Universiteit Utrecht

The conformal limit of inflation

Utrecht University

Master thesis – Theoretical Physics

Author:
Jaap V.S. van Wijck

Supervisor:
Dr. Enrico Pajer

June 30, 2016

Abstract

In this thesis, we study the origin of the primordial fluctuations in the Cosmic Microwave Background by assuming that they originated from quantum fluctuations in the early universe, which stretched to cosmic size during a period of inflationary expansion. We shall consider the case of *slow-roll inflation*, a single field rolling down a flat potential. The main predictions of such models are characterized by two slow-roll parameters ϵ and η. Recent measurements of the temperature and the polarization fluctuations in the CMB suggest that these parameters satisfy $\epsilon \lesssim \eta/3$. This inequality will become much stronger in the absence of the detection of primordial B-modes. Therefore, the natural question to ask is what characterizes this $\epsilon \ll \eta$ limit, or more generically, what characterizes the $\epsilon \to 0$ limit. We will refer to this limit as the *decoupling limit*. As we will see, a new hierarchy in the way of organizing the slow-roll expansion will arise. In addition, the isometries of the background will reduce to that of a *de Sitter* space-time. When the physical wavelength of the perturbations becomes super Hubble size, the *de Sitter-isometries* acting on the inflaton perturbation correlation functions will reduce to those of a *Euclidean conformal field theory*. In this limit, the inflaton perturbation correlation functions become fully fixed by the conformal symmetries. For this reason, we will refer to this limit as the *conformal limit of inflation*. Last but not least, in conformal limit, the *equilateral* non-Gaussianities can be approximated by the *spectral tilt* of the potential, $f_{NL} \sim \alpha_s$.

Contents

Chapter 1

Introduction

1.1 Introduction

Measurement of the *Cosmic Microwave Background* show us that the universe is *homogeneous* and *isotropic* to a very high accuracy. When one splits up the universe into causal patches, one finds that all causal patches in the universe have thermalised to the same temperature to a precision of 1 in 10^5. This fact of the CMB is troubling, since how could all causal patches have known to what temperature they should have thermalized without having ever communicated with one another? This problem is called the *Horizon problem.* It can be solved by assuming the universe has undergone a period of inflationary expansion, which we call inflation. In this thesis, we study the vanilla model of inflation, which is the *slow-roll* model. In this model, a scalar field, the *inflaton*, rolls down a very flat potential. As it rolls down, it picks up kinetic energy and due to a weak coupling to gravity, this energy is converted to the gravitational sector which expanses the universe. The backgrounds of these models are usually described by *Friedmann-Lemaître-Robertson-Walker* metrics. The simplest FLRW-metric is of the form

$$ds^2 = -dt^2 + a(t)^2 r^2 d\Omega^2 \,,$$

here $a(t)$ is called the scale factor and it describes how much the spatial part of the background is expanding at a certain time t. During inflation, the scale factor scales as, $a(t) \equiv e^{H(t)t}$, with H the *Hubble expansion rate.* This parameter describes the rate of acceleration of the universe. Three new important scales arise during inflation, the first one is *Hubble expansion rate*, the second one is the *rate of deceleration* of the expansion of the universe

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \,,$$

and the third scale is

$$\eta \equiv \frac{\dot{\epsilon}}{\epsilon H} \,,$$

which is closely related to the *mass* of the inflaton. The *isotropy* and the *homogeneity* of the CMB suggests that the early universe was very close to be scale invariant. Scale invariance arises naturally from the *de Sitter* isometries, a spacetime in which H is constant. This suggests that the background dynamics the universe are governed during inflation by the relations

$$\epsilon \ll 1 \qquad \text{and} \qquad \eta \ll 1 \,.$$

These parameters, ϵ and η, are called *slow-roll parameters* and all predictions and results of inflationary theories are typically expressed in terms of them.

In the universe, there exists a certain scalar quantity that is conserved when its wavelength becomes super horizon scales, $aH < k$. This quantity is called *the adiabatic mode* and is defined to be

$$\mathcal{R}_i = \left(\frac{\delta\rho}{\bar{\rho} + \bar{p}}\right)_i \,,$$

here $\delta\rho_i$, $\bar{\rho}_i$ and $\bar{p}_i$ are the density perturbations, average density and average pressure respectively of the different components contributing to the energy density of the universe, $i \in$ {matter, radiation, dark energy, etc.}. Due to energy conservation, this quantity is the same for every component ($\mathcal{R}_i = \mathcal{R}_j$) in the universe, which means that we can describe all scalar (curvature) perturbations in the universe by means of one single scalar field.

Recent measurements of the polarizations by the *Planck satellite* constrain the value of ϵ and η to be $\epsilon \lesssim \eta/3$. As the detection of gravitational B-modes stay out, this inequality will only become stronger in the future [12]. It is therefore natural to ask ourselves what this $\epsilon \ll \eta$ limit characterizes, or more generically, what the $\epsilon \to 0$ limit characterizes. It turns out that in the limit in which we have a vanishing first slow roll parameter, ϵ, and when the scalar curvature perturbations have become super Hubble size, the correlation functions are fully constrained by the *de Sitter* isometries. This means that the density perturbations in the early universe, which generated the current matter distribution of the universe, can be fully deduced from symmetries.

When the scalar perturbation $\mathcal{R}$ becomes super-horizon size, it stops evolving in time. Taking the limit of $t \to \infty$, or in conformal time taking $\tau \to 0$ where $dt = a(\tau)d\tau$, does not change the value for $\mathcal{R}$. In the limit of $\tau \to 0$, the isometries of the *de Sitter* spacetime will become equivalent to that of a 3D *Euclidean Conformal field theory* and all *inflaton perturbations* will become fully invariant under these isometries. Therefore we call this limit *the conformal limit of inflation.*

In standard single field inflation, a n-point correlation function can be related to a $(n-1)$-point correlation function by taking one of the external momenta to be soft, $k_i = k_l \to 0$. In the conformal limit, a new physical relation for the inflaton perturbations arises from this relation,

$$\langle \varphi_l \varphi_s \varphi_s \rangle = \langle \varphi_l \varphi_l \rangle \partial_{\varphi_l} \langle \varphi_s \varphi_s \rangle \approx \alpha_s \langle \varphi_l \varphi_l \rangle \langle \varphi_s \varphi_s \rangle \,,$$

where α_s is defined to be the running of the potential. When calculating correlation functions of inflaton perturbations, one is usually resticted by the number of exact calculations one can perform due to the complexity of the mode functions. For example, when the inflaton is not massless, the integral one encounters when calculating the three point correlation function

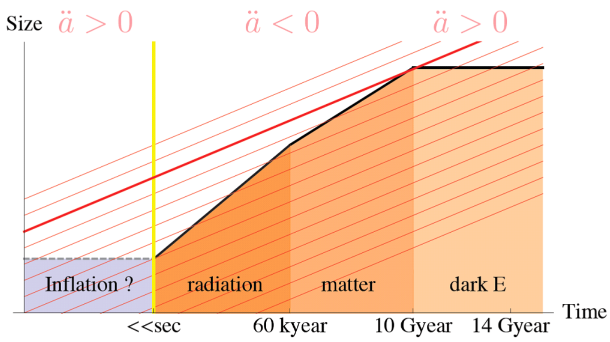

Figure 1.1: The evolution of the physical length of the universe and the physical length of the scalar perturbations. The red lines denote the physical wavelength of the scalar perturbations. The black line seperating the white area and the colored area is the Hubble radius. During inflation, the physical length of the perturbations exceeds the Hubble length and the modes freeze out. As the universe evolves, the Hubble radius changes and eventually catches up with the physical wavelength of the perturbations. At this point the perturbations start oscillating again and can be observed [30].

will become extremely hard to solve. This new physical relation let us calculate (three point) correlation functions in an alternative way, in which we do not need to solve integrals. Also, this new physical relation constrains the equilateral non-Gaussianities that our model predicts to be

$$f_{\rm NL} \sim \frac{\langle \varphi_l \varphi_s \varphi_s \rangle}{\langle \varphi_l^2 \rangle \langle \varphi_s^2 \rangle} \sim \alpha_s \, .$$

This new estimate of non-Gaussianities might just be measurable in the near future.

Outline of this thesis

We start this thesis with an introduction to modern cosmology 2, here we discuss the relation of the *CMB* and the *Homogeneous* and *Isotropic* universe model. We study the *Horizon problem* and discuss why the best paradigm to solve this problem is inflation.

In 3 we move away from the ideal homogeneous and isotropic universe model and we consider perturbations. The methods and techniques that are discussed for perturbation theory concern *linear perturbation theory*, *gauge freedom* concerning couplings to gravity, the *ADM-formalism* in inflationary cosmology, the *Foliation* of spacetime into spacelike hypersurfaces of equal time and the *Keldysh-Schwinger* or *in-in* formalism.

In 4 we study perturbation theory in the *Comoving gauge* along the lines of [13]. We calculate corrections to the so called *Bispectrum* that are one order higher than have previously done in literature. As we will see, current methods and approximations like the ones described in most recent literature, [13][14][19][20] are not sufficient to prove the "freezing out" of the *comoving curvature*-bispectrum. The time dependence is explored by means of a toy model 4.6.1 and a new way in calculating the bispectrum is derived that satisfies the full *Consistency relation* [13] to next order.

In 5 we do perturbation theory in the *Spatially flat gauge.* We consider a certain limit of inflation, similar to the one taken in the effective field theory of inflation [21], where the dynamics of the Goldstone boson associated to time translations decouples from the gravitational fluctuations. In this limit, we have a vanishing first slow-roll parameter, ϵ. We will discuss the results of [22] and show that the results can be explained by the *conformal invariance* of the correlation functions. We derive the *scalar consistency relation* in the spatially flat gauge, similar to [13] and discuss its implications. We show the validity of the consistency relation is the limit that the mass of the inflation is $m^2 = 0$ and $m^2 = 2H^2$, afterwards we will use it to generate a *squeezed-light (arbitrary) mass*-bispectrum. Then we relate the calculations to the ones performed in 4 by means of the δN formalism. Last but not least, we show that the amount of physical non-Gaussianities in our model reduces to the simple expression $f_{\rm NL} \sim \alpha_s$, relating $f_{\rm NL}$ and the running of the spectral index α_s.

In A we relate the definitions used in this thesis, to the most used alternative definitions. In B a full derivation of the gauge transformation between the *spatially flat gauge* and the *Comoving gauge* is given, here [13] is followed closely. In C we calculate the linear order bispectrum to confirm the results of [13]. In D we calculate the two point function in the *de Sitter* spacetime and we give a verification of the result of the two point function as given in [24]. In E we give a discussion and derivation of the *conformal isometries* in momentum space, where we will use them to calculate the bispectrum up to a overall factor.

Chapter 2

Modern Cosmology

2.1 A glance at the early universe

At the beginning of time, the universe was hot and dense, exotic particles were present and particle interactions were frequent and energetic. Matter consisted of atomic nuclei and free electrons. Over time, the universe cooled down, electrons were captured by the nuclei and light elements formed. The universe cooled down even further, photons became less energetic and could not destroy the light elements anymore and were released from the primordial particle plasma and hence, they began to stream freely. Today, we can still observe this afterglow of the primordial universe and we call it the *Cosmic Microwave Background*, abbreviated as the *CMB*. This radiation from the primordial universe is found to be almost completely isotropic, it has a temperature of approximately $T = 2.725$ Kelvin with small deviations of $\Delta T/T \approx \mathcal{O}(10^{-5})$ in every direction [1]. The fluctuations are very small and look insignificant, however, they reflect the presence of small density fluctuations in the matter distribution of the primordial universe, which in turn can give us information about the particles and particle-interactions that were present at the earliest moments of our universe.

In 1965, the CMB was discovered and it's by far the best blackbody spectrum ever measured. This suggests that before recombination, the universe was in thermal equilibrium. After the recombination of protons and electrons into neutral hydrogen, about 380,000 years after the Big Bang [2], the mean free path of the photons became larger than the horizon size and the universe became transparent for the photons produced in the earlier phases of the evolution of the universe. This radiation therefore provides a snapshot of the universe at that time. The collection of points where the photons of the CMB, that are now arriving on the earth had their last scattering before the universe became transparent is called the last scattering surface.

The universe just before recombination was a tightly coupled fluid, where photons scattered off charged particles and since they carried energy, they felt perturbations imprinted in the metric during inflation. The propagation of these small perturbations was very similar to that of sound waves, a train of slight compressions and rarefactions. The gas was heated by the compressions, while the rarefactions cooled it down. This led to a pattern of hot and cold spots, as seen in the CMB as the temperature anisotropies. One can make a distinction between primary and secondary anisotropies. Primary anisotropies arise due to the effects at the time of recombination and secondary anisotropies are generated by scattering along the line of sight. There are three basic primary perturbations, important on respectively large, intermediate, and small angular scales:

- *Gravitational Sachs-Wolfe*, photons released from high density regions at last scattering had to climb out of a higher gravitational-potential well than photons from lower density regions. They are redshifted by $\delta T/T = \delta\Phi$. Here $\delta\Phi$ is the perturbation in gravitational potential.

- *Adiabatic*, recombination occurs "later"[1] in regions of higher density, causing photons coming from denser regions to have smaller redshift from the universal expansion. This redshift corresponding to this effect is given by $\delta T/T = -\delta z/(1+z) = \delta\rho/\rho$.

[1]Recombination happens at a certain temperature and density in the universe, speaking about a certain time might not be the correct choice of words in GR.

- *Doppler*, the primordial plasma before recombination had a velocity at the moment of recombination. This gives a Doppler shift in the frequency of the photons which can be related to the temperature fluctuations via $\delta T/T = \delta \mathbf{v} \cdot \hat{r}/c$, with $\hat{r}$ the direction along the line of sight and $\mathbf{v}$ the mean velocity of the photons in the plasma.

Over the years, these anisotropies have been measured by *Through the Cosmic Background Explorer* (COBE), *Wilkinson Microwave Anisotropy Probe* (WMAP) and more recently by the *Planck* satellite, they were found to be of the order $\delta T/T \sim \mathcal{O}(10^{-5})$. These fluctuations contain a wealth of cosmological information, their angular sizes depend on their physical size at the time of last scattering, but also on the geometry of the universe, through which the photons have been traveling for almost 14 billion years.

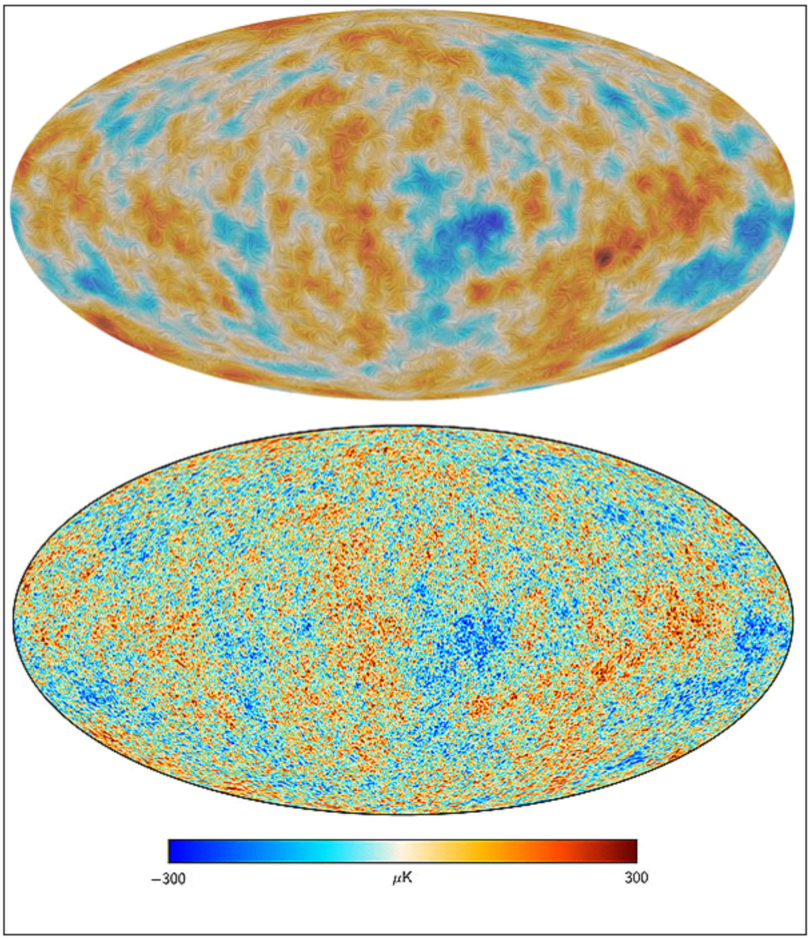

Figure 2.1: A map of the CMB temperature anisotropies of the full sky. The top picture was based on the *COBE* data and the bottom picture is based on most recent data provided by the *Planck* satellite. The avarage temperature of the CMB is 2.725 μK [3].

2.2 Geometric information from CMB Isotropies

When we look at the CMB, we observe a projection of sound-waves onto the sky. A certain mode with a wavelength λ subtends an angle θ on the sky. The observed spectrum of the CMB anisotropies can then be mapped as the magnitude of the temperature fluctuations versus the angular size of the hot and cold spots. This is usually done through a multipole expansion in terms of Legendre polynomials $\mathrm{P}_k(\cos(\theta))$ of a correlation function $C(\theta)$. The order of the polynomial, l, is related to the multipole moment and it plays a similar role in the angular decomposition as the wavenumber $k \sim 1/\lambda$ does for a Fourier decomposition. This means that the value of l is inversely proportional to the characteristic (angular) size of the wave-mode it describes. Therefore we can define the CMB correlation functions, $C(\theta)$, in the following way. Let us define the temperature fluctuations in the CMB from it's mean value in the direction of a unit vector $\hat{n}$. Then $C(\theta)$ is defined as the product of the temperature fluctuations of two points in the sky,

$$C(\theta) \equiv \left\langle \frac{\Delta T(\hat{n}_1)}{T} \frac{\Delta T(\hat{n}_2)}{T} \right\rangle , \tag{2.1}$$

here the angle brackets denote the full-sky avarage over $\hat{n}_1$ and $\hat{n}_2$. Also, it is assumed that the fluctuations are fully Gaussian. Then writing $C(\theta)$ in terms of Legendre polynomials, we obtain

$$C(\theta) = \sum_{l=0}^{\infty} \frac{(2l+1)}{4\pi} C_l P_l(\cos(\theta)) \, . \tag{2.2}$$

We can invert (2.2) for C_l in terms of the correlator, then

$$C_l = \frac{1}{4\pi} \int d^2\hat{n}_1 d^2\hat{n}_2 P(\hat{n}_1 \cdot \hat{n}_2) \left\langle \frac{\Delta T(\hat{n}_1)}{T} \frac{\Delta T(\hat{n}_2)}{T} \right\rangle . \tag{2.3}$$

The multipole coefficient of this correlation function, C_l is plotted in fig.(2.2). The peaks in the CMB power spectrum are being produced by modes caught at the extrema of their

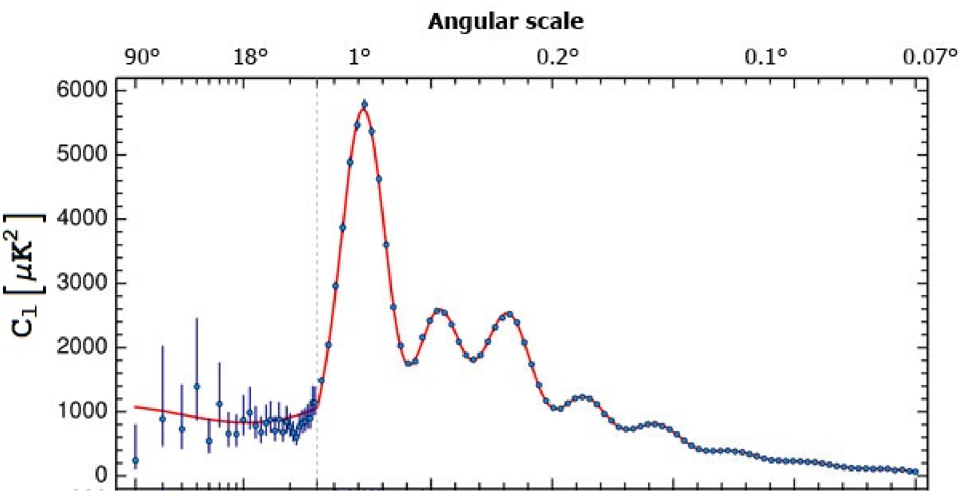

Figure 2.2: The function $l(l+1)C_l/(2\pi)$ plotted against the multipole moment l [4].

oscillations. They form a harmonic series based on the distance the sound waves can travel by recombination. This phenomena is known as the *sound horizon.* The first peak is the mode that compressed once inside the potential wells before recombination, the second peak is the mode that first compressed and later rarefied and the third peak is the mode that first compressed, then rarefied and then compressed again, etc.

C_l can be directly observed. The difference between the theoretical prediction and the actual observed value for C_l is called the *Cosmic variance* and has been defined by

$$\left\langle \left(\frac{C_l - C_l^{\mathrm{obs}}}{C_l} \right)^2 \right\rangle = \frac{2}{2l+1} \,. \tag{2.4}$$

2.2.1 Transfer functions

In order to relate the multipole moment of the CMB temperature fluctuations to the quantum fluctuations in the primordial universe, we introduce a transfer function T_l and a power spectrum $P_{\mathcal{R}}$, then

$$C_l = \frac{2}{\pi} \int dk \; k^2 P_{\mathcal{R}}(k) T_l(k) \,. \tag{2.5}$$

The power spectrum in (2.5) characterizes the quantum fluctuations generated in the primordial universe. The subscript $\mathcal{R}$ refers to a certain parametrization of the quantum fluctuations in the metric that freeze out after the modes surpass the horizon during inflation, we will discuss this in more details in the next couple of chapters. The transfer function T_l is known and accounts for the evolving of the perturbations from the moment they 're-enter the horizon[2]' until the time they are measured.

As it turns out, the acoustic peaks, as shown in 2.2, are produced in the theoretical model by the transfer function. This means that these peaks are not created in the primordial universe and can be related to more recent physics. When one filters these modes from the spectrum, one would find a nearly constant correlation function for C_l. Since the angular moment l is closely related to the solid angle at which we observe the CMB,

$$\Omega = \frac{4\pi}{2l} \,, \tag{2.6}$$

where Ω is the solid angle at which we observe the CMB. This suggests that if we measure the CMB at a larger angle the magnitude of the fluctuations remains the same. This suggests that the primordial universe was close to be scale invariant. Since scale invariance arises naturally from the *de Sitter* isometries, it suggests that the early universe can be described by an approximate de Sitter background.

[2]We come back to this in 2.4

2.3 Friedmann-Lemaître-Robertson-Walker models

When we look far into our universe, we see that the matter distribution in terms of galaxies becomes simpler and that the universe is expanding. If we average over large scales, we see that the universe starts looking more and more the same in every direction, the universe is *isotropic* at large scales. If the universe is also isotropic around all points, i.e. independent of the position we look at, it is also *homogeneous*.

Homogeneity and isotropy single out a unique form of the spacetime geometry. This suggests that the universe can be represented by a time-ordered sequence of three-dimensional spatial slices Σ_t, each of which is homogeneous and isotropic. In terms of isometries, a homogeneous spacetime is invariant under *spatial-translations* and isotropic means that the metric is invariant under *rotations*. A metric that describes a spacetime for an approximate isotropic, homogeneous and expanding space time is the *Friedmann-Lemaître-Robertson-Walker* (FLRW) metric. A line-element in this spacetime is given by

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right], \tag{2.7}$$

here $a(t)$ is called the *scale factor*, which is an arbitrary function depending on time, κ encodes the curvature of the spacetime, with $\kappa = \{0, +1, -1\}$ denote a spatially flat, positively curved and negatively curved spacetime respectively. In this thesis, we will focus on the case in which $\kappa = 0$. In order to study the causal structure of the universe, it is convenient to define

$$d\tau \equiv \frac{dt}{a(t)}, \tag{2.8}$$

where τ is called *conformal time*. When we substitute this into (2.7) the FLRW metric factorizes into a static Minkowski metric $\eta_{\mu\nu}$ metric multiplied by a (conformal)-time dependent scale factor

$$ds^2 = a^2(\tau)\left[-d\tau^2 + dr^2 + r^2 d\Omega^2\right] \equiv a^2(\tau)\eta_{\mu\nu}dx^\mu dx^\nu . \tag{2.9}$$

2.3.1 The Friedmann equations

In order to study the evolution of the universe, we would like to know the solution of the scale factor $a(t)$. Since the universe is a very complicated system, the scale factor of the universe will depend on the distribution of matter, radiation and even dark energy, Λ. Luckily, the influence of all components will not be the same and we can make approximations that some components will dominate the universe and others will give sub-leading contributions. In order to get an expression for $a(t)$, we start by considering the following action

$$S_{\text{total}} = S_{\text{HE}} + S_\phi , \tag{2.10}$$

where S_{HE} is the *Einstein-Hilbert* action describing the coupling to gravity

$$S_{\text{HE}} = \frac{M_{\text{Pl}}^2}{2}\int d^4x \sqrt{-g}R , \tag{2.11}$$

S_ϕ is the action describing matter fields that are minimally coupled to gravity

$$S_\rho = \int d^4x \, \sqrt{-g}\mathcal{L}_\phi \,, \tag{2.12}$$

where $\mathcal{L}_\phi$ is an arbitrary Lagrangian containing the fields (ϕ). Varying (2.10) with respect to the metric $g_{\mu\nu}$ yield the *Einstein-Equations*, formally given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{M_{\mathrm{Pl}}^2} T_{\mu\nu} \,, \tag{2.13}$$

where $\mathcal{R}_{\mu\nu}$ and $\mathcal{R}$ are the *Ricci- tensor* and *scalar* respectively and $T_{\mu\nu}$ is the *stress-energy tensor* defined by

$$T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} \,. \tag{2.14}$$

When assuming that the energy-density distribution of the different components in the universe behaves as a perfect fluid, i.e.

$$T^{00} = \rho(t) \,, \qquad T^{0i} = T^{i0} = 0 \,, \qquad T^{ij} = a^{-2}(t) p(t) \delta^{ij} \,. \tag{2.15}$$

Combining (2.13) and (2.15) we obtain the following equations

$$\frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} = \frac{1}{2M_{\mathrm{Pl}}^2}(\rho - p) \,, \tag{2.16}$$

$$\frac{3\ddot{a}}{a} = -\frac{1}{2M_{\mathrm{Pl}}}(3p + \rho) \,, \tag{2.17}$$

Since the stress-energy tensor is a conserved quantity, we also have

$$\nabla_\nu T^{\mu\nu} = 0 \,, \tag{2.18}$$

this gives us

$$\dot{\rho} + \frac{3\dot{a}}{a}(p + \rho) = 0 \,. \tag{2.19}$$

Then (2.17), (2.17) and (2.19) are called the *Friedmann equations* which together fully characterize the solution of the scalefactor $a(t)$. In FLRW spacetimes we usually define the *Hubble radius*, i.e.

$$r_H(t) = \frac{c}{H(t)} \,, \tag{2.20}$$

as the characteristic length-scale where we defined the *Hubble constant* (H) as

$$H(t) = \frac{\dot{a}(t)}{a(t)} \,. \tag{2.21}$$

2.4 The horizon problem

One of the most remarkable things about the CMB is probably that it's so extremely isotropic. To understand why this is troubling, we have to split up the universe into causal patches, where the size of each causal patch of space is determined by the distance light can travel in a certain amount of time. Since spacetime is isotropic, we can define our coordinate system in such a what that light travels in the radial direction. Then the evolution is determined by a two-dimensional line element

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + dr^2\right] . \tag{2.22}$$

Photons move along null geodesics, i.e. $ds^2 = 0$, therefore their path is defined by

$$\Delta r(\tau) = \pm \Delta \tau \, , \tag{2.23}$$

here the plus and minus sign represent outgoing and incoming photons respectively. Note that in conformal time, the light cones are at 45° in the $r - \tau$ coordinates. With these definitions, we can now define two different types of cosmological horizons. One which limits the distance at which past events can be observed and one which limits the distances at which it will be possible to observe future events.

- Particle Horizon: The greatest comoving distance from which an observer will be able to receice signals travelling at the speed of light is given by [6]

  $$r_{\mathrm{ph}}(\tau) = \tau - \tau_i = \int_{t_i}^{\tau} \frac{dt}{a(t)} \, . \tag{2.24}$$

  This is the *(comoving) particle horizon.* Causal influences have to come from within this region, this means that only particles whose worldlines intersect the past light cone of the observer will be able to communicate.

- Event Horizon: In comoving coordinates, the greatest distance from which an observer at time t_f will receive signals emitted at any time later than t is given by [6]

  $$r_{\mathrm{eh}}(\tau) = \tau_f - \tau = \int_{t}^{t_f} \frac{dt}{a(t)} \, . \tag{2.25}$$

  This horizon is called the *comoving event horizon.*

Now to get back why a such a uniform CMB is problematic, we will have to look at (2.24) and (2.25) for different causal patches that we observe in the sky. As it turns out, almost every spot in the CMB has non-overlapping past light cones and hence could never have been in causal contact with one another. To illustrate this problem, let's consider fig.4.1. In this figure, two causal patches at opposite directions in the sky are considered. The CMB photons that we receive from these directions were emitted at the points labeled p and q and they originated at the initial hypersurface of a moment shortly after recombination. The photons were emitted sufficiently close to the Big Bang singularity such that the past light cones of p and q do not overlap. This implies that no point lies inside the particle horizons of both p and q. The big

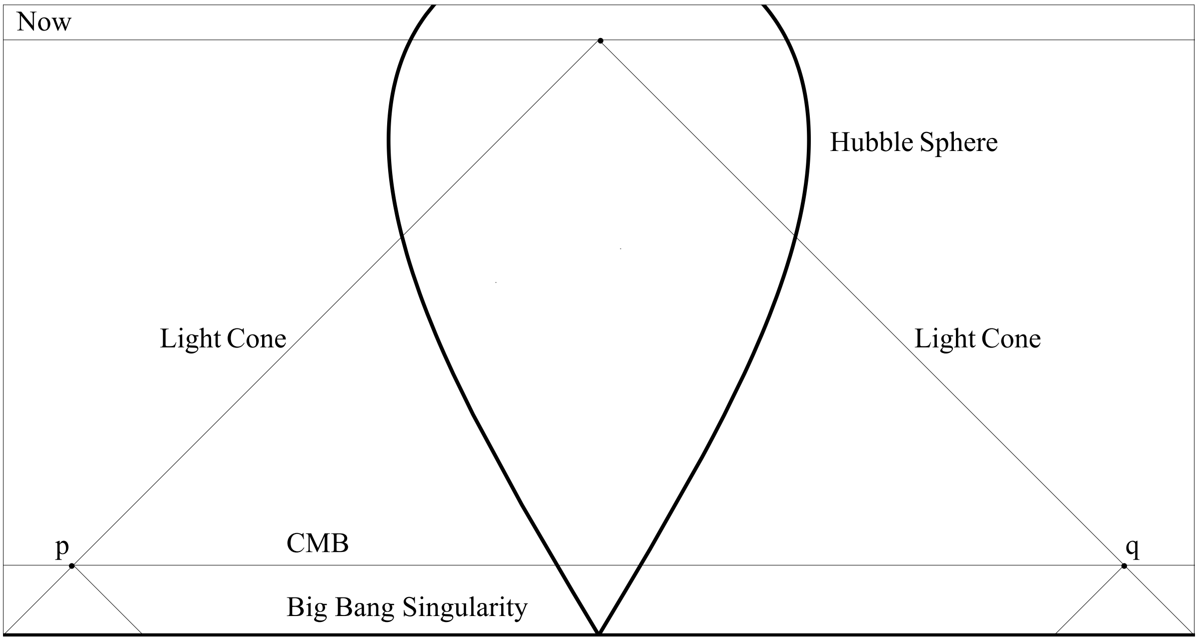

Figure 2.3: A schematic representation of the horizon problem. All events we observe lay on our past light cones. The horizontal line represents the spacelike hypersurface of the observed CMB. Points p and q are two spots in the CMB that are and were causally disconnected from one another. However, both spots have thermalized to the same temperature with a presision of the order 1 in 10^5 K.

question that now arises is: how do the photons coming from p and q know that they should be at almost exactly the same temperature? The same question applies to any two points in the CMB that are separated by more than 1° in the sky [6]. If there was not enough time for these regions to communicate, why do they look so similar? This is called *the horizon problem.*

2.5 Cosmological Inflation as solution to the horizon problem

The relevant horizon for solving the horizon problem is the particle horizon. In order to see this, let us rewrite (2.24) into a more convenient form

$$r_{\text{ph}} = \int_{t_i}^{t} \frac{dt}{a} = \int_{\ln(a_i)}^{\ln(a)} d[\ln(a)] \, \frac{1}{aH} \,, \tag{2.26}$$

here $a_i = 0$ denotes the *Big Bang singularity.* The causal structure of the universe can then be related to the *comoving Hubble radius*, i.e. $(aH)^{-1}$. For a universe dominated by a perfect fluid with a constant equation of state we have [6]

$$\frac{1}{aH} = \frac{1}{H_0} a^{\frac{1}{2}(1+3w)} \,. \tag{2.27}$$

here $w = P/\rho$. A possible solution to the horizon problem is a phase in which the comoving Hubble radius was decreasing in the early universe,

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \,. \tag{2.28}$$

In terms of (2.27), this corresponds to a fluid which violates the *Strong Energy Condition*, i.e. $(1+3w) < 0$. In a phase like this, (2.26) is dominated by the lower (integration) limit. In terms of conformal time this means that the initial (conformal) time,

$$\tau_i = \frac{2H_0^{-1}}{1+3w} a_i^{\frac{1}{2}(1+3w)} \,, \tag{2.29}$$

is being pushed to $\tau_i \to -\infty$. Intuitively, one could conclude from this that there should have been much more time between the Big Bang Singularity and the moment of last scattering. This phase of decreasing comoving Hubble sphere is known as *Inflation*.

A rough estimate can be made about the duration of inflation, if we assume that the universe starts with a Big Bang singularity and ends in a radiation dominated epoch. In a radiation dominated universe we have $H \sim a^{-2}$, then

$$\frac{a_0 H_0}{a_E H_E} \sim \frac{a_E}{a_0} \sim \frac{T_0}{T_E} \sim 10^{-28} \,. \tag{2.30}$$

where we used a numerical estimate of $T_E \sim 10^{15}$ GeV and $T_0 \sim 10^{-3}$ eV. For inflation this implies that $(aH)^{-1}$ should shrink by a factor of 10^{28}. If we assume that $H \approx$ constant during inflation, we have that $H_I \approx H_E$, giving us

$$\ln\left(\frac{a_E}{a_I}\right) > 64 \,. \tag{2.31}$$

This suggests that in order for inflation to solve the horizon problem, inflation should last at least 60 e-folds.

There are two other cosmological problems that inflation could be the answer to, these are *the flatness problem* and *the monopole problem.* The flatness problem is the problem which basically asks why the universe is so flat. One would expect a curvature in spacetime, since in general relativity spacetime is dynamical, curving around matter. Inflation solves this problem by flattening the spacetime during the expansion. The monopole problem is the problem why we do not observe magnetic monopoles, which one would expect when looking at the Maxwell equations. Inflation would solve this problem by reducing the magnetic monopole ratio such that its presence would not be observable today.

2.5.1 Physics of inflation

During inflation, a number of conditions are met. The first one is that the universe has an accelerated expansion. We can see this studying the comoving Hubble radius. Taking the first time derivative we find

$$\frac{d}{dt}\left(\frac{1}{aH}\right) = \frac{d}{dt}\left(\frac{1}{\dot{a}}\right) = -\frac{\ddot{a}}{\dot{a}^2}. \tag{2.32}$$

From this we conclude that $\ddot{a} > 0$.

The second condition is that there are a number of dimensionless parameters which we refer to as *Hubble parameters* that are slowly varying during inflation. These Hubble parameters are defined in the following way. Consider again the total time derivative of the comoving Hubble radius

$$\frac{d}{dt}\left(\frac{1}{aH}\right) = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} \equiv -\frac{1}{a}(1-\epsilon)\,, \tag{2.33}$$

then the first Hubble parameter, ϵ, is defined as

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{1}{H}\frac{dH}{dN}\,, \tag{2.34}$$

here we defined $dN = d\ln(a) = Hdt$, which measures the number of e-foldings N during inflation. Note that since the "Hubble sphere" is shrinking during the period of inflation, we have that $\epsilon < 1$. (2.34) implies that the fractional change of the Hubble parameter is small. Since we want the phase of inflationary expansion to last for at least 60 e-folds, we also want that the higher order derivatives of ϵ remain small during inflation. We can define then higher order Hubble parameters in a similar fashion to ϵ,

$$\eta \equiv \frac{\dot{\epsilon}}{\epsilon H} = \frac{1}{\epsilon}\frac{d\epsilon}{dN}\,, \qquad \xi^{(1)} \equiv \frac{\dot{\eta}}{\eta H}, \qquad , \,\dots\,, \qquad \xi^{(n)} \equiv \frac{\dot{\xi}^{(n-1)}}{\xi^{(n-1)}H}\,, \tag{2.35}$$

here the superscript on $\xi^{(n)}$ refers to the $(n-2)^{\text{th}}$ Hubble parameter. In literature, you can find much more of these kinds of parameters, which are also called ϵ, η, etc.. To be consistent, we will only use Hubble parameters. When another slow-roll parameter is used, we will give it a label. For further definitions and conventions, see A.

For perfect inflation, we have that $\epsilon = 0$. In this limit, the metric reduces to the *de Sitter* metric

$$ds^2 = -dt^2 + e^{2Ht}dx^i dx_i\,, \tag{2.36}$$

where $H = \partial_t \ln(a) = $ constant. Since we are not living in a inflating universe, we know that inflation should have ended. Therefore, it should not correspond to a de Sitter space. However, since we assume $\epsilon \ll 1$, line element of (2.36) will be still a good approximation to the inflationary background. For this reason, the inflationary background during inflation is often referred to as a *quasi-de Sitter* spacetime.

2.5.2 Slow-roll inflation

The toymodel that is often used to parametrize inflation contains a canonical scalar fields rolling down a flat potential. The choice for scalar fields comes from the fact that scalar fields are not located in a particular place, they permeate the universe. In a way they represent a smooth background, which may, however couple to other fields in physically interesting ways. Also, for example when using vector fields or other structures, the CMB would acquire a prefered direction. We do not observe this. In this thesis, we will be considering a model where a single canonical scalar field minimally coupled to gravity rolls down a potential a very flat potential. These models are referred to as *slow-roll inflation*-models. The general action for such models is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{2.37}$$

here ϕ is the canonical scalar field that drives inflation, i.e. the *inflaton*. The spacetime dynamics generated by the inflaton are again characterized by the stress -energy tentor, which in the case of (2.37) is given by

$$T_{\mu\nu} = \left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right]g_{\mu\nu} + \partial_\mu\phi\partial_\nu\phi. \tag{2.38}$$

Neglecting all spatial derivative terms, the different components of (2.38) are given by

$$T^{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \qquad T^{0i} = T^{i0} = 0\,, \qquad T^{ij} = \frac{1}{a^2}\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)\delta^{ij}\,. \tag{2.39}$$

If we now match these components to the one of a perfect fluid (2.15), we find

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad\qquad p = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{2.40}$$

Then the Friedmann equations, (2.17), (2.17) and (2.19), become

$$\dot{H} = -\frac{1}{2M_{\rm Pl}^2}\dot{\phi}^2, \tag{2.41}$$

$$\dot{H} + 3H^2 = \frac{1}{2M_{\rm Pl}^2}V(\phi), \tag{2.42}$$

$$3H = -\frac{1}{\dot{\phi}}\left(\ddot{\phi} + V'(\phi)\right)\,. \tag{2.43}$$

Using (2.43), we can relate the Hubble parameters, ϵ and η, to the canonical scalar field by

$$\begin{aligned} \epsilon &= \frac{\dot{\phi}^2}{2M_{\mathrm{Pl}}^2 H^2} \ll \mathcal{O}(1) & (2.44) \\ \eta &= 2\left(\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2}\right) \ll \mathcal{O}(1). & (2.45) \end{aligned}$$

In the general slow-roll regime, the Friedmann equations (2.43) cannot be solved exactly. However, we can solve the equations when a certain potential is specified, or we express the equations in terms of the lowest order Hubble parameters. Since they are all small, the general dynamics is described by a good approximation.

Chapter 3

The inhomogeneous universe

3.1 Cosmological perturbation theory

So far, we considered that the universe was perfectly homogeneous and isotropic, and to a high accuracy, for deviations of the order 1 in 10^5, this is true. However, as mentioned before, there are certain processes that will produce small inhomogeneities. As long as these perturbations are small compared to the perfectly homogeneous and isotropic situation, we can and will treat them in perturbation theory. In this chapter we will first discuss the linear perturbation theory of the "matter" and "metric" perturbations classically and then generalize it to a quantized theory. The review of this section is based on [5] and [6].

3.1.1 Linear perturbation theory

Since the perturbations around a homogeneous and isotropic universe-model are very small, we can model the perturbations using linear perturbation theory. Lets assume we start with (2.7) with zero curvature and denote this unperturbed metric by $\bar{g}_{\mu\nu}$. We can perturb this metric with a small fluctuation around it

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \,, \tag{3.1}$$

here $g_{\mu\nu}$ is the total perturbed metric and $\delta g_{\mu\nu}$ is the small fluctuation. Assuming that $\delta g_{\mu\nu}$ is a general two tensor, we can decompose $g_{\mu\nu}$ as

$$ds^2 = a(\tau)^2 \left[-(1+2A(\mathbf{x},\tau))d\tau^2 + 2\delta_{ij}B^i(\mathbf{x},\tau)dx^j d\tau + (\delta_{ij} + h_{ij}(\mathbf{x},\tau))\, dx^i dx^j\right] \,, \tag{3.2}$$

here A, B_i and h_{ij} are functions of spacetime. We can further decompose the vector B_i by making a Helmholtz decomposition [5],[6] of the form

$$B_i = \partial_i B + \hat{B}_i \,, \tag{3.3}$$

here $\partial_i B$ is the irrotational of B_i and $\hat{B}_i$ is the incompressible part of B_i satisfying $\partial_i \hat{B}^i = 0$. The tensor h_{ij} can also be further decomposed using the *Scalar-Vector-Tensor* (*SVT*) decomposition

$$h_{ij} = 2C\delta_{ij} + 2\partial_{\langle i}\partial_{j\rangle}E + 2\partial_{(i}\hat{E}_{j)} + 2\hat{\gamma}_{ij} \,, \tag{3.4}$$

where

$$\begin{aligned}
\partial_{\langle i}\partial_{j\rangle}E &= \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial^2\right)E, && (3.5)\\
\partial_{(i}\hat{E}_{j)} &= \frac{1}{2}\left(\partial_i\hat{E}_j + \partial_j\hat{E}_i\right), && (3.6)\\
\hat{\gamma}_i{}^i &= 0, && (3.7)\\
\partial^i\hat{E}_i &= 0\,. && (3.8)
\end{aligned}$$

In this decomposition, it is now easy to count the number of degrees of freedom that we have in our theory, i.e. 10 degrees of freedom. We have 4 scalar degrees of freedom, corresponding to A, B, C and E, we also have 4 vector degrees of freedom, corresponding to the two divergenceless vector quantities $\hat{B}_i$ and $\hat{C}_i$ and last not least, we have 2 tensorial degrees of freedom corresponding to the traceless tensor $\hat{\gamma}_{ij}$. Note that in my conventions, the hatted quantities are divergenceless or traceless. The reason for making a SVT-decomposition is because in the Einstein equations the scalar, vector and tensor fluctuations to not mix at linear order [5],[6], thus the different fluctuations can be treated at linear order separately.

