The edge reconstruction conjecture for graphs

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Abstract

In 1942 Kelly conjectured that any finite, simple, undirected graph having at least 3 vertices is uniquely determined by the multiset of all its subgraphs obtained by deleting a vertex and all edges adjacent to it. In 1964 Harary conjectured analogously that any graph having at least 4 edges is uniquely determined by all its subgraphs obtained by deleting a single edge, which is known as the edge reconstruction conjecture. Both conjectures are still open. In the first part of this thesis we will discuss some of the work done so far and provide some evidence in favour of the reconstruction conjectures. In the second part I will prove that a specific type of tridegreed graphs is edge-reconstructible, using techniques similar to those used by Myrvold, Ellingham and Hoffman to prove that any bidegreed graph is edge-reconstructible.

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Contents

1 Introduction

In this section we will introduce the main conjecture of this thesis. Even though this thesis is mainly about the edge reconstruction conjecture, it is natural to start with the vertex reconstruction conjecture. It is not only the basis for, but as we will show it even implies the edge reconstruction conjecture. Most of the results and statements in this chapter are from [Bon91].

By a graph we will mean a finite, undirected simple graph. That is, a graph G is an ordered pair (V, E) where V is a finite set and E is a set consisting of unordered pairs of (distinct) elements of V. The elements of V are called vertices and the elements of E are called edges. We will often refer to a graph by G, omitting (V, E) from the notation. Furthermore, an edge $\{u, v\} = e \in E$ is usually abbreviated as $e = uv$. We call u and v the endpoints of e. We will denote $v(G) = |V|$ for the number of vertices and $e(G) = |E|$ for the number of edges.

1.1 The vertex reconstruction conjecture

Given a graph G and a vertex $v \in V$, we write $G - v$ for the graph with vertex set $V - \{v\}$ and all the edges of G not having v as one of the endpoints. We call $G - v$ a **vertex-deleted subgraph** of G.

Definition 1.1. *Given a graph G, the multiset of all vertex-deleted subgraphs, up to isomorphism, is called the* ***deck*** *of G. An element of the deck of G is called a* ***card****.*

It is natural to wonder whether or not the set of all vertex-deleted subgraphs uniquely determines the graph G. This gives rise to the following definition.

Definition 1.2. *A graph H is called a* ***reconstruction*** *of G if the edge decks of G and H coincide. A graph G is* ***reconstructible*** *if any reconstruction H of G is isomorphic to G. An invariant of a graph is* ***reconstructible*** *if it is the same for all reconstructions H of G.*

In 1942 Kelly conjectured the following in his PhD thesis [Kel42] (page 73).

Conjecture (Reconstruction conjecture)**.** *Any graph having at least* 3 *vertices is reconstructible.*

Remark. The condition that the graph has at least 3 vertices is necessary, as we can see using the example $G = K_2$ and $H = 2K_1$, both of which have their deck equal to two copies of K_1. Clearly G and H are not isomorphic, showing that neither graph is reconstructible.

1.2 The edge reconstruction conjecture

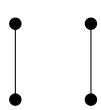
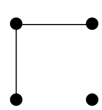
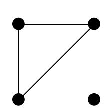
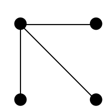
Analogous to before, an **edge-deleted subgraph** $G - e$ of G is the graph having the same vertex set as G and with edge set equal to $E - \{e\}$. Furthermore, we have the following analogous definitions.

Definition 1.3. *Given a graph G, the multiset of all edge-deleted subgraphs, up to isomorphisms, of G is called the* ***edge-deck*** *of G. An* ***edge-reconstruction*** *of G is a graph H such that G and H have the same edge-deck. We say that G is* ***edge-reconstructible*** *if any edge-reconstruction H of G is isomorphic to G. An invariant of a graph is* ***edge-reconstructible*** *if it is the same for all edge-reconstructions H of G.*

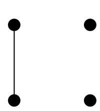
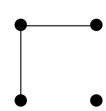
In 1964, Harary conjectured in [Har64] the following analogue of the reconstruction conjecture.

Conjecture (Edge-reconstruction conjecture)**.** *Any graph having at least* 4 *edges is edge-reconstructible.*

Remark. Again the condition on the number of edges is necessary, as can be seen by the following two examples.

 G_1  G_2  H_1  H_2

Here, both G_1 and G_2 have an edge deck consisting of two copies of G, while H_1 and H_2 have an edge deck consisting of three copies of H, where G and H are the graphs shown below.

 G  H

1.3 Overview of this thesis

This thesis consists of two parts. In the first part, which consists of Chapter 2 and Section 3.1 I will highlight some important results proved in the direction of the reconstruction conjectures. In particular I will show Kelly's lemma, Lemma 2.2, which implies that the degree sequence is reconstructible. Furthermore, Corollary 2.7 shows that disconnected graphs are reconstructible. Both of these facts will be used frequently in Chapter 4. After that, Section 3.1 will be devoted to probabilistic proof that almost all graphs are reconstructible.

In the second part I will show some of my own work on this topic. Firstly, in Section 3.2 I will generalize some of the techniques in Section 3.1 to show that with probability tending to 1 the number of automorphisms of a graph is reconstructible. In Chapter 4, which is the largest part of this thesis, I will briefly highlight techniques used by Myrvold, Ellingham and Hoffman in [MEH87] to show that bidegreed graphs are edge-reconstructible. After that, in Section 4.2 I will use similar techniques to show that tridegreed graphs in which all but two vertices have the same degree are edge-reconstructible. In Section 4.3 I will generalise these techniques and proofs to a next type of tridegreed graphs, showing edge-reconstructability in most of the cases.

2 Classical results

In this section we discuss some of the known results and used techniques. Furthermore, we will show that the deck of G is edge-reconstructible, showing that the reconstruction conjecture implies the edge-reconstruction conjecture. Again, most of the material is from [Bon91]. Many of the results will be proven in the setting of the reconstruction conjecture, but the formulations and proofs are easily adapted to the setting of the edge-reconstruction conjecture.

2.1 Counting arguments

Many classical results in the direction of the reconstruction conjecture are based on some counting argument. Before we continue with a simple but powerful lemma, we need one more definition.

Definition 2.1. *Given two graphs F and G, the number of subgraphs of G isomorphic to F is denoted by $s(F, G)$.*

Now we can state and prove the following lemma, which is Lemma 1 in [Kel57].

Lemma 2.2 (Kelly's lemma)**.** *Given two graphs F and G, satisfying $v(F) < v(G)$, we can reconstruct $s(F, G)$ from the deck of G.*

Proof. Consider any subgraph $\widetilde{F}$ isomorphic to F. This subgraph occurs in a vertex-deleted subgraph $G-v$ if and only if v does not belong to $\widetilde{F}$. Since there are precisely $v(G) - v(\widetilde{F}) = v(G) - v(F)$ such vertices, we find

$$s(F, G) = \frac{1}{v(G) - v(F)} \sum_{v \in V(G)} s(F, G - v). \qquad \square$$

As a consequence, we have the following.

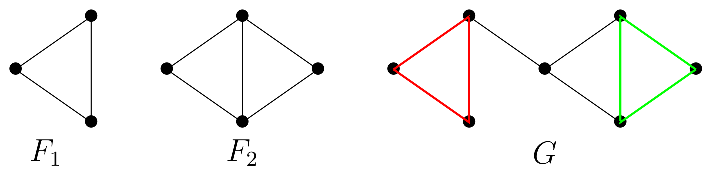
Corollary 2.3. *The number of edges and the degree sequence are reconstructible.*

Proof. For the first statement simply take $F = K_2$ in Kelly's lemma, which is allowed since $v(G) \geq 3 > v(F)$. The second statement follows from the fact that the degree of a vertex v equals the total number of edges minus the number of edges in $G-v$. $\square$

However, we can not only reconstruct the number of subgraphs isomorphic to a given graph. Under certain conditions, we can even construct the number of subgraphs that are maximal in the sense of the following definition.

Definition 2.4. *Let $\mathcal{F}$ be a collection of graphs and G a graph. An* ***$\mathcal{F}$-graph*** *is a member of $\mathcal{F}$. An* ***$\mathcal{F}$-subgraph*** *of G is a subgraph of G isomorphic to an $\mathcal{F}$-graph. Such a subgraph is called a* ***maximal $\mathcal{F}$-subgraph*** *if it is not contained in any other $\mathcal{F}$-subgraph. The number of maximal $\mathcal{F}$-subgraphs of G isomorphic to $F \in \mathcal{F}$ is denoted by $m(F, G)$, omitting the role of $\mathcal{F}$ for simplicity.*

Example. Consider the case in which $\mathcal{F}$ consists of the graphs F_1 and F_2 depicted below.



Then both the green and red triangle are $\mathcal{F}$-subgraphs of G. However, only the red triangle is a maximal $\mathcal{F}$-subgraph, as the green triangle is contained in the subgraph spanned by the four rightmost vertices, which is isomorphic to F_2.

Definition 2.5. *Let $\mathcal{G}$ be a collection of graphs. We say that $\mathcal{G}$ is recognisable if for any $G \in \mathcal{G}$ and any reconstruction H of G, we have $H \in \mathcal{G}$.*

We have the following lemma, the proof of which is again based on a counting argument, although it is more sophisticated than before. It was proven in 1973 in [GH73].

