Bachelor Thesis

FRAUD IN THE LIBOR MARKET

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Chapter 1

Introduction

It was 2012, the world was still recovering from the credit crisis of 2008, during which it became clear that most banks are too big to fail. Some of these banks were still receiving massive support from governments, when the Libor scandal was discovered. For the sole purpose of making more money, financial ethics were thrown overboard, and parameters that are used to determine the Libor rates were manipulated.

Now, in 2016, most of the banks that committed fraud have reached a settlement for about a billion pounds. Submitters and CEO's have been let go, but does that mean the real problem has been tackled? This question is the main motive for writing this thesis.

Since this is a thesis in mathematics, we cannot give an answer to this question right away. In chapter 2 mathematical preliminaries of financial mathematics are covered, after which the models of financial markets in general (chapter 3) and the Libor market in specific (chapter 4) will come to light. Chapter 5 covers the calibration of the Libor market model to actual market data.

After chapter 5 the reader will have enough mathematical background to understand (a simplification of) the fraud in the Libor market, which will be simulated in chapter 6. We will conclude with a discussion.

Chapter 2

Preliminaries

This section will provide the mathematical background that is needed to understand the formulas of Black and Scholes, the transformations in interest rate formulas and the derivation of the final formula for the Libor rates.

For readers that are having their first experience in the world of financial mathematics, it is important to clarify the assumption of no-arbitrage.

Theorem 2.1. *Under the assumption of no-arbitrage, the price of a financial product equals the expected profit.*

This assumption is a crucial one, and basically means that one can not make any money on the money market without any risk. We define the risk-neutral measure $\tilde{\mathbb{E}}$ as the expectation under the no-arbitrage assumption. [1]

Wiener processes, often called Brownian motions, are used in differential equations for bonds and interest rates.

Definition 2.1. *A one-dimensional Brownian motion $W(t)$ is a process in time that satisfies the following conditions:*

- $W(0) = 0$,
- $W(t)$ *is continous,*

[1] For more info, see (Brigo, 2007) [3]

- *$W(t)$ has independent increments, with $W(t) - W(s) \sim \mathcal{N}(0, t-s)$,*

where $\mathcal{N}(\mu, \sigma^2)$ is the cumulative normal distribution with mean μ and variance σ^2.

Definition 2.2. *A d-dimensional Brownian motion $W(t)$ is a process in which, for all $j = 1, \ldots, d$, $W_j(t)$ is a one-dimensional Brownian motion. Furthermore, if i is not j, W_i and W_j are independent.*

The following differential equations differ from standard differential equations, since a stochastic factor $W(t)$ comes in. They are called geometric Brownian motions.

Theorem 2.2. *A geometric Brownian motion follows the equation*

$$dX(t) = \mu dt + \beta \cdot dW(t),$$

in which $W(t)$ is a d-dimensional Brownian motion, β is d-dimensional vector, μ a scalar and the dot denotes the dot-product.

Itô's formula, in specific a corollary that follows from it, will be of major influence in this thesis.[2]

Theorem 2.3 (Corollary of Ito's Lemma)**.** *If X and Y are two independent Brownian motions with μ_X, σ_X, μ_Y and σ_Y, then*

$$\frac{d(X/Y)(t)}{(X/Y)(t)} = (\mu_X(t) - \mu_Y(t) - (\beta_X(t) - \beta_Y(t)) \cdot \beta_Y(t))\, dt + (\beta_X(t) - \beta_Y(t)) \cdot dW(t).$$

[2] For now, one can suffice with this corollary, the Lemma itself can be found in Appendix A of (Pietersz, 2003) [7]

Chapter 3

Financial markets and Black-Scholes

In this thesis, the mathematical Libor Market Model will be introduced. Before digging too deep into it, it is wise to start by sketching the bigger picture in which the Libor interest rates are a very important instrument. In order to do so, the Black-Scholes model will be discussed, with special attention for volatilities.

3.1 The Stock Market

Before 1973, there was no sufficient pricing-formula for options and interest rates, which basically meant, that empirical research was used to predict prices for these financial products. All of this changed, when Fischer Black and Myron Scholes published their paper "The Pricing of Options and Corporate Liabilities" (Black & Scholes, 1973) [2]. Within days, the formulas the gentlemen had introduced were applied in financial institutions. Since it is the basis of the Libor Market as well, it is worthwhile reviewing the model.

The Black-Scholes model prices put- and call-options at time $t = 0$, or, in other words, determines how much you would want to pay for such an option. Let us first start off with a definition of both products: a call-option lets you buy a specified amount of stock at maturity T for price K, whereas a put-option lets you sell a specified amount of stock at maturity T for price K. We will be focussing on the call-option.

Characteristic about the Black-Scholes model is the assumption of fixed interest rates. The risk-free interest rate r equals the rate one gets when depositing money at a bank. It is also used to value prices: when an amount P is guaranteed at time t_1, the present value (at t_0) of such an amount equals $\frac{P}{\int_{t_0}^{t_1} e^{-rt} dt}$ (comparable with inflation).