We can perturb the stress-energy tensor in a similar fashion to the metric

$$T^{\mu}_{\ \nu} = \bar{T}^{\mu}_{\ \nu} + \delta T^{\mu}_{\ \nu} \tag{3.9}$$

here $\bar{T}^{\mu}_{\nu}$ is the unperturbed part of the stress-Energy tensor and δT^{μ}_{ν} is its perturbation. Since $\bar{T}^{\mu}_{\nu}$ has to take the form of an perfect fluid in a homogeneous and isotropic universe, we can define

$$\bar{T}^{\mu}_{\ \nu} = -(\bar{\rho} + \bar{P})\bar{U}^{\mu}\bar{U}_{\nu} - P\delta^{\mu}_{\ \nu}. \tag{3.10}$$

with $\bar{U}_{\mu} = a(t)\delta_{\mu}^{\ 0}$ the comoving four-velocity, ρ the energy density and P the pressure. The pertubation of the stress-energy tensor can be decomposed as

$$\delta T^{\mu}_{\ \nu} = (\delta\rho + \delta P)\bar{U}^{\mu}\bar{U}_{\nu} + (\bar{\rho} + \bar{P})(\delta U^{\mu}\bar{U}_{\nu} + \bar{U}^{\mu}\delta U_{\nu}) - \delta P\delta^{\mu}_{\ \nu} - \Pi^{\mu}_{\ \nu}\,, \tag{3.11}$$

here Π^{μ}_{ν} is called the *anisotropic stress*, which is negligibly small [5],[6]. The spatial part of the anisotropic stress satisfies $\Pi^{i}_{\ i} = 0$, since its trace can be absorbed into the isotropic pressure.

Perturbations in the four-velocity can introduce a non-vanishing energy flux and momentum density, $T^0\ j \neq 0$ and $T^{i}_{\ 0} \neq 0$ respectively. Using $g_{\mu\nu}U^{\mu}U^{\nu} = 1$ we obtain at linear order

$$\delta g_{\mu\nu}\bar{U}^{\mu}\bar{U}_{\nu} + 2\bar{U}_{\mu}U^{\mu} = 0. \tag{3.12}$$

Using our definitions from (3.2) and defining $\delta U^i \equiv v^i/a$, with $v^i \equiv dx^i/d\tau$ the coordinate velocity, we have

$$\begin{aligned} U^{\mu} &= \frac{1}{a(\tau)}[1 - A(\mathbf{x},\tau), v^i]\,, \\ U_{\mu} &= a(\tau)[1 + A, -(v_i + B_i)]\,. \end{aligned} \tag{3.13}$$

Using (3.11), (3.12) and (3.13) we obtain

$$\begin{aligned} \delta T^{0}_{\ 0} &= \delta\rho\,, \\ \delta T^{i}_{\ 0} &= (\bar{\rho} + \bar{P})v^i\,, \\ \delta T^{0}_{\ j} &= -(\bar{\rho} + \bar{P})(v_j + B_j)\,, \\ \delta T^{i}_{\ j} &= -\delta P\delta^{i}_{\ j} - \Pi^{i}_{\ j}\,. \end{aligned} \tag{3.14}$$

When considering a non-negligible small anisotropic stress, it is convenient to decompose the anisotropic stress tensor using the SVT decomposition,

$$\Pi_{ij} = \partial_{\langle i}\partial_{j\rangle}\Pi + \partial_{(i}\hat{\Pi}_{j)} + \hat{\Pi}_{ij}. \tag{3.15}$$

3.1.2 Gauge freedom

The metric (3.2) that we want to use to describe the spacetime has a subtle problem, it is not uniquely defined. This means that one can make certain transformations or can choose certain coordinates that change the perturbation variables. Let us consider for example a spatial translation of the form $x \to x^i + \xi^i(\mathbf{x}, \tau)$. Our unperturbed flat space FLRW metric (2.7) transforms under this translation to

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + 2\partial_\tau \xi_i d\tilde{x}^i d\tau + \left(\delta_{ij} + 2\partial_{(i}\xi_{j)}\right) d\tilde{x}^i d\tilde{x}^j\right] . \tag{3.16}$$

Comparing (3.2) with (3.16), we note that we have introduced the metric perturbations $B_i \equiv \partial_\tau \xi_i$ and $\hat{E}_i = \xi_i$. These modes are called *gauge modes* and can removed by making a coordinate transformation [5],[8]. This subtlety illustrates that we have to find a better way to represent our metric and that we would like to write it in terms of the true *physical* perturbations, i.e. perturbations that cannot be removed by a coordinate transformation.

In order to find this more convenient representation of the metric, we first need to know how the metric transforms under a general coordinate transformation of the form

$$x^\mu \to x^\mu + \xi^\mu(\mathbf{x}, \tau) , \tag{3.17}$$

here we define $\xi^0 \equiv T$ and we decompose the spatial part again with a Helmholtz decomposition to $\xi^i = L^i = \partial^i L + \hat{L}^i$. Again $\hat{L}^i$ is divergenceless. Using the fact that the spacetime interval ds^2 should invariant under the spacetime transformations,

$$ds^2 = g_{\mu\nu}(X)\ dX^\mu dX^\nu = \tilde{g}_{\alpha\beta}(Y)\ dY^\alpha dY^\beta , \tag{3.18}$$

where X and Y represent the spacetime coordinates. Then

$$g_{\mu\nu}(X) = \frac{\partial Y^\alpha}{\partial X^\mu}\frac{\partial Y^\beta}{\partial X^\nu}\tilde{g}_{\alpha\beta}(Y) . \tag{3.19}$$

Then under the coordinate transformation (3.17), the variables $A,\ B,\ \ldots\ ,\ \hat{h}_{ij}$ transform as [6]

$$\begin{aligned}
&A \to A - T' - \frac{a}{a}T , && \\
&B \to B + T - L' , && \hat{B}_i \to \hat{B}_i - \hat{L}_i' , \\
&C \to C - \frac{a'}{a}T - \frac{1}{3}\partial^2 L , && \\
&E \to E - L , && \hat{E}_i \to \hat{E}_i - \hat{L}_i , \\
&\hat{\gamma}_{ij} \to \hat{\gamma}_{ij} , &&
\end{aligned}$$

here we defined $' \equiv \partial_\tau$. Now that we have the transformations of all variables in (3.2) under (3.17) we can construct the following 4 gauge invariant quantities

$$\Psi \equiv A + \frac{a'}{a}(B - E') + (B' - E'') , \tag{3.20}$$

$$\Phi \equiv -C - \frac{a'}{a}(B - E') + \frac{1}{3}\partial^2 E , \tag{3.21}$$

$$\hat{\Phi}_i \equiv \hat{E}_i' - \hat{B}_i , \tag{3.22}$$

$$\hat{\gamma}_{ij} \equiv \hat{\gamma}_{ij} . \tag{3.23}$$

These variables are known as *Bardeen variables.* In terms of the Bardeen variables, the invariant line element can be written as [8]

$$ds^2 = a(\tau)^2 \left[-(1+2\Phi)dt^2 + 2\hat{\Phi}_i \, dx^i d\tau + ((1-2\Psi)\delta_{ij} + \hat{\gamma}_{ij}) \, dx^i dx^j \right] . \tag{3.24}$$

3.1.3 Choice of gauge

In a theory of gravity, choosing a coordinate system is equivalent to choosing a gauge. As shown in [8], using the freedom in the gauge variable $x_i(\mathbf{x}, \tau)$, we can set two degrees of freedom in our theory to zero [5]. Since models where a canonical scalar field is minimally coupled to gravity also contains scalar degrees of freedom, we are also able to remove for example the scalar field perturbations from the theory

$$\phi = \bar{\phi} + \varphi \, , \tag{3.25}$$

here $\bar{\phi}$ is the field avarage of ϕ and φ is its perturbation. Three gauges that are often used when doing perturbation theory in inflationary models are the *Newtonian gauge*, the *Spatially flat gauge* and the *Comoving gauge.*

In the Newtonian gauge, B and E are set to 0 in such a way that we can write the invariant line element as

$$ds^2 = a(\tau)^2 \left[-(1+2\Psi)d\tau^2 + (1-2\Phi)\delta_{ij} dx^i dx^j \right] . \tag{3.26}$$

In this gauge, the hyperslice surfaces of equal time are orthogonal to the worldlines of that of a static observer. In the absence of anisotropic stress, $\Psi \equiv \Phi$ [5].

In the spatially flat gauge, C and E have been set to zero in such a way that we can write the invariant line element as

$$ds^2 = a(\tau)^2 \left[-(1+2\Phi)d\tau^2 + 2\delta_{ij} B^i(\mathbf{x}, \tau) dx^j d\tau + (\delta_{ij} + \hat{\gamma}_{ij}) dx^i dx^j \right] . \tag{3.27}$$

As the name of this gauge suggests and as we will see later, in this gauge $\mathcal{R} = 0$ and the covariant derivative in the spatial direction reduces to $\nabla_i = \partial_i$.

The comoving gauge is a little more subtle, here we choose our hyperslice surfaces in such a way that we remove the scalar perturbations ($\delta\phi$) from the matter fields $\phi = \bar{\phi} + \delta\phi$. In this gauge, we also set $B = 0$, then the invariant line element is given by

$$ds^2 = a(\tau)^2 \left[-(1+2\Psi)d\tau^2 + (1+2\zeta)(\delta_{ij} + h_{ij}) dx^i dx^j \right] . \tag{3.28}$$

As we will see later, fluctuations in the comoving gauge are most naturally connected to the inflationary initial conditions and can directly be related to the number of e-folds. Note that we have 4 scalar degrees of freedom in the metric, 2 in Ψ, ζ and 1 in h_{ij}.

3.2 ADM-formalism in inflationary cosmology

The simplest slow-roll inflation model is given by (2.37). When doing perturbation theory for canonical scalar fields, it is convenient to choose a gauge. When choosing a gauge, we fix our background metric and indirectly we fix a preferred time-slicing of equal time hypersurfaces of our spacetime manifold $\mathcal{M}$. In doing so, the Poincaré symmetry of our theory is broken in the sense that the spacetime diffeomorphism symmetry breaks up into an unbroken spatial diffeomorphism symmetry and an implicitly broken time diffeomorphism symmetry for certain scalar fields. What is meant by implicitly broken time diffeomorphism symmetry is the following. The full theory is still invariant under the full Poincaré group, but not all fields inside the action respect time diffeomorphisms separately. The reason for choosing such a hypersurface slicing is that it is convenient when quantizing the fields. When quantizing the fields, we use the Hamiltonian formalism, which requires a preferred time direction.

A convenient decomposition for the metric which is often used in inflationary slow-roll models is provided by the Arnowitt-Deser-Misner decomposition, i.e.

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{3.29}$$

where N and N^i are functions of spacetime and are called the *lapse* and the *shift* respectively. The reason for these names will become clear in a moment. Since the ADM metric is non-diagonal and we have two extra functions, N and N^i, this action will not look very nice. To get a more convenient form, we are going to slice up the space-time Manifold into hypersurfaces of equal time. By doing this we will see that we can remove the temporal part of the ricci tensor and no object will appear with more than one time derivative [13]. Since in the original Ricci tensor, these kinds of terms do appear we can see this intuitively as we would have partially integrated these terms. In order to account for this partial integration, we have to correct the action with a total (time) derivative term.

3.2.1 Foliation of spacetime

Let us consider a manifold $\mathcal{M}$ that "lives" in a random 4-dimensional spacetime. This manifold can be sliced up into *space-like* hyper-surfaces of equal time, Σ_t. A schematic picture of this is given in fig.(3.1) and fig.(3.2). These pictures show the slicing up of a (random) curved spacetime. The direction of our time-flow is represented by the vector t^α. Note that t^α is perpendicular to both surfaces in a flat space-time and has a tilt in a curved spacetime. In order to proceed, we want to decompose the vector t^α into a part that is perpendicular to Σ_t and a part that is normal to Σ_t. For a flat space-time this is trivial, since we do not have a 'shift'. For the curved space-time this will become [9],[10]

$$t^\alpha = -(g_{\mu\nu}t^\mu n^\nu)n^\alpha + (t^\alpha + (g_{\mu\nu}t^\mu n^\nu n^\alpha) \equiv (N, N^i), \tag{3.30}$$

here n^α is a *time-like* vector normal to Σ_t, i.e. $n^\alpha = g_{\mu\nu}n^\mu n^\nu = -1$. All *space-like* vectors tangent to Σ_t satisfy $h_{ij}t^it^j \geq 0$. From (3.30) and fig.(3.2) we note that N represents the direction of the time lapse and N^i represents shift in t^α as it goes from Σ_t to Σ_{t+dt}.

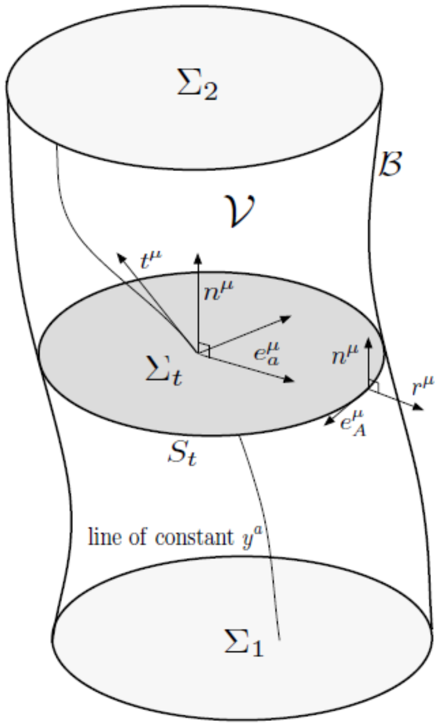

Figure 3.1: A schematic representations of the hypersurface-slicings of a manifold $\mathcal{M}$ into space-like hypersurfaces Σ_t and Σ_{t+dt} of equal time in a curved background. This picture is taken from [11].

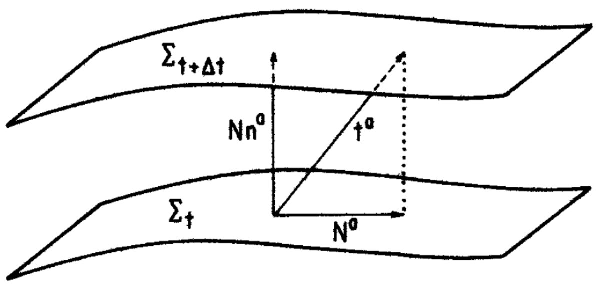

Figure 3.2: A schematic representation of the lapse and shift functions N and N^i respectively. This picture is taken from [9].

To get back to our situation, we would like to make a similar decomposition of our Ricci-scalar $\mathcal{R}$. In order to proceed, we first have to make a decomposition of the Riemann Tensor $R^{\rho}{}_{\sigma\mu\nu}$. The Riemann tensor is defined as

$$R^{\rho}{}_{\sigma\mu\nu}\partial_\rho = \left[\nabla_\sigma, \nabla_\mu\right]\partial_\nu = \nabla_\sigma\nabla_\mu\partial_\nu - \nabla_\mu\nabla_\sigma\partial_\nu\,. \tag{3.31}$$

This object is fully constructed from covariant derivatives ∇_μ, thus in order to make a decomposition of $R^{\rho}{}_{\sigma\mu\nu}$, we have to make a decomposition of $\nabla_\mu v^\alpha$. Here v^α is some vector tangent to Σ_t. Then

$$\begin{aligned} u^\kappa\nabla_\kappa v^\alpha &= -g_{\mu\nu}u^\kappa\nabla_\kappa v^\mu n^\nu n^\alpha + \left(u^\kappa\nabla_\kappa v^\alpha + g_{\mu\nu}u^\kappa\nabla_\kappa t^\mu n^\nu n^\alpha\right) \\ &\equiv K_{ij}u^i u^j n^\alpha + u^\kappa\,{}^{(3)}\nabla_\kappa v^\alpha\,, \end{aligned} \tag{3.32}$$

here K_{ij} is called the *extrinsic curvature*, it describes how the space-time slices Σ_t are embedded into $\mathcal{M}$ and $u^\kappa\,{}^{(3)}\nabla_\kappa$ is the 3-dimensional covariant derivative along some vector field u^κ. If we turn to the basis (n, ∂_i), all indices on tensor structures on $\mathcal{M}$ become Roman indices, since they will coincide with the spatial part of $\mathcal{M}$. Then $\nabla_j\partial_k$ and $\nabla_i\nabla_j\partial_k$ can be written as

$$\begin{aligned} &\nabla_j\partial_k = K_{jk}n + \Gamma^m_{jk}\partial_m \\ &\nabla_i\nabla_j\partial_k = (\partial_i K_{jk} + \Gamma^k_{jk}K_{im}) + (K_{jk}K_i{}^m + \partial_i\Gamma^m_{jk} + \Gamma^n_{kj}\Gamma^l_{in})\partial_m\,, \end{aligned} \tag{3.33}$$

and therefore we can rewrite the Riemann tensor as

$$\begin{aligned} R^{\rho}{}_{ijk}\partial_\rho &= \left[\nabla_i, \nabla_j\right]\partial_k \\ &= \left(\partial_i K_{jk} - \partial_j K_{ik} + \Gamma^m_{jk}K_{im} - \Gamma^m_{ik}K_{jm}\right)n + \\ &\quad + \left(K_{jk}K_i{}^m - K_{ik}K_j{}^m + \partial_i\Gamma^m_{jk} - \partial_j\Gamma^m_{ik} + \Gamma^l_{jk}\,\Gamma^m_{il} - \Gamma^l_{ik}\,\Gamma^m_{jl}\right)\partial_m \\ &\equiv \left(\nabla_i K_{jk} - \nabla_j K_{ik}\right)n + \left(K_{jk}K_i{}^m - K_{ik}K_j{}^m - {}^{(3)}R^m_{ijk}\right)\partial_m\,. \end{aligned} \tag{3.34}$$

The first term is a boundary or surface term. This boundary term is also known as the *Gibbons Hawking York boundary term.* This term becomes important when the spacetime has a boundary such as, for example, in the Schwarzschild description of spacetime around a blackhole or a star. The boundary term in the energy-stress tensor that is often omitted cancels the contribution coming from this term. Since quasi-de Sitter spacetime has a temporal boundary at $\tau \to 0$, we need to subtract this boundary term from the action. Without the boundary term, the Ricci scalar is given by

$$\begin{aligned} R^{(4)} \equiv R^{(4)\ \mu\nu}_{\mu\nu} &= R^{(3)\ ij}_{ij} + K^{ij}K_{ij} - K^i{}_i K^j{}_j \\ &= R^{(3)} + K^{ij}K_{ij} - K^2. \end{aligned} \tag{3.35}$$

We now turn our attention to $\sqrt{-g}$ that appears in the action. To calculate the determinant, it is more convenient to use $\sqrt{|g^{\mu\nu}|^{-1}}$ instead of $\sqrt{|g_{\mu\nu}|}$, since N factorizes and can be pulled out more easily. The inverse metric $g^{\mu\nu}$ is given by

$$g^{\mu\nu} = \begin{pmatrix} \left(\frac{-1}{N^2}\right)_{(1\otimes 1)} & \left(\frac{N^i}{N^2}\right)_{(1\otimes 3)} \\ \left(\frac{N^i}{N^2}\right)_{(3\otimes 1)} & \left(h^{ij} - \frac{N^i N^j}{N^2}\right)_{(3\otimes 3)} \end{pmatrix}, \tag{3.36}$$

here the metric is cut into four pieces, 1 scalar, 2 vectors and 1 tensor, the subscript of $(i \otimes j)$ denotes the dimensions of the object. Then the square root of the determinant can be written as

$$\sqrt{-g} \equiv \sqrt{|g_{\mu\nu}|} = \sqrt{|g^{\mu\nu}|^{-1}} = \sqrt{\left|\left(-\frac{1}{N^2}h^{ij} + \frac{N^iN^j}{N^4}\right) - \frac{N^iN^j}{N^4}\right|} = N\sqrt{|h|}\,, \tag{3.37}$$

here the determinant of $g_{\mu\nu}$ is denoted by $|g_{\mu\nu}| \equiv g$. Using the ADM-metric and using it's decomposition, we can rewrite (2.37) as

$$S = \frac{1}{2}\int d^4x\sqrt{h}\left[M_{\rm Pl}^2 N R^{(3)} - 2NV + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}(\dot{\phi} - N^i\partial_i\phi)^2 - Nh^{ij}\partial_i\phi\partial_j\phi\right]\,, \tag{3.38}$$

here we have defined E_{ij} and E as

$$\begin{aligned} E_{ij} &= \frac{M_{\rm Pl}}{2}(\dot{h}_{ij} - 2\nabla_{(i}N_{j)}) \\ E &= E^i{}_i = h^{ij}E_{ij}\,. \end{aligned} \tag{3.39}$$

and we subtracted the boundary term

$$S_{\text{ADM b.t.}} = \oint_{\partial\Sigma} d^2x\ \sqrt{\gamma}\ \nabla_\alpha(n^\beta\nabla_\beta n^\alpha - n^\alpha K). \tag{3.40}$$

This decomposition and form of the action is used in many papers, for example [13],[14],[19],[20], and shall be used as the starting point of our calculations.

3.2.2 Perturbative expansion constraints

Following section 3.1.1, it turns out that the action (3.38) contains more mathematical degrees of freedom than dynamical ones. The dynamical degrees of freedom are represented by ϕ and h^{ij}, while the non-dynamical degrees of freedom are represented by N and N^i. Therefore we can treat N and N^i as Lagrange multipliers whose equations of motions have to be solved an substituted back into (3.38). It turns out that the solutions for N and N^i cannot be solved exactly. Since we want to express N and N^i in terms of the dynamical scalar perturbations $\delta\phi$ and/or ζ, and both of these quanta are considered of the order $\mathcal{O}\left(10^{-5}\right)$, we can solve them perturbatively order by order. When doing so, a question arises, *If we solve the constraint equations order by order up to order* ζ^n*, to what order in* ζ^n *will our action be correct?*. It turns out that we do not need to solve the constraints to the nth order to get the correct expression for the nth-order action [13],[14],[15],[16]. In [13] it is noted that in order to calculate the cubic order action, S_3, we only need to solve the constraints up to first order in perturbations. Then in [15] it was noted that in order to calculate the nth order action, we could solve the constraints up to order $(n-2)$ and still get the correct action (starting from n=3). In [16], it is wrongfully claimed that in general, starting from $n = 3$, we have to solve the constraints up to order $(n-2)$ in order to get the correct form of the nth order action. In this section we [12] proof that with the $n^{\rm th}$ order constraints, we can actually construct the $(n+1)^{\rm th}$ order action.

Consider a very general Lagrangian that depends on a constraint N and perturbations ζ. Both N and ζ can have derivatives acting on them. The Euler-Lagrange equations of N can be derived in the usual way by varying the action with respect to N

$$
\begin{aligned}
0 = \delta S[N, \partial N, \zeta] &= \int_a^b d^4x \left[\frac{\partial \mathcal{L}}{\partial N}\delta N + \frac{\partial \mathcal{L}}{\partial(\partial_\mu N)}\delta(\partial_\mu N)\right] &(3.41)\\
&= \int_a^b d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial N} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu N)}\right)\right)\delta N + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu N)}\delta N\right)\right], &(3.42)
\end{aligned}
$$

here the last term in (3.42) is a total derivative. This term vanishes after integrating, since the action principle require that all fluctuations of the fields drop at the boundaries, $\delta N(a) = 0 = \delta N(b)$. Then it follows that the *Euler-Lagrange* equations of motion for N are given by

$$
\frac{\partial \mathcal{L}}{\partial N} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu N)}\right) = 0\,. \tag{3.43}
$$

When the field N has no temporal derivatives acting on it, it can be seen as a constraint, i.e. a non-dynamical degree of freedom, in the action. We can eliminate it by solving its equations of motion in terms of dynamical variables, which will turn into constraint equations. After substitution of the solutions of N into the action, we obtain the action parametrized by only dynamical variables.

When we cannot find an exact expression for N in terms of ζ, we can solve N in terms of ζ it perturbatively order by order. If we solve the constraints up to order n, and substitute them back into (3.43) we obtain

$$
\frac{\partial \mathcal{L}}{\partial N}(N_{\le n}, \partial N_{\le n}, \zeta) - \partial_i\left(\frac{\partial \mathcal{L}}{\partial(\partial_i N)}\right)(N_{\le n}, \partial N_{\le n}, \zeta) = 0 + \mathcal{O}(\{N, \partial N, \zeta\}_{\ge n+1})\,, \tag{3.44}
$$

here the first non-zero order corrections appear at order $n+1$ in perturbations. Also, note that the Greek indices turned into Roman indices since we have no temporal ($\partial_0 = \partial_t$) derivatives acting on N (they are non-dynamical). Expanding the Lagrangian in N, we find

$$
\begin{aligned}
\mathcal{L}(N_{\rm sol}, \partial N_{\rm sol}, \zeta) \equiv\; &\mathcal{L}(N_{\le n} + \delta N_{\ge n+1}, \partial N_{\le n} + \partial\delta N_{\ge n+1}, \zeta)\\
=\; &\mathcal{L}(N_{\le n}, \partial N_{\le n}, \zeta) + \delta N_{\ge n+1}\frac{\partial \mathcal{L}}{\partial N}(N_{\le n}, \partial N_{\le n}, \zeta) +\\
&+ \partial_i(\delta N_{\ge n+1})\frac{\partial \mathcal{L}}{\partial(\partial_i N)}(N_{\le n}, \partial N_{\le n}, \zeta) + \mathcal{O}\left((n+1)^2\right)\\
=\; &\mathcal{L}(N_{\le n}, \partial N_{\le n}, \zeta) + \delta N_{\ge n+1}\frac{\partial \mathcal{L}}{\partial N}(N_{\le n}, \partial N_{\le n}, \zeta) -\\
&- \delta N_{\ge n+1}\partial_i\left(\frac{\partial \mathcal{L}}{\partial(\partial_i N)}\right)(N_{\le n}, \partial N_{\le n}, \zeta)\\
&+ \partial_i\left(\delta N_{\ge n+1}\frac{\partial \mathcal{L}}{\partial(\partial_i N)}\right)(N_{\le n}, \partial N_{\le n}, \zeta) + \mathcal{O}\left((n+1)^2\right)\,.
\end{aligned}
$$

In the last step we have made use of a total derivative term, a similar trick to partially integrating the ∂_i. Using (3.44) we obtain the result of interest,

$$\begin{aligned}\mathcal{L}(N_{\text{sol}}, \partial N_{\text{sol}}, \zeta) =& \mathcal{L}(N_{\leq n}, \partial N_{\leq n}, \zeta) + \delta N_{\geq n+1}(0 + \mathcal{O}(\geq n+1)) \\ &+ \partial_i \left(\delta N_{\geq n+1} \frac{\partial \mathcal{L}}{\partial(\partial_i N)} \right) (N_{\leq n}, \partial N_{\leq n}, \zeta) + \mathcal{O}\left((n+1)^2\right) .\end{aligned} \tag{3.45}$$

From this we note that the corrections to the action are correctly solved up to order $2(n+1)-1 = 2n+1$ in perturbations up to a total derivative term. This total boundary term will again give a zero contribution when substituted back into the action, since scalar perturbations damp out at superhorizon scales. Therefore, we conclude that solving the constraints up to n^{th} order in perturbations, we can construct the correct $(2n+1)^{\text{th}}$ order action. This is one of the *new results* of this thesis.

3.2.3 Gauge fixing of the ADM action

At the end of the day, we are interested in quantities that we can relate to the CMB, i.e. quantities that will "freeze out" after their wavelength surpasses the Hubble radius. Weinberg showed [5] that there are two of such quantities, namely $\mathcal{R}$ and γ_{ij}, here $\mathcal{R}$ is the *Comoving curvature perturbation* and γ_{ij} is the *induced spatial metric*. In the language of (3.2), the induced spatial metric is given by

$$\gamma_{ij} = a^2 \left[(1+2C)\delta_{ij} + 2E_{ij}\right] \tag{3.46}$$

In order to find the second conserved quantity, we have to study the scalar perturbations in the metric further, more specifically, we are interested in the three dimensional Ricci scalar $\mathcal{R}^{(3)}$ since this gauge invariant quantity captures the curvature of our induced metric. For scalar perturbations, we have

$$E_{ij} = \partial_{\langle i}\partial_{j\rangle} E \,. \tag{3.47}$$

The three dimensional Ricci scalar that one obtains from (3.46) is then given by

$$a^2 \ {}^{(3)}\mathcal{R} = -4\partial^2 (C - 1/3\partial^2 E) \,. \tag{3.48}$$

From this we define the *curvature perturbations* as

$$\zeta \equiv C - 1/3\partial^2 E \,. \tag{3.49}$$

To get the comoving curvature perturbation, it is convenient to go to the comoving gauge. In this gauge, B and v are zero. Therefore we are free to add any combination of these quantities to the curvature perturbation. As it turns out [6], the gauge invariant combination of B and v is given by $(a'/a)(B+v)$. Adding this to the curvature perturbation, one obtains the *comoving curvature perturbation*

$$\mathcal{R} = C - \frac{1}{3}\partial^2 E + \frac{a'}{a}(B+v) \,. \tag{3.50}$$

In the next chapters of this thesis, we shall study perturbation theory in the comoving and in the *spatially flat gauge*. In the *comoving gauge*, we have that the comoving curvature perturbation is equal to the comoving curvature perturbation, since B and v are zero,

$$\mathcal{R} = C - \frac{1}{3}\partial^2 E \equiv \zeta \,. \tag{3.51}$$

In the spatially flat gauge the comoving curvature perturbation is given by

$$\mathcal{R} = \frac{a'}{a}(B+v)\,. \tag{3.52}$$

To relate $\mathcal{R}$ to the inflaton perturbation φ by the perturbations of the stress-energy tensor, which in the spatially flat gauge is given by

$$\delta T^0_{\ j} = -(\bar{\rho} + P)\partial_j(B+v)\,, \tag{3.53}$$

and

$$\delta T^0_{\ j} = \frac{\bar{\phi}'}{a^2}\partial_j\varphi\,. \tag{3.54}$$

From these two equations, we obtain the relation

$$B + v = -\frac{\varphi}{\bar{\phi}'}\,. \tag{3.55}$$

Substituting (3.55) back into (3.52) we find that the comoving curvature pertubation is given by

$$\mathcal{R} = -\frac{H}{\dot{\bar{\phi}}}\varphi\,. \tag{3.56}$$

Usually, in the spatially flat gauge, we calculate φ correlation functions. Once these perturbations become super Hubble size, one can go over to the comoving gauge by making a gauge transformation of the form $t \to \tilde{t} = t + T(\mathbf{x}, t)$. Up to second order in perturbations, the gauge transformation between the spatially flat and the comoving gauge is given by [13]

$$\begin{aligned}\zeta =& \frac{\dot{\rho}}{\dot{\bar{\phi}}}\varphi - \frac{1}{2}\frac{\dot{\rho}\ddot{\bar{\phi}}}{\dot{\bar{\phi}}^2}\varphi^2 + \frac{\dot{\rho}}{\dot{\bar{\phi}}^2}\varphi\dot{\varphi} + \frac{\ddot{\rho}}{\dot{\bar{\phi}}^2}\varphi^2 - \frac{1}{2\dot{\bar{\phi}}}\partial_k\chi\partial^k\varphi - \frac{1}{4}\frac{e^{-2\rho}}{\dot{\bar{\phi}}^2}\partial_k\varphi\partial^k\varphi \\ &+ \frac{1}{4}\frac{e^{-2\rho}}{\dot{\bar{\phi}}^2}\partial^{-2}\partial_k\partial_l(\partial^k\varphi\partial^l\varphi) + \frac{1}{2\dot{\bar{\phi}}}\partial^{-2}\partial_k\partial_l(\partial^k\varphi\partial^l\varphi)\,.\end{aligned} \tag{3.57}$$

Since the calculation of this gauge transformation is very technical, it is moved to B. In the derivation, [13] is followed closely.

3.3 Quantum field theory in Curved space time

As mentioned, we are interested in calculating (primordial) correlation functions in this thesis. When calculating higher order correlation functions of scattering processes in usual quantum field theories, one usually starts with an initial state at some initial time, t_i then evolves it from $t = -\infty$ to $t = \infty$ and then evaluates its overlap with the final state. The methods that are being used in inflationary cosmologies are not quite the same, but are rather similar to it. In the usual quantum field theory, it is assumed that both states in the expectation value have a fixed value. When calculating correlation functions in a curved spacetime, this can become rather problematic, since both states will become time dependent [18],

$$\langle \Omega(t)|\hat{Q}(t)|\Omega(t)\rangle \,. \tag{3.58}$$

At the start of inflation, we can define a so called Bunch-Davies vacuum state, which is a vacuum state defined in a asymptotically flat background, that we can use to have a fixed value for our vacuum state (more about this later). However, we cannot do this for the final state at $t \to \infty$, since we are working with a time dependent background. Therefore defining a vacuum state at the end of inflation can become rather problematic. The *Keldysh-Schwinger*-Formalism or *In-in*-formalism is constructed to deal with this problem. In the In-in formalism we calculate *expectation values* of operators using only the *initial cauchy data*, in such a way that we do not need to use the final states of the system. In the In-in formalism, the scattering matrix elements are given by

$$\langle \Omega(t_i)|\hat{Q}(t)|\Omega(t_i)\rangle = \sum_f \langle \Omega(t_i)|\Omega(t_f)\rangle\langle \Omega(t_f)|\hat{Q}(t)|\Omega(t_i)\rangle \,, \tag{3.59}$$

here $|\Omega(t_i)\rangle$ is the vacuum state defined at the far past $t \to \infty$ and the summation over $|\Omega(t_f)\rangle$ represents the sum over all possible out states. In this section, I will review this In-in formalism. This review is based on [18] and [14].

3.3.1 Keldysh-Schwinger formalism

To understand how the time-evolution in correlation functions is generated in the *In-in* formalism, we start by considering the action for a generic scalar field ϕ

$$S = \int \mathrm{d}^4x \; \mathcal{L}(\phi(\mathbf{x},t), \dot{\phi}(\mathbf{x},t)) = \int \mathrm{d}t L \,. \tag{3.60}$$

As usual, the Hamiltonian is obtained by

$$H[\phi(t), \pi(t)] = \int \mathrm{d}^3x \; \dot{\phi}\pi_\phi - L \,, \tag{3.61}$$

with π_ϕ the canonical momenta for ϕ,

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \,. \tag{3.62}$$

In quantum theory, the fields ϕ and π satisfy the following equal time commutation relation

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta(\mathbf{x}-\mathbf{y})\,. \tag{3.63}$$

Commutators between two field operators or between two of their canonical momentum operators vanish. Next we decompose our fields in the following way

$$\phi(t,\mathbf{x}) = \bar{\phi}(t) + \delta\phi(t,\mathbf{x}), \tag{3.64}$$

here $\bar{\phi}(t,\mathbf{x})$ is the mean value of $\phi(t,\mathbf{x})$ and $\delta\phi(t,\mathbf{x})$ are perturbations around this value. Then the Hamiltonian splits into a part containing all the field averages, $\bar{H}$ and into a part containing all the perturbations, $\tilde{H}$. Then the Hamiltonian becomes

$$H[\phi(t),\pi(t)] = \bar{H}[\bar{\phi}(t),\bar{\pi}(t)] + \int \mathrm{d}^4x \frac{\partial\mathcal{H}}{\partial\phi(\mathbf{x},t)}\delta\phi(\mathbf{x},t) + \int \mathrm{d}^4x \frac{\partial\mathcal{H}}{\partial\phi(\mathbf{x},t)}\delta\pi(\mathbf{x},t) + \tilde{H}[\delta\phi(t),\delta\pi(t)]\,. \tag{3.65}$$

Next we continue with the part of the Hamiltonian containing the perturbations. We split this perturbed Hamiltonian into a quadratic part (the free part of the perturbed Hamiltonian), which we shall denote by H_0 and into a part that contains all the higher order contributions, which we shall denote by H_I,

$$\tilde{H}[\delta\phi(t),\delta\pi(t)] = H_0[\delta\phi(t),\delta\pi(t)] + H_I[\delta\phi(t),\delta\pi(t)]\,. \tag{3.66}$$

Assuming $\hat{Q}(t)$ in (3.58) is a multiplication of multiple scalar perturbations $\delta\phi$, H_0 is used to evolve $\hat{Q}(t)$. The Heisenberg equations of motion of the perturbations $\delta\phi$ and $\delta\pi$ are given by

$$\begin{aligned} \delta\dot{\phi}(\mathbf{x},t) &= i[H_0[\delta\phi(t),\delta\pi(t)],\delta\phi(\mathbf{x},t)] \\ \delta\dot{\pi}(\mathbf{x},t) &= i[H_0[\delta\phi(t),\delta\pi(t)],\delta\pi(\mathbf{x},t)]\,. \end{aligned} \tag{3.67}$$

The interaction part of $\tilde{H}$, i.e. H_I, is then used to evolve the state $|\Omega(t)\rangle$. The evolution of a state is then found by solving the equation

$$i\frac{d}{dt}|\Omega(t)\rangle = H_I(t)|\Omega(t)\rangle\,. \tag{3.68}$$

(3.68) suggest that we can introduce a time-evolution operator $U_I(t,t')$ that evolves any state from t' to t,

$$|\Omega(t)\rangle = U_I(t,t')|\Omega(t')\rangle\,. \tag{3.69}$$

This operator, U_I, can be found by solving

$$\frac{d}{dt}U_I(t,t') = -iH_I(t)U_I(t,t')\,. \tag{3.70}$$

The solution of (3.70) is given by,

$$U_I(t,t') = \hat{T}\left[e^{-i\int_{t'}^{t} d\tilde{t} H_I(\tilde{t})}\right]\,, \tag{3.71}$$

here $\hat{T}$ denotes the *Time-ordering* operator in such a way that fields occurring at later times are placed to the left of those occuring at earlier times. For example, consider the time ordering of two fields $\zeta(t, \mathbf{x})$ and $\zeta(t', \mathbf{y})$, then

$$\hat{T}\left[\zeta(t, \mathbf{x})\zeta(t', \mathbf{y})\right] = \Theta(t - t')\zeta(t, \mathbf{x})\zeta(t', \mathbf{y}) + \Theta(t' - t)\zeta(t', \mathbf{y})\zeta(t, \mathbf{x}) \,, \tag{3.72}$$

here $\Theta(t - t')$ is the *Heavyside function.* Using (3.68), we can rewrite (3.70) into the following more intuitive form

$$\langle\Omega(t)|\hat{Q}(t)|\Omega(t)\rangle = \langle\Omega(t_0)|U_I^\dagger(t, t_0)\hat{Q}(t)U_I(t, t_0)|\Omega(t_0)\rangle. \tag{3.73}$$

In quantum field theories, we usually write down correlation functions diagrammatically, where time flows from left to right in the diagram. In our case, presenting correlation functions diagrammatically can be rather confusing, since our initial and final time will be the same. Therefore, it is more convenient to stick to the notation of (3.73). Using the fact that $U_I(t, t')$ is an unitary operator, especially the fact that

$$U_I^\dagger(t, t_0) = U_I(t_0, t), \tag{3.74}$$

we note that in (3.73) our system starts in a state at t_0, then it evolves forward until t where the operator $\hat{Q}$ occurs and then the system evolves back to the initial state again at t_0. Using the *unitary* property of the evolution operator,

$$\mathbb{I} = U_I^\dagger(\infty, t)U_I(\infty, t), \tag{3.75}$$

we can extend (3.73) into the infinite future

$$\begin{aligned}
\langle\Omega(t)|\hat{Q}(t)|\Omega(t)\rangle &= \langle\Omega(t_0)|U_I^\dagger(t, t_0)\ \hat{Q}(t)\ U_I(t, t_0)|\Omega(t_0)\rangle \\
&= \langle\Omega(t_0)|U_I^\dagger(t, t_0)\mathbb{I}\ \hat{Q}(t)\ U_I(t, t_0)|\Omega(t_0)\rangle \\
&= \langle\Omega(t_0)|U_I^\dagger(t, t_0)U_I^\dagger(\infty, t)U_I(\infty, t)\ \hat{Q}(t)\ U_I(t, t_0)|\Omega(t_0)\rangle \\
&= \langle\Omega(t_0)|U_I^\dagger(\infty, t_0)U_I(\infty, t)\mathbb{I}\ \hat{Q}(t)\ U_I(t, t_0)|\Omega(t_0)\rangle.
\end{aligned} \tag{3.76}$$

When looking closely to the last line of (3.76), we see that the system starts in an initial state t_0, then evolves to t where the field operators occur, then the systems evolves further to $t \to \infty$ and the goes all the way back to the initial state at t_0. This evolution of the system is represented in fig.(3.3) Note that the forward and backward parts of the contour are treated as two distinct parts of this continuous contour, rather than retreading the same part in two directions. This smart representation of the arcs makes it possible to write the expectation value of operators in terms of one single time-ordered expression.

To make this theorem more explicit for our perturbations $\delta\phi$, let us split the perturbation into two parts

$$\delta\phi(t, \mathbf{x}) = \delta\phi^+(t, \mathbf{x}) + \delta\phi^-(t, \mathbf{x}) \,, \tag{3.77}$$

here $\delta\phi^+(t, \mathbf{x})$ and $\delta\phi^-(t, \mathbf{x})$ are the parts of $\delta\phi(t, \mathbf{x})$ that are located on the positive and negative contour respectively. When calculating correlation functions, we will encounter time-ordered structures of multiple $\delta\phi$. Then the time-ordering of the positive and negative modes

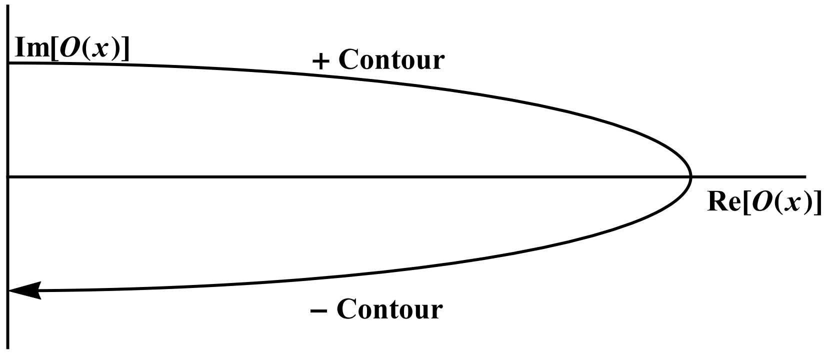

Figure 3.3: The contour in the complex plane over which should be integrated. The contour starts at $0 + i\epsilon$, moves to $= \infty$ and returns at $0 - i\epsilon$. Note that we integrate clockwise.

will be the following

$$\hat{T}\left[\delta\phi^+(t,\mathbf{x})\delta\phi^+(t',\mathbf{y})\right] = \Theta(t-t')\delta\phi^+(t,\mathbf{x})\delta\phi^+(t',\mathbf{y}) + \Theta(t'-t)\delta\phi^+(t',\mathbf{y})\delta\phi^+(t,\mathbf{x})\,, \quad (3.78)$$

$$\hat{T}\left[\delta\phi^+(t,\mathbf{x})\delta\phi^-(t',\mathbf{y})\right] = \delta\phi^-(t',\mathbf{y})\delta\phi^+(t,\mathbf{x})\,, \quad (3.79)$$

$$\hat{T}\left[\delta\phi^-(t,\mathbf{x})\delta\phi^+(t',\mathbf{y})\right] = \delta\phi^-(t,\mathbf{x})\delta\phi^+(t',\mathbf{y})\,, \quad (3.80)$$

$$\hat{T}\left[\delta\phi^-(t,\mathbf{x})\delta\phi^-(t',\mathbf{y})\right] = \Theta(t'-t)\delta\phi^-(t,\mathbf{x})\delta\phi^-(t',\mathbf{y}) + \Theta(t-t')\delta\phi^-(t',\mathbf{y})\delta\phi^-(t,\mathbf{x}). \quad (3.81)$$

Applying this on the evolution operators, we note that all of the operators in $U_I(\infty,t)\ \hat{Q}(t)\ U_I(t,t_0)$ are located on the "+ contour", thus they should all be written as positive fields, i.e.

$$U_I(\infty,t)\ \hat{Q}(t)\ U_I(t,t_0) = \hat{T}\left[Q^+(t)e^{-i\int_{t_0}^{\infty}d\tilde{t}\ H_I^-(\tilde{t})}\right]. \quad (3.82)$$

Similar arguments hold for the conjugate of (3.82), here the operators will be located on the "-contour". Then we can expand (3.58) as

$$\begin{aligned}\langle\Omega(t)|\hat{Q}(t)|\Omega(t)\rangle \approx& \langle\Omega(t_0)|\hat{Q}(t)|\Omega(t_0)\rangle - \\ & - i\int_{t_0}^{\infty} d\tilde{t}\langle\Omega(t_0)|\hat{T}[\hat{Q}^+(t_0)H_I^+(\tilde{t}) - \hat{Q}^+(t_0)H_I^-(\tilde{t})]|\Omega(t_0)\rangle + \mathcal{O}(H_I^2).\end{aligned} \quad (3.83)$$

For each term, one takes the Wick contractions of the fields from all the possible connected diagrams. The difference now compared to the usual *in-out* formalism is that there now are two types of fields, $\zeta^+(t,\mathbf{x})$ and $\zeta^-(t,\mathbf{x})$ and correspondingly four possible Wick contractions

$$\overbracket{\delta\phi^\pm(t,\mathbf{x})\delta\phi^\pm}(t',\mathbf{y}) = \langle\Omega(t_0)|\hat{T}[\delta\phi^\pm(t,\mathbf{x})\delta\phi^\pm(t,\mathbf{x})]|\Omega(t_0)\rangle \equiv G^{\pm\pm}(t,\mathbf{x};t',\mathbf{y}), \quad (3.84)$$

here $G^{\pm\pm}(t,\mathbf{x};t',\mathbf{y})$ has been defined as

$$G^{++}(t,\mathbf{x};t',\mathbf{y}) = \Theta(t-t')G^>(t,\mathbf{x};t',\mathbf{y}) + \Theta(t'-t)G^<(t,\mathbf{x};t',\mathbf{y}), \quad (3.85)$$

$$G^{+-}(t,\mathbf{x};t',\mathbf{y}) = G^<(t,\mathbf{x};t',\mathbf{y}), \quad (3.86)$$

$$G^{-+}(t,\mathbf{x};t',\mathbf{y}) = G^>(t,\mathbf{x};t',\mathbf{y}), \quad (3.87)$$

$$G^{--}(t,\mathbf{x};t',\mathbf{y}) = \Theta(t'-t)G^>(t,\mathbf{x};t',\mathbf{y}) + \Theta(t-t')G^<(t,\mathbf{x};t',\mathbf{y}). \quad (3.88)$$

The functions $G^{<}(t,\mathbf{x};t',\mathbf{y})$ and $G^{>}(t,\mathbf{x};t',\mathbf{y})$ are the positive and negative *Wightman* functions and are defined by

$$G^{>}(t,\mathbf{x};t',\mathbf{y}) = \langle\Omega(t_0)|\delta\phi(t,\mathbf{x})\delta\phi(t',\mathbf{y})|\Omega(t_0)\rangle\,, \tag{3.89}$$

$$G^{<}(t,\mathbf{x};t',\mathbf{y}) = \langle\Omega(t_0)|\delta\phi(t',\mathbf{y})\delta\phi(t,\mathbf{x})|\Omega(t_0)\rangle. \tag{3.90}$$

3.3.2 Choice of vacuum

Inflation was first developed for spacetimes with a globally flat background and since general relativity postulates that it is always possible to choose a locally flat frame for any spacetime point, it might be tempting to impose local flatness in the background. Unfortunately, during inflation, length-scales and locality do not have a definite meaning. This is because it will depend on the time when it is imposed. For example, if you define a length-scale on the wavelength of certain fields you run quickly into trouble, since wavelengths that once might have been small compared with the curvature of the background, might later not be so, since they will have been stretched along with the expansion of space.

