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Master's Thesis

Dynamical interactions in (2+1)D Dirac systems

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Abstract

In this thesis we study the full relativistic and dynamical interaction in (2+1)D Dirac systems. First, we briefly introduce condensed-matter realizations of such systems and discuss some of their key properties. After this, following work by Marino, we project Quantum Electrodynamics (QED) onto a plane, and show that the (2+1)D Pseudo-QED Lagrangian is equivalent to this projection. The properties of this Lagrangian are discussed, and it is shown that it describes unscreened Coulomb interaction in the static limit. A review is made of results of Pseudo-QED in the literature, focusing on obtaining the transverse conductivity. We reproduce results in the literature showing that a quantum valley Hall current may be generated by dynamical interactions for both the massive and massless case, and a quantum Hall current in the massive case. We then couple massive Pseudo-QED to a massive scalar field via a quartic interaction to study the effect on the generated transverse currents. We find that the quantum valley Hall current obtains a non-universal correction dependent on the ratio of the fermionic and scalar field masses. We also consider massless Pseudo-QED coupled to a scalar field and calculate the divergent Feynman diagrams involving the scalar field. These could be used for a renormalization group (RG) analysis of the system to investigate how the RG-flow of massless Pseudo-QED changes under the influence of a scalar field. We end by briefly considering other applications of Pseudo-QED and other projections of QED in the outlook.

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Contents

Introduction

The synthesis of graphene, by Geim and Novoselov in 2005 [1], has brought about a new paradigm in condensed-matter physics. Consisting of a honeycomb lattice of carbon atoms, just one atom thick, graphene possesses some remarkable properties: it is the strongest material ever to be discovered, it is very flexible, light, transparent and is an extremely good conductor. These properties, together with the abundance of carbon, hold great promise for the applications of graphene in a wide range of industries.

The electrons in graphene have a linear dispersion relation, and are massless Dirac fermions that move at a Fermi velocity $v_{\mathrm{F}} = c/300$ [2] that plays the role of the speed of light. They are thus quasi-relativistic particles, and this makes graphene the first relativistic system that can be observed in a tabletop experiment, and where some predictions of relativistic quantum field theories can be tested. Klein tunneling for example, a paradoxical prediction from relativistic quantum field theory which predicts electrons can tunnel through arbitrarily high potential barriers, has been observed in graphene [3] and zitterbewegung, another relativistic effect that causes Dirac electrons to oscillate very rapidly, has also been predicted for graphene [4].

Initially, it was believed that the electrons in graphene were very weakly interacting. However, the measurement of the fractional Quantum Hall Effect [5] and of the renormalization of the Fermi velocity [6, 7] have proven that interactions are indeed important at low temperatures in sufficiently clean samples.

Because of the relativistic nature of graphene, we can treat electron-electron interactions using tools from relativistic quantum field theory, as being mediated by a $U(1)$ gauge field. Standard Quantum Electrodynamics (QED) in (2+1)D does not suffice, however, because in graphene the electromagnetic interaction is unscreened, meaning the electromagnetic gauge field is (3+1)D, while the electrons are confined to a plane.

There is thus a dimensional mismatch, and the (2+1)D effective theory describing this situation is called Pseudo-QED, because of a pseudo-differential operator appearing in the Lagrangian, and it was first described by Marino already in 1993 [8]. With the discovery of graphene and other two-dimensional materials, there has been renewed interest in this theory, and its application to condensed-matter systems. Using Pseudo-QED the longitudinal $T = 0$ DC-conductivity has been calculated, and an interaction-induced valley Hall effect has been predicted [9]. The valley g-factor was also calculated and found to be in agreement with experiments [10].

After the discovery of graphene, many similar two-dimensional materials were proposed and synthesized with atoms different from carbon. Silicene and phosphorene, for example, also consist of honeycomb lattices, but made of silicon and phosphorus atoms respectively. The larger ionic radius of these atoms causes the lattice to deform, which makes it possible to open a gap by applying a perpendicular electric field [11]. Another class of materials that has massive (2+1)D Dirac electrons is Transition Metal Dichalcogenide Monolayers (TMDM). These materials have an intrinsic gap, and no inversion symmetry of the lattice, and this makes them particularly suited to observe the so-called valley degree of freedom in the system. By considering massive electrons, Pseudo-QED can also describe these massive (2+1)D Dirac systems.

While the full dynamical interactions in these systems is often neglected in favour of a static Coulomb interaction, recently it has been shown that the full dynamical description also captures effects that are non-perturbative in v_{F}, and that are thus missed by a static approximation. More specifically an interaction-induced quantum valley Hall effect has been predicted in graphene [9], and also in silicene, where a spontaneous quantum Hall effect is also found [12].

In this thesis we will introduce Dirac materials, derive Pseudo-QED and make a brief review of the literature on the subject. We then study massive Pseudo-QED coupled to a massive scalar field, and calculate corrections to the conductivities from the literature. We also couple massless Pseudo-QED to a scalar field and calculate the divergent Feynman diagrams resulting from this coupling.

The structure of this thesis is as follows: in Chapter 1 we will briefly introduce graphene and other two dimensional Dirac systems found in condensed matter. In Chapter 2 Pseudo-QED is introduced, the formalism to describe relativistic dynamical interactions in these 2D Dirac systems. In Chapter 3 we review Kubo's formula, and reproduce some results from the literature where Kubo's formula is used together with Pseudo-QED to calculate conductivities. In chapter 4 we couple Pseudo-QED to a scalar field and we calculate the Feynman diagrams and conductivity using Kubo's formula.

We end with a conclusion and outlook, in which we consider other applications of Pseudo-QED and different projections of QED.

Chapter 1

Graphene and 2D massive Dirac systems

Massless Dirac electrons were found for the first time in condensed matter in graphene in 2005 by Geim and Novoselov [1], for which they received the Nobel prize in 2010. While a tight-binding description of graphene was already studied in 1947 [13], it was long believed that a single layer of graphene was unstable. The isolation of graphene has opened up many new doors in material science, and it is a remarkable material in itself. It is the best known conductor, the strongest known material and still extremely flexible. The electrons in graphene are, as we will see, quasi-relativistic. They obey the Dirac equation, but with a Fermi velocity $v_{\mathrm{F}} \approx c/300$ replacing the speed of light [2]. This makes it possible to probe relativistic electrons in a tabletop experiment.

After the discovery of graphene, attention quickly turned toward the possibility of creating monolayers of different materials. Graphene has no gap, and it is difficult to one. Hence it is perhaps not very well suited to creating field effect transistors, one of the main components of computers today. Since graphene, many other monolayers have been theoretically studied and experimentally realized, many of which do feature a gap.

In this chapter we will first briefly introduce graphene, its tight-binding model and dispersion relation. We then explain how introducing a sub-lattice potential opens a gap in the system. This is hard to do in graphene, but possible in silicene, phosphorene and transition metal dichalcogenides, which have honeycomb lattices that are deformed in different ways. We will briefly describe these materials and see how a gap can be opened here.

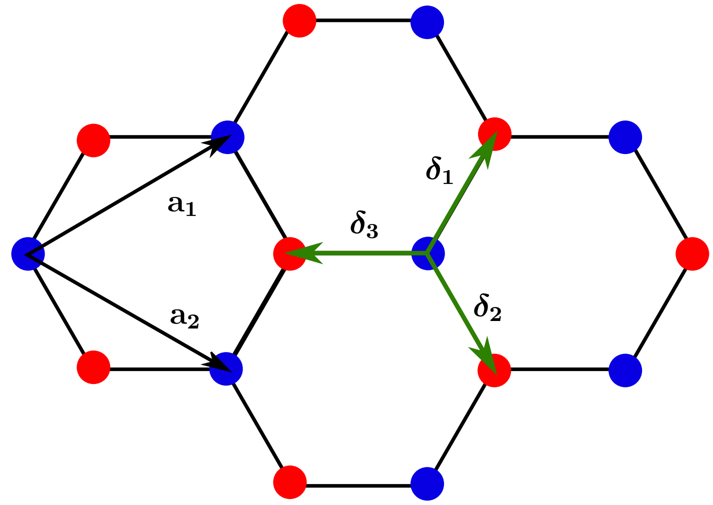


Figure 1.1.1: Schematic drawing of the honeycomb lattice of graphene. The A sublattice is denoted by blue dots, the B sublattice by red dots. The vectors $\mathbf{a_1}$and $\mathbf{a_2}$ denote the primitive lattice vectors, $\boldsymbol{\delta_1}$, $\boldsymbol{\delta_2}$ and $\boldsymbol{\delta_3}$ denote the nearest neighbour vectors of the A sublattice.

1.1 Graphene

In this section, we will briefly review the basic properties of graphene, and in particular its band structure and electronic properties. Graphene consists of a honeycomb lattice of carbon atoms (see Fig. (1.1.1)). An isolated carbon atom has 2 s electrons and 4 p electrons. In graphene, each carbon atom forms a sp^2 bond with its three neighbours, leaving one free p electron, out of the plane. This free p electron is responsible for the electronic properties of graphene.

1.1.1 Tight-binding model

To obtain the dispersion relation of graphene we consider a tight-binding model of the free p electrons with nearest-neighbour hopping, we follow Ref. [2]. The honeycomb lattice has two primitive lattice vectors (along which there is translation symmetry),

$$\mathbf{a_1} = \frac{a}{2}\left(3, \sqrt{3}\right), \qquad \mathbf{a_2} = \frac{a}{2}\left(3, -\sqrt{3}\right), \tag{1.1}$$

and, by the relation $\mathbf{a_i} \cdot \mathbf{b_j} = 2\pi\delta_{ij}$, the reciprocal lattice vectors

$$\mathbf{b_1} = \frac{2\pi}{3a}\left(1, \sqrt{3}\right), \qquad \mathbf{b_2} = \frac{2\pi}{3a}\left(1, -\sqrt{3}\right), \tag{1.2}$$

There are two inequivalent sub-lattices which we will call A and B, that is the unit cell contains two lattice points. The first Brillouin zone is again a hexagon, and has two inequivalent points at the edge, which we can take to be

$$\mathbf{K} = \frac{2\pi}{3a}\left(1, 1/\sqrt{3}\right), \qquad \mathbf{K}' = \frac{2\pi}{3a}\left(1, -1/\sqrt{3}\right). \tag{1.3}$$

These $\mathbf{K}$ and $\mathbf{K}'$ points are related by time-reversal symmetry, and are also called Dirac points, for reasons which will become clear shortly.

To set up our tight-binding model we find the nearest neighbour vectors

$$\boldsymbol{\delta_1} = \pm\frac{a}{2}\left(1, \sqrt{3}\right) \qquad \boldsymbol{\delta_2} = \pm\frac{a}{2}\left(1, -\sqrt{3}\right) \qquad \boldsymbol{\delta_3} = \pm a\left(-1, 0\right), \tag{1.4}$$

with the plus for the A sub-lattice, and the minus for the B sub-lattice, and $a \approx 1.42$Å the lattice spacing. The tight-binding Hamiltonian takes the form

$$H = -t \sum_{\langle i,j\rangle,\sigma} \left(a^\dagger_{\sigma,i} b_{\sigma,j} + b^\dagger_{\sigma,i} a_{\sigma,j}\right), \tag{1.5}$$

where a and b are the annihilation operators of electrons on the A and B sub-lattices respectively, $t \approx 2.8$eV [2] is the hopping amplitude, σ denotes the sum over the spins and $\langle i, j\rangle$ denotes the sum over the nearest neighbours. Ignoring spin for now, we obtain by Fourier transforming the annihilation and creation operators

$$\begin{aligned} H &= -t \sum_{i,j} \frac{1}{N} \int d^2k \int d^2k' \left(a^\dagger_k b_{k'} \exp\left(i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{x}_i\right) \exp\left(-i\mathbf{k} \cdot \delta_j\right) + h.c.\right) \\ &= -t \sum_j \int d^2k \left(a^\dagger_k b_k \exp\left(-i\mathbf{k} \cdot \delta_j\right) + h.c.\right). \end{aligned} \tag{1.6}$$

Denoting $\psi^\dagger = (a^\dagger_k, b_k)$, we have

$$H = \int d^2k \psi^\dagger \begin{pmatrix} 0 & \Delta(k) \\ \Delta^\dagger(k) & 0 \end{pmatrix} \psi, \tag{1.7}$$

with $\Delta(k) = -t\sum_j \exp\left(-i\mathbf{k} \cdot \delta_j\right)$. The eigen-energies are therefore

$$E_\pm = \pm t \left|\sum_j \exp\left(-i\mathbf{k} \cdot \delta_j\right)\right|. \tag{1.8}$$

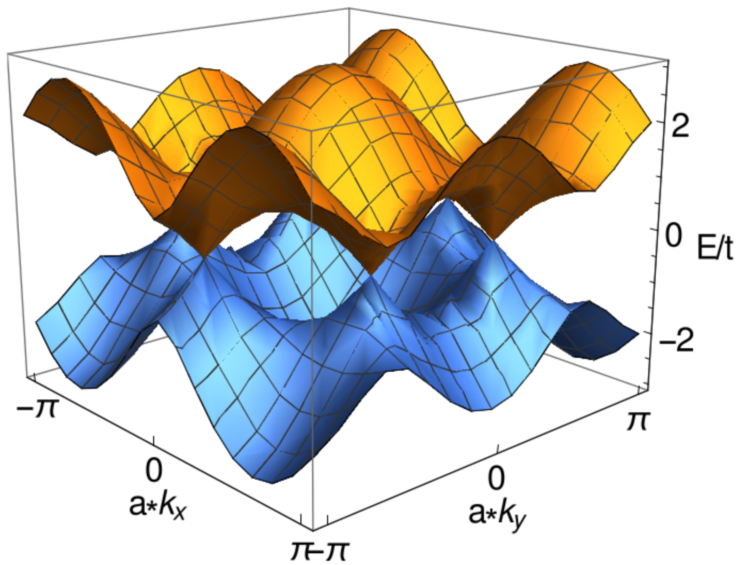


Figure 1.1.2: The valence (blue) and conduction (orange) bands of graphene considering only nearest-neighbour hopping. There are six Dirac cones in a hexagonal configuration.

Substituting Eq. (1.4) into Eq. (1.8), we obtain

$$\begin{aligned} E_\pm(\mathbf{k}) &= \pm t \left| \exp\left[-i\frac{a}{2}\left(k_x + \sqrt{3}k_y\right)\right] + \exp\left[-i\frac{a}{2}\left(k_x - \sqrt{3}k_y\right)\right] + \exp\left(iak_x\right) \right| \\ &= \pm t \left| \exp\left(-i\frac{a}{2}k_x\right) 2\cos\left(\frac{a\sqrt{3}}{2}k_y\right) + \exp\left(iak_x\right) \right| \\ &= \pm t \left\{ 4\cos^2\left(\frac{a\sqrt{3}}{2}k_y\right) + 1 + 2\cos\left(\frac{a\sqrt{3}}{2}k_y\right)\left[\exp\left(-i\frac{3a}{2}k_x\right) + \exp\left(i\frac{3a}{2}k_x\right)\right]\right\}^{1/2} \\ &= \pm t \sqrt{1 + 4\cos^2\left(\frac{a\sqrt{3}}{2}k_y\right) + 4\cos\left(\frac{a\sqrt{3}}{2}k_y\right)\cos\left(\frac{3a}{2}k_x\right)}. \end{aligned} \tag{1.9}$$

The dispersion relation is plotted in Fig. (1.1.2). Note that we have particle-hole symmetry; if we include next-nearest neighbour hopping this symmetry will be broken. At the Dirac points, we have $E_\pm(\mathbf{K}) = E_\pm(\mathbf{K}') = 0$, hence the upper and lower bands are touching. To find the low-energy behaviour of the system, let us expand $\Delta(\mathbf{k})$

around the $\mathbf{K}$ point by writing $\mathbf{k} = \mathbf{K} + \mathbf{q}$ with $\mathbf{q} \ll \mathbf{K}$,

$$\begin{aligned}
\Delta(\mathbf{q}) &= -t \sum_j \exp\left(-i\left(\mathbf{K}+\mathbf{q}\right)\cdot\delta_j\right) \\
&= -t \sum_j \exp\left(-i\mathbf{K}\cdot\delta_j\right)\exp\left(-i\mathbf{q}\cdot\delta_j\right) \\
&\approx -t\left\{\exp\left(-i\frac{2\pi}{3}\right)\left(1-i\mathbf{q}\cdot\delta_1\right)+\left(1-i\mathbf{q}\cdot\delta_2\right)+\exp\left(i\frac{2\pi}{3}\right)\left(1-i\mathbf{q}\cdot\delta_3\right)\right\} \\
&= it\frac{a}{2}\left\{\exp\left(-i\frac{2\pi}{3}\right)\left(q_x+\sqrt{3}q_y\right)+\left(q_x-\sqrt{3}q_y\right)-2q_x\exp\left(i\frac{2\pi}{3}\right)\right\} \\
&= it\frac{a}{2}\left\{q_x\left[\exp\left(-i\frac{2\pi}{3}\right)+1-2\exp\left(i\frac{2\pi}{3}\right)\right]+q_y\left[\exp\left(-i\frac{2\pi}{3}\right)\sqrt{3}-\sqrt{3}\right]\right\} \\
&= it\frac{a}{2}\left\{q_x\left[\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)+1-2\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\right]+q_y\left[\left(-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)\sqrt{3}-\sqrt{3}\right]\right\} \\
&= t\frac{3a}{4}\left\{q_x\left[\sqrt{3}+i\right]+q_y\left[1-i\sqrt{3}\right]\right\}. \qquad (1.10)
\end{aligned}$$

Substituting Eq. (1.10) into Eq. (1.7) , we obtain

$$H_{\mathbf{K}}(\mathbf{q}) = \int d^2q\psi^\dagger \frac{t3a}{2}\begin{pmatrix} 0 & \frac{\left(\sqrt{3}+i\right)}{2}\left(q_x-iq_y\right) \\ \frac{\left(\sqrt{3}-i\right)}{2}\left(q_x+iq_y\right) & 0 \end{pmatrix}\psi. \qquad (1.11)$$

Performing a unitary transformation of the fields

$$\psi_1 \to \mathrm{e}^{-i\frac{\pi}{12}}\psi_1,\ \psi_2 \to \mathrm{e}^{i\frac{\pi}{12}}\psi_2,$$

and defining $v_{\mathrm{F}} = 3ta/2$, we find

$$\begin{aligned}
H_{\mathbf{K}}(\mathbf{q}) &= \int d^2q\,\psi^\dagger v_{\mathrm{F}}\begin{pmatrix} 0 & q_x-iq_y \\ q_x+iq_y & 0 \end{pmatrix}\psi \\
&= \int d^2q\,\psi^\dagger\, v_{\mathrm{F}}\left(\mathbf{q}\cdot\boldsymbol{\sigma}\right)\psi, \qquad (1.12)
\end{aligned}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ are the Pauli matrices acting in the sub-lattice space (not spin space). Expanding the full Hamiltonian around $\mathbf{K}'$, we find by a similar calculation

$$\begin{aligned}
H_{\mathbf{K}'}(\mathbf{q}) &= \int d^2q\,\psi^\dagger v_{\mathrm{F}}\begin{pmatrix} 0 & q_x+iq_y \\ q_x-iq_y & 0 \end{pmatrix}\psi. \\
&= \int d^2q\,\psi^\dagger v_{\mathrm{F}}\left(\mathbf{q}\cdot\boldsymbol{\sigma}^*\right)\psi, \qquad (1.13)
\end{aligned}$$

where $\boldsymbol{\sigma}^* = (\sigma_x, -\sigma_y)$. We see that the valleys are described by the massless 2-component Dirac equation. The dispersion relation around the Dirac points is

$$E_{\pm}(\mathbf{k}) = \pm v_{\mathrm{F}} \left|\mathbf{k}\right|.$$

The full low-energy Hamiltonian consists of the sum of the two valleys.

1.2 Massive Dirac systems

A peculiar property of graphene is the absence of a band-gap. While this is one of the reasons why the carrier mobility in graphene is so high, a band-gap can be desirable, for example to create a field-effect transistor. We are thus led to ask: can we open up a band-gap in graphene, and are there other materials similar to graphene with a band-gap? It turns out that the answer to both questions is yes. In this section we will first demonstrate how breaking the sub-lattice symmetry leads to a gap in graphene. After this, we will introduce other materials that share the honeycomb structure with graphene, but are made of different atoms; namely silicene, phosphorene and dichalcogenides.

1.2.1 Breaking sub-lattice symmetry

Let us return to the tight-binding model of the previous section, but now introduce an on-site potential V with a different sign for each of the sub-lattices (we follow Ref. [14]),

$$H_{on-site} = \frac{V}{2} \int d^2k \left(a^{\dagger}_{\sigma,i} a_{\sigma,i} - b^{\dagger}_{\sigma,i} b_{\sigma,i} \right). \tag{1.14}$$

Combining this with Eq. (1.7), we find the Hamiltonian to be

$$H = \int d^2k \psi^{\dagger} \begin{pmatrix} \frac{V}{2} & \Delta(k) \\ \Delta^{\dagger}(k) & -\frac{V}{2} \end{pmatrix} \psi, \tag{1.15}$$

and the dispersion relation changes to

$$E_{\pm}(\mathbf{k}) = \pm \sqrt{\frac{V^2}{4} + \Delta(\mathbf{k})\Delta^{\dagger}(\mathbf{k})}. \tag{1.16}$$

We know that $\Delta(\mathbf{K}) = \Delta(\mathbf{K}') = 0$, and hence

$$E_{\pm}(\mathbf{K}) = \pm \frac{V}{2}, \tag{1.17}$$

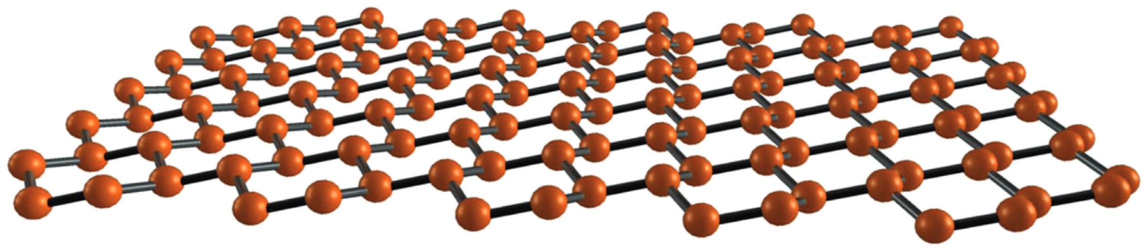


Figure 1.2.1: The buckled honeycomb structure of silicene. Figure extracted from Ref. [15].

By breaking the sub-lattice symmetry we have thus opened up a band-gap of V between the valence and conduction band. We can again expand around the Dirac points to obtain the low-energy Hamiltonian

$$H_{\mathbf{K}}(\mathbf{q}) = \int d^2q\psi^\dagger v_{\mathrm{F}} \begin{pmatrix} \frac{V}{2} & q_x - iq_y \\ q_x + iq_y & -\frac{V}{2} \end{pmatrix} \psi$$
$$= \int d^2q\psi^\dagger \left[v_{\mathrm{F}} \left(\mathbf{q}\cdot\sigma\right) + \sigma_z V\right] \psi. \tag{1.18}$$

1.2.2 Silicene

While we have seen that applying a different on-site potential for each of the two sub-lattices opens up a gap in graphene, it is not clear how to achieve this in an experimental setup. Silicene, a honeycomb lattice of silicon atoms, provides an easier way of opening up a gap. Because of the larger ionic radius of silicon compared to carbon, the lattice of silicene is buckled (see Fig. 1.2.1)[16]. By applying an out-of-plane electric field we can introduce a different potential for each of the two sub-lattices, and thus open up a tunable gap in the spectrum [17]. Another way to achieve the same result is by growing silicene on a substrate [18]. In the case of applying an electric field we need to add a term [19]

$$H_{E-field} = \int d^2k\psi^\dagger \left(-lE_z\sigma_z\right) \psi, \tag{1.19}$$

where $l = 0.23$Å is half the separation between the sub-lattices due to the buckling of the lattice, E_z the strength of the perpendicular electric field and σ_z is the z Pauli matrix acting in sub-lattice space. Notice that Eq. (1.19) has the same structure as Eq. (1.14), and we thus immediately see that a band-gap proportional to E_z will be

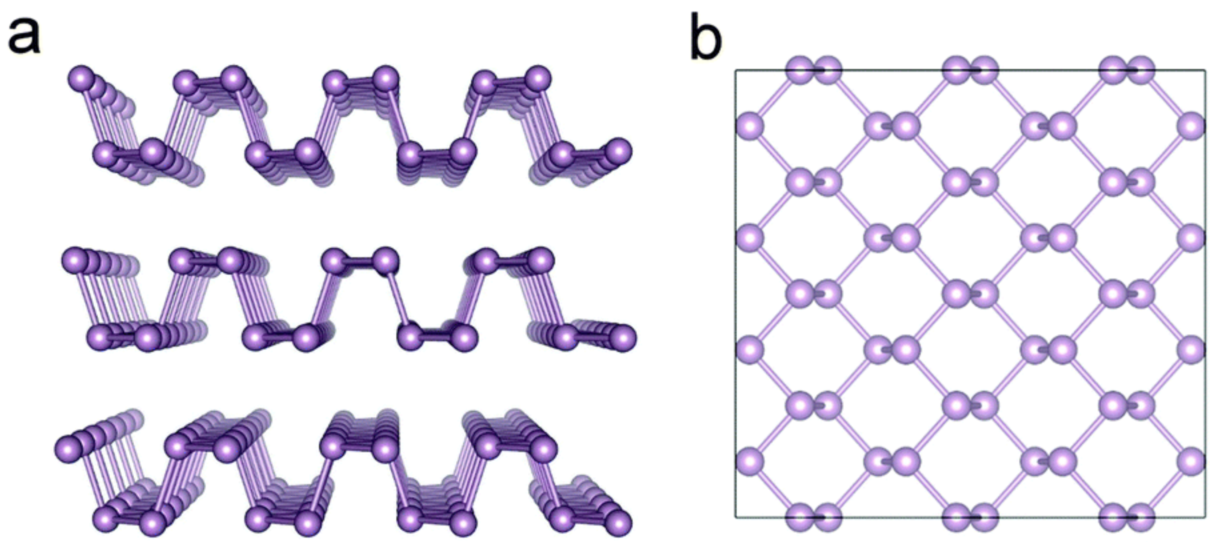


Figure 1.2.2: a) Schematic side-view of three layers of phosphorene. b) Schematic of the lattice of phosphorene, top view. Figure extracted from Ref. [23].

opened. Silicene also has a larger spin-orbit coupling that opens an intrinsic gap, although it is for most purposes still negligible (about 1.55 meV [20]).

Looking at the periodic table of elements, one might wonder what other monolayers would be possible to create, and which ones are Dirac systems? A first guess would be elements similar to carbon and silicon. If we look at the same column in the periodic table, we find: germanium, tin and lead. These elements will have an even larger ionic radius, and thus even more buckling (and more spin-orbit coupling). Germanene (a monolayer of germanium) has already been grown on a substrate [21], and tinene (a monolayer of tin) has been theoretically investigated and is predicted to be a Dirac system with a buckling of 0.7Å[22], much larger than in silicene.

1.2.3 Phosphorene

A monolayer of phosphor, phosphorene, has also been experimentally realized recently [24]. In contrast to the previous materials discussed, however, it is not a Dirac system but has a gap even in the absence of spin-orbit coupling. The reason is that phosphor has five valence electrons instead of four, and this causes the hexagonal lattice to deform in a different way. The lattice is not buckled (as it is for silicene and germanene), but puckered. This means that of a hexagon the leftmost three phosphor atoms lie in a different plane than the rightmost three atoms, as can be seen in Fig. (5.1.2). This breaks inversion times time-reversal symmetry, which is a condition to have a massive Dirac system.

