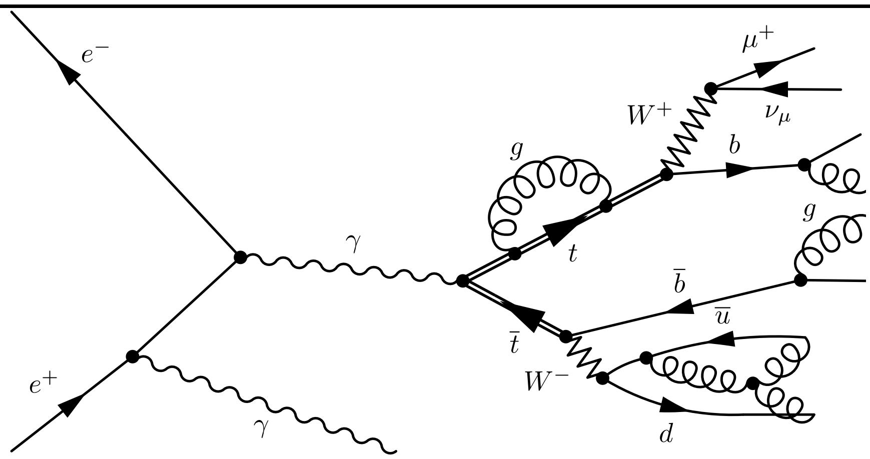
One-Loop Calculations and the Mass of the Top Quark



THESIS

submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE
in
THEORETICAL PHYSICS

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Utrecht & Amsterdam, The Netherlands,
June 30, 2016

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Abstract

The top quark pole mass is affected by perturbative divergences known as renormalons, therefore this mass parameter possesses a theoretical ambiguity of order Λ_{QCD}, the infrared scale of divergence. We subtract a part of the static $t\bar{t}$ potential and obtain the potential-subtracted mass which is renormalon-free and thus measurable with bigger precision. We demonstrate the presence of the renormalon ambiguity in the top quark pole mass and in the static $t\bar{t}$ potential and with this the cancellation of the renormalon ambiguity in the potential-subtracted mass. Before we discuss renormalons, we review how one computes one-loop quantum corrections in QED and the SM, starting from the gauge-principles that underlie these theories. We focus on renormalisation and its non-perturbative implications, we discuss how the conservation of Noether current affects the counter terms in QED and verify the optical theorem explicitly for top quark decay.

Contents

Acknowledgements

This thesis is the culmination of a nine month master research project under supervision of professor Eric Laenen. I want to thank my supervisor for providing me with an interesting problem to work on and his guidance along the way. Most of all, I am grateful for his advice and assistance in my (successful) search after a PhD position.

Chapter 1

Introduction

In quantum field theories we are confronted with a lot of freedom to choose conventions, parametrisations and scheme's. We thus start with a lagrangian and the corresponding Feynman rules, and from that we demonstrate how to compute the one-loop quantum corrections. An understanding of the details in this demonstration, allows us to look at three different mass schemes for the top quark in quantum chromodynamics (QCD), the theory of quarks and gluons.

Motivation

Each mass scheme corresponds to a theoretically defined mass parameter such as: the pole mass m_{pole}, the modified minimal subtracted mass $m_{\overline{MS}}$, the potential subtracted mass m_{PS} [1], the modified potential subtracted mass $m_{\overline{PS}}$ [2] and the $1S$ mass m_{1S} [3]. Not all of these theoretical mass parameters can be extracted from experimental fits with the same precision. We explore how infrared renormalons affect the precision by which the m_{pole}, $m_{\overline{MS}}$ and m_{PS} mass parameters can be measured.

We invest a substantial amount of effort to develop a good understanding of the one-loop quantum corrections. To obtain more accurate predictions it is necessary to compute not only the leading order tree-level contributions but also the leading and eventually the sub-leading quantum corrections, which amounts to an expansion in the coupling constant. Also in these calculations, there is a multitude of possible conventions, parametrisations and schemes which makes it important to recognise what is physically relevant. Therefore, we explore different regularisation and renormalisation schemes in quantum electrodynamics (QED) through explicit step-by-step computations.

Of all the possible particles in the standard model (SM), we choose to focus on the top quark. The top quark is the heaviest fundamental particle in the standard model and although its discovery in 1995 at the Tevatron forwent the discovery of the Higgs particle in 2012 at the LHC by more than a decade, the accuracy by which we know its mass is nearly an order of magnitude less precise. [4] Also the top quark is a vey short-lived particle since it is more massive than the W-boson. It has an open decay to a W-boson and the full decay width $\Gamma_t = 1.41^{+0.19}_{-0.15}$ GeV [4] is so large that a top quark decays before it has time to hadronise, which allows for a wider applicability of the perturbative method. [5, 6]

The electroweak parameters together with a well-determined top quark mass allow one to test the consistency of the SM. The $SU(3) \times SU(2) \times U(1)$ symmetry that underlies the standard model makes that a more precise determination of the top quark mass allows us to determine other parameters with a bigger precision also. The sensitivity to new physics would increase once the effect of top quark quantum corrections is known at bigger precision. Also, due to the large top quark mass there is a strong coupling between the top quark and the Higgs boson, which might provide us with beyond the standard model (BSM) physics sensitivity. [7]

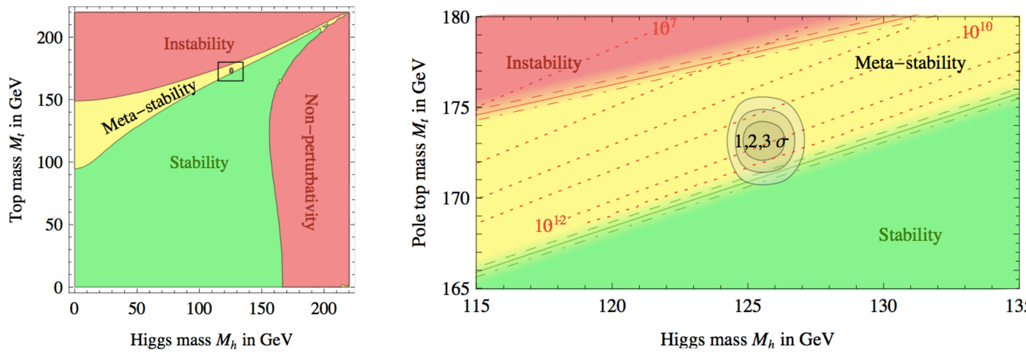


Figure 1.1: *These plots have been taken from Ref. [8]. It shows the for what values of m_t and m_h the SM electroweak vacuum is stable, meta-stable and unstable with a zoom-in on the best measured experimental values of the Higgs and top quark mass at the time. Additionally, the instability scale of the vacuum expectation value of the Higgs field is shown in GeV.*

There are even cosmological implications, the current best measured values of the Higgs and top quark pole masses are $m_H = 125.09 \pm 0.24$ GeV and $m_t = 174.6 \pm 1.9$ GeV [4]. It has been shown at *next-to-next-to-leading-order* (NNLO) that the *standard model* (SM) Higgs potential becomes unstable at energy scales that lie well below the Planck scale, which makes our universe metastable with a lifetime of about the age of the universe. [8, 9] Some claim that metastability is a necessary condition for the universe that is favoured by quantum gravity theories. [10] The largest uncertainty on the conclusions in these works is the top quark pole mass, which can be seen in Fig: 1.1.

Measuring the top quark mass

In this thesis we concern ourselves with the theoretical ambiguity in the top quark pole mass that is caused by so-called renormalons. These renormalons are one of the limiting factors in a precise determination of the theoretical mass parameter known as the top quark pole mass. We show that a different theoretical top quark mass parameter exists that is free of these renormalons. Here, we describe some experimental methods that are used to extract theoretical top quark mass parameters from data.

The top quark mass parameters such as the pole mass, the $\overline{MS}$ mass or the potential-subtracted mass are not physically observable, hence not directly measurable. In particle colliders we only count the number of events with a certain kinematical signature and fit this to theoretical predictions. Then we extract the best-fitting value for the top quark mass parameter m_t with a certain likelihood and uncertainty.

Schematically, the general method is that we try to find the solution for m_t of the following implicit relation as a function of kinematical variables $\{P\}$. $A_{\text{exp}}\left(\{P\}\right)$ is the measured number of events with a certain signature and $A_{\text{th}}\left(m_t, \{P\}\right)$ is the expected number of events with that signature.

$$A_{\text{exp}}\left(\{P\}\right) = A_{\text{th}}\left(m_t, \{P\}\right) \tag{1.1}$$

The first detection and subsequent study of the top quark and its mass have been performed at hadron colliders such as the Tevatron and the LHC. Mass measurements are performed with events wherein a $t\bar{t}$-pair is produced that decays into almost exclusively a W-boson accompanied by a bottom quark. The top quark mass parameter is then extracted from the data by sophisticated methods such as the *template method* or the *matrix element method*.

The QCD-induced production of a $t\bar{t}$-pair happens through either *gluon fusion* or *quark-antiquark annihilation*, whereof the NLO corrections have been computed long ago. [11] Notable is also that due to different collider energies the dominant $t\bar{t}$-production mechanism at the Tevatron is quark-antiquark annihilation whilst at the LHC this is gluon fusion.

The $t\bar{t}$-pair rapidly decays via an open weak decay to a W-boson and

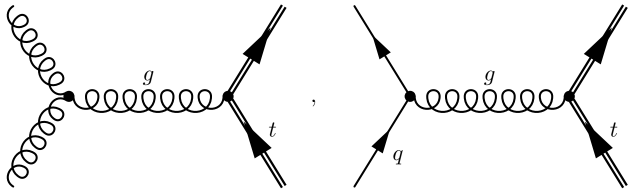


Figure 1.2: *This figure shows the dominant $t\bar{t}$-production channels in hadron colliders. On the left gluon fusion is shown and on the right quark-antiquark annihilation is shown.*

a bottom-quark. The decays to down or strange quark are heavily suppressed as off-diagonal elements in the quark-mixing martrix V_{CKM}, which is nearly diagonal in flavour space. The bottom quarks hadronise into jets of particles whilst the W-bosons have the option to either decay into a jet or into a lepton with a neutrino.

The $t\bar{t}$-decay channels with each their own merits are classified through the decay channels of the two W-boson. The dilepton-channel has two jets from the bottom quarks and a low background relative to other two channels, but is plagued by a low branching fraction and therefore low statistics. In contrast with this the all-hadronic channel with four jets has a large branching fraction, but is plagued by a large background from QCD multi-jet events. In this channel, it is necessary to employ data filtering methods inspired by neural networks that combine kinematic variables in order to construct a discriminant that allows one to filter out $t\bar{t}$-events. Preferably one looks at the lepton + jet channel with three jets, which has less background than the all-hadronic channel and a much bigger branching fraction than the dilepton channel.

The template method is the traditional method employed to extract the top quark mass from collision data. The data is fitted to the kinematics of the event by minimising a χ^2-function and a top quark mass m_t^{rec} is reconstructed from the best-fitting event hypothesis. Additionally, events with different values for m_t are simulated to construct template histograms. One uses these templates to derive a parametrisation of the m_t^{rec} histograms as a function of m_t. Finally, maximum-likelihood methods are used to fit the template function to the histogram of m_t^{rec} that was observed in the collision data.

The most precise top quark measurements have been obtained by use of the matrix-element method. In this method the entire event kinematics is used to improve sensitivity for the top quark mass. One computes the event probability $\mathcal{P}\left(m_t, \{P\}\right)$ as a function of the top quark mass m_t and the measured kinematic variables $\{P\}$. This requires expressions for the $t\bar{t}$-production and decay diagrams at leading order, which are convoluted with parton distribution functions and the detector response. Events with a bigger likelihood to be a $t\bar{t}$-event are given a greater weight in the top quark mass determination, which is central to the effectiveness of this method.

These and other experimental methods to determine the top quark mass in hadron collider and their (dis)advantages with additional references are reviewed in [12], a more accessible treatment is found in [13].

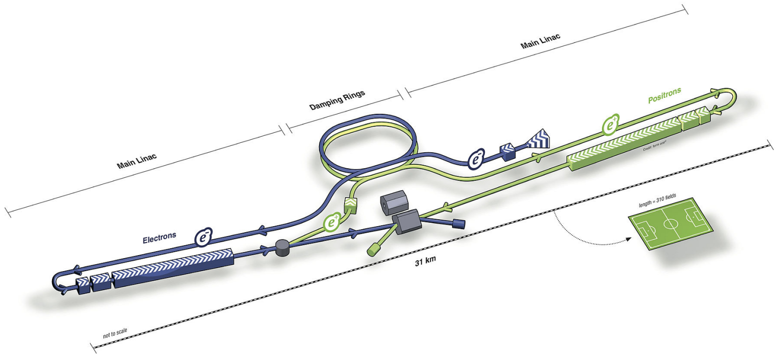


Figure 1.3: This is the design for the proposed International Linear Collider (ILC), a 31 kilometre long electron-positron collider that would reach energies of up to 500 *GeV. The image was taken from:* `www.linearcollider.org`.

However, the top quark mass is expected to be measurable with a higher precision at or below $\sim$ 100 MeV in future lepton colliders such as the proposed International Linear Collider (ILC) [14] and the Compact Linear Collider (CLIC) [7]. At lepton and most notably the above named e^+e^- colliders one collides fundamental particles instead of composite objects like protons, hence the QCD background is reduced because the initial state is fully characterised.

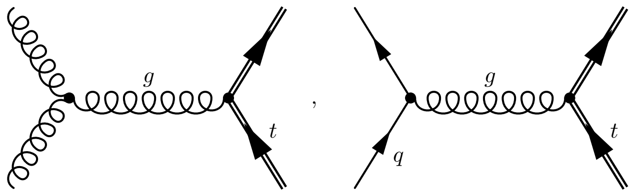


Figure 1.4: This figure shows the dominant $t\bar{t}$-production channels in e^+e^- colliders. On the left the photon exchange is shown and on the right Z-boson exchange is shown.

At a lepton collider the dominant $t\bar{t}$-production goes via virtual photon and Z-boson exchange with backgrounds from *di-boson production* of a Z-boson or W-boson pair. The $t\bar{t}$-pair then decays again into the dilepton channel, the semi-leptonic channel and the all-hadronic channel via an intermediate W-boson.

One way to extract the top quark mass is by means of direct reconstruction above the threshold energy for the production of two top quarks, which is at about $\sqrt{s} \approx 2m_t \sim 350$ GeV . This method is similar to the template method that was described for hadron colliders, but an application of the matrix element method should be feasible as well. Unfortunately, the connection between the mass parameters in theory to experimentally accessible parameters is still lacking. [7]

A threshold scan is expected to outperform the direct reconstruction in the determination of the top quark mass. In a threshold scan one measures the cross-section of $t\bar{t}$-events close to and at the threshold in finite size steps. Also, data points are taken further below threshold to measure the background. To extract from these data points the top quark mass, the data-points are compared to calculated cross-sections for different mass hypotheses via χ^2-fitting.

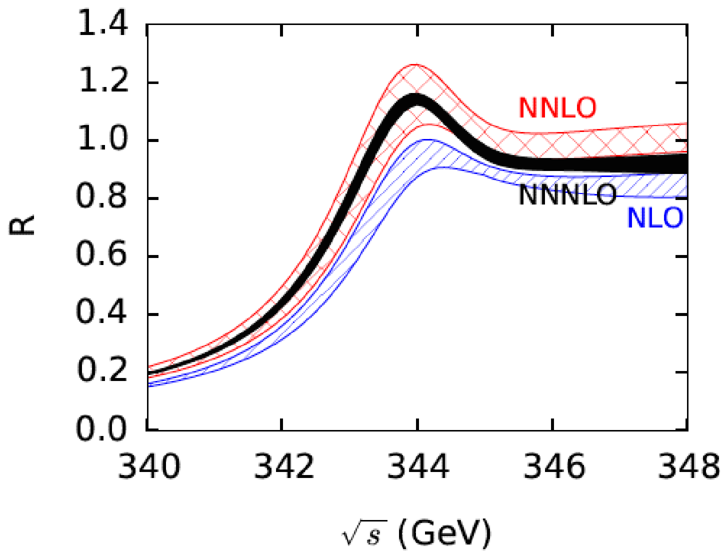


Figure 1.5: *This plot has been taken from Ref. [15]. It shows at successive orders the prediction for the cross-section of $t\bar{t}$-events close to threshold for electron positron scattering as a function of the centre-of-mass energy $\sqrt{s}$ taking $m_{PS}\left(\mu_f = 20\text{ GeV}\right) = 171.5$ GeV, $\Gamma_{CM} = 1.33$ GeV and $\alpha_s\left(m_Z\right) = 0.1185 \pm 0.0006$ as input. The colour bands show the variation of the result as a function of renormalisation scale μ.*

At the threshold energy the quark anti-quark pair have non-relativistic relative momentum, hence the strong interaction drives the system to form sharp bound-state resonances. Only, these sharp bound-state resonances are smeared out by the weak interaction's decay into one broad peak. This process is well-understood theoretically at N^3LO in the QCD corrections by use of effective field theories such as non-relativistic QCD (NRQCD) and potential NRQCD (PNRQCD) in the potential-subtracted mass scheme. [15] The N^3LO computation was necessary because the N^2LO contribution was still large with respect to the NLO prediction as shown in Fig: 1.5.

Structure

To compute quantum loop corrections in QED without too much luggage, we start off with a description of QED and its symmetries in **Ch: 2**. This allows us to introduce our notation, spinor and gamma-matrix conventions and define the Feynman rules with free gauge parameter and a photon regulator mass.

Having understood the origin of the QED Feynman rules, we use them in **Ch: 3** to compute the leading order one-loop quantum corrections to the propagators and vertex function in QED. Along with this computation we define the exact propagator, the vertex-function, the pole mass and compare two infrared regulator schemes. More importantly though, our step-by-step by hand evaluation of the Feynman diagrams allows us to explain the techniques that are necessary for the evaluation and motivate the use of more automated techniques.

The divergences in the self-energies and vertex functions that have been regulated in Ch: 3 need to be cancelled by an appropriate choice for the counter terms Z_i. In **Ch: 4** we compute these counter terms in the $\overline{MS}$ and the OS renormalisation schemes for both infrared regulator schemes. Additionally, we relate the mass parameters in the two schemes and show what non-perturbative information is contained in the Z_i.

In **Ch: 5** we take a side-step and demonstrate that the quantum equivalent of conservation of Noether current in QED leads to constraints on the space-time structure of QED amplitudes and the equality of counter terms in the $\overline{MS}$ and OS renormalisation schemes.

From **Ch. 6** onwards we move on from one-loop calculations in QED to the top quark and the SM. Here, we work the Feynman rules relevant for the top quark interactions and for top quark and gluon one-loop calculations in the SM. We do not simply state the Feynman rules, but also explain which mechanism or principle gives rise to them.

After the significant effort of the by hand evaluation of QED one-loop diagrams, we use a more automated approach in **Ch: 7** to one-loop self-energy diagrams of the gluon and the top quark in the SM. We aim to describe an automisation method that proves to be sufficient for our purposes, not one that is general.

In **Ch: 8** we use the more automated method to show the validity of the optical theorem at leading order for the dominant decay channel of the top quark into a W-boson and a bottom quark in the unitary gauge.

The general concept of renormalons is introduced in **Ch: 9**. We discuss the definition of renormalons in QED, demonstrate their characteristic $n!$-growth in a perturbative expansion and show an effective method of resummation. Finally, we argue for a gauge-invariant prescription that generalises the QED renormalons to QCD.

With the running coupling prescription from Ch: 9, we evaluate in **Ch: 10** how renormalons induce a theoretical ambiguity in the top quark pole mass that is of order $\Lambda_{QCD} \approx (200 - 300)\,\mathrm{MeV}$ [16]. We demonstrate that renormalons induce a similar ambiguity in the static toponium potential, which cancels the pole mass renormalon ambiguity in the Schrödinger equation of the toponium system. This motivates the definition of a renormalon-free mass, the potential-subtracted mass.

Chapter 2

A Description of QED

We describe the lagrangian of quantum electrodynamics (QED), show that this lagrangian is gauge-invariant and deduce the corresponding Feynman rules for later computations. We also introduce notation and define our conventions concerning spinors and gamma-matrices.

We start with a fermion spin-½ field with a local $U(1)$ gauge symmetry. The local gauge-symmetry requires that we covariantise the derivative and introduce an interaction with a gauge-field. For the gauge-field we introduce a kinetic term, a gauge-fixing term and a small regularisation mass.

The Feynman rules that we derive will be used to calculate the self-energies and the vertex-correction. For a more complete treatise of the subject one could look at introductions to quantum field theory such as: Ref. [17, ch. 58] and Ref. [18, 19] or without a rigorous treatment of quantum field theory: Ref. [20].

2.1 The lepton sector

QED is the theory of electromagnetism at the quantum scales, it involves charged fermion particles called leptons. These leptons are electrons, muons, tauons and their respective anti-particles. The (anti)-muons and (anti)-tauons that are produced via electromagnetic or weak interactions quickly decay to (anti)-electrons (and neutrino's) via the weak interaction. Here, we consider only electrons and their anti-particles, the positrons.

The lagrangian of QED contains an electron/positron spin-½ fermion field $\Psi(x)$. These objects are four-component vectors in spin space, and their Dirac conjugate fields are defined as $\overline{\Psi}(x) \equiv \Psi^{\dagger}(x)\gamma^{0}$.

To use fermion fields we need the γ-matrices, four independent traceless 4×4 matrices with a space-time index, that obey a Clifford algebra as shown in Eq: 2.1. As a shorthand notation we also introduce for any vector a^{μ} the shorthand $a^{\mu}\gamma_{\mu} \equiv a_{\mu}\gamma^{\mu} \equiv \not{a}$. Relations and identities involving γ-matrices are tabulated in App: A.

$$\{\gamma^{\mu}, \gamma^{\nu}\} \equiv \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = -2\eta^{\mu\nu} \tag{2.1}$$

The free-field lagrangian for the spinor field $\Psi(x)$ is shown in Eq: 2.2, the origin of which is found in textbooks such as [17, ch. 36]. From the free-field Lagrangian one constructs the propagator in momentum space $\widetilde{S}(\not{p})$ by taking the inverse, as shown in Eq: 2.3. This propagator, as shown in Eq: 2.3 is inserted for every internal electron line in a Feynman diagram. *

$$\mathcal{L}_{\Psi,0} = -\overline{\Psi}(x)\left(-\mathrm{i}\not{\partial} + m\right)\Psi(x) \tag{2.2}$$

$$\frac{1}{\mathrm{i}}\widetilde{S}(\not{p}) \equiv \frac{-\mathrm{i}}{\not{p} + m - \mathrm{i}\epsilon} = -\mathrm{i}\frac{-\not{p} + m}{p^{2} + m^{2} - \mathrm{i}\epsilon} \tag{2.3}$$

The external lines of electrons or positrons in Feynman diagrams are contracted with spinors. The spinors describe the spin-polarisation of the asymptotic incoming and outgoing states. Incoming electrons are contracted with $u_s(\mathbf{p})$, outgoing electrons with the Dirac conjugate spinor $\overline{u}_s(\mathbf{p})$, incoming positrons with $\overline{v}_s(\mathbf{p})$ and outgoing positrons with $v_s(\mathbf{p})$. The s-index is used to denote the spin-state of the spinor.

The $\overline{u}_s(\mathbf{p})$ and $\overline{v}_s(\mathbf{p})$ spinors describe the asymptotic states, therefore they satisfy the classical equations of motion that are derived by varying the free-field fermion lagrangian.

$$\left(-\mathrm{i}\not{\partial} + m\right)\Psi(x) = 0 \tag{2.4}$$

We now construct a plane-wave decomposition for $\Psi(x)$ in terms of positive and negative frequency modes and the spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$.

$$\Psi(x) \propto u_s(\mathbf{p})\, e^{\mathrm{i}px} + v_s(\mathbf{p})\, e^{-\mathrm{i}px} \tag{2.5}$$

*The '$-\mathrm{i}\epsilon$'-prescription in the propagators is usually not written explicitly, expect for in the definition of the Feynman rules.

Since the field $\Psi(x)$ should satisfy Dirac's equation, we find that the spinors should satisfy:

$$\begin{aligned} (\not{p}+m)\,u_s(\mathbf{p}) = 0 \quad (-\not{p}+m)\,v_s(\mathbf{p}) = 0 \\ \overline{u}_s(\mathbf{p})\,(\not{p}+m) = 0 \quad \overline{v}_s(\mathbf{p})\,(-\not{p}+m) = 0 \end{aligned} \tag{2.6}$$

We use a relativistic normalisation of the spinors:

$$\begin{aligned} \overline{u}_{s'}(\mathbf{p})\,u_s(\mathbf{p}) = 2m\delta_{s,s'} \quad \overline{u}_{s'}(\mathbf{p})\,v_s(\mathbf{p}) = 0 \\ \overline{v}_{s'}(\mathbf{p})\,v_s(\mathbf{p}) = -2m\delta_{s,s'} \quad \overline{v}_{s'}(\mathbf{p})\,u_s(\mathbf{p}) = 0 \end{aligned} \tag{2.7}$$

Some additional identities involving spinors contracted with gamma-matrices are tabulated in App: A.

The local $U(1)$ gauge-symmetry

QED has a local $U(1)$ gauge-symmetry and thus the lagrangian is invariant under the transformation $\Psi(x) \to \Psi'(x) = e^{ie\xi(x)}\Psi(x)$, where $e = -0.302822$ the charge of the electron in Heaviside-Lorentz units with $\hbar = c = 1$ and $\xi(x)$ is the gauge parameter field. The free-field lagrangian transforms as:

$$\mathcal{L}_{\Psi,0} \to \mathcal{L}'_{\Psi,0} = -\overline{\Psi}(x)\,(-i\not{\partial}+m)\,\Psi(x) - e\partial_\mu\xi(x)\,\overline{\Psi}(x)\,\gamma^\mu\Psi(x) \tag{2.8}$$

We see that the free-field lagrangian is not invariant under local $U(1)$ gauge transformations. To solve this, we introduce a massless spin-1 gauge field $A^\mu(x)$, a bosonic space-time vector field that represents the photon. This field transforms under gauge transformations as:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\xi(x) \tag{2.9}$$

To ensure that the lagrangian in Eq: 2.2 becomes invariant under local $U(1)$ gauge transformations we replace the ordinary derivative by a covariant derivative, defined as $\partial_\mu \to D_\mu \equiv \partial_\mu - ieA_\mu(x)$. This replacement gives us an interaction term:

$$\mathcal{L}_I = e\overline{\Psi}(x)\,\not{A}(x)\,\Psi(x) \tag{2.10}$$

Upon adding the interaction term to the free-field lagrangian we get:

$$\mathcal{L}_{\Psi,0} + \mathcal{L}_I = -\overline{\Psi}(x)\,(-i\not{D}+m)\,\Psi(x) \tag{2.11}$$

$$= -\overline{\Psi}(x)\,(-i\not{\partial}+m)\,\Psi(x) + e\overline{\Psi}(x)\,\not{A}(x)\Psi(x)\,. \tag{2.12}$$

For every electron-photon vertex in a Feynman diagram we insert a vertex factor that is constructed from $\mathcal{L}_I$. The vertex factor from Eq: 2.10 equals $\mathrm{i}e\gamma^\mu$, which is i times the factor that multiplies the fields in the interaction term.

2.2 The gauge sector

The free-field lagrangian for the gauge field, which is invariant under the $U(1)$ gauge transformations given in Eq: 2.9, is given in Eq: 2.13. The tensor $F_{\mu\nu}$ is the field strength tensor, introduced as a shorthand notation. In QED the components of $F_{\mu\nu}$ correspond to the electric and the magnetic field whilst the vector-field $A_\mu(x)$ corresponds to the four-vector potential.

$$\mathcal{L}_{A,0} = -\frac{1}{4}F^{\mu\nu}(x)\,F_{\mu\nu}(x) = -\frac{1}{4}\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right)^2 \tag{2.13}$$

The local gauge invariance of the free field lagrangian ensures that not all components of the gauge field are physical. It is also not possible to construct a propagator, since the part of the lagrangian quadratic in the fields in momentum space is not invertible. The quadratic part has a null vector along the direction of the momentum, the longitudinal polarisation of the photon.

To construct a propagator for the gauge fields we introduce a term that makes the quadratic part invertible, a so-called gauge-fixing term as shown in Eq: 2.14 with λ as an arbitrary gauge-dependent term. Since λ is arbitrary, physical observables must not be a function of λ.

$$\mathcal{L}_{g.f.} = -\frac{1}{2}\left(\lambda\partial_\mu A^\mu(x)\right)^2 \tag{2.14}$$

We also introduce an infrared photon regulator mass to pre-empt the appearance of infrared divergences due to diagrams where an on-shell electron is indiscernible from an on-shell electron with an additional zero-momentum photon. A small photon regulator mass m_γ regularises infrared divergences. We add the mass term shown in Eq: 2.15 to the QED lagrangian. Formally m_γ equals zero, so it is more a mathematical crutch.

$$\mathcal{L}_{m_\gamma} = -\frac{1}{2}m_\gamma^2 A_\mu(x)\,A^\mu(x) \tag{2.15}$$

The photon mass term and the gauge-fixing term break the gauge invariance of the theory. However, we expect that gauge-invariance is recovered for infinitesimal photon regulator masses and the gauge-fixing term decouples from the interacting sector of the theory. *

With these additions the part quadratic in the gauge-fields becomes in momentum space:

$$S = -\frac{1}{2}\int d^d p\, \overline{A}_\mu(p)\left[\left(p^2+m_\gamma^2\right)\eta^{\mu\nu} - \left(1-\lambda^2\right)p^\mu p^\nu\right]A_\nu(p) \tag{2.16}$$

Now we find the propagator by inverting the term in square brackets of Eq: 2.16. The propagator is given in Eq: 2.18, in Feynman diagrams the complete term shown must be included for every internal photon line.

$$\widetilde{\Delta}_{\mu\sigma}(p)\left[\left(p^2+m_\gamma^2\right)\eta^{\sigma\nu} - \left(1-\lambda^2\right)p^\sigma p^\nu\right] = \delta_\mu^\nu \tag{2.17}$$

$$\Rightarrow \frac{1}{\mathrm{i}}\widetilde{\Delta}_{\mu\nu}(p) = \frac{-i}{p^2+m_\gamma^2-i\epsilon}\left[\eta_{\mu\nu} - \left(1-\lambda^{-2}\right)\frac{p_\mu p_\nu}{p^2+\lambda^{-2}m_\gamma^2}\right] \tag{2.18}$$

2.3 The complete QED lagrangian

When we sum all terms in Eq: 2.13 , Eq: 2.14, Eq: 2.15 , Eq: 2.2 and Eq: 2.10 we finally find the full bare Lagrangian describing QED, wherein we regularise infrared divergences using a small photon mass.

$$\begin{aligned}\mathcal{L}_{QED} &= \mathcal{L}_{A,0} + \mathcal{L}_{g.f.} + \mathcal{L}_{m_\gamma} + \mathcal{L}_{\Psi,0} + \mathcal{L}_I \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\left(\lambda\partial_\mu A^\mu\right)^2 - \frac{1}{2}m_\gamma^2 A_\mu A^\mu - \overline{\Psi}\left(-\mathrm{i}\partial\!\!\!/ + m\right)\Psi + e\overline{\Psi}A\!\!\!/\Psi\end{aligned} \tag{2.19}$$

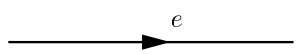
The bare lagrangian with its parameters e and m describes a theory which is not finite, due to infinite loop contributions. This fact motivates us to do some shifting, renaming and rescaling to ensure that the m and e in the Eq: 2.20 are finite.

$$\begin{aligned}\mathcal{L}_{QED} = &-\frac{1}{4}Z_3 F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\lambda^2\left(\partial_\mu A^\mu\right)^2 \\ &- \frac{1}{2}m_\gamma^2 A_\mu A^\mu + iZ_2\overline{\Psi}\partial\!\!\!/\Psi - Z_0 m\overline{\Psi}\Psi + Z_1 e\overline{\Psi}A\!\!\!/\Psi\end{aligned} \tag{2.20}$$

Up to tree-order approximation, without considering quantum loop corrections, we find $Z_i = 1 + \mathcal{O}(e^2)$ since each loop in QED requires at least two vertices. The terms proportional $Z_i - 1$ are called counter terms, since they are introduced to counter unwanted divergences.

The Feynman rules of QED are given by:

*This statement is not trivial and not true in theories like QCD or the SM, see the more detailed treatise of gauge-fixing in QCD in Sec: 6.1 and BRST-symmetry in [17, ch. 74].

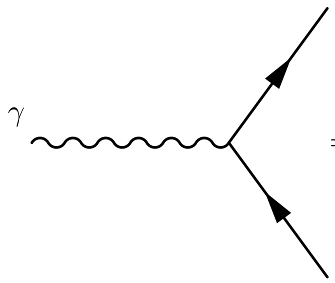


$$=: \frac{-\mathrm{i}}{\not{p} + m}$$

(a)

$$=: \frac{-\mathrm{i}}{p^2 + m_\gamma^2}\left[\eta_{\mu\nu} - (1 - \lambda^{-2})\frac{p_\mu p_\nu}{p^2 + \lambda^{-2} m_\gamma^2}\right]$$

(b)



$$=: \mathrm{i}e\gamma^\mu$$

(c)

Figure 2.1: *The Feynman rules in QED gauge sector: (a) the electron propagator, (b) the photon propagator and (c) the electron-photon vertex.*

Chapter 3

One Loop Calculations in QED

> In this chapter we determine and evaluate the one-loop corrections to the photon propagator, the electron propagator and the vertex function. We also define the exact (electron) propagator and the pole mass and relate it to the self-energy functions.
> During these step-by-step by hand computations, we explain the techniques necessary to evaluate the Feynman diagrams. This includes the use of spinor and gamma-matrix identities, Wick rotations, space-time symmetries of the integral, Feynman parametrisation and regularisation of divergences.

We consider two different infrared regulators, namely the photon regulator mass and full dimensional regularisation (in contrast with dimensional regularisation of solely UV-divergences). At this stage both regulators yield the same unrenormalised photon and electron self-energy functions, only the vertex correction is affected by what regulator we use.

3.1 The photon self-energy

In this section we determine the one-loop quantum corrections to the photon self-energy. We discuss the definition of the self-energy, and its relation to the exact photon propagator. We evaluate the diagrams that contribute to the photon self-energy using a photon regulator mass for the infrared divergences and dimensional regularisation for ultraviolet divergences.

The exact photon propagator

The photon self-energy is made up out of the interactions that a photon undergoes with itself. If we take these quantum corrections into account we can construct the exact propagator, $\widetilde{\mathbf{\Delta}}_{\mu\nu}\left(p\right)$, which is defined as the sum of Feynman diagrams with two photon sources. The self-energy is the subset of diagrams denoted as $\mathrm{i}\Pi^{\mu\nu}\left(p\right)$, which stays connected if any one line is cut. This subset of diagrams, that is *one-particle irreducible* or 1PI for short.

To see how the exact photon propagator, the free-field photon propagator and the self-energy are related let us consider the diagrammatic Fig: 3.1 and the algebraic expression that follows from it in Eq: 3.1.

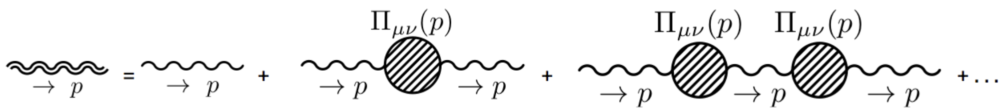


Figure 3.1: *This diagrammatic expression resembles Eq: 3.1. The sum of diagrams wherein a photon undergoes interactions with itself.*

$$\begin{aligned}\frac{1}{\mathrm{i}}\widetilde{\mathbf{\Delta}}_{\mu\nu}\left(p\right) &= \frac{1}{\mathrm{i}}\widetilde{\Delta}_{\mu\nu}\left(p\right) + \frac{1}{\mathrm{i}}\widetilde{\Delta}_{\mu\rho}\left(p\right)\left[\mathrm{i}\Pi^{\rho\sigma}\left(p\right)\right]\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\sigma\nu}\left(p\right)\\ &+ \frac{1}{\mathrm{i}}\widetilde{\Delta}_{\mu\rho}\left(p\right)\left[\mathrm{i}\Pi^{\rho\sigma}\left(p\right)\right]\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\sigma\alpha}\left(p\right)\left[\mathrm{i}\Pi^{\alpha\beta}\left(p\right)\right]\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\beta\nu}\left(p\right) + \ldots\\ &= \frac{1}{\mathrm{i}}\widetilde{\Delta}_{\mu\rho}\left(p\right)\sum_{n=0}^{\infty}\left[\left[\mathrm{i}\Pi^{\rho\sigma}\left(p\right)\right]\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\sigma\nu}\left(p\right)\right]^{n}\end{aligned} \tag{3.1}$$

In the full calculation, we will learn that $\Pi^{\mu\nu}\left(p\right) = p^2\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\Pi\left(p^2\right) = p^2 P_{\mu\nu}\Pi\left(p^2\right)$. Here $P_{\mu\nu}$ is a tensor that projects vectors in the space-time direction transverse to the photon direction of motion, so $p_\mu P^{\mu\nu} = 0$. In anticipation of this result we can rewrite the exact propagator in terms of the self-energy $\Pi\left(p^2\right)$ using the the sum of the geometric series *.

Since $\Pi\left(p^2\right)$ involves corrections of at least two vertices, making up one fermion loop, it is of order $\mathcal{O}\left(e^2\right) \ll 1$ and we can legitimately use the geometric series. We now rewrite the last term in brackets of Eq: 3.1, we have put $m_\gamma^2 \to 0$ since there are no infrared divergences to regularise at this stage:

*The geometric series is given by $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x < 1$

$$
\begin{aligned}
\left[\mathrm{i}\Pi^{\mu\sigma}(p)\right]\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\sigma\nu}(p) &= \Pi^{\mu\sigma}(p)\,\widetilde{\Delta}_{\sigma\nu}(p) \\
&= p^2\left[\eta^{\mu\sigma}-\frac{p^\mu p^\sigma}{p^2}\right]\left[\eta_{\sigma\nu}-\left(1-\lambda^{-2}\right)\frac{p_\sigma p_\nu}{p^2}\right]\frac{1}{p^2-\mathrm{i}\epsilon}\Pi\left(p^2\right) \\
&= p^2\left[\delta^\mu_\nu-\frac{p^\mu p_\nu}{p^2}\right]\frac{1}{p^2-\mathrm{i}\epsilon}\Pi\left(p^2\right) = P^\mu_\nu\Pi\left(p^2\right)
\end{aligned}
\tag{3.2}
$$

Since P^μ_ν is a projector matrix, we find:

$$
\left[\left[\mathrm{i}\Pi^{\mu\sigma}(p)\right]\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\sigma\nu}(p)\right]^n = \left[\delta^\mu_\nu-\frac{p^\mu p_\nu}{p^2}\right]\Pi^n\left(p^2\right) = P^\mu_\nu(p)\,\Pi^n\left(p^2\right) \tag{3.3}
$$

We now apply the geometric series to Eq: 3.1, where we note that the gauge-dependent term that survives comes from the $n=0$ term in the summation. All other gauge-dependent terms vanish when they are multiplied by any of the projectors, the final correction to the photon propagator in terms of $\Pi\left(p^2\right)$ is given by:

$$
\begin{aligned}
\frac{1}{\mathrm{i}}\widetilde{\mathbf{\Delta}}_{\mu\nu}(p) &= \frac{1}{\mathrm{i}}\widetilde{\Delta}_{\mu\rho}(p)\sum_{n=0}^{\infty}\left[\left[\mathrm{i}\Pi^{\rho\sigma}(p)\right]\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\sigma\nu}(p)\right]^n \\
&= \frac{P_{\mu\nu}(p)}{p^2\left[1-\Pi\left(p^2\right)\right]-\mathrm{i}\epsilon}+\lambda^{-2}\frac{\frac{p_\mu p_\nu}{p^2}}{p^2-\mathrm{i}\epsilon}
\end{aligned}
\tag{3.4}
$$

The gauge-dependent part of the propagator decoupled from the self-energy interactions, which verifies the claim that was made earlier.

The contributing diagrams

The diagrams that contribute to $\Pi^{\mu\nu}(p)$ in the one-loop approximation are shown in Fig: 3.2. The first diagram contains a fermion loop which adds a factor -1 and the second one comes from the counter term vertex proportional to (Z_3-1).

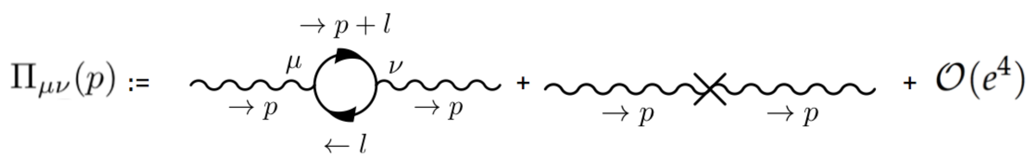


Figure 3.2: *This diagrammatic expression shows us the diagrams that contribute to the photon self-energy up to one loop.*

$$\mathrm{i}\Pi^{\mu\nu}(p) = \overbrace{(-1)\,(\mathrm{i}eZ_1)^2\left(\frac{1}{\mathrm{i}}\right)^2\int\frac{d^4l}{(2\pi)^4}Tr\left[\widetilde{S}\left(\not{l}+\not{p}\right)\gamma^{\mu}\widetilde{S}\left(\not{l}\right)\gamma^{\nu}\right]}^{\boxed{\text{a}}} - \mathrm{i}\,(Z_3-1)\,p^2\left(\eta^{\mu\nu}-\frac{p^{\mu}p^{\nu}}{p^2}\right)+\mathcal{O}\left(e^4\right) \tag{3.5}$$

The evaluation of the diagrams

We first use $Z_1 = 1 + \mathcal{O}\left(e^4\right)$ which simplifies the first term and allows for the evaluation of the $\mathcal{O}(e^2)$ contribution to the photon self-energy. Next we expand the trace of $\boxed{\text{a}}$ from Eq: 3.5:

$$Tr\left[\widetilde{S}\left(\not{l}+\not{p}\right)\gamma^{\mu}\widetilde{S}\left(\not{l}\right)\gamma^{\nu}\right] = \frac{1}{\left((l+p)^2+m^2\right)(l^2+m^2)}Tr\left[\left(-\not{l}-\not{p}+m\right)\gamma^{\mu}\left(-\not{l}+m\right)\gamma^{\nu}\right] \tag{3.6}$$

We now expand the trace from Eq: 3.6 and use the γ-matrix identities from App: A:

$$\begin{aligned} Tr\left[\left(-\not{l}-\not{p}+m\right)\gamma^{\mu}\left(-\not{l}+m\right)\gamma^{\nu}\right] =& l_{\rho}\,(l+p)_{\sigma}\,Tr\left[\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}\right]-(l+p)_{\rho}\,m\,\underbrace{Tr\left[\gamma^{\rho}\gamma^{\mu}\gamma^{\nu}\right]}_{\to 0} \\ & -l_{\rho}m\,\underbrace{Tr\left[\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}\right]}_{\to 0}+m^2\,Tr\left[\gamma^{\mu}\gamma^{\nu}\right] \end{aligned} \tag{3.7}$$

We cancel the trace over an odd number of γ-matrices by applying Eq: A.14. We then apply Eq: A.15 and Eq: A.16 to get rid of the traces, and we obtain:

$$Tr\left[\left(-\not{l}-\not{p}+m\right)\gamma^{\mu}\left(-\not{l}+m\right)\gamma^{\nu}\right] = 4\left[(l+p)^{\mu}l^{\nu}+l^{\mu}(l+p)^{\nu}-\left[l\cdot(l+p)+m^2\right]\eta^{\mu\nu}\right]. \tag{3.8}$$

This is plugged back into Eq: 3.6 and the fermion loop contribution of Eq: 3.5 becomes:

$$\boxed{\text{a}} = -4e^2\int\frac{d^4l}{(2\pi)^4}\frac{(l+p)^{\mu}l^{\nu}+l^{\mu}(l+p)^{\nu}-\left[l\cdot(l+p)+m^2\right]\eta^{\mu\nu}}{(l^2+m^2)\left((l+p)^2+m^2\right)} \tag{3.9}$$

We now introduce *Feynman parameters* to further evaluate Eq: 3.9 by combining the denominators. Details about the Feynman parameter trick can be found in App: B. We apply Eq: B.2 to Eq: 3.9 by choosing $A = (l+p)^2+m^2$ and $B = l^2+m^2$, we then get:

$$\boxed{\text{a}} = -4e^2 \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \frac{(l+p)^\mu l^\nu + l^\mu (l+p)^\nu - \left[l \cdot (l+p) + m^2\right] \eta^{\mu\nu}}{\left(l^2 + 2x\,(l \cdot p) + x^2 p^2 + m^2\right)^2} \tag{3.10}$$

Now we perform a shift in the integration variable from l to $q = l + xp$ which modifies the term in the denominator of Eq: 3.10 as:

$$l^2 + 2x\,(l \cdot p) + x^2 p^2 + m^2)^2 = (l + xp)^2 + x\,(1-x)\,p^2 + m^2 = q^2 + D_0^2 \tag{3.11}$$

Here, we made the definition:

$$D_0^2 \equiv x\,(1-x)\,p^2 + m^2 \tag{3.12}$$

We perform the same shift in integration variable for the numerator:

$$l \cdot (l+p) = (q - xp)\,(q + (1-x)\,p) = q^2 + (p \cdot q) - x\,(1-x)\,p^2 \tag{3.13}$$

$$\begin{aligned} (l+p)^\mu l^\nu &= (q + (1-x)\,p)^\mu\,(q - xp)^\nu \\ &= q^\mu q^\nu + (1-x)\,p^\mu q^\nu - x q^\mu p^\nu - x\,(1-x)\,p^\mu p^\nu \end{aligned} \tag{3.14}$$

All the terms in the numerator that are odd in q will vanish since we integrate them over an even integration domain with a denominator that is even in q. Every contribution at q will cancel against an equal but negative contribution at $-q$, thus we can neglect them.

$$\boxed{\text{a}} = -4e^2 \int \frac{d^4 q}{(2\pi)^4} \int_0^1 dx \frac{2q^\mu q^\nu - 2x\,(1-x)\,p^\mu p^\nu - \left[q^2 - x(1-x)p^2 + m^2\right] \eta^{\mu\nu}}{\left(q^2 + D_0^2\right)^2} \tag{3.15}$$

The momentum integral in Eq: 3.15 diverges since the integrand does not vanish fast enough for large momenta, irrespective of the value that we choose for m_γ. We perform *dimensional regularisation* to regularise this UV-divergence and parametrise the divergence such that the integral can be evaluated.

In dimensional regularisation one goes from $d = 4$ dimensions to $d = 4 - \epsilon$ dimensions. As a consequence, the coupling constant e will get a mass dimension $[e] = \frac{1}{2}\epsilon$ *. We introduce an arbitrary mass scale $\widetilde{\mu}^2 = \frac{e^{\gamma_E}}{4\pi}\mu^2$ ** where $[\mu] = 1$ in mass dimensions. We now write $e \to e\widetilde{\mu}^{\frac{\epsilon}{2}}$ to ensure that e remains a dimensionless parameter.

*This can be verified by demanding that the QED Lagrangian has a dimension of $d = 4 - \epsilon$. From the mass terms and the kinetic terms the dimension of the fields $A^\mu(x)$ and $\Psi(x)$ can be determined after that one can determine $[e]$ from the interaction term.

**This special choice for $\widetilde{\mu}$ in terms of μ motivated by Ref. [17, ch. 14] cancels some terms down the road. $\gamma_E \approx 0.5772...$, the Euler-Mascheroni constant.

If we also switch the order of integration then we get the Eq: 3.15 becomes:

$$\boxed{\text{a}} = -4e^2\widetilde{\mu}^{\epsilon}\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\frac{2q^{\mu}q^{\nu}-2x\left(1-x\right)p^{\mu}p^{\nu}-\left[q^2-x\left(1-x\right)p^2+m^2\right]\eta^{\mu\nu}}{\left(q^2+D_0^2\right)^2} \tag{3.16}$$

We now apply some momentum integral identities such that Eq: 3.16 becomes easier to handle. We first apply Eq: C.2 and separate the integral in two pieces:

$$\begin{aligned}\boxed{\text{a}} = 4e^2\widetilde{\mu}^{\epsilon}\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\Bigg[&\overbrace{\left(1-\frac{2}{d}\right)\eta^{\mu\nu}\frac{q^2}{\left(q^2+D_0^2\right)^2}}^{\boxed{\text{a.1}}} \\ &+\frac{2x\left(1-x\right)p^{\mu}p^{\nu}+\left[m^2-x\left(1-x\right)p^2\right]\eta^{\mu\nu}}{\left(q^2+D_0^2\right)^2}\Bigg]\end{aligned} \tag{3.17}$$

We now apply Eq: C.4 from App: C.2 to $\boxed{\text{a.1}}$:

$$\int d^dq\,\frac{q^2}{\left(q^2+D_0^2\right)^2} = \frac{d}{2}\int d^dq\,\frac{1}{\left(q^2+D_0^2\right)} = \frac{d}{2}\int d^dq\,\frac{\left(q^2+D_0^2\right)}{\left(q^2+D_0^2\right)^2} \tag{3.18}$$

$$\Rightarrow\ \boxed{\text{a.1}} = \left(1-\frac{2}{d}\right)\int d^dq\,\frac{q^2}{\left(q^2+D_0^2\right)^2} = -D_0^2\int d^dq\,\frac{1}{\left(q^2+D_0^2\right)^2} \tag{3.19}$$

We now substitute $\boxed{\text{a.1}}$ back into Eq: 3.17 and fill out $D_0^2 = x\left(1-x\right)p^2 + m^2$, we find:

$$\boxed{\text{a}} = \overbrace{-8e^2\widetilde{\mu}^{\epsilon}}^{\boxed{\text{b}}}\,p^2\left(\eta^{\mu\nu}-\frac{p^{\mu}p^{\nu}}{p^2}\right)\overbrace{\int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\frac{x\left(1-x\right)}{\left(q^2+D_0^2\right)^2}}^{\boxed{\text{b}}} \tag{3.20}$$

We see in Eq: 3.20 the justification of our assumption on the tensorial structure of $\Pi^{\mu\nu}\left(p\right)$. We now zoom in on the fermion loop contribution to $\Pi(p^2)$, namely $\boxed{\text{b}}$. We now perform a *Wick rotation* as explained in App: C.3 to circumvent the singularities of our integration domain by using Eq: C.6.

$$\boxed{\text{b}} = -i8e^2\widetilde{\mu}^{\epsilon}\int_0^1 dx\,x(1-x)\int\frac{d^d\overline{q}}{(2\pi)^d}\frac{1}{\left(\overline{q}^2+D_0^2\right)^2}. \tag{3.21}$$

We now evaluate the momentum integral in Eq: 3.21 by using Eq: C.7 from App: C.4 and setting $d = 4-\epsilon$, this gives:

$$\boxed{b} = -\frac{ie^2}{2\pi^2}\Gamma\left(\frac{\epsilon}{2}\right)\int_0^1 dx\, x\,(1-x)\left(\frac{4\pi\tilde{\mu}^2}{D_0^2}\right)^{\frac{\epsilon}{2}} \tag{3.22}$$

The use of Eq: C.7 constrains the regularisation parameter as $\epsilon > 0$, this is important whenever we use that equation. We now expand the Gamma function for infinitesimal values for ϵ Eq: C.10, fill out $\tilde{\mu}$ in terms of μ and use the fact that $x^\epsilon = 1 + \epsilon \ln x + \mathcal{O}\left(\epsilon^2\right)$, after which we find:

$$\boxed{b} = -\frac{ie^2}{2\pi^2}\left(\frac{2}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)\right)\int_0^1 dx\, x\,(1-x)\left[1 + \frac{\epsilon}{2}\left(\gamma_E - \ln\frac{D_0^2}{\mu^2}\right) + \mathcal{O}\left(\epsilon^2\right)\right] \tag{3.23}$$

$$= -\frac{ie^2}{\pi^2}\left(\frac{1}{\epsilon} - \frac{1}{2}\gamma_E + \mathcal{O}(\epsilon)\right)\left[\frac{1}{6}\left(1 + \frac{\epsilon}{2}\gamma_E\right) - \frac{\epsilon}{2}\int_0^1 dx\, x\,(1-x)\ln\frac{D_0^2}{\mu^2} + \mathcal{O}\left(\epsilon^2\right)\right] \tag{3.24}$$

$$= -\frac{ie^2}{6\pi^2}\left[\frac{1}{\epsilon} - 3\int_0^1 dx\, x\,(1-x)\ln\frac{D_0^2}{\mu^2}\right] \tag{3.25}$$

Now we can substitute $\boxed{b}$ from Eq: 3.25 back into $\boxed{a}$ at Eq: 3.5, and write down the full expression for $\Pi\left(p^2\right)$ in accordance with Eq: 5.30 in Ref. [21].

$$\boxed{\Pi\left(p^2\right) = -\frac{e^2}{6\pi^2}\left[\frac{1}{\epsilon} - 3\int_0^1 dx\, x\,(1-x)\ln\frac{D_0^2}{\mu^2}\right] - (Z_3 - 1) + \mathcal{O}\left(e^4\right)} \tag{3.26}$$

3.2 The electron self-energy

In this section we determine the one-loop quantum corrections to the electron self-energy. In analogy with the photon self-energy we discuss the definition of the electron self-energy and its relation to the electron exact propagator and the pole mass. We evaluate the diagrams that contribute to the electron self-energy using a photon regulator mass for the infrared divergences and dimensional regularisation for the UV-divergences.

The exact electron propagator

The electron self-energy is conceptually defined analogous to the photon self-energy. Only, we now sum the 1PI diagrams with two electron instead of two photon sources and that the electron self-energy has spinor indices.

Let us first see how the electron self-energy $\Sigma(\not{p})$ is defined algebraically, and how it relates to the exact electron propagator $\widetilde{\mathbf{S}}(\not{p})$.

$$\begin{aligned}\frac{1}{\mathrm{i}}\widetilde{\mathbf{S}}(\not{p}) &= \frac{1}{\mathrm{i}}\widetilde{S}(\not{p})\sum_{n=0}^{\infty}\left[\mathrm{i}\Sigma(\not{p})\frac{1}{\mathrm{i}}\widetilde{S}(\not{p})\right]^{n} \\ &= \frac{1}{\mathrm{i}}\widetilde{S}(\not{p})\left[1+\mathrm{i}\Sigma(\not{p})\frac{1}{\mathrm{i}}\widetilde{S}(\not{p})\sum_{n-1=0}^{\infty}\left[\mathrm{i}\Sigma(\not{p})\frac{1}{\mathrm{i}}\widetilde{S}(\not{p})\right]^{n-1}\right] \\ &= \frac{1}{\mathrm{i}}\widetilde{S}(\not{p})\left[1+\mathrm{i}\Sigma(\not{p})\frac{1}{\mathrm{i}}\widetilde{\mathbf{S}}(\not{p})\right]\end{aligned} \tag{3.27}$$

Let us now have a look at the exact inverse propagator in terms of the inverse free-field propagator and the self-energy.

$$\begin{aligned}\left[1-\widetilde{S}(\not{p})\Sigma(\not{p})\right]\widetilde{\mathbf{S}}(\not{p}) &= \widetilde{S}(\not{p}) \\ \widetilde{\mathbf{S}}^{-1}(\not{p}) &= \widetilde{S}^{-1}(\not{p})-\Sigma(\not{p}) = \not{p}+m-\Sigma(\not{p})\end{aligned} \tag{3.28}$$

Therefore the exact electron propagator in terms of the electron self-energy can be written as:

$$\frac{1}{i}\widetilde{\mathbf{S}}(\not{p}) = \frac{-i}{\not{p}+m-i\epsilon-\Sigma(\not{p})} \tag{3.29}$$

The choice of counter terms or i.o.w. the renormalisation scheme affects both m and $\Sigma(\not{p})$. However, the location of the pole $\not{p} = z_{pole}$, which can be complex, is not affected by a different choice of renormalisation scheme. This motivates the definition of a mass parameter that is known as the pole mass m_{pole} through the following self-consistency relation:

$$m_{pole} \equiv -\mathrm{Re}\left[z_{pole}\right] \tag{3.30}$$

When we additionally define $\gamma \equiv 2\mathrm{Im}\left[z_{pole}\right]$ then it is possible to expand Eq: 3.29 around z_{pole} and express the pole mass in terms of the self-energy function, in accordance with Ref. [22]. **

$$\frac{1}{\mathrm{i}}\widetilde{\mathbf{S}}(\not{p}) = \frac{-\mathrm{i}\mathcal{Z}\left(z_{pole}\right)}{\not{p}+m_{pole}-\frac{\mathrm{i}}{2}\gamma}+\mathrm{n.p.} \tag{3.31}$$

A comparison of Eq: 3.31 with Eq: 3.29 allows us to express there parameters in terms of the self-energy function $\Sigma(\not{p})$. The parameter γ is

**The expansion of the exact propagator around z_{pole} requires the following expansion for the self-energy : $\Sigma(\not{p}) = \Sigma\left(z_{pole}\right)+\frac{\partial}{\partial\not{p}}\Sigma(\not{p})\Big|_{\not{p}=z_{pole}}\left(\not{p}-z_{pole}\right)+\mathcal{O}(\left(\not{p}-z_{pole}\right)^{2})$

proportional to the coupling constant squared and in QED just zero, this allows us to set $m_{pole} \to m$ on the right hand side of the self-consistency relation.

$$\begin{aligned} m_{pole} &= m - \mathrm{Re}\left[\Sigma\left(z_{pole}\right)\right] && \approx m - \mathrm{Re}\left[\Sigma\left(-m\right)\right] \\ \gamma &= 2\mathrm{Im}\left[\Sigma_R\left(z_{pole}\right)\right] && \approx 2\mathrm{Im}\left[\Sigma_R\left(z_{pole}\right)\right] \\ Z^{-1}\left(z_{pole}\right) &= 1 - \frac{\partial}{\partial \not{p}}\Sigma\left(\not{p}\right)\Big|_{\not{p}=z_{pole}} && \approx 1 - \frac{\partial}{\partial \not{p}}\Sigma\left(\not{p}\right)\Big|_{\not{p}=-m} \end{aligned} \tag{3.32}$$

The contributing diagrams

The diagrams that contribute to $\Sigma\left(\not{p}\right)$ in the one-loop approximation are shown in Fig: 3.3. The first diagram contains a virtual photon arc, and the second one comes from the counter term vertices involving both Z_0 and Z_2.

