The BMS algebra and black hole information

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Abstract

As a means of working towards solving the information paradox, we discuss conformal diagrams of evaporating black holes. We generalize the Schwarzschild solution to the asymptotically flat Bondi-metric and give a derivation of its asymptotic symmetry algebra, the BMS algebra. We discuss the interpretation as a charge algebra of zero-energy currents. Finally, we establish the centrally extended BMS-algebra as the semi-simple product of the Virasoro algebra acting on a representation.

Contents

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1. Introduction

1.1 Introduction

One of the first non-trivial examples of a metric in General Relativity is the Schwarzschild metric. It models the behaviour of the gravitational field of massive bodies, such as planet earth, or the sun. For example, the metric describing planet earth is equal to the Schwarzschild metric with Schwarzschild radius $r_S = 2GM_\oplus$, outside of earth's matter radius.

The matter radius of astronomical objects is maintained by outward force due to interaction of the matter that composes it. If the gravity is stronger than this outward force, the matter radius decreases. If this matter radius decreases beyond the Schwarzschild radius, a black hole is formed.

As a result of the Unruh effect and the equivalence principle, Stephen Hawking and Jacob Bekenstein discovered an evaporation process of black holes, which applies in particular for the Schwarzschild solution. By emitting radiation, its mass and radius decrease. The discovery of this process has led to a number of interesting questions. One of them concerns black hole information.

We briefly discuss the problem below. In the next chapters, we take on a strategy to solve it. Though not finished, some important progress has been made recently. The goal of this thesis is to understand what has been done so far, and what should be done to finish the work.

1.1.1 Black hole information

For the Schwarzschild solution, and its stationary generalizations (Kerr, Reissner-Nordstrom, Kerr–Newman), it has been shown that the only conserved quantities of are the mass M, the charge Q, and the angular momentum J. This is known as the **no-hair theorem**. It has been conjectured to be true for more general black holes.

The no-hair seems to infer that information about anything that is absorbed by the black hole is lost; for how can one possibly keep track of whatever comes in using just these three parameters?

Black hole evaporation, discovered by Hawking in 1974, [1], poses a problem in this context. How are incoming material and evaporative radiation related if the black hole 'forgets' the information of what came in?

At the time of the discovery of black hole evaporation, there were no ideas as to how to maintain the information of incoming radiation or matter on the Schwarzschild horizon, in conflict with the law of preserved information; which is classically due to Liouville's theorem, and in quantum mechanics equivalent to quantum unitarity. This problem is commonly referred to as the *information paradox.*

1.1.2 Lumpy black holes

In order to give a better description of black holes that evaporate, we should relax the condition of stationarity; we need a time-dependent metric to describe evolving black holes. Due to **Birkhoff's Theorem** such a metric cannot solve Einstein's equations for the vacuum, if it is spherically sym-

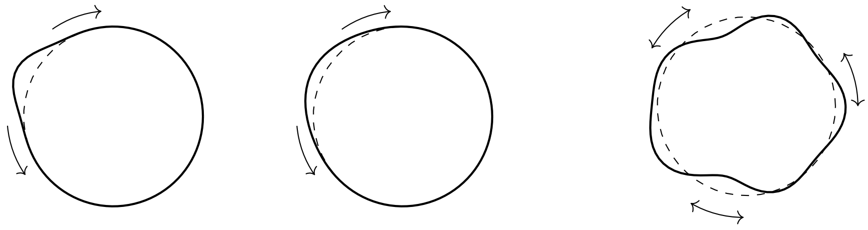


Figure 1.1: A small object has fallen into a black hole. The radial size increment traverses the black hole horizon, (left, center). It turns out in section 4.3, that classically a black hole with propagating waves does not in general become spherical again, (right).

metric. So simultaneously we need to relax this condition. In 1962, Bondi, van der Burg, and Metzner in [2] have generalized black hole solutions, to a class of axially symmetric, asymptotically flat metrics. These metrics describe to a multitude (i.e., possibly more than one) of 'lumpy' black holes that evolve over time. A metric that satisfies said conditions goes by the name of **Bondi metic**. The Minkowski and Schwarzschild metrics are both Bondi metrics.

The relaxation of spherical symmetry seems very reasonable, from a physical point of view. Imagine a small object being thrown into a black hole. Then surely the far end does not 'know' straight away that the radius should increase; the increment in size can only travel at the speed of light, so at least for a while, the black hole has a lump. It turns out in section 4.3, that black holes can be (classically) excited with waves that do not die out.

1.1.3 Asymptotic Symmetries

The group of vector fields that leave the Bondi metric asymptotically invariant, (i.e., the new metric which is the result of flowing the old along the vector field is again a Bondi metric), is the BMS group. It consists of rotations and supertranslations, an infinite class of transformations that is generated similar to the translations.

In 2010, Barnich and Troessaert have made the case in [7], that the BMS group should be expanded so as to contain the local conformal transformations of the sphere, which will be part of the studies in this thesis. These local conformal transformations are called superrotations.

Recently, in [5], Strominger showed that the BMS group is a symmetry group of classical gravitational scattering, and of the $\mathcal{S}$-matrix in quantum gravity. He argues that in a finite neighbourhood of the Minkowski vacuum, classical gravitational scattering is BMS-invariant. Furthermore the $\mathcal{S}$-matrix of asymptotically Minkowskian quantum gravity, has the symmetry

$$X_\varepsilon^+ \mathcal{S} = \mathcal{S} X_\varepsilon^-,$$

where $X_\varepsilon^\pm$ are infinitesimal generators of BMS$^\pm$.

One of the key results of this article is the conservation law of local energy, which is defined by the Bondi mass aspect $m_B(u, x^A)$, at each angle x^A. (The Bondi mass for a Schwarzschild black hole is GM.)

Proposition 1.1 (Strominger). *The total incoming energy flux integrated along any null generator on $\mathscr{I}^-$ equals the total outgoing energy flux integrated along the continuation of this null generator on $\mathscr{I}^+$.*

Global energy conservation is due to a global time translation, which is a supertranslation that does not depend on the angle. Conversely, an angle dependent supertranslation which acts only on one angle, will lead to a conservation law at that one angle; i.e., conservation of local energy.

For a scattering in Minkowski spacetime of particles propagating from $\mathscr{I}^-$ to $\mathscr{I}^+$, this angular energy is maintained by soft gravitons. They have localized energy contributions which insure that at each angle the local energy is conserved, whilst having zero total energy. In [19], these soft particles have been used to construct conserved currents. They have zero total energy, and therefore they are dubbed **soft hairs**, as a contraction of 'hairs' in the no-theorem and 'soft particles'. The BMS vector fields are established at null infinity $\mathscr{I}^+$ or $\mathscr{I}^-$, rather than at spatial infinity i^0, which is the main reason that these modes have remained undiscovered as an ADM-charge.

The asymptotic symmetry algebra shall be the main subject of this thesis. The associated soft hair (or perhaps 'follicle') is found to be a means of storing information. Before deriving the Bondi metric, and the BMS-algebra, we discuss some consequences of black hole evaporation, and we develop some necessary machinery. After having derived the explicit vector fields inducing the supertranslations and superrotations, we investigate central extensions of the BMS algebra, necessary to move from the classical to the quantum picture.

2. Black holes and evaporation

In this chapter we discuss some consequences of the evaporation of black holes. This should help understand the information problem, and where we should look to solve it.

2.1 Black hole evaporation

Under the right circumstances a black hole is formed by a collapsing star. For simplicity, we assume that a black hole of mass M_0 has been formed at time t_0, by a massive spherical object, with little or no outward pressure due to interaction, of mass M_0. We know that the matter sphere exterior has the Schwarzschild metric, and that the black hole starts to evaporate at time t_0.

2.1.1 Evaporation

We use the Stefan-Boltzmann law for black body radiation as an estimate for the radiated power P:

$$P = A\sigma T^4. \tag{2.1}$$

Here $A = 16\pi G^2 M^2$ is the area of a Schwarzschild black hole, and σ is the Stefan-Boltzmann constant, given by

$$\sigma = \frac{\pi^2 k_B^4}{60\hbar^3}.$$

For the temperature we use the Hawking temperature

$$T_H = \frac{\hbar}{8\pi G M k_B}.$$

Substituting the Schwarzschild area and the Hawking temperature into (2.1), we obtain

$$P = \frac{\hbar}{15360\pi G^2}\frac{1}{M^2} = \frac{K_{ev}}{M^2},$$

where we have defined evaporation constant

$$K = \frac{\hbar}{15360\pi G^2}.$$

But the radiation gives rise to a decrease in mass, via $P = -dE/dt = -dM/dt$, $(c = 1)$. This gives rise to the differential equation

$$-\frac{dM}{dt} = \frac{K_{ev}}{M^2}.$$

Under the boundary conditions set above, this gives rise to the time-dependent mass function $m(t)$:

$$m(t) = \begin{cases} M_0, & t - t_0 \leqslant 0 \\ \left(M_0^3 - 3K_{ev}(t-t_0)\right)^{1/3} & 0 < t - t_0 < t_{ev} = \dfrac{M_0^3}{3K_{ev}} \\ 0, & t_{ev} \leqslant t - t_0 \end{cases} \tag{2.2}$$

Let us consider as a first example the emission of a single photon out of a Schwarzschild black hole. Let the emission occur at time t_{em} (em for emission), and let the radius decrease from r_0 to $r_0 - \eta$. We now define an adjusted tortoise coordinate

$$r^* = r + (r_0 - \eta)\ln\left|\frac{r}{r_0 - \eta} - 1\right|,$$

valid for the emitted photon, and any light-like trajectories that are beyond its light-cone. The photon itself carries the 'news' that the black hole has decreased in size. Since the photon must start at $r_0 = t_{em}$, the photon trajectory is defined by

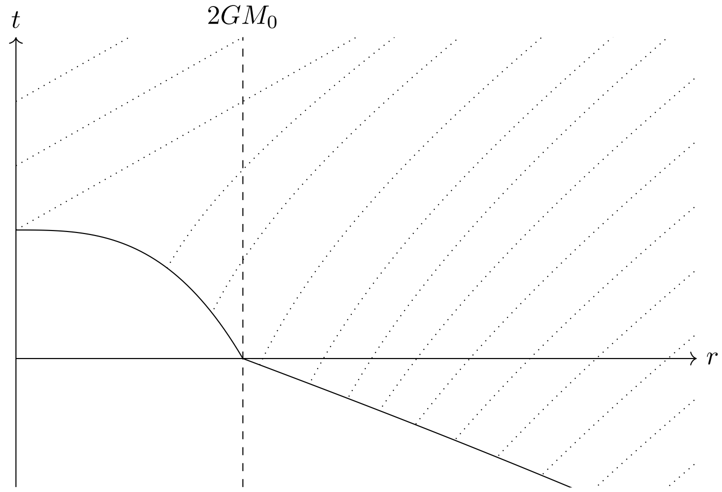
$$t - r^* = u_0, \qquad u_0 = t_{em} - r^*_{em} = t_{em} - r_0 - (r_0 - \eta)\ln\left|\frac{\eta}{r_0 - \eta}\right|.$$

Note that u_0 is finite for non-zero η, so we have a well-defined photon trajectory

$$t - t_{em} = r - r_0 + (r_0 - \eta)\ln\left|\frac{r - (r_0 - \eta)}{\eta}\right|.$$

It coincides with the trajectory of a photon through empty space if $\eta = r_0$, i.e., if the black hole has completely evaporated.

Consider a collapsing star, with some strictly decreasing matter radius $r(t)$, such that $r(t_0) = 2GM_0$. Based on the observation above, we wish to extend the idea to a black hole with the horizon evolution (2.2). Here we propose an adjusted Eddington-Finkelstein diagram, with the added null lines are obtained by interpolation between the Schwarzschild metric with mass M_0 before the star collapses, and the Minkowski metric, after the black hole has evaporated. The diagram should look like this, with the dotted lines depicting null curves.



2.2 Conformal diagrams

Based on the discussion in the previous section, light *does* escape the black hole as it shrinks, and that points on the evolving horizon have different u_0 coordinates.

Proposition 2.1. *The horizon of a homogeneously shrinking black hole is timelike.*

This should be no surprise, for if we wait for a black hole of radius r_0 to evaporate to, say, half the size, then the radial point r_0 lies well outside the black hole. Based on this, we propose the conformal diagram of a collapsing star, forming a black hole, that eventually evaporates.

Next, we propose that the associated conformal diagram be as in 2.1. It is essentially obtained by 'straightening out' the light lines in the adjusted Eddington-Finkelstein diagram above.

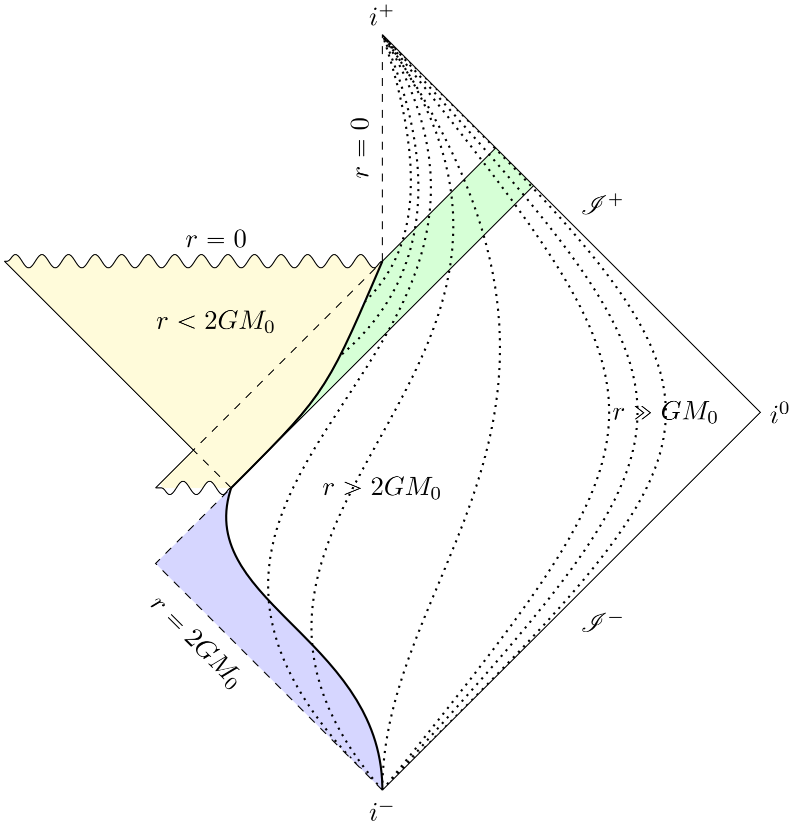


Figure 2.1: Conformal diagram of a confguration of matter (blue) of total mass M_0 collapsing into a black hole (yellow) which evaporates into radiation (green). Dotted lines are of equal radius.

Near past infinity i^0 we have a spherical configuration of matter, confined to a finite region. Outside that region, spacetime is well described by the Schwarzschild metric, with the matter centre of mass as its origin. Due to gravitational interaction, the matter will be attracted inwards. As the massive body shrinks in size, line of equal radius pass out of the outer edge in the diagram.

Eventually, all of the matter will pass beyond the Schwarzschild radius, thus forming a black hole (yellow). It evaporates at a rate (2.2). Similar to the matter sphere before, as the black hole decreases in size, lines of equal radius emanate from the black hole. Note that as these line pass through the horizon, they should approach a 45◦ angle. In addition, different lines cannot touch in the interior of the diagram (i.e., strictly between the horizon, $\mathscr{I}^+$ and $\mathscr{I}^-$). This is yet another reason not to draw the evolving black hole horizon as a null line in this diagram.

Since the black hole horizon evolves in a timelike fashion, lines of equal radius come out, and the radiation constitutes a region, rather than a point, at null infinity $\mathscr{I}^+$. Using the appropriate retarded time coordinate $u = t - r^*$, the stress-energy tensor $T_{\mu\nu}(u)$ will increase, starting at u_0, and becoming constant from u_f onwards.

After the black hole has completely evaporated, the metric should describe Minkowski spacetime, as reflected in the top part of the diagram. Similarly, for sufficiently large r, the digaram should look just like the conformal diagram of Minkowski spacetime. In fact, the far right of the diagram should be (almost) indistinguishable from the far right part of the Minkowski spacetime conformal diagram.

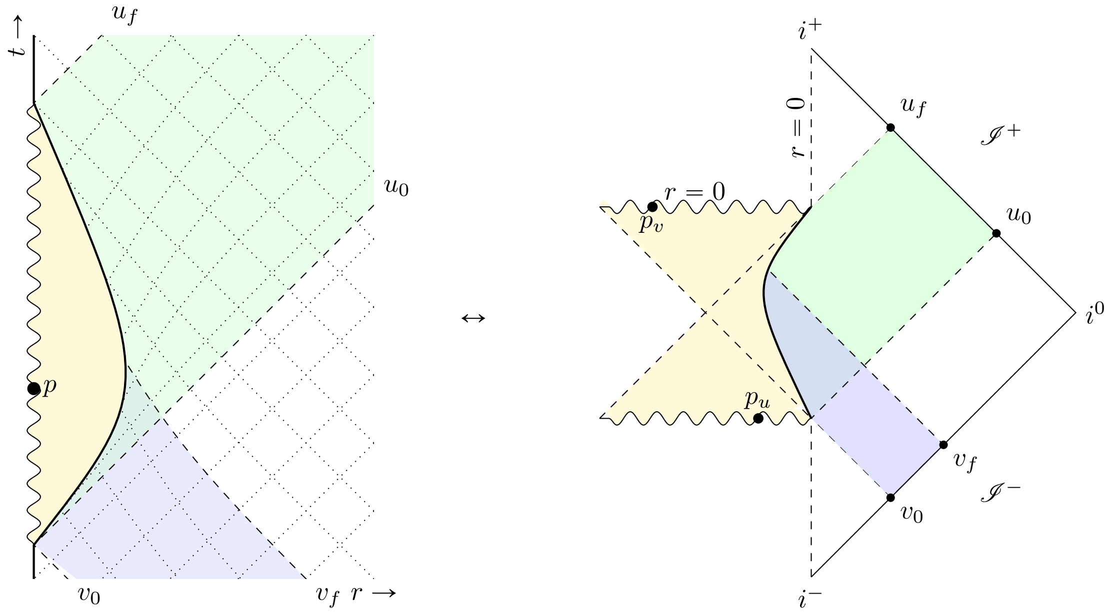


Figure 2.2: Conformal diagram (right) and (r, t)-diagram (left) of radiation 'active' between advanced time v_0 and v_f (blue) forming a black hole (yellow) which evaporates into radiation (green) observed between retarded time u_0 and u_f. Dotted lines are of equal advanced (retarded) null coordinate u (v). The point p is drawn to illustrate that the black hole origin and the white hole origin should be identified (i.e., p_u should be folded p_v to obtain the left diagram from the right)

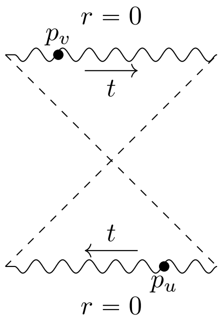
Similarly, we consider a black hole formed by incoming radiation, based on the consideration that the shrinkage allows for null lines to come out of the black hole. The radiation is 'active' between v_0 and v_f. The precise distribution over time of the radiation is not important. What is important is that the black hole increases in size, until the incoming radiation stops (or is less intense than the outgoing radiation). After that the black hole still needs some time to evaporate.

Since a conformal diagram is obtained by 'straightening out' null lines, (and confining infinity conformally to a box), the (r, t)-diagram of such a black hole tells us how to draw the conformal diagram, as seen in . This conceptually proofs the following proposition.

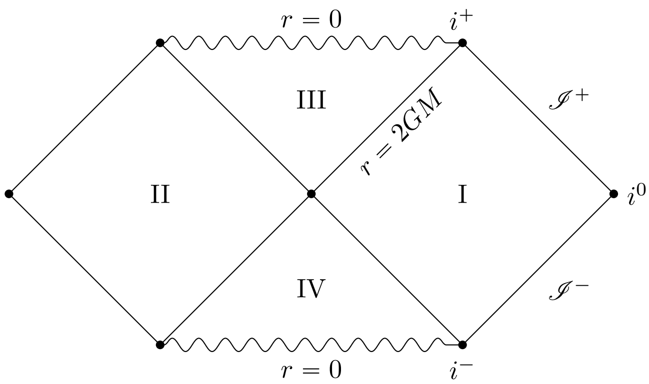
Proposition 2.2. *The black hole origin $r = 0$ and the white hole origin, in the conformal diagram of a black hole, are two copies of the origin of the same object. The black (white) hole origin is well-defined in terms of retarded (advanced) time coordinate u (v), i.e., the appropriate Eddington-Finkelstein coordinate. In order to obtain the (r, t)-diagram, they should be folded over one another, as seen in 2.2.*

2.2.1 Time direction

As a final exercise, we aim our attention at the black hole interior in the conformal diagram in 2.2. Proposition 2.2, infers a time direction close to the two copies of the origin:



Let us consider the limit of an evaporating black hole that corresponds to the Schwarzschild solution. In physical terms, that is either a black hole created at past infinity i^-, and which only starts to (significantly) evaporate at future infinity i^+, or a black hole kept at fixed radius $2GM$ by matching the incoming and outgoing radiation. Its Penrose diagram is



Excluding region II, it should be seen as the limit of diagram 2.2. If we do extend the diagram, it should be clear that Proposition 2.2 implies that the time direction in region II actually is downwards.

Corollary 2.2.A *The time direction in region* II *is opposite to the time direction in region* I.

This corollary also arose in [20] by Gerard 't Hooft, where it was derived using unitarity of near-horizon wave-functions.

3. Introduction to asymptotic symmetry

In this chapter we develop some notions and machinery that are relevant to understanding the main body of this thesis: the asymptotic symmetry algebra.

3.1 Asymptotic Symmetries of the plane

Asymptotic flatness conditions ensure that a manifold behaves like flat space far away from some interior region. This is something that is reasonable to require from a metric describing one or more black holes; far away their gravity should be negligible. The 4 dimensional spacetime case is attended to in the next chapter. Here we consider a 2 dimensional manifold $\mathcal{M}$ with a Riemannian metric g_{ij}. It can always be cast in the polar form

$$ds^2 = dr^2 + f(r,\theta)d\theta^2$$

by solving the system of equations:

$$g_{ij} = \tilde{g}_{\tilde{i}\tilde{j}} \frac{d\tilde{x}^{\tilde{i}}}{dx^i} \frac{d\tilde{x}^{\tilde{j}}}{dx^j}, \qquad (\tilde{g}_{11} = 1, \tilde{g}_{12} = 0).$$

for the three functions $\tilde{x}^1 \equiv r, \tilde{x}^2 \equiv \theta,\ \tilde{g}_{22} \equiv f(r,\theta)$. Here θ is an angular coordinate, i.e., $\theta = \theta + 2\pi$.

Definition 3.1. *A metric of the above form is* ***asymptotically flat*** *if*
(A1) There exists a bounded region $U \subset \mathcal{M}$ outside of which the coordinates r, θ are valid.
(A2) In $\mathcal{M} \backslash U$ the function f goes like

$$f(r,\theta) = r^2 + a(r,\theta),$$

where $a(r,\theta) \in \mathcal{O}_\infty(r^1)$.