Another problem arises from the dynamical scale for the inflatory expansion, which is the Hubble scale H, which is usually chosen to be smaller than the Planck scale. At sub-Planckian length scales, the description of nature should by governed by the laws of *Quantum field theory* and *General Relativity.* However, no definite theory has been constructed so far that explains nature at these scales correctly. The standard prescription is to define modes such that in the infinite past $t_0 \to -\infty$ match with the positive energy modes of flat space. This prescription defines what is called the *Bunch Davies* vacuum state [14],[18],

$$|\Omega(t_0)\rangle = |\Omega(-\infty)\rangle \equiv |0\rangle\,. \tag{3.91}$$

3.3.3 Boundary conditions

When calculating the correlation functions given by (3.83), we split the integral term into three separate time-domains. The first domain is the part where the correlation functions are well within the Horizon, the second domain is around the moment of horizon crossing and the third domain starts a couple of e-folds after horizon crossing and ends at the moment when the modes re-enter the horizon again.

For the first domain, we require that the correlation functions should oscillate very fast and that the total contribution to this part of the integral is zero. We can achieve this by shifting the initial states marginally into the complex plain, this gives an exponentially suppression at early times. This is the reason why the arcs in fig.(3.3) are shifted onto the complex plane. This boundary condition is known as the *Hartle-Hawking boundary condition.*

In the second domain, where we evaluate the correlator around horizon crossing, we assume that the Hubble parameter and all slow-roll parameters can be approximated to stay approximately constant. In the last domain when the modes are well outside the horizon, we have to treat the scalar perturbations in the gauges differently. In the *comoving gauge*, the scalar fluctuations ζ freeze out and do not evolve anymore. This contribution can be safely ignored. In the *Spatially flat gauge*, this is not true anymore.

Chapter 4

Cosmological Perturbation theory

4.1 Introduction to non-Gaussianities

The Cosmic Microwave Background contains a lot of information about the primordial universe, allowing us to make precise test of inflationary models via their predictions of the primordial perturbations. The primordial power spectrum, typically expressed in terms of 2-point correlation functions, is well measured. If there are inhomogeneities in our initial conditions, further information might be obtained from higher order correlation functions. If the perturbations are exactly Gaussian, the N-Point correlation functions vanish when N is odd, and then is fully specified in terms of the 2-point correlation function for even N.

In general, all cosmological inflationary models predict some level of non-Gaussianity. Also, non-linear corrections will generate non-Gaussianities in the CMB, even if the spectrum is purely Gaussian. Therefore, the observed non-linearities are a combination of the primordial non-Gaussianities and of second order couplings between modes. A rough estimation of these non-Gaussianities can be parametrized by a parameter f_{NL}^{local}, which is defined by the *Komatsu-Spergel Local form*

$$\Phi(x) = \Phi_G(x) + f_{NL}^{local}(\Phi_G(x)^2 - \langle\Phi_G(x)\rangle^2)\,, \tag{4.1}$$

here $\Phi(x)$ is the Gaussian gravitational potential as defined in the *Newtonian gauge* [5]. It can be related to the source of the temperature anisotropies in the CMB, $\Delta T/T$, which were roughly of the order $\Delta T/T \sim \mathcal{O}(10^{-5})$, at a point $\mathbf{x}$ in the sky, the subscript G denotes the Gaussian parts of the primordial perturbations and the subscript NL refers to non-linear. Generally, second order couplings between modes will be of the order $f_{NL}^{local} \approx \mathcal{O}(1)$, while it has been shown that for standard single field inflation, $f_{NL}^{local} \sim \epsilon \sim \mathcal{O}(10^{-2})$, we will come back to this later.

Non-Gaussianities can produce higher order N-point correlation functions where N is odd. The lowest order correlation functions with N is odd is the 3-point correlation function. This 3-point correlation function is an independent statistic, and can be calculated in the *interaction picture* using the *in-in formalism* as discussed in 3.3.1. This three point correlation function contains information on both the dependence of the *shape* and the *scale* of the momentum triangle. Also, Maldacena has shown that in the squeezed limit, it is possible to determine how the tilt of the power spectrum is influenced by these three point correlation functions [13]. This relation between the squeezed limit and the tilt is known as *the consistency relation*, more about this later.

In this chapter, we study perturbation theory in the *comoving gauge*, we start by analyzing the action and the constraints, we calculate the mode functions that follow from the equations of motion for ζ, the power spectrum and the bispectrum one order higher in the slow-roll parameters than what has previously been done in literature. We then comment on the correctness using the consistency relation first observed by [13].

4.1.1 Komatsu-Spergel Local form

In order to estimate the non-Gaussianities from higher order N-point correlation functions, for example the three point correlation function, we can study that correlation function in terms of the *Komatsu-Spergel Local form*. Consider again the Gaussian field $\Phi(x)$, its Fourier transform can then be written as

$$\Phi(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\Phi(\mathbf{k})e^{i\mathbf{k}.\mathbf{x}}. \tag{4.2}$$

By defining (4.1), we have split the field $\Phi(\mathbf{x})$ into its linear (Gaussian) part $\Phi_G(\mathbf{x})$ and its non-linear part which is the square of its local value $\Phi(\mathbf{x})^2$ minus the variance of its Gaussian part $\langle\Phi_G(\mathbf{x})^2\rangle$. Using these relations, the Fourier transform of (4.1) becomes

$$\begin{aligned}\Phi(\mathbf{k}) &\equiv \Phi_G(\mathbf{k}) + \Phi_{NL}(\mathbf{k}) \\ &= \Phi_G(\mathbf{k}) + f_{NL}^{local}\left(\int \frac{d^3p}{(2\pi)^3}\Phi_G(\mathbf{p}+\mathbf{k})\Phi_G(-\mathbf{p}) - (2\pi)^3\delta^3(\mathbf{k})\langle\Phi(\mathbf{x})^2\rangle\right).\end{aligned} \tag{4.3}$$

The Gaussian two-point correlation function is given by

$$\langle\Phi_G(\mathbf{x})\Phi_G(\mathbf{y})\rangle \equiv \int \frac{d^3k}{(2\pi)^3}P(\mathbf{k})e^{i\mathbf{k}.(\mathbf{x}-\mathbf{y})}, \tag{4.4}$$

here $P(\mathbf{k})$ is called the power-spectrum which is defined as $P(\mathbf{k}) = |\Phi_G(\mathbf{k})|^2$. In momentum space, invariance under parity imposes that the fields have to be real. The reality condition on $\Phi(\mathbf{x})$, i.e. $\Phi^*(\mathbf{k}) = \Phi(\text{-}\mathbf{k})$, now implies that

$$\langle\Phi_G(\mathbf{k}_1)\Phi_G(\mathbf{k}_2)\rangle = (2\pi)^3P(\mathbf{k}_1)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2). \tag{4.5}$$

At leading order, the three point function is given by

$$\langle\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle = \langle\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi_{NL}(\mathbf{k}_3)\rangle + \text{perms} + \text{higher order corrections}. \tag{4.6}$$

Using Wick's theorem to work out the different permutations in (4.6), one obtains

$$\begin{aligned}&\langle\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle = \\ &\quad= f_{NL}^{local}\left\langle\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\left(\int\frac{d^3p}{(2\pi)^3}\Phi_G(\mathbf{p}+\mathbf{k}_3)\Phi_G(-\mathbf{p}) - (2\pi)^3\delta^3(\mathbf{k}_3)\langle\Phi(\mathbf{x})^2\rangle\right)\right\rangle \\ &\quad= f_{NL}^{local}\langle\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\rangle\left(\int\frac{d^3p}{(2\pi)^3}\langle\Phi_G(\mathbf{p}+\mathbf{k}_3)\Phi_G(-\mathbf{p})\rangle - (2\pi)^3\delta^3(\mathbf{k}_3)\left\langle\Phi(\mathbf{x})^2\right\rangle\right) \\ &\qquad+ f_{NL}^{local}\int\frac{d^3p}{(2\pi)^3}\langle\Phi(\mathbf{k}_1)\Phi(\text{-}\mathbf{p})\rangle\langle\Phi(\mathbf{k}_2+\mathbf{p})\Phi(\mathbf{k}_3)\rangle + 2\text{ perms.}.\end{aligned} \tag{4.7}$$

Using the definition of the power-spectrum, the first term on the right hand side of (4.7) vanishes while the second term on the right hand side can be written as $(2\pi)^3P(\mathbf{k}_1)P(\mathbf{k}_2)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)$. Then the expression one obtains for the three point function in terms of (4.1) is then given by

$$\langle\Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle = f_{NL}^{local}\left[2(2\pi)^3P(\mathbf{k}_1)P(\mathbf{k}_2)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) + 2\text{ sym.}\right]. \tag{4.8}$$

From this we conclude that it is possible to rewrite the three point correlation function as a sum over a product of power spectra times the non-linear coupling parameter f_{NL}^{local}. We will use this template form of the bispectrum, to estimate the amount of non-Gaussianties in 4 and 5.

4.2 Mathematical Kung-Fu in the Comoving gauge

In order to calculate correlation functions, it is crucial to understand the dynamics of the dynamical and non-dynamical degrees of freedom in the action. In this section, we give a comprehensive analysis of the action and the constraints in the *comoving gauge*. As it turns out, when deriving the action perturbatively, we end up with certain boundary terms containing the shift constraint. After substitution of the solutions of the constraints, we will see that these boundary terms are mathematically not really boundary terms, but seem to be dynamical terms. When comparing the solution per order of the action with the zero momentum limit, these terms boundary terms are not present, suggesting that the solutions are not continuously connected.

4.2.1 The constraint equations

We start our analyzation with the ADM-action

$$S = \frac{1}{2}\int d^4x\sqrt{h}\left[M_P^2 N R^{(3)} - 2NV + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}(\dot{\phi} - N^i\partial_i\phi)^2 - Nh^{ij}\partial_i\phi\partial_j\phi\right],$$

again, E_{ij} and E have been defined as

$$\begin{aligned} E_{ij} &= \frac{M_{\rm Pl}}{2}\left(\dot{h}_{ij} - 2\nabla_{(i}N_{j)}\right), \\ E &= E^i{}_i = h^{ij}E_{ij}\,. \end{aligned}$$

We decompose the field $\phi(t,\mathbf{x})$ into its field average, $\bar{\phi}(t)$, and fluctuations around it, $\delta\phi(t,\mathbf{x})$, then

$$\phi(t,\mathbf{x}) = \bar{\phi}(t) + \delta\phi(t,\mathbf{x})\,. \tag{4.9}$$

As mentioned before, the action (3.38) has more 'mathematical' degrees of freedom than dynamical ones. When fixing the gauge into the *comoving gauge*, we have

$$\delta\phi = 0\,, \qquad h_{ij} = e^{2\rho}[(1+2\zeta)\delta_{ij} + \hat{\gamma}_{ij}]\,, \qquad \partial_i\hat{\gamma}_{ij} = 0\,, \qquad \hat{\gamma}_{ii} = 0\,, \tag{4.10}$$

here ζ and $\hat{\gamma}_{ij}$ are the *physical* (dynamical) degrees of freedom, ρ is defined in such a way that $\dot{\rho} \equiv H$, ζ parametrizes the scalar fluctuations and $\hat{\gamma}_{ij}$ parametrizes the tensor fluctuations. The non-dynamical degrees of freedom are parametrized by N and N_i, which means that they turn into *Lagrange Multipliers* and their equations of motions turn into constraint equations. An easy way to see what the dynamical degrees of freedom are is by looking at which objects come with (conformal) time derivatives. The exact action, under the conditions (4.9) and (4.10), is then given by

$$S = \frac{1}{2}\int d^4x\sqrt{h}\left[M_P^2 N R^{(3)} - 2NV + N^{-1}(E_{ij}E^{ij} - E^2 + \dot{\bar{\phi}}^2)\right]. \tag{4.11}$$

Varying (4.11) with respect to N and N^i gives us the two constraint equations

$$M_P^2\mathcal{R}^{(3)} - 2V - N^{-2}\left[E_{ij}E^{ij} - E^2 + \dot{\bar{\phi}}^2\right] = 0\,. \tag{4.12}$$

and

$$\nabla_i \left[N^{-1}\left(E^i{}_j - E\delta^i{}_j\right)\right] = 0\,. \tag{4.13}$$

Note that the exact solution for N is given by

$$N_{\rm sol}^2 = \frac{\left(E_{ij}E^{ij} - E^2 + \dot{\bar{\phi}}^2\right)}{\left(R^{(3)} - 2V\right)}\,. \tag{4.14}$$

Using (4.14) and putting $M_{\rm Pl} = 1$, we can rewrite the action as

$$S = \int d^4x\, \sqrt{h} N_{\rm sol}\left[R^{(3)} - 2V(\bar{\phi})\right]\,. \tag{4.15}$$

Since we solved the constraints exactly, to expand (4.15) to any given order, we need to solve $N_{\rm sol}$ up to that same order. When calculating the action in the zero momentum limit, i.e. $N = N(t, \zeta(t))$ and $N_i(t, \zeta(t)$, (4.15) is convenient to work with. This is because the constraint equations become trivial in the absence of spatial derivatives. We will continue with (4.11) and solve the constraints order by order perturbatively. With the $n^{\rm th}$ order solutions to the constraints, we can construct the $(2n+1)^{\rm th}$ order action, as discussed in 3.2.2.

In order to keep nice and clear, lets first start with a couple of definitions that I will use concerning the expansions of the constraints and in the way I define derivatives. My definition for the expansions of the constraints is given by

$$N = 1 + \sum_{j=1}^{n} N^{(j)}, \tag{4.16}$$

$$N_i = \sum_{j=1}^{3}\left(\partial_i\psi^{(j)} + \hat{N}_i^{(j)}\right)\,, \tag{4.17}$$

here the superscript (j) refers to the order in (ζ) perturbations in which the constraints are expanded. The constraints N_i has been decomposed using the Helmholtz decomposition into a irrotational part, $\partial_i\psi^{(j)}$, and into a incompressible part, $\hat{N}_i^{(j)}$. Here $\hat{N}_i$ is traceless, i.e. it satisfies $\partial_i\hat{N}_i = 0$. Also, I define $\partial_i N^i \equiv h^{ij}\partial_i\partial_j$ and $\partial_i N_i \equiv \delta^{ij}\partial_i N_j$.

Next we expand all objects in (4.11) in terms of ζ, N and N^i, but we do not want to substitute the expansions for N and N^i yet. If we turn of gravity, by putting the tensor degrees of freedom to, zero $\hat{\gamma}_{ij} \to 0$, we can expand the variables that depend on the metric, the time derivative of the metric, the *Christoffel symbol* and the *covariant derivative*, as

$$\dot{h}_{ij} = 2e^{2\rho+2\zeta}(\dot{\rho} + \dot{\zeta})\delta_{ij}\,, \tag{4.18}$$

$$\begin{aligned}\Gamma^k_{ij} &= \frac{1}{2}h^{kl}(\partial_j h_{ik} + \partial_i h_{jl} - \partial_l h_{ij}) \\ &= \delta^{kl}(\partial_j\zeta\delta_{il} + \partial_i\zeta\delta_{jl} - \partial_l\zeta\delta_{ij})\,,\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\nabla_i N_j &= \partial_i N_j - \Gamma^k_{ij}N_k \\ &= \partial_i N_j - \partial_j\zeta N_i - \partial_i\zeta N_j + \partial^k\zeta N_k h_{ij}\,.\end{aligned} \tag{4.20}$$

With these relations, we can calculate the more complicated objects in (4.11), which are the Ricci scalar and the terms related to the extrinsic curvature. Since we turned off the graviton when fixing the gauge, the Ricci scalar reduces to

$$R^{(3)} = -2(2\partial_k\partial^k\zeta + \partial_k\zeta\partial^k\zeta) \tag{4.21}$$

Using (4.19) and (4.20), we can rewrite E_{ij}, E^{ij} and E as

$$\begin{aligned} E_{ij} &= e^{2\rho+2\zeta}(\dot{\rho} + \dot{\zeta} - N_k\partial^k\zeta)\delta_{ij} - \partial_{(i}N_{j)} + 2N_{(i}\partial_{j)}\zeta\,, \\ E^{ij} &= e^{-4\rho-4\zeta}\delta^{ik}\delta^{jl}E_{kl}\,, \\ E &= E_{ij}h^{ij} = 3(\dot{\rho} + \dot{\zeta} - N_k\partial^k\zeta) - \partial_k N^k + 2N_k\partial^k\zeta\,. \end{aligned} \tag{4.22}$$

Then $(E_{ij}E^{ij} - E^2)$ is given by

$$\begin{aligned} (E_{ij}E^{ij} - E^2) &= (h^{ik}h^{jl} - h^{ij}h^{kl})E_{ij}E_{kl} \\ &= -6(\dot{\rho} + \dot{\zeta} - N_k\partial^k\zeta)^2 + 4(\dot{\rho} + \dot{\zeta} - N_k\partial^k\zeta)(\partial_k N^k - 2N_k\partial^k\zeta) \\ &\quad + \partial_{(i}N_{j)}\partial^{(i}N^{j)} - 4\partial_{(i}N_{j)}N^{(i}\partial^{j)}\zeta + 4N_{(i}\partial_{j)}\zeta N^{(i}\partial^{j)}\zeta \\ &\quad - (\partial_k N^k)^2 + 4\partial_k N^k N_l\partial^l\zeta - 4(N_k\partial^k\zeta)^2. \end{aligned} \tag{4.23}$$

This is the backbone of our calculation, we can now continue with expanding the N and N^i constraints and solving (4.13) and (4.13) order by order.

4.2.2 Constraints equations at zeroth order

At zeroth order in perturbations, we can write (4.12) as

$$(6\dot{\rho}^2 - \dot{\bar{\phi}}^2 - 2V) = 0\,, \tag{4.24}$$

and (4.13) as

$$\nabla_j\left(\dot{\rho}\delta_i^{\ j}\right) = \partial_i\dot{\rho} = 0\,. \tag{4.25}$$

At this moment, a little comment is in order. In section 2.5.2, we calculated the Friedmann equations using the stress-energy tensor and the Einstein equations. These were dynamical equations. Now at zeroth order in perturbations, we find that (4.12) reduces to those exact same Friedman equations. This means that in the comoving gauge, the Friedmann equations are not dynamical equations of motion, but are Hamiltonian constraint equations. Also, from (4.13) we conclude that Hubble parameter does not dependent on the spatial coordinate x^i.

Also, a good thing to note is the following. In the zero momentum limit, (4.13) is not present, which means that H is not constraint anymore to be only time dependent.

4.2.3 Solution to the constraint equations at first order in perturbations

At first order in perturbations and after removing all lower order terms by using the zeroth order constraint equations, the constraint equations (4.12) and (4.13) are given by

$$-2\partial_k\partial^k\left(\zeta e^{-2\rho} + \dot{\rho}\psi^{(1)}\right) + \frac{1}{2}\dot{\bar{\phi}}^2 N^{(1)} - 3\dot{\rho}\left[\dot{\rho}N^{(1)} - \dot{\zeta}\right] = 0\,, \tag{4.26}$$

and

$$\partial_j \left[\left(1 - N^{(1)}\right) \left(-2\delta_i^j(\dot\rho - \dot\zeta)\right) \right] - \frac{1}{2}\partial_j\partial^j \hat{N}_i^{(1)} = 0\,. \tag{4.27}$$

Now we have three unknowns and three equations (remember that $\hat{N}_i^{(1)}$ is traceless). We can remove $\hat{N}_i^{(1)}$ from (4.27) by contracting both sides with ∂^i. From this we conclude that the first order contribution of $\hat{N}_i^{(1)}$ must be a constant. Since $\hat{N}^{(1)}$ must be first order in perturbations, the only possible solution for $\hat{N}_i^{(1)}$ is

$$\hat{N}_i^{(1)} = 0\,. \tag{4.28}$$

Using (4.28), (4.27) reduces to

$$2\partial_i(\dot\zeta - \dot\rho N^{(1)}) = 0\,. \tag{4.29}$$

This gives us the following solution for $N^{(1)}$

$$N^{(1)} = \frac{\dot\zeta}{\dot\rho}\,. \tag{4.30}$$

Substituting 4.30 into 4.26, we find that the solution for $\psi^{(1)}$ is given by

$$\psi^{(1)} = -\frac{\zeta}{\dot\rho} + \frac{1}{2}\frac{\dot\phi^2}{\rho^2}\partial^{-2}\dot\zeta\,. \tag{4.31}$$

Theoretically, following 3.2.2 we are now able to construct the correct action up to cubic order in perturbations.

Before we continue, lets first discuss the claim that the solutions to the zero momentum limit are not continuously connected to these solutions. In the zero momentum limit and after using (4.24) and (4.25), at linear order in perturbations (4.14) reduces to

$$N = 1 + \frac{3}{(3-\epsilon)}\frac{\dot\zeta}{\dot\rho} - \frac{3\epsilon}{2(3-\epsilon)^2}\left(\frac{\dot\zeta}{\dot\rho}\right)^2 + \mathcal{O}(\zeta^3)\,. \tag{4.32}$$

Also, N^i is not fixed anymore by the constraint equations. Even worse, $N_i = 0$ at linear order since it cannot depend on position. Since (4.31) has a term that depends on $\dot\zeta$ and (4.23) contains a term $\partial_k N^k$, we loose a $\dot\zeta$ contribution, that otherwise would have shown up.

4.2.4 Solution to the constraint equations at second order in perturbations

The second order constraints can be obtained in the same way as before, expand all "big" object up to second order in ζ and then the zeroth and first order constraint equations can be used to subtract the zeroth and first order terms. Then (4.12) and (4.13) become

$$\begin{aligned}
&\left(-12\dot\rho^2 + 2\dot\phi^2\right)N^{(2)} + \left(2\partial_k\zeta\partial^k\zeta - \frac{4}{\dot\rho}\dot\zeta\partial^2\zeta\right) + 6\dot\zeta^2 - 12\dot\rho\partial_k\psi^{(1)}\partial^k\zeta \\
&- 24\dot\rho\dot\zeta N^{(1)} + 18\dot\rho^2(N^{(1)})^2 - 4\dot\zeta\partial^2\psi^{(1)} - 4\dot\rho\partial^2\psi^{(2)} + 8\dot\rho\partial_k\psi^{(1)}\partial^k\zeta \\
&+ 8\dot\rho\partial^2\psi^{(1)}N^{(1)} - \partial_i\partial_j\psi^{(1)}\partial^i\partial^j\psi^{(1)} + (\partial^2\psi^{(1)})^2 - 3\dot\phi^2(N^{(1)})^2 = 0\,,
\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}
&\nabla_j \left[\delta_i^{\ j}(-2\dot{\zeta} + 2\dot{\rho}N^{(1)}) + \partial^2\psi^{(1)}\delta_i^{\ j} - \partial_i\partial^j\psi^{(1)}\right] + \nabla_j \left[\delta_i^{\ j}(2\partial_k\psi^{(1)}\partial^k\zeta 2\dot{\rho}N^{(2)} + \right.\\
&+2\dot{\zeta}N^{(1)} - 4\dot{\rho}(N^{(1)})^2 + \partial^2\psi^{(2)}\delta_i^{\ j} - \partial_i\partial^j\psi^{(2)} - \partial^2\psi^{(1)}N^{(1)}\delta_i^{\ j} + \partial_i\partial^j\psi^{(1)}N^{(1)} + \\
&\left.+\partial_i\psi^{(1)}\partial^j\zeta + \partial^j\psi^{(1)}\partial_i\zeta - 2\partial_k\psi^{(1)}\partial^k\zeta\delta_i^{\ j} - \frac{1}{2}\partial_i\hat{N}^{(2)j} - \frac{1}{2}\partial^j\hat{N}_i^{(2)}\right] = 0\,.
\end{aligned} \tag{4.34}$$

We can solve (4.33) for $\psi^{(2)}$, then

$$\begin{aligned}
\psi^{(2)} = \frac{1}{4\dot{\rho}}\partial^{-2}&\left[-\frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}^2 + (-2\partial_k\zeta\partial^k\zeta - \frac{4}{\dot{\rho}}\dot{\zeta}\partial^2\zeta) - 2\dot{\rho}\frac{\dot{\phi}^2}{\dot{\rho}^2}\partial_k\partial^{-2}\dot{\zeta}\partial^k\zeta + 4\partial_k\zeta\partial^k\zeta\right.\\
&\left.\quad -\partial_i\partial_j\psi^{(1)}\partial^i\partial^j\psi^{(1)} + (\partial^2\psi^{(1)})^2 - 4VN^{(2)}\right]\,,
\end{aligned} \tag{4.35}$$

note that the solution for $N^{(1)}$ has been substituted, but not the solution for $\psi^{(1)}$, since we have seen from the first order constraint equations that we need to be a little bit more careful with $\psi^{(1)}$. Using the the divergenceless condition for $\hat{N}^{(2)}$, we can solve (4.34) for $N^{(2)}$ and $\hat{N^{(2)}}$. Note when solving these equation, $\nabla_i \sim \partial_i + (\Gamma)_i$, here $(\Gamma)_i$ is a Christoffel symbol, which is first order in ζ. Then only the first line on the right hand side of (4.33) is effected by the Christoffel symbols coming from the covariant derivative. The first line has two kinds of terms, one going as $\delta_i^{\ j}$ times some scalar, for notational convenience lets denote it as M for now, and left over terms that I will denote as $M_i^{\ j}$ for now. Then calculating both terms we find

$$\begin{aligned}
\nabla_j\delta_i^{\ j}M &= \partial_i M + \Gamma^j_{jk}\delta_i^{\ k}M - \Gamma^k_{ji}\delta_k^{\ j}M\\
&= \partial_i M + 3\partial_i\zeta M - 3\partial_i\zeta M\\
&= \partial_i M\,,
\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}
\nabla_j M_i^{\ j} &= \partial_i M + \Gamma^j_{jk}M_i^{\ k} - \Gamma^k_{ji}M_k^{\ j}\\
&= \partial_i M + 3\partial_k\zeta M_i^{\ k} - \partial_i\zeta M_k^{\ k}\,.
\end{aligned} \tag{4.37}$$

Using (4.36) and (4.37), we can rewrite (4.34) as

$$\begin{aligned}
0 =&\partial_i\left[2\dot{\rho}N^{(2)} - \partial^2\psi^{(1)}N^{(1)}\right] + \partial_j\left[\partial_i\partial^j\psi^{(1)}N^{(1)} + \partial_i\psi^{(1)}\partial^j\zeta + \partial^j\psi^{(1)}\partial_i\zeta\right]\\
&- 3\partial_k\zeta\partial_i\partial^k\psi^{(1)} + \partial_i\zeta\partial^2\psi^{(1)} - \frac{1}{2}\partial^2\hat{N}_i^{(2)} - \frac{1}{2}\partial_i\partial_j\hat{N}^{(2)j}.
\end{aligned} \tag{4.38}$$

In order to get a more convenient form for (4.34), we can work out the brackets in (4.34) and pull the second order variables to the left. Then

$$2\dot{\rho}\partial_i N^{(2)} - \frac{1}{2}\partial^2\hat{N}_i^{(2)} = -\left(\partial_k N^{(1)}\partial^k\partial_i\psi^{(1)} - \partial_i N^{(1)}\partial^2\psi^{(1)} + \partial^2\zeta\partial_i\psi^{(1)} + \partial_i\partial^k\zeta\partial_k\psi^{(1)}\right)\,. \tag{4.39}$$

Again, we can split (4.39) into a compressible part and into irrotational part. Acting on (4.39) with ∂^i, we find that the irrotational part of the equation is given by

$$N^{(2)} = -\frac{1}{2\dot{\rho}}\partial^{-2}\partial^i \left(\partial_k N^{(1)}\partial^k\partial_i\psi^{(1)} - \partial_i N^{(1)}\partial^2\psi^{(1)} + \partial^2\zeta\partial_i\psi^{(1)} + \partial_i\partial^k\zeta\partial_k\psi^{(1)}\right) \tag{4.40}$$

and the incompressible part is given by

$$\hat{N}_i^{(2)} = 2\partial^{-2}\left(\partial_k N^{(1)}\partial^k\partial_i\psi^{(1)} - \partial_i N^{(1)}\partial^2\psi^{(1)} + \partial^2\zeta\partial_i\psi^{(1)} + \partial_i\partial^k\zeta\partial_k\psi^{(1)}) - 2\dot{\rho}\partial_i N^{(2)}\right). \tag{4.41}$$

Theoretically, following 3.2.2 we are now able to construct the correct action up to quintic order in perturbations. Also, at second order in perturbations, we can have a non-zero $N_i^{(2)}$. However, it is not constraint by any equations anymore.

4.3 Linear dynamics of ζ

In this section, we use the solutions of the constraints to expand the action to linear and quadratic order. We discuss the treatment of the boundary terms and we calculate the equations of motion for ζ, assuming that the fluctuations and slow-roll parameters are small and therefore can be seen as perturbations around a free theory, as discussed in 3.3.1.

4.3.1 Analyzation of the action

Starting with the usual action (3.38), we expand all terms in the action to linear order in ζ. We obtain the following expression for S_1

$$S_1 = \frac{1}{2}\int d^4x\ e^{3\rho}\left[-4e^{-2\rho}\partial_i\partial_i\zeta - 12\dot{\rho}\dot{\zeta} + (6\dot{\phi}^2 - 36\dot{\rho}^2)\zeta + 4\dot{\rho}e^{-2\rho}\partial_i\partial_i\psi^{(1)}\right]. \tag{4.42}$$

We can rewrite (4.42) into a more convenient form by making use of partial integration. Using

$$-\int d^4x\ e^{3\rho}12\dot{\rho}\dot{\zeta} = \int d^4x\ \left[\partial_0\left(-12\dot{\rho}\zeta e^{3\rho}\right) + 12\ddot{\rho}\zeta e^{3\rho} + 36\dot{\rho}^2\zeta e^{3\rho}\right], \tag{4.43}$$

and using (4.43) and (4.24), we obtain

$$S_1 = \frac{1}{2}\int d^4x\ e^{3\rho}\left(-4e^{-2\rho}\partial_i\partial_i\zeta + 4\dot{\rho}e^{-2\rho}\partial_i\partial_i\psi^{(1)}\right) + \int d^4x\ \partial_0\left[-12\dot{\rho}\zeta e^{3\rho}\right]. \tag{4.44}$$

Naively looking at this expression, we only see boundary terms, which is exactly what was predicted in 3.2.2. However, this is not entirely true. When substituting the solution for $\psi^{(1)}$, there appears to be a term in S_1 that is not a full boundary term. The part of the S_1 action that is not a full boundary term is given by

$$S_1 = \int d^4x\ a^3\left(4\dot{\rho}\frac{1}{2}\frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}\right) = \int d^4x\ a^3\left(4H\epsilon\dot{\zeta}\right). \tag{4.45}$$

Having a non-vanishing S_1 action, we will produce so called *tadpole* corrections to the self-energy of the propagator and tadpole corrections to the three point functions. These tadpole corrections produce *logarithmic divergences* at *tree level* at lowest order in the slow-roll expansion. However, we do not observe these kinds of effects in the power spectrum. Also, when calculating the action in the zero momentum limit, no tadpoles appear and the $\tilde{S}_1$ action is just a total boundary term

$$\tilde{S}_1 = -12\int d^4x\ \partial_0\left[H\zeta e^{3\rho}\right], \tag{4.46}$$

here I denote the zero-momentum limit of the action with a tilde. In literature, [13],[14], these terms are treated as boundary terms and are therefore omitted.

From 3.2.2 we know that in order to calculate S_2, we only have to expand the constraints to first order in perturbations. After performing a couple of partial integrations, we obtain the following cubic order action for ζ

$$S_2 = \int d^4x \left[e^{3\rho} \frac{1}{2} \frac{\dot{\phi}^2}{\dot{\rho}^2} \dot{\zeta}^2 - e^{\rho} \frac{1}{2} \frac{\dot{\phi}^2}{\dot{\rho}^2} \partial_k \zeta \partial_k \zeta \right] + \tag{4.47}$$

$$- \int d^4x \; \partial_0 \left[e^{\rho} \frac{1}{\dot{\rho}} \zeta \partial_k \partial_k \zeta - 2 e^{3\rho} \dot{\rho} (2 + 6\zeta + 9\zeta^2) \right] + \tag{4.48}$$

$$- \int d^4x \; e^{2\rho} \partial_k \left[\left(2 + \zeta + \frac{\ddot{\rho}}{\dot{\rho}^2} \zeta + \frac{\dot{\zeta}}{\dot{\rho}} \right) \partial_k \zeta - \frac{1}{\dot{\rho}} \zeta \partial_k \zeta \right.$$
$$\left. + e^{\rho} \left(4 \dot{\rho} \zeta \partial_k \psi^{(1)} - \partial^2 \psi^{(1)} \partial_k \psi^{(1)} + \partial_i \psi^{(1)} \partial_i \partial_k \psi^{(1)} \right) \right], \tag{4.49}$$

(4.47) is the usual quadratic order action as found in [13],[14], (4.48) are the total temporal boundary terms and (4.49) are the spatial boundary terms. In literature, [13],[14], these terms are again omitted. However, just as for S_1 there appear to be two total boundary terms that are not actually boundary terms, these are the terms $\partial_k \left[-\partial_i \partial_i \psi^{(1)} \partial_k \psi^{(1)} + \partial_i \psi^{(1)} \partial_i \partial_k \psi^{(1)} \right]$.

4.3.2 Equations of motion and quantization

The linear dynamics of the free theory of ζ can be derived by varying (4.47) with respect to zeta. The equations of motion that follow from this are then given by

$$\zeta'' + 2 \frac{(az)'}{az} \zeta' - \partial_i^2 \zeta = 0, \tag{4.50}$$

here the primes denote derivatives with respect to the conformal time τ which is defined as $dt^2 \equiv a(\tau)^2 d\tau^2$, and z has been defined as $z = \bar{\phi}/H$. This equation is known as the *Muhkanov-Sasaki equation* [5]. In order to solve (4.50), it is convenient to decompose ζ into positive and negative Fourier modes

$$\zeta_I(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \left[a_I^\dagger(\mathbf{k}) u_k^*(t) e^{i\mathbf{k}.\mathbf{x}} + a_I(\mathbf{k}) u_k(t) e^{-i\mathbf{k}.\mathbf{x}} \right], \tag{4.51}$$

where $u_k(\tau)$ and $u_k^*(\tau)$ are called mode functions and $a_I(\mathbf{k})$ and $a_I^\dagger(\mathbf{k})$ are the lowering and raising operators, where $a_I^\dagger(\mathbf{k})$ annihilates the Bunch-Davies vacuum,

$$a_I^\dagger(\mathbf{k}) |\Omega(t_0)\rangle = 0\,. \tag{4.52}$$

In order to solve the equation of motion for ζ, we need to substitute (4.51) into (4.50) and rewrite it into a more convenient form

$$\left(\partial_\tau^2 + k^2 - \frac{(az)''}{az} \right) v_k(\tau) = 0\,. \tag{4.53}$$

here $v_k(\tau)$ is the rescaled Muhkanov field which is defined as

$$v_k(\tau) = \mathrm{a}(\tau) z(\tau) u_k(\tau)\,. \tag{4.54}$$

To solve (4.53), we first want to express $\frac{(az)''}{az}$ in terms of the slow-roll parameters. Then

$$(az)' = a\frac{d}{dt}\left[\frac{a\dot{\bar{\phi}}}{H}\right] \approx a^2\left[\dot{\bar{\phi}} + \frac{\ddot{\bar{\phi}}}{H} + \frac{\partial_t^3\bar{\phi}}{2M_{\rm Pl}^2H^2}\right],$$

here the dot denotes a derivative with respect to t, not τ. From this follows that

$$\begin{aligned}\frac{(az)''}{az} &\approx (aH)^2\left[2 + \left(\frac{\dot{\bar{\phi}}^2}{M_{\rm Pl}^2H^2} + \frac{3\ddot{\bar{\phi}}}{\dot{\bar{\phi}}H}\right) + \left(\frac{\partial_t^4\bar{\phi}}{2M_{\rm Pl}^4H^4} + \frac{2\dot{\bar{\phi}}\ddot{\bar{\phi}}}{M_{\rm Pl}^2H^3} + \frac{\partial_t^3\bar{\phi}}{\dot{\bar{\phi}}H}\right)\right]\\ &\approx \frac{2 + (2\epsilon + \frac{\eta}{2}) + (-2\epsilon^2 + 2\epsilon\eta - \xi^{(2)})}{(1-\epsilon)^2\tau^2}\,.\end{aligned}$$

In order to proceed, we have to make an approximation. When doing the slow-roll approximation, we assume that the slow-roll parameters change adiabatically in time, in such a way that we can assume that they can be approximated as constants in (4.53). Then (4.53) resembles Bessel's differential equation. In order to normalize ζ_I we use the Wronskian condition, which is based on the fact that

$$[\zeta(\mathbf{x},\tau), \pi_\zeta(\mathbf{y},\tau)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{4.55}$$

here $\pi_\zeta(\mathbf{x},\tau)$ is the canonical momentum corresponding to ζ, $\pi_\zeta \equiv \partial\mathcal{L}/\partial\dot{\zeta}$. Using the usual commutation relations for the raising and lowering operators,

$$\left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')\right] = (2\pi)^2\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2)\,. \tag{4.56}$$

we obtain

$$u(\tau)\partial_\tau u^*(\tau) - u^*(\tau)\partial_\tau u(\tau) = -i\,. \tag{4.57}$$

This equation is known as the *Wronskian condition*. The solutions for $u_k(\tau)$ are then given by

$$u(k,\tau) = -\frac{1}{a(\tau)}\frac{H}{\dot{\bar{\phi}}}\sqrt{\frac{-\pi\tau}{4}}\mathrm{H}_\nu^{(1)}(-k\tau)\,, \tag{4.58}$$

here $H_\nu^{(1)}(x)$ is the first *Hankel*-function which is a sum of the J and Y *Bessel*-functions,

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)\,. \tag{4.59}$$

Also, in (4.58) we have introduced a new variable ν, as defined in [5],

$$\begin{aligned}\nu^2 = \frac{(az)''}{az} + \frac{1}{4} &\approx \frac{9 - 6\epsilon + 6\eta + \mathcal{O}(\epsilon^2, \epsilon\eta, \eta^2, \xi^2)}{4(1-\epsilon)^2}\\ &\approx \frac{9}{4} + 3\epsilon + \frac{3}{2}\eta + \mathcal{O}(\epsilon^2, \epsilon\eta, \eta^2, \xi^2)\,.\end{aligned} \tag{4.60}$$

At leading order in slow-roll parameters, we have

$$\nu \approx 3/2 + 3\epsilon + \frac{3}{2}\eta\,. \tag{4.61}$$

Equation (4.58) has the following important properties. The mode functions have an oscillatory behavior within the horizon, $k|\tau| \gg 1$. As it gets stretched out of the horizon $k|\tau| \ll 1$, the amplitude becomes constant and freezes. Physically, this means that when we look at the different superhorizon size-comoving patches of the universe, and ignore all short wavelength contributions, they all evolve classically but with different φ. This can be made explicit by relating

$$\zeta \approx -H\frac{\varphi}{\dot{\phi}} \approx H\delta t\,. \tag{4.62}$$

This e-fold difference is the conserved quantity after the modes exit the horizon, and this conservation remains until the mode re-enters the horizon. This quantity can be related to the temperature fluctuations in the CMB, which is the physical quantity that we can measure.

4.4 Scalar two point correlation function

The two point correlation function for the comoving curvature perturbation is defined in the following way

$$\langle\zeta(\mathbf{k}_1,0)\zeta(\mathbf{k}_2,0)\rangle = \langle\zeta(k_1,0)\zeta(k_2,0)\rangle'\ \delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2) \equiv \frac{2\pi^2}{k_1^3}P_\zeta(k_1)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2)\,, \tag{4.63}$$

here $P_\zeta(k)$ is called the *Powerspectrum* and has been defined as

$$P_\zeta(k) = \lim_{\tau_*\to 0}\left[\frac{k^3}{2\pi^2}u(k,\tau_*)u^*(k,\tau_*)\right]\,. \tag{4.64}$$

In (4.63) a momentum conserving delta functions has been pulled out and the power spectrum only depends on the *length* of the momentum vector. The reason for this is that we have assumed that the perturbations are *statistically isotropic*. Physically, it means that all n-point configurations in the CMB are assumed to be drawn from the same distribution, regardless of their orientation.

A more formal argument is the following. Correlation functions inherit the isometries of the background. During inflation, the background can be described as a quasi-de Sitter spacetime. The isometries that this spacetime imposes on the correlation functions are the following. Correlation functions should be invariant under spatial translations, rotations and should be very close to scale invariant. Translational invariance imposes in momentum space that the correlation functions should be proportional to a momentum conserving delta function. Rotational invariance further restricts correlation functions, in the sense that it only depends on the shape and the size. An easy way to see this is the following. For every field inside our correlation function we have d degrees of freedom, corresponding to d-spacetime dimensions. Thus, in an unrestricted n-point correlation function, we have $d*n$ degrees of freedom. There are d possible translations and every correlation function has n possible rotation axes. Then the number of degrees of freedom we are left with is $d*n-d-n$. For power spectrum, this leaves us with $3*2-3-2=1$ degree of freedom, for the three point function (similar to the power spectrum), this leaves us with $3*3-3-3=3$ degrees of freedom.