Lemma 2.6 (Greenwell-Hemminger)**.** *Let $\mathcal{F}$ be a collection of graphs and $\mathcal{G}$ a recognisable class of graphs such that for any $G \in \mathcal{G}$ and any $\mathcal{F}$-subgraph F of G the following two conditions are satisfied:*

(i) $v(F) < v(G)$;

(ii) *F is contained in a unique maximal $\mathcal{F}$-subgraph of G.*

Then $m(F, G)$ is reconstructible for any $F \in \mathcal{F}$ and any $G \in \mathcal{G}$.

Proof. From the second condition we deduce that

$$s(F, G) = \sum_{X \in \mathcal{F}} s(F, X) m(X, G),$$

since the left-hand side of this equation counts the number of subgraphs of G isomorphic to F and the right-hand side counts the same quantity by first considering the unique maximal $\mathcal{F}$-subgraph containing a given copy of F in G. This identity can be inverted to obtain

$$m(F,G) = \sum_{n=0}^{\infty} \sum (-1)^n s(F,X_1)s(X_1,X_2)\cdots s(X_{n-1},X_n)s(X_n,G),$$

where the inner sum runs over all n-tuples $F \subsetneq X_1 \subsetneq \ldots \subsetneq X_n$. Note that this is in fact a finite summation, since there are only finitely many possibilities for X_n such that $s(X_n, G) \neq 0$ and each X_n yields only a finite number of possible tuples. Now, the right-hand side of this equation is reconstructible, since for all possible X_i the first condition implies $v(X_i) < v(G)$, hence $s(X_i, G)$ is reconstructible by Kelly's lemma. Therefore, $m(F, G)$ is reconstructible. □

The Greenwell-Hemminger lemma has the following consequence.

Corollary 2.7. *Disconnected graphs are reconstructible.*

Proof. Note that a graph G is disconnected if and only if G has at most one vertex-deleted subgraph that is connected. Therefore, it is recognisable from the edge-deck whether or not G is disconnected. Now we apply the Greenwell-Hemminger lemma, taking $\mathcal{F}$ to be the collection of connected graphs on at most $v(G) - 1$ vertices and $\mathcal{G}$ the collection of disconnected graphs on $v(G)$ vertices. Clearly $v(F) \leq v(G) - 1 < v(G)$ for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ and by the above $\mathcal{G}$ is recognisable. Since the connected components of G are precisely the maximal $\mathcal{F}$-subgraphs of G, the above lemma shows that we can reconstruct the connected components of G, and hence G itself. □

2.2 Reconstructability implies edge-reconstructability

In this section we will show that the deck of a graph is edge-reconstructible. This will show that if a graph is reconstructible, it is also edge-reconstructible. The first step towards showing this result is to show that we only have to consider graphs without isolated vertices. Note that we cannot apply Corollary 2.7, since the proof presented there does not work for the case of edge-deleted subgraphs.

Lemma 2.8. *The number of isolated vertices is edge-reconstructible.*

Proof. By the edge version of Kelly's lemma we can deduce whether or not G contains a path or cycle of length 3. If it does not, we can also find out whether or not G contains a path of length 2 or not. Now, let m be the minimal number of isolated vertices in an edge-deleted subgraph of G. Clearly, G has at most m isolated vertices. If G contains a path or cycle of length 3, m also equals the number of isolated vertices of G, since removing the middle edge of the path, or one of the edges from the cycle will not increase the number of isolated vertices. Similarly, G has $m - 1$ isolated vertices if it contains a path of length 2 (but no path or cycle of length 3) and $m - 2$ isolated vertices otherwise. □

Corollary 2.9. *The edge-reconstruction conjecture holds if and only if it holds for all graphs without isolated vertices.*

Proof. It is clear that if the edge-reconstruction conjecture holds, it also holds for all graphs having no isolated vertices. For the reverse implication, let G be a graph and H a reconstruction of G. By the previous lemma, G and H have the same number of isolated vertices, so write $G = G' + nK_1$ and $H = H' + nK_1$, where G' and H' have no isolated vertices. Note that any edge of G is in fact an edge of G'. Since G and H have the same number of edges, there is no harm in labeling them with the same set in such a way that $G - e \cong H - e$ for all edges e. Now, by removing n isolated vertices from $G - e$ and $H - e$ we also find $G' - e \cong H' - e$. Therefore, if the edge-reconstruction conjecture holds for graphs without isolated vertices we can conclude $G' \cong H'$, showing $G \cong H$ as well. □

We can now deduce the desired result, using a proof quite similar to the proof of Corollary 2.7. Note that the condition that G has no isolated vertices is necessary to ensure $e(F) < e(G)$ for any $\mathcal{F}$-subgraph F of G.

Theorem 2.10. *Let G be a graph without isolated vertices. Then the deck of G is edge-reconstructible.*

Proof. Let $\mathcal{F}$ be the class of graphs having $v(G) - 1$ vertices and $\mathcal{G}$ the class of edge-reconstructions of G. Clearly $\mathcal{G}$ is edge-recognisable. Since G and its edge-reconstructions have no isolated vertices, we find $e(F) < e(G)$ for any $\mathcal{F}$-subgraph F of G. Furthermore, given an $\mathcal{F}$-subgraph F of G, it is contained in a unique maximal $\mathcal{F}$-subgraph of G, namely the subgraph of G induced by the vertices of F. Note that this equals the graph $G - v$ where v is the vertex not occurring in F.

Therefore, all conditions of the analogue of Lemma 2.6 for edge-deleted subgraphs are satisfied, showing that for any $F \in \mathcal{F}$, the number $m(F, G)$ is reconstructible.

Since $m(F, G)$ is equal to the number of vertex-deleted subgraphs of G isomorphic to F, we can reconstruct the deck of G in this way. $\square$

2.3 Graphs with many edges are edge-reconstructible

Intuitively, since the reconstruction conjecture implies the edge-reconstruction conjecture, the latter should be easier. Although both conjectures are still open, this is true in the sense that in the case of the edge-reconstruction conjecture there are more techniques and more results known. For example, it is known that graphs with a large number of edges are edge-reconstructible. The following theorem and proof are from [Mül77].

Theorem 2.11. *Let G be a graph with $v(G) \geq 6$ and $e(G) > v(G) \cdot (\log_2 v(G) - 1)$. Then G is edge-reconstructible.*

Remark. The condition $v(G) \geq 6$ does not appear in [Mül77], but as we will see below it is required for his proof to work.

Before we can give the proof we need one more definition.

Definition 2.12. *Let G and H be two graphs with $v(G) = v(H)$. Write $\langle G, H \rangle_r$ for the number of bijective maps $f : V(G) \to V(H)$ such that there are precisely r edges $xy \in E(G)$ such that $f(x)f(y) \notin E(H)$. We will call such a map f a **defect-r-homomorphism**. We abbreviate $\langle G, H \rangle = \langle G, H \rangle_0$, which in the case $e(G) = e(H)$ is equal to the number of isomorphisms from G to H.*

Proof of Theorem 2.11. Suppose that G is not edge-reconstructible. Then there exists a reconstruction H of G such that $G \ncong H$, in other words $\langle G, H \rangle = 0$. Write $\overline{G}$ for the complement of G, and abbreviate $m = e(G)$ and $n = v(G)$. We claim that

$$\langle H, \overline{G} \rangle = \langle \emptyset, G \rangle - \sum_{\substack{X \subseteq E(H) \\ |X|=1}} \langle X, G \rangle + \sum_{\substack{X \subseteq E(H) \\ |X|=2}} \langle X, G \rangle + \ldots + (-1)^m \langle H, G \rangle$$

and more generally

$$\langle H, \overline{G} \rangle_r = \sum_{\substack{X \subseteq E(H) \\ |X|=r}} \langle X, G \rangle - \sum_{\substack{X \subseteq E(H) \\ |X|=r+1}} \binom{r+1}{r} \langle X, G \rangle + \ldots + (-1)^{m-r} \binom{m}{r} \langle H, G \rangle.$$

Here, the terms in $\langle H, \overline{G}\rangle_r$ follow from the inclusion-exclusion principle. Clearly, defect-r-homomorphism $f : H \to \overline{G}$ correspond to bijective maps $f : V(H) \to V(G)$ such that f maps precisely r edges of H to edges of G. Suppose that this set of edges is given by $X \subseteq E(H)$, with $|X| = r$, then f induces a homomorphism $f : X \to G$, yielding the contribution $\langle X, G\rangle$. However, extending this homomorphism to a map $f : H \to G$ there could be more edges of H being mapped to edges of G. In particular, if this happens on some set $X' \subseteq E(H)$ having $r+1$ elements, we count this particular map for $\binom{r+1}{r}$ sets X, namely all r-element subsets of X'. This shows that we must subtract the second contribution. Considering the case where even more edges of H are mapped to edges of G we find the other terms in the summation.

