Scalar Quantum Electrodynamics in de Sitter Space from the Non-Perturbative Renormalization Group

by

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Supervised by Dr. Tomislav Prokopec

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Abstract

We apply the non-perturbative renormalization group techniques to integrate out strongly fluctuating infrared modes for a scalar field in de Sitter universe. Using an spontaneously broken potential that is already know, using the two non-perturbative techniques, the stochastic formalism and the renormalization group approach, to spontaneously recover the symmetry due to the strongly fluctuating IR modes. And adding photons interaction to the effective potential we show that the symmetry restoration gets enhanced by the photons. Finally we compare the know results obtained using both methods for the scalar mass and the photon mass, that is know to acquire a non-vanishing mass during inflation, allowing us to compare both methods.

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Contents

Chapter 1

Introduction

1.1 De Sitter space and inflation

The main assumption of modern cosmology is that the universe is isotropic and homogeneous. That assumption implies that space is maximally symmetric, isotropy and homogeneity translate into invariance under rotations and invariance under translations respectively [17, 7]. The Riemann tensor for a maximally symmetric $(d+1)$-dimensional space is,

$$R_{\mu\rho\nu\sigma} = \frac{1}{d(d+1)}(g_{\mu\nu}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R \tag{1.1}$$

where R is the Ricci scalar. De Sitter space is the maximally symmetric space with positive curvature, we use the FLRW metric [7, 19] to describe it,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\mathrm{d}\vec{X}^2, \tag{1.2}$$

where $a(t)$ is the scale factor and $\vec{X} = a(t)\vec{x}$ are the comoving coordinates,

$$a(t) = e^{Ht}, \tag{1.3}$$

and H is the Hubble parameter that measures the expansion rate of the universe and is defined as,

$$H = \frac{1}{a(t)}\frac{\mathrm{d}a(t)}{\mathrm{d}t}. \tag{1.4}$$

We can rewrite the FLRW metric using the conformal time η, defined as $\mathrm{d}t = a(t)\mathrm{d}\eta$,

$$\mathrm{d}s^2 = a(\eta)^2(-\mathrm{d}\eta^2 + \mathrm{d}\vec{X}^2), \tag{1.5}$$

where now,

$$a(\eta) = -\frac{1}{H\eta}, \qquad (-\infty < \eta < 0). \tag{1.6}$$

De Sitter space is also the maximally symmetric space solution of Einstein's equations with a positive cosmological constant. From the equation of state the vacuum energy density, ρ, and pressure, p, for a de Sitter universe are,

$$\rho = -p = \frac{3}{8\pi G}\frac{R}{d(d+1)}, \tag{1.7}$$

where the Ricci scalar for de Sitter space is,

$$R = H^2 d(d+1). \tag{1.8}$$

This space gives a negative pressure (1.7) ($R > 0$), which means that the universe described by this metric is expanding eternally and since H is constant it describes what is called eternal inflation. Inflation is an important phase of the early universe that solves many know problems of the standard Big Bang model [37], like the flatness problem, the horizon problem and the magnetic-monopole problem.

1.2 Stochastic inflation

Here we introduce briefly the stochastic formalism [34] as a method to integrate out strongly fluctuating infrared modes during inflation. For a general scalar field theory with a scalar field φ,

$$S[\varphi] = \int \mathrm{d}t \int \mathrm{d}^d\vec{X}\ \sqrt{-g}\left\{-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi(t,\vec{X})\partial_\nu\varphi(t,\vec{X}) - V(\varphi)\right\}, \tag{1.9}$$

we can formulate an stochastic formalism by splitting the field into short and long wavelength modes. Infrared modes, those with momentum $p < a(t)H$, are treated as stochastic variables (white noise), and ultraviolet modes, those with momentum $p > a(t)H$, are treated perturbatively using standard renormalization techniques and the coupling between short and long wavelength modes is modeled by a Markovian random force. During inflation more and more modes enter the IR regime and these modes are integrated out. To compute expectations values a probability density function (PDF) $\rho(t,\varphi)$ is defined, and it satisfies the Fokker-Planck equation,

$$\frac{\partial\rho(t,\varphi)}{\partial t} = \frac{H^3}{8\pi^2}\frac{\partial^2\rho(t,\varphi)}{\partial\varphi^2} + \frac{1}{3H}\frac{\partial}{\partial\varphi}\left(V'(\varphi)\rho(t,\varphi)\right), \tag{1.10}$$

then the expectations values are computed as,

$$\langle F(\varphi)\rangle = \int \mathrm{d}\varphi\ F(\varphi)\rho(t,\varphi), \tag{1.11}$$

where the late-time equilibrium state is described by the following PDF,

$$\lim_{t\to\infty}\rho(t,\varphi) \propto \exp\left(-\frac{8\pi^2}{3H^4}V(\varphi)\right). \tag{1.12}$$

1.3 Motivation

The stochastic formalism is a technique commonly used to integrate out IR modes during inflation, in the present work we use a different technique to integrate out all the unwanted IR modes, the renormalization group approach. For a Bunch-Davies vacuum, which we will use for this work, IR modes are divergent for (massive) minimally coupled scalars in de Sitter background although they are not really divergent [22], they are divergent for that particular choice of vacuum that is unphysical for IR modes, they are strongly fluctuating and thus give big contributions that we want to integrate out to see how these quantum effects change the dynamics of the field as we progressively integrate them out.

The motivation for this work is to use the renormalization group to study the spontaneous symmetry breaking in de Sitter space and compare the results with the previously obtained ones with both the stochastic formalism and renormalization group approach, and extend the last one with the addition of photons interactions. As a reference, the stochastic formalism described in chapter 1.2 has been applied to an $O(N)$ symmetric scalar field theory with a Higgs-like potential by Prokopec et al. [18]. In that work the authors show that infrared modes restore the symmetry in eternal inflation at late times, this symmetry restoration due to IR modes was discussed previously by Serreau in [32]. They apply the same formalism to scalar quantum electrodynamics in [29] for eternal inflation, there they show that the leading contributions are logarithmic and that the photons acquire mass during inflation. We highlight also the work by Serreau et al. in [33] and [10] where they use a Higgs-like potential in eternal inflation but instead of using the stochastic formalism they use the renormalization group approach. Our task in this thesis is to extend that to scalar quantum electrodynamics (SQED). Serreau shows that there is symmetry restoration and that the late-time equilibrium state of the stochastic formalism is equivalent to integrating out all the infrared modes using the renormalization group approach.

In the present work we use the renormalization group approach to a Higgs-like potential in first place and then we introduce interactions coming from photons and compare the results with those obtained by Prokopec and Serreau. We begin with a review of the formalism needed, first the non-equilibrium QFT in chapter 2, that is needed for an expanding spacetime, in chapter 3 we introduce the effective field theory and renormalization group techniques that will be the key tool for this work, and finally in chapter 4 we present the SQED theory and calculate its effective potential and energy-momentum tensor. The final results are presented in chapter 5 and we conclude this work with some conclusions and outlook in chapter 6.

Chapter 2

Non-equilibrium QFT

It is well known that inflationary cosmology has to be formulated as a non-equilibrium quantum field theory because it does not admit a timelike killing vector [42], for our choice of coordinates, which leads to particle production during the expansion of the universe and the initial vacuum state will no longer be, in general, the same vacuum state for late times and the overlap of the *in* and *out* vacua is not trivial. This means that the usual quantum field theory cannot be used to calculate vacuum amplitudes and we need to use the non-equilibrium QFT formalism that will be presented in this chapter. We first introduce the closed time path formalism and then we obtain a general expression for the non-equilibrium generating functional.

2.1 The closed time path formalism

The closed time path formalism, Schwinger-Keldysh formalism or *in-in* formalism was first introduced by Schwinger [31] and Keldysh [16] to deal with the problem of non-equilibrium processes in quantum field theory. This formalism is also known as the *in-in* formalism rather than the usual *in-out* formalism used in equilibrium quantum field theory.

The main element in the *in-out* formalism is the generating functional $W[J]$, i.e.

the vacuum-to-vacuum amplitude in the presence of a source J,

$$Z[J] = \langle 0_{out} \,|\, 0_{in} \rangle_J \equiv \langle 0; \eta_1 \,|\, 0; \eta_1, \eta_0 \rangle_J \equiv e^{iW[J]}, \tag{2.1}$$

where have defined $|0_{in}\rangle \equiv |0; \eta_0\rangle$ as the *in* vaccum (vacuum state at initial time η_0), $|0_{out}\rangle \equiv |0; \eta_1\rangle$, as the *out* vacuum, (vacuum state at final time η_1), and

$$|0_{in}\rangle_J \equiv |0; \eta_1, \eta_0\rangle_J = T \left[\exp\left(i \int_{\eta_0}^{\eta_1} \mathrm{d}\eta \int \mathrm{d}^d\vec{X} \, \sqrt{-g} \; J(x)\varphi_H(x) \right) \right] |0; \eta_0\rangle \,, \tag{2.2}$$

where T is the time order operator and the subindex H means that the operator is in the Heisenberg picture, so this is the *in* vacuum evolved to the final time η_1 by the source J. This can be writing as a path integral [12],

$$\begin{aligned} Z[J] &= \langle 0_{out}| \, T \left[\exp\left(i \int_{\eta_0}^{\eta_1} \mathrm{d}\eta \int \mathrm{d}^d\vec{X} \, \sqrt{-g} \; J(x)\varphi_H(x) \right) \right] |0_{in}\rangle \\ &= \int \mathcal{D}\varphi \; e^{i\left(S[\varphi] + \int \mathrm{d}^{d+1}x \, \sqrt{-g} \; J(x)\varphi(x)\right)}, \end{aligned} \tag{2.3}$$

where in the last expression all the time integrals go from η_0 to η_1.

In a expanding universe $|0_{in}\rangle$ (vacuum state at $\eta = \eta_0$) and $|0_{out}\rangle$ (vacuum state at $\eta = \eta_1 \gg \eta_0$) states can in general be different [5, 13]. Therefore is not possible to calculate expectations values using this formalism if the *in* and *out* vacuum states are different. It is only possible to calculate the matrix elements $\langle 0_{out} \,|\, 0_{in}\rangle$, rather than the relevant expectation values taken with respect to the same vacuum state. This is solved by evolving the *in* vacuum state with a source J^+ forwards in time from η_0 to the *out* state at time η_1, and then evolving that *out* state backwards in time from η_1 to the initial state at time η_0. Since the state at time η_1 is unknown, we have to sum over all possible *out* states, $|\varphi; \eta_1\rangle$, that are eigenstates of the Heisenberg field operator. Then, the non-equilibrium generating functional is,

$$
\begin{aligned}
Z[J^+, J^-] = {}_{J^-}\langle 0_{in} \,|\, 0_{in}\rangle_{J^+} &= \int \mathcal{D}\varphi \;\; {}_{J^-}\langle 0_{in} \,|\, \varphi; \eta_1 \rangle \, \langle \varphi; \eta_1 \,|\, 0_{in}\rangle_{J^+} \\
= \int \mathcal{D}\varphi \;\; \langle 0_{in} | \, T_- & \left[\exp\left(-i \int_{\eta_0}^{\eta_1} \mathrm{d}\eta \int \mathrm{d}^d \vec{X} \, \sqrt{-g} \; J^-(x) \varphi_H(x) \right) \right] |\varphi; \eta_1 \rangle \\
\langle \varphi; \eta_1 | \, T_+ & \left[\exp\left(i \int_{\eta_0}^{\eta_1} \mathrm{d}\eta \int \mathrm{d}^d \vec{X} \, \sqrt{-g} \; J^+(x) \varphi_H(x) \right) \right] |0_{in}\rangle \, ,
\end{aligned}
\tag{2.4}
$$

where $T_\pm$ are the time and anti-time order operators and the eigenstates satisfy,

$$
\varphi_H(x) \, |\varphi; \eta_1\rangle = \varphi(\vec{X}) \, |\varphi; \eta_1\rangle \, , \tag{2.5a}
$$

$$
\int \mathcal{D}\varphi \;\; |\varphi; \eta_1\rangle \, \langle \varphi; \eta_1| = \mathbb{1}. \tag{2.5b}
$$

In the path integral formalism this represents a sum over all the paths, from the initial vacuum $|0_{in}\rangle$ and time η_0 to the state $|\varphi; \eta_1\rangle$ at time η_1, forward in time in the presence of the source J^+ and then backwards in the presence of the source J^- to the same initial vacuum $|0_{in}\rangle$,

$$
\begin{aligned}
Z[J^+, J^-] &= \int \mathcal{D}\varphi \; e^{i\left(S[\varphi] + \int_{\eta_0}^{\eta_1} \mathrm{d}\eta \int \mathrm{d}^d \vec{X} \, \sqrt{-g} \; J^+(x)\varphi(x) - \int_{\eta_0}^{\eta_1} \mathrm{d}\eta \int \mathrm{d}^d \vec{X} \, \sqrt{-g} \; J^-(x)\varphi(x) \right)} \\
&\equiv \int \mathcal{D}\varphi \; e^{i\left(S[\varphi] + \int_x J(x)\varphi(x) \right)},
\end{aligned}
\tag{2.6}
$$

where we have defined,

$$
\int_x \equiv \int_{\mathcal{C}} \mathrm{d}\eta \int \mathrm{d}^d \vec{X} \, \sqrt{-g}, \tag{2.7}
$$

with the closed time path $\mathcal{C}$ shown in figure 2-1.

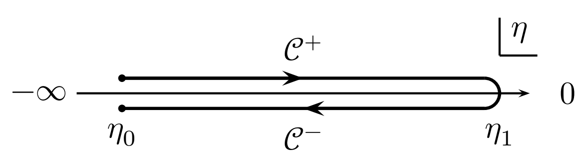

Figure 2-1: Closed time path $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$. The upper branch $\mathcal{C}^+$ goes forward from η_0 to η_1 and the lower branch $\mathcal{C}^-$ goes backward from η_1 to η_0.

This formalism is the standard technique used in non-equilibrium initial valued problems and will be used throughout all the calculations in this work.

2.2 Non-equilibrium generating functional

In a general non-equilibrium system the dynamics is completely specified by the initial state, that in general will not be a vacuum state. This initial state, $|\varphi_{in}\rangle$, is given by the density matrix, ρ_0, at initial time η_0. Once the initial state is specified, the dynamics is completely determined by the non-equilibrium generating functional, this means that all the information is contained in the density matrix ρ_H.

