Uniform Kan cubical sets as a path category

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Abstract

In this thesis we look at the notion of a path category as considered in [vdBM16], and show that an instance of this structure can be found in the category of cubical sets. In particular we investigate the model of dependent type theory in cubical sets as constructed by Bezem, Coquand and Huber in [BCH14], of which we show that its fibrant objects give rise to a path category. First we rehearse the definition of a path category and some major theorems found in [vdBM16], and also prove some additional results beyond those given there. We then recall several of the definitions of and in cubical sets as presented in [BCH14], the most crucial of which will be those of the uniform Kan condition on cubical sets and morphisms, and of equivalences between cubical sets. We demonstrate that uniform Kan fibrations and equivalences can be taken as the fibrations and weak equivalences of a path category whose objects are the uniform Kan cubical sets. Having established this correspondence, we study some properties such as function extensionality on the level of path categories, along with their instantiation in cubical sets.

Contents

Introduction

Over the last several years, the subject of homotopy type theory has attracted increasing attention from a wide range of mathematicians. In part this may be correlated to the appearance of the standard work [HoTT], along with the efforts of the small yet active foundational movement around it. Perhaps a better explanation lies in the nature of the subject, as it finds itself at the remarkable intersection of the fields of homotopy theory, category theory and type theory. While we shall not attempt to provide an adequate historical account of when and where such insights into the relations between these branches of mathematics first arose, we shall say at least something as an introduction for the uninitiated.

The basic connection between homotopy theory and type theory underlying homotopy type theory is that the identity types of dependent type theory in the style of Martin-Löf exhibit the homotopical structure of a groupoid, as discussed in [HS96]. In particular, the objects of a type may be seen as corresponding to the points of a space, so that the terms of the identity type between two objects come out as paths from one such point to another. On the other hand, the connection between category theory and type theory comes from the fact that certain categories carry a structure much like the one belonging to the type formers of dependent type theory, and may in fact serve as models of such a type theory. One may consult for instance [Hof97], [Pit01] or the first chapter of [Hub15] for more about this.

A full-fledged model of dependent type theory (including the univalence axiom) in the category of simplicial sets appeared in [KLV12], in which types are interpreted as Kan fibrations. However, this model proved unsatisfactory for those interested in the constructive aspects of type theory, as it relies at some stages on classical reasoning. A similar model was therefore developed in cubical sets and published in [BCH14], in which types are interpreted as *uniform* Kan fibrations. While this model is indeed entirely constructive, it is so at the expense of the computation rule for the identity type, which holds only propositionally here.

As it turns out, this feature of having only propositional identity types is shared by those models which arise from path categories, the notion of which is introduced in [vdBM16] and further explored in [vdB16]. A path category, short for a category with path objects, is a category equipped with two classes of maps called the fibrations and the weak equivalences which satisfy a number of axioms specifying their behaviour. As such, path categories are related to model categories (which also possess a class of cofibrations), and more closely to categories of fibrant objects (which similarly omit the cofibrations). This correspondence between the homotopical framework of a path category and the type-theoretically motivated model in cubical sets thus presents a further connection between homotopy theory and type theory, with category theory acting as the unifying medium.

In the present work we shall contribute to the understanding of this relation between homotopy theory and type theory in the following way. In the first chapter we present the definition of a path category along with some of the important practical results from [vdBM16]. We then add to these amongst others a result which shows that the path structure within a path category is well-behaved (Proposition 1.13), and more significantly that the weak equivalences may be characterised in a variety of ways (Theorem 1.18), a result which is known to obtain in stronger type theories.

This will serve as a preparation for the second chapter, in which we shall prove that the full subcategory of uniform Kan fibrant cubical sets forms a path category (Theorem 2.28). This requires us to define the parts which make up the structure of a path category, which includes taking the uniform Kan fibrations as the class of fibrations, and demonstrating that the axioms of a path category hold for these. In doing so we consider aspects of the cubical set model which are not treated in [BCH14] or [Hub15], hence we obtain several new results regarding this model (including Proposition 2.20).

In the final chapter we expand our investigation to include function extensionality for exponentials and Π-types in the contexts of path categories and of cubical sets. We prove a few results in path categories which show that function extensionality is equivalent to an alternative condition on exponentials and Π-types. We conclude by giving a more detailed description of the witness to function extensionality for Π-types which may be constructed in cubical sets (Theorem 3.11), thereby expanding on previous results in both contexts.

Chapter 1

Path categories

In this chapter we concern ourselves with the notion of a path category as set out in [vdBM16]. Besides the definition of a path category itself, we consider additional relevant concepts and several important constructions within the context of a path category. As one would expect, a large number of these revolve around homotopies between maps. Hence we look at how homotopies are defined within a path category, and mention some of the results obtained in [vdBM16], which are often (variations of) familiar propositions from related contexts. To these we add a number of propositions which further explicate the homotopical behaviour of certain operations, the most important of which expresses that path objects exhibit a groupoid structure up to homotopy. Using this fact we are able to prove the first major result of this work, namely that the common definitions of equivalence of morphisms are interchangeable within a path category, analogous to a result known in homotopy type theory (cf. Chapter 4 of [HoTT]). We will become able to precisely formulate the content of this statement towards the end of this chapter.

1.1 Essential definitions

We shall start off by presenting the definition of a category with path objects, or path category for short.

Definition 1.1. A *path category* is given by a category $\mathcal{C}$ together with two classes of morphisms in $\mathcal{C}$, the *fibrations* and the *weak equivalences*, satisfying the seven axioms presented below. Here an *acyclic fibration* is any map which is both a fibration and a weak equivalence, and a *path object* on an object X is a factorisation of the diagonal $X \to X \times X$ as a weak equivalence $r : X \to PX$ followed by a fibration $(s, t) : PX \to X \times X$.

1. Fibrations are closed under composition.

2. The pullback of a fibration along any map exists and is a fibration.

3. The pullback of an acyclic fibration along any map is an acyclic fibration.

4. Weak equivalences satisfy 2-out-of-6, that is, if $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$ are composable with gf and hg weak equivalences, then so are f, g, h and hgf.

5. Isomorphisms are acyclic fibrations and every acyclic fibration has a section.

6. For any object X there is at least one path object PX, not necessarily functorial in X.

7. $\mathcal{C}$ has a terminal object 1 and every morphism $X \to 1$ is a fibration.

An immediate consequence of these axioms is the following proposition, which is hardly a noteworthy result in itself, but serves to fix the reference to a kind of object (P_f) which will be employed in other proofs.

Proposition 1.2. *Let $f : Y \to X$ be any map and $(s, t) : PX \to X \times X$ be some path object on X. Then we have the following pullback:*

$$\begin{tikzcd} P_f \arrow[r, "p_2"] \arrow[d, "p_1"'] & PX \arrow[d, "s"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

Proof. Since projection maps are fibrations by Axiom 2, $s : PX \to X$ is a fibration by Axiom 1, hence the pullback above exists by Axiom 2. □

In fact, since $sr = \mathsf{id}_X$ we have that s is an acyclic fibration by Axioms 5 and 4. This can be used to show that any such map f can be factored as a section of an acyclic fibration (id_Y, rf) followed by a fibration tp_2 (cf. Proposition 2.3 in [vdBM16]), but we will not make explicit use of this result. What we will make (indeed essential) use of is the definition of a homotopy between two parallel morphisms, which we come to now.

Definition 1.3. We say that two morphisms $f, g : Y \to X$ are *homotopic* if there is a path object PX of X and a map $h : Y \to PX$ such that $f = sh$ and $g = th$. We denote this by $h : f \simeq g$, or simply $f \simeq g$ if the homotopy h is of less importance.

By exhibiting certain additional maps between the appropriate (path) objects we show later on (Proposition 1.10) that this definition of homotopy is independent of the choice of path object. Moreover, the homotopy relation $\simeq$ thus obtained is a congruence relation on the morphisms of a path category, which is proven in [vdBM16] as Theorem 2.14. Though these facts are certainly not irrelevant, for our purposes we are more interested in the properties of the maps constructed in these proofs, as we need to understand these in order to proof our main result. However, before we do this we also wish to introduce a notion of *fibrewise homotopy*, in which we again follow [vdBM16]. To this end we consider how any object A of the path category $\mathcal{C}$ induces another path category $\mathcal{C}(A)$, which is a full subcategory of the corresponding slice category.

Definition 1.4. For $\mathcal{C}$ a path category and A some object in $\mathcal{C}$ we define the path category $\mathcal{C}(A)$ as follows. Its underlying category has as objects the fibrations (in $\mathcal{C}$) with codomain A, and as morphisms from $q : Y \to A$ to $p : X \to A$ the maps $f : Y \to X$ such that $pf = q$. To make this into a path category, we take such morphisms to be fibrations or weak equivalences in $\mathcal{C}(A)$ precisely when they are such in $\mathcal{C}$.

One can straightforwardly verify that $\mathcal{C}(A)$ thus defined is indeed a path category. We can now state what it means for two maps $f, g : Y \to X$ to be fibrewise homotopic:

Definition 1.5. Let $f, g : Y \to X$ be morphisms and $p : X \to A$ be any fibration, so that X has a path object $(s, t) : P_A(X) \to X \times_A X$ in $\mathcal{C}(A)$. We say that f and g are *fibrewise homotopic* (over A) if there is a map $h : Y \to P_A(X)$ such that $f = sh$ and $g = th$. Similar to before, we denote this by $h : f \simeq_A g$ or simply $f \simeq_A g$ if the homotopy h is of less importance.

We conclude this section with one more definition and some remarks.

Definition 1.6. We call a map $f : Y \to X$ a *homotopy equivalence* if there is a map $g : X \to Y$ such that $fg \simeq \mathsf{id}_X$ and $gf \simeq \mathsf{id}_Y$.

Also for reasons beyond its simplicity, the notion of a homotopy equivalence is quite a natural one to consider. What is interesting in this light is that weak equivalences and homotopy equivalences actually coincide in all path categories (a fact which follows from 2-out-of-6 for weak equivalences: cf. Theorem 2.16 and Remark 2.17 in [vdBM16]). In the course of this chapter we shall add to this result by considering other possible definitions of equivalences and showing that the same holds for those.

1.2 Constructions and proofs

We begin this section by observing a certain interplay between the fibrations and the weak equivalences of a path category, namely that weak equivalences possess a kind of weak lifting property with respect to fibrations. This property is the driving force behind the majority of the constructions which we carry out in the context of path categories, and gives rise to the main proof method which we subsequently use to prove that the notions which we introduce are suitably well-behaved.

1.2.1 The weak lifting property

In [vdBM16] a number of progressively stronger lifting properties of weak equivalences with respect to fibrations are established (Lemma 2.9, Lemma 2.25 and finally Theorem 2.38). We concern ourselves only with the last of these, which is expressed by the following proposition.

Theorem 1.7. *Suppose we have some commutative square as given below, where the map f on the left is a weak equivalence and the map p on the right is a fibration.*

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "f"'] & C \arrow[d, "p"] \\ B \arrow[r, "n"'] & D \end{tikzcd}$$

Then there is a lower filler $l : B \to C$ which is unique up to fibrewise homotopy over D such that $n = pl$ and $lf \simeq_D m$.