Of course, we also have the same equations with H replaced by G. Now, since G and H have the same edge deck, we find that

$$\sum_{\substack{X \subseteq E(G) \\ |X|=r}} \langle X, G\rangle = \sum_{\substack{X \subseteq E(H) \\ |X|=r}} \langle X, G\rangle$$

for any $0 \le r < n$, since the multisets of graphs corresponding to $n-1$-element subsets of $E(G)$ and $E(H)$ coincide, so the same holds for the r-element subsets. Using $\langle H, G\rangle = 0$ we now find that

$$\langle G, \overline{G}\rangle_r - \langle H, \overline{G}\rangle_r = (-1)^{m-r}\binom{m}{r}\langle G, G\rangle$$

for any r. Therefore,

$$\begin{aligned} 2^m \le 2^m\langle G, G\rangle = \langle G, G\rangle \sum_{r=0}^{m}\binom{m}{r} &= \sum_{r=0}^{m} |\langle G, \overline{G}\rangle_r - \langle H, \overline{G}\rangle_r| \\ \le \sum_{r=0}^{m}\left(\langle G, \overline{G}\rangle_r + \langle H, \overline{G}\rangle_r\right) &= 2 \cdot n!, \end{aligned}$$

which is less than $2\left(\frac{n}{2}\right)^n$, provided that $n \ge 6$. We can we rewrite this as $m - 1 < n(\log_2 n - 1)$, contradicting our assumption $m > n(\log_2 n - 1)$. $\square$

2.4 Overview of known results

Of course, the results mentioned above are not the only ones. Below we will give a (certainly not exhaustive) list of results that have been shown.

Theorem 2.13. *The following are reconstructible:*

- *(a) Trees. [Kel57]*
- *(b) The number of spanning trees. [Tut79]*
- *(c) The number of perfect matchings. [Tut79]*
- *(d) The number of Hamilton cycles. [Tut79]*
- *(e) The characteristic polynomial. [Sac64] and [Tut79]*
- *(f) The chromatic polynomial and the chromatic number. [Whi32] and [Tut79]*
- *(g) Graphs having at most* 11 *vertices. [McK97]*

Theorem 2.14. *The following are edge-reconstructible.*

- *(a) Graphs with minimum degree* δ *and average degree* d *satisfying* $d < \delta + 1 - \frac{1}{\delta+1}$. *(Hoffmann 1977, unpublished)*
- *(b) Graphs without an induced subgraph isomorphic to* $K_{1,3}$. *[EPY88]*
- *(c) Graphs with maximal degree* Δ *and average degree* d *satisfying* $2\log_2(2\Delta) \leq d$. *[CNW82]*

3 Probabilistic results

In addition to the deterministic results in the previous section, there are also some probabilistic results in the direction of the reconstruction conjectures. Many results are shown with respect to the following way to generate random graphs.

Definition 3.1 (Erdös - Rényi model)**.** *For any positive integer n and $0 < p < 1$ we denote by $\mathcal{G}(n,p)$ the probability space of graphs on n vertices for which every pair of vertices is joined independently with probability p.*

3.1 Almost all graphs are reconstructible

Although the reconstruction conjectures have shown to be very hard, a probabilistic argument shows that the conjecture holds for almost all graphs. To be precise, we have the following theorem.

Theorem 3.2. *For any $n \in \mathbb{N}$, let p_n be the probability that a graph $G \in \mathcal{G}(n, 1/2)$ is reconstructible. Then $\lim_{n\to\infty} p_n = 1$.*

To prove this theorem, we need the following definition.

Definition 3.3. *Let k be a nonnegative integer. We say that a graph G with n vertices has property A_k if induced subgraphs of G on $n-k$ vertices are pairwise non-isomorphic. In other words, $G - X \ncong G - Y$ for every two distinct k-element subsets $X, Y \subseteq V(G)$.*

The following theorem shows that graphs with property A_3 are worth considering, see also [Bol90].

Theorem 3.4. *Suppose a graph G has property A_3, then G is reconstructible. In fact, G is reconstructible from any 3 vertex-deleted subgraphs.*

Proof. Let u, v and w be vertices of G and consider $G-u$, $G-v$ and $G-w$. We give a proof that G is reconstructible from these graphs. First we show that we can identify v inside $G-u$. Since G has property A_3, it also has property A_2, hence $(G-u)-x \cong (G-v)-y$ if and only if $\{u,x\}$ and $\{v,y\}$ coincide. As $u \neq v$ this implies $x = v$ and $y = u$, hence v is the unique vertex x of $G-u$ such that $(G-u)-x$ is isomorphic to some vertex-deleted subgraph of $G - v$.

Write $X = (G - u) - x$ and $Y = (G - v) - y$ where x and y are the vertices corresponding to v and u respectively. By assumption $X \cong Y$. We will show that there is in fact only one isomorphism from X to Y. Namely, suppose that $f_1, f_2 : X \to Y$ are two different isomorphisms and let $z \in V(X)$ be such that $f_1(z) \neq f_2(z)$. Then $Y - f_1(z) \cong X - z \cong Y - f_2(z)$, hence $G - \{v, y, f_1(z)\} \cong G - \{v, y, f_2(z)\}$ which contradicts property A_3. Therefore, we can label X and Y uniquely and from $G - u$ we can now determine all the neighbors of v inside G and add them to $G - v$. The only exception to this is that we do not know whether or not u and v are connected. However, u and v are both recognisable inside $G-w$, hence we can use this subgraph to find this out. $\square$

Remark. It can be shown that graphs with property A_2 are already reconstructible, which is a stronger result since every graph with property A_3 also has property A_2. However, the above proof also shows that one needs only 3 graphs from the deck, which shows the interesting phenomenon that in most cases very few information from the deck is used.

Furthermore, the subgraph $G - w$ is only used to find out whether or not u and v are connected. Therefore, if the degree sequence (or even the total number of edges) is known, any two subgraphs suffice to reconstruct G.

We will now show that almost every graph has property A_3, which in turn implies Theorem 3.2. A key part of the proof will be the following lemma.

Lemma 3.5. *Let G be a graph with vertex set V and let $W \subseteq V$. Denote $|W| = t$ and $|V| = n$ and let $\rho : W \to V$ be an injective function different from the identity function on W. Write $g = g(\rho)$ for the number of elements $w \in W$ such that $\rho(w) \neq w$. Then there exists a set I_ρ of unordered pairs of elements of W, such that I_ρ consists of at least $g(t-2)/6$ pairs and furthermore $I_\rho \cap \rho(I_\rho) = \emptyset$.*

Proof. Consider all unordered pairs $\{v, w\}$ of elements of W such that $\rho(v) \neq v$ or $\rho(w) \neq w$, of which there are $g(t-g) + \binom{g}{2}$. There are at most $g/2$ such pairs with $\{v, w\} = \{\rho(v), \rho(w)\}$, namely pairs with $\{v, \rho(v)\}$ with $\rho(\rho(v)) = v$. We divide by 2 since every pair $\{w, \rho(w)\}$ is counted for both $v = w$ and $v = \rho(w)$. Therefore, there are at least

$$g(t-g) + \binom{g}{2} - g/2 = g(t - g/2 - 1) \geq g(t/2 - 1)$$

pairs $\{v, w\}$ with $\{\rho(v), \rho(w)\} \neq \{v, w\}$. Let E_ρ be the set of these pairs. We define a graph H_ρ with vertex set E_ρ in which we connect $\{v, w\}$ with $\{\rho(v), \rho(w)\}$ (if $\{\rho(v), \rho(w)\}$ belongs to E_ρ, this might fail if for example $\rho(v)$ or $\rho(w)$ does not belong to W). Now every vertex in H_ρ has degree at most 2, since $\{v, w\}$ can only be connected to $\{\rho(v), \rho(w)\}$ and $\{\rho^{-1}(v), \rho^{-1}(w)\}$. A set I_ρ satisfying $I_\rho \cap \rho(I_\rho) = \emptyset$ now corresponds to an independent set of vertices in H_ρ.

As the degree of every vertex is at most 2, H_ρ consists of isolated vertices, paths and cycles. Therefore, in every connected component we can select at least one third of the vertices forming an independent set. This holds because we can select all isolated vertices, at least one half of the vertices of every path (by starting at an end and choosing vertices alternately) and at least one third in every cycle (again by starting at some vertices and choosing alternately: in case of a triangle we are only able to select one third of the vertices in this way). Therefore, H_ρ has an independent set of size at least $1/3|E_\rho| \geq 1/3g(t/2-1) = g(t-2)/6$, as required. $\square$

We are now ready to prove the following theorem.

Theorem 3.6 (Korshunov, Müller, Bollobás)**.** *Let k be a nonnegative integer and let $G \in \mathcal{G}(n, 1/2)$. Write p_n for the probability that there exists some subset $W \subseteq V(G)$ of size $|W| = n-k$ such that there is some injective* id $\neq \rho : W \to V$ *which is an isomorphism $\rho : G[W] \to G[\rho(W)]$. Then* $\lim_{n\to\infty} p_n = 0$*. In other words, the probability that G has property A_k tends to* 1*.*

Proof. First we fix $W \subseteq V(G)$ of size $n-k$. Note that there are $\binom{n}{k} \leq n^k$ ways to choose such a subset W. Write $t = n-k$, let $\mathrm{id} \neq \rho : W \to V$ be an injective function and let $g = g(\rho)$ be as in the above lemma. Given $1 \leq g \leq t$, there are at most n^{2g} such functions ρ, since ρ is determined by the $w \in W$ such that $\rho(w) \neq w$, for which there are $\binom{k}{g} \leq k^g \leq n^g$ options, and the values it attains for these w, which is also bounded above by n^g, since every w has $n-1 < n$ possible images.