For a given density operator ρ, the ensemble average of an operator O is,

$$\langle O_H \rangle_\rho = \mathrm{Tr}\left\{\rho_H O_H\right\}. \tag{2.8}$$

Then, since the non-equilibrium generating functional is the vacuum expectation value in the presence of two sources with the initial condition given by the (arbitrary) density matrix ρ_0 at initial time η_0, we can define it as [6, 21, 23],

$$\begin{aligned} Z[J^+, J^-, \rho] = {}_{J^-}\langle \varphi_{in} \,|\, \varphi_{in} \rangle_{J^+,\rho} \equiv {}_{J^-}\langle \varphi; \eta_1, \eta_0 \,|\, \varphi; \eta_1, \eta_0 \rangle_{J^+,\rho} = \\ \mathrm{Tr}\left\{ \rho_H T_- \left[\exp\left(-i \int_{\eta_0}^{\eta_1} \mathrm{d}\eta \int \mathrm{d}^d\vec{X}\, \sqrt{-g}\; J^-(x)\varphi_H(x) \right) \right] \right. \\ \left. T_+ \left[\exp\left(i \int_{\eta_0}^{\eta_1} \mathrm{d}\eta \int \mathrm{d}^d\vec{X}\, \sqrt{-g}\; J^+(x)\varphi_H(x) \right) \right] \right\}. \end{aligned} \tag{2.9}$$

We write the trace explicitly,

$$\begin{aligned} Z[J^+, J^-, \rho] = \int \mathcal{D}\varphi \mathcal{D}\varphi' \mathcal{D}\varphi'' \left\{ \langle \varphi; \eta_0 \,|\, \rho_H \,|\, \varphi'; \eta_0 \rangle \right. \\ \langle \varphi'; \eta_0 | \, T_- \left[\exp\left(-i \int_{\eta_0}^{\eta_1} \mathrm{d}\eta \int \mathrm{d}^d\vec{X}\, \sqrt{-g}\; J^-(x)\varphi_H(x) \right) \right] |\varphi''; \eta_1\rangle \\ \left. \langle \varphi''; \eta_1 | \, T_+ \left[\exp\left(i \int_{\eta_0}^{\eta_1} \mathrm{d}\eta \int \mathrm{d}^d\vec{X}\, \sqrt{-g}\; J^+(x)\varphi_H(x) \right) \right] |\varphi; \eta_0\rangle \right\} \end{aligned} \tag{2.10}$$

$$= \int \mathcal{D}\varphi \left\{ \langle \varphi; \eta_0 \,|\, \rho_H \,|\, \varphi; \eta_0 \rangle \, e^{i\left(S[\varphi] + \int_x J(x)\varphi(x) \right)} \right\}.$$

The last task remaining is to parametrize the density matrix, we can do it generally as [6],

$$\langle \varphi; \eta_0 \,|\, \rho_H \,|\, \varphi; \eta_0 \rangle = e^{iK[\varphi]}, \tag{2.11}$$

and expand the functional $K[\varphi]$ in powers of the field,

$$\begin{aligned} K[\varphi] = K_0 + \int_x K_1(x)\varphi(x) + \frac{1}{2}\int_{x,y} K_2(x,y)\varphi(x)\varphi(y) \\ + \frac{1}{6}\int_{x,y,z} K_3(x,y,z)\varphi(x)\varphi(y)\varphi(z) + \cdots . \end{aligned} \tag{2.12}$$

For most of the situations a Gaussian initial density matrix is a very good approximation and we can neglect the terms K_n for $n > 2$. We can absorbe the K_0 term in the measure of the path integral, and the K_1 term can be absorbed in the definition of the source J. Then, the final expression for the non-equilibrium generating functional for a general (gaussian) initial state is,

$$Z[J,K] = e^{iW[J,K]} = \int \mathcal{D}\varphi \; e^{i\left(S[\varphi] + \int_x J(x)\varphi(x) + \frac{1}{2}\int_{x,y} K(x,y)\varphi(x)\varphi(y) \right)}. \tag{2.13}$$

Although expression (2.13) is the most general one, for the rest of this work we will consider an initial vacuum state, so we will neglect the quadratic source and use the generating functional (2.6).

Chapter 3

Effective field theory

In physics we can find interesting phenomena at all scales, from the shortest time scale to the largest time scale, like the age of the universe, or from the lowest energy scale to the largest energy scale, like the Big Bang. Usually we are interested in the physics happening at one particular scale and the dynamics at that scale does not depend on the detailed dynamics at a larger or smaller scale, then we can isolate the set of phenomena at that particular scale from the rest. To do that we set the parameters that are very large or very small compared with the ones that we are studying to be infinity or zero respectively, then the effect of this parameters can be treated as small perturbations to the parameters at the interesting scale. That way of treating the relevant physics that we are interested in is called a effective theory and is widely used in all the branches of physics.

In a effective field theory we do not need to know if it exists a renormalizable field theory at lower (higher) energies than a characteristic scale E_0, we describe the physics at a given energy scale $E \gg E_0$ ($E \ll E_0$) using quantum field theory with a finite number of parameters [9, 4]. To do that we introduce an IR (UV) cutoff and progressively integrate out energy modes starting from that cutoff. The evolution, or flow, of the different parameters of the theory as we integrate out more and more modes is called the renormalization group. In this section we explicitly derive the effective action of our effective field theory and we introduce the regulator and flow equation for the effective action.

3.1 Effective action

We start by constructing the effective action that gives us the equations of motion for the vacuum expectation value of the fields by the variational principle. This is useful if we are interested in studying the spontaneous symmetry breaking of a theory, for which the relevant quantity is the vacuum expectation value of the fields [27, 3].

The vacuum expectation value of the field, $\phi(x)$, is defined from the generating functional (2.13),

$$\frac{1}{\sqrt{-g(x)}}\frac{\delta W[J,K]}{\delta J(x)} = \langle\varphi(x)\rangle_{J,K} \equiv \phi(x). \tag{3.1}$$

The effective action has to be defined then as a functional of ϕ such as the variation with respect to the field ϕ is zero for vanishing external sources, i.e. the variation of the functional gives us the equation of motion for ϕ. That functional is the Legendre transform [41] of the generating functional $W[J,K]$ with respect to J and K [8, 2]. We perform first the Legendre transform with respect to J to obtain the 1PI effective action,

$$\Gamma^{(1)}[\phi] = W[J,K] - \int_x J(x)\frac{1}{\sqrt{-g(x)}}\frac{\delta W[J,K]}{\delta J(x)} = W[J,K] - \int_x J(x)\phi(x). \tag{3.2}$$

The 1PI effective action satisfies the required condition,

$$\frac{1}{\sqrt{-g(x)}}\left.\frac{\delta\Gamma^{(1)}[\phi]}{\delta\phi(x)}\right|_{J=0} = 0. \tag{3.3}$$

Using the two-point correlation function,

$$G(x,y) \equiv \langle\varphi(x)\varphi(y)\rangle_{J,K} = \langle T_{\mathcal{C}}\left[\varphi(x)\varphi(y)\right]\rangle_{J,K} - \langle\varphi(x)\rangle_{J,K}\langle\varphi(y)\rangle_{J,K}, \tag{3.4}$$

where $T_{\mathcal{C}}$ denotes time-ordering along the closed time path $\mathcal{C}$,

$$T_{\mathcal{C}}\left[\varphi(x)\varphi(x')\right] = \theta_{\mathcal{C}}\left(\eta-\eta'\right)\varphi(x)\varphi(x') + \theta_{\mathcal{C}}\left(\eta'-\eta\right)\varphi(x')\varphi(x), \tag{3.5}$$

we have that,

$$\frac{1}{\sqrt{g(x)g(y)}}\frac{\delta W[J,K]}{\delta K(x,y)} = \frac{1}{2}\left\langle T_{\mathcal{C}}\left[\varphi(x)\varphi(y)\right]\right\rangle_{J,K} = \frac{1}{2}\left(\phi(x)\phi(y) + G(x,y)\right). \tag{3.6}$$

Then performing a second Legendre transform with respect to K we obtain the 2PI effective action,

$$\begin{aligned}\Gamma^{(2)}[\phi,G] &= \Gamma^{(1)}[\phi] - \int_{x,y} K(x,y)\frac{1}{\sqrt{g(x)g(y)}}\frac{\delta\Gamma^{(1)}[\phi]}{\delta K(x,y)} \\ &= W[J,K] - \int_x J(x)\phi(x) - \frac{1}{2}\int_{x,y} K(x,y)\left(\phi(x)\phi(y) + G(x,y)\right),\end{aligned} \tag{3.7}$$

which satisfies,

$$\frac{1}{\sqrt{-g(x)}}\left.\frac{\delta\Gamma^{(2)}[\phi,G]}{\delta\phi(x)}\right|_{J=K=0} = 0, \tag{3.8a}$$

$$\frac{1}{\sqrt{g(x)g(y)}}\left.\frac{\delta\Gamma^{(2)}[\phi,G]}{\delta G(x,y)}\right|_{J=K=0} = 0. \tag{3.8b}$$

The 2PI effective action contains more information than the 1PI effective action at the same perturbation order, nevertheless we will use the 1PI effective action for the rest of this work.

3.2 Renormalization group

For a general scalar field theory in de Sitter space [15],

$$S[\varphi] = \int_x \left\{-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi(x)\partial_\nu\varphi(x) - V(\varphi)\right\}, \tag{3.9}$$

the mode function solutions are divergent for low energy modes. Then we need to add a cutoff at an arbitrary low energy scale from which we will be integrating out lower energy modes that may be divergent. We impose the cutoff at the comoving euclidean momentum κ from which we will be integrating out low energy modes. For that we define the average effective action $\Gamma_\kappa[\phi]$ as the 1PI effective action for a vacuum initial state i.e. only one source, J, and including only fluctuations with momentum $p \lesssim \kappa$, which is defined as the following modified Legendre transformation,

$$\Gamma_\kappa[\phi] = W_\kappa[J] - \int_x J(x)\phi(x) - \frac{1}{2}\int_{x,y} R_\kappa(x,y)\phi(x)\phi(y), \tag{3.10}$$

where $W_\kappa[J]$ is defined as,

$$e^{iW_\kappa[J]} = \int \mathcal{D}\varphi\ e^{i\left(S[\varphi]+\int_x J(x)\varphi(x)+\frac{1}{2}\int_{x,y} R_\kappa(x,y)\varphi(x)\varphi(y)\right)}. \tag{3.11}$$

The cutoff momentum-dependent function R_κ is the regulator and takes care that the IR modes are suppressed and the UV modes are not modified, it has to satisfy the following conditions in momentum space to be a good regulator [20],

$$\lim_{p/\kappa\to 0} R_\kappa(p) > 0, \tag{3.12a}$$

$$\lim_{\kappa/p\to 0} R_\kappa(p) \to 0, \tag{3.12b}$$

$$\lim_{\kappa\to\infty} R_\kappa(p) \to \infty. \tag{3.12c}$$

The condition (3.12a) is there to ensure that the cutoff is an IR regulator and all the possible IR divergences remain finite in the limit $p \to 0$. The condition (3.12b) is needed to recover the full effective action when we remove the cutoff, $\Gamma_{\kappa=0}[\phi] = \Gamma[\phi]$, when $\kappa \to 0$. And the condition (3.12c) is necessary to recover the full quantum action, $\Gamma_{\kappa=\infty}[\phi] = S[\phi]$, often referred as the bare action, in the limit $\kappa \to \infty$. Any function that satisfies that conditions can be used as a good regulator, we will use a simple one, the Litim regulator [20],

$$R_\kappa(p) = Z_\kappa\left(\kappa^2 - p^2\right)\theta\left(\kappa^2 - p^2\right), \tag{3.13}$$

where $\sqrt{Z_\kappa}$ is a renormalization factor for the scalar field ϕ that could in principle depend on the moment cutoff κ. An important remark here is that the regulator breaks de Sitter symmetry, but this is something that always happens when we introduce a cutoff by hand in a theory. This cutoff is unphysical and is introduced to study the running of the different parameters of the theory with that cutoff. This cutoff acts like a momentum-dependent mass term that vanishes for large momenta and recovers the original action, and becomes infinity for low momenta making those modes non-dynamical and decoupling from the action.

We use the local potential approximation (LPA') [38] for the average effective action, which is obtained as a derivative expansion of the fields and so preserves the same symmetries as the original action, and we neglect higher derivative terms. We introduce also the field renormalization factor Z_κ,

$$\Gamma_\kappa[\phi] = \int_x \left\{ -\frac{Z_\kappa}{2} g^{\mu\nu} \partial_\mu \phi(x) \partial_\nu \phi(x) - V_\kappa(\phi) \right\}, \tag{3.14}$$

where we have set already the source J to be zero.

We need to calculate the evolution of all the parameters of the theory as we integrate out more and more infrared modes, that flow of the effective action is derived in appendix B [39] and is given by the Wetterich equation,

$$\dot{\Gamma}_\kappa[\phi] = \frac{1}{2} \operatorname{Tr} \left\{ \dot{R}_\kappa(x,y) G_\kappa(y,x) \right\}, \tag{3.15}$$

where we have defined the following notation, $\dot{F} \equiv \kappa \partial_\kappa F$ and $\operatorname{Tr} \equiv \int_{x,y}$, and $G_\kappa(x,y)$ is defined in (B.11). This equation is in most of the cases unsolvable analytically and has to be solved numerically [40].

Chapter 4

Scalar QED

The theory that describes the interaction between complex scalars and photons is called scalar quantum electrodynamics, SQED. The theory is constructed from the QED lagrangian and changing the spinor field for a complex scalar field [27]. The SQED action for a massless, minimally coupled (MMC) scalar is,

$$S = \int_x \left\{ -g^{\mu\nu} \left(D_\mu \phi \right)^\dagger \left(D_\nu \phi \right) - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \right\}, \tag{4.1}$$

where,

$$D_\mu = \partial_\mu - ieA_\mu, \tag{4.2a}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = D_\mu A_\nu - D_\nu A_\mu, \tag{4.2b}$$

and the field ϕ is a complex scalar field. The action (4.1) is invariant under the following gauge transformations,

$$A_\mu(x) \to \tilde{A}_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x), \tag{4.3a}$$

$$\phi(x) \to \tilde{\phi}(x) = \phi(x) e^{-ie\alpha(x)}. \tag{4.3b}$$

The equation of motion for the vector field is,

$$\frac{1}{\sqrt{-g}} \partial_\rho \left(\sqrt{-g} g^{\rho\sigma} g^{\mu\beta} F_{\sigma\beta} \right) - ieg^{\mu\sigma} \left(\phi^\dagger \partial_\sigma \phi + \phi \partial_\sigma \phi^\dagger \right) - 2e^2 g^{\mu\sigma} A_\sigma \phi^\dagger \phi = 0, \tag{4.4}$$

and for the scalar field,

$$\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\left(D_\mu\phi\right)^\dagger\right)+ieg^{\mu\nu}A_\nu\left(D_\mu\phi\right)^\dagger=0. \tag{4.5}$$

4.1 Gauge invariant formulation

The previous formulation of SQED is not gauge invariant, we want to fix those gauge degrees of freedom and for that we decompose the complex scalar field in two real fields, $|\phi(x)|$ and $\theta(x)$,

$$\phi(x)=\frac{|\phi(x)|}{\sqrt{2}}e^{i\theta(x)}. \tag{4.6}$$

In terms of these fields the action is gauge invariant under the transformation,

$$\theta(x)\to\quad\tilde{\theta}(x)=\theta(x)-e\alpha(x). \tag{4.7}$$

To make the action completely gauge invariant we define a modified vector field as,

$$A_\mu^{\text{g.i.}}(x)=A_\mu(x)-\frac{\partial_\mu\theta(x)}{e}, \tag{4.8}$$

then we can write the action in a gauge invariant way,

$$S=\int_x\left\{-\frac{1}{2}g^{\mu\nu}\partial_\mu|\phi|\,\partial_\nu|\phi|-\frac{1}{2}e^2g^{\mu\nu}|\phi|^2A_\mu^{\text{g.i.}}A_\nu^{\text{g.i.}}-\frac{1}{4}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}\right\}. \tag{4.9}$$

The new equation of motion for $A_\mu^{\text{g.i.}}$ is,

$$\frac{1}{\sqrt{-g}}\partial_\rho\left(\sqrt{-g}g^{\rho\sigma}g^{\mu\beta}F_{\sigma\beta}\right)-e^2g^{\mu\sigma}A_\sigma^{\text{g.i.}}|\phi|^2=0, \tag{4.10}$$

and for $|\phi|$ is,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu|\phi|\right)-e^2g^{\mu\nu}A_\mu^{\text{g.i.}}A_\nu^{\text{g.i.}}|\phi|=0. \tag{4.11}$$

If we apply the ∂_μ operator to the equation of motion for the vector field we obtain one extra constrain, the Lorentz condition,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}A_\nu^{\text{g.i.}}e^2\left|\phi\right|^2\right)=0. \tag{4.12}$$

That constrain makes us have in total $d+1$ degrees of freedom, 1 for the scalar field and d for the vector field, because one component is fixed by (4.12). We observe that the current terms drops out from the action and the equations of motion, this means that there are no 3-point vertices in the action, only 4-point vertices, like in figure 4-1.