Though we shall not reproduce its lengthy proof in its entirety here, there are two intermediate results in [vdBM16] which will feature in the other two chapters, which we mention here again without proof. The first of these, there called Proposition 2.31, describes how under some circumstances we may strengthen commutativity up to homotopy.

Proposition 1.8. *Suppose we have a triangle as the one below, where p is a fibration, which commutes up to homotopy $pf \simeq g$.*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, "f"] \arrow[r, "g"'] & X \end{tikzcd}$$

Then there is a map $f' : Z \to Y$ which is homotopic to f for which the triangle commutes strictly, i.e. we have $f' \simeq f$ and $pf' = g$.

The second (Proposition 2.33 there) is a characterisation of the acyclic fibrations, namely as those fibrations with a section which is also a fibrewise homotopy inverse.

Proposition 1.9. *A fibration $f : X \to Y$ is acyclic precisely when it has a section $g : Y \to X$ such that $gf \simeq_Y \mathsf{id}_X$.*

As stated above, these propositions will not become relevant until the next chapter, hence we may forget about these for now. An important consequence of Theorem 1.7 is that we can find a (fibrewise) homotopy of two maps $f, g : Y \to X$ whenever we have a (fibrewise) homotopy $h : fa \simeq ga$ for some weak equivalence $a : Z \to Y$, for then we obtain a lower filler in a commutative square as the one below:

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "a"'] & PX \arrow[d, "{(s,t)}"] \\ Y \arrow[r, "{(f,g)}"'] & X \times X \end{tikzcd}$$

This enables a proof method which can be likened to path induction in homotopy type theory: indeed, if Y is itself some path object PA we can consider the weak equivalence $r : A \to PA$, which corresponds to taking the identity path. Another application of these lower fillers is in defining additional operations on path objects, including those which are used to show that the homotopy relation is an equivalence relation. To begin with, any map $f : X \to Y$ induces a suitable action on paths $Pf : PX \to PY$:

Proposition 1.10. *For any map $f : X \to Y$ and for any path objects PX for X and PY for Y there is a map $Pf : PX \to PY$ such that we have a commutative square:*

$$\begin{tikzcd} PX \arrow[r, "Pf"] \arrow[d, "{(s,t)}"'] & PY \arrow[d, "{(s,t)}"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

In particular, whenever PX and $P'X$ are path objects for X there is a map $g : PX \to P'X$ which commutes with the respective source and target maps. The homotopy relation therefore does not depend on the choice of path objects.

Proof. Both Pf and g arise as lower filler in the following square, where for the particular case we take f to be id_X:

$$\begin{tikzcd} X \arrow[r, "rf"] \arrow[d, "r"'] & PY \arrow[d, "{(s,t)}"] \\ PX \arrow[r, "{(fs,ft)}"'] & Y \times Y \end{tikzcd}$$

□

This proof furthermore tells us that $Pfr_X \simeq_{(Y \times Y)} r_Y f$, a fact which we will keep in mind for later. First, however, we define inversion and composition operations on a path object PX.

Proposition 1.11. *For any object X and for any path object PX for X there is a map $\sigma : PX \to PX$ such that $(s,t)\sigma = (t,s)$.*

Proof. We obtain σ as lower filler in this commutative square:

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "r"'] & PX \arrow[d, "{(s,t)}"] \\ PX \arrow[r, "{(t,s)}"'] & X \times X \end{tikzcd}$$

□

Here the proof contains the additional information that $\sigma r \simeq_{(X \times X)} r$, hence in fact $\sigma \simeq_{(X \times X)} \mathsf{id}_X$. We define the composition map next.

Proposition 1.12. *For any object X and for any path object PX there is a map $\tau : PX \times_X PX \to PX$ such that $(s,t)\tau = (sq_1, tq_2)$.*

Proof. The map τ arises as lower filler in the commutative square

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "{(r,r)}"'] & PX \arrow[d, "{(s,t)}"] \\ PX \times_X PX \arrow[r, "{(sq_1,tq_2)}"'] & X \times X \end{tikzcd}$$

where (r,r) is the unique map given by pullback in the diagram below.

$$\begin{tikzcd} X \arrow[dr, dotted, "{(r,r)}"] \arrow[drr, bend left, "r"] \arrow[ddr, bend right, "r"'] & & \\ & PX \times_X PX \arrow[r, "q_2"'] \arrow[d, "q_1"] & PX \arrow[d, "s"] \\ & PX \arrow[r, "t"'] & X \end{tikzcd}$$

□

From the proof above we conclude that $\tau(r,r) \simeq_{(X\times X)} r$ holds. This expression can be used along with the other two to derive further results regarding these constructed operations, which we will do now.

1.2.2 Establishing further properties

Here we shall demonstrate that homotopies and the previously defined operations interact in a way which can be most succinctly expressed in terms of a groupoid structure up to homotopy on path objects. We specify and prove this statement alongside a related result concerning the action on paths operation. For these we require a lemma which is somewhat of a technical formality and hence considered separately. We are then able to derive a concisely formulated proposition and corollary which serve to abbreviate some of the arguments contained in the final section of this chapter.

As stated above, path objects carry a groupoid structure up to homotopy, namely in the sense of the following proposition.

Proposition 1.13. *Let X be any object and PX be any path object on X. Up to homotopy there is a groupoid structure on PX with respect to the homotopies $h : Y \to PX$ for any Y, by which we mean that the following conditions are satisfied:*

1. *The source and target maps are homotopy sections of the reflexivity map, or $rs \simeq \mathsf{id} \simeq rt$. In particular, if $h : f \simeq g$, then $rf \simeq h \simeq rg$.*
2. *Composition is associative up to fibrewise homotopy, or $\tau(\mathsf{id}\times\tau) \simeq_{(X\times X)} \tau(\tau \times \mathsf{id})$. In particular, we have $\tau(h, \tau(k,l)) \simeq_{(X\times X)} \tau(\tau(h,k), l)$ for any $(h,k,l) : Y \to PX \times_X PX \times_X PX$.*
3. *Composition with the reflexivity path on source or target acts as identity up to fibrewise homotopy, or $\tau(rs, \mathsf{id}) \simeq_{(X\times X)} \mathsf{id} \simeq_{(X\times X)} \tau(\mathsf{id}, rt)$. In particular, if $h : f \simeq g$, then $\tau(rf, h) \simeq_{(X\times X)} h \simeq_{(X\times X)} \tau(h, rg)$.*
4. *Composition with the inverse yields an identity path up to (fibrewise) homotopy, or $\tau(\mathsf{id}, \sigma) \simeq_{(X\times X)} rs$ and $\tau(\sigma, \mathsf{id}) \simeq_{(X\times X)} rt$. In particular, if $h : f \simeq g$, then $\tau(h, \sigma h) \simeq_{(X\times X)} rf$ and $\tau(\sigma h, h) \simeq_{(X\times X)} rg$.*

Furthermore, the action on paths construction turns any map $f : Y \to X$ into a map $Pf : PY \to PX$ which commutes with the groupoid structures up to (fibrewise) homotopy.

Proposition 1.14. *For any map $f : Y \to X$ we have $Pf\sigma_Y \simeq_{(X\times X)} \sigma_X Pf$ and $Pf\tau_Y \simeq_{(X\times X)} \tau_X(Pf \times Pf)$, i.e. Pf commutes up to (fibrewise) homotopy with the respective groupoid structures. Furthermore, action on paths is functorial up to fibrewise homotopy, or $P(fg) \simeq_{(X\times X)} PfPg$ for any maps $g : Z \to Y$ and $f : Y \to X$.*

In some parts of the proofs of these propositions (and in fact in a few other proofs later on in this chapter), we wish to make use of the following way of extending homotopies.

Lemma 1.15. *Let $(h, k) : Y \to PX \times_X PX$, where X and Y are any two objects and PX is any path object for X. Then $h \simeq_{(X\times X)} h'$ implies that $(h, k) \simeq_{(X\times X)} (h', k)$, and $k \simeq_{(X\times X)} k'$ implies that $(h, k) \simeq_{(X\times X)} (h, k')$.*

Proof. We shall only treat the first case, as the proof for the other case is wholly analogous. Take the following pullback:

$$\begin{tikzcd} P_{(X\times X)}(PX) \times_X PX \arrow[r, "u_2"] \arrow[d, "u_1"'] & PX \arrow[d, "s"] \\ P_{(X\times X)}(PX) \arrow[r, "ts"'] & X \end{tikzcd}$$

Since s and ts are acyclic fibrations, u_1 and u_2 are acyclic fibrations as well, though for our purposes it suffices that either of the two is. In turn, consider the commutative square below.

$$\begin{tikzcd} PX \times_X PX \arrow[r, "r"] \arrow[d, "(r\times_X \mathsf{id})"'] & P_{(X\times X)}(PX \times_X PX) \arrow[d, "(s,t)"] \\ P_{(X\times X)}(PX) \times_X PX \arrow[r, "(s\times_X \mathsf{id}, t\times_X \mathsf{id})"'] & (PX \times_X PX) \times_{(X\times X)} (PX \times_X PX) \end{tikzcd}$$

It is easily checked that this is indeed a commutative square, and moreover that $r\times_X \mathsf{id}$ is a weak equivalence by 2-out-of-3, as either of the two projection maps is an acyclic fibration being the pullback of s or t. The latter square therefore has a lower filler $l : P_{(X\times X)}(PX) \times_X PX \to P_{(X\times X)}(PX \times_X PX)$ such that the lower triangle commutes. To see that we are now done, note that for any $(h, k) : Y \to PX \times_X PX$ and $H : h \simeq_{(X\times X)} h'$ we obtain a (unique) map $(H, k) : Y \to P_{(X\times X)}(PX) \times_X PX$ for which we see that $(s, t)l(H, k) = (s \times_X \mathsf{id}, t \times_X \mathsf{id})(H, k) = ((h, k), (h', k))$, hence $l(H, k)$ is our desired homotopy $(h, k) \simeq_{(X\times X)} (h', k)$. $\square$

A slight variation on this proof can be found in [vdB16] which is based on weak equivalences being preserved under pullback along fibrations. In any case, we are able at this point to provide proofs of the Propositions 1.13 and 1.14, which we shall do so in that order.