Now write S_ρ for the event that ρ is an isomorphism $G[W] \to G[\rho(W)]$. Let I_ρ be the set constructed in the previous lemma, then for each $\{v, w\} \in I_\rho$ the event

$$vw \text{ and } \rho(v)\rho(w) \text{ are both edges or both nonedges}$$

has probability $1/2$. Furthermore, all those events are mutually independent, since all pairs involved are different. In order for S_ρ to be true, all of these events must hold, hence the probability that S_ρ occurs is at most

$$\mathbb{P}(S_\rho) \leq (1/2)^{|I_\rho|} \leq (1/2)^{g(t-2)/6}.$$

Therefore, for our given W the probability that we can find a $\mathrm{id} \neq \rho : W \to V$ yielding an isomorphism $G[W] \to G[\rho(W)]$ is given by

$$\sum_{\rho \neq \mathrm{id}} \mathbb{P}(S_\rho) = \sum_{g=1}^{t} \sum_{\rho : g(\rho) = g} \mathbb{P}(S_\rho) \leq \sum_{g=1}^{t} n^{2g} 2^{-g(t-2)/6} = \sum_{g=1}^{t} \left(n^2 2^{(2-t)/6}\right)^g$$

Now, since $t = n - k$ we find that $n^2 2^{(2-t)/6} = n^2 2^{(2+k-n)/6} < n^{-k-2}$ for n large enough. This shows that the above summation is bounded from above by

$$\sum_{\rho \neq \mathrm{id}} \mathbb{P}(S_\rho) \leq \sum_{g=1}^{t} (n^{-k-2})^g = n^{-k-2} \sum_{g=1}^{t} (n^{-k-2})^{g-1} \leq n^{-k-2} \sum_{g=1}^{t} 1 = tn^{-k-2} \leq n^{-k-1}.$$

Taking into account all possibilities for W we find that $p_n \leq n^k \cdot n^{-k-1} \leq n^{-1}$, showing $p_n \to 0$ for $n \to \infty$. □

3.2 Reconstruction of the number of automorphisms

In the proof of Theorem 2.11 we use the estimate $\#\mathrm{Aut}(G) = \langle G, G \rangle \geq 1$. One might wonder if one can use results about $\mathrm{Aut}(G)$ to improve on this bound, or at least say something about the reconstructability of $\#\mathrm{Aut}(G)$. Firstly we note that the case $k = 0$ of Theorem 3.6 shows that with probability tending to 1 we have $\#\mathrm{Aut}(G) = 1$, so in general we will not be able to improve on this bound. However, we can use techniques similar to above to say something about the reconstructability of $\#\mathrm{Aut}(G)$. First we prove the following lemma, which relates the number of automorphisms of G to properties of edge-deleted subgraphs of G.

Lemma 3.7. *For any graph G and any edge e of G write $x(G-e, G)$ for the number of edges we can add to $G - e$ to obtain a graph isomorphic to G. Then*

$$\#\mathrm{Aut}(G) \cdot x(G - e, G) = \#\mathrm{Aut}(G - e) \cdot s(G - e, G).$$

Proof. We will show that both sides of the equation count the number of labeled embeddings of $G - e$ in G. On the one hand, we can first choose $x(G - e, G)$ ways to extend $G - e$ to a graph isomorphic to G, after which we have $\#\mathrm{Aut}(G)$ ways to choose the images of the vertices of $G - e$. On the other hand, we can also first choose the subgraph of G to which we map $G - e$ in $s(G - e, G)$ ways, after which we have $\#\mathrm{Aut}(G - e)$ to embed $G - e$ into this particular subgraph. □

From this lemma we might deduce that $\min\{\#\mathrm{Aut}(G-e)\cdot s(G-e,G)\}$ is an upper bound for $\#\mathrm{Aut}(G)$. Since $\#\mathrm{Aut}(G-e)$ can be computed and $s(G-e,G)$ is reconstructible, this gives a reconstructible upper bound on $\#\mathrm{Aut}(G)$. Now we will prove that we in fact almost always have equality. The proof will be similar to the proof of Theorem 3.6.

Theorem 3.8. *Let $G \in \mathcal{G}(n, 1/2)$ be a graph. Then, with probability tending to 1, $x(G-e,G) = 1$ for all edges e of G. In particular, with probability tending to 1 we have $\#\mathrm{Aut}(G) = \#\mathrm{Aut}(G-e)\cdot s(G-e,G)$ for all edges e.*

Proof. Note that $x(G-e,e) \neq 1$ for some e if and only if there exists a bijection $\mathrm{id} \neq \rho : V \to V$ such that $\rho(u)\rho(v)$ is not an edge of G for exactly one edge uv of G.

Now, let $\mathrm{id} \neq \rho : V \to V$ be any bijection and let I_ρ be as in Lemma 3.5. There is no harm in assuming that I_ρ has precisely $k_g = g(n-2)/6$ elements. Again, for every $\{v, w\} \in I_\rho$ the probability the event

$$vw \text{ and } \rho(v)\rho(w) \text{ are both edges or are both nonedges}$$

has probability $\frac{1}{2}$, independently of all other events. Note that ρ can only satisfy the above conditions if there are at most two events for which this fails, hence the probability that ρ satisfies is bounded from above by $\binom{k_g}{2}2^{2-k_g} \leq k_g^2 2^{2-k_g}$.

Using similar estimates to those in Theorem 3.6 we may bound the probability of the existence of a bijection $\mathrm{id} \neq \rho : V \to V$ such that $\rho(u)\rho(v)$ is not an edge of G for exactly one edge uv of G by

$$\sum_{g=1}^{n} n^{2g} k_g^2 2^{2-k_g} = 4 \sum_{g=1}^{n} k_g^2 (n^2 2^{(2-n)/6})^g.$$

Using $k_g \leq gn \leq n^2$ and $n^2 2^{(2-n)/6} < n^{-6}$ for n large enough we can bound this by

$$4n^4 \sum_{g=1}^{n} n^{-6g} = 4n^{-2} \sum_{g=1}^{n} n^{-6(g-1)} \leq 4n^{-2} \sum_{g=1}^{n} 1 = 4n^{-1},$$

which goes to 0 when n goes to ∞. $\square$

4 Reconstruction of almost regular graphs

From Corollary 2.3 one can easily deduce the following.

Corollary 4.1. *Regular graphs are edge-reconstructible.*

Proof. By the above mentioned corollary it is recognizable whether or not a graph is k-regular. Now suppose we have a k-regular graph G and consider an edge-deleted subgraph $G - e$. This graph has precisely two vertices of degree $k - 1$, and all other vertices have the same degree, so in order to get a reconstruction of G one must add an edge between these two vertices. □

Of course, a similar proof also holds in the setting of vertex-deleted subgraphs. In this section we will consider graphs that are in some sense almost regular and discuss their reconstructability.

4.1 Reconstruction of bidegreed graphs

A bidegreed graph is a graph with two integers occurring in the degree sequence. In 1987 Myrvold, Ellingham and Hoffman proved the following theorem.

Theorem 4.2. *Bidegreed graphs are edge-reconstructible.*

The full proof can be found in their paper [MEH87], but we will highlight the first steps here. We will do this in order to show the analogy with the proof we will give in the next sections. Also, the entire proof is quite lengthy and technical and including the full proof will not be very enlightening.

The proof starts by assuming that we have a bidegreed graph G that is not edge-reconstructible. If the degrees of G differ by more than 1, a similar argument holds as in the case of regular graphs, so we may assume that the degrees are consecutive. Furthermore, if there is only one vertex of the smaller degree, the same argument holds again by considering an edge incident to this vertex. Therefore, we know that the degrees of G are d and $d + 1$ for some d and there are at least two vertices of degree d.

Also, by Corollary 2.7 we know that G must be connected. Finally we can also restrict to $d \geq 2$ since $d = 1$ will yield a path, which we know to be reconstructible. In the following, we will call vertices of degree $d-1$, d, $d+1$ tiny, small and large, respectively. For a bidegreed graph H, define $s(H)$ to be the smallest path between two small vertices.

Lemma 4.3. *For any edge reconstruction H of G we have $s(H) = s(G)$.*

Proof. Write $s = s(G)$ and consider a shortest path a_0, a_1, …, a_s between small vertices a_0 and a_s. Consider $G - a_0a_1$, in which vertex a_0 becomes tiny and vertex a_1 becomes small. Now, to obtain any reconstruction of G we must connect a_0 to a small vertex b. If $b \neq a_1, a_s$ the existence of the path $a_1 - a_s$ in H shows $s(H) < s(G)$ and if $b \in \{a_1, a_s\}$ the existence of the path $a_0 - a_1 - \ldots - a_s$ or $a_0 - a_s - \ldots - a_1$ respectively shows $s(H) \leq s(G)$. In all cases, $s(H) \leq s(G)$. The analogous argument with H and G shows the desired equality. □

Remark. In fact, the above proof shows that the only way to obtain an edge reconstruction of G is to join a_0 with a_1 or a_s, so there is at most one non-isomorphic reconstruction. Since we assumed that there is also at least one, there is exactly one non-isomorphic edge reconstruction, which we will denote by H. Furthermore, from now on we will write $s = s(G) = s(H)$.