If we suppose we have a stock valued S_0 at time t_0, a European call-option with exercise price K at $t = T$, a risk-free interest rate r and a volatility σ, then the following formula for the fair price, which is arbitrage-free, of the option holds:

$$C_0 = S_0 \mathcal{N}(d_1) - K e^{-rT} \mathcal{N}(d_2), \tag{3.1}$$

with d_1, d_2 defined as

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \qquad d_2 = \frac{\ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},$$

and $\mathcal{N}$ as the cumulative standard normal distribution.

Observe that whenever the volatility σ grows, thus the market becomes more unpredictable, the value of the option rises. This seems natural, since one would be willing to ensure the possibility of buying for a specified price K if the stock is more likely to rise far beyond this value.

3.1.1 Implied volatility

Without digging too deep into the mathematics of Black and Scholes, it is important to observe our main characters. Most are deterministic, such as S_0, K, T and, as assumed, r. If we are able to calculate or estimate a value for volatility σ, equation (3.1) can be used to value any kind of call- or put-option.

The problem here is, that the volatility is a parameter that represents the stability of the specific stock market at a given time. This stability is subject to lots of factors, where one could think of examples such as oil prices, exchange rates and other stock markets. Since there are numerous other factors, which might all have more (or less) influence on day 1 than on day 2, it speaks for itself that there exists no such thing as a general formula for volatilities.

If we are unable to calculate these volatilities, the Black-Scholes formula will not do us any good. Sometimes historical data is used to estimate the volatility of today, but this may turn out to be unsatisfactory since markets are extremely dynamic and there is no reason to assume that todays market will operate in the same way it did last year. The solution is implied volatility, which determines the volatility looking at todays market.

Since, while we are unable to determine the volatilities and thus calculate the price of our option, we can determine prices for similar call-options, because these are being traded throughout the day. This means, that if we compare similar looking call-options that are being traded for an average price of C_0^{A}, we are able to approximate the value of C_0.

This means there is just one variable left, namely σ. The Black-Scholes formula will now supply us with an implied volatility σ, where the calculation comes down to trying various values of σ until the price of the caplet equals C_0^{A}.

3.2 Interest rate markets

In contrast to Black and Scholes in 1973, we are now focussing on the interest rate market. Within a model of such a market, we are not interested in the development of the value of a certain stock, but in the value of a certain interest rate. The interest rate market has a prominent role in inter-bank trading, a recent study (Steinrücke, Zagst & Swishchuck, 2013 [8]) stated that is is good for 78-percent of the entire amount traded in 2012.

There are lots of financial products on these interest rate markets. However, in this thesis the focus will be on the caplets and floorlets, that are quite similar to European call- and put-options. These products protect its buyers against interest rates that are either too high or too low. A caplet will pay you the difference between the real interest rate and a prefixed interest rate K as soon as the interest rate exceeds K, while a floorlet pays you the difference between both when the interest rate drops below K.

3.2.1 Caplets and floorlets

In this paragraph, we will derive formulas for the value of caplets and floorlets at time t, without paying attention to the actual formula of interest rate L_i. This in order to make sure the bigger picture is clear, before we dive into the technical instruments the Libor rates are.

We introduce a time series $T_0 < T_1 < .. < T_i < .. < T_{N+1}$, with $\delta = T_{i+1} - T_i$ being constant. Now, we have a caplet$_i(t)$ for all $T_0 < t < T_N$, with prefixed rate K. Let B_t denote the value of a zero-coupon bond, a risk-free bond that returns nothing but interest, at time t. In contrast to the stock market, the interest rate is not assumed to be constant over time, just L_i is constant on interval $[T_i, T_{i+1}]$. Therefore, we will from now on denote the expectation (at time t) of the interest rate during interval $[T_i, T_{i+1}]$ by $L_i(t)$.

If the interest rate L_i increases above a certain level K, you will receive a payment of $(L_i - K)\delta$, since δ is the time during which $L_i > K$. This would then be the payoff of one caplet. If $L_i < K$, it is equal to 0.

Now, the value of caplet$_i$ at time 0, (caplet$_i(0)$), is equal to the discounted value of the expected payoff multiplied by the borrowed amount M, thus

$$\text{caplet}_i = M\frac{\tilde{\mathbb{E}}[\delta(L_i(T_i) - K)^+]}{B_{T_i+1}} \tag{3.2}$$

Note that $\tilde{\mathbb{E}}$ is the risk-neutral measure defined in chapter 2 and that we need to divide by B_{T_i+1} (and not B_{T_i}), since L_i is constant for $[T_i, T_{i+1}]$ and the pay-off takes place at T_{i+1}.