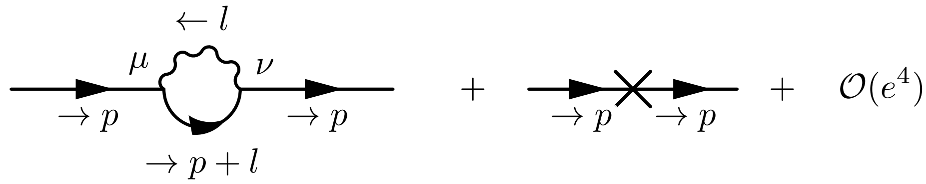


Figure 3.3: *This diagrammatic expression shows us the diagrams that contribute to the electron self-energy at one loop.*

$$\begin{aligned} \mathrm{i}\Sigma\left(\not{p}\right) = \overbrace{\left(\mathrm{i}eZ_1\right)^2 \left(\frac{1}{\mathrm{i}}\right)^2 \int \frac{d^4 l}{\left(2\pi\right)^4} \left[\gamma^\nu \widetilde{S}\left(\not{p} + \not{l}\right)\gamma^\mu\right] \widetilde{\Delta}_{\mu\nu}\left(l\right)}^{\text{a}} \\ -\mathrm{i}\left(Z_2 - 1\right)\not{p} - \mathrm{i}\left(Z_0 - 1\right)m + \mathcal{O}\left(e^4\right) \end{aligned} \tag{3.33}$$

The evaluation of the diagrams with a photon mass

Again, we first use that $Z_1 = 1 + \mathcal{O}(e^2)$ to allow the evaluation of the $\mathcal{O}\left(e^4\right)$ contribution to the electron self-energy and we put $d = 4 - \epsilon$ to apply dimensional regularisation. We first treat the integrand of a from Eq: 3.33 by expanding the propagators:

$$
\begin{aligned}
\left[\gamma^{\nu}\widetilde{S}\left(\not p+\not l\right)\gamma^{\mu}\right]\widetilde{\Delta}_{\mu\nu}\left(l\right) = & \frac{\gamma^{\mu}\left(-\not p-\not l+m\right)\gamma_{\mu}}{\left(\left(p+l\right)^{2}+m^{2}\right)\left(l^{2}+m_{\gamma}^{2}\right)} - \left(1-\lambda^{-2}\right) \\
& \times \overbrace{\left[\not l\,\frac{1}{\not p+\not l+m}\,\not l\right]}^{\boxed{\text{b}}}\frac{1}{\left(l^{2}+m_{\gamma}^{2}\right)\left(l^{2}+\lambda^{-2}m_{\gamma}^{2}\right)}
\end{aligned}
\tag{3.34}
$$

We now first use the γ-matrix identities shown in Eq: A.9 and Eq: A.11 to simplify the numerator of the gauge-independent part of Eq: 3.34:

$$
\gamma^{\mu}\left(-\not p-\not l+m\right)\gamma_{\mu} = -\gamma^{\mu}\left(\not p+\not l\right)\gamma_{\mu}+m\gamma^{\mu}\gamma_{\mu} = -\left(d-2\right)\left(\not p+\not l\right)-dm \tag{3.35}
$$

For $\boxed{\text{b}}$, the gauge-dependent part in square brackets of Eq: 3.34, we first try to cancel factors before we use γ-matrix identities:

$$
\begin{aligned}
\boxed{\text{b}} &= \overbrace{\left[\left(\not p+\not l+m\right)-\left(\not p+m\right)\right]}^{\not l}\frac{1}{\not p+\not l+m}\left[\left(\not p+\not l+m\right)-\left(\not p+m\right)\right] \\
&= \left(\not p+\not l+m\right)-2\left(\not p+m\right)+\left(\not p+m\right)\frac{1}{\not p+\not l+m}\left(\not p+m\right) \\
&= \not l-\left(\not p+m\right)+\frac{\left(\not p+m\right)\left[\left(-\not p+m\right)-\not l\right]\left(\not p+m\right)}{\left(p+l\right)^{2}+m^{2}}
\end{aligned}
\tag{3.36}
$$

Now we can use γ-matrix identities to simplify Eq: 3.36, we especially made use of Eq: A.10 and Eq: A.12:

$$
\begin{aligned}
\boxed{\text{b}} &= \not l-\left(\not p+m\right)+\frac{\left(p^{2}+m^{2}\right)\left(\not p+m\right)-\left(\not p+m\right)\not l\left(\not p+m\right)}{\left(p+l\right)^{2}+m^{2}} \\
&= \not l-\left(\not p+m\right)-\frac{\left(p^{2}+m^{2}\right)\not l}{\left(p+l\right)^{2}+m^{2}}+\left(\not p+m\right)\frac{p^{2}+2\left(p\cdot l\right)+m^{2}}{\left(p+l\right)^{2}+m^{2}} \\
&= \cancelto{0}{\not l}-\frac{\left(p^{2}+m^{2}\right)\not l}{\left(p+l\right)^{2}+m^{2}}-\left(\not p+m\right)\frac{l^{2}}{\left(p+l\right)^{2}+m^{2}}
\end{aligned}
\tag{3.37}
$$

The first term in $\boxed{\text{b}}$ vanishes when we put it back in $\boxed{\text{a}}$ through Eq: 3.34. The integrand will then be an odd function of the integration variable over an even domain.

$$
-e^{2}\widetilde{\mu}^{\epsilon}\left(1-\lambda^{-2}\right)\int\frac{d^{d}l}{\left(2\pi\right)^{d}}\frac{\not l}{\left(l^{2}+m_{\gamma}^{2}\right)\left(l^{2}+\lambda^{-2}m_{\gamma}^{2}\right)} = 0 \tag{3.38}
$$

For the last two simplifications of the integrand of $\boxed{\text{a}}$ we substitute

$l^2 \to \left(l^2 + \lambda^{-2} m^2\right) - \lambda^{-2} m_\gamma^2$ and replace $\not{l} \to \not{p}\frac{(p\cdot q)}{p^2}$ which is justified in Eq: C.1 of App: C.1, we find:

$$
\begin{aligned}
\boxed{\text{a}} =& e^2\widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \frac{(2-d)\left(1+\frac{p\cdot l}{p^2}\right)\not{p} - dm + \left(1-\lambda^{-2}\right)(\not{p}+m)}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)} \\
&+ e^2\widetilde{\mu}^\epsilon \left(1-\lambda^{-2}\right) \int \frac{d^d l}{(2\pi)^d} \frac{\frac{p^2+m^2}{p^2}(p\cdot l)\not{p} + \lambda^{-2} m_\gamma^2 (\not{p}+m)}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)} \\
=& - e^2\widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \frac{\left[(d-2)\left(1+\frac{p\cdot l}{p^2}\right) - \left(1-\lambda^{-2}\right)\right]\not{p} + \left(d+\lambda^{-2}-1\right)m}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)} \\
&- e^2\widetilde{\mu}^\epsilon \left(1-\lambda^{-2}\right) \int \frac{d^d l}{(2\pi)^d} \frac{\left[\lambda^{-2}m_\gamma^2 - \frac{p^2+m^2}{p^2}(p\cdot l)\right]\not{p} + \lambda^{-2}m_\gamma^2 m}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)}
\end{aligned}
\tag{3.39}
$$

Now we plug $\boxed{\text{a}}$ from Eq: 3.39 back into Eq: 3.33. We then decompose the electron self-energy into two functions via $\Sigma(\not{p}) = mA\left(p^2\right) + \not{p}B\left(p^2\right)$. $A\left(p^2\right)$ in Eq: 3.40 contains Z_0 and multiplies the mass term. $B\left(p^2\right)$ in Eq: 3.41 contains Z_2 and multiplies the kinetic term.

$$
\begin{aligned}
A\left(p^2\right) =& \overbrace{\mathrm{i}e^2\widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \frac{d+\lambda^{-2}-1}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)}}^{\boxed{\text{c}}} + \overbrace{\mathrm{i}e^2\widetilde{\mu}^\epsilon\left(1-\lambda^{-2}\right)}^{\boxed{\text{d}}} \\
&\times \overbrace{\int \frac{d^d l}{(2\pi)^d} \frac{\lambda^{-2}m_\gamma^2}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)}}^{\boxed{\text{d}}} - (Z_0-1) + \mathcal{O}\left(e^4\right)
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
B\left(p^2\right) =& \overbrace{\mathrm{i}e^2\widetilde{\mu}^{\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{(d-2)\left(1+\frac{p\cdot l}{p^2}\right)-\left(1-\lambda^{-2}\right)}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)}}^{\text{e}} - \overbrace{\mathrm{i}e^2\widetilde{\mu}^{\epsilon}\left(1-\lambda^{-2}\right)\frac{p^2+m^2}{p^2}}^{\text{f}} \\
&\times\overbrace{\int\frac{d^dl}{(2\pi)^d}\frac{p\cdot l}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)}}^{\text{f}} + \overbrace{\mathrm{i}e^2\widetilde{\mu}^{\epsilon}\left(1-\lambda^{-2}\right)}^{\text{g}} \\
&\times\overbrace{\int\frac{d^dl}{(2\pi)^d}\frac{\lambda^{-2}m_\gamma^2}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)}}^{\text{g}} - (Z_2-1)+\mathcal{O}\left(e^4\right)
\end{aligned}
\tag{3.41}
$$

We perform the momentum integral of c in Eq: 3.40. We use the same tricks as we did for the integral in Eq: 3.9.

$$
\begin{aligned}
\int\frac{d^dl}{(2\pi)^d}\frac{1}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)} &= \int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{1}{\left(l^2+2xp\cdot l+x\left(p^2+m^2\right)+(1-x)\,m_\gamma^2\right)^2} \\
&= \int_0^1 dx\int\frac{d^dq}{(2\pi)^d}\frac{1}{\left(q^2+D_1^2\right)^2}
\end{aligned}
\tag{3.42}
$$

Here, we define for convenience:

$$
D_1^2 \equiv x\,(1-x)\,p^2 + x\left(m^2-m_\gamma^2\right)+m_\gamma^2 \tag{3.43}
$$

We perform a coordinate shift $q = l + xp$, we use Eq: 3.42 in c, we do a Wick rotation and evaluate the momentum integral.

$$
\text{c} = -e^2\widetilde{\mu}^{\epsilon}\int_0^1 dx\int\frac{d^d\overline{q}}{(2\pi)^d}\frac{d+\lambda^{-2}-1}{\left(\overline{q}^2+D_1^2\right)^2} = -e^2\widetilde{\mu}^{\epsilon}\left(d+\lambda^{-2}-1\right)\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\,\Gamma(2)}\int_0^1 dx\left(D_1^2\right)^{-\left(2-\frac{1}{2}d\right)} \tag{3.44}
$$

We substitute $d = 4-\epsilon$ and expand for infinitesimal ϵ:

$$
\begin{aligned}
\boxed{c} &= -\frac{e^2}{16\pi^2}\widetilde{\mu}^{\epsilon}\Gamma\left(\frac{\epsilon}{2}\right)\left(3+\lambda^{-2}-\epsilon\right)\int_0^1 dx\left(\frac{4\pi}{D_1^2}\right)^{\frac{\epsilon}{2}} \\
&= -\frac{e^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right)\left(3+\lambda^{-2}-\epsilon\right)\left(1+\frac{\epsilon}{2}\int_0^1 dx\ln\frac{4\pi\widetilde{\mu}^2}{D_1^2}+\mathcal{O}\left(\epsilon^2\right)\right) \\
&= -\frac{e^2}{8\pi^2}\left(3+\lambda^{-2}\right)\frac{1}{\epsilon}+\frac{e^2}{8\pi^2}\left[1+\frac{1}{2}\left(3+\lambda^{-2}\right)\int_0^1 dx\ln\frac{D_1^2}{\mu^2}\right]+\mathcal{O}(\epsilon)
\end{aligned}
\tag{3.45}
$$

The integral in $\boxed{c}$ can be evaluated by a symbolic math calculator such as Sympy [23], but the result is not very appealing. We were unable to find a compact and meaningful series expansion in terms of $\kappa \equiv \frac{m_\gamma^2}{m^2}$, unless we make additional assumptions for p^2. However, if we set $\kappa = 0$ then we find:

$$
\begin{aligned}
\int_0^1 dx \ln\frac{D_1^2}{\mu^2}\bigg|_{m_\gamma^2=0} &= \int_0^1 dx\ln\left(\frac{x(1-x)p^2+xm^2}{\mu^2}\right) \\
&= \int_0^1 dx\ln x+\int_0^1 dx\ln\left(\frac{p^2+m^2-xp^2}{\mu^2}\right) = -1-\frac{\mu^2}{p^2}\int_{\frac{p^2+m^2}{\mu^2}}^{\frac{m^2}{\mu^2}} du\ln u \\
&= -2-\frac{m^2}{p^2}\ln\frac{m^2}{\mu^2}+\frac{p^2+m^2}{p^2}\ln\left(\frac{p^2+m^2}{\mu^2}\right) \\
&= -2+\ln\frac{m^2}{\mu^2}+\frac{p^2+m^2}{p^2}\ln\left(1+\frac{p^2}{m^2}\right)
\end{aligned}
\tag{3.46}
$$

We now continue with the momentum integral of $\boxed{d}$ in Eq: 3.40, where we neglect the pre-factor again. For this integral we need to use Eq: B.3 with $A = \left((p+l)^2+m^2\right)$, $B = \left(l^2+\lambda^{-2}m_\gamma^2\right)$ and $C = \left(l^2+m_\gamma^2\right)$.

$$
\begin{aligned}
&\int \frac{d^d l}{(2\pi)^d}\frac{1}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)} \\
&= 2\int_0^1 dx\int_0^{1-x} dy\int\frac{d^d l}{(2\pi)^d}\frac{1}{\left(l^2+2xp\cdot l+x\left(p^2+m^2-m_\gamma^2\right)-y\left(1-\lambda^{-2}\right)m_\gamma^2+m_\gamma^2\right)^3} \\
&= 2\int_0^1 dx\int_0^{1-x} dy\int\frac{d^d q}{(2\pi)^d}\frac{1}{\left(q^2+D^2\right)^3}
\end{aligned}
\tag{3.47}
$$

In Eq: 3.47 we define $D^2 \equiv x(1-x)p^2+x\left(m^2-m_\gamma^2\right)-y\left(1-\lambda^{-2}\right)+m_\gamma^2$ and $q = l+xp$. Now we use Eq: 3.47 in $\boxed{d}$ and we perform a Wick rotation and evaluate the momentum integral.

$$
\begin{aligned}
\boxed{\text{d}} &= -2e^2\tilde{\mu}^{\epsilon}\left(1-\lambda^{-2}\right)\int_0^1 dx \int_0^{1-x} dy \int \frac{d^d\bar{q}}{(2\pi)^d}\frac{\lambda^{-2}m_\gamma^2}{\left(\bar{q}^2+D^2\right)^3} \\
&= -\frac{e^2}{16\pi^2}\left(1-\lambda^{-2}\right)\int_0^1 dx \int_0^{1-x} dy \frac{\lambda^{-2}m_\gamma^2}{D^2} \\
&= \frac{e^2}{16\pi^2}\lambda^{-2}\int_0^1 dx \int_{D_1^2}^{\widetilde{D}_1^2}\frac{du}{u} = -\frac{e^2}{16\pi^2}\lambda^{-2}\int_0^1 dx\left[\ln\frac{D_1^2}{m^2}-\ln\frac{\widetilde{D}_1^2}{m^2}\right]
\end{aligned}
\tag{3.48}
$$

Here, we made the additional definition:

$$
\widetilde{D}_1^2 \equiv x(1-x)\,p^2 + x\left(m^2-\lambda^{-2}m_\gamma^2\right)+\lambda^{-2}m_\gamma^2 \tag{3.49}
$$

Now we substitute c from Eq: 3.45 and d from Eq: 3.48 back into Eq: 3.40. In this equation, we can set $m_\gamma^2 = 0$ to simplify our result without running into divergences.

$$
\begin{aligned}
A\left(p^2\right) = -\frac{e^2}{8\pi^2}\left(3+\lambda^{-2}\right)\frac{1}{\epsilon}+\frac{e^2}{8\pi^2}\Bigg[1+\frac{1}{2}\left(3+\lambda^{-2}\right)\int_0^1 dx\ln\frac{D_1^2}{\mu^2} \\
-\frac{1}{2}\lambda^{-2}\int_0^1 dx\left[\ln\frac{D_1^2}{m^2}-\ln\frac{\widetilde{D}_1^2}{m^2}\right]\Bigg]-(Z_0-1)+\mathcal{O}\left(e^4\right)
\end{aligned}
\tag{3.50}
$$

$$
\boxed{
\begin{aligned}
A\left(p^2\right)\Big|_{m_\gamma^2=0} &= -\frac{e^2}{8\pi^2}\left(3+\lambda^{-2}\right)\frac{1}{\epsilon}-\frac{e^2}{8\pi^2}\Bigg[2+\lambda^{-2}-\frac{1}{2}\left(3+\lambda^{-2}\right) \\
&\times\left[\ln\frac{m^2}{\mu^2}+\frac{p^2+m^2}{p^2}\ln\left(1+\frac{p^2}{m^2}\right)\right]\Bigg]-(Z_0-1)+\mathcal{O}\left(e^4\right)
\end{aligned}
}
\tag{3.51}
$$

We now turn our attention to the integrals in $B\left(p^2\right)$ in Eq: 3.41. We first separate e into two parts:

$$
\begin{aligned}
\boxed{\text{e}} = &\overbrace{\mathrm{i}e^2\tilde{\mu}^{\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{d+\lambda^{-2}-3}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)}}^{\text{e.1}} \\
&+\overbrace{\mathrm{i}e^2\tilde{\mu}^{\epsilon}\frac{d-2}{p^2}\int\frac{d^dl}{(2\pi)^d}\frac{p\cdot l}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)}}^{\text{e.2}}
\end{aligned}
\tag{3.52}
$$

If we put $\lambda^{-2}\to\lambda^{-2}-2$ in c from Eq: 3.40, then c equals e.1 in the above expression. So, we do this replacement in Eq: 3.45 and find e.1:

$$\text{e.1} = -\frac{e^2}{8\pi^2}\left(1+\lambda^{-2}\right)\frac{1}{\epsilon} + \frac{e^2}{8\pi^2}\left[1+\frac{1}{2}\left(1+\lambda^{-2}\right)\int_0^1 dx \ln\frac{D_1^2}{\mu^2}\right] + \mathcal{O}(\epsilon) \tag{3.53}$$

We continue with e.2 in Eq: 3.52, we do a Feynman parametrisation and discard the term odd in the integration variable:

$$\begin{aligned}\int \frac{d^d l}{(2\pi)^d}\frac{p\cdot l}{\left((p+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)} &= \int_0^1 dx \int \frac{d^d l}{(2\pi)^d}\frac{p\cdot l}{\left(l^2+2xp\cdot l + x\left(p^2+m^2\right)+(1-x)\,m_\gamma^2\right)^2}\\ &= \int_0^1 dx \int \frac{d^d q}{(2\pi)^d}\frac{p\cdot\left(\cancelto{0}{q} - xp\right)}{\left(q^2+D_1^2\right)^2}\end{aligned} \tag{3.54}$$

The integral in Eq: 3.54 is plugged back into e.2, and then we perform the momentum integral. We expand for infinitesimal values of ϵ and find:

$$\begin{aligned}\text{e.2} &= e^2\tilde{\mu}^{\epsilon}(d-2)\int_0^1 dx\int\frac{d^d\bar{q}}{(2\pi)^d}\frac{x}{\left(\bar{q}^2+D_1^2\right)^2}\\ &= \frac{e^2}{16\pi^2}(d-2)\,\Gamma\left(2-\frac{d}{2}\right)\int_0^1 dx\, x\left(\frac{4\pi\tilde{\mu}^2}{D_1^2}\right)^{\frac{\epsilon}{2}}\\ &= \frac{e^2}{16\pi^2}(2-\epsilon)\left[\frac{2}{\epsilon}-\gamma_E+\mathcal{O}(\epsilon)\right]\left[\frac{1}{2}-\frac{\epsilon}{2}\int_0^1 dx\, x\ln\frac{D_1^2}{4\pi\tilde{\mu}^2}+\mathcal{O}\left(\epsilon^2\right)\right]\\ &= \frac{e^2}{8\pi^2}\frac{1}{\epsilon}-\frac{e^2}{16\pi^2}\left[1+2\int_0^1 dx\, x\ln\frac{D_1^2}{\mu^2}\right]+\mathcal{O}(\epsilon)\end{aligned} \tag{3.55}$$

The integral in e.2 for $m_\gamma^2 \to 0$ is given by:

$$2\int_0^1 dx\, x\ln\frac{D_1^2}{\mu^2}\bigg|_{m_\gamma^2=0} = -2+\ln\frac{m^2}{\mu^2}-\frac{m^2}{p^2}+\left(\frac{p^2+m^2}{p^2}\right)^2\ln\left(1+\frac{p^2}{m^2}\right) \tag{3.56}$$

Continuing to f, we Feyman parametrise the integral in Eq: 3.41, do a coordinate shift and discard the terms odd in the integration variable:

$$
\begin{aligned}
&\int \frac{d^d l}{(2\pi)^d} \frac{p \cdot l}{\left((p+l)^2 + m^2\right)\left(l^2 + m_\gamma^2\right)\left(l^2 + \lambda^{-2} m_\gamma^2\right)} = \\
&2\int_0^1 dx \int_0^{1-x} dy \int \frac{d^d l}{(2\pi)^d} \frac{p \cdot l}{\left(l^2 + 2xp\cdot l + x\left(p^2 + m^2 - m_\gamma^2\right) - y\left(1-\lambda^{-2}\right) m_\gamma^2 + m_\gamma^2\right)^3} \\
&= 2\int_0^1 dx \int_0^{1-x} dy \int \frac{d^d q}{(2\pi)^d} \frac{p \cdot \left(\cancelto{0}{q} - xp\right)}{\left(q^2 + D^2\right)^3}
\end{aligned}
\tag{3.57}
$$

Here, we made use of $q = l + xp$ and $D^2 \equiv x(1-x)p^2 + x\left(m^2 - m_\gamma^2\right) - y\left(1-\lambda^{-2}\right) m_\gamma^2 + m_\gamma^2$. Now we plug the result of the integral in Eq: 3.57 back into the expression for f and compute the momentum integral:

$$
\begin{aligned}
f &= -2e^2 \tilde{\mu}^{\epsilon}\left(1-\lambda^{-2}\right)\left(p^2+m^2\right)\int_0^1 dx \int_0^{1-x} dy \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{x}{\left(\bar{q}^2 + D^2\right)^3} \\
&= -\frac{e^2}{16\pi^2}\left(1-\lambda^{-2}\right)\left(p^2+m^2\right)\int_0^1 dx \int_0^{1-x} dy \frac{x}{D^2} \\
&= \frac{e^2}{16\pi^2}\frac{p^2+m^2}{m_\gamma^2}\int_0^1 dx \int_{D_1^2}^{\tilde{D}_1^2} du \frac{x}{u} = -\frac{e^2}{16\pi^2}\frac{p^2+m^2}{m_\gamma^2}\int_0^1 dx\, x\left[\ln\frac{D_1^2}{m^2} - \ln\frac{\tilde{D}_1^2}{m^2}\right]
\end{aligned}
\tag{3.58}
$$

To evaluate f we can not simply set $m_\gamma^2 \to 0$, since both the factor in the integral and the denominator of the pre-factor vanish. We must take the limit for $m_\gamma^2 \to 0$ by combining the logarithms and taking the first term in the Taylor expansion of the logarithm. For finite values of the integration parameter and the momentum p the validity of this procedure is guaranteed.

$$
\begin{aligned}
\lim_{m_\gamma^2 \to 0}\left[\ln\frac{D_1^2}{m^2} - \ln\frac{\tilde{D}_1^2}{m^2}\right] &= \lim_{m_\gamma^2 \to 0} -\ln\frac{\tilde{D}_1^2}{D_1^2} \\
&= \lim_{m_\gamma^2 \to 0} -\ln\left(1 - \left(1-\lambda^{-2}\right)\frac{(1-x)\,m_\gamma^2}{x(1-x)p^2 + x\left(m^2 - m_\gamma^2\right) + m_\gamma^2}\right) \\
&= \left(1-\lambda^{-2}\right)\frac{(1-x)\,m_\gamma^2}{x(1-x)p^2 + xm^2} + \mathcal{O}\left(m_\gamma^4\right)
\end{aligned}
\tag{3.59}
$$

The validity of this procedure is shown more rigorously by setting $m_\gamma^2 \to 0$ in Eq: 3.57. We then Feynman parametrise, which leads to the same conclusion as Eq: 3.59.

$$\begin{aligned}\frac{p^2+m^2}{m_\gamma^2}\int_0^1 dx\frac{x(1-x)\,m_\gamma^2}{x(1-x)\,p^2+xm^2} &= \left(p^2+m^2\right)\left[1-\int_0^1 dx\frac{m^2}{p^2+m^2-xp^2}\right]\\ =\frac{p^2+m^2}{p^2}\left[1+\frac{m^2}{p^2}\int_{p^2+m^2}^{m^2}\frac{du}{u}\right] &= \frac{p^2+m^2}{p^2}\left[1-\frac{m^2}{p^2}\ln\left(1+\frac{p^2}{m^2}\right)\right]\end{aligned} \tag{3.60}$$

g is identically the same as term **d** in Eq: 3.48. Thus, we now substitute **e** from Eq: 3.53 and Eq: 3.55, **f** from Eq: 3.58 and **g** from Eq: 3.48 into Eq: 3.41:

$$\begin{aligned}B\left(p^2\right) = -\frac{e^2}{8\pi^2}\lambda^{-2}\frac{1}{\epsilon}+\frac{e^2}{16\pi^2}\Bigg[1+\int_0^1 dx\left[1+\lambda^{-2}-2x\right]\ln\frac{D_1^2}{\mu^2}\\ -\int_0^1 dx\left(\lambda^{-2}+\frac{p^2+m^2}{m_\gamma^2}x\right)\left[\ln\frac{D_1^2}{m^2}-\ln\frac{\widetilde{D}_1^2}{m^2}\right]\Bigg]-(Z_2-1)+\mathcal{O}\left(e^4\right)\end{aligned} \tag{3.61}$$

$$\boxed{\begin{aligned}B\left(p^2\right)\Big|_{m_\gamma^2=0} = -\frac{e^2}{8\pi^2}\lambda^{-2}\frac{1}{\epsilon}-\frac{e^2}{8\pi^2}\lambda^{-2}\Bigg[1-\frac{1}{2}\ln\frac{m^2}{\mu^2}-\frac{p^2+m^2}{p^2}\\ \times\left[\frac{1}{2}+\ln\left(1+\frac{p^2}{m^2}\right)\right]+\frac{1}{2}\left(\frac{p^2+m^2}{p^2}\right)^2\ln\left(1+\frac{p^2}{m^2}\right)\Bigg]-(Z_2-1)+\mathcal{O}\left(e^4\right)\end{aligned}} \tag{3.62}$$

3.3 The vertex correction with photon regulator mass

The last one-loop quantum corrections that we need to evaluate are the one-loop quantum corrections to the electron-photon vertex. We first discuss the definition of the vertex function and then determine which diagrams contribute to it. Finally, we evaluate those diagrams with a photon regulator mass for the infrared divergences and dimensional regularisation for the ultraviolet divergences.

The exact vertex function

The exact three-point vertex function $\mathbb{V}^\mu\left(k,p,p'\right)$ is defined as the sum of diagrams with one incoming electron source with momentum p, one outgoing electron source with momentum p' and a photon source with a momentum $k=p-p'$ by conservation of momentum.

We write the exact vertex function as the sum of the tree-level contribution with its counter term, the one-loop correction and higher-order corrections:

$$\mathrm{i}\mathbb{V}^{\mu}\left(k,p,p'\right) = \left(\mathrm{i}eZ_1\right)\gamma^{\mu} + \mathrm{i}\mathbb{V}^{\mu}_{1-loop}\left(p,p'\right) + \mathcal{O}\left(e^5\right) \tag{3.63}$$

The contributing diagrams

The only diagram that contributes to the one-loop vertex correction $\mathbb{V}^{\mu}_{1-loop}\left(p,p'\right)$ is the diagram that is associated to the anomalous magnetic moment of the electron, wherein a virtual photon connects the incoming and outgoing electron legs as shown in Fig: 3.4.

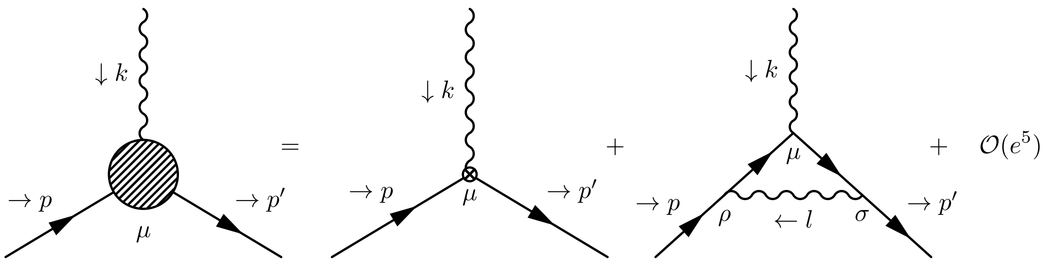


Figure 3.4: *This diagrammatic expression shows us the contribution to the electron-photon vertex up to one loop. The crossed circle vertex has a vertex factor of* $\mathrm{i}Z_1e\gamma^{\mu}$ *instead of the usual* $\mathrm{i}e\gamma^{\mu}$.

$$\mathrm{i}\mathbb{V}^{\mu}_{1-loop}\left(p,p'\right) = \left(\mathrm{i}eZ_1\right)^3\left(\frac{1}{\mathrm{i}}\right)^3\int\frac{d^4l}{(2\pi)^4}\left[\gamma^{\rho}\widetilde{S}\left(\not{p}'+\not{l}\right)\gamma^{\mu}\widetilde{S}\left(\not{p}+\not{l}\right)\gamma^{\sigma}\right]\widetilde{\Delta}_{\rho\sigma}\left(l\right) \tag{3.64}$$

The evaluation of the diagram

Again, we use that $Z_1 = 1 + \mathcal{O}\left(e^2\right)$ to allow the evaluation of the $\mathcal{O}\left(e^3\right)$ contribution to the one-loop vertex correction and we also put $d = 4 - \epsilon$ to apply dimensional regularisation. We then expand Eq: 3.64 by filling out the expressions that we have for the propagators:

$$\left(\frac{\mathbb{V}^{\mu}_{1-loop}(p,p')}{e}\right) = \overbrace{-\,ie^2\widetilde{\mu}^{\epsilon}\int \frac{d^dl}{(2\pi)^d}\gamma^{\rho}\frac{1}{(\not{p}'+\not{l})+m}\gamma^{\mu}\frac{1}{(\not{p}+\not{l})+m}\gamma_{\rho}\frac{1}{l^2+m_{\gamma}^2}}^{\boxed{a}}$$
$$\overbrace{+\,ie^2\widetilde{\mu}^{\epsilon}\left(1-\lambda^{-2}\right)\int\frac{d^dl}{(2\pi)^d}\not{l}\frac{1}{(\not{p}'+\not{l})+m}\gamma^{\mu}\frac{1}{(\not{p}+\not{l})+m}\not{l}\frac{1}{\left(l^2+m_{\gamma}^2\right)\left(l^2+\lambda^{-2}m_{\gamma}^2\right)}}^{\boxed{b}} \tag{3.65}$$

We park the gauge-dependent part b for later evaluation, and continue with the gauge-independent part a:

$$\boxed{a} = -ie^2\widetilde{\mu}^{\epsilon}\int\frac{d^dl}{(2\pi)^d}\gamma^{\rho}\frac{-(\not{p}'+\not{l})+m}{(p'+l)^2+m^2}\gamma^{\mu}\frac{-(\not{p}+\not{l})+m}{(p+l)^2+m^2}\gamma_{\rho}\frac{1}{l^2+m_{\gamma}^2} \tag{3.66}$$

We expand the numerator of a in Eq: 3.66 in terms of nine factors. We take special care to keep factors of $(\not{p}+m)$ and $(\not{p}'+m)$, since these factors vanish when contracted by the spinors of external electron and positron lines.

$$\begin{aligned}
\boxed{a}_N =&\gamma^{\rho}\left[-\left(\not{p}'+\not{l}\right)+m\right]\gamma^{\mu}\left[-\left(\not{p}+\not{l}\right)+m\right]\gamma_{\rho}\\
=&\gamma^{\rho}\left[-\left(\not{p}'+m\right)-\not{l}+2m\right]\gamma^{\mu}\left[-\left(\not{p}+m\right)-\not{l}+2m\right]\gamma_{\rho}\\
=&\overbrace{\gamma^{\rho}\left[\not{p}'+m\right]\gamma^{\mu}\left[\not{p}+m\right]\gamma_{\rho}}^{\boxed{a.1}_N}+\overbrace{\gamma^{\rho}\not{l}\gamma^{\mu}\not{l}\gamma_{\rho}}^{\boxed{a.2}_N}+\overbrace{4m^2\gamma^{\rho}\gamma^{\mu}\gamma_{\rho}}^{\boxed{a.3}_N}\\
&+\overbrace{\gamma^{\rho}\left[\not{p}'+m\right]\gamma^{\mu}\not{l}\gamma_{\rho}}^{\boxed{a.4}_N}+\overbrace{\gamma^{\rho}\not{l}\gamma^{\mu}\left[\not{p}+m\right]\gamma_{\rho}}^{\boxed{a.5}_N}\\
&-\overbrace{2m\gamma^{\rho}\not{l}\gamma^{\mu}\gamma_{\rho}}^{\boxed{a.6}_N}-\overbrace{2m\gamma^{\rho}\gamma^{\mu}\not{l}\gamma_{\rho}}^{\boxed{a.7}_N}-\overbrace{2m\gamma^{\rho}\left[\not{p}'+m\right]\gamma^{\mu}\gamma_{\rho}}^{\boxed{a.8}_N}-\overbrace{2m\gamma^{\rho}\gamma^{\mu}\left[\not{p}+m\right]\gamma_{\rho}}^{\boxed{a.9}_N}
\end{aligned} \tag{3.67}$$

Now we use γ-matrix identities from App: A to bring the $\boxed{a.i}_N$ to the *canonical* form. In the canonical form factors of $(\not{p}+m)$ and $(\not{p}'+m)$ appear completely on the right and left of the tensorial expressions, for the other possible factors we choose a convenient order of terms. In canonical form the terms in Eq: 3.67 become:

$$\begin{aligned}
\boxed{a.1}_N =& 4\gamma^{\mu}\left[p\cdot p'+(d-4)\,m^2\right]+8m\,(p+p')^{\mu}-\left(\not{p}'+m\right)\\
&\times\left[2\,(d-5)\,m\gamma^{\mu}+4p^{\mu}\right]-\left[2\,(d-5)\,m\gamma^{\mu}+4p'^{\mu}\right](\not{p}+m)\\
&+(d-6)\left(\not{p}'+m\right)\gamma^{\mu}\left(\not{p}+m\right)
\end{aligned} \tag{3.68}$$

$$\boxed{a.2}_N = \gamma^{\rho}\not{l}\gamma^{\mu}\not{l}\gamma_{\rho}\ ^{*} \tag{3.69}$$

$$\boxed{\text{a.3}}_N = \gamma^\mu \left[4\left(d-2\right)m^2\right] \tag{3.70}$$

$$\boxed{\text{a.4}}_N = 4\gamma^\mu p' \cdot l - 2\left(d-5\right)m\gamma^\mu \not{l} - 4p'^\mu \not{l} + 8ml^\mu + \left(\not{p}' + m\right)\left[\left(d-6\right)\gamma^\mu \not{l} - 4l^\mu\right] \tag{3.71}$$

$$\boxed{\text{a.5}}_N = 4\gamma^\mu p \cdot l + 2\left(d-5\right)m\gamma^\mu \not{l} - 4p^\mu \not{l} + 4\left(d-3\right)ml^\mu + \left[\left(d-6\right)\not{l}\gamma^\mu - 4l^\mu\right]\left(\not{p} + m\right) \tag{3.72}$$

$$\boxed{\text{a.6}}_N = -2\left(d-4\right)m\gamma^\mu \not{l} - 4\left(d-2\right)ml^\mu \tag{3.73}$$

$$\boxed{\text{a.7}}_N = 2\left(d-4\right)m\gamma^\mu \not{l} - 8ml^\mu \tag{3.74}$$

$$\boxed{\text{a.8}}_N = -4\left(d-3\right)m^2\gamma^\mu - 8mp'^\mu + \left(\not{p}' + m\right)\left[2\left(d-4\right)m\gamma^\mu\right] \tag{3.75}$$

$$\boxed{\text{a.9}}_N = -4\left(d-3\right)m^2\gamma^\mu - 8mp^\mu + \left[2\left(d-4\right)m\gamma^\mu\right]\left(\not{p} + m\right) \tag{3.76}$$

We now sum Eq: 3.68 - Eq: 3.76 to recover $\boxed{\text{a}}_N$ in canonical form, after some cancellations and rearrangements we find:

$$\begin{aligned}\boxed{\text{a}}_N =& \gamma^\rho \not{l} \gamma^\mu \not{l} \gamma_\rho + 4\gamma^\mu \left[p \cdot p' + p' \cdot l + p \cdot l\right] - 4ml^\mu - 4\left(p + p'\right)^\mu \not{l} \\ &+ \left(\not{p}' + m\right)\left[2m\gamma^\mu - 4\left(p + l\right)^\mu + \left(d-6\right)\gamma^\mu \not{l}\right] \\ &+ \left[2m\gamma^\mu - 4\left(p' + l\right)^\mu + \left(d-6\right)\not{l}\gamma^\mu\right]\left(\not{p} + m\right) \\ &+ \left(d-6\right)\left(\not{p}' + m\right)\gamma^\mu\left(\not{p} + m\right)\end{aligned} \tag{3.77}$$

Now that we have worked out the numerator of Eq: 3.66, we introduce Feynman parameters. For the sake of brevity, we refer to the whole numerator in Eq: 3.77 with index abstractly as $\boxed{\text{a}}_N$.

$$\begin{aligned}\boxed{\text{a}} =& -\mathrm{i}e^2\widetilde{\mu}^\epsilon \int \frac{d^d l}{\left(2\pi\right)^d} \frac{\boxed{\text{a}}_N}{\left(\left(p' + l\right)^2 + m^2\right)\left(\left(p + l\right)^2 + m^2\right)\left(l^2 + m_\gamma^2\right)} \\ =& -2\mathrm{i}e^2\widetilde{\mu}^\epsilon \int \frac{d^d l}{\left(2\pi\right)^d} \int_0^1 dx \int_0^{1-x} dy \\ &\times \frac{\boxed{\text{a}}_N}{\left(l^2 + 2xp \cdot l + xp^2 + 2yp' \cdot l + yp'^2 + \left(x + y\right)\left(m^2 - m_\gamma^2\right) + m_\gamma^2\right)^3}\end{aligned} \tag{3.78}$$

We perform a coordinate shift in the denominator of the momentum integral from l to $q = l + xp + yp'$ and change the order of integration.

$$\boxed{\text{a}} = -2\mathrm{i}e^2\widetilde{\mu}^\epsilon \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d q}{\left(2\pi\right)^d} \frac{\boxed{\text{a}}_N\left(l = q - xp - yp'\right)}{\left(q^2 + D_2^2\right)^3} \tag{3.79}$$

Here, we define:

$$D_2^2 \equiv x\left(1-x\right)p^2 + y\left(1-y\right)p'^2 - 2xyp \cdot p' + \left(x + y\right)\left(m^2 - m_\gamma^2\right) + m_\gamma^2 \tag{3.80}$$

---

*This term is left as it is, this form is easier to work with after we have introduced Feynman parameters.

We also need to perform the coordinate shift in the numerator $\boxed{\text{a}}_N$ $(l = q - xp - yp')$ and bring it back to canonical form, but terms odd in q are dropped since they do not contribute. The result after applying γ-matrix identities continuously is given by:

$$
\begin{aligned}
\boxed{\text{a}}_N =&\gamma^\mu \Big[4\left(1 - x - y + \frac{1}{2}(d-2)\,xy\right) p\cdot p' - (d-2)\,y\,(1-y)\,p'^2 - (d-2)\,x\,(1-x)\,p^2 \\
&- (d-6)\,(x+y)\,m^2\Big] + 2\left[(d-2)\,y\,(x+y) - 2x\right] m p'^\mu + 2\Big[(d-2)\,x\,(x+y) \\
&- 2y\Big] m p^\mu + (\not{p}' + m)\Big[(2 + (d-6)\,(x+y))\,m\gamma^\mu + 2\,(d-2)\,y\,(1-y)\,p'^\mu \\
&- 2\left((d-2)\,xy + 2\,(1-x-y)\right) p^\mu\Big] + \Big[(2 + (d-6)\,(x+y))\,m\gamma^\mu \\
&+ 2\,(d-2)\,x\,(1-x)\,p^\mu - 2\left((d-2)\,xy + 2\,(1-x-y)\right) p'^\mu\Big](\not{p} + m) \\
&+ (d-6)\,(\not{p}' + m)\,(1-x-y)\,\gamma^\mu\,(\not{p} + m) + \overbrace{\gamma^\rho \not{q}\gamma^\mu \not{q}\gamma_\rho}^{\boxed{\text{a.I}}_N}
\end{aligned}
\tag{3.81}
$$

We continue evaluation with $\boxed{\text{a.I}}$ and put the remaining term $\boxed{\text{a.II}} = \boxed{\text{a}} - \boxed{\text{a.I}}$ aside. We first apply Eq: C.2 and after that Eq: A.11 twice, we find:

$$
\begin{aligned}
\boxed{\text{a.I}} =& -2\mathrm{i}e^2\widetilde{\mu}^\epsilon \int_0^1 dx \int_0^{1-x} dy \int \frac{d^dq}{(2\pi)^d} \frac{\gamma^\rho \not{q}\gamma^\mu \not{q}\gamma_\rho}{\left(q^2 + D_2^2\right)^3} \\
=& -\frac{2}{d}\mathrm{i}e^2\widetilde{\mu}^\epsilon \int_0^1 dx \int_0^{1-x} dy \int \frac{d^dq}{(2\pi)^d} \frac{q^2\gamma^\rho\gamma^\nu\gamma^\mu\gamma_\nu\gamma_\rho}{\left(q^2 + D_2^2\right)^3} \\
=& -\frac{2\,(d-2)^2}{d}\mathrm{i}e^2\widetilde{\mu}^\epsilon \int_0^1 dx \int_0^{1-x} dy \int \frac{d^dq}{(2\pi)^d} \frac{q^2\gamma^\mu}{\left(q^2 + D_2^2\right)^3}
\end{aligned}
\tag{3.82}
$$

We cancel the numerator term with parts in the denominator and obtain a part with a quadratic denominator and a part with a cubic denominator, subsequently the evaluation proceeds as usual:

$$
\begin{aligned}
\text{a.I} =& \frac{2(d-2)^2}{d} e^2 \widetilde{\mu}^{\epsilon} \gamma^{\mu} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d \bar{q}}{(2\pi)^d} \left[\frac{1}{\left(\bar{q}^2 + D_2^2\right)^2} - \frac{D_2^2}{\left(\bar{q}^2 + D_0^2\right)^3} \right] \\
=& \frac{2(d-2)^2}{d} \frac{e^2}{(4\pi)^{\frac{d}{2}}} \widetilde{\mu}^{\epsilon} \gamma^{\mu} \int_0^1 dx \int_0^{1-x} dy \left[\Gamma\left(2 - \frac{d}{2}\right) - \frac{1}{2} \Gamma\left(3 - \frac{d}{2}\right) \right] (D_2^2)^{-\left(2 - \frac{1}{2}d\right)} \\
=& \frac{e^2}{8\pi^2} \gamma^{\mu} \left(1 - \frac{3}{4}\epsilon + \mathcal{O}\left(\epsilon^2\right)\right) \left(\frac{2}{\epsilon} - \gamma_E - \frac{1}{2} + \mathcal{O}(\epsilon)\right) \\
&\times \left[\frac{1}{2} - \frac{\epsilon}{2} \int_0^1 dx \int_0^{1-x} dy \ln \frac{D_2^2}{4\pi \widetilde{\mu}^2} + \mathcal{O}\left(\epsilon^2\right) \right] \\
=& \frac{e^2}{8\pi^2} \gamma^{\mu} \frac{1}{\epsilon} - \frac{e^2}{8\pi^2} \gamma^{\mu} \left[1 + \frac{1}{2} \ln \frac{m^2}{\mu^2} + \int_0^1 dx \int_0^{1-x} dy \ln \frac{D_2^2}{m^2} \right] + \mathcal{O}(\epsilon)
\end{aligned}
\tag{3.83}
$$

We continue with a.II, we perform the momentum integral and thereafter we set $d = 4 - \epsilon$ dimensions and drop all terms that are of order $\mathcal{O}(\epsilon)$.

$$
\begin{aligned}
\text{a.II} =& -2ie^2 \widetilde{\mu}^{\epsilon} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d q}{(2\pi)^d} \frac{\text{a.II}_N}{\left(q^2 + D_2^2\right)^3} \\
=& \, 2e^2 \widetilde{\mu}^{\epsilon} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d \bar{q}}{(2\pi)^d} \frac{\text{a.II}_N}{\left(\bar{q}^2 + D_2^2\right)^3} \\
=& \frac{e^2}{(4\pi)^{\frac{d}{2}}} \widetilde{\mu}^{\epsilon} \gamma \left(3 - \frac{d}{2}\right) \int_0^1 dx \int_0^{1-x} dy \, \text{a.II}_N \left(D_2^2\right)^{-\left(3 - \frac{1}{2}d\right)} \\
=& \frac{e^2}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{\text{a.II}_{N d=4}}{D_2^2} + \mathcal{O}(\epsilon) \to 0
\end{aligned}
\tag{3.84}
$$

We now fill out $d = 4$ in a.II_N and separate it into five different parts, to make the tensorial structure more explicit.

$$
\begin{aligned}
\text{a.II}_{N d=4} =& -2\gamma^{\mu} \left[x(1-x) p^2 + y(1-y) p'^2 - 2(1-x-y+xy) p \cdot p' - (x+y) m^2 \right] \\
&+ 4\left(y(x+y) - x\right) m p'^{\mu} + 4\left[x(x+y) - y\right] m p^{\mu} \\
&+ 2\left(\not{p}' + m\right) \left[(1-x-y) m \gamma^{\mu} + 2y(1-y) p'^{\mu} - 2(1-x-y+xy) p^{\mu}\right] \\
&+ 2\left[(1-x-y) m \gamma^{\mu} + 2x(1-x) p^{\mu} - 2(1-x-y+xy) p'^{\mu}\right] \left(\not{p} + m\right) \\
&- 2\left(\not{p}' + m\right)(1-x-y) \gamma^{\mu} \left(\not{p} + m\right)
\end{aligned}
\tag{3.85}
$$

$$
\begin{aligned}
\text{a.II}_{N d=4} =& -2\gamma^{\mu} N_0 + 4\left[y(x+y) - x\right] m p'^{\mu} + 4\left[x(x+y) - y\right] m p^{\mu} \\
&+ 2\left(\not{p}' + m\right) N_1'^{\mu} + 2N_1^{\mu} \left(\not{p} + m\right) - 2\left(\not{p}' + m\right)(1-x-y) \gamma^{\mu} \left(\not{p} + m\right)
\end{aligned}
\tag{3.86}
$$

In a.II$_{Nd=4}$ of Eq: 3.86, we define the following three functions:

$$
\begin{aligned}
N_0 &\equiv x\,(1-x)\,p^2 + y\,(1-y)\,p'^2 - 2\,(1-x-y+xy)\,p\cdot p' - (x+y)\,m^2 && (3.87)\\
N_1^\mu &\equiv (1-x-y)\,m\gamma^\mu + 2x\,(1-x)\,p^\mu - 2\,(1-x-y+xy)\,p'^\mu && (3.88)\\
N_1'^\mu &\equiv (1-x-y)\,m\gamma^\mu + 2y\,(1-y)\,p'^\mu - 2\,(1-x-y+xy)\,p^\mu && (3.89)
\end{aligned}
$$

In terms of these functions we write a.II as:

$$
\begin{aligned}
\text{a.II} =& -\frac{e^2}{8\pi^2}\gamma^\mu \int_0^1 dx \int_0^{1-x} dy\, \frac{N_0}{D_2^2} + \frac{e^2 m}{4\pi^2} p'^\mu \int_0^1 dx \int_0^{1-x} dy\, \frac{y\,(x+y)-x}{D_2^2}\\
&+ \frac{e^2 m}{4\pi^2} p^\mu \int_0^1 dx \int_0^{1-x} dy\, \frac{x\,(x+y)-y}{D_2^2}\\
&+ \frac{e^2}{8\pi^2}\left(\not{p}'+m\right) \int_0^1 dx \int_0^{1-x} dy\, \frac{N_1'^\mu}{D_0^2} + \frac{e^2}{8\pi^2} \int_0^1 dx \int_0^{1-x} dy\, \frac{N_1^\mu}{D_2^2}\left(\not{p}+m\right)\\
&- \frac{e^2}{8\pi^2}\left(\not{p}'+m\right) \int_0^1 dx \int_0^{1-x} dy\, \frac{(1-x-y)\,\gamma^\mu}{D_2^2}\left(\not{p}+m\right)
\end{aligned} \tag{3.90}
$$

We continue with the gauge-dependent part b from Eq: 3.65. We first cancel as many propagators as possible, after which we separate b in four constituent pieces.

$$
\begin{aligned}
\text{b} =& \mathrm{i}\left(1-\lambda^{-2}\right) e^2 \widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d}\, \not{l}\, \frac{1}{(\not{p}'+\not{l})+m}\gamma^\mu \frac{1}{(\not{p}+\not{l})+m}\not{l}\\
&\times \frac{1}{\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)}\\
=& \mathrm{i}\left(1-\lambda^{-2}\right) e^2 \widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \left[(\not{p}'+\not{l})+m-(\not{p}'+m)\right] \frac{1}{(\not{p}'+\not{l})+m}\gamma^\mu\\
&\times \frac{1}{(\not{p}+\not{l})+m}\left[(\not{p}+\not{l})+m-(\not{p}+m)\right] \frac{1}{\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)}\\
=& \mathrm{i}\left(1-\lambda^{-2}\right) e^2 \widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \left[1-(\not{p}'+m)\frac{1}{(\not{p}'+\not{l})+m}\right]\gamma^\mu\\
&\times \left[1-\frac{1}{(\not{p}+\not{l})+m}(\not{p}+m)\right] \frac{1}{\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)}
\end{aligned} \tag{3.91}
$$

$$
\begin{aligned}
\text{b} =& \,\mathrm{i}\left(1-\lambda^{-2}\right) e^2 \tilde{\mu}^{\epsilon} \int \frac{d^d l}{(2\pi)^d} \left[\overbrace{\gamma^\mu}^{\text{b.1}} - \overbrace{\frac{\left(\not{p}'+m\right)\left[-\left(\not{p}'+\not{l}\right)+m\right]\gamma^\mu}{\left(p'+l\right)^2+m^2}}^{\text{b.2}} \right. \\
& \left. - \overbrace{\frac{\gamma^\mu\left[-\left(\not{p}+\not{l}\right)+m\right]\left(\not{p}+m\right)}{\left(p+l\right)^2+m^2}}^{\text{b.3}} + \overbrace{\frac{\left(\not{p}'+m\right)\left[-\left(\not{p}'+\not{l}\right)+m\right]\gamma^\mu\left[-\left(\not{p}+\not{l}\right)+m\right]\left(\not{p}+m\right)}{\left(\left(p'+l\right)^2+m^2\right)\left(\left(p+l\right)^2+m^2\right)}}^{\text{b.4}} \right] \\
& \times \frac{1}{\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)}
\end{aligned}
\tag{3.92}
$$

We bring the numerators b.2$_{\mathrm{N}}$, b.3$_{\mathrm{N}}$ and b.4$_{\mathrm{N}}$ into canonical form using γ-matrix identities:

$$
\text{b.2}_{\mathrm{N}} = \left(p'^2+m^2\right)\gamma^\mu - \left(\not{p}'+m\right)\not{l}\,\gamma^\mu \tag{3.93}
$$

$$
\text{b.3}_{\mathrm{N}} = \left(p^2+m^2\right)\gamma^\mu - \gamma^\mu\not{l}\left(\not{p}+m\right) \tag{3.94}
$$

$$
\begin{aligned}
\text{b.4}_{\mathrm{N}} =& \left(p^2+m^2\right)\left(p'^2+m^2\right)\gamma^\mu - \left(p^2+m^2\right)\left(\not{p}'+m\right)\not{l}\,\gamma^\mu \\
& - \left(p'^2+m^2\right)\gamma^\mu\not{l}\left(\not{p}+m\right) + \left(\not{p}'+m\right)\not{l}\,\gamma^\mu\not{l}\left(\not{p}+m\right)
\end{aligned}
\tag{3.95}
$$

We first evaluate b.1 by using the methods that were explained previously:

$$
\begin{aligned}
\text{b.1} =& \,\mathrm{i}\left(1-\lambda^{-2}\right) e^2\tilde{\mu}^{\epsilon}\gamma^\mu \int \frac{d^d l}{(2\pi)^d} \frac{1}{\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)} \\
=& \,\mathrm{i}\left(1-\lambda^{-2}\right) e^2\tilde{\mu}^{\epsilon}\gamma^\mu \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{1}{\left(l^2+\left(1-x\left(1-\lambda^{-2}\right)\right)m_\gamma^2\right)^2} \\
=& -\left(1-\lambda^{-2}\right) e^2\tilde{\mu}^{\epsilon}\gamma^\mu \int_0^1 dx \int \frac{d^d \overline{q}}{(2\pi)^d} \frac{1}{\left(\overline{q}^2+\left(1-x\left(1-\lambda^{-2}\right)\right)m_\gamma^2\right)^2}
\end{aligned}
\tag{3.96}
$$

$$
\begin{aligned}
\boxed{\text{b.1}} =& -\left(1-\lambda^{-2}\right)\frac{e^2\gamma^\mu}{8\pi^2}\left[\frac{2}{\epsilon}-\gamma_E+\mathcal{O}\left(\epsilon\right)\right] \\
&\times\left[1-\frac{\epsilon}{2}\int_0^1 dx\ln\left(\frac{\left(1-x\left(1-\lambda^{-2}\right)\right)m_\gamma^2}{4\pi\widetilde{\mu}^2}\right)+\mathcal{O}\left(\epsilon^2\right)\right] \\
=& -\left(1-\lambda^{-2}\right)\frac{e^2\gamma^\mu}{8\pi^2}\frac{1}{\epsilon}+\left(1-\lambda^{-2}\right)\frac{e^2\gamma^\mu}{16\pi^2}\left[\ln\frac{m_\gamma^2}{\mu^2}+\int_0^1 dx\ln\left(1-x\left(1-\lambda^{-2}\right)\right)\right] \\
=& -\left(1-\lambda^{-2}\right)\frac{e^2\gamma^\mu}{8\pi^2}\frac{1}{\epsilon}-\left(1-\lambda^{-2}\right)\frac{e^2\gamma^\mu}{16\pi^2}\left[1-\ln\frac{m^2}{\mu^2}-\ln\kappa\right]-\frac{e^2\lambda^{-2}}{16\pi^2}\gamma^\mu\ln\lambda^{-2}
\end{aligned}
\tag{3.97}
$$

We now continue with $\boxed{\text{b.2}}$, this requires more efforts since $\boxed{\text{b.2}}_{\text{N}}$ depends on the integration variable.

$$
\begin{aligned}
\boxed{\text{b.2}} =& -\mathrm{i}\left(1-\lambda^{-2}\right)e^2\widetilde{\mu}^\epsilon\int\frac{d^dl}{(2\pi)^d}\frac{\boxed{\text{b.2}}_{\text{N}}}{\left((p'+l)^2+m^2\right)\left(l^2+m_\gamma^2\right)\left(l^2+\lambda^{-2}m_\gamma^2\right)} \\
=& -2\mathrm{i}\left(1-\lambda^{-2}\right)e^2\widetilde{\mu}^\epsilon\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dl}{(2\pi)^d} \\
&\times\frac{\boxed{\text{b.2}}_{\text{N}}}{\left(l^2+2xp'\cdot l+xp'^2+x\left(m^2-m_\gamma^2\right)-y\left(1-\lambda^{-2}\right)m_\gamma^2+m_\gamma^2\right)^3} \\
=& -2\mathrm{i}\left(1-\lambda^{-2}\right)e^2\widetilde{\mu}^\epsilon\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dq}{(2\pi)^d} \\
&\times\frac{\boxed{\text{b.2}}_{\text{N}}\left(l=q-xp'\right)}{\left(q^2+x\left(1-x\right)p'^2+x\left(m^2-m_\gamma^2\right)-y\left(1-\lambda^{-2}\right)m_\gamma^2+m_\gamma^2\right)^3}
\end{aligned}
\tag{3.98}
$$

Now we perform the coordinate shift $q = l + xp'$ in the numerator $\boxed{\text{b.2}}_{\text{N}}$ in Eq: 3.93, where we drop the the part odd in q. Since Eq: 3.93 is linear in the integration variable, we thus make the replacement $l = -xp'$.

$$
\boxed{\text{b.2}}_{\text{N}} = \left(p'^2+m^2\right)\left(1-x\right)\gamma^\mu+\left(\not{p}'+m\right)xm\gamma^\mu \tag{3.99}
$$

The form of $\boxed{\text{b.2}}_{\text{N}}$ in Eq: 3.99 allows us to perform both the momentum integral and the integral over the Feynman parameter y.

$$\begin{aligned}
\text{b.2} = &\left(1-\lambda^{-2}\right)\frac{e^2}{16\pi^2}\int_0^1 dx\int_0^{1-x} dy \\
&\times \frac{\left(p'^2+m^2\right)(1-x)\,\gamma^\mu + \left(\not{p}'+m\right)xm\gamma^\mu}{x(1-x)\,p'^2 + x\left(m^2-m_\gamma^2\right) - y\left(1-\lambda^{-2}\right)m_\gamma^2 + m_\gamma^2} \\
=&\frac{e^2 m_\gamma^{-2}}{16\pi^2}\left(p'^2+m^2\right)\gamma^\mu\int_0^1 dx\,(1-x)\left[\ln\frac{D_1'^2}{m^2}-\ln\frac{\widetilde{D}_1'^2}{m^2}\right] \\
&+\frac{e^2 m_\gamma^{-2}}{16\pi^2}\left(\not{p}'+m\right)\int_0^1 dx\; x\, m\gamma^\mu\left[\ln\frac{D_1'^2}{m^2}-\ln\frac{\widetilde{D}_1'^2}{m^2}\right]
\end{aligned} \tag{3.100}$$

Here, we define the p' analogues of Eq: 3.43 and Eq: 3.49 *:

$$D_1'^2 \equiv x(1-x)\,p'^2 + x\left(m^2-m_\gamma^2\right)+m_\gamma^2 \tag{3.101}$$

$$\widetilde{D}_1'^2 \equiv x(1-x)\,p'^2 + x\left(m^2-\lambda^{-2}m_\gamma^2\right)+\lambda^{-2}m_\gamma^2 \tag{3.102}$$

The result of the full evaluation of b.3 can be argued from the result of b.2. Upon making the replacement $p' \to p$ and interchanging the terms γ^μ and $(\not{p}+m)$ in the numerator of Eq: 3.100, we get the result of b.3. This is also verified by going through the evaluation of the numerator, starting with Eq: 3.94 instead of Eq: 3.93.

$$\begin{aligned}
\text{b.3} =&\frac{e^2 m_\gamma^{-2}}{16\pi^2}\left(p^2+m^2\right)\gamma^\mu\int_0^1 dx\,(1-x)\left[\ln\frac{D_1^2}{m^2}-\ln\frac{\widetilde{D}_1^2}{m^2}\right] \\
&+\frac{e^2 m_\gamma^{-2}}{16\pi^2}\int_0^1 dx\; x\, m\gamma^\mu\left[\ln\frac{D_1^2}{m^2}-\ln\frac{\widetilde{D}_1^2}{m^2}\right]\left(\not{p}+m\right)
\end{aligned} \tag{3.103}$$

Finally, we continue with the evaluation of b.4. This time, the denominator contains four factors and the numerator also contains a quadratic part in the loop momentum.

*Note that for $\lambda = 1$ the term b.2 still vanishes.

$$
\begin{aligned}
\boxed{\text{b.4}} =& i\left(1-\lambda^{-2}\right) e^{2}\widetilde{\mu}^{\epsilon} \int \frac{d^{d}l}{(2\pi)^{d}} \frac{\boxed{\text{b.4}}_{\mathrm{N}}}{\left((p'+l)^{2}+m^{2}\right)\left((p+l)^{2}+m^{2}\right)\left(l^{2}+m_{\gamma}^{2}\right)\left(l^{2}+\lambda^{-2}m_{\gamma}^{2}\right)} \\
=& 6i\left(1-\lambda^{-2}\right) e^{2}\widetilde{\mu}^{\epsilon} \int_{0}^{1} dx \int_{0}^{1-x} dy \int_{0}^{1-x-y} dz \int \frac{d^{d}l}{(2\pi)^{d}} \\
&\times \frac{\boxed{\text{b.4}}_{\mathrm{N}}\left(l=q-xp-yp'\right)}{\left(l^{2}+2xp\cdot l+xp^{2}+2yp'\cdot l+yp'^{2}+(x+y)\left(m^{2}-m_{\gamma}^{2}\right)-z\left(1-\lambda^{-2}\right)m_{\gamma}^{2}+m_{\gamma}^{2}\right)^{4}} \\
=& i6\left(1-\lambda^{-2}\right) e^{2}\widetilde{\mu}^{\epsilon} \int_{0}^{1} dx \int_{0}^{1-x} dy \int_{0}^{1-x-y} dz \int \frac{d^{d}q}{(2\pi)^{d}} \frac{\boxed{\text{b.4}}_{\mathrm{N}}\left(l=q-xp-yp'\right)}{\left(q^{2}+D_{2}^{2}-z\left(1-\lambda^{-2}\right)m_{\gamma}^{2}\right)^{4}}
\end{aligned}
\tag{3.104}
$$

In Eq: 3.104 we used the function D_2^2 that was defined in Eq: 3.80. For the numerator, written in Eq: 3.105, we need to perform the coordinate shift, discard the terms odd in q and bring the it back into the canonical form, we find: *

$$
\begin{aligned}
\boxed{\text{b.4}}_{\mathrm{N}} =& \left(p'^{2}+m^{2}\right)\left(p^{2}+m^{2}\right)(1-x-y)\gamma^{\mu}+\left(p^{2}+m^{2}\right)\left(\not{p}'+m\right)N_{2}'^{\mu} \\
&+\left(p'^{2}+m^{2}\right)N_{2}^{\mu}\left(\not{p}+m\right)+\left(\not{p}'+m\right)N_{3}^{\mu}\left(\not{p}+m\right)+\overbrace{\left(\not{p}'+m\right)\not{q}\gamma^{\mu}\not{q}\left(\not{p}+m\right)}^{\boxed{\text{b.II}}_{\mathrm{N}}}
\end{aligned}
\tag{3.105}
$$

We separated $\boxed{\text{b.4}}$ into $\boxed{\text{b.II}}$, the part quadratic in the integration variable q, and $\boxed{\text{b.I}}=\boxed{\text{b.4}}-\boxed{\text{b.II}}$, the terms that are left over. Furthermore, we make the following definitions for notational convenience:

$$N_{2}^{\mu}=(x+y)\,m\gamma^{\mu}+2xyp^{\mu}-2y(1-y)\,p'^{\mu} \tag{3.106}$$

$$N_{2}'^{\mu}=(x+y)\,m\gamma^{\mu}+2xyp'^{\mu}-2x(1-x)\,p^{\mu} \tag{3.107}$$

$$
\begin{aligned}
N_{3}^{\mu}=&\left(\left(xp+yp'\right)^{2}-\left(p'^{2}+m^{2}\right)y-\left(p^{2}+m^{2}\right)x\right)\gamma^{\mu} \\
&-2(x+y)\,m\left(xp^{\mu}+yp'^{\mu}\right)
\end{aligned}
\tag{3.108}
$$

We first evaluate $\boxed{\text{b.I}}$, we perform both the momentum integral and

*This is actually a quite lengthy though straightforward procedure to do by hand, since it is not that insightful we omit it.

the integral over the Feynman parameter z.

$$\begin{aligned}\text{b.I} =& -\left(1-\lambda^{-2}\right)\frac{e^2}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}dz\frac{\text{b.I}_{\text{N}}}{\left(D_2^2 - z\left(1-\lambda^{-2}\right)m_\gamma^2\right)^2}\\ =&\frac{e^2 m_\gamma^{-2}}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\ \text{b.I}_{\text{N}}\left[\frac{1}{D_2^2}-\frac{1}{\widetilde{D}_2^2}\right]\end{aligned} \tag{3.109}$$

Here, we define:

$$\widetilde{D}_2^2 \equiv x(1-x)\,p^2 + y(1-y)\,p'^2 - 2xy\,p\cdot p' + (x+y)\left(m^2-\lambda^{-2}m_\gamma^2\right)+\lambda^{-2}m_\gamma^2 \tag{3.110}$$

We substitute back the numerator b.I$_{\text{N}}$ from Eq: 3.105 into b.I Eq: 3.109, which gives:

$$\begin{aligned}\text{b.I} =&\frac{e^2 m_\gamma^{-2}}{16\pi^2}\left(p'^2+m^2\right)\left(p^2+m^2\right)\gamma^\mu\int_0^1 dx\int_0^{1-x}dy\ (1-x-y)\left[\frac{1}{D_2^2}-\frac{1}{\widetilde{D}_2^2}\right]\\ &+\frac{e^2 m_\gamma^{-2}}{16\pi^2}\left(p^2+m^2\right)\left(\not{p}'+m\right)\int_0^1 dx\int_0^{1-x}dy\ N_2'^{\mu}\left[\frac{1}{D_2^2}-\frac{1}{\widetilde{D}_2^2}\right]\\ &+\frac{e^2 m_\gamma^{-2}}{16\pi^2}\left(p'^2+m^2\right)\int_0^1 dx\int_0^{1-x}dy\ N_2^{\mu}\left(\not{p}+m\right)\left[\frac{1}{D_2^2}-\frac{1}{\widetilde{D}_2^2}\right]\\ &+\frac{e^2 m_\gamma^{-2}}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\ \left(\not{p}'+m\right)N_3^{\mu}\left(\not{p}+m\right)\left[\frac{1}{D_2^2}-\frac{1}{\widetilde{D}_2^2}\right]\end{aligned} \tag{3.111}$$

We continue with the evaluation of the last integral b.II, with the numerator in Eq: 3.105 and the integral in Eq: 3.104. We handle the numerator b.II$_{\text{N}}$ as the numerator a.I from Eq: 3.83, so we omit those details and write down the result:

$$\begin{aligned}\text{b.II} =&6\mathrm{i}\left(1-\lambda^{-2}\right)e^2\widetilde{\mu}^\epsilon\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}dz\int\frac{d^dq}{(2\pi)^d}\\ &\times\frac{\left(\not{p}'+m\right)\not{q}\gamma^\mu\not{q}\left(\not{p}+m\right)}{\left(q^2+D_2^2-z\left(1-\lambda^{-2}\right)m_\gamma^2\right)^4}\\ =&-6\left(\frac{d-2}{d}\right)\left(1-\lambda^{-2}\right)e^2\widetilde{\mu}^\epsilon\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}dz\int\frac{d^d\overline{q}}{(2\pi)^d}\\ &\times\frac{\left(\not{p}'+m\right)\overline{q}^2\gamma^\mu\left(\not{p}+m\right)}{\left(\overline{q}^2+D_2^2-z\left(1-\lambda^{-2}\right)m_\gamma^2\right)^4}\\ =&-\left(1-\lambda^{-2}\right)\frac{e^2}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}dz\frac{\left(\not{p}'+m\right)\gamma^\mu\left(\not{p}+m\right)}{D_2^2-z\left(1-\lambda^{-2}\right)m_\gamma^2}\\ =&\frac{e^2 m_\gamma^{-2}}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\left(\not{p}'+m\right)\gamma^\mu\left(\not{p}+m\right)\left[\ln\frac{D_2^2}{m^2}-\ln\frac{\widetilde{D}_2^2}{m^2}\right]\end{aligned} \tag{3.112}$$

For reference, we now sum the terms that we have evaluated and write down the full vertex correction up to the one-loop level. Thus, we collect the terms written down in Eq: 3.83, Eq: 3.90, Eq: 3.97, Eq: 3.100, Eq: 3.103, Eq: 3.111 and Eq: 3.112:

$$
\begin{aligned}
\left(\frac{\mathbb{V}^{\mu}_{1-loop}\left(p,p'\right)}{e}\right) &= \frac{e^2\lambda^{-2}}{8\pi^2}\gamma^\mu\frac{1}{\epsilon} - \frac{e^2}{16\pi^2}\gamma^\mu\Bigg[\left(3-\lambda^{-2}\right)-\left(1-\lambda^{-2}\right)\ln\kappa \\
&+\lambda^{-2}\left(\ln\lambda^{-2}+\ln\frac{m^2}{\mu^2}\right)\Bigg] - \frac{e^2}{8\pi^2}\gamma^\mu\int_0^1 dx\int_0^{1-x}dy\Bigg[\ln\frac{D_2^2}{m^2}+\frac{N_0}{D_2^2} \\
&-\frac{1}{2}\frac{\left(p^2+m^2\right)\left(p'^2+m^2\right)}{m_\gamma^2}\left(1-x-y\right)\left(\frac{1}{D_2^2}-\frac{1}{\widetilde{D}_2^2}\right)\Bigg] + \frac{e^2m_\gamma^{-2}}{16\pi^2}\gamma^\mu\int_0^1 dx\left(1-x\right) \\
&\times\left[\left(p^2+m^2\right)\left[\ln\frac{D_1^2}{m^2}-\ln\frac{\widetilde{D}_1^2}{m^2}\right]+\left(p'^2+m^2\right)\left[\ln\frac{D_1'^2}{m^2}-\ln\frac{\widetilde{D}_1'^2}{m^2}\right]\right] \\
&+\frac{e^2m}{4\pi^2}p^\mu\int_0^1 dx\int_0^{1-x}dy\ \frac{x\left(x+y\right)-y}{D_2^2}+\frac{e^2m}{4\pi^2}p'^\mu\int_0^1 dx\int_0^{1-x}dy\ \frac{y\left(x+y\right)-x}{D_2^2} \\
&+\frac{e^2}{8\pi^2}\left(\not{p}'+m\right)\int_0^1 dx\int_0^{1-x}dy\ \frac{N_1'^\mu}{D_2^2}+\frac{e^2m_\gamma^{-2}}{16\pi^2}\left(\not{p}'+m\right)\int_0^1 dx\ x\ m\gamma^\mu \\
&\times\left[\ln\frac{D_1'^2}{m^2}-\ln\frac{\widetilde{D}_1'^2}{m^2}\right]+\frac{e^2m_\gamma^{-2}}{16\pi^2}\left(p^2+m^2\right)\left(\not{p}'+m\right)\int_0^1 dx\int_0^{1-x}dy\ N_2'^\mu \\
&\times\left[\frac{1}{D_2^2}-\frac{1}{\widetilde{D}_2^2}\right]+\frac{e^2}{8\pi^2}\int_0^1 dx\int_0^{1-x}dy\ \frac{N_1^\mu}{D_2^2}\left(\not{p}+m\right)+\frac{e^2m_\gamma^{-2}}{16\pi^2}\int_0^1 dx\ x\ m\gamma^\mu \\
&\times\left[\ln\frac{D_1^2}{m^2}-\ln\frac{\widetilde{D}_1^2}{m^2}\right]\left(\not{p}+m\right)+\frac{e^2m_\gamma^{-2}}{16\pi^2}\left(p'^2+m^2\right)\int_0^1 dx\int_0^{1-x}dy \\
&\times N_2^\mu\left(\not{p}+m\right)\left[\frac{1}{D_2^2}-\frac{1}{\widetilde{D}_2^2}\right]-\frac{e^2}{8\pi^2}\left(\not{p}'+m\right)\int_0^1 dx\int_0^{1-x}dy\ \frac{\left(1-x-y\right)\gamma^\mu}{D_2^2}\left(\not{p}+m\right) \\
&+\frac{e^2m_\gamma^{-2}}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\left(\not{p}'+m\right)\gamma^\mu\left(\not{p}+m\right)\left[\ln\frac{D_2^2}{m^2}-\ln\frac{\widetilde{D}_2^2}{m^2}\right] \\
&+\frac{e^2m_\gamma^{-2}}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\ \left(\not{p}'+m\right)N_3^\mu\left(\not{p}+m\right)\left[\frac{1}{D_2^2}-\frac{1}{\widetilde{D}_2^2}\right]
\end{aligned}
\tag{3.113}
$$

3.4 The vertex correction in full dimensional regularisation

The off-shell vertex correction result in a general gauge and with a finite photon regulator mass is not very practical in use, since it involves multiple unevaluated Feynman parameter integrals and it is not clear which terms are infrared finite due to the complex dependence on the regulator mass.

We thus show the same calculation (omitting some small steps) in full dimensional regularisation where we set $m_\gamma^2 \to 0$ from the beginning and work with on-shell momenta for the electrons. We then see that dimensional regularisation of IR-divergences is more practical and is preferred.