Proof of Proposition 1.13. 1. Since $r = rsr$ and $r = rtr$, we certainly have $r \simeq rsr$ and $r \simeq rtr$, and so $\mathsf{id} \simeq rs$ and $\mathsf{id} \simeq rt$. For the particular case, applying this to the homotopy h we find that $h \simeq rsh = rf$ and $h \simeq rth = rg$ as required.
2. Strictly speaking one would distinguish between $PX \times_X (PX \times_X PX)$ and $(PX \times_X PX) \times_X PX$, but these are easily seen to be isomorphic by the universal property of the pullback, hence we write $PX \times_X PX \times_X PX$. Since the respective projection maps are acyclic fibrations by preservation under pullback, $(r,r,r) : X \to PX \times_X PX \times_X PX$ is again a weak equivalence by 2-out-of-3 on r and the projection maps which are acyclic fibrations. Recall that $\tau(r,r) \simeq_{(X\times X)} r$ by construction, hence by Lemma 1.15 we find

$$\tau(\mathsf{id},\tau)(r,r,r) = \tau(r,\tau(r,r)) \simeq_{(X\times X)} \tau(r,r) \simeq_{(X\times X)} \tau(\tau(r,r),r) = \tau(\tau,\mathsf{id})(r,r,r)$$

so that $\tau(\mathsf{id},\tau)(r,r,r) \simeq_{(X\times X)} \tau(\tau,\mathsf{id})(r,r,r)$ as desired. Thus $\tau(\mathsf{id},\tau) \simeq_{(X\times X)} \tau(\tau,\mathsf{id})$ from which the particular case follows by application to (h,k,l).
3. We have by construction that $\tau(rs,\mathsf{id})r = \tau(r,r) \simeq_{(X\times X)} r$ and also $\tau(\mathsf{id},rt)r = \tau(r,r) \simeq_{(X\times X)} r$, which is enough to give us the described result. For the particular case one can again apply this to the homotopy h given.
4. We have by construction that $\sigma r \simeq_{(X\times X)} r$, so by Lemma 1.15 it follows from this that $(\sigma r, r) \simeq_{(X\times X)} (r,r)$ as well as $(r,\sigma r) \simeq_{(X\times X)} (r,r)$. We therefore find $\tau(id,\sigma)r = \tau(r,\sigma r) \simeq_{(X\times X)} \tau(r,r) \simeq_{(X\times X)} r = rsr$, as well as $\tau(\sigma,\mathsf{id})r = \tau(\sigma r,r) \simeq_{(X\times X)} \tau(r,r) \simeq_{(X\times X)} r = rtr$. Hence the desired results are again found by cancelling the weak equivalence r, and the particular case follows directly by precomposing h with the homotopies just obtained. □

We employ similar arguments to establish the other proposition.

Proof of Proposition 1.14. Recall that $Pfr_Y \simeq_{(X\times X)} r_X f$ for any $f : Y \to X$ by the defining construction for Pf, hence for the first part we find both

$$\begin{aligned} Pf\sigma_Y r_Y \simeq_{(X\times X)} Pfr_Y \simeq_{(X\times X)} r_X f \simeq_{(X\times X)} \sigma_X r_X f \simeq_{(X\times X)} \sigma_X Pfr_Y \text{ and} \\ Pf\tau_Y(r,r) \simeq_{(X\times X)} Pfr_Y \simeq_{(X\times X)} r_X f \simeq_{(X\times X)} \tau_X(r,r)f = \tau_X(rf,rf) \\ \simeq_{(X\times X)} \tau_X(Pfr,Pfr) = \tau_X(Pf\times Pf)(r,r) \end{aligned}$$

by applying Lemma 1.15 twice in the latter case. Furthermore we obtain $P(fg)r_Z \simeq_{(X\times X)} r_X fg \simeq_{(X\times X)} Pfr_Y g \simeq_{(X\times X)} PfPgr_Z$ by the definitions of Pf, Pg and $P(gf)$, so the statements follow by cancelling weak equivalences. □

Having established the truth of Propositions 1.13 and 1.14, we may conclude this section with a result of which the proof is now a relatively simple one. As such, this result is significant mostly because it allows us to shorten the proof of this chapter's main theorem.

Proposition 1.16. *If $f, g : Y \to X$ are two maps such that $h : f \simeq g$, then we have $\tau(Pf, ht) \simeq_{(X\times X)} \tau(hs, Pg)$, where these four are maps $PY \to PX$.*

Proof. By repeated application of Lemma 1.15 and identity we find that

$$\begin{aligned}\tau(Pf, ht)r_Y = \tau(Pfr_Y, h) \simeq_{(X\times X)} \tau(r_X f, h) \simeq_{(X\times X)} h \\ \simeq_{(X\times X)} \tau(h, r_X g) \simeq_{(X\times X)} \tau(h, Pgr_Y) = \tau(hs, Pg)r_Y.\end{aligned}$$

Thus the desired result is found by cancelling the weak equivalence r_Y. □

This proposition has a particular instance which we shall need as well, the derivation of which is not entirely immediate, hence we present it here as a separate result.

Corollary 1.17. *Given any object X and any $f : X \to X$, we find for all maps $h : X \to PX$ which satisfy $h : f \simeq \mathsf{id}$ that $hf \simeq_{(X\times X)} Pfh$ holds.*

Proof. We begin by using Proposition 1.16 and Lemma 1.15 to obtain that

$$\tau(\tau(hf, h), \sigma h) = \tau(\tau(hs, \mathsf{id})h, \sigma h) \simeq_{(X\times X)} \tau(\tau(Pf, ht)h, \sigma h) = \tau(\tau(Pfh, h), \sigma h).$$

This we may use in order to derive the desired result below.

$$\begin{aligned} hf &\simeq_{(X\times X)} \tau(hf, rf) && (\textit{identity}) \\ &\simeq_{(X\times X)} \tau(hf, \tau(h, \sigma h)) && (\textit{inverse},\ 1.15) \\ &\simeq_{(X\times X)} \tau(\tau(hf, h), \sigma h) && (\textit{associativity}) \\ &\simeq_{(X\times X)} \tau(\tau(Pfh, h), \sigma h) && (\textit{see above}) \\ &\simeq_{(X\times X)} \tau(Pfh, \tau(h, \sigma h)) && (\textit{associativity}) \\ &\simeq_{(X\times X)} \tau(Pfh, rf) && (\textit{inverse},\ 1.15) \\ &\simeq_{(X\times X)} Pfh && (\textit{identity}) \end{aligned}$$

□

1.3 Equivalence of the equivalences

With the results of the previous section we can now begin to prove the theorem regarding the interchangeability of homotopy equivalences with other notions of equivalence which we announced earlier. We shall consider three such notions in total, namely the *bi-invertible maps*, the *half-adjoint equivalences* and the *contractible maps*, whose definitions we shall give in a moment. As remarked this is a familiar result in homotopy type theory; parts of our proof can be viewed as a translation of the argument there into the language of path categories. Indeed, our Proposition 1.16 and Corollary 1.17 are analogous to Lemma 2.4.3 and its Corollary 2.4.4 from [HoTT]. Other parts are not so immediately similar, as for instance our definition of a contractible map may at first glance not be recognised as such. In any case, at this stage we are ready to proceed with the direct verification of the main result of this chapter, which is the following theorem.

Theorem 1.18. *For any $f : Y \to X$ the following are equivalent:*
(i) f is a bi-invertible map.
(ii) f is a homotopy equivalence.
(iii) f is a half-adjoint equivalence.
(iv) f is a contractible map.

As one would expect, we prove this by demonstrating that each of these statements implies the next one. While doing so, we will at each step first present a definition of the other notion of equivalence under consideration.

1.3.1 Bi-invertible maps

The definition of a bi-invertible map differs from that of a homotopy equivalence only in that the homotopy inverse may in fact be two different terms, one for each of the identities. To be precise, bi-invertible maps are defined as follows:

Definition 1.19. A morphism $f : Y \to X$ is a *bi-invertible map* whenever we have maps $g, g' : X \to Y$ with $fg \simeq \mathsf{id}_X$ and $g'f \simeq \mathsf{id}_Y$.

One can immediately see that every homotopy equivalence f is a bi-invertible map; showing that bi-invertible maps are in turn homotopy equivalences is not much more difficult.

Proposition 1.20. *Every bi-invertible map is a homotopy equivalence.*

Proof. Let $f : Y \to X$ be a bi-invertible map with maps $g, g' : X \to Y$ given. We see that $gf \simeq (g'f)gf = g'(fg)f \simeq g'f \simeq \mathsf{id}_Y$, hence f is a homotopy equivalence with g as its homotopy inverse. □

We may observe that the proof did not rely on any of the results which we have previously established. This will certainly not be the case for the other two notions of equivalence, which we are about to treat next.

1.3.2 Half-adjoint equivalences

Where bi-invertible maps can be viewed as a weakening of homotopy equivalences, one could regard half-adjoint equivalences rather as a strengthening of these. For half-adjoint equivalences f we require not only a homotopy inverse g, but also a higher homotopy which relates the ways in which the compositions of f and g are homotopic to the respective identities.

Definition 1.21. A morphism $f : Y \to X$ is a *half-adjoint equivalence* if there exists a map $g : X \to Y$ along with homotopies $h : gf \simeq \mathsf{id}_Y$, $k : fg \simeq \mathsf{id}_X$ and $L : Y \to P_{(X\times X)}PX$ such that $L : Pfh \simeq_{(X\times X)} kf$.

Here it is immediate that every half-adjoint equivalence is a homotopy equivalence. The converse statement is not as obvious, which is reflected in the proof's invoking of some of the results from the previous section.

Proposition 1.22. *Every homotopy equivalence is a half-adjoint equivalence.*

Proof. Let $f : Y \to X$ be a homotopy equivalence with homotopy inverse $g : X \to Y$ and homotopies $h : gf \simeq \mathsf{id}_Y$ and $k : fg \simeq \mathsf{id}_X$. We define a new homotopy $k' : fg \simeq \mathsf{id}_X$ by $k' = \tau(\sigma kfg, \tau(Pfhg, k))$: observe that indeed $k' : fg \simeq \mathsf{id}_X$ since $(s,t)\tau(\sigma kfg, \tau(Pfhg, k)) = (tkfg, tk) = (fg, \mathsf{id}_Y)$. Thus f is a half-adjoint equivalence once we have shown $Pfh \simeq k'f$. We find

$$\tau(Pfhgf, kf) \simeq_{(X\times X)} \tau(PfP(gf)h, kf) \simeq_{(X\times X)} \tau(PfPgPfh, kf)$$

by Corollary 1.17, Proposition 1.14 and Lemma 1.15, hence by Lemma 1.15

$$k'f = \tau(\sigma kfgf, \tau(Pfhgf, kf)) \simeq_{(X\times X)} \tau(\sigma kfgf, \tau(PfPgPfh, kf)).$$

In turn, Proposition 1.14 along with Lemma 1.15 and Proposition 1.16 yield

$$\tau(PfPgPfh, kf) = \tau(PfPg, kt)Pfh \simeq_{(X\times X)} \tau(ks, \mathsf{id})Pfh = \tau(kfgf, Pfh).$$

This allows us to conclude our proof with the following derivation:

$$\begin{aligned}
k'f &\simeq_{(X\times X)} \tau(\sigma kfgf, \tau(PfPgPfh, kf)) && \textit{(see above)}\\
&\simeq_{(X\times X)} \tau(\sigma kfgf, \tau(kfgf, Pfh)) && (\textit{Lemma } 1.15)\\
&\simeq_{(X\times X)} \tau(\tau(\sigma kfgf, kfgf), Pfh) && \textit{(associativity)}\\
&\simeq_{(X\times X)} \tau(rfgf, Pfh) && (\textit{inverse},\ 1.15)\\
&\simeq_{(X\times X)} Pfh && \textit{(identity)}
\end{aligned}$$

□

With half-adjoint equivalences covered, all that remains is to take a look at contractible maps.