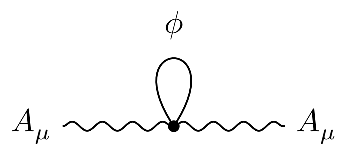

Figure 4-1: 4-point interaction for the SQED action (4.9).

In order to renormalize the theory we introduce counterterms in the action, we do that following [29, 30],

$$\begin{aligned} S=\int_x\Big\{&-\frac{1}{2}\left(1+\delta Z_2\right)g^{\mu\nu}\partial_\mu\left|\phi\right|\partial_\nu\left|\phi\right|-\frac{1}{2}\left(1+\delta Z_2\right)e^2g^{\mu\nu}A_\mu^{\text{g.i.}}A_\nu^{\text{g.i.}}\left|\phi\right|^2\\ &-\frac{1}{4}\left(1+\delta Z_3\right)g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}-\delta\xi\left|\phi\right|^2R-\frac{1}{2}\delta\lambda\left|\phi\right|^4\Big\},\end{aligned} \tag{4.13}$$

where R is the Ricci scalar. We can drop the field renormalizations δZ_2 and δZ_3 because they do not contribute at leading order [30],

$$\delta Z_2=\mathcal{O}\left(e^2\right),\quad \delta Z_3=\mathcal{O}\left(e^2\right),\quad \delta\xi=\mathcal{O}\left(e^2\right),\quad \delta\lambda=\mathcal{O}\left(e^4\right), \tag{4.14}$$

and we obtain the gauge invariant action renormalized at leading order,

$$\begin{aligned} S=\int_x\Big\{&-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}e^2g^{\mu\nu}A_\mu A_\nu\phi^2-\frac{1}{4}g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}\\ &-\delta\xi\phi^2R-\frac{1}{2}\delta\lambda\phi^4\Big\},\end{aligned} \tag{4.15}$$

where we have redefined the fields $\phi \equiv |\phi|$ and $A_\mu \equiv A_\mu^{\text{g.i.}}$.

4.2 Photon propagator

In this section we calculate the photon propagator for the (4.15) action. The inverse propagator for the vector field is,

$$\begin{aligned} iG^{\mu\nu}(x,x')^{-1} = \frac{1}{\sqrt{g(x)g(x')}}\frac{\delta^2 S}{\delta A_\nu(x')\delta A_\mu(x)} = \\ \frac{1}{\sqrt{-g}}\left\{\frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\left(g^{\alpha\sigma}g^{\mu\nu} - g^{\alpha\mu}g^{\nu\sigma}\right)\partial_\sigma\right) - e^2\phi^2 g^{\mu\nu}\right\}\delta_{\mathcal{C}}(x-x'). \end{aligned} \tag{4.16}$$

In order to calculate the inverse of (4.16) we have to make the approximation $e^2\phi^2 \equiv m_\gamma^2$. The detailed calculation is done in appendix C, (C.15), and the final expression for the propagator is [36],

$$\begin{aligned} iG_{\mu\nu}(x,x) = g_{\mu\nu}\frac{d}{2}\frac{H^2}{m_\gamma^2}\frac{H^{d-1}}{(4\pi)^{(d+1)/2}}\left(\frac{\Gamma\left(d\right)}{\Gamma\left(\frac{d+1}{2}+1\right)}\right. \\ \left. -\Gamma\left(-\frac{d+1}{2}\right)\frac{\Gamma\left(\frac{d+2}{2}+\nu\right)\Gamma\left(\frac{d+2}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)}\right), \end{aligned} \tag{4.17}$$

where ν is defined in (C.9).

4.3 Effective potential

The next step is to obtain the effective action $\Gamma[\phi]$ for the scalar field integrating out the vector field,

$$e^{i\Gamma[\phi]} = \int \mathcal{D}A_\mu \; e^{iS}. \tag{4.18}$$

The effective action is then,

$$\begin{aligned} \Gamma[\phi] = \int_x \left\{-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \delta\xi\phi^2 R - \frac{1}{2}\delta\lambda\phi^4\right\} \\ +\frac{i}{2}\log\left(\det\left(\partial_\mu\left(\sqrt{-g}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\rho\nu}\right)\partial_\nu\right) - e^2\sqrt{-g}g^{\rho\sigma}\phi^2\right)\right), \end{aligned} \tag{4.19}$$

where we have omitted the source term. We do not know how to calculate the determinant that appears in (4.19), instead we first variate it with respect to the field to get rid of the determinant and then we can use (4.17) to obtain,

$$\begin{aligned}\frac{1}{\sqrt{-g}}\frac{\delta\Gamma[\phi]}{\delta\phi} &= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - 2\delta\xi\phi(x)R - 2\delta\lambda\phi^3 \\ &-e^2\phi(d+1)\frac{d}{2}\frac{H^2}{m_\gamma^2}\frac{H^{d-1}}{(4\pi)^{(d+1)/2}}\left(\frac{\Gamma(d)}{\Gamma\left(\frac{d+1}{2}+1\right)}\right. \\ &\left.-\Gamma\left(-\frac{d+1}{2}\right)\frac{\Gamma\left(\frac{d+2}{2}+\nu\right)\Gamma\left(\frac{d+2}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)}\right).\end{aligned} \tag{4.20}$$

From (4.20) we can identify the derivative of the effective potential,

$$\begin{aligned}\Gamma[\phi] &= \int_x\left\{-\frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right) - V_{\text{eff}}\left(\phi^2\right)\right\} \\ \Rightarrow \quad \frac{1}{\sqrt{-g}}\frac{\delta\Gamma[\phi]}{\delta\phi} &= \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) - 2\phi V'_{\text{eff}}\left(\phi^2\right),\end{aligned} \tag{4.21}$$

obtaining the leading order first derivative of the effective potential,

$$\begin{aligned}V'_{\text{eff}}\left(\phi^2\right) &= \delta\xi H^2 d(d+1) + \delta\lambda\phi^2 \\ &+\frac{e^2}{2}d(d+1)\frac{H^2}{2m_\gamma^2}\frac{H^{d-1}}{(4\pi)^{(d+1)/2}}\left(\frac{\Gamma(d)}{\Gamma\left(\frac{d+1}{2}+1\right)}\right. \\ &\left.-\Gamma\left(-\frac{d+1}{2}\right)\frac{\Gamma\left(\frac{d+2}{2}+\nu\right)\Gamma\left(\frac{d+2}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)}\right).\end{aligned} \tag{4.22}$$

Now we renormalize the derivative of the effective potential using dimensional regularization [35], to do that we use the following identities,

$$\Gamma(z+1) = z\Gamma(z), \tag{4.23a}$$

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left(\frac{1}{\epsilon} + \Psi(n+1) + \mathcal{O}(\epsilon)\right), \tag{4.23b}$$

$$\Gamma(n+1+\epsilon) = \Gamma(n+1)\left(1 + \epsilon\Psi(n+1) + \mathcal{O}\left(\epsilon^2\right)\right), \tag{4.23c}$$

where $\epsilon = 3 - d$ and $\Psi(x)$ is the digamma function. The renormalized effective potential is, using (1.8),

$$V'_{\text{eff}}\left(\phi^2\right) = \delta\xi H^2 d(d+1) + \delta\lambda\phi^2$$
$$+\frac{e^2}{4}d(d+1)\frac{H^{d-1}}{(4\pi)^{(d+1)/2}}\left(-\frac{2}{\epsilon}\left(1+\frac{e^2\phi^2}{2H^2}\right)+\frac{1}{2}\right. \tag{4.24}$$
$$\left.+\left(1+\frac{e^2\phi^2}{2H^2}\right)\left(\Psi\left(\frac{3}{2}+\nu\right)+\Psi\left(\frac{3}{2}-\nu\right)-\frac{3}{2}+\gamma\right)+\mathcal{O}(\epsilon)\right).$$

We choose the following counterterms to cancel the divergencies,

$$\delta\xi = \frac{e^2 H^{d-3}}{2(4\pi)^{(d+1)/2}}\left(\frac{1}{3-d}+a+\mathcal{O}(\epsilon)\right), \tag{4.25}$$

$$\delta\lambda = \frac{d(d+1)e^4 H^{d-3}}{4(4\pi)^{(d+1)/2}}\left(\frac{1}{3-d}+b+\mathcal{O}(\epsilon)\right), \tag{4.26}$$

where we leave the finite part as generic constants to be specified later, and now we can take the limit $\epsilon \to 0$ and $d = 3$. It is useful to define $z \equiv \frac{e^2\phi^2}{2H^2} = \frac{m_\gamma^2}{2H^2}$ and integrate with respect to z,

$$V_{\text{eff}}(z) = \frac{3H^4}{8\pi^2}\left\{(-1+\gamma+2a)\,z+\left(-\frac{3}{4}+\frac{\gamma}{2}+b\right)z^2\right.$$
$$\left.+\int_0^z \mathrm{d}y\,(1+y)\left(\Psi\left(\frac{3}{2}+\frac{1}{2}\sqrt{1-8y}\right)+\Psi\left(\frac{3}{2}-\frac{1}{2}\sqrt{1-8y}\right)\right)\right\}. \tag{4.27}$$

We introduce another variable x and expand it in power series around $y = 0$,

$$x = \frac{1}{2}-\frac{1}{2}\sqrt{1-8y} = \sum_{n=0}^{\infty}\frac{(2n)!\,2^{n+1}}{n!\,(n+1)!}y^{n+1}. \tag{4.28}$$

We expand as well the digamma function for small z,

$$\Psi^{(n)}(z) = (-1)^{n+1}n!\,\zeta(n+1,z), \tag{4.29}$$

where,

$$\zeta(n,z) = \sum_{m=0}^{\infty}\frac{1}{(m+z)^n}, \tag{4.30}$$

is the Hurwitz zeta function. Then,

$$V_{\text{eff}}(z) = \frac{3H^4}{8\pi^2}\left\{(-\gamma+2a)\,z + \left(-\frac{1}{4}-\frac{\gamma}{2}+b\right)z^2 - \int_0^z \mathrm{d}y\,(1+y)\left(2\sum_{n=1}^{\infty}\zeta(2n+1)x^{2n} - \sum_{n=1}^{\infty}x^n\right)\right\}, \tag{4.31}$$

where $\zeta(n) \equiv \zeta(n,1)$ is the Riemann zeta function. Taking up to order ϕ^{18}, the effective potential is,

$$\begin{aligned} V_{\text{eff}}(\phi) = \frac{3H^4}{8\pi^2}\Bigg\{&(-\gamma+2a)\left(\frac{e^2}{2H^2}\right)\phi^2 + \left(\frac{3}{4}-\frac{\gamma}{2}+b\right)\left(\frac{e^2}{2H^2}\right)^2\phi^4 \\ &+ (10-8\zeta(3))\left(\frac{e^2}{2H^2}\right)^3\frac{\phi^6}{3} + (48-40\zeta(3))\left(\frac{e^2}{2H^2}\right)^4\frac{\phi^8}{4} \\ &+ (264-192\zeta(3)-32\zeta(5))\left(\frac{e^2}{2H^2}\right)^5\frac{\phi^{10}}{5} \\ &+ (1568-1056\zeta(3)-288\zeta(5))\left(\frac{e^2}{2H^2}\right)^6\frac{\phi^{12}}{6} \\ &+ (9792-6272\zeta(3)-2048\zeta(5)-128\zeta(7))\left(\frac{e^2}{2H^2}\right)^7\frac{\phi^{14}}{7} \\ &+ (63360-39168\zeta(3)-14080\zeta(5)-1664\zeta(7))\left(\frac{e^2}{2H^2}\right)^8\frac{\phi^{16}}{8} \\ &+ (420992-253440\zeta(3)-96768\zeta(5)-15360\zeta(7)-512\zeta(9))\left(\frac{e^2}{2H^2}\right)^9\frac{\phi^{18}}{9} \\ &+\mathcal{O}\left(\phi^{20}\right)\Bigg\}. \end{aligned} \tag{4.32}$$

The exact potential is plotted together with the expansion in figure 4-2 for $a = \dfrac{\gamma}{2}$, $b = \dfrac{\gamma}{2} - \dfrac{3}{4}$ and $H = 1$.

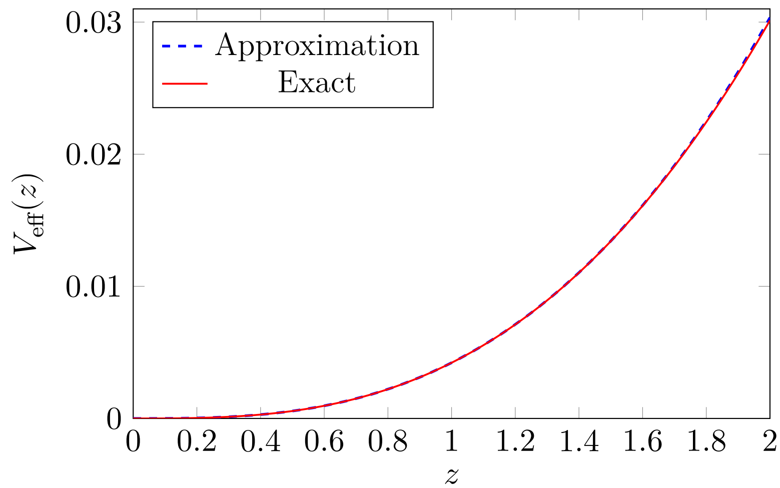

Figure 4-2: Effective potential (4.27) and its power expansion (4.32).

The approximation breaks down for $z \gtrsim 2$ that in terms of the field is $|\phi| \gtrsim 6.6H^2$, and where we have taking $e^2 = 4\pi\alpha \simeq 0.09170123$ [24] and α is the fine-structure constant.