1.2.4 Transition metal dichalcogenide monolayers

An interesting property of the 2D materials we have discussed so far is that they posses a valley degree of freedom. This valley degree of freedom behaves similar to spin and because of this there is the possibility to manipulate this degree of freedom, and thus analogous to spintronics, to build 'valleytronic devices'. A major challenge is of course to access this degree of freedom, and it is a topic of ongoing research.

One class of materials that seems to be particularly suited to valley manipulation is transition metal dichalcogenide monolayers (TMDM). TMDM's consist of a lattice of MX_2 atoms where M is a transition metal atom (most commonly Mo), and X is a chalcogen atom (S, Se or Te). The lattice of MoS_2 is depicted in Fig. 1.2.3. The transition metal atom is sandwiched between two layers of the chalcogen atom, and they form a hexagonal structure when seen from the top. Fig. 1.2.3c shows the dispersion relation of MoS_2, and we see that the low-energy excitations are massive Dirac fermions.

Unlike in graphene, there is no center of inversion in the lattice, which makes accessing the valleys of the system possible. In a recent experiment, researchers probed the valley degree of freedom by circularly polarized photoluminescence [25]. They radiated a sample of MoS_2 monolayer with circularly polarized light, and measured the degree of circular polarization

$$\eta(\mathbf{k}) = \frac{\left|\mathcal{P}_+^{cv}(\mathbf{k})\right|^2 - \left|\mathcal{P}_-^{cv}(\mathbf{k})\right|^2}{\left|\mathcal{P}_+^{cv}(\mathbf{k})\right|^2 + \left|\mathcal{P}_-^{cv}(\mathbf{k})\right|^2}, \tag{1.20}$$

where $\mathcal{P}_+^{cv}$ is the absorption of left-handed light, and $\mathcal{P}_-^{cv}$ that of right-handed light. $\eta(\mathbf{k})$ is thus the difference between the absorption of left and right handed light, divided by the total absorption. The theoretical prediction of $\eta(\mathbf{k})$ over the Brillouin zone can be found in Fig. 1.2.3c. One can see a clear valley polarization, that was also measured [25].

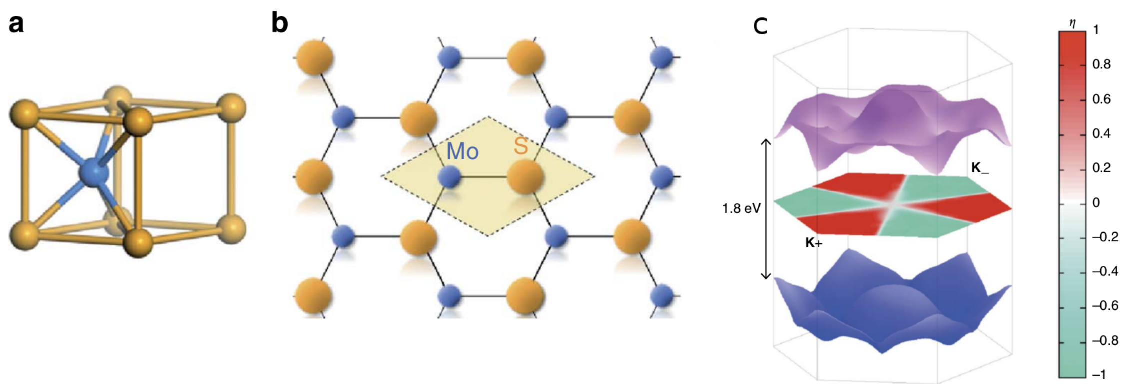


Figure 1.2.3: a) The surroundings of the Mo atoms (blue) in a MoS_2 monolayer. b) The structure of a MoS_2 monolayer as seen from the top. c) The conductance (purple) and valence (blue) band, and in the middle the degree of circular polarization $\eta(\mathbf{k})$. Figures extracted from Ref. [25]

Chapter 2

Pseudo-QED

In the previous chapter we have explored some 2D materials that give rise to massless and massive Dirac systems. The Dirac electrons in these systems also interact with each other via the electromagnetic force. In this chapter we will investigate the electron-electron interactions in these Dirac systems. It was long thought that the electrons in graphene were very weakly interacting, but the observation of the fractional quantum Hall effect [5], and the renormalization of the Fermi velocity [7, 6], have shown that in clean samples and low temperatures, interactions play a crucial role. Electromagnetic interactions in Dirac systems are usually studied in the static approximation, that is, an instantaneous Coulomb force between the electrons is considered.

In this chapter, we aim to introduce another formalism, called Pseudo-QED, that captures the full, relativistic interactions between the electrons. This theory was first written down by E.C. Marino in 1993 [8], but has attracted renewed attention with the rise of 2D Dirac systems in condensed matter. The idea is that, since in graphene and similar materials there is no screening of the electromagnetic field, the photons mediating the interactions are free to move out of the plane, while the electrons are confined to the plane. In other words, there is a dimensional mismatch. The electrons are (2+1)D (they are confined to the plane), and their interactions are mediated by a U(1) gauge field that is (3+1)D. The way to treat this dimensional mismatch is to start with QED in (3+1)D, confine the matter current to a plane, and then integrate out the extra dimension of the gauge field. The effective theory that we obtain by this procedure is called Pseudo-QED, since the Lagrangian contains a pseudo-differential operator. It is sometimes also called reduced QED in the literature [26, 27, 28].

In Section 2.1, we will project QED in (3+1)D onto a plane, and obtain the effective electron-electron interaction in this plane. In Section 2.2, we will show that in the static limit the effective interaction reduces to the Coulomb interaction, and in Section 2.3 we will derive the effective (2+1)D Lagrangian and show that it captures the same information as the projection of QED onto the plane. We give the Feynman rules of this Lagrangian in Section 2.4, and calculate two 1-loop Feynman diagrams to illustrate how calculations proceed.

2.1 Projecting QED onto a plane

In this section, we will first obtain the effective interaction of electrons for QED in (3+1)D. We then confine the electrons to a plane and integrate out the extra dimension of the gauge field. We will see that, contrary to QED in (2+1)D, we are able to describe the correct unscreened electromagnetic interaction in this way.

The standard QED-Lagrangian in (3+1)D is

$$\mathcal{L}_{QED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - m)\psi - eA_\mu j^\mu + \mathcal{L}_{GF}, \tag{2.1}$$

where $F^{\mu\nu}$ is the electromagnetic tensor, ψ is the electron field, m is the bare electron mass, e the electron charge, A_μ the electromagnetic 4-potential, $\mathcal{L}_{GF}$ is a gauge-fixing term and j^μ is the conserved matter current

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{2.2}$$

Let us now obtain the effective current-current interaction by integrating out the A_μ field. The generating functional is given by (we follow Ref. [8])

$$Z_{QED}[j^\mu] = \int \mathcal{D}\bar{\psi}D\psi DA_\mu \mathrm{e}^{iS[\psi,\overline{\psi},A_\mu]}. \tag{2.3}$$

We proceed by completing the square and using the result for a Gaussian integral. First, we write

$$\begin{aligned} S[\psi,\overline{\psi},A_\mu] &= \int d^4z \left\{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - m)\psi - eA_\mu j^\mu - \frac{1}{2}\lambda(\partial^\mu A_\mu)^2\right\} \\ &= \int d^4z \left\{-\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) - \frac{1}{2}\lambda\partial^\mu A_\mu \partial^\nu A_\nu - eA_\mu j^\mu \right. \\ &\quad \left. + \overline{\psi}\left(i\partial\!\!\!/ + m\right)\psi\right\}, \end{aligned} \tag{2.4}$$

where $-\frac{1}{2}(\lambda\partial^\mu A_\mu)^2$ is the gauge-fixing term. We now partially integrate the first two terms to obtain an expression quadratic in A_μ,

$$S[\psi,\overline{\psi},A_\mu] = \int d^4z \left\{\frac{1}{2}A_\mu(\partial_\alpha\partial^\alpha\eta^{\mu\nu} - (1-\lambda)\partial^\mu\partial^\nu)A_\nu - eA_\mu j^\mu + \overline{\psi}(i\partial\!\!\!/ - m)\psi\right\}$$
$$= \int d^4z \left\{\frac{1}{2}A_\mu G^{\mu\nu}A_\nu - eA_\mu j^\mu + \overline{\psi}(i\partial\!\!\!/ - m)\psi\right\}, \qquad (2.5)$$

where $G^{\mu\nu} = (\partial_\alpha\partial^\alpha\eta^{\mu\nu} - (1-\lambda)\partial^\mu\partial^\nu)$ is the inverse photon propagator. The first two terms, which are quadratic and linear in A_μ respectively, can be rewritten by completing the square

$$\int d^4z \left\{\frac{1}{2}A_\mu(G_{\mu\nu})^{-1}A_\nu - eA_\mu j^\mu\right\}$$
$$= \int d^4z \left\{\frac{1}{2}\tilde{A}_\mu(z)(G_{\mu\nu})^{-1}(z)\tilde{A}_\nu(z) + i\int d^4z' \frac{e^2}{2} j^\mu(z)G_{\mu\nu}(z-z')j^\nu(z')\right\}, \qquad (2.6)$$

where we used $G_{\mu\alpha}(z)G^{\alpha\nu}(z') = i\delta^\mu_\nu\delta(z-z')$ and made a redefinition $\tilde{A}_\mu(z) = A_\mu(z) + i\int d^4z' ej^\alpha G_{\alpha\mu}$, which shifts the field by a constant. Finally, we substitute this result back into Eq. (2.3) to obtain

$$Z_{QED}[j^\mu] = \int \mathcal{D}\bar{\psi}D\psi D\tilde{A}_\mu \exp\left\{i\int d^4z \left[\frac{1}{2}\tilde{A}_\mu(G_{\mu\nu})^{-1}\tilde{A}_\nu + i\int d^4z' \frac{e^2}{2} j^\mu G_{\mu\nu}(z-z')j^\nu \right.\right.$$
$$\left.\left. + \overline{\psi}(\partial\!\!\!/ + m)\psi\right]\right\} \qquad (2.7)$$

This expression is quadratic in $\tilde{A}_\mu$, and it is thus a Gaussian integral which can be promptly solved. The result of this Gaussian integral is a constant proportional to $[\det(G_{\mu\nu})]^{-1}$, which we can absorb into the measure of the path integral. The kinetic term of the ψ field does not contribute to the effective current-current interaction, since it does not contain the current so we ignore it. The generating functional then becomes

$$Z^{eff}_{QED} = \int \mathcal{D}\bar{\psi}D\psi \exp\left\{-\frac{e^2}{2}\int d^4z d^4z' \left[j^\mu(z)G_{\mu\nu}(z-z')j^\nu(z')\right]\right\}$$
$$= \int \mathcal{D}\bar{\psi}D\psi \exp\left(-\frac{e^2}{2}\int d^4z d^4z' \left\{j^\mu(z)\left[-\Box\eta_{\mu\nu} + (1-\frac{1}{\lambda})\partial_\mu\partial_\nu\right]\frac{1}{(-\Box)^2}j^\nu(z')\right\}\right), \qquad (2.8)$$

where we used that $G_{\mu\nu} = \left[-\Box\eta_{\mu\nu} + (1-\frac{1}{\lambda})\partial_\mu\partial_\nu\right]\frac{1}{(-\Box)^2}$, with $\Box$ the d'Alembertian operator, is the photon propagator. Remembering that $\partial^\mu j_\mu = 0$, since it is a conserved current, we see that only the first term in Eq. (2.8) contributes and we thus find

$$Z^{eff}_{QED} = \int \mathcal{D}\bar{\psi} D\psi \exp\left[-\frac{e^2}{2}\int d^4z d^4z' j^\mu(z)\left(\frac{\eta_{\mu\nu}}{-\Box}\right) j^\nu(z')\right]. \tag{2.9}$$

To see what the effective interaction of (2+1)D electrons coupled to a (3+1)D U(1) gauge field looks like, we now confine the matter current to a plane by writing [8]

$$j^\mu(x^0, x^1, x^2, x^3) = \begin{cases} j^\mu_{2+1}(x^0, x^1, x^2)\delta(x^3) & \mu = 0, 1, 2 \\ 0 & \mu = 3 \end{cases}. \tag{2.10}$$

Substituting this into Eq. (2.9) we find (note that from now on μ=0,1,2)

$$Z^{eff}_{PQED} = \int \mathcal{D}\bar{\psi} D\psi \exp\left\{-\frac{e^2}{2}\int d^3z d^3z' \left[j^\mu_{2+1}(z)\frac{\eta_{\mu\nu}}{-\Box}\bigg|_{z_3=z_3'=0} j^\nu_{2+1}(z')\right]\right\}. \tag{2.11}$$

The current-current interaction (the term quadratic in the current) is thus proportional to the inverse d'Alembertian. We consider its Fourier transform

$$\frac{1}{-\Box} = \int \frac{d^4k}{(2\pi)^4} \mathrm{e}^{-ik(z-z')}\frac{1}{k^2}. \tag{2.12}$$

Since $z_3 = z_3' = 0$, we can calculate the integral over k_3 as follows

$$\begin{aligned} \frac{1}{-\Box}\bigg|_{z_3=z_3'=0} &= \int \frac{d^3k}{(2\pi)^3}\frac{dk_3}{(2\pi)}\frac{\mathrm{e}^{-i\left[k_0(z_0-z_0')-k_1(z_1-z_1')-k_2(z_2-z_2')\right]}}{k_0^2-k_1^2-k_2^2-k_3^2} \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{i\pi}{(2\pi)}\frac{\mathrm{e}^{-i\left[k_0(z_0-z_0')-k_1(z_1-z_1')-k_2(z_2-z_2')\right]}}{\sqrt{k_0^2-k_1^2-k_2^2}} \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{i}{2}\frac{\mathrm{e}^{-ik(x-y)}}{\sqrt{k^2}} \end{aligned} \tag{2.13}$$

The effective current-current interaction thus has the form

$$e^2 \int \frac{d^3k}{(2\pi)^3}\frac{i}{2}\frac{\mathrm{e}^{-ik(x-y)}}{\sqrt{k^2}} \tag{2.14}$$

2.2 The static limit

To gain some insight into the effective interaction Eq. (2.14), it is helpful to calculate the static limit, which is obtained by setting $\mathbf{j} = 0$. The continuity equation

$$\frac{\partial j^0}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{2.15}$$

then tells us that j^0 is time-independent. Using this, Eq. (2.11) simplifies to ($i = \{1, 2\}$)

$$\begin{aligned} Z^{eff}_{PQED} &= \int \mathcal{D}\bar{\psi} D\psi \exp\left[-i\frac{e^2}{4}\int d^2z d^2z' j^0(z^i)\left(\int \frac{d^3k}{(2\pi)^3} dz^0 dz'^0 \frac{\mathrm{e}^{-ik(z-z')}}{\sqrt{k^2}}\right) j^0(z'^i)\right] \\ &= \int \mathcal{D}\bar{\psi} D\psi \exp\left[-i\frac{e^2}{4}\int d^2z d^2z' j^0(z^i)\left(\int \frac{d^2k}{(2\pi)^2} \frac{\mathrm{e}^{-i\left[-k_1(z_1-z_1')-k_2(z_2-z_2')\right]}}{\sqrt{-k_1^2-k_2^2}}\right) j^0(z'^i)\right] \\ &= \int \mathcal{D}\bar{\psi} D\psi \exp\left[-\frac{1}{2}\int d^2z d^2z' j^0(z^i)\left(\int \frac{d^2k}{(2\pi)^2} \frac{e^2}{2} \frac{\mathrm{e}^{-ik(z-z')}}{\sqrt{k^2}}\right) j^0(z'^i)\right], \end{aligned} \tag{2.16}$$

where in the last line $k = (k_1, k_2)$ is now a 2-component vector. Now the $j^0 - j^0$ interaction is in this limit nothing but the static potential V between two electrons. We can thus calculate

$$\begin{aligned} V &= e^2 \int \frac{d^2k}{(2\pi)^2} \frac{1}{2} \frac{\mathrm{e}^{-ik(z-z')}}{\sqrt{k^2}} \\ &= e^2 \int \frac{dk}{(2\pi)^2} \int_0^{2\pi} d\theta \frac{\|k\|}{2} \frac{\mathrm{e}^{-i\|k\|\|z-z'\|Cos(\theta)}}{\sqrt{k^2}} \\ &= e^2 \int \frac{dk}{(2\pi)^2} \int_0^{2\pi} d\theta \frac{1}{2} \mathrm{e}^{-i\|k\|\|z-z'\|Cos(\theta)} \\ &= e^2 \int \frac{dk}{(4\pi)} J_0(\|k\|\|z-z'\|) \\ &= \frac{1}{4\pi} \frac{e^2}{\|z-z'\|}. \end{aligned} \tag{2.17}$$

In the static limit we thus recover the Coulomb interaction between the electrons, as expected. If we would consider QED in (2+1)D, we would not find this result, but rather a static potential depending on the logarithm the separation between the electrons [29]. Pseudo-QED is thus the correct description of unscreened electron-electron interactions in the plane.

2.3 The Pseudo-QED Lagrangian

To facilitate calculations, it would be nice to have an effective Lagrangian that is (2+1)D from the start that gives us the same correlation functions. This Lagrangian was found by Marino in 1993 [8], and has the form

$$\mathcal{L}_{PQED} = -\frac{1}{2} F^{\mu\nu} \frac{1}{(-\Box)^{\frac{1}{2}}} F_{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - m)\psi - e j^\mu A_\mu. \tag{2.18}$$

Eq. (2.18) is completely (2+1)D, and all indices run from 0 to 2. The difference with QED_{2+1} is the $(-\Box)^{-1/2}$ in the first term. This is a pseudo-differential operator, and makes the electromagnetic interactions in the theory non-local. The theory has nevertheless been proven to be causal [30], and unitary [31]. The fact that the theory is non-local is not a disaster, as it is an effective theory. In the Caldeira-Leggett model, describing dissipative quantum systems, one also obtains a non-local theory by integrating out certain degrees of freedom [32].

2.3.1 The equations of motion and symmetries

The electrons in Pseudo-QED satisfy the (2+1)D Dirac equation

$$(iD_\mu - m)\,\psi = 0, \tag{2.19}$$

where $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative, as one can find by varying Eq. 2.18 with respect to $\overline{\psi}$. This is of course the same as for QED in (2+1)D. For the gauge field, things are different. Let us vary the Pseudo-QED Lagrangian with respect to A_μ, we find

$$\begin{aligned}\delta S &= \int d^3x \left\{ -\frac{1}{2}\left(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu\right) \frac{1}{(-\Box)^{\frac{1}{2}}} F^{\mu\nu} - \frac{1}{2} F^{\mu\nu} \frac{1}{(-\Box)^{\frac{1}{2}}} \left(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu\right) - ej^\nu \delta A_\nu \right\} \\ &= \int d^3x \left\{ \delta A_\nu \frac{1}{(-\Box)^{\frac{1}{2}}} \partial_\mu F^{\mu\nu} + \partial_\mu F^{\mu\nu} \frac{1}{(-\Box)^{\frac{1}{2}}} \delta A_\nu - ej^\nu \delta A_\nu \right\},\end{aligned} \tag{2.20}$$

where we partially integrated in the second line. Now because the inverse d'Alembertian is a pseudo-differential operator, we can generically write it in terms of its kernel

$$\begin{aligned}\int d^3x\, G(x) \frac{1}{\sqrt{-\Box}} H(x) &= \int d^3y \int d^3x\, G(x) K(x-y) H(y) \\ &= \int d^3y \int d^3x\, H(x) K(x-y) G(y),\end{aligned}$$

where $K(x-y)$ is the kernel of the operator. Using this, we see that the first two terms in Eq. (2.20) are equivalent, and we find

$$\delta S = \int d^3x \left\{ \delta A_\nu \frac{2}{(-\Box)^{\frac{1}{2}}} \partial_\mu F^{\mu\nu} - ej^\nu \delta A_\nu \right\}. \tag{2.21}$$

Hence the equation of motion is

$$2 \frac{\partial_\mu F^{\mu\nu}}{(-\Box)^{\frac{1}{2}}} = ej^\nu. \tag{2.22}$$

The Pseudo-QED Lagrangian is still gauge-invariant since $F^{\mu\nu}$ by itself is already gauge-invariant and thus the introduction of the pseudo-differential operator does not break it. If we take the fermions to be massless, the theory also has conformal symmetry.

2.3.2 Equivalence of Pseudo-QED and QED projected onto a plane

To see that Pseudo-QED is equivalent to QED projected onto a plane (as considered in the previous section), we will prove that the correlation functions of the two theories coincide. The correlation functions of a quantum field theory can be obtained in the path integral formalism by coupling each field to a source and taking functional derivatives with respect to the sources. Let us briefly review this procedure, and then compare QED projected onto a plane and Pseudo-QED.

We take our generating functional for QED

$$Z_{QED} = \int \mathcal{D}\bar{\psi} D\psi DA_\mu \exp\left\{ i \left[S_{free} + S_{int} + \int dz^4 \left(J_A^\mu A_\mu + J_\psi \psi + \bar{J}_\psi \overline{\psi} \right) \right] \right\}, \quad (2.23)$$

where we have split our action into a free action and the interacting action

$$\begin{aligned} S_{int} &= -\int dz^4 e j^\mu A_\mu, \\ S_{free} &= \int dz^4 \left\{ -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - m)\psi \right\}. \end{aligned}$$

Note that we have introduced a source term for each field. We can now Taylor expand $\exp(iS_{int})$ and replace the fields by functional derivatives with respect to the sources. This means we can pull this term outside of the integral (we exchange the order of functional integration and differentiation). We obtain

$$\begin{aligned}
Z_{QED} &= \int \mathcal{D}\bar{\psi} D\psi DA_\mu \exp\left\{iS_{int}\left[A_\mu,\psi\right]\right\} \exp\left\{i\left[S_{free} + \int dz^4\left(J_A^\mu A_\mu + J_\psi \psi + \bar{J}_\psi \overline{\psi}\right)\right]\right\} \\
&= \int \mathcal{D}\bar{\psi} D\psi DA_\mu \left(1 + iS_{int}\left[A_\mu,\psi\right] + ...\right) \exp\left\{i\left[S_{free} + \int dz^4\left(J_A^\mu A_\mu + J_\psi \psi + \bar{J}_\psi \overline{\psi}\right)\right]\right\} \\
&= \int \mathcal{D}\bar{\psi} D\psi DA_\mu \left(1 + iS_{int}\left[\frac{\delta}{\delta J_A^\mu}, \frac{\delta}{\delta J_\psi}\right] + ...\right) \\
&\times \exp\left\{i\left[S_{free} + \int dz^4\left(J_A^\mu A_\mu + J_\psi \psi + \bar{J}_\psi \overline{\psi}\right)\right]\right\} \\
&= \left(1 + iS_{int}\left[\frac{\delta}{\delta J_A^\mu}, \frac{\delta}{\delta J_\psi}\right] + ...\right) \int \mathcal{D}\bar{\psi} D\psi DA_\mu \\
&\times \exp\left\{i\left[S_{free} + \int dz^4\left(J_A^\mu A_\mu + J_\psi \psi + \bar{J}_\psi \overline{\psi}\right)\right]\right\} \\
&= \exp\left\{iS_{int}\left[\frac{\delta}{\delta J_A^\mu}, \frac{\delta}{\delta J_\psi}\right]\right\} \int \mathcal{D}\bar{\psi} D\psi DA_\mu \\
&\times \exp\left\{i\left[S_{free} + \int dz^4\left(J_A^\mu A_\mu + J_\psi \psi + \bar{J}_\psi \overline{\psi}\right)\right]\right\}.
\end{aligned} \tag{2.24}$$

Note that S_{free} is quadratic in the fields. We can thus consider S_{free} together with the source terms, which are linear, and complete the square. In this way we obtain (following the same steps as in Eq. (2.6))

$$\begin{aligned}
&S_{free} + J_A^\mu A_\mu + J_\psi \psi + J_{\overline{\psi}} \overline{\psi} \\
&= \int d^4z \left(\frac{1}{2} A_\mu G_A^{\mu\nu} A_\nu + \frac{1}{2}\overline{\psi} G_\psi \psi + J_A^\mu A_\mu + J_\psi \psi + J_{\overline{\psi}} \overline{\psi}\right)
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
&= \int d^4z' \int d^4z \left(A_\mu - i\int d^4z'\, J_A^\rho G_{A\rho\mu}\right) G_A^{\mu\nu} \left(A_\nu - i\int d^4z'\, G_{A\nu\alpha} J_A^\alpha\right) \\
&+ \left(\overline{\psi} - i\int d^4z'\, \overline{J}_\psi G_\psi\right) G_\psi^{-1} \left(\psi_\mu - i\int d^4z'\, G_\psi J_\psi\right) - \frac{1}{2} J_A^\mu G_{A\mu\nu} J_A^\nu - \frac{1}{2} J_{\overline{\psi}} G_\psi J_\psi \\
&= \int d^4z' \int d^4z\, \frac{1}{2} A_\mu G_A^{\mu\nu} A_\nu + \frac{1}{2}\overline{\psi} G_\psi^{-1} \psi - \frac{1}{2} J_A^\mu G_{A\mu\nu} J_A^\nu - \frac{1}{2} J_{\overline{\psi}} G_\psi J_\psi
\end{aligned} \tag{2.26}$$

where $G_{A\mu\nu}$ and G_ψ are the Green's functions of the A_μ and ψ fields respectively and we have absorbed i into the Green's functions. Also, we have made the shifts $A_\mu \to A_\mu + i\int d^4z'\, J_A^\rho(z') G_{A\rho\mu}(z')$, $\overline{\psi} \to \overline{\psi} + i\int d^4z'\, \overline{J}_\psi(z') G_\psi(z')$ and $\psi \to \psi + i\int d^4z'\, G_\psi(z') J_\psi(z')$. Since we have pulled the interaction terms out of the integration, our integrals are now Gaussian integrals, which will give us constants that we can absorb into the measure.

We thus find

$$Z_{QED} = \exp\left\{iS_{int}\left[\frac{\delta}{\delta J_A^\mu}, \frac{\delta}{\delta J_\psi}\right]\right\} \exp\left\{-\frac{1}{2}J_A^\mu G_{A\mu\nu} J_A^\nu - \frac{1}{2}J_{\overline{\psi}} G_\psi J_\psi\right\}. \tag{2.27}$$

Calculating correlation functions amounts to taking functional derivatives of Eq. (2.27) with respect to the sources. Thus, if the free Green's functions and interaction terms of two theories are equivalent, then their correlation functions are equivalent and the theories themselves are equivalent.