Then there is also the approximate scale invariance of the correlation function. For the two point correlation functions, we therefore pull out a factor of $1/k^3$. If the correlation functions was fully scale invariant, P_ζ will not depend on momentum and is just a constant. However, if there are (small) deviations from scale invariance, this will produce a small tilt in the power spectrum. The value that is defined to model this tilt is the spectral index

$$(n_s-1) \equiv \frac{d\ln(P_\zeta)}{d\ln(k)}\,. \tag{4.65}$$

This spectral index has been measured to high accuracy by WMAP and later by the Planck satellite and is given by $n_s = 0.9667 \pm 0.0040$ with $k_* = 0.05$ Mpc^{-1}, here k_* is taken at the moment when the modes become super Hubble size [2].

4.4.1 Asymptotic behavior

In order to catch the late time behavior of the two point correlation function, we need to analytically continuate the Hankel functions around $|k\tau_*| \ll 1$,

$$\mathrm{H}_\nu^{(1)}(-k\tau) = \frac{1}{\pi}\left[-e^{i\pi\nu}\Gamma(-\nu)\left(\frac{-k\tau}{2}\right)^\nu - i\Gamma(\nu)\left(\frac{2}{-k\tau}\right)^\nu\right] + \text{higher contributions}\,.$$

The first term is suppressed as τ^ν when $\tau \ll 1$, therefore the leading order contribution at late times to the mode function is given by

$$u(k,\tau) = \sqrt{\frac{1}{2}\frac{H}{\partial_t\phi}}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\frac{1}{a(\tau)}\sqrt{-\tau}(-k\tau)^{-\nu}\,. \tag{4.66}$$

In this limit, (4.63) becomes

$$\langle\zeta(k_1,\tau)\zeta(k_2,\tau)\rangle = \frac{(2\pi)^5}{2k_1^3}\left(\frac{2^{2\nu-3}}{(2\pi)^2}H^2\left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2(1-\epsilon)^{2\nu-1}\left(\frac{H}{\dot{\phi}}\right)^2\right)\,.$$

The late time behaviour of the power spectrum is then captured by

$$P_\zeta = \left[\frac{2^{2\nu-3}}{(2\pi)^2}H^2\left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2(1-\epsilon)^{2\nu-1}\left(\frac{H}{\dot{\phi}}\right)^2\right]\,. \tag{4.67}$$

The leading order contribution to the power spectrum is given when $\nu = 3/2$. Then 4.67 reduces to

$$P_\zeta = \frac{H^2}{(2\pi)^2}\left(\frac{H}{\dot{\phi}}\right)^2 = \frac{H^2}{8M_{\rm Pl}^2\epsilon} = \frac{V}{24M_{\rm Pl}^4\epsilon}\,. \tag{4.68}$$

This is the usual result as found in [13],[14].

In order to study the behavior at the earliest moments during inflation, have have to expand the mode functions (4.58) for the limit in which $|k\tau| \gg 1$. In this limit (4.58) behaves as

$$u_\nu(k,\tau) \approx \frac{He^{-ik\tau}}{2\pi\sqrt{4k\epsilon}}\left(1 - i\frac{4\nu^2-1}{8k\tau} - \frac{(4\nu^2-1)(4\nu^2-9)}{2!(8k\tau)^2} - ...\right)\,. \tag{4.69}$$

The power spectrum in this limit is given by

$$P(\tau,k) = \frac{H^2}{(2\pi)^3}\frac{1}{4\epsilon}\left(\frac{2\nu+1}{2(2\nu-1)} + (2\nu+1)^2\frac{(4\nu^2-9)^2}{64k^2\tau^2} + ...\right)\,. \tag{4.70}$$

Again, when taking the limit of $\nu = 3/2$ we find that (4.70) reduces to (4.68). Note that in the last step, we used the following relation in order to get rid of the $1/\tau$ suppression in the second term in (4.69)

$$\tau = -\frac{1}{H(\tau)}\left(\nu - \frac{1}{2}\right)\,. \tag{4.71}$$

At the beginning of inflation, we have that τ is pushed to $-\infty$, therefore the second term in (4.69) is suppressed as $\lim_{\tau\to-\infty} 1/\tau^2$.

4.4.2 Spectral index and running

Before we calculate the spectral index, let us first begin with a formal definition of the spectral index or spectral tilt. The spectral tilt is given by $(n_s - 1)$ and the spectral index is defined to be n_s. These quantities are defined as the logarithmic momentum derivative of the power spectrum. The power spectrum can be split into a scale-independent factor and a part that captures the scale-dependence

$$P_\zeta(k) = P_\zeta(k_*) \left(\frac{k}{k_*}\right)^{n(k)} , \tag{4.72}$$

here k_* is defined to be the moment when the modes become super Hubble size and $n(k)$ is a function that depends on the momentum k and $n(k) \ll 1$. We can bring $n(k)$ down by taking the logarithm of it. If we then expand this function $n(k)$ around k_* we find

$$\ln(P_\zeta(k)) = \ln(P_\zeta(k_*) + n(k_*) \ln\left(\frac{k}{k_*}\right) + \frac{1}{2} \left.\frac{dn(k)}{d\ln(k)}\right|_{k=k_*} \ln\left(\frac{k}{k_*}\right)^2 + \mathcal{O}\left(\ln\left(\frac{k}{k_*}\right)^3\right) . \tag{4.73}$$

Using the usual definition of the spectral index and its first derivative, which we call the *running* of the potential, α_s we can rewrite (4.73) as

$$\ln(P_\zeta(k)) = \ln(P_\zeta(k_*) + (n_s - 1) \ln\left(\frac{k}{k_*}\right) + \frac{1}{2}\alpha_s \ln\left(\frac{k}{k_*}\right)^2 + \mathcal{O}\left(\ln\left(\frac{k}{k_*}\right)^3\right) . \tag{4.74}$$

To calculate the leading order contributions in slow-roll parameters to the spectral index, it is sufficient to start from the leading order power spectrum, (4.68),

$$\begin{aligned}(n_s - 1) = \frac{d\ln(P_\zeta)}{d\ln(k)} &= \frac{d\ln(V)}{d\ln(k)} - \frac{d\ln(\epsilon_v)}{d\ln(k)} \\ &= -\frac{M^2(\partial_\phi V)^2}{V^2} - \frac{\partial_\phi V}{3H^2}\frac{\dot\epsilon}{H\epsilon} \\ &= -2\epsilon - \eta ,\end{aligned} \tag{4.75}$$

In order to calculate the higher order contributions to the spectral index, we need to start from taking derivatives of the exact expression for the power spectrum, (4.67),

$$\begin{aligned}(n_s - 1) = \frac{d\ln(P_\zeta)}{d\ln(k)} &= \frac{d}{d\ln(k)} \ln\left[\frac{2^{2\nu-3}(2\pi)^2}{H^2}\left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2 (1-\epsilon)^{2\nu-1}\left(\frac{H}{\dot\phi}\right)^2\right] \\ &= \frac{d}{d\ln(k)} \ln\left[\frac{2^{2\nu-3}(2\pi)^2}{H^2}\left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2 (1-\epsilon)^{2\nu-1}\right] + \frac{d}{d\ln(k)} \ln\left[\left(\frac{H}{2\pi}\right)^2\left(\frac{H}{\dot\phi}\right)^2\right] .\end{aligned}$$

In order to keep this calculation orderly, we split the calculation in the calculation of two terms and calculate them separately. The contribution from the "first"-term is

$$\begin{aligned}
&\frac{d}{d\ln(k)}\ln\left[\frac{2^{2\nu-3}(2\pi)^2}{H^2}\left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2(1-\epsilon)^{2\nu-1}\right]= \\
&=\frac{d}{d\ln(k)}\left[(2\nu-3)\ln(2)+2\ln(\Gamma(\nu)+(1-e\nu)\ln(1-\epsilon_v)\right] \\
&=\frac{1}{H}(-2\dot{\epsilon}-\dot{\eta})\left(-2+\gamma_E+\log(2)\right)+\mathcal{O}(\epsilon^3,\ \epsilon^2\eta,\ \epsilon\eta^2,\ \eta^3,\ ...)\,,
\end{aligned}$$

here γ_E is the *Euler-gamma constant* which can be approximated as $\gamma_E \approx 0.57721$.

To expand the second-term to second order in slow-roll parameters, we also need to expand the Friedmann equations to the next to leading order. At leading order he $\dot{\phi}$ term dominates the background equation (2.43) and $\ddot{\phi}$ is neglected. When calculating the second order corrections to the spectral index, we cannot neglect this term anymore. Its contributions can be approximated as $\ddot{\phi} \approx H\dot{\phi}(\epsilon - \eta_v)$. With this correction, (2.43) becomes

$$(3-\epsilon+\frac{1}{2}\eta)H\dot{\phi}+V'=0\,. \tag{4.76}$$

Also, the derivative to the next to leading order in slow-roll parameters is given by

$$\frac{d\log(k)}{dt}=\frac{d\log(aH)}{dt}=\frac{\dot{a}}{a}+\frac{\dot{H}}{H}=H\left(1+\frac{\dot{H}}{H}\right)=H(1+\epsilon)\,. \tag{4.77}$$

Then the contribution of the second term is given by

$$\begin{aligned}
\frac{d}{d\ln(k)}\ln\left[\left(\frac{H}{2\pi}\right)^2\left(\frac{H}{\dot{\phi}}\right)^2\right]&=\frac{2\epsilon}{1-\epsilon}-\frac{\dot{\epsilon}_H}{(1-\epsilon)H\epsilon} \\
&=-\frac{2\epsilon}{1-\epsilon}-\frac{\eta}{1-\epsilon} \\
&\approx -2\epsilon-\eta-2\epsilon^2-\epsilon\eta\,.
\end{aligned} \tag{4.78}$$

Combining both results, we find the following expression for the spectral index to second order in slow-roll parameters

$$(n_s-1)=-2\epsilon-\eta-2\epsilon^2-\epsilon\eta-\frac{1}{H}(2\dot{\epsilon}+\dot{\eta})\left(-2+\gamma_E+\log(2)\right)+\mathcal{O}(\epsilon^3,\ \epsilon^2\eta,\ \epsilon\eta^2,\ \eta^3,\ ...)\,. \tag{4.79}$$

Following the definition of the running, (4.74), we can also calculate the leading order running of the potential,

$$\alpha_s \equiv \frac{d(n_s-1)}{d\ln(k)}=-2\epsilon\eta-\eta\xi^{(1)}\,. \tag{4.80}$$

4.5 Tensor two point function

So far, our main focus was on the scalar degrees of freedom inside our metric, however, there are also tensorial degrees of freedom. With the tensor power spectrum P_t, we can determine the tensor to scalar ratio in the CMB. As we will see, this will provide a bound on the value for ϵ, which will be the starting point of the next chapter.

The tensor degrees of freedom are parametrized in our gauge by the transverse traceless quantity $\hat{\gamma}_{ij}$. The cubic action for $\hat{\gamma}_{ij}$ is given by

$$S_{gr}[\hat{\gamma}_{ij}] = \frac{M_{\rm Pl}^2}{8} \int d^3x \; d\tau \; a^2 \left[(\partial_\tau \hat{\gamma}_{ij})^2 - (\nabla \hat{\gamma}_{ij})^2 \right] . \tag{4.81}$$

Following the usual formula for canonical momentum, we find that the canonical momentum of the graviton is given by

$$\hat{\pi}_{ij} = \frac{\delta S_{gr}}{\delta(\partial_\tau \hat{\gamma}_{ij})} = \frac{M_{\rm Pl}^2}{4} a^2 \partial_\tau \hat{\gamma}_{ij} \, . \tag{4.82}$$

The proper canonical quantisation is then given by

$$\left[\hat{\gamma}_{ij}(\mathbf{x}, \tau), \hat{\pi}^{kl}(\mathbf{y}, \tau) \right] = \frac{i}{2} \left[\hat{P}_{il}\hat{P}_{jk} + \hat{P}_{ik}\hat{P}_{ik} - \hat{P}_{ij}\hat{P}_{kl} \right] \delta^{(3)}(\mathbf{x} - \mathbf{y}) \, , \tag{4.83}$$

here $\hat{P}_{ij} \equiv \delta_{ij} - \partial_i \partial_j / \nabla^2$ is the transverse projection operator. Since the graviton has two polarizations, usually denoted by $+$ and $\times$, and both polarizations are governed by the same equation of motion, it is convenient to decompose the gravition in the following way [6]

$$\hat{\gamma}_{ij}(x) = \frac{2}{M_{\rm Pl}} \sum_{\alpha = +, \times} \int \frac{d^3k}{(2\pi)^3} \left[h(k,\tau) \epsilon^{\alpha}_{ij} \hat{a}_{\mathbf{k}\alpha} e^{i\mathbf{kx}} + h^*(k,\tau) \epsilon^{\dagger\alpha}_{ij} \hat{a}^{\dagger}_{\mathbf{k}\alpha} e^{-i\mathbf{kx}} \right] . \tag{4.84}$$

here $\hat{a}_{\mathbf{k}\alpha}$ and $\hat{a}^{\dagger}_{\mathbf{k}\alpha}$ are the annihilation and creation operators where $\hat{a}_{\mathbf{k}\alpha}|\Omega(t_0)\rangle = 0$. They following the following commutation relation

$$\left[\hat{a}_{\mathbf{k}\alpha}, \hat{a}^{\dagger}_{\mathbf{k}'\alpha'} \right] = (2\pi)^3 \delta_{\alpha,\alpha'} \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \qquad [\hat{a}_{\mathbf{k}\alpha}, \hat{a}_{\mathbf{k}'\alpha'}] = 0, \qquad \left[\hat{a}^{\dagger}_{\mathbf{k}\alpha}, \hat{a}^{\dagger}_{\mathbf{k}'\alpha'} \right] = 0 \, , \tag{4.85}$$

and ϵ^{α}_{ij} are the two graviton polarization tensors, which characterise a massless spin two particle.

Varying (4.81) with respect to $\hat{\gamma}_{ij}$ we find that the equations of motion for the mode functions $h(\tau, k)$ are given by

$$\left(\partial_\tau^2 + k^2 - \frac{a''}{a} \right) (a h(\tau, k)) = 0 \, . \tag{4.86}$$

The solutions that follow from (4.86) and the Wronskian condition are given by

$$h(\tau, k) = \frac{1}{a} \sqrt{\frac{-\pi\tau}{4}} \mathrm{H}^{(1)}_{\nu}, \qquad \nu = \frac{3 - \epsilon}{2(1 - \epsilon)} \, . \tag{4.87}$$

Then the power spectrum for the tensor perturbations can be defined in a similar fashion to the scalar power spectrum, then

$$P_{gr}(k) = \left[\frac{k^3}{2\pi^2} h(k, \tau_*) h^*(k, \tau_*) \right] . \tag{4.88}$$

Then the exact expression for the gravition power spectrum is then given by

$$P_{gr}(k,\tau) = \frac{H^2}{\pi^3 M_{\rm Pl}} 2^{\frac{3-\epsilon}{1-\epsilon}}\, \Gamma\left(\frac{3-\epsilon}{2(1-\epsilon)}\right)^2 (1-\epsilon)^{\frac{2}{1-\epsilon}} \left(\frac{k}{H}\right)^{-\frac{2\epsilon}{1-\epsilon}} . \tag{4.89}$$

4.5.1 The Lyth bound at the end of inflation

Similar to the spectral tilt the comoving curvature perturbation, the spectral tilt for the graviton is given by

$$(n_t - 1) \equiv \frac{d\ln(P_{gr})}{d\ln(k)} . \tag{4.90}$$

At leading we obtain $(n_t - 1) = -2\epsilon$. Therefore, the slow-roll parameter ϵ can be related to both the scale dependence of the power spectra and the presence of physical perturbations.

In theory, it is possible to measure at some point the ratio between the tensor and scalar perturbations. In [17] a lower bound was derived on the variation in the inflaton field during inflation in terms of the ratio r between tensor and scalar perturbations generated during inflation,

$$r \equiv \frac{P_t}{P_\zeta} \approx 16\epsilon . \tag{4.91}$$

So far, this ratio has not been measured yet. The constraint this gives on the value of ϵ is called the *Lyth bound*. The current constraints on r and ϵ are $r < 0.07$ and $\epsilon < 0.01$.

4.6 Scalar three point correlation function

In this section we calculate the three point correlation function for ζ. In literature this has already been done at leading order in slow-roll parameters, our goal of this section is to produce the next to leading corrections. When performing this calculation in the usual way, we will stumble onto left over time dependence. We do not expect this time dependence since ζ should freeze out after it leaves the horizon, hence all correlators of ζ should do the same. We explore this left over time dependence with a toy model and then will calculate the correct next to leading order bispectrum.

The three point correlation function is a bit more involved to calculate compared to the two point function. This is because we now need to expand the evolution operators of the vacuum in-in states. For a general operator, the evolution can be realized by a unitary transformation, i.e.

$$\langle Q(t)\rangle = \left\langle \left[\bar{T}\exp\left(i\int_{-\infty(1-i\varepsilon)}^{t} \mathrm{dt}\ H_I(t)\right)\right]\hat{Q}_I(t)\left[T\exp\left(-i\int_{-\infty(1+i\varepsilon)}^{t} \mathrm{dt}\ H_I(t)\right)\right]\right\rangle . \quad (4.92)$$

Here T and $\bar{T}$ refer to time ordering and anti-time-ordering, the subscript I refers to the fact that we are calculating the expectation values of the operators in the *interaction picture*. Using this equation, the 3-point correlation function for $\zeta_n(k)$ in leading order of slow-roll parameters is then given by

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = -i\int_{-\infty(1\pm i\varepsilon)}^{\tau} d\tau'\ a(\tau')\left\langle\left[\zeta(\mathbf{k}_1,0)\zeta(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0), H_I(\tau')\right)\right]\right\rangle . \quad (4.93)$$

Just as the two point function could be written in terms of the *Power spectrum*, a momentum conserving delta function and a scaled out momentum factor by assuming statistical isotropy, the three point function can be written in a similar way. Only in this case, we write it in terms of a *Bispectrum* or a *Shape function* instead of a Power spectrum. The Bispectrum and Shape are defined by

$$\begin{aligned}\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle &\equiv (2\pi)^7\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B(k_1,k_2,k_3)\\ &\equiv (2\pi)^7\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{S(k_1,k_2,k_3)}{(k_1k_2k_3)^2}.\end{aligned} \quad (4.94)$$

When studying the symmetries of three point correlation function, the Bispectrum definition is more convenient to use. When studying the momentum dependence of the momentum triangle, the shape is more convenient to use.

In order to calculate three point correlation functions, we need the part of the interaction Hamiltonian that has an odd number of ζ fields. Since S_1 is a full boundary term that does not give a contribution, the leading order contributions will come from S_3. In [13] it was noted at cubic order in perturbations, we have $H_{\text{int}} = -L_{\text{int}}$. A formal prove of this statement is the following.

A toy model

Consider a Lagrangian density containing an arbitrary potential and an additional term with three time derivatives

$$\mathcal{L} = \frac{1}{2}\dot{\zeta}^2 - V(\zeta, \partial\zeta) + \alpha\dot{\zeta}^3 \,. \tag{4.95}$$

The canonical momentum corresponding to ζ is given by

$$\pi = \frac{\delta\mathcal{L}}{\delta\dot{\zeta}} = \dot{\zeta} + 3\alpha\dot{\zeta}^2 \,. \tag{4.96}$$

This relation can be inverted to find a relation for $\dot{\zeta}$,

$$\dot{\zeta} = \pi - 3\alpha\dot{\zeta}^2 \,. \tag{4.97}$$

Calculating the Hamiltonian explicitly we find

$$\begin{aligned}
\mathcal{H} &= \pi\dot{\zeta} - \mathcal{L} \\
&= \pi^2 - 3\alpha\dot{\zeta}^2 - \frac{1}{2}\left(\pi^2 + 9\alpha^2\pi\dot{\zeta}^2 - 6\alpha\pi\dot{\zeta}^2\right) - \alpha\pi^3 + V(\zeta, \partial\zeta) + \mathcal{O}(\zeta^4) \\
&= \frac{1}{2}\pi^2 + V(\zeta, \partial\zeta) - \alpha\pi^3 + \mathcal{O}(\zeta^4) \,.
\end{aligned}$$

Then from the inverted expression (4.97), it follows that

$$\mathcal{H}_{\rm int} = -\alpha\pi^3 - V_{\rm int,3}(\zeta, \partial\zeta) + \mathcal{O}(\zeta^4) = -\mathcal{L}_{\rm int,3} + \mathcal{O}(\zeta^4) \,. \tag{4.98}$$

Notice that this works only if we write the quadratic term as function of π rather than $\dot{\zeta}$, as we should since we are computing the Hamiltonian. This argument is therefore very general.

4.6.1 Analyzation of the cubic action

Before we calculate the bispectrum to the next order in slow-roll parameters, let us first analyze the action. In order to obtain the action at cubic order in perturbations, one needs to expand the action in terms of ζ. In the comoving gauge, this was done in the seminal work [13]. After performing a number of partial integrations, the result obtained for S_3 is given by

$$\begin{aligned}
S_3 = \int \mathrm{dt}\; d^3x \Big[& a^3\epsilon^2\zeta\dot{\zeta}^2 - 2a\epsilon^2\dot{\zeta}(\partial\zeta)(\partial\chi) + a\epsilon^2\zeta(\partial\zeta)^2 \\
& + \frac{a^3}{2}\epsilon\dot{\eta}\zeta^2\dot{\zeta} + \frac{1}{2}\frac{\epsilon}{a}\partial\zeta\partial\psi^{(1)}\partial^2\psi^{(1)} + \frac{\epsilon}{4}\partial^2\zeta(\partial\psi^{(1)})^2 + f(\zeta)\frac{\delta\mathcal{L}_2}{\delta\zeta}\Big] \,,
\end{aligned} \tag{4.99}$$

here $f(\zeta^2)$ has been defined as

$$f(\zeta^2) \approx \frac{\eta}{4}\zeta^2 + \frac{1}{H}\dot{\zeta}\zeta + \mathcal{O}(\partial\zeta) \,. \tag{4.100}$$

This paper has been quoted many and many times in the past by numerous authors, therefore let us use this action as the starting point of our discussion.

When studying (4.99), we note that the last term looks very compact, but after substitution of $f(\zeta)$ it actually is inconveniently large to work with. Normally speaking, a term multiplying the linear equations of motion of a field will not produce a contribution to any Feynman diagram when it is evaluated on-shell. Therefore one naively can drop these terms when only interested in terms that contribute to Feynman diagrams. However, there is one subtlety that has to be kept in mind. The mode functions of ζ contain *Hankel* functions, and these can only be expressed as analytic functions when the index, ν is half integer. One could argue that we can approximate the mode functions as $\nu \approx 3/2$ and still would get the correct result at leading orders in slow-roll parameter. This was done in [13]. However, this approximation of the mode functions corresponds to the mode functions of a *de Sitter* space, and therefore only solve the *Quasi-de Sitter* equation of motion approximately. Therefore, this term will give a finite contribution when calculating this and cannot therefore not simply be omitted. In [13], these terms were removed by means of a field shift of the form [13],

$$\zeta \to \zeta_n + f(\zeta_n^2)\,, \tag{4.101}$$

$$f(\zeta_n^2) \approx \frac{\eta}{4}\zeta_n^2 + \frac{1}{H}\dot{\zeta}_n\zeta_n + \mathcal{O}(\partial\zeta_n)\,. \tag{4.102}$$

Note that when performing a field shift, we will not be calculating correlation functions in terms of ζ anymore but rather correlation functions in terms of ζ_n. While ζ is conserved outside of the horizon, ζ_n is not. This means that ζ_n correlators will contain left over time dependence. A second thing to note is the following; in order to get from (3.38) to (4.99), numerous partial integrations must have been performed. This can been seen to the fact that $\delta\mathcal{L}/\delta\zeta$ contains two derivatives on one zeta. Since no terms starting from (3.38) have this property, these terms can only be produced by means of partial integration.

To gain a better understanding about the time dependences and the field shift of ζ and ζ_n, let us study a toy model where after performing partial integration, we can subtract a term that scales with the linear equations of motion, by means of a field shift.

A Toy model

Let us construct a toy model based on the action (3.38) and (4.99). We would like to start with a term similar to one in (3.38) and then, by means of partial integration, we would like to construct in it a term similar to one found in (4.99) that only has one or both time derivatives acting on a single ζ. Since we are interested in studying the spurious time dependences, let us therefore consider the following toy model-action,

$$S = \int dt\ \epsilon^2 a^3 \zeta\dot{\zeta}^2 = \int dt\ \left[\partial_t(a^3\epsilon^2\zeta^2\dot{\zeta}) - \zeta\partial_t(a^3\epsilon^2\zeta\dot{\zeta})\right], \tag{4.103}$$

here the dots denote time derivatives ∂_t. As already shown in (4.103), we can move the time dependence from one of the ζ fields to, $a(t)$, $\epsilon(t)$ or another $\zeta(t)$ by means of partial integration. In doing so, we can produce higher order slow-roll parameters, starting from η, and we can create spurious divergences in the bispectrum when $\tau \to 0$. To study these cases separately, let us first study a model in which $\epsilon =$ fixed. Such a model corresponds to a *de Sitter spacetime.* Here we only consider the spurious time divergences at the same order of slow-roll parameters, and afterwards we consider the case in which ϵ has a time dependence. Also, in every calculation of any bispectrum term, similar to what is usually done in literature, we approximate the *Quasi-de Sitter* mode functions by the *de Sitter* mode functions, i.e. by approximating $\nu \approx 3/2$.

Using $H_3 = -L_3$ and assuming that ϵ is fixed, we obtain the following contribution to the bispectrum produced by the *left hand side* of (4.103)

$$\begin{aligned}\langle\zeta^3\rangle' &= -\frac{H^4\epsilon}{32\epsilon^2 k_1^3k_2^3k_3^3}\frac{k_1^3+2k_1k_2^2+k_2^2(k_2+k_3)+k_1^2(2k_2+k_3)}{k_t^2} + 2\text{ perm.} \\ &= -\frac{H^4\epsilon}{32\epsilon^2 k_1^3k_2^3k_3^3}\left[\frac{1}{k_t}\sum_{i<j}k_i^2k_j^2 + \frac{k_1k_2k_3}{k_t^2}\sum_{i<j}k_ik_j\right],\end{aligned} \tag{4.104}$$

here we defined $k_t = \sum_{i=1}^3 k_i$ and the prime on the brackets of the correlation function denote the truncation of a momentum conserving delta function and a factor of $(2\pi)^3$. When working out the brackets on the *right hand side* of (4.103) we obtain the following four terms

$$\int d\tau\ \left[\partial_\tau\left(a^2\epsilon^2\zeta^2\zeta'\right) - a^3\epsilon^2\left(\frac{1}{a}\zeta(\zeta')^2 + 3H\zeta^2\zeta' + \zeta^2\partial_\tau\left(\frac{1}{a}\partial_\tau\zeta\right)\right)\right]. \tag{4.105}$$

Note that we switched to conformal time in this expression and the primes denote again ∂_τ. Another important thing to note is that the last two terms in (4.105) are both in the *de Sitter* equations of motion. Calculating the contributions to the bispectrum in the same order as given in (4.105), we obtain

$$\langle\zeta^3\rangle' \supseteq \frac{H^4}{32\epsilon^2k_1^3k_2^3k_3^3}2\epsilon k_3^2 + \text{ perm.}, \tag{4.106}$$

$$\langle\zeta^3\rangle' \supseteq \frac{H^4\epsilon}{32k_1^3k_2^3k_3^3\epsilon^2}\frac{(k_1^3+2k_1k_2^2+k_2^2(k_2+k_3)+k_1^2(2k_2+k_3)}{k_t^2} + 2\text{ perm.}, \tag{4.107}$$

$$\begin{aligned}\langle\zeta^3\rangle' \supseteq &-\frac{6H^4\epsilon}{32\epsilon^2k_1^3k_2^3k_3^3}\frac{k_3^2\left(k_1^2+(k_2+k_3)(k_2+(1-\gamma_E)k_3)+k_1(k_2+(2-\gamma_E)k_3)-k_3k_t\log(-k_t\tau)\right)}{k_t} \\ &+ 2\text{ perm.},\end{aligned} \tag{4.108}$$

and

$$\begin{aligned}\langle\zeta^3\rangle' \supseteq -&\frac{2H^4\epsilon}{32\epsilon^2k_1^3k_2^3k_3^3}\left(2(k_1+k_2)(k_1^2+k_1k_2+k_2^2+\left((7-3\gamma_E)k_1^2+(13-6\gamma_E)k_1k_2\right.\right. \\ &\left.\left.+(7-3\gamma_E)k_2^2\right)k_3 - (-7+6\gamma_E)(k_1+k_2)k_3^2 + (2-3\gamma_E)k_3^3 - 3k_3k_t^3\log(-k_t\tau)\right). \\ &+ 2\text{ perm.}.\end{aligned} \tag{4.109}$$

Note that when we add (4.106)-(4.109) we obtain (4.104). Another very important fact is the following. When we add (4.108) and (4.109), we find that the time dependence, τ, in both contributions cancel, i.e. they are spurious time dependences. Adding the "spatial" kinetic term of the action, the entire contribution vanishes. Therefore we conclude, that when we use the *de Sitter* solutions to the modefunctions, the terms multiplying the *de Sitter* equations of motion will formally give a zero contribution to the bispectrum.

Let us now turn on the time dependence of ϵ. We will assume that ϵ varies slowly in time. Then after partial integration, the *right hand side* of (4.103) will now contain an additional term,

$$\int d\tau \ \left[\partial_\tau \left(a^2\epsilon^2\zeta^2\zeta'\right) - a^3\epsilon^2 \left(3H\zeta^2\zeta' + \frac{1}{a}\zeta(\zeta')^2 + \zeta^2\partial_\tau\left(\frac{1}{a}\partial_\tau\zeta\right) + 2\eta H\zeta^2\zeta'\right)\right]. \tag{4.110}$$

Naively, the extra contribution to the bispectrum the last term produces in (4.110) compared to (4.105) is given by

$$\langle\zeta^3\rangle \supseteq \frac{H^4\epsilon\eta}{16\epsilon^2 k_1^3k_2^3k_3^3}\left((-1+\gamma_E+\log(k_t\tau))\sum_{i=1}^{3}k_i^3 - \sum_{i\neq j}k_ik_j^2 + k_1k_2k_3\right). \tag{4.111}$$

Again, we find a time-dependent part in the bispectrum. Only this time it is not that obvious that this time dependent part is spurious when $\tau \to 0$. One has to realize however that we have been treating H, ϵ and all other slow-roll parameters as constants under integration. When assuming that these variables still contain some very small time dependence, treating them as constants under integration would produce a mismatch in the results when partially integrating the action. Therefore we *should* and *must* consider subleading corrections to these *slowly varying parameters*. For this reason, let us study for example the first slow-roll parameter ϵ. When assuming ϵ varies slowly in time, one can approximate ϵ by

$$\begin{aligned}\epsilon(t) &\approx \epsilon(t_*) + (t-t_*)\dot\epsilon(t_*) + \ ... \\ &\approx \epsilon_* - \frac{1}{H}\log\left(\frac{\tau}{\tau_*}\right)\epsilon_*\eta_*H_* + \ ... \\ &\approx \epsilon_* - \log\left(\frac{\tau}{\tau_*}\right)\epsilon_*\eta_* + \ ....\end{aligned} \tag{4.112}$$

Using this approximation, one obtains the following form for the *left hand side* of (4.103)

$$S = \int dt\ \epsilon^2a^3\zeta\dot\zeta^2 \approx \int dt\left(\epsilon_*^2 - 2\epsilon_*^2\eta_*\log\left(\frac{\tau}{\tau_*}\right)\right)a^3\zeta\dot\zeta^2 \tag{4.113}$$

The contribution to the bispectrum that the additional term in (4.113) produces compared to (4.103) is given by

$$\langle\zeta^3\rangle \supseteq \frac{H^4}{32\epsilon_*^2k_1^3k_2^3k_3^3}\frac{2\epsilon_*\eta_*}{k_t^2}\left(k_t^2\sum_{i\neq j}^{3}k_i^2k_j(-1+2\gamma_E+2\log(-k_t\tau_*)) + \sum_{i\neq j}^{3}k_i^3k_j^2\left(\gamma_E+\log(-k_t\tau_*)\right)\right). \tag{4.114}$$

Note that this term is now fully time independent again. Using this approximation and partially integrating (4.113), we obtain

$$\int dt\ \left[\partial_\tau\left(\left(\epsilon_* - 2\epsilon_*^2\eta_*\log\left(\tau/\tau_*\right)\right)a^2\zeta^2\zeta'\right) - \frac{2H\epsilon_*^2\eta_*}{\tau H}a^2\zeta^2\zeta' - \epsilon_*\zeta'\partial_\tau(a^2\zeta\zeta')\right]. \tag{4.115}$$

Note that in the third term, we can approximate $-1/(H\tau) \approx \mathrm{a}(\tau)$ giving us the term similar to (4.110) again. The extra contribution to the bispectrum coming from the additional boundary term is given by

$$\langle \zeta^3 \rangle \supseteq \frac{H^4 \epsilon_* \eta_*}{32\epsilon^2 k_1^3 k_2^3 k_3^3} \log\left(\frac{\tau}{\tau_*}\right) \sum_{i=1}^{3} k_i^3 \,. \tag{4.116}$$

This term now cancels exactly the time dependence in (4.111) making the contribution to the bispectrum time independent again.

There are a couple of important things that we have learned from this model. The first thing is that when one approximates the *Quasi-de Sitter* wave functions by the *de Sitter* wave functions, the contribution of terms multiplying the *Quasi-de Sitter* equations of motion will be non-zero. One can show that only the contribution from terms multiplying the *de Sitter* equations of motion will produce a zero result, leaving us with contributions that appear from higher order slow-roll corrections. Therefore these higher order slow-roll terms should be pulled out of the $f(\zeta)\delta L/\delta\zeta$ terms in the action when approximating the *Quasi-de Sitter* mode functions by the *de Sitter* ones. The second important thing to note is that higher order slow-roll parameters enter the game when partially integrating the action. When a time derivative is partially integrated onto a slow-roll parameter or H, a time divergence in the bispectrum appears. Total boundary terms together with the *slowly varying parameter* approximation play a crucial role in controlling and canceling these time divergences. This prescription in dealing with the time divergences is one of the (new) fundamental results of this thesis.

Subtleties concerning the field shift

In [19] and [20], it is noted that the derivation of the η contribution to the bispectrum is somewhat misleading, in the sense that (4.99) does not produce the right value for $(n_s - 1)$ using the consistency relation for the bispectrum without this field shift. The reason for this is that all boundary terms in the derivation of the result in [13] were omitted. Since we just learned from the toy model that we cannot simply omit all boundary terms, we have to study these boundary terms better. All temporal boundary terms that were omitted are given by [20]

$$\begin{aligned} S_3^{Boundary} = \int \mathrm{dt}\, d^3x\, \frac{d}{dt}\Bigg[&-9a^3 H\zeta^3 + \frac{a}{H}\zeta(\partial\zeta)^2 - \frac{1}{4aH^3}(\partial\zeta)^2\partial^2\zeta - \frac{a\epsilon}{H}\zeta(\partial\zeta)^2 \\ &- \frac{\epsilon a^3}{H}\zeta\dot{\zeta}^2 + \frac{1}{2aH^2}\zeta\left(\partial_i\partial_j\zeta\partial_i\partial_j\psi^{(1)} - \partial^2\zeta\partial^2\psi^{(1)}\right) \\ &- \frac{\eta a}{2}\zeta^2\partial^2\psi^{(1)} - \frac{1}{2aH}\zeta(\partial_i\partial_j\psi^{(1)}\partial_i\partial_j\psi^{(1)} - \partial^2\psi^{(1)}\partial^2\psi^{(1)})\Bigg] . \end{aligned} \tag{4.117}$$

When using the quasi-de Sitter mode functions, all terms that do not have exactly one derivative on a ζ perturbations will formally produce a zero contribution to the bispectrum. When approximating the mode functions by the *de Sitter* mode functions, all terms that have two or more derivatives acting on ζ will formally produce a zero contribution to the bispectrum.

There is one term that is important when one calculates the η-contribution to the bispectrum, i.e. $\frac{d}{dt}\left[-\frac{\eta a}{2}\zeta^2\partial^2\psi^{(1)}\right]$. In order to calculate its contribution, it is more convenient to work in conformal time, τ, rather than 'normal' time, t. In conformal time, the part of this boundary term that produces a non-zero contribution is given by (4.126)

$$\begin{aligned} -\int \mathrm{dt}\frac{d}{dt}\left(\frac{\epsilon\eta a^3}{2}\zeta^2\dot{\zeta}\right) &= -\int d\tau\ \frac{d}{d\tau}\left(\frac{\epsilon\eta a^2}{2}\zeta^2\zeta'\right) \\ &= -\left(\frac{\epsilon\eta a^2}{2}\zeta^2\zeta'\right). \end{aligned} \tag{4.118}$$

The contribution this term produces to the bispectrum is given by

$$\begin{aligned} \langle\zeta(k_1,\tau)\zeta(k_2,\tau)\zeta(k_3,\tau)\rangle'_\eta =& i\frac{\epsilon\eta}{2}a^2(\tau)\ u(\tau,k_1)u(\tau,k_2)u(\tau,k_3)u^*(\tau,k_1)u^*(\tau,k_2)u^*(\tau,k_3) \\ &+ 5\text{ perm.} + \text{c.c.}, \end{aligned} \tag{4.119}$$

here *perm.* refers to 5 cyclic permutations of $k_1,\ k_2$ and k_3 and *c.c.* refers to complex conjugate. Thus the contribution to the bispectrum we obtain is

$$\langle\zeta(k_1,\tau)\zeta(k_2,\tau)\zeta(k_3,\tau)\rangle'_\eta = \frac{H^4}{16\epsilon^2}\frac{\eta}{2}\frac{(k_1^3+k_2^3+k_3^3)}{(k_1k_2k_3)^3}. \tag{4.120}$$

This result is equivalent to the one found in [13].

When performing the field shift as proposed in [13], this boundary will cancel against another boundary term that we produce during partially integrating the shifted quadratic action to produce terms proportional to $\delta\mathcal{L}/\delta\zeta$,

$$\begin{aligned} &S_2[\zeta\to\zeta_n+f(\zeta_n)] = \int d^4x\ \left[e^{3\rho}\frac{1}{2}\frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}^2 - e^{2\rho}\partial_i\zeta\partial_j\zeta\right] \\ &= \int d^4x\ \left[e^{3\rho}\frac{1}{2}\frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}_n^2 - e^{2\rho}\partial_i\zeta_n\partial_i\zeta_n + e^{3\rho}\frac{\dot{\phi}^2}{\dot{\rho}^2}\left(\dot{\zeta}_n\dot{f}_n - e^{-2\rho}\partial_i\zeta_n\partial_i f_n\right)\right] \\ &= \int d^4x\ \left[e^{3\rho}\frac{1}{2}\frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}_n^2 - e^{2\rho}\partial_i\zeta_n\partial_i\zeta_n - f(\zeta)\left\{\frac{d}{dt}\left(e^{3\rho}\frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}_n\right) - e^{\rho}\frac{\dot{\phi}^2}{\dot{\rho}^2}\partial_i\partial_i\zeta_n\right\}\right] + \\ &\quad + \int d^4x\ \frac{d}{dt}\left[2a^3\epsilon f_n\dot{\zeta}_n\right] \\ &= \int d^4x\ \left[e^{3\rho}\frac{1}{2}\frac{\dot{\phi}^2}{\dot{\rho}^2}\dot{\zeta}_n^2 - e^{2\rho}\partial_i\zeta_n\partial_i\zeta_n - f(\zeta_n)\frac{\delta\mathcal{L}_2}{\delta\zeta}\right] + \int d^4x\ \frac{d}{dt}\left[2a^3\epsilon f_n(\zeta_n)\dot{\zeta}_n\right]. \end{aligned} \tag{4.121}$$

Due to this boundary term, the calculation of [13] is consistent with the calculation when we do not perform a field shift. Then, the action we should use to calculate our bispectrum with is given by

$$\begin{aligned} S_3 = \int \mathrm{dt}\ d^3x\, \Bigg(& a^3\epsilon^2\zeta\dot{\zeta}^2 - 2a\epsilon^2\dot{\zeta}(\partial_k\zeta)(\partial_k\psi^{(1)}) + a\epsilon^2\zeta(\partial\zeta)^2 + f(\zeta)\frac{\delta\mathcal{L}_2}{\delta\zeta}\bigg|_{dS} \\ &+\frac{a^3}{2}(\epsilon\dot{\eta}-\epsilon^2\eta H)\zeta^2\dot{\zeta} + \frac{1}{2}\frac{\epsilon}{a}\partial_k\zeta\partial_k\psi^{(1)}\partial^2\psi^{(1)} + \frac{\epsilon}{4}\partial^2\zeta(\partial\psi^{(1)})^2\Bigg) + \\ &+ \int \mathrm{dt}\ d^3x\ \frac{d}{dt}\left(-\frac{\epsilon\eta}{2}a^3\zeta^2\dot{\zeta} + a^3\epsilon^2\zeta^2\dot{\zeta} + \ \ldots\right), \end{aligned} \tag{4.122}$$

here the dots denote boundary terms that do not give any contribution to any Feynman diagram.

The interaction Hamiltonian of interest

Depending how much people enjoy partially integrating the cubic order action, there are multiple forms one can use for the interaction Hamiltonian [13],[19],[20]. We are however interested in the form presented in [13] with the correction of the boundary terms and all terms that do not appear in the *Quasi-de Sitter* equations of motion pulled out of $\delta L/\delta\zeta$. This form of the action turns out to be the most convenient one when comparing the results of the bispectrum with the results from next chapter. The interaction Hamiltonian we shall use is given by

$$\begin{aligned} H_3 = &-\int \mathrm{dt}\ d^3x \left(a^3\epsilon^2\zeta\dot{\zeta}^2 - 2a\epsilon^2\dot{\zeta}\partial_k\zeta\partial_k\psi^{(1)} + a\epsilon^2\zeta(\partial\zeta)^2 + f(\zeta)\frac{\delta\mathcal{L}_2}{\delta\zeta}\bigg|_{dS} \right. \\ &\left. + \frac{a^3}{2}(\epsilon\dot{\eta} - \epsilon^2\eta H)\zeta^2\dot{\zeta} + \frac{1}{2}\frac{\epsilon}{a}\partial_k\zeta\partial_k\psi^{(1)}\partial^2\psi^{(1)} + \frac{\epsilon}{4}\partial^2\zeta(\partial\psi^{(1)})^2 \right) \\ &+ \int \mathrm{dt}\ d^3x\ \frac{d}{dt}\left[\frac{\epsilon\eta}{2}a^3\zeta^2\dot{\zeta} - a^3\epsilon^2\zeta^2\dot{\zeta} +\ ...\right]. \end{aligned} \tag{4.123}$$

4.6.2 The leading order bispectrum

In order to calculate the bispectrum, we take the leading order corrections in slow-roll parameters to the mode functions, i.e. we approximate ν as $\nu \approx 3/2$. As mentioned before, this choice for ν corresponds to the de Sitter mode functions for ζ. Since the corrections to the mode functions should be at order, error $\sim \mathcal{O}(\eta) \ll 3/2$, this approximation holds at leading orders in slow-roll parameters. In this limit the mode function of ζ reduce to

$$u_k(\tau) = u(\mathbf{k},\tau) = \frac{iH}{\sqrt{4\epsilon k^3}}(1+ik\tau)e^{-ik\tau}. \tag{4.124}$$

The leading order contributions of the three point function, are being produced by the following parts of the interaction Hamiltonian

$$H_3 = \quad -\int \mathrm{dt}\ d^3x \left(a^3\epsilon^2\zeta\dot{\zeta}^2 - 2a\epsilon^2\dot{\zeta}\partial_k\zeta\partial_k\psi^{(1)} + a\epsilon^2\zeta(\partial\zeta)^2 \right) \tag{4.125}$$

$$+\int \mathrm{dt}\ d^3x\ \frac{d}{dt}\left[\frac{\epsilon\eta}{2}a^3\zeta^2\dot{\zeta} +\ ...\right]. \tag{4.126}$$

(4.125) produces the three point function proportional to ϵ. Their contribution to the bispectrum is given by [13]

$$\langle\zeta(k_1,\tau)\zeta(k_2,\tau)\zeta(k_3,\tau)\rangle' \supseteq \frac{H^4}{32\epsilon^2\ k_1^3k_2^3k_3^3}2\epsilon\left[-\sum_{i=1}^{3}k_i^3 + \sum_{i\neq j}^{3}\left(k_i^2k_j + \frac{4}{k_t}k_i^2k_j^2\right)\right], \tag{4.127}$$

here the prime on the brackets of the correlation function denote the truncation of a factor of $(2\pi)^3$ and a momentum conserving delta function. Note that when working out the summations, both terms in the summation of $\sum_{i\neq j}^{3}\left(k_i^2 k_j + \frac{4}{k_t}k_i^2 k_j^2\right)$ will produce six terms. A full derivation of this contribution is given in appendix C.1.