Also, such a situation where there is only one possible replacement for a certain edge is called a **forced move**.

Now, let Γ be G or H and a_0, a_1, …, a_s be a shortest path between two small vertices in Γ. For π a permutation of $\{0, 1, 2, \ldots, s\}$ we write Γ_π for the graph obtained from Γ by deleting the edges $a_{i-1}a_i$ for $1 \leq i \leq s$ and adding the edges $a_{\pi(i-1)}a_{\pi(i)}$. It is readily checked that $(\Gamma_\pi)_\sigma = \Gamma_{\pi\sigma}$ for all π, σ. Let $\theta : i \mapsto i+1 \mod s+1$ and $\phi = (0, 1)$ be two permutations. Considering the forced moves described above one deduces the following.

Lemma 4.4. *If G_π is isomorphic to G, then $G_{\pi\theta}$ and $G_{\pi\theta^{-1}}$ are isomorphic to H.*

Proof. We will show this only for $G_{\pi\theta}$, the other proof is analogous. Consider the shortest path $a_{\pi(0)} - a_{\pi(1)} - \ldots - a_{\pi(s)}$. From the above observations, we know that replacing $a_{\pi(0)}a_{\pi(1)}$ by $a_{\pi(0)}a_{\pi(s)}$ is a forced move, hence the resulting graph is isomorphic to H. A careful consideration of this replacement shows that the resulting graph is equal to $G_{\pi\theta}$. □

Corollary 4.5. *The quantity s is odd.*

Proof. Since θ^{s+1} is the identity permutation, $G_{\theta^{s+1}}$ must be isomorphic to G. By repeated application of the above lemma we find that $s+1$ is even, hence s is odd. $\square$

In a similar vein, we find the following.

Lemma 4.6. *If G_π is isomorphic to G, then $G_{\pi\phi}$ is isomorphic to H.*

Proof. Delete the edge $a_{\pi(1)}a_{\pi(2)}$ from a shortest path $a_{\pi(0)} - a_{\pi(1)} - \ldots - a_{\pi(s)}$ in G_π. The resulting graph F has an path of length 1 between the small vertices $a_{\pi(0)}$ and $a_{\pi(1)}$ and a path of length $s-2$ between the small vertices $a_{\pi(2)}$ and $a_{\pi(s)}$. By reconstructability of the degree sequence, together with the fact that the shortest path between two small vertices must have length s, we find that a replacing edge must have one endpoint from the vertices $\{a_{\pi(0)}, a_{\pi(1)}\}$ and the other from $\{a_{\pi(2)}, a_{\pi(s)}\}$. Therefore, H is isomorphic to one of the three possible graphs. The first option is given

$$F + a_{\pi(1)}a_{\pi(s)} = G_{\pi\phi\theta^2},$$

which is isomorphic to $G_{\pi\phi}$ by the previous lemma. The second option is given by

$$F + a_{\pi(0)}a_{\pi(2)} = G_{\pi\phi},$$

whereas the last option is given by

$$F + a_{\pi(0)}a_{\pi(s)} = G_{\pi\theta^2},$$

which is isomorphic to $G_\pi \cong G$ by the above lemma. Therefore, we can rule out the last case and see that H is isomorphic to $G_{\pi\phi}$ in the two remaining cases. $\square$

Since θ and ϕ generate all permutations of $\{0, 1, \ldots, s\}$, and both permutations are odd, we can combine these results to deduce the following.

Lemma 4.7. *For any permutation π, $G_\pi \cong G$ if and only if π is an even permutation, and $G_\pi \cong H$ if and only if π is an odd permutation.*

With the help of this lemma we can find the exact value of s.

Lemma 4.8. *We have $s = 3$.*

Proof. Using the fact that $G_\pi = G$ for $\pi = (s, s-1, \dots, 1, 0)$ and calculating the sign of π, we find that $s \equiv 3 \mod 4$. Now, if $s > 3$ we can delete the edge a_3a_4 from a shortest path, and any possible replacing edge yields an even permutation, hence gives us back G. Therefore, G is reconstructible in that case, hence by assumption this case is not possible. $\square$

After these preliminaries the proof mostly consists of showing that G (and H) cannot contain various subgraphs as well as showing that they on the other hand must contain some other graphs as a subgraph. For the details we refer to [MEH87]. In their paper they also note that a slight adaptation of their proof also shows the following result.

Theorem 4.9. *Let G be a graph without three consecutive integers occurring in its degree sequence, then G is edge-reconstructible.*

4.2 Reconstruction of tridegreed graphs: a first case

In the view of Theorem 4.2, it is natural to try to prove that tridegreed graphs (that is, graphs with three integers in the degree sequence) are reconstructible, as this might be the first step towards an inductive approach. Since the theorem in the previous section generalizes for graphs without three consecutive integers occurring in their degree sequence, we will assume that we are considering tridegreed graphs with degrees d, $d+1$ and $d+2$ for some $d \geq 1$.

In [KR90] we have the following two theorems which could lead towards such an inductive approach.

Theorem 4.10. *A tridegreed graph with minimal degree at least 8 or average degree larger than $2\log_2(18)$ is edge-reconstructible.*

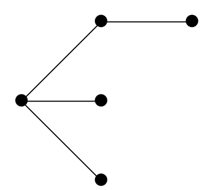
Theorem 4.11. *A fourdegreed graph with minimal degree at least 8 or average degree larger than $2\log_2(68)$ is edge-reconstructible.*

This nearly settles the case of tridegreed graphs, except for the fact that we have some condition on the degrees of the vertices. Below we will give a purely combinatorial proof of reconstructability in a specific case. The advantage of this proof is that it works for all possible degrees, but the disadvantage is that it still does not work for all tridegreed graphs.

Theorem 4.12. *Tridegreed graphs in which all but two vertices have the same degree are edge reconstructible.*

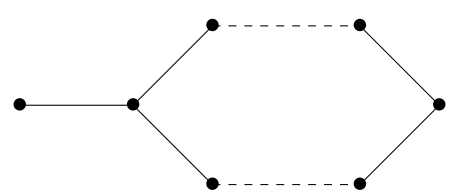
Proof. Suppose we have a graph G of this type which is not edge reconstructible.

Suppose we have more than 1 vertex of degree d. If two vertices of degree d are connected the edge joining them is a forced edge. Otherwise, any vertex of degree d is connected with at most 2 vertices (namely the vertices of degree $d+1$ and $d+2$). Therefore, $d \in \{1, 2\}$. If $d = 1$ connectivity and the degree sequence $1, 1, \ldots, 1, 2, 3$ implies that G must be isomorphic to the following graph.



If $d = 2$, the sum of the degrees of G is equal to $2 + 2 + \ldots + 2 + 3 + 4 \equiv 1 \mod 2$, contradicting that the sum of the degrees is twice the number of edges, so in particular even.

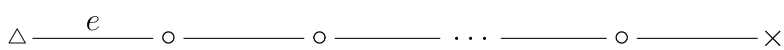
Now assume we have more than 1 vertex of degree $d + 1$. If the vertex of degree d is connected to any vertex of degree $d + 1$, the connecting edge is a forced egde. Otherwise, we must have $d = 1$, so the degree sequence $1, 2, 2, \ldots, 2, 3$ together with connectivity implies that G must be isomorphic to a cycle with one edge joined:



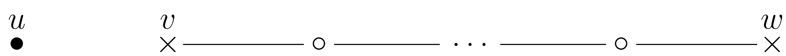
Now assume that we have more than 1 vertex of degree $d + 2$. In the remainder of this proof vertices of degree $d - 1$, d, $d + 1$ and $d + 2$ are denoted by $\bullet$, $\triangle$, $\times$ and $\circ$ respectively. As before, we may assume that the vertices $\triangle$ and $\times$ (which are now unique) are not connected. Write $s(H)$ for the length of the shortest path between the vertices $\triangle$ and $\times$, where H is any connected graph with degree sequence $d, d+1, d+2, d+2, \ldots, d+2$. Similarly to the proof of the bidegreed case, we have the following lemma.

Lemma 4.13. *G has a unique edge reconstruction $H \not\cong G$ and $s(H) = s(G)$.*

Proof. Consider any path

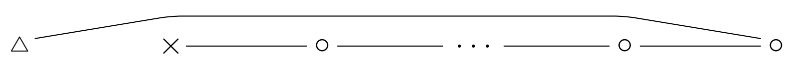


of minimal length and let e be the denoted edge. In $G - e$ the above becomes



Now, to obtain an edge reconstruction of G it is clear that we must add an edge between a vertex of degree $d-1$ and a vertex of degree $d+1$. Therefore, there are at most two options, namely uv and uw. Since G is not reconstructible, there are also at least two options, so there are precisely two options.