The second condition is equivalent to the fall-off conditions:

$$\partial_r f(r,\theta) \in \mathcal{O}_\infty(r^1), \qquad \partial_\theta f(r,\theta) \in \mathcal{O}_\infty(r^1).$$

Consider an asymptotically flat metric. Let $a(r,\theta) = f(r,\theta) - r^2$ so as to obtain:

$$ds^2 = dr^2 + \big(r^2 + a(r,\theta)\big)d\theta^2.$$

Then the inverse metric is

$$g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \big(r^2 + a(r,\theta)\big)^{-1} \end{pmatrix}.$$

The nonvanishing Christoffel symbols are

$$\Gamma^r_{\theta\theta} = -\frac{\partial_r f(r,\theta)}{2} \qquad \Gamma^\theta_{r\theta} = \frac{\partial_r f(r,\theta)}{2f(r,\theta)}, \qquad \Gamma^\theta_{\theta\theta} = \frac{\partial_\theta a(r,\theta)}{2f(r,\theta)}.$$

Definition 3.2. ***Asymptotic isometries*** *are transformations that send the metric g to a metric $\tilde{g}$ that is again asymptotically flat. They are induced by vector fields that maintain the asymptotic form of the metric.*

In our case, this gives rise to the equations

$$\mathcal{L}_X g_{rr} = 0, \qquad \mathcal{L}_X g_{r\theta} = \mathcal{L}_X g_{\theta r} = 0, \qquad \mathcal{L}_X g_{\theta\theta} = 0 + \mathcal{O}_\infty(r^1).$$

Note that these are a weak version of the Killing equations, where $\mathcal{L}_X g_{ij} = 0$ for all i, j. I will therefore refer to these vector fields as **asymptotic Killing vectors**

Calculation: Asymptotic Killing vectors *By metric compatibility the set of equations becomes*

$$\begin{aligned}\partial_r X^r g_{rr} &= 0, \qquad (3.1)\\ \partial_r X^\theta g_{\theta\theta} + \partial_\theta X^r g_{rr} &= \Gamma^\theta_{r\theta} X^\theta g_{\theta\theta} + \Gamma^r_{\theta\theta} X^\theta g_{rr},\\ \partial_\theta X^\theta g_{\theta\theta} &= \Gamma^\theta_{\theta\theta} X^\theta g_{\theta\theta} + \Gamma^\theta_{r\theta} X^r g_{\theta\theta} + \mathcal{O}_\infty(r^1),\end{aligned}$$

First of all, by (3.1), $X^r = T(\theta)$ for some function $T : \mathbb{S}^1 \to \mathbb{R}$. Substituting this, along with the metric entries and Christoffel symbols, we obtain the reduced set of equations:

$$f(r,\theta)\partial_r X^\theta + T'(\theta) = 0, \qquad (3.2)$$

$$2f(r,\theta)\partial_\theta X^\theta + (\partial_\theta a(r,\theta))X^\theta + \partial_r f(r,\theta)T(\theta) = 0 + \mathcal{O}_\infty(r^1), \qquad (3.3)$$

Equation (3.2) is resolved by setting

$$X^\theta = Q(\theta) + T'(\theta)\int_r^\infty \frac{d\tilde{r}}{f(\tilde{r},\theta)}, \qquad (3.4)$$

where $Q(\theta)$ is again an arbitrary function of θ. This is the point where we apply asymptotic flatness of g_{ij}; since $a(r,\theta) \in \mathcal{O}_\infty(r^1)$, we have the geometric series expansion of the integral in (3.4):

$$\int_r^\infty \frac{d\tilde{r}}{\tilde{r}^2 + a(\tilde{r},\theta)} = \int_r^\infty d\tilde{r}\big(\tilde{r}^{-2} - a(\tilde{r},\theta)\tilde{r}^{-4} + a^2(\tilde{r},\theta)\tilde{r}^{-6} - \ldots\big).$$

The terms in the expansion of ever decreasing order in r. The first term of the integral can actually be evaluated

$$\int_r^\infty \frac{d\tilde{r}}{\tilde{r}^2 + a(\tilde{r},\theta)} = r^{-1} + \int_r^\infty d\tilde{r}\frac{a(\tilde{r},\theta)}{\tilde{r}^2 f(\tilde{r},\theta)} = r^{-1} + \mathcal{O}_\infty(r^{-2}).$$

In order to solve the last remaining constraint equation (3.3), we wish to decompose the left hand side of (3.3) into orders of r. First note, that in order $[r^2]$ we have constraint

$$2r^2 Q'(\theta) = 0.$$

So $Q(\theta) \equiv R$ is constant. The rest of the constraint equations is contained in the allowed asymptotic fall-off $\mathcal{O}_\infty(r^1)$. So the general solution is

$$\begin{aligned}X^r &= T(\theta)\\ X^\theta &= R + T'(\theta)\int_r^\infty \frac{d\tilde{r}}{f(\tilde{r},\theta)}.\end{aligned}$$

Note that to first order, the vector field is independent of $f(r,\theta)$:

$$X^\theta = R + \frac{T'(\theta)}{r} + \mathcal{O}_\infty(r^{-2}).$$

The solutions we have obtained are infinitely generated, by arbitrary periodic functions $T(\theta)$. These asymptotic Killing vectors will be referred to as **supertranslations**. The relation to normal translations will become clear in the next section, see e.g. figure 3.1. The R-generated part is just a normal rotation.

Different fall-off conditions The choice of fall-off in the definition for asymptotic flatness, is somewhat arbitrary. In literature sometimes a fall-off $a(r,\theta) = r^2 + \mathcal{O}_\infty(r^{3/2})$. The choice of fall-off in Definition 3.1 leads to a Ricci scalar $R = \mathcal{O}_\infty(r^{-4})$. A weaker fall-off condition would result in an extra constraint in the asymptotic Killing equations, due to (3.3). As a result the Killing vectors and asymptotic Killing vectors then coincide.

Asymptotic flatness in Cartesian coordinates By setting $x = r\cos\theta, y = r\sin\theta$ we obtain an expression for the metric in terms of Cartesian coordinates:

$$g_{ij}(x,y) = \begin{pmatrix} 1 + y^2\frac{a(r,\theta)}{r^4} & -xy\frac{a(r,\theta)}{r^4} \\ -xy\frac{a(r,\theta)}{r^4} & 1 + x^2\frac{a(r,\theta)}{r^4} \end{pmatrix}.$$

Note that this can be written quite nicely to

$$g_{ij}(x,y) = \eta_{ij} + \epsilon_{ik}x^k\epsilon_{jl}x^l p(x,y),$$

where η_{ij} is the Minkowski metric, $\epsilon_{i\tilde{i}}$ is the 2-dimensional Levi-Civita tensor, and $p(x,y) = a(r,\theta)/r^4$. A metric of this form is asymptotically flat, if $p(x,y) \in \mathcal{O}_\infty\Big(\big(x^2+y^2\big)^{-3/2}\Big)$. This is true for terms $x^m y^n$ if $m + n \leqslant -3$.

3.2 The BMS group

The BMS group the transformation group associated to the algebra of asymptotic Killing vectors. Writing Ω for the spherical part of the metric, x^A for the angular coordinate, and $u = t - r^*$ for the retarded time coordinate at $\mathscr{I}^+$, the BMS group is defined as follows.

Definitions 3.3. *The* ***BMS group*** *in any dimension consist of transformations*

$$u \to K(x^A)[u - T(x^A)], \tag{3.5a}$$

$$\Omega \to \Omega'(x^A), \tag{3.5b}$$

where $T(x^A)$ is an arbitrary function of the sphere, and $K(x^A)$ is a conformal scaling function, i.e., $(d\Omega')^2 = K^2 d\Omega^2$. A transformations for which $T \equiv 0$ is a ***superotation****. A transformations for which $\Omega' = \Omega$ is a* ***supertranslation****. A general transformation is called a* ***supertransformation****.*

The conformal transformations (3.5b) consist of rotations and boosts. The rotations have conformal factor 1. The boosts deform the sphere, whilst keeping the spacetime separation s constant. This is reason for the conformal factor in front of the u transformation in (3.5a).

Note that in the definition of the BMS group, the radial coordinate r is not taken into account. This is because the BMS group is fully determined by its action on $\mathscr{I}^+$, and the BMS group has the same action there for any metric. In a sense, the above action of the BMS group is a limit $r \to \infty$ of the action on the whole of spacetime. In the next chapter, we will derive the r-component of the vector fields inducing the transformation of the interior.

3.2.1 BMS in three dimensions

A rotation or boost of $\mathbb{S}^1$, (i.e., an automorphism $a \in \operatorname{Aut}\mathbb{S}^1$), is an arbitrary invertible function $\mathbb{S}^1 \to \mathbb{S}^1$ constrained by periodicity. Hence it is of the form

$$\theta \to \sigma\theta + a_0 + a_r\cos\theta + a_i\sin\theta,$$

for arbitrary scalars $a_0, a_r, a_i \in \mathbb{R}$, and where $\sigma = 1$ for an orientation-preserving transformation, and $\sigma = -1$ for an orientation-reversing transformation. We will restrict to the orientation-preserving case. Similarly, a general orientation-preserving superrotation of $\mathbb{S}^1$ (i.e., a diffeomorphism $\alpha \in \mathrm{Diff}^+ \mathbb{S}^1$), is an arbitrary (periodic) function of the form

$$\theta \to \theta + a_0 + a_{r,1} \cos\theta + a_{i,1} \sin\theta + ... + a_{r,n} \cos n\theta + a_{i,n} \sin n\theta + ...$$

The difference between the automorphism and diffeomorphism groups is discussed further in section 3.4. The variation of such a transformation is equivalent to the complex Fourier series

$$\delta\theta = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}, \tag{3.6}$$

for complex valued α_n with reality condition $\alpha_{-n} = \overline{\alpha}_n$, where $\overline{\cdot}$ denotes complex conjugation. The complex valued scalars $\{\alpha_n\}_{n\in\mathbb{Z}}$ are related to the real scalars $\{a_{r,n}, a_{i,n}\}_n \in \mathbb{N}$ via

$$\begin{aligned}\left(\alpha_n e^{in\theta} + \alpha_{-n} e^{-in\theta}\right) &= (\mathrm{re}\,\alpha_n + i\,\mathrm{im}\,\alpha_n)e^{in\theta} + (\mathrm{re}\,f_n - i\,\mathrm{im}\,f_n)e^{-in\theta} \\ &= 2\,\mathrm{re}\,\alpha_n \cos n\theta - 2\,\mathrm{im}\,\alpha_n \sin n\theta.\end{aligned}$$

So we have

$$\begin{aligned}a_{r,n} &= 2\,\mathrm{re}\,\alpha_n, & a_{i,n} &= -2\,\mathrm{im}\,\alpha_n, \\ \alpha_{\pm n} &= \frac{1}{2}(a_{r,n} \mp i a_{i,n}).\end{aligned}$$

The conformal factor $K(\theta)$ of a superrotation $\theta \to \theta' = \theta + a(\theta)$ is defined by $(d\theta')^2 = K^2(\theta)d\theta^2$, so

$$K(\theta) = \frac{d\theta'}{d\theta} = 1 + a'(\theta) = 1 + \sum_{n=-\infty}^{\infty} \alpha_n i n e^{in\theta}.$$

Similar to the superrotations, any supertranslation has expansion

$$T(\theta) = \sum_{k=-\infty}^{\infty} T_k e^{ik\theta},$$

for arbitrary $\{T_k\}_k$, and satisfying reality condition $T_{-k} = \overline{T}_k$. A rendition of an arbitrary supertranslation is given in figure 3.1.

3.2.2 The BMS algebra in three dimensions

Let $z := e^{i\theta}$. Then we have $\{l_n\}_n$, defined by

$$l_n := -z^{n+1} \frac{d}{dz} = i e^{in\theta} \frac{d}{d\theta},$$

serving as a basis for $\mathrm{Vect}\,\mathbb{S}^1$, the space of arbitrary smooth vector fields of the unit circle. The commutator bracket for these generators (see proof below) is

$$[l_m, l_n] = (m - n) l_{m+n}. \tag{3.7}$$

The algebra of these vector fields is called the **Witt algebra** $\mathfrak{witt} \equiv \mathfrak{vect}\, S^1$.

The BMS algebra consists of the vector fields tangent to the group action near the identity, (which is why we exclude the orientation-reversing action). Ignoring the reality condition, the (orientation-preserving) supertranslation group generators have tangent vector field

$$t_n := e^{in\theta} \partial_u = z^n \partial_u.$$

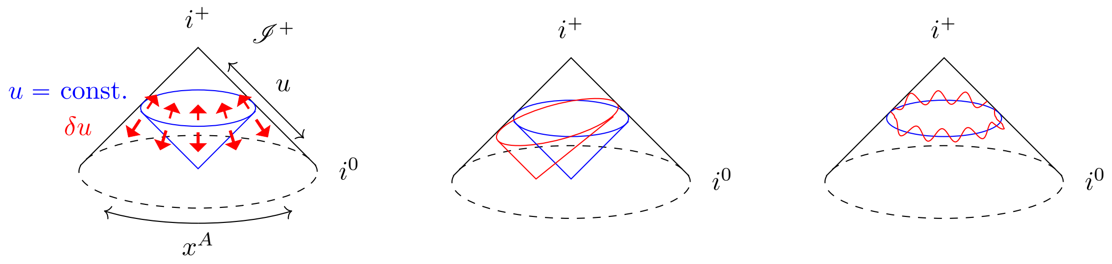


Figure 3.1: Rendition of the future null infinity cone $\mathscr{I}^+ \times \mathbb{S}^1$, null infinity times one of the angular coordinates. Fixing u defines a circle about the cone, which is the $r \to \infty$ limit of a future lightcone of some event. An arbitrary supertranslation transforms these circles (left). For example, the supertranslation $T(x^A) = \cos\theta$ (center) shifts the lightcone endpoints in a way that is equivalent to the translation of the the event from which the lightcone originates. For an arbitrary supertranslation (right) the interior is transformed in such a way that the resulting curve on $\mathscr{I}^+ \times \mathbb{S}^1$ is the limit of a future lightcone. Note that superrotations also transform the sphere itself, resulting in a deformed cone, i.e., stretched/compressed/skewed.

It is immediately clear that these commute. The algebra of supertranslations is denoted $\mathfrak{vect}_{ab}\,\mathbb{S}^1$, since they form an Abelian algebra which, as a vector space, is isomporphic to $\mathfrak{vect}\,\mathbb{S}^1$. Similarly, the (orientation-preserving) superrotation group generators have tangent vector field

$$r_n := e^{in\theta}(\partial_\theta + in\partial u) = iz^{n+1}\partial_z + nz^n\partial_u,$$

where ∂_μ is shorthand for d/dx^μ. Note that l_n and r_n are related by basis transformation $l_n = i(r_n - t_n)$.

Proposition 3.4. *The BMS algebra in three dimensions $\mathfrak{bms}_3$ is generated by $\{l_n\}_n, \{t_k\}_k$, with bracket*

$$\begin{aligned}[l_m, l_n] &= (m-n)l_{m+n},\\ [l_m, t_k] &= -kt_{m+k},\\ [t_j, t_k] &= 0.\end{aligned}$$

Proof *The commutator of the Witt algebra generators is*

$$\begin{aligned}[l_m, l_n] &= \big((z^{m+1}\partial_z + imz^m\partial_u)(-z^{n+1}\partial_z + inz^n\partial_u\big) - (m \leftrightarrow n)\\ &= \big(nz^{m+n+1}\partial_z + z^{m+n+2}\partial_z^2 - in^2z^{m+n}\partial_u - i(n+m)z^{n+m+1}\partial_z\partial_u - mnz^{m+n}\partial_u\big)\\ &\quad - (m \leftrightarrow n)\\ &= (n-m)z^{m+n+1}\partial_z - i(n^2 - m^2)z^{m+n}\partial_u\\ &= (m-n)(-z^{m+n+1}\partial_z + i(n+m)z^{m+n}\partial_u)\\ &= (m-n)l_{n+m}.\end{aligned}$$

The commutator of a superrotation and a supertranslation is

$$\begin{aligned}[l_m, t_k] &= (-z^{m+1}\partial_z + imz^m\partial_u)(z^k\partial_u) - (z^k\partial_u)(-z^{m+1}\partial_z + imz^m\partial_u)\\ &= (-kz^{m+n}\partial_u - z^{m+k+1}\partial_z\partial_u + imz^{m+n}\partial_u^2) - (-z^{k+m+1}\partial_u\partial_z + imz^{k+m}\partial_u^2)\\ &= -kz^{k+m}\partial_u\\ &= -kt_{m+k}.\end{aligned}$$

Note that the result of this bracket is again a supertranslation. Thus we can decompose the BMS algebra into a semisimple sum $\mathfrak{witt} \oplus \mathfrak{vect}_{ab}\,\mathbb{S}^1$, (see 3.6.2). Finally, since the supertranslations are independent of u, their commutator is trivial:

$$[t_j, t_k] = (z^j\partial_u z^k\partial_u) - (z^k\partial_u z^j\partial_u) = 0.$$

3.2.3 BMS in higher dimensions

The same construction can be used to find the higher dimensional BMS groups and algebras. The $\mathbb{S}^2$ analogue of the Fourier series uses an expansion in terms of spherical harmonics $Y_\ell^m(\theta, \phi)$, which is historically how $\mathfrak{bms}_4$ was expanded. However, we shall make use of a stereographic projection onto the complex plane (plus a point added at infinity), similar to the mapping $e^{in\theta} \to z$, which is much less cumbersome.

Rather than postulating the four dimensional BMS group, and deriving the algebra from there, we shall explicitly derive the BMS algebra as an asymptotic vector field algebra, in the next chapter. The advantage is that we can keep track of the r-component, i.e., the BMS-action on the interior of spacetime.

3.3 Spherical Metrics and Conformal Killing vectors

In this section we study conformal Killing vectors of the 2-sphere. They turn up in the 4 dimension BMS algebra, as hinted at in the previous section.

3.3.1 Riemann sphere

The metric of the sphere $\mathbb{S}^2$ in polar angles is

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

We make a change of coordinates, to the pair $(z, \overline{z})$ of complex coordinates, via the **stereographic projection**

$$z := e^{i\phi} \cot \tfrac{\theta}{2}. \tag{3.8}$$

The second coordinate $\overline{z}$ is its complex conjugate, (hence the notation). The following term will occur many times, so often that we give it a name:

$$P \equiv P(z, \overline{z}) := \tfrac{1}{2}(1 + z\overline{z})$$

In terms of θ and ϕ, it has expression

$$\frac{1}{2P} = \frac{1}{1 + z\overline{z}} = \frac{1}{1 + \cot^2\frac{\theta}{2}} = \frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}} = \sin^2\tfrac{\theta}{2}. \tag{3.9}$$

Proposition 3.5. *The unit sphere is isomorphic to the complex plane, plus a point at infinity, $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. This space is known as the* ***Riemann sphere****. It has metric*

$$d\Omega^2 = P^{-2} dz d\overline{z}. \tag{3.10}$$

Proof *The isomorphism is the stereographic projection* (3.8). *The differential dz has expression*

$$dz = d\Big(e^{i\phi} \cot \tfrac{\theta}{2}\Big) = e^{i\phi}\big(i \cot \tfrac{\theta}{2} d\phi - \tfrac{1}{2} \csc^2 \tfrac{\theta}{2} d\theta\big), \tag{3.11}$$

and similarly for $d\bar{z}$. Then, using (3.9), we find

$$\begin{aligned} dzd\bar{z} &= \left(i\cot\frac{\theta}{2}d\phi - \frac{1}{2\sin^2\frac{\theta}{2}}d\theta\right)\left(-i\cot\frac{\theta}{2}d\phi - \frac{1}{2\sin^2\frac{\theta}{2}}d\theta\right) \\ &= \frac{1}{4\sin^4\frac{\theta}{2}}d\theta^2 + \cot^2\frac{\theta}{2}d\phi^2 \\ &= \left(\frac{1}{4\sin^4\frac{\theta}{2}}\right)\left(d\theta^2 + 4\cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2}d\phi^2\right) \\ &= \left(\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}\right)^2\left(d\theta^2+\sin^2\theta d\phi^2\right) \\ &= \frac{(\cot^2\frac{\theta}{2}+1)^2}{4}\left(d\theta^2+\sin^2\theta d\phi^2\right) \\ &= \frac{(z\bar{z}+1)^2}{4}\left(d\theta^2+\sin^2\theta d\phi^2\right). \end{aligned}$$ □

Coordinate transformations For future reference, we provide here the coordinate transformation functions back the usual spherical coordinates:

$$\theta = 2\,\mathrm{arccot}\sqrt{z\bar{z}}, \qquad \phi = \arccos\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\left(=\frac{i}{2}\ln\frac{\bar{z}}{z}\right) \tag{3.12}$$

Using the chain rule, we have

$$\frac{d}{dz} = \frac{d\theta}{dz}\frac{d}{d\theta} + \frac{d\phi}{dz}\frac{d}{d\phi} = -\frac{\sqrt{\bar{z}/z}}{1+z\bar{z}}\frac{d}{d\theta} - \frac{i}{2z}\frac{d}{d\phi} = -e^{-i\phi}\left(\sin^2\tfrac{\theta}{2}\frac{d}{d\theta} + \frac{i}{2}\tan\tfrac{\theta}{2}\frac{d}{d\phi}\right), \tag{3.13}$$

and similar for $\bar{z}$.

3.3.2 Conformal transformations

Later on, we will be interested in transformations that leave the metric $g_{\mu\nu}$ invariant up to a scaling function $\tilde{\lambda} \equiv \tilde{\lambda}(x^\mu)$. The resulting metric $g'_{\mu\nu} = \tilde{\lambda} g_{\mu\nu}$ is equal to $g_{\mu\nu}$ at each point, up to a scaling factor. As a result, angles are preserved by this class of transformations, (which motivates their name: 'conformal').

Definition 3.6. *Vector fields Y satisfying*

$$\mathcal{L}_Y g_{\mu\nu} = \tilde{\lambda} g_{\mu\nu}$$

*are called **conformal Killing vectors**.*

The trace of this equation reveals a necessary relation between $\tilde{\lambda}$ and Y, (called the trace condition),

$$2\nabla_\mu Y^\mu = g^{\mu\nu}2\nabla_{(\mu}Y_{\nu)} = g^{\mu\nu}\mathcal{L}_Y g_{\mu\nu} = g^{\mu\nu}\omega g_{\mu\nu} = n\tilde{\lambda},$$

where n is the dimension of the manifold $(\dim\mathcal{M}, g)$. Hence

$$\tilde{\lambda} = \frac{2}{n}\nabla_\mu Y^\mu. \tag{3.14}$$

Conformal transformations of the Riemann sphere

Arbitrary angular coordinates are denoted x^A, e.g., the Riemann sphere has coordinates $x^A = (z,\bar{z})$ and metric

$$\gamma_{AB} = \tfrac{1}{2}P^{-2}(1-\delta_{AB}) \qquad ds^2 = \frac{4dzd\bar{z}}{(1+z\bar{z})^2}.$$

We are interested in metrics that are conformal to the spherical metric. Let $\tilde{\varphi} = \varphi - \ln P$, for some function $\varphi(x^A)$. So we define the metric of the conformally rescaled Riemann sphere

$$\overline{\gamma}_{AB} := \tfrac{1}{2}e^{2\tilde{\varphi}}(1-\delta_{AB}) \qquad ds^2 = \frac{4e^{2\phi}dzd\overline{z}}{(1+z\overline{z})^2}. \tag{3.15}$$

Note that it coincides with γ_{AB} for $\varphi = 0$. The inverse metric is $\overline{\gamma}^{AB} = 2e^{-2\tilde{\varphi}}(1-\delta_{AB})$. We shall denote its associated Christoffel symbols $\overline{\Gamma}^A_{BC}$, and its covariant derivative $\overline{D}_A$. The non-vanishing Christoffel symbols are

$$\overline{\Gamma}^z_{zz} = 2\partial_z\tilde{\varphi}, \qquad \overline{\Gamma}^{\overline{z}}_{\overline{z}\overline{z}} = 2\partial_{\overline{z}}\tilde{\varphi}. \tag{3.16}$$

Hence, the Christoffel symbol is conviniently written

$$\overline{\Gamma}^A_{BC} = 2\delta^A_B\delta^C_A\partial_C\tilde{\varphi}.$$

As a result, the ofter occuring contraction with a (1,0) tensor, X^A, is

$$\overline{D}_AX^A = \partial_AX^A + \overline{\Gamma}^A_{AB}X^B = \partial_AX^A + 2X^A\partial_A\tilde{\varphi} \tag{3.17}$$

We solve the conformal Killing equation below.

Calculation of conformal Killing vectors *From the trace condition* (3.14)*, we have* $\tilde{\lambda} = \overline{D}_AY^A$. *Then the conformal Killing equation is*

$$\overline{\gamma}_{CB}\overline{D}_AY^C + \overline{\gamma}_{AC}\overline{D}_BY^C = \overline{D}_CY^C\overline{\gamma}_{AB}. \tag{3.18}$$

suppose that $A \neq B$. *Then* (3.18) *becomes*

$$\begin{aligned} 0 &= \overline{\gamma}_{C\overline{z}}\overline{D}_zY^C + \overline{\gamma}_{zC}\overline{D}_{\overline{z}}Y^C - \overline{D}_CY^C\overline{\gamma}_{z\overline{z}} \\ &= \overline{\gamma}_{z\overline{z}}\overline{D}_zY^z + \overline{\gamma}_{z\overline{z}}\overline{D}_{\overline{z}}Y^{\overline{z}} - (\overline{D}_zY^z + \overline{D}_{\overline{z}}Y^{\overline{z}})\overline{\gamma}_{z\overline{z}}, \end{aligned}$$

which is vacuously true. Next, suppose that $A = B = z$. *Then the right hand side vanishes, and* (3.18) *becomes*

$$\begin{aligned} 0 &= \overline{\gamma}_{Cz}\overline{D}_zY^C + \overline{\gamma}_{zC}\overline{D}_zY^C \\ &= \overline{D}_zY_z + \overline{D}_zY_z \\ &= 2\partial_zY_z - 2\overline{\Gamma}^\sigma_{zz}Y_\sigma \\ &= 2\partial_zY_z - 4\partial_z\tilde{\varphi}, \end{aligned}$$

so $Y_z = \frac{1}{2}e^{2\tilde{\varphi}}\overline{f}(\overline{z})$, *for some function* $\overline{f}(\overline{z})$. *As a result* $Y^{\overline{z}} = \overline{\gamma}^{\overline{z}z}Y_z = f(z)$. *Analogously we find* $Y^{\overline{z}}$.