Black-Scholes formulas for caplets and floorlets

Note that when we expand the above to a value of a caplet at time t, it is still equal to the expected difference between Libor rate $L_i(T_i)$ and K, multiplied by period δ and the borrowed amount M. Now caplet$_i(t)$ equals

$$M\delta\frac{B_t}{B_{T_i+1}}\tilde{\mathbb{E}}[\delta(L_i(T_i) - K)^+|\mathcal{F}_t], \text{ for } t < T_i,$$

since the discounting factor becomes $\frac{B_t}{B_{T_i+1}}$. Note that (3.2) is a specific example of this formula, since $B_0 = 1$. If one is not familiar with $\mathcal{F}$-measurability, it might help to look at $\tilde{\mathbb{E}}[X(T_i)|\mathcal{F}_t]$ as the expectation of $X(T_i)$ when $X(t)$ is known. [1]

[1] For more on $\mathcal{F}$-measurability, see (Brigo, 2007) [3]

Now, let $B(t, T_{i+1})$ denote $\frac{B_t}{B_{T_{i+1}}}$. Then the formulas for caplets and floorlets, which are an extension of the pricing of future contracts, introduced by Fischer Black (Black, 1976)[1], that look a lot like those for the European call- and put-options, give us the following time-t prices for Caplets and Floorlets in the Libor Market:

$$\text{caplet}_i(t) = M\delta B(t, T_{i+1})[L_i(t)\mathcal{N}(d_1) - K\mathcal{N}(d_2)] \tag{3.3}$$

$$\text{floorlet}_i(t) = M\delta B(t, T_{i+1})[K\mathcal{N}(-d_2)] - L_i(t)\mathcal{N}(-d_1)] \tag{3.4}$$

with

$$d_1 = \frac{\ln(\frac{L_n(0)}{K}) + \frac{\sigma^2}{2}T_n}{\sigma\sqrt{T_n}} \qquad d_2 = \frac{\ln(\frac{L_n(0)}{K}) - \frac{\sigma^2}{2})T_n}{\sigma\sqrt{T_n}}, \tag{3.5}$$

and $\mathcal{N}(x)$ as the cumulative normal distribution of the standard normal distribution.

Chapter 4

The Libor rates

It is time to take a closer look at interest rates in general, and the Libor-rate in specific. Now that we know how these rates have a major influence on today's trading in financial markets, we will focus on how Libor rates are calculated.

Consider $\frac{B(t,T_i)}{B(t,T_{i+1})}$, and notice this represents the relative growth of a zero-coupon bond during the interval $[T_i, T_{i+1}]$, or at least the one expected at time t. Now, if we borrow 1 euro, or any other currency, at time T_i, the expected debt at T_{i+1} equals $1(1 + \delta L_i(t))$, recalling $L_i(t)$ is the expected interest rate for interval $[T_i, T_{i+1}]$. This leads us to the following formula for L_i:

$$1 + \delta L_i(t) = \frac{B(t,T_i)}{B(t,T_{i+1})}, \forall t : T_0 \leq t \leq T_N. \tag{4.1}$$

Observe that this formula generates $N+1$ different interest rates L_i (for $i = 0, \ldots, N$).

4.1 Wiener processes

From (4.1) it follows that, in order to determine the Libor rates, it is crucial to determine the price of zero-coupon bonds. These are defined by the geometric Brownian motion (as defined in chapter 2)

$$\frac{dB_i(t)}{B_i(t)} = \mu_i(t) + \beta_i(t) \cdot \mathrm{d}W(t),$$

in which $W(t)$ is a d-dimensional Brownian motion, β_i is a $d \times 1$-vector and the dot denotes the dot product. Calculating the dot product, we can rewrite the previous equation to

$$\frac{dB_i(t)}{B_i(t)} = \mu_i(t) + \sum_{j=1}^{d} \beta_{ij}(t) W_j(t). \tag{4.2}$$

In order to understand the use of $W_j(t)$ in the following paragraphs, it might be useful to look at them as factors that influence the bond prices. These factors may have more, less, positive or negative influence on the difference in bond prices, depending on β_{ij}.

The Wiener-process of the Libor Market Model.

Now we have obtained formulas for $B_i(t)$ for $0 \leq t \leq T_N$ we can use $\mathrm{d}L_i(t) = \frac{1}{\delta} d\left(\frac{B_i(t)}{B_{i+1}(t)}\right)$ together with Theorem 2.3, to transform (4.1) into the differential equation

$$\begin{aligned} dL_i(t) \quad &= \quad \frac{1}{\delta}\frac{B_i(t)}{B_{i+1}(t)}[((\mu_i(t) - \mu_{i+1}(t) - (\beta_i(t) - \beta_{i+1}(t))\,\beta_{i+1}(t))\,\mathrm{d}t \\ &\qquad + (\beta_i(t) - \beta_{i+1}(t)) \cdot \mathrm{d}W(t)], 0 \leq t \leq T_i, i = 1, \ldots, N.^1 \end{aligned} \tag{4.3}$$

Now, we would like to define the derivatives of the Libor rates as functions of the factors W_j. In other words, we would like to obtain a function for $\frac{dL_i(t)}{L_i(t)}$ that is of the form $\frac{dL_i(t)}{L_i(t)} = \ldots + \sigma_i \cdot dW(t)$, in which σ_i is a d-dimensional vector.

This is desirable, since we can then write our Libor rates $L_i(t)$ as a function of our factors of uncertainty, W_j. This way, we can determine the effect of one of these factors by looking at σ_{ij}. By doing so, we can predict consequences of changes in other markets, if they are a factor in Brownian motion W, for the Libor market.

Observe that, when

$$L_i(t)\sigma_i(t) = \frac{1}{\delta_i}\frac{B_i(t)}{B_{i+1}(t)}\left(\beta_i(t) - \beta_{i+1}(t)\right), \tag{4.4}$$

this is achieved.

