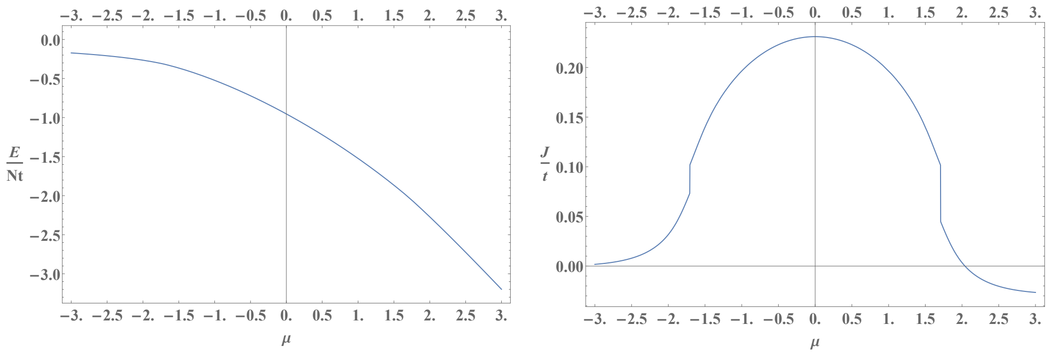

Figure 4.8: Superconducting average ground state energy and current at $\phi = \frac{\pi}{8}$. The transition in regimes is at $\left|\frac{\mu}{t}\right| \approx 1.7$.

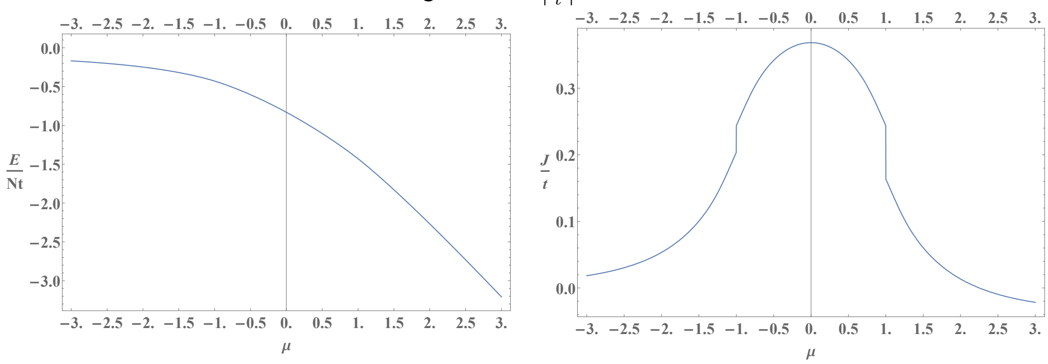

Figure 4.9: Superconducting average ground state energy and current at $\phi = 0$. The transition in regimes is at $\left|\frac{\mu}{t}\right| = 1$.

The behaviour for negative values of ϕ is similar since the current is anti-symmetric in 0. Here we can very clearly see a phase transition occuring due to the jump in the current.

4.3 Thermodynamic limit

Similar to last chapter, we now look at the limiting case where we let $N \to \infty$ and keep the lattice spacing constant at 1. We will show that there will still be a persistent superconducting current in this limit. The average energy in this limit is given by

$$\frac{E_g}{tN} = \int_0^\pi \mathrm{d}k \left(-\mu - 2\cos(k+\phi)\cos(\phi) - \sqrt{(-\mu - 2\cos(k+\phi)\cos(\phi))^2 + 4\sin^2(k)} \right).$$

This integral is not analytically solvable. We differentiate with respect to ϕ to get the current:

$$\frac{J}{t} = \int_0^\pi \mathrm{d}k \{ - \frac{(-2\cos(\phi)\cos(k+\phi) - \mu)(2\sin(\phi)\cos(k+\phi) + 2\cos(\phi)\sin(k+\phi))}{\sqrt{(-2\cos(\phi)\cos(k+\phi) - \mu)^2 + 4\sin^2(k)}} \\ + 2\sin(\phi)\cos(k+\phi) + 2\cos(\phi)\sin(k+\phi))\}.$$

Now we are all out of luck because this integral is also not analytically solvable. However, we can calculate the integrals numerically for fixed values of ϕ and μ and plot the result as a function of ϕ:

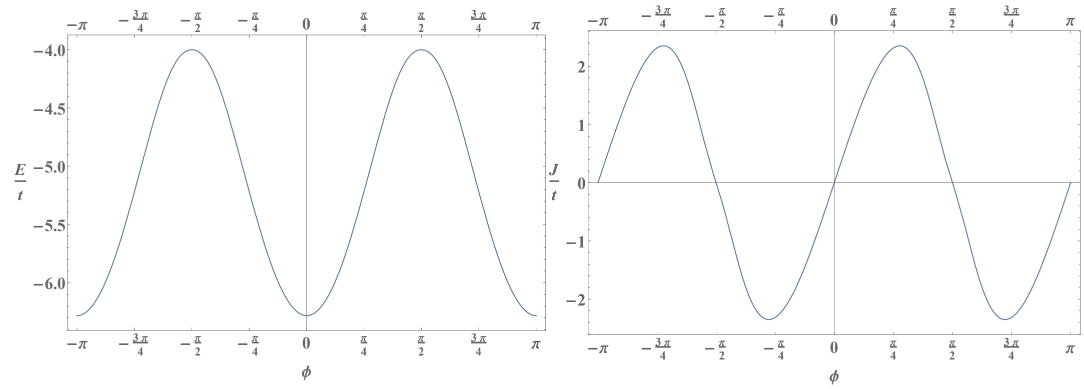

Figure 4.10: Superconducting ground state average energy and current at $\frac{\mu}{t} = 0$ in the thermodynamic limit.

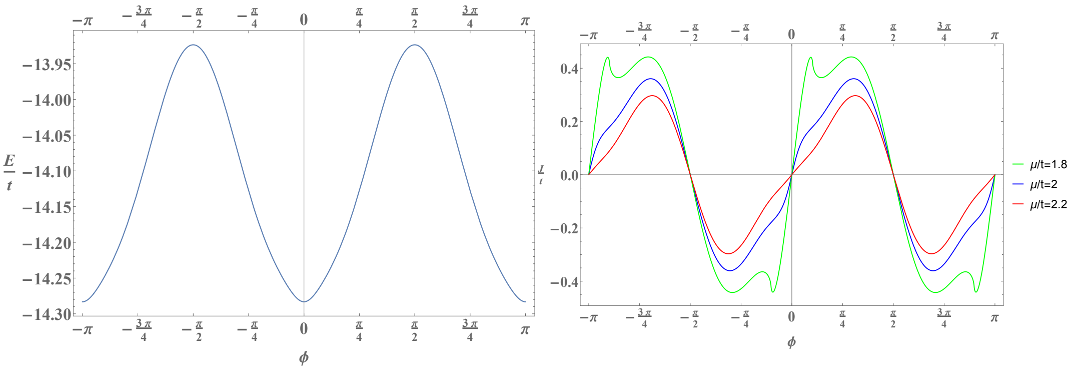

Figure 4.12: Superconducting ground state average energy at $\frac{\mu}{t} = 2$ in the thermodynamic limit. The current is plotted for values of $\frac{\mu}{t}$ around the transition point.

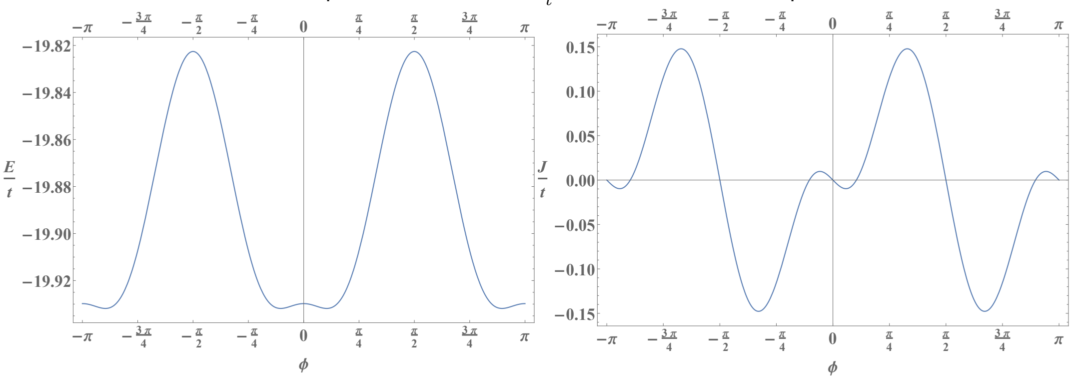

Figure 4.13: Superconducting ground state average energy and current at $\frac{\mu}{t} = 3$ in the thermodynamic limit. The energy has a ground state for flux nonzero, but the effect is less apparent in the thermodynamic limit than for the earlier results with $N = 50$.