So the conformal Killing vectors are arbitrary functions

$$Y^z \equiv Y^z(z), \qquad Y^{\overline{z}} \equiv Y^{\overline{z}}(\overline{z}). \tag{3.19}$$

They have conformal factor

$$\tilde{\lambda} = \overline{D}_AY^A = \partial_AY^A + 2Y^A\partial_A\tilde{\varphi}$$

3.4 Local vs Global transformations

At this point, it is relevant to make the distinction between the local (infinitesimal) and the global version of the conformal group. The global version consist of globally well-defined transformations. In the local version, we allow for any holomorphic (not necessarily invertible) function of the Riemann sphere. We discuss the two classes below. In general, the local symmetry group of a space corresponds to its diffeomorphism group, and the global symmetry group of a space corresponds to its automorphism group.

As advocated by Barnich and Troessaert in [7], choosing to work with the global transformations, results in a much more flexible algebra, and is desirable from a mathematical point of view. We discuss both cases below.

3.4.1 Global transformations: Lorentz group

The global transformations are those that are automorphisms of the Riemann-sphere $\mathrm{Aut}(\hat{\mathbb{C}})$. Recalling that the coordinates of the Riemann sphere are established by stereographically projecting a unit sphere. Geometrically, the image $Y(\mathbb{C})$ of a global transformation $Y \in \mathrm{Aut}(\hat{\mathbb{C}})$, is again a stereographic projection of a unit sphere, as though the original sphere has been moved and rotated.

Since the transformation has to be an automorphism of the Riemann sphere, the image must contain $\{0\}$ and $\{\infty\}$. Thus the holomorphic function should have a simple zero and a simple pole. Thus, it has to be a rational function of the form

$$f(z) = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0, \tag{3.20}$$

for some $a, b, c, d \in \mathbb{C}$, and with the zero at $-b/a$ and the pole at $-d/c$. These transformations are known as the **Möbius transformations**. If $ad = bc$ the above function is constant, so this case has to be disregarded. In addition, note that the functions generated this way, provide a double cover of the transformations of the Riemann-sphere, since the elements generated by a, b, c, d and $-a, -b, -c, -d$ give rise to the same transformation. We will utilize this fact in the following propositions.
First we calculate the composition of two such transformations:

$$f' \circ f(z) = \frac{(aa' + b'c)z + a'b + b'd}{(ac' + cd')z + bc' + dd'}, \tag{3.21}$$

which is again a Möbius transformation. The group of these transformations is also referred to as the Möbius group.

Using the metric (3.15) and adopting the notation from section 3.2, the sphere Ω is transformed into

$$\begin{aligned} d\Omega'^2 &= \frac{4dz'd\bar{z}'}{(1+z'\bar{z}')^2} \\ &= 4\left(\frac{|cz+d|^2}{|cz+d|^2+|az+b|^2}\right)^2 \left|\frac{-ad+bc}{(cz-d)^2}\right|^2 dzd\bar{z} \\ &= \left(\frac{1+z\bar{z}}{|cz+d|^2+|az+b|^2}\right)^2 \frac{4dzd\bar{z}}{(1+z\bar{z})^2} \\ &= K^2(z,\bar{z})d\Omega^2, \end{aligned}$$

so that the conformal factor of a Möbius transformation is given by

$$K(z,\bar{z}) = \frac{(1+z\bar{z})}{|az+b|^2+|cz+d|^2}. \tag{3.22}$$

Proposition 3.7. *The global transformation group* $\mathrm{Aut}(\hat{\mathbb{C}})$ *is isomorphic to the projective linear group* $\mathrm{PGL}(2,\mathbb{C})$.

Proof *From* (3.21) *note that*

$$\phi : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{Aut}(\hat{\mathbb{C}}); \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f$$

is a group homomorphism. Any two matrices which differ by a global factor λ *have the same image, so* $\mathrm{Ker}\,\phi = \mathbb{C}^\times I$. *By the first isomorphism theorem;* $\mathrm{Aut}(\hat{\mathbb{C}}) \cong \mathrm{GL}(2,\mathbb{C})/(\mathbb{C}^\times I) = \mathrm{PGL}(2,\mathbb{C})$.

Proposition 3.8. *The global tranformation group* $\mathrm{Aut}(\hat{\mathbb{C}})$ *is isomorphic to the projective special linear group* $\mathrm{PSL}(2,\mathbb{C})$.

> ***Proof*** *The group homomorphism* ϕ*, when restricted to matrices with unit determinant 1, is still surjective onto* $\mathrm{Aut}\,\hat{\mathbb{C}}$*. The kernel is* $\pm I$*, and so* $\mathrm{Aut}(\hat{\mathbb{C}}) \cong \mathrm{SL}(2,\mathbb{C})/(\pm I) = \mathrm{PSL}(2,\mathbb{C})$.

Corollary 3.8.A *The group of global transformations of the Riemann sphere is isormophic to the proper, orthochronous Lorentz group* $\mathrm{SL}^{\uparrow}(1,3)$.

3.4.2 Infinitesimal Möbius transformations and the Lorentz algebra

To obtain the algebra associated to the Möbius group, we consider the infinitesimal group action. An arbitrary infinitesimal transformation of z is of the form

$$z + \varepsilon_0 + \varepsilon_1 z + \varepsilon_2 z^2 + ...$$

Here $\varepsilon_i \in {}^*\mathbb{C}$ are complex infinitesimals, i.e., $\forall_{i,j} : \varepsilon_i\varepsilon_j = 0$. It is clear that $b \sim \varepsilon_0$, and that a must be of the form $a = 1 + \varepsilon$ for some infinititesimal ε. It turns out that a, b, c, and d are related to the infinitesimals via

$$a = 1 + \frac{\varepsilon_1}{2}, \qquad b = \varepsilon_0, \qquad c = -\varepsilon_2, \qquad d = 1 - \frac{\varepsilon_1}{2}.$$

Substituting these solutions into an arbitrary Möbius transformation then gives:

$$z \to \frac{(1+\frac{\varepsilon_1}{2})z + \varepsilon_0}{1 - \frac{\varepsilon_1}{2} - \varepsilon_2 z} = \big((1+\tfrac{\varepsilon_1}{2})z + \varepsilon_0\big)\big(1 + \tfrac{\varepsilon_1}{2} + \varepsilon_2 z\big) = z + \varepsilon_0 + \varepsilon_1 z + \varepsilon_2 z^2, \tag{3.23}$$

and similar for $\overline{z}$. Note that a general Möbius transformation has three complex degrees of freedom, from which it should be clear that (3.23) indeed gives the most general infinitesimal Möbius transformation.

The associated conformal factor is

$$K(z,\overline{z}) = 1 + \frac{1 - z\overline{z}}{1 + z\overline{z}}\,\mathfrak{re}\,\varepsilon_1 - \frac{2}{1 + z\overline{z}}\,\mathfrak{re}((\overline{\varepsilon}_0 - \varepsilon_2)z), \tag{3.24}$$

or, in terms of θ, ϕ,

$$K(\theta,\phi) = 1 + \cos\theta\,\mathfrak{re}\,\varepsilon_1 - 2\sin^2\tfrac{\theta}{2}\,\mathfrak{re}((\overline{\varepsilon}_0 - \varepsilon_2)z) \tag{3.25}$$

In particular, the conformal factor is 1 if $\mathrm{re}\,\varepsilon_1 = 0$ and $\varepsilon_0 = \overline{\varepsilon}_2$. It follows from (3.23) that the complex Lorentz algebra is (isomorphic to) the algebra generated by

$$z^0\partial_z, \qquad z^1\partial_z, \qquad z^2\partial_z, \qquad \overline{z}^0\partial_{\overline{z}}, \qquad \overline{z}^1\partial_{\overline{z}}, \qquad \overline{z}^2\partial_{\overline{z}}.$$

Writing $l_m = z^{m+1}\partial_z$, and $\bar{l}_m = \overline{z}^{m+1}\partial_{\overline{z}}$ the commutators are

$$[l_m, l_n] = (m-n)l_{m+n}, \qquad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \qquad [l_m, \bar{l}_n] = 0.$$

For a proof, see the proof of Proposition 3.4. Below, we give an example of an infinitesimal Möbius transformation in terms of the more familiar θ, ϕ.

> ***Example*** *Consider the transformation* $z \to z' = z + \eta z, \overline{z} \to \overline{z} + \overline{\eta z}$*, for a complex infinitesimal* η*. Note that for infinitesimals* $e^{i\,\mathrm{im}\,\eta} = 1 + i\,\mathrm{im}\,\eta$*. As a result, we have*

$$\begin{aligned} z \to z' &= z(1+\eta) \\ &= \cot\frac{\theta}{2}e^{i\phi}(1 + \mathrm{re}\,\eta + i\,\mathrm{im}\,\eta) \\ &= (1 + \mathrm{re}\,\eta)\cot\frac{\theta}{2}e^{i(\phi + \mathrm{im}\,\eta)} \\ &= \cot\mathrm{arccot}\left((1+\mathrm{re}\,\eta)\cot\frac{\theta}{2}\right)e^{i(\phi+\mathrm{im}\,\eta)} \\ &= \cot\left(\mathrm{arccot}\left(\cot\frac{\theta}{2}\right) - \frac{\cot\frac{\theta}{2}\,\mathrm{re}\,\eta}{1+\cot^2\frac{\theta}{2}}\right)e^{i(\phi+\mathrm{im}\,\eta)} \\ &= \cot\left(\frac{\theta - \sin\theta\,\mathrm{re}\,\eta}{2}\right)e^{i(\phi+\mathrm{im}\,\eta)}, \end{aligned}$$

where we have used the series expansion $\operatorname{arccot} x + a = \operatorname{arccot} x - a/(1+x^2) + ...$ *So we have correspondence*

$$z \to z(1+\eta), \qquad \overline{z} \to \overline{z}(1+\overline{\eta}); \qquad \Leftrightarrow \qquad \phi \to \phi + \operatorname{im}\eta, \qquad \theta \to \theta - \sin\theta \operatorname{re}\eta.$$

3.4.3 Local transformations: Witt algebra

If we choose our conformal Killing vectors to only be locally well-defined, the resulting functions are arbitrary (holomorphic) functions, $Y^z \equiv Y^z(z), Y^{\overline{z}} \equiv Y^{\overline{z}}(\overline{z})$. Hence the conformal Killing vectors admit Laurent expansion

$$Y = \sum_{n=-\infty}^{\infty} \left(\alpha_n z^{n+1}\partial_z,\ \overline{\alpha}_n \overline{z}^{n+1}\partial_{\overline{z}}\right)$$

with arbitrary parameters α_n, $\overline{\alpha}_n$. Define $l_n := -z^{n+1}\partial_z$, and $\bar{l}_n := -\overline{z}^{n+1}\partial_\zeta$. The Lie algebra of Killing vectors is generated by the basis $(l_n)_n, (\bar{l}_n)_n$. The elements have commutation relations

$$[l_m, l_n] = (m-n)l_{m+n}, \qquad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \qquad [l_m, \bar{l}_n] = 0.$$

The conformal Killing algebra is isomorphic to two (independent) copies of the Witt algebra, (see (3.7)). It was seen in section 3.2.2 that the local conformal transformations of the circle give rise to a single copy the Witt algebra.

The restriction $(l_m, \bar{l}_n)|_{n,m=-1,0,1}$, establishes the Lorentz algebra as a subalgebra, which is immediately clear upon comparing the generators.

3.5 Vector Field preliminaries

In this section we study some relevant properties of vector fields. The definitions throughout this section are taken from [15].

3.5.1 Vector fields as operators

An **integral curve** of a vector field X is a differentiable map $\gamma : I \to M$ with the property that our vector field X is its derivative at all points in some open U. This translates to the condition

$$\gamma : I \to M, \qquad \partial_\tau \gamma(\tau) \equiv \dot{\gamma}(\tau) = X_{\gamma(\tau)}.$$

In covariant notation, this just means that we have the differential equations:

$$\partial_\tau \gamma^\mu(\tau) = X^\mu(\gamma^\sigma(\tau))$$

Its value in zero (if it is in the domain), $\gamma(0) \in M$, is called it **starting point**. Suppose that for each point $p \in \mathcal{M}$ the vector field $X \in \mathfrak{X}(\mathcal{M})$ has a unique integral curve starting at p, and defined for all $\tau \in \mathbb{R}$. Denote the collection of these ingegral curves $\varphi^{(p)} : \mathbb{R} \to M$. Then we define the **flow** of the vector field X as

$$\varphi_\tau : M \to M; \qquad \varphi_\tau(p) := \varphi^{(p)}(\tau).$$

A particularly important example is the **exponential map**:

$$\exp : T_pM \to M; \qquad \exp(X) = \varphi_1(p)$$

Example *Consider the vector field* $X = -x\partial_y + y\partial_x$ *on the manifold* $\mathbb{R}^2$*. Then its integral curve is the solution of the equations*

$$\partial_\tau \gamma^x(\tau) = -\gamma^y(\tau), \qquad \partial_\tau \gamma^y(\tau) = \gamma^x(\tau),$$

solved by

$$\gamma(\tau) = (a\cos\tau - b\sin\tau, a\sin\tau + b\cos\tau).$$

Note that $\gamma(0) = (a, b)$*, so that the flow of* X *is given by*

$$\varphi_\tau(x, y) = (x\cos\tau - y\sin\tau, x\sin\tau + y\cos\tau).$$

Finally, the exponent of X *at* $p = (x, y)$ *is*

$$\exp(X) = (x\cos 1 - y\sin 1, x\sin 1 + y\cos 1).$$

Action of a vector field on a tensor We define the action of a vector field on a covariant tensor:

$$X.T|_p = \varphi_\tau^*(T(p)) = (T_{\mu_1\ldots\mu_k} \circ \varphi_\tau)(p) d(x^{\mu_1} \circ \varphi_\tau) \otimes \ldots \otimes d(x^{\mu_k} \circ \varphi_\tau)$$

For instance, the action on the metric $g_{\mu\nu}dx^\mu dx^\nu$ is

$$X.(g_{\mu\nu}|_p dx^\mu dx^\nu) = g_{\mu\nu}|_{\varphi_\tau(p)} d(\varphi_\tau^\mu) d(\varphi_\tau^\nu) = \left(g_{\tilde\mu\tilde\nu}|_{\varphi_\tau(p)} \frac{\partial\varphi_\tau^{\tilde\mu}}{\partial x^\mu} \frac{\partial\varphi_\tau^{\tilde\nu}}{\partial x^\nu}\right) dx^\mu dx^\nu.$$

Note that this coincides with the familiar transformation law for metrics.

3.5.2 Vector field algebra

X(f) to be the element of C∞(M) whose value at a point p is the directional derivative of f at p in the direction X(p)

3.6 Algebra preliminaries

3.6.1 Lie group - Lie algebra correspondence

In this subsection we recall some important features of Lie groups and their corresponding algebras. Lie groups are denoted by capital letters (e.g., G, H, ...) and Lie algebras by their lower case Fraktur counterpart (e.g., $\mathfrak{g}, \mathfrak{h}, \ldots$). The Lie algebra is the linearization of the Lie group, in the tangent space at the identity. That is, elements $X \in \mathfrak{g}$ are identified to the derivative of the exponential

$$\mathfrak{g} \ni X \equiv \frac{d}{dt} e^{tX}\bigg|_{t=0} \in T_e G$$

and given a t these correspond to some element in the group G:

$$e^{tX} \sim g \in G$$

Proposition 3.9. *If* $\mathfrak{g}$ *is the Lie algebra of a Lie group* G*, then* $\mathrm{ad}_X(Y) = [X, Y]$*.*

Proof *The (left) adjoint action is defined as the conjugation:*

$$\mathrm{Ad} : G \times G \to G; \qquad (a, g) \mapsto a \cdot g \cdot a^{-1}$$
$$\mathrm{Ad}_a \equiv L_a R_{a^{-1}} : G \to G; \qquad g \mapsto a \cdot g \cdot a^{-1}$$

Here L_g, R_g *denote the left and right action of the Lie group element* g*, respectively. Since* $\mathrm{Ad}_a e = e$*, its differential at the unit is an action on the Lie algebra,*

$$\dot{\mathrm{Ad}}_a := (d\,\mathrm{Ad}_a)_e : \mathfrak{g} \to \mathfrak{g};$$

We identify $X \in \mathfrak{g}$ *with the speed of its associated flow (writing* $\exp(tX) := \phi^t_{X^L}(e)$*):*

$$\mathfrak{g} \ni X \equiv \frac{d}{dt} \exp(tX)\bigg|_{t=0} \in T_e G.$$

And thus, by the (differential of the) adjoint action, it is sent to

$$\begin{aligned} \dot{\mathrm{Ad}}_g(X) &= \frac{d}{dt}\big(g\cdot\exp(tX)\cdot g^{-1}\big)\Big|_{t=0} && \in T_eG,\\ &= \frac{d}{dt}\big(\exp(tgXg^{-1})\big)\Big|_{t=0} && \in T_eG,\\ &\equiv gXg^{-1} && \in \mathfrak{g}, \end{aligned}$$

where we used $\exp(taX) = a\exp(tX)$ *and* $\exp(tXb) = \exp(tXb)$*. Now, we introduce the adjoint representation of the Lie group* G*, along with its associated representation of the Lie algebra* $\mathfrak{g}$*:*

$$\begin{aligned} \mathrm{Ad} :&G \to GL(\mathfrak{g}); && g \mapsto \dot{\mathrm{Ad}}_g\\ \mathrm{ad} :&\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}); && X \mapsto \mathrm{ad}_X \end{aligned}$$

The adjoint action then acts on an arbitrary element $Y \in \mathfrak{g}$ *via:*

$$\begin{aligned} \mathrm{ad}(X)(Y) &\equiv \mathrm{ad}_X(Y) && \in \mathfrak{g}\\ &\equiv (d\dot{A}d_{\frac{d}{dt}\exp(tX)|_{t=0}}Y)_e && \in T_eG\\ &= (dR_{\exp(-tX)})(dL_{\exp(tX)})(Y_e)\\ &= (dR_{\exp(-tX)})(Y_{\exp(tX)})\\ &= \frac{d}{dt}\Big(d\phi^{-t}_{X^L}(Y_{\phi^t_{X^L}(e)})\Big)\Big|_{t=0}\\ &= (\mathcal{L}_XY)_e\\ &= [X,Y]_e\\ &\equiv [X,Y] && \in \mathfrak{g}\,. \end{aligned}$$

Here we used that $\exp(tX)^{-1} = \exp(-tX)$*, and the correspondence between the flow and the left and right action. We see that indeed* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is a representation of the Lie algebra* $\mathfrak{g}$ *on itself, and that* $\mathrm{ad}_X Y = [X,Y]$*.* □

From group to algebra

Let V be a vector space of dimension n over a field k. The general linear group $\mathrm{GL}(V)$ is the group of bijective linear transformations of the vector space V. The group operation is composition. Given a basis $(e_j)_{1\leqslant j\leqslant n}$ for V a transformation T is the map

$$Te_k = \sum_{j=1}^{n} a_k{}^j e_j.$$

This shows that $\mathrm{GL}(V)$ is isomorphic to the space of $(n\times n)$-matrices excluding non-invertibles,

$$\mathrm{GL}_n \cong \mathcal{M}^{\times}_{n\times n} = \{A \in \mathcal{M}_{n\times n} |\det A \neq 0\}.$$

In this form, the group operation is just matrix multiplication. Note that it is isomorphic (the isomorphism being picking a basis) to the group of n dimensional GL_n. Subgroups of GL_n are called **linear groups**.

Now we move from the group to the algebra, where we follow [16, § 2.14]. The tangent space of GL_n is canonically denoted in lower case Fraktur, i.e., $T_I\,\mathrm{GL}_n =: \mathfrak{gl}_n$. Any element $X \in \mathfrak{gl}_n$ is identified with the speed at $t = 0$ of $t \mapsto I + tX$. Note that for small enough t, the determinant is nonzero, so that $\mathrm{gl}_n \cong \mathcal{M}_{n\times n}$. Now, X gives rise to left-invariant vector field

$$X^L(A) = \frac{d}{dt}A(I+tX)\Big|_{t=0} \in T_A\,\mathrm{GL}_n$$

for all $A \in \mathrm{GL}_n$. Let $X, Y \in \mathrm{gl}_n$, and $[X,Y]_{\mathfrak{g}}$ the bracket in $\mathfrak{g}$ defined via $[X,Y]^L_{\mathfrak{g}} = [X^L, Y^L]_{\mathfrak{X}(\mathrm{GL_n})}$, where the last bracket is the bracket of vector fields on GL_n. We compute

$$\mathcal{L}_{[X^L,Y^L]}(f) = \mathcal{L}_{X^L}\mathcal{L}_{Y^L}(f) - \mathcal{L}_{Y^L}\mathcal{L}_{X^L}(f), \qquad \forall f : \mathrm{GL}_n \to k.$$

Write ${u_j}^i : \mathfrak{gl}_n \to k$ for the coordinate functions on $\mathfrak{gl}_n$. Then we have

$$\mathcal{L}_{X^L}(f)(A) = \frac{d}{dt} f(A(I + tX)) = \sum_{i,j,k} \frac{df(A)}{d{u_j}^i} {A_k}^i {X_j}^k.$$

In particular

$$\mathcal{L}_{X^L}({u_j}^i)(A) = \sum_k {A_k}^i {X_j}^k.$$

So we have

$$\mathcal{L}_{Y^L}\mathcal{L}_{X^L}({u_j}^i) = \sum_k \mathcal{L}_{Y^L}({u_k}^i){X_j}^k = \sum_{k,l} {u_l}^u {Y_k}^l {X_j}^k = \mathcal{L}_{(YX)^L}({u_j}^i).$$

It follows that $[X, Y]_{\mathfrak{g}} = XY - YX$, i.e., the Lie-bracket for linear algebras is the commutator.

Other linear groups

For other linear groups, we move to the algebra by 'taking the derivative' of the extra condition, i.e.,

$$\mathrm{O}_n = \{A \in \mathrm{GL}_n \,|AA^T = I\}$$

Consider the map

$$f : \mathrm{GL}_n \to \mathcal{M}_{n\times n}; \qquad f(A) = A \cdot A^T$$

Then $\mathrm{O}_n = f^{-1}(\{I\}) \subset \mathrm{GL}_n$ can be seen as the preimage of the identity of f. It has the tangent map

$$(df)_A : \mathfrak{gl}_n \to \mathfrak{gl}_n; \qquad X \mapsto \frac{d}{dt} f(A + tX)\Big|_{t=0} = AX^T + A^T X.$$

As a Lie algebra, $\mathfrak{o}_n = T_I\,\mathrm{O}_n$, (so $A = I$) is just the restriction

$$\{X \in \mathfrak{gl}_n \,|X^T + X = 0\} \cong (df)^{-1}(0) \subset \mathfrak{gl}_n\,.$$

Example: The Lorentz group *The Lorentz group can be defined as follows: We define* $\eta = \mathrm{diag}(-1, 1, 1, ..., 1) \in \mathcal{M}_{n\times n}$ *to be the metric matrix (of Minkowski space-time). Then the (general) Lorentz group consists of transformations satisfying*

$$\mathrm{O}(1, n-1) := \{\Lambda \in \mathrm{GL}_n \,|\Lambda^t \eta \Lambda = \eta\},$$

i.e., coordinate transformations that leave the metric invariant. (Physically this condition means that 'the laws of physics' should look the same in any frame of reference). Let us split time and space components of x^μ. *Then we write*

$$\Lambda = \begin{pmatrix} a & \vec{v}_1^t \\ \vec{v}_2 & S \end{pmatrix}, \qquad x = \begin{pmatrix} x^0 \\ \vec{x} \end{pmatrix}$$

where a *is a scalar,* $\vec{v}_i$ *are* $n-1$*-vectors, and* S *is an* $(n \times n)$*-matrix. A Lorentz transformation* $x \to \Lambda x$ *gives*

$$x^0 \to ax^0 + \vec{v}_1 \cdot \vec{x}, \qquad \vec{x} \to \vec{v}_2 x^0 + S^t \vec{x}.$$

In this form, the condition $\Lambda^t\eta\Lambda = \eta$ *(and as a consequence* $\Lambda\eta\Lambda^t = \eta$*), is just*

$$\begin{pmatrix} a & \vec{v}_2^t \\ \vec{v}_1 & S^t \end{pmatrix}\begin{pmatrix} -1 & \vec{0}^t \\ \vec{0} & I \end{pmatrix}\begin{pmatrix} a & \vec{v}_1^t \\ \vec{v}_2 & S \end{pmatrix} = \begin{pmatrix} -a^2 + |\vec{v}_2|^2 & -a\vec{v}_1^t + \vec{v}_2^t S \\ -a\vec{v}_1 + S^t\vec{v}_2 & -\vec{v}_1 \otimes \vec{v}_1^t + S^t S \end{pmatrix} \overset{!}{=} \begin{pmatrix} -1 & \vec{0}^t \\ \vec{0} & I \end{pmatrix}.$$

So we have conditions

$$\begin{aligned} |\vec{v}_2|^2 - a^2 &= -1, & -a\vec{v}_1 + S^t\vec{v}_2 &= \vec{0}, & -\vec{v}_1 \otimes \vec{v}_1^t + S^t S &= I, \\ |\vec{v}_1|^2 - a^2 &= -1, & -a\vec{v}_2 + S^t\vec{v}_1 &= \vec{0}, & -\vec{v}_2 \otimes \vec{v}_2^t + S^t S &= I. \end{aligned}$$

Note that matrices of the form

$$\{\Lambda \in \mathrm{O}(1, n-1) | a = 1, \vec{v}_i = \vec{0}\} \cong O(n-1),$$

form a subgroup. ***Algebra*** *Passing on to the algebra we get the 'derivative' of the conditions*

$$\mathfrak{o}(1, n-1) := \{\Lambda \in \mathfrak{gl}_n | \Lambda^t \eta + \eta \Lambda = 0\}.$$

In the above form, this is just

$$a = 0, \qquad \vec{v}_1 = \vec{v}_2, \qquad S = -S^t$$

Again, we have the subalgebra

$$\{\Lambda \in \mathfrak{o}(1, n-1) | \vec{v}_i = \vec{0}\} \cong \mathfrak{o}(n-1).$$

The Lorentz algebra acts on the space $\mathbb{R}^n$ *via*

$$x \to \Lambda x.$$

Note that since the general Lorentz group consists of 4 disconnected components, the algebras of the general/ proper/orthochronous/proper orthochronous Lorentz groups coincide.