4.2 Change of measure

In chapter 5 the calibration of the Libor Market Model to actual data sets will be covered. Looking at (4.3) and (4.4), this would mean having to estimate the parameters μ, β and σ.

In order to ease the method of calibration, it is useful to take a closer look at these parameters. Can one or more of them be excluded from the model by a change of measure?

Theorem 4.1. *If there exists a risk-neutral measure Q^B, in other words X/B is a martingale for any (combination of) asset(s) X, then for each other measure N there exists a measure $Q^N \sim Q^B$, such that X is a martingale under Q^N.*

As we did in the stock market, we will assume that our Libor market model is free of arbitrage as well. Observe that the price of bond i is, fluctuations due to the volatilities aside, expected to increase with $\mu_i(t)$ at time t. This value is, under the assumption of no arbitrage, the price of the risks that one takes investing in a bond. These risks can come from different factors, W_{ij}, with different weight β_{ij}. This would mean $\mu_i(t)$ can be written as a function of β_{ij} and some process that expresses the price of each factor W_{ij}.

Definition 4.1 (No arbitrage assumption)**.** *There exists a d-dimensional process ϕ^{RNM} such that*

$$\mu_i(t) = \beta_i(t)\cdot\phi^{RNM}(t),$$

for $0 \leq t \leq T_{i+1}$ and $1 \leq i \leq N+1$.

Now, each component $\phi_j^{RNM}(t)$ can be looked at as the market price of risk component W_j. Notice that $\phi_j^{RNM}(t)$ does not depend on i, which means the market price of component W_j at time t is the same for all L_i.

The process $\phi^{RNM}(t)$ denotes the market prices of risk under the standard measure, $W(t)$. Now, by expressing a new ϕ^N, for some measure N, in terms of ϕ^{RNM}, we will be able to rewrite equation (4.3) to equations more suitable for calibration.

In the next sections, we will do so for the Spot Libor measure and the Terminal measure.

4.2.1 Spot Libor measure

Consider the following investment strategy: at time 0, the price of a bond is $B(0)$. Invest 1 of any currency by buying $\frac{1}{B_1(0)}$ bonds. This will return $\frac{1}{B(0)}$ of currency at time T_1, which you again invest in bonds (which now have price $B_2(T_1)$). Thus, you will be able to buy $\frac{1}{B_1(0)}/B_2(T_1) = \frac{1}{B_1(0)B_2(T_1)}$ bonds.

Extending the above to the general case, we conclude that during the interval $[T_i, T_{i+1}]$, the portfolio holds $1/\Pi_{j=1}^{i+1} B_j(T_{j-1})$ bonds. This leads us to the conclusion that at time t the portfolio has value

$$X(t) = \frac{B_{i+1}(t)}{\Pi_{j=1}^{i+1} B_j(T_{j-1})}, \text{ for } T_i \le t \le T_{i+1}.$$

The stochastic differential equation of the spot Libor measure, recalling that $i(t)$ is the unique integer satisfying $T_i < t \le T_{i+1}$, equals

$$\frac{dX(t)}{X(t)} = \mu_{i(t)}(t)dt + \beta_{i(t)}(t) \cdot \mathrm{d}W(t), \text{ for } 0 \le t \le T_{i+1}.$$

Then, by Ito's Lemma, the geometrical Brownian motion of bond price $B_i(t)$ in terms of portfolio $X(t)$ becomes

$$\begin{aligned} \mathrm{d}\left(\frac{B(t)}{X(t)}\right) &= [\left(\left(\mu_i(t) - \mu_{i(t)}(t) - \left(\beta_i(t) - \beta_{i(t)}(t)\right)\beta_{i(t)}(t)\right)\mathrm{d}t \\ &\quad + \left(\beta_i(t) - \beta_{i(t)}(t)\right) \cdot \mathrm{d}W(t)], \text{for } \text{ for } 0 \le t \le T_i, i = 1, \ldots, N(4.5) \end{aligned}$$

If we define ϕ^{spot} as the d-dimensional process that satisfies

$$\phi^{\text{spot}} = \phi^{\text{RNM}} - \beta_{i(t)}(t), \text{ for } 0 \le t \le T_{n+1},$$

multiplication with $(\beta_i(t) - \beta_j(t))$, where i, j can be any indices between 0 and n, then, using $\beta_i \cdot \phi^{RNM} = \mu_i$, gives us:

$$\mu_i(t) - \mu_j(t) - (\beta_i(t) - \beta_j(t)) \cdot \beta_{i(t)}(t) = (\beta_i(t) - \beta_j(t)) \cdot \phi^{\text{spot}}(t) \tag{4.6}$$