The second contribution at this order in slow-roll parameters is produced by (4.126). Its contribution is given by (4.120)

$$\langle\zeta(k_1,\tau)\zeta(k_2,\tau)\zeta(k_3,\tau)\rangle' \supseteq \frac{H^4}{16\epsilon^2}\frac{\eta}{2}\frac{(k_1^3+k_2^3+k_3^3)}{k_1^3k_2^3k_3^3}.$$

This result is equivalent to the result as found in [13] where a field shift was used of the form

$$\begin{aligned}\zeta &\to \zeta_n + f(\zeta_n^2)\\ f(\zeta_n^2) &\approx \frac{\eta}{4}\zeta_n^2 + \frac{1}{H}\dot{\zeta}_n\zeta_n + \mathcal{O}(\partial\zeta_n).\end{aligned}$$

When converting a ζ_n correlation function back into a ζ correlation function, we have to take into account the amount of time the different modes have evolved with respect to each other before they are converted to ζ. They should be converted back when the last mode leaves the horizon, $k_* = a(\tau_*)H_*$. Since the correlators are invariant under permutations of k_1, k_2 and k_3, we are free to choose $k_* = k_3$. Also during the conversion, we will pick up a superhorizon part due to the second order relation between ζ and ζ_n, then

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle = \langle\zeta_n(\mathbf{k}_1)\zeta_n(\mathbf{k}_2)\zeta_n(\mathbf{k}_3)\rangle + \langle\zeta_n(\mathbf{k}_1)\zeta_n(\mathbf{k}_2) : \zeta_n(\mathbf{k}_3)\zeta_n(\mathbf{k}_3) :\rangle + 5 \text{ perm.} \tag{4.128}$$

It is this superhorizon part in [13] that produced the η contribution to the bispectrum.

4.6.3 The next to leading order bispectrum

In the next chapter we will study a certain limit of inflation where the contributions multiplying higher order slow-roll parameters will play a dominant role. If we are to compare those results to the results in the comoving gauge, we have to compute these higher order corrections to the bispectrum. For this reason, we shall treat the calculation of the $\dot{\eta}$ contribution more explicitly, afterwards we state all other terms of order ϵ^2 and $\epsilon\eta$ that appear in this action. First, we have to note the following; higher order corrections could be the appearance of higher order slow-roll parameters, for example η, $\xi^{(1)}$, terms that multiply more slow-roll parameters or both.

As discussed in the previous sections, there are two methods to calculate the bispectrum, one where we perform a field shift and fix the time dependence at the end, and one where we do not use a field shift but take really good care on all the boundary terms and variation of the slow-roll parameters. We shall perform both calculations and check the results. The term of the cubic action that will produce the $\dot{\eta}$ contribution is given by

$$S_3 \supseteq \int d\tau\ d^3x\left(\frac{a^3}{2}\epsilon\dot{\eta}\zeta^2\zeta'\right). \tag{4.129}$$

Note again that we have converted the expression to conformal time. The contribution to truncated bispectrum is then given by

$$\begin{aligned}&\langle\zeta_n(\mathbf{k}_1)\zeta_n(\mathbf{k}_2)\zeta_n(\mathbf{k}_3)\rangle_{\dot{\eta}} =\\ &= i\left[\Pi_i u_i(\tau_{end})\right]\int_{-\infty(1\pm i\varepsilon)}^{\tau_{end}} d\tau\ \left[a^3\epsilon\dot{\eta}u^*(k_1,\tau)u^*(k_2,\tau)\partial_\tau u^*(k_3,\tau) + 2\text{ perm.}\right] + \text{c.c.}\,,\end{aligned} \tag{4.130}$$

here we used Wicks theorem to contract the different mode functions and the complex conjugate is produced by the commutator in (4.93). Wicks theorem produces six permutations, but since every term shows up twice, we can cancel the factor 1/2 and just write two permutations.

$$
\begin{aligned}
&\langle\zeta_n(\mathbf{k}_1)\zeta_n(\mathbf{k}_2)\zeta_n(\mathbf{k}_3)\rangle_{\dot\eta} = \\
&= i\left[\Pi_i u_i(\tau_{end})\right]\int_{-\infty(1+i\epsilon)}^{\tau_{end}} d\tau\ \left[a^3\epsilon\dot\eta u^*(k_1,\tau)u^*(k_2,\tau)\partial_\tau u^*(k_3,\tau) + 2\text{ perm.}\right] + \text{c.c.} \\
&= \frac{iH^3}{64\epsilon^3k_1^3k_2^3k_3^3}\epsilon\dot\eta\int_{-\infty(1+i\epsilon)}^{\tau_{end}} d\tau\ \left[\frac{-1}{\tau^3}(\tau k_3^2 - i(k_1+k_2)k_3^2\tau^2 - k_1k_2k_3^2\tau^3)e^{ik_t\tau} + 2\text{ perm}\right] \\
&+ \frac{-iH^3}{64\epsilon^3k_1^3k_2^3k_3^3}\epsilon\dot\eta\int_{-\infty(1-i\epsilon)}^{\tau_{end}} d\tau\ \left[\frac{-1}{\tau^3}(\tau k_3^2 + i(k_1+k_2)k_3^2\tau^2 - k_1k_2k_3^2\tau^3)e^{-ik_t\tau} + 2\text{ perm}\right],
\end{aligned}
$$

here we defined $k_t \equiv k_1 + k_2 + k_3$. Since our regularization scheme makes sure that our perturbation theory does not produce a contribution at the initial state, we will put both of the integrals into the same integral,

$$
\begin{aligned}
&\langle\zeta_n(\mathbf{k}_1)\zeta_n(\mathbf{k}_2)\zeta_n(\mathbf{k}_3)\rangle'_{\epsilon\dot\eta} = \\
&= \frac{iH^3}{64\epsilon^3k_1^3k_2^3k_3^3}\epsilon\dot\eta\int_{-\infty}^{\tau_{end}} d\tau\ \left[\frac{k_3^2}{\tau^2}(-2i\sin(k_t\tau)) + \frac{2}{\tau}i(k_1+k_2)k_3^2\cos(k_t\tau) + 2k_1k_2k_3^2 i\sin(k_t\tau) + 2\text{ perm}\right] \\
&= \frac{H^4}{16\epsilon^2k_1^3k_2^3k_3^3}\frac{\dot\eta}{2H}\left[\text{CosInt}(k_t\tau_{end})(k_1^3+k_2^3+k_3^3) - (k_1^2+k_2^2+k_3^2)\frac{\sin(k_t\tau_{end})}{\tau_{end}} + k_1k_2k_3\cos(k_t\tau_{end})\right],
\end{aligned}
$$

here CosInt(x) is the cosine integral, $\int_{-\infty}^{x} dt\cos(t)/t$. At the end of inflation, we have $k\tau_{end} \ll 1$. Then all three terms in the expression will behave differently. The last terms behave trivial, $\cos(k_t\tau_{end}) \approx 1$ and $\frac{\sin(k_t\tau_{end})}{\tau_{end}} = k_t$. The CosIntegral however behaves a little bit different and analytical continuation is needed around small values of $k_t\tau_{end}$. In this region, the CosIntegral has the following behavior

$$
\text{CosInt}(k_t\tau_{end}) \approx \gamma_E + \log(-k_t\tau_{end}) - \frac{k_t^2\tau_{end}^2}{4} + \mathcal{O}((k_t\tau_{end})^4). \tag{4.131}
$$

Substituting this into our equations we find that the contribution to the ζ_n-bispectrum is given by

$$
\begin{aligned}
&\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle' \supseteq \\
&\supseteq \frac{H^4}{16\epsilon^2k_1^3k_2^3k_3^3}\frac{\dot\eta}{2H}\left[(-1+\gamma_E+\log(-k_t\tau_*))\sum_{i=1}^{3}k_i^3 - \sum_{i\neq j}k_ik_j^2 + k_1k_2k_3\right],
\end{aligned} \tag{4.132}
$$

here we have defined $\tau_{end} \equiv \tau_*$, with τ_* is a few e-folds after the last mode leaves the horizon. As discussed earlier, we are free to choose $\tau_{end} = -\frac{1}{k_3}$.

Let us now study the calculation of this $\dot\eta$ term when we do not perform a field shift. As it turns out, the integral we need to calculate when not doing a field shift is very similar, we can see this by expanding η around τ_* in the action

$$\begin{aligned} S_3 &\supseteq \int d\tau\ d^3x \left(\frac{a^3}{2}\epsilon\dot{\eta}\zeta^2\zeta'\right) \\ &\approx \int d\tau\ d^3x \frac{a^3}{2}\left(\epsilon_* - \frac{\dot{\epsilon}_*}{H}\log\left(\frac{\tau}{\tau_*}\right)\right)\frac{d}{a\ d\tau}\left(\eta_* - \frac{\dot{\eta}_*}{H_*}\log\left(\frac{\tau}{\tau_*}\right)\right)\zeta^2\zeta' \\ &\approx -\int d\tau\ d^3x \frac{a^2}{2}\frac{\epsilon_*\dot{\eta}_*}{H\tau}\zeta^2\zeta', \end{aligned} \tag{4.133}$$

here we approximated $-1/H\tau \approx a(\tau)$ giving us our original integral back. Note that $\dot{\epsilon} = H\epsilon\eta$ is one order higher in slow-roll parameters, therefore we will not be considering this contribution. Also, note that we actually did not need to expand the slow-roll parameters, since due to the expansion they are self consistent and we could just have used $\eta(t)$ and $\dot{\eta}$. The contribution to the bispectrum coming from this term is then given by

$$\begin{aligned} &\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle' \supseteq \\ &\supseteq \frac{H^4}{16\epsilon^2 k_1^3k_2^3k_3^3}\frac{\dot{\eta}_*}{2H}\left[(-1+\gamma_E+\log(-k_t\tau))\sum_{i=1}^{3}k_i^3 - \sum_{i\neq j}k_ik_j^2 + k_1k_2k_3\right], \end{aligned} \tag{4.134}$$

The subtle difference between (4.134) and (4.132) is that (4.134) still contains time dependence while (4.132) does not. Just like we have seen in our toy model, to remove this time dependence, we have to use the boundary term given by (4.126) [20] and use it in the same way as we did in our toy model 4.6.1

$$\begin{aligned} S &\supseteq \int \mathrm{dt}\ d^3x\ \frac{d}{dt}\left[-\frac{\epsilon\eta}{2}a^3\zeta^2\dot{\zeta} + \ ...\right] \\ &\supseteq \int d\tau\ d^3x\ \frac{d}{d\tau}\left[-\frac{\epsilon}{2}\left(\eta_* - \frac{\dot{\eta}*}{H_*}\log\left(\frac{\tau}{\tau_*}\right)\right)a^2(\tau)\zeta^2\zeta'\right] \\ &\approx \int d\tau\ d^3x\ \frac{d}{d\tau}\left[-\left(\frac{\epsilon_*\eta_*}{2} - \frac{\epsilon_*\dot{\eta}_*}{2H_*}\log\left(\frac{\tau}{\tau_*}\right)\right)a^2\zeta^2\zeta'\right]. \end{aligned} \tag{4.135}$$

The contribution to the bispectrum this term produces is given by

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle' \supseteq \frac{H^4}{16\epsilon^2}\left(\eta_* - \frac{\dot{\eta}_*}{2H_*}\log\left(\frac{\tau}{\tau_*}\right)\right)\frac{k_1^3+k_2^3+k_3^3}{k_1^3k_2^3k_3^3}. \tag{4.136}$$

Just as we have seen in our toy model, when adding this boundary term contribution (4.140) with the dynamical term contribution (4.134) we note that the time dependence gets canceled. Again we obtain the same result as the one obtained when using a field shift, i.e. we find (4.132).

As discussed in the toy model, by means of partial integration we can get higher order slow-roll parameters. Let us therefore partially integrate (4.129) once more in time

$$S_3 \supseteq \int d\tau\ d^3x \left(\frac{a^3}{2}\epsilon\dot{\eta}\zeta^2\zeta'\right) = \int d\tau\ d^3x\ \left[\frac{-a^3}{6}\left(3H\epsilon\dot{\eta} + \dot{\epsilon}\dot{\eta} + \epsilon\ddot{\eta}\right)\zeta^3 + \partial_\tau\left(\frac{a^3\epsilon\dot{\eta}}{6}\zeta^3\right)\right]. \tag{4.137}$$

The contribution of first term on the right hand side of (4.137) to the bispectrum is given by

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle' \supseteq$$
$$\supseteq \frac{H^4}{16\epsilon^2 k_1^3k_2^3k_3^3}\left(\frac{\dot{\eta}_*}{2H}+\frac{\ddot{\eta}_*}{6H^2}+\frac{\eta_*\dot{\eta}_*}{6H}\right)\left[(-1+\gamma_E+\log(-k_t\tau))\sum_{i=1}^{3}k_i^3-\sum_{i\neq j}k_ik_j^2+k_1k_2k_3\right], \tag{4.138}$$

Note again the time dependence in the correlation function. The time dependence of the $\dot{\eta}$ contribution is annihilated by the boundary term (4.126). Note that when partially integrating, this term is being produces by hitting the scale factor with the time derivative and therefore, we do not have to make a slow varying parameter approximation. The time dependence in the $\ddot{\eta}$ and $\dot{\epsilon}\dot{\eta}$ contributions are being a annihilated by the expanding the boundary term in (4.137) to

$$\int d\tau\ d^3x\ \partial_\tau\left[\frac{1}{6}a^3\left(\epsilon_*-\frac{\dot{\epsilon}_*}{2H_*}\log\left(\frac{\tau}{\tau_*}\right)\right)\left(\dot{\eta}_*-\frac{\ddot{\eta}_*}{2H_*}\log\left(\frac{\tau}{\tau_*}\right)\right)\zeta^3\right]. \tag{4.139}$$

Then the contribution to the bispectrum comming from this boundary term that annihilate the time dependence in the $\ddot{\eta}$ and $\dot{\epsilon}\dot{\eta}$ contributions to the bispectrum are then given by

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle' \supseteq -\frac{H^4}{16\epsilon^2}\left[\frac{\ddot{\eta}_*}{6H_*}\log\left(\frac{\tau}{\tau_*}\right)+\frac{\eta_*\dot{\eta}_*}{6H_*}\log\left(\frac{\tau}{\tau_*}\right)\right]\frac{k_1^3+k_2^3+k_3^3}{k_1^3k_2^3k_3^3}. \tag{4.140}$$

Adding all contributions, we obtain

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle' \supseteq$$
$$\supseteq \frac{H^4}{16\epsilon^2 k_1^3k_2^3k_3^3}\left(\frac{\dot{\eta}_*}{2H}+\frac{\ddot{\eta}_*}{6H^2}+\frac{\eta_*\dot{\eta}_*}{6H}\right)\left[(-1+\gamma_E+\log(-k_t\tau_*))\sum_{i=1}^{3}k_i^3-\sum_{i\neq j}k_ik_j^2+k_1k_2k_3\right]. \tag{4.141}$$

The leftover terms in the action which we did not use , but that still give a contribution to the bispectrum give contributions at order ϵ^2 and $\epsilon\eta$. The contribution to the bispectrum is at order ϵ^2 and is given by

$$\langle\zeta^3\rangle \supseteq \frac{H^4}{16\epsilon^2k_1^3k_2^3k_3^3}\frac{\epsilon^2}{2}\left[\sum_{i=1}^{3}\frac{k_i^5}{k_t^2}+\sum_{i\neq j}\frac{k_i^4k_j}{k_t^2}-2\sum_{i<j<l}\frac{k_i^2k_j^2k_l}{k_t^2}-2\sum_{i\neq j}\frac{k_i^2k_j^3}{k_t^2}\right]. \tag{4.142}$$

A more explicit derivation of this calculation can be found in C.2. The contribution at order $\epsilon\eta$ is given by,

$$\langle\zeta^3\rangle \supseteq \frac{H^4}{16\epsilon_*^2k_1^3k_2^3k_3^3}\frac{\epsilon_*\eta_*}{k_t^2}\left[(-1+\gamma_E+\log(-k_t\tau_*))\sum_{i=1}^{3}k_i^3-\sum_{i\neq j}k_ik_j^2+k_1k_2k_3\right]. \tag{4.143}$$

The derivation of this term is similar to the one performed in the discussion of our toy model.

4.7 Shape and a relation Non-Gaussianities

As mentioned earlier, the bispectrum contains information about the shape of the momentum configuration in the modes. To estimate the non-Gaussianities, a number of templates can be used to compare the shape with. We however, will make an estimate based on the *Komatsu-Spergel local form*

$$\langle \Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle \equiv (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) f_{NL}^{\text{local}} \left[2P(k_1)P(k_2) + \text{perm.}\right],$$

In this definition, the gravitational potential, Φ and the scalar curvature ζ are related by $\Phi(\mathbf{k}) \equiv -\frac{5}{3}\zeta(\mathbf{k})$. For a scale invariant power spectrum, we have

$$P(k) \approx \frac{H^2}{4\epsilon M_{\text{Pl}}^2}\frac{1}{k^3}\,. \tag{4.144}$$

Then (4.8) can be approximated as

$$\langle \Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle' \approx f_{NL}^{\text{local}} \frac{H^4}{16\epsilon^2 M_{\text{Pl}}^4}\frac{(k_1^3+k_2^3+k_3^3)}{(k_1k_2k_3)^3}\,. \tag{4.145}$$

If we compare (4.145) to the other contributions to the bispectrum, we note that the momentum configurations are sometimes different. Also, when taking specific configurations of the momentum triangle, for example when we take one of the momenta very much smaller than the others, we see that the numerical prefactor f_{NL} can change. In general, there are three kinds of shapes. These are the *equilateral*-, *sqeezed*- and *Folded* shapes. These peak when $k_1 \sim k_2 \sim k_3$,

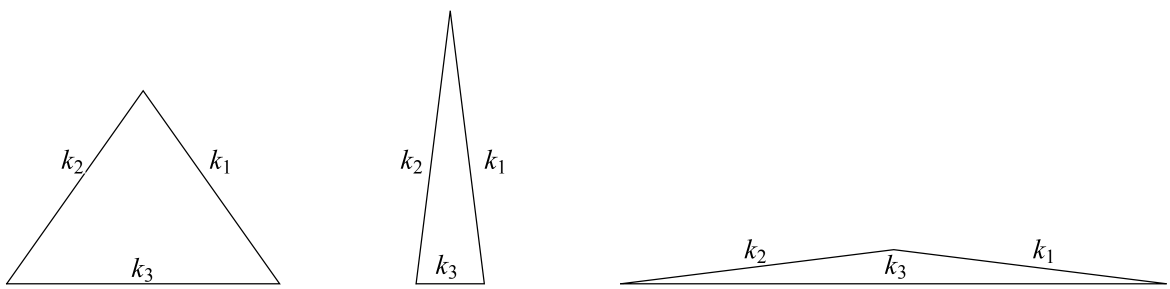

Figure 4.1: The three possible momentum triangle which we consider, from left to right the *equilateral*, the *sqeezed* and the *folded* shape.

$k_1 \ll k_2 \sim k_3$ or $k_1 \gg k_2 \sim k_3$ respectively. We can also split up the different kind of terms one encounters in the bispectrum. In general, there are four kinds of terms a bispectrum can contain, these are *Local*-, *equilateral*, *folded*- and *left over* terms. The local bispectrum is given by

$$B_{loc}(k_1,k_2,k_3) = \frac{(k_1^3+k_2^3+k_3^3)}{(k_1k_2k_3)^3}, \tag{4.146}$$

the equilateral bispectrum is given by

$$B_{equil}(k_1,k_2,k_3) = \frac{1}{k_1^3k_2^3k_3^3}\left[-\sum_{i=1}^{3}k_i^3 - \sum_{i\neq j}k_ik_j^2 + k_1k_2k_3,\right], \tag{4.147}$$

the folded bispectrum is given by

$$B_{folded}(k_1,k_2,k_3) = \frac{1}{k_1^3k_2^3k_3^3}\left[6\sum_{i=1}^{3}k_i^3 - 6\sum_{i\neq j}k_ik_j^2 + 18,\right]. \tag{4.148}$$

and the left over terms are terms containing for example $\log(k_t\tau_*)$ terms. The local and equilateral configurations are the most common. The folded configuration for the bispectrum can be found when assuming for example that the universe is not in the Bunch-Davies vacuum but in an exited state.

In order to estimate the order of non-gaussianities in our situation, we have to look at the leading order contributions of the bispectrum. A general slow-roll model bispectrum can be put in the form of (4.8) by taking it's squeezed limit, this is because the squeezed bispectrum shape coincides with the local shape. If we turn to the leading order bispectrum result [13], we find that in the squeezed limit it is given by

$$\frac{\eta_*}{8} + \frac{\epsilon_*}{8}\left(-\sum_{i=1}^{3}k_i^3 + \sum_{i\neq j}^{3}k_ik_j^2 + \frac{8}{k_t}\sum_{i<j}^{3}k_i^2k_j^2\right) \to \frac{\eta+2\epsilon}{8}\sum_{i=3}^{3}k_i^3\,. \tag{4.149}$$

Then comparing this result with (4.8) we note that the order of the Non-Gaussianities is given by

$$f_{NL} = \frac{\eta_*}{2} + \epsilon_*\,. \tag{4.150}$$

We can do the same thing to the $\dot{\eta}$ contribution to the bispectrum, in the squeezed limit (4.141) becomes

$$\alpha\left((-1+\gamma_E+\log(-k_t\tau_*))\sum_{i=1}^{3}k_i^3 - \sum_{i\neq j}k_ik_j^2 + k_1k_2k_3\right) \to \alpha\left(-2+\gamma_E+\log(2)\right)\sum_{i=1}^{3}k_i^3\,, \tag{4.151}$$

here α has been defined as $\alpha \equiv \frac{\dot{\eta}_*}{H} + \frac{\ddot{\eta}_*}{3H^2} + \frac{\eta_*\dot{\eta}_*}{3H}$, we assumed that $\tau_* = -1/k_s$ since this mode becomes super Hubble size last and that $k_t \approx 2k_s$. Comparing this result with (4.8) we note that the contribution to the Non-Gaussianities is given by

$$f_{NL} = \left(\frac{\dot{\eta}_*}{H} + \frac{\ddot{\eta}_*}{3H^2} + \frac{\eta_*\dot{\eta}_*}{3H}\right)\left(-2+\gamma_E+\log(2)\right) \sim \mathcal{O}(\eta_*\xi_*^{(1)})\,. \tag{4.152}$$

Doing a similar calculation to the bispectrum scaling with ϵ^2 and $\epsilon\eta$ show

$$f_{NL} = 2\epsilon^2 + 2\epsilon_*\eta_*(-2+\gamma_E+\log(2)) \sim \mathcal{O}(\epsilon_*^2, \epsilon_*\eta_*)\,. \tag{4.153}$$

4.7.1 Consistency relation

Beside the fact that we can use the squeezed limit as easy way to estimate the non-gaussianities in a model, it has another important consequence. When one of the modes in the bispectrum has a significant smaller momentum than the other two modes, this mode will have a significant larger wavelength than the other two modes. We therefore refer to this mode as the *long* mode and the other two the *small* modes. A mode with a longer wavelength leaves the horizon earlier than modes with a smaller wavelength and thus freezes out earlier, hence the long mode will act as a perturbation of the background for the two short modes. This changes the time when the other two modes will become super Hubble size by a small amount, $\delta t_* \approx -\frac{\zeta}{H}$, as a first order correction.

Since ζ is a metric perturbation, taking on of the modes to be long in a (n+1)-point correlation function can be seen as a coordinate transformation of a n-point correlation function of ζ, i.e.

$$\langle \zeta_s(\mathbf{x}_1) \cdot ... \cdot \zeta_s(\mathbf{x}_n) | \zeta_l(\mathbf{z}) \rangle = \langle \tilde{\zeta}_s(\mathbf{x}_1) \cdot ... \cdot \tilde{\zeta}_s(\mathbf{x}_n) \rangle \,, \tag{4.154}$$

here $\tilde{\zeta}$ is the shifted field, and the subscripts s and l refer to short and long modes respectively. We can make this statement more explicit. When multiplying both sides with ζ_L from the left and take the average we find

$$\langle \zeta_l(\mathbf{x}) \langle \zeta_s(\mathbf{x}_1) \cdot ... \cdot \zeta_s(\mathbf{x}_n) | \zeta_l(\mathbf{z}) \rangle \rangle = \langle \zeta_l(\mathbf{x}) \langle \tilde{\zeta}_s(\mathbf{x}_1) \cdot ... \cdot \tilde{\zeta}_s(\mathbf{x}_n) \rangle \rangle \,. \tag{4.155}$$

If we assume that this can also be produced by a coordinate transformation of the form

$$x^i \to \tilde{x}^i = x^i + \zeta_l(\mathbf{z}) x^i = x^i + \delta x^i \,, \tag{4.156}$$

we can write

$$\begin{aligned} \langle \tilde{\zeta}_s(\mathbf{x}_1) \cdot ... \cdot \tilde{\zeta}_s(\mathbf{x}_n) \rangle &= \langle \zeta_s(\mathbf{x}_1) \cdot ... \cdot \zeta_s(\mathbf{x}_n) \rangle + \delta x^i \partial_i \langle \zeta_s(\mathbf{x}_1) \cdot ... \cdot \zeta_s(\mathbf{x}_n) + \ ... \\ &= \langle \zeta_s(\mathbf{x}_1) \cdot ... \cdot \zeta_s(\mathbf{x}_n) \rangle + \zeta_L x^i \partial_i \langle \zeta_s(\mathbf{x}_1) \cdot ... \cdot \zeta_s(\mathbf{x}_n) \rangle + \ ... \ . \end{aligned} \tag{4.157}$$

Substituting (4.157) into (4.155) we get

$$\begin{aligned} &\langle \zeta_l(\mathbf{x}) \langle \zeta_s(\mathbf{x}_1) \cdot ... \cdot \zeta_s(\mathbf{x}_n) | \zeta_l(\mathbf{z}) \rangle \rangle = \\ &= \langle \zeta_l(\mathbf{x}) \langle \zeta_s(\mathbf{x}_1) \cdot ... \cdot \zeta_s(\mathbf{x}_n) \rangle \rangle + \langle \zeta_l(\mathbf{x}) \zeta_l(\mathbf{z}) x^i \partial_i \langle \zeta_s(\mathbf{x}_1) \cdot ... \cdot \zeta_s(\mathbf{x}_n) \rangle \rangle + \ ... \\ &= \langle \zeta_l(\mathbf{x}) \zeta_l(\mathbf{z}) x^i \partial_i \langle \zeta_s(\mathbf{x}_1) \cdot ... \cdot \zeta_s(\mathbf{x}_n) \rangle \rangle \\ &= \langle \zeta_l(\mathbf{x}) \zeta_l(\mathbf{z}) \rangle x^i \partial_i \langle \zeta_s(\mathbf{x}_1) \cdot ... \cdot \zeta_s(\mathbf{x}_n) \rangle \,. \end{aligned} \tag{4.158}$$

The first term on the right hand side of (4.158) vanishes because we can only take Wick contractions between two long modes and two short modes. To get the final result for the consistency relation, we want to make a Fourier transformation of (4.158). Note that we cannot just simply replace $x^i \partial_i$ by $k^i \partial_i$, since the we get an additional term due to partial integration. Then consistency relation in Fourier space becomes

$$\begin{aligned} \langle \zeta(\mathbf{k}_1) \zeta(\mathbf{k}_2) \zeta(\mathbf{k}_3) \rangle|_{k_1 \ll k_2, k_3} &= -\langle \zeta_l(\mathbf{k}_1) \zeta_l(-\mathbf{k}_1) \rangle \left[3(n-1) + k^i \partial_{k_i} \right] \langle \zeta_s(\mathbf{k}_2) \zeta_s(\mathbf{k}_3) \rangle \\ &= -\langle \zeta_l(\mathbf{k}_1) \zeta_l(-\mathbf{k}_1) \rangle (n_s - 1) \langle \zeta_s(\mathbf{k}_2) \zeta_s(\mathbf{k}_3) \rangle \,. \end{aligned} \tag{4.159}$$

Here n is the number of spatial dimensions. A very important thing to note here is the following. When acting with $\left[3(n-1)+k^i\partial_{k_i}\right]$ on a power spectrum, it produces a factor of (n_s-1). In this way, we can relate the spectral index of the two point function to the squeezed limit of the bispectrum. This relation is called the *Consistency relation* as first derived in [13].

In [13] it was proven that at leading order in slow-roll parameters, the consistency relation holds. When computing the squeezed limit of (4.132), we find that at leading order in k_l

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle'_{\dot\eta}=\frac{H^4}{16\epsilon^2k_s^3k_l^3}\frac{\dot\eta_*}{H}(-2+\gamma_E+\log(2))\,, \tag{4.160}$$

here we assumed that

$$k_t\tau_*=-\frac{k_t}{k_3}=-\frac{k_1+k_2+k_3}{k_3}\approx-\frac{2k_3}{k_3}=-2\,. \tag{4.161}$$

The squeezed limit at leading order in k_l of (4.142) is given by

$$\langle\zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle'_{\epsilon^2}=\frac{H^4}{16\epsilon^2k_s^3k_l^3}2\epsilon_*^2\,. \tag{4.162}$$

Following the definition of (4.159), we find at subleading order in slow-roll parameters, that the spectral tilt is given by

$$(n_s+1)=-2\epsilon_*^2-2\epsilon_*\eta_*-\left(\epsilon_*\eta_*+\eta_*\xi_*^{(1)}+\frac{\eta_*\xi_*^{(1)}\xi_*^{(2)}}{3}+\frac{\eta_*^2\xi_*^{(1)}}{3H}\right)(-2+\gamma_E+\log(2))\,. \tag{4.163}$$

At this moment a comment is in order. So far, we have calculated the value of the spectral index in two different ways. First we calculated it by taking derivatives of the power spectrum. This was a very clean and relatively simple method and this result can be used as a check. Secondly, we calculated the same subleading contribution to the spectral index by using the consistency relation and the bispectrum. In the sense that we could calculate the bispectrum of the quasi-de Sitter modes and that we needed to make approximations. If we now compare (4.79) and (4.163) we note that the terms that we have, indeed match in both equations.

Chapter 5

The conformal limit of inflation

5.1 The back-reacted decoupling limit

We are interested in the simplest model of single field inflation; namely, the vanilla model of a scalar field rolling down a potential and interacting with gravity. To study this model, we start from the usual Einstein-Hilbert action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{M_{\rm Pl}^2}{2}R - \frac{(\nabla\phi)^2}{2} - V(\phi)\right) \ , \tag{5.1}$$

here we kept the factors of $M_{\rm Pl}$ explicit, as we wish to consider a particular "decoupling or de Sitter limit of the theory. Our universe seems to have a small, perhaps negligible, tensor to scalar ratio, which suggests that $\epsilon \ll 1$. However, taking simply the limit of $\epsilon \to 0$ can be rather confusing, since the scalar amplitude scales with $1/\epsilon$ and the amplitude that we observe is finite. However, when we compare the scalar and the tensor amplitudes we note that this ϵ scaling is absent for tensors

$$\langle\zeta(k)^2\rangle \sim \frac{H^2}{\epsilon M_{pl}^2} = \text{ finite} \quad \text{and} \quad \langle\gamma_{ij}(k)^2\rangle \sim \frac{H^2}{M_{pl}^2} \ll 1\,.$$

Since tensors perturbations have not been measured so far, and in order to keep the amplitude of the scalar fluctuations fixed, it is convenient to consider the following limit

$$\epsilon \to 0\ , \quad \frac{H^2}{M_{\rm Pl}^2} \to 0 \quad \text{and} \quad \frac{H^2}{\epsilon M_{\rm Pl}^2} = \text{fixed}\,. \tag{5.2}$$

We call this limit, the *back-reacted decoupling limit.* In this limit, we have a nearly vanishing first slow-roll parameter $\varepsilon \to 0$, while the other slow-roll parameters are kept small but nonzero. As it turns out, in this limit, the second slow-roll parameter η is related to the mass $\eta \sim m^2/H^2$. Also, it is good to note that we kept H fixed. Therefore the powerspectra for the scalar is kept finite, while the scalar to tensor ratio approaches zero, $r = 16\epsilon \to 0$.

Let us write the background equations of motion for the attractor FRW solution, which will be quasi-de Sitter space in our case. They are

$$\begin{aligned}
&\dot{\bar\phi}^2 = -2M_{\rm Pl}^2\dot H\,,\\
&V(\bar\phi) = M_{\rm Pl}^2(3H^2+\dot H) \approx 3M_{\rm Pl}^2H^2 \quad \text{and}\\
&\ddot{\bar\phi} + 3H\dot{\bar\phi} + V'(\bar\phi) = 0\,.
\end{aligned} \tag{5.3}$$

We see that in the limit (5.2) the potential dominates over the kinetic energy of the background field. Nonetheless, we consider to have a finite kinetic energy, which is why we can interpret the scalar fluctuations as scalar curvature fluctuations by means of a proper gauge transformation.

5.2 Mathematical Taekwondo in the spatially flat gauge

In this section, we solve the constraints for the non-dynamical components of the metric. We use the ADM formalism and work in the spatially flat gauge, in this gauge $\zeta = 0$ and the scalar perturbations are parametrized by the inflaton ϕ. We first solve the solution for the contraints N and N^i, afterwards we will show that in the decoupling limit (5.2), the constraint solutions become trivial, in the sense that

$$N = 1 + \mathcal{O}(M_{\rm Pl}^{-2}) \quad \text{and} \quad N^i = \mathcal{O}(M_{\rm Pl}^{-2})\,. \tag{5.4}$$

In the *spatially flat* gauge, the dynamical scalar fluctuations are contained in the field ϕ, while the tensor fluctuations remain in the metric h_{ij},

$$\phi = \bar{\phi}(t) + \varphi(\mathbf{x}, t) \quad \text{and} \quad h_{ij} = a(t)^2 (e^{\hat{\gamma}})_{ij} \equiv a(t)^2 \left(\delta_{ij} + \hat{\gamma}_{ij} + \frac{1}{2}\hat{\gamma}_{ik}\hat{\gamma}_{kj} + \cdots \right), \tag{5.5}$$

with $\partial_i \hat{\gamma}_{ij} = 0$ and $\hat{\gamma}_{ii} = 0$. Writing the action (5.1) in terms of the fluctuations and Lagrange multipliers, we obtain

$$S = \int \mathrm{d}t\mathrm{d}^3x\, N a^3 \left(\frac{M_{\rm Pl}^2}{2}\mathcal{R} + \frac{1}{2}\left(N^{-2}(\dot{\phi} - N^i\partial_i\phi)^2 - h^{ij}\partial_i\phi\partial_j\phi \right) - V(\phi) \right) = \tag{5.6}$$

$$\begin{aligned} = & \int \mathrm{d}t\mathrm{d}^3x\, a^3 \left[\frac{M_{\rm Pl}^2}{2}\left(N\mathcal{R}^{(3)} + N^{-1}h^{ij}h^{kl}(E_{ik}E_{jl} - E_{ij}E_{kl}) - 6NH^2 \right) + \right. \\ & + \frac{1}{2}\left(N^{-1}(\dot{\bar{\phi}} + \dot{\varphi} - N^i\partial_i\varphi)^2 - Nh^{ij}\partial_i\varphi\partial_j\varphi \right) + \\ & \left. + N\frac{\dot{\bar{\phi}}^2}{2} - N\left(V'(\bar{\phi})\varphi + \frac{V''(\bar{\phi})}{2}\varphi^2 + \cdots \right) \right]. \end{aligned} \tag{5.7}$$

Let us study (5.6) and (5.7) for a moment. We split up (5.7) into different pieces containing different factors of $M_{\rm Pl}$. In the last line of (5.7), a large piece of the action has been separated with a prefactor of $M_{\rm Pl}^2$. These terms will produce the most dominant contributions to the action. The left-over pieces of the action do not have a prefactor of $M_{\rm Pl}^2$. This might not be obvious at first sight. We can make this explicit by expanding the potential around the background solution,

$$V(\phi) = V(\bar{\phi}) + V'(\bar{\phi})\varphi + \frac{1}{2}V''(\bar{\phi})\varphi^2 + \frac{1}{6}V'''(\bar{\phi})\varphi^3 + \ \ldots\,, \tag{5.8}$$

here a piece of the background value $V(\bar{\phi})$ is of $\mathcal{O}(M_{\rm Pl}^2)$ and therefore converges in the limit (5.2). This contribution can be absorbed into the cosmological constant. The subleading piece is proportional to $\dot{\bar{\phi}}^2$ and carries no $M_{\rm Pl}^2$. This separation has been made explicit in the last line of (5.7). The other terms in $V(\phi)$ involve fluctuations and derivatives of the potential. In our decoupling limit, these derivatives are finite, and thus belong to the small piece of the action.

Suppose that we switch off the graviton degree of freedom by setting $h_{ij} = a^2\delta_{ij}$. Writing out the constraint equations, one obtains

$$M_{\rm Pl}^2\left(R^{(3)} - 6H^2 - N^{-2}h^{ij}h^{kl}(E_{ik}E_{jl} - E_{ij}E_{kl})\right) +$$
$$-M_{\rm Pl}^0\left(N^{-2}(\dot{\bar{\phi}} + \dot{\varphi} - N^i\partial_i\varphi)^2 + (-\dot{\bar{\phi}}^2 + 2V'(\bar{\phi})\varphi + \cdots) + h^{ij}\partial_i\varphi\partial_j\varphi\right) = 0\,. \qquad (5.9)$$
$$M_{\rm Pl}^2\left(\nabla_a\left(N^{-1}(h^{ab}E_{bi} - \delta_i{}^a h^{bc}E_{bc})\right)\right) + M_{\rm Pl}^0\left(N^{-1}\partial_i\varphi(N^j\partial_j\varphi - \dot{\bar{\phi}} - \dot{\varphi})\right) = 0\,. \qquad (5.10)$$

We note that the leading order terms in the constraint equations have no explicit φ dependence, therefore we expect that they are solved self-consistently at leading order in $M_{\rm Pl}^2$. We note that the constraints are solved by $N = 1$ and $N^i = 0$ up to terms of order $\mathcal{O}(M_{\rm Pl}^2)$. Therefore we have that the solutions to the constraint equations must following the ansatz:

$$N = M_{\rm Pl}^0(1 + \mathcal{O}(\gamma^2)) + M_{\rm Pl}^{-2}O(\varphi, \gamma^2) + O(M_{\rm Pl}^{-4})\,, \qquad (5.11)$$
$$N^i = M_{\rm Pl}^0(\mathcal{O}(\gamma^2)) + M_{\rm Pl}^{-2}O(\varphi, \gamma^2) + O(M_{\rm Pl}^{-4})\,. \qquad (5.12)$$

Note that the graviton even appears in the strict $M_{\rm Pl}^2 \to \infty$ limit. This is because in this limit, we should recover the limit of pure gravity in the de Sitter space and, in this limit, the solutions to the constraint equations are non-trivial. The graviton fluctuations are absent due to our gauge fixing conditions of transversality and tracelessness. Higher order corrections to the constraints, corrections of $\mathcal{O}(M_{\rm Pl}^2)$, are determined by successive approximations; the $\mathcal{O}(M_{\rm Pl}^2)$ corrections are used in the constraint equations at order $\mathcal{O}(M_{\rm Pl}^{-2})$, the $\mathcal{O}(M_{\rm Pl}^0)$ corrections are used in the constraint equations to $\mathcal{O}(1)$, etc. Calculating the the constraint up to linear order corrections in perturbations, one obtains

$$N = 1 + M_{\rm Pl}^{-2}\frac{\dot{\bar{\phi}}}{2H}\varphi + O(\varphi^2) \quad \text{and} \quad N^i = -M_{\rm Pl}^{-2}\frac{\dot{\bar{\phi}}}{2H}\frac{\partial_i}{\partial^2}\left[\dot{\varphi} + \varphi H\frac{\eta}{2}\right] + \mathcal{O}(\varphi^2)\,. \qquad (5.13)$$

There is an interesting simplification of the constraint equations in the scalar sector, if we use $M_{\rm Pl}^{-2}$ as a small expansion parameter, rather than φ. In other words, consider setting $\gamma_{ij} = 0$, keeping all powers and derivatives of φ, but expand the lapse and shift to $O(M_{\rm Pl}^{-2})$,

$$N(\varphi, \gamma = 0) = 1 + M_{\rm Pl}^{-2}N^{(1)}(\varphi) + \mathcal{O}(M_{\rm Pl}^{-4})\,,$$
$$N_i(\varphi, \gamma = 0) = M_{\rm Pl}^{-2}N_i^{(1)}(\varphi) + \mathcal{O}(M_{\rm Pl}^{-4})\,. \qquad (5.14)$$

Again, it will be convenient to use the Helmholtz decomposition of a vector field, $N_i^{(1)} = \partial_i\psi^{(1)} + \widetilde{N}_i^{(1)}$. In the flat gauge, the geometric quantities simplify considerably,

$$R^{(3)} = 0, \quad E_{ij} = Hh_{ij} - \frac{1}{2}(\partial_i N_j + \partial_j N_i) \quad \text{and} \quad E = 3\dot{\rho} - \partial_k N^k\,. \qquad (5.15)$$

To the order we are interested in, the constraint equations read

$$6HN^{(1)} + a^{-2}\partial_i N_i^{(1)} + L = 0\,,$$
$$2H\partial_i N^{(1)} + \tfrac{a^{-2}}{2}\left(\partial_i\partial_j N_j^{(1)} - \partial^2 N_i^{(1)}\right) + P_i = 0\,, \qquad (5.16)$$

with

$$\begin{aligned} L &\equiv \frac{1}{H}\left(\dot{\bar{\phi}}\dot{\varphi} + \frac{1}{2}\dot{\varphi}^2 + \frac{1}{2}h^{ij}\partial_i\varphi\partial_j\varphi + (V'(\bar{\phi})\varphi + \cdots)\right) \,, \\ P_i &\equiv -\partial_i\varphi(\dot{\bar{\phi}} + \dot{\varphi}) \,. \end{aligned} \tag{5.17}$$

Solving the constraints (5.16) in a similar fashion as we did in perturbation theory in the field fluctuation, we obtain

$$\psi^{(1)} = a^2\partial^{-2}\left(3\partial^{-2}\partial_i P_i - L\right) \,, \tag{5.18}$$

$$\tilde{N}_i^{(1)} = 2a^2\partial^{-2}\left(P_i - \partial^{-2}\partial_i\partial_j P_j\right) \,, \tag{5.19}$$

$$N^{(1)} = -\frac{1}{2H}\partial^{-2}\partial_i P_i \,. \tag{5.20}$$

Substituting the solutions (5.13) into (5.6) and expand the action up to second order in perturbations, we find

$$S_2 = \frac{1}{2}\int dtd^3x\, a^3\left[\dot{\varphi}^2 - \frac{\partial\varphi^2}{a^2} - V''\varphi^2 + \left(6\varepsilon + 2\varepsilon\eta - 2\varepsilon^2\right)\ H^2\varphi^2\right] \tag{5.21}$$

$$= \frac{1}{2}\int dtd^3x\, a^3\left[\dot{\varphi}^2 - \frac{\partial\varphi^2}{a^2} - V''\varphi^2 + \right. \tag{5.22}$$

$$\left. -2\frac{\dot{\bar{\phi}}}{HM_{\rm Pl}}\frac{V'}{M_{\rm Pl}}\varphi^2 - \left(\frac{\dot{\bar{\phi}}}{HM_{\rm Pl}}\right)^2\frac{V}{M_{\rm Pl}^2}\varphi^2\right] + \tag{5.23}$$

$$+\frac{1}{2}\int d^4x\ e^{3\rho}\left(\partial_k\left[\partial_j\psi^{(1)}\partial^j\partial^k\psi^{(1)} - \partial^k\psi^{(1)}\partial^2\psi^{(1)} - 2\dot{\bar{\phi}}\partial^k\psi^{(1)}\varphi\right]e^{-2\rho} + \right.$$

$$\left. -\partial_0\left[\frac{\dot{\bar{\phi}}}{2\dot{\rho}}\varphi^2\right]\right) \,, \tag{5.24}$$

here we made the Planck suppression explicit for all terms in the action. All terms that have vanishing contributions due to this Planck suppression are given in (5.23). Also, note that the boundary terms given the first line of (5.24) are not actual boundary terms again after substitution of the solution for $\psi^{(1)}$.