We now compare QED projected onto a plane to Pseudo-QED. The first thing we note is that the kinetic part of the fermions is the same. Both theories simply contain the Dirac Lagrangian in (2+1)D, hence the fermion propagator will be the same. The second thing we note is that the interaction terms in both theories are also equivalent, and are simply

$$-ej^\mu A_\mu,$$

where $\mu = 0, 1, 2$. The final thing we need to show is that the free Green's functions of the gauge fields are the same. In the case of QED projected onto a plane, we couple the A_μ field to a source term J^μ. We thus add a source term of the form

$$A_\mu J^\mu.$$

But since the source also couples linearly to the gauge field, like the matter current j^μ, we can follow the exact same steps as in Eqs. (2.4)-(2.8) to find that the quadratic term in J_A^μ in the action becomes

$$S_{free} = \int d^4z' \int d^4z \left\{-\frac{1}{2}J_A^\mu \left[-\Box\eta_{\mu\nu} + \left(1 - \frac{1}{\lambda}\right)\partial_\mu\partial_\nu\right] \frac{1}{(-\Box)^2} J_A^\nu\right\}. \tag{2.28}$$

We now also need to confine this source current to the plane, since the electrons source the gauge field. We thus write

$$J_A^\mu = \begin{cases} J_A^\mu(x^0, x^1, x^2)\delta(x^3) & \mu = 0, 1, 2 \\ 0 & \mu = 3 \end{cases}. \tag{2.29}$$

Let us for simplicity set $\lambda = 1$, we can then follow the steps in Eqs. (2.11)-(2.14) to conclude

$$\begin{aligned} S_{free} &= \int d^4z' \int d^4z \left\{-\frac{1}{2}J_A^\mu \left(\int \frac{d^2k}{(2\pi)^2} \frac{1}{2} \frac{\mathrm{e}^{-ik(z-z')}}{\sqrt{k^2}}\right) J_A^\nu\right\} \\ &= \int d^4z' \int d^4z \left\{-\frac{1}{2}J_A^\mu \left(\frac{1}{2}\frac{\eta_{\mu\nu}}{\sqrt{-\Box}}\right) J_A^\nu\right\}. \end{aligned} \tag{2.30}$$

Starting from the Pseudo-QED Lagrangian, and coupling the gauge field to a source term J^μ_A, we find (ignoring terms involving the fermions and introducing a gauge-fixing term with $\lambda = 1$)

$$\begin{aligned} S_{PQED} &= \int d^4z \left[-\frac{1}{2} F^{\mu\nu} \frac{1}{(-\Box)^{\frac{1}{2}}} F_{\mu\nu} + A_\mu J^\mu_A - \frac{1}{2} \lambda \partial^\mu A_\mu \frac{1}{(-\Box)^{\frac{1}{2}}} \partial^\mu A_\mu \right] \\ &= \int d^4z' \int d^4z \left\{ -\frac{1}{2} J^\mu_A \left(\frac{1}{2} \frac{\eta_{\mu\nu}}{\sqrt{-\Box}} \right) J^\nu_A \right\}, \end{aligned} \tag{2.31}$$

where we have completed the square and not written down the term quadratic in A_μ. We see that the free photon propagators in the two theories also coincide, and we thus conclude that QED and projected onto a plane is equivalent to the Pseudo-QED Lagrangian.

2.3.3 Dimensional analysis

Let us briefly examine the mass dimensions of the fields and coupling constants of Pseudo-QED. We are working in units where $[m] = 1$ and $[dx] = -1$. Since in the generating function the action appears in the exponent, the action must be a dimensionless quantity. The action has the form

$$S = \int d^3x \mathcal{L},$$

and thus for the action to be dimensionless we require

$$[\mathcal{L}] = 3.$$

By looking at Eq. (2.18), we deduce from the fermion term

$$\begin{aligned} [\overline{\psi}][m][\psi] &= 3 \\ [\psi] &= 1. \end{aligned}$$

For the kinetic term of the electromagnetic field we have

$$\begin{aligned} [\partial_\mu][A_\mu][\Box^{-1/2}][\partial_\mu][A_\mu] &= 3 \\ [A_\mu][\Box^{-1/2}][A_\mu] &= 1 \\ [A_\mu] &= 1, \end{aligned}$$

where we have used that $[\partial_\mu] = 1$ and $[\Box^{-1/2}] = -1$. We can now deduce the dimension of the coupling constant e by considering the coupling term

$$\begin{aligned}[e][\overline{\psi}][\psi][A_\mu] &= 3 \\ [e] &= 0.\end{aligned}$$

We see that the electric charge e is a dimensionless quantity in Pseudo-QED. This means that the theory is strictly re-normalizable, it can be re-normalized order-by-order. For QED in (2+1)D the electric charge has dimension one, and this theory is thus super re-normalizable, requiring only a finite amount of counter-terms to make the theory finite. QED in (3+1)D also has a dimensionless electric charge, and by projecting the theory onto a plane we see that the dimension of the coupling constant remains the same. As we will see in the outlook, this is a general feature of projecting out one dimension.

2.3.4 The anisotropic case

In the previous sections, we have derived Pseudo-QED for fully relativistic fermions. As we have seen in Chapter 1, however, the Dirac fermions in condensed-matter systems are quasi-relativistic. This means that they have a Fermi velocity v_{F} that plays the role of the speed of light c. The kinetic term of the fermions will thus become

$$\overline{\psi}\left(i\gamma^0\partial_0 + v_{\mathrm{F}} i\gamma^i\partial_i - m\right)\psi. \tag{2.32}$$

To couple the fermions to the gauge field, we replace the partial derivatives by covariant derivatives

$$D_\mu = \partial_\mu - ieA_\mu, \tag{2.33}$$

which ensures a local U(1) gauge symmetry. Inserting the covariant derivatives and adding the kinetic term of the A_μ field, we obtain the Lagrangian for anisotropic Pseudo-QED

$$\mathcal{L} = -\frac{1}{2}F^{\mu\nu}\frac{1}{(-\Box)^{\frac{1}{2}}}F_{\mu\nu} + \overline{\psi}\left(i\gamma^0\partial_0 + v_{\mathrm{F}} i\gamma^i\partial_i - m\right)\psi - e\overline{\psi}\gamma^0\psi A_0 - ev_{\mathrm{F}}\overline{\psi}\gamma^i\psi A_i. \tag{2.34}$$

Since we are concerned with applications to condensed-matter systems, we will mainly use anisotropic Pseudo-QED in the rest of this thesis. The isotropic case can easily be obtained by setting $v_{\mathrm{F}} = 1$. If we take $v_{\mathrm{F}} \ll c$ we obtain the static case again, since the interaction term of Eq. (2.34) becomes

$$-e\overline{\psi}\gamma^0\psi A_0 - e\frac{v_{\mathrm{F}}}{c}\overline{\psi}\gamma^i\psi A_i \approx -e\overline{\psi}\gamma^0\psi A_0 = -ej^0 A_0, \tag{2.35}$$

where we restored c. Thus we see there is only a j_0 interaction left, which is exactly the case we investigated in Sec. (2.2). The value of the Fermi velocity in graphene is $v_{\mathrm{F}} \approx c/300$, and therefore it is tempting to consider only the static case since v_{F} is still quite small compared to c. As we will see however, there are effects which we can only find by taking into account the full interaction, and they may even become independent of v_{F} in certain limits. If we consider renormalization group flow, one can find that for the Fermi velocity, there is one stable point which is $v_{\mathrm{F}} = c$ [10].

2.4 1-loop diagrams

The Feynman rules of anisotropic Pseudo-QED, corresponding to the Lagrangian in Eq. (2.34), can be found in Table 2.1. The difference with regular QED lies in the photon propagator. In Pseudo-QED, the photon propagator is proportional to $\sim |p|^{-1}$, while in regular QED the photon propagator is proportional to $\sim p^{-2}$.

In this section, we will compute the polarization tensor and electron self-energy to one-loop order. As we will see, the polarization tensor one-loop diagram is equivalent to that of anisotropic QED, because there are no internal photon propagators. We will use dimensional regularization to regularize the diagrams, and this has the nice property that in odd dimensions it may make the diagrams finite. For the polarization tensor, there is a big difference between choosing two-component and four-component spinors. As we have seen in Chapter 1, the most natural choice for graphene is the two-component representation. To illustrate the calculational techniques involved, we will use the four-component representation in this chapter, and in the next chapter we will examine the consequence of choosing the two-component representation.

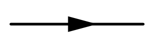
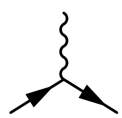
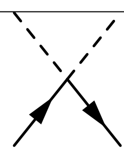
	$\dfrac{i(\gamma^0 p_0 + v_{\mathrm{F}}\gamma^i p_i)}{p_0^2 - v_{\mathrm{F}}^2 \mathbf{p}^2}$
	$\dfrac{i}{p^2 - m^2}$
	$\dfrac{-ig_{\mu\nu}}{\sqrt{p^2}}$
	$ie\overline{\gamma}^{\mu} = \begin{cases} -ie\gamma^0 & \mu = 0 \\ -iev_{\mathrm{F}}\gamma^i & \mu = i \end{cases}$
	ig

Table 2.1: Feynman rules corresponding to the Pseudo-QED Lagrangian Eq. (2.34)

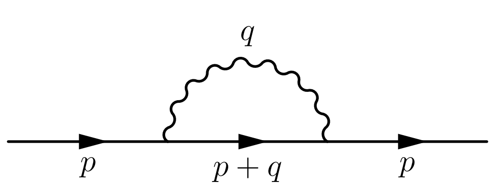


Figure 2.4.1: 1-loop electron self-energy diagram

2.4.1 Electron self-energy

The one-loop diagram contributing to the electron self-energy is depicted in Fig. 2.4.1, the full expression is

$$\begin{aligned} -i\Sigma &= (ie)^2 \int \frac{d^3q}{(2\pi)^3} \overline{\gamma}^{\mu} S_F(p-q) \overline{\gamma}^{\nu} \frac{-ig_{\mu\nu}}{\sqrt{q^2}} \\ &= (ie)^2 \int \frac{d^3q}{(2\pi)^3} \overline{\gamma}^{\mu} \frac{\left[\gamma^{\alpha}\left(\overline{p}-\overline{q}\right)\right]}{\left[(q_0-p_0)^2 - v_{\mathrm{F}}^2\left(\mathbf{q}-\mathbf{p}\right)^2\right]} \overline{\gamma}_{\mu} \frac{1}{\sqrt{q^2}}. \end{aligned} \tag{2.36}$$

First, we need to calculate the product of the gamma matrices, we rewrite

$$\begin{aligned} \overline{\gamma}^{\mu}\gamma^0\overline{\gamma}_{\mu} &= \gamma^0\gamma^0\gamma_0 + v_{\mathrm{F}}^2\gamma^i\gamma^0\gamma_i \\ &= \gamma^0 - v_{\mathrm{F}}^2\gamma^0\gamma^i\gamma_i \\ &= \gamma^0\left(1-2v_{\mathrm{F}}^2\right), \end{aligned} \tag{2.37}$$

$$\begin{aligned} \overline{\gamma}^{\mu}\gamma^i\overline{\gamma}_{\mu} &= \gamma^0\gamma^i\gamma_0 + v_{\mathrm{F}}^2\gamma^j\gamma^i\gamma_j \\ &= -\gamma^i + v_{\mathrm{F}}^2\left(2g^{ij}\gamma_j - \gamma^i\gamma^j\gamma_j\right) \\ &= -\gamma^i + v_{\mathrm{F}}^2\left(2\gamma^i - 2\gamma^i\right) \\ &= -\gamma^i, \end{aligned} \tag{2.38}$$

where we repeatedly used the anti-commutation relation

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}. \tag{2.39}$$

Substituting Eqs. (2.37) and (2.38) into Eq. (2.36), we obtain

$$-i\Sigma = (ie)^2 \int \frac{d^3q}{(2\pi)^3} \frac{\left[\gamma^0\left(1-2v_{\mathrm{F}}^2\right)(p_0-q_0) - \gamma^i v_{\mathrm{F}}\left(p_i-q_i\right)\right]}{\left[(q_0-p_0)^2 - v_{\mathrm{F}}^2\left(\mathbf{q}-\mathbf{p}\right)^2\right]} \frac{1}{\sqrt{q^2}}. \tag{2.40}$$

The next step is to combine the denominators using the Feynman trick

$$\frac{1}{AB^{\frac{1}{2}}} = \frac{1}{2}\int_0^1 dx \frac{(1-x)^{-\frac{1}{2}}}{\left[Ax + B(1-x)\right]^{\frac{3}{2}}}. \tag{2.41}$$

The denominator becomes

$$\begin{aligned}
Ax + B(1-x) &= (q_0 - p_0)^2 - v_F^2 (\mathbf{q} - \mathbf{p})^2 + x \left[q_0^2 - \mathbf{q}^2 - (q_0 - p_0)^2 + v_F^2 (\mathbf{q} - \mathbf{p})^2\right] \\
&= q_0^2 + p_0^2 - 2p_0 q_0 - v_F^2 \left[\mathbf{q}^2 + \mathbf{p}^2 - 2\mathbf{pq}\right] \\
&+ x \left[-\mathbf{q}^2 \left(1 - v_F^2\right) - p_0^2 + 2p_0 q_0 + v_F^2 \left(\mathbf{p}^2 + 2\mathbf{pq}\right)\right] \\
&= q_0^2 - 2p_0 q_0 (1-x) + \left[-x - v_F^2(1-x)\right] \mathbf{q}^2 + 2v_F^2 \mathbf{pq} (1-x) \\
&+ \left(p_0^2 - v_F^2 \mathbf{p}^2\right)(1-x) \\
&= [q_0 + p_0(1-x)]^2 - p_0^2 (1-x)^2 + \underbrace{\left[-x - v_F^2(1-x)\right]}_{\equiv \alpha} \mathbf{q}^2 \\
&+ 2v_F^2 \mathbf{pq} (1-x) + \left(p_0^2 - v_F^2 \mathbf{p}^2\right)(1-x) \\
&= [q_0 - p_0(1-x)]^2 - p_0^2 (1-x)^2 + \alpha \left[\mathbf{q} + \mathbf{p}\frac{v_F^2}{\alpha}(1-x)\right]^2 \\
&- \mathbf{p}^2 \frac{v_F^4}{\alpha}(1-x)^2 + \left(p_0^2 - v_F^2 \mathbf{p}^2\right)(1-x) \\
&= [q_0 - p_0(1-x)]^2 + \alpha \left[\mathbf{q} + \mathbf{p}\frac{v_F^2}{\alpha}(1-x)\right]^2 + p_0^2 x(1-x) \\
&- v_F^2 \mathbf{p}^2 (1-x) \left[1 + (1-x)\frac{v_F^2}{\alpha}\right] \\
&= [q_0 - p_0(1-x)]^2 + \alpha \left[\mathbf{q} + \mathbf{p}\frac{v_F^2}{\alpha}(1-x)\right]^2 - \Delta_1,
\end{aligned} \tag{2.42}$$

where

$$\Delta_1 = -p_0^2 x(1-x) + v_F^2 \mathbf{p}^2 (1-x) \left[1 + (1-x)\frac{v_F^2}{\alpha}\right].$$

We now substitute Eq. (2.42) into Eq. (2.40) to find

$$\begin{aligned}
-i\Sigma &= (ie)^2 \frac{1}{2} \int_0^1 dx\, (1-x)^{-\frac{1}{2}} \int \frac{d^3 q}{(2\pi)^3} \frac{\gamma^0 \left(1 - 2v_F^2\right)(p_0 - q_0) - \gamma^i v_F (p_i - q_i)}{\left[[q_0 - p_0(1-x)]^2 + \alpha \left[\mathbf{q} + \mathbf{p}\frac{v_F^2}{\alpha}(1-x)\right]^2 - \Delta_1\right]^{3/2}} \\
&= (ie)^2 \frac{1}{2} \int_0^1 dx\, (1-x)^{-\frac{1}{2}} \int \frac{d^3 q}{(2\pi)^3} \frac{\gamma^0 \left(1 - 2v_F^2\right) x p_0 - \gamma^i v_F p_i \left[1 + \frac{v_F^2}{\alpha}(1-x)\right]}{\left[q_0^2 + \alpha \mathbf{q}^2 - \Delta_1\right]^{3/2}},
\end{aligned} \tag{2.43}$$

where the terms odd in q do not contribute. The q_0 integral has the form

$$\int \frac{dq_0}{2\pi} \frac{1}{\left(q_0^2 - M\right)^{3/2}} = \frac{2}{-M} \frac{1}{2\pi}, \tag{2.44}$$

using this we find

$$-i\Sigma = (ie)^2 \frac{1}{2\pi} \int_0^1 dx\,(1-x)^{-\frac{1}{2}} \int \frac{d^2q}{(2\pi)^2} \frac{\gamma^0\,(1-2v_{\mathrm{F}}^2)\,xp_0 - \gamma^i v_{\mathrm{F}} p_i \left[1+\frac{v_{\mathrm{F}}^2}{\alpha}(1-x)\right]}{(\alpha\mathbf{q}^2-\Delta_1)}$$
$$= (ie)^2 \frac{1}{2\pi} \int_0^1 dx\,(1-x)^{-\frac{1}{2}} \int \frac{d^2q}{(2\pi)^2} \frac{\gamma^0\,(1-2v_{\mathrm{F}}^2)\,xp_0 - \gamma^i v_{\mathrm{F}} p_i \left[1+\frac{v_{\mathrm{F}}^2}{\alpha}(1-x)\right]}{\alpha\left(\mathbf{q}^2-\frac{\Delta_1}{\alpha}\right)}. \tag{2.45}$$

The remaining integral over $\mathbf{q}$ is linearly divergent, and we employ dimensional regularization to extract the divergence. For details on this procedure see Appendix A. The diverging part of the integral is

$$\int d^n q \frac{1}{(\mathbf{q}^2-\Delta)} = -\frac{2\pi}{\epsilon}, \tag{2.46}$$

Substituting Eq. (2.46) into Eq. (2.45), we obtain

$$-i\Sigma = -\,(ie)^2 \frac{1}{\epsilon} \frac{1}{(2\pi)^2} \int_0^1 dx\,(1-x)^{-\frac{1}{2}} \frac{\gamma^0\,(1-2v_{\mathrm{F}}^2)\,xp_0 - \gamma^i v_{\mathrm{F}} p_i \left[1+\frac{v_{\mathrm{F}}^2}{\alpha}(1-x)\right]}{\alpha}. \tag{2.47}$$

All that is left to do now is the parametric integral. We compute

$$\int_0^1 dx \frac{(1-x)^{-\frac{1}{2}}\,x}{-x-v_{\mathrm{F}}^2(1-x)} = 2\left[\frac{1}{v_{\mathrm{F}}^2-1} + \frac{v_{\mathrm{F}}^2 \arctan\left(\sqrt{v_{\mathrm{F}}^2-1}\right)}{(v_{\mathrm{F}}^2-1)^{3/2}}\right] \equiv F_1, \tag{2.48}$$

$$\int_0^1 dx \frac{(1-x)^{-\frac{1}{2}}\left[1+\frac{v_{\mathrm{F}}^2}{\alpha}(1-x)\right]}{-x-v_{\mathrm{F}}^2(1-x)} = \int_0^1 dx \frac{(1-x)^{-\frac{1}{2}}+\frac{v_{\mathrm{F}}^2}{\alpha}(1-x)^{\frac{1}{2}}}{-x-v_{\mathrm{F}}^2(1-x)}$$
$$= \frac{1}{v_{\mathrm{F}}^2-1}\left[-1-\frac{(v_{\mathrm{F}}^2-2)}{\sqrt{v_{\mathrm{F}}^2-1}} \arcsin\left(\frac{\sqrt{v_{\mathrm{F}}^2-1}}{v_{\mathrm{F}}}\right)\right] = F_2. \tag{2.49}$$

Using these integrals, the final result is

$$-i\Sigma = e^2 \frac{1}{\epsilon} \frac{1}{(2\pi)^2} \left[\gamma^0\left(1-2v_{\mathrm{F}}^2\right) F_1 p_0 - \gamma^i F_2 v_{\mathrm{F}} p_i\right]. \tag{2.50}$$

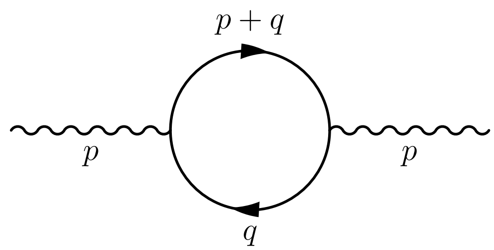


Figure 2.4.2: 1-loop diagram contributing to the polarization tensor

2.4.2 Polarization tensor

The 1-loop diagram contributing to the polarization tensor is shown in Fig. 2.4.2. The full expression is

$$\begin{aligned} i\Pi^{\mu\nu} &= -(ie)^2 \int \frac{d^3q}{(2\pi)^3} \mathrm{Tr}\left[\overline{\gamma}^{\mu} S_F(p+q)\overline{\gamma}^{\nu} S_F(q)\right] \\ &= (ie)^2 \int \frac{d^3q}{(2\pi)^3} \frac{\mathrm{Tr}\left[\overline{\gamma}^{\mu}\gamma^{\alpha}\overline{\gamma}^{\nu}\gamma^{\beta}\right]\overline{q}_{\beta}\left(\overline{q}+\overline{p}\right)_{\alpha}}{\left[(p_0+q_0)^2 - v_{\mathrm{F}}^2\left(\mathbf{q}+\mathbf{p}\right)^2\right]\left(q_0^2 - v_{\mathrm{F}}^2\mathbf{q}\right)}. \end{aligned} \tag{2.51}$$

Because the Lorentz symmetry is broken by the Fermi velocity, it is most convenient to consider the components of the polarization tensor separately. We will compute the spatial component here, where $\mu = i$, $\nu = j$. The trace then becomes

$$\mathrm{Tr}\left[\gamma^i\gamma^{\alpha}\gamma^j\gamma^{\beta}\right] = 4\left(g^{i\alpha}g^{j\beta} - g^{ij}g^{\alpha\beta} + g^{i\beta}g^{j\alpha}\right). \tag{2.52}$$

Using this we find

$$\begin{aligned} i\Pi^{ij} &= (ie)^2 v_{\mathrm{F}}^2 4 \int \frac{d^3q}{(2\pi)^3} \frac{\left(g^{i\alpha}g^{j\beta} - g^{ij}g^{\alpha\beta} + g^{i\beta}g^{j\alpha}\right)\overline{q}_{\beta}\left(\overline{q}+\overline{p}\right)_{\alpha}}{\left[(p_0+q_0)^2 - v_{\mathrm{F}}^2\left(\mathbf{p}+\mathbf{q}\right)^2\right]\left(q_0^2 - v_{\mathrm{F}}^2\mathbf{q}^2\right)} \\ &= (ie)^2 v_{\mathrm{F}}^2 4 \int \frac{d^3q}{(2\pi)^3} \frac{\left[v_{\mathrm{F}}^2\left(q+p\right)^i q^j - g^{ij}\overline{q}^{\alpha}\left(\overline{q}+\overline{p}\right)_{\alpha} + v_{\mathrm{F}}^2\left(q+p\right)^j q^i\right]}{\left[(p_0+q_0)^2 - v_{\mathrm{F}}^2\left(\mathbf{p}+\mathbf{q}\right)^2\right]\left(q_0^2 - v_{\mathrm{F}}^2\mathbf{q}^2\right)}. \end{aligned} \tag{2.53}$$

We now again combine the denominators using the Feynman trick in the form

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{\left[Ax + B(1-x)\right]^2}. \tag{2.54}$$

The denominator becomes

$$
\begin{aligned}
Ax + B(1-x) &= q_0^2 - v_\mathrm{F}^2\mathbf{q}^2 + x\left[(p_0+q_0)^2 - v_\mathrm{F}^2\left(\mathbf{p}+\mathbf{q}\right)^2 - q_0^2 + v_\mathrm{F}^2\mathbf{q}^2\right] \\
&= q_0^2 - v_\mathrm{F}^2\mathbf{q}^2 + x\left[p_0^2 + 2p_0q_0 - v_\mathrm{F}^2\mathbf{p}^2 - 2v_\mathrm{F}^2\mathbf{p}\mathbf{q}\right] \\
&= \left(q_0 + xp_0\right)^2 - v_\mathrm{F}^2\left(\mathbf{q}+x\mathbf{p}\right)^2 + x\left(1-x\right)\left[p_0^2 - v_\mathrm{F}^2\mathbf{p}^2\right]. \qquad (2.55)
\end{aligned}
$$

Substituting Eq. (2.55) into Eq. (2.53) we find

$$
i\Pi^{ij} = (ie)^2 v_\mathrm{F}^2 4\int_0^1 dx \int \frac{d^3q}{(2\pi)^3}\frac{\left[v_\mathrm{F}^2\left(q+p\right)^i q^j - g^{ij}\overline{q}^\alpha\left(\overline{q}+\overline{p}\right)_\alpha + v_\mathrm{F}^2\left(q+p\right)^j q^i\right]}{\left[\left(q_0+xp_0\right)^2 - v_\mathrm{F}^2\left(\mathbf{q}+x\mathbf{p}\right)^2 + x\left(1-x\right)\left(p_0^2 - v_\mathrm{F}^2\mathbf{p}^2\right)\right]^2}. \qquad (2.56)
$$

We shift $q_0 \to q_0 - xp_0$ and $\mathbf{q} \to \mathbf{q} - x\mathbf{p}$, the terms odd in q do not contribute and we find

$$
i\Pi^{ij} = (ie)^2 v_\mathrm{F}^2 4\int_0^1 dx \int \frac{d^3q}{(2\pi)^3}\frac{\left[2v_\mathrm{F}^2\left(q^iq^j - x(1-x)p^ip^j\right) - g^{ij}\overline{q}^2 + g^{ij}x(1-x)\left(p_0^2 - v_\mathrm{F}^2\mathbf{p}^2\right)\right]}{\left[q_0^2 - v_\mathrm{F}^2\mathbf{q}^2 + x\left(1-x\right)\left(p_0^2 - v_\mathrm{F}^2\mathbf{p}^2\right)\right]^2}. \qquad (2.57)
$$

We can use Lorentz invariance to find

$$
\int d^2q \frac{q^iq^j}{F(q^2)} = -\frac{g^{ij}}{2}\int d^2q \frac{\mathbf{q}^2}{F(q^2)}, \qquad (2.58)
$$

where $F(p^2)$ is any (nice enough) denominator. Using this identity two terms cancel in Eq. (2.57), and we obtain

$$
i\Pi^{ij} = (ie)^2 v_\mathrm{F}^2 4\int_0^1 dx \int \frac{d^3q}{(2\pi)^3}\frac{\left[-2v_\mathrm{F}^2 x(1-x)p^ip^j - g^{ij}q_0^2 + g^{ij}x(1-x)\left(p_0^2 - v_\mathrm{F}^2\mathbf{p}^2\right)\right]}{\left[q_0^2 - v_\mathrm{F}^2\mathbf{q}^2 + x\left(1-x\right)\left(p_0^2 - v_\mathrm{F}^2\mathbf{p}^2\right)\right]^2}. \qquad (2.59)
$$

We now compute the q_0 integral using

$$
\begin{aligned}
\int \frac{dq_0}{2\pi}\frac{1}{\left(q_0^2 - \Delta\right)^2} &= \frac{i\pi}{2\left(\Delta\right)^{3/2}}, \\
\int \frac{dq_0}{2\pi}\frac{q_0^2}{\left(q_0^2 - \Delta\right)^2} &= \frac{i\pi}{2\sqrt{\Delta}}. \qquad (2.60)
\end{aligned}
$$

We obtain

$$
\begin{aligned}
i\Pi^{ij} &= (ie)^2\, v_{\mathrm{F}}^2 i \int_0^1 dx \int \frac{d^2q}{(2\pi)^2} \left\{ \frac{\left[-2v_{\mathrm{F}}^2 x(1-x)p^i p^j + g^{ij}x(1-x)\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right)\right]}{\left[v_{\mathrm{F}}^2\mathbf{q}^2 - x\,(1-x)\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right)\right]^{3/2}} \right. \\
&\quad \left. + \frac{-g^{ij}}{\left[v_{\mathrm{F}}^2\mathbf{q}^2 - x\,(1-x)\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right)\right]^{1/2}} \right\} \\
&= (ie)^2\, v_{\mathrm{F}}^2 i \int_0^1 dx \int \frac{d^2q}{(2\pi)^2} \left\{ \frac{\left[-2v_{\mathrm{F}}^2 x(1-x)p^i p^j + g^{ij}x(1-x)\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right)\right]}{v_{\mathrm{F}}^3\left[\mathbf{q}^2 - \Delta_1\right]^{3/2}} \right. \\
&\quad \left. + \frac{-g^{ij}}{v_{\mathrm{F}}\left[\mathbf{q}^2 - \Delta_1\right]^{1/2}} \right\}, \qquad (2.61)
\end{aligned}
$$

where we defined

$$
\Delta_1 = \frac{x\,(1-x)}{v_{\mathrm{F}}^2}\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right).
$$

The $\mathbf{q}$ integrals we can calculate using dimensional regularization, and we obtain

$$
\begin{aligned}
\int d^2q \frac{1}{\left[\mathbf{q}^2 - \Delta_1\right]^{1/2}} &= 2\pi\left(-\Delta_1\right)^{1/2}, \\
\int d^2q \frac{1}{\left[\mathbf{q}^2 - \Delta_1\right]^{3/2}} &= 2\pi\left(-\Delta_1\right)^{-1/2}.
\end{aligned}
$$

Evaluating the $\mathbf{q}$ integrals the polarization tensor becomes

$$
\begin{aligned}
i\Pi^{ij} &= (ie)^2\, v_{\mathrm{F}}^2 i \int_0^1 dx \frac{1}{2\pi} \left\{ \frac{x(1-x)\left[g^{ij}\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right) - 2v_{\mathrm{F}}^2 p^i p^j\right]}{v_{\mathrm{F}}^2\left[-x\,(1-x)\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right)\right]^{1/2}} \right. \\
&\quad \left. + \frac{-g^{ij}}{v_{\mathrm{F}}^2}\left(-x\,(1-x)\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right)\right)^{1/2} \right\}. \qquad (2.62)
\end{aligned}
$$

The parametric integrals have the form

$$
\int_0^1 dx \sqrt{x(1-x)} = \frac{\pi}{8},
$$

so the result for the polarization tensor is

$$
\begin{aligned}
i\Pi^{ij} &= -e^2 i \frac{1}{16} \left\{ \frac{\left[g^{ij}\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right) - 2v_{\mathrm{F}}^2 p^i p^j\right]}{\left(v_{\mathrm{F}}^2\mathbf{p}^2 - p_0^2\right)^{1/2}} \right. \\
&\quad \left. - g^{ij}\left(v_{\mathrm{F}}^2\mathbf{p}^2 - p_0^2\right)^{1/2} \right\}. \\
&= -e^2 i \frac{1}{8} \left\{ \frac{g^{ij}\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right) - v_{\mathrm{F}}^2 p^i p^j}{\left(v_{\mathrm{F}}^2\mathbf{p}^2 - p_0^2\right)^{1/2}} \right\}. \qquad (2.63)
\end{aligned}
$$

The other components of the polarization tensor follow by a similar calculation.