1.3.3 Contractible maps

The definition of a contractible map presented here is perhaps less intuitive than those of the other two notions of equivalence, yet it is still a natural one to consider. We shall first give the definition and then comment on how it may be more easily understood.

Definition 1.23. A morphism $f : Y \to X$ is a *contractible map* if we have a map $g : X \to Y$ along with homotopies $k : fg \simeq \mathsf{id}_X$, $H : P_f \to PY$ with $H : gtp_2 \simeq p_1$ and $K : P_f \to P_{(X\times X)}PX$ with $K : \tau(PfH, p_2) \simeq_{(X\times X)} ktp_2$.

Hence $f : Y \to X$ is a contractible map if it has a homotopy section g which moreover satisfies the following condition: every homotopy in X can be (inversely) lifted along the fibres of f, where the source will be the image under g of the original target (so in particular every fibre of f is contractible to the image of the homotopy section g), in such a way that mapping this homotopy back along f and composing it with the original homotopy is homotopic relative endpoints to the witness that g is a homotopy section of f. Thus it is clear that such contractible maps are deserving of the name.

What makes these maps natural to consider in the context of a path category is that using Proposition 1.9 one may show for any f that the existence of these g, k, H and K is equivalent to the fibration tp_2 in the factorisation $f = tp_2(\mathsf{id}, rf)$ being acyclic. By 2-out-of-3 on this factorisation this condition is precisely that of f being a weak equivalence, which would render further proofs unnecessary as we already know the latter to be the homotopy equivalences. However, proving the equivalence of these two definitions of a contractible map is not less difficult than showing half-adjoint equivalences to be contractible maps in the current sense, hence this is what we shall do.

Proposition 1.24. *Every half-adjoint equivalence is a contractible map.*

Proof. Suppose that $f : Y \to X$ is a half-adjoint equivalence, i.e. we have a map $g : X \to Y$ with homotopies $h : gf \simeq \mathsf{id}_Y$, $k : fg \simeq \mathsf{id}_X$ and $L : Pfh \simeq_{(X\times X)} kf$. We claim that we find a suitable map $H : P_f \to PY$, $H : gtp_2 \simeq p_1$ as lower filler of the following square.

$$\begin{tikzcd} Y \arrow[r, "\tau(Pgkf,h)"] \arrow[d, "(\mathsf{id},\sigma kf)"'] & PY \arrow[d, "(s,t)"] \\ P_f \arrow[r, "(gtp_2,p_1)"'] & Y \times Y \end{tikzcd}$$

We find $(gtp_2, p_1)(\mathsf{id}, \sigma Pfh) = (gsPfh, \mathsf{id}) = (gfgf, \mathsf{id})$, and on the other hand $(s,t)\tau(Pgkf, h) = (sPgkf, th) = (gfgf, \mathsf{id})$ as well, hence this square commutes. Now $(\mathsf{id}, \sigma t L)$ is a weak equivalence by 2-out-of-3 on $p_1(\mathsf{id}, \sigma t L) = \mathsf{id}$, hence we obtain our $H : gtp_2 \simeq p_1$ as a lower filler. This leaves us to show that $\tau(PfH, p_2) \simeq_{(X\times X)} ktp_2$, or equivalently $PfH \simeq_{(X\times X)} \tau(kt,\sigma)p_2$ by Proposition 1.13 and Lemma 1.15. To this end we extend the square considered earlier with another commutative square:

$$\begin{tikzcd}
Y \arrow[r, "\tau(Pgkf,h)"] \arrow[d, "(\mathsf{id},\sigma kf)"'] & PY \arrow[r, "Pf"] \arrow[d, "(s,t)"] & PX \arrow[d, "(s,t)"] \\
P_f \arrow[r, "(gtp_2,p_1)"'] \arrow[ur, dotted, "H"] & Y \times Y \arrow[r, "f\times f"'] & X \times X
\end{tikzcd}$$

If we are able to prove that $\tau(kt,\sigma)p_2$ acts as a lower filler for the outer square, then we have $PfH \simeq_{(X\times X)} \tau(kt,\sigma)p_2$ since these are unique up to fibrewise homotopy. Observe first that $(s,t)\tau(kt,\sigma)p_2 = (sktp_2, sp_2) = (fgtp_2, fp_1)$, hence it makes the lower outer triangle commute. The following derivation furthermore shows that $\tau(kt,\sigma)p_2$ makes the upper outer triangle commute up to fibrewise homotopy, hence it is indeed a lower filler, and we are done.

$$\begin{aligned}
\tau(ks,\sigma\sigma)kf &\simeq_{(X\times X)} \tau(ks,\mathsf{id})kf && (\textit{Lemma } 1.15)\\
&\simeq_{(X\times X)} \tau(P(fg),kt)kf && (1.16 \text{ and } 1.15)\\
&\simeq_{(X\times X)} \tau(PfPgkf,kf) && (1.14 \text{ and } 1.15)\\
&\simeq_{(X\times X)} \tau(PfPgkf,Pfh) && (L \text{ and } 1.15)\\
&\simeq_{(X\times X)} Pf\tau(Pgkf,h) && (\textit{Proposition } 1.14)
\end{aligned}$$

□

This means we have completed the proof of Theorem 1.18 after giving a routine verification of the fact that contractible maps are bi-invertible maps.

Proposition 1.25. *Every contractible map is a bi-invertible map.*

Proof. Let $f : Y \to X$ be a contractible map with maps g, k, H, K given. In order to show that f is bi-invertible, it suffices to provide a homotopy $gf \simeq \mathsf{id}_Y$. To this end, observe that we have a map $(\mathsf{id}_Y, rf) : Y \to P_f$. Now $(s,t)H(\mathsf{id}_Y, rf) = (gtp_2(\mathsf{id}_Y, rf), p_1(\mathsf{id}_Y, rf)) = (gtrf, \mathsf{id}_Y) = (gf, \mathsf{id}_Y)$, so $H(\mathsf{id}_Y, rf) : gf \simeq \mathsf{id}_Y$ as required. Note that this proof in fact shows the stronger statement that contractible maps are homotopy equivalences. □

While Theorem 1.18 is a valid result in itself, it will also play a background role in the next chapter. Though it is of less relevance there, the notion of equivalence between cubical sets considered in [BCH14] is that of a contractible map. Knowing that this definition is equal to that of homotopy equivalence also from the perspective of a path category, we may use the latter in identifying a path category within the category of cubical sets.

Chapter 2

Cubical sets

In this chapter we shall look into the presheaf category of cubical sets, or more precisely, we will consider the model of dependent type theory in cubical sets as described in [BCH14]. The definition of a cubical set contained therein differs from the usual one in some important regards, and several other variations have been proposed and investigated over the last few years. We therefore begin with a detailed treatment of the notion of cubical set used at present, where we have adopted some notation from [Hub15] over that of [BCH14], and at certain inessential points introduced our own. Once this is done, we proceed by discussing the equally crucial notion of uniform Kan fibration, which are used to interpret types in the model under consideration. The remainder of this chapter will then be dedicated to determining a submodel within this model of which we shall demonstrate that it is an instance of a path category. This is done by identifying the necessary structural elements and proving that the axioms of a path category are satisfied, which requires us to establish a number of results which are actually derivable from these axioms.

2.1 Essential definitions, part two

This section is split into two parts. In the first of these we specify the definition of cubical sets with which we shall concern ourselves, along with some notational conventions. In the second one we provide an extensive treatment of the uniform Kan condition for maps between cubical sets, which is extended to cubical sets themselves.

2.1.1 Cubical sets as covariant presheaves

In defining cubical sets we first fix a *name space* $\mathcal{N}$, which is a countably infinite discrete set not containing 0 and 1 whose elements we denote with lowercase letters $x, y, z, \ldots$. Based on this name space we introduce a category $\mathcal{C}_\mathcal{N}$ as below.

Definition 2.1. Given a name space $\mathcal{N}$ we define the category $\mathcal{C}_\mathcal{N}$ in the following way. The objects $I, J, K, \ldots$ of this category are the decidable finite subsets of $\mathcal{N}$, while the morphisms $f : I \to J$ are functions $f : I \to J \cup \{0, 1\}$ which are injective on $f^{-1}(J)$. Composition of morphisms $f : I \to J$ and $g : J \to K$ is given by $gf(x) = g(f(x))$ for $x \in f^{-1}(J)$ and $gf(x) = f(x)$ for $x \in I - f^{-1}(J)$.

We omit the trivial verification that $\mathcal{C}_\mathcal{N}$ is indeed a category. Note that by the *domain* of a morphism $f : I \to J$, on which we say that f is defined, we shall mean $f^{-1}(J)$ and not necessarily I itself. Furthermore, when considering the objects of $\mathcal{C}_\mathcal{N}$ we leave out set brackets, so that for instance $I, x - y$ is short for $(I \cup \{x\}) - \{y\}$. Finally, we assume that to any object I of $\mathcal{C}_\mathcal{N}$ there is associated some particular $x_I \in \mathcal{N}$ which does not lie in I, and write $y_I = x_{I,x_I}$, $z_I = x_{I,x_I,y_I}$ and so on.

There are a number of canonical or otherwise relevant maps in $\mathcal{C}_\mathcal{N}$, for which we provide the following definitions.

Definition 2.2. For any object I, any $x \in I$ and any $a \in \{0, 1\}$ we have a *face map* $(x = a) : I \to I - x$ which sends x to a and fixes the rest of I. For any $x \notin I$ we have instead an *inclusion map* $\imath_x : I \hookrightarrow I, x$. For any $x, y \in I$ we have a *swap map* $(x\ y) : I \to I$ which sends x and y to one another and fixes the rest of I. Lastly, morphisms $f : I \to J$ which are defined on the whole of I are called *degeneracy maps*.

Thus any inclusion map is also a degeneracy map, and to any object I are associated a total of $2^{|I|}$ different face maps. Besides these special kinds of maps there are certain important operations which we may perform using a given morphism. For any morphism $f : I \to J$ and any $x \in I$ we can construct the morphism $f - x : I - x \to J - f(x)$, which takes the same values as f does. One should keep in mind that $J - f(x)$ may actually be the same as J, namely when f is not defined on x. On the other hand, for any $x \notin I$ and $y \notin J$ we can instead construct the morphism $(f, x = y) : I, x \to J, y$ which extends f by $f(x) = y$. Yet the most crucial definition for our current purposes is of course that of a cubical set, which we come to now.