It can now be shown that the measure $W^{Q_{\text{spot}}}$ follows a martingale under the risk-neutral measure[2], when it is defined as

$$W^{Q_{\text{spot}}}(t) = W(t) + \int_0^t \phi^{\text{spot}}(s)\mathrm{d}s.$$

Then, observing that then $\mathrm{d}W(t)$ equals $\mathrm{d}W^{Q_{\text{spot}}}(t) - \phi^{\text{spot}}(t)\mathrm{d}t$ and recalling equation (4.6), we rewrite the price of a bond in equation (4.5):

$$\begin{aligned}
\mathrm{d}\left(\frac{B(t)}{X(t)}\right) &= \Big((\mu_i(t) - \mu_{i(t)}(t) - \big(\beta_i(t) - \beta_{i(t)}(t)\big)\, \beta_{i(t)}(t)\Big)\,\mathrm{d}t \\
&\quad + \big(\beta_i(t) - \beta_{i(t)}(t)\big) \cdot (\mathrm{d}W^{Q_{\text{spot}}}(t) - \phi^{\text{spot}}(t)\mathrm{d}t), \\
&= \big(\beta_i(t) - \beta_{i(t)}(t)\big) \cdot \mathrm{d}W^{Q_{\text{spot}}}(t)
\end{aligned} \tag{4.7}$$

Now, the same substitution for $W(t)$ in equation (4.3), again recalling equation (4.6) gives us

$$\begin{aligned}
\mathrm{d}L_i(t) &= \frac{1 + \delta L_i(t)}{\delta}\Big(\big((\beta_i(t) - \beta_{i+1}(t)) \cdot \big(\beta_{i(t)}(t) - \beta_{i+1}(t)\big)\big)\, dt \\
&\quad + (\beta_i(t) - \beta_{i+1}(t)) \cdot \mathrm{d}W^{Q_{\text{spot}}}(t)\Big)
\end{aligned}$$

Now, observe that (4.1) implies that

$$\frac{1 + \delta L_i(t)}{\delta} = \frac{B_i(t)}{B_{i+1}(t)}. \tag{4.8}$$

We then conclude that, in terms of the measure $W^{Q_{spot}}$, $L_i(t)$ is defined by the geometrical Brownian motion

$$\frac{dL_i(t)}{L_i(t)} = \sum_{j=i(t)}^{i} \frac{\delta L_j(t)\sigma_j(t) \cdot \sigma_i(t)}{1 + \delta L_j(t)}\mathrm{d}t + \sigma_i(t) \cdot \mathrm{d}W^{Q_{spot}}. \tag{4.9}$$

[2] As shown in (Pietersz, 2003) [7]

4.2.2 Terminal measure

Instead of introducing a new portfolio to revalue financial products, one could also choose one of the bonds B_i, for example B_{n+1}. If we do so, the value of B_i, measured in terms of $B_{T_{n+1}}$, at time t becomes $\frac{B_i(t)}{B_{n+1}(t)}$. The differential equation for $\frac{B_i(t)}{B_{n+1}(t)}$ equals

$$\mathrm{d}\left(\frac{(B_i(t)/B_{n+1}(t))}{(B_i(t)/B_{n+1}(t))}\right) = (\mu_i(t) - \mu_{n+1}(t) - (\beta_i(t) - \beta_{n+1}(t)) \cdot \beta_{n+1}(t))\,\mathrm{d}t \\ + (\beta_i(t) - \beta_{n+1}(t)) \cdot \mathrm{d}W(t)$$

The derivation of the Libor rates formula in terms of bond B_{n+1}, numéraire $W^{T_{n+1}}$, is quite similar to that of the Libor rates in terms of the Spot libor measure, where

$$\phi^{T_{n+1}} = \phi^{RNM} - \beta_{n+1} \qquad \text{and} \qquad \mathrm{d}W(t) = \mathrm{d}W^{T_{n+1}} - \phi^{T_{n+1}}\mathrm{d}t.$$

With analogical calculations we conclude that, under the terminal Libor measure, the following geometrical Brownian motion defines the Libor rate:

$$\frac{dL_i(t)}{L_i(t)} = \begin{cases} -\sum_{j=i+1}^{k} \frac{\delta_j L_j(t)\sigma_j(t)\cdot\sigma_i(t)}{1+\delta_j L_j(t)}dt + \sigma_i(t)\cdot dW^{Q_{T_{n+1}}} & \text{if } j < k \\ \sigma_i(t)\cdot dW^{Q_{T_{n+1}}}(t) & \text{if } j = k \\ \sum_{j=i+1}^{k} \frac{\delta_j L_j(t)\sigma_j(t)\cdot\sigma_i(t)}{1+\delta_j L_j(t)}dt + \sigma_i(t)\cdot dW^{Q_{T_{n+1}}} & \text{if } j > k. \end{cases} \tag{4.10}$$

Chapter 5

Calibration

There are several methods to calibrate mathematical models to reality. Like volatilities in the formula of Black and Scholes, the entire Libor market model used to be calibrated considering historical data. This has the disadvantage that there is no reason to assume that todays market was similar to yesterdays market. Another method, which is used in this thesis, is calibrating to market prices of financial products. In this section, we will do so with caplet prices.

5.1 Theoretical calibration

When calibrating the Libor market model, we try to find estimates for σ_i. In order to do so, we will derive the model prices C_i^{model} for a chosen financial product and compare it with market prices. We will do so for caplets, recalling equation (3.3), in particular the Black-Scholes price of caplet i, which we will from now on denote by C_i^{Black}.