Chapter 3

Calculating transport properties using Pseudo-QED

Now that we have derived the Feynman rules of Pseudo-QED and examined how calculations can be done within this formalism, we are ready to turn to actually calculating physical observables. In this Chapter we will focus on transport properties of Pseudo-QED. First, we will briefly review the theory of linear response theory and in particular Kubo's formula for conductivity. After this, we will apply linear response theory to Pseudo-QED and introduce the notion of valley conductivity, which arises due to the valley degree of freedom in graphene-like systems. We will then review the results of Ref. [9], where an interaction-driven quantum valley Hall effect is found in massless Dirac systems, and those of Ref. [12], where the authors found an interaction-driven quantum Hall effect in massive Dirac systems, in addition to a quantum valley Hall effect.

3.1 Kubo's formula

In this section we will derive Kubo's formula for the electrical conductivity, following Refs. [33, 34]. Kubo's formula gives us the response function of a perturbation to a given system. The idea is that we consider a system in equilibrium, and then at a certain time turn on the perturbation. We are then interested in the response of this system to the perturbation. For example, if we apply a potential to a system in equilibrium, a current may flow through this system as a response. The response functions characterize the transport properties of a system, and are thus important quantities to calculate. In general it is very hard to predict how a system will respond to a perturbation, but making the assumption that the perturbation is small, and thus linearly coupled to the response, simplifies the problem a lot. In the case of QED (and Pseudo-QED), the electrons are coupled linearly to the A_μ field, and linear-response theory is thus enough to calculate the conductivity.

Let us assume that we have a linear perturbation of our Hamiltonian of the form (again we follow Ref. [34])

$$H_{int}(t) = \int d^{d-1}x\, \phi_j(x,t)\mathcal{O}_j(x,t), \tag{3.1}$$

where ϕ is the source term and $\mathcal{O}$ is the observable being perturbed. The question is now, how does the expectation value of the observable change under the perturbation? This can be expressed as

$$\delta \langle \mathcal{O}_i (x,t) \rangle = \int d^{d-1}x' dt' \chi_{ij} (x - x', t - t') \phi_j (x', t'), \tag{3.2}$$

where χ_{ij} is the response function, and where we have assumed space- and time-translational invariance. To identify χ_{ij}, we compute the expectation value of the operator defined as

$$\langle \mathcal{O}_i (x,t) \rangle = \mathrm{Tr} \left[\rho(x,t) \mathcal{O}_i(x,t) \right], \tag{3.3}$$

where ρ is the density matrix. If we work in the interaction picture (see Ref. [35]), the time-evolution operator is given by

$$U(t,t_0) = T \left[\exp \left(-i \int\limits_{t_0}^{t} H_{int}(t')dt' \right) \right], \tag{3.4}$$

where T is the time-ordering operator. If we let $\rho_0(x)$ denote the density matrix at time $t = -\infty$, and define $U(t) = U(t, -\infty)$, we can write

$$\rho(x,t) = U(t)\rho_0(x)U^{-1}(t). \tag{3.5}$$

Substituting Eq. (3.5) into Eq. (3.3), we obtain

$$\begin{aligned}
\langle \mathcal{O}_i\,(x,t)\rangle &= \mathrm{Tr}\left[\rho_0(x)U^{-1}(t)\mathcal{O}_i(x,t)U(t)\right] \\
&= \mathrm{Tr}\left[\rho_0(x)\left(\mathcal{O}_i(x,t) + i\int\limits_{-\infty}^{t} dt'\,[H_{int}(t'),\mathcal{O}_i(x,t)] + \ldots\right)\right] \\
&\approx \langle \mathcal{O}_i\,(x,t)\rangle\,|_{\phi=0} + i\int\limits_{-\infty}^{t} dt'\,\langle[H_{int}(t'),\mathcal{O}_i(x,t)]\rangle \\
&= \langle \mathcal{O}_i\,(x,t)\rangle\,|_{\phi=0} + \delta\,\langle \mathcal{O}_i\,(x,t)\rangle\,,
\end{aligned} \tag{3.6}$$

where we expanded the exponential in the time-evolution operator up to first order. We thus see that turning on the source term gives the following change in the expectation value of the operator

$$\begin{aligned}
\delta\,\langle \mathcal{O}_i\,(x,t)\rangle &= i\int\limits_{-\infty}^{t} dt'\,\langle[H_{int}(t'),\mathcal{O}_i(x,t)]\rangle \\
&= i\int\limits_{-\infty}^{t} dt' \int d^{d-1}x\,\langle[\mathcal{O}_j(x',t'),\mathcal{O}_i(x,t)]\rangle\,\phi_j(x',t') \\
&= i\int\limits_{-\infty}^{+\infty} dt' \int d^{d-1}x\,\theta(t-t')\,\langle[\mathcal{O}_j(x',t'),\mathcal{O}_i(x,t)]\rangle\,\phi_j(x',t'),
\end{aligned} \tag{3.7}$$

where θ is the Heaviside step function. Comparing Eq. (3.7) to Eq. (3.2), we find that the response function is

$$\chi_{ij}\,(x-x',t-t') = -i\theta(t-t')\,\langle[\mathcal{O}_i(x,t),\mathcal{O}_j(x',t')]\rangle\,. \tag{3.8}$$

This is the Kubo formula in its general form.

3.1.1 Conductivity

Let us now apply the Kubo formula of Eq. (3.8) to the specific case of electrical conductivity. The perturbation term of the Hamiltonian is in this case

$$H_{int}(t) = e\int d^{d-1}x\,A_\mu j^\mu \tag{3.9}$$

We can write the response of the system to the perturbation as

$$\delta \langle j_i(\mathbf{k},\omega)\rangle = \sigma_{ij}(\mathbf{k},\omega)E_j(\mathbf{k},\omega), \tag{3.10}$$

where σ_{ij} is the conductivity and E_j is the electric field. Eq. (3.10) is Ohm's law restated in the language of quantum field theory. Let us pick the gauge $A_0 = 0$ for simplicity. In this case $E_i = -\partial_0 A_i$, or in Fourier space

$$A_i(\mathbf{k},\omega) = \frac{E_i(\mathbf{k},\omega)}{i\omega}. \tag{3.11}$$

Combining Eqs. (3.11) and (3.10) with (3.2) we obtain

$$\delta \langle j_i(\mathbf{k},\omega)\rangle = e^2\frac{\chi_{ij}(\mathbf{k},\omega)}{i\omega}A_i(\mathbf{k},\omega), \tag{3.12}$$

and we thus read off

$$\sigma_{ij}(\mathbf{k},\omega) = e^2\frac{\chi_{ij}(\mathbf{k},\omega)}{i\omega}. \tag{3.13}$$

Considering the Kubo formula in momentum space now yields

$$\begin{aligned}\chi_{ij}(\mathbf{k},\omega) &= -i\int dxdt e^{i(\omega t-kx)}\theta(t)\langle[j_i(x,t),j_j(0,0)]\rangle . \\ &= \langle j_i j_j\rangle^{ret}(\mathbf{k},\omega),\end{aligned} \tag{3.14}$$

which is the retarded Green's function of the current. At $T = 0$ and $\omega = 0$, which are the cases we will consider here, this Green's function coincides with time-ordered (Feynman) Green's function, which is the quantity we compute in a quantum field theory. Substituting Eq. (3.14) into Eq. (3.13), we obtain the expression for the conductivity

$$\sigma_{ij}(\mathbf{k},\omega) = -e^2\frac{i\langle j_i j_j\rangle}{\omega}, \tag{3.15}$$

where $\langle j_i j_j\rangle$ is now the time-ordered correlation function.

The current-current correlation function is defined in the path-integral formalism as

$$\langle j_i j_j\rangle = N\int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}A_\mu j_i j_j e^{i\int d^3x\, S_{free}-eA_\mu j^\mu}, \tag{3.16}$$

where N is the proper normalization factor and S_{free} is the quadratic part of the Pseudo-QED Lagrangian. We can now replace the currents in this expression with functional derivatives with respect to $A_\mu$to find

$$\langle j_i j_j\rangle = -\frac{1}{e^2}N\int \mathcal{D}\overline{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \frac{\delta}{\delta A_i}|_{j=0}\ \frac{\delta}{\delta A_i}|_{i=0}\ e^{i\int d^3x\, S_{free}-eA_\mu j^\mu}, \tag{3.17}$$

but this is the 2-point function of the $A_\mu$field, or in other words the polarization tensor Π_{ij}. We can thus rewrite the Kubo formula as

$$\sigma_{ij}(\mathbf{k},\omega) = -e^2 \frac{i\langle j_i j_j \rangle}{\omega} = \frac{i\Pi_{ij}}{\omega}, \tag{3.18}$$

where Π_{ij} is the polarization tensor. We will be interested in the zero frequency and momentum limit (the DC-conductivity), which we denote

$$\sigma_{ij} = \lim_{\omega\to 0,\mathbf{k}\to 0} \frac{i\Pi_{ij}}{\omega}. \tag{3.19}$$

3.2 Conductivity in Dirac systems

In graphene-like systems we a have a valley degree of freedom. This arises because of the lattice structure, and, as we have seen in Chapter 1, it can be seen as a 'pseudo-spin' degree of freedom. The two valleys in the system are related by time-reversal symmetry, and there are two valley currents. This allows us to define the total conductivity

$$\sigma^{ij}_{tot} = \lim_{\omega\to 0,\mathbf{p}\to 0} \left\{ \frac{i\langle j^i j^j \rangle}{\omega} + \frac{i\langle j^i j^j \rangle^{\mathrm{T}}}{\omega} \right\} = \sigma^{tot}_{xx}\delta^{ij} + \sigma^{tot}_{xy}\epsilon^{ij}, \tag{3.20}$$

which is the sum of the two valley currents, and the valley conductivity,

$$\sigma^{ij}_{val} = \lim_{\omega\to 0,\mathbf{p}\to 0} \left\{ \frac{i\langle j^i j^j \rangle}{\omega} - \frac{i\langle j^i j^j \rangle^{\mathrm{T}}}{\omega} \right\} = \sigma^{val}_{xx}\delta^{ij} + \sigma^{val}_{xy}\epsilon^{ij}, \tag{3.21}$$

which is the difference of the two valley currents, where ϵ^{ij} is the Levi-Civita symbol and we have split the conductivities into a longitudinal and a transverse part. The valley conductivity is analogous to the spin conductivity, which is a measure of the spin current. One key difference to keep in mind between spin and valley, is that valley is a degree of freedom coming from the geometry of the lattice, and thus does not exist outside of the lattice (unlike spin).

In this section we will calculate the longitudinal and transverse conductivities of graphene (following Ref. [9]), and calculate the transverse conductivity of massive Dirac systems (following Ref. [12]), all at $T = 0$ and using Pseudo-QED.

3.2.1 Conductivity in graphene

We now apply the Kubo formula for conductivity, Eq. (3.19), to massless electrons to study the conductivity of graphene using the Pseudo-QED formalism. We follow

Ref. [9] in this section. As we have seen in Sec. 3.1.1, to obtain the conductivity, we need to compute the polarization tensor of the theory.

The one-loop diagram of the polarization tensor is shown in Fig.2.4.2, and it is in fact equal to the result from QED in (2+1)D. This is because there are no internal photon lines, and the fermion propagators of the two theories coincide. We can thus use the result from QED in (2+1)D for the 1-loop diagram. We will use the 2-component spinor representation, as this arises naturally out of the tight-binding description in graphene (see Chapter 1). In this representation the result for the polarization tensor is different from the one that we have calculated in Chapter 2. The 2-component diagram has already been calculated a while ago by Coste and Lüscher [36], and the result is

$$\Pi_{ij}^{(1)}(p) = -ie^2 A(p) P_{ij} - e^2 i B \epsilon_{ij0} p^0, \tag{3.22}$$

where $A(p) = \sqrt{p_0^2 - v_{\mathrm{F}}^2 \mathbf{p}^2}/16$, $B = (1/2\pi)(n + 1/2)$ with n integer and $P_{\mu\nu} = \eta_{\mu\nu} - v_{\mathrm{F}}^2 p_\mu p_\nu / \sqrt{p_0^2 - v_{\mathrm{F}}^2 \mathbf{p}^2}$, which is the transverse projection operator. The second term in Eq. (3.22) is a topological term (it does not depend on the specifics of our metric), and gives a topological mass to the photon. This effect is specific to QED in (2+1)D and only arises when choosing a two-component spinor representation, and it is referred to in the literature as a Chern-Simons term, or a 'toplogical mass' term. Coste and Lüscher found this result using lattice regularization, and found the integer n to be the winding number of the free fermion propagator. Using other regularization methods produces the expression only for a specific n, and Eq. (3.22) is thus the most general result.

The 2-loop result is specific to Pseudo-QED, as the diagrams include internal photon lines. It was calculated in Ref. [37], and found to be

$$\Pi_{ij}^{(2)}(p) = -ie^2 \frac{\sqrt{p_0^2 - v_{\mathrm{F}}^2 \mathbf{p}^2}}{16} C_\alpha \alpha_g P_{\mu\nu}, \tag{3.23}$$

with $C_\alpha = \left(92 - 9\pi^2\right)/18\pi$ and $\alpha_g \approx 300/137$. The topological term does not receive any corrections from the 2-loop diagrams, and in fact does not receive any corrections at all according to the Coleman-Hill theorem [38]. We now have the polarization tensor up to 2-loop order, and we can thus calculate the total and valley DC-conductivities using Eqs. (3.20) and (3.21).

3.2.1.1 Longitudinal conductivity

Let us start with the longitudinal component of the total conductivity. We combine the longitudinal parts of Eq. (3.23) and (3.22) to find

$$i\Pi_{ij}^{longitudinal} = \left\{ \frac{e^2\sqrt{p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2}P_{ij}}{16\omega}\left[1 + \frac{(92-9\pi^2)}{18\pi}\alpha_g\right]\right\}. \tag{3.24}$$

We now have to take the zero momentum and frequency limit, and the order is quite important [39] (the momentum limit should be taken first). We note that in the zero momentum limit $P_{ij} \to \eta_{ij}$. Under time reversal ($i \to -i$, $p_0 \to p_0$, $\mathbf{p} \to -\mathbf{p}$), Eq. (3.24) does not change and thus the two valleys contribute evenly. Taking also into account a factor of two for the spins, and restoring factors of $\hbar$, we find by replacing Eq. (3.24) into Eq. (3.20)

$$\begin{aligned}\sigma_{xx}^{tot}\delta_{ij} &= \lim_{\omega\to 0,\mathbf{p}\to 0}\left\{\frac{4e^2\sqrt{p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2}P_{ij}}{\hbar 16\omega}\left[1 + \frac{(92-9\pi^2)}{18\pi}\alpha_g\right]\right\}\\ &= \lim_{\omega\to 0}\left\{\frac{4e^2\sqrt{p_0^2}\delta_{ij}}{\hbar 16\omega}\left[1 + \frac{(92-9\pi^2)}{18\pi}\alpha_g\right]\right\}\\ &= \delta_{ij}\left(\frac{2e^2}{\pi h}\right)\left[1 + \frac{(92-9\pi^2)}{18\pi}\alpha_g\right].\end{aligned} \tag{3.25}$$

Thus the longitudinal total conductivity is

$$\begin{aligned}\sigma_{xx}^{tot} &= \left(\frac{2e^2}{\pi h}\right)\left[1 + \frac{(92-9\pi^2)}{18\pi}\alpha_g\right]\\ &\approx 1.76\frac{e^2}{h}.\end{aligned} \tag{3.26}$$

The experimental result for the conductivity, extrapolated to zero temperature is $\sigma_{xx}^{tot} = 2.16e^2/h$. This indicates that there are other factors not included in this model, that also contribute to conductivity. A big assumption in the model here is that it is a perfect lattice, and perhaps including disorder would improve the result [9].

Looking at the definition of the valley conductivity Eq. (3.21) and the result Eq. (3.24), we can immediately see that the longitudinal valley conductivity vanishes, since the longitudinal part of the polarization tensor is invariant under time-reversal, and thus the contributions from the two valleys will cancel each other.

3.2.1.2 Transverse conductivity

As we can read off from Eqs. (3.20) and (3.21), the transverse conductivity will be proportional the Levi-Civita symbol ϵ^{ij}. We have already seen that such a term can only come from the 1-loop diagram, and its expression is

$$\Pi_{ij}^{transverse} = -i\frac{e^2}{2\pi}(n+\frac{1}{2})\epsilon_{ij0}p^0. \tag{3.27}$$

This expression is *not* invariant under time reversal symmetry, but changes sign. We can thus conclude that there is no total transverse conductivity, since the contributions from the two valleys will cancel each other out. There is, however, a transverse *valley* conductivity. Substituting Eq. (3.27) into Eq. (3.21), taking into account a factor of two for the spins and restoring $\hbar$, we obtain

$$\begin{aligned} \sigma_{xy}^{tot}\epsilon^{ij} &= \lim_{\omega\to 0,\mathbf{p}\to 0}\left\{\frac{2e^2}{\hbar\pi}(n+\frac{1}{2})\epsilon^{ij}\frac{p_0}{\omega}\right\} \\ &= \frac{4e^2}{h}(n+\frac{1}{2})\epsilon^{ij}. \end{aligned} \tag{3.28}$$

Hence one obtains a universal, quantized, transverse valley current. This is thus a quantum valley Hall effect, induced by the electromagnetic interactions in graphene.

3.2.2 Conductivity in massive Dirac systems

Let us now turn to massive Dirac systems. This section follows Ref. [12]. We now add a mass term to the electrons such that the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{2}F^{\mu\nu}\frac{1}{(-\Box)^{\frac{1}{2}}}F_{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - m_\psi)\psi - ej^\mu A_\mu, \tag{3.29}$$

we define $m_\psi = \xi m_0$, with $m_0 > 0$ and $\xi = \pm 1$ depending on the valley we are considering.

Our goal is now to apply the analysis from the previous section to massive Dirac systems. For this we need the result for the 1-loop polarization tensor diagram at finite mass, this was also calculated by Coste and Lüscher and reads

$$\Pi_{ij} = e^2 A(\mathbf{p},\omega)P_{ij} - e^2 iB(\mathbf{p},\omega)\epsilon_{ij}p^0, \tag{3.30}$$

where now

$$A(\mathbf{p},\omega,\Delta) = \frac{1}{2\pi}\int_0^1 dt \frac{t(1-t)\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right)}{\sqrt{m_\psi^2 - t(1-t)\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right)}},$$

$$B(\mathbf{p},\omega,\Delta) = \frac{n}{2\pi} + \frac{1}{4\pi}\int_0^1 dt\left(1 + \frac{m_\psi}{\sqrt{m_\psi^2 - t(1-t)\left(p_0^2 - v_{\mathrm{F}}^2\mathbf{p}^2\right)}}\right). \tag{3.31}$$

Since the valleys are related by time-reversal symmetry, the way we introduced the mass in Eq. (3.29) make it so that the mass flips sign under this transformation. We see that $A(\mathbf{p},\omega)$ is invariant under time-reversal, while $B(\mathbf{p},\omega)$ is not, due to the presence of the m_ψ. We now compute the parametric integrals to find

$$\lim_{\omega\to 0,\mathbf{p}\to 0} A(\mathbf{p},\omega,m_\psi) = 0,$$

$$\lim_{\omega\to 0,\mathbf{p}\to 0} B(\mathbf{p},\omega,m_\psi) = \begin{cases} \frac{1}{2\pi}(n+1) & m_\psi > 0 \\ \frac{1}{2\pi}n & m_\psi < 0 \end{cases}. \tag{3.32}$$

Since $A(\mathbf{p},\omega)$ is zero, the longitudinal conductivity will be zero. To find the transverse conductivity we follow the same procedure as for the massless case. The transverse part of the polarization tensor is

$$\Pi_{ij}^{transverse} = -i\epsilon_{ij}p^0 B(m_\psi), \tag{3.33}$$

and its time-reversal is

$$\left(\Pi_{ij}^{transverse}\right)^T = i\epsilon_{ij}p^0 B(-m_\psi).$$

Substituting this into Eqs. (3.20) and (3.21), taking into account a factor of two for the spins and reintroducing $\hbar$, we obtain

$$\begin{aligned}\sigma_{xy}^{tot} &= \frac{4\pi e^2}{h}\left[B(m_\psi) - B(-m_\psi)\right] \\ &= 2\frac{e^2}{h},\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}\sigma_{xy}^{val} &= \frac{4\pi e^2}{h}\left[B(m_\psi) + B(-m_\psi)\right] \\ &= \frac{4e^2}{h}(n + \frac{1}{2}).\end{aligned} \tag{3.35}$$

We thus find the same quantum valley Hall effect as in the massless case, but now there is also a quantum Hall effect. This quantum Hall effect is again universal, and is found in the absence of any magnetic field whatsoever.

Chapter 4

Coupling massive Pseudo-QED to a scalar field

In this chapter, we investigate the change of the QHE and QVHE in massive Dirac systems described in the previous chapter, in the presence of a massive scalar field σ, which we couple to the fermions via a quartic interaction. The Lagrangian takes the form

$$\mathcal{L} = -\frac{1}{2}\frac{(F^{\mu\nu})^2}{\sqrt{-\Box}} + \bar{\psi}(i\gamma^0\partial_0 + iv_{\mathrm{F}}\gamma^i\partial_i - m_\psi)\psi + \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma - \frac{1}{2}m_\sigma^2\sigma^2 - e\bar{\psi}\gamma^\mu\psi A_\mu + g\bar{\psi}\psi\sigma^2, \tag{4.1}$$

where $F^{\mu\nu}$ is the electromagnetic tensor, ψ is the electron field, σ the scalar field, m_ψ the electron mass, m_σ the scalar field mass, g the new coupling between electrons and the scalar field, e the electron charge and A_μ the electromagnetic 4-potential.

In addition to standard anisotropic Pseudo-QED, as considered in the previous chapters, there are now two extra Feynman rules, depicted in Table 4.1. We now have two dimensionless coupling constants, the electric charge e and the new coupling constant g governing the interaction between the electrons and the new scalar field. Since $[g] = [e] = 0$, i.e. the coupling constants are dimensionless, our theory is still renormalizable.