The on-shell vertex function

The *on-shell vertex function* $\Lambda^\mu(p', p)$ with $k = p' - p$ and on-shell electron momenta is defined in terms of the one-loop vertex correction from the previous section:

$$\begin{aligned}\Lambda^\mu(p', p) =& Z_1\gamma^\mu + \left(\frac{\mathbb{V}^\mu_{1-loop}(p', p)}{e}\right)\Bigg|_{\substack{p^2=p'^2=-m^2 \\ m_\gamma^2=0}} + \mathcal{O}\left(e^4\right) \\ =& Z_1\gamma^\mu - \overbrace{\mathrm{i}e^2\widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d}\gamma^\rho \frac{1}{(\not{p}' + \not{l}) + m}\gamma^\mu \frac{1}{(\not{p} + \not{l}) + m}\gamma_\rho \frac{1}{l^2}}^{\boxed{a}} \\ & \overbrace{+\,\mathrm{i}e^2\widetilde{\mu}^\epsilon\left(1 - \lambda^{-2}\right)\int \frac{d^d l}{(2\pi)^d}\not{l}\frac{1}{(\not{p}' + \not{l}) + m}\gamma^\mu \frac{1}{(\not{p} + \not{l}) + m}\not{l}\frac{1}{l^4}}^{\boxed{b}} + \mathcal{O}\left(e^4\right)\end{aligned} \tag{3.114}$$

The evaluation of the integrals

The evaluation of $\boxed{a}$ has many similarities with what we saw in the previous section. The numerator $\boxed{a}_N$ is again given by Eq: 3.77, the denominators however look different on-shell:

$$\boxed{a} = -\mathrm{i}e^2\widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d}\frac{\boxed{a}_N}{(l^2 + 2p \cdot l)\,(l^2 + 2p \cdot l)\,l^2} \tag{3.115}$$

The next step is to introduce Feynman parameters, followed by a special coordinate shift of $u = \frac{y}{x+y}$ and $v = x + y$ which brings the integral in a particularly convenient form:

$$\begin{aligned}\boxed{\mathrm{a}} &= -2\mathrm{i}e^2\widetilde{\mu}^{\epsilon}\int_0^1 dx\int_0^{1-x} dy\int\frac{d^dl}{(2\pi)^d}\frac{\boxed{\mathrm{a}}_N}{\left(l^2+2l\cdot(xp+yp')\right)^3}\\ &= -2\mathrm{i}e^2\widetilde{\mu}^{\epsilon}\int_0^1 du\int_0^1 dv\, v\int\frac{d^dl}{(2\pi)^d}\frac{\boxed{\mathrm{a}}_N}{\left(l^2+2l\cdot((1-u)\,vp+uvp')\right)^3}\end{aligned} \tag{3.116}$$

The validity of the coordinate shift and the accompanying change in the boundary conditions is easier to see by interposing the coordinate shifts $\{x\ ,\ y\} \to \{s = x - y\ ,\ v = x + y\} \to \{t = s/v\ ,\ v\} \to \{u = \frac{1}{2}(1-t)\ ,\ v\}$:

$$\begin{aligned}\int_0^1 dx\int_0^{1-x} dy\, f(x\,,\,y) &= \frac{1}{2}\int_0^1 dv\int_{-v}^{v} ds\, f\left(x=\frac{1}{2}(v+s)\ ,\ y=\frac{1}{2}(v-s)\right)\\ &= \frac{1}{2}\int_{-1}^1 dt\int_0^1 dv\, vf\left(x=\frac{1}{2}v(1+t)\ ,\ y=\frac{1}{2}v(1-t)\right)\\ &= \int_0^1 du\int_0^1 dv\, vf\left(x=(1-u)\,v\ ,\ y=uv\right)\end{aligned} \tag{3.117}$$

We also do a coordinate shift in the momentum variable $l = q - (1-u)\,vp - uvp'$ and rearrange terms:

$$\begin{aligned}\boxed{\mathrm{a}} &= -2\mathrm{i}e^2\widetilde{\mu}^{\epsilon}\int_0^1 du\int_0^1 dv\, v\int\frac{d^dq}{(2\pi)^d}\frac{\boxed{\mathrm{a}}_N\left(l=q-(1-u)\,vp-uvp'\right)}{\left(q^2-v^2\left((1-u)\,p+up'\right)^2\right)^3}\\ &= -2\mathrm{i}e^2\widetilde{\mu}^{\epsilon}\int_0^1 du\int_0^1 dv\, v\int\frac{d^dq}{(2\pi)^d}\frac{\boxed{\mathrm{a}}_N\left(l=q-v\,(p+uk)\right)}{\left(q^2+u\,(1-u)\,v^2k^2+v^2m^2\right)^3}\end{aligned} \tag{3.118}$$

We work out the numerator $\boxed{\mathrm{a}}_{\mathrm{N}}(l = q - v\,(p+uk))$ in canonical form and divide the terms into three constituents, the terms independent of the Feynman parameter v, the terms linear in v and the terms quadratic in v. This is easier with a specialised symbolic mathematics calculator such as FORM [24], we find:

$$
\begin{aligned}
\left\{ \text{a}_N\,,\ \mathcal{O}\left(v^0\right)\right\} =& -2\gamma^\mu\left(k^2+2m^2\right)+\left(\not p'+m\right)\left[2m\gamma^\mu-4p^\mu\right]+\left[2m\gamma^\mu-4p'^\mu\right]\\
&\times\left(\not p+m\right)+\left(d-6\right)\left(\not p+m\right)\gamma^\mu\left(\not p+m\right)+\overbrace{\gamma^\rho \not q\gamma^\mu \not q\gamma_\rho}^{\text{a.I}_N}\\
\left\{ \text{a}_N\,,\ \mathcal{O}\left(v^1\right)\right\} =& \, v\Big\{2\gamma^\mu\left(k^2+4m^2\right)-4mup^\mu-4m\left(1-u\right)p'^\mu+\left(\not p'+m\right)\\
&\times\left[\left(d-6\right)m\gamma^\mu+4p^\mu+2\left(d-2\right)up'^\mu\right]+\left[\left(d-6\right)m\gamma^\mu\right.\\
&\left.+2\left(d-2\right)\left(1-u\right)p^\mu+4p'^\mu\right]\left(\not p+m\right)-\left(d-6\right)\left(\not p'+m\right)\\
&\times\gamma^\mu\left(\not p+m\right)\Big\}\\
\left\{ \text{a}_N\,,\ \mathcal{O}\left(v^2\right)\right\} =& -v^2\Big\{\left(d-2\right)\gamma^\mu\left(u\left(1-u\right)k^2+m^2\right)-2\left(d-2\right)\left(1-u\right)mp^\mu\\
&-2\left(d-2\right)ump'^\mu+\left(\not p'+m\right)\left[2\left(d-2\right)u\left(\left(1-u\right)p^\mu+up'^\mu\right)\right]\\
&+\left[2\left(d-2\right)\left(1-u\right)\left(\left(1-u\right)p^\mu+up'^\mu\right)\right]\left(\not p'+m\right)\Big\}
\end{aligned}
\tag{3.119}
$$

We now fully evaluate a.I and leave a.II $=$ a $-$ a.I, the term that is quadratic in q and the only UV-divergent term, in a way analogous to what we have shown in the previous section. This time we must only be careful to evaluate the integral over the Feynman parameter v for finite values of ϵ. At the end we expand in the infinitesimal parameter $\epsilon_{UV} = \epsilon > 0$ and take the leading order terms, we find:

$$
\begin{aligned}
\text{a.I} =& \frac{2\left(d-2\right)^2}{d}\frac{e^2}{\left(4\pi\right)^{\frac{d}{2}}}\widetilde{\mu}^\epsilon\gamma^\mu\left[\Gamma\left(2-\frac{d}{2}\right)-\frac{1}{2}\Gamma\left(3-\frac{d}{2}\right)\right]\int_0^1 dv\\
&\times\frac{1}{v^{3-d}}\int_0^1 du\left(u\left(1-u\right)k^2+m^2\right)^{-\left(2-\frac{1}{2}d\right)}\\
=&\frac{e^2}{8\pi^2}\gamma^\mu\frac{1}{\epsilon_{UV}}-\frac{e^2}{16\pi^2}\gamma^\mu\left[1+\ln\left(\frac{m^2}{\mu^2}\right)+\int_0^1 du\ln\left(u\left(1-u\right)\frac{k^2}{m^2}+1\right)\right]+\mathcal{O}\left(\epsilon\right)
\end{aligned}
\tag{3.120}
$$

For the other terms that we get when we plug the numerator Eq: 3.119 back into the integral of Eq: 3.118, we first perform the integral over q. The integral over the Feynman parameter v is now straightforward, the decomposition in Eq: 3.119 for v^x with $x = 0, 1, 2$ gives us:

$$
\int_0^1 dv\frac{v^{x+1}}{\left(v^2\right)^{3-\frac{1}{2}d}}=\int_0^1 dv\, v^{x-\left(1+\epsilon\right)}=\frac{1}{x-\epsilon}\ \text{with}\ \epsilon<0\ \text{if}\ x=0 \tag{3.121}
$$

The integral over the Feynman parameter v as shown in Eq: 3.121 implies that in the decomposition we made for the numerator the terms of order $\mathcal{O}\left(v^0\right)$ are the IR-divergent ones, and the other terms are the IR-

and UV-finite terms.

Two tricks that help to simplify the end-result are related to the symmetry of part of the integrand under a coordinate shift of $u \to 1-u$:

$$\begin{aligned} \int_0^1 du\ (1-u)^2 f(u(1-u)) &= \int_0^1 du\, u^2 f(u(1-u)) \\ \int_0^1 du\, u f(u(1-u)) &= \int_0^1 du\, (1-u) f(u(1-u)) \\ \Rightarrow \int_0^1 du\, u f(u(1-u)) &= \frac{1}{2}\int_0^1 du\, f(u(1-u)) \end{aligned} \tag{3.122}$$

With this information we set $\epsilon = 0$ for the non-divergent terms and expand in ϵ for the infrared divergent terms, in terms of $D_0^2 \equiv u(1-u)k^2 + m^2$ we get*:

$$\begin{aligned} \text{a.II} =& \frac{e^2}{16\pi^2}\left[\left(\frac{1}{\epsilon_{IR}} + \frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)\right)\int_0^1 du \frac{1}{D_0^2} + \frac{1}{2}\int_0^1 du \frac{1}{D_0^2}\ln\left(\frac{D_0^2}{m^2}\right)\right]\left\{\text{a.II}_N\, ,\, \mathcal{O}\left(v^0\right)\right\}\Big|_{v=1} \\ &+ \frac{e^2}{16\pi^2}\int_0^1 du \frac{1}{D_0^2}\left(\left\{\text{a.II}_N\, ,\, \mathcal{O}\left(v^1\right)\right\} + \frac{1}{2}\left\{\text{a.II}_N\, ,\, \mathcal{O}\left(v^2\right)\right\}\right)\Big|_{v=1} \end{aligned} \tag{3.123}$$

We continue with the b, after cancelling as many terms from the numerator as possible we get in analogous to Eq: 3.92 from the previous section:

$$\begin{aligned} \text{b} =& i\left(1-\lambda^{-2}\right)e^2\tilde{\mu}^{\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^4}\Bigg[\overbrace{\gamma^\mu}^{\text{b.1}} - \overbrace{\frac{(\not{p}'+m)\left[-(\not{p}'+\not{l})+m\right]\gamma^\mu}{l^2+2p'\cdot l}}^{\text{b.2}} \\ &- \overbrace{\frac{\gamma^\mu\left[-(\not{p}+\not{l})+m\right](\not{p}+m)}{l^2+2p\cdot l}}^{\text{b.3}} + \overbrace{\frac{(\not{p}'+m)\left[-(\not{p}'+\not{l})+m\right]\gamma^\mu\left[-(\not{p}+\not{l})+m\right](\not{p}+m)}{(l^2+2p'\cdot l)(l^2+2p\cdot l)}}^{\text{b.4}}\Bigg] \end{aligned} \tag{3.124}$$

We first evaluate b.1, which is made difficult by the fact that the integral is both IR- and UV-divergent. To cope with this we can not use the standard formula Eq: C.7 since the right hand side would be infinite. Instead of this we need Wick rotate and break up the integral in an angular part and a radial part, the angular part evaluates to $2\pi^{\frac{d}{2}}/\Gamma\left(\frac{d}{2}\right)$ and the radial part needs to be broken up again. We break up the radial part in a part that is regularised in the IR-domain and a part that is regularised in

*We work it out explicitly at the end, and sum it with the gauge-dependent terms.

the UV-domain.

$$
\begin{aligned}
\text{b.1} =& -\left(1-\lambda^{-2}\right) e^2 \widetilde{\mu}^{\epsilon} \gamma^{\mu} \int \frac{d^d \bar{l}}{(2\pi)^d} \frac{1}{\bar{l}^4} \\
=& -2\left(1-\lambda^{-2}\right) \frac{e^2}{(4\pi)^{\frac{d}{2}}} \gamma^{\mu} \frac{\widetilde{\mu}^{\epsilon}}{\Gamma\left(\frac{d}{2}\right)} \left[\int_0^{\mu} d\bar{l} \frac{1}{\bar{l}^{5-d}} + \int_{\mu}^{\infty} d\bar{l} \frac{1}{\bar{l}^{5-d}}\right] \\
=& -\left(1-\lambda^{-2}\right) \frac{e^2}{8\pi^2} \gamma^{\mu} \left(4\pi\widetilde{\mu}^2\right)^{\epsilon} \left(1+\frac{\epsilon}{2}\left(\gamma_E+1\right)\right) \left[-\frac{1}{\epsilon}\bar{l}^{-\epsilon}\Big|_0^{\mu} - \frac{1}{\epsilon}\bar{l}^{-\epsilon}\Big|_{\mu}^{\infty}\right] \\
=& -\left(1-\lambda^{-2}\right) \frac{e^2}{8\pi^2} \gamma^{\mu} \left(\frac{1}{\epsilon_{UV}} + \frac{1}{\epsilon_{IR}}\right)
\end{aligned}
\tag{3.125}
$$

For the evaluation of b.2 we use the numerator that was determined in Eq: 3.93 which simplifies due to the on-shell condition, and after a coordinate shift becomes Eq: 3.99.

$$
\begin{aligned}
\text{b.2} =& \mathrm{i}\left(1-\lambda^{-2}\right) e^2 \widetilde{\mu}^{\epsilon} \int \frac{d^d l}{(2\pi)^d} \frac{\left(\not{p}'+m\right) \not{l} \gamma^{\mu}}{\left(l^2+2p'\cdot l\right) l^4} \\
=& \mathrm{i}\left(1-\lambda^{-2}\right) e^2 \widetilde{\mu}^{\epsilon} \int_0^1 dx \int \frac{d^d l}{(2\pi)^d} \frac{\left((1-x)\not{p}'+m\right) \not{l} \gamma^{\mu}}{\left(l^2+2xp'\cdot l\right)^3} \\
=& \left(1-\lambda^{-2}\right) \frac{e^2}{16\pi^2} \left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)^{\frac{\epsilon}{2}} \Gamma\left(1+\frac{\epsilon}{2}\right) \frac{\not{p}'+m}{m} \gamma^{\mu} \int_0^1 dx\, x(1-x) \left(x^2\right)^{-\left(1+\frac{1}{2}\epsilon\right)} \\
=& \left(1-\lambda^{-2}\right) \frac{e^2}{16\pi^2} \frac{\left(\not{p}'+m\right)\gamma^{\mu}}{m} \left[\frac{1}{\epsilon_{IR}} - 1 + \frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)\right]
\end{aligned}
\tag{3.126}
$$

Just as in the previous section we argue for b.3 based on the fact that it is the same integral with a slightly different tensorial structure in the numerator, we find:

$$
\text{b.3} = \left(1-\lambda^{-2}\right) \frac{e^2}{16\pi^2} \frac{\gamma^{\mu}\left(\not{p}+m\right)}{m} \left[\frac{1}{\epsilon_{IR}} - 1 + \frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)\right] \tag{3.127}
$$

Finally, we continue with the evaluation of b.4, starting in an analogue fashion as in the last section such as to use as much of those results as possible. We first introduce Feynman parameters and perform a coordinate shift:

$$
\begin{aligned}
\boxed{\text{b.4}} =&6\mathrm{i}\left(1-\lambda^{-2}\right)e^2\widetilde{\mu}^{\epsilon}\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dl}{(2\pi)^d}\frac{(1-x-y)\,\boxed{\text{b.4}}_N}{\left(l^2+2l\cdot(xp+yp')\right)^4}\\
=&6\mathrm{i}\left(1-\lambda^{-2}\right)e^2\widetilde{\mu}^{\epsilon}\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dq}{(2\pi)^d}\frac{(1-x-y)\,\boxed{\text{b.4}}_N\,(l=q-xp-yp')}{\left(q^2+xyk^2+(x+y)^2m^2\right)^4}\\
=&6\mathrm{i}\left(1-\lambda^{-2}\right)e^2\widetilde{\mu}^{\epsilon}\int_0^1 du\int_0^{1}dv\int\frac{d^dq}{(2\pi)^d}\frac{(1-v)\,\boxed{\text{b.4}}_N\,(l=q-v\left[(1-u)\,p+up'\right])}{\left(q^2+v^2\left(u\left(1-u\right)k^2+m^2\right)\right)^4}
\end{aligned}
\tag{3.128}
$$

We turn our attention to the numerator and perform the same coordinate shifts, we start with Eq: 3.105 enforce the on-shell condition and replace $x\rightarrow v\,(1-u)$ and $y\rightarrow uv$, we find:

$$
\begin{aligned}
\boxed{\text{b.4}}_N =& -\overbrace{v^2\left(\not{p}'+m\right)\left[\left(u\left(1-u\right)k^2+m^2\right)\gamma^{\mu}+2mu\left(p^{\mu}+p'^{\mu}\right)\right]\left(\not{p}+m\right)}^{\boxed{\text{b.I}}_N}\\
&+\underbrace{}_{}\overbrace{\left(\not{p}'+m\right)\not{q}\gamma^{\mu}\not{q}\left(\not{p}+m\right)}^{\boxed{\text{b.II}}_N}
\end{aligned}
\tag{3.129}
$$

Without repeating tricks that were shown full out previously we perform the momentum integral, the integral over the Feynman parameter v and expand in terms of ϵ. In particular, we used Eq: 3.122 to simplify $\boxed{\text{b.I}}$ and treated the q dependence of the numerator in $\boxed{\text{b.II}}$ as in the previous section, we find:

$$
\begin{aligned}
\boxed{\text{b.I}} =& \left(1-\lambda^{-2}\right)\frac{e^2}{16\pi^2}\frac{\left(\not{p}'+m\right)\gamma^{\mu}\left(\not{p}+m\right)}{m^2}\left[\left(\frac{1}{\epsilon_{IR}}-\frac{1}{2}\left(3-\ln\left(\frac{m^2}{\mu^2}\right)\right)\right)\right.\\
&\times\left.\int_0^1 du\frac{m^2}{D_0^2}+\frac{1}{2}\int_0^1 du\frac{m^2}{D_0^2}\ln\left(\frac{D_0^2}{m^2}\right)\right]+\left(1-\lambda^{-2}\right)\frac{e^2}{16\pi^2}\frac{\left(\not{p}'+m\right)\left(p^{\mu}+p'^{\mu}\right)\left(\not{p}+m\right)}{m^3}\\
&\times\left[\left(\frac{1}{\epsilon_{IR}}-\frac{1}{2}\left(3-\ln\left(\frac{m^2}{\mu^2}\right)\right)\right)\int_0^1 du\left(\frac{m^2}{D_0^2}\right)^2+\frac{1}{2}\int_0^1 du\left(\frac{m^2}{D_0^2}\right)^2\ln\left(\frac{D_0^2}{m^2}\right)\right]
\end{aligned}
\tag{3.130}
$$

$$
\begin{aligned}
\boxed{\text{b.II}} =& -\left(1-\lambda^{-2}\right)\frac{e^2}{16\pi^2}\frac{\left(\not{p}'+m\right)\gamma^{\mu}\left(\not{p}+m\right)}{m^2}\left[\left(\frac{1}{\epsilon_{IR}}-\frac{1}{2}\left(1-\ln\left(\frac{m^2}{\mu^2}\right)\right)\right)\right.\\
&\times\left.\int_0^1 du\frac{m^2}{D_0^2}+\frac{1}{2}\int_0^1 du\frac{m^2}{D_0^2}\ln\left(\frac{D_0^2}{m^2}\right)\right]
\end{aligned}
\tag{3.131}
$$

We now sum al the contributions from $\boxed{\text{a.I}}$ in Eq: 3.120, $\boxed{\text{a.II}}$ in Eq:

3.123, b.1 in Eq: 3.125, b.2 in Eq: 3.126, b.3 in Eq: 3.127, b.I in Eq: 3.130 and b.II in Eq: 3.131. This gives us a final result for $\Lambda^\mu(p,p')$ in accordance with Eq: 5.66 and Eq: 5.66a in [21]:

$$\begin{aligned}\Lambda^\mu(p,p') =& \Lambda_0\left(k^2\right)\gamma^\mu + \Lambda_1\left(k^2\right)\frac{p^\mu+p'^\mu}{m} + \frac{\not{p}'+m}{m}\Lambda_2^\mu(p,p')\\ &+\Lambda_3^\mu(p,p')\frac{\not{p}+m}{m} + \frac{\not{p}'+m}{m}\Lambda_4^\mu(p,p')\frac{\not{p}+m}{m} + \mathcal{O}\left(e^4\right)\end{aligned} \tag{3.132}$$

$$\begin{aligned}\Lambda_0\left(k^2\right) =& Z_1 + \frac{e^2}{8\pi^2}\lambda^{-2}\frac{1}{\epsilon_{UV}} - \frac{e^2}{8\pi^2}\frac{1}{\epsilon_{IR}}\left[\left(1-\lambda^{-2}\right)+\left(2+\frac{k^2}{m^2}\right)\right.\\ &\times\left.\int_0^1 du\ln\left(\frac{D_0^2}{m^2}\right)\right] - \frac{e^2}{16\pi^2}\left[2+\ln\left(\frac{m^2}{\mu^2}\right)+\int_0^1 du\ln\left(\frac{D_0^2}{m^2}\right)+\left(2+\frac{k^2}{m^2}\right)\right.\\ &\times\left.\left[\int_0^1 du\frac{m^2}{D_0^2}\ln\left(\frac{D_0^2}{m^2}\right)+\ln\left(\frac{m^2}{\mu^2}\right)\int_0^1 du\frac{m^2}{D_0^2}\right]-2\left(4+\frac{k^2}{m^2}\right)\int_0^1 du\frac{m^2}{D_0^2}\right]\end{aligned} \tag{3.133}$$

$$\Lambda_1\left(k^2\right) = -\frac{e^2}{16\pi^2}\int_0^1 du\frac{m^2}{D_0^2} \tag{3.134}$$

$$\begin{aligned}\Lambda_2^\mu(p,p') =& \frac{e^2}{16\pi^2}\frac{1}{\epsilon_{IR}}\left[\left(3-\lambda^{-2}\right)\gamma^\mu-4\frac{p^\mu}{m}\right]\int_0^1 du\frac{m^2}{D_0^2}-\frac{e^2}{16\pi^2}\left[\left(1-\lambda^{-2}\right)\right.\\ &\times\left[1-\frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)\right]-\left[\gamma^\mu-2\frac{p^\mu}{m}\right]\left(\int_0^1 du\frac{m^2}{D_0^2}\ln\left(\frac{D_0^2}{m^2}\right)\right.\\ &\left.\left.+\ln\left(\frac{m^2}{\mu^2}\right)\int_0^1 du\frac{m^2}{D_0^2}\right)+2\int_0^1 du\frac{\gamma^\mu-\frac{1}{2}p^\mu-2p'^\mu+u(1-u)k_\mu}{D_0^2}\right]\end{aligned} \tag{3.135}$$

$$\Lambda_3^\mu(p,p') = \Lambda_2(p',p) \quad \text{NB: ! } p \leftrightarrows p' \Rightarrow k\to -k\text{ !} \tag{3.136}$$

$$\begin{aligned}\Lambda_4^\mu(p,p') =& -\frac{e^2}{8\pi^2}\gamma^\mu\frac{1}{\epsilon_{IR}}\int_0^1 du\frac{m^2}{D_0^2}+\left(1-\lambda^{-2}\right)\frac{e^2}{16\pi^2}\frac{p^\mu+p'^\mu}{m}\frac{1}{\epsilon_{IR}}\\ &\times\int_0^1 du\left(\frac{m^2}{D_0^2}\right)^2-\frac{e^2}{32\pi^2}\frac{p^\mu+p'^\mu}{m}\left[\int_0^1 du\left(\frac{m^2}{D_0^2}\right)^2\ln\left(\frac{D_0^2}{m^2}\right)\right.\\ &\left.-\left(3-\ln\left(\frac{m^2}{\mu^2}\right)\right)\int_0^1 du\left(\frac{m^2}{D_0^2}\right)^2\right]\end{aligned} \tag{3.137}$$

Chapter 4

Renormalisation of QED

In this chapter, we renormalise QED by computing the counter terms Z_0, Z_1, Z_2 and Z_3 in the on-shell and $\overline{MS}$ renormalisation schemes. Additionally, we find that the on-shell (OS) renormalisation scheme is affected by the IR-regulator scheme that we use.
The counter terms allow us to determine the non-perturbative running of the electron mass parameter m and the coupling constant e at higher momentum scales. We demonstrate that the renormalised electron mass and charge respectively decrease and increase in magnitude at higher momentum scales.

As we showed in the previous chapter, the addition of quantum corrections to the vertices and propagators in QED yields divergences. We regulated the divergences using dimensional regularisation of the UV-divergences and both dimensional regularisation and a photon regulator mass for the IR-divergences. Renormalisation is now necessary to cancel the regulated divergences in physical observables by an appropriate choice for the counter terms.

Differences choices for the finite parts of the counter terms define different renormalisation schemes, and thus different definitions of the renormalised mass parameter. We consider here the $\overline{MS}$ and OS mass schemes for the electron mass, of these only the $\overline{MS}$ mass is renormalisation scale dependent.

We aim to show that renormalisation is more than a necessary procedure to counter divergences. The freedom to choose different values for

μ allows us sum certain logarithmic terms to all orders and even in cases gives us non-perturbative information about our theory. Renormalisation allows us to apply perturbation theory at a whole range of different energy scales parametrised by μ and thus extends the applicability of the perturbative method. More details and a conceptual interpretation of renormalisation in terms of Wilson's approach using cutoff regularisation can be found in Ref. [18, ch. 12].

4.1 The $\overline{MS}$ renormalisation scheme

In the modified minimally subtracted renormalisation scheme, denoted as $\overline{MS}$, the counter terms are chosen as to cancel only the divergent contributions to the photon self-energy, the electron self-energy and the electron-photon vertex function. *

The photon self-energy shown in Eq: 3.26, the electron self-energy functions $A\left(p^2\right)$ shown in Eq: 3.51 and $B\left(p^2\right)$ shown in Eq: 3.62 and the photon-electron vertex function shown in both Eq: 3.113 and Eq: 3.133 for different infrared regulators become UV-finite for the following choices for the counter terms:

$$\begin{aligned}
Z_{0,\overline{MS}} &= 1 - \frac{e^2}{8\pi^2}\left(3+\lambda^{-2}\right)\frac{1}{\epsilon} + \mathcal{O}\left(e^4\right) \\
Z_{1,\overline{MS}} &= 1 - \frac{e^2\lambda^{-2}}{8\pi^2}\frac{1}{\epsilon} + \mathcal{O}\left(e^4\right) \\
Z_{2,\overline{MS}} &= 1 - \frac{e^2}{8\pi^2}\lambda^{-2}\frac{1}{\epsilon} + \mathcal{O}\left(e^4\right) \\
Z_{3,\overline{MS}} &= 1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon} + \mathcal{O}\left(e^4\right)
\end{aligned} \tag{4.1}$$

We note the remarkable equality of the counter terms $Z_{1,\overline{MS}} = Z_{2,\overline{MS}}$. The results in Eq: 4.1 are consistent with the UV-divergent part of the results in [17, ch. 62] and the minimally subtracted results in [21, ch. 5] in the Feynman gauge $\lambda = 1$ **.

The $\overline{MS}$ photon self-energy function $\Pi_{\overline{MS}}\left(p^2\right)$ is given by:

$$\Pi_{\overline{MS}}\left(p^2\right) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x\,(1-x)\ln\left(\frac{D_0^2}{\mu^2}\right) + \mathcal{O}\left(e^4\right) \tag{4.2}$$

*The scheme is referred to as modified because we choose $\widetilde{\mu}^2 = \frac{e^{\gamma_E}}{4\pi}\mu^2$ and minimal since we only cancel the divergent part.

**To compare the result in [21] with our result we note that they use a mostly minus metric instead of our mostly positive metric.

The $\overline{MS}$ electron self-energy, we write as $\Sigma_{\overline{MS}}(\not{p}) = mA_{\overline{MS}}(p^2) + \not{p}B_{\overline{MS}}(p^2)$. The functions $A_{\overline{MS}}(p^2)$ and $B_{\overline{MS}}(p^2)$ are given by:

$$\begin{aligned} A_{\overline{MS}}\left(p^2\right) = -\frac{e^2}{8\pi^2}\Bigg[2+\lambda^{-2}-\frac{1}{2}\left(3+\lambda^{-2}\right)\Bigg[\ln\frac{m^2}{\mu^2} \\ +\frac{p^2+m^2}{p^2}\ln\left(1+\frac{p^2}{m^2}\right)\Bigg]\Bigg]+\mathcal{O}\left(e^4\right) \end{aligned} \tag{4.3}$$

$$\begin{aligned} B_{\overline{MS}}\left(p^2\right) = -\frac{e^2}{8\pi^2}\lambda^{-2}\Bigg[1-\frac{1}{2}\ln\frac{m^2}{\mu^2}-\frac{p^2+m^2}{p^2}\left[\frac{1}{2}+\ln\left(1+\frac{p^2}{m^2}\right)\right] \\ +\frac{1}{2}\left(\frac{p^2+m^2}{p^2}\right)^2\ln\left(1+\frac{p^2}{m^2}\right)\Bigg]+\mathcal{O}\left(e^4\right) \end{aligned} \tag{4.4}$$

4.2 The on-shell renormalisation scheme

An appropriate choice for the finite parts of the counter terms allows us to let the mass parameter m and e correspond to the physical mass and charge of the electron, which is what the OS renormalisation scheme does. We require the exact propagators for on-shell particles to have a singularity with residue one whenever the momentum squared equals minus the on-shell mass squared. Also, the electron-photon vertex function must be of the form: $ie\gamma^\mu + \mathcal{O}\left(e^4\right)$ when all in and out-going particles are on-shell.

The photon self-energy

For $Z_{3,OS} = Z_{3,\overline{MS}} + Z_{3,fin}$ where $Z_{3,\overline{MS}}$ is given by Eq: 4.1, we need to fix $Z_{3,fin}$ such that the on-shell conditions are satisfied. For the photon exact propagator, the tensorial structure of $\Pi^{\mu\nu}(p)$ guarantees that we have a pole at $p^2 = 0$, so we only need to fix $\Pi(0) = 0$ to guarantee that the exact propagator has a pole with a residue of one.

$$\frac{P_{\mu\nu}(p)}{[1-\Pi(0)]} = P_{\mu\nu} \Rightarrow \Pi(0) = 0 \tag{4.5}$$

We apply the constraint on Eq: 3.26, we find:

$$\begin{aligned}\Pi_{OS}(0) &= -\frac{e^2}{6\pi^2}\left[\frac{1}{\epsilon} - 3\int_0^1 dx\, x(1-x)\ln\left(\frac{m^2}{\mu^2}\right)\right] - (Z_{3,OS}-1) + \mathcal{O}\left(e^4\right) \\ &= \frac{e^2}{12\pi^2}\ln\left(\frac{m^2}{\mu^2}\right) - Z_{3,fin} + \mathcal{O}\left(e^4\right) = 0 \\ \Rightarrow Z_{3,fin} &= \frac{e^2}{12\pi^2}\ln\left(\frac{m^2}{\mu^2}\right) + \mathcal{O}(e^4)\end{aligned} \tag{4.6}$$

The end result in the *OS* renormalisation scheme after substituting $Z_{3,fin}$ back into the self-energy function $\Pi\left(p^2\right)$ and the counter term is given by:

$$\Pi_{OS}\left(p^2\right) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\ln\left(\frac{D_0^2}{m^2}\right) + \mathcal{O}\left(e^4\right) \tag{4.7}$$

$$Z_{3,OS} = 1 - \frac{e^2}{6\pi^2}\frac{1}{\epsilon} + \frac{e^2}{12\pi^2}\ln\left(\frac{m^2}{\mu^2}\right) + \mathcal{O}(e^4) \tag{4.8}$$

The results in Eq: 4.7 and Eq: 4.8 are gauge-independent and therefore consistent with the on-shell results in Ref. [17, ch. 62] that were obtained using the Feynman gauge $\lambda = 1$.

In Fig: 4.1 we show how $\Pi_{OS}\left(p^2\right)$ varies as a function of the photon momentum. We see in particular that for $p^2 < -4m^2$ the photon propagator gets an imaginary component. This means that the off-shell photon becomes unstable and can decay into an on-shell electron and an on-shell positron.

The electron self-energy with photon regulator mass

For $Z_{0,OS} = Z_{0,\overline{MS}} + Z_{0,fin}$ and $Z_{2,OS} = Z_{2,\overline{MS}} + Z_{2,fin}$ where $Z_{0,\overline{MS}}$ and $Z_{2,\overline{MS}}$ are given by Eq: 4.1, we need to fix $Z_{0,fin}$ and $Z_{2,fin}$ such that the on-shell conditions are satisfied. To obtain a pole in the exact electron propagator at $\not{p} = -m$ with a residue of one we must enforce on the self-energy the conditions: $\Sigma(-m) = 0^*$ and $\frac{\partial}{\partial \not{p}}\Sigma(\not{p})\,|_{\not{p}=-m} = 0$.

We apply the constraints on the self energy functions $A\left(p^2\right)$ in Eq: 3.50 and $B\left(p^2\right)$ in Eq: 3.61 with $m_\gamma^2 \neq 0$:

$$\Sigma(-m) = m\left(A\left(-m^2\right) - B\left(-m^2\right)\right) = 0 \;\Rightarrow\; A\left(-m^2\right) - B\left(-m^2\right) = 0 \tag{4.9}$$

*Here $\not{p} = -m \;\Rightarrow p^2 = -\not{p}^2 = -m^2$.

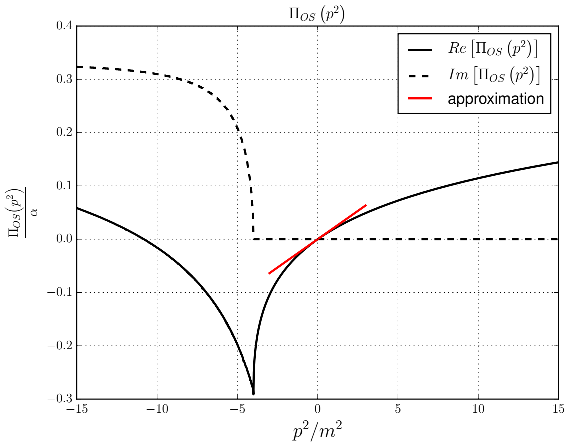


Figure 4.1: *This figure shows us the behaviour of $\Pi_{OS}(p^2)$ in units of $\alpha = \frac{e^2}{4\pi}$ and an approximation to $\Pi_{OS}(p^2)$ where we approximated the logarithm inside the integral by the linear term in its Taylor expansion.*

We set $p^2 = -m^2$ in Eq: 3.50 and Eq: 3.61 and subtract them from each other.

$$\begin{aligned} A\left(-m^2\right) - B\left(-m^2\right) =& \frac{e^2}{16\pi^2}\left[1 + 3\ln\frac{m^2}{\mu^2} + 2\int_0^1 dx\,(1+x)\ln\left(x^2 + (1-x)\frac{m_\gamma^2}{m^2}\right)\right. \\ &\left. - \lambda^{-2}\int_0^1 dx\ln\left(1 - \left(1-\lambda^{-2}\right)\frac{(1-x)\,m_\gamma^2}{x^2m^2 + (1-x)\,m_\gamma^2}\right)\right] \\ &- \left(Z_{0,fin} - Z_{2,fin}\right) + \mathcal{O}\left(e^4\right) = 0 \end{aligned} \tag{4.10}$$

The expression in Eq: 4.10 allows us to determine $\left(Z_{0,fin} - Z_{2,fin}\right)$ for $m_\gamma^2 \neq 0$, a cumbersome expression. Fortunately, the expression in Eq: 4.10 is a finite function in terms of $\kappa = \frac{m_\gamma^2}{m^2}$, which we see from the series expansions in Eq: C.11 and Eq: C.12 of App: C.5. The result for $m_\gamma^2 = 0$ is given by:

$$\left(Z_{0,fin} - Z_{2,fin}\right)\Big|_{m_\gamma^2=0} = -\frac{e^2}{4\pi^2}\left(1 - \frac{3}{4}\ln\frac{m^2}{\mu^2}\right) + \mathcal{O}\left(e^4\right) \tag{4.11}$$

We now continue to determine the second constraint on the counter terms $\frac{\partial}{\partial \not{p}}\Sigma\left(\not{p}\right)|_{\not{p}=-m} = 0$:

$$\begin{aligned} \frac{\partial}{\partial \not{p}} \Sigma(\not{p})_{\not{p}=-m} &= \frac{\partial}{\partial \not{p}} \left[mA\left(p^2\right) + \not{p} B\left(p^2\right) \right]_{\not{p}=-m} \\ &= 2m^2 \frac{\partial}{\partial p^2} \left[A\left(p^2\right) - B\left(p^2\right) \right]_{p^2=-m^2} + B\left(-m^2\right) = 0 \end{aligned} \tag{4.12}$$

We first determine the value of the derivative term:

$$\begin{aligned} A\left(p^2\right) - B\left(p^2\right) =& \frac{e^2}{16\pi^2} \left[1 + 3\ln\frac{m^2}{\mu^2} + 2\int_0^1 dx\,(1+x)\ln\frac{D_1^2}{m^2} + \frac{p^2+m^2}{m_\gamma^2} \right. \\ & \left. \times \int_0^1 dx\, x \left[\ln\frac{D_1^2}{m^2} - \ln\frac{\widetilde{D}_1^2}{m^2} \right] \right] - \left(Z_{0,fin} - Z_{2,fin} \right) + \mathcal{O}\left(e^4\right) \end{aligned} \tag{4.13}$$

Now we substitute the result we found in Eq: 4.11:

$$\begin{aligned} A\left(p^2\right) - B\left(p^2\right) =& \frac{e^2}{16\pi^2} \left[5 + 2\int_0^1 dx\,(1+x)\ln\frac{D_1^2}{m^2} + \frac{p^2+m^2}{m_\gamma^2} \right. \\ & \left. \times \int_0^1 dx\, x \left[\ln\frac{D_1^2}{m^2} - \ln\frac{\widetilde{D}_1^2}{m^2} \right] \right] + \mathcal{O}\left(e^4\right) \end{aligned} \tag{4.14}$$

Before we take the full derivative of Eq: 4.14 with respect to p^2 evaluated at $p^2 = -m^2$, we act it on the first integrand separately:

$$m^2 \frac{\partial}{\partial p^2} \ln\frac{D_1^2}{m^2}\bigg|_{p^2=-m^2} = \frac{x(1-x)}{x(1-x)\frac{p^2}{m^2} + x + (1-x)\kappa}\bigg|_{p^2=-m^2} = \frac{x(1-x)}{x^2+(1-x)\kappa} \tag{4.15}$$

Using Eq: 4.15, we find that the full derivative of Eq: 4.14 with respect to p^2 evaluated at $p^2 = -m^2$ is given by:

$$\begin{aligned} 2m^2 \frac{\partial}{\partial p^2} \left[A\left(p^2\right) - B\left(p^2\right) \right]_{p^2=-m^2} =& \frac{e^2}{8\pi^2} \left[2\int_0^1 dx \frac{x(1-x^2)}{x^2+(1-x)\kappa} \right. \\ & \left. - \frac{1}{\kappa} \int_0^1 dx\, x \ln\left(1 - \left(1-\lambda^{-2}\right) \frac{(1-x)\kappa}{x^2+(1-x)\kappa} \right) \right] + \mathcal{O}\left(e^4\right) \end{aligned} \tag{4.16}$$

It is not possible to set $\kappa = 0$, since the integrals contain logarithmic divergences in κ. We therefore take the limit for κ approaching zero, and we use the series expansions that are tabulated in App: C.5, we find:

$$\lim_{\kappa\to 0} 2m^2 \frac{\partial}{\partial p^2}\left[A\left(p^2\right) - B\left(p^2\right)\right]_{p^2=-m^2} = -\frac{e^2}{8\pi^2}\left[1 + \ln\kappa + \frac{1}{2\kappa}\left(\left(1-\lambda^{-2}\right)\right.\right.$$
$$\left.\left. \times\left(1+\ln\kappa\right)\kappa - \lambda^{-2}\ln\lambda^{-2}\kappa\right) + \mathcal{O}\left(\sqrt{\kappa}\right)\right] + \mathcal{O}\left(e^4\right) \tag{4.17}$$
$$= -\frac{e^2}{16\pi^2}\left[\left(3-\lambda^{-2}\right)\left(1+\ln\kappa\right) - \lambda^{-2}\ln\lambda^{-2}\right] + \mathcal{O}\left(\sqrt{\kappa}\right) + \mathcal{O}\left(e^4\right)$$

Now, we fill out Eq: 4.17 and $B\left(-m^2\right)$ by using Eq: 3.62 into Eq: 4.12, do some rearranging and find ourselves an expression for $Z_{2,fin}$:

$$\lim_{\kappa\to 0} Z_{2,fin} = -\frac{e^2}{16\pi^2}\left[\left(3+\lambda^{-2}\right) - \lambda^{-2}\ln\frac{m^2}{\mu^2} + \left(3-\lambda^{-2}\right)\ln\kappa\right.$$
$$\left. -\lambda^{-2}\ln\lambda^{-2} + \mathcal{O}\left(\sqrt{\kappa}\right)\right] + \mathcal{O}\left(e^4\right) \tag{4.18}$$

Now, we use Eq: 4.11 in Eq: 4.18 to find $Z_{0,fin}$:

$$\lim_{\kappa\to 0} Z_{0,fin} = -\frac{e^2}{16\pi^2}\left[7 + \lambda^{-2} - \left(3+\lambda^{-2}\right)\ln\frac{m^2}{\mu^2} + \left(3-\lambda^{-2}\right)\ln\kappa\right.$$
$$\left. -\lambda^{-2}\ln\lambda^{-2} + \mathcal{O}\left(\sqrt{\kappa}\right)\right] + \mathcal{O}\left(e^4\right) \tag{4.19}$$

The end-result in the OS renormalisation scheme after substituting $Z_{0,fin}$ and $Z_{2,fin}$ back into the self-energy functions $A\left(p^2\right)$ and $B\left(p^2\right)$ and the counter terms in terms of κ as infrared regulator are given by:

$$A_{OS}\left(p^2\right) = \frac{e^2}{16\pi^2}\left[\left(3-\lambda^{-2}\right)\left[1+\ln\kappa\right] - \lambda^{-2}\ln\lambda^{-2}\right] \tag{4.20}$$
$$+ \frac{e^2}{16\pi^2}\left(3+\lambda^{-2}\right)\frac{p^2+m^2}{p^2}\ln\left(1+\frac{p^2}{m^2}\right) + \mathcal{O}\left(e^4\right)$$

$$B_{OS}\left(p^2\right) = \frac{e^2}{16\pi^2}\left[\left(3-\lambda^{-2}\right)\left[1+\ln\kappa\right] - \lambda^{-2}\ln\lambda^{-2}\right] + \frac{e^2}{16\pi^2}\lambda^{-2}\left[\frac{p^2+m^2}{p^2}\right. \tag{4.21}$$
$$\left. \times\left[1+2\ln\left(1+\frac{p^2}{m^2}\right)\right] - \left(\frac{p^2+m^2}{p^2}\right)^2\ln\left(1+\frac{p^2}{m^2}\right)\right] + \mathcal{O}\left(e^4\right)$$

$$Z_{0,\,OS} = 1 - \frac{e^2}{8\pi^2}\left(3+\lambda^{-2}\right)\frac{1}{\epsilon} - \frac{e^2}{16\pi^2}\left[7 + \lambda^{-2} - \left(3+\lambda^{-2}\right)\ln\frac{m^2}{\mu^2}\right. \tag{4.22}$$

$$+\left(3-\lambda^{-2}\right)\ln\kappa-\lambda^{-2}\ln\lambda^{-2}\Big]+\mathcal{O}\left(e^4\right)$$

$$Z_{2,\,OS}=1-\frac{e^2}{8\pi^2}\lambda^{-2}\frac{1}{\epsilon}-\frac{e^2}{16\pi^2}\Big[\left(3+\lambda^{-2}\right)-\lambda^{-2}\ln\frac{m^2}{\mu^2} \tag{4.23}$$

$$+\left(3-\lambda^{-2}\right)\ln\kappa-\lambda^{-2}\ln\lambda^{-2}\Big]+\mathcal{O}\left(e^4\right)$$

$A_{OS}\left(p^2\right)$ and $B\left(p^2\right)$ still depend on: μ, κ, and λ but the functions in Eq: 4.24 do not:

$$\Delta A_{OS}\left(p^2\right)\Big|_{\lambda=1}=A_{OS}\left(p^2\right)-A_{OS}\left(-m^2\right)\Big|_{\lambda=1}=\frac{\alpha}{\pi}\frac{p^2+m^2}{p^2}\ln\left(1+\frac{p^2}{m^2}\right)+\mathcal{O}\left(e^4\right)$$

$$\Delta B_{OS}\left(p^2\right)\Big|_{\lambda=1}=B_{OS}\left(p^2\right)-B_{OS}\left(-m^2\right)\Big|_{\lambda=1}=\frac{\alpha}{4\pi}\left[\frac{p^2+m^2}{p^2}\left[1+2\ln\left(1+\frac{p^2}{m^2}\right)\right]\right.$$

$$\left.-\left(\frac{p^2+m^2}{p^2}\right)^2\ln\left(1+\frac{p^2}{m^2}\right)\right]+\mathcal{O}\left(e^4\right) \tag{4.24}$$

In Fig: 4.2 we show how $\Delta A_{OS}\left(p^2\right)\Big|_{\lambda=1}$ and $\Delta B_{OS}\left(p^2\right)\Big|_{\lambda=1}$ vary as a function of the electron momentum.

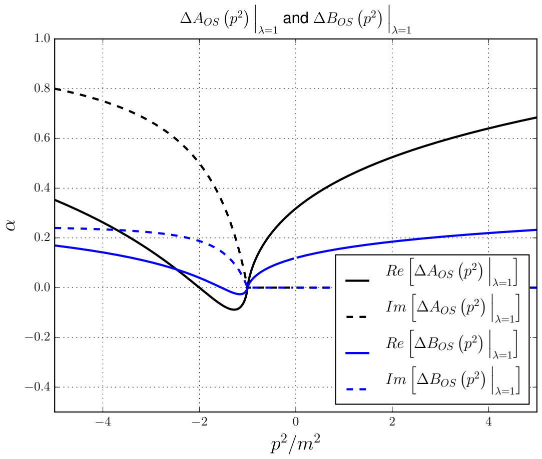


Figure 4.2: *This figure demonstrates the behaviour of $\Delta A_{OS}\left(p^2\right)\Big|_{\lambda=1}$ and $\Delta B_{OS}\left(p^2\right)\Big|_{\lambda=1}$ in units of $\alpha=\frac{e^2}{4\pi}$. When the gauge is changed the graphs will scale with different scale factors namely: $\left(3-\lambda^{-2}\right)$ and λ^{-2} respectively.*

The electron self-energy in full dimensional regularisation

A photon regulator mass regularises the IR-divergences in the OS condition but the expansion in terms of κ is quite difficult. In a different approach we handle the IR-divergence using dimensional regularisation by delaying the expansion in the regularisation parameter ϵ to the very end.

We have already regularised the UV-divergences in terms of $\epsilon > 0$, now we use $\epsilon < 0$ to regularise the IR-divergence in the OS condition. To discern between the two types of divergences we introduce $\epsilon_{UV} \equiv \epsilon > 0$ and $\epsilon_{IR} \equiv -\epsilon > 0$.

All IR-finite expressions such as $A_{\overline{MS}}\left(p^2\right)$ in Eq: 4.3 and $B_{\overline{MS}}\left(p^2\right)$ in Eq: 4.4 are unaffected when we employ full dimensional regularisation. However to evaluate the derivative term in Eq: 4.12 we must now take the derivative before we expand in ϵ.

We first put $m_\gamma^2 = 0$ in the expression for $A\left(p^2\right)$ shown in Eq: 3.40:

$$A\left(p^2\right) = \overbrace{\mathrm{i}e^2\widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \frac{d+\lambda^{-2}-1}{\left((p+l)^2+m^2\right) l^2}}^{\boxed{C}} - (Z_0 - 1) + \mathcal{O}\left(e^4\right) \tag{4.25}$$

We decompose the integral as before, and take the derivative of the expression for $\boxed{C}$ as shown in Eq: 3.45:

$$\begin{aligned} 2m^2 \frac{\partial}{\partial p^2} \boxed{C}\Big|_{p^2=-m^2} =& -\frac{e^2}{8\pi^2}\Gamma\left(\frac{\epsilon}{2}\right)\left(3+\lambda^{-2}-\epsilon\right)\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)^{\frac{\epsilon}{2}} \\ &\times \int_0^1 dx\, m^2 \frac{\partial}{\partial p^2}\left[x(1-x)\frac{p^2}{m^2}+x\right]^{-\frac{\epsilon}{2}}\Big|_{p^2=-m^2} \\ =& \frac{e^2}{8\pi^2}\Gamma\left(\frac{\epsilon}{2}\right)\left(3+\lambda^{-2}-\epsilon\right)\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)^{\frac{\epsilon}{2}}\left(\frac{\epsilon}{2}\right)\int_0^1 dx\, x(1-x)\left(x^2\right)^{-\left(1+\frac{1}{2}\epsilon\right)} \end{aligned} \tag{4.26}$$

Now we perform the integral over the Feynman parameter x, where we need $\epsilon < 0$ for the integral to be well-defined:

$$\begin{aligned} \int_0^1 dx\,(1-x)\,x^{-1-\epsilon} &= -\frac{1}{\epsilon}(1-x)\,x^{-\epsilon}\Big|_0^1 - \frac{1}{\epsilon}\int_0^1 dx\, x^{-\epsilon} \\ &= \frac{1}{\epsilon}(0)^{-\epsilon} - \frac{1}{\epsilon(1-\epsilon)}\Big|_{\epsilon<0} = -\frac{1}{\epsilon(1-\epsilon)} \end{aligned} \tag{4.27}$$

The factor $\frac{1}{\epsilon}(0)^{-\epsilon} = 0$ only for $\epsilon < 0$ and is else undefined. We plug the integral of Eq: 4.27 back into expression $\boxed{C}$ of Eq: 4.26, expand for ϵ, rearrange and then find in agreement with [21, ch. 5]:

$$2m^2 \frac{\partial}{\partial p^2} A\left(p^2\right)\Big|_{p^2=-m^2} = \frac{e^2}{8\pi^2}\left(3+\lambda^{-2}\right)\frac{1}{\epsilon_{IR}} - \frac{e^2}{8\pi^2}\Big[2+\lambda^{-2} - \frac{1}{2}\left(3+\lambda^{-2}\right)\ln\frac{m^2}{\mu^2}\Big] + \mathcal{O}\left(e^4\right) \tag{4.28}$$

We continue with $B\left(p^2\right)$ by putting $m_\gamma^2 = 0$ in Eq: 3.41:

$$B\left(p^2\right) = \overbrace{\mathrm{i}e^2\widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d}\frac{(d-2)\left(1+\frac{p\cdot l}{p^2}\right)-\left(1-\lambda^{-2}\right)}{\left((p+l)^2+m^2\right)l^2}}^{\text{e}} - \overbrace{\mathrm{i}e^2\widetilde{\mu}^\epsilon\left(1-\lambda^{-2}\right)\frac{p^2+m^2}{p^2}}^{\text{f}} \times \overbrace{\int \frac{d^d l}{(2\pi)^d}\frac{p\cdot l}{\left((p+l)^2+m^2\right)l^4}}^{\text{f}} - (Z_2-1) + \mathcal{O}\left(e^4\right) \tag{4.29}$$

We decompose the integral as before, and continue with the $m_\gamma^2 = 0$ analogue of the decomposition of e in terms of e.1 and e.2 as in Eq: 3.52. The result for the derivative of e.1 again follows from the substitution of $\lambda^{-2} \to \lambda^{-2} - 2$ in c:

$$2m^2\frac{\partial}{\partial p^2}\,\text{e.1}\Big|_{p^2=-m^2} = \frac{e^2}{8\pi^2}\left(1+\lambda^{-2}\right)\frac{1}{\epsilon_{IR}} - \frac{e^2}{8\pi^2}\Big[\lambda^{-2} - \frac{1}{2}\left(1+\lambda^{-2}\right)\ln\frac{m^2}{\mu^2}\Big] \tag{4.30}$$

The derivative of e.2 follows from Eq: 3.55:

$$\begin{aligned} 2m^2\frac{\partial}{\partial p^2}\,\text{e.2}\Big|_{p^2=-m^2} =& \frac{e^2}{8\pi^2}(2-\epsilon)\,\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)^{\frac{\epsilon}{2}} \\ &\times \int_0^1 dx\, xm^2\frac{\partial}{\partial p^2}\left[x(1-x)\frac{p^2}{m^2}+x\right]^{\frac{\epsilon}{2}}\Big|_{p^2=-m^2} \\ =& -\frac{e^2}{8\pi^2}(2-\epsilon)\,\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)^{\frac{\epsilon}{2}}\left(\frac{\epsilon}{2}\right)\int_0^1 dx\, x^2(1-x)\left(x^2\right)^{-\left(1+\frac{1}{2}\epsilon\right)} \end{aligned} \tag{4.31}$$

Now we perform the integral over the Feynman parameter x, we see that unlike Eq: 4.27 this integral is finite for $\epsilon = 0$:

$$\int_0^1 dx\,(1-x)\,x^{-\epsilon} = \frac{1}{1-\epsilon}\,(1-x)\,x^{1-\epsilon}\Big|_0^1 + \frac{1}{1-\epsilon}\int_0^1 dx\,x^{1-\epsilon} = \frac{1}{(1-\epsilon)\,(2-\epsilon)} \tag{4.32}$$

(The boundary term is struck through and marked $\rightarrow 0$.)

We plug the integral of Eq: 4.32 back into e.2 of Eq: 4.31, expand for ϵ, rearrange and then find:

$$2m^2 \frac{\partial}{\partial p^2}\,\text{e.2}\Big|_{p^2=-m^2} = -\frac{e^2}{8\pi^2} \tag{4.33}$$

Finally we put $m_\gamma^2 = 0$ in the expression for f in Eq: 3.58. We keep ϵ and integrate over the now redundant Feynman parameter y, we then find:

$$\text{f} = -\frac{e^2}{16\pi^2}\left(1-\lambda^{-2}\right)\Gamma\left(1+\frac{\epsilon}{2}\right)\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)^{\frac{\epsilon}{2}}\frac{p^2+m^2}{m^2} \times \int_0^1 dx\,x\,(1-x)\left(x\,(1-x)\,\frac{p^2}{m^2}+x\right)^{-\left(1+\frac{1}{2}\epsilon\right)} \tag{4.34}$$

We take the derivative of f as shown in Eq: 4.34 and evaluate it for $p^2 = -m^2$. We substitute Eq: 4.27 for the integral over x, rearrange, and then find:

$$2m^2\frac{\partial}{\partial p^2}\,\text{f}\Big|_{p^2=-m^2} = -\frac{e^2}{8\pi^2}\left(1-\lambda^{-2}\right)\left(\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{4\pi\widetilde{\mu}^2}{m^2}\right)^{\frac{\epsilon}{2}}\int_0^1 dx\,x\,(1-x)\left(x^2\right)^{-\left(1+\frac{1}{2}\epsilon\right)} = -\frac{e^2}{8\pi^2}\left(1-\lambda^{-2}\right)\frac{1}{\epsilon_{IR}} + \frac{e^2}{8\pi^2}\left(1-\lambda^{-2}\right)\left[1-\frac{1}{2}\ln\frac{m^2}{\mu^2}\right] \tag{4.35}$$

We plug the integrals e.1 of Eq: 4.30, e.2 of Eq: 4.33 and f of Eq: 4.35 into the derivative of $B\left(p^2\right)$, then find in agreement with [21, ch. 5]:

$$2m^2\frac{\partial}{\partial p^2}B\left(p^2\right)\Big|_{p^2=-m^2} = \frac{e^2}{4\pi^2}\lambda^{-2}\frac{1}{\epsilon_{IR}} - \frac{e^2}{4\pi^2}\lambda^{-2}\left[1-\frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)\right] + \mathcal{O}\left(e^4\right) \tag{4.36}$$

We now write down the complete derivative term by subtracting the contribution in Eq: 4.36 from Eq: 4.28:

$$2m^2 \frac{\partial}{\partial p^2} \left[A\left(p^2\right) - B\left(p^2\right) \right] \Big|_{p^2=-m^2} = \frac{e^2}{8\pi^2}\left(3-\lambda^{-2}\right)\frac{1}{\epsilon_{IR}} - \frac{e^2}{8\pi^2}\Big[2-\lambda^{-2} \\ -\frac{1}{2}\left(3-\lambda^{-2}\right)\ln\frac{m^2}{\mu^2}\Big] + \mathcal{O}\left(e^4\right) \tag{4.37}$$

The expression in Eq: 4.37 and $B\left(-m^2\right)$ from Eq: 4.4 allow us to find $Z_{0,OS}$ and $Z_{2,OS}$ and the self-energy functions $A\left(p^2\right)$ and $B\left(p^2\right)$:

$$A_{OS}\left(p^2\right) = -\frac{e^2}{8\pi^2}\left(3-\lambda^{-2}\right)\left[\frac{1}{\epsilon_{IR}} + \frac{1}{2}\ln\frac{m^2}{\mu^2}\right] + \frac{e^2}{8\pi^2}\Bigg[2-\lambda^{-2} \tag{4.38}$$

$$+\frac{1}{2}\left(3+\lambda^{-2}\right)\frac{p^2+m^2}{p^2}\ln\left(1+\frac{p^2}{m^2}\right)\Bigg] + \mathcal{O}\left(e^4\right)$$

$$B_{OS}\left(p^2\right) = -\frac{e^2}{8\pi^2}\left(3-\lambda^{-2}\right)\left[\frac{1}{\epsilon_{IR}} + \frac{1}{2}\ln\frac{m^2}{\mu^2}\right] + \frac{e^2}{8\pi^2}\Bigg[2-\lambda^{-2}+\lambda^{-2}\frac{p^2+m^2}{p^2} \tag{4.39}$$

$$\times\left[\frac{1}{2}+\ln\left(1+\frac{p^2}{m^2}\right)\right] - \left(\frac{p^2+m^2}{p^2}\right)^2\ln\left(1+\frac{p^2}{m^2}\right)\Bigg] + \mathcal{O}\left(e^4\right)$$

$$Z_{0,\,OS} = 1 - \frac{e^2}{8\pi^2}\left(3+\lambda^{-2}\right)\frac{1}{\epsilon_{UV}} + \frac{e^2}{8\pi^2}\left(3-\lambda^{-2}\right)\frac{1}{\epsilon_{IR}} \tag{4.40}$$

$$-\frac{e^2}{4\pi^2}\left[2-\frac{3}{2}\ln\frac{m^2}{\mu^2}\right] + \mathcal{O}\left(e^4\right)$$

$$Z_{2,\,OS} = 1 - \frac{e^2}{8\pi^2}\lambda^{-2}\frac{1}{\epsilon_{UV}} + \frac{e^2}{8\pi^2}\left(3-\lambda^{-2}\right)\frac{1}{\epsilon_{IR}} - \frac{e^2}{8\pi^2}\left[2-\frac{3}{2}\ln\frac{m^2}{\mu^2}\right] + \mathcal{O}\left(e^4\right) \tag{4.41}$$

The vertex function with a photon regulator mass

For $Z_{1,OS} = Z_{1,\overline{MS}} + Z_{1,fin}$ where $Z_{1,\overline{MS}}$ is given by Eq: 4.1, we need to fix $Z_{1,fin}$ such that the OS condition is satisfied. The OS condition for the vertex-function can formally be written as:

$$\overline{u}_{s'}\left(\mathbf{p}'\right)\mathbb{V}^{\mu}\left(p,p'\right)u_s\left(\mathbf{p}\right)\Bigg|_{\substack{p^2=p'^2=-m^2\\(p-p')^2=0}} = \overline{u}_{s'}\left(\mathbf{p}'\right)\gamma^{\mu}u_s\left(\mathbf{p}\right)\Bigg|_{\substack{p^2=p'^2=-m^2\\(p-p')^2=0}} \tag{4.42}$$

We use the spinor identity given in Eq: A.17 on the right-hand side and the fact that $p = p'$ with $p^2 = p'^2 = -m^2$:

$$\overline{u}_s\left(\mathbf{p}\right)\left(\frac{\mathbb{V}^{\mu}_{1-loop,fin}\left(p,p\right)}{e}\right)u_s\left(\mathbf{p}\right) = -2p^{\mu}Z_{1,fin} \tag{4.43}$$

We now expand the left-hand side of the above equation:

$$\begin{aligned}\overline{u}_s(\mathbf{p})\Bigg(&-\frac{e^2}{16\pi^2}\gamma^\mu\left[\left(3-\lambda^{-2}\right)-\left(1-\lambda^{-2}\right)\ln\kappa+\lambda^{-2}\left(\ln\lambda^{-2}-\ln\left(\frac{\mu^2}{m^2}\right)\right)\right]\\&-\frac{e^2}{8\pi^2}\gamma^\mu\int_0^1 dx\int_0^{1-x}dy\left[\ln\left(\frac{D_2^2}{m^2}\right)+\frac{N_0}{D_2^2}\right]\\&+\frac{e^2m}{4\pi^2}p^\mu\int_0^1 dx\int_0^{1-x}dy\,\frac{(x+y)^2-(x+y)}{D_2^2}\Bigg)u_s(\mathbf{p})=-2p^\mu Z_{1,fin}\end{aligned} \tag{4.44}$$

Now we use the spinor identities to get an equation for $Z_{1,fin}$. The on-shell conditions give us that all terms proportional to $\left(p^2+m^2\right)$ or $\left(p'^2+m^2\right)$ vanish and the terms involving $(\not p+m)$ and $(\not p'+m)$ drop out due to the contraction with spinors. The on-shell condition allows us to simplify Eq: 3.80 and Eq: 3.87 into:

$$D_2^2\Big|_{\substack{p=p'\\p^2=-m^2}}=(x+y)^2m^2-(x+y)\,m_\gamma^2+m_\gamma^2\equiv\mathbb{D}_2^2 \tag{4.45}$$

$$N_0\Big|_{\substack{p=p'\\p^2=-m^2}}=\left[(x+y)^2-4(x+y)+2\right]m^2 \tag{4.46}$$

The use of which allows us to write for $Z_{1,fin}$:

$$\begin{aligned}Z_{1,fin}=&\frac{e^2}{16\pi^2}\left[\left(3-\lambda^{-2}\right)-\left(1-\lambda^{-2}\right)\ln\kappa+\lambda^{-2}\left(\ln\lambda^{-2}+\ln\left(\frac{m^2}{\mu^2}\right)\right)\right]\\&+\frac{e^2}{8\pi^2}\int_0^1 dx\int_0^{1-x}dy\left[\ln\left(\frac{\mathbb{D}_2^2}{m^2}\right)-m^2\frac{(x+y)^2+2(x+y)-2}{\mathbb{D}_2^2}\right]+\mathcal{O}\left(e^4\right)\end{aligned} \tag{4.47}$$

We perform a coordinate shift $u=(x+y)$ and $v=(x-y)$ on the integrals in Eq: 4.47 and use the series expansions given in Eq: C.12, Eq: C.13 and Eq: C.14 to evaluate the integrals:

$$\begin{aligned}\int_0^1 dx\int_0^{1-x}dy\ln\left(\frac{\mathbb{D}_2^2}{m^2}\right)&=\int_0^1 du\,u\ln\left(u^2+(1-u)\,\kappa\right)\\&\approx-\frac{1}{2}-\frac{1}{2}(1+\ln\kappa)\,\kappa+\mathcal{O}\left(\kappa^2\right)\end{aligned} \tag{4.48}$$

$$\int_0^1 dx\int_0^{1-x}dy\frac{\left[(x+y)^2+2(x+y)-2\right]m^2}{\mathbb{D}_2^2}=\int_0^1 du\frac{u\left(u^2+2u-2\right)}{u^2+(1-u)\,\kappa} \tag{4.49}$$

$$\approx \frac{5}{2} + \ln\kappa - \frac{3\pi}{2}\sqrt{\kappa} + \frac{1}{2}\left(5 - \ln\kappa\right)\kappa + \mathcal{O}\left(\kappa^{\frac{3}{2}}\right)$$

After some rearranging, these integrals give us $Z_{1,fin}$:

$$\lim_{\kappa\to 0} Z_{1,fin} = -\frac{e^2}{16\pi^2}\left[\left(3+\lambda^{-2}\right) - \lambda^{-2}\ln\left(\frac{m^2}{\mu^2}\right) + \left(3-\lambda^{-2}\right)\ln\kappa \right. \\ \left. - \lambda^{-2}\ln\lambda^{-2} + \mathcal{O}\left(\sqrt{\kappa}\right)\right] + \mathcal{O}\left(e^4\right) \tag{4.50}$$

We substitute $Z_{1,fin}$ from Eq: 4.52 into the expression for $Z_{1,OS}$, this allows us to write down the one-loop vertex correction. We keep κ only to parametrise IR divergences and write down the part of the vertex correction that does not contain unevaluated Feynman parameter integrals. We note that the identity $Z_{1,OS} = Z_{2,OS}$ also holds for the OS renormalisation scheme.

$$\mathrm{i}\mathbb{V}^{\mu}_{OS}\left(p,p'\right) = \mathrm{i}e\gamma^{\mu} - \frac{\mathrm{i}e^3}{8\pi^2}\gamma^{\mu}\left[3+\ln\kappa\right] + \text{ integrals } + \mathcal{O}\left(e^5\right) \tag{4.51}$$

$$Z_{1,OS} = 1 - \frac{e^2\lambda^{-2}}{8\pi^2}\frac{1}{\epsilon} - \frac{e^2}{16\pi^2}\left[\left(3+\lambda^{-2}\right) - \lambda^{-2}\ln\left(\frac{m^2}{\mu^2}\right)\right. \\ \left. + \left(3-\lambda^{-2}\right)\ln\kappa - \lambda^{-2}\ln\lambda^{-2}\right] + \mathcal{O}\left(e^4\right) \tag{4.52}$$

The vertex function in full dimensional regularisation

The vertex-function with a photon regulator mass yields the correct values for the counter terms but it is easier in full dimensional regularisation. The OS renormalisation prescription in terms of $\Lambda^{\mu}\left(p,p'\right)$ is given by:

$$\overline{u}_{s'}\left(\mathbf{p}'\right)\Lambda^{\mu}\left(p,p'\right)u_s\left(\mathbf{p}\right)\Big|_{\substack{p^2=p'^2=-m^2\\k^2=0}} = \overline{u}_{s'}\left(\mathbf{p}'\right)\gamma^{\mu}u_s\left(\mathbf{p}\right)\Big|_{\substack{p^2=p'^2=-m^2\\k^2=0}} \tag{4.53}$$

The constraints $p^2 = p'^2 = -m^2$ and $k^2 = 0$ again imply $p = p'$, using this and the spinor identities and normalisation from App: A we fix $Z_{1,fin}$:

$$\overline{u}_{s'}\left(\mathbf{p}'\right)\Lambda^{\mu}\left(p,p'\right)u_s\left(\mathbf{p}\right)\Bigg|_{\substack{p^2=p'^2=-m^2\\k^2=0}} = \Lambda_0\left(0\right)\underbrace{\overline{u}_{s'}\left(\mathbf{p}\right)\gamma^{\mu}u_s\left(\mathbf{p}\right)}_{2p^{\mu}\delta_{s,s'}} + \frac{2p^{\mu}}{m}\Lambda_1\left(0\right)\underbrace{\overline{u}_{s'}\left(\mathbf{p}\right)u_s\left(\mathbf{p}\right)}_{2m\delta_{s,s'}} = 2p^{\mu}$$
$$\Rightarrow \Lambda_0\left(0\right)+2\Lambda_1\left(0\right)+\mathcal{O}\left(e^4\right) = 1 \tag{4.54}$$

From Eq: 4.54, we solve for $Z_{1,fin}$ and substitute this back into $Z_{1,OS}$. We find again as in the previous section that $Z_{1,OS} = Z_{2,OS}$ and the expression is consistent with the result in Ref. [21, ch. 5].

$$Z_{1,OS} = 1-\frac{e^2}{8\pi^2}\lambda^{-2}\frac{1}{\epsilon_{UV}}+\frac{e^2}{8\pi^2}\left(3-\lambda^{-2}\right)\frac{1}{\epsilon_{IR}}-\frac{e^2}{8\pi^2}\left[2-\frac{3}{2}\ln\left(\frac{m^2}{\mu^2}\right)\right]+\mathcal{O}\left(e^4\right) \tag{4.55}$$

Now that we have fixed $Z_{1,OS}$ we write down the vertex-function renormalised in the OS renormalisation prescription with OS electrons.

$$\Lambda^{\mu}_{OS}\left(p,p'\right) = \Lambda_{0,OS}\left(k^2\right)\gamma^{\mu}+\Lambda_1\left(k^2\right)\frac{p^{\mu}+p'^{\mu}}{m}+\frac{\not{p}'+m}{m}\Lambda^{\mu}_2\left(p,p'\right) + \Lambda^{\mu}_3\left(p,p'\right)\frac{\not{p}+m}{m}+\frac{\not{p}'+m}{m}\Lambda^{\mu}_4\left(p,p'\right)\frac{\not{p}+m}{m}+\mathcal{O}\left(e^4\right) \tag{4.56}$$

$$\begin{aligned}\Lambda_{0,OS}\left(k^2\right) =& 1+\frac{e^2}{8\pi^2}\frac{1}{\epsilon_{IR}}\left[2-\left(2+\frac{k^2}{m^2}\right)\int_0^1 du\ln\left(\frac{D_0^2}{m^2}\right)\right]-\frac{e^2}{16\pi^2}\Bigg[6\\ &-2\ln\left(\frac{m^2}{\mu^2}\right)+\int_0^1 du\ln\left(\frac{D_0^2}{m^2}\right)+\left(2+\frac{k^2}{m^2}\right)\left[\int_0^1 du\frac{m^2}{D_0^2}\ln\left(\frac{D_0^2}{m^2}\right)\right.\\ &\left.+\ln\left(\frac{m^2}{\mu^2}\right)\int_0^1 du\frac{m^2}{D_0^2}\right]-2\left(4+\frac{k^2}{m^2}\right)\int_0^1 du\frac{m^2}{D_0^2}\Bigg]\end{aligned} \tag{4.57}$$

An alternative more popular decomposition of the vertex-function in terms of two form factors $F_1\left(k^2\right)$ and $F_2\left(k^2\right)$, where $F_1\left(k^2\right)$ is referred to as the charge form factor and $F_2\left(k^2\right)$ as the anomalous magnetic moment form factor, is given by:

$$\overline{u}_{s'}\left(\mathbf{p}'\right)\Lambda^{\mu}\left(p,p'\right)u_s\left(\mathbf{p}\right)\Bigg|_{p^2=p'^2=-m^2} = F_1\left(k^2\right)\overline{u}_{s'}\left(\mathbf{p}'\right)\gamma^{\mu}u_s\left(\mathbf{p}\right)+\frac{\mathrm{i}k_{\nu}}{m}F_2\left(k^2\right)\overline{u}_{s'}\left(\mathbf{p}'\right)S^{\mu\nu}\left(\mathbf{p}\right) \tag{4.58}$$

By making use of the proof of Eq: A.17 we determine the two form factors:

$$F_1\left(k^2\right) = \Lambda_0\left(k^2\right) + 2\Lambda_1\left(k^2\right) \Rightarrow F_1\left(k^2\right) = 1$$
$$F_2\left(k^2\right) = -2\Lambda_1\left(k^2\right) \Rightarrow F_2\left(0\right) = \frac{e^2}{8\pi^2} \tag{4.59}$$

These expressions for the form factors are consistent with the results in Ref. [21, ch. 5]. We note that $F_1\left(k^2\right)$ was fixed at $k^2 = 0$ by the renormalisation procedure but is still IR-divergent for all $k^2 \neq 0$, $F_2\left(k^2\right)$ on the other hand was not affected by the renormalisation. Because of the IR-divergences, it is not useful to try and plot $F_1\left(k^2\right)$ but it is possible to plot $F_2\left(k^2\right)$ as shown in Fig: 4.3:

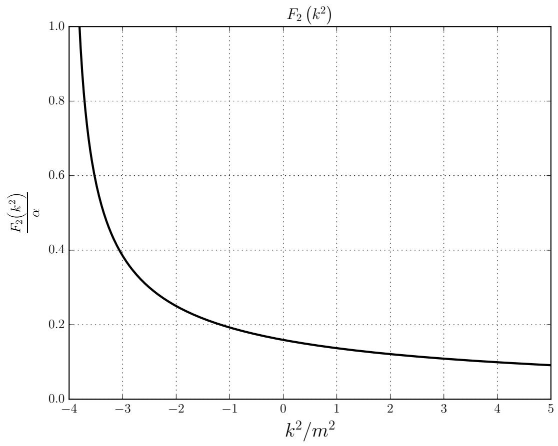


Figure 4.3: *This figure shows us the behaviour of $F_2\left(k^2\right)$ in units of $\alpha = \frac{e^2}{4\pi}$. This result is invariant of the choice of gauge or renormalisation.*

4.3 Relating IR-regularisation schemes

We have now successfully renormalised QED in both the $\overline{MS}$ and OS-renormalisation schemes by means of two different IR-regulator procedures. Namely, we have considered the extension of dimensional regularisation to handle also IR-divergences and we have introduced a fictitious photon mass term in the original lagrangian back in Sec: 2.2. Here, we now compare the different IR-regularisation schemes using the previous results.