Definition 2.3. The category $\mathcal{C}_{\mathcal{N}}\mathsf{Set}$ of cubical sets is the category of covariant presheaves on $\mathcal{C}_{\mathcal{N}}$, i.e. it is the category of functors $\mathcal{C}_{\mathcal{N}} \to \mathsf{Set}$ and natural transformations between them. We use capital Greek letters $\Gamma, \Delta, \Theta, \ldots$ to denote cubical sets, lowercase Greek letters $\alpha, \beta, \omega, \ldots$ to denote elements of $\Gamma(I)$ for some Γ and I, and $\sigma, \tau, \varphi, \ldots$ to denote natural transformations.

Since the particular name space $\mathcal{N}$ is of no special significance, we will no longer mention it from now on, so that we may simply write $\mathcal{C}$ and $\mathcal{C}\mathsf{Set}$. As a presheaf category, results known for these will also hold for cubical sets, hence in particular we ought to have a model of dependent type theory. However, we are interested in a restricted version of this naturally arising model, which involves the uniform Kan condition introduced in [BCH14]. This condition will be the subject of the remaining part of this section.

2.1.2 The uniform Kan condition

The uniform Kan condition describes certain well-behaved filling procedures in the context of a cubical set or natural transformations between these. What exactly is being filled are called open boxes, which are collections of data belonging to cubical sets satisfying a particular coherence requirement. In [BCH14] (Section 4) and [Hub15] (Section 3.1) the uniform Kan condition is defined separately for cubical sets and their morphisms, after which one recognises that a cubical set is uniform Kan whenever the map to a terminal object in $\mathcal{C}\mathsf{Set}$ is. Here we shall rather restrict ourselves to giving the uniform Kan condition for morphisms, which we then use to define uniform Kan cubical sets. We begin by determining what we mean by an open box shape.

Definition 2.4. Let I and J be any objects of $\mathcal{C}$ and x any name such that $J, x \subseteq I$ with $x \notin J$. An *open box shape* on I is a triple $S = ((x, a); J; I)$, which has indices $\langle S \rangle = \{(x, 1-a)\} \cup (J \times \{0, 1\})$. We call S a $+$-shape or a $-$-shape when $a = 1$ or $a = 0$ respectively.

Having defined open box shapes, we can state what an open box for a morphism between cubical sets is. After some additional remarks, we may finally come to the uniform Kan condition for morphisms.

Definition 2.5. Let $\sigma : \Delta \to \Gamma$ be any morphism, $S = ((x, a); J; I)$ be any open box shape and take any $\alpha \in \Gamma(I)$. An S*-open box* for σ over α is then an $\langle S \rangle$-indexed family $\vec{u}$ consisting of $u_{yb} \in \sigma^{-1}_{I-y}(\Gamma(y = b)(\alpha))$ for every $(y, b) \in \langle S \rangle$, which moreover satisfies $\Delta(z = c)(u_{yb}) = \Delta(y = b)(u_{zc})$ for all pairs of indices $(y, b), (z, c) \in \langle S \rangle$ such that $y \neq z$. We shall refer to the latter requirements as the *adjacency conditions* of an open box.

Note that whenever $\vec{u}$ is an S-open box for σ over α and $f : I \to K$ is defined on J, x, there is an open box shape $fS = ((f(x), a), f(J), K)$ on K. Correspondingly we also find an fS-open box $f\vec{u}$ for σ over $\Gamma(f)(\alpha)$ given by $(f\vec{u})_{yb} = \Delta(f - y)(u_{yb})$. Such open boxes are essential to the definition of the uniform Kan condition for morphisms, which we are now able to present.

Definition 2.6. Let $\sigma : \Delta \to \Gamma$ be any morphism. We say that σ is a *uniform Kan fibration* if the following holds. For any open box shape S on I and any S-open box $\vec{u}$ for σ over α, there is a uniform Kan filler $\sigma^{-1}(\alpha) \uparrow_S \vec{u}$ or $\sigma^{-1}(\alpha) \downarrow_S \vec{u}$ in $\sigma_I^{-1}(\alpha)$ (if S is a $+$-shape or a $-$-shape respectively) such that $\Delta(y = b)(\sigma^{-1}(\alpha) \uparrow_S \vec{u}) = u_{yb}$ for any $(y, b) \in \langle S \rangle$. These fillers must satisfy the *uniformity condition* that for any $f : I \to K$ which is defined on J, x we have that $\Delta(f)(\sigma^{-1}(\alpha) \uparrow_S \vec{u}) = \sigma^{-1}(\Gamma(f)(\alpha)) \uparrow_{fS} f\vec{u}$, and similarly for $-$-shapes and the corresponding fillers $\downarrow_S$, $\downarrow_{fS}$.

Now if $\sigma : \Gamma \to \mathbf{1}$ where $\mathbf{1}$ is a terminal object in $\mathcal{C}\mathsf{Set}$, then we need not specify the element α over which an open box lies, as there can be only one for all open box shapes on a particular I. Thus in such cases we may drop the mention of this data, with the following caveat. The terminal object in $\mathcal{C}\mathsf{Set}$ is only unique up to isomorphism, hence even though a uniform Kan structure on one terminal map induces such a structure on all terminal maps from Γ, it may be possible that distinct terminal maps are endowed with different uniform Kan structures. The following definition therefore presupposes a specific choice of terminal object, say $\mathbf{1}(I) = \{I\}$, in order to unambiguously fix a uniform Kan structure.

Definition 2.7. A cubical set Γ is called *uniform Kan* if the morphism $1^\Gamma : \Gamma \to \mathbf{1}$ is a uniform Kan fibration. We write $\Gamma \uparrow_S \vec{u}$ and $\Gamma \downarrow_S \vec{u}$ as appropriate for the uniform Kan fillers provided.

As any identity morphism is trivially (and uniquely) a uniform Kan fibration, we recognise that the terminal object is itself uniform Kan. For completeness' sake we mention that the model of dependent type theory studied in [BCH14] uses the uniform Kan fibrations to interpret types, but allows all cubical sets as contexts. We shall see in the next section that we must restrict ourselves to uniform Kan cubical sets in order to obtain a path category. Finally, we trust that the reader is by now sufficiently aware of the uniformity condition that from here on we may simply write "Kan" to mean "uniform Kan".

2.2 A path category in cubical sets

Now that we have a basic understanding of cubical sets, Kan structures and the model of dependent type theory which they give rise to, we shall work towards showing that it has a submodel which is also a path category. This requires us to specify amongst other things what the fibrations and weak equivalences of this path category should be. The obvious candidates for the former are the Kan fibrations which we have just defined, which we will indeed take to be the fibrations. It is then immediate that we need to consider a proper submodel of the model in question which is given by the Kan cubical sets, for Axiom 7 of a path category demands that all objects are fibrant, i.e. that every cubical set in the underlying category is Kan. Thus this restriction to Kan cubical sets is necessary and sufficient to guarantee the truth of Axiom 7 for the path category which we aim to construct (which we shall therefore refer to as $\mathcal{K}$), since as remarked the terminal object is Kan. At this stage we may already verify that Axioms 1 and 2 hold as well, but we postpone this until we have defined the path category $\mathcal{K}$ in its entirety. First, we shall take a look at what the weak equivalences of $\mathcal{K}$ shall be.

As mentioned earlier, the notion of an equivalence between cubical sets which is considered in [BCH14] is derived from that of a contractible map. However, we have seen that such maps are interchangeable with homotopy equivalences from the perspectives of both type theory and path categories (see Theorem 1.18). Thus we would like to define a map $\sigma : \Delta \to \Gamma$ to be a weak equivalence if it is a homotopy equivalence, which means we shall have to define a homotopy relation in the context of cubical sets. We therefore need to look at what the path objects of $\mathcal{K}$ could be, which we do next.

2.2.1 Defining the path structure

Here we shall define a homotopy relation, show that it is a congruence relation, and take the weak equivalences of $\mathcal{K}$ to be maps which are equivalences with respect to this relation. Afterwards we also prove a few results which are necessary for checking that the axioms of a path category now hold, amongst which the characterisation of acyclic fibrations expressed in Proposition 1.9. First, we must introduce the notion of a path object on a cubical set.

Definition 2.8. Let Γ be a cubical set. We define a path object $\mathrm{P}\Gamma$ on Γ by $\mathrm{P}\Gamma(I) = \Gamma(I, x_I)$ and $\mathrm{P}\Gamma(f : I \to J) = \Gamma(f, x_I = x_J)$. The reflexivity map $r : \Gamma \to \mathrm{P}\Gamma$ is defined by $r_I = \Gamma(\imath_{x_I})$, while the source and target maps $s, t : \mathrm{P}\Gamma \to \Gamma$ are defined by $s_I = \Gamma(x_I = 0)$ and $t_I = \Gamma(x_I = 1)$ respectively.

Though it is easy to see that $\mathrm{P}\Gamma$ is well-defined as a cubical set, we require moreover that $\mathrm{P}\Gamma$ is Kan whenever Γ is. We may conclude that this is the case once we verified that $\mathrm{P}\Gamma$ is indeed a path object on Γ in the context of a path category. For if we know that $(s,t) : \mathrm{P}\Gamma \to \Gamma \times \Gamma$ is a Kan fibration, and that Kan fibrations are closed under pullback and conjunction, then Γ being Kan implies that $\Gamma \times \Gamma$ and in turn $\mathrm{P}\Gamma$ are Kan. With this in mind, we proceed by adapting the notion of homotopy to the setting of cubical sets.

Definition 2.9. Two parallel morphisms $\sigma, \tau : \Delta \to \Gamma$ between cubical sets are defined to be homotopic if there is a map $\ell : \Delta \to \mathrm{P}\Gamma$ such that $s\ell = \sigma$ and $t\ell = \tau$. In this case we write $\ell : \sigma \simeq \tau$, or $\sigma \simeq \tau$ if the homotopy is of less importance.

Now to show that this notion of homotopy again gives rise to a congruence relation we must construct inversion and composition operations like those which we have seen for path categories. To streamline our discussion we introduce some notational devices. Whenever $\sigma : \Delta \to \Gamma$ we shall write $\mathrm{P}\sigma : \mathrm{P}\Delta \to \mathrm{P}\Gamma$ for the morphism given by $(\mathrm{P}\sigma)_I = \sigma_{I,x_I}$ (like a strict version of Proposition 1.10). Similarly, when $f : I \to J$ we shall use $\mathrm{P}f$ to mean the morphism $(f, x_I = x_J) : I, x_I \to J, x_J$, Lastly, we shall use $\omega : \alpha \to \beta$ to denote the case when $\omega \in \mathrm{P}\Gamma(I)$ is such that $s\omega = \alpha$ and $t\omega = \beta$. In fact, we will sometimes even use the latter notation for arbitrary elements, where it should be clear from the context in which name x the source and target are related. Furthermore, we will regularly omit indices and (even more frequently) variable corrections, especially in visual representations, as including these would lead to rather unwieldy diagrams. With these formalities out of the way, we establish the existence of an inversion map.