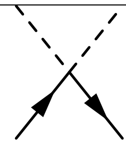
	$\frac{i}{p^2 - m^2}$
	ig

Table 4.1: Feynman rules involving the scalar field corresponding to Lagrangian Eq. (4.1)

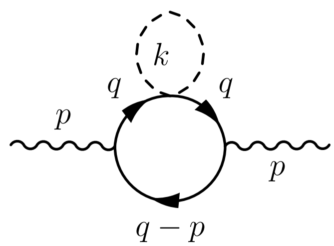


Figure 4.1.1: Two loop diagram involving the scalar field contributing to the polarization tensor

4.1 Transverse conductivity

We calculate the contribution to the transverse current to lowest order in g. There is one Feynman diagram contributing, which is depicted in Fig. (4.1.1). The full expression is

$$i\Pi^{ij} = -2(ie)^2(ig)v_{\mathrm{F}}^2 \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left\{\frac{i}{k^2 - m_\sigma^2}\mathrm{Tr}\left[\gamma^i S_F(q)^2\gamma^j S_F(q-p)\right]\right\}, \tag{4.2}$$

where we put a minus sign for the fermion loop and a symmetry factor of 2. Since the momentum of the bosonic loop does not mix with anything else, we can immediately calculate the integral

$$\begin{aligned}\int \frac{d^3k}{(2\pi)^3}\frac{i}{k^2 - m^2} &= \frac{-2\sqrt{\pi}}{(4\pi)^{3/2}}m_\sigma \\ &= -\frac{m_\sigma}{4\pi},\end{aligned} \tag{4.3}$$

using dimensional regularization. Substituting Eq. (4.3) into Eq. (4.41) we find

$$i\Pi^{ij} = -2(ie)^2(ig)v_{\mathrm{F}}^2\left(\frac{-m_\sigma}{4\pi}\right)\int\frac{d^3q}{(2\pi)^3}\left\{\mathrm{Tr}\left[\gamma^i S_F(q)^2\gamma^j S_F(q-p)\right]\right\}. \tag{4.4}$$

We again consider the two-component representation of the spinors, in which we take the gamma matrices to equal the Pauli matrices. We then have

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\right] = 2i\epsilon^{\mu\nu\rho}. \tag{4.5}$$

Our full expression is thus (we use the notation $\overline{qp} = q_0p_0 - v_{\mathrm{F}}^2\mathbf{qp}$ and $\overline{q} = (q_0, v_{\mathrm{F}}\mathbf{q})$ again),

$$i\Pi^{ij} = -2(ie)^2(ig)v_{\mathrm{F}}^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)\int\frac{d^3q}{(2\pi)^3}\left\{\frac{\mathrm{Tr}\left[\gamma^i\left(\gamma^\alpha\overline{q}_\alpha + m_\psi\right)^2\gamma^j\left(\gamma^\beta(\overline{q}-\overline{p})_\beta + m_\psi\right)\right]}{\left(\overline{q}^2 - m_\psi^2\right)^2\left[(\overline{q}-\overline{p})^2 - m_\psi^2\right]}\right\}. \tag{4.6}$$

Now the transverse conductivity will be proportional to $\epsilon^{ij0}p_0$, terms like this will in our case only arise from terms with 3 or 5 gamma matrices. We only consider these terms, since we are interested in the transverse conductivity. The terms with 3 gamma matrices are

$$\begin{aligned}
&2m_\psi^2\mathrm{Tr}\left[\gamma^i\gamma^0\gamma^j\right]q_0 + m_\psi^2\mathrm{Tr}\left[\gamma^i\gamma^j\gamma^0\right](q_0 - p_0)\\
=&2m_\psi^2\mathrm{Tr}\left[\gamma^i\gamma^0\gamma^j\right]q_0 - m_\psi^2\mathrm{Tr}\left[\gamma^i\gamma^0\gamma^j\right](q_0 - p_0)\\
=&m_\psi^2\mathrm{Tr}\left[\gamma^i\gamma^0\gamma^j\right](q_0 + p_0)\\
=&2m_\psi^2 i\epsilon^{i0j}(q_0 + p_0).
\end{aligned} \tag{4.7}$$

The term with 5 gamma matrices is

$$\begin{aligned}
&\mathrm{Tr}\left[\gamma^i\gamma^\alpha\gamma^\beta\gamma^j\gamma^\delta\right]\overline{q}_\alpha\overline{q}_\beta(\overline{q}-\overline{p})_\delta\\
=&-\mathrm{Tr}\left[\gamma^i\gamma^\beta\gamma^\alpha\gamma^j\gamma^\delta\right]\overline{q}_\alpha\overline{q}_\beta(\overline{q}-\overline{p})_\delta + 2g^{\alpha\beta}\mathrm{Tr}\left[\gamma^i\gamma^j\gamma^\delta\right]\overline{q}_\alpha\overline{q}_\beta(\overline{q}-\overline{p})_\delta\\
=&-\mathrm{Tr}\left[\gamma^i\gamma^\alpha\gamma^\beta\gamma^j\gamma^\delta\right]\overline{q}_\alpha\overline{q}_\beta(\overline{q}-\overline{p})_\delta + 4i\epsilon^{ij0}\overline{q}^2(q_0 - p_0).
\end{aligned} \tag{4.8}$$

Here we have used the fact that $\overline{q}$ commutes with itself and renamed the dummy indices. It follows that

$$\mathrm{Tr}\left[\gamma^i\gamma^\alpha\gamma^\beta\gamma^j\gamma^\delta\right]\overline{q}_\alpha\overline{q}_\beta(\overline{q}-\overline{p})_\delta = 2i\epsilon^{ij0}\overline{q}^2(q_0 - p_0) \tag{4.9}$$

So our complete expression, keeping only the terms contributing to the transverse conductivity, becomes

$$i\Pi^{ij} = -2(ie)^2(ig)v_{\mathrm{F}}^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)\int\frac{d^3q}{(2\pi)^3}\left\{\frac{2i\epsilon^{ij0}\left[\overline{q}^2(q_0-p_0)-m_\psi^2(q_0+p_0)\right]}{\left(\overline{q}^2-m_\psi^2\right)^2\left[(\overline{q}-\overline{p})^2-m_\psi^2\right]}\right\}. \tag{4.10}$$

The next step is to combine the denominators, we use the Feynman trick

$$\begin{aligned}\frac{1}{A^2B} &= \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)}\int_0^1 dydx\frac{y\delta\left(1-x-y\right)}{\left(yA+xB\right)^3}\\ &= 2\int_0^1 dx\frac{(1-x)}{\left[(1-x)A+xB\right]^3}.\end{aligned} \tag{4.11}$$

The denominator then becomes

$$\begin{aligned}(1-x)A+xB &= (1-x)\left[q_0^2-v_{\mathrm{F}}^2\mathbf{q}^2-m_\psi^2\right]+x\left[\left(q_0-p_0\right)^2-v_{\mathrm{F}}\left(\mathbf{q}-\mathbf{p}\right)^2-m_\psi^2\right]\\ &= q_0^2-v_{\mathrm{F}}^2\mathbf{q}^2-m_\psi^2+x\left[p_0^2-2p_0q_0-v_{\mathrm{F}}^2\mathbf{p}^2+2v_{\mathrm{F}}^2\mathbf{pq}\right]\\ &= q_0^2-2xp_0q_0-v_{\mathrm{F}}^2\mathbf{q}^2-m_\psi^2+x\left[p_0^2-v_{\mathrm{F}}^2\mathbf{p}^2+2v_{\mathrm{F}}^2\mathbf{pq}\right]\\ &= \left(q_0-xp_0\right)^2+x(1-x)p_0^2-v_{\mathrm{F}}^2\mathbf{q}^2-m_\psi^2+x\left[-v_{\mathrm{F}}^2\mathbf{p}^2+2v_{\mathrm{F}}^2\mathbf{pq}\right]\\ &= q_0^2-\Delta_1,\end{aligned} \tag{4.12}$$

where $\Delta_1 = -x(1-x)p_0^2 + v_{\mathrm{F}}^2\mathbf{q}^2 + m_\psi^2 + x\left[v_{\mathrm{F}}^2\mathbf{p}^2 - 2v_{\mathrm{F}}^2\mathbf{pq}\right]$. We have made a shift $q_0 \to q_0 + xp_0$, we also have to do this in the numerator. Substituting everything our full expression is then

$$\begin{aligned}i\Pi^{ij} = &-2(ie)^2(ig)v_{\mathrm{F}}^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)2\int_0^1 dx\int\frac{d^3q}{(2\pi)^3}\\ &\times\left\{\frac{2i\epsilon^{ij0}\left[\overline{q}^2(q_0-p_0)-m_\psi^2(q_0+p_0)\right](1-x)}{\left[\left(q_0-xp_0\right)^2-\Delta_1\right]^3}\right\}\end{aligned} \tag{4.13}$$

We now have to make the shift $q_0 \to q_0 + xp_0$. This makes the denominator even in q_0, hence odd terms in q_0 in the numerator do not contribute. The numerator becomes

$$
\begin{aligned}
\overline{q}^2(q_0 - p_0) - m_\psi^2(q_0 + p_0) \to & \left[(q_0 + xp_0)^2 - v_\mathrm{F}^2\mathbf{q}^2\right](q_0 + (x-1)p_0) - m_\psi^2\left[q_0 + (1+x)p_0\right] \\
&= q_0^2(x-1)p_0 + 2xp_0q_0^2 + x^2(x-1)p_0^3 - v_\mathrm{F}^2\mathbf{q}^2(x-1)p_0 \\
&- m_\psi^2(1+x)p_0 \\
&= q_0^2\left[(x-1)p_0 + 2xp_0\right] + x^2(x-1)p_0^3 - v_\mathrm{F}^2\mathbf{q}^2(x-1)p_0 \\
&- m_\psi^2(1+x)p_0 \\
&= q_0^2 C + D, \qquad (4.14)
\end{aligned}
$$

where

$$
\begin{aligned}
C &\equiv (x-1)p_0 + 2xp_0 \\
D &\equiv x^2(x-1)p_0^3 - v_\mathrm{F}^2\mathbf{q}^2(x-1)p_0 - m_\psi^2(1+x)p_0. \qquad (4.15)
\end{aligned}
$$

Using this notation our full expression becomes

$$
i\Pi^{ij} = -2(ie)^2(ig)v_\mathrm{F}^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)2\int_0^1 dx \int \frac{d^3q}{(2\pi)^3}\left\{\frac{2i\epsilon^{ij0}\left[q_0^2C + D\right](1-x)}{\left[q_0^2 - \Delta_1\right]^3}\right\}. \qquad (4.16)
$$

We can now calculate the q_0 integrals using

$$
\begin{aligned}
\int \frac{dq_0}{(2\pi)}\frac{q_0^2}{\left(q_0^2 - \Delta_1\right)^3} &= \frac{-i}{16}\Delta_1^{-3/2}, \\
\int \frac{dq_0}{(2\pi)}\frac{1}{\left(q_0^2 - \Delta_1\right)^3} &= i\frac{3}{16}\Delta_1^{-5/2}. \qquad (4.17)
\end{aligned}
$$

Substituting the integrals into Eq. (4.16) leads to

$$
\begin{aligned}
i\Pi^{ij} = -2(ie)^2(ig)v_\mathrm{F}^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)2\int_0^1 dx \int \frac{d^2q}{(2\pi)^2}&\left\{\frac{-2i\epsilon^{ij0}C(1-x)}{\Delta_1^{3/2}}\frac{i}{16}\right. \\
&\left. + \frac{2i\epsilon^{ij0}D(1-x)}{\Delta_1^{5/2}}\frac{3i}{16}\right\}. \qquad (4.18)
\end{aligned}
$$

We proceed by rewriting

$$\begin{aligned}
\Delta_1 &= -x(1-x)p_0^2 + v_{\rm F}^2\mathbf{q}^2 + m_\psi^2 + x\left[v_{\rm F}^2\mathbf{p}^2 - 2v_{\rm F}^2\mathbf{pq}\right] \\
&= v_{\rm F}^2\left[\mathbf{q}^2 - 2x\mathbf{pq} + x\mathbf{p}^2 + \frac{m_\psi^2}{v_{\rm F}^2} - x(1-x)p_0^2\frac{1}{v_{\rm F}^2}\right] \\
&= v_{\rm F}^2\left[(\mathbf{q}-x\mathbf{p})^2 + x(1-x)\mathbf{p}^2 + \frac{m_\psi^2}{v_{\rm F}^2} - x(1-x)p_0^2\frac{1}{v_{\rm F}^2}\right] \\
&= v_{\rm F}^2\left[(\mathbf{q}-x\mathbf{p})^2 - \Delta_2\right], \qquad (4.19)
\end{aligned}$$

where

$$\Delta_2 \equiv -x(1-x)\mathbf{p}^2 - \frac{m_\psi^2}{v_{\rm F}^2} + x(1-x)p_0^2\frac{1}{v_{\rm F}^2}.$$

We now have to shift $\mathbf{q} \to \mathbf{q} + x\mathbf{p}$. This also affects the numerator. Note that C is independent of $\mathbf{q}$. D becomes

$$\begin{aligned}
D &\to x^2(x-1)p_0^3 - v_{\rm F}^2x^2\mathbf{p}^2(x-1)p_0 - m_\psi^2(1+x)p_0 - v_{\rm F}^2\mathbf{q}^2(x-1)p_0, \\
&= E - v_{\rm F}^2\mathbf{q}^2(x-1)p_0 \qquad (4.20)
\end{aligned}$$

where $E = x^2(x-1)p_0^3 - v_{\rm F}^2x^2\mathbf{p}^2(x-1)p_0 - m_\psi^2(1+x)p_0$, and where the terms odd in $\mathbf{q}$ do not contribute. Our full expression after the shift is

$$\begin{aligned}
i\Pi^{ij} = -2(ie)^2(ig)v_{\rm F}^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)2\int_0^1 dx\int\frac{d^2q}{(2\pi)^2}\Bigg\{\frac{-2i\epsilon^{ij0}C(1-x)}{v_{\rm F}^3\left(\mathbf{q}^2-\Delta_2\right)^{3/2}}\frac{i}{16} \\
+\frac{2i\epsilon^{ij0}\left(E - v_{\rm F}^2\mathbf{q}^2(x-1)p_0\right)(1-x)}{v_{\rm F}^5\left(\mathbf{q}^2-\Delta_2\right)^{5/2}}\frac{3i}{16}\Bigg\}. \qquad (4.21)
\end{aligned}$$

The $\mathbf{q}$ integrals are

$$\begin{aligned}
\int\frac{d^2q}{(2\pi)^2}\frac{1}{\left(\mathbf{q}^2-\Delta_2\right)^{3/2}} &= \frac{-i}{2\pi}\frac{1}{\sqrt{\Delta_2}} \\
\int\frac{d^2q}{(2\pi)^2}\frac{\mathbf{q}^2}{\left(\mathbf{q}^2-\Delta_2\right)^{5/2}} &= \frac{-i2}{2\pi}\frac{1}{3}\frac{1}{\sqrt{\Delta_2}} \\
\int\frac{d^2q}{(2\pi)^2}\frac{1}{\left(\mathbf{q}^2-\Delta_2\right)^{5/2}} &= \frac{i}{2\pi}\frac{1}{3}\frac{1}{\left(\Delta_2\right)^{3/2}}. \qquad (4.22)
\end{aligned}$$

Substituting Eq. (4.22) into Eq. (4.21), we obtain

$$i\Pi^{ij} = -2(ie)^2(ig)v_{\mathrm{F}}^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)2\frac{1}{2\pi}\frac{1}{16}\int_0^1 dx\left\{\underbrace{\frac{2i\epsilon^{ij0}\left[-C-2(x-1)p_0\right](1-x)}{v_{\mathrm{F}}^3\left(\Delta_2\right)^{1/2}}}_{I_1}\right.$$
$$\left. - \underbrace{\frac{2i\epsilon^{ij0}E(1-x)}{v_{\mathrm{F}}^5\left(\Delta_2\right)^{3/2}}}_{I_2}\right\}. \tag{4.23}$$

Let us first compute integral I_1,

$$\int_0^1 dx\frac{2i\epsilon^{ij0}\left[-C-2(x-1)p_0\right](1-x)}{v_{\mathrm{F}}^3\left(\Delta_2\right)^{1/2}} = 2i\epsilon^{ij0}\frac{1}{v_{\mathrm{F}}^2}\int_0^1 dx\frac{\left[(1-3x)+2(1-x)\right](1-x)p_0}{\left[x(1-x)\left(p_0^2-v_{\mathrm{F}}^2\mathbf{p}^2\right)-m_\psi^2\right]^{1/2}}$$
$$= 2i\epsilon^{ij0}p_0\frac{1}{v_{\mathrm{F}}^2}\int_0^1 dx\frac{(3-5x)(1-x)}{\left[x(1-x)\overline{p}^2-m_\psi^2\right]^{1/2}}. \tag{4.24}$$

Calculating the integral we find

$$I_1 = 2i\epsilon^{ij0}p_0\frac{1}{v_{\mathrm{F}}^2}\left\{-i\frac{20}{8}\frac{|m_\psi|}{\overline{p}^2} - i\frac{\left(20m_\psi^2-7\overline{p}^2\right)}{8\overline{p}^3}\left[\ln\left(2\sqrt{-m_\psi^2}-i\overline{p}\right)-\ln\left(2\sqrt{-m_\psi^2}+i\overline{p}\right)\right]\right\}$$
$$= 2i\epsilon^{ij0}p_0\frac{1}{v_{\mathrm{F}}^2}\left\{-i\frac{20}{8}\frac{|m_\psi|}{\overline{p}^2} - i\frac{\left(20m_\psi^2-7\overline{p}^2\right)}{8\overline{p}^3}\ln\left(\frac{2\,|m_\psi|-\overline{p}}{2\,|m_\psi|+\overline{p}}\right)\right\}$$
$$= 2i\epsilon^{ij0}p_0\frac{1}{v_{\mathrm{F}}^2}F_1(m_\psi,p), \tag{4.25}$$

with

$$F_1(M,p) = -i\frac{20}{8}\frac{|m_\psi|}{\overline{p}^2} - i\frac{\left(20m_\psi^2-7\overline{p}^2\right)}{8\overline{p}^3}\ln\left(\frac{2\,|m_\psi|-\overline{p}}{2\,|m_\psi|+\overline{p}}\right). \tag{4.26}$$

The second integral becomes

$$
\begin{aligned}
I_2 &= \int_0^1 dx \frac{2i\epsilon^{ij0}E(1-x)}{v_{\rm F}^5\left(\Delta_2\right)^{3/2}} \\
&= 2i\epsilon^{ij0}\frac{1}{v_{\rm F}^5}\int_0^1 dx(1-x)\frac{\left[x^2(x-1)p_0^3 - v_{\rm F}^2 x^2\mathbf{p}^2(x-1)p_0 - m_\psi^2(1+x)p_0\right]}{\left(\Delta_2\right)^{3/2}} \\
&= 2i\epsilon^{ij0}\frac{1}{v_{\rm F}^2}\int_0^1 dx(1-x)\frac{\left[x^2(x-1)p_0^3 - v_{\rm F}^2 x^2\mathbf{p}^2(x-1)p_0 - m_\psi^2(1+x)p_0\right]}{\left[-x(1-x)\mathbf{p}^2 v_{\rm F}^2 - m_\psi^2 + x(1-x)p_0^2\right]^{3/2}} \\
&= 2i\epsilon^{ij0}\frac{1}{v_{\rm F}^2}\int_0^1 dx\frac{\left[-x^2(1-x)^2\left(p_0^2 - v_{\rm F}^2\mathbf{p}^2\right)p_0 - m_\psi^2(1-x^2)p_0\right]}{\left[x(1-x)\left(p_0^2 - \mathbf{p}^2 v_{\rm F}^2\right) - m_\psi^2\right]^{3/2}} \\
&= 2i\epsilon^{ij0}\frac{1}{v_{\rm F}^2}\int_0^1 dx\frac{\left[\overbrace{-x^2(1-x)^2\overline{p}^2 p_0}^{I_{2A}} - \overbrace{m_\psi^2(1-x^2)p_0}^{I_{2B}}\right]}{\left[x(1-x)\overline{p}^2 - m_\psi^2\right]^{3/2}}.
\end{aligned}
\tag{4.27}
$$

Doing the two terms I_{2A} and I_{2B} separately, we find

$$
\begin{aligned}
I_{2A} &= \int_0^1 dx\frac{-x^2(1-x)^2\overline{p}^2 p_0}{\left[x(1-x)\overline{p}^2 - m_\psi^2\right]^{3/2}} \\
&= \frac{-1}{4m_\psi^4\overline{p}^3 - m_\psi^2\overline{p}^5}\left\{2i\left|m_\psi\right|\overline{p}(2m_\psi^2 + \overline{p}^2) + im_\psi^2(4m_\psi^2 - \overline{p}^2)\ln\left[\frac{2\left|m_\psi\right| - \overline{p}}{2\left|m_\psi\right| + \overline{p}}\right]\right\}\overline{p}^2 p_0 \\
&= F_{2A}(m_\psi, p)\overline{p}^2 p_0,
\end{aligned}
\tag{4.28}
$$

and

$$
\begin{aligned}
I_{2B} &= \int_0^1 dx\frac{-(1-x^2)m_\psi^2 p_0}{\left[x(1-x)\overline{p}^2 - m_\psi^2\right]^{3/2}} \\
&= \frac{-1}{8\overline{p}^5(-4m_\psi^2 + \overline{p}^2)}\left\{4i\left|m_\psi\right|\overline{p}\left(-12m_\psi^2 + \overline{p}^2\right) - i\left(48m_\psi^4 - 8m_\psi^2\overline{p}^2 - \overline{p}^4\right)\right. \\
&\quad \left.\times\ln\left[\frac{2\left|m_\psi\right| - \overline{p}}{2\left|m_\psi\right| + \overline{p}}\right]m_\psi^2 p_0\right\} \\
&= F_{2B}(m_\psi, p)m_\psi^2 p_0,
\end{aligned}
\tag{4.29}
$$

where

$$F_{2A}(m_\psi, p) = \frac{-1}{4m_\psi^4\overline{p}^3 - m_\psi^2\overline{p}^5}\left\{2i\,|m_\psi|\,\overline{p}(2m_\psi^2 + \overline{p}^2) + im_\psi^2(4m_\psi^2 - \overline{p}^2)\ln\left[\frac{2\,|m_\psi| - \overline{p}}{2\,|m_\psi| + \overline{p}}\right]\right\},$$

$$\begin{aligned} F_{2B}(m_\psi, p) = \frac{-1}{8\overline{p}^5(-4m_\psi^2 + \overline{p}^2)}\Bigg\{&4i\,|m_\psi|\,\overline{p}\left(-12m_\psi^2 + \overline{p}^2\right) - i\left(48m_\psi^4 - 8m_\psi^2\overline{p}^2 - \overline{p}^4\right) \\ &\times \ln\left[\frac{2\,|m_\psi| - \overline{p}}{2\,|m_\psi| + \overline{p}}\right]\Bigg\}. \end{aligned} \tag{4.30}$$

Putting everything together we find

$$\begin{aligned} i\Pi^{ij} &= -\frac{e^2 g}{32\pi^2} m_\sigma 2i\epsilon^{ij0} p_0\left[F_1(m_\psi, p) + F_{2A}(m_\psi, p)\overline{p}^2 + F_{2B}(m_\psi, p)m_\psi^2\right] \\ &= -\frac{m_\sigma e^2 g}{(4\pi)^2} i\epsilon^{ij0} p_0\left[F_1(m_\psi, p) + F_{2A}(m_\psi, p)\overline{p}^2 + F_{2B}(m_\psi, p)m_\psi^2\right]. \end{aligned} \tag{4.31}$$

In order to calculate the contribution to the transverse conductivity of this diagram we apply the Kubo formula, for which we have to calculate

$$\begin{aligned} \lim_{p_0\to 0, \mathbf{p}\to 0}\frac{\Pi^{ij}}{p_0} &= \lim_{p_0\to 0, \mathbf{p}\to 0} -\frac{m_\sigma e^2 g}{(4\pi)^2}\epsilon^{ij0}\left[F_1(m_\psi, p) + F_{2A}(m_\psi, p)\overline{p}^2 + F_{2B}(m_\psi, p)m_\psi^2\right] \\ &= \lim_{p_0\to 0} -\frac{m_\sigma e^2 g}{(4\pi)^2}\epsilon^{ij0}\left[F_1(m_\psi, p) + F_{2A}(m_\psi, p)\overline{p}^2 + F_{2B}(m_\psi, p)m_\psi^2\right] \end{aligned}$$

It turns out that the limit $p \to 0$ is finite for each term separately, so we can calculate the limit term by term. To correctly take the limit, we have to Taylor expand the logarithm around $p_0 = 0$. This will provide terms that cancel the divergences. A Taylor expansion gives

$$\ln\left[\frac{2\,|m_\psi| - \overline{p}}{2\,|m_\psi| + \overline{p}}\right] = -\frac{p_0}{|m_\psi|} - \frac{1}{12}\frac{p_0^3}{|m_\psi|^3} - \frac{1}{80}\frac{p_0^5}{|m_\psi|^5} + O\left(p_0^6\right). \tag{4.32}$$

Using this we can now calculate the limit $p_0 \to 0$. The first term becomes

$$\begin{aligned} \lim_{p_0\to 0} F_1(m_\psi, p_0) &= \lim_{p_0\to 0} -i\frac{20}{8}\frac{|m_\psi|}{p_0^2} - i\frac{\left(20m_\psi^2 - 7\overline{p}^2\right)}{8\overline{p}^3}\ln\left(\frac{2\,|m_\psi| - \overline{p}}{2\,|m_\psi| + \overline{p}}\right) \\ &= \lim_{p_0\to 0} -i\frac{20}{8}\frac{|m_\psi|}{p_0^2} - i\frac{\left(20m_\psi^2 - 7\overline{p}^2\right)}{8\overline{p}^3}\left(-\frac{p_0}{|m_\psi|} - \frac{1}{12}\frac{p_0^3}{|m_\psi|^3}\right) + O(p_0) \\ &= \lim_{p_0\to 0} -i\frac{20}{8}\frac{|m_\psi|}{p_0^2} + i\frac{20}{8}\frac{|m_\psi|}{p_0^2} + \frac{5}{24}\frac{i}{|m_\psi|} - \frac{7}{8}\frac{i}{|m_\psi|} + O(p_0) \\ &= \lim_{p_0\to 0} -\frac{16}{24}\frac{i}{|m_\psi|} \\ &= -\frac{2}{3}\frac{i}{|m_\psi|} \end{aligned} \tag{4.33}$$

The second term gives

$$
\begin{aligned}
\lim_{p_0\to 0} F_{2A}(m_\psi,p_0)p_0^2 &= \frac{-1}{4m_\psi^4\overline{p}^3 - m_\psi^2\overline{p}^5}\left\{2i\left|m_\psi\right|\overline{p}(2m_\psi^2+\overline{p}^2) + im_\psi^2(4m_\psi^2-\overline{p}^2)\ln\left[\frac{2\left|m_\psi\right|-\overline{p}}{2\left|m_\psi\right|+\overline{p}}\right]\right\}\\
&= \lim_{p_0\to 0}\frac{-p_0^2}{m_\psi^2 p_0^3\left(4m_\psi^2-p_0^2\right)}\left\{4i\left|m_\psi\right|^3 p_0 + 2i\left|m_\psi\right|p_0^3 + im_\psi^2(4m_\psi^2-\overline{p}^2)\ln\left[\frac{2\left|m_\psi\right|-\overline{p}}{2\left|m_\psi\right|+\overline{p}}\right]\right\}\\
&= \lim_{p_0\to 0}\frac{-4i\left|m_\psi\right|}{\left(4m_\psi^2-p_0^2\right)} - \frac{2ip_0^2}{\left|m_\psi\right|\left(4m_\psi^2-p_0^2\right)} - \frac{i}{p_0}\left(-\frac{p_0}{\left|m_\psi\right|}\right) + O(p_0)\\
&= \frac{-4i\left|m_\psi\right|}{4m_\psi^2} + \frac{i}{\left|m_\psi\right|}\\
&= 0.
\end{aligned}
\tag{4.34}
$$

The final term yields

$$
\begin{aligned}
F_{2B}(m_\psi,p) &= \frac{-1}{8\overline{p}^5(-4m_\psi^2+\overline{p}^2)}\Bigg\{4i\left|m_\psi\right|\overline{p}\left(-12m_\psi^2+\overline{p}^2\right) - i\left(48m_\psi^4 - 8m_\psi^2\overline{p}^2 - \overline{p}^4\right)\\
&\quad\times\ln\left[\frac{2\left|m_\psi\right|-\overline{p}}{2\left|m_\psi\right|+\overline{p}}\right]\Bigg\}\\
&= \lim_{p_0\to 0}\frac{-\left[4i\left|m_\psi\right|\left(p_0^2-12m_\psi^2\right)\right]m_\psi^2}{8p_0^4(-4m_\psi^2+p_0^2)} + \frac{i\left(48m_\psi^4-8m_\psi^2p_0^2-p_0^4\right)m_\psi^2}{8p_0^5(-4m_\psi^2+p_0^2)}\ln\left[\frac{2\left|m_\psi\right|-\overline{p}}{2\left|m_\psi\right|+\overline{p}}\right]\\
&= \lim_{p_0\to 0}\frac{-4im_\psi^3}{8p_0^2(p_0^2-4m_\psi^2)} + \frac{6im_\psi^5}{p_0^4(p_0^2-4m_\psi^2)} + \Bigg[\frac{i6m_\psi^6}{p_0^5(p_0^2-4m_\psi^2)} - \frac{im_\psi^4}{p_0^3(p_0^2-4m_\psi^2)}\\
&\quad - \frac{im_\psi^2}{8p_0(p_0^2-4m_\psi^2)}\Bigg]\times\left(-\frac{p_0}{\left|m_\psi\right|} - \frac{1}{12}\frac{p_0^3}{\left|m_\psi\right|^3} - \frac{1}{80}\frac{p_0^5}{\left|m_\psi\right|^5}\right) + O(p_0)\\
&= \lim_{p_0\to 0}\frac{-4im_\psi^3}{8p_0^2(p_0^2-4m_\psi^2)} + \frac{6im_\psi^5}{p_0^4(p_0^2-4m_\psi^2)} - \frac{i6m_\psi^5}{p_0^4(p_0^2-4m_\psi^2)} - \frac{i6m_\psi^3}{12p_0^2(p_0^2-4m_\psi^2)}\\
&\quad - \frac{i6\left|m_\psi\right|}{80(p_0^2-4m_\psi^2)} + \frac{im_\psi^3}{p_0^2(p_0^2-4m_\psi^2)} + \frac{i\left|m_\psi\right|}{12(p_0^2-4m_\psi^2)} + \frac{i\left|m_\psi\right|}{8(p_0^2-4m_\psi^2)} + O(p_0)\\
&= \lim_{p_0\to 0}\frac{i\left|m_\psi\right|}{12(p_0^2-4m_\psi^2)} + \frac{i\left|m_\psi\right|}{8(p_0^2-4m_\psi^2)} - \frac{i6\left|m_\psi\right|}{80(p_0^2-4m_\psi^2)} + O(p_0)\\
&= \lim_{p_0\to 0}\frac{2i\left|m_\psi\right|}{15(p_0^2-4m_\psi^2)} + O(p_0)\\
&= -\frac{1}{30}\frac{i}{\left|m_\psi\right|}
\end{aligned}
\tag{4.35}
$$