5.3 Linear dynamics

In order to calculate correlation functions for the scalar perturbations, we have to find the mode functions that correspond to φ. In order to do this, we have to vary the non-vanishing quadratic terms in (5.22) with respect to φ and then solve the equation. Varying (5.22) with respect to φ yields the *Mukhanov-Sasaki equation* for a massive scalar field,

$$\varphi'' - \frac{2}{\tau}\varphi' + k^2\varphi + \frac{m^2}{H^2\tau^2}\varphi = 0\,, \tag{5.25}$$

here the mass has been defined as

$$m^2 = -\partial_\phi^2 V(\phi) + \left(6\varepsilon + 2\varepsilon\eta - 2\varepsilon^2\right) H^2, \tag{5.26}$$

k is the magnitude of the comoving momentum $\mathbf{k}$, τ is the conformal time, $dt \equiv a(\tau)d\tau$, and the prime denotes the derivative with respect to τ. In order to calculate the φ correlators, the scalar perturbation is again promoted to an operator that can be expanded in terms of creation and annihilation operators,

$$\varphi(t, \mathbf{k}) = u(k)\hat{a}_{\mathbf{k}} + u^*(k)\hat{a}^\dagger_{\mathbf{k}}, \tag{5.27}$$

where we assume that $\hat{a}_{\mathbf{k}}$ annihilates the Bunch-Davies vacuum and that $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ satisfy the usual commutation relation,

$$\left[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')\right] = (2\pi)^2\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2). \tag{5.28}$$

The dymamics of φ is then determined by (5.25) together with the Wronskian condition,

$$\partial_x u(x)u^*(x) - \partial_x u^*(x)u(x) = -i. \tag{5.29}$$

In contrary to the ζ equations of motion of last chapter we now have an additional mass term in the equation of motion. This mass term depends on the slow-roll parameters and on the running of the potential. Usually, this mass term is considered to be small and can be positive or negative. As it turns out, in order to test the decoupling limit, a more general treatment of the mass term is needed.

The mode functions can be split up into two regimes, where we look at the value of the mass. For $m^2/H^2 \leq 9/4$ the dynamics of the mode functions is captured by

$$u(\tau, k) = -ie^{-i\frac{\pi}{2}(\nu+\frac{1}{2})}\frac{\sqrt{\pi}}{2}H(-\tau)^{\frac{3}{2}}\mathrm{H}^{(1)}_\nu(-k\tau), \tag{5.30}$$

here we defined $\nu = \sqrt{9/4 - m^2/H^2}$. For $m^2/H^2 > 9/4$ the dynamics of the mode functions is captured by

$$u(\tau, k) = -ie^{-\frac{\pi}{2}\tilde{\nu}+\frac{i\pi}{4}}\frac{\sqrt{\pi}}{2}H(-\tau)^{\frac{3}{2}}\mathrm{H}^{(1)}_{i\tilde{\nu}}(-k\tau), \tag{5.31}$$

here we defined $\tilde{\nu} = \sqrt{m^2/H^2 - 9/4}$. The normalization of both mode functions has been chosen such that, when the momentum k/a is much larger than the Hubble parameter H and the mass m, we recover the Bunch-Davies vacuum

$$u_k \to -\frac{H}{\sqrt{2k}}\tau e^{-ik\tau}. \tag{5.32}$$

Our main interest lays in the region where $\tau \to \tau_*$, which is the moment around horizon crossing of the modes. After analytically continuation of the mode functions in this limit, we obtain

$$u(\tau,k) = i\frac{2^{\nu-1}}{\sqrt{\pi}}\Gamma(\nu)\frac{H}{k^{\nu}}(-\tau)^{-\nu+\frac{3}{2}}, \qquad \frac{m^2}{H^2} < \frac{9}{4}. \tag{5.33}$$

$$u(\tau,k) = i\frac{1}{\sqrt{\pi}}H(-\tau)^{3/2}\ln(-k\tau), \qquad \frac{m^2}{H^2} = \frac{9}{4}, \tag{5.34}$$

$$u(\tau,k) = e^{-\frac{\pi}{2}\tilde{\nu}}\frac{\sqrt{\pi}}{2}H(-\tau)^{\frac{3}{2}}\left[\frac{1}{\Gamma(i\tilde{\nu}+1)}\left(\frac{-k\tau}{2}\right)^{i\tilde{\nu}} - \frac{i\Gamma(i\tilde{\nu})}{\pi}\left(\frac{-k\tau}{2}\right)^{-i\tilde{\nu}}\right] \qquad \frac{m^2}{H^2} > \frac{9}{4}. \tag{5.35}$$

The decay factors indicate that the perturbations for massive fields eventually roll back in the potential to zero, (5.35) has an oscillation facor of $\tau^{\pm i\tilde{\nu}}$. Also, as the value of the mass m^2 increases, the behavior of the perturbations changes from that of an over-damped oscillator corresponding to $(m^2/H^2 < 9/4)$ to that of an under-damped oscillator corresponding to $(m^2/H^2 > 9/4)$. In the under-damped case, the contributions to correlation functions are suppressed by a factor of $\sim e^{-m/H}$ per mode. Because of this suppression, this limit will not be interesting for us and we will concentrate at the case in which the mode functions are described by (5.33).

5.3.1 Exact expressions for the mode functions

There are certain limits, for which the Bessel equations can be solved exactly that determine the modefunctions. These mode functions correspond to mode functions with a half-integer value for ν. If we only look at the (conformal) time- and momentum dependence in the mode functions we have

$$u(-k\tau) \sim \tau^{\frac{3}{2}}H_{\nu}^{(1)}(-k\tau) = \tau^{\frac{3}{2}}\left(J_{\nu}^{(1)}(-k\tau) + iY_{\nu}^{(1)}(-k\tau)\right). \tag{5.36}$$

There are two exact solutions for the mode functions for which $0 < \nu \leq 3/2$, i.e. for which ν is equal to $\nu = 1/2$ and $\nu = 3/2$. The mode functions for these values scale as

$$\begin{aligned} u(-k\tau) &\sim \tau^{\frac{3}{2}}H_{1/2}^{(1)}(-k\tau) = -i\sqrt{\frac{2}{\pi k}}\tau\, e^{ik\tau}, \\ u(-k\tau) &\sim \tau^{\frac{3}{2}}H_{3/2}^{(1)}(-k\tau) = -i(1-ik\tau)\sqrt{\frac{2}{\pi k^3}}e^{ik\tau}. \end{aligned} \tag{5.37}$$

To see which masses correspond to these limits, we have to study ν. We can expand ν in terms of the slow-roll parameters and the mass

$$\nu = \sqrt{\frac{9}{4} + 3\epsilon + \frac{3}{2}\eta} \equiv \sqrt{\frac{9}{4} + 9\epsilon - 3\eta_v} \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}},$$

here we have defined the following *Potential slow-roll* parameter $\eta_v \equiv M_{\rm Pl}\frac{V''}{V} \ll \mathcal{O}(1)$ [14]. This suggests that $\nu = 1/2$ corresponds to a very heavy massive scalar field with a mass of $m^2 = 2H^2$ and that $\nu = 3/2$ corresponds to a massless scalar field, $m^2 = 0$. Both of these expressions can be used to calculate the Bispectrum, since these are the only ones which we can integrate exactly.

5.4 Scalar Power spectrum

In this section, we derive the usual tilt of the ζ-power spectrum by converting the scalar perturbation of φ into ζ at the same time τ_*, i.e. at a moment when all modes are well outside the horizon, for every k mode. In the conversion $\zeta = \varphi/\sqrt{2\epsilon_*}$ and the slow-roll parameter $\epsilon \equiv \dot{H}/H^2$ are evaluated at the time τ_* for any k and therefore induces no corrections to the tilt.

When calculating the two point function in the decoupling limit (5.2), we have that $\epsilon \ll 1$. In this limit, we have a vanishing positive vacuum expectation value, $\dot{\bar{\phi}} \to 0$, meaning that our slow-roll expansion does not break the dS isometries anymore. It is also important to notice that higher slow-roll parameters can have a non-zero values and at the same time that the correlators still enjoy the full isometry group of de Sitter.

Consider the mode functions in the late time limit with a mass in the range $0 < m^2/H^2 < 9/4$ (5.35), then the truncated two point function for φ is given by

$$\langle \phi(k,\tau)\phi(-k,\tau')'\rangle = \frac{H^2}{M_{\rm Pl}^2}\frac{(\tau\tau')^{3/2}}{4\pi}\left[\Gamma(\nu)^2\left(\frac{k^2\tau\tau'}{4}\right)^{-\nu} + c.c.\right] . \tag{5.38}$$

From (2.35), (5.2) and (5.26), we note that during slow-roll inflation, the mass is generally very small $m \ll H$. Therefore, we will will expand ν and work at leading order in m,

$$\nu = \frac{3}{2}\left[1 - \frac{2}{9}\frac{m^2}{H^2} + \mathcal{O}\left(\frac{m^4}{H^4}\right)\right] \approx \frac{3}{2} - \eta_V . \tag{5.39}$$

At this moment, a comment is in order. It is not directly clear from our definitions, but in (5.39) we have a negative mass. This can be made explicit when relating the *Hubble* and *Potential* slow-roll parameters to each other by $\eta_V = 2\epsilon - \eta/2 \approx -\eta/2$. Then using $\Gamma(3/2) = \sqrt{\pi}/2$ and keeping the leading order m/H term only in the exponents, we find the following ζ power spectrum

$$\langle \zeta(k,\tau_*)\zeta(-k,\tau_*)\rangle = \frac{1}{2\epsilon M_{\rm Pl}}\frac{H^2}{2\epsilon M_{\rm Pl}}\tau_*^{2\eta_V} k^{-3+2\eta_V} . \tag{5.40}$$

Using $\Gamma(3/2) = \sqrt{\pi}/2$ and keeping the leading order m/H term only in the exponents we obtain

$$\langle \zeta(k,\tau_*)\zeta(-k,\tau_*)\rangle = \frac{1}{2\epsilon M_{\rm Pl}}\frac{H^2}{2\epsilon M_{\rm Pl}}\tau_*^{2\eta_V} k^{-3+2\eta_V} . \tag{5.41}$$

The tilt derived from this expression agrees with the standard result [14],

$$(n_s - 1) = -6\epsilon_V + 2\eta_V , \tag{5.42}$$

here ϵ_V has been approximated at leading order in slow-roll as $\epsilon \sim \epsilon_V \to 0$.

5.5 Three point functions

In the next sections, we show that in the decoupling limit (5.2) the interaction part of the action simplifies. We give the expression for the bispectrum in the spatially flat gauge and work out the transformation of the bispectrum of a general massive scalar field φ to the comoving gauge and we show that this transformation can be related to the δN-formalism. Then we relate the running of the potential to the ζ-bispectrum. Last, but not least, we check our results by taking the decoupling limit of our the calculation performed in 4.6.

As non-Gaussianities arise from departures from non-linear couplings, we need to calculate the action to cubic order in perturbations in order to calculate the bispectrum. When assuming the effects of interactions are small, the higher correlation functions can be obtained by a perturbative expansion of the interaction Hamiltonian around the free theory. These higher correlation functions can again be calculated using the *in-in formalism*,

$$\left\langle \prod_a \varphi_a(t) \right\rangle = \left\langle \left[\bar{T} \exp\left(i \int_{t_0}^{t} dt \; H_{int}(t') \right) \right] \prod_a \hat{\varphi}_a^I(t) \left[\bar{T} \exp\left(-i \int_{t_0}^{t} dt \; H_{int}(t') \right) \right] \right\rangle, \quad (5.43)$$

where the superscript I signifies that these modes are evolved using the linear (free) equations of motion, and T and $\bar{T}$ refer to time-ordering and anti-time-ordering.

5.5.1 The Spatially-flat gauge bispectrum

In order to calculate the bispectrum, we have to expand the action to cubic order in perturbations. As shown earlier, with the n$^{\text{th}}$ order solutions of the constraints, we can build the $(2n+1)^{\text{th}}$ order action. For the cubic action, this means that we only need the solutions of N and N_i to first order in perturbations. Then

$$\begin{aligned} S_3 = \int d^4x \; a^3 \Bigg[& \frac{a^{-4}}{2M_{\text{Pl}}^6} \left(-\partial_i N_j^{(1)} \partial^i N^{j(1)} + \partial_i N^{i(1)} \partial_j N^{j(1)} \right) N^{(1)} \\ & + a^{-2} \left(\frac{2H}{M_{\text{Pl}}^6} \partial_i N^{i(1)} (N^{(1)})^2 - \frac{1}{M_{\text{Pl}}^2} N_i^{(1)} \partial^i \varphi \dot{\varphi} + \frac{1}{M_{\text{Pl}}^4} \dot{\bar{\phi}} \; N^{(1)} N_i^{(1)} \partial^i \varphi - \frac{1}{M_{\text{Pl}}^2} N^{(1)} (\partial\varphi)^2 \right) \\ & + \frac{(6H^2 - \dot{\bar{\phi}}^2)}{2M_{\text{Pl}}^6} (N^{(1)})^3 - \frac{1}{M_{\text{Pl}}^2} N^{(1)} \dot{\varphi}^2 + \frac{1}{M_{\text{Pl}}^4} \dot{\bar{\phi}} \; (N^{(1)})^2 \dot{\varphi} - \frac{1}{6} V'''(\bar{\phi}) \varphi^3 - \frac{V''(\bar{\phi})}{2M_{\text{Pl}}^2} \varphi^2 N^{(1)} \Bigg]. \end{aligned} \quad (5.44)$$

In the limit (5.2), both $N^{(1)}$ and $N^{i(1)}$ scale with ϵ. Then the only term in S_3 that is not Planck suppressed comes from expanding the potential around the background value of ϕ,

$$S_3 = \int d^4x \; a^3 \left[-\frac{1}{6} V'''(\bar{\phi}) \varphi^3 \right]. \quad (5.45)$$

In [22] it was noted that the Bispectrum of a scalar field φ can be calculated exactly when ν is half-integer, since only then an exact solution can be found for (5.33). There are two cases that lay within the range $0 < \nu \leq 3/2$, namely $\nu = 3/2$ and $\nu = 1/2$. These values for μ correspond to scalar fields with a mass $m = 0$ and $m = \sqrt{2}H$ respectively. The bispectrum of a massless scalar field is then given by [22]

$$\langle\varphi(\mathbf{k}_1,\tau_*)\varphi(\mathbf{k}_2,\tau_*)\varphi(\mathbf{k}_3,\tau_*)\rangle' = \\ = \frac{H^2}{k_1^3k_2^3k_3^3}\frac{V'''(\bar{\phi})}{12}\left[(-1+\gamma_E+\log(-k_t\tau_*))\sum_{i=1}^{3}k_i^3 - \sum_{i\neq j}k_i^2k_j + k_1k_2k_3\right], \tag{5.46}$$

here $k_t \equiv k_1 + k_2 + k_3$ and τ_* is the moment inflation ends. The bispectrum of a scalar field with $m = \sqrt{2}H$ is given by [22]

$$\langle\varphi(\mathbf{k}_1,\tau_*)\varphi(\mathbf{k}_2,\tau_*)\varphi(\mathbf{k}_3,\tau_*)\rangle' = \frac{\pi H^2 V'''(\bar{\phi})}{8}\frac{\tau_*^3}{k_1k_2k_3}, \tag{5.47}$$

here the prime on the brackets indicate the removal of the momentum conserving delta function and a factor of $(2\pi)^3$. When calculating (5.46) and (5.47), it is very important to take the limit of $k_*\tau_* \ll 1$ at the very end of the calculation to avoid relative prefactor mistakes between the different momentum functions in (5.46). Both of these expressions can be used as a check when building a general formalism to calculate three point correlation functions for an arbitrary mass.

5.5.2 Conformal analysis of the three point function

There are two methods to construct primordial correlation functions. The first and most common way is to start from a general action, solving the free equations of motion and then use the higher order action as perturbative corrections to the theory. The second method is by looking at which isometries the background imposes on the correlation functions and use that to constrain the general expression for the correlation function. However, with the isometries, we can only build correlation functions up to a certain prefactor. For example if $A(k_i)$ is an isometry of the background, the correlation function should satisfy

$$\sum_{a=1}^{n} A_a\langle\varphi(k_1)\cdot ... \cdot\varphi(k_n)\rangle = 0\,, \tag{5.48}$$

here the sum over a corresponds to a sum over all different fields with momentum k_a.

In the limit (5.2), we expect that our spacetime reduces to a pure de Sitter spacetime and that all correlation functions should satisfy it's SO(1,3) group. So far, we have only used the *translation*- and the *rotation* isometries to constrain the degrees of freedom of our correlation function, since these were also isometries of the Quasi-de Sitter background. In 4.4 we saw that the correlation functions should contain a momentum conserving Dirac delta function and could only depend on the length of the momenta vectors. In a de Sitter/Conformal spacetime, correlation functions should also be invariant under the *Dilation* and *Special conformal* isometries. These will constrain the overal momentum and conformal time dependence and will also constrain the shape of the function.

Let's start our conformal analyzation of (5.46) and (5.47) by studying the *dilation* isometry. The differential form of the dilation isometry in momentum space of a truncated three point function is given by [24]

$$\left(\sum_{a=1}^{3} D_a - (d-1)\right) \langle\varphi(k_1)\cdot ... \cdot\varphi(k_n)\rangle' = \left(-3(\Delta-3) + k_a\partial^a - d + 1\right) \langle\varphi(k_1)\cdot ... \cdot\varphi(k_n)\rangle' \,, \tag{5.49}$$

here a is not a spacetime index but is again a label for the momentum, n is the number different momenta in the momentum conserving delta function that has been truncated and Δ is the conformal scaling dimension which can be related to ν by $\Delta = 3/2 - \nu$ [22]. Also, it is good to note why we have subtracted n in this equation from the usual dilation isometry; we are looking at an stripped correlator and the n is the correction for the truncation. More explicitly, consider the delta function containing three momenta

$$k_i\partial^i\delta^{(3)}\left(\sum_{i=1}^{3}\mathbf{k}_i\right) = (-1-1-1)\;\delta^{(3)}\left(\sum_{i=1}^{3}\mathbf{k}_i\right)\,. \tag{5.50}$$

For three point correlation functions, we should subtract 3 from D.

Applying (5.49) on (5.47) we obtain

$$\begin{aligned}\left(\sum_{i=1}^{3} D_i - 3\right)\left[\frac{\tau_*^3}{k_1k_2k_3}\right] &= \left(-3(\Delta-3) + k_i\partial^i - 3\right)\left[\frac{\tau_*^3}{k_1k_2k_3}\right] \\ &= (-3+9-3-3)\left[\frac{\tau_*^3}{k_1k_2k_3}\right] \\ &= 0\,.\end{aligned}$$

note that $\Delta = 1$ for $\nu = 1/2$. Thus (5.47) satisfies the Dilation isometry. Doing the same for (5.46) we find

$$\begin{aligned}\left(\sum_{i=1}^{3} D_i - 3\right)\left[\frac{k_1^3+k_2^3+k_3^3}{(k_1^3k_2^3k_3^3)}\right] &= \left(-3(\Delta-3) + k_i\partial^i - 3\right)\left[\frac{(k_1^3+k_2^3+k_3^3)}{k_1^3k_2^3k_3^3}\right] \\ &= (9-6-3)\left[\frac{(k_1^3+k_2^3+k_3^3)}{k_1^3k_2^3k_3^3}\right] \\ &= 0\,,\end{aligned} \tag{5.51}$$

note that for $\nu = 3/2$ we have $\Delta = 0$. Here we only acted with $D-3$ on the local part of the correlation function. We can do this because the dilation isometry only looks at the overall momentum and (conformal) time dependence of the correlator, since (5.46) is invariant under cyclic exchange of k_1, k_2 and k_3.

The analyse the three point functions with *special conformal* isometry is a little bit more involved than the analysis with the dilation isometry. The reason for this is that the special conformal isometry is a vector which depends on $\mathbf{k}_a^\mu$. The differential form of the special conformal isometry is given by [24]

$$\sum_{a=1}^{3} b_i K_a^i = \sum_{a=1}^{3} b_i k_a^i \left[\partial_a^2 + \frac{(d+1-2\Delta_a)}{k_a}\partial_a\right], \tag{5.52}$$

here we have contracted K_a^μ with a arbitrary vector b_i to get a scalar operator. By making smart use of rotational invariance of the three point correlation function, it can be shown E.2 that (5.52) can be reduced when action on (5.47) and (5.46) to

$$\sum_{a<b}^{3} b_i K_a^i = \left(\left[\partial_a^2 + \frac{(d+1-2\Delta_a)}{k_a}\partial_a\right] - \left[\partial_b^2 + \frac{(d+1-2\Delta_b)}{k_b}\partial_b\right]\right), \tag{5.53}$$

with $a \neq b$. A full derivation is given in appendix E.2.

When acting with (5.53) on (5.47) and (5.46), we find that both expressions are annihilated [22]. At this moment, a rather important comment is in order. The shape of (5.47) is relatively simple and consists out of one single term and nothing out of the ordinary is happening here, the shape of (5.46) however is much more complicated and consists of *local*, *equilateral* and local times *logarithmic terms*. Therefore is not directly obvious that the full massless bispectrum is annihilated by all isometries. As usual, the *local* terms are annihilated by all isometries. However, the equilateral shape (4.147) and the logarithmic term multiplying a local term separately, are not. When we added together, both contributions cancel eachother. The reason why this is happening can be found when doing a more formal analysis of (5.52). As we show in E.2, this combination of terms appears as higher order corrections when solving the (5.52) using dimensional regularization. This means that this shape follows from the conformal isometries and therefore is invariant. For this reason we can define this new shape as the *conformal shape*,

$$B_{SCT} \sim \frac{\log(-k_t\tau_*)\sum_{i=1}^{3} k_i^3 - \sum_{i\neq j} k_i^2 k_j + k_1 k_2 k_3}{k_1^3 k_2^3 k_3^3}. \tag{5.54}$$

A plot of this shape is given in 5.1

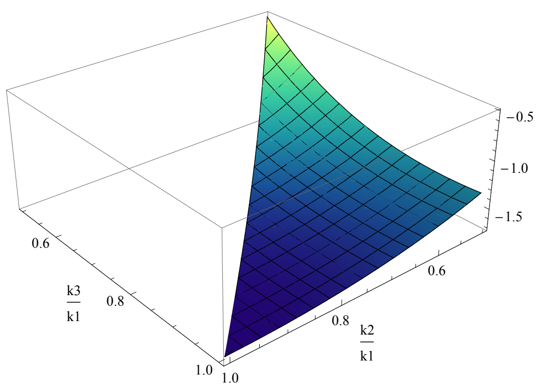

Figure 5.1: The conformal shape plotted as function of k_2/k_1 and k_3/k_1 and with k_1 fixed.

5.6 A scalar consistency relation

Just like in the comoving gauge where we had a consistency relation for ζ, we can define a consistency relation in the spatially flat gauge. This time however, it will be a consistency relation for φ rather than ζ. In the comoving gauge, we saw that the squeezed limit of a correlation function corresponded to a coordinate transformation. In this case, when we are working in the spatially flat gauge, we can see a long φ mode as a homogeneous background fluctuation, to leading order in derivatives.

Before we continue to the effects on correlation functions, let us study the background equations of motion for δH and $\delta \dot{H}$ first. To study the effects on the unperturbed Hubble parameters and its time derivative(s), we need to solve pertubatively these background equations of motion (5.3). To linear order, one obtains

$$V'(\bar{\phi})\varphi = M_{\rm Pl}^2(6H\delta H + \delta\dot{H}) \quad \text{and} \tag{5.55}$$

$$2\dot{\bar{\phi}}\dot{\varphi} = -2M_{\rm Pl}^2\delta\dot{H}\,. \tag{5.56}$$

Solving these equations for δH and $\delta\dot{H}$, we obtain

$$\delta H = \frac{1}{6M_{\rm Pl}^2 H}\left[(-\ddot{\bar{\phi}} - 3H\dot{\bar{\phi}})\varphi + \dot{\bar{\phi}}\dot{\varphi}\right]\,, \text{ and} \tag{5.57}$$

$$\delta\dot{H} = -\frac{\dot{\bar{\phi}}\dot{\varphi}}{M_{\rm Pl}^2}\,, \tag{5.58}$$

here we used (5.3) to write $V'(\bar{\phi})$ in terms of time derivatives of $\bar{\phi}$. As the numerators are all finite, while the denominator is proportional to $M_{\rm Pl}^2$, we see that the background evolution is not affected by a homogeneous φ fluctuation in the limit (5.2).

Now that we have seen that the background equations of motion remain uneffected by a long φ mode, let us continue to the effect on correlation functions. Taking one of the modes to be long for a n-point correlation function, corresponds to taking the squeezed limit, i.e. $\lim_{k_i\to 0}\langle\varphi(\mathbf{k},\tau)^n\rangle$. The effect on this correlation is intuitively the following; as one of the modes will have a significantly smaller wavelength than the others, the wavelength of the longer mode will surpass the Hubble radius much earlier than the short modes. This mode function will therefore become super Hubble size much earlier than the other and will contribute to the background for the short modes. This long mode can therefore effectively be seen as a field shift in the background fields for the short modes. Following this argumentation, one can write

$$\langle\varphi_s(\mathbf{x}_1,\tau)\cdot\ldots\cdot\varphi_s(\mathbf{x}_n,\tau)|\varphi_l(\mathbf{z},\tau)\rangle = \langle\tilde{\varphi}_s(\mathbf{x}_1,\tau)\cdot\ldots\cdot\tilde{\varphi}_s(\mathbf{x}_n,\tau)\rangle|_{\varphi_l}\,. \tag{5.59}$$

Here the subscript s stands for *small wavelength*-modes and the subscript l stands for *long wavelength*-modes. The last term is evaluated in the background of a $\varphi_l(z)$ mode. I assume that $\mathbf{z}$ is of the same order as all the other $\mathbf{x}_i$ of the short mode fields. Now we can expand the right hand side of (5.59) as

$$\begin{aligned}\langle\tilde{\varphi}(\mathbf{x}_1,\tau)\cdot\ldots\cdot\tilde{\varphi}(\mathbf{x}_n,\tau)\rangle|_{\varphi_l} =& \langle\varphi_s(\mathbf{x}_1,\tau)\cdot\ldots\cdot\varphi_s(\mathbf{x}_n,\tau)\rangle \\ &+ \varphi_l(z)\left[\frac{\delta}{\delta\varphi_l(z)}\langle\varphi_s(\mathbf{x}_1,\tau)\cdot\ldots\cdot\varphi_s(\mathbf{x}_n,\tau)\rangle\right]_{\varphi_l\to 0} + \mathcal{O}(\varphi_l^2)\,.\end{aligned} \tag{5.60}$$

To prove that there is a consistency relation, we have to use a little trick. We will multiply both sides of (5.59) from the left with $\varphi_l(\mathbf{x},\tau)$ and then take the average, i.e.

$$\langle \varphi_l(\mathbf{x},\tau) \langle \varphi_s(\mathbf{x}_1,\tau) \cdot ... \cdot \varphi_s(\mathbf{x}_n,\tau) | \varphi_l(\mathbf{z},\tau) \rangle \rangle = \langle \varphi_l(\mathbf{x},\tau) \ \langle \tilde{\varphi}_s(\mathbf{x}_1,\tau) \cdot ... \cdot \tilde{\varphi}_s(\mathbf{x}_n,\tau) \rangle |_{\varphi_l} \rangle \,. \quad (5.61)$$

Since we are considering that the long-modes become super Hubble size much earlier than the short-modes, we can only make Wick contractions between two short modes and between two long modes, so no cross Wick contractions between a φ_s and a φ_l are allowed. This means that the first term of (5.60) will give a zero contribution, which means that the leading contributions come from the second term. Then substituting (5.60) into (5.61) we find that the squeezed limit of the three point function becomes

$$\begin{aligned} \langle \varphi_l(\mathbf{x}_1,\tau) | \varphi_s(\mathbf{x}_2,\tau) \varphi_s(\mathbf{x}_3) \rangle &\approx \langle \varphi_l(\mathbf{x}_1,\tau) \langle \varphi_s(\mathbf{x}_2,\tau) \varphi_s(\mathbf{x}_3) \rangle_{\varphi_l} \rangle \\ &= \left\langle \varphi(\mathbf{x}_1,\tau) \varphi_l(\mathbf{z},\tau) \ \frac{\delta}{\delta \varphi_l(z)} \langle \varphi_s(\mathbf{x}_2,\tau) \varphi_s(\mathbf{x}_3) \rangle_{\varphi_l} \Big|_{\varphi_l \to 0} \right\rangle \\ &= \langle \varphi_l(\mathbf{x}_1,\tau) \varphi_l(\mathbf{z},\tau) \rangle \left[\frac{\delta}{\delta \varphi_l(z)} \langle \varphi_s(\mathbf{x}_2,\tau) \varphi_s(\mathbf{x}_3) \rangle_{\varphi_l} \right]_{\varphi_l \to 0} . \end{aligned} \quad (5.62)$$

Since we prefer working in momentum space, since the calculations are much easier there when solving the bispectrum, we have to make Fourier transformation of (5.62). Then the consistency relation for φ in momentum space takes the following form

$$\begin{aligned} \langle \varphi(\mathbf{k}_1,\tau) \varphi(\mathbf{k}_2,\tau) \varphi(\mathbf{k}_3,\tau) \rangle' &\approx \langle \varphi_l(\mathbf{k}_1,\tau) \varphi_l(-\mathbf{k}_1,\tau) \rangle' \frac{\delta}{\delta \varphi_l} \ \langle \varphi_s(\mathbf{k}_2,\tau) \varphi_s(\mathbf{k}_3,\tau) \rangle' \big|_{\varphi_l \to 0} \\ &= \alpha_s \langle \varphi_l(\mathbf{k}_1,\tau) \varphi_l(-\mathbf{k}_1,\tau) \rangle' \ \langle \varphi_s(\mathbf{k}_2,\tau) \varphi_s(\mathbf{k}_3,\tau) \rangle' \big|_{\varphi_l \to 0} \end{aligned} \quad (5.63)$$

with $k_1 \ll k_2, k_3$.

5.7 Squeezed three point functions

In general, it is very difficult to calculate three point correlation functions for fields with a non-zero mass or when $m^2 \neq 2H^2$. This is because only when $m^2 = 0$ and $m^2 = 2H^2$, we can find exact expressions for the mode functions that we are able to integrate. However, via the squeezed limit of a correlation function we can calculate the local non-Gaussianities $f_{\rm NL}^{\rm local}$ of three point functions without having to do integrals. Therefore we want to test the validity, explore and and try to exploit (5.63) in the limit (5.2) to see whether it is possible to construct (more) general massive three point correlation functions. Also, the consistency relation for scalar-fields in the *spatially flat gauge* has never been fully tested for the entire bispectrum. Since in (5.2) the cubic interaction action reduces to only a single term, we can and will show that this consistency relation holds for particles with a mass of $m = 0$ or $m = \sqrt{2}H$.

A natural question one might ask at this point is the following, *do we loose information about the interactions when using the scalar consistency relation to compute three point functions?*. The answer to this question is no. The reason is the following; when taking one of the legs to be soft of the interaction vertex, corresponds to introducing an effective mass term for the short modes, φ_s,

$$-\frac{V'''}{6}\varphi^3 \to -\frac{V'''}{6}\left(\varphi_s^3 + 3\varphi_l\varphi_s^2 + 3\varphi_l^2\varphi_s + \varphi_l^3\right) . \tag{5.64}$$

Note that the terms with different modes will have a different relative prefactor due to symmetry factors. Since φ is real, we obtain the following effective mass for the short mode

$$m_{\rm eff}^2(\tau) = V'''(\bar{\phi})\varphi_l(\tau) . \tag{5.65}$$

It is very important to notice that this effective mass term has a time dependence. Contrary to the scalar perturbation ζ that freezes and becomes constant in (conformal) time after it leaves the horizon, the scalar perturbation φ does not freeze. Therefore, it seems that the the squeezed limit breaks conformal invariance, in the sense that we could not derive the correct squeezed bispectrum by only using conformal symmetries. However, "squeezing" a conformal three point correlation function in position space shows

$$\begin{aligned}\langle O_{\Delta_1}(x_1)O_{\Delta_2}(x_2)O_{\Delta_3}(x_3)\rangle \sim & \frac{1}{x_{12}^{\Delta_3-\Delta_1-\Delta_2}x_{13}^{\Delta_2-\Delta_1-\Delta_3}x_{23}^{\Delta_1-\Delta_2-\Delta_3}} + \\ & - \delta(x_{12})\frac{1}{x_{13}^{\Delta_2-\Delta_1-\Delta_3}} + 2 \text{ perm.} + \text{higher corrections} ,\end{aligned} \tag{5.66}$$

here we defined $x_{ij} \equiv |x_i - x_j|$. Making now a Fourier transformation to momentum space can give rise to non-trivial corrections that can explain the reason why we can not seemingly construct the squeezed bispectrum only from the momentum isometries. However, these higher correction terms have not been studied in literature in momentum space. The study of these terms in momentum space is therefore considered to beyond the scope of this thesis concerning Conformal field theories.

The equation of motion for φ_s one obtains by taking one of the legs to be soft of the interaction vertex is given by

$$\partial_\tau^2\varphi(\tau) - \frac{2}{\tau}\partial_\tau\varphi(\tau) + \left(k^2 + \frac{m^2}{\tau^2H^2}\right)\varphi(\tau) = -\frac{V'''(\bar{\phi})}{H^2\tau^2}\hat{\varphi}_l(k_l)\tau^{3/2-\nu}\varphi(\tau) , \tag{5.67}$$

here we dropped the subscript s of the short mode for notational convenience, $\hat{\varphi}_l(k_l)$ is the time independent part of the mode function and k_l is the momentum of the long mode. We shall refer to (5.67) as the *Effective-Mukhanov-Sasaki*-equation, or just *EMS*. Also, note that we used the late time behavior of the modefunctions, i.e. we used (5.33).

In general, Bessels equation with an additional time dependent term cannot be solved exactly. Therefore in order to find a solution to (5.67), we will iteratively solve (5.67) using Greensfunction methods. We first show that for $m = 0$ and $m = \sqrt{2}H$ the leading order solutions that follow from (5.63) and (5.67) are the same as the squeezed limits of (5.47) and (5.46). Then we will perturbatively solve (5.67) for the case that $m = 0 + m_0$ and $m = \sqrt{2}H + m_0$, where we assume $m_0 \ll 1$.

5.7.1 Perturbative solutions of the EMS

Let us first start with a brief review of Greens functional methods. A general inhomogeneous differential equation has the form

$$\hat{\mathcal{L}}(\tau)\phi(\tau) = f(\tau), \tag{5.68}$$

here $\hat{\mathcal{L}}$ is a differential equation in terms of τ and $f(\tau)$ is a random function depending on τ. To find a solution of $\phi(\tau)$, we split up the solution for ϕ into a *homogeneous* and a *particular* solution;

$$\phi(\tau) = \phi_H(\tau) + \phi_p(\tau)\,, \tag{5.69}$$

where ϕ_H solves (5.68) for $f(\tau) \to 0$. To solve the particular part of the solution for ϕ, we can look at particular solutions of the form

$$\phi_p(\tau) = \int_a^b d\tau'\; G(\tau,\tau')f(\tau')\,, \tag{5.70}$$

here $G(\tau,\tau')$ is the Green's function and a and b define the domain for which the particular solution of ϕ should be valid. Of course, when we act on this particular solution with the differential operator $\hat{\mathcal{L}}$, we should obtain $f(\tau)$ again,

$$\int_a^b d\tau'\; \hat{\mathcal{L}}(\tau)G(\tau,\tau')f(\tau') = f(\tau)\,. \tag{5.71}$$

Therefore, the Green's function should satisfy the following relation

$$\hat{\mathcal{L}}(\tau)G(\tau,\tau') = \delta(\tau-\tau')\,, \tag{5.72}$$

here $\delta(\tau-\tau')$ is the dirac delta function. To get back to our differential equation (5.67), there is one extra subtlety. When the mass of the particle increases, the time dependence of the effective mass increases. The problem that now arises is the following, the effective mass term is of the form $V'''(\bar{\phi})\varphi_l\varphi_s$. Then increasing the mass changes the φ_s mode function, changing the right hand side of (5.67). Therefore, to catch the full time dependence of φ, we need to iteratively solve φ,

$$\left(\partial_\tau^2\varphi^{(n+1)}(\tau) - \frac{2}{\tau}\partial_\tau\varphi^{(n+1)} + \left(k^2 + \frac{m^2}{\tau^2H^2}\right)\varphi^{(n+1)}\right) = -\frac{V'''(\bar{\phi})}{H^2\tau^2}\hat{\varphi}_l\tau^{3/2-\nu}\varphi^{(n)}\,. \tag{5.73}$$

here the subscript $(n+1)$ and (n) refer to the order of the correction to the Green's function and the full solution of φ will be of the form

$$\varphi(\tau,k)=\sum_{i=1}^{\infty}\varphi^{(n)}\,. \tag{5.74}$$

To prove my claim that we only need the linear order correction to the mode functions to catch the leading order contribution to the bispectrum in the squeezed limit, we shall calculate first explicitly the linear order corrections to the mode functions corresponding to a mass of $m=0$ and $m=\sqrt{2}H$ respectively and show that with these corrections we can calculate leading order squeezed bispectrum of (5.46) and (5.47).

In order to calculate the particular solution for a general massive scalar field φ, we first need to determine the corresponding Green's functions. The Green's function satisfies the following equation of motion for a massive particle in a de Sitter space

$$\sum_{i=1}^{\infty}\left(\Box-m^2\right)G^{(n)}(\tau,\tau')=\frac{1}{\sqrt{-g}}\delta(\tau-\tau')\,, \tag{5.75}$$

here the superscript (n) on the Green's function signifies that it is the linear order correction to the full Green's function. Working out the D'Alembertian explicitly, we obtain

$$\left(\partial_\tau^2 G^{(1)}(\tau,\tau')-\frac{2}{\tau}\partial_\tau G^{(1)}(\tau,\tau')+\left(k^2-\frac{m^2}{\tau^2H^2}\right)G^{(1)}(\tau,\tau')\right)=\tau^2H^2\delta(\tau-\tau')\,. \tag{5.76}$$

Note that the right hand side of (5.76) is only non-vanishing when $\tau=\tau'$, therefore we can write the function outside the delta function as a function only depending on τ or τ'. Also, in the language of (5.68), the function $f(\tau)$ is given by

$$f(\tau)\equiv-\frac{V'''(\phi_0)}{H^2\tau^2}\hat{\varphi}_l\tau^{3/2-\nu}\varphi^{(0)}(\tau)\,. \tag{5.77}$$

Since (5.67) is a second order differential equation, we can find that in general the solutions for $G^{(n)}(\tau,\tau')$ are given by

$$G^{(1)}(\tau,\tau')=i\Theta(\tau-\tau')\left(\varphi_1^{(0)}(\tau)\varphi_2^{(0)}(\tau')-\varphi_1^{(0)}(\tau')\varphi_2^{(0)}(\tau)\right)\,, \tag{5.78}$$

with $\varphi_1^{(0)}(\tau)$ and $\varphi_1^{(0)}(\tau)$ the two homogeneous solutions of (5.67) and $\Theta(\tau-\tau')$ the *Heaviside* step-function. After substitution of the homogeneous solutions to the mode functions (5.30), we obtain

$$G^{(1)}(\tau,\tau')=\frac{H^2\pi}{2}(\tau\tau')^{3/2}\left(\mathrm{J}_\nu(-k\tau')\mathrm{Y}_\nu(-k\tau)-\mathrm{J}_\nu(-k\tau)\mathrm{Y}_\nu(-k\tau')\right)\Theta(\tau-\tau')\,, \tag{5.79}$$

here $\mathrm{J}_\nu(-k\tau)$ and $\mathrm{Y}_\nu(-k\tau)$ are Bessel functions and $\nu\equiv\sqrt{9/4-m^2/H^2}$. The Green's functions for the massless and the massive $m=\sqrt{2}H$ case respectively are given by

$$G_{3/2}^{(1)}(\tau,\tau')=\frac{H^2}{k^3}\left[(1+k^2\tau\tau')\sin\left(k(\tau-\tau')\right)-k(\tau-\tau')\cos\left(k(\tau-\tau')\right)\right]\Theta(\tau-\tau')\,, \tag{5.80}$$

$$G_{1/2}^{(1)}(\tau,\tau')=\frac{H^2}{k}\tau\tau'\sin\left(k(\tau-\tau')\right)\Theta(\tau-\tau')\,, \tag{5.81}$$

here the subscript on the Green's functions of 3/2 and 1/2 refer to the values of ν of the Besselfunctions from which the Green's function is constructed. Then the particular solution at leading order in k_l for the massless scalar field φ is given by

$$\varphi_p^{(1)}(\tau, k_s) = -\frac{V'''(\phi_0)\hat{\varphi}_l(k_l)}{3\sqrt{2k_s^3}H}\left(2 + ie^{2ik_s\tau}(i + k_s\tau)\text{ExpEi}(-2ik_s\tau)\right)e^{-ik_s\tau}\,. \tag{5.82}$$

Following (5.63), we obtain the following expression for the squeezed massless three point function

$$\langle\varphi(\mathbf{k}_s, \tau_*)\varphi(\mathbf{k}_s, \tau_*)\varphi(\mathbf{k}_l, \tau_*)\rangle' \approx \frac{H^2V'''}{6}\frac{(-2 + \gamma_E + \log(-2k_s\tau_*))}{k_s^3k_l^3}\,. \tag{5.83}$$

This result is equal to the result when taking the squeezed limit of (5.46).

The particular solution at leading order in k_l for the massive $m = \sqrt{2}H$ is given by

$$\varphi_p^{(1)}(\tau, k_s) = \frac{iV'''(\bar{\phi})\hat{\varphi}_l(k_l)\tau e^{-ik_s\tau}}{2\sqrt{2k_s^3}}\left(e^{2ik_s\tau}\text{ExpEi}(-2ik_s\tau) - 2\log(k_l\tau_*)\right) \tag{5.84}$$

The squeezed three point correlation function that we obtain from (5.84) is given by

$$\langle\varphi(\mathbf{k}_s, \tau_*)\varphi(\mathbf{k}_s, \tau_*)\varphi(\mathbf{k}_l, \tau_*)\rangle' = \frac{\pi H^2V'''(\bar{\phi})}{8}\frac{\tau_*^3}{k_s^2k_l}. \tag{5.85}$$

This result is also equal to the result we obtain by taking the squeezed limit of (5.47). We have thus proven that the scalar consistency relation holds at least at leading orders in k_l.

We are interested in correlation functions that correspond to scalar fields which have a very small mass, i.e. $\nu \approx 3/2 + \alpha$ with $\alpha \ll 1$. Since these massive correlations functions correspond to correlation functions with mode functions where ν lays withing the range $1/2 < \nu < 3/2$, and we have just proven that the linear order corrections to the mode functions capture the leading order contributions to the squeezed three point functions, we can be confident that the linear order corrections to the mode functions of interest also capture the correct leading order squeezed bispectrum behavior.