4.4 Energy-momentum tensor

Lastly we compute the SQED energy-momentum tensor which is defined as,

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{4.33}$$

then,

$$\begin{aligned} T_{\mu\nu} = &\left(\delta^\alpha_\mu \delta^\beta_\nu - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\right)\left(\partial_\alpha\phi\partial_\beta\phi + e^2 A_\alpha A_\beta \phi^2\right) \\ &+ \left(\delta^\alpha_\mu \delta^\beta_\nu g^{\gamma\delta} - \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\right) F_{\alpha\gamma}F_{\beta\delta} \\ +2\delta\xi\phi^2 &\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + g_{\mu\nu}\nabla_\rho\nabla^\rho\phi^2 - \nabla_\mu\nabla_\nu\phi^2\right) - \frac{\delta\lambda}{2}\phi^4 g_{\mu\nu}. \end{aligned} \tag{4.34}$$

We can get rid of the covariant derivatives of the vector fields because they do not contribute at leading order and rewrite it as,

$$\begin{aligned} T_{\mu\nu} &= \left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\right) D_{\alpha}\phi D_{\beta}\phi \\ &+ \left(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}g^{\gamma\delta} - \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\right) F_{\alpha\gamma}F_{\beta\delta} - \delta\xi\phi^2 g_{\mu\nu}H^2 d(d-1) - \frac{\delta\lambda}{2}\phi^4 g_{\mu\nu}. \end{aligned} \tag{4.35}$$

Now we have to integrate out the vector field, we do it in the same way as we did to obtain the effective action. In general we can integrate out the vector field of any operator in the following way,

$$\int \mathcal{D}A_{\mu}\ e^{iS}\mathcal{O}\left(\phi, A_{\mu}\right) = e^{i\Gamma[\phi]}\tilde{\mathcal{O}}\left(\phi\right). \tag{4.36}$$

We can see that,

$$\begin{aligned} \int \mathcal{D}A_{\mu}\ e^{iS} &= \exp\left(i\int_x \left\{-\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - \delta\xi\phi^2 R - \frac{1}{2}\delta\lambda\phi^4\right\}\right) \\ &\cdot \int \mathcal{D}A_{\mu}\ \exp\left(\frac{i}{2}\int_x \left\{A_{\rho}(x)\mathcal{D}^{\rho\sigma}(x,x)A_{\sigma}(x)\right\}\right), \end{aligned} \tag{4.37}$$

where $\mathcal{D}^{\rho\sigma}(x,x)$ is defined in appendix C, then,

$$\int \mathcal{D}A_{\mu}\ e^{iS}A_{\mu}(x)A_{\nu}(x) = \frac{-2i}{\sqrt{-g}}\frac{\delta}{\delta\mathcal{D}^{\mu\nu}(x,x)}\int \mathcal{D}A_{\mu}\ e^{iS} = e^{i\Gamma[\phi]}iG_{\mu\nu}(x,x). \tag{4.38}$$

This implies that we have to calculate the following operators to integrate out the vector field of the energy-momentum tensor,

$$\mathcal{O} = A_{\mu}(x)A_{\nu}(x) \quad \rightarrow \quad \tilde{\mathcal{O}} = \left\langle A_{\mu}(x)A_{\nu}(x)\right\rangle = iG_{\mu\nu}(x,x), \tag{4.39}$$

$$\begin{aligned} \mathcal{O} = F_{\alpha\gamma}F_{\beta\delta} \quad &\rightarrow \quad \tilde{\mathcal{O}} = \left\langle F_{\alpha\gamma}F_{\beta\delta}\right\rangle = D_{\alpha}D_{\beta}iG_{\gamma\delta}(x,x) \\ -D_{\alpha}D_{\delta}iG_{\gamma\beta}(x,x) &- D_{\gamma}D_{\beta}iG_{\alpha\delta}(x,x) + D_{\gamma}D_{\delta}iG_{\alpha\beta}(x,x), \end{aligned} \tag{4.40}$$

$$\begin{aligned} \mathcal{O} = D_{\alpha}\phi D_{\beta}\phi &= \partial_{\alpha}\phi\partial_{\beta}\phi + e^2 A_{\alpha}(x)A_{\beta}(x)\phi^2 \\ \rightarrow \quad \tilde{\mathcal{O}} &= \left\langle D_{\alpha}\phi D_{\beta}\phi\right\rangle = e^2\phi^2 iG_{\alpha\beta}(x,x). \end{aligned} \tag{4.41}$$

So, we need to calculate $iG_{\mu\nu}(x,x)$ and $D_\alpha D_\beta iG_{\mu\nu}(x,x)$, this is done in appendix C. Using (C.15) and (C.16) we obtain,

$$\begin{gathered}\left\langle D_\alpha\phi D_\beta\phi\right\rangle = e^2\phi^2\gamma(0)g_{\alpha\beta} = \\ g_{\alpha\beta}\frac{d}{2}\frac{H^{d+1}}{(4\pi)^{\frac{d+1}{2}}}\left\{\frac{\Gamma\left(d\right)}{\Gamma\left(\frac{d+1}{2}+1\right)} - \Gamma\left(-\frac{d+1}{2}\right)\frac{\Gamma\left(\frac{d}{2}+1+\nu\right)\Gamma\left(\frac{d}{2}+1-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)}\right\},\end{gathered} \tag{4.42}$$

and,

$$\begin{gathered}\left\langle F_{\alpha\gamma}F_{\beta\delta}\right\rangle = 4H^2\left(-\frac{d+3}{d}\gamma'(0)+\gamma(0)\right)\left(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\delta}g_{\gamma\beta}\right) = \\ \left(g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\delta}g_{\gamma\beta}\right)\frac{H^{d+1}}{(4\pi)^{\frac{d+1}{2}}}\Gamma\left(-\frac{d+1}{2}\right)\frac{\Gamma\left(\frac{d}{2}+1+\nu\right)\Gamma\left(\frac{d}{2}+1-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)}.\end{gathered} \tag{4.43}$$

We have now everything needed to obtain the energy-momentum tensor integrated out the vector field,

$$\begin{gathered}\left\langle T_{\mu\nu}\right\rangle = \left(\delta^\alpha_\mu\delta^\beta_\nu - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\right)\left\langle D_\alpha\phi D_\beta\phi\right\rangle \\ +\left(\delta^\alpha_\mu\delta^\beta_\nu g^{\gamma\delta} - \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\right)\left\langle F_{\alpha\gamma}F_{\beta\delta}\right\rangle - \delta\xi\phi^2 g_{\mu\nu}H^2 d(d-1) - \frac{\delta\lambda}{2}\phi^4 g_{\mu\nu} = \\ -g_{\mu\nu}\frac{1}{4}d(d-1)\frac{H^{d+1}}{(4\pi)^{\frac{d+1}{2}}}\left\{\frac{\Gamma\left(d\right)}{\Gamma\left(\frac{d+1}{2}+1\right)} - \right. \\ \left.\Gamma\left(-\frac{d+1}{2}\right)\frac{\Gamma\left(\frac{d}{2}+1+\nu\right)\Gamma\left(\frac{d}{2}+1-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)}\right\} \\ -g_{\mu\nu}\frac{d}{4}\left(d-3\right)\frac{H^{d+1}}{(4\pi)^{\frac{d+1}{2}}}\Gamma\left(-\frac{d+1}{2}\right)\frac{\Gamma\left(\frac{d}{2}+1+\nu\right)\Gamma\left(\frac{d}{2}+1-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)} \\ -\delta\xi\phi^2 g_{\mu\nu}H^2 d(d-1) - \frac{\delta\lambda}{2}\phi^4 g_{\mu\nu}.\end{gathered} \tag{4.44}$$

Taking the counterterms (4.25) and (4.26) and performing the dimensional regularization, we obtain,

$$\begin{gathered}\left\langle T_{\mu\nu}\right\rangle = -g_{\mu\nu}\frac{3H^4}{8\pi^2}\left\{\left(-1+\frac{\gamma}{2}+a\right)z + \left(-\frac{5}{4}+\frac{\gamma}{2}+b\right)z^2\right. \\ \left. +\frac{1}{2}\left(z+z^2\right)\left(\Psi\left(\frac{3}{2}+\nu\right)+\Psi\left(\frac{3}{2}-\nu\right)\right)\right\}.\end{gathered} \tag{4.45}$$

The power series expansion is,

$$\begin{aligned}\langle T_{\mu\nu}\rangle = -g_{\mu\nu}\frac{3H^4}{8\pi^2}\Bigg\{\left(-1+\frac{\gamma}{2}+a\right)z+\left(-\frac{5}{4}+\frac{\gamma}{2}+b\right)z^2 \\ -\frac{1}{2}\left(z+z^2\right)\left(2\gamma-1+2\sum_{n=1}^{\infty}\zeta(2n+1)x^{2n}-\sum_{n=1}^{\infty}x^n\right)\Bigg\},\end{aligned} \tag{4.46}$$

that as function of ϕ and up to order ϕ^{18} becomes,

$$\begin{aligned}\langle T_{\mu\nu}\rangle = -g_{\mu\nu}\frac{3H^4}{8\pi^2}\Bigg\{\left(-\frac{1}{2}-\frac{\gamma}{2}+a\right)\left(\frac{e^2}{2H^2}\right)\phi^2 \\ +\left(\frac{1}{4}-\frac{\gamma}{2}+b\right)\left(\frac{e^2}{2H^2}\right)^2\phi^4+(5-4\zeta(3))\left(\frac{e^2}{2H^2}\right)^3\phi^6 \\ +(24-20\zeta(3))\left(\frac{e^2}{2H^2}\right)^4\phi^8+(132-96\zeta(3)-16\zeta(5))\left(\frac{e^2}{2H^2}\right)^5\phi^{10} \\ +(784-528\zeta(3)-144\zeta(5))\left(\frac{e^2}{2H^2}\right)^6\phi^{12} \\ +(4896-3136\zeta(3)-1024\zeta(5)-64\zeta(7))\left(\frac{e^2}{2H^2}\right)^7\phi^{14} \\ +(31680-19584\zeta(3)-7040\zeta(5)-832\zeta(7))\left(\frac{e^2}{2H^2}\right)^8\phi^{16} \\ +(210496-126720\zeta(3)-48384\zeta(5)-7680\zeta(7)-256\zeta(9))\left(\frac{e^2}{2H^2}\right)^9\phi^{18} \\ +\mathcal{O}\left(\phi^{20}\right)\Bigg\}.\end{aligned} \tag{4.47}$$

As pointed out in [29] this energy-momentum tensor is proportional to the metric tensor,

$$\langle T_{\mu\nu}\rangle = -g_{\mu\nu}V_{\text{em}}\left(\phi\right), \tag{4.48}$$

and the relation between $V_{\text{eff}}(z)$ and $V_{\text{em}}(z)$ is,

$$V_{\text{eff}}(z) = \frac{3H^4}{8\pi^2}f(z), \tag{4.49a}$$

$$V_{\text{em}}(z) = \frac{3H^4}{8\pi^2}g(z), \tag{4.49b}$$

with $g(z) = \frac{1}{2}\left(zf'(z) - z - z^2\right)$. The potential $V_{\text{em}}(z)$ is plotted together with its expansion in figure 4-3 for $a = \frac{\gamma}{2}$, $b = \frac{\gamma}{2} - \frac{3}{4}$ and $H = 1$.

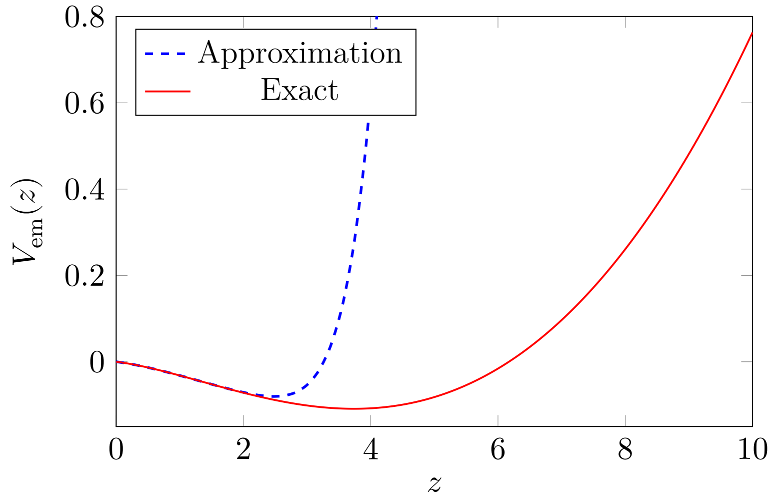

Figure 4-3: Effective potential for the energy-momentum tensor (4.48).

As we can observe, the approximation breaks down again for $z \gtrsim 2$ as this is a small field expansion for the energy-momentum tensor. To have a better approximation we will need to include more terms in the series expansion, but order ϕ^{18} will be enough for the purpose of this thesis.

Chapter 5

Renormalization group flow

In this section we show the results obtained after applying the renormalization group flow equation for a Higgs potential first, and then adding the photons interactions on top of it. Finally we calculate the flow of the SQED energy-momentum tensor with the vector field integrated out. We start by obtaining the flow equation for the effective potential and for small κ. We need to change from real to momentum space, this is equivalent to perform a Fourier transform, so the regulator in momentum representation is, using isotropy and homogeneity of the spatial coordinates,

$$R_\kappa(x,y) = R_\kappa(\eta,\eta',|\vec{X}-\vec{Y}|) = \int \frac{\mathrm{d}^d\vec{K}}{(2\pi)^d} e^{i\vec{K}\cdot(\vec{X}-\vec{Y})}\tilde{R}_\kappa\left(\eta,\eta',K\right), \qquad (5.1)$$

where $\vec{K}$ is the comoving spatial momentum and we use the notation $\tilde{F}$ for the Fourier transform of F.

De Sitter symmetries imply that the Green's function only depend on the invariant distance $y(x,x')$ defined in (C.5) [26], and is convenient to rescale conformally all the quantities,

$$\phi(x) \to a(\eta)^{\frac{d-1}{2}}\phi(x) = (-\eta H)^{-\frac{d-1}{2}}\phi(x), \qquad (5.2)$$

$$F(x,x') \to \left(a(\eta)a(\eta')\right)^{d_F} F(\eta,\eta',|\vec{X}-\vec{X}'|) = \left(H^2\eta\eta'\right)^{-d_F} F(\eta,\eta',|\vec{X}-\vec{X}'|), \qquad (5.3)$$

where d_F is the conformal dimension of the quantity F. Then, as shown in [26], in the momentum representation $p=-KH\eta$, $p'=-KH\eta'$ and introducing the notation

$\widehat{F}$ for dimensionless functions, extracting all the scale and dimensional factors,

$$\tilde{R}_{\kappa}\left(\eta,\eta',K\right)=\left(H^2\eta\eta'\right)^{\frac{d+3}{2}}K^3\hat{R}_{\kappa}\left(p,p'\right), \tag{5.4}$$

$$\hat{R}_{\kappa}(p,p')=H\frac{\delta_{\mathcal{C}}\left(p-p'\right)}{p^2}R_{\kappa}(p), \tag{5.5}$$

where $R_{\kappa}(p)$ suppresses modes with $p<\kappa$ and is chosen to be (3.13).