Adding up all of the contributions we obtain

$$\begin{aligned}\lim_{p_0\to 0,\mathbf{p}\to 0}\frac{\Pi^{ij}}{p_0} &= \lim_{p_0\to 0} -\frac{m_\sigma e^2 g}{(4\pi)^2}\epsilon^{ij0}\left[F_1(m_\psi,p)+F_{2A}(m_\psi,p)\overline{p}^2+F_{2B}(m_\psi,p)m_\psi^2\right]\\ &= -\frac{m_\sigma e^2 g}{(4\pi)^2}\epsilon^{ij0}\left(-\frac{2}{3}\frac{i}{|m_\psi|}-\frac{1}{30}\frac{i}{|m_\psi|}\right)\\ &= i\frac{\epsilon^{ij0}}{(4\pi)^2}\frac{7}{10}\frac{m_\sigma}{|m_\psi|}e^2 g. \end{aligned} \tag{4.36}$$

Under time-reversal in silicene, the mass we have introduced transforms as $m_\psi \to -m_\psi$, and the frequency (p_0) as $i\omega \to -i\omega$. Using Kubo's formula, we obtain a non-universal correction to the transverse valley conductivity,

$$\delta\sigma^{ij}_{val} = \lim_{\omega\to 0,\mathbf{p}\to 0}\left\{\frac{i\langle j^i j^j\rangle}{\omega}-\frac{i\langle j^i j^j\rangle^{\mathrm{T}}}{\omega}\right\} = -4\frac{\epsilon^{ij0}}{(4\pi)^2}\frac{7}{10}\frac{m_\sigma}{|m_\psi|}e^2 g, \tag{4.37}$$

where we have picked up a factor of 2 for the spins. Restoring the dimensionality, we find

$$\delta\sigma^{val}_{xy} = -\frac{1}{(2\pi)^2}\frac{7}{10}\frac{m_\sigma}{|m_\psi|}g\frac{e^2}{h}, \tag{4.38}$$

for the valley Hall current. Combining this with the result from Ref. [12], we obtain

$$\sigma^{val}_{xy} = 2\frac{e^2}{h}\left(2n+1-\frac{1}{(2\pi)^2}\frac{7}{20}g\frac{m_\sigma}{|m_\psi|}\right). \tag{4.39}$$

For the Hall current, we find no correction since

$$\begin{aligned}\delta\sigma^{ij}_{val} &= \lim_{\omega\to 0,\mathbf{p}\to 0}\left\{\frac{i\langle j^i j^j\rangle}{\omega}-\frac{i\langle j^i j^j\rangle^{\mathrm{T}}}{\omega}\right\}\\ &= -2\frac{\epsilon^{ij0}}{(4\pi)^2}\frac{7}{10}\frac{m_\sigma}{|m_\psi|}e^2 g+2\frac{\epsilon^{ij0}}{(4\pi)^2}\frac{7}{10}\frac{m_\sigma}{|m_\psi|}e^2 g\\ &= 0. \end{aligned} \tag{4.40}$$

4.1.1 Massless fermions case

If we have no fermion mass ($m_\psi = 0$), we find starting from Eq. (4.6)

$$
\begin{aligned}
i\Pi^{ij} &= -2(ie)^2(ig)v_F^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)\int\frac{d^3q}{(2\pi)^3}\left\{\frac{\mathrm{Tr}\left[\gamma^i\left(\gamma^\alpha\overline{q}_\alpha\right)^2\gamma^j\left(\gamma^\beta(\overline{q}-\overline{p})_\beta\right)\right]}{\left(\overline{q}^2\right)^2\left[(\overline{q}-\overline{p})^2\right]}\right\} \\
&= -2(ie)^2(ig)v_F^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)\int\frac{d^3q}{(2\pi)^3}\left\{\frac{\mathrm{Tr}\left[\gamma^i\overline{q}^2\gamma^j\left(\gamma^\beta(\overline{q}-\overline{p})_\beta\right)\right]}{\left(\overline{q}^2\right)^2\left[(\overline{q}-\overline{p})^2\right]}\right\} \\
&= -2(ie)^2(ig)v_F^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)\int\frac{d^3q}{(2\pi)^3}\left\{\frac{\overline{\mathrm{q}}^2\mathrm{Tr}\left[\gamma^i\gamma^j\gamma^0\right](q-p)_0}{\left(\overline{q}^2\right)^2\left[(\overline{q}-\overline{p})^2\right]}\right\} \\
&= -2(ie)^2(ig)v_F^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)\int\frac{d^3q}{(2\pi)^3}\left\{\frac{2i\epsilon^{ij0}(q-p)_0}{\overline{q}^2\left[(\overline{q}-\overline{p})^2\right]}\right\}. \qquad (4.41)
\end{aligned}
$$

Combining the denominators using the Feynman trick we find

$$
\begin{aligned}
Ax + (1-x)B &= q_0^2 - v_F^2\mathbf{q}^2 + p_0^2x(1-x) - v_F^2\mathbf{p}^2x(1-x) \\
&= q_0^2 - \Delta_1. \qquad (4.42)
\end{aligned}
$$

Where we have shifted $q_0 \to q_0 + xp_0$ and $\mathbf{q} \to \mathbf{q} + x\mathbf{p}$. We also have to do this in the numerator. Performing the shifts, the full expression becomes

$$
i\Pi^{ij} = -2(ie)^2(ig)v_F^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)\int_0^1 dx\int\frac{d^3q}{(2\pi)^3}\left\{\frac{2i\epsilon^{ij0}(x-1)p_0}{(q_0^2-\Delta_1)^2}\right\}. \qquad (4.43)
$$

We now use

$$
\int\frac{dq_0}{2\pi}\frac{1}{\left(q_0^2-\Delta_1\right)^2} = \frac{1}{4}\frac{1}{(-\Delta_1)^{3/2}}, \qquad (4.44)
$$

which gives us

$$
\begin{aligned}
i\Pi^{ij} &= -2(ie)^2(ig)v_{\mathrm{F}}^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)\frac{1}{4}\int_0^1 dx\int\frac{d^2q}{(2\pi)^2}\left\{\frac{2i\epsilon^{ij0}(x-1)p_0}{(-\Delta_1)^{3/2}}\right\} \\
&= -2(ie)^2(ig)v_{\mathrm{F}}^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)\frac{1}{4}\int_0^1 dx\int\frac{d^2q}{(2\pi)^2} \\
&\times\left\{\frac{2i\epsilon^{ij0}(x-1)p_0}{(-v_{\mathrm{F}}^2)^{3/2}\left(\mathbf{q}^2-\frac{p_0^2}{v_{\mathrm{F}}^2}x(1-x)+\mathbf{p}^2x(1-x)\right)^{3/2}}\right\} \\
&= -2(ie)^2(ig)v_{\mathrm{F}}^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)\frac{1}{4}\int_0^1 dx\int\frac{d^2q}{(2\pi)^2}\left\{\frac{2i\epsilon^{ij0}(x-1)p_0}{(-v_{\mathrm{F}}^2)^{3/2}(\mathbf{q}^2-\Delta_2)^{3/2}}\right\}, \qquad (4.45)
\end{aligned}
$$

where

$$
\Delta_2 \equiv \frac{p_0^2}{v_{\mathrm{F}}^2}x(1-x)-\mathbf{p}^2x(1-x).
$$

We now calculate the $\mathbf{q}$ integral

$$
\begin{aligned}
\int\frac{d^2q}{(2\pi)^2}\frac{1}{(q^2-\Delta_2)^{3/2}} &= \frac{1}{(2\pi)}\int_0^\infty dq\frac{q}{(q^2-\Delta_2)^{3/2}} \\
&= \frac{1}{2\pi}\frac{1}{\sqrt{-\Delta_2}}. \qquad (4.46)
\end{aligned}
$$

Our complete expression becomes

$$
\begin{aligned}
i\Pi^{ij} &= -2(ie)^2(ig)v_{\mathrm{F}}^2\left(\frac{-m_\sigma}{4\pi}\right)(-i)\frac{1}{4}\frac{1}{2\pi}\int_0^1 dx\left\{\frac{2i\epsilon^{ij0}(x-1)p_0}{(-v_{\mathrm{F}}^2)^{3/2}(-\Delta_2)^{1/2}}\right\} \\
&= -2(ie)^2(ig)\left(\frac{-m_\sigma}{4\pi}\right)(-i)\frac{1}{4}\frac{1}{2\pi}\int_0^1 dx\left\{\frac{2i\epsilon^{ij0}(x-1)p_0}{\sqrt{-\overline{p}^2x(1-x)}}\right\} \\
&= -2(ie)^2(ig)\left(\frac{-m_\sigma}{4\pi}\right)(-i)\frac{1}{4}\frac{1}{2\pi}2i\epsilon^{ij0}p_0\frac{(-i)}{\sqrt{p_0^2-v_{\mathrm{F}}^2\mathbf{p}^2}}\left(-\frac{\pi}{2}\right) \\
&= -\frac{m_\sigma e^2 g}{16\pi}\epsilon^{ij0}\frac{p_0}{\sqrt{p_0^2-v_{\mathrm{F}}^2\mathbf{p}^2}}. \qquad (4.47)
\end{aligned}
$$

We obtain the same result starting from Eq. (4.23) and setting $m_\psi = 0$. Note that applying the Kubo formula, and taking the limit $p \to 0$ now gives a divergence. This is not unexpected because the mass dimension of the diagram will be 1. Due to the boson loop, we already have a m_σ in front of the expression, and the part contributing to the transverse conductivity will be proportional to p_0. On these grounds, our expression will be proportional to $\sim p_0 m_\sigma$ and thus we need to divide by a term with mass dimension 1. The only quantities available in the massless case are p_0 and $v_{\mathrm{F}}\mathbf{p}$, but dividing by these will cause a divergence when taking the limits in the Kubo formula. If we have a fermion mass, we could also divide by m_ψ to get something finite in the limit $\overline{p} \to 0$, and indeed this is exactly what happens when we introduce a fermion mass, as we can see from the result in Eq. (4.36).

Chapter 5

Coupling massless Pseudo-QED to a scalar field

In this chapter we will examine massless Pseudo-QED coupled to a massive scalar field σ, with mass m_σ. The Lagrangian is equivalent to Eq. (4.1) with $m_\psi = 0$, we have

$$\mathcal{L} = -\frac{1}{2}\frac{(F^{\mu\nu})^2}{\sqrt{-\Box}} + \bar{\psi}(i\gamma^0\partial_0 + iv_{\mathrm{F}}\gamma^i\partial_i)\psi + \frac{1}{2}\partial^\mu\sigma\partial_\mu\sigma - \frac{1}{2}m_\sigma^2\sigma^2 - e\bar{\psi}\gamma^\mu\psi A_\mu + g\bar{\psi}\psi\sigma^2, \quad (5.1)$$

Our goal is to examine the renormalization group flow of this theory, and to compare it with Pseudo-QED without a scalar field. To this end we compute the divergences in the Feynman diagrams up to two loop order involving the scalar field for the electron self-energy and the electron-photon vertex. We have calculated the divergent part of these diagrams using dimensional regularization, the RG-analysis remains to be done.

5.1 Electron self-energy

There are two diagrams contributing to the electron self-energy at two-loop order involving the scalar field. We calculate the divergent parts of these diagrams.

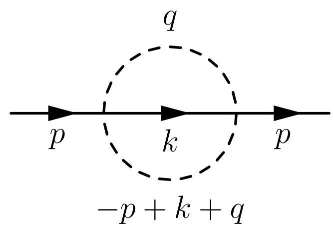


Figure 5.1.1: 2-loop contribution involving to the electron self-energy involving the scalar field.

5.1.1 First diagram

The first diagram is depicted in Figure 5.1.1. The complete expression of the diagram is

$$\begin{aligned} -i\Sigma &= -g^2 \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{i(k_0\gamma^0 + v_\mathrm{F}\gamma^i k_i)}{k_0^2 - v_\mathrm{F}^2 \mathbf{k}^2} \frac{i}{q^2 - m_\sigma^2} \frac{i}{(k+q-p)^2 - m_\sigma^2} \\ &= +ig^2 \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{(k_0\gamma^0 + v_\mathrm{F}\gamma^i k_i)}{k_0^2 - v_\mathrm{F}^2 \mathbf{k}^2} \frac{1}{q^2 - m^2} \frac{1}{(k+q-p)^2 - m_\sigma^2}. \end{aligned} \tag{5.2}$$

Our next step is to combine the two scalar propagators using the Feynman trick. We first want to calculate the q integral, to do this we combine the denominators involving q using the Feynman trick

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + (1-x)B]^2}. \tag{5.3}$$

In this way we can combine

$$\begin{aligned} \frac{1}{q^2 - m_\sigma^2} \frac{1}{(k+q-p)^2 - m_\sigma^2} &= \int_0^1 dx \frac{1}{[(k+q-p)^2 x - m_\sigma^2 x + (1-x)(q^2 - m_\sigma^2)]^2} \\ &= \int_0^1 dx \frac{1}{\left[(q + x(k-p))^2 - m_\sigma^2 + x\,(1-x)\,(k-p)^2\right]^2}, \end{aligned} \tag{5.4}$$

where we have completed the square and rearranged some terms. We can now make a shift $q \to q - x\,(k-p)^2$ to simplify the expression even further. Looking at Eq. (5.2) we see that there are no factors of q in the numerator. Hence, by substituting Eq. (5.4) into Eq. (5.2) we obtain

$$
\begin{aligned}
-i\Sigma &= +ig^2 \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{(k_0\gamma^0+v_{\mathrm{F}}\gamma^i k_i)}{k_0^2-v_{\mathrm{F}}^2\mathbf{k}^2}\frac{1}{\left[q^2-m_\sigma^2+x(1-x)(k-p)^2\right]^2} \\
&= +ig^2 \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{(k_0\gamma^0+v_{\mathrm{F}}\gamma^i k_i)}{k_0^2-v_{\mathrm{F}}^2\mathbf{k}^2}\frac{1}{\left[q^2-\Delta\right]^2},
\end{aligned} \tag{5.5}
$$

where we have defined $\Delta \equiv m_\sigma^2 - x(1-x)(k-p)^2$. We can now compute the q integral. The result of the integral is

$$
\begin{aligned}
\int \frac{d^3q}{(2\pi)^3}\frac{1}{\left(q^2-\Delta\right)^2} &= \frac{i}{(4\pi)^{3/2}}\sqrt{\frac{\pi}{\Delta}} \\
&= \frac{i}{8\pi}\frac{1}{\sqrt{\Delta}}.
\end{aligned} \tag{5.6}
$$

Substituting Eq. (5.6) into Eq. (5.5), we obtain

$$
\begin{aligned}
-i\Sigma &= ig^2\frac{i}{8\pi}\int_0^1 dx \int \frac{d^3k}{(2\pi)^3}\frac{(k_0\gamma^0+v_{\mathrm{F}}\gamma^i k_i)}{k_0^2-v_{\mathrm{F}}^2\mathbf{k}^2}\frac{1}{\Delta^{\frac{1}{2}}} \\
&= ig^2\frac{i}{8\pi}\int_0^1 dx \int \frac{d^3k}{(2\pi)^3}\frac{(k_0\gamma^0+v_{\mathrm{F}}\gamma^i k_i)}{k_0^2-v_{\mathrm{F}}^2\mathbf{k}^2}\frac{1}{\left[m_\sigma^2-x(1-x)(k-p)^2\right]^{\frac{1}{2}}}.
\end{aligned} \tag{5.7}
$$

Next, we want to calculate the k integral. Since the Lorentz symmetry is broken by the Fermi velocity v_{F}, we have to treat the k_0 and $\mathbf{k}$ integrals separately; we will do the k_0 integral first. As before, we combine the denominators using the Feynman trick. In this case, it looks like

$$
\frac{1}{AB^{\frac{1}{2}}} = \frac{1}{2}\int_0^1 dy \frac{(1-y)^{-\frac{1}{2}}}{(Ay+B(1-y))^{\frac{3}{2}}}. \tag{5.8}
$$

Let us first calculate the denominator

$$
\begin{aligned}
Ay + B(1-y) &= yk_0^2 - yv_{\mathrm{F}}^2\mathbf{k}^2 + (1-y)m_\sigma^2 - (1-y)x(1-x)(k-p)^2 \\
&= yk_0^2 - yv_{\mathrm{F}}^2\mathbf{k}^2 + (1-y)m_\sigma^2 - x(1-x)\left[(k_0-p_0)^2 - c^2(\mathbf{k}-\mathbf{p})^2\right] \\
&\quad + yx(1-x)\left[(k_0-p_0)^2 - c^2(\mathbf{k}-\mathbf{p})^2\right] \\
&= k_0^2\left[y-(1-y)x(1-x)\right] + 2k_0p_0\left[(1-y)x(1-x)\right] \\
&\quad - yv_{\mathrm{F}}^2\mathbf{k}^2 + (1-y)x(1-x)c^2(\mathbf{k}-\mathbf{p})^2 + (1-y)m_\sigma^2 - (1-y)x(1-x)p_0^2,
\end{aligned} \tag{5.9}
$$

and introduce the shorter notation

$$\begin{aligned}\alpha &\equiv y-(1-y)x(1-x)\\ \delta &\equiv (1-y)x(1-x).\end{aligned} \tag{5.10}$$

Using this notation we find

$$\begin{aligned}Ay+B(1-y) &= k_0^2\alpha + 2k_0p_0\delta - yv_{\mathrm{F}}^2\mathbf{k}^2 + c^2(\mathbf{k}-\mathbf{p})^2\delta + (1-y)m_\sigma^2 - \delta p_{0.}^2\\ &= \alpha\left[k_0^2 + 2k_0p_0\frac{\delta}{\alpha} - \frac{y}{\alpha}v_{\mathrm{F}}^2\mathbf{k}^2 + c^2(\mathbf{k}-\mathbf{p})^2\frac{\delta}{\alpha} + \frac{(1-y)}{\alpha}m_\sigma^2 - \frac{\delta}{\alpha}p_{0.}^2\right]\\ &= \alpha\left[\left(k_0 + p_0\frac{\delta}{\alpha}\right)^2 - \frac{\delta}{\alpha}\left(1+\frac{\delta}{\alpha}\right)p_0^2 - \frac{y}{\alpha}v_{\mathrm{F}}^2\mathbf{k}^2 + c^2(\mathbf{k}-\mathbf{p})^2\frac{\delta}{\alpha}\right.\\ &\left.+\frac{(1-y)}{\alpha}m_\sigma^2\right]\\ &= \alpha\left[\left(k_0 + p_0\frac{\delta}{\alpha}\right)^2 - \Delta_2\right],\end{aligned} \tag{5.11}$$

where we defined

$$\Delta_2 \equiv \frac{\delta}{\alpha}\left(1+\frac{\delta}{\alpha}\right)p_0^2 + \frac{y}{\alpha}v_{\mathrm{F}}^2\mathbf{k}^2 - c^2(\mathbf{k}-\mathbf{p})^2\frac{\delta}{\alpha} - \frac{(1-y)}{\alpha}m_\sigma^2.$$

Therefore, the combined denominators become

$$\frac{1}{(k_0^2 - v_{\mathrm{F}}^2\mathbf{k}^2)\left[m_\sigma - x(1-x)(k-p)^2\right]^{\frac{1}{2}}} = \frac{1}{2}\int_0^1 dy \frac{(1-y)^{-\frac{1}{2}}}{\alpha^{\frac{3}{2}}\left[\left(k_0 + p_0\frac{\delta}{\alpha}\right)^2 - \Delta_2\right]^{\frac{3}{2}}}. \tag{5.12}$$

Substituting this into Eq. (5.7), we obtain

$$\begin{aligned}\Sigma &= ig^2\frac{i}{8\pi}\frac{1}{2}\int_0^1 dy\int_0^1 dx\int \frac{d^3k}{(2\pi)^3}\frac{(1-y)^{-\frac{1}{2}}(k_0\gamma^0 + v_{\mathrm{F}}\gamma^i k_i)}{\alpha^{\frac{3}{2}}\left[\left(k_0 + p_0\frac{\delta}{\alpha}\right)^2 - \Delta_2\right]^{\frac{3}{2}}}\\ &= -\frac{g^2}{16\pi}\int_0^1 dy\int_0^1 dx\int \frac{d^3k}{(2\pi)^3}\frac{(1-y)^{-\frac{1}{2}}\left[(k_0 - p_0\frac{\delta}{\alpha})\gamma^0 + v_{\mathrm{F}}\gamma^i k_i\right]}{\alpha^{\frac{3}{2}}\left[k_0^2 - \Delta_2\right]^{\frac{3}{2}}}\\ &= -\frac{g^2}{16\pi}\int_0^1 dy\int_0^1 dx\int \frac{d^2k}{(2\pi)^2}\frac{dk_0}{(2\pi)}\frac{(1-y)^{-\frac{1}{2}}\left(-p_0\frac{\delta}{\alpha}\gamma^0 + v_{\mathrm{F}}\gamma^i k_i\right)}{\alpha^{\frac{3}{2}}\left[k_0^2 - \Delta_2\right]^{\frac{3}{2}}},\end{aligned} \tag{5.13}$$

where we have made a shift $k_0 \to k_0 - p_0\frac{\delta}{\alpha}$ and noticed that the terms odd in k_0 do not contribute. The k_0 integral, which is finite, now reads

$$\int \frac{dk_0}{(2\pi)} \frac{1}{(k_0^2 - \Delta_2)^{\frac{3}{2}}} = -\frac{2}{(2\pi)\Delta_2}. \tag{5.14}$$

Substituting this result into Eq. (5.13) yields

$$\begin{aligned}\Sigma &= -\frac{g^2}{16\pi}\left(-\frac{1}{\pi}\right)\int_0^1 dy \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{(1-y)^{-\frac{1}{2}}}{\alpha^{\frac{3}{2}}} \frac{\left(-p_0\frac{\delta}{\alpha}\gamma^0 + v_{\mathrm{F}}\gamma^i k_i\right)}{\Delta_2} \\ &= \frac{g^2}{16\pi^2}\int_0^1 dy \int_0^1 dx \int \frac{d^2k}{(2\pi)^2} \frac{(1-y)^{-\frac{1}{2}}}{\alpha^{\frac{3}{2}}} \frac{\left(-p_0\frac{\delta}{\alpha}\gamma^0 + v_{\mathrm{F}}\gamma^i k_i\right)}{\Delta_2}. \end{aligned} \tag{5.15}$$

Before we calculate the $\mathbf{k}$ integral let us rewrite Δ_2 as

$$\begin{aligned}\Delta_2 &= \frac{\delta}{\alpha}\left(1 + \frac{\delta}{\alpha}\right)p_0^2 + \frac{y}{\alpha}v_{\mathrm{F}}^2\mathbf{k}^2 - c^2(\mathbf{k}-\mathbf{p})^2\frac{\delta}{\alpha} - \frac{(1-y)}{\alpha}m_\sigma^2 \\ &= \mathbf{k}^2\left(\frac{y}{\alpha}v_{\mathrm{F}}^2 - c^2\frac{\delta}{\alpha}\right) + 2c^2\mathbf{k}\mathbf{p}\frac{\delta}{\alpha} + c^2\mathbf{p}^2\frac{\delta}{\alpha} - \frac{(1-y)}{\alpha}m_\sigma^2 + \frac{\delta}{\alpha}\left(1+\frac{\delta}{\alpha}\right)p_{0.}^2 \end{aligned} \tag{5.16}$$

Defining

$$\rho \equiv \left(\frac{y}{\alpha}v_{\mathrm{F}}^2 - c^2\frac{\delta}{\alpha}\right),$$

we can further simplify

$$\begin{aligned}\Delta_2 &= \rho\left[\mathbf{k}^2 + 2c^2\mathbf{k}\mathbf{p}\frac{\delta}{\alpha\rho} + c^2\mathbf{p}^2\frac{\delta}{\rho\alpha} - \frac{(1-y)}{\rho\alpha}m_\sigma^2 + \frac{\delta}{\rho\alpha}\left(1+\frac{\delta}{\alpha}\right)p_{0.}^2\right] \\ &= \rho\left[\left(\mathbf{k} + \mathbf{p}\frac{c^2\delta}{\alpha\rho}\right)^2 + \mathbf{p}^2\frac{c^2\delta}{\rho\alpha}\left(1 + \frac{c^2\delta}{\rho\alpha}\right) - \frac{(1-y)}{\rho\alpha}m_\sigma^2 + \frac{\delta}{\rho\alpha}\left(1+\frac{\delta}{\alpha}\right)p_{0.}^2\right] && (5.17) \\ &= \rho\left[\left(\mathbf{k} + \mathbf{p}\frac{c^2\delta}{\alpha\rho}\right)^2 - \Delta_3\right]. && (5.18)\end{aligned}$$

We now plug Eq. (5.17) into Eq. (5.15) and perform the shift

$$\mathbf{k} \to \mathbf{k} - \mathbf{p}\frac{c^2\delta}{\alpha\rho},$$

to obtain

$$\Sigma = \frac{g^2}{16\pi^2}\int_0^1 dy \int_0^1 dx \int \frac{d^2k}{(2\pi)^2}\frac{(1-y)^{-\frac{1}{2}}}{\alpha^{\frac{3}{2}}}\frac{\left(-p_0\frac{\delta}{\alpha}\gamma^0 + v_{\mathrm{F}}\gamma^i k_i\right)}{\rho\left[\left(\mathbf{k}+\mathbf{p}\frac{c^2\delta}{\alpha\rho}\right)^2 - \Delta_3\right]}$$

$$= \frac{g^2}{16\pi^2}\int_0^1 dy \int_0^1 dx \int \frac{d^2k}{(2\pi)^2}\frac{(1-y)^{-\frac{1}{2}}}{\alpha^{\frac{3}{2}}}\frac{\left(-p_0\frac{\delta}{\alpha}\gamma^0 + v_{\mathrm{F}}\gamma^i\left(k - p\frac{c^2\delta}{\alpha\rho}\right)_i\right)}{\rho\left(\mathbf{k}^2 - \Delta_3\right)}$$

$$= \frac{g^2}{16\pi^2}\int_0^1 dy \int_0^1 dx \int \frac{d^2k}{(2\pi)^2}\frac{(1-y)^{-\frac{1}{2}}}{\alpha^{\frac{3}{2}}}\frac{\delta}{\alpha}\frac{\left(-p_0\gamma^0 - v_{\mathrm{F}}\gamma^i p_i\frac{c^2}{\rho}\right)}{\rho\left(\mathbf{k}^2 - \Delta_3\right)}, \tag{5.19}$$

where we have eliminated the term odd in k_i. We can now perform the $\mathbf{k}$ integral using dimensional regularization. The result is as before

$$\int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{1}{(\mathbf{k}^2 - \Delta_3)} = -\frac{1}{2\pi}\frac{1}{\epsilon} + finite\, terms, \tag{5.20}$$

where we keep only the divergent term. If we substitute Eq. (5.20) result in Eq. (5.19) we find

$$-i\Sigma = \frac{g^2}{16\pi^2}\left(-\frac{1}{2\pi}\frac{1}{\epsilon}\right)\int_0^1 dy \int_0^1 dx (1-y)^{-\frac{1}{2}}\frac{\delta}{\alpha^{\frac{5}{2}}\rho}\left(-p_0\gamma^0 - v_{\mathrm{F}}\gamma^i p_i \frac{c^2}{\rho}\right)$$

$$= \frac{g^2}{16\pi^2}\frac{i^3}{4\pi}\left(-\frac{1}{2\pi}\frac{1}{\epsilon}\right)\int_0^1 dy \int_0^1 dx (1-y)^{-\frac{1}{2}}\frac{\delta}{\alpha^{\frac{3}{2}}\left(yv_{\mathrm{F}}^2 - c^2\delta\right)}\left(-p_0\gamma^0 - v_{\mathrm{F}}\gamma^i p_i \frac{c^2}{\rho}\right). \tag{5.21}$$