The photon regulator mass introduces extra integrals proportional to m_γ^2 in the evaluation of the self-energy functions, namely the integrals d in $A\left(p^2\right)$ of Eq: 3.40 and g in $B\left(p^2\right)$ of Eq: 3.41. Also, in the evaluation of f in $B\left(p^2\right)$ we have to introduce an extra Feynman parameter to account for the regulator and it does not seem possible to simplify the result for the vertex correction such that we only have one Feynman parameter integral.

The application of $\overline{MS}$ renormalisation scheme was not affected by the use of regulator scheme, but the OS renormalisation scheme was. In dimensional regularisation, it was possible to isolate the $\propto \frac{1}{\epsilon_{IR}}$ divergent terms by hand, whilst the isolation of $\propto \ln \kappa$ terms was only possibly by using computer algebra software.

The use of full dimensional regularisation has the advantage that it does not require the introduction of extra terms in the lagrangian. Hence, dimensional regularisation does also not introduce additional terms that break gauge-invariance like the photon mass term in Eq: 2.15. Furthermore, the use of full dimensional regularisation does not require a separate implementation, because dimensional regularisation is already used for the regularisation of the UV-divergences. We only need to delay the expansion in terms of ϵ until after all other calculations have been completed.

When we compare the result for the derivative terms in Eq: 4.37 in full dimensional regularisation with the result we found using a photon regulator mass as shown in Eq: 4.17, then we find:

$$\frac{2}{\epsilon_{IR}} = -\ln\frac{m_\gamma^2}{\mu^2} + \frac{1}{3-\lambda^{-2}}\left[1-\lambda^{-2}\left(1-\ln\lambda^{-2}\right)\right] \tag{4.60}$$

This correspondence between the photon mass and the dimensional regulator was also found in Ref. [21, ch. 5] in the Feynman gauge ($\lambda = 1$).

Concluding, we consider the extension of dimensional regularisation of UV-divergences to also regularise IR-divergences in the same framework preferable with respect to the inclusion of a photon mass regulator. The only drawback of this method is that intuitively the departure from $d = 4$ to $d = 4 - \epsilon$ dimensions, where ϵ can be both positive and negative, is hard to imagine whilst giving a photon an asymptotically small mass can be imagined.

4.4 Relating the $\overline{MS}$ and OS mass schemes

We have given the electron self-energy in both the $\overline{MS}$ and the OS renormalisation scheme. The pole mass m_{pole} as defined in Eq: 3.30 equals the on-shell mass m_{OS} and since this definition is renormalisation scheme independent, we can express $m_{pole} = m_{OS}$ in terms of $m_{\overline{MS}}$.

$$\frac{1}{\mathrm{i}}\widetilde{\mathbf{S}}\left(\not{p}\right) = \frac{-\mathrm{i}}{\not{p} + m_{\overline{MS}} - \Sigma_{\overline{MS}}\left(\not{p}\right)} = \frac{-\mathrm{i}}{\not{p} + m_{OS} - \Sigma_{OS}\left(\not{p}\right)} \tag{4.61}$$

$$\not{p} + m_{OS} - \Sigma_{OS}\left(\not{p}\right)\Big|_{\not{p}=-m_{OS}} = 0 \;\Rightarrow\; \not{p} + m_{\overline{MS}} - \Sigma_{\overline{MS}}\left(\not{p}\right)\Big|_{\not{p}=-m_{OS}} = 0 \tag{4.62}$$

We first determine the $\mathcal{O}\left(\alpha^0\right)$ solution of Eq: 4.62 and plug that into the $\mathcal{O}\left(\alpha\right)$ equation to find the first order solution:

$$\begin{aligned}\mathcal{O}\left(\alpha^0\right): \quad -m_{OS} + m_{\overline{MS}} + \mathcal{O}\left(\alpha\right) = 0 \;\Rightarrow\; m_{OS} =& m_{\overline{MS}} + \mathcal{O}\left(\alpha\right) \\ =& m_{\overline{MS}} + \Delta m + \mathcal{O}\left(\alpha^2\right)\end{aligned} \tag{4.63}$$

Here we defined Δm as the $\mathcal{O}\left(\alpha\right)$ contribution to m_{OS}. Plugging the zeroth order solution back into the equation we obtain the first order equation:

$$\begin{aligned}\mathcal{O}\left(\alpha\right): \quad -\Delta m + m_{\overline{MS}}\left[B\left(-m^2_{\overline{MS}}\right) - A\left(-m^2_{\overline{MS}}\right)\right] + \mathcal{O}\left(\alpha^2\right) = 0 \\ \Rightarrow \Delta m = \frac{\alpha}{2\pi} m_{\overline{MS}}\left[2 - \frac{3}{2}\ln\frac{m^2_{\overline{MS}}}{\mu^2}\right] + \mathcal{O}\left(\alpha^2\right)\end{aligned} \tag{4.64}$$

We now write down the relation between the OS mass and the $\overline{MS}$ mass up to $\mathcal{O}\left(\alpha\right)$ using the one-loop self-energy result. This result is invertible by the same order by order expansion method.

$$\begin{aligned} m_{OS} =& m_{\overline{MS}}\left[1 + \frac{\alpha}{2\pi}\left(2 - \frac{3}{2}\ln\frac{m^2_{\overline{MS}}}{\mu^2}\right)\right] + \mathcal{O}\left(\alpha^2\right) \\ m_{\overline{MS}} =& m_{OS}\left[1 - \frac{\alpha}{2\pi}\left(2 - \frac{3}{2}\ln\frac{m^2_{OS}}{\mu^2}\right)\right] + \mathcal{O}\left(\alpha^2\right)\end{aligned} \tag{4.65}$$

The accuracy by which a measurement of the in QED physically observable m_{OS} allows us to determine $m_{\overline{MS}}\left(\mu\right)$ depends on whether at higher loop corrections the analogue of the series expansion in Eq: 4.65 converges or not.

The higher order corrections up to the three-loop order in QED are given in Eq: 66 of Ref. [25] and the $\mathcal{O}\left(\alpha\right)$ result is in agreement what we find in Eq: 4.65. In QCD the relation between the heavy quark mass defined in the $\overline{MS}$ and OS renormalisation schemes has been computed in perturbative QCD at the four-loop order by Ref. [26].

4.5 Renormalisation scale dependence in $\overline{MS}$

The bare lagrangian in Eq: 2.19 and the renormalised lagrangian in Eq: 2.20 describe the same theory in different parametrisations. Only the parameters of the bare lagrangian do not depend on the used renormalisation scheme whilst the renormalised lagrangian does depend on the used renormalisation scheme and scale. By use of the method and notation shown in Ref. [17, ch. 66] we explore the implications of this.

In QED we can write down the bare electron charge and the bare mass in terms of the renormalised electron charge and mass and the counter terms. The $\widetilde{\mu}$-derivative and also the μ-derivative of these bare parameters must vanish because the original theory is not $\widetilde{\mu}$-dependent or μ-dependent.

The bare electron charge is given by Eq: 4.66 denoted with a subscript zero, we take the logarithm of this equation and then the $\ln \mu$-derivative.

$$e_0 = \frac{Z_1}{\sqrt{Z_3} Z_2} \widetilde{\mu}^{\frac{\epsilon}{2}} e \tag{4.66}$$

$$\ln e_0 = E\left(e\right) + \ln\left(e\right) + \frac{\epsilon}{2} \ln \widetilde{\mu} \quad \text{with} \quad E\left(e, \epsilon\right) = \sum_{n=1}^{\infty} \frac{E_n\left(e\right)}{\epsilon^n} = \ln \frac{Z_1}{\sqrt{Z_3} Z_2} \tag{4.67}$$

We take the $\ln \mu$-derivative of the above expression:

$$\begin{aligned} 0 =& \frac{\partial E\left(e, \epsilon\right)}{\partial e} \frac{\partial e}{\partial \ln \mu} + \frac{1}{e} \frac{\partial e}{\partial \ln \mu} \frac{\epsilon}{2} \\ 0 =& \left(1 + \frac{e}{\epsilon} \frac{\partial E_1\left(e\right)}{\partial e} + \ldots\right) \frac{\partial e}{\partial \ln \mu} + \frac{\epsilon}{2} e \end{aligned} \tag{4.68}$$

We choose $\frac{\partial e}{\partial \ln \mu} = -\frac{\epsilon}{2} e + \beta_e\left(e\right)$ and fix the function $\beta_e\left(e\right)$ order by order in the renormalised electron charge using the expression for the counter terms as derived for $\overline{MS}$.

$$\ln \frac{Z_1}{\sqrt{Z_3} Z_2} = \frac{e^2}{12\pi^2} \frac{1}{\epsilon} + \mathcal{O}\left(e^4\right) \quad \Rightarrow \quad \beta_e\left(e\right) = \frac{e^2}{2} \frac{\partial E_1\left(e\right)}{\partial e} + \ldots = \frac{e^3}{12\pi^2} + \mathcal{O}\left(e^5\right) \tag{4.69}$$

With the explicit expression for the beta-function $\beta_e\left(e\right)$ in terms of the renormalised electron charge e we can solve a differential equation and relate the electron charge at different scales non-perturbatively. The solution of this equation shows that QED becomes more strongly coupled higher energies.*

*The pole in the solution of the beta-function is commonly referred to as the Landau

$$e(\mu_2) = e(\mu_1)\left(1 - \frac{e^2(\mu_1)}{6\pi^2}\ln\frac{\mu_2}{\mu_1}\right)^{-\frac{1}{2}} \tag{4.70}$$

Every loop order in the quantum corrections of a process introduces at least two additional interactions, therefore one usually considers the beta-function of the fine-structure constant $\alpha_e = \frac{e^2}{4\pi}$. With our knowledge of $\beta_e(e)$, we find:

$$\beta_{\alpha_e}(\alpha_e(\mu)) = \frac{\partial \alpha_e}{\partial \ln\mu} = \frac{\partial \alpha_e}{\partial e}\beta_e(e(\mu)) = \overbrace{\frac{2}{3\pi}}^{\beta_0}\alpha_e^2 + \mathcal{O}\left(\alpha_e^3\right) \tag{4.71}$$

The same procedure gives us non-perturbative information about the electron $\overline{MS}$ mass. The term $\gamma_m(e) = \frac{1}{m}\frac{\partial m}{\partial \ln\mu}$ is referred to as the anomalous dimension of the electron mass.

$$m_0 = \frac{Z_0}{Z_2}m \tag{4.72}$$

$$\ln m_0 = M(e,\epsilon) + \ln m \quad \text{with} \quad M(e,\epsilon) = \sum_{n=1}^{\infty}\frac{M_1(e)}{\epsilon^n} = \ln\frac{Z_0}{Z_2} \tag{4.73}$$

Now we take the $\ln\mu$-derivative and substitute in the beta-function we found previously:

$$\begin{aligned} 0 &= \frac{1}{m}\frac{\partial m}{\partial \ln\mu} + \frac{\partial M(e,\epsilon)}{\partial e}\frac{\partial e}{\partial \ln\mu} \\ 0 &= \gamma_m(e) + \left(\frac{\partial M_1(e)}{\partial e}\frac{1}{\epsilon} + ...\right)\left(-\frac{\epsilon}{2}e + \beta_e(e)\right) \end{aligned} \tag{4.74}$$

We now substitute what we now about the values of the counter terms and the beta-function to find the anomalous dimension of the electron mass*:

$$\ln\frac{Z_0}{Z_2} = -\frac{3e^2}{8\pi^2}\frac{1}{\epsilon} + \mathcal{O}\left(e^4\right) \quad \Rightarrow \quad \gamma_m(e) = -\frac{3e^2}{8\pi^2} + \mathcal{O}\left(e^4\right) \tag{4.75}$$

With the expressions for both $\beta(e)$ and $\gamma_m(e)$ we can solve a differential equation and relate the electron mass at different scales non-perturbatively. The solution shows that the effective mass of an electron become less at

pole. For $\mu_1 \approx 1$ eV the Landau pole lies at $\mu_2 = \mu_1 \exp\frac{6\pi^2}{e^2(\mu_1)} \approx 10^{286}$ eV. Neither QED or perturbation theory is valid at those energy scales.

*This result can also be obtained by taking the $\ln\mu$-derivative of m_{OS} as shown in Eq: 4.65.

higher energies.

$$m(\mu_2) = m(\mu_1)\left(\frac{e(\mu_1)}{e(\mu_2)}\right)^{\frac{9}{2}} = m(\mu_1)\left(1 - \frac{e^2(\mu_1)}{6\pi^2}\ln\frac{\mu_2}{\mu_1}\right)^{\frac{9}{4}} \tag{4.76}$$

The Callan-Symanzik equation

Not only bare parameters are μ-independent, also the correlation functions of bare fields need to be μ-independent.

$$\begin{aligned} G^{n,m}(e,m,\widetilde{\mu}) &= \langle 0|\overbrace{A^{\mu_1}(x_1)\ldots A^{\mu_n}(x_n)}^{n}\overbrace{\overline{\Psi}(x_{n+1})\ldots\Psi(x_{n+m})}^{m}|0\rangle \\ &= (Z_3)^{-\frac{n}{2}}(Z_2)^{-\frac{m}{2}}\overbrace{\langle 0|A_0^{\mu_1}(x_1)\ldots A_0^{\mu_n}(x_n)\overline{\Psi}_0(x_{n+1})\ldots\Psi_0(x_{n+m})|0\rangle}^{\mu\text{-independent}} \end{aligned} \tag{4.77}$$

We now define the anomalous dimensions for the photon field $\gamma_A(e)$ and the electron field as $\gamma_\Psi(e)$ and evaluate them as we did for the electron mass in the previous section:

$$\begin{aligned} \gamma_A(e) =& \frac{1}{2}\frac{\partial\ln Z_3}{\partial\ln\mu} = \frac{e^2}{12\pi^2} + \mathcal{O}\left(e^4\right) \\ \gamma_\Psi(e) =& \frac{1}{2}\frac{\partial\ln Z_2}{\partial\ln\mu} = \frac{e^2\lambda^{-2}}{16\pi^2} + \mathcal{O}\left(e^4\right) \end{aligned} \tag{4.78}$$

These definitions allow us to construct a differential equation that describes the μ-dependence of correlation function's in QED non-perturbatively:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta_e(e)\frac{\partial}{\partial e} + m\gamma_m(e)\frac{\partial}{\partial m} + n\gamma_A(e) + m\gamma_\Psi(e)\right)G^{n,m}(e,m,\mu) = 0 \tag{4.79}$$

In some cases, the above differential equation provides a solution at values for the external momenta where perturbation theory breaks down. This could happen when there are large logarithms in the perturbative expansion that do not converge. It is an example of non-perturbative physics that is found through renormalisation of a perturbative expansion.

Chapter 5

The Consequences of Noether-Current Conservation

In this chapter, we provide an explanation for the seemingly coincidental equality of $Z_{1,\overline{MS}} = Z_{2,\overline{MS}}$ and $Z_{1,OS} = Z_{2,OS}$ that we found when we renormalised QED up to one-loop in Ch: 4 based on the treatise in Refs. [17, ch. 22, 67 and 68]. The equality of the counter terms in QED is brought about by the *Ward-Takahashi* identities, which also constrain the space-time structure of amplitudes. However, we demonstrate that at a more fundamental level these are actually both consequences of the quantum equivalent of the conservation of Noether currents in classical mechanics.

5.1 The conservation of Noether current in QFT

We consider a theory of scalar fields ϕ_a instead of QED aiming for conceptual understanding instead of mathematical rigour. In a classical field theory the variational principle by assumption dictates that the field configuration that is realised in nature is characterised as an extremum of the action. To satisfy this, the fields must obey the equations of motion.

$$S = \int d^d x \mathcal{L} \left(\phi_a, \partial_\mu \phi_a\right) \tag{5.1}$$

$$\frac{\delta S}{\delta \phi_a} = \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi_a\right)} \right) = 0 \tag{5.2}$$

We perform an infinitesimal transformation of the fields that corresponds to a continuous symmetry of the action, in other words a transformation $\phi_a \to \phi_a + \delta\phi_a$ such that $\delta S = 0$ with a vanishing $\delta\phi_a$ at the boundaries of the integration range. If $\delta S = 0$ then the lagrangian density can at most transform by a total derivative $\delta\mathcal{L} = \partial_\mu K^\mu$. This observation allows us to define a current j^μ that is conserved when the equations of motion are satisfied, the Noether current:

$$\delta\mathcal{L} = \frac{\delta S}{\delta\phi_a}\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_a\right)}\delta\phi_a\right) = \partial_\mu K^\mu \tag{5.3}$$

$$\partial_\mu j^\mu = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_a\right)}\delta\phi_a - K^\mu\right) = -\frac{\delta S}{\delta\phi_a}\delta\phi_a \tag{5.4}$$

In QFT the fundamental object is not the action but the path integral $Z(J)$, which does not choose one specific field configuration but includes all of them with their own weight. Unless the path integral measure is affected by the transformation of the fields, any transformation of the fields ϕ_a leaves $\delta Z(J) = 0$ because we are integrating over all of them.

$$Z(J) = \int \mathcal{D}\phi e^{\mathrm{i}\left[S+\int d^d y\, J_a\phi_a\right]} \tag{5.5}$$

$$\delta Z(J) = \mathrm{i}\int \mathcal{D}\phi e^{\mathrm{i}\left[S+\int d^d y\, J_a\phi_a\right]}\int d^d x\left(\frac{\delta S}{\delta\phi_a} + J_a\right)\delta\phi_a \tag{5.6}$$

The Schwinger-Dyson equations follow when we take n functional derivatives with respect to the source J_a and afterwards set $J_a = 0$, $\delta\phi_a$ is factored out as an arbitrary change in the fields.

$$\begin{aligned} 0 = \int \mathcal{D}\phi e^{\mathrm{i}S}\int d^d x\Big(&\mathrm{i}\frac{\delta S}{\delta\phi_a}\phi_{a_1}(x_1)\ldots\phi_{a_n}(x_n) \\ &+\sum_{j=1}^{n}\phi_{a_1}(x_1)\ldots\delta_{aa_j}\delta^d\left(x-x_j\right)\ldots\phi_{a_n}(x_n)\Big)\delta\phi_a(x) \end{aligned} \tag{5.7}$$

$$\begin{aligned} 0 = \mathrm{i}\langle 0|\,T\frac{\delta S}{\delta\phi_a}\phi_{a_1}(x_1)\ldots\phi_{a_n}(x_n)\,|0\rangle +& \\ \sum_{j=1}^{n}\langle 0|\,T\phi_{a_1}(x_1)\ldots\delta_{aa_j}\delta^d\left(x-x_j\right)\ldots\phi_{a_n}(x_n)\,|0\rangle& \end{aligned} \tag{5.8}$$

The terms in the summation of Eq: 5.8 are called *contact terms*, they only become important when $x_i = x_j$ and do not contribute to scattering amplitudes in the LSZ formula so the classical equation of motions inside correlation functions are conserved up to contact terms. The *Ward-Takahashi identity* follows from Eq: 5.4 if the field transformation corresponds to a continuous symmetry:

$$\mathrm{i}\partial_{\mu}\left\langle 0\right| Tj^{\mu}\left(x\right)\phi_{a_1}\left(x_1\right)...\phi_{a_n}\left(x_n\right)\left|0\right\rangle = \sum_{j=1}^{n}\left\langle 0\right| T\phi_{a_1}\left(x_1\right)...\delta\phi_{a_j}\delta^{d}\left(x-x_j\right)...\phi_{a_n}\left(x_n\right)\left|0\right\rangle \tag{5.9}$$

The Noether currents inside correlation functions are thus conserved up to contact terms under the assumption that the path integral measure is invariant under the field transformation. This finding is also valid for theories involving fermions or vector bosons though for fermions the anti-commutation relations insert some minus signs in the contact terms.The above treatment is presented in more detail with examples in Ref. [17, ch. 22] and Ref. [18, ch. 9].

5.2 The constraint of WT-identities on QED amplitudes

We must first determine how to get from the mathematically abstract n-point correlation function to physical scattering amplitudes to apply the Ward-Takahashi identity in QED scattering amplitudes. Exactly this is what is done by the *Lehmann- Symanzik-Zimmermann* formula or LSZ-formula, shown in Eq: 5.10 for scalar particles. To make Eq: 5.10 applicable also for fermions and vector-bosons one needs to include spin-polarisation vectors for all initial and final states.

$$\begin{aligned}\left\langle f\mid i\right\rangle =&\mathrm{i}^{n+n'}\left(Z\right)^{-\frac{n+n'}{2}}\int d^{d}x_1 e^{\mathrm{i}k_1x_1}\left(-\partial_1^2+m^2\right)...d^{d}x_1' e^{\mathrm{i}k_1'x_1'}\left(-\partial_{1'}^2+m^2\right)\\ &\times\left\langle 0\right| T\phi\left(x_1\right)...\phi\left(x_1'\right)...\left|0\right\rangle\end{aligned} \tag{5.10}$$

The LSZ-formula is the consequence of integrating the n-point correlation functions over wave packets that describe asymptotic initial and final states with definite momenta at time $T=\pm\infty$. [17, ch. 41] [18, sec. 7.2] The field-strength normalisation factors that appear in the LSZ-formula follow from matrix elements that describe the probability amplitude to create a properly normalised one-particle state from the vacuum; $\sqrt{Z} = \left\langle 0\right|\phi\left(0\right)\left|\lambda_0\right\rangle$ where λ_0 denotes a zero-momentum one-particle state. The field equations in combination with $\sqrt{Z}$ do not completely cancel the exact propagators of the external lines they leave a factor of $\sqrt{Z}$.

We consider a transformation of the fields Ψ and $\overline{\Psi}$ with a space-time independent constant ξ of the form $\Psi\left(x\right)\rightarrow\Psi'\left(x\right) = e^{-\mathrm{i}e\xi}\Psi\left(x\right)$, a continuous symmetry of the QED lagrangian by construction. The conserved

Noether current that is associated to this symmetry is given by:

$$\partial_\mu j^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Psi\right)} \delta \Psi \right) = e\xi \partial_\mu \left(\overline{\Psi} \gamma^\mu \Psi \right) = 0 \tag{5.11}$$

Since this is true for any ξ, we drop ξ and find:

$$J^\mu = e\overline{\Psi}\gamma^\mu \Psi \ \Rightarrow \ \partial_\mu J^\mu = 0 \tag{5.12}$$

Now we write down the Lagrangian for the photon sector of the theory:

$$\begin{aligned} \mathcal{L}_{QED,A} =& -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} F^2 + J^\mu A_\mu \\ =& -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu - \frac{1}{2}\lambda^2 \partial_\mu A^\mu \partial_\nu A^\nu + J_\mu A^\mu \end{aligned} \tag{5.13}$$

We determine the equations of motion by the variational principle and note that the factor in square brackets is the inverse photon propagator *:

$$\left[\Box \eta_{\mu\nu} - \left(1 - \lambda^2\right) \partial_\mu \partial_\nu \right] A^\nu + J_\mu = 0 \tag{5.14}$$

We now use Eq: 5.8 to apply the equations of motion for the photon field to the scattering amplitude of a process involving an incoming photon:

$$\begin{aligned} \langle f \mid i \rangle =& \mathrm{i}^{n+n'} \left(Z\right)^{-\frac{n+n'}{2}} \int d^d x_1 \, \epsilon_1^\mu e^{-\mathrm{i}k_1 x_1} \left[\Box \eta_{\mu\nu} - \left(1-\lambda^2\right) \partial_\mu \partial_\nu \right] ... \langle 0 | \, T A^\nu \left(x_1\right) ... | 0 \rangle \\ =& - \mathrm{i}^{n+n'} \left(Z\right)^{-\frac{n+n'}{2}} \int d^d x_1 \, \epsilon_1^\mu e^{-\mathrm{i}k_1 x_1} ... \left(\langle 0 | \, T J_\mu \left(x_1\right) ... | 0 \rangle + \text{contact terms} \right) \end{aligned} \tag{5.15}$$

The contact terms do not contribute to the left hand side because the contact terms do not have the correct singularities and are killed by the wave operators. We now show that the scattering amplitude of processes involving longitudinally polarised photons with polarisation vectors $\propto k^\mu$ vanish by replacing $\epsilon^\mu \rightarrow k^\mu$ and doing a partial integration where surface terms integrate to zero:

*If we contract the equations of motion with ∂_μ we find that the photon field classically has no longitudinal polarisation; $\lambda^2 \Box \partial_\nu A^\nu + \cancelto{0}{\partial_\mu J^\mu} = 0 \ \Rightarrow \ \partial_\nu A^\nu = 0$

$$\begin{aligned}\langle f \mid i\rangle =& -\mathrm{i}^{n+n'}(Z)^{-\frac{n+n'}{2}}\int d^d x_1 k_1^\mu e^{-\mathrm{i}k_1x_1}...\langle 0|\,TJ_\mu(x)...|0\rangle \\ =&\mathrm{i}^{n+n'-1}(Z)^{-\frac{n+n'}{2}}\int d^d x_1 \partial_1^\mu\left(e^{-\mathrm{i}k_1x_1}\right)...\langle 0|\,TJ_\mu(x)...|0\rangle \\ =& -\mathrm{i}^{n+n'-1}(Z)^{-\frac{n+n'}{2}}\int d^d x_1 e^{-\mathrm{i}k_1x_1}...\partial_1^\mu\langle 0|\,TJ_\mu(x)...|0\rangle + \text{surface terms} \\ =& -\mathrm{i}^{n+n'-1}(Z)^{-\frac{n+n'}{2}}\int d^d x_1 e^{-\mathrm{i}k_1x_1}...\text{contact terms} + \text{surface terms} = 0\end{aligned} \tag{5.16}$$

We use the general Ward-Takahashi identity as shown in Eq: 5.9 to go from the third line to the fourth line.

5.3 The equality of counterterms in QED

One could ask now, what happens when we introduce counter terms to the QED Lagrangian. The Noether current associated to the gauge symmetry is then modified as: $J^\mu \to J'^\mu = Z_2 J^\mu$, whilst the equations of motion for the photon field are modified as:

$$\left[\Box\eta_{\mu\nu} - \left(1-\lambda^2\right)\partial_\mu\partial_\nu\right]A^\nu + \frac{Z_1}{Z_3}J_\mu = 0 \tag{5.17}$$

More importantly though, we find that the Ward-Takahashi identities and eventually gauge invariance imply that $Z_1 = Z_2$ to all orders in perturbation theory in the $\overline{MS}$- and OS-renormalisation schemes. From this it becomes apparent that the kinetic term $\mathrm{i}Z_2\overline{\Psi}\partial\!\!\!/\Psi$ and the interaction term $Z_1 e\overline{\Psi}A\!\!\!/\Psi$ can be renormalised as one term: $\mathrm{i}Z_1\overline{\Psi}D\!\!\!\!/\Psi$, with the covariant instead of regular derivative.

To show this more rigorously we consider the Fourier transform of the correlation function between two fermion fields connected by a vertex: $Z_1 J^\mu(x)$ for an external gauge field. By contracting the fermion fields inside the current J^μ with the explicit fermion fields in the correlation function this can be rewritten in terms of the exact propagators and the exact vertex function:

$$\begin{aligned}C^\mu_{\alpha\beta} =& \mathrm{i}Z_1\int d^4x d^4y d^4z e^{\mathrm{i}kx-\mathrm{i}p'y+\mathrm{i}pz}\langle 0|\,TJ^\mu(x)\Psi_\alpha(y)\overline{\Psi}_\beta(z)\,|0\rangle \\ =& (2\pi)^4\delta^4\left(k+p-p'\right)\left[\frac{1}{\mathrm{i}}\widetilde{\mathbf{S}}\left(p'\right)\mathrm{i}\mathbb{V}^\mu\left(p',p\right)\frac{1}{\mathrm{i}}\widetilde{\mathbf{S}}\left(p\right)\right]_{\alpha\beta}\end{aligned} \tag{5.18}$$

Now we rewrite $k_\mu C^\mu_{\alpha\beta}$ in terms of the Noether current J'^μ associated to the infinitesimal field transformations $\delta\Psi(x) = -\mathrm{i}e\Psi(x)$ and $\delta\overline{\Psi}(x) =$

$\mathrm{i}e\overline{\Psi}(x)$. We then discard a surface term and rewrite the derivative of the Noether current in terms of the contact terms shown in Eq: 5.9:

$$\begin{aligned} k_\mu C^\mu_{\alpha\beta} &= -Z_1 \int d^4x d^4y d^4z e^{\mathrm{i}kx - \mathrm{i}p'y + \mathrm{i}pz} \partial_\mu \langle 0 | T J^\mu(x) \Psi_\alpha(y) \overline{\Psi}_\beta(z) | 0 \rangle \\ &= -\mathrm{i}\frac{Z_1}{Z_2} e (2\pi)^4 \delta^4(k + p - p') \left[\widetilde{\mathbf{S}}(p) - \widetilde{\mathbf{S}}(p') \right]_{\alpha\beta} \end{aligned} \tag{5.19}$$

We now have two expressions for $k_\mu C^\mu_{\alpha\beta}$ which must be equal, these expressions can be evaluated in different renormalisation schemes.

$$\begin{aligned} (p - p')_\mu \widetilde{\mathbf{S}}(p') \mathbb{V}^\mu(p', p) \widetilde{\mathbf{S}}(p) &= \frac{Z_1}{Z_2} e \left[\widetilde{\mathbf{S}}(p) - \widetilde{\mathbf{S}}(p') \right] \\ \Rightarrow \ (p - p')_\mu \mathbb{V}^\mu(p', p) &= \frac{Z_1}{Z_2} e \left[\widetilde{\mathbf{S}}^{-1}(p') - \widetilde{\mathbf{S}}^{-1}(p) \right] \end{aligned} \tag{5.20}$$

In the $\overline{MS}$-scheme we know that the vertex function and the exact propagator are finite whilst any contribution to the counter terms is infinite therefore $Z_{1,\overline{MS}} = Z_{2,\overline{MS}}$. In the OS-scheme we know that for $p^2 = p'^2 = -m^2$ we have $\mathrm{i}\mathbb{V}^\mu(p, p') = \mathrm{i}e\gamma^\mu$ and $\widetilde{\mathbf{S}}^{-1}(p) = \not{p} + m$ therefore again $Z_{1,OS} = Z_{2,OS}$. *

We demonstrated that the equality of counter terms in QED is a consequence of the conservation of Noether current. This result shows that fundamental physical principles can leave hints in small subtleties of the theory.

*A more diagrammatic derivation of the equality of counter terms from the Ward-Takahashi identities is given in Ref. [18, sec. 7.4].

Chapter 6

The Standard Model Interactions of the Top Quark

> In this chapter, we work out the Feynman rules for parts of the standard model relevant for top quark and gluon one-loop calculations in the renormalisable gauge. The standard model is the gauge theory with an underlying gauge symmetry $SU(3) \times SU(2) \times U(1)$, which at the moment best describes our world.
> We first focus on the $SU(3)$ part of the SM before we continue to electroweak part with the underlying $SU(2) \times U(1)$ symmetry. We demonstrate the gauge-fixing procedure, the BEH-mechanism and derive the vertex factors and propagators in the quark sector.

More details about QCD in the context of nonabelian gauge theories is found in Ref. [17, ch. 69,72]. A more thorough explanation of the Higgs mechanism and spontaneous symmetry breaking in the SM and a few other examples is found in Ref. [17, ch. 86-87, 89], Ref. [18, ch. 20] or more brief and accessible in Ref. [19, ch. 46-47].

In our short description of the SM, we used Ref. [17] notation and conventions as much as possible. The Feynman rules of the standard model in the mostly negative metric and using different conventions is found in Ref. [27]. The great multitude of possible conventions that is explained in Ref. [27], motivates why we want to make no mistake and derive them from the lagrangian terms ourselves.

6.1 The $SU(3)$ part of the standard model

The $SU(3)$ part of the standard model is commonly referred to as quantum chromodynamics, the theory of quarks and gluons. The interaction associated to the $SU(3)$ gauge symmetry is the strong-interaction, which acts on particles that transform under $SU(3)$ gauge transformations e.g. particles with so-called colour charge.

The quark lagrangian

The quark lagrangian describes colour charged spin-½ fermions and anti-fermions named quarks respectively anti- quarks. The formalism that is used to handle the fermion nature of these particles was treated in Sec: 2.1. There are three generations of quarks that each contain one quark with an electric charge of $+\frac{2}{3}|e|$ and a quark with an electric charge of $-\frac{1}{3}|e|$, together this makes that there are six different quark *flavours*. The quarks with a positive electric charge are called u, c and t (up, charm and top) and the negatively charged quarks are d, s and b (down, strange and bottom).

These quarks are described by Dirac fields $\Psi_{i\ I}(x)$ that transform under the fundamental representation of $SU(3)$ so: $\Psi_{i\ I}(x) \rightarrow \Psi'_{i\ I}(x) = U_{ij}(x)\Psi_{j\ I}(x)$ where i, j are the colour indices summed from one to three and I is a flavour index which is summed from one to six. Any $SU(3)$ transformation $U_{ij}(x)$ can be formed by eight traceless hermitian generator matrices T^a_{ij} that together span the group, which allows us to write: $U_{ij}(x) = e^{-\mathrm{i}g_3\theta^a(x)T^a}$ where g_3 is the $SU(3)$ coupling constant.

These generator matrices T^a_{ij} obey commutation relations of the form shown in Eq: 6.1, where f^{abc} are totally antisymmetric structure constants. The generators can be conveniently normalised according to Eq: 6.2 where $T(R)$ is a representation dependent number called the index of the group, we choose it to be one half.

$$\left[T^a, T^b\right] = \mathrm{i}f^{abc}T^c \tag{6.1}$$

$$Tr\left[T^a T^b\right] = T(R)\,\delta^{ab} = \frac{1}{2}\delta^{ab} \tag{6.2}$$

We need to introduce a gauge-field that transforms as in Eq: 6.3 in the adjoint representation of the group to make the Dirac lagrangian shown in Eq: 2.2 invariant under $SU(3)$ transformations of the fields $\Psi_{i\ I}(x)$. The gauge-field of QCD is known as the gluon field and it is a massless spin-1 boson field, that carries colour charge but no electric charge.

$$A_\mu(x) \to A'_\mu(x) = U(x)\,A_\mu(x)\,U^\dagger(x) + \frac{\mathrm{i}}{g_3}U(x)\,\partial_\mu U^\dagger(x) \tag{6.3}$$

From the gauge-field we construct a covariant derivative $D_{ij\ \mu}$:

$$\left(D_\mu\right)_{ij} = \delta_{ij}\partial_\mu - \mathrm{i}g_3 A^a_\mu T^a_{ij} \tag{6.4}$$

We now write down the gauge-invariant quark lagrangian from which we can deduce the quark propagator and the quark-gluon vertex factor:

$$\begin{aligned}\mathcal{L}_q &= -\,\overline{\Psi}_{i\ I}\left(-\mathrm{i}\not{D}_{ij} + m_I\delta_{ij}\right)\Psi_{j\ I}\\ &= -\,\overline{\Psi}_{i\ I}\left(-\mathrm{i}\not{\partial} + m_I\right)\Psi_{i\ I} + g_3\overline{\Psi}_{i\ I}\not{A}^a T^a_{ij}\Psi_{j\ I}\end{aligned} \tag{6.5}$$

By following the treatment in Sec: 2.1 we can write down the quark propagator as:

$$\frac{1}{\mathrm{i}}\widetilde{S}_{ij\ I}(\not{p}) \equiv \frac{-\mathrm{i}\delta_{ij}}{\not{p} + m_I - \mathrm{i}\epsilon} \tag{6.6}$$

From the quark lagrangian in Eq: 6.5 we can deduce the vertex factor that describes the coupling of quarks to gluons in the Feynman rules. Except for the colour and flavour factors the Feynman rules for electrons in Sec: 2.1 also apply to quarks.

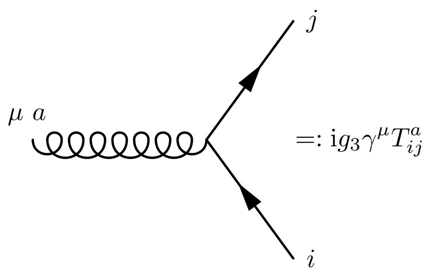


Figure 6.1: *The quark-gluon vertex that follows from the quark lagrangian shown in Eq: 6.5.*

The gluon lagrangian

The covariant derivative shown in Eq: 6.4 allows us to construct the field-strength tensor $F^{\mu\nu}$, which transforms covariantly in the adjoint representation of the group in other words: $F_{\mu\nu}(x) \to F'_{\mu\nu}(x) = U(x)\,F_{\mu\nu}(x)\,U^\dagger(x)$. In terms of the gauge-field the field-strength tensor is given by:

$$F_{\mu\nu} = \frac{\mathrm{i}}{g_3}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g_3\left[A_\mu, A_\nu\right] \tag{6.7}$$

It is apparent that the simplest gauge-invariant term involving only the gauge-field A_μ is given by the trace of two contracted field-strength tensors as shown in Eq: 6.8. This is the classical lagrangian for a nonabelian gauge theory. We see that this theory is complicated by the fact that the gauge field now couples to itself as well.

$$\begin{aligned}\mathcal{L}_{cl.g} =& -\frac{1}{4T(R)}Tr\left[F_{\mu\nu}F^{\mu\nu}\right] = -\frac{1}{2}Tr\left[F_{\mu\nu}F^{\mu\nu}\right] \\ =& -\frac{1}{2}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - g_3 f^{abc}A^{a\mu}A^{b\nu}\partial_\mu A^c_\nu - \frac{1}{4}g_3^2 f^{abe}f^{cde}A^{a\mu}A^{b\nu}A^c_\mu A^d_\nu\end{aligned} \tag{6.8}$$

Just as in the QED case, we cannot construct a propagator from the lagrangian shown in Eq: 6.8. The lagrangian in Eq: 6.8 is manifestly Lorentz invariant, which is an advantage but has a downside. The downside is that our description has a redundancy associated to gauge transformations. In the path-integral description we are still integrating over an infinite number of field-configurations that are connected via gauge transformations parametrised by $\theta^a(x)$ and are thus physically equal.

Gauge-fixing and ghosts

To correctly fix the gauge, we use Faddeev-Popov gauge-fixing following closely [17, ch. 71]. We first add a source term to the lagrangian in Eq: 6.8 and write down the path integral:

$$Z(J) \propto \int \mathcal{D}A\, e^{\mathrm{i}S_{cl.g}(A,J)} \text{ with } S_{cl.g}(A,J) = \int d^4x\left[-\frac{1}{2}Tr\left[F_{\mu\nu}F^{\mu\nu}\right] + J^{a\mu}A^a_\mu\right] \tag{6.9}$$

To restrict the integration of the path integral in Eq: 6.9 only to physical distinct field configurations at every space-time point x, we must introduce delta functions that only picks out one gauge slice parametrised by one specific $\theta^a_0(x)$ *. Assuming the $\mathcal{D}A$ is an integration over physical distinct configurations and physical non-distinct configurations are parametrised by $\theta^a(x)$:

$$Z(J) \propto \int \mathcal{D}A \prod_{x,a}\delta\left(\theta^a(x) - \theta^a_0(x)\right) e^{\mathrm{i}S_{cl.g}(A,J)} \tag{6.10}$$

*If one considers the original lagrangian as the fundamental object of the theory, then we should also include an integration over physical non-distinct states. This introduces a pre-factor that is formally infinite but does not contribute to correlation functions.

To evaluate this integral we must define a $\theta_0^a(x)$ and find what field configurations are physically distinct. We pick a gauge by demanding it to be the solution of a gauge-fixing function $G^a(x) = 0$ where we use $G^a(x) = \partial_\mu A^{a\mu} - \omega^a$ in terms of a still arbitrary field ω^a. Expanding the gauge-fixing function around the solution for $\theta_0^a(x)$ allows a coordinate shift involving a determinant, which results in:

$$Z(J) \propto \int \mathcal{D}A \ \det\left(\frac{\delta G^a(x)}{\delta\theta^b(y)}\right) \prod_{x,a} \delta\left(G^a(x)\right) e^{\mathrm{i}S_{cl.g}(A,J)} \tag{6.11}$$

The original path integral does not depend on the field $\omega^a(x)$, so the inclusion of the term $\int \mathcal{D}\omega \ e^{-\frac{\mathrm{i}}{2\xi}\int d^4x\omega^a\omega^a}$ only changes the overall (irrelevant) normalisation of $Z(J)$. The delta-functions allow us to evaluate the ω integral and give us the gauge-fixing term:

$$Z(J) \propto \int \mathcal{D}A \ \det\left(\frac{\delta G^a}{\delta\theta^b}\right) e^{\mathrm{i}S_{cl.g}(A,J)+\mathrm{i}S_{g.f.}(A)} \text{ with } S_{g.f.}(A) = -\frac{1}{2\xi}\int d^4x \left(\partial_\mu A^{a\mu}\right)^2 \tag{6.12}$$

The gauge-fixing term allows us to construct the gluon propagator but we still need to evaluate the determinant. Under an infinitesimal gauge transformation the gluon field transforms as: $A^{a\mu}(x) \to A'^{a\mu}(x) = A^{a\mu}(x) - D_\mu^{ac}\theta^c(x)$ where the covariant derivative of the adjoint representation is given by: $D_\mu^{ac} = \delta^{ac}\partial_\mu + g_3 f^{abc}A_\mu^b$. With this information we evaluate the functional derivative of the gauge fixing function as: $\frac{\delta G^a(x)}{\delta\theta^b(y)} = -\partial^\mu D_\mu^{ab}\delta^4(x-y)$. In [18, ch. 9] it is shown that this functional determinant is equal to the path-integral of Grassman-valued fields:

$$\begin{aligned} &\det\left(\frac{\delta G^a(x)}{\delta\theta^b(y)}\right) \propto \int \mathcal{D}c\mathcal{D}\bar{c} \ e^{\mathrm{i}S_{gh}(A)} \\ &\quad \text{with } S_{gh}(A) = \int d^4x \ \bar{c}^a\partial^\mu D_\mu^{ab}c^b = -\int d^4x \left[\partial^\mu\bar{c}^a\partial_\mu c^a - g_3 f^{abc}\partial^\mu\bar{c}^a c^b A_\mu^c\right] \end{aligned} \tag{6.13}$$

The full gauge-fixing procedure has introduced unphysical polarisations of the gluons and unphysical ghost states that are both dependent on an arbitrary gauge-fixing function $G^a(x)$. Since the theory is gauge invariant, these unphysical states do not appear in final or initial states. BRST-symmetry arguments as elaborated on in [17, ch. 74] lie at the basis of these assertions.

The Feynman rules for gluon interactions

Now, we add the gauge-fixing and ghost parts to the classical gluon lagrangian in Eq: 6.8. This gives us the propagators for the gluons and the ghost fields, the gluon-ghost vertex and the gluon self-interaction vertices:

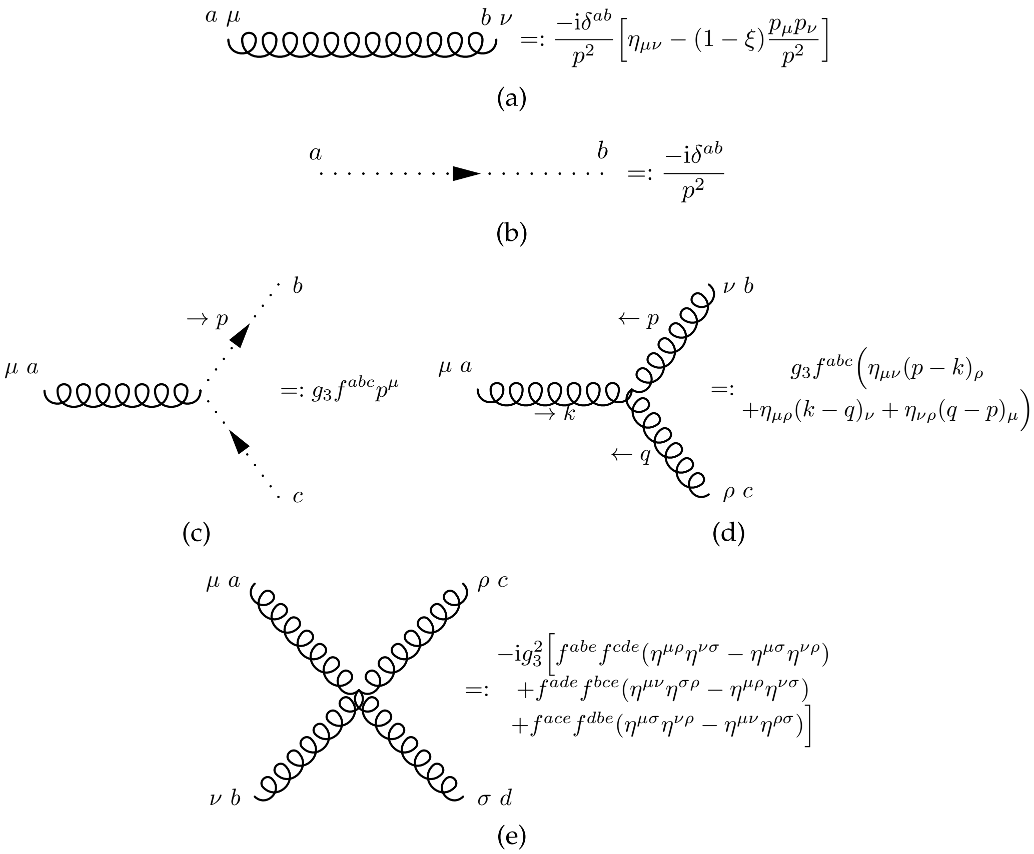


Figure 6.2: *The Feynman rules in QCD gauge sector: (a) the gauge-fixed gluon propagator, (b) the ghost propagator, which shows the direction of the flow of ghost number, (c) the gluon-ghost-ghost vertex, (d) the triple gluon vertex and (e) the quartic gluon vertex.*

6.2 The $SU(2) \times U(1)$ gauge-sector

The $SU(2) \times U(1)$ part of the standard model is commonly referred to as the electroweak part of the standard model or GWS-theory named after Glashow-Weinberg and Salam. The phenomenology of this theory is to a great extend determined by a complex scalar $SU(2)$-doublet ϕ referred to as the *Higgs*-field. The non-zero vacuum expectation value of the Higgs-field spontaneously breaks $SU(2) \times U(1)$ to $U(1)$. This remaining symmetry constitutes QED.

In this section, we describe how the BEH-mechanism spontaneously breaks the symmetry that underlies GWS-theory and how it affects the quadratic part of the Higgs-field and the gauge-fields. We do not explicitly work out all possible interactions among the Higgs-field, gauge-fields, Goldstone-fields and ghost fields.

The Higgs-field and the BEH-mechanism

The *Higgs*-field is a two-component complex scaler doublet that transforms under $SU(2) \times U(1)$ as $\phi_i(x) \to \phi'_i(x) = U_{ij}(x)\,\phi_j$ where $U_{ij}(x)$ is given by:

$$U_{ij}(x) = \overbrace{\exp\left(-\mathrm{i}g_2\theta^a(x)\,T^a\right)}^{SU(2)}\overbrace{\exp\left(-\mathrm{i}g_1\xi(x)\,Y\right)}^{U(1)} \tag{6.14}$$

We choose $T^a = \frac{1}{2}\sigma^a$ as the generators of the $SU(2)$ symmetry such that Eq: 6.1 and Eq: 6.2 are obeyed. The factor $g_1 Y$ in the $U(1)$ part is reminiscent of the charge in QED (see Sec: 2.1) and therefore Y is known as the *hypercharge* of the field. For the Higgs-field $Y = -\frac{1}{2}\mathbb{1}_2$.

We also introduce a set of gauge-fields $A_\mu(x)$ * that transforms in the $SU(2)$ equivalent of Eq: 6.3 and a gauge-field $B_\mu(x)$ that transforms in the same way as the photon field in Eq: 2.9. These gauge fields allow us to construct a covariant derivative for the Higgs-field, given by:

$$\begin{aligned}(D_\mu)_{ij} =& \delta_{ij}\partial_\mu - \mathrm{i}\left[g_2 A^a_\mu T^a_{ij} + g_1 B_\mu Y_{ij}\right] \\ =& \begin{pmatrix} \partial_\mu - \frac{\mathrm{i}}{2}\left(g_2 A^3_\mu - g_1 B_\mu\right) & -\frac{\mathrm{i}g_2}{2}\left(A^1_\mu - \mathrm{i}A^2_\mu\right) \\ -\frac{\mathrm{i}g_2}{2}\left(A^1_\mu + \mathrm{i}A^2_\mu\right) & \partial_\mu + \frac{\mathrm{i}}{2}\left(g_2 A^3_\mu + g_1 B_\mu\right) \end{pmatrix}\end{aligned} \tag{6.15}$$

A general gauge-invariant lagrangian for the Higgs-field that can be constructed with the covariant derivative and a potential $V\left(\phi^\dagger\phi\right)$ is given by:

$$\mathcal{L}_\phi = -\left(D_\mu\phi\right)^\dagger\left(D^\mu\phi\right) - V\left(\phi^\dagger\phi\right) \tag{6.16}$$

*The field $A_\mu(x)$ is not to be confused with the gluon field from the previous section, gauge-fields are often called $A_\mu(x)$ regardless of the symmetry group.

The general form of the potential which is renormalisable, has a nonzero vacuum expectation value and is bounded from below is given by:

$$V\left(\phi^{\dagger}\phi\right) = -\mu^2\phi^{\dagger}\phi + \lambda\left(\phi^{\dagger}\phi\right)^2 \tag{6.17}$$

The field configuration of the vacuum of this theory should minimise the potential energy, the minus sign in front of the quadratic term in the potential shown in Eq: 6.17 creates a local at $|\phi| = 0$ so this is not the correct vacuum to use in perturbation theory. The true vacuum is found at the other zero of the derivative of the potential with respect to $|\phi|$:

$$\begin{gathered} \frac{dV}{d|\phi|} = 0 \quad \Rightarrow \quad -2\mu^2|\phi| + 4\lambda|\phi|^3 = 0 \\ |\phi_0| = \frac{\pm}{\sqrt{2}}v \quad \text{with} \quad v = \sqrt{\frac{\mu^2}{\lambda}} \end{gathered} \tag{6.18}$$

We now decompose the complex scalar Higgs-field ϕ in terms of two scalar fields $H(x)$ and $\phi_0(x)$ and one complex field $\phi_-(x)$ with $(\phi_-(x))^{\dagger} = \phi_+(x)$:

$$\phi(x) = \begin{pmatrix} \frac{1}{\sqrt{2}}\left(v + H(x) + \mathrm{i}\phi_0(x)\right) \\ \phi_-(x) \end{pmatrix} \tag{6.19}$$

Upon substitution of Eq: 6.19 in the potential shown in Eq: 6.17 we find that up to terms quadratic in the fields the potential is given by:

$$V\left(\phi^{\dagger}\phi\right) = \frac{-m_H^4}{16\lambda} + \frac{1}{2}m_H^2 H^2 + ... \quad \text{with} \quad m_H^2 = 2\lambda v^2 = 2\mu^2 \tag{6.20}$$

The constant term in the potential does not affect the physics of the theory so it can be discarded. More importantly though the field $H(x)$,the scalar Higgs field, has a mass m_H whilst the other fields ϕ_0 and $\phi_{\pm}$, the Goldstone bosons, are massless.

The electroweak gauge-fields

For the electroweak gauge fields $A_\mu(x)$ and $B_\mu(x)$ we introduce the kinetic terms analogue to those shown previously in Eq: 6.8 and in Eq: 2.13, which gives us:

$$\mathcal{L} = -\frac{1}{2}Tr\left[F_{\mu\nu}F^{\mu\nu}\right] - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{6.21}$$

Here we introduced the definition $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$. The vacuum expectation value of the of the complex scalar Higgs-field ϕ in the covariant derivative shown in Eq: 6.15 generates mass terms for the $A^a_\mu(x)$ and $B_\mu(x)$ gauge fields.

$$D_\mu \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} = \frac{-\mathrm{i}}{\sqrt{2}}\begin{pmatrix} \frac{v}{2}\left(g_2 A^3_\mu - g_1 B_\mu\right) \\ \frac{g_2 v}{2}\left(A^1_\mu + \mathrm{i}A^2_\mu\right) \end{pmatrix} \tag{6.22}$$

Upon substituting Eq: 6.22 into the lagrangian for the complex scalar Higgs-field in Eq: 6.16 three linear combinations of the four gauge-fields will acquire masses, the complex scalar W-boson field and the Z-boson field. The only field that does not acquire a mass corresponds to the photon field.

$$\begin{aligned} W^\pm_\mu &= \frac{1}{\sqrt{2}}\left(A^1_\mu \mp \mathrm{i}A^2_\mu\right) \quad \text{with} \quad m_W = \frac{g_2 v}{2} \\ Z_\mu &= \frac{1}{\sqrt{g_1^2+g_2^2}}\left(g_2 A^3_\mu - g_1 B_\mu\right) \quad \text{with} \quad m_Z = \frac{\sqrt{g_1^2+g_2^2}\,v}{2} \\ A_\mu &= \frac{1}{\sqrt{g_1^2+g_2^2}}\left(g_1 A^3_\mu + g_2 B_\mu\right) \quad \text{with} \quad m_A = 0 \end{aligned} \tag{6.23}$$

Now we substitute the linear combinations of Eq: 6.23 back into Eq: 6.21, this generates the kinetic terms that we need and also interaction and self-interaction terms involving only gauge-fields.

$$\mathcal{L} = -\frac{1}{2}|\partial_\mu W^+_\nu - \partial_\nu W^+_\mu|^2 - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + ... \tag{6.24}$$

Here, we introduced the definition $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$, used $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and show only the quadratic terms that we need to construct the propagators. Now we introduce the conventional weak-mixing angle $\cos\theta_W = g_2/\sqrt{g_1^2+g_2^2} = m_W/m_Z$ and the electron charge $e = -g_1\cos\theta_W$ and substitute the new decomposition of the gauge-fields in the general covariant derivative of Eq: 6.15, we find the $SU(2)$ generators that belong to new decomposition:

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - \frac{\mathrm{i}g_2}{\sqrt{2}}\left[W^+_\mu T^- + W^-_\mu T^+\right] - \mathrm{i}\frac{g_2}{\cos\theta_W}\left(T^3 - \sin^2\theta_W Q\right)Z_\mu - \mathrm{i}|e|QA_\mu \tag{6.25}$$

The $SU(2)$ generators for the W-boson are $T^\pm = T^1 \mp \mathrm{i}T^2$ and the generator for the electromagnetic charge is given by $Q = T^3 + Y$. The scalar Higgs-field $H(x)$ is thus neutral whilst the $\phi^-(x)$ Goldstone-boson has a negative electromagnetic charge.

Gauge-fixing and ghosts

We now fix the gauge as explained for the $SU(3)$ part of the standard model in Sec: 6.1. Since we are free to choose a gauge-fixing function, we can cancel the following unwanted terms that appear in the covariant derivative of Eq: 6.16:

$$\begin{aligned}\boxed{a} &= -\frac{1}{\sqrt{2}}\left[\partial_\mu \begin{pmatrix} \frac{1}{\sqrt{2}}(H + i\phi_0) \\ \phi^- \end{pmatrix}\right]^\dagger D_\mu \begin{pmatrix} v \\ 0 \end{pmatrix} + \text{h.c.} \\ &= m_Z \partial_\mu \phi_0 Z_\mu + i m_W \left(\partial_\mu \phi^+ W^{\mu\,-} - \partial_\mu \phi^- W^{\mu\,+}\right)\end{aligned} \tag{6.26}$$

The gauge-fixing function G^a with $a \in \{1, ..., 4\}$ that cancels these terms through the gauge-fixing term $\mathcal{L}_{g.f.} = -\frac{1}{2\xi}G^2$, is given by the functions:

$$\begin{aligned} G^1 &= \partial_\mu A^1_\mu - \frac{i m_W \xi}{\sqrt{2}}\left(\phi^+ - \phi^-\right) \qquad & G^2 &= \partial_\mu A^2_\mu + \frac{m_W \xi}{\sqrt{2}}\left(\phi^+ + \phi^-\right) \\ G^3 &= \partial_\mu A^3_\mu - m_W \xi \phi_0 & G^4 &= \partial_\mu B_\mu - \tan\theta_W m_W \xi \phi_0 \end{aligned} \tag{6.27}$$

The other terms in the gauge-fixing term make it possible to construct propagators for the gauge bosons and they give gauge-dependent and thus unphysical masses to the Goldstone-bosons.

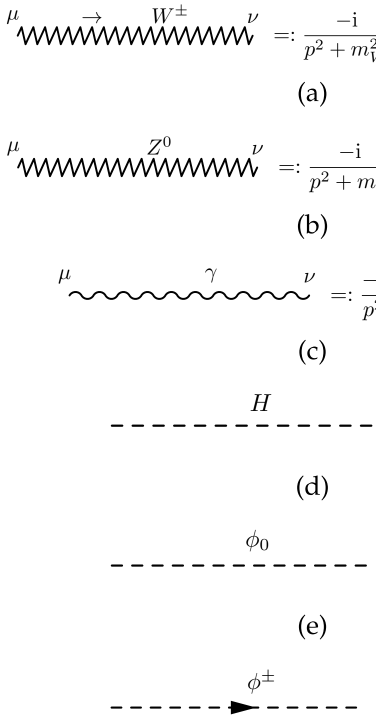
$$\begin{aligned}\mathcal{L}_{g.f.} = -\frac{1}{2\xi}G^2 = &-\frac{1}{2\xi}\left(2|\partial_\mu W^{\mu\,+}|^2 + \left(\partial_\mu Z^\mu\right)^2 + \left(\partial_\mu A^\mu\right)^2\right) - \boxed{a} + \text{tot. deriv.} \\ &- \xi m_W^2 \phi^+ \phi^- - \frac{1}{2}\xi m_Z^2 \phi_0^2\end{aligned} \tag{6.28}$$

The ghost term shown for $SU(3)$ in Sec: 6.1 in $SU(2)$ gets an additional term from the last terms in gauge-fixing functions shown in Eq: 6.27. Leaving the explicit details of the calculation this introduces gauge-dependent masses for the ghost-fields * and additional interactions coupling the ghost-fields to the scalar Higgs-field and the Goldstone-bosons.

The Feynman rules for the electroweak gauge sector

We now have all the ingredients to write down the propagators of the gauge-fields, the scalar Higgs field and the Goldstone- bosons by inverting the parts that are quadratic in the associated fields.

*When the ghost-fields are written in the analogue linear combinations of Eq: 6.23 then their masses will be equal to ξ times the mass of the gauge-field that they are associated to.



$$=: \frac{-\mathrm{i}}{p^2+m_W^2}\left[\eta_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2+\xi m_W^2}\right]$$

(a)

$$=: \frac{-\mathrm{i}}{p^2+m_Z^2}\left[\eta_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2+\xi m_Z^2}\right]$$

(b)

$$=: \frac{-\mathrm{i}}{p^2}\left[\eta_{\mu\nu}-(1-\xi)\frac{p_\mu p_\nu}{p^2}\right]$$

(c)

$$=: \frac{-\mathrm{i}}{p^2+m_H^2}$$

(d)

$$=: \frac{-\mathrm{i}}{p^2+\xi m_Z^2}$$

(e)

$$=: \frac{-\mathrm{i}}{p^2+\xi m_W^2}$$

(f)

Figure 6.3: *The propagators in the electroweak gauge sector: (a) $W^\pm$-boson, (b) Z^0-boson, (c) photon, (d) Higgs-boson, (e) ϕ_0 Goldstone-boson and (f) $\phi_\pm$ Goldstone-boson.*

6.3 The $SU(2) \times U(1)$ quark-sector

The complex scalar Higgs-field ϕ that we considered in the previous section is not the only field that transforms non-trivially under the $SU(2) \times U(1)$-symmetry of the standard model. The left-handed and right-handed components of the three generations of quark fields and lepton fields also transform independently in different representations of $SU(2) \times U(1)$-symmetry.

In this section we describe how the left-handed and right-handed quark fields are organised to form the quark lagrangian discussed in Sec: 6.1, Eq: 6.5. We will work out the BEH-mechanism affected interactions between the quark fields and the gauge-fields, scalar Higgs-field and Goldstone fields. We do not consider the lepton-sector in detail since it is fairly analogue to the quark-sector and we disregard the $SU(3)$ colour indices.