Unitarity

Let $\mathfrak{g}$ be a real, (or complex), Lie algebra. An operation $\omega : \mathfrak{g} \to \mathfrak{g}$ is called an **anti-linear anti-involution** on $\mathfrak{g}$ if for all $X, Y \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$, (or $\mathbb{C}$), it satisfies

$$\omega(\lambda X) = \overline{\lambda}\omega(X), \qquad \omega([X, Y]) = [\omega(X), \omega(Y)].$$

Let $(V, \langle\cdot|\cdot\rangle)$ be a representation of $\mathfrak{g}$, equipped with a positive-definite Hermitian form $\langle\cdot|\cdot\rangle$, compatible [is dit altijd waar?] with the algebra in the sense

$$\langle Xu|v\rangle = -\langle u|Xv\rangle.$$

The Hermitian form is called **contravariant** if

$$\langle Xu|v\rangle = \langle u|\omega(X)v\rangle.$$

If it is non-degenerate this means

$$X^\dagger = \omega(X), \qquad \text{for all } X \in \mathfrak{g}.$$

Here $X^\dagger$ denotes the Hermitian conjugate of X.

A representation is called **unitary** if in addition

$$\langle v|v\rangle > 0, \qquad \text{for all} v \in V, v \neq 0.$$

3.6.2 Semi-direct product of groups

Construction

Let G be a group, with subgroups $H, N < G$. We denote the identity element $\{e\}$. Then G is a **semi-direct product** of H **acting on** N, denoted

$$G = H \ltimes N$$

if the following conditions are met:
(s1) N is a normal subgroup[1] $N \lhd G$
(s2) $G = NH$, (i.e., $\forall_{g \in G} \exists_{n \in N, h \in h} g = nh$)
(s3) $N \cap H = \{e\}$.

[1] $N \lhd G$ is a normal subgroup precisely if $\forall_{n \in N} \forall_{g \in G} gng^{-1} \in N$

Proposition 3.10. *Let $G = H \ltimes N$ be a semidirect product. Any $g \in G$ admits a unique expression in terms of elements in H and N.*

> ***Proof*** *Suppose $n, n' \in N, h, h' \in H$ satisfying $g = nh = n'h'$, (existence by (s2)). Then $n'n^{-1} = h'^{-1}h$. But $n'^{-1}n \in N, h'h^{-1} \in H$, and by (s3) since the coincide they must equal the identiy, i.e., $n'n^{-1} = e = h'h^{-1}$. It follows that $n' = n, h' = h$.* □

Note that we have the natural isomorphism $(H, e_N) \cong H$. Given an element $h \in H$ we shall write $\hat{h}$ for the inclusion $\hat{h} = \iota_{H \rightarrowtail H \ltimes N}(h) = (h, e_N)$. Similarly, for elements $n \in N$ we write $\hat{n} = (e_H, n)$.

Note that this semi-direct product given H and N is not unique, for example,

$$\mathbb{Z}/6 = \mathbb{Z}/3 \rtimes \mathbb{Z}/2, \qquad S_3 = \mathbb{Z}/3 \rtimes \mathbb{Z}/2.$$

In order for $H \ltimes N$ uniquely determine G, we need to specify how H acts on the normal subgroup N, (proven below). So we have the extra condition (making the semi-direct product unique):

(s4') There is a homomorphism $\sigma : H \to \mathrm{Aut}(N); h \mapsto \sigma_h$, such that $\hat{h}\hat{n}\hat{h}^{-1} = \hat{\sigma}_h(n)$.

We denote the semi-direct product of H acting on N by σ as $G = H \ltimes_\sigma N$. Taking this a step further, we write $\mathrm{Ad}_{\hat{h}}(\hat{n}) = \hat{h}\hat{n}\hat{h}^{-1} \equiv (e_H, \sigma_h(n))$

Proposition 3.11. *Let H, N be two groups. Conjugation by h is a homomorphism: $\phi : H \to \mathrm{Aut}(N)$ with assignment $\phi(h)(\cdot) \equiv \phi_h(\cdot) = h \cdot h^{-1}$.*

> ***Proof*** *For any $h, h' \in H, n \in N$ we have*
>
> $$\phi_h\phi_{h'}(n) = h(h'nh'^{-1})h^{-1} = (hh')n(hh')^{-1} = \phi_{hh'}(n).$$ □

Proposition 3.12. *Given two groups H, N and a homomorphism $\sigma : H \to \mathrm{Aut}(N)$, there exists a unique semi-direct product, $G = H \ltimes_\sigma N$, satisfying (s1),(s2),(s3),(s4').*

> ***Proof*** *Write elements of G as the Cartesian product (h, n), and endow it with multiplication*
>
> $$(h, n)(h', n') = (hh', n\sigma_h(n')).$$
>
> *First, we show that this multiplication is associative:*
>
> $$\begin{aligned} \big[(h, n)(h', n')\big](h'', n'') &= \big(hh', n\sigma_h(n')\big)(h'', n'') \\ &= \big(hh'h'', n\sigma_h(n')\sigma_{hh'}(n'')\big) \\ &= \big(hh'h'', n\sigma_h(n'\sigma_{h'}(n''))\big) \\ &= (n, h)\big(h'h'', n'\sigma_{h'}(n'')\big) \\ &= (n, h)\big[(h', n')(h'', n'')\big]. \end{aligned}$$
>
> *The identity is $e_G = (e_H, e_N)$, which is quickly verified by noting that $\sigma_{e_H}(n) = n$ and $\sigma_h(e_N) = e_N$ by the homomorphism properties of σ. Element (h, n) has inverse*
>
> $$(h, n)^{-1} = (h^{-1}, \sigma_{h^{-1}}(n^{-1})).$$
>
> *We give a quick verification:*
>
> $$(h, n)(h^{-1}, \sigma_{h^{-1}}(n^{-1})) = (hh^{-1}, n\sigma_h(\sigma_{n^{-1}}(h^{-1})) = (hh^{-1}, n\sigma_{e_H}(n^{-1})) = (e_H, e_N).$$
>
> *We conclude that G is a group, with subgroups $(H, e_N) \cong H, (e_H, N) \cong N$. Note that the multiplication has been chosen so as to satisfy (s4'): $hnh^{-1} = \sigma_h(n)$, i.e.,*
>
> $$(h, e_N)(e_H, n)(h^{-1}, e_N) = (h, \sigma_h(n))(h^{-1}, e_N) = (hh^{-1}, \sigma_h(n)\sigma_h(e_N)) = (e_H, n).$$
>
> *Next, we show that G is the semi-direct product of H acting on N (by σ).*
>
> *(s1) By the above calculation $hnh^{-1} = (e_H, \sigma_h(n))$, we know that N is the kernel of the homomorphism $\phi : G \to \mathrm{Aut}(N)$, assigning $(h, n) \mapsto \sigma_h$, i.e., $\phi(e_H, n) = \sigma_{e_H}$. Hence it is normal in G.*
> *(s2) $HN \cong (H, e_N)(e_H, N) = (H, N) \cong G$.*
> *(s3) The intersection is $H \cap N \cong (H, e_N) \cap (e_H, N) = \{(e_H, e_N)\}$.*
>
> *We conclude that $G = H \ltimes_\sigma N$.* □

Examples

We provide a few examples of semi-direct products. Note that for our purposes, the normal group is usually Abelian, in which case we shall denote its multiplication by a +.

Cyclic group $C_n \ltimes C_m$ *Consider the cyclic groups* $C_n = \langle a|a^n = e\rangle, \langle b|b^m = e\rangle$*. Then they admit a range of semi-direct products, uniquely determined by the relation* $aba^{-1} = b^k$*, (with* k, n *coprime).*

Linear group GL_n *Claim: Given a field* k*, we have* $GL(n, k) \cong SL(n, k) \rtimes k^\times$*.*

Euclidean group $\mathrm{SO}_n \ltimes \mathbb{R}^n$ *The special orthogonal group* SO_n *is canonically represented by the orthogonal* $n \times n$*-matrices with determinant 1. So the semi-direct product consists of pairs* (M, v)*, where* SO_n *acts via matrix multiplication:* $(M, v)(M', v') = (MM', v + Mv')$

Lorentz group *The general Lorentz group consists of 4 disconnected components. It can be written as the semi-direct product of the proper orthochronous Lorentz group, and the discrete group* $\{I, P, T, PT\}$*:*

$$\mathrm{O}(1, n-1) \cong \mathrm{SO}(1, n-1)^\uparrow \rtimes \{I, P, T, PT\}.$$

via

$$(\Lambda, X)(\Lambda', X') = (\Lambda X \Lambda', XX').$$

Poincaré group $\mathrm{SO}(n-1, n)^\uparrow \ltimes \mathbb{R}^n$ $\mathrm{SO}(n-1, n)^\uparrow$ *is the group of proper orthochronous Lorentz transformations, acting on the translations:*

$$(\Lambda^\mu{}_\nu, x^\mu)(\Lambda^\mu{}_\nu, \tilde{x}^\mu) = (\Lambda^\mu{}_\lambda \Lambda^\lambda{}_\nu, x^\mu + \Lambda^\mu{}_\nu \tilde{x}^\nu).$$

3.6.3 Semi-direct sum of algebras

In the case where H, N are Lie groups, the semi-direct product $G = H \ltimes_\sigma N$ is naturally carried over to the semi-direct sum of Lie-algbras: $\mathfrak{g} = \mathfrak{h} \oplus_\Sigma \mathfrak{n}$. Moreover, if the normal group is an Abelian Lie-group A, it is isomorphic to its algebra, $\mathfrak{a} \cong A$.

Definition

We first propose a definition and then show that it is indeed the Lie-algebra counterpart. Let $\mathfrak{h}, \mathfrak{n}$ be two Lie-algebras, and a Lie-algebra homomorphism[2] $\Sigma : \mathfrak{h} \to \mathrm{Der}(\mathfrak{n}); H \mapsto \Sigma_H$. Here $\mathrm{Der}(\cdot)$ denotes the space of derivations[3]. We define a Lie-bracket on $\mathfrak{g} := \mathfrak{h} \oplus_\Sigma \mathfrak{n}$ by

$$[(X, v), (Y, w)]_\mathfrak{g} := ([X, Y]_\mathfrak{h}, [v, w]_\mathfrak{n} + \Sigma_X(w) - \Sigma_Y(v)), \qquad (\forall X, Y \in \mathfrak{h}; v, w \in \mathfrak{n}.)$$

Some remarks on the notation So as to avoid confusion between elements of Lie algebras, and the Lie groups, we write X, Y, Z for elements of $\mathfrak{h}$, and v, w, u for elements in $\mathfrak{n}$. (This, as opposed to the usual upper case letter $H \in \mathfrak{h}$).

Note that we have the isomorphism $(\mathfrak{h}, 0_\mathfrak{n}) \cong \mathfrak{h}$. Given an element $X \in \mathfrak{h}$ we shall write $\hat{X}$ for the inclusion $\hat{X} = \iota_{\mathfrak{h} \rightarrowtail \mathfrak{h} \oplus \mathfrak{n}}(X) = (X, 0_\mathfrak{n}) \in \mathfrak{h} \oplus \mathfrak{n}$. Similarly, for elements $v \in \mathfrak{n}$ we write $\hat{v} = \iota_{\mathfrak{n} \rightarrowtail \mathfrak{h} \oplus \mathfrak{n}}(v) = (0_\mathfrak{h}, v) \in \mathfrak{h} \oplus \mathfrak{n}$.

We shall omit subscripts of commutators whenever this wouldn't cause confusion.

Proposition 3.13. *The above commutator gives rise to a Lie-algebra.*

[2] A map $f : \mathfrak{g} \to \mathfrak{h}$ is a Lie-algebra homomorphism if $f([X, Y]) = [f(X), f(Y)]$ for all $X, Y \in \mathfrak{g}$.

[3] A derivation $D \in \mathrm{Der}(A)$ on an algebra A, (over k) is a (k-)linear map $D : A \to A$ satisfying the Leibniz rule $D(ab) = D(a)b + aD(b)$. Note that the 'product' in the Lie-algebra is denoted $[\cdot, \cdot]$, so that we rather should write $D([a, b]) = [D(a), b] + [a, D(b)]$.

Proof *As a space $\mathfrak{g} = \mathfrak{h} \times \mathfrak{n}$ is just the Cartesian product, so clearly it is a vector space. Note that the brackets $[\cdot,\cdot]_\mathfrak{h}, [\cdot,\cdot]_\mathfrak{n}$ satisfy all desired properties, so we only need to check them for the assignment $[(X,v),(Y,w)] \leftrightarrow \Sigma_X(w) - \Sigma_Y(v)$.*

- *Bilinearity:*

$$\begin{aligned}[a(X,v)+b(X',v'),(X'',v'')] &\leftrightarrow \Sigma_{aX+bX'}(v'') - \Sigma_{X''}(av+bv') \\ &= a\Sigma_X(v'') + b\Sigma_{X'}(v'') - a\Sigma_{X''}(v) - b\Sigma_{X''}(v') \\ &= a(\Sigma_X(v'') - \Sigma_{X''}(v)) + b(\Sigma_{X'}(v'') - \Sigma_{X''}(v')) \\ &\leftrightarrow a[(X,v),(X'',v'')] + b[(X',v'),(X'',v'')],\end{aligned}$$

and similar for the second argument of the bracket.

- *Anticommutativity:*

$$\begin{aligned}[(X,v),(Y,w)] &\leftrightarrow \Sigma_X(w)\Sigma_Y(v) \\ &= -(\Sigma_Y(v) - \Sigma_X(w)) \\ &\leftrightarrow -[(Y,w),(X,v)].\end{aligned}$$

- *Jacobi identity; Since we take consecutive brackets, we need to be more careful, i.e., we shall have to get dirty hands. First we calculate one of the three brackets:*

$$[(X,v),[(X',v'),(X'',v'')]] = \big[(X,v),\big([X',X'']_\mathfrak{h},[v',v'']_\mathfrak{n} + \Sigma_{X'}(v'') - \Sigma_{X''}(v')\big)\big]$$

Call $w = \Sigma_{X'}(v'') - \Sigma_{X''}(v')$, so that we obtain

$$\begin{aligned}&[(X,v),[(X',v'),(X'',v'')]] = \\ &\quad = \big[(X,v),\big([X',X'']_\mathfrak{h},[v',v'']_\mathfrak{n} + w\big)\big] \\ &\quad = \Big([X[X',X'']_\mathfrak{h}]_\mathfrak{h},[v,[v',v'']_\mathfrak{n}]_\mathfrak{n} + [v,w]_\mathfrak{n} + \Sigma_X([v',v'']_\mathfrak{n}) + \Sigma_X(w) - \Sigma_{[X',X'']_\mathfrak{h}}(v)\Big)\end{aligned}$$

Now, we check the Jacobi identity in pieces. Note that $[X[X',X'']_\mathfrak{h}]_\mathfrak{h}$ and $[v,[v',v'']_\mathfrak{n}]_\mathfrak{n}$ satisfy it already. As for the other pieces, consider first:

$$[v,w]_\mathfrak{n} + \Sigma_X([v',v'']_\mathfrak{n}) = [v,\Sigma_{X'}(v'')]_\mathfrak{n} - [v,\Sigma_{X''}(v')]_\mathfrak{n} - [v'',\Sigma_X(v')]_\mathfrak{n} + [v',\Sigma_X(v'')]_\mathfrak{n}$$

Here we used the fact that for a given Y, Σ_Y is derivation on $\mathfrak{n}$. Now, we shall perform different cyclic permutations on the other terms, that is, exchanging terms from the written part with terms in the '+cyclic permutations*' part:*

$$\begin{aligned}&[v,w]_\mathfrak{n} + \Sigma_X([v',v'']_\mathfrak{n}) + \text{cyclic permutations} = \\ &\quad = [v,\Sigma_{X'}(v'')]_\mathfrak{n} - [v,\Sigma_{X''}(v')]_\mathfrak{n} - [v'',\Sigma_X(v')]_\mathfrak{n} + [v',\Sigma_X(v'')]_\mathfrak{n} \\ &\qquad + \text{cyclic permutations} \\ &\quad = [v,\Sigma_{X'}(v'')]_\mathfrak{n} - [v,\Sigma_{X''}(v')]_\mathfrak{n} - [v,\Sigma_{X'}(v'')]_\mathfrak{n} + [v,\Sigma_{X''}(v')]_\mathfrak{n} \\ &\qquad + \text{cyclic permutations} \\ &\quad = 0\end{aligned}$$

For the last piece, we have

$$\begin{aligned}&\Sigma_X(w) - \Sigma_{[X',X'']_\mathfrak{h}}(v) + \text{cyclic permutations} = \\ &\quad = \Sigma_X(\Sigma_{X'}(v'') - \Sigma_{X''}(v')) - \Sigma_{[X',X'']_\mathfrak{h}}(v) + \text{cyclic permutations} \\ &\quad = \Sigma_X(\Sigma_{X'}(v'')) - \Sigma_X(\Sigma_{X''}(v')) - \Sigma_{[X',X'']_\mathfrak{h}}(v) + \text{cyclic permutations} \\ &\quad = \Sigma_{X'}(\Sigma_{X''}(v)) - \Sigma_{X''}(\Sigma_{X'}(v)) - \Sigma_{[X',X'']_\mathfrak{h}}(v) + \text{cyclic permutations} \\ &\quad = \Sigma_{[X',X'']_\mathfrak{h}}(v) - \Sigma_{[X',X'']_\mathfrak{h}}(v) + \text{cyclic permutations}\end{aligned}$$

Here we once more performed some cyclic permutations, and we used the fact that $Y \to \Sigma_Y$ is a Lie algebra homomorphism.

$$\begin{aligned}&\Sigma_X(w) - \Sigma_{[X',X'']_\mathfrak{h}}(v) + \text{cyclic permutations} = \\ &\quad = \Sigma_X(\Sigma_{X'}(v'') - \Sigma_{X''}(v')) - \Sigma_{[X',X'']_\mathfrak{h}}(v) + \text{cyclic permutations}\end{aligned}$$

□

Proposition 3.14. *Let H, N be two linear Lie groups, $\sigma : H \to \mathrm{Aut}(N)$ a Lie group homomorphism, and let $G = H \ltimes_\sigma N$ be the semi-direct product. Let $\mathfrak{g}, \mathfrak{h}, \mathfrak{n}$ be the Lie-algebras corresponding to G, H, N, respectively, and $\Sigma : \mathfrak{h} \to \mathrm{Der}\,\mathfrak{n}$ the differential of σ (at the origin of H), that is*

$$\Sigma_X(v) = \frac{d}{dt}\frac{d}{ds}\sigma_{e^{t\hat{X}}}(e^{s\hat{v}})\bigg|_{t=0,v=0}.$$

Then $\mathfrak{g} = \mathfrak{h} \oplus_\Sigma \mathfrak{n}$.

Proof *We calculate the commutator $[(X,0),(0,v)]_\mathfrak{g}$, and infer the other commutators from this one. Via the exponential map, we move to the Lie-group, (see 3.6.1).*

$$[\hat{X},\hat{v}]_\mathfrak{g} = \mathrm{ad}_{\hat{X}}\,\hat{v} = \frac{d}{dt}\,\mathrm{Ad}_{e^{t\hat{X}}}\,\hat{v}\bigg|_{t=0}. = \frac{d}{dt}\frac{d}{ds}\,\mathrm{Ad}_{e^{t\hat{X}}}\,e^{s\hat{v}}\bigg|_{t=0,s=0}.$$

Note that by definition of the semi-direct product $\mathrm{Ad}_{\hat{h}}\,\hat{n} = \sigma_h n$, so we now know

$$[\hat{X},\hat{v}]_\mathfrak{g} = \frac{d}{dt}\frac{d}{ds}\sigma_{e^tX}\,e^{sv}\bigg|_{t=0,s=0} = \Sigma_X(v).$$

By anticommutativty then $[\hat{u},\hat{Y}]_\mathfrak{g} = -\Sigma_Y(u)$. Finally, by noting that $(H, e_N) \cong H$, and $(e_H, N) \cong N$, we know from 3.6.1 that $[\hat{X},\hat{Y}] = [X,Y]_\mathfrak{h}$, and $[\hat{u},\hat{v}] = [u,v]_\mathfrak{n}$. We conclude that

$$[(X,u),(Y,v)]_\mathfrak{g} = ([X,Y]_\mathfrak{h}, [u,v]_\mathfrak{n} + \Sigma_X(v) - \Sigma_Y(u)).$$

So the Lie-bracket of $\mathfrak{g}$ corresponds to the Lie-bracket of $\mathfrak{h} \oplus_\Sigma \mathfrak{n}$. □

Poincaré Algebra

We proceed to calculate the Poincaré Algebra. We start with the group $SO(n-1,1) \ltimes \mathbb{R}^n$, with elements (Λ, x). Note that the Lie-algebra of $\mathbb{R}^n$ is just $\mathbb{R}^n$ with the trivial commutator; we shall write P_μ for the algebra elements. The Lorentz algebra consists of matrices

$$\Lambda = \begin{pmatrix} 0 & \vec{v}^t \\ \vec{v} & S \end{pmatrix}, \qquad\qquad S = -S^t.$$

with the commutator as Lie-bracket. Since the translations have a trivial bracket we have

$$[(\Lambda, x), (\Lambda, x')] = (\Lambda\Lambda' - \Lambda'\Lambda, \Lambda x' - \Lambda' x).$$

BMS algebra

The BMS algebra turns out to be $\mathfrak{bms}_n = \mathfrak{vect}(\mathbb{S}^{n-2}) \oplus_{\mathrm{ad}} \mathfrak{vect}(\mathbb{S}^{n-2})_{ab}$, ($ab$ denotes the abelianisation, i.e., treat the algebra as a vector space with addition/trivial bracket). Since as vector spaces, the algebras coincide, the superrotation action can be taken to be the adjoint action, the commutator of vector fields.

$$[(Y_1,f_1),(Y_2,f_2)] = ([Y_1,Y_2], f_1 + \mathrm{ad}_X(f_2) - \mathrm{ad}_Y(f_1)) = (Y_1Y_2 - Y_2Y_1, Y_1(f_2) - Y_2(f_1)).$$

4. Asymptotically flat spacetimes and supertransformations

4.1 Bondi Metric

In this section we will consider a class of metrics of the 4-dimensional space-time, that are asymptotically flat, axially symmetric, and reflection symmetric. An important example is the Schwarzschild metric. After casting it in a canonical form, due to [2], we will derive the algebra of vector fields that leave this class invariant.

4.1.1 Canonical Bondi metric

We will use coordinates (u, r, θ, ϕ), where θ and ϕ are the usual radial coordinates. Since the metric is assumed to be axially symmetric, we readily have $g_{u\phi} = g_{r\phi} = g_{\theta\phi} = 0$.

Next, we define time-like coordinate u, and the angles, in such a way, that they are constant along light rays. For light-like separation ($ds^2 = 0$), this means that $du = d\theta = d\phi = 0$. In particular, this implies that $g_{rr}dr^2 = ds^2 = 0$, so $g_{rr} = 0$.