Now, the unique edge reconstruction H contains



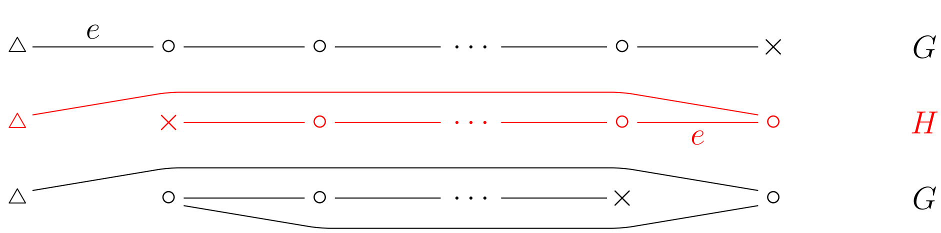
showing that $s(H) \leq s(G)$. Completely analogously we obtain the reverse inequality, so we have in fact equality. ♣

Write $s = s(G) = s(H) \geq 2$ and consider a shortest path $\triangle = a_0, a_1, \ldots, a_s = \times$ in G. For $\pi \in S_n = S_{\{1,2,\ldots,s\}}$ we write G_π for the graph obtained by deleting $a_i a_{i+1}$ for $1 \leq i \leq s-1$ and adding $a_{\pi(i)} a_{\pi(i+1)}$ instead.

First assume that $s \geq 4$. Then we have the following.

Lemma 4.14. *For $\pi = (s, \ldots, 2, 1)$ we have $G_\pi \cong G$.*

Proof. This is immediate from the following sequence of two forced moves.

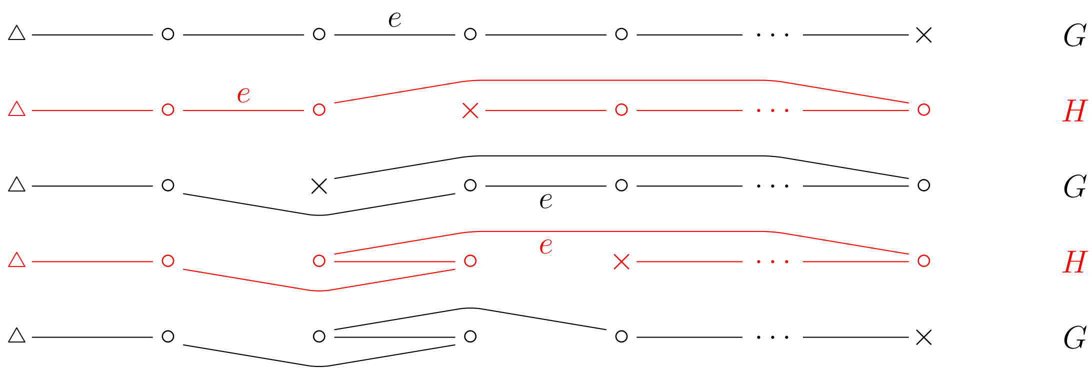


Here, the second forced moves follows since the edge on has to add must join two vertices of degree $d+1$ (after deletion of e), hence it must join two of a_1, a_{s-1}, a_s. If it joins a_{s-1} and a_s we get back H and if it joins a_1 and a_{s-1} the new graph contains a shorter path between the vertices of degree d and $d+1$ (in this case a path of length 1). Therefore, we must replace e by $a_1 a_s$. ♣

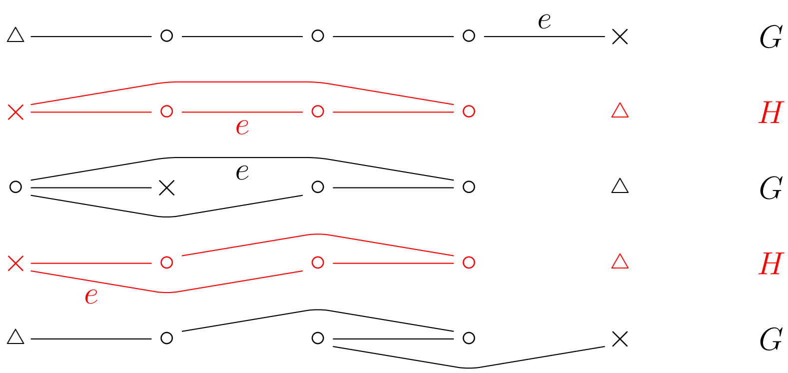
Analogously, we can find the following.

Lemma 4.15. *For* $\pi = (2,3)$ *we have* $G_\pi \cong G$.

Proof. When $s \geq 5$ we can consider the following series of 4 forced moves showing the required. In every step, the move is forced due to reasons completely analogous to those used in the previous lemma.



When $s = 4$ we have the following sequence of forced moves.

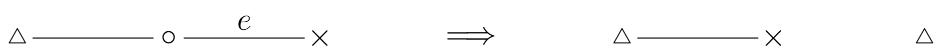


Here, the first and last forced moves occur since (after deleting the edge) we have two vertices of degree d and one of $d+1$, so the new edge must join the vertex of degree

$d+1$ with the other vertex of degree d. The two forced moves inbetween follow since after deleting the edge we have three vertices of degree $d+1$ and we need to connect two of them. In both cases one pair is already connected and the other pair is just deleted, so we must add the third pair. ♣

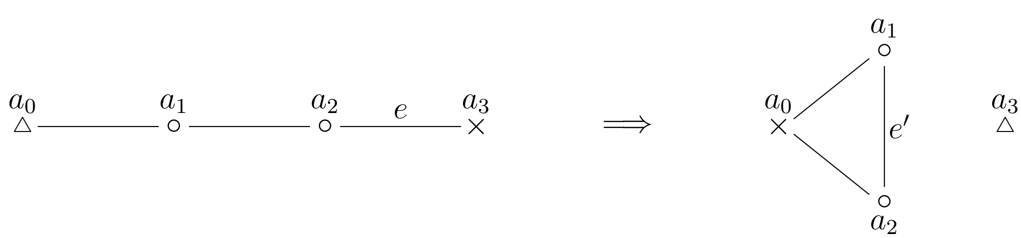
Now, since $(s, \ldots, 2, 1)$ and $(2, 3)$ generate S_s we find $G_\pi \cong G$ for any $\pi \in S_s$. However, the second graph drawn in Lemma 4.14 is of the form G_π for some π, hence $H \cong G_\pi \cong G$, a contradiction.

Therefore, we only need to consider $s = 2$ and $s = 3$. However, if $s = 2$ deleting the designated edge e from a shortest path gives the following situation.



Now we must join a vertex of degree d with a vertex of degree $d+1$. Since we have only two possible pairs and one pair is already joined, the edge e is forced.

If $s = 3$ we have the forced move shown below.



Now, deletion of the edge e' in the graph on the right gives a graph with 3 vertices of degree $d+1$ of which we have to connect 2. However, two of the three pairs are already joined, so e' is a forced edge, showing that the rightmost graph is uniquely edge reconstructible. □

4.3 Reconstruction of tridegreed graphs: a second case

In this section we will focus on tridegreed graphs with a unique vertex of degree d, two vertices of degree $d+1$ and the remaining vertices of degree $d+2$. We will try to apply the above techniques to this specific case. As before, denote the vertices of degree $d-1$, d, $d+1$ and $d+2$ by $\bullet$, $\triangle$, $\times$ and $\circ$. Suppose we have a non-reconstructible graph G and write $s(H)$ for the length of the shortest path from the vertex of degree d to one of the vertices of degree $d+1$, where H is any graph having the specified degree sequence.

Lemma 4.16. *For any edge-reconstruction H of G we have $s(H) = s(G)$.*

Proof. Consider the edge-reconstruction H with $s(H)$ minimal. We know that the vertex of degree d cannot be connected to a vertex of degree $d+1$, so we know $s(H) \geq 2$. If $s(H) = 2$ we have the leftmost situation and deletion of the indicated edge e gives the rightmost situation.

$$\triangle \overset{e}{\rule{1.5cm}{0.4pt}} \circ \rule{1.5cm}{0.4pt} \times \qquad \Longrightarrow \qquad \bullet \qquad\qquad \times \rule{1.5cm}{0.4pt} \times$$

Now we note that it can be seen from the edge deck whether or not the two vertices of degree $d+1$ in H are connected, since they are connected if and only if there is some edge-deleted subgraph. Since the degree sequence is reconstructible, it is clear that we must connect the vertex of degree $d-1$ with some vertex of degree $d+1$. If the two vertices of degree $d+1$ are not connected it is clear that we must join the vertex of degree $d-1$ with one of the two drawn vertices of degree $d+1$, giving us a reconstruction H' with some path of length 2. By minimality $2 = s(H) \leq s(H') \leq 2$, showing equality. If the vertices of degree $d+1$ are connected in the original graph, the edge-deleted subgraph contains the following situation

$$\bullet \qquad\qquad \times \rule{1.5cm}{0.4pt} \times \rule{1.5cm}{0.4pt} \times$$

It is clear that joining the vertex of degree $d-1$ with any of the vertices of degree $d+1$ yields a path of length 2 between the vertex of degree d and a vertex of degree $d+1$, again showing that any reconstruction H' of H has $s(H') = 2$ as well.