Next, we compute the parametric integrals over x and y that were introduced by the Feynman trick. Let us compute the p_0 term first. This integral reads

$$I_1 = -\int_0^1 dy \int_0^1 dx (1-y)^{-\frac{1}{2}}\frac{\delta}{\alpha^{\frac{3}{2}}\left(yv_{\mathrm{F}}^2 - c^2\delta\right)} p_0\gamma^0$$

$$= -\int_0^1 dy \int_0^1 dx \frac{(1-y)^{-\frac{1}{2}}(1-y)x(1-x)}{[y-(1-y)x(1-x)]^{\frac{3}{2}}\left(yv_{\mathrm{F}}^2 - c^2(1-y)x(1-x)\right)} p_0\gamma^0$$

$$= -\frac{1}{c^2}\int_0^1 dy \int_0^1 dx \frac{(1-y)^{\frac{1}{2}}x(1-x)}{[y-(1-y)x(1-x)]^{\frac{3}{2}}\left(y\frac{v_{\mathrm{F}}^2}{c^2} - (1-y)x(1-x)\right)} p_0\gamma^0. \tag{5.22}$$

We will compute the y integral first, and write for notational convenience $\beta = v_{\mathrm{F}}^2/c^2$ and $\lambda = x(1-x)$. Continuing with the integral we find

$$\begin{aligned}
I_1 &= -\frac{1}{c^2}\int_0^1 dy \int_0^1 dx \frac{(1-y)^{\frac{1}{2}}\lambda}{\left[y-(1-y)\lambda\right]^{\frac{3}{2}}\left[y\beta-(1-y)\lambda\right]} p_0\gamma^0. \\
&= -\frac{1}{c^2}\int_0^1 dx \left[\frac{2i}{\sqrt{\lambda}(1-\beta)} - \frac{2}{\sqrt{\lambda}}\frac{\beta^{\frac{1}{2}}}{(\beta-1)^{\frac{3}{2}}}\tan^{-1}\left(i\sqrt{\frac{\beta+1}{\beta}}\right)\right] p_0\gamma^0 \\
&= -\frac{1}{c^2}\left[\frac{2i}{(1-\beta)} - 2\frac{\beta^{\frac{1}{2}}}{(\beta-1)^{\frac{3}{2}}}\tan^{-1}\left(i\sqrt{\frac{\beta+1}{\beta}}\right)\right]\int_0^1 dx \frac{1}{\sqrt{\lambda}} p_0\gamma^0 \\
&= -\frac{1}{c^2}\left[\frac{2i}{(1-\beta)} - 2\frac{\beta^{\frac{1}{2}}}{(\beta-1)^{\frac{3}{2}}}\tan^{-1}\left(i\sqrt{\frac{\beta+1}{\beta}}\right)\right]\int_0^1 dx \frac{1}{\sqrt{x(1-x)}} p_0\gamma^0 \\
&= -\frac{i\pi}{c^2}\left[\frac{2}{(1-\beta)} - 2\frac{\beta^{\frac{1}{2}}}{(\beta-1)^{\frac{3}{2}}}\tanh^{-1}\left(\sqrt{\frac{\beta+1}{\beta}}\right)\right] p_0\gamma^0 \\
&\equiv -F_1(\beta) p_0\gamma^0.
\end{aligned} \tag{5.23}$$

The other integral proceeds in the same way. We find here

$$
\begin{aligned}
I_2 &= -\int_0^1 dy \int_0^1 dx (1-y)^{-\frac{1}{2}} \frac{\delta}{\alpha^{\frac{3}{2}}\left(y v_{\mathrm{F}}^2 - c^2\delta\right)} v_{\mathrm{F}}\gamma^i p_i \frac{c^2}{\rho} \\
&= -\int_0^1 dy \int_0^1 dx \frac{(1-y)^{\frac{1}{2}} x(1-x)}{\left[y-(1-y)x(1-x)\right]^{\frac{3}{2}}\left(y\beta-(1-y)x(1-x)\right)} v_{\mathrm{F}}\gamma^i p_i \frac{1}{\rho} \\
&= -\frac{1}{c^2}\int_0^1 dy \int_0^1 dx \frac{(1-y)^{\frac{1}{2}} x(1-x)}{\left[y-(1-y)x(1-x)\right]^{\frac{1}{2}}\left[y\beta-(1-y)x(1-x)\right]^2} v_{\mathrm{F}}\gamma^i p_i \\
&= -\frac{1}{c^2}\int_0^1 dy \int_0^1 dx \frac{(1-y)^{\frac{1}{2}} \lambda}{\left[y-(1-y)\lambda\right]^{\frac{3}{2}}\left[y\beta-(1-y)\lambda\right]^2} v_{\mathrm{F}}\gamma^i p_i \\
&= -\frac{1}{c^2}\int_0^1 dx \frac{1}{\sqrt{\lambda}} \left\{ \frac{i}{\beta-1} + \frac{1}{\sqrt{\beta}(\beta-1)^{\frac{3}{2}}} \tan^{-1}\left[i\sqrt{\frac{\beta-1}{\beta}} \right] \right\} v_{\mathrm{F}}\gamma^i p_i \\
&= -\frac{i\pi}{c^2} \left\{ \frac{1}{\beta-1} + \frac{1}{\sqrt{\beta}(\beta-1)^{\frac{3}{2}}} \tanh^{-1}\left[\sqrt{\frac{\beta-1}{\beta}} \right] \right\} v_{\mathrm{F}}\gamma^i p_i \\
&\equiv -F_2(\beta) v_{\mathrm{F}}\gamma^i p_i .
\end{aligned}
\tag{5.24}
$$

Substituting I_1 and I_2 into Eq. (5.21), we finally find

$$
\begin{aligned}
-i\Sigma &= \frac{g^2}{16\pi^2}\left(-\frac{1}{2\pi}\frac{1}{\epsilon}\right)\left[-p_0\gamma^0 F_1(\beta) - v_f\gamma^i p_i F_2(\beta)\right] \\
&= -\frac{g^2}{32\pi^3}\frac{1}{\epsilon}\left[p_0\gamma^0 F_1(\beta) + v_f\gamma^i p_i F_2(\beta)\right].
\end{aligned}
\tag{5.25}
$$

Note that this is only the divergent part of diagram.

5.1.2 Second diagram

The second diagram is shown in Fig. 5.1.2. The complete expression becomes

$$
\begin{aligned}
-i\Sigma &= (ie)^2 (ig) i^2 \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{(-ig^{\mu\nu})}{\sqrt{(q-p)^2}}\frac{i}{k^2-m_\sigma^2}\frac{\gamma_\mu\left(\gamma^0 q_0 + v_{\mathrm{F}} q_i\gamma^i\right)^2\gamma_\nu}{\left(q_0^2 - v_{\mathrm{F}}^2\mathbf{q}^2\right)^2} \\
&= e^2 g \int \frac{d^3k}{(2\pi)^3}\frac{i}{k^2-m_\sigma^2}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{(q-p)^2}}\frac{\gamma_\mu\left(\gamma^0 q_0 + v_{\mathrm{F}} q_i\gamma^i\right)^2\gamma^\mu}{\left(q_0^2 - v_{\mathrm{F}}^2\mathbf{q}^2\right)^2}.
\end{aligned}
\tag{5.26}
$$

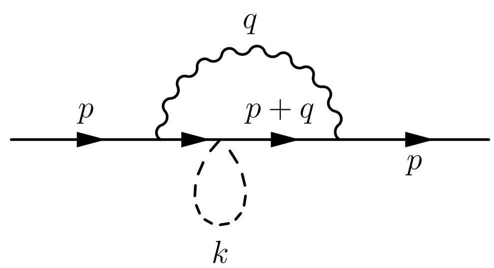


Figure 5.1.2: The second 2-loop diagram contributing to the electron self-energy involving the scalar field

The k integral we have done before, and we immediately find

$$-i\Sigma = e^2 g\left(-\frac{m_\sigma}{4\pi}\right)\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{(q-p)^2}}\frac{\gamma_\mu\left(\gamma^0 q_0 + v_{\mathrm{F}} q_i \gamma^i\right)^2\gamma^\mu}{\left(q_0^2 - v_{\mathrm{F}}^2\mathbf{q}^2\right)^2}. \tag{5.27}$$

Turning to the gamma matrices we find

$$\begin{aligned}\gamma_\mu\left(\gamma^0 q_0 + v_{\mathrm{F}} q_i\gamma^i\right)^2\gamma^\mu &= \gamma_\mu\left(q_0^2 + v_{\mathrm{F}}^2 q_i q_j\gamma^i\gamma^j\right)\gamma^\mu \\ &= nq_0^2 + v_{\mathrm{F}}^2\gamma_\mu\frac{1}{2}q_i q_j\left(\gamma^i\gamma^j + \gamma^j\gamma^i\right)\gamma^\mu \\ &= nq_0^2 + v_{\mathrm{F}}^2\gamma_\mu\gamma^\mu q_i q_j g^{ij} \\ &= n\left(q_0^2 + v_{\mathrm{F}}^2\mathbf{q}^2\right).\end{aligned} \tag{5.28}$$

Where we have used the anti-commutation relation of the gamma matrices and $n = 1 - 2v_{\mathrm{F}}^2$. Substituting Eq. (5.28) into Eq. (5.27) we obtain

$$-i\Sigma = e^2 g\left(-\frac{m_\sigma}{4\pi}\right) n\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{(q-p)^2}}\frac{q_0^2 + v_{\mathrm{F}}^2\mathbf{q}^2}{\left(q_0^2 - v_{\mathrm{F}}^2\mathbf{q}^2\right)^2}. \tag{5.29}$$

Our next step is again to combine the denominators using the Feynman trick

$$\begin{aligned}\frac{1}{A^2 B^{\frac{1}{2}}} &= \frac{\Gamma(5/2)}{\Gamma(2)\Gamma(1/2)}\int_0^1 dy dx \frac{y x^{-\frac{1}{2}}\delta(1-x-y)}{(yA + xB)^{\frac{5}{2}}} \\ &= \frac{3}{4}\int_0^1 dx \frac{(1-x)x^{-\frac{1}{2}}}{[(1-x)A + xB)]^{\frac{5}{2}}}.\end{aligned} \tag{5.30}$$

The denominator becomes

$$\begin{aligned}(1-x)A+xB &= (1-x)\left(q_0^2-v_{\mathrm{F}}^2\mathbf{q}^2\right)+x\left[(q_0-p_0)^2-c^2\left(\mathbf{q}-\mathbf{p}\right)^2\right] \\ &= q_0^2-v_{\mathrm{F}}^2\mathbf{q}^2+x\left[\left(p_0^2-2p_0q_0\right)-\mathbf{q}^2\left(c^2-v_{\mathrm{F}}^2\right)-c^2\mathbf{p}^2x+2xc^2\mathbf{pq}\right] \\ &= q_0^2-2xp_0q_0-\mathbf{q}^2\left[v_{\mathrm{F}}^2+x\left(c^2-v_{\mathrm{F}}^2\right)\right]+2xc^2\mathbf{pq}+xp_0^2-xc^2\mathbf{p}^2 \\ &= (q_0-xp_0)^2-\mathbf{q}^2\left[v_{\mathrm{F}}^2+x\left(c^2-v_{\mathrm{F}}^2\right)\right]+2xc^2\mathbf{pq}+x(1-x)p_0^2-xc^2\mathbf{p}^2 \\ &= (q_0-xp_0)^2-\Delta_1, \end{aligned} \tag{5.31}$$

with $\Delta_1=\mathbf{q}^2\left[v_{\mathrm{F}}^2+x\left(c^2-v_{\mathrm{F}}^2\right)\right]-2xc^2\mathbf{pq}-x(1-x)p_0^2+xc^2\mathbf{p}^2$. Now we shift $q_0\to q_0+xp_0$, which for the full expression yields

$$\begin{aligned}-i\Sigma &= e^2g\left(-\frac{m_\sigma}{4\pi}\right)n\frac{3}{4}\int_0^1 dx(1-x)x^{-\frac{1}{2}}\int\frac{d^3q}{(2\pi)^3}\frac{(q_0+xp_0)^2+v_{\mathrm{F}}^2\mathbf{q}^2}{\left(q_0^2-\Delta_1\right)^{5/2}} \\ &= e^2g\left(-\frac{m_\sigma}{4\pi}\right)n\frac{3}{4}\int_0^1 dx(1-x)x^{-\frac{1}{2}}\int\frac{d^3q}{(2\pi)^3}\frac{q_0^2+x^2p_0^2+v_{\mathrm{F}}^2\mathbf{q}^2}{\left(q_0^2-\Delta_1\right)^{5/2}}, \end{aligned} \tag{5.32}$$

where the terms odd in q_0 do not contribute. Now we can calculate the q_0-integrals

$$\begin{aligned}\int\frac{dq_0}{2\pi}\frac{1}{\left(q_0^2-\Delta_1\right)^{5/2}} &= \frac{2}{3\pi}\frac{1}{\Delta_1^2} \\ \int\frac{dq_0}{2\pi}\frac{q_0^2}{\left(q_0^2-\Delta_1\right)^{5/2}} &= -\frac{1}{3\pi}\frac{1}{\Delta_1}. \end{aligned} \tag{5.33}$$

The complete expression then becomes

$$-i\Sigma = e^2g\left(-\frac{m_\sigma}{4\pi}\right)n\frac{3}{4}\int_0^1 dx(1-x)x^{-\frac{1}{2}}\int\frac{d^2q}{(2\pi)^2}\frac{1}{3\pi}\left[2\frac{x^2p_0^2+v_{\mathrm{F}}^2\mathbf{q}^2}{\Delta_1^2}-\frac{1}{\Delta_1}\right]. \tag{5.34}$$

We proceed by rewriting Δ_1 as

$$\begin{aligned}\Delta_1 &= \mathbf{q}^2\left[v_{\mathrm{F}}^2+x\left(c^2-v_{\mathrm{F}}^2\right)\right]-2xc^2\mathbf{pq}-x(1-x)p_0^2+xc^2\mathbf{p}^2 \\ &= \alpha\mathbf{q}^2-2xc^2\mathbf{pq}-x(1-x)p_0^2+xc^2\mathbf{p}^2 \\ &= \alpha\left[\mathbf{q}^2-2xc^2\mathbf{pq}\frac{1}{\alpha}-\frac{x(1-x)}{\alpha}p_0^2+\frac{xc^2\mathbf{p}^2}{\alpha}\right] \\ &= \alpha\left[\left(\mathbf{q}-x\frac{c^2}{\alpha}\mathbf{p}\right)^2-\frac{x(1-x)}{\alpha}p_0^2+\frac{x(1-x)c^2\mathbf{p}^2}{\alpha}\right] \\ &= \alpha\left[\left(\mathbf{q}-x\frac{c^2}{\alpha}\mathbf{p}\right)^2-\Delta_2\right], \end{aligned} \tag{5.35}$$

where we defined $\alpha \equiv v_{\mathrm{F}}^2 + x\,(c^2 - v_{\mathrm{F}}^2)$. Now we shift $\mathbf{q} \to \mathbf{q} + x\mathbf{p}c^2/\alpha$. Substituting this into Eq. (5.34), we obtain

$$
\begin{aligned}
-i\Sigma &= e^2 g \left(-\frac{m_\sigma}{4\pi}\right) n \frac{3}{4} \int_0^1 dx (1-x) x^{-\frac{1}{2}} \int \frac{d^2q}{(2\pi)^2} \frac{1}{3\pi} 2 \left[\frac{x^2 p_0^2 + v_{\mathrm{F}}^2 \left(\mathbf{q}^2 + x^2 \frac{c^4}{\alpha^2} \mathbf{p}^2\right)}{\alpha^2 \left(\mathbf{q}^2 - \Delta_2\right)^2} \right. \\
&\quad \left. - \frac{1}{\alpha \left(\mathbf{q}^2 - \Delta_2\right)} \right] \\
&= e^2 g \left(-\frac{m_\sigma}{4\pi}\right) n \frac{3}{4} \frac{1}{3\pi} \int_0^1 dx (1-x) x^{-\frac{1}{2}} \int \frac{d^2q}{(2\pi)^2} \left[2 \frac{x^2 p_0^2 + v_{\mathrm{F}}^2 \left(\mathbf{q}^2 + x^2 \frac{c^4}{\alpha^2} \mathbf{p}^2\right)}{\alpha^2 \left(\mathbf{q}^2 - \Delta_2\right)^2} \right. \\
&\quad \left. - \frac{1}{\alpha \left(\mathbf{q}^2 - \Delta_2\right)} \right] \\
&= e^2 g \left(-\frac{m_\sigma}{4\pi}\right) n \frac{3}{4} \frac{1}{3\pi} \int_0^1 dx (1-x) x^{-\frac{1}{2}} \int \frac{d^2q}{(2\pi)^2} \left[\underbrace{2 \frac{x^2 p_0^2 + v_{\mathrm{F}}^2 \left(x^2 \frac{c^4}{\alpha^2} \mathbf{p}^2\right)}{\alpha^2 \left(\mathbf{q}^2 - \Delta_2\right)^2}}_{I_1} + \underbrace{2 \frac{v_{\mathrm{F}}^2 \mathbf{q}^2}{\alpha^2 \left(\mathbf{q}^2 - \Delta_2\right)^2}}_{I_2} \right. \\
&\quad \left. - \underbrace{\frac{1}{\alpha \left(\mathbf{q}^2 - \Delta_2\right)}}_{I_3} \right] .
\end{aligned}
\tag{5.36}
$$

So we see we have three distinct integrals to solve. Note that integral I_1 is finite, while I_2 and I_3 are log divergent. We are interested in the diverging part of the diagram, hence we drop the I_1 term. We compute I_3 again using dimensional regularization

$$
\begin{aligned}
I_3 &= \frac{1}{\alpha} \int \frac{d^2q}{(2\pi)^2} \frac{1}{\left(\mathbf{q}^2 - \Delta_2\right)} \\
&= -\frac{1}{\alpha} \frac{1}{2\pi} \frac{1}{\epsilon} + finite\, terms
\end{aligned}
$$

where we have used dimensional regularization as before, and omitted the limit notation and the finite part. Now the integral I_2,

$$
\begin{aligned}
I_2 &= \frac{2v_{\mathrm{F}}^2}{\alpha^2}\int \frac{d^2q}{(2\pi)^2}\frac{\mathbf{q}^2}{(\mathbf{q}^2-\Delta_2)^2} \\
&= \frac{2v_{\mathrm{F}}^2}{\alpha^2}\int \frac{d^2q}{(2\pi)^2}\frac{\mathbf{q}^2-\Delta_2+\Delta_2}{(\mathbf{q}^2-\Delta_2)^2} \\
&= \frac{2v_{\mathrm{F}}^2}{\alpha^2}\int \frac{d^2q}{(2\pi)^2}\left\{\frac{1}{(\mathbf{q}^2-\Delta_2)}+\frac{\Delta_2}{(\mathbf{q}^2-\Delta_2)^2}\right\} \\
&= -\frac{1}{2\pi}\frac{1}{\epsilon}\frac{2v_{\mathrm{F}}^2}{\alpha^2}
\end{aligned}
$$

where again we have only kept the divergent terms and omitted the limit notation. Substituting everything into Eq. (5.36), we find

$$
\begin{aligned}
\Sigma &= e^2 g\left(-\frac{m_\sigma}{4\pi}\right)n\frac{3}{4}\frac{1}{3\pi}\int_0^1 dx(1-x)x^{-\frac{1}{2}}\left[-\frac{1}{2\pi}\frac{1}{\epsilon}\frac{2v_{\mathrm{F}}^2}{\alpha^2}+\frac{1}{2\pi}\frac{1}{\epsilon}\frac{1}{\alpha}\right] \\
&= e^2 g\left(-\frac{m_\sigma}{4\pi}\right)n\frac{3}{4}\frac{1}{3\pi}\frac{1}{2\pi}\frac{1}{\epsilon}\int_0^1 dx\frac{(1-x)x^{-\frac{1}{2}}}{\alpha}\left[\underbrace{1}_{I_4}-\underbrace{\frac{2v_{\mathrm{F}}^2}{\alpha}}_{I_5}\right].
\end{aligned}
\tag{5.37}
$$

Now we compute the x-integrals. We have

$$
\begin{aligned}
I_4 &= \int_0^1 dx\frac{(1-x)x^{-\frac{1}{2}}}{\alpha} \\
&= \frac{1}{v_{\mathrm{F}}^2}\int_0^1 dx\frac{(1-x)x^{-\frac{1}{2}}}{1+x\left(\frac{c^2}{v_{\mathrm{F}}^2}-1\right)} \\
&= \frac{2}{v_{\mathrm{F}}^2}\left[\frac{-1}{\left(\frac{c^2}{v_{\mathrm{F}}^2}-1\right)}+\frac{c^2}{v_{\mathrm{F}}^2}\frac{\arctan\left(\sqrt{\frac{c^2}{v_{\mathrm{F}}^2}-1}\right)}{\left(\frac{c^2}{v_{\mathrm{F}}^2}-1\right)^{3/2}}\right],
\end{aligned}
\tag{5.38}
$$

and

$$\begin{aligned} I_5 &= 2v_\mathrm{F}^2 \int_0^1 dx \frac{(1-x)x^{-\frac{1}{2}}}{\alpha^2} \\ &= \frac{2}{v_\mathrm{F}^2} \int_0^1 dx \frac{(1-x)x^{-\frac{1}{2}}}{\left[1 + x\left(\frac{c^2}{v_\mathrm{F}^2} - 1\right)\right]^2} \\ &= \frac{2}{v_\mathrm{F}^2} \left[\frac{1}{\left(\frac{c^2}{v_\mathrm{F}^2} - 1\right)} + \frac{\left(\frac{c^2}{v_\mathrm{F}^2} - 2\right) \arctan\left(\sqrt{\frac{c^2}{v_\mathrm{F}^2} - 1}\right)}{\left(\frac{c^2}{v_\mathrm{F}^2} - 1\right)^{3/2}} \right]. \end{aligned} \tag{5.39}$$

Substituting this into Eq. (5.37), we find

$$\begin{aligned} -i\Sigma &= e^2 g \left(-\frac{m_\sigma}{4\pi}\right) n \frac{3}{4} \frac{1}{3\pi} \frac{1}{2\pi} \frac{1}{\epsilon} \frac{4}{v_\mathrm{F}^2} \left[\frac{\arctan\left(\sqrt{\frac{c^2}{v_\mathrm{F}^2} - 1}\right)}{\left(\frac{c^2}{v_\mathrm{F}^2} - 1\right)^{3/2}} - \frac{1}{\frac{c^2}{v_\mathrm{F}^2} - 1} \right] \\ &= -e^2 g \frac{m_\sigma}{8\pi^3} \left(1 - v_\mathrm{F}^2\right) \frac{1}{\epsilon} \frac{1}{v_\mathrm{F}^2} \left[\frac{\arctan\left(\sqrt{\frac{c^2}{v_\mathrm{F}^2} - 1}\right)}{\left(\frac{c^2}{v_\mathrm{F}^2} - 1\right)^{3/2}} - \frac{1}{\frac{c^2}{v_\mathrm{F}^2} - 1} \right]. \end{aligned} \tag{5.40}$$

Again this is only the divergent part of the diagram.