The quark-fields and gauge-coupling

Let us denote the positively charged quark-fields in a generation by u_I and the negatively charged quark field by d_I e.g. the up and down quark fields for $I = 1$. The $SU(2)$ acts only on the left-handed components of these quark fields whilst the $U(1)$ acts on all field with different values for the hypercharge Y. The u_{LI} and d_{LI} components organise themselves in a $SU(2)$ doublet q_I with hypercharge $Y = \frac{1}{6}$. The u_{RI} and d_{RI} components are $SU(2)$ singlets with hypercharges $Y = \frac{2}{3}$ and $Y = -\frac{1}{3}$ respectively. *

$$Y\left(q_I = \begin{pmatrix} u_{LI} \\ d_{LI} \end{pmatrix}\right) = \frac{1}{6} \qquad Y(u_{RI}) = \frac{2}{3} \qquad Y(d_{LI}) = -\frac{1}{3} \tag{6.29}$$

The chiral fields u_L, d_L, u_R and d_R can be constructed from the Dirac fields by using chiral projectors.

$$\begin{aligned} u_L &= \left(\frac{1-\gamma^5}{2}\right) u \qquad u_R = \left(\frac{1+\gamma^5}{2}\right) u_R \\ d_L &= \left(\frac{1-\gamma^5}{2}\right) d \qquad d_R = \left(\frac{1+\gamma^5}{2}\right) d_R \end{aligned} \tag{6.30}$$

The covariant derivative for q_I is shown in Eq: 6.15 where i and j run over the doublets entries and $Y = \frac{1}{6}$. The covariant derivatives for $u_{R\ I}$ and $d_{R\ I}$ are given by:

$$D_\mu u_{RI} = \partial_\mu u_{RI} - \mathrm{i} g_1 \left(\frac{2}{3}\right) B_\mu u_{RI} \qquad D_\mu d_{RI} = \partial_\mu d_{RI} - \mathrm{i} g_1 \left(-\frac{1}{3}\right) B_\mu d_{RI} \tag{6.31}$$

We write down the kinetic terms for the fermions, plug in the linear combinations of Eq: 6.23 and combine the chiral fermion fields where possible to form non-chiral fields e.g. $u_I = u_{RI} + u_{LI}$ and $d_I = d_{RI} + d_{LI}$.

$$\begin{aligned} \mathcal{L}_{kin} =& \mathrm{i}\bar{q}_I \not{D} q_I + \mathrm{i}\bar{u}_{RI} \not{D} u_{RI} + \mathrm{i}\bar{d}_{RI} \not{D} d_{RI} \\ =& \mathrm{i}\bar{u}_I \not{\partial} u_I + \mathrm{i}\bar{d}_I \not{\partial} d_I + \frac{g_2}{\sqrt{2}} \bar{u}_{LI} \not{W}^+ d_{LI} + \frac{g_2}{\sqrt{2}} \bar{d}_{LI} \not{W}^- u_{LI} \\ &+ \frac{g_2}{\cos\theta_W} \bar{u}_I \not{Z} \left(\frac{1}{2}\left(\frac{1-\gamma^5}{2}\right) - \frac{2}{3}\sin^2\theta_W\right) u_I \\ &+ \frac{g_2}{\cos\theta_W} \bar{d}_I \not{Z} \left(-\frac{1}{2}\left(\frac{1-\gamma^5}{2}\right) + \frac{1}{3}\sin^2\theta_W\right) d_I + \frac{2}{3}|e|\bar{u}_I \not{A} u_I - \frac{1}{3}|e|\bar{d}_I \not{A} d_I \end{aligned} \tag{6.32}$$

*In the lepton sector the doublet has a hypercharge of $Y = -\frac{1}{2}$ and the upper singlet e.g. electron, muon and tauon have $Y = -1$ whilst the lower singlet e.g. the corresponding neutrinos might not exist.

We do not draw all vertices and determine their vertex factors, since all the interaction terms have a similar structure. Let G_μ denote a gauge-field, α some arbitrary pre factor, Γ some spinor structure, $\overline{v}$ the outgoing spinor-field and w the incoming spinor field. All interaction terms in Eq: 6.32 have the structure $\mathcal{L}_{int} = \alpha G_\mu \overline{v} \Gamma^\mu w$ and thus a vertex-factor: $\mathrm{i}\alpha\Gamma^\mu$.

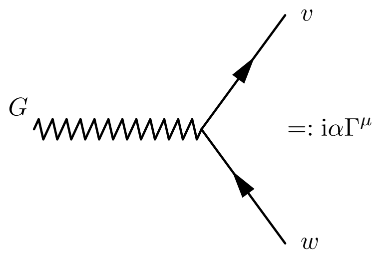


Figure 6.4: *The general vertex between a gauge-field and two spinor-fields that follows from interaction terms of the form:* $\mathcal{L}_{int} = \alpha G_\mu \overline{v} \Gamma^\mu w$.

The Yukawa-terms

The mass terms for the quarks in Eq: 6.5 require us to introduce terms that couple the left-handed and right-handed chiral fields. The mass term can be decomposed as: $\overline{u}u = \overline{u}_L u_R + \overline{u}_R u_L$ It is necessary to couple these terms to the complex scalar Higgs field, if we want to introduce mass terms for the fermions. These terms are called Yukawa terms, and the most general form of them is given by:

$$\mathcal{L}_{Yuk} = -\epsilon_{ij}\phi_i^\dagger \left(\overline{q}_I\right)_j y'_{IJ} d_{RJ} - \phi_i \left(\overline{q}_I\right)_i y''_{IJ} u_{RJ} + \text{h.c.} \tag{6.33}$$

We coupled the left-handed quark doublet in two different ways to the complex scalar Higgs-field, with an antisymmetric epsilon tensor and a dot-product using indices i and j. The indices I and J allow for the possibility of coupling chiral fermion fields from different generations.

The Yukawa terms in Eq: 6.34 will include terms proportional to both $\overline{d}_{LI} y'_{IJ} d_{RJ}$ and $\overline{u}_{LI} y''_{IJ} u_{RJ}$. We now reparametrise the quark fields using the unitary matrices U_L, U_R, D_L and D_R in such a way that the mass terms are diagonal with positive real coefficients y'_K and y''_K as follows:

$$\left.\begin{array}{lll} u_{LI} \to (U_L)_{IJ} u_{LJ} & \text{and} & \overline{u}_{LI} \to \overline{u}_{LJ} \left(U_L^\dagger\right)_{JI} \\ u_{RI} \to (U_R)_{IJ} u_{RJ} & \text{and} & \overline{u}_{RI} \to \overline{u}_{RJ} \left(U_R^\dagger\right)_{JI} \end{array}\right\} \quad \text{such that} \quad y''_{IJ} = (U_L)_{IK} y''_K \left(U_R^\dagger\right)_{KJ}$$
$$\left.\begin{array}{lll} d_{LI} \to (D_L)_{IJ} d_{LJ} & \text{and} & \overline{d}_{LI} \to \overline{d}_{LJ} \left(D_L^\dagger\right)_{JI} \\ d_{RI} \to (D_R)_{IJ} d_{RJ} & \text{and} & \overline{d}_{RI} \to \overline{d}_{RJ} \left(D_R^\dagger\right)_{JI} \end{array}\right\} \quad \text{such that} \quad y'_{IJ} = (D_L)_{IK} y'_K \left(D_R^\dagger\right)_{KJ} \tag{6.34}$$

Going back to the gauge-coupling interactions that follow from Eq: 6.32, the only affected terms are to be found in the terms that involve the $W^{\pm}$-boson. We define the Cabibbo-Kobayashi-Maskawa matrix as: $V_{CKM} = U_L^{\dagger} D_L$ and write down the new gauge-coupling terms to the $W^{\pm}$-boson:

$$\mathcal{L}_{W^{\pm}\,int} = \frac{g_2\,(V_{CKM})_{IJ}}{\sqrt{2}} \overline{u}_{LI} \not{W}^{+} d_{LJ} + \frac{g_2\,(V^{\dagger}_{CKM})_{IJ}}{\sqrt{2}} \overline{d}_{LI} \not{W}^{-} u_{LJ} \tag{6.35}$$

We now use the decomposition of the complex scalar Higgs field shown in Eq: 6.19 and the diagonalisation of the coupling terms in generation space from Eq: 6.34 to rewrite the Yukawa terms and show how it generates masses for the quark fields.

$$\begin{aligned}\mathcal{L}_{Yuk} = &- m_{u_K}\overline{u}_K u_K - m_{d_K}\overline{d}_K d_K - \frac{m_{u_K}}{v} H\overline{u}_K u_K - \frac{m_{d_K}}{v} H\overline{d}_K d_K - \mathrm{i}\frac{m_{u_K}}{v}\phi_0 \overline{u}_K \gamma^5 u_K \\ &+ \mathrm{i}\frac{m_{d_K}}{v}\phi_0 \overline{d}_K \gamma^5 d_K + \frac{\sqrt{2}\,(V_{CKM})_{IK}\, m_{d_K}}{v}\phi^{+}\overline{u}_{LI} d_{RK} - \frac{\sqrt{2} m_{u_K}\,(V_{CKM})_{KI}}{v}\phi^{+}\overline{u}_{RK} d_{LI} \\ &+ \frac{\sqrt{2} m_{d_K}\,(V^{\dagger}_{CKM})_{KI}}{v}\phi^{-}\overline{d}_{RK} u_{LI} - \frac{\sqrt{2}\,(V^{\dagger}_{CKM})_{IK}\, m_{u_K}}{v}\phi^{-}\overline{d}_{LI} u_{RK}\end{aligned} \tag{6.36}$$

The quark mass terms appear in Eq: 6.36 as was expected by means of the BEH-mechanism. The vertices between the quark spinors and the scalar Higgs field and the Goldstone boson's will introduce additional vertices. Let ψ denote either the scalar Higgs field or the Goldstone boson fields, α some arbitrary pre factor, Γ some spinor structure, $\overline{v}$ the outgoing spinor-field and w the incoming spinor field. All interaction terms in Eq: 6.36 have the structure $\mathcal{L}_{int} = \alpha\psi\overline{v}\Gamma w$ and thus a vertex-factor: $\mathrm{i}\alpha\Gamma$.

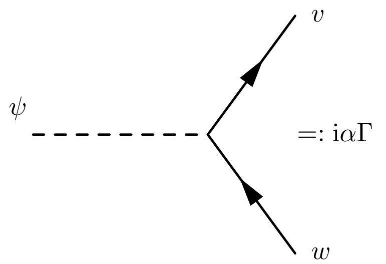


Figure 6.5: *The general vertex between a component of the complex scalar Higgs field and two spinor-fields that follows from interaction terms of the form:* $\mathcal{L}_{int} = \alpha\psi\overline{v}\Gamma w$.

Chapter 7

One-loop calculations in QCD

In this chapter, we evaluate the exact one-loop corrections to gluon and top quark propagators in *full* dimensional regularisation and renormalisable gauge. We employ more automated techniques by use of FORM [24] to cope with spinor and space-time indices. We demonstrate how the number of integrals that must be evaluated is reduced in a more systematic way, through the use of symmetries and the identification of a set of standard integrals. We aim to give a description of an automisation method that proves to be sufficient for our purposes, not one that is general.

7.1 Implementation of Automated Techniques

In this section, we describe the semi-automatic procedure, that we used to evaluate the QCD one-loop diagrams. In broad lines, we first perform the colour algebra using standard relations, then we handle the momentum and spinor indices using projectors to construct scalar integrals and at the end we write our results in terms of a small set of standard integrals.

STEP 1: The colour algebra

In one loop calculations we do not encounter strings of more than two $SU(3)$ generators in either fundamental or adjoint representation, thus we only need two group invariants to perform the summation over colour indices.

The normalisation of the $SU(3)$ generators in the fundamental representation in Eq: 6.2 already tells us:

$$T^a_{ij} T^b_{ji} = T(R)\,\delta^{ab} \quad \text{where} \quad T(R) = \frac{1}{2} \tag{7.1}$$

The following commutation relation shows us that $\left(T^a T^a\right)_{ij}$ must be proportional to the identity since it commutes with all elements generators of the group. The proportionality constant $C(R)$ is known as the quadratic Casimir number in the fundamental representation.

$$\left[T^a T^a, T^b\right] = \mathrm{i} f^{abc}\left(T^a T^c + T^c T^a\right) = 0 \;\Rightarrow\; \left(T^a T^a\right)_{ij} = C(R)\,\delta_{ij} \tag{7.2}$$

The value for the quadratic Casimir number $C(R)$ follows from the normalisation relation in Eq: 7.1. The $SU(3)$ gauge group has eight generators and the trace over δ_{ij} is equal to number of colours, from this we find $C(R) = \frac{4}{3}$.

The structure constants are the generators of the adjoint representation of the group: $\left(T^a_A\right)^{bc} = \mathrm{i} f^{abc}$. By the Jacobi identity of the fundamental representation, they satisfy a commutation relation similar to Eq: 6.1. The adjoint representation also has a normalisation $T(A)$, and the invariant Casimir number $C(A)$ is equal to this.

$$\begin{aligned} \left(T^a_A T^b_A\right)^{cd} \delta^{cd} &= f^{acd} f^{bcd} = T(A)\,\delta^{ab} \\ \left(T^a_A T^a_A\right)^{cd} &= f^{ace} f^{ade} = C(A)\,\delta^{cd} \end{aligned} \tag{7.3}$$

The choice for the normalisation $T(R)$ of the fundamental representation fixes $C(A) = T(A)$. In our case, we have $C(A) = N_c = 3$, which is already fixed by $T(R) = \frac{1}{2}$. The value for $C(A)$ can be derived by using the diagrammatic approach to colour algebra, explained in Ref. [28], on the eight gluon states defined in Ref. [20].

STEP 2: The Lorentz algebra

The use of the symmetries of our integrals removes the space-time indices from our expressions. We project our integrals onto orthogonal subspaces by using projectors that are constructed from the external momenta.

By conservation of momenta our one-loop integrals $I^{\mu\nu}(p)$ only depend on one external momenta, say p. We then decompose our integral as:

$$\begin{aligned} I^{\mu\nu}(p) &= \left(\eta^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2}\right)p^2 F_1\left(p^2\right) + p^{\mu}p^{\nu}F_2\left(p^2\right) \\ \text{where} \quad F_1\left(p^2\right) &= \frac{1}{(d-1)}\frac{1}{p^2}\left(\eta_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2}\right)I^{\mu\nu}(p) \quad , \quad F_2\left(p^2\right) = \frac{p_{\mu}p_{\nu}}{p^4}I^{\mu\nu}(p) \end{aligned} \tag{7.4}$$

This step is fully automated by FORM [24] by considering the space-time indices as vector indices. At this stage the use of some automisation is recommended to prevent bookkeeping errors, especially for expressions that involve vertex factors with a complicated space-time structure.

STEP 3: The spinor algebra

We remove free spinor indices from our expressions by means of spinor projectors using traces of spinor structures. The resulting and already present spinor traces are evaluated by use of the relations that are listed in App: A. Fortunately, FORM already has a built-in definition of the gamma-matrices and methods to contract and trace them. The gamma-matrices as defined by FORM are related to our definition by the following transformation rule: $\gamma^{\mu} = \pm \mathrm{i}\gamma^{\mu}_{FORM}$.

A general spinor matrix $\widetilde{M}$ has $4 \times 4 = 16$ degrees of freedom which can be decomposed in the following basis of gamma-matrix bilinears that are orthogonal under the trace operation: *

$$\widetilde{M} = \overbrace{\widetilde{M}_{(0)}}^{\text{1 comp.}} + \overbrace{\widetilde{M}_{(1)}}^{\text{1 comp.}}\gamma^5 + \overbrace{\widetilde{M}_{(2)\ \mu}}^{\text{4 comp.}}\gamma^{\mu} + \overbrace{\widetilde{M}_{(3)\ \mu}}^{\text{4 comp.}}\gamma^{\mu}\gamma^5 + \overbrace{\widetilde{M}_{(4)\ \mu\nu}}^{\text{6 comp.}}S^{\mu\nu} \quad ^{= \text{16 comp.}} \tag{7.5}$$

When one considers a spinor matrix $M(\not{p}, m)$ that only depends on one external momentum p and the mass of the particle then the term proportional to $S^{\mu\nu}$ vanishes by the antisymmetry of this spinor bilinear. We only need to use the full chiral decomposition when the underlying theory treats the two chiralities of the fermion on unequal footing such as the weak interaction in the SM.

$$\begin{aligned} \text{non-chiral:} \quad M(\not{p}, m) &= mA\left(p^2\right) + \not{p}B\left(p^2\right) \\ \text{chiral:} \quad M(\not{p}, m) &= \sum_{R,L} P_{R,L}\left(mA_{L,R}\left(p^2\right) + \not{p}B_{L,R}\left(p^2\right)\right) \\ \text{where:} \quad P_{R,L} &= \frac{1}{2}\left(1 \pm \gamma^5\right) \end{aligned} \tag{7.6}$$

*We define $S^{\mu\nu} \equiv \frac{\mathrm{i}}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right]$

Here, we note that $A\left(p^2\right) = A_L\left(p^2\right) + A_R\left(p^2\right)$ and $B\left(p^2\right) = B_L\left(p^2\right) + B_R\left(p^2\right)$. The associated projectors are given by:

$$\begin{aligned} A\left(p^2\right) &= \frac{1}{4m} Tr\left[M\left(\not p, m\right)\right] \qquad & A_{R,L}\left(p^2\right) &= \frac{1}{2m} Tr\left[M\left(\not p, m\right) P_{R,L}\right] \\ B\left(p^2\right) &= -\frac{1}{4p^2} Tr\left[M\left(\not p, m\right) \not p\right] \qquad & B_{R,L}\left(p^2\right) &= -\frac{1}{2p^2} Tr\left[M\left(\not p, m\right) \not p P_{R,L}\right] \end{aligned} \tag{7.7}$$

STEP 4: Obtaining a basis set of integrals

We rewrite the multitude of possible terms in terms of ordered sets of integrals that are evaluated in the next step. We eliminate the appearances of the internal momenta in the numerator by rewriting them in terms of denominators, external momenta and particle masses.

Case 1: We first consider one-loop integrals with one mass scale in the propagators and one external momentum, which appear in fermion loop diagrams. We express these integrals in terms of the following denominators: $D_0 \equiv l^2 + m^2$ and $D_1 \equiv (p+l)^2 + m^2$.

We now rewrite all appearances of the internal loop momenta in the numerator in terms of the denominators D_0 and D_1. We thus write: $p \cdot l = \frac{1}{2}\left(D_1 - D_0 - p^2\right)$ and $l^2 = D_0 - m^2$. The terms that we are left with, will then contain integrals of the following form:

$$J_1\left(\alpha, \beta\right) \equiv \widetilde{\mu}^{\epsilon} \int \frac{d^d l}{(2\pi)^d} \frac{1}{D_0^{\alpha} D_1^{\beta}} \tag{7.8}$$

The set of $J_1\left(\alpha, \beta\right)$ is not linearly independent because there exists relations between them. Let us first consider the relations that originate from linear transformations in the integration variable and space-time symmetries.

$$\int \frac{d^d l}{(2\pi)^d} \frac{1}{D_0^{\alpha} D_1^{\beta}} \overbrace{=}^{l \to -l-p} \int \frac{d^d l}{(2\pi)^d} \frac{1}{D_0^{\beta} D_1^{\alpha}} \quad \Rightarrow \quad J_1\left(\alpha, \beta\right) = J_1\left(\beta, \alpha\right) \tag{7.9}$$

This uses the fact that we are integrating over an even integration domain, so the shift in integration variable $l \to -l$ does not affect the value of the integral. Furthermore the integral of an odd integrand over an even integration domain vanishes:

$$\int \frac{d^d l}{(2\pi)^d} \frac{p \cdot l}{D_0^{\alpha}} = 0 \quad \Rightarrow \quad J_1\left(\alpha, -1\right) = J_1\left(\alpha - 1, 0\right) + p^2 J_1\left(\alpha, 0\right) \tag{7.10}$$

Finally, we use the analytic continuations to relate our integrals, as shown in App: C.2. In our case this gives us the following relation:

$$J_1\left(\alpha,0\right) = \frac{\alpha m^2}{\alpha - \frac{d}{2}} J_1\left(\alpha+1,0\right) \tag{7.11}$$

We use the above expression to bring all integrals $J_1\left(\alpha,0\right)$ to $J_1\left(2,0\right)$. This integral is one of the integrals in our preliminary basis of independent integrals to which we try to bring our expressions.

Case 2: For massless particles in the loop the relations for J_1 still hold. Nevertheless, we define a separate set of integrals J_0 with $D_0 \equiv l^2$ and $D_1 \equiv \left(p+l\right)^2$ as follows:

$$J_0\left(\alpha,\beta\right) \equiv \widetilde{\mu}^{\epsilon} \int \frac{d^d l}{\left(2\pi\right)^d} \frac{1}{D_0^{\alpha} D_1^{\beta}} \tag{7.12}$$

The most conspicuous change is that Eq: 7.11 now implies that $J_0\left(\alpha,0\right) = J_0\left(0,\alpha\right) = 0$ for $\alpha > 0$, the statement however is stronger since Eq: C.3 implies that $J_0\left(\alpha,0\right) = J_0\left(0,\alpha\right) = 0$ for any value of α. In dimensional regularisation scaleless integrals evaluate to zero. Intuitively we would say that the integral has a mass dimension but after the integration there are none available. *

When we use Eq: 7.11 on a general $J_0\left(\alpha,\beta\right)$ then we arrive after some rearrangements at:

$$J_0\left(\alpha,\beta\right) = \frac{\beta}{d-2\alpha-\beta} J_0\left(\alpha-1,\beta+1\right) - \frac{p^2\beta}{d-2\alpha-\beta} J_0\left(\alpha,\beta+1\right) \tag{7.13}$$

This shows that for $\beta < 0$ which in combination with $J_0\left(\beta,\alpha\right) = J_0\left(\alpha,\beta\right)$ also applies to α. We find that a repeated application of Eq: 7.13 leads us to $J_0\left(\alpha,\beta\right) = 0$ for either $\alpha \leq 0$ or $\beta \leq 0$.

Case 3: We also consider loop diagrams with two different particles in the loop of which one is massless, which appear in the fermion self-energy. We rewrite our integrals in terms of the denominators $D_0 \equiv l^2$ and $D_1 \equiv \left(p+l\right)^2 + m^2$. For appearances of the loop momentum in the numerator, we write: $p \cdot l = \frac{1}{2}\left(D_1 - D_0 - \left(p^2+m^2\right)\right)$ and $l^2 = D_0$. The terms that we are left with, then contain integrals of the following form:

$$J_2\left(\alpha,\beta\right) \equiv \widetilde{\mu}^{\epsilon} \int \frac{d^d l}{\left(2\pi\right)^d} \frac{1}{D_0^{\alpha} D_1^{\beta}} \tag{7.14}$$

*A rigorous argument for the so-called *'t Hooft-Veltman conjecture* is found in Ref. [29].

We note that Eq: 7.10 also produces a relation for integrals J_2, namely:

$$J_2(\alpha, -1) = J_2(\alpha - 1, 0) + \left(p^2 + m^2\right) J_2(\alpha, 0) \tag{7.15}$$

We do not make the possible replacements $J_2(0, \alpha) = J_1(\alpha, 0)$ and $J_2(\alpha, 0) = J_0(\alpha, 0) = 0$. Instead we introduce two new sets of integrals and corresponding replacement rules, which significantly reduces the number of integrals that we need to evaluate.

$$J_r(\alpha) \equiv \int \frac{d^d l}{(2\pi)^d} \frac{p \cdot l}{D_0^\alpha D_1} \qquad J_s(\alpha) \equiv \int \frac{d^d l}{(2\pi)^d} \frac{p \cdot l}{D_0 D_1^\alpha} \tag{7.16}$$

$$\begin{aligned} J_2(0, \alpha) &= J_2(1, \alpha - 1) - 2J_s(\alpha) - \left(p^2 + m^2\right) J_2(1, \alpha) \\ J_2(\alpha, 0)|_{\alpha > 1} &= J_2(\alpha - 1, 1) + 2J_r(\alpha) + \left(p^2 + m^2\right) J_2(\alpha, 1) \end{aligned} \tag{7.17}$$

One fully eliminates all appearances of either $J_2(0, \alpha)$ and $J_2(\alpha, 0)$ when one applies Eq: 7.17 with in addition $J_2(1, 0) = J_0(1, 0) = 0$.

Case 4: Another possibility that arises in loop diagrams of the fermion self-energy is the appearance of two massive particles in the loop, of which one is a massive gauge particle with mass M. We rewrite our integrals in terms of denominators $D_0 \equiv l^2 + M^2$, $D_1 \equiv (p + l)^2 + m^2$ and $D_2 = l^2 + \xi M^2$. The massive gauge particle thus introduces an additional gauge dependent denominator. For appearances of the loop momentum in the numerator, we write: $p \cdot l = \frac{1}{2}\left(D_1 - D_0 + M^2 - \left(p^2 + m^2\right)\right)$ and $l^2 = D_0 - M^2$. The terms that are left, then contain integrals of the form:

$$\begin{aligned} J_3(\alpha, \beta, \gamma) &\equiv \widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \frac{1}{D_0^\alpha D_1^\beta D_2^\gamma} \\ J_4(\alpha, \beta) &\equiv J_3(\alpha, \beta, 0) \end{aligned} \tag{7.18}$$

We apply an odd integrand relation to eliminate the integrals $J_3(\alpha, -1, \gamma)$:

$$\int \frac{d^d l}{(2\pi)^d} \frac{p \cdot l}{D_0^\alpha D_2^\gamma} = 0 \;\Rightarrow\; J_3(\alpha, -1, \gamma) = J_3(\alpha - 1, 0, \gamma) + \left(p^2 + m^2 - M^2\right) J_3(\alpha, 0, \gamma) \tag{7.19}$$

To eliminate integrals such as $J_3(0, \beta, \gamma)$, $J_4(\alpha, 0)$ and $J_4(0, \beta)$ for $\beta > 1$ we define two additional sets of integrals with replacement rules.

$$\begin{aligned} \widetilde{J}_3(\alpha, \beta, \gamma) &\equiv \widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \frac{p \cdot l}{D_0^\alpha D_1^\beta D_2^\gamma} \\ \widetilde{J}_4(\alpha, \beta) &\equiv \widetilde{J}_3(\alpha, \beta, 0) \end{aligned} \tag{7.20}$$

$$
\begin{aligned}
J_3(0,\beta,\gamma) &= -2\widetilde{J}_3(1,\beta,\gamma) + J_3(1,\beta-1,\gamma) + \left(M^2 - p^2 - m^2\right) J_3(1,\beta,\gamma) \\
J_4(\alpha,0)\Big|_{\alpha>1} &= 2\widetilde{J}_4(\alpha,1) + J_4(\alpha-1,1) + \left(p^2 + m^2 - M^2\right) J_4(\alpha,1) \\
J_4(0,\beta) &= -2\widetilde{J}_4(1,\beta) + J_4(1,\beta-1) + \left(M^2 - p^2 - m^2\right) J_4(1,\beta) \\
J_4(1,-1) &= J_4(0,0) - \left(M^2 - p^2 - m^2\right) J_4(1,0)
\end{aligned}
\tag{7.21}
$$

Additionally we observe that the integral $J_4(1,0)$ equals $J_1(1,0)$ when we set $m^2 \to M^2$ in Eq: 7.8.

STEP 5: Evaluating the integrals

After the preceding steps, we obtain an expression in terms of small sets of integrals. We now evaluate these sets of integrals in terms of elementary functions such as exponents and powers and special functions such as the gamma function and (generalised) hypergeometric functions by use of Feynman parametrisation.

There is still a lot of redundancy in these expressions with gamma functions and (generalised) hypergeometric functions. We can often bring our expressions into a simpler, more aesthetic, expression, by using special properties of these functions, partial integrations of the integral representations of the hypergeometric functions and Gauss' relations for contiguous functions that are tabulated in Ref. [30].

Case 1: We define $x_\pm\left(m^2\right) = \frac{1}{2}\left(1 \pm \sqrt{1 + 4\frac{m^2}{p^2}}\right)$ and evaluate the integral set J_1 in terms of elementary and special functions:

$$
\begin{aligned}
J_1(0,2) &= \frac{\mathrm{i}}{16\pi^2}\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{m^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} \mathrm{e}^{\frac{\gamma_E\epsilon}{2}} \\
J_1(\alpha,\beta) &= \frac{\mathrm{i}}{16\pi^2}\frac{\Gamma\left(\frac{\epsilon}{2}+\alpha+\beta-2\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{\Gamma(\alpha+\beta)}\left(\frac{m^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(m^2\right)^{2-\alpha-\beta} \\
&\quad\times \mathcal{F}_1\left(\alpha;\frac{\epsilon}{2}+\alpha+\beta-2,\frac{\epsilon}{2}+\alpha+\beta-2;\alpha+\beta;\frac{1}{x_+},\frac{1}{x_-}\right)
\end{aligned}
\tag{7.22}
$$

The function $\mathcal{F}_1$ is Appell's first hypergeometric function as defined in Ref. [31, ch. 9], which is a generalisations of the standard hypergeometric functions. The first Appell's hypergeometric function has the following integral representation, where B is the Euler-beta function:

$$
B(\alpha\,,\,\gamma-\alpha)\,\mathcal{F}_1(\alpha\,;\,\beta\,,\,\beta'\,;\,\gamma\,;\,y\,,\,z) = \int_0^1 dx\, x^{\alpha-1}(1-x)^{\gamma-\alpha-1}(1-yx)^{-\beta}(1-zx)^{-\beta'}
\tag{7.23}
$$

For aesthetical reasons in the calculation of fermion loops we substitute $\mathcal{F}_1\left(2;\frac{\epsilon}{2},\frac{\epsilon}{2};4;\frac{1}{x_+},\frac{1}{x_-}\right)$ and eliminate $\mathcal{F}_1\left(1;\frac{\epsilon}{2},\frac{\epsilon}{2};2;\frac{1}{x_+},\frac{1}{x_-}\right)$ by expanding the partial integration of the following integral:

$$\int_0^1 dx\left(\frac{x\,(1-x)}{4x_+x_-}+1\right)^{1-\frac{\epsilon}{2}} = 1-\frac{1-\frac{\epsilon}{2}}{4x_+x_-}\int_0^1 dx\,x\,(1-2x)\left(\frac{x\,(1-x)}{4x_+x_-}+1\right)^{-\frac{\epsilon}{2}} \tag{7.24}$$

We find the following relation between Appell's first hypergeometric functions:

$$\frac{3-\epsilon}{6}\mathcal{F}_1\left(2;\frac{\epsilon}{2},\frac{\epsilon}{2};4;\frac{1}{x_+},\frac{1}{x_-}\right)=\left(\frac{1}{2}\left(1-\frac{\epsilon}{2}\right)+x_+x_-\right)\mathcal{F}_1\left(1;\frac{\epsilon}{2},\frac{\epsilon}{2};2;\frac{1}{x_+},\frac{1}{x_-}\right)-x_+x_- \tag{7.25}$$

Case 2: The integral set $J_0\left(\alpha,\beta\right)$ is evaluated in terms of elementary and special functions in the following expression, the $m^2\to 0$ limit of Eq: 7.22:

$$J_0\left(\alpha,\beta\right)=\frac{\mathrm{i}}{16\pi^2}\frac{\Gamma\left(\frac{\epsilon}{2}+\alpha+\beta-2\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}B\left(2-\alpha-\frac{\epsilon}{2},2-\beta-\frac{\epsilon}{2}\right)\left(\frac{p^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(p^2\right)^{2-\alpha-\beta} \tag{7.26}$$

The structure of $J_0\left(\alpha,\beta\right)$ in terms of gamma functions makes it possible to parametrise all member of the set $J_0\left(\alpha,\beta\right)$ for $\alpha>0$ and $\beta>0$ in terms of one member of the set say: $J_0\left(1,1\right)$.

$$\begin{aligned}
J_0\left(1,1\right)&=\frac{\mathrm{i}}{16\pi^2}\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}B\left(1-\frac{\epsilon}{2},1-\frac{\epsilon}{2}\right)\left(\frac{p^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\\
J_0\left(1,2\right)&=-\frac{1-\epsilon}{p^2}J_0\left(1,1\right)\\
J_0\left(2,2\right)&=\frac{2+\epsilon}{p^2}J_0\left(1,2\right)=-\frac{\left(2+\epsilon\right)\left(1-\epsilon\right)}{p^4}J_0\left(1,1\right)
\end{aligned} \tag{7.27}$$

Case 3: The integral set $J_2\left(\alpha,\beta\right)$ is evaluated in terms of elementary and special functions in the following expression:

$$\begin{aligned}
J_2\left(\alpha,\beta\right)=&\frac{\mathrm{i}}{16\pi^2}\frac{\Gamma\left(\frac{\epsilon}{2}+\alpha+\beta-2\right)\Gamma\left(2-\alpha-\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{\Gamma\left(\beta\right)\Gamma\left(2-\frac{\epsilon}{2}\right)}\\
&\times{}_2F_1\left(\frac{\epsilon}{2}+\alpha+\beta-2\,,\,\alpha\,;\,2-\frac{\epsilon}{2}\,;\,-\frac{p^2}{m^2}\right)\left(\frac{m^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(m^2\right)^{2-\alpha-\beta}
\end{aligned} \tag{7.28}$$

The function ${}_2F_1$ is the hypergeometric function as defined in Ref. [30, ch. 15], the standard hypergeometric function. The standard hypergeometric function has the following integral representation:

$$B\left(b\,,\,c-b\right){}_2F_1\left(a\,,\,b\,;\,c\,;\,z\right)=\int_0^1 dx\, x^{b-1}\left(1-x\right)^{c-b-1}\left(1-zx\right)^{-a} \tag{7.29}$$

The additional $J_r(\alpha)$ and $J_s(\alpha)$ evaluated in terms of elementary and special functions are given by the following expressions:

$$\frac{1}{p^2}J_r\left(\alpha\right)=-\frac{\mathrm{i}}{16\pi^2}\frac{\Gamma\left(\frac{\epsilon}{2}+\alpha-1\right)\Gamma\left(3-\alpha-\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{\Gamma\left(3-\frac{\epsilon}{2}\right)} \tag{7.30}$$
$$\times\,{}_2F_1\left(\frac{\epsilon}{2}+\alpha-1\,,\,\alpha\,;\,3-\frac{\epsilon}{2}\,;\,-\frac{p^2}{m^2}\right)\left(\frac{m^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(m^2\right)^{1-\alpha}$$

$$\frac{1}{p^2}J_s\left(\alpha\right)=-\frac{\mathrm{i}}{8\pi^2}\frac{\Gamma\left(\frac{\epsilon}{2}+\alpha-1\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{\Gamma\left(\alpha\right)\left(4-\epsilon\right)} \tag{7.31}$$
$$\times\,{}_2F_1\left(\frac{\epsilon}{2}+\alpha-1\,,\,1\,;\,3-\frac{\epsilon}{2}\,;\,-\frac{p^2}{m^2}\right)\left(\frac{m^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(m^2\right)^{1-\alpha}$$

From Gauss' relations for contiguous functions in Ref. [30, ch. 15] we see that there is a large redundancy in integrals containing hypergeometric functions. We chose to express our results in terms of (in short-hand notation):

$$F_{012}\left(m^2\right)\equiv{}_2F_1\left(\frac{\epsilon}{2}\,,\,1\,;\,2-\frac{\epsilon}{2}\,;\,-\frac{p^2}{m^2}\right) \tag{7.32}$$

$$F_{023}\left(m^2\right)\equiv{}_2F_1\left(\frac{\epsilon}{2}\,,\,2\,;\,3-\frac{\epsilon}{2}\,;\,-\frac{p^2}{m^2}\right) \tag{7.33}$$

We could express the following three hypergeometric functions entirely in terms of the the hypergeometric functions in Eq: 7.32 and Eq: 7.44 by using Eq: 15.2.14, Eq: 15.2.17 and Eq: 15.2.19 from Ref. [30, ch. 15]. This list is far from exhaustive but it sufficed for our purposes.

$${}_2F_1\left(\frac{\epsilon}{2}\,,\,1\,;\,3-\frac{\epsilon}{2}\,;\,-\frac{p^2}{m^2}\right)=\frac{1}{2-\epsilon}\left(\left(4-\epsilon\right)F_{012}\left(m^2\right)-2F_{023}\left(m^2\right)\right) \tag{7.34}$$

$${}_2F_1\left(1+\frac{\epsilon}{2}\,,\,1\,;\,3-\frac{\epsilon}{2}\,;\,-\frac{p^2}{m^2}\right)=-\frac{2}{\epsilon}\left(\left(2-\frac{\epsilon}{2}\right)F_{012}\left(m^2\right)-2F_{023}\left(m^2\right)\right) \tag{7.35}$$

$$\left(1+\frac{p^2}{m^2}\right){}_2F_1\left(1+\frac{\epsilon}{2}\,,\,2\,;\,3-\frac{\epsilon}{2}\,;\,-\frac{p^2}{m^2}\right)=\frac{2}{\epsilon}\left(\left(2-\frac{\epsilon}{2}\right)F_{012}\left(m^2\right)-\left(2-\epsilon\right)F_{023}\left(m^2\right)\right) \tag{7.36}$$

Case 4: For our calculations we do not need the preliminary base set $J_3(\alpha,\beta,\gamma)$ for general values of α, β and γ which yields an overly complex expression. The expression for a few special cases suffices, such as $\alpha = \beta = \gamma = 1$:

$$M^2 J_3(1,1,1) = -\frac{\mathrm{i}}{16\pi^2}\frac{\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{1-\xi}\left(\mathcal{F}_1\left(1;\frac{\epsilon}{2},\frac{\epsilon}{2};2;\frac{1}{y_+},\frac{1}{y_-}\right)-\mathcal{F}_1\left(1;\frac{\epsilon}{2},\frac{\epsilon}{2};2;\frac{1}{\widetilde{y}_+},\frac{1}{\widetilde{y}_-}\right)\right)\left(\frac{m^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}$$

$$\text{with}\quad y_\pm \equiv \frac{1}{2}\left(1+\frac{M^2-m^2}{p^2}\pm\sqrt{1+2\frac{M^2+m^2}{p^2}+\left(\frac{M^2-m^2}{p^2}\right)^2}\right)\qquad \widetilde{y}_\pm \equiv y_\pm\Big|_{M^2\to\xi M^2} \tag{7.37}$$

In case $\beta = 0$ we can write a general expression for $J_3(\alpha,0,\gamma)$:

$$\begin{aligned}J_3(\alpha,0,\gamma) = {}& \frac{\mathrm{i}}{16\pi^2}\frac{\Gamma\left(\frac{\epsilon}{2}+\alpha+\gamma-2\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{\Gamma(\alpha+\gamma)}\\ &\times {}_2F_1\left(\frac{\epsilon}{2}+\alpha+\gamma-2\,,\,\gamma\,;\,\alpha+\gamma\,;\,(1-\xi)\right)\left(\frac{M^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(M^2\right)^{2-\alpha-\gamma}\end{aligned} \tag{7.38}$$

When additionally $\alpha = 1$ and $\gamma = 1$, then the hypergeometric function can be evaluated and we find:

$$\begin{aligned}{}_2F_1\left(\frac{\epsilon}{2}\,,\,1\,;\,2\,;\,(1-\xi)\right) = {}& \frac{\xi^{1-\frac{\epsilon}{2}}-1}{\left(1-\frac{\epsilon}{2}\right)(1-\xi)}\\ \Rightarrow J_3(1,0,1) = {}& \frac{\mathrm{i}}{16\pi^2}\frac{\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}\left(\xi^{1-\frac{\epsilon}{2}}-1\right)}{\left(1-\frac{\epsilon}{2}\right)(1-\xi)}\left(\frac{M^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\end{aligned} \tag{7.39}$$

The additional integral $\widetilde{J}_3(1,1,1)$ in terms of special and elementary functions is given by:

$$\frac{M^2}{p^2}\widetilde{J}_3(1,1,1) = \frac{\mathrm{i}}{32\pi^2}\frac{\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{1-\xi}\left(\mathcal{F}_1\left(1;\frac{\epsilon}{2},\frac{\epsilon}{2};3;\frac{1}{y_+},\frac{1}{y_-}\right)-\mathcal{F}_1\left(1;\frac{\epsilon}{2},\frac{\epsilon}{2};3;\frac{1}{\widetilde{y}_+},\frac{1}{\widetilde{y}_-}\right)\right)\left(\frac{m^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} \tag{7.40}$$

The set of integrals $J_4(\alpha,\beta) \equiv J_3(\alpha,\beta,0)$ in terms of elementary and special functions is given by the following expression:

$$\begin{aligned}J_4(\alpha,\beta) = {}& \frac{\mathrm{i}}{16\pi^2}\frac{\Gamma\left(\frac{\epsilon}{2}+\alpha+\beta-2\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{\Gamma(\alpha+\beta)}\\ &\times\mathcal{F}_1\left(\beta;\frac{\epsilon}{2}+\alpha+\beta-2,\frac{\epsilon}{2}+\alpha+\beta-2;\alpha+\beta;\frac{1}{y_+},\frac{1}{y_-}\right)\left(\frac{m^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(m^2\right)^{2-\alpha-\beta}\end{aligned} \tag{7.41}$$

The additional integral set $\widetilde{J}_4(\alpha,\beta) \equiv \widetilde{J}_3(\alpha,\beta,0)$ evaluated in terms of elementary and special functions is given by the following expression:

$$\frac{1}{p^2}\widetilde{J}_4(\alpha,\beta) = -\frac{\mathrm{i}}{16\pi^2}\frac{\alpha\Gamma\left(\frac{\epsilon}{2}+\alpha+\beta-2\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{\Gamma(\alpha+\beta+1)} \times \mathcal{F}_1\left(\beta;\frac{\epsilon}{2}+\alpha+\beta-2,\frac{\epsilon}{2}+\alpha+\beta-2;\alpha+\beta+1;\frac{1}{y_+},\frac{1}{y_-}\right)\left(\frac{m^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(m^2\right)^{2-\alpha-\beta} \tag{7.42}$$

We choose to express our Appell's first hypergeometric functions in terms of the following representatives, to counter the redundancy in expressions with Appell's first hypergeometric functions. We also show their hypergeometric analogues when one sets $m^2 \to 0$.

$$\mathcal{F}_{12}\left(m^2,M^2\right) \equiv \mathcal{F}_1\left(1;\frac{\epsilon}{2},\frac{\epsilon}{2};2;\frac{1}{y_+},\frac{1}{y_-}\right)\Big|_{m^2\to 0} = \left(\frac{m^2}{M^2}\right)^{\frac{\epsilon}{2}}\frac{\Gamma\left(1-\frac{\epsilon}{2}\right)}{\Gamma\left(2-\frac{\epsilon}{2}\right)}{}_2F_1\left(\frac{\epsilon}{2},1,2-\frac{\epsilon}{2},-\frac{p^2}{M^2}\right) \tag{7.43}$$

$$\mathcal{F}_{23}\left(m^2,M^2\right) \equiv \mathcal{F}_1\left(2;\frac{\epsilon}{2},\frac{\epsilon}{2};3;\frac{1}{y_+},\frac{1}{y_-}\right)\Big|_{m^2\to 0} = \left(\frac{m^2}{M^2}\right)^{\frac{\epsilon}{2}}\frac{\Gamma\left(1-\frac{\epsilon}{2}\right)}{\Gamma\left(3-\frac{\epsilon}{2}\right)}{}_2F_1\left(\frac{\epsilon}{2},2,3-\frac{\epsilon}{2},-\frac{p^2}{M^2}\right) \tag{7.44}$$

The analogue of Eq: 7.34 in terms of Appell's first hypergeometric functions is given by:

$$\mathcal{F}_1\left(1;\frac{\epsilon}{2},\frac{\epsilon}{2};3;\frac{1}{y_+},\frac{1}{y_-}\right) = 2\mathcal{F}_{12}\left(m^2,M^2\right) - \mathcal{F}_{23}\left(m^2,M^2\right) \tag{7.45}$$

7.2 The Gluon Self-Energy

In this section, we compute the one-loop quantum corrections to the gluon self-energy. We first determine the contributing diagrams and then evaluate them using the automated techniques from Sec: 7.1 in full dimensional regularisation and renormalisable gauge.

We will repeatedly make use of the following decomposition of the self-energy contributions for diagram $i \in \{1,...,4\}$ instead of the general decomposition in Eq: 7.4:

$$\Pi^{(i)\mu\nu ab}(p) = \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)p^2\Pi^{(i)ab}\left(p^2\right) + p^\mu p^\nu \Delta^{(i)ab}\left(p^2\right) \tag{7.46}$$

The diagrams that contribute to the gluon self-energy at the one-loop order are pure QCD graphs. From Sec: 6.1 we learned that gluons only couple to the colour-charged fields, which are: the quark fields, the gluon fields and the ghost-fields. Only at two-loop order can gluons couple to electromagnetically or weakly charged fields through quark interactions.

The quark contribution

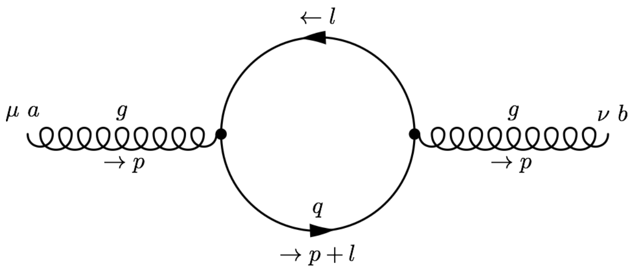


Figure 7.1: *The quark loop contribution to the gluon self-energy.*

We now apply the Feynman rules for the diagram in Fig: 7.1 as we have derived in Sec: 6.1, we find:

$$\begin{aligned} \mathrm{i}\Pi^{(1)\mu\nu ab}(p) &= (-1)\sum_q \int \frac{d^4 l}{(2\pi)^4} Tr\left[\left(\mathrm{i}g_3\gamma^\mu T^a_{ij}\right)\left(\frac{1}{\mathrm{i}}\widetilde{S}^q_{il}(\not{l})\right)\left(\mathrm{i}g_3\gamma^\nu T^b_{kl}\right)\left(\frac{1}{\mathrm{i}}\widetilde{S}^q_{jk}(\not{p}+\not{l})\right)\right] \\ &= -g_3^2\widetilde{\mu}^\epsilon\left(T^a_{ij}T^b_{ji}\right)\sum_q \int \frac{d^d l}{(2\pi)^d} Tr\left[\gamma^\mu \widetilde{S}^q(\not{l})\gamma^\nu\widetilde{S}^q(\not{p}+\not{l})\right] \end{aligned} \tag{7.47}$$

By the automatic procedure as described in Sec: 7.1 we find the following results in Eq: 7.48 and Eq: 7.49. Both the colour-charge and the transversality of the gluon are conserved in this interaction. This result is the exact analogue of Eq: 3.26, which was obtained by hand.

$$\begin{aligned} &\Pi^{(1)ab}\left(p^2\right) = -\frac{\alpha_s}{3\pi}T(R)\,\delta^{ab}\,\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}\sum_q\left(\frac{m_q^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\mathcal{F}_1\left(2;\frac{\epsilon}{2},\frac{\epsilon}{2};4;\frac{1}{x_+\left(m_q^2\right)},\frac{1}{x_-\left(m_q^2\right)}\right) \\ &\Delta^{(1)ab}\left(p^2\right) = 0 \end{aligned} \tag{7.48}$$

The expansion of Eq: 7.48 in terms of the parameter ϵ, which is required to find the values for the $\overline{MS}$ counter terms, is given by:

$$\begin{aligned}\Pi^{(1)ab}\left(p^2\right) &= -\frac{2\alpha_s}{3\pi}T\left(R\right)n_f\delta^{ab}\frac{1}{\epsilon}+\frac{2\alpha_s}{\pi}T\left(R\right)\delta^{ab}\sum_q\int_0^1 dx\, x\left(1-x\right)\ln\left(x\left(1-x\right)\frac{p^2}{\mu^2}+\frac{m_q^2}{\mu^2}\right)+\mathcal{O}\left(\epsilon\right)\\ |_{m_q^2\ll p^2} &= -\frac{2\alpha_s}{3\pi}T\left(R\right)n_f\delta^{ab}\frac{1}{\epsilon}+\frac{\alpha_s}{3\pi}T\left(R\right)n_f\delta^{ab}\left(-\frac{5}{3}+\ln\frac{p^2}{\mu^2}\right)+\mathcal{O}\left(\epsilon\right)\\ \Delta^{(1)ab}\left(p^2\right) &= 0\end{aligned} \tag{7.49}$$

The gluon four-point vertex contribution

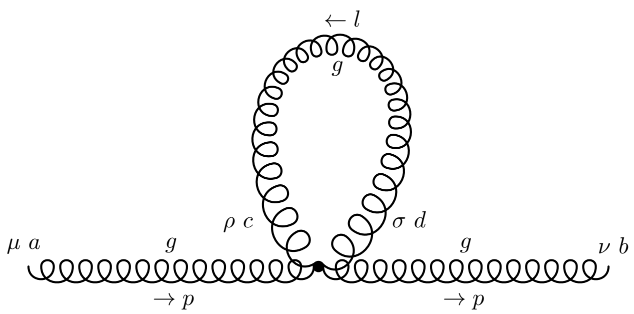


Figure 7.2: *The gluon four-point interaction loop contribution to the gluon self-energy.*

We now apply the Feynman rules for the diagram in Fig: 7.2 as we have derived in Sec: 6.1, we find:

$$\begin{aligned}\mathrm{i}\Pi^{(2)\mu\nu ab}\left(p\right) = \Big(-\mathrm{i}g_3^2\Big[&f^{abe}f^{cde}\left(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}\right)+f^{ade}f^{bce}\left(\eta^{\mu\nu}\eta^{\sigma\rho}-\eta^{\mu\rho}\eta^{\nu\sigma}\right)\\ &+f^{ace}f^{dbe}\left(\eta^{\mu\sigma}\eta^{\nu\rho}-\eta^{\mu\nu}\eta^{\rho\sigma}\right)\Big]\Big)\int\frac{d^4l}{\left(2\pi\right)^4}\left(\frac{1}{i}\widetilde{\Delta}^{dc}_{\rho\sigma}\left(l\right)\right)\end{aligned} \tag{7.50}$$

We could use the automatic procedure to evaluate $\Pi^{(2)\mu\nu ab}\left(p\right)$ but in this case a short argument will already show us that $\Pi^{(2)\mu\nu ab}\left(p\right)=0$ in dimensional regularisation because it contains only scaleless integrals. To see this let us disregard the complex vertex factor:

$$\widetilde{\mu}^{\epsilon}\int\frac{d^dl}{\left(2\pi\right)^d}\widetilde{\Delta}^{dc}_{\rho\sigma}\left(l\right)=\delta^{dc}\widetilde{\mu}^{\epsilon}\int\frac{d^dl}{\left(2\pi\right)^d}\frac{1}{l^2}\left(\eta_{\rho\sigma}-\left(1-\xi_s\right)\frac{l_\rho l_\sigma}{l^2}\right)=\boxed{\mathrm{a}}\;\eta_{\rho\sigma}\delta^{dc} \tag{7.51}$$

There are no external momenta in the problem so the integral must be proportional to $\eta_{\rho\sigma}$, using this:

$$\boxed{\text{a}} = \frac{d-1+\xi_s}{d}\widetilde{\mu}^{\epsilon}\int\frac{d^d l}{(2\pi)^d}\frac{1}{l^2} = 0 \tag{7.52}$$

The gluon three-point contribution

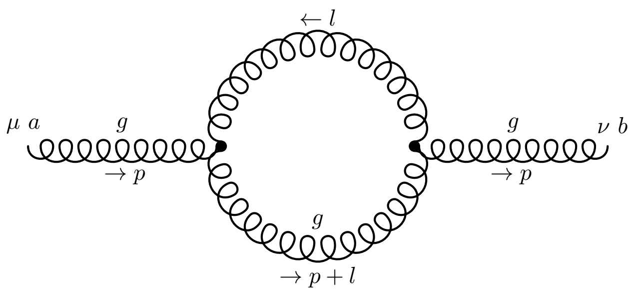


Figure 7.3: *The gluon three-point interaction loop contribution to the gluon self-energy.*

We now apply the Feynman rules for the diagram in Fig: 7.3 as we have derived in Sec: 6.1. We must not forget to include a symmetry factor of one half that corresponds to an interchange of the two propagators in the loop, we find:

$$\begin{aligned} \mathrm{i}\Pi^{(3)\mu\nu ab}(p) &= \frac{1}{2}\int\frac{d^4 l}{(2\pi)^4}\left[g_3 f^{abc}\left(\eta^{\mu\rho}(l-p)^{\sigma}+\eta^{\mu\sigma}(2p+l)^{\rho}-\eta^{\rho\sigma}(p+2l)^{\mu}\right)\right]\left(\frac{1}{\mathrm{i}}\widetilde{\Delta}^{df}_{\sigma\kappa}(p+l)\right) \\ &\times\left[g_3 f^{bef}\left(\eta^{\nu\lambda}(-l+p)^{\kappa}-\eta^{\nu\kappa}(2p+l)^{\lambda}+\eta^{\lambda\kappa}(p+2l)^{\nu}\right)\right]\left(\frac{1}{\mathrm{i}}\widetilde{\Delta}^{ce}_{\rho\lambda}(l)\right) \\ &= \frac{g_3^2}{2}\left(f^{acd}f^{bcd}\right)\widetilde{\mu}^{\epsilon}\int\frac{d^d l}{(2\pi)^d}\left[\left(\eta^{\mu\rho}(l-p)^{\sigma}+\eta^{\mu\sigma}(2p+l)^{\rho}-\eta^{\rho\sigma}(p+2l)^{\mu}\right)\right] \\ &\times\left[\left(\eta^{\nu\lambda}(l-p)^{\kappa}+\eta^{\nu\kappa}(2p+l)^{\lambda}-\eta^{\lambda\kappa}(p+2l)^{\nu}\right)\right]\widetilde{\Delta}_{\sigma\kappa}(p+l)\,\widetilde{\Delta}_{\rho\lambda}(l) \end{aligned} \tag{7.53}$$

By the automatic procedure as described in Sec: 7.1 we find the following results in Eq: 7.54 and Eq: 7.55.

$$\begin{aligned} \Pi^{(3)ab}\left(p^2\right) &= -\frac{\alpha_s T(R) N_c}{16(3-\epsilon)\pi}\delta^{ab}\Big(-50+37\epsilon-7\epsilon^2+2\left(6-11\epsilon+3\epsilon^2\right)\xi_s \\ &\quad -(3-\epsilon)\epsilon\xi_s^2\Big)\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}B\left(1-\frac{\epsilon}{2},1-\frac{\epsilon}{2}\right)\left(\frac{p^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} \end{aligned} \tag{7.54}$$

$$\Delta^{(3)ab}\left(p^2\right) = -\frac{\alpha_s}{8\pi}T(R)N_c\delta^{ab}\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}B\left(1-\frac{\epsilon}{2},1-\frac{\epsilon}{2}\right)\left(\frac{p^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}$$

The expansion of Eq: 7.54 in terms of the parameter ϵ, which is required to find the values for the $\overline{MS}$ counter terms, is given by:

$$\begin{aligned}\Pi^{(3)ab}\left(p^2\right) &= \frac{\alpha_s}{12\pi}T\left(R\right)N_c\delta^{ab}\frac{1}{\epsilon}\left(25-6\xi_s\right)\\ &+\frac{\alpha_s}{72\pi}T\left(R\right)N_c\left(89-75\ln\frac{p^2}{\mu^2}+18\left(1+\ln\frac{p^2}{\mu^2}\right)\xi_s+9\xi_s^2\right)+\mathcal{O}\left(\epsilon\right)\\ \Delta^{(3)ab}\left(p^2\right) &= -\frac{\alpha_s}{4\pi}T\left(R\right)N_c\delta^{ab}\frac{1}{\epsilon}-\frac{\alpha_s}{8\pi}T\left(R\right)N_c\left(2-\ln\frac{p^2}{\mu^2}\right)+\mathcal{O}\left(\epsilon\right)\end{aligned} \tag{7.55}$$

The ghost contribution

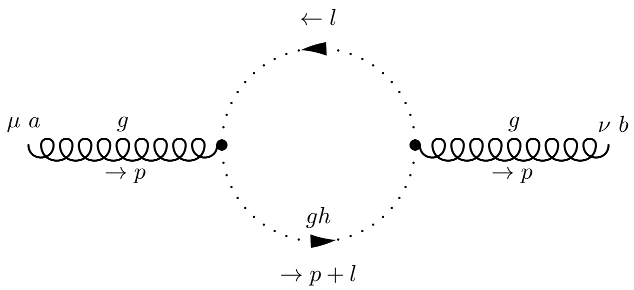


Figure 7.4: *The ghost loop contribution to the gluon self-energy.*

We now apply the Feynman rules for the diagram in Fig: 7.4 as we have derived in Sec: 6.1. We must not forget to include a minus sign for the fact that the ghost fields are fermion fields, we find:

$$\begin{aligned}\mathrm{i}\Pi^{(4)\mu\nu ab}\left(p\right) &= \left(-1\right)\left(g_3 f^{adc}\left(p+l\right)^{\mu}\right)\left(g_3 f^{bef}l^{\nu}\right)\int\frac{d^4l}{\left(2\pi\right)^4}\left(\frac{1}{\mathrm{i}}\widetilde{\Delta}^{df}\left(p+l\right)\right)\left(\frac{1}{\mathrm{i}}\widetilde{\Delta}^{ce}\left(l\right)\right)\\ &= -g_3^2\left(f^{acd}f^{bcd}\right)\widetilde{\mu}^{\epsilon}\int\frac{d^dl}{\left(2\pi\right)^d}\frac{\left(p+l\right)^{\mu}l^{\nu}}{\left(p+l\right)^2 l^2}\end{aligned} \tag{7.56}$$

By the automatic procedure as described in Sec: 7.1 we find the following results in Eq: 7.57 and Eq: 7.58.

$$\begin{aligned}\Pi^{(4)ab}\left(p^2\right) &= \frac{\alpha_s T\left(R\right)N_c}{8\left(3-\epsilon\right)\pi}\delta^{ab}\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}B\left(1-\frac{\epsilon}{2},1-\frac{\epsilon}{2}\right)\left(\frac{p^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\\ \Delta^{(4)ab}\left(p^2\right) &= \frac{\alpha_s}{8\pi}T\left(R\right)N_c\delta^{ab}\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}B\left(1-\frac{\epsilon}{2},1-\frac{\epsilon}{2}\right)\left(\frac{p^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\end{aligned} \tag{7.57}$$

The expansion of Eq: 7.57 in terms of the parameter ϵ, which is required to find the values for the $\overline{MS}$ counter terms, is given by:

$$\begin{aligned}\Pi^{(4)ab}\left(p^2\right) &= \frac{\alpha_s}{12\pi}T\left(R\right)N_c\delta^{ab}\frac{1}{\epsilon} + \frac{\alpha_s}{72\pi}T\left(R\right)N_c\left(8 - 3\ln\frac{p^2}{\mu^2}\right) + \mathcal{O}\left(\epsilon\right) \\ \Delta^{(4)ab}\left(p^2\right) &= \frac{\alpha_s}{4\pi}T\left(R\right)N_c\delta^{ab}\frac{1}{\epsilon} + \frac{\alpha_s}{8\pi}T\left(R\right)N_c\left(2 - \ln\frac{p^2}{\mu^2}\right) + \mathcal{O}\left(\epsilon\right)\end{aligned} \tag{7.58}$$

Conclusion

The sum of the four diagrams evaluated in Eq: 7.48, Eq: 7.54 and Eq: 7.57 together make up the gluon self-energy. The expansion in terms of ϵ for $p^2 \ll m_q^2$ gives us:

$$\begin{aligned}\Pi^{ab}\left(p^2\right) &= -\frac{\alpha_s}{2\pi}T\left(R\right)\delta^{ab}\frac{1}{\epsilon}\left(\frac{4}{3}n_f - \left(\frac{13}{3} - \xi_s\right)N_c\right) + \frac{\alpha_s}{3\pi}T\left(R\right)\left(n_f\left(-\frac{5}{3} + \ln\frac{p^2}{\mu^2}\right)\right. \\ &\quad \left. + \frac{1}{24}N_c\left(97 - 78\ln\frac{p^2}{\mu^2}\right) + \frac{3}{4}N_c\left(1 + \ln\frac{p^2}{\mu^2}\right)\xi_s + \frac{3}{8}N_c\xi_s^2\right) - \left(Z_3 - 1\right) + \mathcal{O}\left(\epsilon\right) \\ \Delta^{ab}\left(p^2\right) &= 0\end{aligned} \tag{7.59}$$

The divergent part of this result agrees with Ref. [17, ch. 73] for $\xi_s = 1$. We see that the longitudinal component to the self-energy that was introduced by the three-point gluon interaction loop in Eq: 7.54 and Eq: 7.55 is cancelled by the contribution of the ghost loop in Eq: 7.57 and Eq: 7.58. Therefore the introduction of the ghost fields that were necessary for the gauge-fixing procedure conserve the transversality of the gluon.

7.3 The Top-Quark Self-Energy

In this section, we compute the one-loop quantum corrections to the quark self-energy. We first determine the contributing diagrams and then evaluate them using the automated techniques from Sec: 7.1 in full dimensional regularisation and renormalisable gauge.

The diagrams that contribute to the top-quark self-energy at one-loop order involve besides the pure QCD graphs also a QED graph and various weak interaction graphs. Quarks couple directly to gluons, photons, W-bosons, Z-bosons, Higgs-bosons, Goldstone-bosons and quarks with different flavours.