Since the value of caplet$_i$ at time t only depends on $L_i(t)$, we will use the terminal measure to derive C_i^{model}. This means the differential equation of $L_n(t)$ is the second from (4.10), which has solution

$$L_n(t) = e^{\int_0^t \sigma_n(s) dW^{Q_{T_{n+1}}}(s) - \frac{1}{2}\int_0^t ||\sigma_n(s)||^2 ds}, 0 \leq t \leq T_n.[1]$$

[1] Which follows from Itô's Lemma

Under $Q^{T_{n+1}}$, $L_n(T_n)$ can be written as $L_n(0)e^Z$, where Z is an $\mathcal{F}(T_n)$-measurable function, normally distributed (under the measure $Q^{T_{n+1}}$) with $Z \sim \mathcal{N}(-\frac{1}{2}\tau^2, \tau^2)$, in which

$$\tau^2 = \int_0^{T_n} ||\sigma_n(s)||^2 ds.$$

Now, we want to calculate the price $C_i^{\text{model}}(T_n, K)$. Recalling (3.2), we observe that it equals

$$\begin{aligned} C_i^{\text{model}}(T_n, K) &= M\sigma B_{n+1}(0)\mathbb{E}^{Q_{T_{n+1}}}\left[\frac{(L_n(T_n)-K)_+}{B_{n+1}(T_{n+1})}\right] \\ &= M\sigma B_{n+1}(0)\mathbb{E}^{Q_{T_{n+1}}}\left[(L_n(T_n)-K)_+\right], \end{aligned}$$

because $B_{n+1}(T_{n+1}) = 1$.

To simplify $\mathbb{E}^{Q_{T_{n+1}}}\left[(L_n(T_n)-K)_+\right]$, we must first introduce a new lemma.

Lemma 1. *If $Z : \Omega \to \mathbb{R}$ is normally distributed ($Z \sim \mathcal{N}(\alpha, \beta^2)$) and f is an integrable function, then*

$$\mathbb{E}\left[e^Z f(Z)\right] = e^{\alpha+\frac{1}{2}\beta^2}\mathbb{E}\left[f(Z+\beta^2)\right].$$

Then the price of caplet i at time T_n equals

$$C_i^{\text{model}}(T_n, K) = M\sigma B_{n+1}(0)(L_n(0)\mathcal{N}(d_1) - K\mathcal{N}(d_2)), \tag{5.1}$$

where

$$d_1 = \frac{\ln(\frac{L_n(0)}{K}) + \frac{\tau^2}{2}}{\tau} \qquad d_2 = \frac{\ln(\frac{L_n(0)}{K}) - \frac{\tau^2}{2})}{\tau}. \tag{5.2}$$

Comparing (5.2) and (3.5), we conclude the following relation between the implied volatility of the Black Scholes model and the one of the Libor market holds:

$$\sigma_n^{Black} = \sqrt{\frac{1}{T_n}\int_0^{T_n} ||\sigma_n(s)||^2 ds.} \tag{5.3}$$

This concludes our mathematical calibration of the Libor market model.

5.1.1 Calculation of implied caplet price

Now that it is clear how C_i^{Black} influences the σ_i in the Libor market, it is worthy discussing how it is calculated, which is comparable with the method of implied volatility as in section 3.1.1. Some of the biggest banks, so-called panel banks, have to provide their prices $C_i^{\text{Panel}_j}$, which is the price they, bank j, would be willing to pay for a specified caplet i (in our case, could be any financial product to which the market is calibrated).

The lowest 25% as well as the highest 25% of each $C_i^{\text{Panel}_j}$ are filtered out, after which the mean of the remaining 50%, which makes C_i^{Panel}, is calculated. Now, this C_i^{Panel} is used to approximate the value of C_i^{Black}.

5.2 Characters in calibration

Now that it is clear how to determine the σ_i, let us take a look at the calibration in practice. Assume σ_i is constant and one-dimensional, thus depends on just one source of uncertainty. Now, let

$$C^{\text{Panel}_j} = \begin{pmatrix} C_1^{\text{Panel}_j} \\ \vdots \\ C_n^{\text{Panel}_j} \end{pmatrix}$$

be the $n \times 1$-vector that consists of the submitted caplet prices from bank j for maturities T_2 up until T_{n+1}. Now, taking the mean of the mid-50% will provide us with

$$C^{\text{Panel}} = \begin{pmatrix} C_1^{\text{Panel}} \\ \vdots \\ C_n^{\text{Panel}} \end{pmatrix} = \begin{pmatrix} C_1^{\text{Black}} \\ \vdots \\ C_n^{\text{Black}} \end{pmatrix} = C^{\text{Black}}$$

where, as explained in section 5.1.1, C^{Panel} is used to estimate C^{Black} of the specific caplet the market is calibrated with.

Next the Black-Scholes model can be used to determine the implied volatilities σ_i^{Black}, for $1 \leq i \leq n$. Now, when

$$\sigma^{\text{Black}} = \begin{pmatrix} \sigma_1^{\text{Black}} \\ \vdots \\ \sigma_n^{\text{Black}} \end{pmatrix},$$

equation (5.3) will give us the implied σ_i for the Libor market model, which we take together in vector

$$\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}.$$

5.3 Example of calibration

Consider a two year cap-option with $K = 1.3\%$ that is divided into caplets of three months. Then the interval of caplet$_i$ is $[T_i, T_{i+1}] = [\frac{i-1}{4}, \frac{i}{4}]$, with T_i in years and $1 \leq i \leq 8$. Assume the initial interest rate L_0 is constant and equals 1.1%.