5.7.2 The massive bispectra

As mentioned before, when trying to solve φ_s for a general mass, one might encounter problems that no exact solution to (5.67) exists. As previously shown, there are two limits in which we can solve the EMS exactly. This was because we wrote down our differential equation as Bessels equation that equaled a arbitrary function. However, when our mass is considered to be very small, $m^2/H^2 \ll 1$, we can also rewrite (5.67) into a more convenient form

$$\partial_\tau^2\varphi(\tau) - \frac{2}{\tau}\partial_\tau\varphi(\tau) + k^2\varphi(\tau) = -\frac{1}{H^2\tau^2}\left(m_0^2 + V'''(\bar{\phi})\hat{\varphi}_l\tau^{\frac{m_0^2}{3H^2}}\right)\varphi(\tau)\,, \tag{5.86}$$

Mathematically, (5.67) and (5.86) look very similar. There is one difference however, when solving (5.86) we take the mass term into the inhomogeneous part of the solution for φ_s. Meaning that we will be looking for solutions of the form

$$\varphi(\tau, m + m_0 + \delta m(\tau)) \approx \varphi(\tau, m) + m_0^2\;\varphi(\tau, m_0) + \delta m(\tau)^2\;\varphi(\tau, m)\,, \tag{5.87}$$

here m is the "standard" unperturbed mass, m_0 is the small perturbation of the mass, and $\delta m(\tau)$ is the time dependent mass that the interactions induce. Before we continue with the calculation of the squeezed bispectrum of a particle with a small mass, we will first discuss the momentum dependence that the correlation function should have based on the isometries. The dilation operator that annihilates a general massive three point function is given by

$$\sum_{a=1}^{3} D_a - 3 = \left(-3(\Delta - 3) + k_i\partial^i - 3\right)\,, \tag{5.88}$$

here we have subtracted -3 in order to account for the truncation of the Dirac-delta function. Remember that $\Delta = \frac{3}{2} + i\mu = \frac{3}{2} - \nu$. This suggests that our three point function should scale as

$$\langle\varphi^3\rangle' \sim \frac{\tau^{3(\frac{3}{2}-\nu)}}{k^{\frac{3}{2}+3\nu}}\,. \tag{5.89}$$

We can use this scaling as a check to see whether the solution we found meets the isometry conditions.

To calculate the light mass bispectrum, we will make the following assumptions. Following our definition of (5.87) we set $m = 0$ and $m_0 \ll 1$. In this case, the leading order correction to the Green's function is given by (5.80). Then the particular solution of φ is then given by

$$\begin{aligned}
&\varphi_p^{(1)}(\tau, k) = \\
&= -\frac{1}{24\sqrt{2}H^5k^4(-\tau)^{5/2}\sqrt{-k\tau}}\left(\frac{1}{3H^4 - 4H^2m_0^2 + m_0^4}\;\left(16H^4k^3m_0^2\tau^3\left(3H^4 - 4H^2m_0^2 + m_0^4\right) - \right.\right.\\
&\quad - 3H^2V'''(-\tau)^{\frac{m_0^2}{H^2}}\left(2iH^2m_0^4(k\tau + i)^2 + m_0^6(k\tau + i)\; - H^4m_0^2(k\tau + i)(2k\tau - i)(2k\tau + i)\right.\\
&\quad \left.\left. +2H^6(k\tau - i)(-1 + 3k\tau(2k\tau + i))\right)\hat{\varphi}_l\right) + \\
&\quad e^{2ik\tau}(k\tau + i)\left(3V'''\hat{\varphi}_l\left(2H^4 + H^2m_0^2 - m_0^4\right)(-\tau)^{\frac{\mathrm{m0}^2}{H^2}}E_{4-\frac{\mathrm{m0}^2}{H^2}}(2ik\tau) + 8iH^4k^3m_0^2\tau^3\mathrm{Ei}(-2ik\tau)\right)\Bigg)\,.
\end{aligned} \tag{5.90}$$

This is the linear order correction to the Green's function. We can check the validity of (5.90) by taking the limit of $m_0 \to 0$. In this limit, we obtain the following expression for the particular solution

$$\varphi_p(k, \tau_*) = \frac{V''' \hat{\varphi}_l e^{-ik\tau} \left(2 + e^{2ik\tau}(1 - ik\tau)E_1(2ik\tau)\right)}{3\sqrt{2}Hk^{3/2}} . \tag{5.91}$$

This result is equivalent to (5.82) up to an overall phase factor. The squeezed limit one obtains by using (5.91) is equal to (5.83).

Then after a lot of "taking the real part juggling Kung-Fu", we end up with the following expression

$$\begin{aligned}
\langle \varphi(\mathbf{k}_s, \tau_*)\varphi(\mathbf{k}_s, \tau_*)\varphi(\mathbf{k}_l, \tau_*)\rangle' &\approx \frac{H^2 V'''}{6} \frac{(-2 + \gamma_E + \log(-2k_s\tau_*))}{k_s^3 k_l^3} + \\
+ \frac{m_0^2 V'''}{k_l^3 k_s^3} \Bigg[\frac{1}{9}[-2 + \gamma_E + \log(-2k_s\tau_*)] \log\left(\frac{k_l}{k_s}\right) &- \frac{7}{108}[-2 + \gamma_E + \log(-2k_s\tau_*)] \\
+ \frac{1}{6}[-2 + \gamma_E + \log(-2k_s\tau_*)][-2 + \gamma_E + \log(-2k_s\tau_*)] &- \frac{25}{432}\pi^2 \Bigg] .
\end{aligned} \tag{5.92}$$

At this point, let us comment on this result. When solving a massive correlation function, we expect the momentum and (conformal) time scalings to change. However, this does not seem to be the case in (5.92). The overall scaling is still the one inherited from the massless bispectrum, since we assumed that the mass was very small. Also, note that all logarithm terms in this expressions are invariant under the massless dilation and special conformal isometry.

What we can do however is the following, by increasing the value of the mass m_0 and the value of k_l, there are a number of terms that start to dominate over the others. We can then make an educated guess about the "un-squeezed" momentum configuration by exponentiating the dominant terms. In doing that, we obtain for (5.92)

$$\begin{aligned}
&\langle \varphi(\mathbf{k}_s, \tau_*)\varphi(\mathbf{k}_s, \tau_*)\varphi(\mathbf{k}_l, \tau_*)\rangle' \approx \\
&\qquad \approx \frac{H^2 V'''}{6} \frac{(-2 + \gamma_E + \log(-2k_s\tau_*))}{k_s^3 k_l^3} \left[1 + \frac{m_0^2}{H^2} \left(\frac{2}{3} \log\left(\frac{k_l}{k_s}\right) + \log(-2k_s\tau_*) \right) \right] \\
&\qquad = \frac{H^2 V'''}{6} (-2 + \gamma_E + \log(-2k_s\tau_*)) \frac{(-2\tau_*)^{\frac{m_0^2}{H^2}}}{k_s^{3 - \frac{m_0^2}{3H^2}} k_l^{3 - \frac{2m_0^2}{3H^2}}} \\
&\qquad \sim \frac{\tau_*^{\frac{m_0^2}{H^2}}}{k^{6 - \frac{m_0^2}{H^2}}} .
\end{aligned} \tag{5.93}$$

The overall scaling of (5.93) is the scaling that we would expect of a massive correlation function in our situation. If we very boldly make an educated guess based on (5.46) how this part of

(5.92) would look like in the equilateral configuration it would be

$$\langle\varphi(\mathbf{k}_1,\tau_*)\varphi(\mathbf{k}_2,\tau_*)\varphi(\mathbf{k}_3,\tau_*)\rangle' \approx$$
$$\approx \frac{H^2V'''}{12}\frac{(-2\tau_*)^{\frac{m_0^2}{H^2}}}{(k_1k_2k_3)^{3-\frac{2m_0^2}{3H^2}}}\left[(-1+\gamma_E+\log(-K\tau_*))\sum_{i=1}^{3}k_i^{3-\frac{m_0^2}{H^2}}-\sum_{i\neq j}^{3}\left(k_i^2k_j\right)^{1-\frac{m_0^2}{3H^2}}+(k_1k_2k_3)^{1-\frac{m_0^2}{3H^2}}\right]. \tag{5.94}$$

Note however, that a number of terms were omitted in (5.94) that, when added to (5.94), produce slight deviation from conformal invariance. Whether it is possible to re-sum the left over terms in (5.92) is not entirely clear, but since we made approximations in deriving this result, we would expect small deviation form conformal invariance in (5.92) anyway.

We can also perform a similar calculation for fields with $m_{\text{tot}} = \sqrt{2}H + m_0$. Instead of perturbing around $m = 0$, we will solve (5.67) by perturbing around $m = \sqrt{2}H$. Again we assume $m_0 \ll 1$. The result at quadratic order in m_0 is given by

$$\langle\varphi(\mathbf{k}_s,\tau_*)\varphi(\mathbf{k}_s,\tau_*)\varphi(\mathbf{k}_l,\tau_*)\rangle' \approx \frac{\pi H^2V'''}{8}\frac{\tau_*^3}{k_lk_s^2}\left[1+\frac{2m_0^2}{H^2}\left(\log\left(\frac{k_l}{k_s}\right)+\gamma_E+\log(-2k_s\tau_*)\right)\right]. \tag{5.95}$$

Note that this expression is much simpler than the massless one, the reason for this is that the mode functions for $\nu = 1/2$ are much simpler than for the $\nu = 3/2$ case. This expression however, does not satisfy at leading order in m_0 and k_l the expected momentum and (conformal) time scaling that we would have expected using the dilation isometry. Therefore we have to conclude that the approximation we have made has broken down and the result seems not to be valid anymore.

5.8 Gauge transformation and the δN formalism

To relate the correlation functions to the CMB, we want to express them in terms of comoving curvature perturbations ζ, since ζ stops evolving after horizon exit. The transformation between φ and ζ correspond to a gauge transformation between the spatially flat gauge and the comoving gauge. There are two methods to relate these gauges to one another. The first method is a direct gauge transformation of the form $t \to t + T(\varphi)$. This was explored in [13] and B. The second method is by relating ζ to φ via the δN-formalism. In this section, we shall explore this second scheme.

In the δN-formalism, it is realized that ζ is equal to a perturbation to the number of e-folds, N, that arise from perturbing the initial scalar field ϕ in the spatially flat gauge. Scalar perturbations to the spatial metric on a fixed time slice t can be written as a local perturbation to the scale factor,

$$a(\mathbf{x}, t) \equiv a(t) e^{\zeta(\mathbf{x},t)} . \tag{5.96}$$

The local number of e-folds a field makes before it exits the horizon is

$$N = \int_{t_0}^{t_*} dt' \; H(t') = \ln \left(\frac{a(t_*)}{a(t_0)} \right) = H(t_* - t_0) , \tag{5.97}$$

here t_0 is an arbitrary initial time and t_* is the moment the mode leaves the horizon. If we define $N(\mathbf{x}, t)$ as the number of e-folds from a fixed flat slice to a comoving curvature slice at time t, then

$$\zeta(\mathbf{x}, t) = \delta N(\mathbf{x}, t) . \tag{5.98}$$

To relate $\zeta(\mathbf{x}, t)$ to the inflaton perturbations $\varphi(\mathbf{x}, t)$, we assume that the inflaton field has become superhorizon at some initial time. At this time, we choose a spatially flat time-slice on which there are no scalar fluctuations in the metric, but only fluctuations in the matter fields $\phi(\mathbf{x}, t) = \bar{\phi}(t) + \varphi(\mathbf{x}, t)$. We then choose the final time slice to be coinciding with a comoving curvature slice, where the scalar fluctuations are all inside the metric. In order to go from one slice to the other, we evolve the unperturbed and the perturbed fields classically to the final slice separately. The difference between the two results is then equal to the differents in the number of e-folds a field makes,

$$\zeta = \delta N = N(\bar{\phi} + \varphi) - N(\bar{\phi}) . \tag{5.99}$$

Expanding (5.99) around $\bar{\phi}$ we can obtain an expression for $\zeta(\mathbf{x}, t)$ in terms of the scalar fluctuations φ and derivatives of N defined on the initial slice.

$$\zeta \approx N'(\bar{\phi})\varphi + \frac{1}{2} N''(\bar{\phi})\varphi^2 + ... , \tag{5.100}$$

here the primes denote derivatives with respect to ϕ. Note that the δN formula is merely an identity which expresses ζ in terms of geometrical quantities and can be used as a "book-keeping" object which describes how much one region of the universe has expanded ($\zeta > 0$) or contracted $\zeta < 0$ relative to the mean expansion. As implicitly stated earlier, the mean expansion is specified as the region of the universe which is supposed to be described by the unperturbed background. Also, for this reasoning, ζ is a *pure gauge mode* in an exact de Sitter spacetime.

To convert the power spectrum, a first order relation in φ is sufficient. However, for the bispectrum a transformation to second order in φ is needed [13]. To first order in perturbations, we have

$$\frac{\partial\phi}{\partial N} = -\frac{\dot{\phi}}{H} = -\left(\frac{\dot{\bar{\phi}}}{H} + \frac{\dot{\varphi}}{H}\right), \tag{5.101}$$

here the dot denotes a derivative with respect to t. Inverting this expression and taking another ϕ-derivative we find

$$\left.\frac{\partial N}{\partial\phi}\right|_{\bar{\phi}} = -\left(\frac{H}{\dot{\bar{\phi}}} - \frac{H}{\dot{\bar{\phi}}}\dot{\varphi}\right) \quad \text{and} \quad \left.\frac{\partial N^2}{\partial\phi^2}\right|_{\bar{\phi}} = -\left(\frac{\dot{H}}{H^2} - \frac{\ddot{\bar{\phi}}}{H\dot{\bar{\phi}}}\right)\left(\frac{H}{\dot{\bar{\phi}}}\right)^2, \tag{5.102}$$

here we assumed that $\varphi \ll \bar{\phi}$. Also, although when evaluating ϕ we assumed that $\varphi = 0$, but not $\dot{\varphi}$. Then making use of (2.35) we find that ζ and φ are related to second order by

$$\zeta = -\frac{1}{\sqrt{2\epsilon}M_{\rm Pl}}\varphi + \frac{\eta}{8\epsilon M_{\rm Pl}^2}\varphi^2 + \frac{1}{2\epsilon M_{\rm Pl}^2 H}\varphi\dot{\varphi}\ ...\ , \tag{5.103}$$

which is equivalent to the transformation found in [13] in the limit (5.2). In literature, the second order contribution $\varphi\dot{\varphi}$ is often omitted or not taken to be into account. When calculating three point correlation functions, this term will be τ_*^ν suppressed at the end of inflation and can therefore be safely ignored.

Constant ζ in time is a solution of it's equation of motion to all orders in powers of ζ outside the horizon [13], meaning that any three point function of $\zeta(\tau_* \to 0)$ should freeze out after all modes have passed the horizon. To convert the φ correlation functions to ζ correlation functions, we need to choose a convenient final slice. As we already observed in (5.46), our result can contains logaritmic terms. Taking the final slice to be $\tau \to 0$ can then be rather confusing since we would encounter a fictional divergence. This divergence is fictional in the sense that one no observer can really set $\tau_* \to 0$, since observations must be made at a finite time which gives a cutoff scale for τ_*. An alternative and better choice would be the moment of horizon crossing, the moment at which the logaritmic divergence are controllable,

$$k_* \equiv a(\tau_*)H(\tau_*)\,, \tag{5.104}$$

here τ_* is the moment when the modes leaves the horizon. However, since not all modes inside the bispectrum need to carry the same momenta, modes with a longer wavelength will exit the horizon earlier than modes with a shorter wavelength. To account for the difference in evolution before the modes become super Hubble size, we will convert the bispectrum at the moment when the last mode leaves the horizon. Since the bispectrum is invariant under permutations of the different modes, we choose $k_1 \leq k_2 \leq k_3$ and so we set $\tau_* = -1/k_3$. Then, the general ζ-bispectrum is related to the φ-bispectrum via

$$\begin{aligned}\langle\zeta(\mathbf{k}_1,0)\zeta(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle' = & \frac{-1}{(2\epsilon)^{\frac{3}{2}}M_{\mathrm{Pl}}^3}\langle\varphi(\mathbf{k}_1,\tau_*)\varphi(\mathbf{k}_2,\tau_*)\varphi(\mathbf{k}_3,\tau_*)\rangle' + \\ & + \frac{1}{2\epsilon^2 M_{\mathrm{Pl}}^4}\frac{\eta}{8}\langle\varphi(\mathbf{k}_1,\tau_*)\varphi(\mathbf{k}_2,\tau_*)[\varphi(\mathbf{k}_3,\tau_*)\bullet\varphi(\mathbf{k}_3,\tau_*)] + 2\text{ perm.}\rangle',\end{aligned} \tag{5.105}$$

here $\bullet$ denotes a convolution product. The first term in (5.105) is the connected part of the φ correlator and the last term is the contribution due to the superhorizon part of the gauge transformation between comoving- and the spatially flat gauge. For an arbitrary mass, this superhorizon part is given by

$$\begin{aligned}&\frac{\eta}{16\epsilon^2 M_{\mathrm{Pl}}^4}\langle\varphi(\mathbf{k}_1,\tau_*)\varphi(\mathbf{k}_2,\tau_*)[\varphi(\mathbf{k}_3,\tau_*)\bullet\varphi(\mathbf{k}_3,\tau_*)] + 2\text{ perm.}\rangle' = \\ &\qquad = \frac{\eta H^4}{16\epsilon^2 M_{Pl}^4}\,\frac{\Gamma(-i\mu)^4}{4^{2+2i\mu}\pi^2}\,\frac{\sum_{i=1}^3 k_i^{-2i\mu}}{(k_1k_2k_3)^{-2i\mu}}(-\tau_*)^{6+4i\mu}\,.\end{aligned} \tag{5.106}$$

With this prescription at hand to calculate ζ correlators from φ correlators, we can convert the bispectrum to the comoving gauge. Converting (5.46) one obtains

$$\begin{aligned}&\langle\zeta(\mathbf{k}_1,0)\zeta(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle' = \\ &\quad = \frac{H^2}{(2\epsilon)^{3/2}M_{Pl}^3}\frac{V'''(\bar\phi)}{12}\frac{1}{k_1^3k_2^3k_3^3}\left[(-1+\gamma_E+\log(-k_t\tau_*))\sum_{i=1}^3 k_i^3 - \sum_{i\neq j}k_i^2k_j + k_1k_2k_3\right] \\ &\quad + \frac{H^4}{16\epsilon^2 M_{Pl}^4}\frac{\eta}{2}\frac{k_1^3+k_2^3+k_3^3}{k_1^3k_2^3k_3^3}\,,\end{aligned} \tag{5.107}$$

and converting (5.47) to the comoving gauge, one obtains

$$\langle\zeta(\mathbf{k}_1,0)\zeta(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle' = \frac{\pi H^2 V'''(\bar\phi)}{8}\frac{\tau_*^3}{k_1k_2k_3} + \frac{H^4}{16\epsilon^2 M_{Pl}^4}\frac{\eta}{2}\frac{(k_1+k_2+k_3)}{k_1k_2k_3}\tau_*^4\,. \tag{5.108}$$

Let us examine (5.107) closely, we note that it is fully constructed from *conformal* and *local* terms and therefore is invariant under the full conformal isometry group.

When examining (5.108) however, we note a new shape, the second term. This term is invariant under the *special conformal isometry* but does not seem to be invariant under the *Dilation isometry*. There are a few subtleties one has to consider when making a statement about this term. In the *equilateral configuration*, we have that $k_t \sim -1/\tau_*$. Therefore we can argue that in the equilateral limit, the correlator still obeys the conformal isometries. Also, for a massive correlation function, the time dependence strongly suppresses the massive contribution to the bispectrum as $\tau_* \to 0$. Therefore one would expect a negligible contribution to the bispectrum comming from massive scalar fields producing again a conformal invariant bispectrum.

If we want to compare this result to the one performed in the comoving gauge we need to convert the third derivative of the potential to slow-roll parameters, since interactions in the comoving gauge arise from different couplings. Starting from (5.3) and taking three ϕ-derivatives we obtain

$$V'''(\bar{\phi}) = \frac{H^2}{\sqrt{2\epsilon}M_{\rm Pl}}\left[-\frac{3}{2}\frac{\dot{\eta}}{H} - \frac{\ddot{\eta}}{2H^2} - \frac{\eta\dot{\eta}}{2H} + 3\eta\epsilon - \frac{9}{2}\frac{\epsilon\dot{\eta}}{H} + \frac{5}{2}\epsilon\eta^2 + 3\epsilon^2\eta - 8\epsilon^3\right]. \tag{5.109}$$

In the decoupling limit (5.2), this expression simplifies to

$$V'''(\bar{\phi}) = \frac{H^2}{\sqrt{2\epsilon}M_{\rm Pl}}\left[-\frac{3}{2}\frac{\dot{\eta}}{H} - \frac{\ddot{\eta}}{2H^2} - \frac{\eta\dot{\eta}}{2H}\right]. \tag{5.110}$$

From (2.35) we note that the first term in (5.110) is leading order in slow-roll parameters. At leading order in slow-roll parameters (5.107) becomes

$$\begin{aligned}
&\langle\zeta(\mathbf{k}_1,0)\zeta(\mathbf{k}_2,0)\zeta(\mathbf{k}_3,0)\rangle' = \\
&= \frac{H^4}{16\epsilon^2 M_{\rm Pl}^4}\left(\frac{\dot{\eta}}{2H} - \frac{\ddot{\eta}}{6H^2} - \frac{\eta\dot{\eta}}{6H}\right)\frac{1}{k_1^3k_2^3k_3^3}\left[(-1+\gamma_E+\log(-k_t\tau_*))\sum_{i=1}^{3}k_i^3 - \sum_{i\neq j}k_i^2k_j + k_1k_2k_3\right] + \\
&\quad + \frac{H^4}{16\epsilon^2 M_{\rm Pl}^4}\frac{\eta}{2}\frac{k_1^3+k_2^3+k_3^3}{k_1^3k_2^3k_3^3}.
\end{aligned} \tag{5.111}$$

When we take the full bispectrum result of the previous chapter, including the $\epsilon_*\eta_*$ contribution as given in [19], and take the limit (5.2), we obtain the exact same result as (5.111). This verifies the validity of the results and calculations done in this chapter.

5.9 Estimate of non-Gaussianities

Just as in the comoving gauge calculation where we had estimated the amount of non-Gaussianities the slow-roll model contains using the *Komatsu-Spergel local form*, we can estimate the (equilateral) non-Gaussianities in our decoupling limit model in a similar fashion. It turns out that in the limit (5.2), a very elegant simplification arises for the estimate of the connected part of φ correlation function, namely

$$f_{\rm NL} \sim \alpha_s \,. \tag{5.112}$$

This relation can be obtained in the following way. In general, the spectral tilt of the power spectrum is given by

$$n_s - 1 = -2\epsilon + \frac{\epsilon_{,N}}{\epsilon}\,, \tag{5.113}$$

here we expressed the slow-roll parameter η in terms of the "e-folds"-derivatives of ϵ. In our decoupling limit, it is now possible to directly relate the running of the potential to the third derivative of the potential with respect to ϕ,

$$\begin{aligned} \alpha_s &\equiv (1-n_s)_{,N} = 2\epsilon_{,N} - \frac{\epsilon_{,NN}}{\epsilon} + \left(\frac{\epsilon_{,N}}{\epsilon}\right)^2 \\ &= (1-n_s) - 6\epsilon\,(1-n_s) + 8\epsilon - \frac{\epsilon_{NN}}{\epsilon} \\ &\simeq (1-n_s)\,\xi + \mathcal{O}(\epsilon) \\ &\simeq 2M_{\rm Pl}^4\frac{V'''V'}{V^2} + \mathcal{O}(\epsilon_V)\,, \end{aligned} \tag{5.114}$$

where again $\xi \equiv \dot\eta/(\eta H) = -\eta_{,N}/\eta$ and $\epsilon_V \ll 1$. Consider the connected part of the correlation function, we can now write

$$\langle\zeta^3\rangle \supset (2\epsilon)^{-3/2} M_{\rm Pl}^{-3}\langle\delta\phi^3\rangle \propto \frac{V'''H^2}{(2\epsilon)^{3/2}M_{\rm Pl}^3} = \alpha_s \times \text{numbers}\,. \tag{5.115}$$

After bringing the superhorizon part of the ζ correlator into the form of (4.145) we find that the non-Gaussianities produced by this term are given by

$$f_{\rm NL}^{\rm local} \sim \frac{\eta}{2} \sim \frac{m^2}{H^2}\,, \tag{5.116}$$

which is the same term appears in the leading contributions to the non-Gaussianities produced by the boundary term [20] in the comoving gauge. Note that in contrary to the comoving gauge calculation, this η contribution arises from the superhorizon part of the gauge transformation.

In [25] it is noted that terms in the bispectrum of the form $(n_s-1)\times$local will not contribute to any non-Gaussianities, since we cannot observe locally whether our measurements are effected by a long mode in the background. The reason for this is that the results of our calculation are measured in *comoving coordinates*, but the actual distance is measured with the full metric. Therefore, one needs to put back the $(1+2\zeta)$ term back in h_{ij}. This transformation cancels exactly the $(n_s-1)\times$ local contributions in the bispectrum.

This can be made explicit by writing the comoving curvature perturbation as

$$\zeta(x) = \zeta_s(x) + \zeta_l(x)\,, \tag{5.117}$$

here the ζ-perturbation has been separated into a long- and short-wavelength piece on the scale of the patch within the correlation function is measured. When we transform ζ to *Fermi normal coordinates* [25], we loose the ζ_l contribution of (5.117) giving us

$$\begin{aligned}\bar{\zeta}(x_F) &= \zeta_x(x) + \mathcal{O}(\partial_i\partial_j\zeta_l)\\ &\approx \zeta_s(x)\,,\end{aligned} \tag{5.118}$$

here the bar on ζ signifies the fact that ζ is given in Fermi normal coordinates. When we now transform the squeezed ζ-bispectrum, since we are interested in the "equilateral" non-Gaussianities, to Fermi normal coordinates, we obtain we obtain [25]

$$B_{\bar{\zeta}\bar{\zeta}\bar{\zeta}}(\mathbf{k}_l, \mathbf{k}_1\mathbf{k}_2) = P_\zeta(k_l)P_\zeta(k_s)\frac{\partial \ln\left(k_s^3 P_\zeta(k_s)\right)}{\partial \ln(k_s)} + B_{\zeta\zeta\zeta}(\mathbf{k}_l, \mathbf{k}_1\mathbf{k}_2) \tag{5.119}$$

As previously shown,

$$B_{\zeta\zeta\zeta}(\mathbf{k}_l, \mathbf{k}_1\mathbf{k}_2) = -(n_s - 1)P_\zeta(k_l)P_\zeta(k_s) \tag{5.120}$$

therefore all terms proportional to $(n_s - 1) \times$ local cancel. Therefore, we conclude that the physical (equilateral) non-Gaussianities in this model follow (5.112).

Chapter 6

Conclusion

6.1 Summary

We started this thesis with a brief introduction to modern cosmology where we related the *CMB* and the *Horizon problem*. We argued that the most promising paradigm that solves this problem was inflation. We then moved away from the ideal homogeneous and isotropic universe picture and we studied perturbation theory. We gave a formal way to foliate the spacetime of the primordial universe and we gave a formal (new) proof to what order we need to perturbatively solve the ADM-constraints in order for our action to be valid.

Then we did cosmological perturbation theory in the *comoving gauge*. We solved the spectral tilt to the next order and we proved that the consistency relation holds up to second order for the bispectrum. During this calculation, we derived a new formal treatment how to calculate the bispectrum when approximating the *Quasi-de Sitter* mode functions as the *de Sitter* mode functions and showed that the dangerous logarithmic divergences that appear in the bispectrum can be controlled by the use of boundary terms.

And last not least, we did perturbation theory in the *spatially flat gauge*. Here we studied our *decoupling limit*. At the moment when the modes left the horizon, we found that the isometries of the (Quasi)-de Sitter space reduced to that of an conformal field theory and that all three point correlation functions enjoy the full conformal symmetry. The conformal field theory that corresponded with this dS/CFT was Euclidean. We constructed a scalar consistency relation and showed that it holds in our decoupling limit. We then calculated the squeezed bispectrum for a more general massive particle. For very light massive particles the approximation seems to hold at leading orders in k_l, but it seems that the approximation breaks down for the calculation when we perturbed around a mass of $m^2 = 2H^2$. We then converted the result of the spatially flat gauge to the comoving gauge and showed that the δN formalism method is equivalent to a direct gauge transformation of the time coordinate t. We compared the results between both gauges and proved that the calculations are consistent. Finally we estimated the physical non-Gaussianities in our model to $f_{NL} \sim \alpha_s$.

6.2 Beyond this thesis

Loop corrections

Another interesting application arises when one is interested in calculating the one loop correction to the propagator. In the limit, (5.2) also the quartic interaction action becomes trivial in the sense that we can write the full interaction action up to quartic order in perturbations as

$$S_{\text{int}} = \int d^4x \; \left[\frac{1}{6}V'''(\bar{\phi})\varphi^3 + \frac{1}{24}V''''(\bar{\phi})\varphi^4\right]. \tag{6.1}$$

A simplification arises when one wants to convert the corrections to the *comoving* gauge. We are free to choose our moment of conversion as long as we do it consistently. By choosing the moment when we convert the loop correction, we can control the dangerous logarithmic temporal divergence. In any case, we can make sure that the loop correction that arises from the quartic vertex dominates over the loop correction that arises when using two cubic vertices, since the latter is suppressed by two sets of slow-roll parameters. Also, one would expect that there will be three loop corrections arising from the fact that there is a second order relation between φ and ζ. Then the integrals one has to perform to get the full loop correction are then given by

$$\begin{aligned}
\langle \varphi(\tau,k)\varphi(\tau,k)\rangle' \;\approx\; & \left[-2\,\frac{H^6}{\dot{\bar{\phi}}^6}\text{Im}\left[-\int_{-\infty}^{t_*} d\tilde{t}\;\left\langle [H^{(4)}(\tilde{t}),\varphi(t,k)^2\right\rangle'\right] + \right. & (6.2)\\
& + \frac{\partial N}{\partial\phi}\frac{\partial^2 N}{\partial\phi^2}\int \frac{d^3q}{(2\pi)^3}B(k,k-q,q) + & (6.3)\\
& + \frac{1}{2}\frac{\partial^2 N}{\partial\phi^2}\int \frac{d^3q}{(2\pi)^3}P(k-q)P(q) + & (6.4)\\
& \left. + \frac{\partial N}{\partial\phi}\frac{\partial^3 N}{\partial\phi^3}\int \frac{d^3q}{(2\pi)^3}P(k)P(q)\right]_{\tau\to\tau_*}, & (6.5)
\end{aligned}$$

here (6.2) is the loop correction arising from the interactions and (6.3)-(6.5) arise from the second order relation between φ and ζ.

Higher order consistency relations

With the consistency relation, as given in 5.6, we can study higher correlation functions in the presence of a long background mode. In appendix F, a simple (new) calculation is performed to calculate the tri-spectrum for the massless case and the massive $m^2 = 2H^2$ case. As we see in this appendix, the more massive the particle becomes, the more distorted the original three point correlation function becomes. Similar to what we discussed in 5.7.1, the mode functions for the short modes will shift again due to the introduction of a time dependent mass. However, since we produce a equilateral shaped three point function when taking one of the legs soft of a four point function, suggests that when squeezing the four point function, we also find the solution, at leading orders in k_l, to a very non-trivial integral.

Acknowledgements

First of all I would like to thank my supervisor, Enrico Pajer, for all the enjoyable months that I was working on this master thesis. Besides all the interesting things I learned about the primordial universe, he also showed and encouraged me to slow down with the formulas and to try to gain a better physical intuition behind the theory. In the end, it gives you a better scope of the situation making problems easier to solve.

I also want to thank Gui Leite Pimentel for sharing ideas and the nice discussions we had about my thesis and the draft of future paper we are working on together with Enrico Pajer.

Then I want to thank Richard Dijk and Bernardo Finelli-de Moraes, which were already working on a closely related subject when I started my thesis, for sharing ideas, thoughts and for all the interesting discussions during our weekly meetings.

Last and most certainly not least, I want to thank all master students in the master room(s) for the enjoyable time we had during lunches, events and drinks.

Appendix A

Definitions and conventions

A.1 A connection to literature

In literature, you might come across different definitions of the slow-roll parameters. This is a brief overview of the slow-roll parameters that are often used in literature and the conversion factors between them.

$$\epsilon_V \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V_{,\phi}}{V}\right)^2 , \qquad \eta_V \equiv M_{\rm Pl}^2 \frac{V_{,\phi\phi}}{V} , \qquad \xi_V^{(1)} \equiv M_{\rm Pl}^4 \frac{V_{,\phi}V_{,\phi\phi\phi}}{V^2} , \tag{A.1}$$

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \approx \epsilon_V , \qquad \eta \equiv \frac{\dot{\epsilon}}{\epsilon H} = -\frac{\epsilon_{,N}}{\epsilon} \approx 4\epsilon_V - 2\eta_V , \tag{A.2}$$

$$\delta \equiv \frac{\ddot{\phi}}{\dot{\phi} H} , \qquad \xi_\phi \equiv \frac{\partial_t^3 \phi}{\phi H^2} , \tag{A.3}$$

$$\epsilon_V = \epsilon\left(\frac{3+\eta/2-\epsilon}{3-\epsilon}\right)^2 = \epsilon\left(\frac{3-\delta}{3-\epsilon}\right)^2 , \tag{A.4}$$

$$\eta_V = \frac{6\epsilon - 3/2\eta - \xi_\phi}{3-\epsilon} = \frac{3(\epsilon+\delta)-\xi_\phi}{3-\epsilon} . \tag{A.5}$$

The potential slow-roll parameters are defined in (A.1) and are denoted by a subscript V, the Hubble slow-roll parameters are defined in (A.2) and do not carry a subscript and the potential slow-roll parameters as defined in [5] are given by (A.3) and are denoted with a subscript ϕ. The remaining equations (A.4) and (A.5) are the conversion factors between the different slow-roll parameters.

The leading order corrections to the spectral tilt and running are defined by

$$(n_s - 1) = \frac{\partial P}{\partial \log(k)} = -2\epsilon - \eta = -6\epsilon_V + 2\eta_V , \tag{A.6}$$

$$\alpha_s = \frac{\partial n_s}{\partial \ln(k)} = -2\epsilon\eta - \eta\xi^{(1)} = -16\epsilon_V\eta_V + 24\epsilon_V^2 + 2\xi_V^{(1)} , \tag{A.7}$$

here the subscript s denotes the fact tat we are looking at *scalar* perturbations. The potential slow-roll parameters are defined in (A.1) and are denoted by a subscript V, the Hubble slow-roll parameters are defined in (A.2)

A.2 Potential in terms of slow-roll parameters

When comparing results from different gauges with eachother, it is convenient to express all quantities that only depend on the background equations of motion of the fields in terms of slow-roll parameters. For example, one can express the potential and its derivatives as

$$V(\bar{\phi}) = M_{\rm Pl}^2 H^2 (3-\epsilon)\,, \tag{A.8}$$

$$V'(\bar{\phi}) = M_{\rm Pl} H^2 \left[-\sqrt{\frac{\epsilon}{2}}\eta - 3\sqrt{2\epsilon} + \sqrt{2\epsilon}\epsilon\right]\,, \tag{A.9}$$

$$V''(\bar{\phi}) = H^2 \left[-\frac{3}{2}\eta + \frac{5}{2}\epsilon\eta - \frac{1}{4}\eta^2 - \frac{1}{2}\frac{\dot{\eta}}{H} + 4\epsilon^2\right]\,, \tag{A.10}$$

$$V'''(\bar{\phi}) = \frac{H^2}{\sqrt{2\varepsilon}M_{\rm Pl}} \left[-\frac{3}{2}\frac{\dot{\eta}}{H} - \frac{\ddot{\eta}}{2H^2} - \frac{\eta\dot{\eta}}{2H} + 9\varepsilon\eta + 3\frac{\varepsilon\dot{\eta}}{H} + 3\varepsilon\eta^2 - 9\varepsilon^2\eta + 4\varepsilon^3 - 12\varepsilon^2\right] \tag{A.11}$$

here the factors of $M_{\rm Pl}$ have been kept explicit for the reasons discussed in 5.

Appendix B

Gauge transformation between the Comoving and Spatially flat gauge

In this section we rederive the gauge transformation between the spatially flat and the comoving gauge, we follow [13] closely. Consider again a scalar field $\phi(t, \mathbf{x})$ that is dependent on space and time. We can decompose this field into it's mean value which slowly varies in time, $\bar{\phi}(t)$ and perturbations around this mean value with we will denote by $\varphi(t, \mathbf{x})$. Then we can write $\phi(t, \mathbf{x})$ as

$$\phi(t, \mathbf{x}) = \bar{\phi}(t) + \varphi(t, \mathbf{x})\,. \tag{B.1}$$

The coordinate transformation that corresponds to this gauge transformation is given by the following time translation

$$t \to \tilde{t} = t + T(t, \mathbf{x})\,, \tag{B.2}$$

here $T(t, \mathbf{x})$ is a gauge variable that depends on space and time. We would like to find a relation (order by order) between φ and T and a relation between ζ and T to connect φ and ζ. In order to do this, we split T into

$$T = T_1 + T_2 + \ ... \ + T_n\,, \tag{B.3}$$

here the subscripts 1, 2, ... n refer to the order in perturbation in φ or ζ. To find a relation between φ and T, we can transform the field $\phi(t, \mathbf{x})$ under (B.2),

$$\begin{aligned} \phi(t, \mathbf{x}) \to \tilde{\phi}(\tilde{t}, \mathbf{x}) &= \phi(t, \mathbf{x}) + (\tilde{t} - t)\frac{\partial \phi}{\partial t} + \frac{1}{2}(\tilde{t} - t)^2 \frac{\partial^2 \phi}{\partial t^2} + \ ... \\ &= \bar{\phi} + \varphi + \dot{\bar{\phi}} T + \dot{\varphi} T + \frac{1}{2}\ddot{\bar{\phi}} T^2\,. \end{aligned} \tag{B.4}$$

Substituting (B.3) into (B.4) we find

$$T_1 = -\frac{1}{\dot{\bar{\phi}}}\varphi\,, \tag{B.5}$$

and

$$T_2 = \frac{1}{\dot{\bar{\phi}}^2}\varphi\dot{\varphi} - \frac{1}{2}\frac{\ddot{\bar{\phi}}}{\dot{\bar{\phi}}^3}\varphi^2\,. \tag{B.6}$$

Substituting these equations back into (B.3) we find the relation between T and φ to second order

$$T = -\frac{1}{\dot{\bar{\phi}}}\varphi + \frac{1}{\dot{\bar{\phi}}^2}\varphi\dot{\varphi} - \frac{1}{2}\frac{\ddot{\bar{\phi}}}{\dot{\bar{\phi}}^3}\varphi^2 + \ \dots \,. \tag{B.7}$$

To find the relation between T and ζ, we have to realize that $\zeta(t, \mathbf{x})$ is a scalar perturbation in the metric, thus in order to find the relation between T and ζ we also want to know how the metric transforms under (B.2). We start with the usual ADM-metric

$$ds^2 = \left[N^2 - h_{ij}N^iN^j\right]dt^2 - 2N_i dt dx^i - h_{ij}dx^i dx^j \,. \tag{B.8}$$

Then under (B.2) the time element transforms as

$$d\tilde{t} = dt + \partial_i T dx^i \,. \tag{B.9}$$

Substituting this into (B.8) we find

$$\begin{aligned} ds^2 = &\left[\tilde{N}^2 - \tilde{h}_{ij}\tilde{N}^i\tilde{N}^j\right]dt^2 - 2\left[\tilde{N}_i - \tilde{N}^2\partial_i T + \tilde{h}_{jk}\tilde{N}^j\tilde{N}^k\partial_i T\right]dtdx^i \\ &- \left[\tilde{h}_{ij} - \tilde{N}_i\partial_j T + \tilde{N}_j\partial_i T - \tilde{N}^2\partial_i T\partial_j T + \tilde{h}_{kl}\tilde{N}^k\tilde{N}^l\partial_i T\partial_j T\right]dx^i dx^j \,. \end{aligned} \tag{B.10}$$

The (old) spatial metric can be recast in terms of the transformed variables as

$$\begin{aligned} h_{ij} &= e^{2\rho(\tilde{t})}\delta_{ij} + \tilde{N}_i\partial_j T + \tilde{N}_j\partial_i T - \partial_i T\partial_j T + \ \dots \\ &= e^{2\rho(t)}\left(\delta_{ij}\left[1 + 2\dot{\rho}T + \ddot{\rho}T^2 + 2\dot{\rho}^2T^2 + \dots\right] + \partial_i\chi\partial_j T + \partial_i T\partial_j\chi - e^{-2\rho(t)}\partial_i T\partial_j T + \ \dots\right) \,. \end{aligned} \tag{B.11}$$

For notational convenience, let us define

$$\mu_{ij} \equiv \partial_i\chi\partial_j T + \partial_i T\partial_j\chi - e^{-2\rho(t)}\partial_i T\partial_j T \,. \tag{B.12}$$

Then there is still some gauge freedom left in the spatial part of the metric. Transforming the spatial coordinate x^i of the metric under

$$x^i \to \tilde{x}^i = x^i + \epsilon^i(t, \mathbf{x}) \,, \tag{B.13}$$

we find that the spatial part of the metric becomes

$$h_{ij} = e^{2\rho(t)}\left(\delta_{ij}\left[1 + 2\dot{\rho}T + \ddot{\rho}T^2 + 2\dot{\rho}^2T^2 + \dots\right] + \partial_i\epsilon_j + \partial_j\epsilon_i + \mu_{ij} + \ \dots\right) \,. \tag{B.14}$$

The part of the metric that transforms as scalars generally has to be of the form

$$h_{ij} = e^{2\zeta}\delta_{ij} + \partial_i\partial_j\xi \,. \tag{B.15}$$

Expanding both forms for h_{ij} and expanding in perturbations, we can relate T to ζ as

$$\left[1 + 2\zeta_1 + 2\zeta_2 + 2\zeta_1^2\right]\delta_{ij} + \partial_i\partial_j\xi = \delta_{ij}\left[1 + 2\dot{\rho}T_1 + 2\dot{\rho}T_2 + \ddot{\rho}T_1^2 + 2\dot{\rho}^2T_1^2\right] + \partial_i\epsilon_j + \partial_j\epsilon_i + \mu_{ij} \,,$$

here ϵ_i and μ_{ij} are second order variables. Then at linear order in perturbations, ζ, T_1 and ζ are related by

$$\zeta_1 = \dot{\rho} T_1 = -\frac{\dot{\rho}}{\dot{\phi}} \varphi \tag{B.16}$$

$$\varphi = -\frac{\dot{\phi}}{\dot{\rho}} \zeta + \ ... \ . \tag{B.17}$$

In order to find the second order relation between ζ and φ, we have to get rid of the ξ term. This can be done applying the differential operator $\partial_i \partial_j$ on it and by taking a trace of (B). Then

$$6\zeta_2 = 3\left[2\dot{\rho}T_2 + \ddot{\rho}T_1^2\right] + 2\partial_k \epsilon^k + \mu_k{}^k \tag{B.18}$$

$$2\zeta_2 = 2\dot{\rho}T_2 + \ddot{\rho}T_1^2 + 2\partial_k \epsilon^k + \partial^{-2}\left(\partial_i \partial_j \mu^{ij}\right) . \tag{B.19}$$

Note that in both equations we have $\partial_k \epsilon^k$, pulling them out and subtracting the results equals zero, thus

$$\zeta_2 = \dot{\rho}T_2 + \frac{1}{2}\ddot{\rho}T_1^2 + \frac{1}{4}\pi_k{}^k - \frac{1}{4}\partial^{-2}\left(\partial^i \partial^j \mu_{ij}\right) . \tag{B.20}$$

Then the gauge transformation between φ and ζ is then given by [13]

$$\begin{aligned} \zeta =& \frac{\dot{\rho}}{\dot{\phi}}\varphi - \frac{1}{2}\frac{\dot{\rho}\ddot{\phi}}{\dot{\phi}^2}\varphi^2 + \frac{\dot{\rho}}{\dot{\phi}^2}\varphi\dot{\varphi} + \frac{\ddot{\rho}}{\dot{\phi}^2}\varphi^2 - \frac{1}{2\dot{\phi}}\partial_k \chi \partial^k \varphi - \frac{1}{4}\frac{e^{-2\rho}}{\dot{\phi}^2}\partial_k \varphi \partial^k \varphi \\ &+ \frac{1}{4}\frac{e^{-2\rho}}{\dot{\phi}^2}\partial^{-2}\partial_k \partial_l (\partial^k \varphi \partial^l \varphi) + \frac{1}{2\dot{\phi}}\partial^{-2}\partial_k \partial_l (\partial^k \varphi \partial^l \varphi) . \end{aligned} \tag{B.21}$$

Appendix C

Calculation of the Bispectrum

In this section, we give a explicit derivation of the leading order bispectrum as first calculated in [13]. We will calculate the bispectrum from the action in which we performed the field shift of $\zeta \to \zeta_n + f(\zeta_n^2) + ...$