In the case of $s(H) \geq 3$ we can have a similar argument, considering deletion of the edge e indicated below.

$$\triangle \rule{1.5cm}{0.4pt} \cdots \rule{1.5cm}{0.4pt} \circ \overset{e}{\rule{1.5cm}{0.4pt}} \circ \rule{1.5cm}{0.4pt} \times$$

$$\Downarrow$$

$$\triangle \rule{1.5cm}{0.4pt} \cdots \rule{1.5cm}{0.4pt} \times \qquad\qquad \times \rule{1.5cm}{0.4pt} \times$$

From the degree sequence it follows that we must join two vertices of degree $d+1$. By minimality of $s(H)$, one of which must be the leftmost of the three in the above situation. Now, if the vertices of degree $d+1$ are not connected, we must join this vertex with one of the two leftmost vertices, yielding a path of length $s(H)$ in the reconstruction H', which similarly to above suffices to prove the lemma. Also, if the two vertices of degree $d+1$ are connected in the original graph we can apply the same reasoning as above. $\square$

As before, write $s = s(G)$, hence $s = s(H)$ for any edge-reconstruction H of G. Note that the above proof also shows that in the case the two vertices of degree $d+1$ are not connected we have a unique edge-reconstruction H of G with $H \not\cong G$. Let us first handle most of the cases where the two vertices of degree $d+1$ are connected.

Lemma 4.17. *Suppose that in G the two vertices of degree $d+1$ are connected and that $s \geq 3$. Then G is edge-reconstructible.*

Proof. Consider a shortest path of length s and remove the edge e indicated below.

$$\triangle \overset{e}{\text{———}} \circ \text{———} \cdots \text{———} \circ \text{———} \times \text{———} \times$$

$$\Downarrow$$

$$\bullet \qquad\qquad \times \text{———} \cdots \text{———} \circ \text{———} \times \text{———} \times$$

From reconstructability of the degree sequence, we see that we must join the vertex of degree $d-1$ with one of the vertices of degree $d+1$. However, if we connect it with one of the two rightmost vertices, we get a path of length 2 from the vertex of degree d to a vertex of degree $d+1$, contradicting the fact that $s \geq 3$. Therefore, the only way to get an edge-reconstruction of G is to draw e again, showing that G is edge-reconstructible. $\square$

Unfortunately, the above proof misses the case $s = 2$. From now on, we will restrict to graphs where the vertices of degree $d+1$ are not connected. Write $m(G)$ for the length of the shortest path from any path $\triangle = a_0 - a_1 - \ldots - a_{s-1} - a_s = \times$ of length s to the vertex of degree $d+1$ not on this path. We will first show that the parameter $m(G)$ is reconstructible. Remember that we assumed that the vertices of degree $d+1$ are not connected, hence we have a unique nonisomorphic edge-reconstruction H of G.

Lemma 4.18. *We have $m(H) = m(G)$.*

Proof. When proving $s(H) = s(G)$ in Lemma 4.16 we have shown that H can be obtained from G by deleting some edge from the shortest path $a_0 - a_1 - \ldots - a_s$ and inserting some edge $a_i a_j$. Since the shortest path of length $m(G)$ does not contain the deleted edge, we still have a path of length $m(G)$ from the vertex of degree $d+1$ to the path of length s (which consists of the vertices a_0, a_1, …, a_s in some order), showing $m(H) \leq m(G)$. Analogously, we have the reverse inequality, showing the desired equality. □

Therefore, from now on, we will write $m = m(G) = m(H)$. Note that by definition the length of the shortest path between the two vertices of degree $d + 1$ is at least m. The following lemma shows that at least one of s and m must be small.

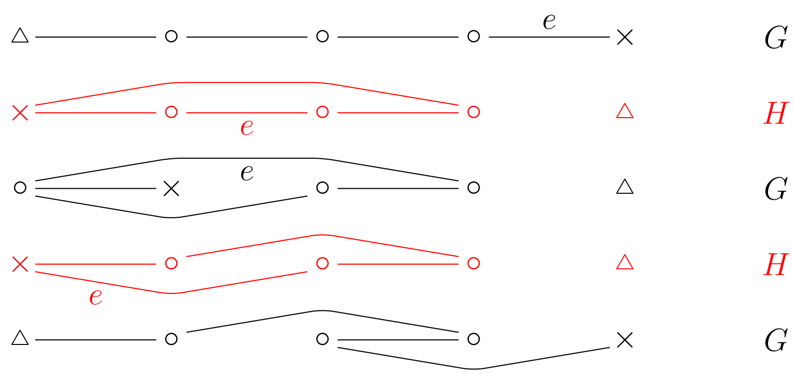
Lemma 4.19. *Suppose that $m, s \geq 5$, then G is edge-reconstructible.*

Proof. Consider any shortest path $\triangle = a_0 - a_1 - \ldots - a_s$. Similarly Lemma 4.14 and Lemma 4.15 we will show that $G_\pi \cong G$ for $\pi = (s-4, s-3, s-2, s-1, s)$ and $\pi = (s-4, s-3)$. This will yield $G_\pi \cong G$ for any $\pi \in S_{\{s-4,s-3,s-2,s-1,s\}}$, which will lead to a contradiction to the same reasons as before. For the proof that $G_\pi \cong G$ for $\pi = (s, s-1, s-2, s-3, s-4)$ the first step will be to remove $a_{s-5}a_{s-4}$ and add $a_{s-5}a_s$. Here, we must add $a_{s-5}a_s$ since the path $\times = a_{s-4} - a_{s-3} - a_{s-2} - a_{s-1} - a_s = \times$ is a path of length $4 < m$ between to vertices of degree $d + 1$, hence the replacing edge must have at least one of these vertices as an endpoint. Also, if $s = 5$ the vertex $a_{s-5} = a_0$ now has degree $d - 1$, hence we must join this vertex with either a_{s-4} or a_s. Since we removed $a_{s-5}a_{s-4}$ we must indeed add $a_{s-5}a_s$. If $s \geq 6$ the path $\triangle = a_0 - a_1 - \ldots - a_{s-5} = \times$ is a path of length $s - 5 < s$ between the vertex of degree d and a vertex of degree $d + 1$, so the replacing edge must have a_{s-5} as its endpoint (as the replacing edge must have two vertices of degree d as its endpoints). Since we removed $a_{s-5}a_{s-4}$, we find again that we must add $a_{s-5}a_s$. □

In fact, we can do the same things for the case where $s = 4$, where Lemma 4.15 required a different proof.

Lemma 4.20. *Suppose that $m \geq s = 4$. Then G is edge-reconstructible.*

Proof. Just as in the above lemma we can show the analogue of Lemma 4.14 in this case, so it suffices to prove the analogue of Lemma 4.15. In fact, with a little bit more consideration, the same series of forced moves can be used to show that we still have $G_{(2,3)} \cong G$. For clarity, we will again show this series of moves.



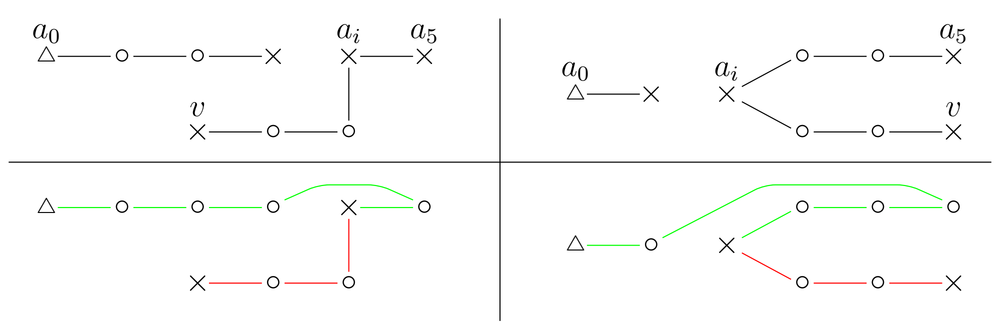
We will now discuss why this series of moves is still forced. As usual, enumerate the vertices from left to right as $\triangle = a_0 - a_1 - a_2 - a_3 - a_4 = \times$. Let v be the vertex of degree $d+1$ not equal to a_4. Now, after deletion of $e = a_3a_4$, the vertex a_3 will have degree $d+1$ and the vertex a_4 will have degree d. Since we know the degree sequence, the replacing edge must join vertices of degree d and $d+1$ respectively. Therefore, the replacing edge is one of a_0v, a_4v, a_0a_3 (we exclude a_4a_3 since we just deleted that edge). Now, if we add a_0v we get a path $\triangle = a_0 - a_1 - a_2 - a_3 = \times$ of length $3 < s$ between the vertex of degree d and a vertex of degree $d+1$, whereas adding a_4v will yield the path $\times = a_0 - a_1 - a_2 - a_3 = \times$ of length $3 < m$ between the vertices of degree $d+1$. Therefore, the replacing edge must be a_0a_3. By exactly the same reasoning the last move is forced. Now we discuss why the second move is forced, the third move is analogous. After deleting a_1a_2, we have four vertices of degree $d+1$, namely a_0, a_1, a_2 and v, of which we must connect two. Since a_0 and a_1 are connected, the replacing edge must have either a_0 or a_1 as its endpoint. Also, the path $a_0 - a_3 - a_2$ is a path of length $2 < m$ between two vertices of degree $d+1$, so the replacing edge must also have either a_0 or a_2 as an endpoint. Therefore, if a_0 is not an endpoint of the replacing edge, the replacing edge must be a_1a_2, but that is the edge we just deleted. Therefore, a_0 must be an endpoint. Since a_0 and a_1 are already connected, the replacing edge is either a_0v or a_0a_2. However, if we add a_0v, the path $\times = a_1 - a_0 - a_3 - a_2$ is a path of length $3 < m$ between the two vertices of degree $d+1$, a contradiction. $\square$

Now we consider the case where s is large, but m is small. We can show the desired result except for the case $(s, m) = (5, 4)$.