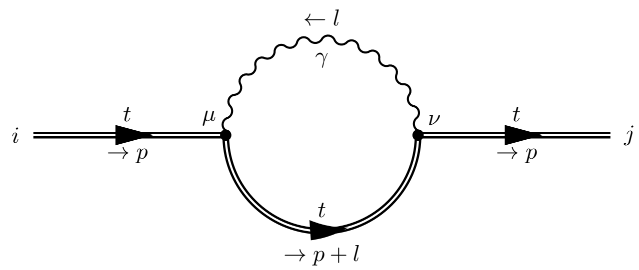


Figure 7.5: *The photon contribution to the top-quark self-energy.*

The photon contribution

We now apply the Feynman rules for the diagram in Fig: 7.5 as we have derived in Sec: 6.3. The electromagnetic charge of the top quark relative to the positron charge is denoted by $q_t = \frac{2}{3}$.

$$\begin{aligned} \mathrm{i}\Sigma_{ij}^{(1)}(\not p) =& \int \frac{d^4 l}{(2\pi)^4} \left(-\mathrm{i}q_t e\gamma^\nu\right) \left(\frac{1}{\mathrm{i}}\widetilde{S}_{ij}(\not p + \not l)\right) \left(-\mathrm{i}q_t e\gamma^\mu\right) \left(\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\mu\nu}(l)\right) \\ =& q_t^2 e^2 \widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \left[\gamma^\nu \widetilde{S}(\not p + \not l)\gamma^\mu\right] \widetilde{\Delta}_{\mu\nu}(l)\,\delta_{ij} \end{aligned} \tag{7.60}$$

By the automatic procedure as described in Sec: 7.1 we find the following results in Eq: 7.61 and Eq: 7.65. This result is the exact analogue of Eq: 3.51 and Eq: 3.62, which were obtained by hand.

$$\begin{aligned} A_{ij}^{(1)}\left(p^2\right) &= -\frac{q_t^2\alpha_e}{2\pi}\left(3 + \xi_{EW} - \epsilon\right) \frac{\Gamma\left(\frac{\epsilon}{2}\right) \mathrm{e}^{\frac{\gamma_E \epsilon}{2}}}{2-\epsilon} \left(\frac{m_t^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} F_{012}\left(m_t^2\right)\delta_{ij} \\ B_{ij}^{(1)}\left(p^2\right) &= -\frac{q_t^2\alpha_e}{\pi}\xi_{EW} \frac{\Gamma\left(\frac{\epsilon}{2}\right) \mathrm{e}^{\frac{\gamma_E \epsilon}{2}}}{4-\epsilon} \left(\frac{m_t^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} F_{023}\left(m_t^2\right)\delta_{ij} \end{aligned} \tag{7.61}$$

The expansion of Eq: 7.61 in terms of the parameter ϵ, which is required to find the values for the $\overline{MS}$ counter terms, is given by:

$$\begin{aligned} A_{ij}^{(1)}\left(p^2\right) =& -\frac{q_t^2\alpha_e}{2\pi}\left(3+\xi_{EW}\right)\delta_{ij}\frac{1}{\epsilon} - \frac{q_t^2\alpha_e}{2\pi}\Big[2 + \xi_{EW} - \frac{1}{2}\left(3+\xi_{EW}\right)\ln\frac{m_t^2}{\mu^2} \\ &+ \frac{1}{2}\left(3+\xi_{EW}\right)\ln\left(1+\frac{p^2}{m_t^2}\right)\Big]\delta_{ij} + \mathcal{O}(\epsilon) \\ B_{ij}^{(1)}\left(p^2\right) =& -\frac{q_t^2\alpha_e}{2\pi}\xi_{EW}\delta_{ij}\frac{1}{\epsilon} - \frac{q_t^2\alpha_e}{2\pi}\xi_{EW}\Big[1 - \frac{1}{2}\ln\frac{m_t^2}{\mu^2} - \frac{p^2+m_t^2}{p^2}\left[\frac{1}{2} + \ln\left(1+\frac{p^2}{m_t^2}\right)\right] \\ &+ \frac{1}{2}\left(\frac{p^2+m_t^2}{p^2}\right)\ln\left(1+\frac{p^2}{m_t^2}\right)\Big]\delta_{ij} + \mathcal{O}(\epsilon) \end{aligned} \tag{7.62}$$

The gluon contribution

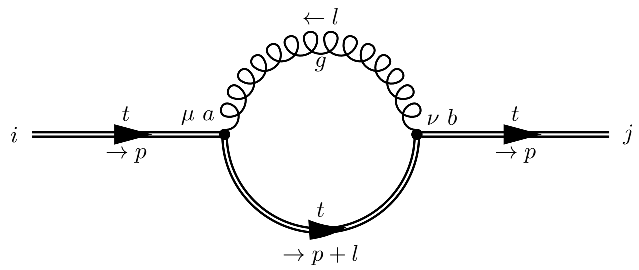


Figure 7.6: *The gluon contribution to the top-quark self-energy.*

We now apply the Feynman rules for the diagram in Fig: 7.6 as we have derived in Sec: 6.1, we find:

$$\begin{aligned} \mathrm{i}\Sigma^{(2)}_{ij}(\not{p}) &= \int \frac{d^4 l}{(2\pi)^4} \left(\mathrm{i}g_3 \gamma^\mu T^a_{ik}\right) \left(\frac{1}{\mathrm{i}} \widetilde{S}_{kl}(\not{p} + \not{l})\right) \left(\mathrm{i}g_3 \gamma^\nu T^b_{lj}\right) \left(\frac{1}{\mathrm{i}} \widetilde{\Delta}^{ab}_{\mu\nu}(l)\right) \\ &= g_3^2 \left(T^a_{ik} T^b_{kj}\right) \tilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \left[\gamma^\nu \widetilde{S}(\not{p} + \not{l}) \gamma^\mu\right] \widetilde{\Delta}^{ab}_{\mu\nu}(l) \end{aligned} \tag{7.63}$$

We can either use the automatic procedure from Sec: 7.1 or recognise that Eq: 7.63 only differs from Eq: 7.60 in the colour factors. Either way, we find:

$$\begin{aligned} A^{(2)}_{ij}\left(p^2\right) &= -\frac{\alpha_s C(R)}{2\pi} (3 + \xi_s - \epsilon) \frac{\Gamma\left(\frac{\epsilon}{2}\right) \mathrm{e}^{\frac{\gamma_E \epsilon}{2}}}{2 - \epsilon} \left(\frac{m_t^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} F_{012}\left(m_t^2\right) \delta_{ij} \\ B^{(2)}_{ij}\left(p^2\right) &= -\frac{\alpha_s C(R)}{\pi} \xi_s \frac{\Gamma\left(\frac{\epsilon}{2}\right) \mathrm{e}^{\frac{\gamma_E \epsilon}{2}}}{4 - \epsilon} \left(\frac{m_t^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} F_{023}\left(m_t^2\right) \delta_{ij} \end{aligned} \tag{7.64}$$

The expansion of Eq: 7.48 in terms of the parameter ϵ, which is required to find the values for the $\overline{MS}$ counter terms, is given by:

$$\begin{aligned} A^{(2)}_{ij}\left(p^2\right) =& -\frac{\alpha_s C(R)}{2\pi} (3 + \xi_s) \delta_{ij} \frac{1}{\epsilon} - \frac{\alpha_s C(R)}{2\pi} \Big[2 + \xi_s - \frac{1}{2}(3 + \xi_s) \ln \frac{m_t^2}{\mu^2} \\ & + \frac{1}{2}(3 + \xi_s) \ln\left(1 + \frac{p^2}{m_t^2}\right)\Big] \delta_{ij} + \mathcal{O}(\epsilon) \\ B^{(2)}_{ij}\left(p^2\right) =& -\frac{\alpha_s C(R)}{2\pi} \xi_s \delta_{ij} \frac{1}{\epsilon} - \frac{\alpha_s C(R)}{2\pi} \xi_{EW} \Big[1 - \frac{1}{2} \ln \frac{m_t^2}{\mu^2} - \frac{p^2 + m_t^2}{p^2} \left[\frac{1}{2} + \ln\left(1 + \frac{p^2}{m_t^2}\right)\right] \\ & + \frac{1}{2}\left(\frac{p^2 + m_t^2}{p^2}\right) \ln\left(1 + \frac{p^2}{m_t^2}\right)\Big] \delta_{ij} + \mathcal{O}(\epsilon) \end{aligned} \tag{7.65}$$

The W^+-boson/quark contribution

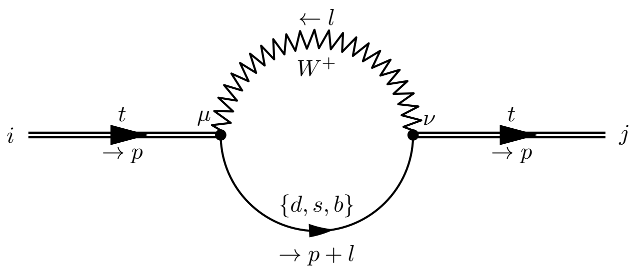


Figure 7.7: *The W^+/quark contribution to the top-quark self-energy. The quark in the loop can either be a down, strange or bottom quark.*

We now apply the Feynman rules for the diagram in Fig: 7.7 as we have derived in Sec: 6.2 and Sec: 6.3, we find:

$$\begin{aligned} \mathrm{i}\Sigma^{(3)}_{ij}(\not p) &= \sum_{q\in\{d,s,b\}} \int \frac{d^4 l}{(2\pi)^4} \left(\frac{\mathrm{i}g_2 V_{tq}}{\sqrt{2}}\gamma^\nu \frac{1-\gamma^5}{2}\right)\left(\frac{1}{\mathrm{i}}\widetilde{S}_{ij}(\not p + \not l)\right)\left(\frac{\mathrm{i}g_2 V^*_{tq}}{\sqrt{2}}\gamma^\mu \frac{1-\gamma^5}{2}\right)\left(\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\mu\nu}(l)\right) \\ &= \sum_{q\in\{d,s,b\}} \frac{\pi\alpha_W |V_{tq}|^2}{2}\widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d}\left[\gamma^\nu\left(1-\gamma^5\right)\widetilde{S}(\not p + \not l)\gamma^\mu\left(1-\gamma^5\right)\right]\widetilde{\Delta}_{\mu\nu}(l)\,\delta_{ij} \end{aligned} \tag{7.66}$$

We use the automatic procedure from Sec: 7.1 using the chiral decomposition, the only nonzero self-energy function is given by:

$$\begin{aligned} B^{(3)}_{R\;ij}\left(p^2\right) = &-\sum_{q\in\{d,s,b\}} \frac{\alpha_W |V_{tq}|^2}{16\pi}\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}\left(\frac{m_q^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\Bigg[(2-\epsilon)\,\mathcal{F}_{23}\left(m_q^2, m_W^2\right) \\ &+ \frac{4\left(1-\xi_{EW}^{1-\frac{\epsilon}{2}}\right)}{2-\epsilon}\left(\frac{m_q^2}{m_W^2}\right)^{\frac{\epsilon}{2}} - 4\frac{m_q^2}{m_W^2}\left(\mathcal{F}_{12}\left(m_q^2, m_W^2\right) - \mathcal{F}_{12}\left(m_q^2, \xi_{EW} m_W^2\right)\right) \\ &- \frac{p^2 - m_q^2}{m_W^2}\left(\mathcal{F}_{23}\left(m_q^2, m_W^2\right) - \mathcal{F}_{23}\left(m_q^2, \xi_{EW} m_W^2\right)\right)\Bigg]\delta_{ij} \end{aligned} \tag{7.67}$$

We define $D^2 \equiv x(1-x)\,p^2 + x\left(m_W^2 - m_q^2\right) + m_q^2$ and also $\widetilde{D}^2$ with $m_W^2 \to \xi_{EW} m_W^2$. The expansion of Eq: 7.54 in terms of the parameter ϵ, which is required to find the values for the $\overline{MS}$ counter terms, is given by:

$$
\begin{aligned}
B_{R}^{(3)}{}_{ij}\left(p^2\right) = &-\sum_{q\in\{d,s,b\}} \frac{\alpha_W|V_{tq}|^2}{4\pi}\left(2-\xi_{EW}\right)\delta_{ij}\frac{1}{\epsilon} + \sum_{q\in\{d,s,b\}} \frac{\alpha_W|V_{tq}|^2}{8\pi}\left[\xi_{EW}\left(1-\ln\left(\xi_{EW}\frac{m_W^2}{\mu^2}\right)\right)\right.\\
&+\ln\frac{m_W^2}{\mu^2} + 2\int_0^1 dx\, x \ln\frac{D^2}{\mu^2} - 2\frac{m_q^2}{m_W^2}\int_0^1 dx\left(\ln\frac{D^2}{\mu^2} - \ln\frac{\widetilde{D}^2}{\mu^2}\right)\\
&\left.-\frac{p^2-m_q^2}{m_W^2}\int_0^1 dx\, x\left(\ln\frac{D^2}{\mu^2} - \ln\frac{\widetilde{D}^2}{\mu^2}\right)\right]\delta_{ij} + \mathcal{O}\left(\epsilon\right)
\end{aligned}
\tag{7.68}
$$

The ϕ^+-boson/quark contribution

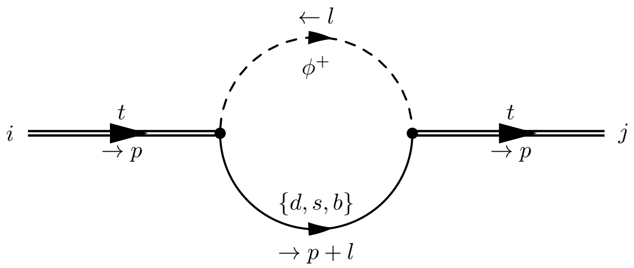


Figure 7.8: *The ϕ^+/quark contribution to the top-quark self-energy. The quark in the loop can either be a down, strange or bottom quark.*

We now apply the Feynman rules for the diagram in Fig: 7.8 as we have derived in Sec: 6.2 and Sec: 6.3, we find:

$$
\begin{aligned}
\mathrm{i}\Sigma_{ij}^{(4)}\left(\not{p}\right) = &\sum_{q\in\{d,s,b\}}\int\frac{d^4 l}{(2\pi)^4}\left(\frac{\mathrm{i}\sqrt{2}V_{tq}}{v}\left(m_q\frac{1-\gamma^5}{2} - m_t\frac{1+\gamma^5}{2}\right)\right)\left(\frac{1}{\mathrm{i}}\widetilde{S}_{ij}\left(\not{p}+\not{l}\right)\right)\\
&\times\left(\frac{\mathrm{i}\sqrt{2}V_{tq}^*}{v}\left(m_q\frac{1+\gamma^5}{2} - m_t\frac{1-\gamma^5}{2}\right)\right)\left(\frac{1}{\mathrm{i}}\widetilde{\Delta}\left(l\right)\right)\\
= &\sum_{q\in\{d,s,b\}}\pi\alpha_W|V_{tq}|^2\widetilde{\mu}^{\epsilon}\int\frac{d^d l}{(2\pi)^d}\left[\left(\frac{m_q}{m_W}\left(1+\gamma^5\right) - \frac{m_t}{m_W}\left(1-\gamma^5\right)\right)\widetilde{S}\left(\not{p}+\not{l}\right)\right.\\
&\left.\times\left(\frac{m_q}{m_W}\left(1-\gamma^5\right) - \frac{m_t}{m_W}\left(1+\gamma^5\right)\right)\right]\widetilde{\Delta}\left(l\right)\delta_{ij}
\end{aligned}
\tag{7.69}
$$

We use the automatic procedure from Sec: 7.1 using the chiral decomposition for the mass self-energy function and the non-chiral decomposition for the kinetic self-energy function:

$$
\begin{aligned}
A_R^{(4)}{}_{ij}\left(p^2\right) &= \sum_{q\in\{d,s,b\}} \frac{\alpha_W |V_{tq}|^2}{4\pi}\Gamma\left(\frac{\epsilon}{2}\right) \mathrm{e}^{\frac{\gamma_E \epsilon}{2}} \left(\frac{m_q^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} \frac{m_q m_t}{m_W^2}\mathcal{F}_{12}\left(m_q^2, \xi_{EW} m_W^2\right)\delta_{ij} \\
A_L^{(4)}{}_{ij}\left(p^2\right) &= \sum_{q\in\{d,s,b\}} \frac{\alpha_W |V_{tq}|^2}{4\pi}\Gamma\left(\frac{\epsilon}{2}\right) \mathrm{e}^{\frac{\gamma_E \epsilon}{2}} \left(\frac{m_q^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} \frac{m_q^3}{m_t m_W^2}\mathcal{F}_{12}\left(m_q^2, \xi_{EW} m_W^2\right)\delta_{ij} \quad (7.70)\\
B^{(4)}{}_{ij}\left(p^2\right) &= \sum_{q\in\{d,s,b\}} \frac{\alpha_W |V_{tq}|^2}{4\pi}\Gamma\left(\frac{\epsilon}{2}\right) \mathrm{e}^{\frac{\gamma_E \epsilon}{2}} \left(\frac{m_q^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} \frac{m_q m_t}{m_W^2}\mathcal{F}_{23}\left(m_q^2, \xi_{EW} m_W^2\right)\delta_{ij}
\end{aligned}
$$

We define $D^2 \equiv x(1-x)\,p^2 + x\left(\xi_{EW} m_W^2 - m_q^2\right) + m_q^2$. The expansion of Eq: 7.70 in terms of the parameter ϵ, which is required to find the values for the $\overline{MS}$ counter terms, is given by:

$$
\begin{aligned}
A_R^{(4)}{}_{ij}\left(p^2\right) &= \sum_{q\in\{d,s,b\}} \frac{\alpha_W |V_{tq}|^2}{2\pi}\frac{m_q m_t}{m_W^2}\delta_{ij}\frac{1}{\epsilon} - \sum_{q\in\{d,s,b\}} \frac{\alpha_W |V_{tq}|^2}{4\pi}\frac{m_q m_t}{m_W^2}\int_0^1 dx \ln\frac{D^2}{\mu^2}\delta_{ij} + \mathcal{O}(\epsilon) \\
A_L^{(4)}{}_{ij}\left(p^2\right) &= \sum_{q\in\{d,s,b\}} \frac{\alpha_W |V_{tq}|^2}{2\pi}\frac{m_q^3}{m_t m_W^2}\delta_{ij}\frac{1}{\epsilon} - \sum_{q\in\{d,s,b\}} \frac{\alpha_W |V_{tq}|^2}{4\pi}\frac{m_q^3}{m_t m_W^2}\int_0^1 dx \ln\frac{D^2}{\mu^2}\delta_{ij} + \mathcal{O}(\epsilon) \\
B^{(4)}{}_{ij}\left(p^2\right) &= \sum_{q\in\{d,s,b\}} \frac{\alpha_W |V_{tq}|^2}{2\pi}\frac{m_q m_t}{m_W^2}\delta_{ij}\frac{1}{\epsilon} - \sum_{q\in\{d,s,b\}} \frac{\alpha_W |V_{tq}|^2}{2\pi}\frac{m_q m_t}{m_W^2}\int_0^1 dx\, x \ln\frac{D^2}{\mu^2}\delta_{ij} + \mathcal{O}(\epsilon)
\end{aligned}
\quad (7.71)
$$

The Z^0-boson contribution

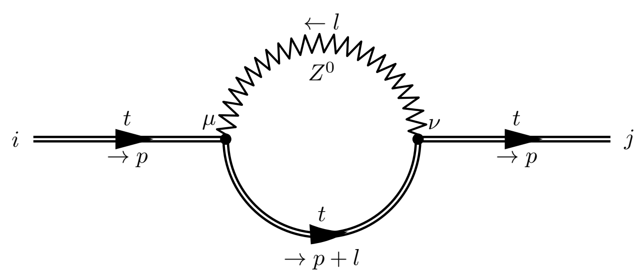


Figure 7.9: *The Z^0 contribution to the top-quark self-energy.*

We now apply the Feynman rules for the diagram in Fig: 7.9 as we have derived in Sec: 6.2 and Sec: 6.3, we find:

$$
\begin{aligned}
\mathrm{i}\Sigma^{(5)}_{ij}\left(\not p\right) = & \int \frac{d^4 l}{(2\pi)^4}\left(\frac{\mathrm{i}g_2}{\cos\theta_W}\gamma^{\nu}\left(\frac{1}{2}\left(\frac{1-\gamma^5}{2}\right)-q_t\sin^2\theta_W\right)\right)\left(\frac{1}{\mathrm{i}}\widetilde{S}_{ij}\left(\not p+\not l\right)\right)\\
& \times\left(\frac{\mathrm{i}g_2}{\cos\theta_W}\gamma^{\mu}\left(\frac{1}{2}\left(\frac{1-\gamma^5}{2}\right)-q_t\sin^2\theta_W\right)\right)\left(\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\mu\nu}\left(l\right)\right)\\
= & \frac{4\pi\alpha_W}{\cos^2\theta_W}\widetilde{\mu}^{\epsilon}\int\frac{d^d l}{(2\pi)^d}\left[\gamma^{\nu}\left(\frac{1}{2}\left(\frac{1-\gamma^5}{2}\right)-q_t\sin^2\theta_W\right)\widetilde{S}\left(\not p+\not l\right)\right.\\
& \left.\times\,\gamma^{\mu}\left(\frac{1}{2}\left(\frac{1-\gamma^5}{2}\right)-q_t\sin^2\theta_W\right)\right]\widetilde{\Delta}_{\mu\nu}\left(l\right)\delta_{ij}
\end{aligned}
\tag{7.72}
$$

We use the automatic procedure from Sec: 7.1 using the chiral decomposition for the kinetic self-energy function and the non-chiral decomposition for the mass self-energy function:

$$
\begin{aligned}
A^{(5)}{}_{ij}\left(p^2\right) = & \frac{q_t\alpha_W}{4\pi}\tan^2\theta_W\left(1-2q_t\sin^2\theta_W\right)\frac{\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{2-\epsilon}\left(\frac{m_Z^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(5-\xi_{EW}^{1-\frac{\epsilon}{2}}\right)\delta_{ij}\\
& +\frac{q_t\alpha_W}{8\pi}\tan^2\theta_W\left(1-2q_t\sin^2\theta_W\right)\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}\left(\frac{m_t^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left[(4-\epsilon)\mathcal{F}_{12}\left(m_t^2,m_Z^2\right)\right.\\
& +\frac{p^2-m_t^2}{m_Z^2}\left(\mathcal{F}_{12}\left(m_t^2,m_Z^2\right)-\mathcal{F}_{12}\left(m_t^2,\xi_{EW}m_Z^2\right)\right)-\frac{p^2}{m_Z^2}\left(\mathcal{F}_{23}\left(m_t^2,m_Z^2\right)\right.\\
& \left.\left.-\mathcal{F}_{23}\left(m_t^2,\xi_{EW}m_Z^2\right)\right)\right]\delta_{ij}\\
B_R^{(5)}{}_{ij}\left(p^2\right) = & -\frac{\alpha_W}{8\pi}\frac{\left(1-2q_t\sin^2\theta_W\right)^2}{\cos^2\theta_W}\frac{\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{2-\epsilon}\left(\frac{m_Z^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(1-\xi_{EW}^{1-\frac{\epsilon}{2}}\right)\delta_{ij}\\
& -\frac{\alpha_W}{32\pi}\frac{\left(1-2q_t\sin^2\theta_W\right)^2}{\cos^2\theta_W}\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}\left(\frac{m_t^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left[(2-\epsilon)\mathcal{F}_{23}\left(m_t^2,m_Z^2\right)\right.\\
& -4\frac{m_t^2}{m_Z^2}\left(\mathcal{F}_{12}\left(m_t^2,m_Z^2\right)-\mathcal{F}_{12}\left(m_t^2,\xi_{EW}m_Z^2\right)\right)-\frac{p^2-m_t^2}{m_Z^2}\left(\mathcal{F}_{23}\left(m_t^2,m_Z^2\right)\right.\\
& \left.\left.-\mathcal{F}_{23}\left(m_t^2,\xi_{EW}m_Z^2\right)\right)\right]\delta_{ij}\\
B_L^{(5)}{}_{ij}\left(p^2\right) = & -\frac{q_t^2\alpha_W}{2\pi}\sin^2\theta_W\tan^2\theta_W\frac{\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{2-\epsilon}\left(\frac{m_Z^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(1-\xi_{EW}^{1-\frac{\epsilon}{2}}\right)\delta_{ij}\\
& -\frac{q_t^2\alpha_W}{8\pi}\sin^2\theta_W\tan^2\theta_W\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}\left(\frac{m_t^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left[(2-\epsilon)\mathcal{F}_{23}\left(m_t^2,m_Z^2\right)\right.\\
& -4\frac{m_t^2}{m_Z^2}\left(\mathcal{F}_{12}\left(m_t^2,m_Z^2\right)-\mathcal{F}_{12}\left(m_t^2,\xi_{EW}m_Z^2\right)\right)-\frac{p^2-m_t^2}{m_Z^2}\left(\mathcal{F}_{23}\left(m_t^2,m_Z^2\right)\right.
\end{aligned}
$$

$$- \mathcal{F}_{23}\left(m_t^2, \xi_{EW} m_Z^2\right)\Big)\Big]\delta_{ij} \tag{7.73}$$

We define $D^2 \equiv x(1-x)p^2 + x\left(m_Z^2 - m_t^2\right) + m_t^2$ and also $\widetilde{D}^2$ with $m_Z^2 \to \xi_{EW} m_Z^2$. The expansion of Eq: 7.73 in terms of the parameter ϵ, which is required to find the values for the $\overline{MS}$ counter terms, is given by:

$$\begin{aligned}
A^{(5)}{}_{ij}\left(p^2\right) =& \frac{q_t\alpha_W}{4\pi}\tan^2\theta_W\left(1-2q_t\sin^2\theta_W\right)\left(1-\xi_{EW}\right)\delta_{ij}\frac{1}{\epsilon} - \frac{q_t\alpha_W}{8\pi}\tan^2\theta_W\left(1-2q_t\sin^2\theta_W\right)\\
&\times\left[2-\left(1-\xi_{EW}\right)\left(1-\ln\frac{m_Z^2}{\mu^2}\right)-\xi_{EW}\ln\xi_{EW}+4\int_0^1 dx\ln\frac{D^2}{\mu^2}\right.\\
&\left.+\frac{p^2-m_t^2}{m_Z^2}\int_0^1 dx\left(\ln\frac{D^2}{\mu^2}-\ln\frac{\widetilde{D}^2}{\mu^2}\right)-2\frac{p^2}{m_Z^2}\int_0^1 dx\,x\left(\ln\frac{D^2}{\mu^2}-\ln\frac{\widetilde{D}^2}{\mu^2}\right)\right]\delta_{ij}+\mathcal{O}\left(\epsilon\right)\\
B_R^{(5)}{}_{ij}\left(p^2\right) =& -\frac{\alpha_W}{8\pi}\frac{\left(1-2q_t\sin^2\theta_W\right)^2}{\cos^2\theta_W}\left(2-\xi_{EW}\right)\delta_{ij}\frac{1}{\epsilon}+\frac{\alpha_W}{16\pi}\frac{\left(1-2q_t\sin^2\theta_W\right)^2}{\cos^2\theta_W}\\
&\times\left[1-\left(1-\xi_{EW}\right)\left(1-\ln\frac{m_Z^2}{\mu^2}\right)-\xi_{EW}\ln\xi_{EW}+2\int_0^1 dx\,x\ln\frac{D^2}{\mu^2}\right.\\
&\left.-2\frac{m_t^2}{m_Z^2}\int_0^1 dx\left(\ln\frac{D^2}{\mu^2}-\ln\frac{\widetilde{D}^2}{\mu^2}\right)-\frac{p^2-m_t^2}{m_Z^2}\int_0^1 dx\,x\left(\ln\frac{D^2}{\mu^2}-\ln\frac{\widetilde{D}^2}{\mu^2}\right)\right]\delta_{ij}+\mathcal{O}\left(\epsilon\right)\\
B_L^{(5)}{}_{ij}\left(p^2\right) =& -\frac{q_t^2\alpha_W}{2\pi}\tan^2\theta_W\sin^2\theta_W\left(2-\xi_{EW}\right)\delta_{ij}\frac{1}{\epsilon}+\frac{q_t^2\alpha_W}{4\pi}\tan^2\theta_W\sin^2\theta_W\\
&\times\left[1-\left(1-\xi_{EW}\right)\left(1-\ln\frac{m_Z^2}{\mu^2}\right)-\xi_{EW}\ln\xi_{EW}+2\int_0^1 dx\,x\ln\frac{D^2}{\mu^2}\right.\\
&\left.-2\frac{m_t^2}{m_Z^2}\int_0^1 dx\left(\ln\frac{D^2}{\mu^2}-\ln\frac{\widetilde{D}^2}{\mu^2}\right)-\frac{p^2-m_t^2}{m_Z^2}\int_0^1 dx\,x\left(\ln\frac{D^2}{\mu^2}-\ln\frac{\widetilde{D}^2}{\mu^2}\right)\right]\delta_{ij}+\mathcal{O}\left(\epsilon\right)
\end{aligned} \tag{7.74}$$

The Higgs-boson contribution

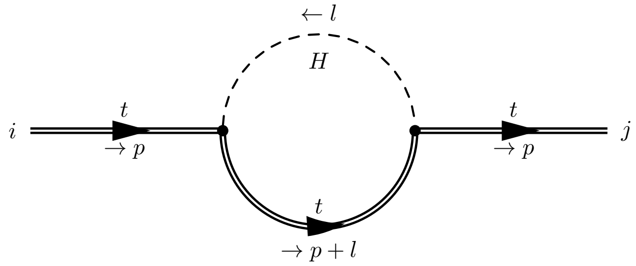


Figure 7.10: *The Higgs-boson contribution to the top-quark self-energy.*

We now apply the Feynman rules for the diagram in Fig: 7.10 as we have derived in Sec: 6.2 and Sec: 6.3, we find:

$$\begin{aligned} \mathrm{i}\Sigma_{ij}^{(6)}\left(\not p\right) =& \int \frac{d^4 l}{(2\pi)^4} \left(-\mathrm{i}\frac{m_t}{v}\right)\left(\frac{1}{\mathrm{i}}\widetilde{S}_{ij}\left(\not p + \not l\right)\right)\left(-\mathrm{i}\frac{m_t}{v}\right)\left(\frac{1}{\mathrm{i}}\widetilde{\Delta}\left(l\right)\right) \\ =& \pi\alpha_W \frac{m_t^2}{m_W^2}\widetilde{\mu}^{\epsilon}\int \frac{d^d l}{(2\pi)^d}\widetilde{S}\left(\not p + \not l\right)\widetilde{\Delta}\left(l\right)\delta_{ij} \end{aligned} \tag{7.75}$$

By the automatic procedure as described in Sec: 7.1 we find the following results in Eq: 7.76 and Eq: 7.77.

$$\begin{aligned} A_{ij}^{(6)}\left(p^2\right) &= \frac{\alpha_W}{16\pi}\frac{m_t^2}{m_W^2}\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}\mathcal{F}_{12}\left(m_t^2, m_H^2\right)\delta_{ij} \\ B_{ij}^{(6)}\left(p^2\right) &= -\frac{\alpha_W}{32\pi}\frac{m_t^2}{m_W^2}\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}\mathcal{F}_{23}\left(m_t^2, m_H^2\right)\delta_{ij} \end{aligned} \tag{7.76}$$

We define $D^2 \equiv x\left(1-x\right)p^2 + x\left(m_H^2 - m_t^2\right) + m_t^2$. The expansion of Eq: 7.76 in terms of the parameter ϵ, which is required to find the values for the $\overline{MS}$ counter terms, is given by:

$$\begin{aligned} A_{ij}^{(6)}\left(p^2\right) =& \frac{\alpha_W}{8\pi}\frac{m_t^2}{m_W^2}\delta_{ij}\frac{1}{\epsilon} - \frac{\alpha_W}{16\pi}\frac{m_t^2}{m_W^2}\int_0^1 dx \ln\frac{D^2}{\mu^2}\delta_{ij} + \mathcal{O}\left(\epsilon\right) \\ B_{ij}^{(6)}\left(p^2\right) =& -\frac{\alpha_W}{16\pi}\frac{m_t^2}{m_W^2}\delta_{ij}\frac{1}{\epsilon} + \frac{\alpha_W}{16\pi}\frac{m_t^2}{m_W^2}\int_0^1 dx\, x \ln\frac{D^2}{\mu^2}\delta_{ij} + \mathcal{O}\left(\epsilon\right) \end{aligned} \tag{7.77}$$

The ϕ_0-boson contribution

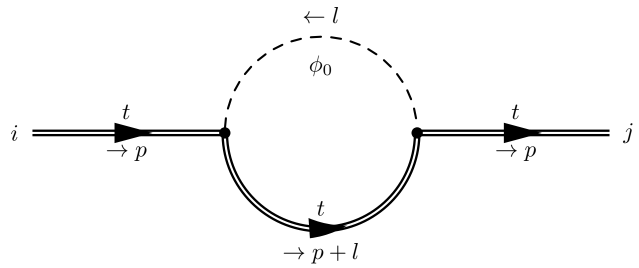


Figure 7.11: *The ϕ_0 contribution to the top-quark self-energy.*

We now apply the Feynman rules for the diagram in Fig: 7.11 as we have derived in Sec: 6.2 and Sec: 6.3, we find:

$$\begin{aligned} \mathrm{i}\Sigma_{ij}^{(7)}\left(\not{p}\right) &= \int \frac{d^4 l}{(2\pi)^4} \left(\frac{m_t}{v}\gamma^5\right) \left(\frac{1}{\mathrm{i}}\widetilde{S}_{ij}\left(\not{p}+\not{l}\right)\right) \left(\frac{m_t}{v}\gamma^5\right) \left(\frac{1}{\mathrm{i}}\widetilde{\Delta}\left(l\right)\right) \\ &= -\pi\alpha_W \frac{m_t^2}{m_W^2}\widetilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d} \left[\gamma^5 \widetilde{S}\left(\not{p}+\not{l}\right)\gamma^5\right] \widetilde{\Delta}\left(l\right)\delta_{ij} \end{aligned} \tag{7.78}$$

By the automatic procedure as described in Sec: 7.1 we find the following results in Eq: 7.79 and Eq: 7.80.

$$\begin{aligned} A_{ij}^{(7)}\left(p^2\right) &= -\frac{\alpha_W}{16\pi}\frac{m_t^2}{m_W^2}\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}\mathcal{F}_{12}\left(m_t^2, \xi_{EW}m_Z^2\right)\delta_{ij} \\ B_{ij}^{(7)}\left(p^2\right) &= -\frac{\alpha_W}{32\pi}\frac{m_t^2}{m_W^2}\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}\mathcal{F}_{23}\left(m_t^2, \xi_{EW}m_Z^2\right)\delta_{ij} \end{aligned} \tag{7.79}$$

We define $D^2 \equiv x\left(1-x\right)p^2 + x\left(\xi_{EW}m_Z^2 - m_t^2\right) + m_t^2$. The expansion of Eq: 7.79 in terms of the parameter ϵ, which is required to find the values for the $\overline{MS}$ counter terms, is given by:

$$\begin{aligned} A_{ij}^{(7)}\left(p^2\right) &= -\frac{\alpha_W}{8\pi}\frac{m_t^2}{m_W^2}\delta_{ij}\frac{1}{\epsilon} + \frac{\alpha_W}{16\pi}\frac{m_t^2}{m_W^2}\int_0^1 dx \ln\frac{D^2}{\mu^2}\delta_{ij} + \mathcal{O}\left(\epsilon\right) \\ B_{ij}^{(7)}\left(p^2\right) &= -\frac{\alpha_W}{16\pi}\frac{m_t^2}{m_W^2}\delta_{ij}\frac{1}{\epsilon} + \frac{\alpha_W}{16\pi}\frac{m_t^2}{m_W^2}\int_0^1 dx\, x \ln\frac{D^2}{\mu^2}\delta_{ij} + \mathcal{O}\left(\epsilon\right) \end{aligned} \tag{7.80}$$

Conclusion

The seven evaluated diagrams in Eq: 7.61, Eq: 7.64, Eq: 7.67, Eq: 7.70, Eq: 7.73, Eq: 7.76 and Eq: 7.79 together make up the top quark self-energy at the one-loop level. A change of parameters: $m_t \to m_q$, $q_t \to q_q$ and the component of V_{CKM} also allows us to obtain the full one-loop self-energy for the up and charm quark.

We could sum the seven diagrams that contribute to top quark self-energy but it yields us no exact cancellations or simplifications like the ones we encountered for the gluon. The cancellation of the divergent parts of $A_{ij}^{(6)}\left(p^2\right)$ and $A_{ij}^{(7)}\left(p^2\right)$ therefore seems to be only coincidental.

In Ch: 8, the W-boson contribution to the self-energy is discussed in relation to the weak decay of the top quark. Then in the subsequent chapters, Ch: 9 and Ch: 10, we discuss how a specific class of higher order contributions affect the top quark self-energy.

Chapter 8

The Weak Decay of the Top Quark

> In this chapter, we verify the optical theorem for the decay of the top quark into a W-boson and mainly a bottom but also a strange or down quark. We first explain how the optical theorem is a consequence of unitarity of the scattering matrix, before we compute the the leading order decay rate and self-energy contribution of the W-boson and quark. In this chapter, we use the unitary gauge to eliminate the contributions of charged Goldstone-bosons.

A general formula for the decay rate is derived in Ref. [17, ch. 11] and Ref. [18, ch. 4] upon assuming that the LSZ formula, which relates the scattering amplitudes to Green's functions, is valid also for particles that are not one-particle eigenstates of the hamiltonian. The decay rate in a general frame for a particle with momentum p and mass m in terms of the amplitude of the contributing diagrams that describe the decay of that particle to n other particles $\mathcal{M}_{1\to n}$, the Lorentz-invariant phase space measure for the outgoing particles denoted as $d\mathrm{LIPS}_n\left(p\right)$ and a symmetry factor $S = \prod_i n_i!$ to account for identical particles in the final state is given by:

$$\begin{aligned} \Gamma\left(p^2\right) &= \frac{1}{2SE\left(p^2\right)} \int \left|\mathcal{M}_{1\to n}\right|^2 d\mathrm{LIPS}_n\left(p\right) \\ \text{with } d\mathrm{LIPS}_n\left(p\right) &\equiv \left(2\pi\right)^4 \delta^4\left(p - \sum_i^n k_i\right) \prod_{j=1}^n \widetilde{dk_j} \end{aligned} \tag{8.1}$$

Here, the Lorentz invariant momentum measure is defined as $\widetilde{dk_j} \equiv d^3\mathbf{k}_j/\left(2\pi\right)^3 2k_j^0$. We write: $\Gamma_{CM} = \frac{E(p^2)}{m}\Gamma\left(p^2\right)$ to go to the centre-of-mass frame.

8.1 The optical theorem

The optical theorem is a nonlinear relation between amplitudes of a process $\mathcal{M}_{i\to f}$, where i denotes the initial state and f denotes the final state, and the amplitudes of the processes: $\mathcal{M}_{i\to n}$ and $\mathcal{M}_{n\to f}$, where n denote all possible intermediate states. We denote the total incoming and out going momenta by $k_i = k_f$, then the optical theorem can be stated as:

$$\mathrm{Im}\left[\mathcal{M}_{i\to f}\right] = \frac{1}{2S}\sum_n \int \mathcal{M}^*_{f\to n}\mathcal{M}_{i\to n} d\mathrm{LIPS}_n\left(k_i\right) \tag{8.2}$$

The optical theorem is a consequence of the unitarity of the scattering matrix or S-matrix, which relates the initial states $|i\rangle$ at time $T \to -\infty$ with the final states $|f\rangle$ at time $T \to +\infty$.

$$\begin{aligned} |f\rangle &= S^\dagger |i\rangle \\ |i\rangle &= S\,|f\rangle \end{aligned} \tag{8.3}$$

The initial state is normalised such that $\langle i\,|\,i\rangle = 1$, conservation of probability requires that also $\langle f\,|\,f\rangle = 1$. To ensure this, the S matrix must be unitary:

$$1 = \langle i\,|\,i\rangle = \langle f|\,S^\dagger S\,|f\rangle = \langle f\,|\,f\rangle = 1 \quad \Rightarrow S^\dagger S = I \tag{8.4}$$

In the absence of interactions the initial state and the final state are equal. This motivates the following decomposition in terms of the scattering matrix in terms of the identity matrix and a T-matrix, which contains the interactive part of the scattering process:

$$S = I + \mathrm{i}T \tag{8.5}$$

Using this decomposition we show that in accordance with [18, sec. 7.3] unitarity of the S-matrix results in a nonlinear constraint to the T-matrix:

$$I = S^\dagger S = I + \mathrm{i}\left(T - T^\dagger\right) + T^\dagger T \quad \Rightarrow \mathrm{i}\left(T - T^\dagger\right) = -T^\dagger T \tag{8.6}$$

When we express this relation in terms of the invariant amplitude $\mathcal{M}_{a\to b}$

and insert an identity element between T and $T^\dagger$ on the righthand side as the summation over all possible intermediate states n with momenta k_j we obtain Eq: 8.2. Note that $\langle b| T^\dagger |a\rangle = \langle a| T |b\rangle$, furthermore one needs:

$$(2\pi)^4 \delta^4 (k_a - k_b) \mathcal{M}_{a\to b} = \langle b| T |a\rangle \qquad I = \sum_n \int \prod_{j=1}^n \widetilde{dk_j} |n\rangle \langle n| \tag{8.7}$$

The optical theorem relates the decay rate to the imaginary part of the self-energy. If we choose the final and initial state in Eq: 8.2 to equal that of a propagating unstable particle *, then the left-hand side equals the imaginary part of the self-energy contracted with the initial and final polarisation vectors or spinors. The right-hand side becomes the decay rate in the centre of mass frame multiplied with the mass of the particle, see Eq: 8.1. For an unstable fermion, we find:

$$\mathrm{Im}\left[\overline{u}(p)\,\Sigma(\not{p})\,u(p)\right] = m\Gamma\left(p^2\right) \tag{8.8}$$

The above expression can be related to the definition of γ in Sec: 3.2. The relation that we find depends on the chiral structure of the self-energy function.

$$\begin{aligned} \gamma &= \Gamma_{CM}\mathbb{1}_4 \qquad && \text{non-chiral: } \Sigma(\not{p}) \\ \gamma &= 2\Gamma_{CM}P_{R,L} \qquad && \text{chiral: } \Sigma(\not{p}) \end{aligned} \tag{8.9}$$

8.2 The decay rate for $t \to bW$

In this section, we compute the physical decay rate for a top quark into a W-boson and mainly a bottom quark**. This is a strong and dominant decay channel because the top quark is so massive that the W-boson need not be off-shell such as in the decays of lighter quarks and leptons. In this tree-level calculation we consider both the incoming and outgoing particles to be on-shell, although this need not be the case.

The Feynman rules that we use here were derived in Sec: 6.2, they arise from the electroweak gauge coupling of the quark fields and are adjusted by the diagonalisation of the Yukawa couplings between the quark fields and the complex scalar Higgs doublet.

*Actually, unstable particles have no asymptotic states, though it is possible to define states that solve classical equations of motion

**The branching ratio of top quark decay into W-bosons and bottom quarks with respect to decay into a W-boson and a bottom, strange or down quark according to Ref. [4] is given by: $\frac{\Gamma(t\to bW)}{\Gamma(t\to qW)} = 0.957 \pm 0.034$ where $q \in \{d, s, b\}$.

The squared decay amplitude evaluated

The Feynman diagram that describes the $t \to bW$ decay is shown in Fig: 8.1. The corresponding amplitude is given by the vertex factor with polarisation spinors for the incoming top quark and the outgoing bottom quark and a polarisation vector for the W-boson.

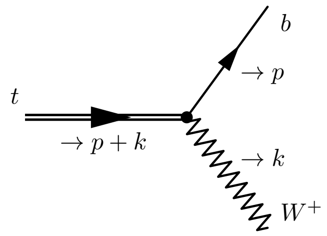


Figure 8.1: *The Feynman diagram that corresponds to the process of a top quark decaying into a bottom quark and a W^+-boson. The vertex factor is equal to:* $\frac{ig_2 V_{tb}^*}{\sqrt{2}} \gamma^\mu \left(\frac{1-\gamma^5}{2}\right)$

$$\mathcal{M} = \frac{ig_2 V_{tb}^*}{\sqrt{2}} \overline{u}_b(p) \overbrace{\gamma_\mu \left(\frac{1-\gamma^5}{2}\right)}^{\Gamma} u_t(p+k)\, \epsilon_\lambda^\mu(k) \tag{8.10}$$

We define $\left\langle |\mathcal{M}|^2 \right\rangle$ as the absolute squared amplitude averaged over the initial spins and summed over the final spins.

$$\begin{aligned} \left\langle |\mathcal{M}|^2 \right\rangle = \pi \alpha_W |V_{tb}|^2 \sum_{\text{spins}} & \left[\overline{u}_b(p)_\alpha \left(\Gamma_\mu\right)_{\alpha\beta} u_t(p+k)_\beta\right] \\ & \times \left[\overline{u}_t(p+k)_\rho \left(\gamma^0 \Gamma_\nu^\dagger \gamma^0\right)_{\rho\sigma} u_b(p)_\sigma\right] \epsilon_\lambda^{\nu*}(k)\, \epsilon_\lambda^\mu(k) \end{aligned} \tag{8.11}$$

We employ *Casimir's trick* e.g. Eq: A.8 and the equivalent formula for the W-boson in Eq: 8.12 to remove the explicit spinor and polarisation tensor dependencies in Eq: 8.11.

$$\sum_\lambda \epsilon_\lambda^{\mu*}(k)\, \epsilon_\lambda^\nu(k) = \eta^{\mu\nu} + \frac{k^\mu k^\nu}{m_W^2} \tag{8.12}$$

The replacement of the spinors and polarisation tensors leaves us with a trace over a string of gamma-matrices that we have performed using *FORM* [24].

$$\begin{aligned}\left\langle |\mathcal{M}|^2 \right\rangle =& \pi\alpha_W |V_{tb}|^2 Tr\left[\left(-\not{p}+m_b\right)\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\left(-\not{p}-\not{k}+m_t\right)\left(\frac{1+\gamma^5}{2}\right)\gamma^\nu\right]\left(\eta_{\mu\nu}+\frac{k_\mu k_\nu}{m_W^2}\right)\\ =& -2\pi\alpha_W|V_{tb}|^2\left(p\cdot(p+k)-2\frac{(k\cdot(p+k))\,(k\cdot p)}{m_W^2}\right)\end{aligned} \tag{8.13}$$

We use that all the incoming and outgoing particles are on-shell so $p^2 = -m_b^2$, $k^2 = -m_W^2$ and $(p+k)^2 = -m_t^2$ by momentum conservation so the final expression for $\langle |\mathcal{M}|^2 \rangle$ after some rearranging becomes:

$$\left\langle |\mathcal{M}|^2 \right\rangle = \pi\alpha_W |V_{tb}|^2 \frac{m_t^4}{m_W^2}\left[\left(1-\frac{m_W^2}{m_t^2}\right)\left(1+2\frac{m_W^2}{m_t^2}\right)-\frac{m_b^2}{m_t^2}\left(2-\frac{m_W^2}{m_t^2}-\frac{m_b^2}{m_t^2}\right)\right] \tag{8.14}$$

The decay rate evaluated

We use Eq: 8.1 in the CM-frame of the top quark such that $k_1 = (m_t, 0, 0, 0)$ to compute the decay rate with the squared amplitude as written in Eq: 8.14. We first expand the Lorentz-invariant phase space measure, then we use the spatial delta-functions to perform three integrations over the spatial momenta of the bottom quark and since we have no angular dependencies we also perform the angular integrations of the W-boson momenta, we find:

$$\begin{aligned}\Gamma_{CM} =& \frac{1}{2m_t}\int\left\langle |\mathcal{M}|^2\right\rangle d\text{LIPS}_n(k_1) = \frac{\langle|\mathcal{M}|^2\rangle}{32\pi^2 m_t}\int\frac{d^3k}{k^0}\frac{d^3p}{p^0}\delta\left(k^0+p^0-m_t\right)\delta^3(\mathbf{k}+\mathbf{p})\\ =& \frac{\langle|\mathcal{M}|^2\rangle}{32\pi^2 m_t}\int\frac{d^3k}{k^0p^0}\delta\left(k^0+p^0-m_t\right) = \frac{\langle|\mathcal{M}|^2\rangle}{8\pi m_t}\int d|\mathbf{k}|\frac{|\mathbf{k}|^2}{k^0p^0}\delta\left(k^0+p^0-m_t\right)\end{aligned} \tag{8.15}$$

In Eq: 8.15 the on-shell condition for the W-boson and bottom quark gives us: $k^0 = \sqrt{|\mathbf{k}|^2 + m_W^2}$ and $p^0 = \sqrt{|\mathbf{k}|^2 + m_b^2}$. The last integral over $|\mathbf{k}|$ is nonzero for only one specific value of $|\mathbf{k}|$ that conserves momentum but we must remember that integrating over this delta-function means that we must divide with the absolute value of the derivative of the argument with respect to the integration variable:

$$\int dx\,\delta(f(x)) = \sum_i \left|\frac{df(x_i)}{dx}\right|^{-1} \quad \text{with} \quad f(x_i) = 0 \tag{8.16}$$

The integration over the delta-function simplifies the expression for the decay rate significantly:

$$\frac{\partial}{\partial|\mathbf{k}|}\left(k^0+p^0-m_t\right)=\frac{|\mathbf{k}|\left(k^0+p^0\right)}{k^0p^0} \quad \Rightarrow \quad \Gamma=\frac{\langle|\mathcal{M}|^2\rangle}{8\pi m_t^2}|\mathbf{k}| \tag{8.17}$$

We solve: $\sqrt{|\mathbf{k}|^2+m_W^2}+\sqrt{|\mathbf{k}|^2+m_b^2}-m_t=0$ for $|\mathbf{k}|$ to find the final expression for the decay rate. At the solution for $|\mathbf{k}|^2$ the arguments of the roots in this equation must be the squares of something to cancel m_t, the solution is:

$$|\mathbf{k}|=\frac{1}{2m_t}\sqrt{m_t^4-2\left(m_W^2+m_b^2\right)m_t^2+\left(m_W^2-m_W^2\right)^2} \tag{8.18}$$

Now, we plug the solution for $|\mathbf{k}|$ and the full expression for $\langle|\mathcal{M}|^2\rangle$ into the expression we have for the decay rate and conclude*:

$$\begin{aligned}\Gamma_{CM}=&\frac{\alpha_W|V_{tb}|^2m_t^3}{16m_W^2}\left[\left(1-\frac{m_W^2}{m_t^2}\right)\left(1+2\frac{m_W^2}{m_t^2}\right)-\frac{m_b^2}{m_t^2}\left(2-\frac{m_W^2+m_b^2}{m_t^2}\right)\right]\\&\times\sqrt{1-2\frac{m_W^2+m_b^2}{m_t^2}+\frac{\left(m_W^2-m_b^2\right)^2}{m_t^4}}\end{aligned} \tag{8.19}$$

If we replace the CKM-matrix element V_{tb} and m_b by V_{ts} and m_s or V_{td} and m_d we obtain the decay rates of the top quark to strange and down quarks instead of bottom quarks. We mention the decays of the top quark to a down or a strange quark for completeness, although in practise they are significantly suppressed because $\frac{|V_{td}|^2}{|V_{tb}|^2}\sim 10^{-4}$ and $\frac{|V_{ts}|^2}{|V_{tb}|^2}\sim 10^{-3}$.

We take $\alpha_W=(3.392772\pm 0.000002)\cdot 10^{-2}$, $|V_{tb}|=1.021\pm 0.032$, $m_W=80.385\pm 0.015$ GeV, $m_b=4.18\pm 0.03$ GeV and $m_t=160^{+5}_{-4}$ GeV from Ref. [4], where m_t is the $\overline{MS}$ mass parameter. Since the strange and down quark contributions are negligible, the tree level result in Eq: 8.19 predicts for the full decay rate:

$$\Gamma_{CM}=(1.18\pm 0.14)\cdot\left(\frac{m_t}{160\text{ GeV}}\right)^{3.7}\text{ GeV} \tag{8.20}$$

The value for Γ_{CM} in Eq: 8.20 agrees well with $\Gamma_t=1.41^{+0.19}_{-0.15}$ GeV [4] from experimental fits. The given error is the error induced only by the un-

*If we are prepared to make a 3.8% error we can set $m_b^2=0$ (0.3%) and $|V_{tb}|^2=1$ (4.0%) and conclude $\Gamma_{CM}\approx\frac{\alpha_W m_t^3}{16m_W^2}\left(1-\frac{m_W^2}{m_t^2}\right)^2\left(1+2\frac{m_W^2}{m_t^2}\right)$ in agreement with [18, ch. 21].

certainties in the input parameters which is dominated by the error in the top quark mass and the CKM-matrix element, therefore it underestimates the true error.

Additional systematic errors are expected from:

- This is the tree-level result, quantum-loop corrections at higher-orders in perturbation theory have not been taken into account.
- The input parameters: α_W, m_t, m_b and m_W are renormalised parameters as a function of the renormalisation scale μ. The running of the coupling and the masses as a function of the renormalisation scale has not been taken into account.

8.3 Relating the decay rate to self-energy contributions

In this section, we present the calculation of the $\hat{W}^+$-boson and bottom quark contribution to the top quark self-energy in the unitary gauge $\xi_{EW} \to \infty$**. We re-evaluate the contributing diagram because it is not simply the limit of Eq: 7.68 for $\xi_{EW} \to \infty$. Afterwards, we compare the result with the decay rate that we have determined in the last section.

The evaluation of the diagram

We evaluate the diagram shown in Fig: 8.2, it has a bubble with a bottom quark and a W-boson. We use the unitary gauge $\xi_{EW} \to \infty$ to eliminate Goldstone-bosons.

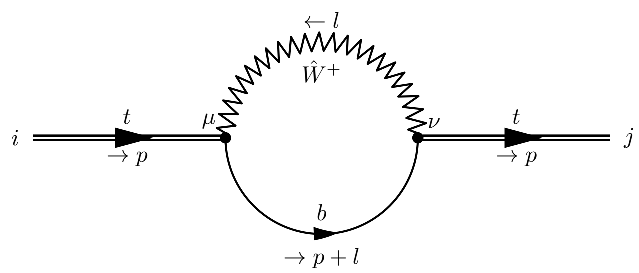


Figure 8.2: *The Feynman diagram that corresponds to the contribution of the $\hat{W}^+$-boson and the bottom quark at the one-loop level.*

**We denote the $\hat{W}^+$-boson in the unitary gauge $\xi_{EW} \to \infty$ with a hat.

We apply the Feynman rules for the diagram in Fig: 8.2 as we have derived in Sec: 6.2, with $\xi_{EW} \to \infty$. This yields Eq: 8.21 for the unitary $\hat{W}^+$-boson propagator, which does not fall of for large momenta as the renormalisable gauge W-boson propagator in Fig: 6.3(a), therefore we expect that the UV-divergences in the self-energy become worse.

$$\frac{1}{\mathrm{i}}\hat{\tilde{\Delta}}_{\mu\nu}(p) = \frac{-\mathrm{i}}{p^2+m_W^2}\left[\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m_W^2}\right] \tag{8.21}$$

In terms of this propagator, we find:

$$\begin{aligned} \mathrm{i}\hat{\Sigma}_{ij}(\not{p}) &= \int \frac{d^4 l}{(2\pi)^4}\left(\frac{\mathrm{i}g_2 V_{tb}}{\sqrt{2}}\gamma^\nu \frac{1-\gamma^5}{2}\right)\left(\frac{1}{\mathrm{i}}\tilde{S}_{ij}(\not{p}+\not{l})\right)\left(\frac{\mathrm{i}g_2 V_{tb}^*}{\sqrt{2}}\gamma^\mu \frac{1-\gamma^5}{2}\right)\left(\frac{1}{\mathrm{i}}\hat{\tilde{\Delta}}_{\mu\nu}(l)\right) \\ &= \frac{\pi\alpha_W |V_{tb}|^2}{2}\tilde{\mu}^\epsilon \int \frac{d^d l}{(2\pi)^d}\left[\gamma^\nu\left(1-\gamma^5\right)\tilde{S}(\not{p}+\not{l})\gamma^\mu\left(1-\gamma^5\right)\right]\hat{\tilde{\Delta}}_{\mu\nu}(l)\,\delta_{ij} \end{aligned} \tag{8.22}$$

We use the automatic procedure from Sec: 7.1 using the chiral decomposition, the only nonzero self-energy function is given by:

$$\begin{aligned} \hat{B}_{Rij}\left(p^2\right) =& \frac{\alpha_W |V_{tb}|^2}{4\pi}\frac{\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{2-\epsilon}\left(\frac{m_W^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\delta_{ij} + \frac{\alpha_W |V_{tb}|^2}{16\pi}\Gamma\left(\frac{\epsilon}{2}\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}\left(\frac{m_b^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} \\ &\times\left[4\frac{m_b^2}{m_W^2}\mathcal{F}_{12}\left(m_b^2, m_W^2\right) + \left(\frac{p^2-m_b^2}{m_W^2} - 2 + \epsilon\right)\mathcal{F}_{23}\left(m_b^2, m_W^2\right)\right]\delta_{ij} \end{aligned} \tag{8.23}$$

We expand Eq: 8.23 in terms of the parameter ϵ and define $D^2 = x(1-x)\,p^2 + x\left(m_W^2 - m_b^2\right) + m_b^2$:

$$\begin{aligned} \hat{B}_{Rij}\left(p^2\right) =& \frac{\alpha_W |V_{tb}|^2}{8\pi}\left(\frac{p^2+3m_b^2}{m_W^2}\right)\delta_{ij}\frac{1}{\epsilon} + \frac{\alpha_W |V_{tb}|^2}{8\pi}\left[2 - \ln\frac{m_W^2}{\mu^2} + \right. \\ &\left.\left(2 - \frac{p^2}{m_W^2}\right)\int_0^1 dx\, x \ln\frac{D^2}{\mu^2} - \frac{m_b^2}{m_W^2}\int_0^1 dx\,(2-x)\ln\frac{D^2}{\mu^2}\right]\delta_{ij} + \mathcal{O}(\epsilon) \end{aligned} \tag{8.24}$$

The expressions in Eq: 8.23 and Eq: 8.24 in the unitary gauge corresponds to the general gauge results in Eq: 7.67 and Eq: 7.68. In fact the gauge independent terms in Eq: 7.67 can all be found also in Eq: 8.23. One must put $(2-\xi_{EW}) \to -\xi_{EW}$ and $\tilde{D}^2 \to x\xi_{EW}\frac{m_W^2}{\mu^2}$ in Eq: 7.68 to see what roughly happens with the gauge dependent terms. Evaluating the Feynman integrals with these approximations that are valid for $\xi_{EW} \to \infty$ we recover the UV-divergent terms in Eq: 7.68 up to finite terms when we make the replacement:

$$\ln\left(\xi_{EW}\frac{m_W^2}{\mu^2}\right) \simeq \frac{2}{\epsilon} \tag{8.25}$$

The full self-energy function is given by:

$$\hat{\Sigma}_{ij}(\not{p}) = \left(\frac{1+\gamma^5}{2}\right)\not{p}\hat{B}_{Rij}\left(p^2\right) \tag{8.26}$$

The self-energy contribution of the W-boson and the bottom quark in Eq: 8.24 has not been renormalised, the real part is still divergent. We ignore this, and instead relate the imaginary part of the self-energy to the decay rate that was found earlier in Eq: 8.19 by using Eq: 8.8.

$$\mathrm{Im}\left[\overline{u}(p)\,\hat{\Sigma}(\not{p})\,u(p)\right]\Big|_{p^2=-m_t^2} = m_t\Gamma_{CM}(p) \tag{8.27}$$

We contract the spinors and gamma matrix structure by applying the spinor definitions and normalisation of Eq: A.6 and Eq: A.7 together with the anti-commutation relation of the γ^5 matrix of Eq: A.5.

$$\begin{aligned}\overline{u}(p)\left(\frac{1+\gamma^5}{2}\right)\not{p}u(p) =& \frac{1}{4}\left(\overline{u}(p)\not{p}\left(1-\gamma^5\right)u(p)+\overline{u}(p)\left(1+\gamma^5\right)\not{p}u(p)\right)\\ =&-\frac{m_t}{4}\left(\overline{u}(p)\left(1-\gamma^5\right)u(p)+\overline{u}(p)\left(1+\gamma^5\right)u(p)\right)\\ =&-\frac{m_t}{2}\overline{u}(p)\,u(p) = -m_t^2\end{aligned} \tag{8.28}$$

Since the spinor contraction yields a real constant, the imaginary part of Eq: 8.24 must reside in the integration over the logarithms. The complex-valued logarithm written in terms of its real and imaginary part is given by:

$$\ln x \equiv \ln|x| + \mathrm{i}\mathrm{Arg}x \tag{8.29}$$

The complex part is proportional to the phase of x, so when x becomes negative this part contributes a factor π. Thus we examine where in our integration domain the argument of the logarithm becomes negative.

$$\frac{D^2}{\mu^2}\Big|_{p^2=-m_t^2} = \frac{m_t^2}{\mu^2}x^2 + \left(\frac{m_W^2-m_b^2-m_t^2}{\mu^2}\right)x + \frac{m_b^2}{\mu^2} \leq 0 \tag{8.30}$$

When the argument of the logarithm becomes negative the phase of the argument is: $\mathrm{Arg}\frac{D^2}{\mu^2}\Big|_{p^2=-m_t^2} = -\pi$ for a branch cut on the negative axis since we absorbed an infinitesimal complex value in the masses; $m_W^2 \equiv$

$m_W^2 - \mathrm{i}\epsilon$ and $m_b^2 \equiv m_b^2 - \mathrm{i}\epsilon$. The roots of the argument of the logarithm are given by:

$$x_{\pm} = \frac{1}{2}\left(\overbrace{\left(1 - \frac{m_W^2 - m_b^2}{m_t^2}\right)}^{\alpha} \pm \overbrace{\sqrt{1 - 2\frac{m_W^2 + m_b^2}{m_t^2} + \frac{\left(m_W^2 - m_b^2\right)^2}{m_t^4}}}^{\beta}\right) \tag{8.31}$$

Both roots are real and in the integration range in our case since $m_t^2 \geq \left(m_W^2 + m_b^2\right)^2$ and $m_W^2 \gg m_b^2$. So we now have all that we need to calculate the imaginary part of the following integrals:

$$\mathrm{Im}\left[\int_0^1 dx \ln \frac{D^2}{\mu^2}\Big|_{p^2=-m_t^2}\right] = -\pi \int_{\frac{1}{2}(\alpha-\beta)}^{\frac{1}{2}(\alpha+\beta)} dx = -\pi\beta \tag{8.32}$$

$$\mathrm{Im}\left[\int_0^1 dx\, x \ln \frac{D^2}{\mu^2}\Big|_{p^2=-m_t^2}\right] = -\pi \int_{\frac{1}{2}(\alpha-\beta)}^{\frac{1}{2}(\alpha+\beta)} dx\, x = -\frac{\pi\alpha\beta}{2} \tag{8.33}$$

We substitute the result of the above integrals and the spinor contraction in Eq: 8.27 and find:

$$\begin{aligned} \mathrm{Im}\left[\overline{u}\,(p)\,\hat{\Sigma}\,(\not{p})\,u\,(p)\right]\Big|_{p^2=-m_t^2} &= \frac{\alpha_W |V_{tb}|^2 m_t^2}{16}\left(\left(2 + \frac{m_t^2}{m_W^2}\right)\alpha - \frac{m_b^2}{m_W^2}\,(4-\alpha)\right)\beta \\ &= \frac{\alpha_W |V_{tb}|^2 m_t^4}{16 m_W^2}\left[\left(1 - \frac{m_W^2}{m_t^2}\right)\left(1 + 2\frac{m_W^2}{m_t^2}\right) - \frac{m_b^2}{m_t^2}\left(2 - \frac{m_W^2 + m_b^2}{m_t^2}\right)\right]\beta \end{aligned} \tag{8.34}$$

When we compare the expression above with Eq: 8.19, then we find the verification of Eq: 8.27. This is the generalisation to spin-½ of $\mathrm{Im}\left[\Pi\left(p^2\right)\right]\Big|_{p^2=-m_t^2} = m_t \Gamma_{CM}$, where $\Pi\left(p^2\right)$ is the scalar particle's self-energy, which is worked out in Ref. [17, ch. 25]. This was also verified for the top quark decay into a W-boson and a bottom quark in Ref. [32]. We also see that although $\hat{\Sigma}\,(\not{p})$ depends on the renormalisation scale μ, this μ-dependence disappears in physical observables such as the decay rate Γ_{CM}.

Chapter 9

The Definition of Renormalons

In this chapter, we introduce renormalons and demonstrate their presence in the all-order summation of QED bubble chain diagrams. We define and apply the Borel resummation technique to resummate and construct an analytical representation of this class of diagrams.
The definition of renormalons in terms of quark bubble graphs in QCD would neglect the equally sized contributions of gluon and ghost bubbles. We explore how the definition of renormalons in QED is extended to QCD and leads to a prescription that is usable to compute the renormalon contribution to a diagram.

Up until now, we studied one-loop contributions to the top quark self-energy. These calculations can be extended to two-loop and three-loop orders but the computations become increasingly more difficult and soon incalculable. A different approach is to compute the first few orders and then estimate the higher order contribution.

Some gauge-invariant subsets of the higher-order corrections in QED were found, that with some approximations are computable at arbitrary loop-order in perturbation theory. [33] These subsets are the bubble-chain diagrams, wherein a photon propagator is dressed by an arbitrary number of fermion bubbles. It was also found that these diagrams posses $n!$-growth, which induces divergent behaviour of the perturbative expansion.

These perturbative divergences with a characteristic $n!$-growing series expansion in renormalisable field theories such as QED but also in QCD are known as renormalons. Renormalons also significantly affect the top quark self-energy, which is why we consider renormalons here, and how they affect the top quark mass in Ch: 10. A more extensive study of renormalons in different physical processes is found in Ref. [34].

9.1 The renormalon prescription in QED

We first discuss the definition of renormalons in QED before treating the more intricate definition of renormalons in QCD. Unlike for QCD it is possible in QED to identify an explicit gauge invariant set of diagrams that displays renormalon behaviour, namely the bubble-chain diagrams. The renormalon contribution to a diagram in QED such as an electron self-energy diagram or a electron-electron scattering diagram is found by replacing the photon propagator by the photon exact propagator.

Here, we shall take a closer look at the renormalon contribution to the electron self-energy, the diagram in Fig: 9.1(a), since this diagram is directly connected to the definition of the electron pole mass through Eq: 3.30 or more practically Eq: 3.32.

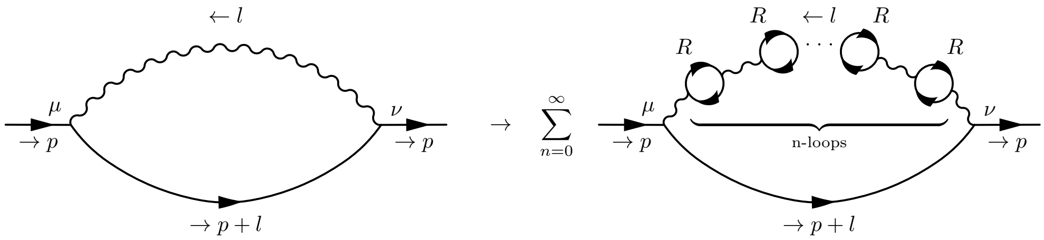
When we apply the Feynman rules, then we find the following expression for the diagram in Fig: 9.1(a) which contains the photon one-loop exact propagator:

$$\begin{aligned} \mathrm{i}\Sigma_{\widetilde{r}}(\not{p}) =& \sum_{n=0}^{\infty}\int\frac{d^4l}{(2\pi)^4}\left(\mathrm{i}e\gamma^{\nu}\right)\left(\frac{1}{\mathrm{i}}\widetilde{S}(\not{p}+\not{l})\right)\left(\mathrm{i}e\gamma^{\mu}\right)\left(\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\mu\rho}(l)\right)\left[\left(\mathrm{i}\Pi^{\rho\sigma}_{\overline{MS}}(l)\right)\left(\frac{1}{\mathrm{i}}\widetilde{\Delta}_{\sigma\nu}(l)\right)\right]^n \\ =& e^2\widetilde{\mu}^{\epsilon}\int\frac{d^dl}{(2\pi)^d}\left[\gamma^{\nu}\widetilde{S}(\not{p}+\not{l})\gamma^{\mu}\right]\left(\overbrace{\sum_{n=0}^{\infty}\frac{1}{l^2}P_{\mu\nu}(l)\,\Pi^n_{\overline{MS}}\left(l^2\right)+\xi_{EW}\frac{l_\mu l_\nu}{l^4}}^{\widetilde{\mathbf{\Delta}}_{\mu\nu}(l)}\right) \end{aligned} \tag{9.1}$$

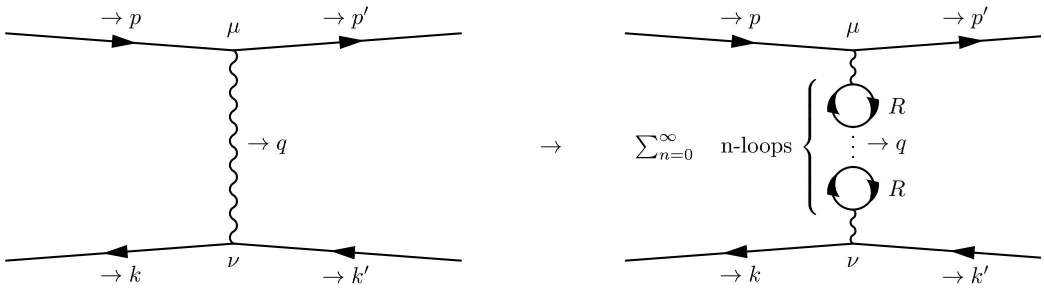
We make the approximation $\mu^2 \gg m^2$ to get a more tractable expression for the photon self-energy function in the above expression then the one given in Eq: 4.2. We also recall that $\beta_0 = \frac{2}{3\pi}$ from Eq: 4.71 and make the definition $C = -\frac{5}{3}$, we find:

$$\Pi_{\overline{MS}}\left(p^2\right) \approx \frac{\beta_0\alpha_e}{2}\ln\left(\frac{p^2}{\mu^2}\mathrm{e}^C\right) + \mathcal{O}\left(\alpha_e^2\right) \tag{9.2}$$

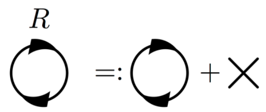
The renormalon contribution to a QED diagram is thus obtained by the exact propagator prescription which adds a factor of $\sum_{n=0}^{\infty}\Pi_{\overline{MS}}\left(l^2\right)$ to the integrand of Eq: 9.1. This addition corresponds to a gauge invariant subset of the quantum corrections to the self-energy which exhibits $n!$-growth, as



(a) The electron one-loop self-energy diagram.



(b) The tree-level contribution to the electron-electron scattering.



(c) The electron bubble as the sum of an unrenormalised electron bubble and the counter term.

Figure 9.1: *This figure shows the prescription which yields renormalons in the shown two QED diagrams. In this prescription, one replaces the photon propagator by the photon exact propagator, renormalisation of the electron bubble insertions has thus already been performed.*

we will show. The factor e^C is renormalisation scheme dependent and only of subleading interest, exhibiting at most $(n-1)!$-growth.

In QED there exist a different prescription that yields up to the subleading factor e^C the same renormalon behaviour. In violation with Feynman rules we then make the replacement $\alpha_e(\mu) \to \alpha_e\left(l^2\right)$ in the integrand, where $\alpha_e\left(l^2\right)$ or the running coupling is the solution of Eq: 4.71, given by:

$$\alpha_e\left(l^2\right) = \frac{\alpha_e(\mu)}{1 - \frac{\beta_0 \alpha_e(\mu)}{2} \ln \frac{l^2}{\mu^2}} = \alpha_e(\mu) \sum_{n=0}^{\infty} \left(\frac{\beta_0 \alpha_e(\mu)}{2}\right)^n \ln^n \frac{l^2}{\mu^2} \tag{9.3}$$

The running coupling prescription adds a factor $\sum_{n=0}^{\infty} \Pi_{\overline{MS}}\left(l^2 \mathrm{e}^{-C}\right)$ to the integrand of Eq: 9.1. This addition thus corresponds to the same gauge-invariant subset of the quantum corrections to the self-energy as for the exact propagator prescription.

9.2 The $n!$-growth of the individual diagrams

Here, we explicitly show the $n!$-growth of the the individual n-bubble diagrams $\Sigma_{n,\widetilde{r}}\left(\not p\right)$ as shown in Fig: 9.1(a). For the moment we are just interested in the leading order behaviour and therefore we set the external momentum p to be on-shell as a simplification and assume $n > 0$, so that we do not need to consider the gauge-dependent part of the photon exact propagator which only contributes to the $n = 0$ contribution.