Proposition 2.10. *Whenever Γ is Kan, we have an inversion morphism $\mathfrak{p} : \mathrm{P}\Gamma \to \mathrm{P}\Gamma$ such that $(s,t)\mathfrak{p} = (t,s)$ and $r \simeq \mathfrak{p}r$.*

Proof. For arbitrary I, consider the $+$-shape $\dot{S}^I = ((y_I, 1); x_I; I, x_I, y_I)$. Given any $\omega \in \mathrm{P}\Gamma(I)$ where $\omega : \alpha \to \beta$ we define an $\dot{S}^I$-open box $\vec{u}^\omega$ in $\mathrm{P}\Gamma$ by $u^\omega_{x_I 0} = \Gamma(x_I = y_I)(\omega)$, $u^\omega_{x_I 1} = \Gamma(x_I = y_I) r_I(\alpha)$ and $u^\omega_{y_I 0} = r_I(\alpha)$. We then obtain a Kan filler $\Gamma \uparrow_{\dot{S}^I} \vec{u}^\omega \in \mathrm{PP}\Gamma(I)$ represented by the square below, which we use to define $\mathfrak{p}_I(\omega) = \mathrm{P}t_I(\Gamma \uparrow_{\dot{S}^I} \vec{u}^\omega)$.

$$\begin{tikzcd}
\beta \arrow[r, dotted, "\mathfrak{p}\omega"] & \alpha \\
\alpha \arrow[u, "\omega"] \arrow[r, "r\alpha"'] \arrow[ur, phantom, "\Gamma\uparrow_{\dot{S}^I}\vec{u}^\omega"] & \alpha \arrow[u, "r\alpha"']
\end{tikzcd}$$

To check that $\mathfrak{p}$ thus defined is natural, let $f : I \to J$ be any morphism. We write ω' for $\mathrm{P}\Gamma(f)(\omega)$, so that $\mathfrak{p}_J \mathrm{P}\Gamma(f)(\omega) = \mathrm{P}t_J(\Gamma \uparrow_{\dot{S}^J} \vec{u}^{\omega'})$, where we see that $\dot{S}^J = \mathrm{PP}f\dot{S}^I$ and $\vec{u}^{\omega'} = \mathrm{PP}f\vec{u}^{\omega}$ by naturality. Thus by uniformity we find that $(\Gamma \uparrow_{\dot{S}^J} \vec{u}^{\omega'}) = \Gamma(\mathrm{PP}f)(\Gamma \uparrow_{\dot{S}^I} \vec{u}^{\omega})$, so we have $\mathfrak{p}_J \mathrm{P}\Gamma(f)(\omega) = \mathrm{P}t_J\Gamma(\mathrm{PP}f)\Gamma(X \uparrow_{\dot{S}^I} \vec{u}^{\omega}) = \Gamma(\mathrm{P}f)\mathrm{P}t_I\Gamma(X \uparrow_{\dot{S}^I} \vec{u}^{\omega}) = \mathrm{P}\Gamma(f)\mathfrak{p}_I(\omega)$. Therefore $\mathfrak{p}$ is indeed natural, while $(s,t)\mathfrak{p} = (t,s)$ is immediate from the definition. Finally, that $r \simeq \mathfrak{p}r$ is witnessed by $\hat{\mathfrak{p}}r$, where $\hat{\mathfrak{p}} : \mathrm{P}\Gamma \to \mathrm{PP}\Gamma$ is the morphism such that $\hat{\mathfrak{p}}_I(\omega) = \Gamma \uparrow_{\dot{S}^I} \vec{u}^{\omega}$. $\square$

Once we have defined the notion of a fibrewise homotopy in the context of cubical sets, we see that in fact $\hat{\mathfrak{p}}r : r \simeq_{(\Gamma\times\Gamma)} \mathfrak{p}r$, hence $\mathfrak{p}$ corresponds exactly to the map σ as defined in Proposition 1.11. The same remark holds for the composition map which we exhibit next in relation to the map τ from Proposition 1.12.

Proposition 2.11. *Whenever Γ is Kan, we have a composition morphism $\mathfrak{q} : \mathrm{P}\Gamma \times_\Gamma \mathrm{P}\Gamma \to \mathrm{P}\Gamma$ such that $(s,t)\mathfrak{q} = (sd_1, td_2)$ and $r \simeq \mathfrak{q}(r,r)$, where $\mathrm{P}\Gamma \times_\Gamma \mathrm{P}\Gamma$ arises as the following pullback.*

$$\begin{tikzcd}
\mathrm{P}\Gamma \times_\Gamma \mathrm{P}\Gamma \arrow[r, "d_2"] \arrow[d, "d_1"'] & \mathrm{P}\Gamma \arrow[d, "s"] \\
\mathrm{P}\Gamma \arrow[r, "t"'] & \Gamma
\end{tikzcd}$$

Proof. For arbitrary I, consider again the $+$-shape $\dot{S}^I = ((y_I, 1); x_I; I, x_I, y_I)$. Given any $(\omega, \eta) \in \mathrm{P}\Gamma \times_\Gamma \mathrm{P}\Gamma$ where $\omega : \alpha \to \beta$ and $\eta : \beta \to \gamma$, define the $\dot{S}^I$-open box $\vec{u}^{\omega\eta}$ in $\mathrm{P}\Gamma$ by $u^{\omega\eta}_{x_I 0} = X(x_I = y_I)r_I(\alpha)$, $u^{\omega\eta}_{x_I 1} = X(x_I = y_I)(\eta)$ and $u^{\omega\eta}_{y_I 0} = \omega$. We then obtain a Kan filler $\Gamma \uparrow_{\dot{S}^I} \vec{u}^{\omega\eta} \in \mathrm{PP}\Gamma(I)$ represented by the square below, which we use to define $\mathfrak{q}_I(\omega, \eta) = \mathrm{P}t_I(\Gamma \uparrow_{\dot{S}^I} \vec{u}^{\omega\eta})$.

$$\begin{tikzcd}
\alpha \arrow[r, dotted, "\mathfrak{q}(\omega,\eta)"] & \gamma \\
\alpha \arrow[u, "r\alpha"] \arrow[r, "\omega"'] \arrow[ur, phantom, "\Gamma\uparrow_{\dot{S}^I}\vec{u}^{\omega\eta}"] & \beta \arrow[u, "\eta"']
\end{tikzcd}$$

Showing that $\mathfrak{q}$ is natural proceeds along the same lines as for $\mathfrak{p}$ in Proposition 2.10. Let $f : I \to J$ be arbitrary, and write ω' and η' for $\mathrm{P}\Gamma(f)(\omega)$ and $\mathrm{P}\Gamma(f)(\eta)$ respectively. We have $\dot{S}^J = \mathrm{PP}f\dot{S}^I$ and $\vec{u}^{(\omega'\eta')} = \mathrm{PP}f\vec{u}^{(\omega\eta)}$ as before by naturality. Therefore we find $(\Gamma \uparrow_{\dot{S}^J} \vec{u}^{(\omega'\eta')}) = \Gamma(\mathrm{PP}f)(\Gamma \uparrow_{\dot{S}^I} \vec{u}^{(\omega\eta)})$ by uniformity, from which the naturality of $\mathfrak{q}$ almost immediately follows. That $(s,t)\mathfrak{q} = (sd_1, td_2)$ is again apparent from the construction; the fact that $r \simeq \mathfrak{q}(r,r)$ is here witnessed by $\hat{\mathfrak{q}}(r,r)$, where $\hat{\mathfrak{q}} : \mathrm{P}\Gamma \times_\Gamma \mathrm{P}\Gamma \to \mathrm{PP}\Gamma$ is the morphism such that $\hat{\mathfrak{q}}_I(\omega,\eta) = \Gamma \uparrow_{\dot{S}^I} \vec{u}^{\omega\eta}$. $\square$

Thus the homotopy relation given in Definition 2.9 is an equivalence relation. No further proofs are required to show that it is also a congruence, as whenever $\ell : \sigma \simeq \tau$ we have for any appropriate φ that $\ell\varphi : \sigma\varphi \simeq \tau\varphi$ and $\mathrm{P}\varphi\ell : \varphi\sigma \simeq \varphi\tau$, which is sufficient. Having established that our notion of homotopy is sound, we define the notion of a homotopy equivalence of Kan cubical sets.

Definition 2.12. We say that $\sigma : \Delta \to \Gamma$ is a homotopy equivalence of Kan cubical sets whenever we have a map $\tau : \Gamma \to \Delta$ such that $\sigma\tau \simeq \mathsf{id}_\Gamma$ and $\tau\sigma \simeq \mathsf{id}_\Delta$.

This provides us with a means of defining the last structural element of a path category, namely the weak equivalences, which means we are now able to precisely specify the data which should make $\mathcal{K}$ into a path category.

Definition 2.13. We define the category $\mathcal{K}$ to be the full subcategory of $\mathcal{C}\mathsf{Set}$ consisting of the uniform Kan cubical sets. In this category $\mathcal{K}$, the fibrations are the uniform Kan fibrations, the weak equivalences are the homotopy equivalences, and the path objects are as defined in Definition 2.8.

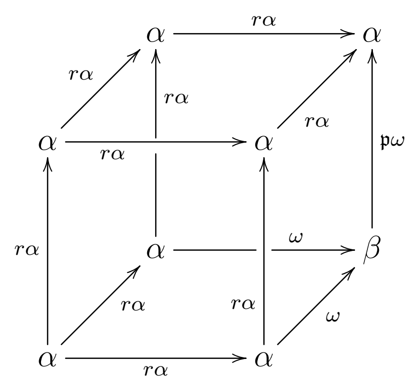
The rest of this chapter is dedicated to proving that $\mathcal{K}$ is indeed a path category. As we have remarked earlier, Axiom 7 holds by definition, and some of the others may already be verified at this point. However, Axiom 3 in particular can be dealt with more easily if we have some additional results at hand; proving these shall be done in the next subsection.

2.2.2 Quasi-connections and fibrewise homotopies

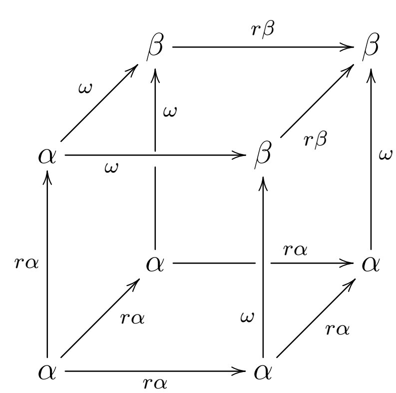
Our main goal for this subsection is to prove the characterisation of acyclic fibrations in $\mathcal{K}$ as in Proposition 1.9: this result will be Proposition 2.20. However, this involves the notion of a fibrewise homotopy, which means we have to extend our treatment in order to include it. Before we do this, we first discuss a class of maps which arises in a similar fashion as $\mathfrak{p}$ and $\mathfrak{q}$, or more precisely as $\hat{\mathfrak{p}}$ and $\hat{\mathfrak{q}}$ did. We shall refer to these maps as quasi-connections, since they are strongly reminiscent of the additional degeneracy maps which are part of a connection structure on a cubical set. The only practical difference between them is that the quasi-connections arise for each particular Kan cubical set, which means they need not commute with morphisms between cubical sets. Fortunately, as in the case of $\mathfrak{p}$ and $\mathfrak{q}$, this fact does not preclude us from using the quasi-connections as we mean to.