Remark *In the case of the Schwarzschild metric, the coordinates are the well-known outgoing Eddington-Finkelstein coordinates:* $u = t - r^*$ *and* $r \equiv r^*$.

Finally, we show that $g_{r\theta}$ necessarily vanishes.

Calculation *The (preliminary) inverse metric is*

$$g^{\mu\nu} = \frac{1}{\det g}\begin{pmatrix} -g_{r\theta}^2 g_{\phi\phi} & (g_{r\theta}g_{u\theta} - g_{ur}g_{\theta\theta})g_{\phi\phi} & g_{r\theta}g_{ur}g_{\phi\phi} & 0 \\ * & g_{uu}g_{\theta\theta}g_{\phi\phi} - g_{u\theta}^2 g_{\phi\phi} & g_{ur}g_{u\theta}g_{\phi\phi} - g_{r\theta}g_{uu}g_{\phi\phi} & 0 \\ * & * & -g_{ur}^2 g_{\phi\phi} & 0 \\ 0 & 0 & 0 & 2g_{ur}g_{u\theta}g_{r\theta} - g_{r\theta}^2 g_{uu} - g_{ur}^2 g_{\theta\theta} \end{pmatrix} \tag{4.1}$$

Since u, θ *and* ϕ *are constant along light rays, the geodesic equations for light-like geodesics are*

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{rr}\frac{dx^r}{d\lambda}\frac{dx^r}{d\lambda} = 0. \tag{4.2}$$

Note that

$$\Gamma^\mu_{rr} = g^{\mu\lambda}\partial_r g_{r\lambda}, \tag{4.3}$$

because $g_{rr} = 0$. *Since* $dx^\mu/d\lambda = 0$ *for* $\mu = u, \theta, \phi$, *the Christoffel symbols* Γ^μ_{rr} *must vanish for those* μ. *Note that* Γ^ϕ_{rr} *is automatically zero. From (4.2) and (4.3) we obtain the necessary conditions:*

$$g^{u\lambda}\partial_r g_{r\lambda} = 0, \qquad g^{\theta\lambda}\partial_r g_{r\lambda} = 0. \tag{4.4}$$

Using the inverse metric, the constraints (4.4) become

$$\frac{g_{\phi\phi}}{\det g}(g_{r\theta}g_{ur}\partial_r g_{r\theta} - g_{r\theta}^2\partial_r g_{ur}) = 0, \qquad \frac{g_{\phi\phi}}{\det g}(g_{r\theta}g_{ur}\partial_r g_{ur} - g_{ur}^2\partial_r g_{r\theta}) = 0. \tag{4.5}$$

Both $g_{\phi\phi}$ and $\det g$ must be finite, otherwise the metric would be degenerate. Rearranging the terms in the brackets among each other, the constraints are equivalent to

$$g_{r\theta}(g_{r\theta}\partial_r g_{ur} - g_{ur}\partial_r g_{r\theta}) = 0, \qquad g_{ur}(g_{r\theta}\partial_r g_{ur} - g_{ur}\partial_r g_{r\theta}) = 0. \tag{4.6}$$

The equations are solved by

$$(i): \ g_{ur} = 0, \qquad \text{or} \qquad (ii): \ g_{r\theta} = 0, \qquad \text{or} \qquad (iii): \ g_{r\theta}\partial_r g_{ur} = g_{ur}\partial_r g_{r\theta}. \tag{4.7}$$

First we consider case (i). Suppose g_{ur} is zero. Consider an infinitesimal separation, in such a way that $d\theta = d\phi = 0$. Then $ds^2 = g_{uu}du^2$. Now for time-like, and space-like separations, this must have the opposite signature, which is a contradiction. So we disregard (i).
Next, note that:

$$\partial_r \frac{g_{r\theta}}{g_{ur}} = \frac{g_{ur}\partial_r g_{r\theta} - g_{r\theta}\partial_r g_{ur}}{(g_{ur})^2},$$

so (iii) is equivalent to the condition that $g_{r\theta}/g_{ur}$ is independent of r. In this case, then, there exist $\tilde{u}(u,\theta), \lambda(u,\theta)$ defined by

$$\lambda d\tilde{u} = \left(du + \frac{g_{r\theta}}{g_{ur}}d\theta\right).$$

In terms of the new coordinate, the metric is

$$\begin{aligned}
ds^2 &= g_{uu}du^2 + 2g_{ur}dudr + 2g_{u\theta}dud\theta + 2g_{r\theta}drd\theta + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \\
&= \lambda^2 g_{uu}d\tilde{u}^2 - 2\lambda\frac{g_{uu}g_{r\theta}}{g_{ur}}d\tilde{u}d\theta + \frac{g_{uu}g_{r\theta}^2}{g_{ur}^2}d\theta^2 + \\
&\quad + 2(g_{ur}dr + g_{u\theta}d\theta)\left(\lambda d\tilde{u} - \frac{g_{r\theta}}{g_{ur}}d\theta\right) + 2g_{r\theta}drd\theta + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \\
&= \lambda^2 g_{uu}d\tilde{u}^2 + 2\lambda\left(g_{u\theta} - \frac{g_{uu}g_{r\theta}}{g_{ur}}\right)d\tilde{u}d\theta + 2\lambda g_{ur}d\tilde{u}dr + \\
&\quad + (2g_{r\theta} - 2g_{r\theta})drd\theta + \left(\frac{g_{uu}g_{r\theta}^2}{g_{ur}^2} - 2\frac{g_{u\theta}g_{r\theta}}{g_{ur}} + g_{\theta\theta}\right)d\theta^2 + g_{\phi\phi}d\phi^2.
\end{aligned}$$

In these coordinates we have $g_{r\theta}(\tilde{u}, r, \theta) = 0$, so that conditions (ii) and (iii) actually coincide. Hence the metric is characterized by

$$g_{rr} = g_{r\theta} = 0.$$

The Bondi metric is of the general form

$$ds^2 = g_{uu}du^2 + 2g_{ur}dudr + 2g_{u\theta}dud\theta + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \tag{4.8}$$

4.1.2 Einstein's equations

In this setup, we wish the metric to satisfy the Einstein equations for the vacuum. That is, $G_{\mu\nu} = 0$. Because we cannot hope to solve the equation in full generality, we solve it up to the relevant order of r. We will actually go about it in steps, adding parameters and solving, until we have reached the most general form (4.8). We will start off assuming that the metric is asymptotically flat, and that we are in the flat limit ($r \to \infty$).

Conformal rescaling

First, consider a conformal rescaling of the angular part, by $e^{2\varphi}\gamma_{AB}$. (See also 3.3.2). At this point, we assume $\partial_r\varphi = 0$. Any lower order corrections (of order $\mathcal{O}_\infty(r^1)$) to the angular part of the metric will be allowed later on.[1] It turns out that the only metric is

$$ds^2 = g_{uu}du^2 - 2dudr^2 + r^2 e^{2\varphi}\gamma_{AB}dx^A dx^B, \tag{4.9}$$

[1] Alternatively, one could introduce with an r-dependency to φ and observe that it must be r-dependent for G_{rr} to vanish, for any asymptotic metric.

where x^A denotes the angular coordinate, and γ_{AB} is the metric of the (flat) 2-sphere. Then the non-zero Einstein tensor components are G_{uu}, G_{ur}, G_{uA} and G_{AB}. Without showing the calculation, I present the relevant tensor components in a convenient order:

$$G_{AB} = -\tfrac{1}{2} r \gamma_{AB} e^{2\varphi} \big(2\partial_r g_{uu} + r\partial_r^2 g_{uu} + 4\partial_u \varphi \big). \tag{4.10}$$

Equation (4.10) is solved by

$$g_{uu} = -2r\partial_u \phi + f_0 + \frac{f_1}{r},$$

where f_0 and f_1 are arbitrary functions of u, x^A. Next, by

$$G_{uA} = -\frac{\partial_A f_1}{2r^2}, \tag{4.11}$$

f_1 must be independent of r. Next, by choosing (temporarily) $g_{AB} = \frac{1}{2} r^2 e^{2\tilde{\varphi}}(1 - \delta_{AB})$, we find

$$G_{ur} = \frac{1}{r^2}(f_0 - \overline{\Delta}\tilde{\varphi}),$$

where $\overline{\Delta} = \overline{D}^A \overline{D}_A$ is the Laplacian of the conformally rescaled metric of the Riemann sphere $\overline{\gamma}_{AB}$. Note that $\overline{\Delta}(-\ln P) = -1$, so that we indeed recover the Minkowski metric if we conformally rescale the angular part of the metric by 1. Thus we have

$$g_{uu} = -2r\partial_u \tilde{\varphi} + \overline{\Delta}\tilde{\varphi} + \frac{f_1}{r},$$

Here we have changed $\partial_u \varphi$ to $\partial_u \tilde{\varphi}$, since $\partial_u(\tilde{\varphi} - \varphi) = 0$. Finally, we need to solve $G_{uu} = 0$. It turns out, that this cannot be achieved, for any φ. We will not go into the details of the most general $\tilde{\varphi}$ that satisfies Einstein's equations. A large class of functions that satisfy the equation consists of functions of the form

$$\tilde{\varphi} = p(u) + q(x^A), \tag{4.12}$$

where p, and q are arbitrary functions of u and x^A. In this case

$$G_{uu} = \frac{f_1'(u)}{r^2},$$

so f_1 has to be a constant. In fact, the constant f_1 gives rise to a black hole mass. By setting $f_1 = 2GM$ one retrieves the Schwarzschild solution. Summarizing: In accordance with literature, let

$$\begin{aligned} \frac{V}{r} &= -2r\partial_u \tilde{\varphi} + \overline{\Delta}\tilde{\varphi} + \frac{f_1}{r}, \qquad (4.13)\\ \tilde{\varphi} &:= \varphi - \ln P = \alpha(u) + \beta(x^A) - \ln P, \\ \overline{\gamma}_{AB} &= \tfrac{1}{2} e^{2\tilde{\varphi}}(1 - \delta_{AB}), \end{aligned}$$

for f_1 a constant. Then

$$ds^2 = \frac{V}{r} du^2 - 2dudr + r^2 \overline{\gamma}_{AB} dx^A dx^B,$$

satisfies $G_{\mu\nu} = 0$, and it equal to the Schwarzschild metric for $\varphi = 0, f_1 = 2GM$.
Note that a change of $\varphi \to \varphi + \tilde{\omega}$ induces a conformal rescaling of the angular part of the metric, i.e.,

$$\delta g_{AB} = 2\tilde{\omega} g_{AB} \delta\tilde{\varphi}.$$

So the conformal factor is $\lambda = 2\tilde{\omega}$. Coincidentally, g_{uu} has variation

$$\begin{aligned} \delta g_{uu} &= \delta(-2r\partial_u \tilde{\varphi} + \overline{\Delta}\tilde{\varphi}) \\ &= -2r\partial_u \tilde{\omega}\delta\tilde{\varphi} + \delta\overline{\gamma}^{AB}(\overline{D}_A \partial_B \partial_C \tilde{\varphi}) \\ &= -2r\partial_u \tilde{\omega}\delta\tilde{\varphi} - 2\tilde{\omega}\overline{\gamma}^{AB}(\overline{D}_A \partial_B \tilde{\varphi})\delta\tilde{\varphi} + \overline{\gamma}^{AB}\delta(\overline{D}_A \partial_B \tilde{\varphi}) \\ &= -2r\partial_u \tilde{\omega}\delta\tilde{\varphi} - 2\tilde{\omega}\overline{\Delta}\tilde{\varphi}\delta\tilde{\varphi} + \overline{\Delta}\tilde{\omega}\delta\tilde{\varphi}. \qquad (4.14) \end{aligned}$$

In the last step, we used equation (3.16), to see that $\overline{\gamma}^{AB}\overline{\Gamma}^C_{AB} = 0$.

Non-flat interior

At this point, we introduce functions that have the appropriate limit as $r \to \infty$, i.e., the metric becomes the Minkowski metric for sufficiently large r. First off, note that g_{ur} must be strictly negative, (see case (i) of (4.7)), so let $g_{ur} = e^{2\beta}$. In addition, let $g_{uA} = U_A$. We leave g_{uu} arbitrary for now. A priory, we have asymptotic flatness, if

$$g_{uu} = -1 + \mathcal{O}_\infty(r^{-1}), \quad g_{ur} = -1 + \mathcal{O}_\infty(r^{-1}), \quad g_{uA} = \mathcal{O}_\infty(r^0), \quad g_{AB} = r^2\overline{\gamma}_{AB} + \mathcal{O}_\infty(r^1),$$

(Recall that the metric components scale with a factor r for each angular index.) Because we use Bondi coordinates, g_{rr} and g_{rA} remain zero for all r. It turns out, that $[r^{-1}]g_{ur}$ needs to be zero in order to solve Einstein's equations for the vacuum, and $g_{uu} = e^{2\beta}\frac{V}{r} + U_A U^A$. Of course, the functions U_A, β, g_{AB} are constrained by the Einstein equations, but since at this point we are only interested in the asymptotic part of the metric, assessing their orders is sufficient for now. Thus we have rewritten the metric to

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + g_{AB}(dx^A + U^A du)(dx^B + U^B du),$$

where

$$\begin{aligned} g_{AB} &= r^2\overline{\gamma}_{AB} + \mathcal{O}_\infty(r^1) \\ &= \tfrac{1}{2}r^2 e^{2\tilde{\varphi}}(1 - \delta_{AB}) + \mathcal{O}_\infty(r^1) \\ U^A, \beta &= \mathcal{O}_\infty(r^{-2}), \\ \frac{V}{r} &= -2r\partial_u\tilde{\varphi} + \overline{\Delta}\tilde{\varphi} + \mathcal{O}_\infty(r^{-1}). \end{aligned} \tag{4.15}$$

In our calculation of the supertransformations later on (in order to simplify (4.34)), we will further assume $\det g_{AB} = \frac{1}{4}r^4 e^{4\tilde{\varphi}}$. The metric component g_{uu} has expansion

$$g_{uu} = -2r\partial_u\tilde{\varphi} + \overline{\Delta}\tilde{\varphi} + \frac{2m_B}{r} + \mathcal{O}_\infty(r^{-2})$$

Although the term $m_B \equiv m_B(u, x^A)$ is subordinate to the asymptotic fall-off, it is worth mentioning. It is referred to as the **Bondi mass**, because it gives rise to the (possibly u-dependent) ADM-mass of the metric. The Schwarzschild metric is established as a Bondi metric by setting

$$U^A = \beta = 0, \qquad g_{AB} = r^2\overline{\gamma}_{AB}, \qquad \tilde{\varphi} = -\ln P, \qquad m_B = GM.$$

With this choice V/r is fixed by Einstein's equation to be $V/r = -1 + 2GM$, (cf. (4.13)).

4.2 Supertransformations

The asymptotic Killing vectors and conformal rescalings of the angular part of the metric, are simultaneously found, by solving the equations

$$\mathcal{L}_X g_{rr} = \mathcal{L}_\xi g_{rA} = 0 \tag{4.16a}$$
$$\mathcal{L}_X g_{ur} = \mathcal{O}_\infty(r^{-2}) \tag{4.16b}$$
$$\mathcal{L}_X g_{uu} = -2r\partial_u\tilde{\omega} - 2\tilde{\omega}\overline{\Delta}\tilde{\varphi} + \overline{\Delta}\tilde{\omega} + \mathcal{O}_\infty(r^{-1}) \tag{4.16c}$$
$$\mathcal{L}_X g_{uA} = \mathcal{O}_\infty(r^0) \tag{4.16d}$$
$$\mathcal{L}_X g_{AB} = 2\tilde{\omega}g_{AB} + \mathcal{O}_\infty(r^1). \tag{4.16e}$$

The conformal scaling function $2\tilde{\omega}$ is constrained by the trace condition $g^{AB}\mathcal{L}_X g_{AB} = 4\tilde{\omega}$. Note that generally it is different from the conformal factor $\tilde{\lambda}$ of the conformal rescaling $\mathcal{L}_Y\overline{\gamma}_{AB} = \tilde{\lambda}\overline{\gamma}_{AB}$, i.e., $2\tilde{\omega} \neq \tilde{\lambda} = \overline{D}_A Y^A$. Although $\tilde{\omega}$ may depend on u, it is assumed to be independent of r, (for otherwise it would meddle with the asymptotic flatness).

Definition 4.1. *The vector fields satisfying equations (4.16) are called the* ***supertransformations****. The asymptotic Killing vectors, the* ***supertranslations****, are those satisfying $\tilde{\omega} = 0$. The conformal rescalings of the angular part of the metric,* ***the superrotations****, are the supertransformations modulo the supertranslations.*

Calculation of supertransformations *We start by (exactly) solving the equations that allow no falloff,* (4.16a). *By*

$$\mathcal{L}_X g_{rr} = -2e^{2\beta}\partial_r X^u = 0, \tag{4.17}$$

X^u is independent of r, i.e,. $X^u = f$ for some function f satisfying

$$\partial_r f = 0. \tag{4.18}$$

Then, the other exact condition becomes

$$\mathcal{L}_X g_{rA} = -e^{2\beta}\partial_A f + g_{AB}\partial_r X^B = 0.$$

It is solved by imposing

$$X^A = Y^A + I^A, \qquad I^A = -\partial_B f \int_r^\infty dr' \Big(e^{2\beta} g^{AB}\Big). \tag{4.19}$$

The function Y^A depends only on the angular coordinates, i.e., $Y^A \equiv Y^A(x^A)$. The resulting vector component has r-expansion

$$X^A = Y^A - \frac{\overline{\gamma}^{AB}\partial_B f}{r} + \mathcal{O}_\infty(r^2). \tag{4.20}$$

The rest of the equations (apart from the trace condition) need only be solved up to the appropriate order of r. It is useful at this point to assess the maximal orders of r of all elements of the vector field. Let us take a look at the condition $\mathcal{L}_X g_{AB} = \mathcal{O}_\infty(r^1)$. From the falloff conditions (4.15), *in addition with* (4.18) *and* (4.19) *we know*

$$X^u \sim r^0, \qquad X^A = \mathcal{O}_\infty(r^0), \qquad g_{AB} = \mathcal{O}_\infty(r^2), \qquad U_A \equiv g_{AB}U^B = \mathcal{O}_\infty(r^0). \tag{4.21}$$

Using (4.19) *and $g_{uA} = 0$, the Lie derivative of the angular part of the metric is asymptotically*

$$\begin{aligned}\mathcal{L}_X g_{AB} &= X^u\partial_u g_{AB} + X^r\partial_r g_{AB} + X^C\partial_C g_{AB} + (U_A\partial_B + U_B\partial_A)X^u + (g_{AC}\partial_B + g_{BC}\partial_A)X^C \\ &= X^u\partial_u g_{AB} + X^r\partial_r g_{AB} + (\partial_C g_{AB} + g_{AC}\partial_B + g_{BC}\partial_A)X^C + \mathcal{O}_\infty(r^1).\end{aligned} \tag{4.22}$$

So (4.16e), *is resolved if*

$$[r^{\geqslant 2}]\Big(X^u\partial_u g_{AB} + X^r\partial_r g_{AB} + X^C\partial_C g_{AB} + (g_{AC}\partial_B + g_{BC}\partial_A)X^C - 2\tilde{\omega} g_{AB}\Big) = 0. \tag{4.23}$$

Using the order assesment (4.21), *this implies the order of X^r to be at most $[r^1]$.*

$$X^r \in \mathcal{O}_\infty(r^1). \tag{4.24}$$

As a result, the often occuring contraction $X^\sigma\partial_\sigma$ is of order $\mathcal{O}_\infty(r^0)$. Before we solve (4.23), *we resolve* (4.16b). *We have*

$$\begin{aligned}\mathcal{L}_X g_{ur} &= X^\sigma\partial_\sigma g_{ur} + g_{\sigma r}\partial_u X^\sigma + g_{u\sigma}\partial_r X^\sigma \\ &= g_{ur}\partial_u X^u + g_{uu}\partial_r X^u + g_{ur}\partial_r X^r + g_{uA}\partial_r X^A + \mathcal{O}_\infty(r^{-2}) \\ &= \partial_u f - \partial_r X^r + U_A\partial_r X^A + \mathcal{O}_\infty(r^{-2}),\end{aligned} \tag{4.25}$$

where we have used $\partial_\sigma g_{ur} = \mathcal{O}_\infty(r^{-2})$, and the fall-off (4.21). *In addition, note that $\partial_r X^A = \partial_r I^A = \mathcal{O}_\infty(r^{-2})$.. Hence equation* (4.16b) *is solved if $[r^1]X^r = -\partial_u f r$. It follows*

$$X^r = -r\partial_u f + \mathcal{O}_\infty(r^0), \tag{4.26}$$

With this information, consider (4.16d). *Using $\partial_u Y^A = 0$ and* (4.20), *we have*

$$\begin{aligned}\mathcal{L}_X g_{uA} &= g_{AB}\partial_u X^B + g_{uu}\partial_A X^u + g_{ur}\partial_A X^r + \mathcal{O}_\infty(r^0) \\ &= -r\overline{\gamma}_{AB}\partial_u(\overline{\gamma}^{BC}\partial_C f) - 2r(\partial_u\tilde{\varphi})\partial_A f + r\partial_A\partial_u f + \mathcal{O}_\infty(r^0) \\ &= -r\overline{\gamma}_{AB}[\overline{\gamma}^{BC}\partial_u\partial_C f - 2\overline{\gamma}^{BC}(\partial_u\tilde{\varphi})\partial_C f] - 2r(\partial_u\tilde{\varphi})\partial_A f + r\partial_A\partial_u f + \mathcal{O}_\infty(r^0) \\ &= \mathcal{O}_\infty(r^0).\end{aligned}$$

So it is automatically resolved. Using (4.26), equation (4.23) gives the condition

$$\begin{aligned}0 &= [r^2]\Big(X^u\partial_u g_{AB} + X^r\partial_r g_{AB} + X^C\partial_C g_{AB} + (g_{AC}\partial_B + g_{BC}\partial_A)X^C - 2\tilde{\omega} g_{AB}\Big)\\ &= X^u\partial_u\overline{\gamma}_{AB} + 2[r^1]X^r\overline{\gamma}_{AB} + Y^C\partial_C\overline{\gamma}_{AB} + (\overline{\gamma}_{AC}\partial_B + \overline{\gamma}_{BC}\partial_A)Y^C - 2\tilde{\omega}\overline{\gamma}_{AB}\\ &= 2\overline{\gamma}_{AB}[f(\partial_u\tilde{\varphi}) - \partial_u f - \tilde{\omega}] + (\overline{\gamma}_{AC}\partial_B Y^C + \overline{\gamma}_{BC}\partial_A Y^C + 2\overline{\gamma}_{AB}Y^C\partial_C\tilde{\varphi})\end{aligned} \tag{4.27}$$

Let $A = B$ in (4.27). Since $\overline{\gamma}_{zz} = \overline{\gamma}_{\overline{z}\overline{z}} = 0$, we have constraint

$$0 = \overline{\gamma}_{AC}\partial_B Y^C + \overline{\gamma}_{BC}\partial_A Y^C$$

So we require

$$\partial_z Y^{\overline{z}} = \partial_{\overline{z}} Y^z = 0. \tag{4.28}$$

Comparing with (3.19), we see that imposing (4.28) is equivalent to having Y^A be a conformal Killing vector of the Riemann sphere. The final three terms in (4.27) vanish for $A = B$. For $A \neq B$, they are

$$\begin{aligned}&(\overline{\gamma}_{AC}\partial_B Y^C + \overline{\gamma}_{BC}\partial_A Y^C + 2\overline{\gamma}_{AB}Y^C\partial_C\tilde{\varphi})_{A\neq B}\\ &\quad= \overline{\gamma}_{z\overline{z}}(\partial_{\overline{z}}Y^{\overline{z}} + \partial_z Y^z + 2Y^C\partial_C\tilde{\varphi})\\ &\quad= \overline{\gamma}_{z\overline{z}}(\overline{D}_C Y^C),\end{aligned}$$

where we have used (3.17) in the last step. Equation (4.27) is then conveniently rewritten

$$0 = 2\overline{\gamma}_{AB}\Big[f(\partial_u\tilde{\varphi}) - \partial_u f - \tilde{\omega} + \tfrac{1}{2}\overline{D}_C Y^C\Big]. \tag{4.29}$$

So now we know

$$f(\partial_u\tilde{\varphi}) - \partial_u f - \tilde{\omega} + \tfrac{1}{2}\overline{D}_C Y^C = 0 \tag{4.30}$$