C.1 Bispectrum at first order in slow-roll parameters

The following terms in the action will produce the leading order ϵ contribution to the bispectrum

$$S_3 = \int \mathrm{dt}\ d^3x\ \left[a^3\epsilon^2\zeta\dot{\zeta}^2 + a\epsilon^2\zeta(\partial\zeta)^2 - 2a\epsilon\dot{\zeta}(\partial\zeta)(\partial\chi)\right].$$

Using the definition of $\psi^{(1)}$, which is given by $\psi^{(1)} = a^2\epsilon\partial^{-2}\dot{\zeta}$ and using that the cubic action terms will contribute as -$\langle\zeta^3\rangle \sim i\langle[H,\zeta^3]\rangle$, we find the following three expressions that contribute to the bispectrum at order ϵ^2

$$\begin{aligned}
&\langle\zeta\zeta\zeta\rangle' = \\
&= -2i\ u(k_1,0)u(k_2,0)u(k_3,0)\int_{-\infty}^{0} d\tau\ a^2\epsilon^2\left(u^*(k_1,\tau)\frac{du^*(k_2,\tau)}{d\tau}\frac{du^*(k_3,\tau)}{d\tau} + 2\text{ perm}\right) + \text{c.c.} \\
&\quad + 2i\ u(k_1,0)u(k_2,0)u(k_3,0)\int_{-\infty}^{0} d\tau\ (a^2\epsilon^2 u^*(k_1,\tau)u^*(k_2,\tau)u^*(k_3,\tau)\mathbf{k}_1.\mathbf{k}_2 + 2\text{ perm}) + \text{c.c.} \\
&\quad + 2i\ u(k_1,0)u(k_2,0)u(k_3,0)\int_{-\infty}^{0} d\tau\ a^2\epsilon^2(u^*(k_1,\tau)\frac{du^*(k_2,\tau)}{d\tau}\frac{du^*(k_3,\tau)}{d\tau}\mathbf{k}_1.\mathbf{k}_2\left(\frac{1}{k_2^2}+\frac{1}{k_3^2}\right) \\
&\qquad\qquad + 2\text{ perm}) + \text{c.c.}\ .
\end{aligned} \tag{C.1}$$

These expressions oscillate rapidly when $\tau \to -\infty$, which can produce a non-zero result at this boundary, even worse, it can even diverge. Since the mode functions at this boundary are well within the Horizon ($k \gg aH$), we require that their contribution cancels out. We must therefore regularize our integral at this boundary [14]. We can regularize this boundary by shifting it slightly into the complex plane, as discussed in 3.3.1, then

$$\int_{-\infty}^{\tau_{end}} \to \int_{-\infty(1\pm i\epsilon)}^{\tau_{end}} . \tag{C.2}$$

Evaluating all integrals separately we find

term 1 =

$$\begin{aligned}
&= -2i\ u(k_1,0)u(k_2,0)u(k_3,0)\int_{-\infty(1+i\epsilon)}^{0} d\tau\ a^2\epsilon^2\left(u^*(k_1,\tau)\frac{du^*(k_2,\tau)}{d\tau}\frac{du^*(k_3,\tau)}{d\tau} + 2\ \text{perm}\right) + \text{c.c.}\\
&= -2\text{Re}\left[\frac{-iH^3}{\sqrt{64\epsilon^3\ \Pi_i k_i^3}}\int_{-\infty(1+i\epsilon)}^{0} d\tau\ \frac{2i}{H^2\tau^2}\frac{iH^3}{\sqrt{64\epsilon^3\ \Pi_i k_i^3}}(1-ik_1\tau)k_2^2k_3^2\tau^2 e^{ik\tau} + 2\ \text{perm}\right]\\
&= -\frac{H^4}{32\epsilon^2\ \Pi_i k_i^3}2\epsilon\ \text{Re}\left[ie^{ik\tau}k_2^2k_3^2\left(-i\frac{(k_t+k_1)}{k_t^2} - \frac{k_1\tau}{k_t}\right) + 2\ \text{perm}\right]_{\tau\to 0}\\
&= -\frac{H^4}{32\epsilon^2\ \Pi_i k_i^3}2\epsilon\ \left[k_2^2k_3^2\left(\frac{k_t+k_1}{k_t^2}\right) + 2\ \text{perm}\right]\\
&= -\frac{H^4}{32\epsilon^2\ \Pi_i k_i^3}2\epsilon\ \left[\frac{1}{k_t}\sum_{i<j}k_i^2k_j^2 + \frac{k_1k_2k_3}{k_t^2}\sum_{i<j}k_ik_j\right] .
\end{aligned}$$

To deal with the momentum vectors in the next two terms, we can use the following trick in combination with momentum conservation, i.e. $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$,

$$\begin{aligned}
&(\mathbf{k}_1+\mathbf{k}_2).(\mathbf{k}_1+\mathbf{k}_2) = (\mathbf{k}_1+\mathbf{k}_2)^2 = k_1^2 + 2\mathbf{k}_1.\mathbf{k}_2 + k_2^2\\
&\Rightarrow (\mathbf{k}_1+\mathbf{k}_2)^2 = (-\mathbf{k}_3)^2 = k_3^2 = k_1^2 + 2\mathbf{k}_1.\mathbf{k}_2 + k_2^2\\
&\Rightarrow \mathbf{k}_1.\mathbf{k}_2 = \frac{1}{2}\left(k_3^2 - k_1^2 - k_2^2\right) .
\end{aligned}$$

Similar expressions can be obtained for $\mathbf{k}_1.\mathbf{k}_3$ and $\mathbf{k}_2.\mathbf{k}_3$ by cyclic permutations of the $\mathbf{k}_1.\mathbf{k}_2$ result. Then term 2 and term 3 become

term 2 =

$$\begin{aligned}
&= +2i\ u(k_1,0)u(k_2,0)u(k_3,0)\int_{-\infty(1+i\epsilon)}^{0} d\tau\ a^2\epsilon^2 u^*(k_1,\tau)u^*(k_2,\tau)u^*(k_3,\tau)(\mathbf{k}_1.\mathbf{k}_2 + 2\ \text{perm}) + \text{c.c.}\\
&= -\frac{H^4}{32\epsilon^2\ \Pi_i k_i^3}2\epsilon\ \text{Re}\left[\int_{-\infty(1+\imath\epsilon)}^{0} d\tau\ \frac{i}{\tau^2}(1-ik_1\tau)(1-ik_2\tau)(1-ik_3\tau)e^{ik_t\tau}(\mathbf{k}_1.\mathbf{k}_2 + 2\ \text{perm})\right]\\
&= -\frac{H^4}{32\epsilon^2\ \Pi_i k_i^3}2\epsilon\ \left[\frac{k_tk_1k_2 + k_1k_2k_3 + k_t(k_1+k_2)k_3}{k_t^2}(\mathbf{k}_1.\mathbf{k}_2 + 2\ \text{perm})\right]\\
&= -\frac{H^4}{32\epsilon^2\ \Pi_i k_i^3}2\epsilon\ \left[\sum_{i\neq j}k_i^2k_j + \frac{1}{k_t}\sum_{i=1}^{3}k_i^4 - k_1k_2k_3\left(1 + \frac{1}{k_t^2}\sum_{i=1}^{3}k_i^2\right)\right] ,
\end{aligned}$$

term 3 =

$$= +2i\ u(k_1,0)u(k_2,0)u(k_3,0)\int_{-\infty(1+i\epsilon)}^{0} d\tau\ a^2\epsilon^2(u^*(k_1,\tau)\frac{du^*(k_2,\tau)}{d\tau}\frac{du^*(k_3,\tau)}{d\tau}\mathbf{k}_1.\mathbf{k}_2\left(\frac{1}{k_2^2}+\frac{1}{k_3^2}\right)$$
$$+ 2\text{ perm}) + \text{c.c.}$$
$$= -\frac{H^4}{32\epsilon^2\ \Pi_i k_i^3}(-2\epsilon)\text{Re}[\frac{-iH^3}{\sqrt{64\epsilon^3\ \Pi_i k_i^3}}\int_{-\infty(1+i\epsilon)}^{0} d\tau\ \left(\frac{i}{\tau^2}(1-ik_1\tau)k_2^2k_3^2\tau^2 e^{ik\tau}\mathbf{k}_1.\mathbf{k}_2\left(\frac{1}{k_2^2}+\frac{1}{k_3^2}\right)\right)$$
$$+ 2\text{ perm})] + \text{c.c.}$$
$$= \frac{H^4}{32\epsilon^2\ \Pi_i k_i^3}2\epsilon\left[\frac{k_2^2k_3^2(k_t+k_1)}{k_t^2}\frac{\mathbf{k}_1.\mathbf{k}_2}{k_2^2}+5\text{ perm}\right]$$
$$= \frac{H^4}{32\epsilon^2\ \Pi_i k_i^3}2\epsilon\left[-\sum_{i=1}^{3}k_i^3+\frac{6}{k_t}\sum_{i<j}k_i^2k_j^2-\frac{1}{k_t}\sum_{i=1}^{3}k_i^4+\frac{2k_1k_2k_3}{k_t^2}\sum_{i=1}^{3}k_i^2\right].$$

Combining all three terms, we find

$$\langle\zeta\zeta\zeta\rangle'_{\epsilon^2} = \frac{H^4}{32\epsilon^2\ \Pi_i k_i^3}2\epsilon\left[-\sum_{i=1}^{3}k_i^3+\sum_{i\neq j}^{3}\left(k_i^2k_j+\frac{4}{k_t}k_i^2k_j^2\right)\right]. \tag{C.3}$$

Note that both terms in the summation of $\sum_{i\neq j}^{3}\left(k_i^2k_j+\frac{4}{k_t}k_i^2k_j^2\right)$ produce six terms. This result is known and was first calculated by Maldacena ref.([13]).

At leading order, there is also a contribution multiplying η to the bispectrum. This contribution is produced by the superhorizon part of our field shift 4.101. The contribution of this term is given by

$$\langle\zeta(\mathbf{k}_1,\tau_*)\zeta(\mathbf{k}_2,\tau_*) : \zeta(\mathbf{k}_3,\tau_*)\zeta(\mathbf{k}_3,\tau_*) :\rangle' +\ 2\text{ perm} = \frac{H^4}{16\epsilon^2}\frac{\eta}{2}\frac{k_1^3+k_2^3+k_3^3}{k_1^3k_2^3k_3^3}. \tag{C.4}$$

This term can be interpreted in the following way. When doing a field shift, we "move away" from the conserved comoving curvature perturbation fluctuation ζ. At the moment we convert back to the comoving curvature field, we have a second order relation between ζ_n and ζ. The linear order relation produces the interaction diagrams which we calculate using the *in-in* formalism. The second order relation produces a superhorizon contribution, which is the correction for moving away from the comoving curvature hyperslicings.

C.2 Bispectrum at next to leading order in slow-roll parameters

The terms in the action that produce an ϵ^2 contribution to the bispectrum are given by

$$S_3 = \int \mathrm{dt}\ d^3x \left(\frac{1}{2}\frac{\epsilon}{a}\partial\zeta\partial\psi^{(1)}\partial^2\psi^{(1)} + \frac{\epsilon}{4}\partial^2\zeta(\partial\psi^{(1)})^2 \right) \tag{C.5}$$

Note that $\psi^{(1)}$ is build from two terms, but only one of them multiplies a factor of ϵ. Then the contribution at order ϵ^2 is obtained when taking both of the terms that are multiplying a factor of ϵ. Their contributions are then given by

$$\begin{aligned}\langle\zeta^3\rangle_{\epsilon^2} &= \\ &= \left(\prod_{i=1}^{3} u_i(0)\right) \frac{i}{2}\int_{-\infty}^{\tau_{end}} d\tau\ a^2\epsilon^2 \left[u_1^*(\tau)\frac{du_2^*(\tau)}{d\tau}\frac{du_3^*(\tau)}{d\tau}\left(\frac{\mathbf{k}_1.\mathbf{k}_2}{k_2^2} + \frac{\mathbf{k}_1.\mathbf{k}_3}{k_3^2}\right) + 2 \text{ perm.}\right] + \text{c.c.} \\ &+ \left(\prod_{i=1}^{3} u_i(0)\right) \frac{i}{2}\int_{-\infty}^{\tau_{end}} d\tau\ a^2\epsilon^2 \left[u_1^*(\tau)\frac{du_2^*(\tau)}{d\tau}\frac{du_3^*(\tau)}{d\tau}\frac{k_1^2}{k_2^2k_3^2}\mathbf{k}_2.\mathbf{k}_3 + 2 \text{ perm.}\right] + \text{c.c.}\end{aligned} \tag{C.6}$$

Then their contributions separately are given by

$$\begin{aligned}\langle\zeta^3\rangle_{\epsilon^2,1} &\equiv \\ &\equiv \left(\prod_{i=1}^{3} u_i(0)\right) \frac{i}{2}\int_{-\infty}^{\tau_{end}} d\tau\ a^2\epsilon^3 \left[u_1^*(\tau)\frac{du_2^*(\tau)}{d\tau}\frac{du_3^*(\tau)}{d\tau}\left(\frac{\mathbf{k}_1.\mathbf{k}_2}{k_2^2} + \frac{\mathbf{k}_1.\mathbf{k}_3}{k_3^2}\right) + 2 \text{ perm.}\right] + \text{c.c.} \\ &= \frac{H^4}{16\epsilon^2\prod_{i=1}^{3}k_i^3}\frac{\epsilon^2}{4}\left[\sum_{i=1}^{3}\frac{k_i^5}{k_t^2} + \frac{3}{2}\sum_{i\neq j}\frac{k_i^4k_j}{k_t^2} - 3\sum_{i<j<l}\frac{k_i^2k_j^2k_l}{k_t^2} - \frac{5}{2}\sum_{i\neq j}\frac{k_i^2k_j^3}{k_t^2}\right],\end{aligned} \tag{C.7}$$

and

$$\begin{aligned}\langle\zeta^3\rangle_{\epsilon^2,2} &\equiv \left(\prod_{i=1}^{3} u_i(0)\right) \frac{i}{2}\int_{-\infty}^{\tau_{end}} d\tau\ a^2\epsilon^3 \left[u_1^*(\tau)\frac{du_2^*(\tau)}{d\tau}\frac{du_3^*(\tau)}{d\tau}\frac{k_1^2}{k_2^2k_3^2}\mathbf{k}_2.\mathbf{k}_3 + 2 \text{ perm}\right] + \text{c.c.} \\ &= \frac{H^4}{16\epsilon^2\prod_{i=1}^{3}k_i^3}\frac{\epsilon^2}{4}\left[\sum_{i=1}^{3}\frac{k_i^5}{k_t^2} + \frac{1}{2}\sum_{i\neq j}\frac{k_i^4k_j}{k_t^2} - \frac{3}{2}\sum_{i<j<l}\frac{k_i^2k_j^2k_l}{k_t^2} - \frac{3}{2}\sum_{i\neq j}\frac{k_i^2k_j^3}{k_t^2}\right],\end{aligned} \tag{C.8}$$

here the subscript 1 and 2 refer to the first and second term on the right hand side of (C.6). Summing both results we obtain

$$\langle\zeta^3\rangle_{\epsilon^2} = \frac{H^4}{16\epsilon^2\prod_{i=1}^{3}k_i^3}\frac{\epsilon^2}{4}\left[2\sum_{i=1}^{3}\frac{k_i^5}{k_t^2} + 2\sum_{i\neq j}\frac{k_i^4k_j}{k_t^2} - 4\sum_{i<j<l}\frac{k_i^2k_j^2k_l}{k_t^2} - 4\sum_{i\neq j}\frac{k_i^2k_j^3}{k_t^2}\right]. \tag{C.9}$$

Appendix D

Two point function in a pure de Sitter- spacetime

In this section, we calculate the two point correlation function, starting directly in a pure de Sitter spacetime, rather than starting in a quasi-de Sitter spacetime and then taking the appropriate limit. We verify the result of sections 2 and 3 of [24] in this appendix.

D.1 Two point function - momentum space

We start with the action for a minimally coupled (massive) scalar field,

$$S = \frac{1}{2}\int \mathrm{dt}\ d^3x\ \sqrt{|g|}\left(\partial_\mu\phi\partial^\mu\phi - m^2|\phi|^2\right) \tag{D.1}$$

In order to solve the equation of motion, it is convenient to make the following field redefinition,

$$\chi = a^{\frac{n-1}{2}}\phi, \tag{D.2}$$

here n is defined as the number of spatial dimensions. Then (D.1) becomes

$$S = \frac{1}{2}\int \mathrm{dt}\ d^3x\ \left((\partial_\tau\chi)^2 + \frac{1}{2}\left(\frac{(n^2-1)}{4\tau^2} - \frac{m^2}{H^2\tau^2} - k^2\right)\chi^2\right) \tag{D.3}$$

The equation of motion for χ is given by

$$\partial_\tau^2\chi + \left(\frac{m^2}{H^2\tau^2} + k^2 - \frac{(n^2-1)}{4\tau^2}\right)\chi = 0 \tag{D.4}$$

This equation resembles the *Mukhanov-Sasaki* equation and the solutions for χ are given by

$$\chi(\tau) = \hat{a}\sqrt{\tau}\ \mathrm{H}_\nu^{(1)}(k\tau) + \hat{a}^\dagger\sqrt{\tau}\ \mathrm{H}_\nu^{(2)}(k\tau), \tag{D.5}$$

here $\nu = \sqrt{n^2/4 - m^2/H^2}$. Note that we can make the usual $(\tau \to -\tau)$ change if we want to, then the leading order contribution of the two point function in the $\tau \to 0$ limit is then given by

$$\langle\phi(\tau,k)\phi(\tau',-k)\rangle' = \frac{(\tau\tau')}{4\pi}\left[\Gamma(\nu)^2\left(\frac{k^2\tau\tau'}{4}\right)^{-\nu} + \Gamma(-\nu)^2\left(\frac{k^2\tau\tau'}{4}\right)^{\nu}\right] + \ ... \tag{D.6}$$

D.2 Two point function - configuration space

We can acquire the two point correlation function in configuration space from making a direct Fourier transformation of (D.6). Then the two point function is given by

$$\begin{aligned}
\langle\phi(x)\phi(y)\rangle &= \frac{\tau\pi}{4(2\pi)^n a^{n-1}} \int d^n k\ e^{ik(x-y)} \mathrm{H}_\nu^{(1)}(k\tau)\mathrm{H}_\nu^{(2)}(k\tau) \\
&= \frac{\tau\pi}{4(2\pi)^n a^{n-1}} \int d^n k\ e^{ik(x-y)} (J_\nu(k\tau)+iY_\nu(k\tau))(J_\nu(k\tau)-iY_\nu(k\tau)) \\
&= \frac{\tau\pi}{4(2\pi)^n a^{n-1}} \int d^n k\ e^{ik(x-y)} (J_\nu^2(k\tau)+Y_\nu^2(k\tau)) \\
&= \frac{\tau\pi}{4(2\pi)^n a^{n-1}} \int d^n k\ e^{ikr} \left(\frac{2}{\pi}\int_0^\infty \frac{\mathrm{dz}}{z} e^{\frac{-k^2}{2z}+\tau^2 z} K_\nu(\tau^2 z) e^{\tau^2 z-\frac{r^2}{2}z}\right) \\
&= \frac{\tau\pi}{(4(2\pi)^n a^{n-1}} \int d^n k\ e^{ikr} \left(\frac{2}{\pi}\int_0^\infty \mathrm{dz}\ z^{\frac{n}{2}-1} K_\nu(\tau^2 z) e^{\tau^2 z-\frac{r^2}{2}z} r^{\frac{n}{2}-1}\right) \\
&= \frac{\tau\pi}{4(2\pi)^n a^{n-1}} \int d^n k\ e^{ikr} \sqrt{\frac{2}{\pi^3\tau^2 r}}\Gamma\left(\frac{n}{2}-\nu\right)\Gamma\left(\frac{n}{2}+\nu\right)\left(\tau^2-\frac{r^2}{4}\right)^{\frac{1-n}{4}} P_{\nu-\frac{1}{2}}^{\frac{1-n}{2}}\left(\frac{r^2}{2\tau^2}-1\right) \\
&= \frac{1}{(4\pi r)^{\frac{n+1}{2}} a^{n-1}} \frac{\Gamma\left(\frac{n}{2}-\nu\right)\Gamma\left(\frac{n}{2}+\nu\right)}{4\tau^2-r^2} P_{\nu-\frac{1}{2}}^{\frac{1-n}{2}}\left(\frac{r^2}{2\tau^2}-1\right) \\
&= \frac{H^2}{(4\pi)^{\frac{n+1}{2}}} \frac{\Gamma\left(\frac{n}{2}-\nu\right)\Gamma\left(\frac{n}{2}+\nu\right)}{\Gamma\left(\frac{n+1}{2}\right)} \mathrm{F}\left(\frac{n}{2}-\nu, \frac{n}{2}+\nu; \frac{n+1}{2}; 1-\frac{r^2}{2\tau^2}\right)\ ,
\end{aligned} \tag{D.7}$$

here r has been defined as $r \equiv |x-y|$. This result is equivalent to (3.14) of [24].

Appendix E

Solving Correlation functions using conformal symmetries

In this section, we calculate explicitly the momentum dependence of the two and three point functions for scalar fields up to a overall multiplicative factor. We show that, at least at the conformal boundary, a new way arises in order to organize the different shapes from which correlators are build.

E.1 Two point correlation function of conformal scalars

We can derive an explicit form for the two point correlation function of two conformal (scalar) fields by imposing the Dilation and the SCT on the two point correlation function. We shall do this in momentum space. In momentum space, the Poincarè symmetries imply that the two point correlation functions only depend on the magnitude one single momentum vector. Then the Dilation isometry, D, and the SCT isometry, K, then give the following two equations which contrain the form of the correlation function [24]

$$0 = D\langle \mathcal{O}(\mathbf{p})\mathcal{O}(-\mathbf{p})\rangle' = \left[d - \Delta_1 - \Delta_2 + p\frac{\partial}{\partial p}\right]\langle \mathcal{O}(\mathbf{p})\mathcal{O}(-\mathbf{p})\rangle' \tag{E.1}$$

and

$$0 = K\langle \mathcal{O}(\mathbf{p})\mathcal{O}(-\mathbf{p})\rangle' = \left[\partial_p^2 - \frac{(\Delta_1 + \Delta_2 - d - 1)}{p}\partial_p\right]\langle \mathcal{O}(\mathbf{p})\mathcal{O}(-\mathbf{p})\rangle'. \tag{E.2}$$

These identities are also known as *Ward identities*. From (E.1) we conclude that

$$\langle \mathcal{O}(\mathbf{p})\mathcal{O}(-\mathbf{p})\rangle = c_{12}p^{\Delta_1+\Delta_2-d}, \tag{E.3}$$

here c_{12} is a integration constant. Using (E.2) we note that it only holds iff $\Delta_1 = \Delta_2$. Then our two point correlation up to an overall multiplicative factor is given by

$$\langle \mathcal{O}(\mathbf{p})\mathcal{O}(-\mathbf{p})\rangle = c_{12}\; p^{2\Delta_1-d}. \tag{E.4}$$

E.2 Three point correlation function of conformal scalars

For the three point correlation functions of scalar fields, the Poincarè symmetries imply again that our correlation function $\langle \mathcal{O}(\mathbf{p}_1)\mathcal{O}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle'$ can be expressed in terms of the length of our momenta, i.e. in terms of p_1, p_2 and p_3. Then the *Ward identities* for the three point function become [24],[26]

$$D\langle \mathcal{O}(\mathbf{p}_1)\mathcal{O}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle' = \left[2d - \Delta_t + \sum_{a=1}^{3} p_a \frac{d}{dp_a}\right] \langle \mathcal{O}(\mathbf{p}_1)\mathcal{O}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle' \,, \tag{E.5}$$

and

$$\left[b_\mu \sum_{a=1}^{3} K_a^\mu\right] \langle \mathcal{O}(\mathbf{p}_1)\mathcal{O}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle' = \sum_{a=1}^{3} \left[b_\mu p_a^\mu \left(\frac{\partial^2}{\partial p_a^2} - \frac{(2\Delta_a - d - 1)}{p_a}\frac{\partial}{\partial p_a}\right)\right] \langle \mathcal{O}(\mathbf{p}_1)\mathcal{O}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle' . \tag{E.6}$$

The first ward identity (E.5) constraints the overal momentum scaling of the three point function to

$$\langle \mathcal{O}(\mathbf{p}_1)\mathcal{O}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle' \sim p_1^{\Delta_t - 2d} F\left(\frac{p_2}{p_1}, \frac{p_3}{p_1}\right), \tag{E.7}$$

here $F\left(\frac{p_2}{p_1}, \frac{p_3}{p_1}\right)$ is an arbitrary function with no overall momentum scaling. To constrain the three point function with the SCT isometry, i.e. (E.6), we need to get rid of the overall scaling $b_\mu p_a^\mu$. This can be achieved by using the Poincarè symmetries again. Since our correlation function should also be invariant under rotations, we can choose our vector b_μ in such a way that it removes one of the three K_a^μ of (E.6). If we choose for example $b_\mu p_3^\mu = 0$, together with momentum conservation we can write

$$b_\mu \left(p_1^\mu + p_2^\mu + p_3^\mu\right) = 0 \qquad \Rightarrow \qquad b_\mu p_1^\mu = -b_\mu p_2^\mu. \tag{E.8}$$

This reduces (E.6) to

$$K_{ab}\langle \mathcal{O}(\mathbf{p}_1)\mathcal{O}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle' \equiv \left[K_a - K_b\right] \langle \mathcal{O}(\mathbf{p}_1)\mathcal{O}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle'. \tag{E.9}$$

here $a, b \in_{cyclic} \{1, 2, 3\}$ and

$$K_a \equiv \left[\frac{\partial^2}{\partial p_a^2} - \frac{(2\Delta_a - d - 1)}{p_a}\frac{\partial}{\partial p_a}\right]. \tag{E.10}$$

Note that K_a is again the D'Alembert operator for a $\mathbb{R}^{d+2-2\Delta_a}$ spacetime with radial coordinate p_a. In order to solve (E.9), we have to realize that every single K_a only acts on the momentum corresponding to one of the conformal scalar fields. This suggests that in order to solve the three point function, we can use separation of variables and therefore we can look at solutions of the form

$$\langle \mathcal{O}(\mathbf{p}_1)\mathcal{O}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle' = f(p_1, p_2, p_3) = f(p_1)f(p_2)f(p_3)\,. \tag{E.11}$$

Then the functions $f(p_a)$, with $a \in \{1,2,3\}$ are determined by

$$\frac{K_1 f(p_1)}{f(p_1)} = \frac{K_2 f(p_2)}{f(p_2)} = \frac{K_2 f(p_3)}{f(p_3)} = C\,, \tag{E.12}$$

here C is a constant. The equation for $f(p_a)$ then becomes

$$\left[\frac{\partial^2}{\partial p_a^2} - \frac{(2\Delta_a - d - 1)}{p_a}\frac{\partial}{\partial p_a}\right] f(p_a) = Cf(p_a)\,. \tag{E.13}$$

The solution of (E.13) is known, since this is equation is equivalent to Bessels equation. The solution of $f(p_a)$ is given by

$$f(p_a) = p^{\Delta_a - d/2}\left(a_K \mathrm{K}_{\Delta_a - d/2} + a_I \mathrm{I}_{\Delta_a - d/2}\right)\,, \tag{E.14}$$

here $\mathrm{K}_{\Delta_a - d/2}$ and $\mathrm{I}_{\Delta_a - d/2}$ are *modified-Bessel functions* and a_K and a_I are constants which need to be determined. In order to find a solution for the D and the SCT isometry, we can use a *Mellin transformation* of the form

$$\int_0^\infty dx\; x^{-(d+1+\Delta_t)} f(p_1 x, p_2 x, p_3 x)\,. \tag{E.15}$$

The integral in (E.15) only diverges as at least one of the $f(p_a)$ is given by $p^{\Delta_a - d/3} a_K \mathrm{K}_{\Delta_a - d/2}$. Since we require that the correlation function is invariant under permutation of p_1, p_2 and p_3 the only *physically* relevant solution is when all three $f(p_a)$ functions are modified bessel K-functions. Then the final solution for the three point function, up to a multiplicative constant, is given by

$$\begin{aligned}&\langle \mathcal{O}(\mathbf{p}_1)\mathcal{O}(\mathbf{p}_2)\mathcal{O}(\mathbf{p}_3)\rangle' \sim \\ &\qquad \sim p_1^{\Delta_1 - d/2} p_2^{\Delta_1 - d/2} p_3^{\Delta_1 - d/2} \int_0^\infty dx\; x^{d/2-1} \mathrm{K}_{\Delta_1 - \frac{d}{2}}(p_1 x) \mathrm{K}_{\Delta_2 - \frac{d}{2}}(p_2 x) \mathrm{K}_{\Delta_3 - \frac{d}{2}}(p_3 x)\,.\end{aligned} \tag{E.16}$$

This solution is known as the *Triple-K-Integral* [26]. A very comprehensive analysis of the implications of conformal invariance for three point functions is given by [26]. An interesting thing to note when comparing (E.17) to calculations performed in chapters 4 and 5 is that the (conformal) time integral resembles the dummy variable that is used in the Mellin transformation. Due to this, the triple K integral is almost exactly the same as the integral that needs to be calculated in order to derive (5.46) and (5.47).

E.3 Solutions of the Ward identities

From a conformal symmetry point of view, the full solution to the truncated three point function is given by [26]

$$\begin{aligned}\langle \mathcal{O}(\mathbf{k}_1)\mathcal{O}(\mathbf{k}_2)\mathcal{O}(\mathbf{k}_3)\rangle &= C_{123}\pi^{\frac{3d}{2}}2^{3d-\Delta_t}\prod_i \frac{\Gamma(\delta_i)}{\Gamma\left(\frac{d}{2}-\delta_j\right)}\int d^dk\ \frac{1}{|\mathbf{k}|^{2\delta_3}|\mathbf{k}_1-\mathbf{k}|^{2\delta_2}|\mathbf{k}_2+\mathbf{k}|^{2\delta_1}}\\ &= \frac{c_{123}\pi^d 2^{4+\frac{3d}{2}-\Delta_t}}{\Gamma\left(\frac{\Delta_t-d}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_3-\Delta_2}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_3-\Delta_1}{2}\right)}\times\\ &\quad\times k_1^{\Delta_1-\frac{d}{2}}k_2^{\Delta_2-\frac{d}{2}}k_3^{\Delta_3-\frac{d}{2}}\int_0^\infty dx\ x^{\frac{d}{2}-1}K_{\Delta_1-\frac{d}{2}}(k_1x)K_{\Delta_2-\frac{d}{2}}(k_2x)K_{\Delta_3-\frac{d}{2}}(k_3x)\,,\end{aligned} \tag{E.17}$$

here δ_j has been defined as $\delta_j = \frac{d-\Delta_t}{2}+\Delta_j$, where $j=1,2,3$, d is the total number of spatial dimensions and C_{123} and c_{123} are two constant. Just as before, only the modified Bessel functions with half-integer $\Delta_i - \frac{d}{2}$ give us exact expressions for $K_{\Delta_i-\frac{d}{2}}(k_ix)$. This limits us again when searching for exact expressions for the three point function.

Another problem that arise is the fact that the integral part of (E.17) might not converge and that some sort of regularization scheme is necessary. In general, and assuming that all variables are real, the integral converges for

$$\frac{d}{2} > \sum_{j=1}^{3}\left|\Delta_j - \frac{d}{2}\right| + 2. \tag{E.18}$$

When we encounter a divergence that is not logarithmic, we can use analytic continuation in order to regularize our integral by putting [26]

$$d \to d+2\varepsilon, \qquad \Delta_i \to \Delta_i + \varepsilon. \tag{E.19}$$

Note that ε is not related to any Hubble parameter, it is just a small phase factor ($\varepsilon \ll 1$) used to regularize the integral. If the divergence is logarithmic, renormalisation is needed. After renomalisation, the correlators exhibit anomalous scaling transformations: they theory suffers from conformal anomalies.

E.3.1 Solution to the massive case

We want to calculate the bispectrum term using the ward identities and its solution, i.e. (E.17) and compare it to (5.47). In this case, we have that $d=3$, $\Delta_1=\Delta_2=\Delta_3=1$. Then the modified K-Bessel function will become

$$K_{1/2}(x) = K_{-1/2}(x) = \sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x}}. \tag{E.20}$$

The integral will not converge, so we need to regularize is by using (E.19). Then the "Triple K-integral" becomes

$$\begin{aligned}&\int_0^\infty dx\; x^{-1+\varepsilon} K_{-1/2+2\varepsilon}(k_1 x) K_{-1/2+2\varepsilon}(k_2 x) K_{-1/2+2\varepsilon}(k_3 x) = \\ &= \lim_{\varepsilon\to 0}\left[\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \frac{(k_1+k_2+k_3)^{-\varepsilon}}{\sqrt{k_1 k_2 k_3}} \Gamma(\varepsilon)\right] .\end{aligned} \tag{E.21}$$

Combining this with the rest of (E.17) we find

$$\langle \mathcal{O}(k_1)\mathcal{O}(k_2)\mathcal{O}(k_3)\rangle \sim \frac{1}{k_1 k_2 k_3} . \tag{E.22}$$

E.3.2 Solution to the massless case

For the massless case we have $d = 3$, $\Delta_1 = \Delta_2 = \Delta_3 = 0$. The modified K-Bessel function becomes

$$K_{3/2}(x) = K_{-3/2} = \sqrt{\frac{\pi}{2}}(1+x)\frac{e^{-x}}{x^{\frac{3}{2}}} . \tag{E.23}$$

Again, analytic continuation is needed to regularize this integral.

$$\begin{aligned}&\int_0^\infty dx\; x^{\frac{3}{2}-1+\varepsilon} K_{-3/2+2\varepsilon}(k_1 x) K_{-3/2+2\varepsilon}(k_2 x) K_{-3/2+2\varepsilon}(k_3 x) = \\ &= \lim_{\varepsilon\to 0}\left[\left(\frac{\pi}{2}\right)^{\frac{3}{2}} \frac{(k_1+k_2+k_3)^{-\varepsilon}}{(k_1 k_2 k_3)^{\frac{3}{2}}}(-2+\varepsilon)\left(\sum_{i=1}^{3} k_i^3 + \varepsilon\left(\sum_{i\neq j}^{3} k_i^2 k_j - k_1 k_2 k_3\right) + \varepsilon^2 k_1 k_2 k_3\right)\Gamma(-3+\varepsilon)\right] .\end{aligned} \tag{E.24}$$

Combining this with the rest of (E.17) we find in the leading order limit in ε

$$\langle \mathcal{O}(k_1)\mathcal{O}(k_2)\mathcal{O}(k_3)\rangle \sim \frac{k_1^3+k_2^3+k_3^3}{k_1^3 k_2^3 k_3^3}. \tag{E.25}$$

This is the *Local* shape for massless scalars. Expanding (E.24) to the next to leading order in ε produces

$$\langle \mathcal{O}(k_1)\mathcal{O}(k_2)\mathcal{O}(k_3)\rangle \sim \frac{\log(-K\tau_*)\sum_{i=1}^3 k_i^3 - \sum_{i\neq j} k_i^2 k_j + k_1 k_2 k_3}{k_1^3 k_2^3 k_3^3} . \tag{E.26}$$

Note that an additional factor in the log term has been put, τ_*. This has been done to make the factor inside the log-term dimensionless. Its interpretation is that I acts as a cutoff scale to which this approximation is valid. I shall refer to this shape as the *conformal shape*. These terms are exactly the left-over terms with the correct relative prefactors as found in (5.46). This suggests that we actually can build the scalar bispectrum up to a relative prefactor between the *local* and the *conformal shape*.

Appendix F

Higher order correlation functions beyond the three point function

In the decoupling limit, (5.2), the action expanded in terms of field perturbations beyond cubic order also simplifies considerably,

$$S = \int d^4x \; a^3 \left[-\frac{V^{(4)}}{24}\varphi^4 - \frac{V^{(5)}}{120}\varphi^5 \right] . \tag{F.1}$$

At tree level, we can have two kinds of diagrams, the first diagram is constructed from two three point vertices and the second diagram is constructed from only a single four point vertex. Let us study the squeezed limit of the trispectrum contribution of the contribution to the bispectrum produced by the single four-vertex diagram.

F.1 The squeezed four point function

It will be interesting to see how the correlation function of the three point function changes by a field shift of the background fields. As already mentioned in 5.7.1, the mode functions in this correlation functions will get an effective mass and therefore, the three point contribution should get non-trivial contributions. Even worse, we will not be able to solve the integrals with these time dependent mass corrected mode functions. Therefore, by taking the squeezed limit of the four point function, we are also finding (a general) solution to a very difficult integral.

Let us study this squeezed limit of the contribution to the bispectrum coming from the four-vertex diagram. The general solution to this contribution is given by

$$\begin{aligned}\langle\varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\rangle = \frac{H^4\; V^{(4)}(\bar{\phi})}{48k_t k_1^3 k_2^3 k_3^3 k_4^3}\Bigg(2\sum_{i=4}^{4} k_i^4\left(-1+\gamma_E+\log(-k_t\tau_*)\right) + \\ + \sum_{i\neq j}^{3} k_i k_j^3(-4+2\gamma_E+2\log(k\tau_*) + 4\sum_{i\neq j} k_i^2k_j^2 - 2\sum_{i\neq j\neq l} k_i^2 k_j k_l + 2k_1k_2k_3k_4\Bigg)\end{aligned} \tag{F.2}$$

If we now take the squeezed limit of this four point function by setting $k_4 \equiv k_l \ll 1$, but keeping all other momenta to be different, so we do not redefine $k_1, k_2, k_3 \equiv k_s$ we obtain for the squeezed limit of (F.2)

$$\begin{aligned}&\langle\varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\rangle|_{k_4 \ll k_{1,2,3}} = \\ &= \frac{H^4\ V^{(4)}(\bar{\phi})}{24\ k_1^3 k_2^3 k_3^3 k_l^3}\left[(-1+\gamma_E+\log(-k_t\tau_*))\sum_{i=1}^{3}k_i^3 - \sum_{i\neq j}k_i^2 k_j + k_1 k_2 k_3\right].\end{aligned} \tag{F.3}$$

Interestingly enough, we observe that the squeezed limit of this diagram has precisely the momentum scaling that one would have obtained by multiplying the "regular" two point (5.38) and three point functions (5.46). This means that the leading order contributions to the bispectrum are not effected by the long mode which shifts the background dynamics.

Similar to the massless case, we obtain the following contribution to the massive $m^2 = 2H^2$ trispectrum

$$\langle\varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\rangle = \frac{H^4\ V^{(4)}(\bar{\phi})}{8}\frac{\tau_*^4}{k_1 k_2 k_3 k_4 k_t}. \tag{F.4}$$

Taking the squeezed limit and using definitions similar to (F.5), we obtain

$$\langle\varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\rangle|_{k_4 \ll k_{1,2,3}} = \frac{H^4\ V^{(4)}(\bar{\phi})}{8}\frac{\tau_*^4}{k_1 k_2 k_3 (k_1+k_2+k_3)\ k_l}. \tag{F.5}$$

Interestingly enough, if we divide the long mode two point function, we note that the shifted three point function in this expression does not resemble (5.47) anymore. The conformal time and momentum scaling of this "shifted three point function" are now given by

$$\langle\varphi(k_1)\varphi(k_2)\varphi(k_3)\rangle_{\varphi_l} \sim \frac{\tau_*^2}{k_1 k_2 k_3 (k_1+k_2+k_3)}. \tag{F.6}$$

F.2 Five point function

The contribution to the five point function that is produced by a single five point vertex is given by

$$\begin{aligned}&\langle\varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4)\varphi(k_5)\rangle = \\ &= \frac{H^6 V^{(5)}}{48 k_1^5 k_2^5 k_3^5 k_4^5 k_5^5}\left(\sum_{i=1}^{5}k_i^3(-1+\gamma_E+\ln(-k_t\tau_*) - \sum_{i\neq j}k_i k_j^2 - \frac{3}{k_t^2}\sum_{i\neq j\neq m\neq n}k_i^2 k_j k_m k_n\right).\end{aligned} \tag{F.7}$$

Bibliography

[1] D.J. Fixsen, *The Temperature of the Cosmic Microwave Background*, arXiv:0911.1955v2 [astro-ph.CO].

[2] C. L. Bennett et all, *Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Final Maps and Results*, arXiv:1212.5225 [astro-ph.CO].

[3] NASA WMAP Science Team, url: http://map.gsfc.nasa.gov/.

[4] NASA WMAP Science Team, url: http://wmap.gsfc.nasa.gov/media/111133/index.html.

[5] S. Weinberg, *Cosmology*, 2008.

[6] D. Baumann, *TASI 2009*, TASI Lectures on Inflation.

[7] L. Senatore, *Tasi 2012*, Tasi 2012, lecture on inflation.

[8] J.M. Bardeen, *Gauge invariant perturbations*, PhysRevD.22.1882.

[9] R.M. Wald, *General Relativity*, QC173.6.W35 1984.

[10] M. Blau, Lecture Notes on General Relativity, version of 17 October, 2014.

[11] E. Gourgoulhon, *3+1 Formalism and Bases of Numerical Relativity*, (2007), arXiv:gr-qc/0703035.

[12] E. Pajer, L. Pimentel, J.V.S. van Wijck, draft: *The era of sub-microKelvin polarimetry and the conformal limit of inflation.*

[13] J.M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, JHEP **0305** (2003) arXiv:astro-ph/0210603.

[14] X. Chen, *Primordial Non-Gaussianities from Inflation Models*, Adv.Astron. 2010 (2010) 638979 arXiv:1002.1416.

[15] X.Chen, M.Huang, S. Kachru, G. Shiu, *Observational Signatures and Non-Gaussianities of General Single Field Inflation*, JCAP 0701:002, 2007, arXiv:hep-th/0605045.

[16] F. Arroja, K. Koyama, *Non-gaussianity from the trispectrum in general single field inflation*, Phys.Rev.D77:083517, 2008, arXiv:0802.1167 [hep-th].

[17] D.H. Lyth, *What would we learn by detecting a gravitational wave signal in the cosmic microwave background anisotropy?*, Phys. Rev. Lett. 78, 1861, arXiv:hep-ph/9606387.

[18] S. Weinberg, *Quantum Contributions to Cosmological Correlations*, Phys.Rev.D72:043514 (2005), arXiv:hep-th/0506236.

[19] C. Burrage, R.H. Ribeiro, D. Seery, *Primordial non-gaussianities from multiple-field inflation*, JCAP07 (2011) 032, arXiv:astro-ph/0210603.

[20] F. Arroja, T. Tanaka, *A note on the role of the boundary terms for the non-Gaussianity in general k-inflation*, JCAP 1105:005, 2011[astro-ph.CO], arXiv:astro-ph/0210603.

[21] C. Cheung, P. Creminelli, A. Liam Fitzpatrick, J. Kaplan, L. Senatore, *The Effective Field Theory of Inflation*, arXiv:0709.0293 [hep-th].

[22] P. Creminelli, *Conformal invariance of scalar perturbations in inflation*, Phys.Rev. D85 (2012) 041302, arXiv:1108.0874 [hep-th].

[23] D. Seery, J.E. Lidsey,*Primordial non-gaussianities from multiple-field inflation* JCAP0509:011,2005, arXiv:astro-ph/0506056.

[24] N. Arkani-Hamed, J. Maldacena, *Cosmological Collider Physics*, arXiv:1503.08043v1 [hep-th].

[25] E. Pajer, F. Schmidt, M. Zaldarriaga, *The Observed Squeezed Limit of Cosmological Three-Point Functions*, arXiv:1305.0824 [astro-ph.CO]

[26] A. Bzowski, P. McFadden and K. Skenderis, *Implications of conformal invariance in momentum space*, Doi: 10.1007/JHEP03 (2014) 111, arXiv:1304.7760 [hep-th].

[27] P. Di Francesco, P.Mathieu, D. Snchal, *Conformal Field Theory*, 1996.

[28] B. Ratha, *Restoration of spontaniously broken continuous symmetries in De Sitter spacetime*, 1968.

[29] H. Osborn and A. Petkos, *Implications of Conformal Invariance in Field Theories for General Dimensions*, Ann.Phys.231:311-362 1994, arXiv:hep-th/9307010v2 [hep-th].

[30] E. Pajer, *Cosmic (pie) evolution*, National Theory HEP seminar, UvA, NL