The markets caplet-prices C_i^{Panel} and maturities T_{i+1} that are used to determine σ_i^{Black} are given in the following table. [2]

i	Maturity (T_{i+1})	C_i^{Panel}	$L_i(0)$
1	$\frac{1}{4}$	0.5	1.1
2	$\frac{1}{2}$	0.9	1.1
3	$\frac{3}{4}$	1.3	1.1
4	1	1.8	1.1
5	$\frac{5}{4}$	2.4	1.1
6	$\frac{3}{2}$	3	1.1
7	$\frac{7}{4}$	3.3	1.1
8	2	3.8	1.1

Table 5.1: Maturities, market prices and libor rates of caplet$_i$.

Now, under our assumptions, from equation (5.3) the computed implied volatilities σ_i^{Black} equal the σ_i's of the Libor market.

[2] These are made-up numbers, but are based on real-life rent rates, maturities and values of options

i	$\sigma_i^{\text{Black}} = \sigma_i$
1	0.5268
2	0.5115
3	0.5252
4	0.5692
5	0.6316
6	0.69
7	0.6921
8	0.7321

Table 5.2: Implied volatilities σ_i^{Black}.

Observe that when i increases, σ_i increases as well. This is what we would expect intuitively, since it seems natural that we would be better at approximating a certain value for a few months from now, than that we would for a year (or two) from now. Thus, the insecurity about the approximated value for $L_i(T_i)$ increases whenever the maturity T_i lies further in the future. This corresponds with an increase in σ_i.

Chapter 6

Fraud in the Libor market

Considering the Libor market model of chapter 4 and the calibration of chapter 5, we will now focus on possible fraud in the Libor market.

6.1 The weak spot

When the Libor market parameters are calibrated using the Black-Scholes formula, the estimated market prices $C_i^{\text{Panel}_j}$ of panel bank j play an important role. These prices are submitted by so-called "submitters" from the panel banks, who are just a few floors away from the traders that are trying to make money on the same market.

In a bonus-based culture where you can get fired every five minutes, according to Joris Luyendijk (Luyendijk, 2015 [4]), the temptation to submit slightly different numbers, which will make your friend more on his deal of the week, can get huge.

Although one could argue about (possible remedies for) these temptations, the weakest spot of the Libor market model and its calibration would remain $C_i^{\text{Panel}_j}$, since this is the part were some guessing comes in. In order to determine σ and use our model to calculate Libor rates, we must somehow estimate C_i^{Black}. In a world without any fraud, there would still be human errors that lead to false estimations of market prices of caplets.

6.2 Simulation of fraud

When an employee of bank A commits fraud on the Libor market, he or she tries to manipulate volatility σ_i by submitting a false $C_i^{\text{Panel}_A}$. For now, we will be looking at possible motives to influence the price of caplets. It would be tempting to manipulate the volatility in such a way that σ_i^{Fraud} is lower than σ_i, if one is planning on buying a caplet, since then the price of the option, looking at (3.3), would be cheaper than in the original situation. Equivalently, a fraud would want to increase σ_i if his or her bank is planning on selling a caplet, since then the price of the option would increase as well.

Let us assume that the market is calibrated to prices of caplets with maturities of every three months and strike value $K = 1.3\%$, as in Table 5.1. Then the panel banks submit $C_i^{\text{Panel}_j}$, thus the price they (bank j) would be willing to pay for each caplet i with $K = 1.3\%$. Now, according to section 5.1.1, if the submitted price of bank A for caplet$_i$, $C_i^{\text{Panel}_A}$, is within 25 percent of the median, its submitter is able to influence C_i^{Panel} (and thus σ_i).

Suppose this is the case for all i and that his colleague, a trader, is planning to buy a cap[1] with volume M, strike rate $K = 1.4\%$ and maturities $T_i = \frac{i}{4}, 1 \leq i \leq 8$. Then, the fraud would want to manipulate $C_i^{\text{Panel}_A}$ in such a way that the mean of all submitted prices, C_i^{Panel}, and σ_i decrease, since then the price of caplet$_i$ of the cap his colleague is buying, would be cheaper than the original market price.

Let C_i^{Fraud} denote the approximated price of the caplet to which the Libor market is calibrated, in the case the market is manipulated. Now, let V_i be the value of caplet$_i$ of the above described option with $K = 1.4\%$ without any fraud, and V_i^{Fraud} the value with as much fraud such that $C_i^{\text{Fraud}} = C_i^{\text{Panel}} - 0.1$ for all i. Then the following tables display the outcome of a minor fraud.