The Green's function in momentum representation with the appropriate scale factor is,

$$G_{\kappa}(x,y)=G_{\kappa}(\eta,\eta',|\vec{X}-\vec{Y}|)=\int\frac{\mathrm{d}^d\vec{K}}{(2\pi)^d}e^{i\vec{K}\cdot(\vec{X}-\vec{Y})}\tilde{G}_{\kappa}\left(\eta,\eta',K\right), \tag{5.6}$$

$$\tilde{G}_{\kappa}\left(\eta,\eta',K\right)=\frac{\left(H^2\eta\eta'\right)^{\frac{d-1}{2}}}{K}\hat{G}_{\kappa}\left(p,p'\right). \tag{5.7}$$

Now we need an expression for the effective potential flow, for that we use equation (3.14) and perform the derivative with respect to κ, $\kappa\partial_{\kappa}$, at a constant field ϕ,

$$\dot{\Gamma}_{\kappa}[\phi]\big|_{\phi=\text{const.}}=\int_x\left\{-\dot{V}_{\kappa}(\phi)\right\}\Big|_{\phi=\text{const.}}. \tag{5.8}$$

The integral over x is just the volume factor $\Omega\equiv\int_x$, then,

$$\dot{V}_{\kappa}(\phi)=-\Omega^{-1}\dot{\Gamma}_{\kappa}[\phi]\big|_{\phi=\text{const.}}. \tag{5.9}$$

Using (3.15),

$$\dot{V}_{\kappa}(\phi)=-\frac{1}{2\Omega}\operatorname{Tr}\left\{\dot{R}_{\kappa}(x,y)G_{\kappa}(y,x)\right\}=\frac{1}{2}\int\frac{\mathrm{d}^dp}{(2\pi)^d}\dot{R}_{\kappa}\left(p\right)\frac{\widehat{F}_{\kappa}\left(p,p\right)}{p}, \tag{5.10}$$

where we have defined the 2-point function as,

$$G_{\kappa}(x,x')=\langle T_{\mathcal{C}}\phi(x)\phi(x')\rangle=F_{\kappa}(x,x')-\frac{i}{2}\operatorname{sign}_{\mathcal{C}}(\eta-\eta')\rho_{\kappa}(x,x'), \tag{5.11}$$

and we have defined the statistical, F_κ, and spectral, ρ_κ, functions as,

$$F_\kappa(x,x') \equiv \frac{1}{2}\left\langle\{\phi(x),\phi(x')\}\right\rangle, \tag{5.12a}$$

$$\rho_\kappa(x,x') \equiv i\left\langle[\phi(x),\phi(x')]\right\rangle. \tag{5.12b}$$

In momentum space the decomposition of the Green's function becomes,

$$\hat{G}_\kappa(p,p') = \hat{F}_\kappa(p,p') - \frac{i}{2}\,\mathrm{sign}_{\mathcal{C}}\,(p-p')\,\hat{\rho}_\kappa\,(p,p'). \tag{5.13}$$

We use the modified inverse Green's function for the presence of the regulator, given by (B.11), and the defining property of the Green's functions in curved spacetime,

$$\int_z G_\kappa(x,z)^{-1}G_\kappa(z,y) = \frac{\delta_{\mathcal{C}}(x-y)}{\sqrt{-g(x)}}. \tag{5.14}$$

Then, we obtain the equation for the Green's function in the momentum representation,

$$\left(\partial_p^2 + \frac{1}{H^2} - \left(\nu_\kappa^2 - \frac{1}{4} - \frac{R_\kappa(p)}{Z_\kappa H^2}\right)\frac{1}{p^2}\right)\hat{G}_\kappa\,(p,p') = \frac{i\delta_{\mathcal{C}}\,(p-p')}{Z_\kappa H}, \tag{5.15}$$

with,

$$\nu_\kappa = \sqrt{\frac{d^2}{4} - \frac{V_\kappa''(\phi)}{Z_\kappa H^2}}. \tag{5.16}$$

Taking the decomposition of the Green's function in its statistical and spectral parts, the right-hand side is set to zero for each of them. It is convenient to write them in terms of a new function $u_\kappa(p)$ that only depends on one momentum p, $\hat{F}_\kappa(p,p') = Z_\kappa^{-1}\,\mathrm{Re}\,\{u_\kappa(p)u_\kappa^*(p')\}$ and $\hat{\rho}_\kappa(p,p') = -2Z_\kappa^{-1}\,\mathrm{Im}\,\{u_\kappa(p)u_\kappa^*(p')\}$. Doing that and inserting the expression for the regulator, the equation for the Green's function is,

$$\left(\partial_p^2 + \frac{1}{H^2} - \left(\nu_\kappa^2 - \frac{1}{4}\right)\frac{1}{p^2}\right)u_\kappa(p) = 0 \qquad (p \geq \kappa), \tag{5.17}$$

$$\left(\partial_p^2 + \frac{1}{H^2} - \left(\nu_\kappa^2 - \frac{1}{4} - \frac{\kappa^2}{H^2} + \frac{p^2}{H^2}\right)\frac{1}{p^2}\right)u_\kappa(p) = 0 \qquad (p \leq \kappa), \tag{5.18}$$

where we have defined $\bar{\nu}_\kappa = \sqrt{\nu_\kappa^2 - \frac{\kappa^2}{H^2}}$. The solutions for those differential equations are [10],

$$u_\kappa(p) = \frac{1}{2}\sqrt{\pi\frac{p}{H}} e^{i\gamma_\kappa} H_{\nu_\kappa}\left(\frac{p}{H}\right) \qquad (p \geq \kappa), \tag{5.19}$$

$$u_\kappa(p) = \frac{1}{2}\sqrt{\pi\frac{p}{H}} e^{i\gamma_\kappa} \left[c_\kappa^- \frac{\kappa^{\bar{\nu}_\kappa}}{p^{\bar{\nu}_\kappa}} + c_\kappa^+ \frac{p^{\bar{\nu}_\kappa}}{\kappa^{\bar{\nu}_\kappa}}\right] \qquad (p \leq \kappa), \tag{5.20}$$

where $H_{\nu_\kappa}(z)$ is the Hankel function of the first kind and $\gamma_\kappa = \frac{\pi}{2}\left(\nu_\kappa + \frac{1}{2}\right)$. Requiring continuity of the function and its first derivative we obtain,

$$c_\kappa^\pm = \frac{1}{2}\left[H_{\nu_\kappa}\left(\frac{\kappa}{H}\right) \pm \frac{\kappa}{\bar{\nu}_\kappa} H'_{\nu_\kappa}\left(\frac{\kappa}{H}\right)\right]. \tag{5.21}$$

From equation (5.10) the flow of the effective potential as function of $u_\kappa(p)$ is,

$$\dot{V}_\kappa(\phi) = \frac{\Omega_d}{2(2\pi)^d} \int_0^\kappa \mathrm{d}p\, p^{d-2} \left((2-\eta_\kappa)\,\kappa^2 + \eta_\kappa p^2\right) |u_\kappa(p)|^2 , \tag{5.22}$$

where we have defined $\eta_\kappa = -\dot{Z}_\kappa / Z_\kappa$ and,

$$\begin{aligned} \Omega_d &= \int_0^{2\pi} \mathrm{d}\theta_{d-1} \int_0^{\pi} \mathrm{d}\theta_{d-2} \cdots \int_0^{\pi} \mathrm{d}\theta_1 \; \sin(\theta_1)^{d-2} \sin(\theta_2)^{d-3} \cdots \sin(\theta_{d-2}) \\ &= \frac{2\pi^{d/2}}{\Gamma(d/2)}, \end{aligned} \tag{5.23}$$

where $\Gamma(z)$ is the Gamma function. Expanding $u_\kappa(p)$ for small κ,

$$H_{\nu_\kappa}(\kappa) = \frac{2^{\nu_\kappa}\Gamma(\nu_\kappa)}{i\pi\kappa^{\nu_\kappa}} \left(1 + \mathcal{O}\left(\kappa^2\right)\right) \qquad \left(\kappa^2 \ll 1\right), \tag{5.24}$$

the flow equation for the effective potential and small κ is,

$$\dot{V}_\kappa(\phi) = \frac{\Omega_d F_{\nu_\kappa}}{2(2\pi)^d} \frac{\kappa^{d+2-2\nu_\kappa}}{H^{1-2\nu_\kappa}} \left[\frac{2-\eta_\kappa}{d-2\bar{\nu}_\kappa} + \frac{\eta_\kappa}{d+2-2\bar{\nu}_\kappa}\right], \tag{5.25}$$

where we have defined,

$$F_{\nu_\kappa} = \frac{\left(2^{\nu_\kappa}\Gamma(\nu_\kappa)\right)^2}{4\pi}. \tag{5.26}$$

We make one last approximation, $|V''_\kappa(\phi)| \ll 1$ and $Z_\kappa = 1$ ($\eta_\kappa = 0$), to obtain the final expression for the flow of the effective potential,

$$\dot{V}_\kappa(\phi) = 2A_d \frac{\kappa^2 H^{d+1}}{V''_\kappa(\phi) + \kappa^2}, \tag{5.27}$$

with,

$$A_d = \frac{d\Omega_d F_{d/2}}{4(2\pi)^d} = \frac{d\Gamma(d/2)}{8\pi^{d/2+1}}. \tag{5.28}$$

5.1 Higgs potential

In this section we compute the flow for a Higgs-like potential in $d = 3$ with the following ansatz for equation (5.27),

$$V_\kappa(\phi) = \frac{\alpha_\kappa}{2}\phi^2 + \frac{\lambda_\kappa}{24}\phi^4 + \sum_{n=3} \frac{c_{2n_\kappa}}{(2n)!}\phi^{2n}, \tag{5.29}$$

where the couplings α_κ and λ_κ are defined as,

$$\alpha_\kappa = V''_\kappa(0), \quad \lambda_\kappa = V''''_\kappa(0), \quad c_{2n_\kappa} = V^{(2n)}_\kappa(0) \tag{5.30}$$

and the powers of the field higher than ϕ^4 are taking to be zero initially but they can run and be non-zero for small κ and also affect the running of α_κ and λ_κ, we include up to order ϕ^{18}. Taking $H = 1$ and with the initial conditions $\alpha_{\kappa_0} = -\lambda_{\kappa_0} = -0.0001$, $\kappa_0 = 1$ and $c_{2n_{\kappa_0}} = 0$ ($n = 3, 4, \dots, 9$). The flow of the coupling constants is shown in figures 5-1—5-9.

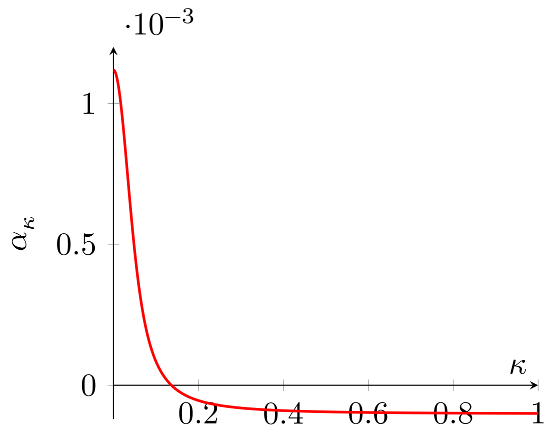

Figure 5-1: Flow of α_κ.

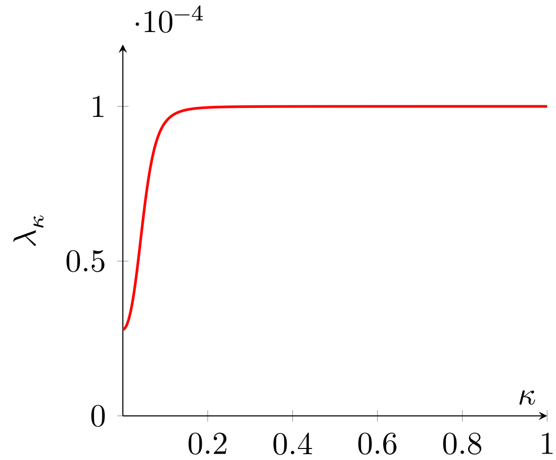

Figure 5-2: Flow of λ_κ.

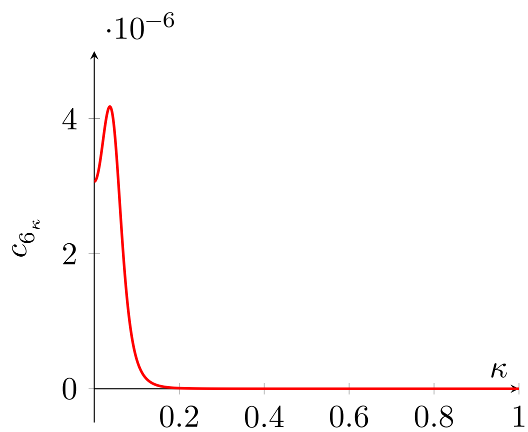

Figure 5-3: Flow of c_{6_κ}.

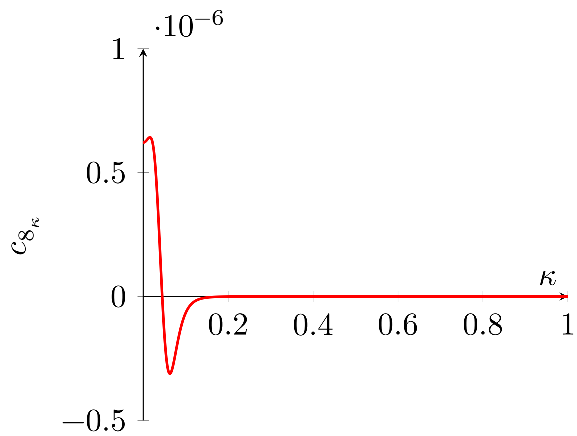

Figure 5-4: Flow of c_{8_κ}.

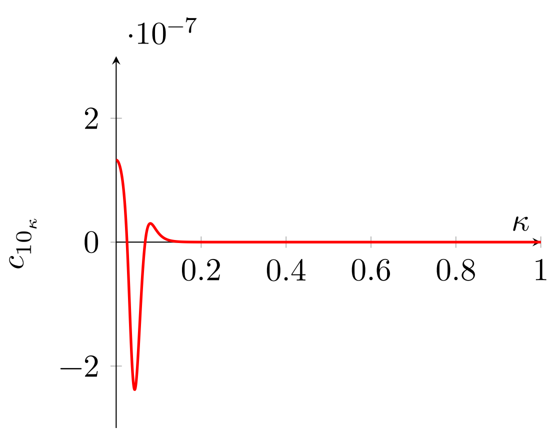

Figure 5-5: Flow of c_{10_κ}.

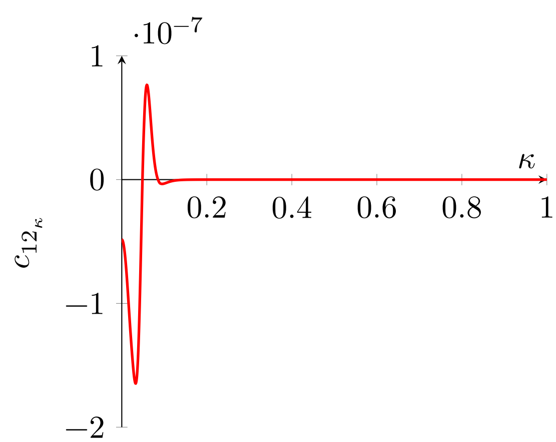

Figure 5-6: Flow of c_{12_κ}.