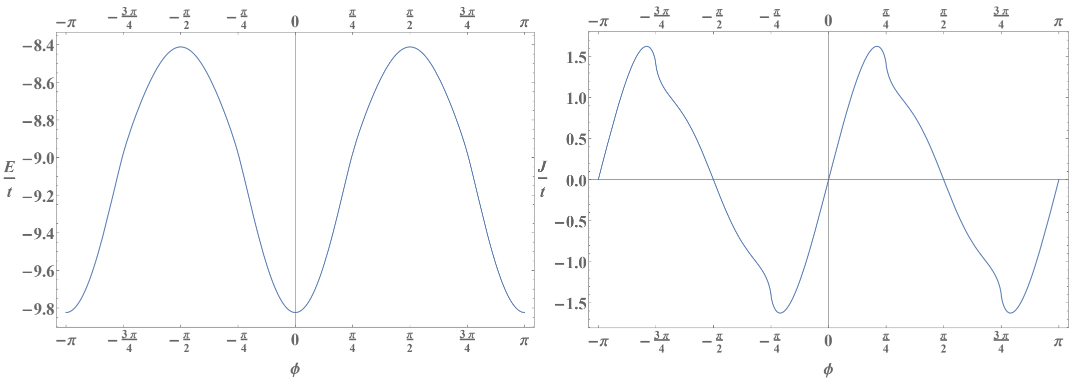

Figure 4.11: Superconducting ground state average energy and current at $\frac{\mu}{t} = 1$ in the thermodynamic limit.

We conclude that there is still a flux in the thermodynamic limit in the superconducting system. This is what we would expect, because the superfluid of Cooper pairs is a Bose-Einstein Condensate. A Bose-Einstein Condensate can be described by a single global wave function, so even in the thermodynamic limit, this wavefunction for the Cooper pairs stays global so a flux quantum can always be added.

Furthermore, comparing to Figure 4.3 - Figure 4.6, the energy and current have the same form but the jumps in current have now become smooth. When we increase $\frac{\mu}{t}$ to 2.6 we start seeing a ground state with nonzero flux. At $\frac{\mu}{t} = 3$ it is already well visible that flux zero is now a local maximum of the energy. In relative magnitude, this maximum is lower than the maximum at flux zero in Figure 4.6 and the current has not shifted completely. For larger values of $\frac{\mu}{t}$, this does not change. So the current does not shift completely in the thermodynamic limit.

5 Conclusion and outlook

In the second chapter we analyzed the behaviour of a normal conducting ring. We found that for a finite ring a persistent current arises. We simulated the finite ring for $N = 50$ and $\frac{\mu}{t} < 2$ and found that the energy and current behave in a bumpy way due to the falling in and out of the Brillouin zone of the points that contribute to the energy. When we looked at the thermodynamic limit, we concluded that there can be no persistent current. In chapter three we introduced the theory to describe a superconducting ring and found that the superconductor has two distinct topological phases. We calculated the energy and current assuming incorrectly that the Cooper pairs were stationary and saw the first signs of strange behaviour around the transition points of the topological regimes indicating a phase transition occurring. This gave us the topological phase for values $\left|\frac{\mu}{t}\right| < 2|\cos(\phi)|$ and the topologically trivial phase $\left|\frac{\mu}{t}\right| > 2|\cos(\phi)|$. When we calculated the energy and current in the thermodynamic limit under the false assumptions, we found a nonzero current. Finally, in the fourth chapter we corrected our model and incorporated the movement of the Cooper pairs. This gave the new topological phase $\left|\frac{\mu}{t}\right| < \left|2\cos(\phi)^2\right|$ and topologically trivial phase $\left|\frac{\mu}{t}\right| > \left|2\cos(\phi)^2\right|$. We observed a jump in current on the transition points between topological regimes, indicating of a phase transition. Furthermore, in the topological regime flux zero gives a ground state of the energy whereas in the topologically trivial regime flux zero is no longer a ground state. Finally, we calculated the energy and current in the thermodynamic limit and found that there will be a persistent current with behaviour resembling that of the finite ring.

There are some points where further research could be done to better understand the model. Firstly, we would like to understand better why there is a nonzero flux ground state energy in the topologically trivial phase while there is a zero flux ground state energy in the topological phase. Secondly, the model can be altered to replace some part of the chain with another part that has different chemical potential and hopping amplitude. For instance such that one part of the chain is in the topological phase and the other in the topologically trivial phase. Or one could make one part of the chain superconducting and other part normal conducting. This can be done in many ways and these configurations are called SQUIDs. It is theorized that Majorana fermions appear on the edge states of these SQUIDs, influencing the tunneling of electrons from one state to the other [7].

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