[1] Which should not be confused with the cap the Libor market is calibrated with.

i	C_i^{Panel}	σ_i	V_i	i	C_i^{Fraud}	σ_i^{Fraud}	V_i^{Fraud}
1	0.5	0.5268	$0.0031M$	1	0.4	0.4740	$0.0023M$
2	0.9	0.5115	$0.0067M$	2	0.8	0.4777	$0.0058M$
3	1.3	0.5252	$0.0105M$	3	1.2	0.4986	$0.0095M$
4	1.8	0.5692	$0.0154M$	4	1.7	0.5464	$0.014M$
5	2.4	0.6316	$0.0213M$	5	2.3	0.6111	$0.020M$
6	3	0.69	$0.0274M$	6	2.9	0.6709	$0.026M$
7	3.3	0.6921	$0.0305M$	7	3.2	0.6742	$0.029M$
8	3.8	0.7321	$0.0356M$	8	3.7	0.7150	$0.034M$

Table 6.1: Implied volatilities and caplet-prices for markets with and without fraud.

Observe that the table confirms our intuition: the fraud has, by submitting false $C_i^{\text{Panel}_A}$, been able to manipulate the (fair) price C_i^{Panel}, in such a way that the approximated prices of all caplet$_i$ have decreased to C_i^{Fraud}. Since the market is calibrated using the prices of this specific caplet, the implied volatilities following from 3.3 have decreased as well. It has already been noticed that these equals the volatilities of the Libor market, σ_i.

Since σ_i has dropped, the price V_i of caplet$_i$ with $K = 1.4\%$ has dropped to V_i^{Fraud}. The value V of a cap equals the sum of all caplet$_i$'s, thus the non-fraud price equals

$$V = \sum_{i=1}^{8} C_i = 0.1505M,$$

and the price V^{Fraud} equals

$$V^{\text{Fraud}} = \sum_{i=1}^{8} C_i^{\text{Fraud}} = 0.1406M.$$

We conclude the fraud has reached his goal, namely to manipulate the Libor market such that the bank was able to buy a caplet for a price lower than its market price.

The consequenses.

The fraud led to a 6.5% decrease in price with just a minor action, numerically speaking, of fraud. This may not seem like too big of a deal, but it is important to bear in mind that M typically is a number in millions or billions and financial products like these are being traded on a large scale all day.

Further on, the implied volatilities σ_i^{Fraud} are used in equations like (4.10) to determine Libor rates $L_i^{\text{Fraud}}(t)$, which will in their turn also differ from $L_i(t)$. Libor rates are an important instrument throughout the entire financial sector. They influence rent rates on deposits and mortgages and influence the Stock market as well[2]. Due to the fraud of one submitter of bank A, these financial products will all be misvalued.

In the recent credit crisis of 2008, we have seen what kind of consequences misvaluation of financial products can have.

6.3 The solution?

As discussed in section 6.2, the weakest spots of the Libor market are the submitted prices. Calibrated the right way, these are the only external variables the Libor market model. It is argued that the financial sector needs to be rebooted, which would be excluding all formulas that are sensible from the market, since it triggers frauds like these in the Libor market.

However, the Libor market, like the banks in chapter 1, has become "too big to fail". Rebooting the system will have disastrous consequences compared to the relatively small ones the fraud has, and there are no models for financial products that can be completely exempted from fraud, since there will always have to be some kind of calibration to market data.

This does not mean that better regulations, such as a bigger physical distance between traders and submitters, more compliance etc., could not reduce the chance that fraud like the one in 2012 will take place.

[2]As can be deduced from the Black-Scholes formula in equation (3.1)

Chapter 7

Discussion

In this thesis, the mathematical dynamics of the Libor market came to light. Starting from the price of bonds $B(t)$ and d sources of uncertainty brought together in Brownian motion $W(t)$, the mathematical model of the Libor market was described. We were able to calibrate the model to reality with estimated prices of caplets and the Black-Scholes formula.

Some assumptions had to be made along the way. The first, which is a general one in financial mathematics, was that markets were arbitrage-free. Although it is a desirable assumption to make in this field, it is false by definition. If the prices of financial products equalled the expected value, one would think that the expected profits of huge financial institutions are around zero[1]. It needs no further explanation to state that this surely is not the case, and thus this assumption is false.

Further on, it was assumed in section 5.2 that σ_i was one-dimensional and constant over time, which meant that $W(t)$ had to be one-dimensional as well. This does not coincide with the theory of σ_i, but was a desired simplification looking at (5.3). If we had not made this assumption, we would not have been able to simulate the calibration of section 5.3, neither the fraud of section 6.2.

It is worthwhile discussing an extension to the Libor market, which is suggested by Steinrücke, Zagst and Swishchuk. In their paper of 2013 (Steinrücke, Zagst & Swishchuk, 2013 [8]) they extend the Libor market in a way such that it is possible

[1]Since they trade in these financial products on such a large scale, one could assume so according to the Law of large numbers

to implement "structural breaks and changes in the overall economic climate". They also cover possible changes in dynamics in case of a change of measure. Although the practice of this paper is beyond the scope of this thesis, it is a fine example of how up-to-date the field of financial mathematics in general and the Libor market model in specific are.

Now that the assumptions have once more been stressed, it is time to draw our conclusion. Sadly, we must conclude that the Libor market model and our financial climate still invite to commit fraud. This may seem inevitable, since in all financial models a calibration to market data has to be used, otherwise we would not be able to work with the model. However, one might argue that there is still progress to be made by better regulation and a greater sense of responsibility from the biggest banks in the world, which could be subject to other fields of study.

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