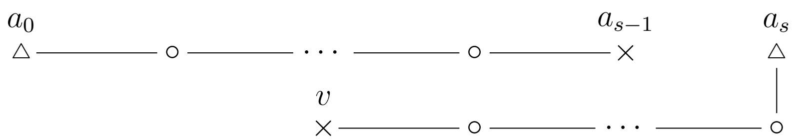
Lemma 4.21. *Suppose that we have $(s,m) = (4,2)$ or $s \geq 5$, $m \leq 4$ and $(s,m) \neq (5,4)$. Then G is reconstructible.*

Proof. Denote by $\triangle = a_0 - a_1 - \ldots - a_s = \times$ a path of length s such that the distance from the vertex v of degree $d+1$ not on this path to this path is m. Suppose that this path of length m is between v and a_i. Since $a_0 - a_1 - \ldots - a_i - \ldots - v$ is a path of length $i+m$ from the vertex of degree d to a vertex of degree $d+1$, we must have $i+m \geq s$, hence $i \geq m-s$. By assumption, we have $s-m \geq 2$.

We will first show that we can reduce to the case where $i = s$. Henceforth, suppose that $2 \leq i < s$ and delete the edge $e = a_{i-1}a_i$. After deletion of this edge we will have four vertices of degree $d+1$, namely a_{i-1}, a_i, a_s and v, of which we must connect 2. The existence of the path $\triangle = a_0 - a_1 - \ldots - a_{i-1} = \times$ shows that a_{i-1} must be one of the endpoints. If $i > m-s$ the path $\times = a_i - \ldots - a_s$ has length less then m, hence the other endpoint must be either a_i or a_s. Since we removed $a_{i-1}a_i$ we see that the replacing edge must be $a_{i-1}a_s$. If $i = m-s$ the roles of a_s and v are symmetric symmetric, since both have distance m from a_i. Since the replacing edge is $a_{i-1}a_s$ or $a_{i-1}a_v$ we may assume without loss of generality that it is $a_{i-1}a_s$. Now, in this new situation, we in fact have $i = s$. Below one can see these situations for $(s,m) = (5,3)$ for $i = 4 > s-m$ and $i = 2 = s-m$ respectively. The upper drawings are the situation after deleting $a_{i-1}a_i$, whereas at the bottom one can see the situation after adding in a_ia_s, after which the shortest path of length s is depicted in green, and the path of length m to the ultimate vertex of this path is shown in red.



Now, in the case that $i = s$, remove the edge $a_{s-1}a_s$, to give the situation shown below. Here the length of the path between a_s and v is equal to m.



Now, we must connect one of the vertices of degree d with one of the vertices of degree $d+1$, hence one of a_0, a_s with one of a_{s-1}, a_v. However, since both $\triangle = a_0 - \ldots - a_{s-1} = \times$ and $\triangle = a_s - \ldots - v = \times$ are paths between a vertex of degree d and a vertex of degree $d+1$, and their respective lengths are $s-1 < s$ and $m < s$, one endpoint must be among a_0, a_{s-1} and the other endpoint among a_s, v. Since we just deleted $a_{s-1}a_s$ this gives a_0v as the only option. But then the path $\triangle = a_s - \ldots - v - a_0$ is a path of length $m+1 < s$ between the vertex of degree d and a vertex of degree $d+1$, which is a contradiction. Therefore, the edge $a_{s-1}a_s$ is a forced edge and G is reconstructible. $\square$

There are only few cases remaining, which we can summarize in the following theorem.

Theorem 4.22. *Let G be a graph with one vertex of degree d, two vertices of degree $d+1$ and all other vertices of degree $d+2$. Then G is edge-reconstructible, except for possibly the following cases:*

(a) The vertices of degree $d+1$ are connected and $s = 2$.

(b) The vertices of degree $d+1$ are not connected and $s \in \{2, 3\}$.

(c) The vertices of degree $d+1$ are not connected and $(s, m) = (4, 3)$ or $(s, m) = (5, 4)$.

A Graph theory glossary

Below one finds an overview of some of the notation and concepts used throughout this thesis. Some of the definitions below are specific to finite, simple, undirected graphs, the only type of graph considered. Below G and H will always be graphs.

$G + H$	The graph with $V(G+H) = V(G) \sqcup V(H)$ and $E(G+H) = E(G) \sqcup E(H)$
nG	$G + G + \ldots + G$ (n times)
Bidegreed graph	A graph with exactly two integers in the degree sequence
Complete graph K_n	The graph on n vertices with all possible edges; $V(K_n) = \{1, 2, \ldots, n\}$, $E(K_n) = \{ij \mid 1 \leq i < j \leq n\}$
Connected graph	A graph with a path between any two vertices
Connected component	A connected subgraph not strictly contained in any other connected subgraph
Degree of a vertex v	The number of edges having v as an endpoint; The number of $u \in V$ such that $uv \in E$
Degree sequence	The non-increasing sequence of the vertex degrees
E, $E(G)$	The edge set of G
$e(G)$	The number of edges of G
Homomorphism $G \to H$	A map $f : V(G) \to V(H)$ such that $f(x)f(y) \in E(H)$ for all $xy \in E(G)$
Isomorphism	An invertible homomorphism
Automorphism	An isomorphism from a graph to itself
Isolated vertex	A vertex of degree zero
Path	A sequence of distinct vertices $v_0, v_1, \ldots, v_n$ (with possibly $v_0 = v_n$) such that $v_{i-1}v_i \in E$ for all $1 \leq i \leq n$
Regular graph	A graph in which every vertex has the same degree
k-regular graph	A graph in which every vertex has degree k
Subgraph H of G	A graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$
Subgraph induced by W, $G[W]$	The graph H with $V(H) = W$ and $E(H)$ all pairs of elements of W contained in $E(G)$
Tridegreed graph	A graph with exactly three integers in the degree sequence
V, $V(G)$	The vertex set of G
$v(G)$	The number of edges of G

References

[Bol90] Béla Bollobás. Almost every graph has reconstruction number three. *J. Graph Theory*, 14(1):1–4, 1990.

[Bon91] J. A. Bondy. A graph reconstructor's manual. In *Surveys in combinatorics, 1991 (Guildford, 1991)*, volume 166 of *London Math. Soc. Lecture Note Ser.*, pages 221–252. Cambridge Univ. Press, Cambridge, 1991.

[CNW82] J. Caunter and C. St. J. A. Nash-Williams. Degree conditions for edge reconstruction. preprint, 1982.

[EPY88] M. N. Ellingham, L. Pyber, and Xingxing Yu. Claw-free graphs are edge reconstructible. *J. Graph Theory*, 12(3):445–451, 1988.

[GH73] D. L. Greenwell and R. L. Hemminger. Reconstructing the n-connected components of a graph. *Aequationes Math.*, 9:19–22, 1973.

[Har64] F. Harary. On the reconstruction of a graph from a collection of subgraphs. In *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963)*, pages 47–52. Publ. House Czechoslovak Acad. Sci., Prague, 1964.

[Kel42] Paul Joseph Kelly. *ON ISOMETRIC TRANSFORMATIONS*. ProQuest LLC, Ann Arbor, MI, 1942. Thesis (Ph.D.)–The University of Wisconsin - Madison.

[Kel57] Paul J. Kelly. A congruence theorem for trees. *Pacific J. Math.*, 7:961–968, 1957.

[KR90] I. Krasikov and Y. Roditty. Recent applications of Nash-Williams lemma to the edge-reconstruction conjecture. *Ars Combin.*, 29(A):215–224, 1990. Twelfth British Combinatorial Conference (Norwich, 1989).

[McK97] Brendan D. McKay. Small graphs are reconstructible. *Australas. J. Combin.*, 15:123–126, 1997.

[MEH87] W. J. Myrvold, M. N. Ellingham, and D. G. Hoffman. Bidegreed graphs are edge reconstructible. *J. Graph Theory*, 11(3):281–302, 1987.

[Mül77] Vladimír Müller. The edge reconstruction hypothesis is true for graphs with more than $n \cdot \log_2 n$ edges. *J. Combinatorial Theory Ser. B*, 22(3):281–283, 1977.

[Sac64] Horst Sachs. Beziehungen zwischen den in einem Graphen enthaltenen Kreisen und seinem charakteristischen Polynom. *Publ. Math. Debrecen*, 11:119–134, 1964.

[Tut79] W. T. Tutte. All the king's horses. A guide to reconstruction. In *Graph theory and related topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977)*, pages 15–33. Academic Press, New York-London, 1979.

[Whi32] Hassler Whitney. A logical expansion in mathematics. *Bull. Amer. Math. Soc.*, 38(8):572–579, 1932.