Proposition 2.14. *Whenever Γ is Kan, we have two quasi-connections $\acute{\mathfrak{c}}, \grave{\mathfrak{c}} : \mathrm{P}\Gamma \to \mathrm{PP}\Gamma$, called the forward and backward connection maps respectively. The former satisfies $\acute{\mathfrak{c}} : rs \simeq \mathsf{id}$ and $\overline{(x\ y)}\acute{\mathfrak{c}} : rs \simeq \mathsf{id}$, while the latter satisfies $\grave{\mathfrak{c}} : \mathsf{id} \simeq rt$ and $\overline{(x\ y)}\grave{\mathfrak{c}} : \mathsf{id} \simeq rt$, where $\overline{(x\ y)} : \mathrm{PP}\Gamma \to \mathrm{PP}\Gamma$ is given by $\overline{(x\ y)}_I = \Gamma(x_I\ y_I)$. Recall that $(x_I\ y_I)$ is the swap map from Definition 2.2.*

Proof. We first construct $\acute{\mathfrak{c}}$, which we use in turn to obtain $\grave{\mathfrak{c}}$. For arbitrary I, consider the $-$-shape $\ddot{S}^I_- = ((z_I, 0); x_I, y_I; I, x_I, y_I, z_I)$. Now for any given $\omega \in \mathrm{P}\Gamma(I)$ such that $\omega : \alpha \to \beta$ we let $\acute{\vec{u}}^{\omega}$ be the $\ddot{S}^I_-$-open box in Γ which is obtained by taking $\acute{u}^{\omega}_{x_I 0} = \Gamma(x_I = z_I)(\mathrm{P}rr)_I(\alpha)$, $\acute{u}^{\omega}_{x_I 1} = \Gamma(x_I = z_I)\hat{\mathfrak{p}}_I(\omega)$, $\acute{u}^{\omega}_{y_I 0} = \Gamma(y_I = z_I)(\mathrm{P}rr)_I(\alpha)$, $\acute{u}^{\omega}_{y_I 1} = \Gamma(y_I = z_I)\Gamma(x_I\ y_I)\hat{\mathfrak{p}}_I(\omega)$ and as last one $\acute{u}^{\omega}_{z_I 1} = (\mathrm{P}rr)_I(\alpha)$. We then find a Kan filler $\Gamma \downarrow_{\ddot{S}^I_-} \acute{\vec{u}}^{\omega} \in \mathrm{PPP}\Gamma(I)$ like the hypercube below and define $\acute{\mathfrak{c}}_I(\omega)$ as its bottom face $(\mathrm{PP}t)_I(\Gamma \downarrow_{\ddot{S}^I_-} \acute{\vec{u}}^{\omega})$.



One can readily check that these $\acute{\mathfrak{c}}_I$ taken together form a natural transformation $\acute{\mathfrak{c}} : \mathrm{P}\Gamma \to \mathrm{PP}\Gamma$ with the desired properties. In order to define the backward connection map $\grave{\mathfrak{c}} : \mathrm{P}\Gamma \to \mathrm{PP}\Gamma$ we shall proceed as follows. Consider the $+$-shape $\ddot{S}^I_+ = ((z_I, 1); x_I, y_I; I, x_I, y_I, z_I)$, so that for any $\omega \in \mathrm{P}\Gamma(I)$ with $\omega : \alpha \to \beta$ we may take the $\ddot{S}^I_+$-open box $\grave{\vec{u}}^{\omega}$ in Γ by $\grave{u}^{\omega}_{x_I 0} = \Gamma(x_I = z_I)\acute{\mathfrak{c}}_I(\omega)$, $\grave{u}^{\omega}_{x_I 1} = \Gamma(x_I = z_I)\mathrm{P}r_I(\omega)$, $\grave{u}^{\omega}_{y_I 0} = \Gamma(y_I = z_I)\acute{\mathfrak{c}}_I(\omega)$, $\grave{u}^{\omega}_{y_I 1} = \Gamma(y_I = z_I)\Gamma(x_I\ y_I)\mathrm{P}r_I(\omega)$ and $\grave{u}^{\omega}_{z_I 0} = (\mathrm{P}rr)_I(\alpha)$. Here we find a Kan filler $\Gamma \uparrow_{\ddot{S}^I_+} \grave{\vec{u}}^{\omega} \in \mathrm{PPP}\Gamma(I)$ represented by the hypercube below, whose top face $(\mathrm{PP}t)_I(\Gamma \uparrow_{\ddot{S}^I_+} \grave{\vec{u}}^{\omega})$ is taken to define $\grave{\mathfrak{c}}_I(\omega)$. We claim that this again results in the required natural transformation $\grave{\mathfrak{c}} : \mathrm{P}\Gamma \to \mathrm{PP}\Gamma$.



$\square$

Observe that these connection maps can be regarded as specific witnesses to an equivalent of the first statement of Proposition 1.13, which was that $rs \simeq \mathsf{id} \simeq rt$. In order to arrive at our desired characterisation of acyclic fibrations, we shall in fact prove (stronger) versions of Propositions 1.13 and 1.14 for our current setup, which of course requires us to specify what our fibrewise homotopies are.

Definition 2.15. Given any Kan fibration $\varphi : \Gamma \to \Theta$, we construct the path object $\mathrm{P}_\Theta(\Gamma)$ with respect to φ as the pullback of $\mathrm{P}\varphi$ and $r : \Theta \to \mathrm{P}\Theta$. We say that $\sigma, \tau : \Delta \to \Gamma$ are fibrewise homotopic over Θ if these exists $\ell : \Delta \to \mathrm{P}_\Theta(\Gamma)$ such that $s\ell = \sigma$ and $t\ell = \tau$. In that case we write $\ell : \sigma \simeq_\Theta \tau$, or $\sigma \simeq_\Theta \tau$ if the homotopy is of less importance.

With a slight abuse of notation, we will use this relation for particular elements of a path object as well. It is easily seen that the definition again gives rise to a congruence relation, as the only difference is that in order to define inversion and composition we now use $\hat{\mathfrak{p}}_I(\omega) = \varphi^{-1}((\mathrm{P}r\varphi)_I(\omega)) \uparrow_{\dot{S}^I} \vec{u}^\omega$ and $\hat{\mathfrak{q}}_I(\omega, \eta) = \varphi^{-1}((\mathrm{P}r\varphi)_I(\omega)) \uparrow_{\dot{S}^I} \vec{u}^{\omega\eta}$. However, there is a lot more to say in the particular case where φ is the Kan fibration $(s,t) : \mathrm{P}\Gamma \to \Gamma \times \Gamma$ for some Γ. As mentioned earlier, we shall establish Proposition 2.19 as a counterpart to Propositions 1.13 and 1.14, though for this we require a pair of lemmas which we prove first.

Lemma 2.16. *For any Γ which is Kan we have a morphism $\check{\mathfrak{p}} : \mathrm{P}\Gamma \to \mathrm{PP}\Gamma$ such that $\check{\mathfrak{p}}_I(\omega)$, where $\omega : \alpha \to \beta$, is of the form represented by the following square.*

$$\begin{tikzcd}
\alpha \arrow[r, "\omega"] & \beta \\
\beta \arrow[u, "\mathfrak{p}\omega"] \arrow[r, "r\beta"'] \arrow[ur, phantom, "\check{\mathfrak{p}}_I(\omega)"] & \beta \arrow[u, "r\beta"']
\end{tikzcd}$$

Proof. Given arbitrary I and $\omega \in \mathrm{P}\Gamma(I)$ such that $\omega : \alpha \to \beta$ we consider the $\ddot{S}^I_+$-open box $\check{\vec{u}}^\omega$ in Γ given by $\check{u}^\omega_{x_I 0} = \Gamma(x_I = z_I)\Gamma(x_I\ y_I)\hat{\mathfrak{p}}_I(\omega)$, $\check{u}^\omega_{x_I 1} = \Gamma(x_I = z_I)\mathrm{P}r_I(\omega)$, $\check{u}^\omega_{y_I 0} = \Gamma(y_I = z_I)\Gamma(x_I\ y_I)\mathrm{P}r_I(\omega)$, along with $\check{u}^\omega_{y_I 1} = \Gamma(y_I = z_I)\acute{\mathfrak{c}}_I(\omega)$ and $\check{u}^\omega_{z_I 0} = (\mathrm{P}rr)_I(\alpha)$. Then we have a Kan filler in $\mathrm{PPP}\Gamma(I)$ which we use to define $\check{\mathfrak{p}}_I(\omega) = (\mathrm{PP}t)_I(\Gamma \uparrow_{\ddot{S}^I_+} \check{\vec{u}}^\omega)$. That is, it arises as the top face of the hypercube below. We omit the routine verification that $\check{\mathfrak{p}}$ thus defined is a natural transformation.

$$\begin{tikzcd}
& \alpha \arrow[rr, "\omega"] & & \beta \\
\beta \arrow[ur, "\mathfrak{p}\omega"] \arrow[rr, "r\beta"'] & & \beta \arrow[ur, "r\beta"'] & \\
& \alpha \arrow[uu, "r\alpha"] \arrow[rr, "r\alpha"] & & \alpha \arrow[uu, "\omega"'] \\
\alpha \arrow[uu, "\omega"] \arrow[ur, "r\alpha"'] \arrow[rr, "r\alpha"'] & & \alpha \arrow[uu, "\omega"] \arrow[ur, "r\alpha"'] &
\end{tikzcd}$$

$\square$

Informally speaking, $\check{\mathfrak{p}}$ reverses the order of composing a path and its inverse in $\hat{\mathfrak{p}}$. We have a similar result where the composition of two paths is composed with with reflexivity at the target rather than at the source.