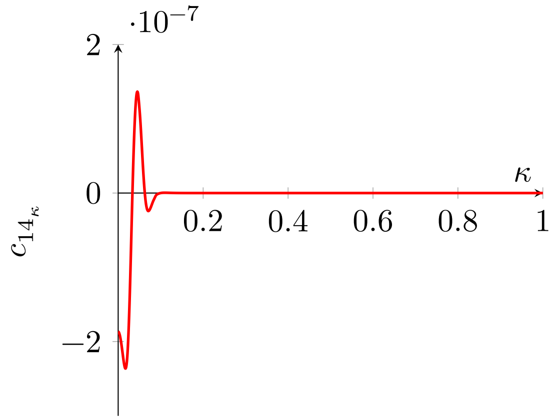

Figure 5-7: Flow of c_{14_κ}.

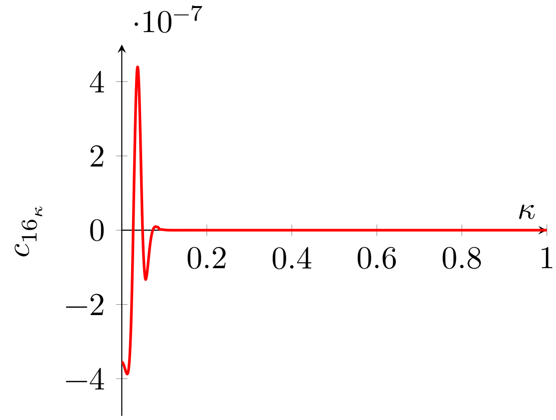

Figure 5-8: Flow of c_{16_κ}.

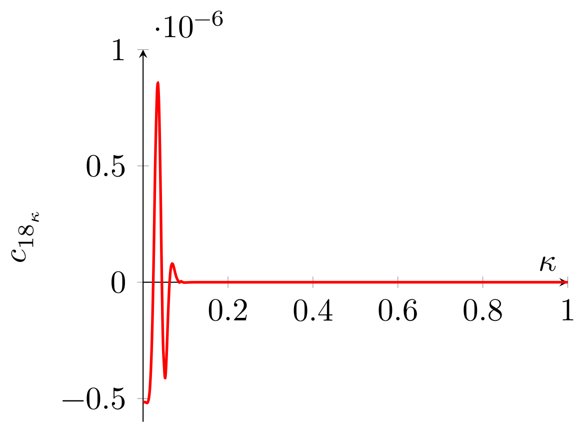

Figure 5-9: Flow of c_{18_κ}.

The coupling α_κ changes the sign for small κ, therefore restoring the spontaneously broken symmetry. To see exactly how the symmetry gets restored we show the flow of the minimum of the potential, $\bar{\phi}$, in figure 5-10 and how does the scalar field mass, $m_\kappa^2 = V_\kappa''(\phi)|_{\phi=\bar{\phi}}$, runs in figure 5-11.

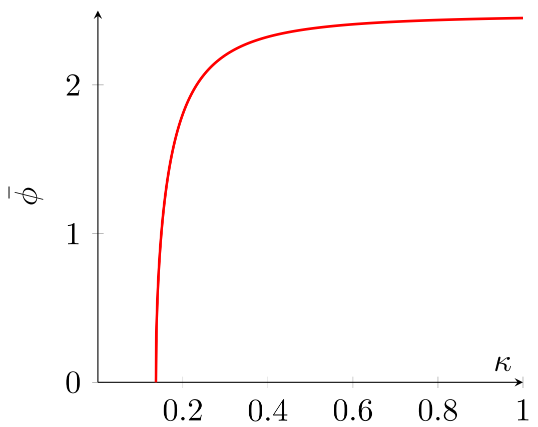

Figure 5-10: Flow of the minimum of the effective potential, $\bar{\phi}$.

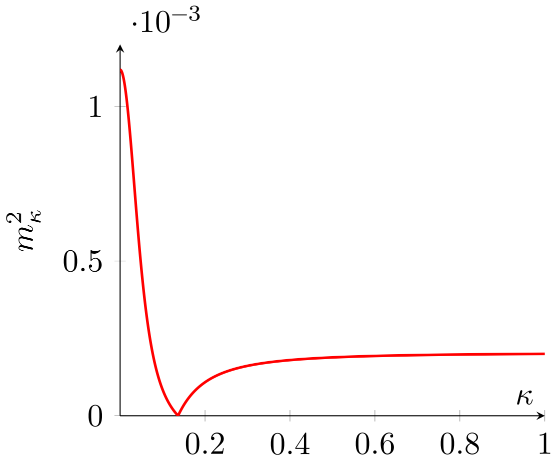

Figure 5-11: Flow of the mass.

Finally the flow of the effective potential is shown in figure 5-12 for different values of κ, where we observe explicitly that the symmetry gets restored for small values of κ in Hubble units.

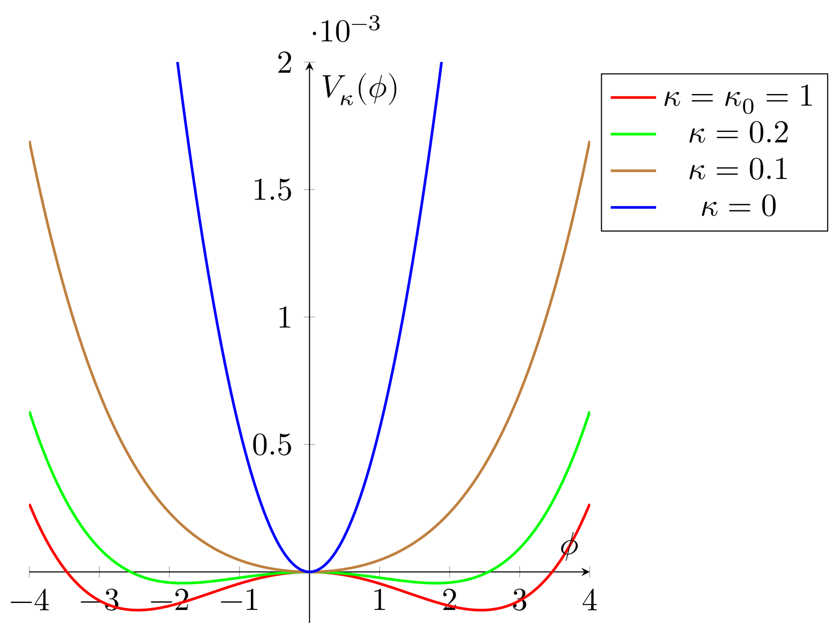

Figure 5-12: Flow of effective potential (5.29).

5.2 Higgs potential with photons interactions

The next step is to introduce photons interactions, coming from (4.32), on top of the Higgs potential. The ansatz used now for the effective potential is,

$$V_\kappa(\phi) = \frac{c_{2_\kappa}}{2}\phi^2 + \frac{c_{4_\kappa}}{24}\phi^4 + \frac{c_{6_\kappa}}{6!}\phi^6 + \frac{c_{8_\kappa}}{8!}\phi^8 + \frac{c_{10_\kappa}}{10!}\phi^{10} + \frac{c_{12_\kappa}}{12!}\phi^{12} + \frac{c_{14_\kappa}}{14!}\phi^{14} + \frac{c_{16_\kappa}}{16!}\phi^{16} + \frac{c_{18_\kappa}}{18!}\phi^{18} + \mathcal{O}\left(\phi^{20}\right) \tag{5.31}$$

The couplings are defined as,

$$c_{n_\kappa} = V_\kappa^{(n)}(0), \qquad (n = 2, 4, \dots, 18). \tag{5.32}$$

We take the counterterms to vanish the quadratic and quartic interactions coming from the photons, $a = \frac{\gamma}{2} + \frac{\alpha_{\kappa_0}}{2e^2}$, $b = \frac{\gamma}{2} - \frac{3}{4} + \frac{\lambda_{\kappa_0}}{6e^4}$ with $H = 1$, $\alpha_{\kappa_0} = -\lambda_{\kappa_0} = -0.0001$, $\kappa_0 = 1$, and we take $c_{n_{\kappa_0}}$ to be the coefficients of ϕ^n in (4.32). In this way we have the same Higgs potential (5.29) with photons interactions on top of it, that come at order ϕ^6 and higher. The flow of the coupling constants is shown in figures 5-13—5-21.

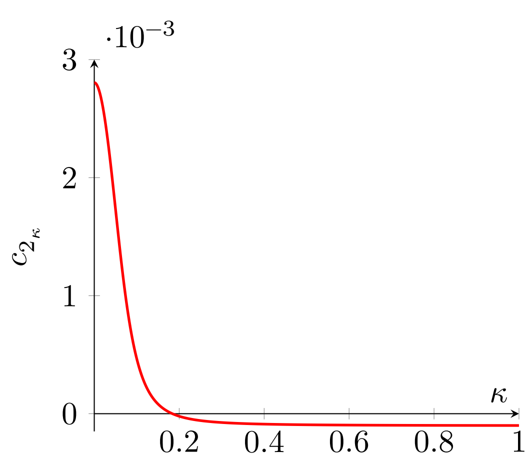

Figure 5-13: Flow of c_{2_κ}.

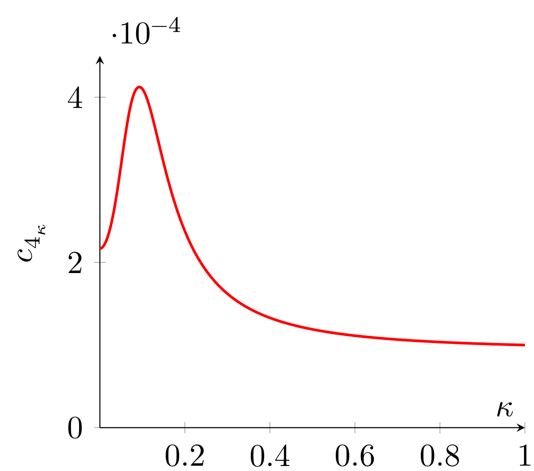

Figure 5-14: Flow of c_{4_κ}.

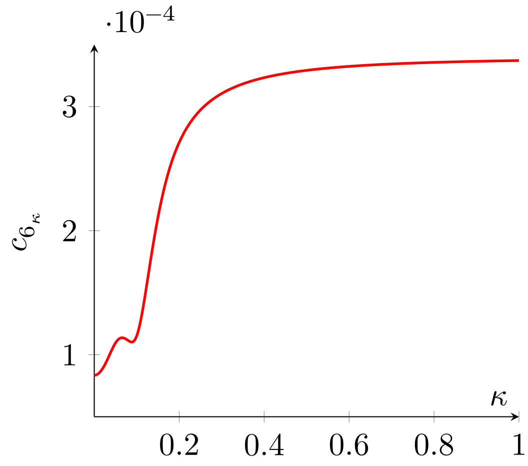

Figure 5-15: Flow of c_{6_κ}.

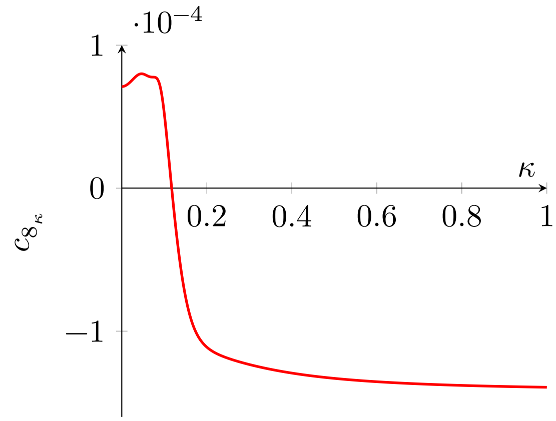

Figure 5-16: Flow of c_{8_κ}.

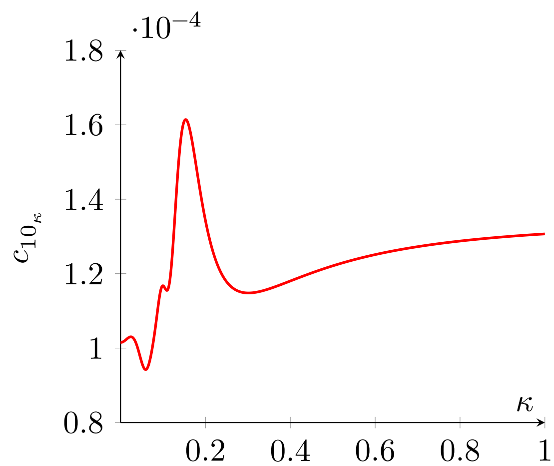

Figure 5-17: Flow of c_{10_κ}.

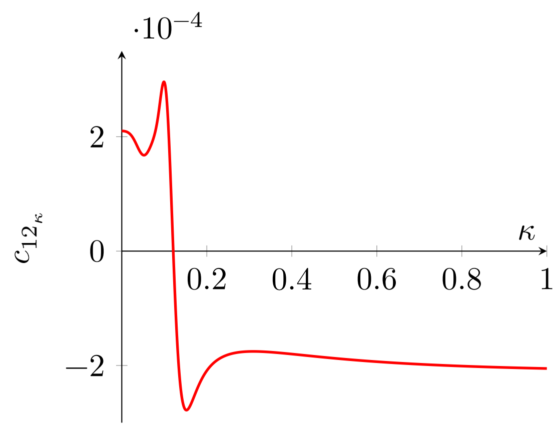

Figure 5-18: Flow of c_{12_κ}.

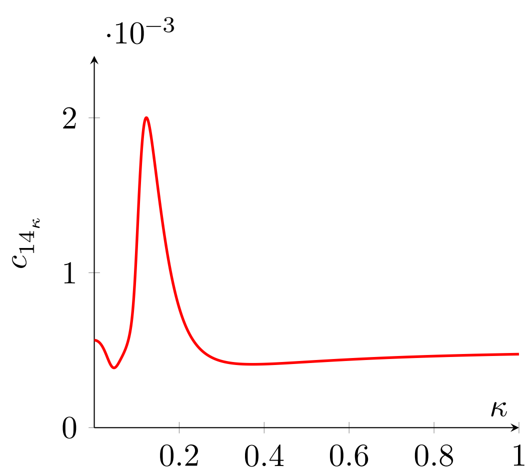

Figure 5-19: Flow of c_{14_κ}.

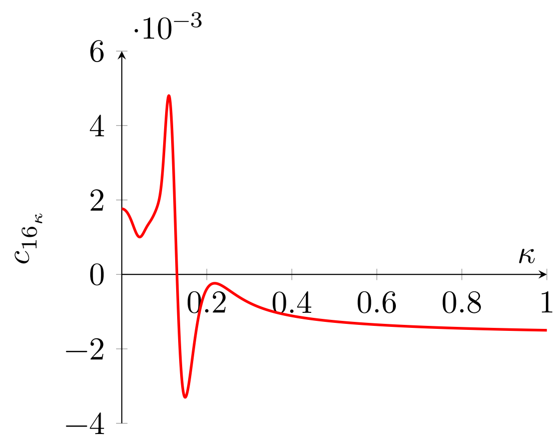

Figure 5-20: Flow of c_{16_κ}.