We follow the techniques described in Sec: 7.1 up until *STEP 4* using the fact that all relations for the J_2 integrals except for analytic continuations work in the same way for integrals that have an additional factor of $\sim \ln^n \frac{p^2}{\mu^2}$. We then set $p^2 = -m^2$ and find:

$$A_{n,\widetilde{r}}\left(p^2\right) = \overbrace{\mathrm{i}e^2\left(d-1\right)\left(\frac{\beta_0\alpha_e}{2}\right)^n \widetilde{\mu}^{\epsilon}\int\frac{d^dl}{\left(2\pi\right)^d}\frac{\ln^n\left(\frac{l^2}{\mu^2}\mathrm{e}^C\right)}{\left(l^2+2p\cdot l\right)l^2}}^{\boxed{a}} \tag{9.4}$$

$$B_{n,\widetilde{r}}\left(p^2\right) = \overbrace{\mathrm{i}e^2\left(\frac{\beta_0\alpha_e}{2}\right)^n \widetilde{\mu}^{\epsilon}\int\frac{d^dl}{\left(2\pi\right)^d}\frac{\left(\left(d-3\right)-\left(d-2\right)\frac{p\cdot l}{m^2}\right)\ln^n\left(\frac{l^2}{\mu^2}\mathrm{e}^C\right)}{\left(l^2+2p\cdot l\right)l^2}}^{\boxed{b}} \tag{9.5}$$

We need to apply the following trick to evaluate the integrals in a and b:

$$\ln^n x = \left(\frac{\partial}{\partial\delta}\right)^n x^{\delta}\Big|_{\delta=0} \tag{9.6}$$

We apply this trick to a, after which we introduce Feynman parameters, perform the momentum integral and recognise the integral representation of the Euler beta function.

$$\begin{aligned}\boxed{a} =& \mathrm{i}\left(d-1\right)e^2\widetilde{\mu}^{\epsilon}\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\left(\frac{\mathrm{e}^C}{\mu^2}\right)^{\delta}\int\frac{d^dl}{\left(2\pi\right)^d}\frac{1}{\left(l^2+2p\cdot l\right)\left(l^2\right)^{1-\delta}}\Bigg|_{\delta=0}\\ =& -\left(3-\epsilon\right)\frac{\alpha_e}{4\pi}\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\frac{\Gamma\left(\frac{\epsilon}{2}-\delta\right)\Gamma\left(1+2\delta-\epsilon\right)}{\Gamma\left(2+\delta-\epsilon\right)}\left(\frac{m^2}{\mu^2}\right)^{-\left(\frac{\epsilon}{2}-\delta\right)}\mathrm{e}^{\frac{\gamma_E\epsilon}{2}+\delta C}\Bigg|_{\delta=0}\end{aligned} \tag{9.7}$$

We now use the duplication formula Eq: 6.1.17 from Ref. [30] on $\Gamma\left(1+2\delta-\epsilon\right)$ in $\boxed{a}$, followed by the reflection formula Eq: 6.1.18 from Ref. [30] on $\Gamma\left(\frac{\epsilon}{2}-\delta\right)\Gamma\left(1+\delta-\frac{\epsilon}{2}\right)$. This brings $\boxed{a}$ in the following form:

$$\begin{aligned}\boxed{a} = &-\left(3-\epsilon\right)\frac{\alpha_e}{4\pi}\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\left(\frac{\pi}{\sin\left(\pi\left(\frac{\epsilon}{2}-\delta\right)\right)}\right)\\ &\times\left(\frac{e^{\frac{\gamma_E\epsilon}{2}+\delta C-2\left(\frac{\epsilon}{2}-\delta\right)\ln 2}\Gamma\left(\frac{1}{2}+\delta-\frac{\epsilon}{2}\right)}{\sqrt{\pi}\Gamma\left(2+\delta-\epsilon\right)}\right)\left(\frac{m^2}{\mu^2}\right)^{-\left(\frac{\epsilon}{2}-\delta\right)}\Bigg|_{\delta=0}\end{aligned} \tag{9.8}$$

We continue the evaluation with $\boxed{b}$ in the same manner:

$$\begin{aligned}\boxed{b} =& ie^2\tilde{\mu}^\epsilon\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\left(\frac{e^C}{\mu^2}\right)^\delta\int\frac{d^dl}{\left(2\pi\right)^d}\frac{\left(d-3\right)-\left(d-2\right)\frac{p\cdot l}{m^2}}{\left(l^2+2p\cdot l\right)\left(l^2\right)^{1-\delta}}\Bigg|_{\delta=0}\\ =&\frac{\alpha_e}{4\pi}\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\left(1-\left(2-\epsilon\right)\frac{1-\delta}{2+\delta-\epsilon}\right)\\ &\times\frac{\Gamma\left(\frac{\epsilon}{2}-\delta\right)\Gamma\left(1+2\delta-\epsilon\right)}{\Gamma\left(2+\delta-\epsilon\right)}\left(\frac{m^2}{\mu^2}\right)^{-\left(\frac{\epsilon}{2}-\delta\right)}e^{\frac{\gamma_E\epsilon}{2}+\delta C}\Bigg|_{\delta=0}\end{aligned} \tag{9.9}$$

We again apply the duplication and reflection formulas from Ref. [30] on the Γ-functions in $\boxed{b}$ and find:

$$\begin{aligned}\boxed{b} =&\frac{\alpha_e}{4\pi}\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\left(1-\left(2-\epsilon\right)\frac{1-\delta}{2+\delta-\epsilon}\right)\left(\frac{\pi}{\sin\left(\pi\left(\frac{\epsilon}{2}-\delta\right)\right)}\right)\\ &\times\left(\frac{e^{\frac{\gamma_E\epsilon}{2}+\delta C-2\left(\frac{\epsilon}{2}-\delta\right)\ln 2}\Gamma\left(\frac{1}{2}+\delta-\frac{\epsilon}{2}\right)}{\sqrt{\pi}\Gamma\left(2+\delta-\epsilon\right)}\right)\left(\frac{m^2}{\mu^2}\right)^{-\left(\frac{\epsilon}{2}-\delta\right)}\Bigg|_{\delta=0}\end{aligned} \tag{9.10}$$

To find the leading order behaviour of the self-energy functions $A_{n,\tilde{r}}\left(p^2\right)$ and $B_{n,\tilde{r}}\left(p^2\right)$ we neglect terms of order $\mathcal{O}\left(\epsilon,\delta\right)$.

$$\begin{aligned}A_{n,\tilde{r}}\left(p^2\right) =& -\left(3+\mathcal{O}\left(\delta,\epsilon\right)\right)\frac{\alpha_e}{4\pi}\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\left(\frac{1}{\frac{\epsilon}{2}-\delta}+\mathcal{O}\left(\delta,\epsilon\right)\right)\left(1+\mathcal{O}\left(\delta,\epsilon\right)\right)\left(\frac{m^2}{\mu^2}\right)^{-\left(\frac{\epsilon}{2}-\delta\right)}\Bigg|_{\delta=0}\\ =& -\frac{3\alpha_e}{4\pi}\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\left(\frac{1}{\frac{\epsilon}{2}-\delta}+\mathcal{O}\left(\delta^0,\epsilon^0\right)\right)\left(\frac{m^2}{\mu^2}\right)^{-\left(\frac{\epsilon}{2}-\delta\right)}\Bigg|_{\delta=0}\end{aligned} \tag{9.11}$$

$$
\begin{aligned}
B_{n,\tilde{r}}\left(p^2\right) =& \frac{\alpha_e}{4\pi}\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\left(\frac{3}{2}\delta+\mathcal{O}\left(\delta^2,\epsilon^2,\delta\epsilon\right)\right)\left(\frac{1}{\frac{\epsilon}{2}-\delta}+\mathcal{O}\left(\alpha,\epsilon\right)\right)\\
&\times\left(1+\mathcal{O}\left(\alpha,\epsilon\right)\right)\left(\frac{m^2}{\mu^2}\right)^{-\left(\frac{\epsilon}{2}-\delta\right)}\Bigg|_{\delta=0}\\
=&\frac{3\alpha_e}{8\pi}\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\left(\frac{\delta}{\frac{\epsilon}{2}-\delta}+\mathcal{O}\left(\delta,\epsilon\right)\right)\left(\frac{m^2}{\mu^2}\right)^{-\left(\frac{\epsilon}{2}-\delta\right)}\Bigg|_{\delta=0}
\end{aligned}
\tag{9.12}
$$

We now expand $\left(\frac{m^2}{\mu^2}\right)^{-\left(\frac{\epsilon}{2}-\delta\right)}$ in terms of $\left(\frac{\epsilon}{2}-\delta\right)$ and perform the derivative to find the dominant contribution to the self-energy functions.

$$
\begin{aligned}
&\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\left(\frac{1}{\frac{\epsilon}{2}-\delta}+\mathcal{O}\left(\delta^0,\epsilon^0\right)\right)\left(\frac{m^2}{\mu^2}\right)^{-\left(\frac{\epsilon}{2}-\delta\right)}\Bigg|_{\delta=0}\\
&\quad=\left(\frac{\beta_0\alpha_e}{2}\right)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\partial}{\partial\delta}\right)^n\left(\left(\frac{\epsilon}{2}-\delta\right)^{k-1}\ln^k\frac{m^2}{\mu^2}+\mathcal{O}\left(\left(\frac{\epsilon}{2}-\delta\right)^k\ln^k\frac{m^2}{\mu^2}\right)\right)\\
&\quad=n!\left(\frac{\beta_0\alpha_e}{\epsilon}\right)^n\left(\frac{2}{\epsilon}\right)+\frac{1}{n+1}\left(\frac{\beta_0\alpha_e}{2}\right)^n\ln^{n+1}\frac{m^2}{\mu^2}+\mathcal{O}\left(\left(\frac{2}{\epsilon}\right)^n,\ln^n\frac{m^2}{\mu^2}\right)
\end{aligned}
\tag{9.13}
$$

In Eq: 9.13, the regularisation parameter ϵ is such that $\left|\frac{2}{\epsilon}\right| \gg \left|\ln\frac{m^2}{\mu^2}\right|$. For finite and non-zero values of $\frac{m^2}{\mu^2}$ we can thus neglect the mass dependent term. Nevertheless, we include the mass dependent term since for either $\frac{m^2}{\mu^2} \gg 1$ or $\frac{m^2}{\mu^2} \ll 1$ the highest power of $\ln\frac{m^2}{\mu^2}$ is the leading order finite term that survives after the $\left(\frac{2}{\epsilon}\right)^k$ terms are cancelled by the counter terms through renormalisation.

$$
\begin{aligned}
&\left(\frac{\beta_0\alpha_e}{2}\frac{\partial}{\partial\delta}\right)^n\left(\frac{\delta}{\frac{\epsilon}{2}-\delta}+\mathcal{O}\left(\delta,\epsilon\right)\right)\left(\frac{m^2}{\mu^2}\right)^{-\left(\frac{\epsilon}{2}-\delta\right)}\Bigg|_{\delta=0}\\
&\quad=-\left(\frac{\beta_0\alpha_e}{2}\right)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{\partial}{\partial\delta}\right)^n\Bigg(\left(\frac{\epsilon}{2}-\delta\right)^k\ln^k\frac{m^2}{\mu^2}-\frac{\epsilon}{2}\left(\frac{\epsilon}{2}-\delta\right)^{k-1}\ln^k\frac{m^2}{\mu^2}\\
&\qquad+\mathcal{O}\left(\{\delta,\epsilon\}\left(\frac{\epsilon}{2}-\delta\right)^k\ln^k\frac{m^2}{\mu^2}\right)\Bigg)\\
&\quad=n!\left(\frac{\beta_0\alpha_e}{\epsilon}\right)^n-\left(\frac{\beta_0\alpha_e}{2}\right)^n\ln^n\frac{m^2}{\mu^2}+\mathcal{O}\left(\left(\frac{2}{\epsilon}\right)^{n-1},\ln^{n-1}\frac{m^2}{\mu^2}\right)
\end{aligned}
\tag{9.14}
$$

In Eq: 9.14, the mass dependent term cancels the contribution from the

UV-divergence for $n = 0$. The term is also not suppressed by a factor $n+1$ as in Eq: 9.13. Substituting Eq: 9.13 and Eq: 9.14 into the self-energy functions $A_{n,\tilde{r}}\left(p^2\right)$ and $B_{n,\tilde{r}}\left(p^2\right)$, we find:

$$A_{n,\tilde{r}}\left(p^2\right)\Big|_{n\geq 0} = -\frac{9}{4}n!\left(\frac{\beta_0\alpha_e}{\epsilon}\right)^{n+1} + \mathcal{O}\left(\left(\frac{2}{\epsilon}\right)^{n}\right) \tag{9.15}$$

$$B_{n,\tilde{r}}\left(p^2\right)\Bigg|_{n>0} = \frac{9}{8}n!\left(\frac{\beta_0\alpha_e}{\epsilon}\right)^{n+1}\frac{\epsilon}{2} + \mathcal{O}\left(\left(\frac{2}{\epsilon}\right)^{n-1}\right) \tag{9.16}$$

The leading order term that appears in the self-energy functions indeed grows factorially with the number of electron loops. The divergent feature of this class of diagrams compels us to include all of them in a perturbative treatment to make meaningful predictions, the smallness of the coupling constant does not counter the factorial growth of the number of electron loops.

In the general case of a $n!$-growing perturbative series of the following form:

$$R = \sum_{n=0}^{\infty} r_n = \sum_{n=0}^{\infty} n!\alpha^n c_n \tag{9.17}$$

In the above expression, α is the coupling constant and c_n is an arbitrary constant that does not depend heavily on the value of n. If we assume $c_n \sim c_{n+1}$, then at a certain order in perturbation theory one reaches the smallest correction, this happens when $|r_{n+1}| > |r_n|$:

$$\frac{|r_{n+1}|}{|r_n|} \geq 1 \quad \Rightarrow \quad (n+1)\,\alpha\,\overbrace{\frac{|c_{n+1}|}{|c_n|}}^{\sim 1} \geq 1 \quad \Rightarrow \quad n \geq \frac{1}{\alpha} - 1 \tag{9.18}$$

After renormalisation at the same order as the number of electron loops the terms proportional to $\left(\frac{2}{\epsilon}\right)^k$ are cancelled by counter term contributions and the factorial growth is hidden in this particular class of diagrams. One thus usually considers the Adler function in the discussion of renormalons such as in Ref. [34].

9.3 The Borel transform

The method of choice to handle factorial growth is the Borel resummation technique. By using the *Borel transform*, it is possible to define the representation of the exact photon propagator in Borel space, that can be used as a Feynman rule in Borel space. Analogous to Ref. [34] we define the *Borel transform* of a series as:

$$R = \pm \sum_{n=0}^{\infty} r_n \alpha^{n+1} \Rightarrow B[R](t) = \sum_{n=0}^{\infty} r_n \frac{t^n}{n!} \quad \text{for sgn}(\alpha) = \pm 1 \tag{9.19}$$

The original series R might not be convergent but the Borel transform often is, the $n!$-divergent behaviour has been regularised through this transform. We use the ratio test to compute $\mathcal{R}_R$, the radius of convergence for the original series R. If the limit exist then we find:

$$\mathcal{R}_R = \lim_{n\to\infty} \left| \frac{r_n}{r_{n+1}} \right| \quad \Rightarrow \quad \mathcal{R}_{B(R)} = \lim_{n\to\infty} \left| \frac{r_n (n+1)!}{r_{n+1} n!} \right| = \mathcal{R}_R \overbrace{\left(\lim_{n\to\infty} (n+1) \right)}^{\infty} \tag{9.20}$$

We thus find that the Borel transform of a series R with a finite radius of convergence $\mathcal{R}_R$ has an infinite radius of convergence. If the original series is asymptotic with a radius of convergence $\mathcal{R}_R = 0$ and $|r_n| \sim n! A^n$ for $n \to \infty$ then the radius of convergence of the Borel transform is finite $\mathcal{R}_{B(R)} \geq \frac{1}{A}$.

We also define the *Borel integral* $\widetilde{R}$ which is the inverse Borel transform. This analytical representation of R has the same series expansion as the original series R. It is important to note that the sign of α dictates what part of the Borel transform is relevant for R or $\widetilde{R}$.

$$\widetilde{R} = \begin{cases} \int_0^{\infty} dt\, e^{-\frac{t}{|\alpha|}} B[R](t) & , \alpha > 0 \\ \int_{-\infty}^{0} dt\, e^{\frac{t}{|\alpha|}} B[R](t) & , \alpha < 0 \end{cases} \tag{9.21}$$

With the above formalism we now compute the Borel transform of the leading-order self-energy functions $A_{n,\widetilde{r}}\left(p^2\right)$ from Eq: 9.15 and $B_{n,\widetilde{r}}\left(p^2\right)$ from Eq: 9.16 choosing $\alpha = \frac{\beta_0 \alpha_e}{2}$, we find:

$$B\left[\sum_{n=0}^{\infty} A_{n,\widetilde{r}}\left(p^2\right)\right](t) \approx -\frac{9}{4}\frac{1}{\epsilon - t} \sim -\frac{9}{4}\frac{1}{(-t)} \tag{9.22}$$

$$B\left[\sum_{n=1}^{\infty} B_{n,\widetilde{r}}\left(p^2\right)\right](t) \approx \frac{9}{16}\frac{t}{\epsilon - t} \sim \frac{9}{16}\frac{t}{(-t)} \tag{9.23}$$

The expression in Eq: 9.22 and Eq: 9.23 shows us that we have a divergence at $t = 0$ which dominates the summation before renormalisation. This divergence corresponds to the $\left(\frac{2}{\epsilon}\right)^k$ terms in both Eq: 9.15 and Eq: 9.16 and is thus related to the UV-behaviour of the diagrams, hence a UV

renormalon. The Borel parameter now regulates the divergence and provides us with an analytical representation.

We construct the Borel transform of the gauge-independent part of the exact photon propagator to determine the sub-leading renormalons in the self-energy functions, we find:

$$R = \sum_{n=0}^{\infty} \Pi^{n}_{\overline{MS}}\left(l^2\right) \;\Rightarrow\; B\left[R\right](t) = \frac{2}{\beta_0 \alpha_e}\left(\mathrm{e}^{C}\frac{l^2}{\mu^2}\right)^{t} \tag{9.24}$$

$$\widetilde{R} = \frac{2}{\beta_0 \alpha_e}\int_0^{\infty} dt\, \mathrm{e}^{-\frac{2t}{\beta_0 \alpha_e}}\left(\mathrm{e}^{C}\frac{l^2}{\mu^2}\right)^{t} \tag{9.25}$$

This leads us to define the following Feynman rule for the exact photon propagator in Borel space:

$$B\left[\widetilde{\mathbf{\Delta}}_{\mu\nu}(l)\right](t) = \frac{2}{\beta_0 \alpha_e}\left[\left(\frac{\mathrm{e}^{C}}{\mu^2}\right)^{t}\frac{1}{\left(l^2\right)^{1-t}}P_{\mu\nu}(l) + \frac{\xi_{EW} l_\mu l_\nu}{l^4}\right] \tag{9.26}$$

The Borel resummation of the one-loop exact photon propagator in term of the inverse Borel transform of Eq: 9.26 does not invalidate the Dyson resummated expression in Eq: 3.4. However, the Borel resummated expression allows the evaluation of the momentum integral and regulates the collective divergence of the all-order summation in terms of the Borel parameter t.

9.4 The exact evaluation of the renormalon diagram

We now perform the exact evaluation of $\Sigma_{\widetilde{r}}(\not{p})$ as shown in Eq: 9.1 using the exact photon propagator in Borel space as shown in Eq: 9.26. This calculation can be done with the procedure in Sec: 7.1 upon making a small redefinition of $J_s(\alpha)$ and deriving new relations to handle the t dependence of the hypergeometric functions.

The gauge-dependence in $\Sigma_{\widetilde{r}}(\not{p})$ comes from the $n = 0$ term in the summation of Eq: 9.1. The calculation of the $n = 0$ term has already been performed in Sec: 3.2 for general values of ξ_{EW}, so we can set $\xi_{EW} = 0$ for now and add the gauge-dependent part of $A\left(p^2\right)$ in Eq: 3.51 and $B\left(p^2\right)$ in Eq: 3.62 afterwards, if necessary. We find that the $n > 0$ terms constitute a gauge-independent contribution to the renormalon diagram.

The regularisation parameter ϵ, that we use to dimensionally regularise our integrals is not strictly necessary when we calculate the Borel transforms of $A_{\widetilde{r}}\left(p^2\right)\Big|_{\xi_{EW}=0}$ and $B_{\widetilde{r}}\left(p^2\right)\Big|_{\xi_{EW}=0}$. We set $\epsilon = 0$ and treat the Borel parameter t as our regularisation parameter. We also renormalise in terms of the Borel parameter. To remain in the $\overline{MS}$ renormalisation scheme we need to insert two scheme-dependent polynomials $R_A(t)$ and $R_B(t)$, the explicit form of which is not important for the discussion of the singularities according to Ref. [16].

The redefinition of $J_s(\alpha)$ which is more suited to this particular calculation reads for $D_0 \equiv l^2$ and $D_1 \equiv (p+l)^2 + m^2$:

$$\widetilde{J}_s(t,\alpha) \equiv \int \widetilde{\mu}^{\epsilon} \frac{d^d l}{(2\pi)^d} \frac{p \cdot l}{D_0^{1-t} D_1^{\alpha}} \tag{9.27}$$

$$J_2(-t,\alpha-t) = J_2(1-t,\alpha-t) - 2\widetilde{J}_s(t,\alpha) - \left(p^2+m^2\right) J_2(1-t,\alpha) \tag{9.28}$$

The evaluation of $\widetilde{J}_s(t,\alpha)$ in terms of elementary and special functions is given by:

$$\begin{aligned}\frac{1}{p^2}\widetilde{J}_s(t,\alpha) = -\frac{\mathrm{i}}{16\pi^2} & \frac{\Gamma\left(\frac{\epsilon}{2}-t+\alpha-1\right)\Gamma\left(2-\frac{\epsilon}{2}+t\right)\mathrm{e}^{\frac{\gamma_E\epsilon}{2}}}{\Gamma(\alpha)\Gamma\left(3-\frac{\epsilon}{2}\right)} \\ & \times {}_2F_1\left(\frac{\epsilon}{2}-t+\alpha-1\,,\,1-t\,;\,3-\frac{\epsilon}{2}\,;\,-\frac{p^2}{m^2}\right)\left(\frac{m^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\left(m^2\right)^{1+t-\alpha}\end{aligned} \tag{9.29}$$

We express our result in terms of the following hypergeometric functions that are regularised with the Borel parameter and where the dimensional regularisation parameter ϵ has been set to zero (in short-hand notation):

$$\widetilde{F}_{012}(t) \equiv {}_2F_1\left(-t\,,\,1-t\,;\,2\,;\,-\frac{p^2}{m^2}\right) \tag{9.30}$$

$$\widetilde{F}_{023}(t) \equiv {}_2F_1\left(-t\,,\,2-t\,;\,3\,;\,-\frac{p^2}{m^2}\right) \tag{9.31}$$

By applying Eq: 15.2.14, Eq: 15.2.17 and Eq: 15.2.19 from Ref. [30, ch. 15] we find expressions for the following hypergeometric functions:

$${}_2F_1\left(-t\,,\,1-t\,;\,3\,;\,-\frac{p^2}{m^2}\right) = \frac{1}{1+t}\left(2\widetilde{F}_{012}(t) - (1-t)\,\widetilde{F}_{023}(t)\right) \tag{9.32}$$

$${}_2F_1\left(1-t\,,\,1-t\,;\,3\,;\,-\frac{p^2}{m^2}\right) = \frac{1}{t(1+t)}\left(2\widetilde{F}_{012}(t) - (2+t)\,\widetilde{F}_{023}(t)\right) \tag{9.33}$$

$$\left(1+\frac{p^2}{m^2}\right){}_2F_1\left(1-t\,,\,1-t\,;\,3\,;\,-\frac{p^2}{m^2}\right) = -\frac{1}{t}\left(2\widetilde{F}_{012}(t) + (2+t)(1-t)\,\widetilde{F}_{023}(t)\right) \tag{9.34}$$

When we take into account the above modifications, we find the following result by the procedure in Sec: 7.1:

$$B\left[A_{\widetilde{r}}\left(p^2\right)\Big|_{\xi_{EW}=0}\right](t) = -\frac{3}{2\pi\beta_0}\left(\mathrm{e}^C\frac{m^2}{\mu^2}\right)^t \Gamma(-t)\,\Gamma(1+t) \tag{9.35}$$
$$\times\, {}_2F_1\left(-t\,,\,1-t\,,\,2,-\frac{p^2}{m^2}\right) - \frac{1}{2\pi\beta_0}\left[\frac{3}{t} - R_A(t)\right]$$

$$B\left[B_{\widetilde{r}}\left(p^2\right)\Big|_{\xi_{EW}=0}\right](t) = \frac{3}{4\pi\beta_0}\left(\mathrm{e}^C\frac{m^2}{\mu^2}\right)^t t\,\Gamma(-t)\,\Gamma(1+t) \tag{9.36}$$
$$\times\, {}_2F_1\left(-t\,,\,2-t\,,\,3,-\frac{p^2}{m^2}\right) + \frac{1}{2\pi\beta_0}R_B(t)$$

The second term in $B\left[A_{\widetilde{r}}\left(p^2\right)\Big|_{\xi_{EW}=0}\right](t)$ of Eq: 9.35 and Eq: 9.36 has been included to renormalises the gauge-independent part of the Borel transform of the self-energy function at $t \sim 0$ in the $\overline{MS}$ renormalisation scheme. It equals the gauge-independent part of the Borel transform of $(Z_0 - 1)$.

The location of renormalons

We have evaluated the self-energy contribution in QED of the whole class of diagrams shown in Fig: 9.1(a). We obtained Eq: 9.35 and Eq: 9.36 with the decomposition $\Sigma_{\widetilde{r}}(\not{p}) = mA_{\widetilde{r}}\left(p^2\right) + \not{p}B_{\widetilde{r}}\left(p^2\right)$ which is the QED analogue of the result in Eq: 3.9 of Ref. [16] for QCD.

The hypergeometric functions ${}_2F_1\left(-t\,,\,1-t\,,\,2,-\frac{p^2}{m^2}\right)$ and ${}_2F_1\left(-t\,,\,2-t\,,\,3,-\frac{p^2}{m^2}\right)$ absolutely converge for $t > -\frac{1}{2}$ and $|p^2| < m^2$ but there is a pole at $t = -\frac{1}{2}$ according to Ref. [30]. This is not the whole story, the Γ-functions introduce additional poles on every positive integer $t > 0$ whenever the hypergeometric function is nonzero.

The pole at $t = -\frac{1}{2}$ is an infrared renormalons and does not trouble us in QED since the inverse Borel transform integration domain is between $t = 0$ and $t = +\infty$. The poles for positive integers are the UV-renormalons that do affect QED. The inverse Borel transform can not be performed without a modification of the integration contour and therefore the perturbation series is not Borel-summable, as concluded also in the original work by Ref. [33]. This signals us that the QED perturbative series is only an asymptotic series as explained in App: D.

9.5 The renormalon prescription in QCD

In this section, we evaluate the QED renormalon prescriptions in QCD. We demonstrate that the exact propagator prescription does not constitute a gauge-invariant set of diagrams in QCD. This motivates the use of the running coupling prescription in QCD which is by its definition gauge-invariant.

The exact propagator prescription

When we use the exact propagator prescription, we first need to determine the $\overline{MS}$ self-energy function analogue of Eq: 9.2 by use of the result in Eq: 7.59 for $m_q \sim 0$. It is possible to bring the result in a similar form but it is no longer possible to relate the pre-factor to the β_0 coefficient.

$$\begin{gathered}\Pi^{ab}_{\overline{MS}}\left(p^2\right) \approx \frac{\alpha_s T\left(R\right)}{3\pi}\left(a + b\ln\frac{p^2}{\mu^2}\right)\delta^{ab} \approx \frac{\alpha_s T\left(R\right)}{3\pi} b\ln\left(\mathrm{e}^{\frac{a}{b}}\frac{p^2}{\mu^2}\right)\delta^{ab} \\ a \equiv -\frac{5}{3}n_f + \left(\frac{11}{3} + \frac{3}{8}\left(1+\xi_s\right)^2\right)N_c \quad , \quad b \equiv n_f - \left(\frac{5}{2} + \frac{3}{4}\left(1-\xi_s\right)\right)N_c\end{gathered} \tag{9.37}$$

Unlike for what we saw for QED, we note that the one-loop corrections that contribute to the exact gluon propagator in QCD are not gauge-invariant. This has the consequence that it is possible to have either $b < 0$, $b = 0$ or $b > 0$, therefore this analysis does not allow us to say how renormalons in QCD affect physical observables.

Regardless of whether $b > 0$ or $b < 0$ we construct the Borel transform of the power series of the self-energy function to be:

$$R = \sum_{n=0}^{\infty} \Pi^{n}_{\overline{MS}}\left(l^2\right) \;\Rightarrow\; B\left[R\right]\left(t\right) = \frac{3\pi}{\alpha_s T\left(R\right)\left|b\right|}\left(\mathrm{e}^{\frac{a}{b}}\frac{l^2}{\mu^2}\right)^t \tag{9.38}$$

The sign of b determines what the integration domain of the inverse Borel transform is:

$$\widetilde{R} = \begin{cases} \frac{3\pi}{\alpha_s T(R)|b|}\int_0^{\infty} dt\, \mathrm{e}^{-\frac{3\pi t}{\alpha_s T(R)|b|}}\left(\mathrm{e}^{\frac{a}{b}}\frac{l^2}{\mu^2}\right)^t & ,b > 0 \\ \frac{3\pi}{\alpha_s T(R)|b|}\int_{-\infty}^{0} dt\, \mathrm{e}^{+\frac{3\pi t}{\alpha_s T(R)|b|}}\left(\mathrm{e}^{\frac{a}{b}}\frac{l^2}{\mu^2}\right)^t & ,b < 0 \end{cases} \tag{9.39}$$

$$B\left[\widetilde{\Delta}^{ab}_{\mu\nu}\left(l\right)\right]\left(t\right) = \frac{3\pi}{\alpha_s T\left(R\right)\left|b\right|}\left[\left(\frac{\mathrm{e}^{\frac{a}{b}}}{\mu^2}\right)^t \frac{1}{\left(l^2\right)^{1-t}}P_{\mu\nu}\left(l\right) + \frac{\xi_s l_\mu l_\nu}{l^4}\right]\delta^{ab} \tag{9.40}$$

We obtain the Borel transform of the QCD equivalent of the diagram shown in Fig: 9.1(a) by making the replacements $C \to \frac{a}{b}$, $\alpha_e \to \alpha_s$, $\xi_{EW} \to \xi_s$ and $\beta_0 \to \frac{2T(R)}{3\pi}|b|$ in Eq: 9.35 and Eq: 9.36.

The location of infrared and ultraviolet renormalons in the Borel plane now depends on the value of the gauge parameter ξ_s through b and a. It is not even clear whether we are affected by either IR or UV renormalons. The exact propagator prescription does not yield a gauge invariant set of diagrams and therefore has no well-defined physical interpretation.

The running coupling prescription

We use the analogue of Eq: 9.3 in QCD for $\alpha_s(\mu)$ with a $\beta_0 < 0$ * to apply the running coupling prescription $\alpha_s(\mu) \to \alpha_s(l^2)$ in the evaluation of the QCD analogue of Fig: 9.1(a).

$$\alpha_s\left(l^2\right) = \alpha_s(\mu)\frac{1}{1 + \frac{|\beta_0|\alpha_s(\mu)}{2}\ln\frac{l^2}{\mu^2}} = \alpha_s(\mu)\sum_{n=0}^{\infty}\left(-\frac{|\beta_0|\alpha_s(\mu)}{2}\right)^n \ln^n\frac{l^2}{\mu^2} \tag{9.41}$$

The momentum-scale at which the running-coupling diverges is Λ_{QCD}, it relates to both $\alpha_s(\mu)$ and β_0:

$$1 = \frac{|\beta_0|\alpha_s(\mu)}{2}\ln\frac{\mu^2}{\Lambda^2_{QCD}} \quad \Rightarrow \quad \Lambda_{QCD} = \mu\, e^{-\frac{1}{|\beta_0|\alpha_s(\mu)}} \tag{9.42}$$

The numerical value for the infrared scale of divergence lies at $\Lambda_{QCD} \approx (200-300)$ MeV [16]. The running coupling can be conveniently rewritten in terms of Λ_{QCD}:

$$\alpha_s\left(l^2\right) = \frac{2}{|\beta_0|\ln\frac{l^2}{\Lambda^2_{QCD}}} \tag{9.43}$$

The Feynman rule in Borel space that enforces the running coupling prescription for a gluon exchange between two quarks, is given by:

$$B\left[\widetilde{\boldsymbol{\Delta}}^{ab}_{\mu\nu}(l)\right](t) = \frac{2}{|\beta_0|\alpha_s(\mu)}\left[\left(\frac{l^2}{\mu^2}\right)^t \frac{1}{l^2}P_{\mu\nu}(l) + \frac{\xi_s l_\mu l_\nu}{l^4}\right]\delta^{ab} \tag{9.44}$$

The renormalon contribution to the top quark self-energy diagram in Eq: 7.64 with the above identification is given by an analogue of Eq: 9.35 and Eq: 9.36. We replace $\alpha_e(\mu) \to \alpha_s(\mu)$, $\xi_{EW} \to \xi_s$, $\beta_0^{(QED)} \to |\beta_0^{(QCD)}|$,

*In QCD $\beta_0 = -\frac{1}{6\pi}\left(11N_c - 4n_f T(R)\right)$ acc. to Ref. [17, ch. 73] which is negative for $n_F \leq 16$ when $N_c = 3$ and $T(R) = \frac{1}{2}$.

$C \to 1$, $m^2 \to m_t^2$ and integrate the Borel parameter t over the negative axis, since $\beta_0^{(QCD)} < 0$. Additionally, we multiply with an overall colour factor of $C(R)$ in accordance with the one-loop result in Eq: 7.64, we find:

$$B\left[A_{\widetilde{r}}\left(p^2\right)\Big|_{\xi_s=0}\right](t) = -\frac{3C(R)}{2\pi|\beta_0|}\left(\frac{m_t^2}{\mu^2}\right)^t \Gamma(-t)\,\Gamma(1+t) \times {}_2F_1\left(-t\,,\,1-t\,,\,2,-\frac{p^2}{m_t^2}\right) - \frac{C(R)}{2\pi|\beta_0|}\left[\frac{3}{t} - R_A(t)\right] \tag{9.45}$$

$$B\left[B_{\widetilde{r}}\left(p^2\right)\Big|_{\xi_s=0}\right](t) = \frac{3C(R)}{4\pi|\beta_0|}\left(\frac{m_t^2}{\mu^2}\right)^t t\,\Gamma(-t)\,\Gamma(1+t) \times {}_2F_1\left(-t\,,\,2-t\,,\,3,-\frac{p^2}{m_t^2}\right) + \frac{C(R)}{2\pi|\beta_0|}R_B(t) \tag{9.46}$$

We observe that the inverse Borel transform of Eq: 9.45 and Eq: 9.46 has the same t-dependence as the QED result and therefore the leading order contribution to the series expansion exhibits the same $n!$-growth as Eq: 9.15 and Eq: 9.16. Also the β_0 coefficient in QCD is a gauge-invariant quantity as it was in QED, which means that the renormalons in this prescription are not affected by an arbitrary choice of gauge. We thus verify that the running coupling prescription in QCD captures the characteristic $n!$-growth and retains gauge invariance.

Renormalon behaviour is not exclusive to the diagrams in Fig: 9.1, we expect all processes in either QED or QCD to be affected by them through the running coupling prescription. The renormalon bubble chain diagrams define a divergent subset of the higher-order quantum corrections in the interaction between two electrons in QED, or diagrammatically:

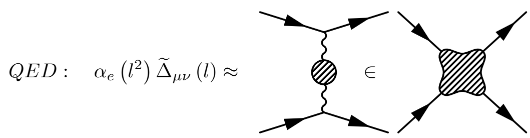
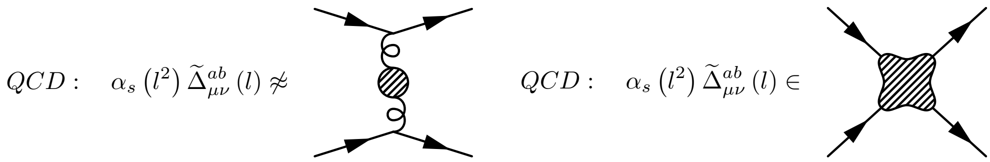


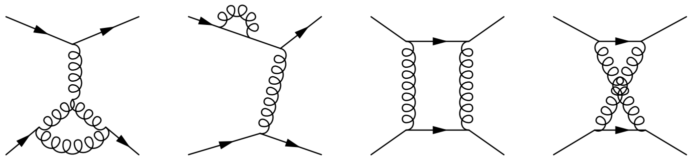
Figure 9.2: *The running coupling prescription in QED captures roughly the contribution of the photon exact propagator for $m^2 \ll \mu^2$ in the interaction of two electrons. However, this is only a subset of all higher order corrections.*

We find that in QCD the running coupling prescription in the interaction between quarks is no longer similar to the contribution of the exact gluon propagator even when $m_q^2 \ll \mu^2$. There are other diagrams that contribute, so it is still possible that a subset of diagrams exists that exhibits the same renormalon behaviour as the QED bubble chain diagrams. * Although it might be possible to find a gauge-invariant subset of diagrams that exhibits renormalon behaviour at one-loop or two-loop, this does not guarantee that it is possible to find a calculable all-order definition of this set of diagrams. A few one-loop diagrams that do contribute to the quark interaction but are not included in the exact propagator prescription are shown in Fig: 9.3(c-f).



(a) The running coupling prescription is not roughly similar to the exact propagator prescription in QCD.

(b) The running coupling prescription is assumed to be still a subset of the interaction between quarks.



(c) (d) (e) (f)

Figure 9.3: *The running coupling prescription in QCD might include at the one-loop order* $\propto \alpha_s(\mu)$ *(c) vertex corrections, (d) quark self-energy contributions or (e-f) two-gluon exchange. These are just a few examples, at the one-loop order already there are* $14 + n_f$ *number of diagrams that contribute to the interaction between two quarks.*

The use of the running coupling prescription is arguable through the qualitative interpretation of the running coupling. The running coupling is like an effective coupling at a certain energy or momentum scale. This is because one always chooses a renormalisation scale at about the same scale as the involved momenta to not produce large logarithms which invalidate the perturbative expansion. The running coupling prescription aims to include already a lot of higher order corrections by integrating over this renormalisation scale.

A more rigorous argumentation for the running-coupling prescription is given by Ref. [36]. They generalise their prescription from fermion

*According to Ref. [35] one-gluon exchange can be mimicked by two-gluon exchange as a reason for why one-gluon exchange is not gauge-invariant.

bubble insertions, which we used in Sec: 9.1 for QED at one-loop and dubbed the exact propagator prescription. Therefore, they also include the factor $\propto \mathrm{e}^{C}$ in their prescription.

The fermion bubble chain diagrams form a gauge-invariant subset in QCD but are not expected to be a good approximation of the multi-loop result because of equally sized contributions from gluon and ghost-loop insertions. In a QCD-like theory with a large number of flavours $n_f \to \infty$, the so-called $\frac{1}{n_f}$-expansion of QCD*, these contribution can be neglected and the self-energy function of the gluon becomes (in analogy with the QED result in Eq: 9.2):

$$\Pi_{\overline{MS}}\left(p^2\right) \approx \frac{\beta_0 \alpha_s}{2} \ln\left(\frac{p^2}{\mu^2}\mathrm{e}^{C}\right) + \mathcal{O}\left(\alpha_s^2\right) \quad \text{with} \quad \beta_0 \approx \frac{2 n_f T\left(R\right)}{3\pi} \tag{9.47}$$

The generalisation to QCD is completed by putting $\beta_0^{1/n_f} \to \left|\beta_0^{QCD}\right|$ in a procedure referred to as *naive nonabelianisation*. The full prescription then reads $\alpha_s\left(\mu\right) \to \alpha_s\left(l^2 \mathrm{e}^{C}\right)$, which in QED is equivalent to our exact propagator prescription.

In Ref. [36] that in many QCD observables such as the pole mass, when one applies the running coupling prescription in the one-loop result the $\propto \alpha_s^2\left(\mu\right)$ term is a good approximation to the full two-loop order result. A justification of the validity of this prescription in QCD at an arbitrary order can only be given afterwards, if one computes all contributing diagrams at that order and compares.

*An application of the $\frac{1}{n_f}$-expansion to the calculation of the quark pole mass is found in Ref. [16].

Chapter 10

Different Mass Schemes and Toponium

In this chapter, we demonstrate the presence of the renormalon ambiguity in the (top) quark pole mass and the renormalon cancellation in the potential-subtracted mass or PS mass. The PS mass is defined in terms of the toponium potential, which is the QCD equivalent of the Coulomb potential in QED.
We explicitly compute the leading order contribution to the static toponium potential before we apply the running coupling prescription to determine the renormalon contribution. We demonstrate the renormalon ambiguity in the potential and show that despite this, the energy levels of possible bound states would be unambiguous.

10.1 The renormalon ambiguity in the top quark pole mass

We defined the pole mass m_{pole} for an electron in Eq: 3.30 as minus the real part of the complex pole z_{pole} in the electron propagator. This definition is also valid for the top quark even though $\gamma \neq 0$ because the top quark mass $m_t \gg \gamma \sim \Gamma_{CM}$. We again assume $m_{pole} = m_{\overline{MS}}(\mu) + \mathcal{O}\left(\alpha_s m_{\overline{MS}}\right)$.

$$
\begin{aligned}
m_{pole} &= m_{\overline{MS}}(\mu) - \text{Re}\left[\Sigma_{\tilde{r}}\left(-m_{pole}\right)\right] \\
m_{pole} &= m_{\overline{MS}}(\mu) - \text{Re}\left[m_{\overline{MS}}(\mu)\, A_{\tilde{r}}\left(-m^2_{pole}\right) - m_{pole} B_{\tilde{r}}\left(-m^2_{pole}\right)\right]
\end{aligned}
\tag{10.1}
$$

We solve the above self-consistency relation at next-to-leading order by setting $m_{pole} \to m_{\overline{MS}}(\mu)$ on the right-hand side of Eq: 10.1. We first consider the Borel transform because it is then possible to plug in Eq: 9.45 and Eq: 9.46 for the Borel transform of the self-energy functions. When $\xi_s \neq 0$, the self-energy functions obtain gauge-dependent one-loop contributions that cancel each in the determination of the pole mass, just as in Sec: 4.4.

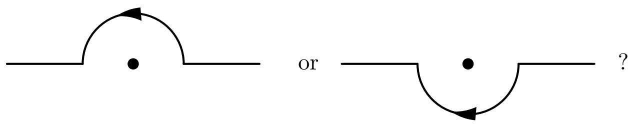
$$
\begin{aligned}
\Delta m(\mu, t) =& B\left[m_{pole} - m_{\overline{MS}}(\mu)\right](t) \\
=& m_{\overline{MS}}(\mu)\, B\left[B_{\tilde{r}}\left(-m^2_{\overline{MS}}(\mu)\right) - A_{\tilde{r}}\left(-m^2_{\overline{MS}}(\mu)\right)\right](t)
\end{aligned}
\tag{10.2}
$$

We substitute the Borel-transforms of the self-energy functions evaluated at $p^2 = -m^2_{\overline{MS}}(\mu)$ which by use of Eq: 15.1.20 of Ref. [30] allows us to write $\Delta m(\mu, t)$ as:

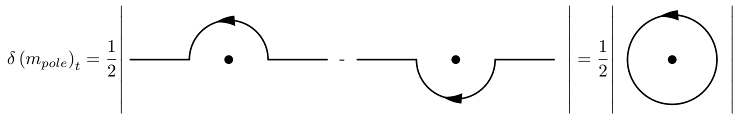
$$
\Delta m(\mu, t) = \frac{C(R)\, m_{\overline{MS}}(\mu)}{2\pi|\beta_0|}\left[6\left(\frac{m^2_{\overline{MS}}(\mu)}{\mu^2}\right)^t \frac{\Gamma(-t)\,\Gamma(1+2t)}{\Gamma(3+t)}(1+t) + \left[\frac{3}{t} + R_B(t) - R_A(t)\right]\right]
\tag{10.3}
$$

We can now not apply the inverse Borel transform to find the mass difference between the pole mass and the $\overline{MS}$ mass, because the above expression has poles in the integration domain of the Borel parameter at $t = -\left(\frac{1}{2} + k\right)$ for $k \in \mathbb{N}$ and additionally at $t = -2$. We need to modify the integration contour around these poles to perform the inverse transform.

There is no preferred way to modify the integration contour around the poles. This introduces a theoretical ambiguity in the definition of the top quark pole mass, known as the renormalon ambiguity. The size of this ambiguity is estimated to be half the difference of the integral taken by moving the integration contour via $\text{Im}[t] > 0$ and via $\text{Im}[t] < 0$ around the pole, the simplest two prescriptions. The ambiguity should then be equal to the magnitude of the pole residue of the integrand in the inverse Borel transform of Eq: 10.3.



(a) The simplest two modification of the integration contour around the poles.



(b) The definition of the renormalon ambiguity due to a renormalon at location t in terms of the two integration contour modifications.

Figure 10.1: *This figure defines diagrammatically the renormalon ambiguity in the pole mass due to a pole at t in terms of the simplest two integration contour modifications.*

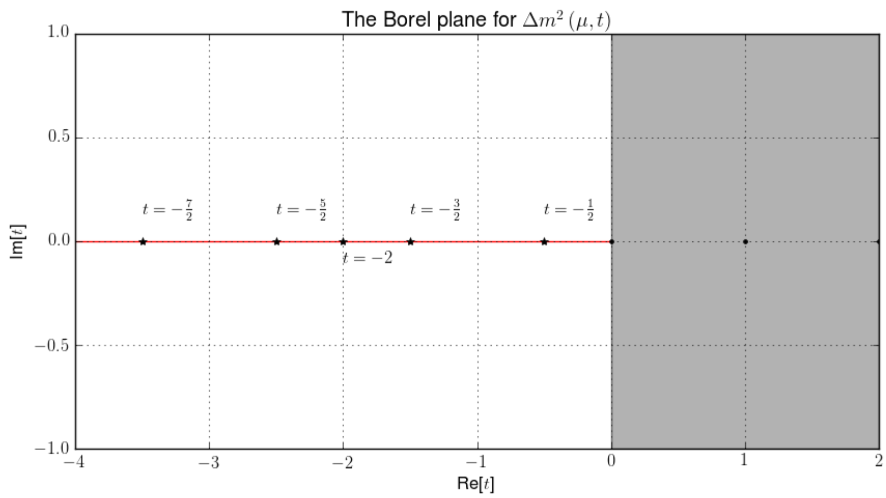


Figure 10.2: *In this figure we show the location of the poles of $\Delta m\left(\mu, t\right)$, denoted by stars and dots, in the complex plane of the Borel parameter t, the Borel plane. The dot singularities are UV-renormalons, all but the $t = 0$ renormalon lie outside the integration range for $\beta_0 < 0$. The $t = 0$ renormalon is cancelled by the renormalisation counter term at $t = 0$. The IR-renormalons that affect $\Delta m\left(\mu, t\right)$ lie at $t = -\left(\frac{1}{2} + k\right)$ for $k \in \mathbb{N}$ and additionally at $t = -2$.*

We expand $\Delta m\left(\mu,t\right)$ around the poles to find the pole residues and their relative sizes:

$$\begin{aligned}\lim_{t\to-\frac{1}{2}}\Delta m\left(\mu,t\right)&=\frac{C\left(R\right)m_{\overline{MS}}\left(\mu\right)}{\pi|\beta_0|}\left(\frac{m_{\overline{MS}}^2}{\mu^2}\right)^{-\frac{1}{2}}\frac{1}{t+\frac{1}{2}}+\text{n.p.}\\ \lim_{t\to-\frac{3}{2}}\Delta m\left(\mu,t\right)&=-\frac{3C\left(R\right)m_{\overline{MS}}\left(\mu\right)}{8\pi|\beta_0|}\left(\frac{m_{\overline{MS}}^2}{\mu^2}\right)^{-\frac{3}{2}}\frac{1}{t+\frac{3}{2}}+\text{n.p.}\\ \lim_{t\to-2}\Delta m\left(\mu,t\right)&=\frac{C\left(R\right)m_{\overline{MS}}\left(\mu\right)}{4\pi|\beta_0|}\left(\frac{m_{\overline{MS}}^2}{\mu^2}\right)^{-2}\frac{1}{t+2}+\text{n.p.}\end{aligned} \tag{10.4}$$

The other poles of $\Delta m\left(\mu,t\right)$ with $k\in\mathbb{N}\backslash\{0\}$ have residues:

$$\text{Res}\left[\frac{\Delta m\left(\mu,t\right)}{m_{\overline{MS}}\left(\mu\right)};t=-\left(\frac{1}{2}+k\right)\right]=\frac{(-1)^k}{\left(1-\frac{2}{3}k\right)2^{4k-2}\left(2k\right)!}\left(\frac{(2k-1)!}{(k-1)!}\right)^2\frac{C\left(R\right)}{\pi|\beta_0|}\left(\frac{m_{\overline{MS}}^2}{\mu^2}\right)^{-\left(\frac{1}{2}+k\right)} \tag{10.5}$$

The relative size of the pole residues for adjacent values k-values for $k>1$ is given by:

$$\frac{\text{Res}\left[\frac{\Delta m(\mu,t)}{m_{\overline{MS}}(\mu)};t=-\left(\frac{1}{2}+(k+1)\right)\right]}{\text{Res}\left[\frac{\Delta m(\mu,t)}{m_{\overline{MS}}(\mu)};t=-\left(\frac{1}{2}+k\right)\right]}=-\frac{1}{4}\left(\frac{k+\frac{1}{2}}{k+1}\right)\left(\frac{m_{\overline{MS}}^2}{\mu^2}\right)^{-1} \tag{10.6}$$

We use the inverse Borel transform to determine the theoretical ambiguity $\delta\left(m_{pole}\right)$:

$$m_{pole}-m_{\overline{MS}}\left(\mu\right)=\text{Re}\left[\int_{-\infty}^0 dt\,\Delta m\left(\mu,t\right)\mathrm{e}^{\frac{2t}{\alpha_s(\mu)|\beta_0|}}\right]=\text{Re}\left[\int_{-\infty}^0 dt\,\Delta m\left(\mu,t\right)\left(\frac{\Lambda_{QCD}^2}{\mu^2}\right)^{-t}\right] \tag{10.7}$$

For $\Lambda_{QCD}\ll m_{\overline{MS}}\left(\mu\right)$ the ambiguity caused by the pole at $t=-\frac{1}{2}$ dominates all others by at least a factor $\frac{m_{\overline{MS}}^2}{\Lambda_{QCD}^2}\gg 1$.

$$\delta\left(m_{pole}\right)_{t=-\frac{1}{2}}=C\left(R\right)\frac{\Lambda_{QCD}}{|\beta_0|} \tag{10.8}$$

The ambiguity due to the infrared renormalon is given by $\delta\left(m_{pole}\right)\propto\Lambda_{QCD}\sim(200-300)$ MeV in Eq: 10.8. The big uncertainty in the top quark pole mass is thus partly caused by the renormalon ambiguity. In future e^+e^- colliders we will require a mass definition that is not affected by the renormalon ambiguity to be able to make better predictions.

10.2 The LO static toponium potential

Here, we define the static toponium potential and explicitly compute the leading order contribution. This demonstrates by what method we compute the potential and what Feynman diagrams make up this potential. The toponium static potential is defined in terms of the scattering amplitude in the limit of static quarks. We also determine the leading order relativistic correction, which allows us to determine the range of validity of the potential.

The definition of the static toponium potential

We define the static toponium potential between a top and an anti-top colour singlet due to QCD interactions in momentum space as a function of the transferred squared momentum q^2 to equal minus the scattering amplitude in the static limit, we find:

$$\begin{aligned} \langle p', k' | \, \mathrm{i}T_{q\bar{q}} \, | p, k \rangle &= \mathrm{i}\mathcal{M} \, (2\pi)^4 \, \delta^4 \left((p+k) - (p'+k') \right) \\ \Rightarrow \widetilde{V}_{t\bar{t}} \left(q^2 \right) &= \frac{1}{4m^2} \mathcal{M}_{t\bar{t}} \left(q^2 \right) |_{\text{static}} \end{aligned} \tag{10.9}$$

The colour singlet state of a quark and an anti-quark unlike the colour octet states forms bound state mesons for the lighter quarks, if these involve a quark and an anti-quark with the same flavour this is called quarkonium. The rapid weak decay of the top quark prevents the formation of a bound-state toponium, but the would-be toponium resonance does leave an imprint on the threshold cross-section as was shown in Fig: 1.5.

We take the colour traces over the incoming and outgoing quarks and divide by a normalisation factor of three. This prescription enforces the colour singlet condition and corresponds to the following expression for the colour state with r, b and g the eigenvectors of the colour representation and $\bar{r}$, $\bar{b}$ and $\bar{g}$ of the anti-colour representation.

$$|Q\rangle_{\text{singl.}} = \frac{1}{\sqrt{3}} \left(|r\bar{r}\rangle + \left| b\bar{b} \right\rangle + |g\bar{g}\rangle \right) \tag{10.10}$$

The static potential describes the interaction between the quark and anti-quark with relative velocities in the non-relativistic limit like the Coulomb potential in QED. We later demonstrate that in this limit dynamical effects such as spin-flips and angular enhancements are suppressed.

The scattering amplitude

The diagrams that contribute to the static toponium potential at leading order are shown in Fig: 10.3. The first diagram is trivial since a gluon has no colour singlet state. Therefore we only need to compute the second diagram.

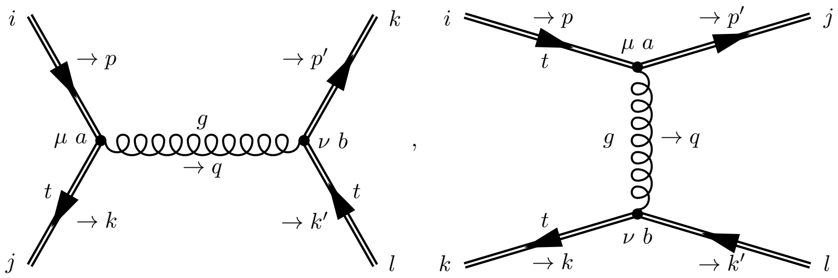


Figure 10.3: *These two diagrams would contribute at leading order, which is at tree level, to the static toponium potential. The first diagram is trivial however since there are no gluon singlets.*

The first diagram in Fig: 10.3, that corresponds to quark annihilation is zero for an incoming quark anti-quark singlet. This is shown mathematically by first writing down the vertex factor:

$$\mathrm{i}g_3\gamma^\mu T^a_{ij} \tag{10.11}$$

This vertex factor is contracted with $\frac{\delta_{ij}}{\sqrt{3}}$ by the singlet condition:

$$\mathrm{i}g_3\gamma^\mu T^a_{ij}\frac{\delta_{ij}}{\sqrt{3}} = \frac{\mathrm{i}g_3\gamma^\mu}{\sqrt{3}} \cancelto{0}{T^a_{ij}\delta_{ij}} = 0 \tag{10.12}$$

This is true by the tracelessness of the $SU(3)$ generator matrices, hence the whole diagram vanishes.

For the second diagram we use the Feynman rules that we have derived in Sec: 6.1. We also enforce the colour singlet condition but not yet the non-relativistic limit on the scattering amplitude to obtain the static potential, we find:

$$\begin{aligned} i\mathcal{M}_{t\bar{t}}\left(q^2\right) &= \left[\overline{u}_{s'}\left(\mathbf{p}'\right)\left(ig_3\gamma^\mu T^a_{ik}\right)u_s\left(\mathbf{p}\right)\right]\left(\frac{1}{i}\widetilde{\Delta}_{ab}\left(q\right)\right)\left[\overline{v}_r\left(\mathbf{k}\right)\left(ig_3\gamma^\nu T^b_{lj}\right)v_{r'}\left(\mathbf{k}'\right)\right]\frac{\delta_{ij}\delta_{kl}}{3} \\ &= ig_3^2\left(\frac{T^a_{ik}T^a_{ki}}{3}\right)\frac{1}{q^2}\left[\overline{u}_{s'}\left(\mathbf{p}'\right)\gamma^\mu u_s\left(\mathbf{p}\right)\right]\left[\overline{v}_r\left(\mathbf{k}\right)\gamma_\nu v_{r'}\left(\mathbf{k}'\right)\right] \\ &= i\frac{4\pi\alpha_s C\left(R\right)}{q^2}\overbrace{\left[\overline{u}_{s'}\left(\mathbf{p}'\right)\gamma^\mu u_s\left(\mathbf{p}\right)\right]\left[\overline{v}_r\left(\mathbf{k}\right)\gamma_\nu v_{r'}\left(\mathbf{k}'\right)\right]}^{\boxed{a}} \end{aligned} \quad (10.13)$$

We note that this scattering amplitude is gauge-invariant due to the spinor contractions and the only thing we still need to do is determine $\boxed{a}$ in the non-relativistic limit. The simplest way is to set all momenta to be $p = m\left(1,\vec{0}\right)$, $p' = m\left(1,\mathbf{0}\right)$, $k = m\left(1,\mathbf{0}\right)$ and $k' = m\left(1,\mathbf{0}\right)$.

Formally, there is no momentum transfer in this limit so $q^2 = 0$ and thus $\mathcal{M}_{t\bar{t}}\left(q^2\right)$ diverges. It is not mathematically correct to set $q^2 \to 0$ in $\boxed{a}$ but not in the rest of $\mathcal{M}_{t\bar{t}}\left(q^2\right)$. In the next subsection we consider the non-relativistic limit in more detail and show that the used prescription leads to the right conclusion. For now, we apply the Gordon decomposition identities shown in Eq: A.17 and Eq: A.18, we find:

$$i\mathcal{M}_{t\bar{t}}\left(q^2\right) = -i\frac{4\pi\alpha_s C\left(R\right)}{q^2}\left(2m\right)^2\delta_{s,s'}\delta_{r,r'} \quad \Rightarrow \quad \widetilde{V}_{t\bar{t}}\left(q^2\right) = -\frac{4\pi\alpha_s C\left(R\right)}{q^2} \quad (10.14)$$

The LO relativistic correction

Here, we compute the leading order relativistic correction to the spinor structure $\boxed{a}$ from the previous subsection, which proves the validity of our adhoc prescription that brought us to Eq: 10.14. Furthermore, it shows us when we need to start worrying about relativistic effects.

The static potential is derived from the scattering amplitude $\mathcal{M}\left(q^2\right)$ instead of the squared scattering amplitude $\left|\mathcal{M}\left(q^2\right)\right|^2$ and therefore we need to choose a basis for our $u_s\left(\mathbf{p}\right)$ and $v_r\left(\mathbf{k}\right)$ spinors. This is the only instance where the specifications for the spinors given in App: A are not enough.

We use the basis of spinors that are eigenspinors of the spin operator in the z-direction in the static limit from Ref. [17, ch. 38]. We use a transformation matrix to boost the static spinors in an arbitrary direction to obtain spinors with arbitrary momenta, which is shown in Ref. [17, ch. 38] namely Eq: 38.11. For convenience we reproduce these definitions.

The static spinors from Ref. [17, ch. 38] Eq: 38.6 are given by:

$$
\begin{aligned}
u_+(\mathbf{0}) &= \sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \quad , \quad u_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix} \\
v_+(\mathbf{0}) &= \sqrt{m}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix} \quad , \quad v_-(\mathbf{0}) = \sqrt{m}\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}
\end{aligned}
\tag{10.15}
$$

The expression Eq: 38.12 in Ref. [17, ch. 38] boosts these spinors in the $\mathbf{p}$ or $\mathbf{k}$ direction in terms of the boost matrix $K^j = \frac{\mathrm{i}}{2}\gamma^j\gamma^0$ and the rapidity $\eta \equiv \sinh^{-1}\left(\frac{|\mathbf{p}|}{m}\right)$, we find:

$$
\begin{aligned}
u_s(\mathbf{p}) &= \exp\left(\mathrm{i}\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right) u_s(\mathbf{0}) \\
v_s(\mathbf{p}) &= \exp\left(\mathrm{i}\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right) v_s(\mathbf{0})
\end{aligned}
\tag{10.16}
$$

We now expand the on-shell momentum four-vector up until the leading velocity-dependent term in each component and find: $p_0 = m + \frac{1}{2}m\mathbf{v}^2$ and $\mathbf{p} = m\mathbf{v}$ for $|\mathbf{v}| \ll 1$. In polar coordinates we parametrise each velocity vector as:

$$
\mathbf{v} = \begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix} \quad \Rightarrow \quad \hat{\mathbf{v}}\cdot\boldsymbol{\sigma} = \begin{pmatrix}\cos\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta\\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta & \cos\theta\end{pmatrix}
\tag{10.17}
$$

The transformation matrix in Eq: 10.16 is brought in a more convenient form:

$$
\exp\left(\mathrm{i}\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right) = \exp\left(-\frac{\eta}{2}\begin{pmatrix}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} & 0\\ 0 & -\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}\end{pmatrix}\right)
\tag{10.18}
$$

We expand the exponent in a Taylor series and separate the odd from the even terms, we find:

$$
\begin{pmatrix}\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} & 0\\ 0 & -\hat{\mathbf{v}}\cdot\boldsymbol{\sigma}\end{pmatrix}^2 = \hat{v}^i\hat{v}^j\begin{pmatrix}\frac{1}{2}\{\sigma^i,\sigma^j\} & 0\\ 0 & \frac{1}{2}\{\sigma^i,\sigma^j\}\end{pmatrix} = \mathbb{1}_4
\tag{10.19}
$$

$$
\Rightarrow \quad \exp\left(\mathrm{i}\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right) = \cosh\left(\frac{\eta}{2}\right)\mathbb{1}_4 + \sinh\left(\frac{\eta}{2}\right)\begin{pmatrix}-\hat{\mathbf{v}}\cdot\boldsymbol{\sigma} & 0\\ 0 & \hat{\mathbf{v}}\cdot\boldsymbol{\sigma}\end{pmatrix}
\tag{10.20}
$$

We go to the non-relativistic limit by expanding the hyperbolic functions in terms of η up to leading order in $|\mathbf{v}|$:

$$
\begin{aligned}
|\mathbf{v}| = \sinh\eta \quad &\Rightarrow \quad \cosh\eta = \sqrt{1+\mathbf{v}^2} = 1 + \frac{1}{2}\mathbf{v}^2 + \mathcal{O}\left(\mathbf{v}^4\right) \\
\cosh\left(\frac{\eta}{2}\right) &= \sqrt{\frac{\cosh(\eta)+1}{2}} = \sqrt{1+\frac{\mathbf{v}^2}{4}+\mathcal{O}\left(\mathbf{v}^4\right)} = 1 + \mathcal{O}\left(\mathbf{v}^2\right) \\
\sinh\left(\frac{\eta}{2}\right) &= \sqrt{\frac{\cosh(\eta)-1}{2}} = \sqrt{\frac{\mathbf{v}^2}{4}+\mathcal{O}\left(\mathbf{v}^4\right)} = \frac{\mathbf{v}}{2} + \mathcal{O}\left(\mathbf{v}^2\right)
\end{aligned} \tag{10.21}
$$

We now write down the transformation matrix of the spinor with the leading order relativistic correction:

$$
\exp\left(i\eta\hat{\mathbf{p}}\cdot\mathbf{K}\right) = \begin{pmatrix} \mathbb{1}_2 - \mathbf{v}\cdot\boldsymbol{\sigma} & 0 \\ 0 & \mathbb{1}_2 + \mathbf{v}\cdot\boldsymbol{\sigma} \end{pmatrix} + \mathcal{O}\left(\mathbf{v}^2\right) \tag{10.22}
$$

We now use the above expression in Eq: 10.22 to find the leading order relativistic correction to the static spinors shown in Eq: 10.15 and with our definition of the γ-matrices in App: A we can determine the leading order relativistic correction to a. The only leftover issue is that there are still four free spins and nine free parameters related to the magnitude and direction of the incoming and outgoing momenta in a.

We first align the z-axis to the spin-direction of particle i *, then particle i is fixed to have spin up $s \to +$. A rotation around the z-axis allows us to eliminate the initial azimuthal angle $\phi_i \to 0$ of particle i. Additionally we decide to look at a in the center-of-mass frame which is defined by: $\mathbf{k} + \mathbf{p} = 0$ and $\mathbf{k}' + \mathbf{p}' = 0$.

This convenient reference frame allows us to parametrise a (s, s', r, r') in terms of the magnitude of the velocity of particle i and three angles. The first angle and initial direction θ_i is the angle between the z-axis and the velocity direction of particle i. The second and third angle are the declination and azimuth of particle j, the final direction denoted as θ_f and ϕ_f.

After this reparametrisation we find two spin combination that do not flip spins and contribute to the static potential:

*Here, we label the particles in Fig: 10.3 by their colour index letter: i, j, l or k.

$$
\begin{aligned}
-\frac{1}{4m^2}\,\text{a}\,(++++) = 1 + &\Big[1-\frac{3}{2}\left(\cos\left(2\theta_i\right)+\cos\left(2\theta_f\right)\right)+\cos\phi_f \\
&\times\left(\cos\left(\theta_f-\theta_i\right)-\cos\left(\theta_i+\theta_f\right)\right)\Big]\mathbf{v}^2+\mathcal{O}\left(\mathbf{v}^4\right) \\
-\frac{1}{4m^2}\,\text{a}\,(++--) = 1 + &\Big[1+\frac{1}{2}\left(\cos\left(2\theta_i\right)+\cos\left(2\theta_f\right)\right)+2\left(1-\frac{3}{2}\mathrm{e}^{-\mathrm{i}\phi_f}\right)\cos\left(\theta_i+\theta_f\right) \\
&+2\left(1+\frac{3}{2}\mathrm{e}^{\mathrm{i}\phi_f}\right)\cos\left(\theta_f-\theta_i\right)\Big]\mathbf{v}^2+\mathcal{O}\left(\mathbf{v}^4\right)
\end{aligned}
\tag{10.23}
$$

The instances that do flip spin are of order $\mathcal{O}\left(\mathbf{v}^2\right)$ with a pre-factor involving θ_i, θ_f and ϕ_f. The static potential remains accurate as long as $\mathbf{v}^2 \ll 1$ and thus the q momentum that is transferred need not be zero for the validity of Eq: 10.14.

The potential in position space

We have computed the amplitude for a quark scattering of an anti-quark, from which we obtained the toponium potential in momentum space, as shown in Eq: 10.14. We then demonstrated the validity of this potential in the non-relativistic limit by determining the leading-order relativistic corrections. Now, we make contact with the Coulomb potential in QED by determining its QCD analogue for a quark anti-quark singlet.

We take the Fourier transform of Eq: 10.14 and reintroduce the implicit $'-\mathrm{i}\epsilon'$-term in the gluon propagator to obtain the quarkonium potential in position space. Also, in the non-relativistic limit we have $q^2 = (p-p')^2 = (\mathbf{p}-\mathbf{p}')^2 = \mathbf{q}^2$.

$$
\begin{aligned}
V_{t\bar{t}}\left(\mathbf{r}\right) &= -4\pi\alpha_s C\left(R\right)\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2-\mathrm{i}\epsilon} = -\frac{\alpha_s C\left(R\right)}{\pi}\int_0^\infty dq\frac{q^2}{q^2-\mathrm{i}\epsilon}\int_0^\pi d\theta\sin\theta\,\mathrm{e}^{\mathrm{i}qr\cos\theta} \\
&= -\frac{2\alpha_s C\left(R\right)}{\pi r}\int_0^\infty dq\frac{q\sin\left(qr\right)}{q^2-\mathrm{i}\epsilon} = -\frac{\alpha_s C\left(R\right)}{\mathrm{i}\pi r}\int_{-\infty}^\infty dq\frac{q\mathrm{e}^{\mathrm{i}qr}}{q^2-\mathrm{i}\epsilon}
\end{aligned}
\tag{10.24}
$$

The above integral has complex poles at $q^2 = \mathrm{i}\epsilon \Rightarrow q_\pm = \pm\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\sqrt{\epsilon}$. The integration contour from $q=-\infty$ to $q=\infty$ can be closed via the positive complex half-plane. This contour envelops only the pole at q_+.

$$
V_{t\bar{t}}\left(\mathbf{r}\right) = -\frac{\alpha_s C\left(R\right)}{2\mathrm{i}\pi r}\int_C dq\frac{\mathrm{e}^{\mathrm{i}qr}}{q-q_+} = -\frac{\alpha_s C\left(R\right)}{r}
\tag{10.25}
$$

We can associate a distance to the non-relativistic limit $|\mathbf{v}| \ll 1$ from the previous subsection. The relativistic modes correspond to $q \sim 2m$ since $\mathbf{q} = 2m\mathbf{v}$. When these modes oscillate rapidly in the exponent of the Fourier transform they do not contribute to the potential in position space, therefore we require:

$$qr \gg 1 \quad \Rightarrow \quad 2mr \gg 1 \quad \Rightarrow \quad r \gg \frac{1}{2m} \tag{10.26}$$

10.3 Renormalons in the toponium potential

Here, we use the running coupling prescription to find the renormalon contribution to the static toponium potential in both momentum and position space. The renormalons further constrain the range where the potential formulation is adequate.