Solved by

$$\partial_u f = f\partial_u\tilde{\varphi} + \tfrac{1}{2}\overline{D}_C Y^C - \tilde{\omega} \tag{4.31}$$

or, equivalently

$$f = e^{\tilde{\varphi}}\left[T(x^A) + \int_0^u du' e^{-\tilde{\varphi}}(\tfrac{1}{2}\overline{D}_C Y^C - \tilde{\omega})\right],$$

for some function $T(x^A)$. Summarizing, the remaining constraint equations are (4.16c) and the trace condition, and the thus far obtained solutions are of the form

$$\begin{cases} X^u = f, & f = e^{\tilde{\varphi}}\big[T + \int_0^u du' e^{-\tilde{\varphi}}(\tfrac{1}{2}\overline{D}_C Y^C - \tilde{\omega})\big],\\ X^r = -r\partial_u f + \mathcal{O}_\infty(r^0), & \partial_u f = f\partial_u\tilde{\varphi} + \tfrac{1}{2}\overline{D}_C Y^C - \tilde{\omega}\\ X^A = Y^A + I^A, & I^A = -\partial_B f\int_r^\infty dr' e^{2\beta}g^{AB}.\end{cases} \tag{4.32}$$

for arbitrary function $T \equiv T(x^A)$, and where $Y^A \equiv Y^A(x^A)$ is a conformal Killing vector of the Riemann sphere. Using the general form of the solutions (4.32), along with (4.30) to substitute $\tilde{\omega}$, the trace condition becomes

$$\begin{aligned}0 &= g^{AB}[X^\sigma\partial_\sigma g_{AB} + (U_A\partial_B + U_B\partial_A)X^u + (g_{AC}\partial_B + g_{BC}\partial_A)X^C] - 4\tilde{\omega}\\ &= X^\sigma g^{AB}\partial_\sigma g_{AB} + 2U^C\partial_C f + 2\partial_C(X^C) - 4\tilde{\omega}.\end{aligned} \tag{4.33}$$

At this point our life is made much easier by the assumption $\det g_{AB} = \frac{1}{4}r^4 e^{4\tilde{\varphi}}$. This implies that $g^{AB}\partial_\sigma g_{AB} = \partial_\sigma \ln\frac{1}{4}r^4 e^{4\tilde{\varphi}}$, and so (4.34) becomes

$$0 = 4X^u\partial_u\tilde{\varphi} + \tfrac{4}{r}X^r + 4X^C\partial_C\tilde{\varphi} + 2U^C\partial_C f + 2\partial_C(Y^C + I^C) - 4\tilde{\omega}. \tag{4.34}$$

And so we arrive at an exact solution for X^r:

$$\begin{aligned}X^r &= -r\left[X^u\partial_u\tilde{\varphi} + \frac{1}{2}U^C\partial_C f + \frac{1}{2}(\partial_C(X^C) + 2X^C\partial_C\tilde{\varphi}) - \tilde{\omega}\right]\\ &= -r\left[f\partial_u\tilde{\varphi} + \frac{1}{2}U^C\partial_C f + \frac{1}{2}\overline{D}_C X^C - \tilde{\omega}\right].\end{aligned}$$

The highest order terms of X^r are

$$[r^1]X^r = -r\partial_u f = -r\Big(f\partial_u\tilde{\varphi} + \frac{1}{2}\overline{D}_C Y^C - \tilde{\omega}\Big), \tag{4.35}$$

as a result of (4.26) *and* (4.31)*. The final constraint is* (4.16c)*. Using* (4.20) *the r^0-order part of X^r is*

$$[r^0]X^r = \frac{-r}{2}\Big([r^{-1}]\overline{D}_C X^C\Big) = \frac{1}{2}\overline{D}_C\overline{\gamma}^{CB}\partial_B f = \frac{1}{2}\overline{\gamma}^{CB}\overline{D}_C\partial_B f = \frac{1}{2}\overline{\gamma}^{CB}\partial_C\partial_B f. \tag{4.36}$$

In the last step we have used that $\overline{\gamma}^{CB}$ is anti-symmetric in BC, while the Christoffel symbol $\overline{\Gamma}^A_{BC}$ is symmetric in BC. To somewhat reduce clutter let us denote $h := [r^0]X^r$ in the remaining of the calculation. Note that

$$\begin{aligned}\partial_u h &= \frac{1}{2}\partial_u e^{-2\tilde{\varphi}}\gamma^{CB}\partial_C\partial_B f\\ &= -2h\partial_u\tilde{\varphi} + \frac{1}{2}\overline{\gamma}^{CB}\partial_C\partial_B\partial_u f.\end{aligned} \tag{4.37}$$

For the same symmetry reason, $\overline{\gamma}^{CB}\partial_C\partial_B\lambda = \overline{\Delta}\lambda$ for any scalar λ. In preparation for solving the last constraint equation, note that from (4.31) *it follows*

$$\begin{aligned}-2(h\partial_u\tilde{\varphi} + \partial_u h) &= 2h\partial_u\tilde{\varphi} - \overline{\Delta}\partial_u f\\ &= (\overline{\Delta}f)\tilde{\varphi} - \overline{\Delta}(f\partial_u\tilde{\varphi} + \tfrac{1}{2}\overline{D}_C Y^C - \tilde{\omega})\\ &= -f\partial_u\overline{\Delta}\tilde{\varphi} - \tfrac{1}{2}\overline{\Delta}\overline{D}_C Y^C + \overline{\Delta}\tilde{\omega}\end{aligned} \tag{4.38}$$

The final constraint is (4.16c)*. The Lie derivative of g_{uu} is*

$$\begin{aligned}\mathcal{L}_X g_{uu} &= X^\sigma\partial_\sigma g_{uu} + 2g_{u\sigma}\partial_u X^\sigma\\ &= X^\sigma\partial_\sigma g_{uu} + 2g_{uu}\partial_u X^u - 2\partial_u X^r + \mathcal{O}_\infty(r^{-1}).\end{aligned} \tag{4.39}$$

Using the previous order assessments, the derivatives of V/r have expansion

$$\begin{aligned}X^\sigma\partial_\sigma g_{uu} &= X^\sigma\partial_\sigma\Big(e^{2\beta}\frac{V}{r} + U_A U^A\Big)\\ &= X^\sigma\Big((\partial_\sigma\beta)e^{2\beta}\frac{V}{r} + e^{2\beta}\partial_\sigma\frac{V}{r} + \partial_\sigma U_A U^A\Big)\\ &= X^\sigma\partial_\sigma\frac{V}{r} + \mathcal{O}_\infty(r^{-1})\\ &= X^\sigma\partial_\sigma\big(-2r\partial_u\tilde{\varphi} + \overline{\Delta}\tilde{\varphi}\big) + \mathcal{O}_\infty(r^{-1})\\ &= -2rX^u\partial_u^2\tilde{\varphi} + X^u(\partial_u\overline{\Delta}\tilde{\varphi}) - 2X^r\partial_u\tilde{\varphi} - 2rX^A(\partial_A\partial_u\tilde{\varphi}) + X^A\partial_A\overline{\Delta}\tilde{\varphi} + \mathcal{O}_\infty(r^{-1})\\ &= -2r\Big[f\partial_u^2\tilde{\varphi} - (\partial_u f)\partial_u\tilde{\varphi} + Y^A(\partial_A\partial_u\tilde{\varphi})\Big]\\ &\quad + \Big[f(\partial_u\overline{\Delta}\tilde{\varphi}) - 2h\partial_u\tilde{\varphi} - 2rI^A(\partial_A\partial_u\tilde{\varphi}) + Y^A\partial_A\overline{\Delta}\tilde{\varphi}\Big]\\ &\quad + \mathcal{O}_\infty(r^{-1})\end{aligned} \tag{4.40}$$

The other two terms in (4.39) *have expansion*

$$\begin{aligned}g_{uu}\partial_u X^u - \partial_u X^r &= \big(-2r\partial_u\tilde{\varphi} + \overline{\Delta}\tilde{\varphi}\big)(\partial_u f) - \partial_u(-r\partial_u f + h) + \mathcal{O}_\infty(r^{-1})\\ &= -r\big[2\partial_u\tilde{\varphi}\partial_u f - \partial_u^2 f\big]\\ &\quad + \big[(\overline{\Delta}\tilde{\varphi})\partial_u f - \partial_u h\big]\\ &\quad + \mathcal{O}_\infty(r^{-1})\end{aligned} \tag{4.41}$$

Using (4.41) *and* (4.40)*, equation* (4.39) *becomes*

$$\begin{aligned}\mathcal{L}_X g_{uu} = -2r&\Big[f\partial_u^2\tilde{\varphi} - (\partial_u f)\partial_u\tilde{\varphi} + Y^A(\partial_A\partial_u\tilde{\varphi}) + 2\partial_u\tilde{\varphi}\partial_u f - \partial_u^2 f\Big]\\ &+ \Big[f(\partial_u\overline{\Delta}\tilde{\varphi}) - 2h\partial_u\tilde{\varphi} - 2rI^A(\partial_A\partial_u\tilde{\varphi}) + Y^A\partial_A\overline{\Delta}\tilde{\varphi} + 2(\overline{\Delta}\tilde{\varphi})\partial_u f - 2\partial_u h\Big]\\ &+ \mathcal{O}_\infty(r^{-1}).\end{aligned}$$

And so, by (4.16c), *we have constraints*

$$f\partial_u^2\tilde{\varphi} - (\partial_u f)\partial_u\tilde{\varphi} + Y^A(\partial_A\partial_u\tilde{\varphi}) + 2\partial_u\tilde{\varphi}\partial_u f - \partial_u^2 f = \partial_u\tilde{\omega} \tag{4.42a}$$

$$f(\partial_u\overline{\Delta}\tilde{\varphi}) - 2h\partial_u\tilde{\varphi} - 2rI^A(\partial_A\partial_u\tilde{\varphi}) + Y^A\partial_A\overline{\Delta}\tilde{\varphi} + 2(\overline{\Delta}\tilde{\varphi})\partial_u f - 2\partial_u h = -2\tilde{\omega}\overline{\Delta}\tilde{\varphi} + \overline{\Delta}\tilde{\omega} \tag{4.42b}$$

The left hand side of, (4.42a), *using* (4.31) *to write out the derivatives of* f,

$$\begin{aligned} &f\partial_u^2\tilde{\varphi} + Y^A(\partial_A\partial_u\tilde{\varphi}) + \partial_u\tilde{\varphi}\partial_u f - \partial_u[f\partial_u\tilde{\varphi} + \tfrac{1}{2}\overline{D}_C Y^C - \tilde{\omega}] \\ &= Y^A(\partial_A\partial_u\tilde{\varphi}) - \tfrac{1}{2}\partial_u(\partial_C Y^C + 2Y^C\partial_C\tilde{\varphi}) + \partial_u\tilde{\omega} \\ &= \partial_u\tilde{\omega}. \end{aligned}$$

where we used that $\partial_u Y^A = 0$, *and* (3.17). *So* (4.42a) *is automatically satisfied. As for* (4.42b), *we have left hand side*

$$\begin{aligned} &-2(h\partial_u\tilde{\varphi} + \partial_u h) + f(\partial_u\overline{\Delta}\tilde{\varphi}) - 2rI^A(\partial_A\partial_u\tilde{\varphi}) + Y^A\partial_A\overline{\Delta}\tilde{\varphi} + 2(\overline{\Delta}\tilde{\varphi})\partial_u f \\ &= -\tfrac{1}{2}\overline{\Delta}\overline{D}_C Y^C + \overline{\Delta}\tilde{\omega} + Y^A\partial_A\overline{\Delta}\tilde{\varphi} + 2(\overline{\Delta}\tilde{\varphi})\partial_u f \\ &= -\tfrac{1}{2}\overline{\Delta}(\partial_C Y^C + 2Y^C\partial_C\tilde{\varphi}) + Y^A\partial_A\overline{\Delta}\tilde{\varphi} + 2(\overline{\Delta}\tilde{\varphi})(f\partial_u\tilde{\varphi} + \tfrac{1}{2}\overline{D}_C Y^C - \tilde{\omega}) + \overline{\Delta}\tilde{\omega} \\ &= \overline{\Delta}\omega - 2\tilde{\omega}\overline{\Delta}\tilde{\varphi}, \end{aligned} \tag{4.43}$$

after substituting h *using* (4.38), *applying the assumption* (4.12) *that* $\partial_u\partial_A\tilde{\varphi} = 0$. *This expression,* (4.43) *coincides with the right hand side of* (4.42b). *Thus all of the constraints have been met.*

The solutions are

$$\begin{cases} X^u = f, & f = e^{\tilde{\varphi}}\big[T + \int_0^u du' e^{-\tilde{\varphi}}(\tfrac{1}{2}\overline{D}_C Y^C - \tilde{\omega})\big], \\ X^r = -r\big[f\partial_u\tilde{\varphi} + \tfrac{1}{2}U^C\partial_C f + \tfrac{1}{2}\overline{D}_C X^C - \tilde{\omega}\big], & \\ X^A = Y^A + I^A, & I^A = -\partial_B f\int_r^\infty dr' e^{2\beta} g^{AB}. \end{cases}$$

for arbitrary functions $T \equiv T(x^A)$, and where $Y^A \equiv Y^A(x^A)$ is a conformal Killing vector of $\overline{\gamma}_{AB}$. Although the full description of the vector fields does depend on the choice of the parameters in the metric (β, U^A, etc.), the fact that the solutions are generated by T and Y^A does not.

Remark *The algebra of vector fields obtained is this way is established using retarded time coordinate* $u = t - r$, *i.e., the vector fields live on* $\mathscr{I}^+$. *Therefore the associated transformation group is denoted* BMS$^+$, *and its algebra* $\mathfrak{bms}^+$. *Completely analogous, one might choose to solve the same equations, using advanced time coordinate* $v = t + r$, *to find the vector fields on* $\mathscr{I}^-$, *with group* BMS$^-$, *and algebra* $\mathfrak{bms}^-$.

4.3 The action of supertransformations

Infinitesimally, the action of a supertransformation is

$$X.g = g + \epsilon\mathcal{L}_X g.$$

The resulting action on the Schwarzschild metric is

Example: supertranslation of the Schwarzschild metric *The Schwarzschild metric, in the coordinates as above, is*

$$ds^2 = -\left(1 - \frac{2m_B}{r}\right) - 2dudr + r^2\frac{dzd\overline{z}}{P^2}.$$

In this case $g_{AB} = r^2\overline{\gamma}_{AB}$, *so* $\tilde{\omega} = \frac{1}{2}\overline{D}_C Y^C$; *in addition,* $\beta = U^A = 0$, *and* $V/r = -1 + 2m_B/r$. *So the supertransformations of the Schwarzschild metric, generated by* T, Y^A, *are of the form*

$$X^u = f, \qquad X^r = -\frac{r}{2}\overline{D}_A I^A, \qquad X^A = Y^A + I^A,$$

$$f = \frac{T}{P} \qquad\qquad I^A = -\frac{1}{r}\overline{\gamma}^{AB}\partial_B f.$$

In fact, using $\tilde{\varphi} = -\ln P$, and the the derivative of the Riemann sphere $\overline{D}$, we can simplify

$$\begin{aligned}\overline{D}_A I^A &= -\frac{1}{r}\overline{D}_A\overline{\gamma}^{AB}\partial_B\frac{T}{P} \\ &= -\frac{1}{r}\left(\partial_A\overline{\gamma}^{AB}\partial_B\frac{T}{P} - 2(\partial_A \ln P)\overline{\gamma}^{AB}\partial_B\frac{T}{P}\right)\end{aligned} \tag{4.44}$$

The first term in (4.44) simplifies to

$$\begin{aligned}\partial_A\overline{\gamma}^{AB}\partial_B\frac{T}{P} &= 2\partial_z P^2\partial_{\overline{z}}\frac{T}{P} + 2\partial_{\overline{z}}P^2\partial_z\frac{T}{P} \\ &= 2(\partial_z P^2)\partial_{\overline{z}}\frac{T}{P} + 2(\partial_{\overline{z}}P^2)\partial_z\frac{T}{P} + 4P^2\partial_z\partial_{\overline{z}}\frac{T}{P} \\ &= 4\Bigg[(\partial_z P)(\partial_{\overline{z}}T) + (\partial_{\overline{z}}P)(\partial_z T - 2T\frac{(\partial_z P)(\partial_{\overline{z}}P)}{P} \\ &\quad + P\partial_z\partial_{\overline{z}}T - (\partial_z T)(\partial_{\overline{z}}P) - (\partial_{\overline{z}}T)(\partial_z P) - T(\partial_z\partial_z P) + 2T\frac{(\partial_z P)(\partial_{\overline{z}}P)}{P}\Bigg] \\ &= 4P\partial_z\partial_{\overline{z}}T - 4T\partial_z\partial_{\overline{z}}P.\end{aligned} \tag{4.45}$$

Similarly, the second term in (4.44) simplifies to

$$\begin{aligned}-2(\partial_A \ln P)\overline{\gamma}^{AB}\partial_B\frac{T}{P} &= -4P^2\left((\partial_{\overline{z}}\ln P)\partial_z\frac{T}{P} + (\partial_z \ln P)\partial_{\overline{z}}\frac{T}{P}\right) \\ &= 4\left(-(\partial_z P)(\partial_{\overline{z}}T) - (\partial_{\overline{z}}P)(\partial_z T) + 2T\frac{(\partial_z P)(\partial_{\overline{z}}P)}{P})\right).\end{aligned} \tag{4.46}$$

Combining (4.45) and (4.46), we obtain

$$\overline{D}_A I^A = -\frac{4}{r}\left(P\partial_z\partial_{\overline{z}}T - T\partial_z\partial_{\overline{z}}P - (\partial_z P)(\partial_{\overline{z}}T) - (\partial_{\overline{z}}P)(\partial_z T) + 2T\frac{(\partial_z P)(\partial_{\overline{z}}P)}{P}).\right)$$

For $T = z^m\overline{z}^n$, and $P = \frac{1}{2}(1 + z\overline{z})$ then

$$-\frac{2}{1+z\overline{z}}\left(z^m\overline{z}^n - ((mz^{m-1} + (m-1)z^m\overline{z})(n\overline{z}^{n-1} + (n-1)z\overline{z}^n))\right)$$

The infinitesimal supertranslations, i.e., $Y^A = 0$ have (non-zero) Lie derivatives

$$\begin{aligned}\mathcal{L}_X g_{uu} &= \frac{2m_B}{r^2}\left(\frac{(1-z\overline{z})}{2P}T + z\partial_z T + \overline{z}\partial_{\overline{z}}T - 2P\partial_z\partial_{\overline{z}}T\right) \\ \mathcal{L}_X g_{uz} &= z\partial_z^2 T - 2P\partial_{\overline{z}}\partial_z^2 T + \frac{2m_B}{r}\left(\frac{\partial_z T}{P} - \frac{\overline{z}T}{2P^2}\right), \\ \mathcal{L}_X g_{zz} &= -\frac{2r\partial_z^2 T}{P},\end{aligned}$$

and similar for $\overline{z}$.

For a general metric, we calculate the an example superrotation in terms of the angular coordinates θ, ϕ.

***Example:** $Y^z(z) = az^2$ Let $Y^z(z) = az^2$ for some complex scalar $a \in \mathbb{C}$. Then $Y^{\overline{z}}(\overline{z}) = \overline{a}\overline{z}^2$. The resulting Asymptotic Killing vector field is*

$$X(u,r,z,\overline{z}) = \begin{pmatrix} -\dfrac{u(az+\overline{az})}{1+z\overline{z}} \\ \dfrac{(r+u)(az+\overline{az})}{1+z\overline{z}} \\ \dfrac{\overline{a}u + az^2(2r-u)}{2r} \\ \dfrac{au + \overline{az}^2(2r-u)}{2r} \end{pmatrix}.$$

Using (3.12), *the factor* $(az + \overline{az})$ *is written in terms of* θ, ϕ

$$\begin{aligned}(az + \overline{az}) &= \cot\tfrac{\theta}{2}\Big(ae^{i\phi} + \overline{a}e^{-i\phi}\Big)\\ &\cot\tfrac{\theta}{2}\Big(\operatorname{re} a(e^{i\phi} + e^{-i\phi}) + i \operatorname{im} a(e^{i\phi} - e^{-i\phi})\Big)\\ &= 2\cot\tfrac{\theta}{2}(\operatorname{re} a\cos\phi - \operatorname{im} a\sin\phi).\end{aligned}$$

Then, using the transformation rules for vector fiels, (3.11), *and* (3.13), *it is a straightforward calculation to find*

$$X(u,r,\theta,\phi) = \begin{pmatrix} -u\sin\theta(\operatorname{re} a\cos\phi - \operatorname{im} a\sin\phi)\\ (r+u)\sin\theta(\operatorname{re} a\cos\phi - \operatorname{im} a\sin\phi)\\ -(1+(1-\frac{u}{r})\cos\theta)(\operatorname{re} a\cos\phi - \operatorname{im} a\sin\phi)\\ \big(1-\frac{u}{r}+\cos\theta\big)\csc\theta(\operatorname{im} a\cos\phi - \operatorname{re} a\sin\phi)\end{pmatrix}.$$

4.4 Commutators

In this section, we calculate the commutators of the supertransformations. To calculate the algebra, we may safely ignore the r-component of the vector fields, because $\partial_r f = \partial_r Y^A = 0$, i.e., it has no implications on the determining functions T, Y. So we let $X \equiv X^m\partial_m$, where the index m runs over u, x^A. First, consider two supertranslations

$$[X_{T_1}, X_{T_2}] = \big[e^{\tilde{\varphi}}T_1\partial_u, e^{\tilde{\varphi}}T_2\partial_u\big] = 0$$

because $\partial_u T_i = 0$. Two superroations give rise to the bracket

$$\begin{aligned}[X_{Y_1}, X_{Y_2}] &= \big[f_1\partial_u + Y_1^A\partial_A, f_2\partial_u + Y_2^A\partial_A\big]\\ &= (f_1\partial_u + Y_1^B\partial_B)(f_2\partial_u + Y_2^A\partial_A) - (f_2\partial_u + Y_2^B\partial_B)(f_1\partial_u + Y_1^A\partial_A)\\ &= (Y_1^B\partial_B Y_2^A - Y_2^B\partial_B Y_1^A)\partial_A\end{aligned}$$

Next, consider a superrotation and a superrotation, i.e., X_1 determined by $Y_1 = 0$, and X_2 determined by $Y_2^A = 0$. Then we have

$$\begin{aligned}X_{Y_1} &:= X_1^m\partial_m = f_1\partial_u + Y_1^A\partial_A,\\ X_{T_2} &:= X_2^m\partial_m = e^{\tilde{\varphi}}T_2\partial_u,\end{aligned}$$

where f_1 is determined by Y_1 as above. The commutation relation is

$$\begin{aligned}[X_{Y_1}, X_{T_2}] &= \big[f_1\partial_u + Y_1^A\partial_A, e^{\tilde{\varphi}}T_2\partial_u\big]\\ &= f_1\partial_u e^{\tilde{\varphi}}T_2\partial_u + Y_1^A\partial_A e^{\tilde{\varphi}}T_2\partial_u - e^{\tilde{\varphi}}T_2\partial_u f_1\partial_u - e^{\tilde{\varphi}}T_2\partial_u Y_1^A\partial_A\\ &= Y_1^A e^{\tilde{\varphi}}(\partial_A T_2)\partial_u - e^{\tilde{\varphi}}T_2\big((\partial_u f_1) - f_1(\partial_u\tilde{\varphi}) - Y_1^A(\partial_A\tilde{\varphi})\big)\partial_u\\ &= Y_1^A e^{\tilde{\varphi}}(\partial_A T_2)\partial_u\end{aligned}$$

However, in order for the Lie-bracket to be faithful, it needs to account for the change induced in the metric by conformal vector fields (those containing nonzero Y^A), i.e., the variation induced by the transformation $\tilde{\varphi} \to \tilde{\varphi} + \tilde{\omega}$, which we denote $\delta_{\tilde{\omega}}$. In the above notation:

$$\begin{aligned}\delta_{\tilde{\omega}_1}X_2 &= \delta_{\tilde{\omega}_1}e^{\tilde{\varphi}}T_2\partial_u = \tilde{\omega}_1 e^{\tilde{\varphi}}T_2\partial_u,\\ \delta_{\tilde{\omega}_2}X_1 &= 0.\end{aligned}$$

Thus we modify the Lie-bracket to be the sum of the commutator and the variations.

$$\begin{aligned}[X_1, X_2]_M &= [X_1, X_2] + \delta_{\tilde{\omega}_2}X_1 - \delta_{\tilde{\omega}_1}X_2\\ &= Y_1^A e^{\tilde{\varphi}}(\partial_A T_2)\partial_u - \tfrac{1}{2}(\partial_A Y_1^A)e^{\tilde{\varphi}}T_2\partial_u\\ &= e^{\tilde{\varphi}}\big(Y_1^A(\partial_A T_2)\partial_u - \tfrac{1}{2}(\partial_A Y_1^A)T_2\big)\partial_u\end{aligned}$$

Now the bracket introduced above is established as a semi-direct sum

$$\mathfrak{vect}\,\mathbb{S}^2 \oplus_\Sigma \mathfrak{vect}\,\mathbb{S}^2_{ab}$$

with elements (Y^A, T) and bracket

$$[(Y_1^A, T_1), (Y_2^B, T_2)] = ([Y_1^A, Y_2^B], \Sigma_{Y_1^A} T_2 - \Sigma_{Y_2^B} T_1)$$

where $\Sigma_{Y^A} T = -\tilde{\omega} T$.