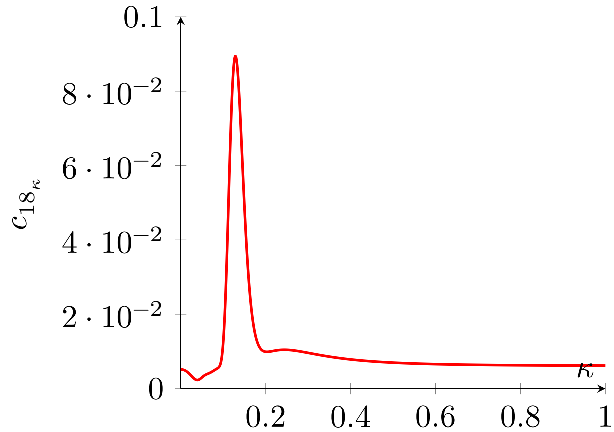

Figure 5-21: Flow of c_{18_κ}.

The flow of the minimum of the potential and the scalar field mass are shown in figures 5-22 and 5-23 respectively.

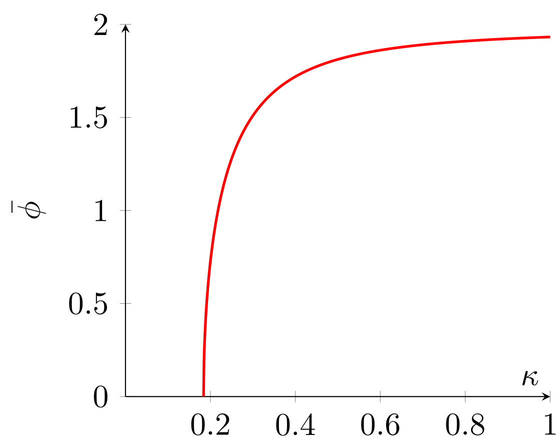

Figure 5-22: Flow of the minimum of the effective potential, $\bar{\phi}$.

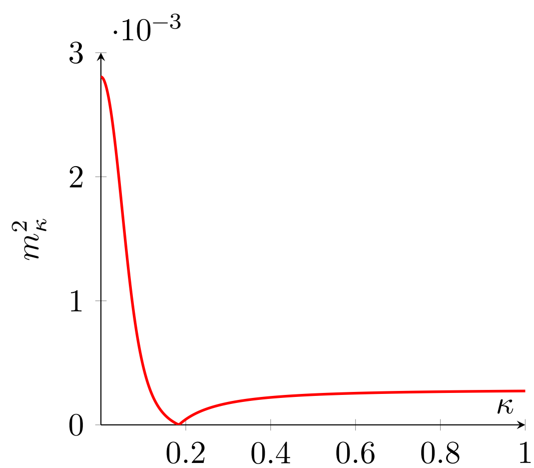

Figure 5-23: Flow of the scalar field mass.

The photon mass, as pointed out in [29], acquires mass during inflation. The flow of the photon mass is plotted in figure 5-24, and has been computed using (1.11) and (1.12) as $m_\gamma^2 = \langle e^2 \phi^2 \rangle$.

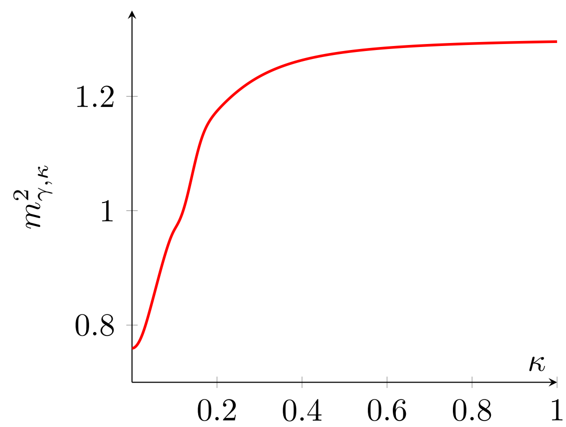

Figure 5-24: Flow of the photon mass m^2_γ.

The photon mass decreases for $\kappa \to 0$, this may indicate that the coupling between photons and scalar UV modes give the main contribution to the photon mass and when we introduce more and more strong perturbations coming from the IR modes, these modes screen that strong coupling between photons and UV modes consequently decreasing the photon mass. This is a possible explanation for the decrease of the photon mass, but we do not really know if this is a realistic explanation. Finally, the effective potential again gets symmetry restoration as shown in figure 5-25.

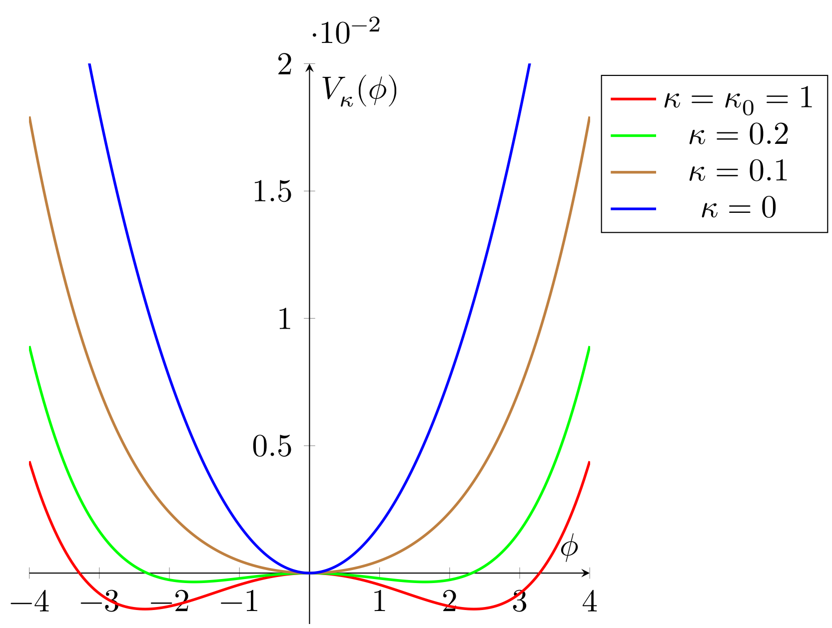

Figure 5-25: Flow of the effective potential (5.31).

The photons interactions accelerate the symmetry restoration (5-22) and as a consequence the mass of the scalar field gets a bigger increase because it has more momentum space to grow than without the photons interactions. Is important to point out that for all these calculations we have neglected the dependence of the coupling e with κ, we have assumed that it does not run or the running is neglectable in comparison with the running of the rest of parameters of theory.

5.3 Scalar QED energy-momentum tensor

As a last step we calculate the flow of the SQED energy-momentum tensor given by expression (4.48), and the running effective potential takes the same form as (5.31) where the initial conditions $c_{n_{\kappa_0}}$ are the coefficients of ϕ^n in (4.47). To calculate the flow we use the simple relation between $V_{\text{eff}}(z)$ and $V_{\text{em}}(z)$, this is shown in figure 5-26 with the counterterms $a = \frac{\gamma}{2}$ and $b = \frac{\gamma}{2} - \frac{3}{4}$.

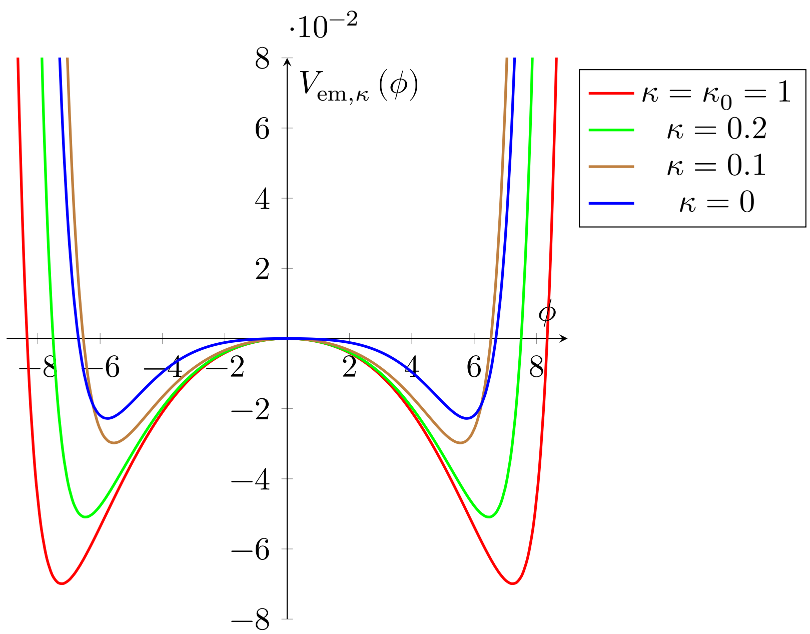

Figure 5-26: Flow of the SQED effective energy-momentum tensor.

The shift of energy in the energy-momentum tensor while we run $\kappa \to 0$ translates into a change in the pressure from the equation of state, which is obtained as $\nabla_\mu T^\mu_0 = 0$. It is important to point out that for $|\phi| \gtrsim 6.6$ the expansion is not longer a good approximation, and that with a different choice of counterterms the shape of the potential can change in an important way for small values of the field ϕ.

5.4 Comparison with previous results

Once we have obtained the flow of all the parameters, we can compare the results with those obtained previously using the stochastic formalism. As explained in [10] the late-time solutions of the stochastic formalism ($t \to \infty$) coincide with the limit $\kappa \to 0$ of the renormalization group approach. First, the mass for the scalar field obtained in [18] is $m^2 = 1.118\,68 \cdot 10^{-3} H^2$ with a Higgs-like potential using the stochastic formalism, in comparison we have obtained for the same potential, $m^2 = 1.118\,05 \cdot 10^{-3} H^2$ for $\kappa = 0$ without photons interactions. We have obtained a very good agreement for the mass of the scalar field using these two different techniques.

Second, the mass for the photons obtained in [29] is $m_\gamma^2 = 3.2991 H^2$ while the value obtained here is $m_\gamma^2 = 0.7762 H^2$, on both cases there is a non-zero photon mass during inflation. Finally, the mass for the scalar field with photons interactions obtained in [29] is $m^2 = 3.122 \cdot 10^{-3} H^2$ while we have obtained a lower mass, $m^2 = 2.707 \cdot 10^{-3} H^2$, for the same interaction and counterterms. The significant difference for both results suggests that, as pointed out in [29], the stochastic formalism gives a good description of the physics at the Hubble scale, but is not that good in the deep IR, although we have obtained in every case masses of the same order of magnitude for both methods.

The masses that we have obtained with the renormalization group approach are lower that the masses obtained with the stochastic formalism in [29], while the mass that we have obtained for [18] is the same as the mass obtained using the renormalization group approach. We have have also reproduced the results obtained by Serreau and Guilleux using the same renormalization group technique [33, 10].

Chapter 6

Conclusion and outlook

We have studied the running under the renormalization group, first of a Higgs-like potential obtaining symmetry restoration coming from IR modes, reproducing the results obtained already by Guilleux and Serrearu [10] and comparing them also with the results obtained by Lazzari and Prokopec [18] using the stochastic formalism. Later we added SQED interactions to see how does it affect the previous potential, resulting into an earlier symmetry restoration and a bigger mass for the scalar field, and we compared that results with those obtained by Prokopec et al. [29] with the stochastic formalism. One important remark is that the stochastic formalism and the renormalization group approach are different and is unclear which one gives better results or up to what point the renormalization group approach can be used as a substitute for the stochastic formalism, although the results shown in chapter 5.4 suggests that the stochastic formalism does not describe well the physics happening in the deep IR.

We can sum up the main conclusions of this work in the following sentence: Infrared modes restore the spontaneously broken symmetry for a scalar field in de Sitter space, and if we add photons interactions to the theory the symmetry restoration gets enhanced by the photons interactions.

In this thesis we have been working with eternal inflation but one important question to ask is, what happens if we apply the same renormalization group approach to slow-roll inflation instead of eternal inflation? and, how would the slow-roll parameters change? The slow-roll parameters depend on the potential and its first and second derivatives [1], the first slow-roll parameter is defined as,

$$\epsilon \equiv -\frac{1}{H^2}\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{m_{\mathrm{P}}^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2, \tag{6.1}$$

and the second slow-roll parameter is defined as,

$$\eta \equiv \epsilon - \frac{1}{2H}\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = m_{\mathrm{P}}^2\frac{V''(\phi)}{V(\phi)}, \tag{6.2}$$

where $m_{\mathrm{P}} = 1/\sqrt{8\pi G}$ is the reduced Planck mass, and the necessary conditions for inflation with these parameters are $\epsilon \ll 1$ and $|\eta| \ll 1$. Therefore, if $V(\phi)$ changes to $V_\kappa(\phi)$, the running of the potential will affect, maybe significantly, to inflation. As well, it would change the numbers of e-folds,

$$N = \int_{\phi_{\mathrm{end}}}^{\phi_{\mathrm{start}}} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon}}, \tag{6.3}$$

and to solve the flatness and horizon problems we need $N \gtrsim 60$. So, we need to know how the number of e-folds would be affected as well.

During inflation low energy modes, $p < aH$, exit the horizon and freeze out, but at some point after inflation those modes will re-enter the horizon. The moment in which the re-entry happens will be affected by how long does inflation last, that could be changed due to those IR modes, this would lead to possible changes on the CMB that could be, in principle, measurable. This is another open question left for future research.

As a final thought we could also study $O(N)$ scalar field theories, we have studied in this work an $O(1)$ theory. This has been done for a Higgs-like potential using the same renormalization group technique in [33, 10], and would be interesting to do the same calculations adding the photons interactions and also for slow-roll inflation. In those works the authors find that the same symmetry restoration due to IR modes that happens in a $O(1)$ theory for a spontaneously broken potential, also occurs for $O(N)$ theories and they have qualitatively the same behavior for any value of N.

Appendix A

Conventions

In this work we use the following conventions.

The determinant of the metric is defined as $g \equiv \det(g_{\mu\nu})$, the signature of the metric is $(-,+,\dots,+)$ in $d+1$ dimensions and we use natural units $\hbar = c = k_\beta = 1$.

The notation for the coordinates is t for the time and η for the conformal time, and for the spatial components we use $\vec{x}$ for the physical coordinates and $\vec{X}$ for the comoving coordinates. As a shorthand notation we use $x = (\eta, \vec{X})$ for the coordinates that will be used for all the calculations in this thesis.