5.2 Vertex correction

The lowest order correction to the electron-photon vertex is of order eg^2 and is depicted in Fig. (5.2.1). To simplify the calculation, we break the Lorentz invariance of the scalar field in the same way as for the electrons. That is, we replace c by v_F. The full expression becomes

$$i\Gamma^\mu = 2ieg^2 \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \left[S_F(p+q-k) \overline{\gamma}^\mu S_F(p'+q-k) \frac{i}{\overline{k}^2 - m_\sigma^2} \frac{i}{\overline{q}^2 - m_\sigma^2} \right], \tag{5.41}$$

where we picked up a symmetry factor of 2. Since we are interested only in the divergent part, and since this will come from the loop integrals, we can set $p = p' = 0$ to simplify

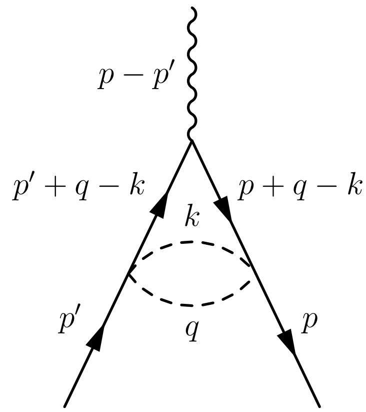


Figure 5.2.1: Correction to the electron-photon vertex due to the scalar field

the calculation. In this way

$$\begin{aligned} i\Gamma^\mu &= 2ieg^2 \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left[S_F(q-k)\overline{\gamma}^\mu S_F(q-k)\frac{1}{\overline{k}^2-m_\sigma^2}\frac{1}{\overline{q}^2-m_\sigma^2}\right] \\ &= 2ieg^2 \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left[\gamma^\alpha\overline{\gamma}^\mu\gamma^\beta \frac{\left(\overline{q}-\overline{k}\right)_\alpha\left(\overline{q}-\overline{k}\right)_\beta}{\left[(q_0-k_0)^2-v_\mathrm{F}^2\left(\mathbf{q}-\mathbf{k}\right)^2\right]^2}\frac{1}{\overline{k}^2-m_\sigma^2}\frac{1}{\overline{q}^2-m_\sigma^2}\right]. \end{aligned} \tag{5.42}$$

We can write the product of the three gamma matrices as

$$\gamma^\alpha\overline{\gamma}^\mu\gamma^\beta = \gamma^\alpha\left(-\gamma^\beta\overline{\gamma}^\mu + 2g^{\mu\beta}\Theta^\mu\right), \tag{5.43}$$

where $\Theta^0 = 1$ and $\Theta^i = v_\mathrm{F}$ otherwise. Substituting Eq. (5.43) into Eq. (5.42) we find

$$i\Gamma^\mu = 2ieg^2 \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left[\frac{-\overline{\gamma}^\mu\left(\overline{q}-\overline{k}\right)^2 + 2\gamma^\alpha\left(\overline{q}-\overline{k}\right)_\alpha\left(\overline{q}-\overline{k}\right)^\mu\Theta^\mu}{\left[(q_0-k_0)^2-v_\mathrm{F}^2\left(\mathbf{q}-\mathbf{k}\right)^2\right]^2}\frac{1}{\overline{k}^2-m_\sigma^2}\frac{1}{\overline{q}^2-m_\sigma^2}\right]. \tag{5.44}$$

We combine the denominators using

$$\frac{1}{A^2B} = 2\int_0^1 dx \frac{x}{[Ax+(1-x)B]^3}. \tag{5.45}$$

We have

$$\begin{aligned}
Ax+(1-x)B &= \overline{q}^2 - m_\sigma^2 + x\left[q_0^2 + k_0^2 - 2q_0k_0 - v_{\mathrm{F}}^2\mathbf{q}^2 - v_{\mathrm{F}}^2\mathbf{k}^2 + 2v_{\mathrm{F}}^2\mathbf{qk} - q_0^2 + v_{\mathrm{F}}^2\mathbf{q}^2 + m_\sigma^2\right] \\
&= \overline{q}^2 - m_\sigma^2 + x\left[k_0^2 - 2q_0k_0 - v_{\mathrm{F}}^2\mathbf{k}^2 + 2v_{\mathrm{F}}^2\mathbf{qk} + m_\sigma^2\right] \\
&= \overline{q}^2 - 2x\overline{q}\overline{k} - m_\sigma^2(1-x) + x\left[\overline{k}^2 + m_\sigma^2\right] \\
&= \left(\overline{q} - x\overline{k}\right)^2 - m_\sigma^2(1-x) + \overline{k}^2 x(1-x) \\
&= \left(\overline{q} - x\overline{k}\right)^2 - \delta_1,
\end{aligned} \tag{5.46}$$

with

$$\delta_1 = m_\sigma^2(1-x) - x(1-x)\overline{k}^2.$$

Substituting Eqs. (5.45) and (5.46) into Eq. (5.45) we obtain

$$\begin{aligned}
i\Gamma^\mu &= 2ieg^2 2\int_0^1 dx\, x \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left[\frac{-\overline{\gamma}^\mu\left(\overline{q}-\overline{k}\right)^2 + 2\gamma^\alpha\left(\overline{q}-\overline{k}\right)_\alpha\left(\overline{q}-\overline{k}\right)^\mu\Theta^\mu}{\left(\overline{q}-x\overline{k}\right)^2 - \delta_1}\frac{1}{\overline{k}^2 - m_\sigma^2}\right] \\
&= 2ieg^2 2\int_0^1 dx\, x \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left[\frac{1}{\overline{k}^2 - m_\sigma^2}\right. \\
&\times \left.\frac{-\overline{\gamma}^\mu\left(\overline{q}-\overline{k}(1-x)\right)^2 + 2\gamma^\alpha\left(\overline{q}-\overline{k}(1-x)\right)_\alpha\left(\overline{q}-\overline{k}(1-x)\right)^\mu\Theta^\mu}{\left(\overline{q}^2-\delta_1\right)^3}\right] \\
&= 2ieg^2 2\int_0^1 dx\, x \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left\{\frac{1}{\overline{k}^2 - m_\sigma^2}\right. \\
&\times \left.\frac{-\overline{\gamma}^\mu\left[\overline{q}^2 + \overline{k}^2(1-x)^2\right] + 2\gamma^\alpha\left[\overline{q}_\alpha\overline{q}^\mu + \overline{k}_\alpha\overline{k}^\mu(1-x)^2\right]\Theta^\mu}{\left(\overline{q}^2-\delta_1\right)^3}\right\}.
\end{aligned} \tag{5.47}$$

Using

$$\int \frac{d^3q}{(2\pi)^3}\overline{q}^\mu\overline{q}^\alpha G(\overline{q}^2) = \frac{g^{\mu\alpha}}{3}\int \frac{d^3q}{(2\pi)^3}\overline{q}^2 G(\overline{q}^2), \tag{5.48}$$

where $G(\overline{q}^2)$ is the denominator (the identity follows from Lorentz invariance), we find

$$i\Gamma^\mu = 2ieg^2 2\int_0^1 dx\, x \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left\{\frac{-\overline{\gamma}^\mu\left[\overline{q}^2+\overline{k}^2(1-x)^2\right]+\frac{2}{3}\overline{\gamma}^\mu\left[\overline{q}^2+\overline{k}^2(1-x)^2\right]}{\left(\overline{q}^2-\delta_1\right)^3}\right.$$
$$\left.\times\frac{1}{\overline{k}^2-m_\sigma^2}\right\}$$
$$= 2ieg^2 2\int_0^1 dx\, x \int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\left\{\frac{-\frac{1}{3}\overline{\gamma}^\mu\left[\overline{q}^2+\overline{k}^2(1-x)^2\right]}{\left(\overline{q}^2-\delta_1\right)^3}\frac{1}{\overline{k}^2-m_\sigma^2}\right\}. \tag{5.49}$$

The result for the q integral is

$$\int\frac{d^3q}{(2\pi)^3}\frac{1}{\left(\overline{q}^2-\delta_1\right)^3} = -\frac{i}{v_{\rm F}^2}\frac{1}{32\pi\delta_1^{3/2}},$$
$$\int\frac{d^3q}{(2\pi)^3}\frac{\overline{q}^2}{\left(\overline{q}^2-\delta_1\right)^3} = \frac{i}{v_{\rm F}^2}\frac{3}{32\pi\delta_1^{1/2}}.$$

Using this we obtain

$$i\Gamma^\mu = -\frac{2}{3}\frac{i}{v_{\rm F}^2 16\pi}\overline{\gamma}^\mu ieg^2\int_0^1 dx\, x\int\frac{d^3k}{(2\pi)^3}\left\{\left[\frac{3}{\delta_1^{1/2}}-\frac{\overline{k}^2(1-x)^2}{\delta_1^{3/2}}\right]\frac{1}{\overline{k}^2-m_\sigma^2}\right\}$$
$$= -\frac{2}{3}\frac{i}{v_{\rm F}^2 16\pi}\overline{\gamma}^\mu ieg^2\int_0^1 dx\, x\int\frac{d^3k}{(2\pi)^3}\left\{\frac{1}{\overline{k}^2-m_\sigma^2}\right.$$
$$\left.\times\left[\frac{3}{i\sqrt{x(1-x)}\left(\overline{k}^2-\frac{m_\sigma^2}{x}\right)^{1/2}}+\frac{\overline{k}^2(1-x)^2}{i\left[x(1-x)\right]^{3/2}\left(\overline{k}^2-\frac{m_\sigma^2}{x}\right)^{3/2}}\right]\right\} \tag{5.50}$$

Next, we combine the remaining denominators using

$$\frac{1}{A^{1/2}B} = \frac{1}{2}\int_0^1 dy\frac{y^{-1/2}}{\left[Ay+B(1-y)\right]^{3/2}}, \tag{5.51}$$

$$\frac{1}{A^{3/2}B} = \frac{3}{2}\int_0^1 dy\frac{y^{1/2}}{\left[Ay+B(1-y)\right]^{5/2}}. \tag{5.52}$$

The denominator becomes

$$Ay+B(1-y) = \overline{k}^2-m_\sigma^2+y\left[-\frac{m_\sigma^2}{x}+m_\sigma^2\right]$$
$$= \overline{k}^2-m_\sigma^2\left(1-y+\frac{y}{x}\right). \tag{5.53}$$

The full expression becomes

$$i\Gamma^\mu = -\frac{i}{v_{\mathrm{F}}^2 16\pi}\overline{\gamma}^\mu ieg^2 \int\limits_0^1 dy \int\limits_0^1 dx\, x \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{y^{-1/2}}{i\sqrt{x(1-x)}\left[\overline{k}^2 - m_\sigma^2\left(1-y+\frac{y}{x}\right)\right]^{3/2}} \right.$$
$$\left. + \frac{y^{1/2}\overline{k}^2(1-x)^{1/2}}{ix^{3/2}\left[\overline{k}^2 - m_\sigma^2\left(1-y+\frac{y}{x}\right)\right]^{5/2}} \right\}. \tag{5.54}$$

The divergent part of the first k integral

$$\begin{aligned}\int \frac{d^3k}{(2\pi)^3} \frac{1}{\left[\overline{k}^2 - m^2\left(1-y+\frac{y}{x}\right)\right]^{3/2}} &= \frac{1}{v_{\mathrm{F}}^2(2\pi)^3}\frac{\Gamma(-\frac{\epsilon}{2})}{\Gamma(3/2)}\frac{i}{(-1)^{3/2}}\pi^{3/2}\left[m^2\left(1-y+\frac{y}{x}\right)\right]^{\frac{\epsilon}{2}} \\ &= -\frac{1}{v_{\mathrm{F}}^2(2\pi)^2}\Gamma(-\frac{\epsilon}{2})\left[m^2\left(1-y+\frac{y}{x}\right)\right]^{\frac{\epsilon}{2}} \\ &= \frac{2}{v_{\mathrm{F}}^2(2\pi)^2}\frac{1}{\epsilon} + finite\, terms.\end{aligned} \tag{5.55}$$

The second k integral has the same divergent part, to see this we write

$$\begin{aligned}\int \frac{d^3k}{(2\pi)^3} \frac{\overline{k}^2}{\left[\overline{k}^2 - m^2\left(1-y+\frac{y}{x}\right)\right]^{5/2}} &= \int \frac{d^3k}{(2\pi)^3} \frac{\overline{k}^2 - m^2\left(1-y+\frac{y}{x}\right) + m^2\left(1-y+\frac{y}{x}\right)}{\left[\overline{k}^2 - m^2\left(1-y+\frac{y}{x}\right)\right]^{5/2}} \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left[\overline{k}^2 - m^2\left(1-y+\frac{y}{x}\right)\right]^{3/2}} \\ &\times + \int \frac{d^3k}{(2\pi)^3} \frac{m^2\left(1-y+\frac{y}{x}\right)}{\left[\overline{k}^2 - m^2\left(1-y+\frac{y}{x}\right)\right]^{5/2}}.\end{aligned} \tag{5.56}$$

The second integral in the last line of Eq. (5.56) is finite, hence all diverging terms will come from the first integral, but this integral is exactly Eq. (5.55). Hence we have

$$\int \frac{d^3k}{(2\pi)^3} \frac{\overline{k}^2}{\left[\overline{k}^2 - m^2\left(1-y+\frac{y}{x}\right)\right]^{5/2}} = \frac{2}{v_{\mathrm{F}}^2(2\pi)^2}\frac{1}{\epsilon} + finite\, terms. \tag{5.57}$$

Using these results for the k integrals, Eq. (5.54) becomes

$$
\begin{aligned}
i\Gamma^\mu &= -\frac{2}{v_\mathrm{F}^2 16\pi}\overline{\gamma}^\mu ieg^2 \int_0^1 dy \int_0^1 dx\, x \frac{1}{v_\mathrm{F}^2 (2\pi)^2}\frac{1}{\epsilon}\left\{\frac{y^{-1/2}}{\sqrt{x(1-x)}} + y^{1/2}\frac{(1-x)^{1/2}}{x^{3/2}}\right\} \\
&= -\frac{2}{v_\mathrm{F}^2 16\pi}\overline{\gamma}^\mu ieg^2 \int_0^1 dx\, x \frac{1}{v_\mathrm{F}^2 (2\pi)^2}\frac{1}{\epsilon}\left\{2\frac{1}{\sqrt{x(1-x)}} + \frac{2}{3}\frac{(1-x)^{1/2}}{x^{3/2}}\right\} \\
&= -\frac{2}{v_\mathrm{F}^2 16\pi}\overline{\gamma}^\mu ieg^2 \frac{1}{v_\mathrm{F}^2 (2\pi)^2}\frac{1}{\epsilon}\frac{4\pi}{3} \\
&= -\frac{ieg^2}{v_\mathrm{F}^4 24\pi^2}\overline{\gamma}^\mu \frac{1}{\epsilon}.
\end{aligned}
\tag{5.58}
$$

Chapter 6

Conclusions and Outlook

In this thesis we have examined dynamical interactions in (2+1)D Dirac systems. In Chapter 1, we have briefly explored condensed-matter systems where Dirac fermions occur, such as graphene and silicene. In Chapter 2, we examined dynamical interactions in such (2+1)D systems. Starting from QED in (3+1)D, we confined the matter current to a plane, and proved that the (2+1)D Pseudo-QED formalism is equivalent to this, and that it reproduces a Coulomb interaction in the static limit. We briefly analyzed the Pseudo-QED Lagrangian: it has non-local interactions, is strictly renormalizable and is unitary and causal. We stated the Feynman rules and to illustrate calculations calculated two one-loop diagrams.

In Chapter 3, we introduced the Kubo formalism to obtain the conductivity using Feynman diagrams, and reviewed two articles that employ Pseudo-QED to obtain conductivities in both massless and massive Dirac systems. We reproduced their results and found the longitudinal conductivity and a quantum valley Hall effect in graphene. In massive Dirac systems, we reproduced the same quantum valley Hall effect, and also a spontaneous total Hall effect.

In Chapter 4, we coupled massive Pseudo-QED to a scalar field to study how the transverse conductivities, reproduced in Chapter 3, changed under the influence of a new interaction. We found a non-universal correction, depending on the ratio of the scalar-field and fermion masses, to the quantum valley Hall effect. The total Hall conductivity remained unchanged, and in the massless case the limit $p \to 0$ was not well-defined.

In Chapter 5, we examined massless Pseudo-QED coupled to a scalar field, and calculated the divergent parts of three 2-loop Feynman diagrams containing this newly introduced scalar field. Using these diverging parts, together with the 1-loop diagrams, we can analyze the RG-flow of this system, although this was not completed.

We have seen that Pseudo-QED describes relativistic, dynamical interactions in (2+1)D Dirac systems, that are possible to create in condensed-matter. Using this description we are able to capture anomalies, particle hole symmetry, and it is possible to obtain effects non-perturbative in v_{F}. It is thus a more complete description of interactions in (2+1)D Dirac systems, and although not always computationally convenient, it could improve our understanding of (2+1)D Dirac systems.

Outlook

We briefly discuss some other possible applications of Pseudo-QED, and other possible projections of QED that might be considered in the future.

Projection of QED onto two planes

One may think of having two sheets of electrons, separation of d. We follow the same steps as in the derivation of Pseudo-QED in Chapter 2, but now we use the following matter current:

$$j^{\mu}(x^0,x^1,x^2,x^3) = \begin{cases} j^{+\mu}(x^0,x^1,x^2)\delta(x^3-\frac{d}{2}) + j^{-\mu}(x^0,x^1,x^2)\delta(x^3+\frac{d}{2}) & \mu = 0,1,2 \\ 0 & \mu = 3 \end{cases}, \tag{6.1}$$

where j^{+} denotes the matter current in the top plane, and j^{-} the matter current in the bottom plane. We have to insert this current into Eq. (2.11). We then obtain four terms, one for $j^{+}-j^{+}$,$j^{-}-j^{-}$,$j^{+}-j^{-}$ and $j^{-}-j^{+}$ current-current interactions. As expected, the $j^{+}-j^{+}$ and $j^{-}-j^{-}$ interactions, that is the interactions in the planes amongst the electrons, yield the same expressions as found for Pseudo-QED. In other words, the in-plane interactions remain the same. The difference lies thus in the $j^{+}-j^{-}$ and $j^{-}-j^{+}$ terms. This is the interaction between the two planes. Since they will be equivalent (as also expected), we will consider only the $j^{+}-j^{-}$term. This term describes the interaction between electrons in different planes

To find the current-current interaction, we have to evaluate Eq. (2.12) not at $z_3 =$

$z'_3 = 0$, as in Chapter 2, but for $z_3 = -z'_3 = d/2$. This yields (in Euclidean space)

$$\begin{aligned}
\left.\frac{1}{-\Box_E}\right|_{z_3=-z'_3=d} &= \int \frac{d^3k}{(2\pi)^3}\frac{dk_3}{(2\pi)}\frac{\mathrm{e}^{-i\left[k_0(z_0-z'_0)+k_1(z_1-z'_1)+k_2(z_2-z'_2)\right]}\mathrm{e}^{-idk_3}}{k_0^2+k_1^2+k_2^2+k_3^2} \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{\pi}{2\pi}\frac{\mathrm{e}^{-i\left[k_0(z_0-z'_0)+k_1(z_1-z'_1)+k_2(z_2-z'_2)\right]}\mathrm{e}^{-d\|k\|}}{\sqrt{k_0^2+k_1^2+k_2^2}} \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}\frac{\mathrm{e}^{-ik(z-z')}\mathrm{e}^{-d\|k\|}}{\sqrt{k^2}},
\end{aligned} \tag{6.2}$$

where now k is a three-component vector. Compared to Pseudo-QED, there is now an extra factor $\mathrm{e}^{-d\|k\|}$. We can follow the same steps to find the static interaction and we find (the k_0 integral now vanishes since the currents do not depend on it)

$$\begin{aligned}
V &= e^2 \int \frac{d^2k}{(2\pi)^2}\frac{1}{2}\frac{\mathrm{e}^{-ik(z-z')}\mathrm{e}^{-|2d|\|k\|}}{\sqrt{k^2}}. \\
&= e^2 \int \frac{dk}{(2\pi)^2}\int_0^{2\pi} d\theta \frac{\|k\|}{2}\frac{\mathrm{e}^{-i\|k\|\|z-z'\|Cos(\theta)}\mathrm{e}^{-|2d|\|k\|}}{\sqrt{k^2}} \\
&= e^2 \int \frac{dk}{(2\pi)^2}\int_0^{2\pi} d\theta \frac{1}{2}\mathrm{e}^{-i\|k\|\|z-z'\|Cos(\theta)}\mathrm{e}^{-|2d|\|k\|} \\
&= e^2 \int \frac{dk}{4\pi}\mathrm{e}^{-|d|\|k\|}J_0(\|k\|\|z-z'\|) \\
&= \frac{1}{4\pi}\frac{e^2}{\sqrt{(z-z')^2+d^2}},
\end{aligned} \tag{6.3}$$

which shows that we recover a Coulomb interaction between two electrons in different planes in the static limit.

The next step is to find an effective (2+1)D Lagrangian that is equivalent to this projection. Here there still lie some difficulties. There are now two different interactions in the system, those between electrons in the plane, and those between electrons from different planes. It is not clear if these should be mediated by the same gauge field, with an extra structure accounting for these two interactions, or that there perhaps should be two gauge fields in the system. Secondly, the kinetic term for the gauge field mediating the interactions between the planes would take the following form in the effective Lagrangian

$$\frac{F^{\mu\nu}\mathrm{e}^{d\sqrt{-\Box}}F_{\mu\nu}}{\sqrt{-\Box}},$$

but it is not clear that this is unitary. Perhaps one could justify an expansion in the separation d as a small parameter, and in this way obtain a simpler form of the

Lagrangian. The first term of such an expansion would correspond to Pseudo-QED, the second one to normal QED.

Projecting (3+1)D QED onto a wire

Instead of projecting QED onto a plane, one could also consider projecting down QED onto a wire. We would then try to confine the matter current to (1+1)D by writing

$$j^\mu(x) = \begin{cases} j^\mu_{1+1}(x^0, x^1)\delta(x^3)\delta(x^2) & \mu = 0, 1 \\ 0 & \mu = 2, 3 \end{cases}. \tag{6.4}$$

It turns out there are some bad divergences if we proceed in this way. In Ref. [40] the regularized the δ-functions by replacing them with

$$f(x) = \sqrt{\frac{a}{\pi}} \mathrm{e}^{-ax^2}.$$

This reduces to the δ-function in the limit $a \to \infty$. In this way we are considering the gauge field to propagate in a wire of finite width, and they found the effective gauge field propagator to be (in the Feynman-'t Hooft gauge) [40]

$$D_{\mu\nu}(k) = -\delta_{\mu\nu} \frac{e^2}{4\pi} \exp\left(\frac{k^2}{2a}\right) \mathrm{Ei}\left(-\frac{k^2}{2a}\right),$$

where $\mathrm{Ei}(x)$ is the integral exponential function. This propagator does not have a nice form, but perhaps other regulators, or certain limits of a or introducing a cut-off will improve the situation. Here it would be interesting to check what the RG-flow of such a theory would be, and find its fixed points.

Projecting (2+1)D QED onto a wire

We can also think about projecting (2+1)D QED onto a wire. Perhaps this could describe a situation where the electromagnetic field is strongly screened, such that it is confined to a plane, and if we then have a conducting wire lying in this plane. We can follow essentially the same steps as with the projection of (3+1)-dimensional QED onto a plane. We start from the generating functional with the A_μ field integrated out. We find (in Euclidean space)

$$Z^{eff}_{QED} = \exp\left[e^2 \int d^3z d^3z' j^\mu(z) \left(\frac{\delta_{\mu\nu}}{-\Box_{\mathrm{E}}}\right) j^\nu(z')\right]. \tag{6.5}$$

The only difference is that there is one spatial dimension less. Continuing in the same fashion, we confine the matter current to a wire by writing

$$j^\mu(x^0,x^1,x^2) = \begin{cases} j^\mu_{1+1}(x^0,x^1)\delta(x^2) & \mu = 0,1 \\ 0 & \mu = 2 \end{cases}. \tag{6.6}$$

Substituting Eq. (6.6) into Eq. (6.5), and noting that from now on all indices are either 0 or 1, we find

$$Z^{eff}_{QED} = = \exp\left\{ e^2 \int d^2z d^2z' \left[j^\mu_{1+1}(z) \frac{\delta^{\mu\nu}}{-\Box_E}\bigg|_{z_3=z'_3=0} j_{1+1}(z') \right] \right\}.$$

We rewrite the inverse d'Alembertian as

$$\begin{aligned} \frac{1}{-\Box_E}\bigg|_{z_2=z'_2=0} &= \int \frac{d^2k}{(2\pi)^2}\frac{dk_2}{(2\pi)}\frac{\mathrm{e}^{-i\left[k_0(z_0-z'_0)+k_1(z_1-z'_1)\right]}}{k_0^2+k_1^2+k_2^2} \\ &= \int \frac{d^2k}{(2\pi)^2}\frac{\pi}{(2\pi)}\frac{\mathrm{e}^{-i\left[k_0(z_0-z'_0)+k_1(z_1-z'_1)\right]}}{\sqrt{k_0^2+k_1^2}} \\ &= \int \frac{d^2k}{(2\pi)^2}\frac{1}{2}\frac{\mathrm{e}^{-i\left[k_0(z_0-z'_0)+k_1(z_1-z'_1)\right]}}{\sqrt{k_0^2+k_1^2}}. \end{aligned}$$

This has the exact same form as the one we have found before when projecting from QED in (3+1)-dimensions onto the plane. We can immediately infer, following the same reasoning, that the Lagrangian

$$\mathcal{L} = -\frac{1}{2}F^{\mu\nu}\frac{1}{(-\Box)^{\frac{1}{2}}}F_{\mu\nu} + -\overline{\psi}(\partial\!\!\!/ - m_\psi)\psi + ej^\mu A_\mu, \tag{6.7}$$

is the effective (1+1)-dimensional Lagrangian. Our arguments for showing that the correlation functions are equivalent can all still be applied. The essential difference is of course, that Eq. (6.7) is (1+1)-dimensional.

The first thing we want to check in this theory, is the dimension of the fields and the coupling constants. Starting from the kinetic term of the fermions, and working in mass dimensions, we find

$$[\overline{\psi}][m][\psi] = 2.$$

Thus $[\psi] = \frac{1}{2}$. From the kinetic part of the A_μ field we conclude

$$[\partial]^2[A_\mu]^2[\frac{1}{\sqrt{-\Box}}] = 2.$$

We know $[\partial] = 1$ and thus $\left[\frac{1}{\sqrt{-\Box}}\right] = -1$. We then find $[A_\mu] = \frac{1}{2}$. Using this we can infer from the interaction term that in this theory $[e] = \frac{1}{2}$. Since our coupling constant is positive, we are dealing with a super-renormalizable theory.

QED in (1+1)D is quite different from QED in higher dimensions. The photon has not propagating degrees of freedom in this theory (since there are no transverse modes in 1D), and thus Maxwell theory in (1+1)D is a topological theory. It would be interesting to consider instead of Maxwell theory, the first term of Eq. (6.7), and see how the non-locality changes this analysis.

Appendix

In this Appendix we briefly review the concept of dimensional regularization. The divergences encountered in quantum field theories can be systematically extracted by computing the integrals in n dimensions, and taking the limit of n to the dimension in which we are interested. We follow Ref. [41] in this appendix.

Let us see how dimensional regularization works by calculating the integral

$$\int d^n q \frac{1}{(\mathbf{q}^2 - \Delta)^\alpha}, \tag{6.8}$$

where we will be interested in the case $n = 2$. Note that this is a spatial vector, and thus we are already in Euclidean space. The first step is to go to spherical coordinates in n dimension. The result is

$$\int d^n q \frac{1}{(\mathbf{q}^2 - \Delta)^\alpha} = \int_0^\infty dq \frac{q^{n-1}}{(q^2 - \Delta)^\alpha} \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_2 \ldots \int_0^{\pi} d\theta_{n-1} \sin^{n-2}\theta_{n-1}.$$

To calculate the angular integrals, we need the result

$$\int_0^\pi d\theta \sin^k \theta = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}k\right)}{\Gamma\left(1 + \frac{1}{2}k\right)}.$$

Using this result we find

$$\begin{aligned}
\int d^n q \frac{1}{(\mathbf{q}^2 - \Delta)^\alpha} &= \int_0^\infty dq \frac{q^{n-1}}{(q^2 - \Delta)^\alpha} 2\pi \frac{\Gamma\left(\frac{1}{2}\right)^{n-2}}{\Gamma\left(1 + \frac{1}{2}(n-2)\right)} \\
&= \int_0^\infty dq \frac{q^{n-1}}{(q^2 - \Delta)^\alpha} \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.
\end{aligned}$$

To compute the q integral, we need the integral

$$\int_0^\infty dx \frac{x^{2\beta-1}}{(x^2 + a^2)^\alpha} = \frac{\Gamma(\beta)\Gamma(\alpha - \beta)}{\Gamma(\alpha)} \frac{1}{2\,(a^2)^{\alpha-\beta}},$$

which is valid for $\alpha > \beta > 0$. We can however use the analytic continuation of the Gamma function to extend the range in which it is valid. In our case we have

$$\int_0^\infty dq \frac{q^{n-1}}{(q^2-\Delta)^\alpha} = \frac{\Gamma(\frac{n}{2})\Gamma(\alpha-\frac{n}{2})}{\Gamma(\alpha)} \frac{1}{2\,(-\Delta)^{\alpha-n/2}}.$$

The complete integral then becomes

$$\int d^n q \frac{1}{(\mathbf{q}^2-\Delta)^\alpha} = \frac{\Gamma(\frac{n}{2})\Gamma(\alpha-\frac{n}{2})}{\Gamma(\alpha)} \frac{1}{(-\Delta)^{\alpha-n/2}} \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

At this point we take $n = 2+\epsilon$, where ϵ is a small parameter which take to 0 in order to obtain the result for 2 dimensions. Note that because the result is expressed in Gamma functions, by using the analytic continuation of this function, we can make the result well-defined for non-integer values of n. Let us find the compute the integral for $\alpha = 1$,

$$\int d^n q \frac{1}{(\mathbf{q}^2-\Delta)} = \pi\Gamma(-\frac{\epsilon}{2})\left(-\Delta\pi\right)^{\epsilon/2}.$$

The divergence if we take $\epsilon \to 0$ is now contained in the Gamma function, which has a pole at 0. We expand the expression in ϵ using

$$\Gamma(-\frac{\epsilon}{2}) = -\frac{2}{\epsilon} - \gamma_{\mathrm{E}} + \mathcal{O}(\epsilon),$$
$$\left(-\Delta\pi\right)^{\epsilon/2} = \exp\left(\frac{\epsilon}{2}\ln(-\Delta\pi)\right) = 1 + \epsilon\ln(-\Delta\pi),$$

where $-\Delta$ must be positive (which is indeed the case for the cases we encounter in this thesis) and γ_{E} is Euler's constant. Substituting these expansion into the integral, we find

$$\int d^2 q \frac{1}{(\mathbf{q}^2-\Delta)} = -2\pi\left[\frac{1}{\epsilon} + \frac{1}{2}\gamma_{\mathrm{E}} + \frac{1}{2}\ln(-\Delta\pi) + \mathcal{O}(\epsilon)\right].$$

This expression is, however, not dimensionally correct. We can introduce an arbitrary reference mass μ, to solve this problem. The full expression then becomes

$$\int d^2 q \frac{1}{(\mathbf{q}^2-\Delta)} = -2\pi\mu^\epsilon\left[\frac{1}{\epsilon} + \frac{1}{2}\gamma_{\mathrm{E}} + \frac{1}{2}\ln(\frac{-\Delta\pi}{\mu^2}) + \mathcal{O}(\epsilon)\right].$$

In the main text we will only focus on the divergent term of the diagram, and omit the μ^ϵ.

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