The renormalon contribution in momentum space

We first make the replacement $\alpha_s(\mu) \to \alpha_s(q^2)$ which in the non-relativistic limit amounts to $\alpha_s(\mu) \to \alpha_s(\mathbf{q}^2)$ since in that case we have $q^2 = \mathbf{q}^2$.

$$\widetilde{V}_{t\bar{t}}\left(\mathbf{q}^2\right) = -\frac{4\pi\alpha_s(\mu)\,C(R)}{\mathbf{q}^2}\sum_{n=0}^{\infty}\left(\frac{\beta_0\alpha_s(\mu)}{2}\ln\frac{\mathbf{q}^2}{\mu^2}\right)^n \tag{10.27}$$

As before, this summation can be resummated by the Borel transform:

$$B\left[\widetilde{V}_{t\bar{t}}\left(\mathbf{q}^2\right)\right](t) = -\frac{8\pi C(R)}{|\beta_0|}\frac{1}{\mathbf{q}^2}\left(\frac{\mathbf{q}^2}{\mu^2}\right)^t \tag{10.28}$$

The renormalon contribution in position space

The Fourier transform of Eq: 10.27 simply diverges, but the divergence is regularised by the Borel transform. We thus take the Fourier transform of Eq: 10.31 first, the inverse Borel transform gets us back to the toponium potential in position space with a renormalon induced ambiguity.

The Fourier transform of Eq: 10.31 equals:

$$\begin{aligned} B\left[V_{t\bar{t}}(\mathbf{r})\right](t) &= -\frac{8\pi C(R)}{|\beta_0|}\mu^{-2t}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^{2(1-t)}} = -\frac{2C(R)}{i\pi|\beta_0|r}\mu^{-2t}\int_{-\infty}^{\infty}dq\frac{e^{iqr}}{q^{1-2t}} \\ &= -\frac{4C(R)}{|\beta_0|r}(\mu r)^{-2t}\overbrace{\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}dx\frac{e^{ix}}{x^{1-2t}}\right]}^{b} \end{aligned} \tag{10.29}$$

The integral b is evaluated with *Mathematica* [37] for a restricted range of the Borel parameter t:

$$\text{b} = \frac{1}{\pi}\int_0^\infty dx \frac{\sin x}{x^{1-2t}} = \frac{\Gamma(2t)\sin(\pi t)}{\pi} \quad , \quad -\frac{1}{2} < \text{Re}[t] < \frac{1}{2} \tag{10.30}$$

The divergence at $t = -\frac{1}{2}$ corresponds to the leading order IR-renormalon that also appeared in the top quark pole mass in Sec: 10.1. We rewrite this integral exclusively in terms of Γ-functions and we also analytically continue the integral to the full domain of the Borel parameter t. We then find for the Borel transform of the static toponium potential in position space:

$$B\left[V_{t\bar{t}}(\mathbf{r})\right](t) = -\frac{2C(R)}{\pi|\beta_0|r}(\mu r)^{-2t}\frac{\Gamma\left(\frac{1}{2}-t\right)\Gamma\left(\frac{1}{2}+t\right)}{\Gamma(1-2t)} \tag{10.31}$$

We again need a prescription on how to integrate around the poles at $t = -\left(\frac{1}{2}+k\right)$ for $k \in \mathbb{N}$ to apply the inverse Borel transform. Since there is no preferred way to deform the integration contour away from the poles via either $\text{Im}(t) > 0$ or $\text{Im}(t) < 0$, we obtain a theoretical ambiguity. This is again the renormalon ambiguity we encountered in Sec: 10.1, which is defined to be as big as half the pole residue.

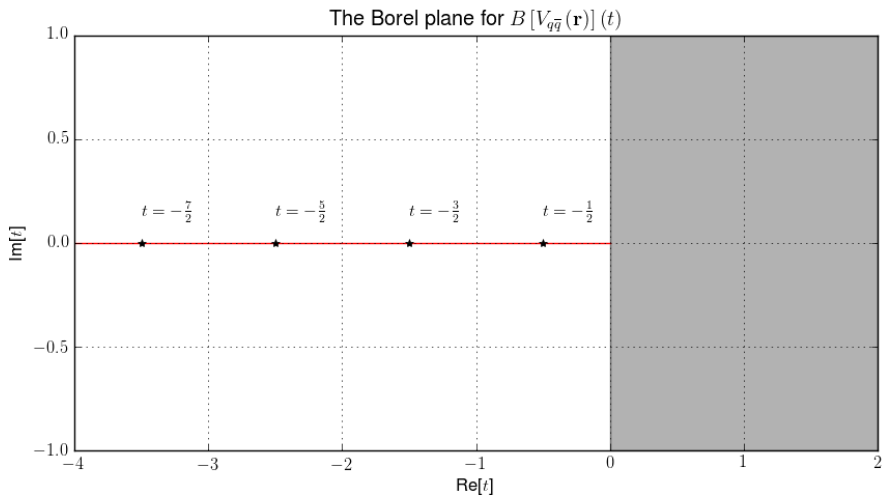


Figure 10.4: *This figure shows the complex plane of the Borel parameter t, the Borel plane. The location of the singularities in $B\left[V_{t\bar{t}}(\mathbf{r})\right](t)$ are denoted with stars. The toponium potential is only affected by IR-renormalons, which lie at $t = -\left(\frac{1}{2}+k\right)$ for $k \in \mathbb{N}$.*

For the leading order and sub-leading IR-renormalons, we expand the Borel transformed potential around $t = -\frac{1}{2}$ and $t = -\frac{3}{2}$ respectively, we find:

$$\begin{aligned}\lim_{t\to-\frac{1}{2}} B\left[V_{t\bar{t}}(\mathbf{r})\right](t) &= -\frac{2C(R)\,\mu}{\pi|\beta_0|}\frac{1}{t+\frac{1}{2}} + \text{n.p.}\\ \lim_{t\to-\frac{3}{2}} B\left[V_{t\bar{t}}(\mathbf{r})\right](t) &= \frac{C(R)\,\mu}{3\pi|\beta_0|}(\mu r)^2\frac{1}{t+\frac{3}{2}} + \text{n.p.}\end{aligned} \tag{10.32}$$

The other poles of $B\left[V_{t\bar{t}}(\mathbf{r})\right](t)$ with $k \in \mathbb{N}$ have the following residues with relative sizes:

$$\begin{aligned}\text{Res}\left[B\left[V_{t\bar{t}}(\mathbf{r})\right](t)\,;t=-\left(\frac{1}{2}+k\right)\right] &= \frac{(-1)^{k+1}}{1+2k}\left(\frac{2C(R)\,\mu}{\pi|\beta_0|}\right)\frac{(\mu r)^{2k}}{(2k)!}\\ \frac{\text{Res}\left[B\left[V_{t\bar{t}}(\mathbf{r})\right](t)\,;t=-\left(\frac{1}{2}+(k+1)\right)\right]}{\text{Res}\left[B\left[V_{t\bar{t}}(\mathbf{r})\right](t)\,;t=-\left(\frac{1}{2}+k\right)\right]} &= -\frac{(\mu r)^2}{(2+2k)(3+2k)}\end{aligned} \tag{10.33}$$

We now use the inverse Borel transform to determine the theoretical uncertainty $\delta\left(V_{t\bar{t}}(\mathbf{r})\right)$:

$$V_{t\bar{t}}(\mathbf{r}) = \text{Re}\left[\int_{-\infty}^{0} dt\, B\left[V_{t\bar{t}}(\mathbf{r})\right](t)\, e^{\frac{2t}{\alpha_s(\mu)|\beta_0|}}\right] = \text{Re}\left[\int_{-\infty}^{0} dt\, B\left[V_{q\bar{q}}(\mathbf{r})\right](t)\left(\frac{\Lambda_{QCD}^2}{\mu^2}\right)^{-t}\right] \tag{10.34}$$

For $r \ll \frac{1}{\Lambda_{QCD}}$ the ambiguity caused by the pole at $t = -\frac{1}{2}$ dominates all others by at least a factor $\frac{1}{\left(r\Lambda_{QCD}\right)^2} \gg 1$.

$$\delta\left(V_{t\bar{t}}(\mathbf{r})\right)_{t=-\frac{1}{2}} = 2C(R)\frac{\Lambda_{QCD}}{|\beta_0|} \tag{10.35}$$

Furthermore, we can give a range of validity for the quarkonium potential where relativistic correction as shown in Eq: 10.26 and sub-leading IR-renormalons are small. This shows us that for $m \sim \Lambda_{QCD}$ the potential formulation is wholly inadequate.

$$\frac{1}{2m} \ll r \ll \frac{1}{\Lambda_{QCD}} \tag{10.36}$$

10.4 The Toponium Spectrum

The toponium potential is affected by a renormalon ambiguity with a magnitude of about Λ_{QCD}. This ambiguity affects the precision by which we can extract the the pole mass m_{pole} from the toponium energy or mass spectrum. We review some non-relativistic quantum mechanics to understand how one would extract the pole mass from the toponium spectrum and how the renormalon ambiguity affects the precision by which this is possible.

This discussion is not fully applicable to the toponium system, because it ignores the weak decay of the top quark which stops the formation of bound states. We still include the discussion since it motivates the definition of a renormalon-free mass. The proper way of treating the toponium system at threshold in the non-relativistic limit to obtain the $t\bar{t}$ cross-section up to leading-logarithmic order (LL) is described in Ref. [6].

The Schrödinger equation for toponium

In non-relativistic quantum mechanics particles are described by wave functions and the evolution of these wave-functions is described by the Schrödinger equation. A wave-function is a complex valued probability amplitude from which observables are computed. Unlike the spinor amplitudes that we encountered previously this wave function is normalised to one:

$$\int d^3\mathbf{r}\, |\Psi(\mathbf{r},\mathbf{t})|^2 = 1 \tag{10.37}$$

The Schrödinger equation describes the evolution of these wave functions in terms of the hamiltonian operator $\hat{\mathcal{H}}$:

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \hat{\mathcal{H}}\Psi(\mathbf{r},t) \tag{10.38}$$

In the bound-state problem of toponium, we are interested in the energy spectrum. This corresponds to wave function that are eigenfunctions of the hamiltonian operator with as eigenvalues the energy of the system E_{bind}. Through separation of variables, we define time-independent Schrödinger equation as:

$$\Psi(\mathbf{r},t) \equiv \psi(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}Et} \tag{10.39}$$

$$\hat{\mathcal{H}}\psi(\mathbf{r}) = E_{bind}\psi(\mathbf{r}) \tag{10.40}$$

The hamiltonian operator is defined as the hamiltonian energy function under the replacements $\mathbf{p} \rightarrow \hat{\mathbf{p}}$ and $\mathbf{r} \rightarrow \hat{\mathbf{r}}$. The operators $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$ are again defined in terms of $\mathbf{r}$ as $\hat{\mathbf{p}} = -\mathrm{i}\nabla$ and $\hat{\mathbf{r}} = \mathbf{r}$. For quarkonium, a system with two particles and a potential, we find the hamiltonian:

$$H = \frac{1}{2}m_{pole}\mathbf{v}_1^2 + \frac{1}{2}m_{pole}\mathbf{v}_2^2 + V_{t\bar{t}}\left(|\mathbf{r}_1 - \mathbf{r}_2|\right) \tag{10.41}$$

The above hamiltonian is a function of the three spatial and three velocity coordinates of particle one and the three spatial and velocity coordinates of particle two; twelve coordinates in total. The potential only depends on the inter-particle separation and we are only interested in the mass eigenstates of toponium and not the translational kinetic energy of the whole system. We thus consider the system in the centre-of-mass frame, which means that we only need the relative velocities and positions and this reduces the number of coordinates to three spatial coordinates and three velocity coordinates.

We introduce the relative coordinates $\mathbf{r}$ and $\mathbf{v}$:

$$\begin{aligned} \mathbf{r} &\equiv \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{v} &\equiv \mathbf{v}_1 - \mathbf{v}_2 \end{aligned} \tag{10.42}$$

The centre-of-mass frame conditions allow us to simplify these statements:

$$\begin{aligned} m_{pole}\mathbf{r}_1 + m_{pole}\mathbf{r}_2 &= 0 \quad \Rightarrow \quad \mathbf{r} = \frac{1}{2}\mathbf{r}_1 = -\frac{1}{2}\mathbf{r}_2 \\ \mathbf{p}_1 + \mathbf{p}_2 &= m_{pole}\mathbf{v}_1 + m_{pole}\mathbf{v}_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \frac{1}{2}\mathbf{v}_1 = -\frac{1}{2}\mathbf{v}_2 \end{aligned} \tag{10.43}$$

In terms of the new coordinates the hamiltonian becomes:

$$H = \frac{1}{4}m_{pole}\mathbf{v}^2 + V_{t\bar{t}}\left(|\mathbf{r}|\right) \tag{10.44}$$

The hamiltonian in Eq: 10.44 in terms of the relative coordinates equals the hamiltonian of one particle with a reduced mass of $m_{red} = \frac{m_{pole}}{2}$ that moves around in a potential fixed at the origin.

Now we replace the coordinates $\mathbf{p} = m_{red}\mathbf{v}$ and $\mathbf{r}$ by their operators $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$ to obtain the hamiltonian operator. This gives us a differential equation that needs to be solved to find the spectrum of energy eigenvalues.

$$\begin{aligned}\hat{\mathcal{H}} &= \frac{\hat{\mathbf{p}}^2}{2m_{red}} + V_{t\bar{t}}\left(|\hat{\mathbf{r}}|\right) = -\frac{\nabla^2}{m_{pole}} + V_{t\bar{t}}\left(|\mathbf{r}|\right) \\ \Rightarrow &\quad \left(-\frac{\nabla^2}{m_{pole}} + V_{t\bar{t}}\left(|\mathbf{r}|\right) - E_{bind}\right)\psi\left(\mathbf{r}\right) = 0\end{aligned} \tag{10.45}$$

It is an enormous task to find all the eigenfunction $\psi_i\left(\mathbf{r}\right)$ and corresponding eigenvalues E_i for a general radial potential. Especially, when we are only interested in the spectrum of binding-energy eigenvalues E_i and not in the expressions for $\psi_i\left(\mathbf{r}\right)$. We assume that there exists a complete set of orthonormal eigenfunctions $\psi_i\left(\mathbf{r}\right)$, in this case any quantum state $\psi\left(\mathbf{r}\right)$ is a linear combination of $\psi_i\left(\mathbf{r}\right)$'s.

$$\begin{aligned}\text{completeness:} &\quad \sum_i \psi_i^*\left(\mathbf{r}\right)\psi_i\left(\mathbf{r}'\right) = \delta^3\left(\mathbf{r}-\mathbf{r}'\right) \\ \text{orthonormality:} &\quad \int d^3\mathbf{r}\;\psi_i^*\left(\mathbf{r}\right)\psi_j\left(\mathbf{r}\right) = \delta_{ij}\end{aligned} \tag{10.46}$$

Because the eigenfunctions $\psi_i\left(\mathbf{r}\right)$ solve the differential equation in Eq: 10.45, the solution of the following differential equation is also known:

$$\begin{aligned}\left(-\frac{\nabla^2}{m_{pole}} + V_{t\bar{t}}\left(|\mathbf{r}|\right) - E_{bind}\right) G\left(\mathbf{r},\mathbf{r}',E_{bind}\right) &= \delta^3\left(\mathbf{r}-\mathbf{r}'\right) \\ \Rightarrow \quad G\left(\mathbf{r},\mathbf{r}',E_{bind}\right) &= \sum_i \frac{\psi_i^*\left(\mathbf{r}\right)\psi_i\left(\mathbf{r}'\right)}{E_i - E_{bind}}\end{aligned} \tag{10.47}$$

If we find the so-called Green's function $G\left(\mathbf{r},\mathbf{r}',E\right)$ by solving the differential equation in Eq: 10.47 then the poles of this function at $E = E_i$ give us the spectrum of energies. If it is possible to solve Eq: 10.47 instead of Eq: 10.45, it spares us the effort of having to compute all the eigenfunctions $\psi_i\left(\mathbf{r}\right)$, but we can do better still.

For a radial potential that does not depend on the specific direction of $\mathbf{r}$ such as our toponium potential $V_{t\bar{t}}\left(r\right)$ where $r \equiv |\mathbf{r}|$, we notice that our differential equation in Eq: 10.45 is rotationally invariant. It is possible to rewrite ∇^2 in radial coordinates and use separation of variables on $\psi\left(\mathbf{r}\right)$ to solve the angular part, as is demonstrated in Ref. [38, ch. 4].

We separate variables and write our eigenfunctions as $\psi_i\left(\mathbf{r}\right) = \frac{u_{nl}(r)}{r} Y_l^m\left(\theta,\phi\right)$ with the set $i = \{n, l, m\}$ in terms of the standard spherical harmonics that have the following normalisation.

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\; \left[Y_l^m\left(\theta,\phi\right)\right]^* \left[Y_{l'}^{m'}\left(\theta,\phi\right)\right] = \delta_{ll'}\delta_{mm'} \tag{10.48}$$

The function $u_{nl}(r)$ is normalisation and satisfies a radial differential equation, compared to Eq: 10.45 this equation has an additional centrifugal term.

$$\int_0^\infty dr\, |u(r)|^2 = 1 \quad , \quad \left(-\frac{1}{m_{pole}} \frac{\partial^2}{\partial r^2} + \left[V_{t\bar{t}}(r) + \frac{l(l+1)}{m_{pole} r^2} - E_{bind} \right] \right) u(r) = 0 \quad (10.49)$$

Finally, we use the decomposition of $\psi_i(\mathbf{r})$ in Eq: 10.47, choose $\mathbf{r}$ and $\mathbf{r}'$ to be aligned and then integrate over all angles. This yields a Green's function $g(r, r', E_{bind})$ in terms of fewer parameters and which still has poles at $E_{bind} = E_i \equiv E_{nl}$.

$$\begin{aligned} \left(-\frac{1}{m_{pole}} \frac{\partial^2}{\partial r^2} + \left[V_{t\bar{t}}(r) + \frac{l(l+1)}{m_{pole} r^2} - E_{bind} \right] \right) g(r, r', E_{bind}) &= r r' \delta(r - r') \\ \Rightarrow \quad g(r, r', E_{bind}) &= \sum_{nl} \frac{u^*_{nl}(r)\, u_{nl}(r')}{E_{nl} - E_{bind}} \end{aligned} \quad (10.50)$$

The differential equation in Eq: 10.50 corresponds to the result in Eq: 4.4 of [6] for $l = 0$ * and $\Gamma_{CM} \neq 0$, which also gives some hints on how to solve for $g(r, r', E_{bind})$. The Green's function $g(r, r', E_{bind})$ illustrates that for a radial potential $V_{t\bar{t}}(r)$ the solution of the angular part of $G(\mathbf{r}, \mathbf{r}', E_{bind})$ in Eq: 10.47 can be written down immediately, which saves effort. In what comes, we shall not make further use of Eq: 10.50 and $g(r, r', E_{bind})$.

Extraction of the pole mass

We first need to measure the required *centre-of-mass energy* $\sqrt{s} = E_{bind} + 2m_{pole}$, that is necessary to produce the different $\{n, l\}$ toponium states, the experimental input. On the theory side, we need to further solve Eq: 10.50 to find the bound-state eigenvalues E_{nl} for different values of $\{n, l\}$.

We replace the binding energy E_{bind} in Eq: 10.47 by the relativistically invariant centre-of-mass energy $\sqrt{s}$.

$$\left(-\frac{\nabla^2}{m_{pole}} + 2m_{pole} + V_{t\bar{t}}(|\mathbf{r}|) - \sqrt{s} \right) G\left(\mathbf{r}, \mathbf{r}', \sqrt{s} - 2m_{pole}\right) = \delta^3(\mathbf{r} - \mathbf{r}') \quad (10.51)$$

The advantage of this representation is that the poles of the Green's function in $\sqrt{s}$ corresponds to the invariant toponium mass states. The combination $2m_{pole} + V_{t\bar{t}}(|\mathbf{r}|)$ corresponds to the static energy E_{static} of the toponium system.

*The $t\bar{t}$ cross-section as treated in [6] in the non-relativistic limit is dominated by the $l = 0$ or 1S state.

The renormalon contribution in $V_{t\bar{t}}(r)$ is included through Eq: 10.34. The ambiguity due to the leading order IR-renormalon makes that it is not possible to obtain m_{pole} with an accuracy better than:

$$\delta\left(m_{pole}\right) = \frac{1}{2}\delta\left(V_{t\bar{t}}(\mathbf{r})\right) = C(R)\frac{\Lambda_{QCD}}{|\beta_0|} \tag{10.52}$$

This result coincides with what we found in Sec: 10.1 and Eq: 10.8, where we determined the renormalon induced ambiguity in the definition of the top pole mass.

Although both the pole mass and the toponium potential are ambiguous due to IR-renormalons the toponium spectrum is not. The toponium spectrum depends on the combination $E_{static} = 2m_{pole} + V_{t\bar{t}}(|\mathbf{r}|)$, which is free of the $\propto \Lambda_{QCD}$ renormalon, which was first observed by Refs. [1, 39]. The expansion around $t = -\frac{1}{2}$ of the Borel expansions, shown in Eq: 10.32 and Eq: 10.4, cancel up to finite terms:

$$\begin{aligned}\lim_{t\to-\frac{1}{2}} B\left[E_{static}\right](t) =& 2\left(\frac{C(R)\, m_{\overline{MS}}(\mu)}{\pi|\beta_0|}\left(\frac{m^2_{\overline{MS}}}{\mu^2}\right)^{-\frac{1}{2}}\frac{1}{t+\frac{1}{2}} + \text{n.p.}\right)\\ &+\left(-\frac{2C(R)\,\mu}{\pi|\beta_0|}\frac{1}{t+\frac{1}{2}} + \text{n.p.}\right) = \text{n.p.}\end{aligned} \tag{10.53}$$

10.5 The Potential-Subtracted Mass

The observation that E_{static} is free of the $\propto \Lambda_{QCD}$ IR-renormalon due to a cancellation of contributions to both m_{pole} and $V_{t\bar{t}}(r)$ motivates the definition of the *potential-subtracted mass* or PS mass. We work out the definition of the PS mass and show that it is indeed unaffected by the leading-order IR-renormalon and allows for a more precise determination of the $\overline{MS}$ mass.

The definition of the potential-subtracted mass

The PS mass for a heavy quark is defined by adding the IR part of the relevant static toponium potential to the pole mass following Ref. [1]. There is a certain arbitrariness in what we choose as the IR part of the potential parametrised by a momentum scale μ_f which is known as the factorisation scale.

$$m_{PS} \equiv m_{pole} - \Delta m\left(\mu_f\right) \quad \text{with} \quad \Delta m\left(\mu_f\right) = -\frac{1}{2}\int_{|\mathbf{q}|<\mu_f} \frac{d^3\mathbf{q}}{(2\pi)^3} \widetilde{V}_{t\bar{t}}\left(\mathbf{q}^2\right) \tag{10.54}$$

The mass correction $\Delta m\left(\mu_f\right)$ is not exactly equal to half the IR part of the static toponium potential because we have approximated $\mathrm{e}^{\mathbf{iq\cdot r}} \sim 1$. Therefore $\Delta m\left(\mu_f\right)$ only corresponds to the contribution of the IR part of the potential when $r \ll \frac{1}{\mu_f}$. The details of the definition are not that essential though, one could choose not to do the approximation $\mathrm{e}^{\mathbf{iq\cdot r}} \sim 1$ at the expense of making the calculation of $\Delta m\left(\mu_f\right)$ harder.

The important thing to verify is that the leading order IR-renormalon in the pole mass is being cancelled, and that thus the PS mass is renormalon free. We evaluate $\Delta m\left(\mu_f\right)$ in Borel space with the expression for $B\left[\widetilde{V}_{t\bar{t}}\left(q^2\right)\right](t)$ given in Eq: 10.31 to show the cancellation:

$$\begin{aligned} B\left[\Delta m\left(\mu_f\right)\right](t) =& -\frac{1}{2}\int_{|\mathbf{q}|<\mu_f} \frac{d^3\mathbf{q}}{(2\pi)^3} B\left[\widetilde{V}_{t\bar{t}}\left(\mathbf{q}^2\right)\right](t) \\ =& \frac{4\pi C(R)}{|\beta_0|}\int_{|\mathbf{q}|<\mu_f} \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{\mathbf{q}^2}\left(\frac{\mathbf{q}^2}{\mu^2}\right)^t = \frac{2C(R)}{\pi|\beta_0|}\int_0^{\mu_f} dq\left(\frac{q^2}{\mu^2}\right)^t \\ =& \frac{2C(R)\mu}{\pi|\beta_0|}\frac{1}{2t+1}\left(\frac{q}{\mu}\right)^{2t+1}\Bigg|_0^{\mu_f} = \frac{C(R)\mu}{\pi|\beta_0|}\frac{1}{t+\frac{1}{2}}\left(\frac{\mu_f}{\mu}\right)^{2t+1} \end{aligned} \tag{10.55}$$

We see that $B\left[\Delta m\left(\mu_f\right)\right](t)$ becomes independent of μ_f in Borel space exactly at the leading order IR-renormalon at $t = -\frac{1}{2}$. When we expand $B\left[\Delta m\left(\mu_f\right)\right](t)$ we see that it has the same expansion up to non-pole terms as $\Delta m(\mu, t)$ in Eq: 10.4, and therefore there is a cancellation.

$$\lim_{t\to-\frac{1}{2}} B\left[\Delta m\left(\mu_f\right)\right](t) = \frac{C(R)\mu}{\pi|\beta_0|}\frac{1}{t+\frac{1}{2}} + \text{n.p.} \tag{10.56}$$

Constraining the factorisation scale

We only need $\Delta m\left(\mu_f\right)$ to cancel the IR-renormalon at $t = -\frac{1}{2}$ but the mass correction contributes in the whole integration. The mass correction does not cancel the subleading renormalons at $t = -\left(\frac{3}{2}+k\right)$ for $k \in \mathcal{N}$ in Borel space.

It is possible to choose μ_f in such a way that the residual finite contribution of the mass correction to the PS mass from the integration range from $t = -\infty$ to the first subleading renormalon at $t = -\frac{3}{2}$ is finite and not bigger than the leading order renormalon in size.

We plug the Borel transform of the mass correction $\Delta m\left(\mu_f\right)$ in the inverse Borel transform with a modified integration range from $t=-\infty$ to $t=-\frac{3}{2}$:

$$\begin{aligned}\int_{\infty}^{-\frac{3}{2}} dt\, B\left[\Delta m\left(\mu_f\right)\right](t)\, \mathrm{e}^{\frac{2t}{\alpha_s(\mu)|\beta_0|}} &= \int_{\infty}^{-\frac{3}{2}} dt\, B\left[\Delta m\left(\mu_f\right)\right](t)\left(\frac{\Lambda_{QCD}^2}{\mu^2}\right)^{-t} \\ &= \frac{C(R)\,\mu_f}{\pi|\beta_0|}\int_{-\infty}^{-\frac{3}{2}} \frac{dt}{t+\frac{1}{2}}\left(\frac{\Lambda_{QCD}^2}{\mu_f^2}\right)^{-t}\end{aligned} \tag{10.57}$$

We do a linear coordinate transformation $s=t+\frac{1}{2}$ followed by $r=-s$ and then $u=\ln\left(\frac{\mu_f^2}{\Lambda_{QCD}^2}\right)r$:

$$\begin{aligned}&= \frac{C(R)\,\Lambda_{QCD}}{\pi|\beta_0|}\int_{-\infty}^{-1} \frac{ds}{s}\left(\frac{\Lambda_{QCD}^2}{\mu_f^2}\right)^{-s} = -\frac{C(R)\,\Lambda_{QCD}}{\pi|\beta_0|}\int_{1}^{\infty}\frac{dr}{r}\mathrm{e}^{-\ln\left(\frac{\mu_f^2}{\Lambda_{QCD}^2}\right)r} \\ &= -\frac{C(R)\,\Lambda_{QCD}}{\pi|\beta_0|}\int_{\ln\left(\frac{\mu_f^2}{\Lambda_{QCD}^2}\right)}^{\infty}\frac{du}{u}\mathrm{e}^{-u} = \frac{C(R)\,\Lambda_{QCD}}{\pi|\beta_0|}\mathrm{Ei}\left[-\ln\left(\frac{\mu_f^2}{\Lambda_{QCD}^2}\right)\right]\end{aligned} \tag{10.58}$$

We demand this to be smaller in absolute value than the ambiguity of the leading order renormalon shown in Eq: 10.8:

$$\begin{aligned}\left|\frac{C(R)\,\Lambda_{QCD}}{\pi|\beta_0|}\mathrm{Ei}\left[-\ln\left(\frac{\mu_f^2}{\Lambda_{QCD}^2}\right)\right]\right| &< \frac{C(R)\,\Lambda_{QCD}}{|\beta_0|} \\ \Rightarrow \quad & \mu_f > 1.013\,\Lambda_{QCD}\end{aligned} \tag{10.59}$$

The function $\mathrm{Ei}[x]$ is the exponential integral which has a singularity at $x=0$, it is the reason why the residual piece of the mass correction is already smaller than the dominant IR-renormalon ambiguity at the given value for $\mu_f \sim \Lambda_{QCD}$. In Fig: 10.5 we see that μ_f must be larger than Λ_{QCD} though it need not be very much larger.

Extraction of the potential-subtracted mass

Since the potential-subtracted mass is free of the renormalon ambiguity, it is expected that it can be extracted from the toponium spectrum with better accuracy than the pole mass through Eq: 10.51. After a few modifications in Eq: 10.51 we solve for the bound-state eigenvalues E_{nl} in terms of $m_{PS}\left(\mu_f\right)$ instead of m_{pole}.

To replace all the occurrences of the pole mass in Eq: 10.51 by the potential-subtracted mass we introduce the subtracted potential $V_{t\bar{t}}\left(r,\mu_f\right)$:

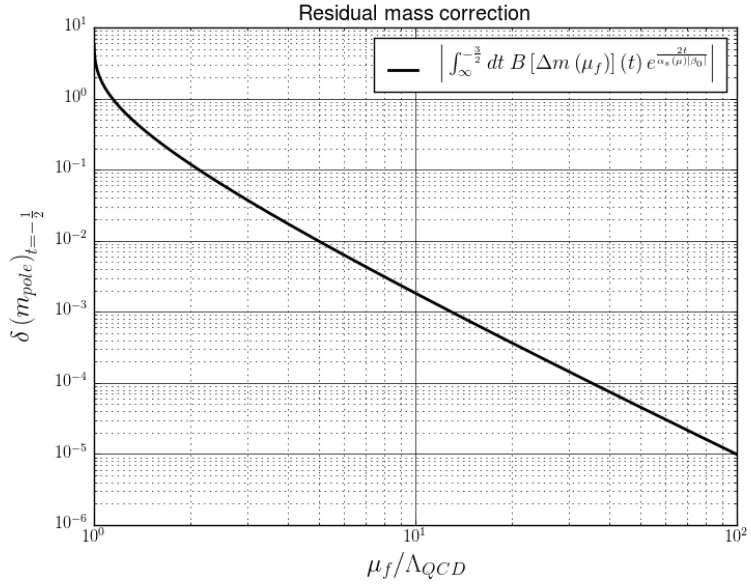


Figure 10.5: *This figure shows the absolute size of the residual mass correction in units of* $\delta\left(m_{pole}\right)_{t=-\frac{1}{2}}$*, the dominant* IR*-renormalon ambiguity at* $t = -\frac{1}{2}$*, in terms of* μ_f/Λ_{QCD}*.*

$$V_{t\bar{t}}\left(r, \mu_f\right) = V_{t\bar{t}}(r) + 2\Delta m\left(\mu_f\right) \tag{10.60}$$

We also replace $m_{pole} \to m_{PS}$ in the kinetic term, which introduces corrections that are given by:

$$-\frac{\nabla^2}{m_{pole}} = -\frac{\nabla^2}{m_{PS}}\left(1 - \frac{\Delta m\left(\mu_f\right)}{m_{PS}} + \mathcal{O}\left(\left(\frac{\Delta m\left(\mu_f\right)}{m_{PS}}\right)^2\right)\right) \tag{10.61}$$

We want to neglect the correction that we show in Eq: 10.61, which is only possible when $\Delta m\left(\mu_f\right) \ll m_{PS}$. We assume that the leading order potential, shown in Eq: 10.14, is the dominant contribution to $\Delta m\left(\mu_f\right)$. This allows us to estimate the size of $\Delta m\left(\mu_f\right)$ as a function of μ_f:

$$\Delta m\left(\mu_f\right) \sim 2\pi\alpha_s(\mu)\, C(R) \int_{|\mathbf{q}|<\mu_f} \frac{d^3\mathbf{q}}{(2\pi)^3}\frac{1}{\mathbf{q}^2} = \frac{\alpha_s(\mu)\, C(R)}{\pi}\int_0^{\mu_f} dq = \frac{\alpha_s(\mu)\, C(R)}{\pi}\mu_f \tag{10.62}$$

This result agrees with the first term in the NNLO expression that is computed by Ref. [2]. If we neglect factors of order unity we see that

$\Delta m\left(\mu_f\right) \sim \alpha_s\left(\mu\right)\mu_f$ and our constraint amounts to $\alpha_s\left(\mu\right)\mu_f \ll m_{PS}$. This upper constraint is still too loose according to Ref. [1], they give a stricter upper bound for $\mu_f < m_{PS}v$ by assuming $\alpha_s\left(\mu\right) \sim v$.

We perform the replacement $m_{pole} \to m_{PS}$ and neglect the corrections in the kinetic term by a suitable small choice for the factorisation scale. The equation of which the solution yields us the bound-state eigenvalues E_{nl} is thus given by:

$$\left(-\frac{\nabla^2}{m_{PS}} + 2m_{PS} + V_{t\bar{t}}\left(|\mathbf{r}|, \mu_f\right) - \sqrt{s}\right) G\left(\mathbf{r}, \mathbf{r}', \sqrt{s} - 2m_{PS}\right) = \delta^3\left(\mathbf{r} - \mathbf{r}'\right) \qquad (10.63)$$

If the top quark were less massive, than this method of determining energy levels of the toponium bound-system would predict at which centre-of-mass energies $\sqrt{s}$ the top anti-top scattering cross-section peaks. In reality the spectrum is smeared out by the strong weak decay of the top quark, which renders the above method inadequate for top anti-top scattering.

Nevertheless, the threshold cross-section of the top quark can still be computed by use of perturbative QCD and effective field theory methods as was done by Ref. [6] at LL in terms of m_{pole}, by Ref. [2] at NNLO in terms of both m_{pole} and $m_{\overline{PS}}$* as shown in Fig: 10.6 and by Ref. [15] at NNNLO using m_{PS} as was shown in Fig: 1.5. In Fig: 10.6, we see an improved convergence of the peak in the cross-section in terms of $m_{\overline{PS}} \sim m_{PS}$ with respect to m_{pole}, this implies that the renormalon-free mass definition can be extracted from a fit with better precision.

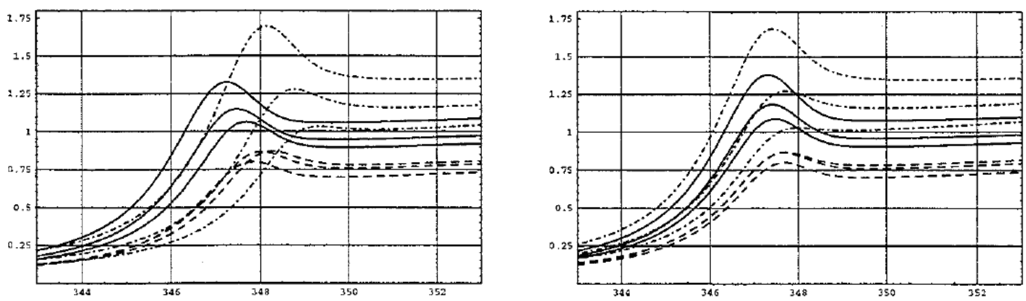


Figure 10.6: *These plots have been taken from Ref. [2]. They show as a function of centre-of-mass energy $\sqrt{s}$ the threshold behaviour of $e^+e^- \to t\bar{t}$ in terms of m_{pole} (left) and $m_{\overline{PS}}$ (right) at LO (dashed-dotted), NLO (dashed) and NNLO (solid) for $m_{\overline{MS}}\left(m_{\overline{MS}}\right) = 160$ GeV, 165 GeV and 170 GeV.*

*The $m_{\overline{PS}}$ definition is very similar to the m_{PS} definition. It subtracts an extra relativistic and non-abelian correction compared to the regular potential-subtracted mass.

We expand $m_{PS}\left(\mu_f\right) = m_{pole} - \Delta m\left(\mu_f\right)$ up until a given order in perturbation theory to use measurements of $m_{PS}\left(\mu_f\right)$ to determine the theoretical mass parameter $m_{\overline{MS}}$. This power series in $\alpha_s\left(\mu\right)$ is expected to have better convergence properties than the power series of m_{pole}.

Appendix A

γ-matrix and spinor identities

Here we give a list of γ-matrix and spinor identities with their proofs. The γ-matrices that were first introduced in Sec: 2.1 were defined as four independent traceless 4×4-matrices that satisfy the following Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\eta^{\mu\nu} \tag{A.1}$$

The explicit matrix representation of the gamma matrices in terms of the σ^i Pauli matrices is given by:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{A.2}$$

For convenience we define a fifth γ-matrix:

$$\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \tag{A.3}$$

By applying Eq: A.1 to commute individual γ-matrices we can show the following properties of γ^5:

$$\left(\gamma^5\right)^2 = 1 \tag{A.4}$$

$$\left\{\gamma^5, \gamma^\mu\right\} = 0 \tag{A.5}$$

Spinors were defined in Sec: 2.1 such that the spin-½ fermion field $\Psi(x)$ satisfies the Dirac equation. This implies for the spinors that they satisfy:

$$\begin{aligned} (\not{p} + m)\, u_s(\mathbf{p}) = 0 \quad & (-\not{p} + m)\, v_s(\mathbf{p}) = 0 \\ \overline{u}_s(\mathbf{p})\,(\not{p} + m) = 0 \quad & \overline{v}_s(\mathbf{p})\,(-\not{p} + m) = 0 \end{aligned} \tag{A.6}$$

We also normalise the spinors according to:

$$\begin{aligned} \overline{u}_{s'}(\mathbf{p})\, u_s(\mathbf{p}) &= 2m\delta_{s,s'} \quad \overline{u}_{s'}(\mathbf{p})\, v_s(\mathbf{p}) = 0 \\ \overline{v}_{s'}(\mathbf{p})\, v_s(\mathbf{p}) &= -2m\delta_{s,s'} \quad \overline{v}_{s'}(\mathbf{p})\, u_s(\mathbf{p}) = 0 \end{aligned} \tag{A.7}$$

Explicit expressions for the eigenspinors of the z-component of the spin matrix can be constructed and boosted to an arbitrary value of the momentum as shown in [17, ch. 38], [18, ch. 3]. The explicit solutions that are stated in [18, ch. 3] verify the following spin sums:

$$\begin{aligned} \sum_{s=\pm} u_s(\mathbf{p})\, \overline{u}_s(\mathbf{p}) &= -\not{p} + m \\ \sum_{s=\pm} v_s(\mathbf{p})\, \overline{v}_s(\mathbf{p}) &= -\not{p} - m \end{aligned} \tag{A.8}$$

Identities involving contractions of γ-matrices

$$\boxed{\gamma^\mu \gamma_\mu = -d} \tag{A.9}$$

To show this we contract the Eq: A.1 with the metric.

$$\begin{aligned} \left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right) \eta_{\mu\nu} &= -2\eta^{\mu\nu}\eta_{\mu\nu} \\ 2\gamma^\mu \gamma_\mu &= -2d \end{aligned}$$

$$\boxed{\not{p}\not{p} = -p^2} \tag{A.10}$$

To show this one contracts Eq: A.1 with two vectors, the result follows from the fact that the term is symmetric under interchange.

$$\not{p}\not{p} = \frac{1}{2} p_\mu p_\nu \left\{\gamma^\mu, \gamma^\nu\right\} = -p_\mu p_\nu \eta^{\mu\nu} = -p^2$$

$$\boxed{\gamma^\mu \gamma^\rho \gamma_\mu = (d-2)\,\gamma^\rho} \qquad \boxed{\gamma^\mu \not{p} \gamma_\mu = (d-2)\,\not{p}} \tag{A.11}$$

To show this we use Eq: A.1 to commute the γ-matrices and Eq: A.9 to simplify the result:

$$\gamma^\mu \gamma^\rho \gamma_\mu = \gamma^\mu \gamma^\rho \gamma^\nu \eta_{\mu\nu} = \gamma^\mu \left(-\gamma^\nu \gamma^\rho - 2\eta^{\nu\rho}\right) \eta_{\mu\nu} = -\left[\gamma^\mu \gamma_\mu\right] \gamma^\rho - 2\gamma^\rho = (d-2)\,\gamma^\rho$$

$$\boxed{\not{p}\not{l}\not{p} = p^2 \not{l} - 2(p \cdot l)\not{p}} \tag{A.12}$$

To show this we use Eq: A.1 in combination with Eq: A.10:

$$\not p \not l \not p = \not p l_\mu p_\nu \gamma^\mu \gamma^\nu = \not p l_\mu p_\nu \left(-\gamma^\nu \gamma^\mu - 2\eta^{\mu\nu}\right)$$
$$= \not p \left(-\not p \not l - 2\left(p \cdot l\right)\right) = p^2 \not l - 2\left(p \cdot l\right) \not p$$

For longer strings of γ-matrices the basic strategy is to use Eq: A.1 to interchange the γ-matrices until one can simplify them by means of one of the above relations.

Identities involving traces of γ-matrices

$$\boxed{Tr\left[\gamma^5\right] = 0} \tag{A.13}$$

To show this we use the cyclic identity to put γ^3 in front, then we commute it back onto its place:

$$Tr\left[\gamma^5\right] = \mathrm{i}Tr\left[\gamma^0\gamma^1\gamma^2\gamma^3\right] = \mathrm{i}Tr\left[\gamma^3\gamma^0\gamma^1\gamma^2\right] = -\mathrm{i}Tr\left[\gamma^0\gamma^1\gamma^2\gamma^3\right] = -Tr\left[\gamma^5\right]$$
$$\Rightarrow Tr\left[\gamma^5\right] = -Tr\left[\gamma^5\right]$$

$$\boxed{Tr\left[\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_{2n+1}}\right] = 0 \text{ with } n \in \mathbb{N}} \tag{A.14}$$

To show this we first add $1 = \left(\gamma^5\right)$ and the cyclic identity of the trace:

$$Tr\left[\gamma^{i_1}\gamma^{i_2}...\gamma^{i_{2n+1}}\right] = Tr\left[\gamma^{i_1}\gamma^{i_2}...\gamma^{i_{2n+1}}\left(\gamma^5\right)^2\right] = Tr\left[\gamma^5\gamma^{i_1}\gamma^{i_2}...\gamma^{i_{2n+1}}\gamma^5\right]$$

Now we anti-commute the other γ^5 matrix towards the beginning picking up $2n+1$ minus signs:

$$Tr\left[\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_{2n+1}}\gamma^5\right] = (-1)^{2n+1}\,Tr\left[\left(\gamma^5\right)^2\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_{2n+1}}\right] = -Tr\left[\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_{2n+1}}\right]$$
$$\Rightarrow Tr\left[\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_{2n+1}}\right] = -Tr\left[\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_{2n+1}}\right] = 0$$

$$\boxed{Tr\left[\gamma^\mu\gamma^\nu\right] = -4\eta^{\mu\nu}} \tag{A.15}$$

To show this use the cyclic property of the trace on half of the factor in the trace:

$$Tr\left[\gamma^\mu\gamma^\nu\right] = \frac{1}{2}\left(Tr\left[\gamma^\mu\gamma^\nu\right] + Tr\left[\gamma^\nu\gamma^\mu\right]\right) = \frac{1}{2}Tr\left[\{\gamma^\mu,\gamma^\nu\}\right] = -\eta^{\mu\nu}Tr\left[\mathbb{1}_4\right] = -4\eta^{\mu\nu}$$

$$\boxed{Tr\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left[\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right]} \tag{A.16}$$

- If all indices in Eq: A.16 were different then the trace is proportional to the trace of the γ^5 matrix which vanishes by Eq: A.13.
- If $\mu = \nu$ then we are left with $-\eta^{\mu\nu} Tr\left[\gamma^\rho \gamma^\sigma\right] = 4\eta^{\mu\nu}\eta^{\rho\sigma}$

We thus have three situations to discern: $\mu = \nu$, $\mu = \rho$ or $\mu = \sigma$. We know from the above that $\mu = \nu$ comes with a plus sign, similarly $\mu = \sigma$ should come with a plus sign due to the cyclic identity of the trace. If we use the defining Clifford algebra identity of Eq: A.1 to switch γ^ρ and γ^ν, we get two terms: the first one involves only three γ-matrices so it vanishes and the second comes with a minus sign.

Identities involving spinor contractions

$$\boxed{\overline{u}_{s'}\left(\mathbf{p}'\right)\gamma^\mu u_s\left(\mathbf{p}\right) = \left(p^\mu + p'^\mu\right)\delta_{s,s'} + \frac{\mathrm{i}q_\nu}{m}\overline{u}_{s'}\left(\mathbf{p}'\right)S^{\mu\nu}u_s\left(\mathbf{p}\right)} \tag{A.17}$$

This is the Gordon decomposition for the $u_s\left(\mathbf{p}\right)$ type spinors, to show it we use the Eq: A.6 and Eq: A.7, the vector $q \equiv p' - p$ and the tensor object $S^{\mu\nu} \equiv \frac{\mathrm{i}}{4}\left[\gamma^\mu, \gamma^\nu\right]$:

$$\begin{aligned}
\overline{u}_{s'}\left(\mathbf{p}'\right)\gamma^\mu u_s\left(\mathbf{p}\right) &= -\frac{1}{2m}\left[\overline{u}_{s'}\left(\mathbf{p}'\right)\not{p}'\gamma^\mu u_s\left(\mathbf{p}\right) + \overline{u}_{s'}\left(\mathbf{p}'\right)\gamma^\mu \not{p} u_s\left(\mathbf{p}\right)\right] \\
&= -\frac{1}{4m}\left(p_\nu + p'_\nu\right)\overline{u}_{s'}\left(\mathbf{p}'\right)\left\{\gamma^\mu, \gamma^\nu\right\}u_s\left(\mathbf{p}\right) - \frac{q_\nu}{4m}\overline{u}_{s'}\left(\mathbf{p}'\right)\left[\gamma^\mu, \gamma^\nu\right]u_s\left(\mathbf{p}\right) \\
&= \left(p^\mu + p'^\mu\right)\delta_{s,s'} + \frac{\mathrm{i}q_\nu}{m}\overline{u}_{s'}\left(\mathbf{p}'\right)S^{\mu\nu}u_s\left(\mathbf{p}\right)
\end{aligned}$$

$$\boxed{\overline{v}_r\left(\mathbf{k}\right)\gamma^\mu v_{r'}\left(\mathbf{k}'\right) = \left(k^\mu + k'^\mu\right)\delta_{r,r'} - \frac{\mathrm{i}q_\nu}{m}\overline{v}_r\left(\mathbf{k}\right)S^{\mu\nu}v_{r'}\left(\mathbf{k}'\right)} \tag{A.18}$$

This is the Gordon decomposition for the $v_{r'}\left(\mathbf{k}'\right)$ type spinors, to show it we use the Eq: A.6 and Eq: A.7, the vector $q \equiv k' - k$ and the tensor object $S^{\mu\nu} \equiv \frac{\mathrm{i}}{4}\left[\gamma^\mu, \gamma^\nu\right]$:

$$\begin{aligned}
\overline{v}_r\left(\mathbf{k}\right)\gamma^\mu v_{r'}\left(\mathbf{k}'\right) &= \frac{1}{2m}\left[\overline{v}_r\left(\mathbf{k}\right)\not{k}\gamma^\mu v_{r'}\left(\mathbf{k}'\right) + \overline{v}_r\left(\mathbf{k}\right)\gamma^\mu \not{k}' v_{r'}\left(\mathbf{k}'\right)\right] \\
&= \frac{1}{4m}\left(k_\nu + k'_\nu\right)\overline{v}_r\left(\mathbf{k}\right)\left\{\gamma^\mu, \gamma^\nu\right\}v_{r'}\left(\mathbf{k}'\right) + \frac{q_\nu}{4m}\overline{v}_r\left(\mathbf{k}\right)\left[\gamma^\mu, \gamma^\nu\right]v_{r'}\left(\mathbf{k}'\right) \\
&= \left(k^\mu + k'^\mu\right)\delta_{r,r'} - \frac{\mathrm{i}q_\nu}{m}\overline{v}_r\left(\mathbf{k}\right)S^{\mu\nu}v_{r'}\left(\mathbf{k}'\right)
\end{aligned}$$

Appendix B

The Feynman parameter trick

The Feynman parameter trick is a mathematical procedure by which one can combine denominators by introducing additional parameters over which one needs to integrate. This method is applied almost anytime in momentum integrals that are products of different denominators.

Generally for any number of different denominator we make use of Eq: B.1. The x_i are the *Feynman parameters*.

$$\boxed{\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2} ... A_n^{\alpha_n}} = \frac{\Gamma\left(\alpha_1 + \alpha_2 + ... + \alpha_n\right)}{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)...\Gamma\left(\alpha_n\right)} \int_0^1 dx_1 dx_2 ... dx_n \frac{x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} ... x_n^{\alpha_n - 1} \delta\left(1 - x_1 - x_2 - ... - x_n\right)}{\left(x_1 A_1 + x_2 A_2 + ... x_n A_n\right)^{\alpha_1 + \alpha_2 + ... + \alpha_n}}} \tag{B.1}$$

For the case $n = 2$, $\alpha_1 = \alpha_2 = 1$ and $A_1 = A$, $A_2 = B$:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{\left(xA + (1 - x) B\right)^2} \tag{B.2}$$

For the case $n = 3$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $A_1 = A$, $A_2 = B$, $A_3 = C$:

$$\frac{1}{ABC} = 2 \int_0^1 dx \int_0^{1-x} dy \frac{1}{\left(xA + yB + (1 - x - y) C\right)^3} \tag{B.3}$$

For the case $n = 3$, $\alpha_1 = 1$, $\alpha_2 = 2$ and $A_1 = A$, $A_2 = B$:

$$\frac{1}{AB^2} = 2 \int_0^1 dx \frac{1 - x}{\left(xA + (1 - x) B\right)^3} \tag{B.4}$$

An example application and additional details can be found in [18, sec. 6.3]. A more elaborate treatise with details about this and a related parametrisation known as the α-parametrisation is given in [40, ch. 2].

Appendix C

Momentum integral identities

Here we give a list of identities that can be used to simplify or fully evaluate momentum integrals in dimensional regularisation, which implies that $d = 4 - \epsilon$. In the following $f(q)$ or $f\left(q^2\right)$ will be an arbitrary well-behaved function dependent on q resp. q^2.

C.1 Lorentz invariant simplifications

$$\boxed{\int d^d q\, q^\mu f(q,p) = \frac{p^\mu}{p^2} \int d^d q\ (p \cdot q)\, f(q,p)} \tag{C.1}$$

To show this we use the fact that the result of the momentum integral is a vector independent of q, since there is only one such constant vector around we find for a still undetermined constant a:

$$\int d^d q\, q^\mu f(q,p) = a p^\mu$$

We now contract with the constant vector p_μ:

$$\int d^d q\ (p \cdot q)\, f(q,p) = a p^2 \Rightarrow a = \frac{1}{p^2} \int d^d q\ (p \cdot q)\, f(q,p)$$

$$\boxed{\int d^d q\, q^\mu q^\nu f\left(q^2\right) = \frac{\eta^{\mu\nu}}{d} \int d^d q\, q^2 f\left(q^2\right)} \tag{C.2}$$

To show this we use the fact that the result of the momentum integral is a symmetric two-indexed object independent of q, since there is only one such object around we find for a still undetermined constant a:

$$\int d^d q\, q^\mu q^\nu f\left(q^2\right) = a\eta^{\mu\nu}$$

We now contract with $\eta_{\mu\nu}$ to evaluate a:

$$\int d^d q\, q^2 f\left(q^2\right) = da \Rightarrow a = \frac{1}{d}\int d^d q\, q^2 f\left(q^2\right)$$

C.2 Analytic continuations

$$\boxed{\int d^d q\, f(q) = -\frac{1}{d}\int d^d q\, q^\mu \left(\frac{\partial}{\partial q^\mu} f(q)\right)} \tag{C.3}$$

To make this procedure reasonable, let us consider the chain rule:

$$\int d^d q\, \frac{\partial}{\partial q^\mu}\left(q^\mu f(q)\right) = \int d^d q\, q^\mu \left(\frac{\partial}{\partial q^\mu} f(q)\right) + d\int d^d q\, f(q)$$

Here we used that $\frac{\partial q^\mu}{\partial q^\mu} = d$. The left-hand side is a surface term that can be dropped if $f(q)$ vanishes sufficiently rapidly at infinity. The point of an analytic continuation is that we drop the surface term regardless of the asymptotic behaviour.

We can now consider the case that $f(q) = \frac{1}{(q^2+m^2)^\alpha}$ which gives:

$$\boxed{\int d^d q\, \frac{1}{(q^2+m^2)^\alpha} = \frac{2\alpha}{d}\int d^d q\, \frac{q^2}{(q^2+m^2)^{\alpha+1}}} \tag{C.4}$$

Writing $q^2 = \left(q^2+m^2\right) - m^2$ gives after some rearranging:

$$\boxed{\int d^d q\, \frac{1}{(q^2+m^2)^\alpha} = \frac{\alpha m^2}{\alpha - \frac{d}{2}}\int d^d q\, \frac{1}{(q^2+m^2)^{\alpha+1}}} \tag{C.5}$$

C.3 Wick rotation

Here, we largely follow [17, ch. 14]. Let us consider a momentum integral that we encounter often such as, for an arbitrary positive integer value for α:

$$\int d^d q \frac{1}{(q^2+m^2)^\alpha}$$

When we would try to perform the q^0 integral from $-\infty$ to $+\infty$ we see that there are two singularities in the integration domain at $q^0 = \pm\left(\sqrt{\mathbf{q}^2+m^2} - \mathrm{i}\epsilon\right)$, where we reintroduced the '$-\mathrm{i}\epsilon$'-prescription $m^2 \to m^2 - \mathrm{i}\epsilon$.

We now use some elementary results from complex analysis to deform the integration contour of q^0 from $-\infty$ to $+\infty$ towards $-\mathrm{i}\infty$ to $\mathrm{i}\infty$. This procedure is called a *Wick rotation* and it does not change the value of the integral, as long as the contribution of the arcs of the contours at infinity in the first and third quadrant vanish and the contours are not deformed in such a way that they cross a singularity. We also perform a coordinate shift to introduce Euclidean coordinates, coordinates with an Euclidean metric, via $q^0 \to \overline{q}^d$, $q^j \to \overline{q}^j$ and the measure $d^dq \to \mathrm{i}d^d\overline{q}$. For the momentum integrals, that we consider:

$$\boxed{\int d^dq\, \frac{1}{\left(q^2+m^2-\mathrm{i}\epsilon\right)^\alpha} = \mathrm{i}\int d^d\overline{q}\frac{1}{\left(\overline{q}^2+m^2-\mathrm{i}\epsilon\right)^\alpha}} \tag{C.6}$$

The validity of Eq: C.6 is guaranteed for $\frac{1}{\left(\overline{q}^2+m^2-\mathrm{i}\alpha\right)^\alpha} \to 0$ * faster than $\frac{1}{\overline{q}^d}$ as $\overline{q} \to \infty$. In other words, this formula holds only for $\alpha > d$.

C.4 Full evaluation

After we have simplified, analytically continued and Wick rotated our momentum integral, we are often in the position to fully evaluate our integral. We first introduce radial coordinates, evaluate the angular integrals and then do the radial integral, if d were integer-valued. For non-integer values for d we need to generalise the result of the calculation for integer-value d to the non-integers. This is possible by essentially replacing factorials by the Euler gamma function.

The result of taking the above actions is stated in Eq: C.7, furthermore we give some properties of the Euler gamma function for a nonnegative integer n and infinitesimal value for ϵ:

$$\boxed{\int \frac{d^d\overline{q}}{(2\pi)^d}\frac{1}{\left(\overline{q}^2+m^2\right)^\alpha} = \frac{\Gamma\left(\alpha-\frac{1}{2}d\right)}{(4\pi)^{\frac{d}{2}}\,\Gamma\left(\alpha\right)}\left(m^2\right)^{-\left(\alpha-\frac{1}{2}d\right)}} \tag{C.7}$$

with $\Gamma\left(x\right)$ satisfying:

*Note that $q^2 = -q_0^2+q_1^2+...+q_{d-1}^2 = \overline{q}_d^2+\overline{q}_1^2+...+\overline{q}_{d-1}^2 = \overline{q}^2$

$$\Gamma(n+1) = n! \tag{C.8}$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!}{n!2^{2n}}\sqrt{\pi} \tag{C.9}$$

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} - \gamma_E + \sum_{k=1}^{n}\frac{1}{k} + \mathcal{O}(\epsilon)\right] \tag{C.10}$$

where $\gamma_E \approx 0.5772...$ is the Euler-Mascheroni constant.

C.5 Useful series expansions

Here, we provide some useful series expansions of certain integrals that repeatedly come up in calculations that involve a small regularisation parameter. These series expansions were determined using the python library for symbolic mathematics called Sympy [23].

$$\boxed{\lim_{\kappa\to 0}\int_0^1 dx \ln\left(x^2 + (1-x)\kappa\right) \approx -2 + \pi\sqrt{\kappa} - \left(1 - \frac{1}{2}\ln\kappa\right)\kappa + \mathcal{O}\left(\kappa^{\frac{3}{2}}\right)} \tag{C.11}$$

$$\boxed{\lim_{\kappa\to 0}\int_0^1 dx\, x\ln\left(x^2 + (1-x)\kappa\right) \approx -\frac{1}{2} - \frac{1}{2}(1+\ln\kappa)\kappa + \mathcal{O}\left(\kappa^{\frac{3}{2}}\right)} \tag{C.12}$$

$$\boxed{\lim_{\kappa\to 0}\int_0^1 dx \frac{x(1-x)}{x^2+(1-x)\kappa} \approx -1 - \frac{1}{2}\ln\kappa + \frac{3}{4}\pi\sqrt{\kappa} - \left(\frac{3}{4} - \frac{1}{2}\ln\kappa\right)\kappa + \mathcal{O}\left(\kappa^{\frac{3}{2}}\right)} \tag{C.13}$$

$$\boxed{\lim_{\kappa\to 0}\int_0^1 dx \frac{x^2(1-x)}{x^2+(1-x)\kappa} \approx \frac{1}{2} - \frac{1}{2}\pi\sqrt{\kappa} - \left(\frac{1}{2} + \ln\kappa\right)\kappa + \mathcal{O}\left(\kappa^{\frac{3}{2}}\right)} \tag{C.14}$$

Appendix D

Asymptotic Series

Here, we give some explanation about asymptotic series in contrast with convergent series. This concept is important in perturbation series since both the QED and QCD perturbation series are asymptotic. This is already seen in the subset of bubble chain diagrams in QED of which the Borel transform has poles on the positive real axis. The all-order resummation of this set of diagrams is therefore divergent. We demonstrate the concept of an asymptotic series by the following mathematical example.

D.1 Formal definition of asymptotic series

Two functions $f(z)$ and $g(z)$ with $z, z_0 \in \mathbb{C}$ are asymptotically equal, denoted as $f(z) \sim g(z)$ as $z \to z_0$, when:

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 1$$

Depending on the value of z_0 one typically expands $f(z)$ in terms of a different local series of linear independent functions $\phi_n(z)$. For three typical values of z_0 the following functions are typically used to expand:

$$\phi_n(z) = \begin{cases} (z - z_0)^n & \text{if } z_0 \neq \pm\infty \\ \frac{1}{z^n} & \text{if } z_0 = \pm\infty \end{cases}$$

It is usually possible to make a reparametrisation such that it is possible to pick either of these sets of functions or yet an entirely different set of functions. A Taylor expansion around the point $z = z_0$ with $z_0 \neq \pm\infty$ uses precisely $\phi_n(z) = (z - z_0)^n$ the above given sets of functions with the coefficients $a_n = \frac{f^n(z_0)}{n!}$. As we will see in the example sometimes a different more natural expansion exists. Any expansion such as:

$$f(z) \sim \sum_{k=0}^{n} a_k \phi_k(z)$$

is valid when the following condition is met:

$$\lim_{z \to z_0} \frac{1}{\phi_n(z)} \left[f(z) - \sum_{k=0}^{n} a_k \phi_k(z) \right] = 0$$

From this we learn that the coefficients a_n are given by:

$$a_n = \lim_{z \to z_0} \frac{1}{\phi_n(z)} \left[f(z) - \sum_{k=0}^{n-1} a_k \phi_k(z) \right]$$

Furthermore it tells us that the absolute value difference between the function $f(z)$ and the series expansion up until order $\mathcal{O}(\phi_n(z))$ is of order $\mathcal{O}(\phi_{n+1}(z))$ close to z_0.

$$\lim_{z \to z_0} \left| f(z) - \sum_{k=0}^{n} a_k \phi_k(z) \right| = |a_{n+1} \phi_{n+1}(z)| = \mathcal{O}(\phi_{n+1}(z))$$

We give a few characteristics of asymptotic series:

- A given function $f(z)$ has a unique expansion around $z = z_0$ in terms of a series of functions $\phi_n(z)$.
- An asymptotic expansion around $z = z_0$ does not uniquely define a particular function $f(z)$.
- The asymptotic expansion of a product of functions $f(z)$ and $g(z)$ that both have an asymptotic expansion $f(z) \sim \sum_{k=0}^{n} a_n \phi_n(z)$ and $g(z) \sim \sum_{k=0}^{n} b_n \phi_n(z)$ around $z = z_0$ in terms of the same series of functions $\phi_n(z)$ is given by $f(z) g(z) \sim \sum_{k=0}^{n} c_n \phi_n(z)$ with $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.
- A convergent (power) series is always an asymptotic series though an asymptotic series need not be convergent, it can be divergent in any neighbourhood of $z = z_0$ except for at $z = z_0$.

It is possible for an asymptotic series (such as a QFT) that the first few orders converge to the $f(z)$ after which subsequent higher order calculations diverge from $f(z)$ we will see an example of this in the what comes next.

D.2 Example of asymptotic series

A nice example of a divergent asymptotic series is given by the error function, the integral over a normalised gaussian which often appears in statistics since it describes the normal distribution.

$$\mathrm{Erf}\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\; e^{-t^2} = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty dt\; e^{-t^2}$$

The error function $\mathrm{Erf}\,(x)$ has a natural expansion for $x \to \infty$ which is constructed by repeated partial integration of the above integral:

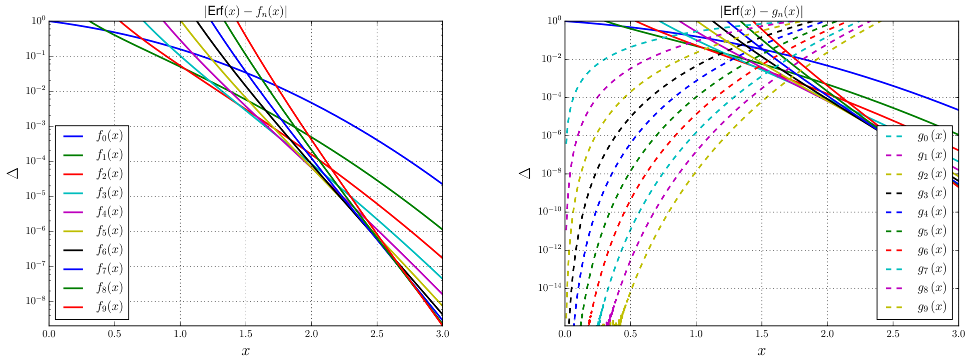
$$\int_x^\infty dt\, \frac{e^{-t^2}}{t^n} = \frac{e^{-x^2}}{2x^{n+1}} + \frac{(-1)\,(n+1)}{2} \int_x^\infty dt\, \frac{e^{-t^2}}{t^{n+2}}$$

Repeated application of the above formula allows us to expand the error function in terms of a series of functions $\phi_n\,(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \left(\frac{1}{x^{2n+1}}\right)$ and coefficients $a_n = (-1)^{n+1}\, \frac{(2n-1)!!}{2^n}$:

$$\begin{aligned} \mathrm{Erf}\,(x) &\sim 1 + \frac{e^{-x^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^{n+1}\, \frac{(2n-1)!!}{2^n} \frac{1}{x^{2n+1}} \\ &\sim 1 + \sum_{n=0}^{\infty} a_n \phi_n\,(x) \end{aligned}$$

This alternating asymptotic series diverges as can be confirmed using the ratio test, the ratio between the coefficients a_{n+1} and a_n diverges for $n \to \infty$. Furthermore the absolute difference between the error function and asymptotic series up to $\mathcal{O}\,(\phi_n\,(x))$ is of order $\mathcal{O}\,(\phi_{n+1}\,(x))$. The accuracy by which the asymptotic series approximates the error function is shown in Fig: D.1(a).

In Fig: D.1(a) one sees that for a given $x > 0.43$ the approximations of the $f_n\,(x)$ first converge towards the error function, but eventually a maximum in accuracy will be reached and the approximations begin to diverge again. In contrast one sees that for the convergent Taylor series in Fig: D.1(b) around $x = 0$ every approximation is better than the previous one.



(a) The divergent asymptotic series.

(b) The convergent Taylor series with the asymptotic series as a comparison.

Figure D.1: *This figure shows the accuracy and divergent behaviour of the asymptotic series of the error function and the accuracy and convergent behaviour of the Taylor series of the error function around* $x = 0$. $\Delta = \left| Erf(x) - f_n(x) \right|$ *or* $\Delta = \left| Erf(x) - g_n(x) \right|$ *as a function of* x *for consecutive approximations.*

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