Appendix B

Derivation of the Wetterich equation

The exact flow evolution equation for the effective potential was derived by Wetterich [39], here is the derivation of that equation. We start by applying the operator $\kappa\partial_\kappa$ to the effective action (3.10) [11],

$$\kappa\partial_\kappa \Gamma_\kappa[\phi] = \kappa\partial_\kappa W_\kappa[J] - \frac{1}{2}\int_{x,y} (\kappa\partial_\kappa R_\kappa(x,y))\,\phi(x)\phi(y). \tag{B.1}$$

Now we need to calculate the derivative of $W_\kappa[J]$, to do that we insert the identity $\exp(-iW_\kappa[J])\exp(iW_\kappa[J]) = 1$,

$$\begin{aligned}\kappa\partial_\kappa W_\kappa[J] &= \exp\left(-iW_\kappa[J]\right)\exp\left(iW_\kappa[J]\right)\kappa\partial_\kappa W_\kappa[J] \\ &= -i\exp\left(-iW_\kappa[J]\right)\kappa\partial_\kappa\exp\left(iW_\kappa[J]\right),\end{aligned} \tag{B.2}$$

and using the integral representation for $W_\kappa[J]$,

$$
\begin{aligned}
\kappa\partial_\kappa W_\kappa[J] &= \exp\left(-iW_\kappa[J]\right)\int \mathcal{D}\varphi\ \frac{1}{2}\int_{x,y}\left(\kappa\partial_\kappa R_\kappa(x,y)\right)\varphi(x)\varphi(y)\exp\left(iW_\kappa[J]\right)\\
&= \frac{1}{2}\int_{x,y}\left(\kappa\partial_\kappa R_\kappa(x,y)\right)\exp\left(-iW_\kappa[J]\right)\int \mathcal{D}\varphi\ \varphi(x)\varphi(y)\exp\left(iW_\kappa[J]\right)\\
&= \frac{1}{2}\int_{x,y}\left(\kappa\partial_\kappa R_\kappa(x,y)\right)\exp\left(-iW_\kappa[J]\right)\frac{-1}{\sqrt{g(x)g(y)}}\frac{\delta^2}{\delta J(y)\delta J(x)}\exp\left(iW_\kappa[J]\right),
\end{aligned}
\tag{B.3}
$$

and using (3.4) we can write,

$$
\begin{aligned}
&\exp\left(-iW_\kappa[J]\right)\frac{-1}{\sqrt{g(x)g(y)}}\frac{\delta^2}{\delta J(y)\delta J(x)}\exp\left(iW_\kappa[J]\right)\\
&= \frac{1}{\sqrt{g(x)g(y)}}\frac{\delta^2 W_\kappa[J]}{\delta J(y)\delta J(x)} + \frac{1}{\sqrt{-g(y)}}\frac{\delta W_\kappa[J]}{\delta J(y)}\frac{1}{\sqrt{-g(x)}}\frac{\delta W_\kappa[J]}{\delta J(x)}\\
&= G_\kappa(x,y) + \phi(x)\phi(y).
\end{aligned}
\tag{B.4}
$$

This leads to,

$$
\kappa\partial_\kappa W_\kappa[J] = \frac{1}{2}\int_{x,y}\left(\kappa\partial_\kappa R_\kappa(x,y)\right)\left(G_\kappa(x,y)+\phi(x)\phi(y)\right), \tag{B.5}
$$

finally inserting (B.5) back into (B.1) we obtain the Wetterich equation,

$$
\begin{aligned}
\kappa\partial_\kappa \Gamma_\kappa[\phi] &= \frac{1}{2}\int_{x,y}\left(\kappa\partial_\kappa R_\kappa(x,y)\right)\left(G_\kappa(x,y)+\phi(x)\phi(y)\right)\\
-\frac{1}{2}\int_{x,y}\left(\kappa\partial_\kappa R_\kappa(x,y)\right)\phi(x)\phi(y) &= \frac{1}{2}\int_{x,y}\left(\kappa\partial_\kappa R_\kappa(x,y)\right)G_\kappa(x,y).
\end{aligned}
\tag{B.6}
$$

To obtain an expression for $G_\kappa(x,y)$ we start by calculating the variation of the effective action,

$$
\frac{1}{\sqrt{-g(x)}}\frac{\delta\Gamma_\kappa[\phi]}{\delta\phi(x)} = -J(x) - \int_y R_\kappa(x,y)\phi(y), \tag{B.7}
$$

then, the second variation is,

$$\frac{1}{\sqrt{g(x)g(y)}}\frac{\delta^2\Gamma_\kappa[\phi]}{\delta\phi(y)\delta\phi(x)} = -\frac{1}{\sqrt{-g(y)}}\frac{\delta J(x)}{\delta\phi(y)} - R_\kappa(x,y), \tag{B.8}$$

and using now the delta function as,

$$\begin{aligned}\frac{\delta_{\mathcal{C}}(x-y)}{\sqrt{-g(x)}} &= \int_z \frac{1}{\sqrt{-g(x)}}\frac{\delta\phi(x)}{\delta J(z)}\frac{1}{\sqrt{-g(y)}}\frac{\delta J(z)}{\delta\phi(y)} \\ = \int_z \frac{1}{\sqrt{g(x)g(z)}}\frac{\delta^2 W_\kappa[J]}{\delta J(x)\delta J(z)} &\left(-\frac{1}{\sqrt{g(z)g(y)}}\frac{\delta^2\Gamma_\kappa[\phi]}{\delta\phi(z)\delta\phi(y)} - R_\kappa(z,y)\right),\end{aligned} \tag{B.9}$$

we can identify the propagator and the inverse propagator in the presence of the regulator,

$$iG_\kappa(x,y) = \frac{1}{\sqrt{g(x)g(y)}}\frac{\delta^2 W_\kappa[J]}{\delta J(y)\delta J(x)}, \tag{B.10}$$

$$iG_\kappa(x,y)^{-1} = \frac{1}{\sqrt{g(x)g(y)}}\frac{\delta^2\Gamma_\kappa[\phi]}{\delta\phi(y)\delta\phi(x)} + R_\kappa(x,y). \tag{B.11}$$

Appendix C

Derivation of the massive vector field propagator

Defining $\mathcal{D}^{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\partial_\rho \left(\sqrt{-g}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\rho\mu}g^{\nu\sigma}\right)\partial_\sigma\right) - m_\gamma^2 g^{\mu\nu}$, the propagator satisfies the usual Green's function equation,

$$\mathcal{D}_\mu{}^\rho iG_{\rho\nu}(x,x') = ig_{\mu\nu}\frac{\delta_{\mathcal{C}}(x-x')}{\sqrt{-g}}, \tag{C.1}$$

but the propagator has to satisfy the Lorentz (transversality) condition,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}iG_{\nu\rho}(x,x')\right) = 0, \tag{C.2}$$

then (C.1) gets modified as [36, 25],

$$\mathcal{D}_\mu{}^\rho iG_{\rho\nu}(x,x') = ig_{\mu\nu}\frac{\delta_{\mathcal{C}}(x-x')}{\sqrt{-g}} + \partial_\mu\partial'_\nu iG(x,x'), \tag{C.3}$$

where $G(x,x')$ is the propagator of a MMC scalar field and satisfies the following condition,

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu iG(x,x')\right) = i\frac{\delta_{\mathcal{C}}(x-x')}{\sqrt{-g}}. \tag{C.4}$$

Any de Sitter invariant vector propagator can be expressed in terms of the invariant distance $y(x,x')$ [36, 29],

$$y(x,x') \equiv aa'H^2 \left(||\vec{X} - \vec{X}'||^2 - (|\eta - \eta'| - i\epsilon)^2 \right). \tag{C.5}$$

in the following way,

$$iG_{\mu\nu}(x,x') = B(y)\frac{\partial^2 y}{\partial x^\mu \partial x'^\nu} + C(y)\frac{\partial y}{\partial x^\mu}\frac{\partial y}{\partial x'^\nu}. \tag{C.6}$$

From the Lorentz condition we can obtain an expression for $B(y)$ and $C(y)$ in terms of a function $\gamma(y)$,

$$B(y) = \frac{1}{4dH^2}\left(-(4y - y^2)\gamma'(y) - d(2-y)\gamma(y)\right), \tag{C.7a}$$

$$C(y) = \frac{1}{4dH^2}\left((2-y)\gamma'(y) - d\gamma(y)\right), \tag{C.7b}$$

where the function $\gamma(y)$ is,

$$\gamma(y) = -\frac{d}{2}\frac{H^2}{m_\gamma^2}\frac{H^{d-1}}{(4\pi)^{\frac{d+1}{2}}}\left\{ -\frac{\Gamma(d)}{\Gamma\left(\frac{d+1}{2}+1\right)}{}_2F_1\left(d, 2, \frac{d+1}{2}+1; 1-\frac{y}{4}\right) + \right.$$
$$\left. \frac{\Gamma\left(\frac{d}{2}+1+\nu\right)\Gamma\left(\frac{d}{2}+1-\nu\right)}{\Gamma\left(\frac{d+1}{2}+1\right)}{}_2F_1\left(\frac{d}{2}+1+\nu, \frac{d}{2}+1-\nu, \frac{d+1}{2}+1; 1-\frac{y}{4}\right)\right\}, \tag{C.8}$$

and where ${}_2F_1$ are the hypergeometric functions [28] and we have defined,

$$\nu = \sqrt{\left(\frac{d}{2}-1\right)^2 - \frac{m_\gamma^2}{H^2}}. \tag{C.9}$$

It is useful to make a Laurent expansion for $\gamma(y)$,

$$\gamma(y) = -\frac{d}{2}\frac{H^2}{m_\gamma^2}\frac{H^{d-1}}{(4\pi)^{\frac{d+1}{2}}}\left\{ -\frac{m_\gamma^2}{H^2}\Gamma\left(\frac{d+1}{2}-1\right)\left(\frac{4}{y}\right)^{\frac{d+1}{2}-1} \right.$$
$$+ \sum_{n=0}^{\infty}\left(-(n+1)\frac{\Gamma(n+d)}{\Gamma\left(n+\frac{d+1}{2}+1\right)}\left(\frac{y}{4}\right)^n \right. \tag{C.10}$$

$$+\left(n-\frac{d+1}{2}+3\right)\frac{\Gamma\left(n+\frac{d+1}{2}+1\right)}{\Gamma\left(n+3\right)}\left(\frac{y}{4}\right)^{n-\frac{d+1}{2}+2}\Bigg)$$
$$+\frac{\Gamma\left(\frac{d+1}{2}-1\right)\Gamma\left(2-\frac{d+1}{2}\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)}\sum_{n=0}^{\infty}\Bigg(\frac{\Gamma\left(n+\frac{d}{2}+1+\nu\right)\Gamma\left(n+\frac{d}{2}+1-\nu\right)}{\Gamma\left(n+\frac{d+1}{2}+1\right)\Gamma\left(n+1\right)}\left(\frac{y}{4}\right)^{n}$$
$$-\frac{\Gamma\left(n+\frac{5}{2}+\nu\right)\Gamma\left(n+\frac{5}{2}-\nu\right)}{\Gamma\left(n+3\right)\Gamma\left(n-\frac{d+1}{2}+3\right)}\left(\frac{y}{4}\right)^{n-\frac{d+1}{2}+2}\Bigg)\Bigg\},$$

and also for the first derivative,

$$\gamma'(y)=-\frac{d}{2}\frac{H^2}{m_\gamma^2}\frac{H^{d-1}}{(4\pi)^{\frac{d+1}{2}}}\Bigg\{-\frac{m_\gamma^2}{H^2}\Gamma\left(\frac{d+1}{2}-1\right)(2-2d)\left(\frac{4}{y}\right)^{\frac{d+1}{2}-2}$$
$$+\sum_{n=0}^{\infty}\Bigg(-\frac{n\left(n+1\right)}{4}\frac{\Gamma\left(n+d\right)}{\Gamma\left(n+\frac{d+1}{2}+1\right)}\left(\frac{y}{4}\right)^{n-1}$$
$$+\left(n-\frac{d+1}{2}+3\right)\left(n-\frac{d+1}{2}+2\right)\frac{\Gamma\left(n+\frac{d+1}{2}+1\right)}{4\Gamma\left(n+3\right)}\left(\frac{y}{4}\right)^{n-\frac{d+1}{2}+1}\Bigg)$$
$$+\frac{\Gamma\left(\frac{d+1}{2}-1\right)\Gamma\left(2-\frac{d+1}{2}\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)}\sum_{n=0}^{\infty}\Bigg(n\frac{\Gamma\left(n+\frac{d}{2}+1+\nu\right)\Gamma\left(n+\frac{d}{2}+1-\nu\right)}{4\Gamma\left(n+\frac{d+1}{2}+1\right)\Gamma\left(n+1\right)}\left(\frac{y}{4}\right)^{n-1}$$
$$-\left(n-\frac{d+1}{2}+2\right)\frac{\Gamma\left(n+\frac{5}{2}+\nu\right)\Gamma\left(n+\frac{5}{2}-\nu\right)}{4\Gamma\left(n+3\right)\Gamma\left(n-\frac{d+1}{2}+3\right)}\left(\frac{y}{4}\right)^{n-\frac{d+1}{2}+1}\Bigg)\Bigg\}. \tag{C.11}$$

We are interested in the coincidence limit $x'\to x$, where $y\to 0$ and only the powers y^0 survive, and the divergent powers of y vanish using dimensional regularization. Then, we obtain,

$$\gamma(0)=\frac{d}{2}\frac{H^2}{m_\gamma^2}\frac{H^{d-1}}{(4\pi)^{\frac{d+1}{2}}}\Bigg\{\frac{\Gamma\left(d\right)}{\Gamma\left(\frac{d+1}{2}+1\right)}$$
$$-\Gamma\left(-\frac{d+1}{2}\right)\frac{\Gamma\left(\frac{d}{2}+1+\nu\right)\Gamma\left(\frac{d}{2}+1-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)}\Bigg\}, \tag{C.12}$$

and,

$$\gamma'(0)=\frac{d^2}{2(d+3)}\frac{H^2}{m_\gamma^2}\frac{H^{d-1}}{(4\pi)^{\frac{d+1}{2}}}\Bigg\{\frac{\Gamma\left(d\right)}{\Gamma\left(\frac{d+1}{2}+1\right)}$$
$$-\frac{\Gamma\left(-\frac{d+1}{2}\right)\Gamma\left(\frac{d}{2}+2+\nu\right)\Gamma\left(\frac{d}{2}+2-\nu\right)}{2d\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)}\Bigg\}. \tag{C.13}$$

Using the following expressions [14],

$$\lim_{x'\to x}\frac{\partial y}{\partial x^{\mu}}\frac{\partial y}{\partial x'^{\nu}}=0, \tag{C.14a}$$

$$\lim_{x'\to x} \frac{\partial^2 y}{\partial x^\mu x'^\nu} = -2H^2 g_{\mu\nu}, \tag{C.14b}$$

$$D_\mu \frac{\partial y}{\partial x^\nu} = H^2(2-y) g_{\mu\nu}, \tag{C.14c}$$

$$D_\mu \frac{\partial^2 y}{\partial x^\nu \partial x'^\rho} = -H^2 g_{\mu\nu} \frac{\partial y}{\partial x'^\rho}, \tag{C.14d}$$

that also hold interchanging $x \leftrightarrow x'$. We obtain finally,

$$\lim_{x'\to x} iG_{\mu\nu}(x,x') = \gamma(0) g_{\mu\nu}, \tag{C.15}$$

$$\begin{aligned} \lim_{x'\to x} D_\alpha D'_\beta iG_{\mu\nu}(x,x') &= H^2 \left(-2\frac{d+2}{d}\gamma'(0) + \gamma(0)\right) g_{\alpha\beta} g_{\mu\nu} \\ &+ H^2 \left(\frac{2}{d}\gamma'(0)\right) g_{\alpha\mu} g_{\beta\nu} + H^2 \left(\frac{2}{d}\gamma'(0) - \gamma(0)\right) g_{\beta\mu} g_{\alpha\nu}. \end{aligned} \tag{C.16}$$

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