As seen in 3.4.3, the functions Y are conveniently expanded in terms of

$$Y = (\alpha_m z^{m+1} \partial_z, \overline{\alpha}_n \overline{z}^{n+1} \partial_{\overline{z}}) = (\alpha_m l_m, \overline{\alpha}_n \overline{l}_n).$$

It will turn out in 5.1.A, that if one chooses basis

$$l_m = -z^{m+1} \partial_z, \qquad \overline{l}_n = -\overline{z}^{n+1} \partial_{\overline{z}}$$

for the superrotations, ($Y^z = \sum \alpha_m l_m$, etc.) and

$$T_{j,k} = z^{j+\frac{1}{2}} \overline{z}^{k+\frac{1}{2}} (dz d\overline{z})^{-1/2},$$

for the supertranslations, the commutator algebra of Y and T, has is isomorphic to the vector field algebra with the modified bracket.

5. Central extension of the BMS-algebra

In this chapter, we study algebraic properties of a slightly generalized version of the BMS-algebra. Our purpose is to give a full description of possible central extensions.

Centrally extended algebras show up in physics as a result of a quantization process. The central charge c then, is an anomaly which occurs when the Weyl symmetry of a quantum theory is broken.

5.1 Representations of the Witt algebra

We have noted before, in 3.6.3, that the BMS-algebra is a semi-direct sum $\mathfrak{bms}_3 = \mathfrak{witt} \oplus \mathfrak{vect}_{ab}\,\mathbb{S}^1$, splitting over the Abelian algebra of supertranslations. Similarly, as seen in 4.4, $\mathfrak{bms}_4 = (\mathfrak{witt} \oplus \mathfrak{vect}_{ab}\,\mathbb{S}^1)\times(\mathfrak{witt} \oplus \mathfrak{vect}_{ab}\,\mathbb{S}^1)$, i.e., the four dimensional BMS-algebra consists of two independent copies of the Witt algebra acting on an Abelian algebra.

The Abelian normal subalgebra of the semi-direct sum, is isomorphic to a vector space V with trivial bracket, which will be established as a representation of $\mathfrak{witt}$ in the following section. Thus we generalize our study to (independent copies of) $\mathfrak{witt} \oplus$ V, where V is an arbitrary representation of $\mathfrak{witt}$, as suggested by Barnich and Oblak in [3]. In studying the algebra, we closely follow Kac and Raina's lectures [17], where they study similar properties of the Witt (and Virasoro) algebra, without the semi-direct Abelian term. This motivates the following lemma.

Lemma 5.1. *Let $V_{\alpha,\beta}$ denote the space of 'densities' of the form $P(z)z^\alpha(dz)^\beta$, with $\alpha,\beta\in\mathbb{C}$, and $P(z)$ some Laurent polynomial in $\mathbb{C}[z,z^{-1}]$. It is spanned by elements of the form*

$$\nu_k = z^{k+\alpha}(dz)^\beta, \qquad k\in\mathbb{Z}.$$

Then the action of $l_n \in \mathfrak{vect}(\mathbb{S}^1) \equiv \mathfrak{witt}$ on $v_k \in V_{\alpha,\beta}$ is given by

$$l_n(\nu_k) = -(k+\alpha+\beta+\beta n)\nu_{n+k}. \tag{5.1}$$

Proof *Let infinitesimal group elements $\gamma \in \mathrm{Diff}^+\,\mathbb{S}^1$ act on functions $f(z)$ via*

$$\rho_\gamma f(z) = f(\gamma^{-1}(z)),$$

Note that it has an expansion in terms of infinitesimals ϵ_n, i.e., $\gamma : z \mapsto z + \sum_n \epsilon_n z^n$, and so

$$\rho_\gamma f(z) = f(z - \sum_n \epsilon_n z^n) = f(z) + \sum_n \epsilon_n z^{n+1}\partial_z f(z) = (1+\epsilon_n l^n) f(z).$$

So we see that the action of $\gamma(z) = z + \epsilon_n z^n$ (no summation) corresponds to the (action of the) Lie algebra generator l^n. Then we find how infinitesimal elements act on this basis:

$$\begin{aligned}
\rho_\gamma \nu_k &= (\gamma^{-1}(z))^k + \alpha(d\gamma^{-1}(z))^\beta \\
&= \left(z - \sum_n \epsilon_n z^{n+1}\right)^{k+\alpha}\left(\left(1-\sum_n \epsilon_n (n+1) z^n\right)dz\right)^\beta \\
&= \left(z - (k+\alpha)\sum_n \epsilon_n z^n\right)\left(1-\beta\sum_n \epsilon_n(n+1)z^n\right)z^{k+\alpha}(dz)^\beta \\
&= \left(z - \sum_n (k+\alpha+\beta n+\beta)\epsilon_n z^n\right)z^{k+\alpha}(dz)^\beta
\end{aligned}$$

so that

$$l^n(\nu_k) = -(k+\alpha+\beta+\beta n)\nu_{n+k}.$$

The proof above uses the infinitesimal group action, rather than a direct calculation. Based on integration by parts, we derive the calculation rules for tensor densities below. If $y = y(x)$ we have

$$dy = \frac{dy}{dx}dx, \qquad \int u'(x)v(x)dx + \int u(x)v'(x)dx = u(x)v(x), \qquad \int vdu + \int udv = uv,$$

where the latter is just short hand notation for the second identity. Dropping the integral signs from our notation, the density $(dx)y\partial_x$ acts on an arbitrary test function $f = f(x)$ as

$$((dx)y\partial_x)f = ydf = yf - fdy = (y - y'(dx))f, \tag{5.2}$$

where we used that $\partial_z dz = dz\partial_z = 1$. Using the rule (5.2) for arbitrary tensor densities $(dz)^\beta$, we have, in accordance with Lemma 5.1,

$$\begin{aligned}[-z^{m+1}\partial_z, z^{k+\alpha}(dz)^\beta] &= -z^{m+1}\partial_z z^{k+\alpha}(dz)^\beta + z^{k+\alpha}(dz)^\beta z^{m+1}\partial_z \\ &= -(k+\alpha)z^{m+k+\alpha}(dz)^\beta - z^{k+\alpha}\beta(dz)^{\beta-1}(dz^{m+1}) \\ &= -(k+\alpha)z^{m+k+\alpha}(dz)^\beta - z^{k+\alpha+m}\beta(m+1)(dz)^\beta \\ &= -(k+\alpha+\beta(m+1))(dz)^\beta.\end{aligned}$$

The Lemma, applied to the BMS algebra in four dimensions, leads to the following corollary:

Corollary 5.1.A *In the four dimensional BMS-algebra* $\mathfrak{bms}_4$, *if the superrotations are generated by* $l^m = -z^{m+1}\partial_z$, *and* $\overline{l}^n = -\overline{z}^{n+1}$, *and the supertranslation by* $v_j = z^{j+\frac{1}{2}}(dz)^{-\frac{1}{2}}$, *and* $\overline{v}_k z^{k+\frac{1}{2}}(d\overline{z})^{-\frac{1}{2}}$, *then the superrotations have the adjoint action on the supertranslations:*

$$\mathfrak{bms}_4 = (\mathfrak{witt} \oplus_{ad} V_{\frac{1}{2},-\frac{1}{2}}) \times (\mathfrak{witt} \oplus_{ad} V_{\frac{1}{2},-\frac{1}{2}}).$$

5.2 Central Extensions

5.2.1 Quantum anomalies

Central extensions to an algebra are naturally in physics, when quantizing a classical symmetry algebra to an algebra of operators. In quantization the Poisson brackets are promoted to a bracket, i.e.,

$$\{A, B\} \to \frac{1}{i\hbar}[A, B].$$

Consider, for example an algebra of classical transformations that admit an expansion $\sum_n \alpha_n e^{in\theta}$, with Fourier modes α_n, (e.g., supertranslations (3.6)). In canonical quantization, these modes are promoted to quantum operators a_n, with commutation relation

$$[a_m, a_n] = m\delta_{m+n}.$$

The operators act on quantum states, denoted $|\psi\rangle$, increasing the energy $a_n^\dagger$ (for $n > 0$), or decreasing the energy a_n, (for $n > 0$). The Hermitian conjugation relation $a_n^\dagger = a_{-n} = \overline{a}_n$ is due to the reality condition on the Fourier expansion. In particular, the vacuum state, which has the lowest possible energy, is destroyed by the annihilation operators: $a_n|0\rangle = 0$.

Let $\tilde{L}_0$ be the operator

$$\tilde{L}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} a_{-n}a_n = \frac{a_0^2}{2} + \frac{1}{2}\sum_{n>0} a_n^\dagger a_n + \frac{1}{2}\sum_{n>0} a_n a_n^\dagger.$$

Acting on the vacuum will pose a problem; there are either infinitely many excitations, by the infinite sum over $a_n a_n^\dagger$. A way around this problem is by moving all the annihilation operators to the right, known as **normal ordering**. Doing so will lead to

$$\begin{aligned}\tilde{L}_0 &= \frac{1}{2}a_0^2 + \sum_{n>0} a_n^\dagger a_n + \frac{1}{2}\sum_{n>0}[a_n, a_{-n}] \\ &= \frac{1}{2}a_0^2 + \sum_{n>0} a_n^\dagger a_n + \frac{1}{2}\sum_{n>0} n.\end{aligned}$$

The infinite sum is shown to give $\sum_{n>0} n = -\frac{1}{12}$, for instance by analytical continuation of the Riemann zeta function. See e.g., [23]. Such a non-zero added constant, e.g. $-\frac{1}{24}$, is referred to as a **quantum anomaly**. The redefinition of $\tilde{L}_0$ to the normal ordered

$$L_0 := \frac{1}{2}a_0^2 + \sum_{n>0} a_n^\dagger a_n$$

gets rid of this anomaly. However, with the redefinition of the operator, the algebra changes too. As will become clear in 5.3, the algebra is effectively centrally extended by cancelling out the quantum anomaly.

Anticipating such a procedure for the BMS algebra, this section is dedicated to finding all possible central extensions of $\mathfrak{witt} \oplus V_{\alpha,\beta}$.

5.2.2 Central extension of the Witt algebra

First we consider the Witt algebra. We will extend the algebra $\mathfrak{g}$ to $\hat{\mathfrak{g}}$ by a 1-dimensional centre $c\mathbb{C}$. A priori we have the new relations

$$[l^m, l^n] = (m-n)l^{m+n} + f(m,n)c \qquad [l^m, c] = 0. \tag{5.3}$$

We will consecutively impose the Jacobi identity for l^0, l^m, l^n, antisymmetry of the bracket, and the Jacobi identity for general generators. Note first that, after a change of basis

$$l^0 \to l^0, \qquad l^n \to l^n - \frac{f(0,n)}{n}c \qquad (n \neq 0)$$

we obtain from (5.3), that

$$\left[l^0, l^n\right] = -n\left(l^n - \frac{f(0,n)}{n}c\right) + f(0,n)c = -nl^n. \tag{5.4}$$

Then, form the Jacobi identity we obtain

$$\begin{aligned}[l^0, [l^m, l^n]] &= [[l^0, l^m], l^n] + [l^m, [l^0, l^n]] \\ &= -(m+n)[l^m, l^n].\end{aligned} \tag{5.5}$$

Combining (5.3, 5.4, 5.5), we obtain

$$\begin{aligned}-(m+n)(m-n)l^{m+n} &= [l^0, (m-n)l^{m+n} + f(m,n)c] \\ &= [l^0, [l^m, l^n]] \\ &= -(m+n)(m-n)l^{m+n} + (m+n)f(m,n)c\end{aligned}$$

So $(m+n)f(m,n)c = 0$, implying $f(m,n) = \delta_{m,-n}f(m)$. Here $\delta_{m,-n} \equiv \delta_{m+n}$ denotes the Kronecker delta. Then (5.3) becomes

$$[l^m, l^n] = (m-n)l^{m+n} + \delta_{m,-n}f(m)c, \tag{5.6}$$

where, by anticommutativity $f(-m) = -f(m)$. Now consider for l^k, l^m, l^n

$$\begin{aligned}[l^k,[l^m,l^n]] &= (m-n)[l^k,l^{m+n}] + [l^k, \delta_{m,-n} f(m)c] \\ &= (m-n)(k-(m-n))l^{k+m+n} + (m-n)\delta_{k,-(m+n)} f(k)c \quad (+0) \\ &= (m-n)(k-m+n)l^{k+m+n} + (m-n)\delta_{k+m+n} f(k)c\end{aligned}$$

By the Jacobi identity the sum of cyclic permutations of these brackets must vanish. Note first that for any k, m, n:

$$(m-n)(k-m+n) + (n-k)(m-n+k) + (k-m)(n-k+m) = 0$$

So f must satisfy

$$\delta_{k+m+n}[(m-n)f(k) + (n-k)f(m) + (k-m)f(n)] = 0.$$

This is vacuously true unless $k+m+n$, in which case

$$(n-m)f(m+n) + (m+2n)f(m) - (n+2m)f(n) = 0. \tag{5.7}$$

In particular, for $n = 1$;

$$(1-m)f(m+1) = (1+2m)f(1) - (m+2)f(m). \tag{5.8}$$

We have arrived at a linear recursion relation, determined in full by $f(1), f(2)$, so the solution space of (5.7) is at most 2-dimensional. (Recall $f(0) = 0$ and $f(-m) = -f(m)$.) Note that $f(m) = c_1 m, f(m) = c_3 m^3$, satisfy (5.7), for any $c_1, c_3 \in \mathbb{C}$. So any central extension of the Witt algebra is given by

$$f(m,n)c = \delta_{m+n}(c_1 m + c_3 m^3)c.$$

The Virasoro algebra

The centrally extended Witt algebra (with $c_3 = -c_2 = \frac{1}{12}$) is called the **Virasoro algebra**:

$$\begin{aligned}\mathfrak{vir} &:= \mathbb{C}c + \sum_{n\in\mathbb{Z}} \mathbb{C}l^n, \\ [l^n, c] &= 0, \\ [l^m, l^n] &= (m-n)l^{m+n} + \delta_{m+n}\frac{m^3-m}{12}c.\end{aligned}$$

Proposition 5.2. *Every non-trivial central extension of the Witt algebra by a 1-dimensional centre is isomorphic to the Virasoso algebra $\mathfrak{vir}$.*

Proof *Consider a central extension where $c_3 = 0$, i.e., $f(m) = c_1 m$. Now consider the change of basis*

$$\text{(for all } k \neq 0), \qquad l'^k = l^k, \qquad l'^0 = l^0 + \frac{c_1}{2}.$$

As a result

$$(m-n)l^{m+n} = (m-n)\Big(l'^{m+n} - \delta_{m+n}\frac{c_1}{2}\Big) = (m-n)l^{m+n} + \delta_{m+n}c_1 mc.$$

So the bracket reduces to the non-centrally extended case

$$[l'^m, l'^n] = (m-n)l'^{m+n}.$$

We will refer to any such extension $f(m) = c_1 m$ as a ***trivial central extension****. The above calculation shows in particular that the value c_1 can be modified arbitrarily by a basis transformation. This means that we can always put $c_1 = -c_3$, so that $f(m) = c_3(m^3 - m)$. Finally, we can fix c_3 to be $\frac{1}{12}$ by scaling c.*

Corollary 5.2.A *If $[l^m, l^n] = (m-n)l^{m+n} + \delta_{m+n} f(m)c$ defines a Lie algebra, then $f(m) = c_1 m + c_3 m^3$, for some $c_1, c_3 \in \mathbb{C}$.*

5.2.3 BMS algebra and other representations of the Witt algebra

In this section we consider the algebra the Witt algebra action on the representation $V_{\alpha,\beta}$, via the adjoint action. So we have $\mathfrak{g} = (l^n, v_k) \in \text{Witt} \ltimes_{\text{ad}} V_{\alpha,\beta}$, with commutation relations

$$\begin{aligned}[][l^m, l^n] &= (m-n)l^{m+n} \\ [l^n, v_k] &= -(k+\alpha+\beta(n+1))v_{k+n} \\ [v_k, v_l] &= 0.\end{aligned}$$

We will extend the algebra $\mathfrak{g}$ to $\hat{\mathfrak{g}}$ by a 1-dimensional center $c\mathbb{C}$. A priori we have the new relations

$$\begin{aligned}[][l^m, l^n] &= (m-n)l^{m+n} + f(m,n)c \\ [l^n, v_k] &= -(k+\alpha+\beta(n+1))v_{k+n} + g(n,k)c \quad (5.9)\\ [v_j, v_k] &= h(k,l)c \quad (5.10)\end{aligned}$$

Now we need conditions such that the three 2-parameter functions f, g, h obey the Lie algebra axioms. Note that f is just as in the previous paragraph, i.e., $f(m,n) = \delta_{m+n}(m^3 - m)/12$.

Proposition 5.3. *For a general α, β, the central extension $g(m,k)c$ (of the Witt algebra acting on the representation $V_{\alpha,\beta}$) must satisfy relation*

$$(n-m)g(n+m) = (n(\beta-1)-m)g(m) - (m(\beta-1)-n)g(n), \quad (5.11)$$

where $g(n) = \delta_{n+k+\alpha+\beta}g(n,k)$.

Note that $g(m) = m$ satifies (5.11), for any α, β. We will call this the **trivial solution**.

Proof *First note that, as before, we make a basis transformation (for all $k \neq -(\alpha+\beta)$):*

$$v'_k = v_k - \frac{g(0,k)}{(k+\alpha+\beta)}c, \qquad v'_{-(\alpha+\beta)} = v_{-(\alpha+\beta)}.$$

So then we have

$$[l^0, v'_k] = -(k+\alpha+\beta)\left(v_k - \frac{g(0,k)}{(k+\alpha+\beta)}c\right) = -(k+\alpha+\beta)v'_k. \quad (5.12)$$

From the Jacobi identity we obtain (using equations (5.4,5.9,5.12))

$$\begin{aligned}[][l^0, [l^n, v_k]] &= [[l^0, l^n], v_k] + [l^n, [l^0, v_k]] \\ &= -(n+k+\alpha+\beta)[l^n, v_k] \\ &= (n+k+\alpha+\beta)((k+\alpha+\beta(n+1))v_{k+n} - g(n,k)c) \quad (5.13)\end{aligned}$$

Combing (5.9,5.12,5.13) we obtain

$$\begin{aligned}&(k+n+\alpha+\beta)(k+\alpha+\beta(n+1))v_{k+n} = \\ &\quad = [l^0, -(k+\alpha+\beta(n+1))v_{k+n} + g(n,k)c] \\ &\quad = [l^0, [l^n, v_k]] \\ &\quad = (k+n+\alpha+\beta)(k+\alpha+\beta(n+1))v_{k+n} - (n+k+\alpha+\beta)g(n,k)c\end{aligned}$$

So $(n+k+\alpha+\beta)g(n,k)c = 0$, inferring g is zero for $n+k+\alpha+\beta \neq 0$, and that g is fully determined by the first argument otherwise, i.e.,

$$g(n,k) = \delta_{n+k+\alpha+\beta}g(n),$$

so that (5.9) *becomes*

$$[l^n, v_k] = -(k+\alpha+\beta(n+1))v_{k+n} + \delta_{n+k+\alpha+\beta}g(n)c \quad (5.14)$$

The Jacobi identity can be written

$$[l^m, [l^n, v_k]] = [[l^m, l^n], v_k] + [l^n, [l^m, v_k]] \quad (5.15)$$

The c-dependent part of the left hand side of (5.15) *reads*

$$[\sim c][l^m,[l^n,v_k]] = -(k+\alpha+\beta(n+1))[\sim c][l^m,v_{k+n}]$$
$$= -\delta_{m+n+k+\alpha+\beta}(k+\alpha+\beta(n+1))g(m)c.$$

The c-dependent part of the right hand side of (5.15) *reads*

$$[\sim c][[l^m,l^n],v_k]+[l^n,[l^m,v_k]] = [\sim c](m-n)[l^{m+n},v_k]-[\sim c](k+\alpha+\beta(m+1))[l^n,v_{k+m}]$$
$$= \delta_{m+n+k+\alpha+\beta}((m-n)g(m+n)c-(k+\alpha+\beta(m+1))g(n)c)$$

The v_{k+m+n}-terms cancel by virtue of the Jacobi identity. For the central extension to satisfy the Jacobi identity as well, the c-dependent must obey

$$-\delta_{m+n+k+\alpha+\beta}(k+\alpha+\beta(n+1))g(m)c = \delta_{m+n+k+\alpha+\beta}(m-n)g(m+n)c$$
$$-\delta_{n+m+k+\alpha+\beta}(k+\alpha+\beta(m+1))g(n)c$$

For $k+m+n+\alpha+\beta \neq 0$ this is vacuously true. So let $k+m+n+\alpha+\beta = 0$. Then the constraint equation becomes

$$(n-m)g(m+n) = (n(\beta-1)-m)g(m)-(m(\beta-1)-n)g(n).$$

Note in particular the coincidence with (5.7)*, when $\beta = -1$.*

Corollary 5.3.A *Let α arbitrary. Then we have*

$$g(0) = 0, \qquad \text{(for any } \beta \neq 1)$$
$$g(-m) = -g(m), \qquad \text{(for any } \beta \neq 0,1)$$
$$2g(m) = g(2m), \qquad \text{(for any } \beta \neq -1,0,1)$$

In particular this implies there are no nontrivial solutions for $\beta \neq -1,0,1$.

Proof *Substitute $n = 0$ in* (5.11) *to obtain $m(\beta-1)g(0) = 0$. Now, substitute $n = -m$ in* (5.11) *to obtain $2mg(0) = m\beta g(m)+m\beta g(-m)$. Finally, substituting $n = -2m$,* (5.11) *becomes*

$$-3mg(-m) = m(1-2\beta)g(m)-m(\beta+1)g(-2m).$$

Now we apply $g(-m) = -g(m)$ for any $\beta \neq 0,1$. It follows

$$m(1+\beta)g(2m) = 2m(1+\beta)g(m).$$

Corollary 5.3.B *The central extensions are of the form*

$$g(m) = g_1 m + g_0, \qquad \text{(for } \beta = 1),$$
$$g(m) = g_1 m + g_2 m^2, \qquad \text{(for } \beta = 0),$$
$$g(m) = g_1 m + g_3 m^3, \qquad \text{(for } \beta = -1),$$
$$g(m) = g_1 m \qquad \text{(for any } \beta \neq -1,0,1).$$

for arbitrary constants $g_i \in \mathbb{C}$.

Proof *Plug the solutions into* (5.11) *to see that the satisfy the relation. The solutions must obey the recursion, (obtained by setting $n = 1$ in* (5.11)*),*

$$(1-m)g(m+1) = ((\beta-1)-m)g(m)-(m(\beta-1)-1)g(1),$$

which fully characterizes $g(m)$ by $g(1), g(2)$ for $m \geqslant 2$. For $\beta \neq 0,1$ the relation $g(-m) = -g(m)$ fixes the negative values also. For $\beta = 0,1$ plug in $m = \pm 1, n = \mp 2$ to obtain

$$(2\beta-1)g(\mp 1) = (\beta+1)g(\pm 2)-3g(\pm 1)$$

fixing $g(m)$ for $m \leqslant -1$. Finally, for $\beta = -1$, the value of $g(0)$ is determined by plugging in $m = 1, n = -1$:

$$2g(0) = g(1)+g(-1).$$

If follows that for all β the solution space for g is at most 2-dimensional. For $\beta \neq -1,0,1$, by corollary 5.3.A, the solution must be linear.

Central Extension of the Abelian bracket

Finally, we look for $h(j,k)$ as in (5.10). Consider first l^0 acting on arbitrary bracket. By the Jacobi identity:

$$[l^0,[v_j,v_k]] = [[l^0,v_j],v_k] + [v_j,[l^0,v_k]] \tag{5.16}$$

The left hand side of (5.16) gives, using that c commutes with everything,

$$[l^0,[v_j,v_k]] = [l^0,h(j,k)c] = 0.$$

The right hand side of (5.16) gives, applying (5.12),

$$[[l^0,v_j],v_k] + [v_j,[l^0,v_k]] = -(j+k+2(\alpha+\beta))[v_j,v_k] = -(j+k+2\alpha+2\beta)h(j,k).$$

So $h(j,k)$ must vanish for $j+k+2(\alpha+\beta)\neq 0$. So the central extension becomes

$$[v_j,v_k] = \delta_{j+k+2(\alpha+\beta)}h(j)c,$$

where have intruduced $h(j) := h(j,-j)$. Note that by antisymmetry of the bracket, $h(-j) = -h(j)$. Any such function must be a polynomial in odd powers $h_1j+h_3j^3+...+h_pj^p$. In particular, by basis transformation $v_j \to h_1v_j/(h_1+h_3j^2+...+h_pj^{p-1})$ it can always be cast in the form $h(j) = h_1j$. Next, we act with l^m, yielding once again by the Jacobi identity

$$[l^m,[v_j,v_k]] = [[l^m,v_j],v_k] + [v_j,[l^m,v_k]] \tag{5.17}$$

The left hand side of (5.17) gives 0, again using that c commutes with everything. The right hand side of (5.17) reads

$$\begin{aligned}[[l^m,v_j],v_k] + [v_j,[l^m,v_k]] &= -(j+k+2(\alpha+\beta(m+1)))[v_j,v_k]\\ &= -(j+k+2(\alpha+\beta(m+1)))\delta_{j+k+2(\alpha+\beta)}h(j)c.\end{aligned} \tag{5.18}$$

In the first line we ignored the central extension terms inside the bracket, as they vanish. Combining (5.17,5.18), and assuming $j+k+2(\alpha+\beta)=0$, we obtain constraint equation for $h(j)$:

$$-2\beta m h(j)c = 0. \tag{5.19}$$

This implies that for $\beta\neq 0$ no central extension $h(j,k)c$ exists. For $\beta = 0$, the central extension is

$$h(j,k)c = \delta_{j+k+2\alpha}h_1jc,$$

where $h(j) = h_1j$ up to a basis transformation. Note in particular that it is only non-trivial if $\alpha\in\frac{1}{2}\mathbb{Z}$.

5.2.4 Conculsion

For convenience, we summarize the result obtained in this section. The Witt algebra acting on $V_{\alpha,\beta}$ has central extension:

$$\begin{aligned}[l^m,l^n] &= (m-n)l^{m+n} + \delta_{m+n}f_mc_1\\ [l^m,v_k] &= -(k+\alpha+\beta(m+1))v_{k+m} + \delta_{m+k+\alpha+\beta}g(m)c_2\\ [v_j,v_k] &= \delta_{j+k+2\alpha}h(j)c_3,\end{aligned} \tag{5.20}$$

where

$$f(m) = f_1m + f_3m^3$$

$$g(m) = \begin{cases} g_1m+g_0, & \text{if } \beta = 1,\\ g_1m+g_2m^2, & \text{if } \beta = 0,\\ g_1m+g_3m^3, & \text{if } \beta = -1,\\ g_1m & \text{if } \beta\neq -1,0,1\end{cases}$$

$$h(j) = \begin{cases} h_1j, & \text{if } \beta = 0,\\ 0, & \text{if } \beta\neq 0,\end{cases}$$

5.3 Oscillator algebra

5.3.1 Oscillators

Let $\mathfrak{a}$ be the **oscillator**, or **Heisenberg algbra**, with basis $\{a_n | n \in \mathbb{Z}\} \cup \{\hbar\}$. and commutation relations

$$[\hat{\hbar}, a_n] = 0, \qquad \text{(for all } n \in \mathbb{Z}),$$
$$[a_m, a_n] = m\delta_{m+n}\hbar, \qquad \text{(for all } m, n \in \mathbb{Z}). \tag{5.21}$$

Since $[a_0, a_n] = 0$, we know a_0 is a central element (or zero mode) . We define an antilinear anti-involution $\cdot^\dagger$ on $\mathfrak{a}$:

$$a_n^\dagger = a_n, \qquad \hat{\hbar}^\dagger = \hat{\hbar}.$$

For $n > 0$, we will refer to a_n as **annihilation operators** and $a_n^\dagger$ as **creation operators**.

5.3.2 Oscillators acting on the Fock space

Let $\mathcal{F} = \mathbb{C}[x_1, x_2, ...]$ be the **Fock space**, the space of polynomials in infinite variables. Let $\mu, \hbar \in \mathbb{R}$ be arbitrary. Then we define a representation of $\mathfrak{a}$ on $\mathcal{F}$ by

$$a_n^\dagger = \frac{n\hbar}{\epsilon_n} x_n, \qquad a_n = \epsilon_n \partial_n, \qquad (n \in \mathbb{N})$$
$$a_0 = \mu, \qquad \hat{\hbar} = \hbar.$$

Here $\partial_n = \frac{\partial}{\partial x_n}$ denotes derivation with respect to the n-th variable. The ϵ_n denote arbitrary (real) scale factors. Note that the lower two operation are just multiplication by the given constants $\mu, \hbar$. Since δ_{m+n} is not an operator acting on the a's, they mutually commute. It follows that powers of a_n commute as

$$\begin{aligned}
[a_m^p, a_n] &= a_m^p a_n - a_n a^p \\
&= a_m^{p-1}[a_m, a_n] + a_m^{p-2}[a_m, a_n]a_m + ... + [a_m, a_n]a_m^{p-1} \\
&= p[a_m, a_n]a_m^{p-1}
\end{aligned} \tag{5.22}$$

Lemma 5.4. *If $\hbar \neq 0$ the above representation of $\mathfrak{a}$ is irreducible.*

> ***Proof*** *Any polynomial in $\mathcal{F}$ can be reduced to the constant polynomial, by repeated annihilation. Then any other polynomial can be obtained via repeated creation, as long as $\hbar > 0$.*

5.3.3 Fock space gradation and entropy

The **degree** of a monomial in $\mathcal{F}$ is defined to be

$$\deg x_1^{n_1} ... x_k^{n_k} := n_1 + 2n_2 + ... + kn_k.$$

In physical applications, the monomial $x_1^{n_1}...x_k^{n_k}$ represents n_1 oscillators in state 1, n_2 in state 2, etc. The powers n_j of a monomial in $\mathcal{F}$ are displayed in the form of a 'ket':

$$x_1^{n_1}...x_k^{n_k} \equiv |n_1, ..., n_k, 0, 0...\rangle \in \mathcal{F}.$$

Often the zeroes at the end are omitted. The constant polynomial is often abbreviated

$$|0, 0...\rangle \equiv |\Omega\rangle \equiv |0\rangle.$$

A number n_j at the j-th place is to be interpreted as 'n_j particles having energy j'. So the degree of a Fock state corresponds to its energy. The degeneracy $W(n)$ of a state with total energy (or degree) n corresponds to the entropy via $S(n) = k_B \ln W(n)$.

Let $\mathcal{F}_n$ be the subspace of $\mathcal{F}$ spanned by monomials of degree n. Then $\dim \mathcal{F}_n = P_n$, where P_n denotes the number of (unlabelled) partitions of $n \in \mathbb{Z}_{\geqslant 0}$ into sums of positive integers. (0 is defined to have 1 partition). For example, the 5 partitions of 4, and their corresponding monomials, are:

$$\begin{array}{lllll} 4, & 1+3, & 2+2, & 1+1+2, & 1+1+1+1, \\ x_1^4, & x_1x_3, & x_2^2, & x_1^2x_2, & x_1^4. \end{array}$$

The **principal gradation of** $\mathcal{F}$ is the decomposition

$$\mathcal{F} = \bigoplus_{n \geqslant 0} \mathcal{F}_n.$$

The z**-dimension** is of $\mathcal{F}$ is defined as

$$\dim_z \mathcal{F} := \sum_{n \geqslant 0} (\dim \mathcal{F}_n) z^n.$$

But the dimension of the subspace $\mathcal{F}_n$ is P_n, so that

$$\dim_z \mathcal{F} = \sum_{n \geqslant 0} P_n z^n = \prod_{n \in \mathbb{N}} \frac{1}{1 - z^n}.$$

By using a saddle point method for combinatorial classes, (see for instance [21, p.574]), one finds the large n-limit

$$P_n \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi \sqrt{\frac{2n}{3}}\right).$$

Then, using $W(n) = P_n$, the entropy has large n expression

$$S(n) \sim k\left(\sqrt{\frac{2n}{3}}\pi - \ln\left(4\sqrt{3}n\right)\right).$$

J. Cardy has calculated the same quantity in [22], $S \sim 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)}$, known as the **Cardy formula**, where c is the central charge and L_0 the energy.

5.4 Oscillator representations of $\mathfrak{vir}$

We define **normal ordering**, $: \cdot :$, of a pair $a_i a_j$ as

$$: a_i a_j := \begin{cases} a_i a_j, & \text{if } i \leqslant j, \\ a_j a_i, & \text{if } i > j, \end{cases}$$

and similar for higher numbers. So we effectively 'sort' the elements in increasing order of index, thus putting the creation operator to the left of the annihilation operators. For all $k \in \mathbb{Z}$ we define operator

$$L_m = \frac{1}{2} \sum_{j \in \mathbb{Z}} : a_{-j} a_{j+m} : . \tag{5.23}$$

Due to the normal ordering, when applied to any polynomial in $\mathcal{F}$, only a finite number of terms contribute. (Those for which the polynomial 'survives' the annihilation, before the creation operator has its turn). For example, when action on the mononial x_i^p:

$$L_k x_i^p = a_{k-i} a_i x_i^p = a_{k-i} \epsilon_i x_i^{p-1} = \begin{cases} \hbar \frac{\epsilon_i}{\epsilon_{k-i}} x_{k-i} x_i^{p-1}, & \text{if } k-i<0, \\ \mu \epsilon_i x_i^{p-1}, & \text{if } k-i=0, \\ 0, & \text{if } k-i>0 \end{cases} \tag{5.24}$$

Due to the commutation relation of the oscillators a_n, the normal ordering is only relevant when $m = 0$. So without normal ordering notation, the operators L_m can be written as

$$L_m = \frac{\eta_m}{2} a_{m/2}^2 + \sum_{j>-m/2} a_{-j} a_{j+m}, \qquad \eta_m = \begin{cases} 0, & \text{if } m \text{ is odd} \\ 1, & \text{if } m \text{ is even} \end{cases}$$

In particular the **energy operater** is given by

$$L_0 = \frac{\mu^2}{2} + \sum_{j>0} a_{-j} a_j.$$

In the remaining of this chapter, we will no longer distinguish the operators a_0 and $\hat{\hbar}$ from the scalars μ and $\hbar$ in our notation.

Proposition 5.5. *L_m acts on an oscillator as*

$$[L_m, a_k] = -k a_{m+k}. \tag{5.25}$$

Proof *First, let $m = 0$. Then we have (using (5.22))*

$$\begin{aligned} [L_0, a_k] &= \frac{1}{2}[a_0^2, a_k] + \sum_{j>0} [a_{-j} a_j, a_k] \\ &= 0 \delta_k a_0 + \sum_{j>0} -j \delta_{k-j} a_j + j \delta_{k+j} a_{-j} \\ &= \sum_{j \in \mathbb{Z}} j \delta_{k+j} a_{-j} \\ &= -k a_k \end{aligned}$$

Next, let $m \neq 0$. Then normal ordering doesn't matter and we simply have

$$\begin{aligned} [L_m, a_k] &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [a_{-j} a_{j+m}, a_k] \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} (a_{-j} [a_{j+m}, a_k] + [a_{-j}, a_k] a_{j+m}) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} (-k \delta_{j+m+k} a_{-j} - k \delta_{k-j} a_{j+m}) \\ &= -k a_{m+k} \end{aligned}$$

where we have used distributivity law $[AB, C] = A[B, C] + [A, C]B$ for the Lie bracket in the second line.

Proposition 5.6. *The operators $\{L_m\}_m$ satisfy commutation relations*

$$[L_m, L_n] = (m-n) L_{m+n} + \delta_{m+n} \frac{m^3 - m}{12} \hbar. \tag{5.26}$$

Thus, they provide a representation of the Virasoro algebra, with central charge $c_1 = \hbar$.

Proof *By making the transformation $L_k \to \hbar L_k$ we can reduce to the case $\hbar = 1$. So let $\hbar = 1$. Then, using distributivity law $[A, BC] = [A, B]C + B[A, C]$, and (5.25), we have*

$$\begin{aligned}
[L_m, L_n] &= \frac{\eta_n}{2}[L_m, a_{n/2}^2] + \sum_{j>-n/2} [L_m, a_{-j}a_{j+n}] \\
&= \frac{\eta_n}{2}\left([L_m, a_{n/2}]a_{n/2} + a_{n/2}[L_m, a_{n/2}]\right) + \sum_{j>-n/2} [L_m, a_{-j}]a_{j+n} + \sum_{j>-n/2} a_{-j}[L_m, a_{j+n}] \\
&= -\frac{n\eta_n}{4}\left(a_{n/2+m}a_{n/2} + a_{n/2}a_{n/2+m}\right) + \sum_{j>-n/2} j a_{m-j}a_{j+n} - \sum_{j>-n/2} (j+n)a_{-j}a_{j+m+n} \\
&= \frac{1}{2}\sum_{j+m>-n/2} (j+m)a_{-j}a_{j+m+n} + \frac{1}{2}\sum_{j+m\geqslant -n/2} (j+m)a_{-j}a_{j+m+n} \\
&\quad - \frac{1}{2}\sum_{j>-n/2} (j+n)a_{-j}a_{j+m+n} - \frac{1}{2}\sum_{j\geqslant -n/2} (j+n)a_{-j}a_{j+m+n} \\
&= \frac{1}{2}\sum_{k<n/2+m} (m-k)a_k a_{m+n-k} + \frac{1}{2}\sum_{j\geqslant -n/2-m} (j+m)a_{-j}a_{j+m+n} \qquad (5.27a) \\
&\quad - \frac{1}{2}\sum_{k<n/2} (n-k)a_k a_{m+n-k} - \frac{1}{2}\sum_{j\geqslant -n/2} (j+n)a_{-j}a_{j+m+n}. \qquad (5.27b)
\end{aligned}$$

In the fourth step, we have absorbed the odd center terms into the summation; the prefactors pose no problem here, because the factors j and $-(j+n)$ both have value $-n/2$ for $j = -n/2$. Additionally we have shifted the summation index $j \to j + m$ in the first two summations, allowing to collect the sums later.
The sums in equation (5.27) that run over k are normal ordered if $k \leqslant (m+n)/2$. Similarly, the sums that run over j are normal ordered if $j \geqslant -(m+n)/2$. In all other terms, the order is reversed, yielding commutators:

$$\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{k<n/2+m} (m-k) : a_k a_{m+n-k} : + \frac{1}{2}\sum_{j\geqslant -n/2-m} (j+m) : a_{-j}a_{j+m+n} : \\
&\quad + \frac{1}{2}\sum_{(n+m)/2<k<n/2+m} (m-k)[a_k, a_{m+n-k}] + \frac{1}{2}\sum_{-(m+n)/2>j\geqslant -n/2-m} (j+m)[a_{-j}, a_{j+m+n}] \\
&\quad - \frac{1}{2}\sum_{k<n/2} (n-k) : a_k a_{m+n-k} : - \frac{1}{2}\sum_{j\geqslant -n/2} (j+n) : a_{-j}a_{j+m+n} : \\
&\quad - \frac{1}{2}\sum_{(n+m)/2<k<n/2} (n-k)[a_k, a_{m+n-k}] - \frac{1}{2}\sum_{-(n+m)/2>j\geqslant -n/2} (j+n)[a_{-j}, a_{j+m+n}] \\
&= (m-n)\sum_{j\in\mathbb{Z}} : a_{-j}a_{j+m+n} : \\
&\quad - \frac{\delta_{m+n}}{2}\sum_{(n+m)/2<k<n/2+m} k(m-k) + \frac{\delta_{m+n}}{2}\sum_{-(m+n)/2>j\geqslant -n/2-m} j(j+m) \\
&\quad + \frac{\delta_{m+n}}{2}\sum_{(n+m)/2<k<n/2} k(n-k) - \frac{\delta_{m+n}}{2}\sum_{-(n+m)/2>j\geqslant -n/2} j(j+n) \\
&= (m-n)\sum_{j\in\mathbb{Z}} : a_{-j}a_{j+m+n} : \\
&\quad - \frac{\delta_{m+n}}{2}\sum_{0<k<m/2} k(m-k) + \frac{\delta_{m+n}}{2}\sum_{0>j\geqslant -m/2} j(j+m) \\
&\quad + \frac{\delta_{m+n}}{2}\sum_{0<k<-m/2} k(-m-k) - \frac{\delta_{m+n}}{2}\sum_{0>j\geqslant m/2} j(j-m) \\
&= (m-n)\sum_{j\in\mathbb{Z}} : a_{-j}a_{j+m+n} : + \frac{m^3-m}{12}.
\end{aligned}$$

5.4.1 Oscillator representation of $\mathfrak{vir} \ltimes \hat{V}_{0,0}$

In this section, we consider once again the Heisenberg operators a_m, where we have set $\hbar = 1$, and $a_0 = \mu$. Next, we consider the operators

$$\begin{aligned} \tilde{L}_0 &:= \frac{(\mu^2 + \lambda^2)}{2} + \sum_{j>0} a_{-j} a_j \\ \tilde{L}_m &:= \frac{1}{2} \sum_{j \in \mathbb{Z}} a_{-j} a_j + i\lambda m a_m, \qquad (m \neq 0) \\ \tilde{V}_k &:= a_k \end{aligned}$$

In terms of the previous operators L_m, they have expression

$$\begin{aligned} \tilde{L}_0 &= L_0 + \frac{\lambda^2}{2} \\ \tilde{L}_m &= L_m + i\lambda m \alpha_m. \qquad (m \neq 0) \end{aligned}$$

or, useful for some calculations

$$\tilde{L}_m = L_m + i\lambda m \alpha_m + \delta_m \frac{\lambda^2}{2} \tag{5.28}$$

Proposition 5.7. *The above operators satisfy the commutation relations*

$$\begin{aligned} [\tilde{L}_m, \tilde{L}_n] &= (m-n)\tilde{L}_{m+n} + \delta_{m+n} \frac{m^3 - m}{12} (1 + 12\lambda^2), \\ [\tilde{L}_m, \tilde{V}_k] &= -k\tilde{V}_{m+k} - i\lambda k^2 \delta_{m+k}, \\ [\tilde{V}_j, \tilde{V}_k] &= -k\delta_{j+k} \end{aligned} \tag{5.29}$$

Proof *For $[\tilde{L}_m, \tilde{L}_n]$ Using equations (5.21,5.26,5.28), we have*

$$[\tilde{L}_m, \tilde{L}_n] = [L_m, L_n] + i\lambda n [L_m, \alpha_n] + i\lambda m [\alpha_m, L_n] - \lambda^2 mn [\alpha_m, \alpha_n]$$

[stap te doen]. Next, we calcuate $[\tilde{L}_m, \tilde{V}_k]$. First, let $m = 0$. Then

$$[\tilde{L}_0, \tilde{V}_k] = [L_0, a_k] + \frac{\mu^2 + \lambda^2}{2} [1, a_k] = -k\tilde{V}_k,$$

coinciding with (5.29)*, for $m = 0$. Next, let $m \neq 0$. Then*

$$[\tilde{L}_m, \tilde{V}_k] = [L_m, a_k] + i\lambda m [a_m, a_k] = -k\tilde{V}_k - i\lambda k^2 \delta_{m+k}.$$

This too coincides with (5.29) *(again for $m \neq 0$). Finally, it follows directly that $[\tilde{V}_j, \tilde{V}_k] = j\delta_{j+k}$, by definition of the oscillators* (5.21)*.*

Corollary 5.7.A *The algebra is isomorphic to the centrally extended algebra $\hat{\mathfrak{witt}} \ltimes \hat{V}_{0,0}$,*

$$\begin{aligned} [l^m, l^n] &= (m-n) l^{m+n} + \delta_{m+n} \frac{m^3 - m}{12} c_1 \\ [l^m, v_k] &= -k v_{m+k} + \delta_{m+k} m^2 c_2 \\ [v_j, v_k] &= \delta_{j+k} j c_3 \end{aligned}$$

with central charges $c_1 = 1 + 12\lambda^2, c_2 = -i\lambda, c_3 = 1$.

5.5 Grassmann variables

For $\delta = 0$ (the Ramond sector) or $\delta = \frac{1}{2}$ (the Neveu-Schwarz sector) we define the space of Grassmann variables

$$G_\delta := \Lambda(\{\theta_m | m \in \mathbb{Z}_{\geqslant 0} + \delta\})$$

as the exterior algebra generated by the 'Grassmann numbers' θ_i. Wedges are (almost always) omitted from the notation. They satisfy

$$\theta_m \theta_n = -\theta_n \theta_m \tag{5.30}$$

for all m, n. As a result

$$\theta_m^2 = 0, \qquad \frac{\partial^2}{\partial \theta_m^2} = 0$$

From here, we introduce **fermionic oscillators**

$$\begin{aligned} \psi_m &:= \frac{\partial}{\partial \theta_m}, && \text{for } m > 0 \\ \psi_{-m} &:= \theta_m, && \text{for } m > 0 \\ \psi_0 &:= \frac{1}{\sqrt{2}}\left(\theta_0 + \frac{\partial}{\partial \theta_0}\right), \end{aligned}$$

the last of which occurs in the Ramon sector only. To see what it squares to, we let it act on a test function $a + b\theta_0$:

$$\psi_0^2 = \frac{1}{2}\left(\frac{\partial}{\partial \theta_0}\theta_0 + \theta_0 \frac{\partial}{\partial \theta_0}\right)(a + b\theta_0) = \frac{1}{2}\left(\frac{\partial a \theta_0}{\partial \theta_0} + \theta_0 \frac{\partial}{\partial \theta_0} b\theta_0\right) = \frac{1}{2}(a + b\theta_0) \tag{5.31}$$

So $\psi_0^2 = \frac{1}{2}$. In the case $m > 0, n > 0$ we have

$$\{\psi_m, \psi_{-n}\} \equiv \theta_m \frac{d}{d\theta_n} + \frac{d}{d\theta_n}\theta_m = \theta_m \frac{d}{d\theta_n} + \frac{d\theta_m}{d\theta_n} - \theta_m \frac{d}{d\theta_n} = \delta_{m+n}. \tag{5.32}$$

As a direct consequence of (5.30), (5.31), and (5.32), the fermionic operators satisfy the anticommutation relations

$$\{\psi_m, \psi_n\} \equiv \psi_m \psi_n + \psi_n \psi_m = \delta_{m+n}.$$

Define operators L_k in V_δ:

$$L_m = \delta_m \frac{1 - 2\delta}{16} + \frac{1}{2} \sum_{j \in \mathbb{Z} + \delta} j : \psi_{-j} \psi_{j+m} : . \tag{5.33}$$

Since the operators ψ_i anticommute, the normal ordering is defined by

$$: \psi_j \psi_k := \begin{cases} \psi_j \psi_k & j \leqslant k, \\ -\psi_k \psi_j & k < j. \end{cases}$$

Proposition 5.8. *They satisfy commutation relations*

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \delta_{m+n} \frac{m^3 - m}{24} \\ [L_m, \psi_k] &= -\left(k + \frac{m}{2}\right)\psi_{m+k} \\ [\psi_j, \psi_k] &= 2\psi_j \psi_k - \delta_{jk}. \end{aligned}$$

The proof is similiar to the harmonic oscillator case.

Representations of $V_{0,0}$ and $V_{-\frac{1}{2},\frac{1}{2}}$

By a simple modification by a Grassmann variable ϑ, i.e., $\vartheta^2 = 0$, the previously found algebras are modified to match (5.20). Let L_m be such that it has the Virasoro algebra bracket, e.g., as in (5.23). Let $V_k := L_k\vartheta$. Then

$$\begin{aligned} [L_m, V_k] &= [L_m, L_k]\vartheta = (m-k)V_{m+k} + \delta_{m+k}\frac{m^3-m}{12}\vartheta c_1, \\ [V_j, V_k] &= [L_j, L_k]\vartheta^2 = 0. \end{aligned}$$

The resulting algebra represents $\mathfrak{vir} \oplus V_{1,-1}$, with $c_2 = \vartheta c_1$.

Similarly, let L_m be as in (5.33). Let $V_k = \psi_k\vartheta$, where ϑ is a Grassmann variable. Then the bracket becomes

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \delta_{m+n}\frac{m^3-m}{24} \\ [L_m, V_k] &= -\Big(k + \frac{m}{2}\Big)V_{m+k} \\ [V_j, V_k] &= 0 \end{aligned}$$

This algebra coincides with $\mathfrak{vir} \oplus \hat{V}_{-\frac{1}{2},\frac{1}{2}}$, with central charge $c_1 = \frac{1}{2}$.

6. Conclusions and Outlook

In this thesis we have constructed the asymptotic symmetry group of spacetimes that are described by a Bondi metric, the BMS group. They classically give rise to waves propagating over the black hole horizon. The supertranslation currents give room to store information. The remaining problem is to match the Hawking-Bekenstein entropy $S = A/4$, which will likely be achieved by a cut-off procedure.

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