Lemma 2.17. *For any* Γ *which is Kan we have a map* $\check{\mathfrak{q}} : \mathrm{P}\Gamma \times_\Gamma \mathrm{P}\Gamma \to \mathrm{PP}\Gamma$ *such that* $\check{\mathfrak{q}}_I(\omega, \eta)$*, where* $\omega : \alpha \to \beta$ *and* $\eta : \beta \to \gamma$*, is of the form represented by the following square.*

$$\begin{tikzcd}
\gamma \arrow[r, "r\gamma"] & \gamma \\
\alpha \arrow[u, "\mathfrak{q}(\omega,\eta)"] \arrow[r, "\omega"'] \arrow[ur, phantom, "\check{\mathfrak{q}}_I(\omega,\eta)"] & \beta \arrow[u, "\eta"']
\end{tikzcd}$$

Proof. For arbitrary I and $(\omega, \eta) \in (\mathrm{P}\Gamma \times_\Gamma \mathrm{P}\Gamma)(I)$ such that $\omega : \alpha \to \beta$ and $\eta : \beta \to \gamma$, we have the $\ddot{S}^I_+$-open box $\check{\ddot{u}}^{\omega\eta}$ in Γ given by the data $\check{u}^{\omega\eta}_{x_I 0} = \Gamma(x_I = z_I)\Gamma(x_I\ y_I)\hat{\mathfrak{q}}_I(\omega, \eta)$, $\check{u}^{\omega\eta}_{x_I 1} = \Gamma(x_I = z_I)\acute{\mathfrak{c}}_I(\eta)$, $\check{u}^{\omega\eta}_{y_I 0} = \Gamma(y_I = z_I)\mathrm{P}r_I(\omega)$, $\check{u}^{\omega\eta}_{y_I 1} = \Gamma(y_I = z_I)\Gamma(x_I\ y_I)\mathrm{P}r_I(\eta)$ and $\check{u}^{\omega\eta}_{z_I 0} = \grave{\mathfrak{c}}_I(\omega)$. Here we find a Kan filler in $\mathrm{PPP}\Gamma(I)$ which we use to define $\check{\mathfrak{q}}_I(\omega, \eta) = (\mathrm{PP}t)_I(\Gamma \uparrow_{\ddot{S}^I_+} \check{\ddot{u}}^{\omega\eta})$. In other words, it arises as the top face of the following hypercube, where we again omit the verification that $\check{\mathfrak{p}}$ thus defined is a natural transformation.

$$\begin{tikzcd}
& \gamma \arrow[rr, "r\gamma"] & & \gamma \\
\alpha \arrow[ur, "\mathfrak{q}(\omega,\eta)"] \arrow[rr, "\omega"'] & & \beta \arrow[ur, "\eta"'] & \\
& \beta \arrow[uu, "\eta"] \arrow[rr, "r\beta"] & & \beta \arrow[uu, "\eta"'] \\
\alpha \arrow[uu, "r\alpha"] \arrow[ur, "\omega"'] \arrow[rr, "\omega"'] & & \beta \arrow[uu, "r\beta"] \arrow[ur, "r\beta"'] &
\end{tikzcd}$$

$\square$

For both of these lemmas (and any of the upcoming results) the same remark applies as for $\hat{\mathfrak{p}}$ and $\hat{\mathfrak{q}}$, namely that in the case of fibrewise homotopy we relativise the filler to the fibration under consideration. Besides these two lemmas we need to demonstrate the truth of one more proposition which is noteworthy enough on its own. One may have been tempted to regard the diagrams used to represent higher-dimensional paths as though they were commutative diagrams: the following shows that this perspective is justified up to fibrewise homotopy.

Proposition 2.18. *Let Γ be Kan. Whenever $\kappa \in \mathrm{PP}\Gamma(I)$ is of the form*

$$\begin{tikzcd} \delta \arrow[r, "\zeta"] & \gamma \\ \alpha \arrow[u, "\xi"] \arrow[r, "\omega"'] \arrow[ur, phantom, "\kappa"] & \beta \arrow[u, "\eta"'] \end{tikzcd}$$

we have that $\mathfrak{q}_I(\omega, \eta) \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}_I(\xi, \zeta)$.

Proof. In order to obtain the desired witness to $\mathfrak{q}_I(\omega, \eta) \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}_I(\xi, \zeta)$ using a Kan filler, we must in fact carry out such a filler construction three times, since we require two intermediate paths. These paths have no special significance of their own and shall therefore not be considered as particular natural transformations. The first of these is

$$\begin{tikzcd} \alpha \arrow[r, "\xi"] & \delta \\ \alpha \arrow[u, "r\alpha"] \arrow[r, "\omega"'] \arrow[ur, phantom, "M(\kappa)"] & \beta \arrow[u, "\mathfrak{q}(\mathfrak{p}\omega{,}\xi)"'] \end{tikzcd}$$

which is obtained as $(\mathrm{PP}t)_I(\Gamma \uparrow_{\ddot{S}^I_+} \vec{m}^\kappa)$, where $\vec{m}^\kappa$ is the $\ddot{S}^I_+$-open box in Γ given by $m^\kappa_{x_I0} = \Gamma(x_I = z_I)(\mathrm{P}rr)_I(\alpha)$, $m^\kappa_{x_I1} = \Gamma(x_I = z_I)\Gamma(x_I\ y_I)\hat{\mathfrak{q}}_I(\mathfrak{p}\omega, \xi)$, $m^\kappa_{y_I0} = \Gamma(y_I = z_I)\mathrm{P}r_I(\omega)$, $m^\kappa_{y_I1} = \Gamma(y_I = z_I)\acute{\mathfrak{c}}_I(\xi)$ and $m^\kappa_{z_I0} = \Gamma(x_I\ y_I)\hat{\mathfrak{p}}_I(\omega)$. In terms of diagrams, it is the top face of the hypercube below.

$$\begin{tikzcd} & \alpha \arrow[rr, "\xi"] & & \delta \\ \alpha \arrow[ur, "r\alpha"] \arrow[rr, "\omega"'] & & \beta \arrow[ur, "\mathfrak{q}(\mathfrak{p}\omega{,}\xi)"'] & \\ & \alpha \arrow[uu, "r\alpha"] \arrow[rr, "r\alpha"] & & \alpha \arrow[uu, "\xi"'] \\ \alpha \arrow[uu, "r\alpha"] \arrow[ur, "\alpha"] \arrow[rr, "\omega"'] & & \beta \arrow[uu, "r\beta"] \arrow[ur, "\mathfrak{p}\omega"'] & \end{tikzcd}$$

The second path which we need is

$$\begin{tikzcd}
\delta \arrow[r, "\zeta"] & \gamma \\
\beta \arrow[u, "\mathfrak{q}(\mathfrak{p}\omega,\xi)"] \arrow[r, "\eta"'] \arrow[ur, phantom, "N(\kappa)"] & \gamma \arrow[u, "r\gamma"']
\end{tikzcd}$$

which is obtained as $(\mathrm{PP}t)_I(\Gamma \uparrow_{\ddot{S}^I_+} \vec{n}^\kappa)$. Here $\vec{n}^\kappa$ is the $\ddot{S}^I_+$-open box in Γ given by $n^\kappa_{x_I0} = \Gamma(x_I = z_I)\Gamma(x_I\ y_I)\hat{\mathfrak{q}}_I(\mathfrak{p}_I(\omega), \xi)$, $n^\kappa_{x_I1} = \Gamma(x_I = z_I)\mathrm{P}r_I(\eta)$, $n^\kappa_{y_I0} = \Gamma(y_I = z_I)\acute{\mathfrak{c}}_I(\eta)$, $n^\kappa_{y_I1} = \Gamma(y_I = z_I)(\kappa)$ and $n^\kappa_{z_I0} = \check{\mathfrak{p}}_I(\omega)$. Thus $N(\kappa)$ is the top face of the following hypercube.

$$\begin{tikzcd}
& \delta \arrow[rr, "\zeta"] & & \gamma \\
\beta \arrow[ur, "\mathfrak{q}(\mathfrak{p}\omega,\xi)"] \arrow[rr, "\eta"'] & & \gamma \arrow[ur, "r\gamma"'] & \\
& \alpha \arrow[uu, "\xi"'] \arrow[rr, "\omega"] & & \beta \arrow[uu, "\eta"'] \\
\beta \arrow[uu, "r\beta"] \arrow[ur, "\mathfrak{p}\omega"'] \arrow[rr, "r\beta"'] & & \beta \arrow[uu, "\eta"] \arrow[ur, "r\beta"'] &
\end{tikzcd}$$

Note that instead of $\mathfrak{q}_I(\mathfrak{p}_I(\omega), \xi)$ we could also have considered $\mathfrak{q}_I(\eta, \mathfrak{p}_I(\zeta))$. We are now able to find our desired witness to $\mathfrak{q}_I(\omega, \eta) \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}_I(\xi, \zeta)$, which will be $(\mathrm{PP}t)_I(\Gamma \uparrow_{\ddot{S}^I_+} \vec{w}^\kappa)$. Here $\vec{w}^\kappa$ is the $\ddot{S}^I_+$-open box in Γ which is given by the elements $w^\kappa_{x_I0} = \Gamma(x_I = z_I)(\mathrm{P}rr)_I(\alpha)$, $w^\kappa_{x_I1} = \Gamma(x_I = z_I)N(\kappa)$, $w^\kappa_{y_I0} = \Gamma(y_I = z_I)\hat{\mathfrak{q}}_I(\omega, \eta)$, $w^\kappa_{y_I1} = \Gamma(y_I = z_I)\hat{\mathfrak{q}}_I(\xi, \zeta)$ and $w^\kappa_{z_I0} = M(\kappa)$. Regarded as the top face of this hypercube, we see immediately that it is of the required form.

$$\begin{tikzcd}
& \alpha \arrow[rr, "\mathfrak{q}(\xi,\zeta)"] & & \gamma \\
\alpha \arrow[ur, "r\alpha"] \arrow[rr, "\mathfrak{q}(\omega,\eta)"'] & & \gamma \arrow[ur, "r\gamma"'] & \\
& \alpha \arrow[uu, "r\alpha"'] \arrow[rr, "\xi"] & & \delta \arrow[uu, "\zeta"'] \\
\alpha \arrow[uu, "r\alpha"] \arrow[ur, "r\alpha"'] \arrow[rr, "\omega"'] & & \beta \arrow[uu, "\eta"] \arrow[ur, "\mathfrak{q}(\mathfrak{p}\omega,\xi)"'] &
\end{tikzcd}$$

□

We are now able to compensate for the tedium of the previous proofs with streamlined ones of a few important results. We begin with providing the promised counterpart to Propositions 1.13 and 1.14, which we in turn use to derive Proposition 2.20, the characterisation of acyclic fibrations in $\mathcal{K}$.

Proposition 2.19. *Whenever* Γ *is Kan, we have that the following holds:*

1. *Composition is associative up to fibrewise homotopy, which is to say that* $\mathsf{q}(\mathsf{id} \times \mathsf{q}) \simeq_{(\Gamma\times\Gamma)} \mathsf{q}(\mathsf{q} \times \mathsf{id})$*.*

2. *Composition with reflexivity acts as identity up to fibrewise homotopy, or* $\mathsf{q}(rs, \mathsf{id}) \simeq_{(\Gamma\times\Gamma)} \mathsf{id}$ *and* $\mathsf{q}(\mathsf{id}, rt) \simeq_{(\Gamma\times\Gamma)} \mathsf{id}$*.*

3. *Composition with inverse yields an identity path up to (fibrewise) homotopy, or* $\mathsf{q}(\mathsf{id}, \mathfrak{p}) \simeq_{(\Gamma\times\Gamma)} rs$ *and* $\mathsf{q}(\mathfrak{p}, \mathsf{id}) \simeq_{(\Gamma\times\Gamma)} rt$*.*

4. *Morphisms commute with the path structure up to fibrewise homotopy, or for any* $\sigma : \Delta \to \Gamma$ *between Kan cubical sets we have* $\mathrm{P}\sigma\mathfrak{p} \simeq_{(\Gamma\times\Gamma)} \mathfrak{p}\mathrm{P}\sigma$ *and* $\mathrm{P}\sigma\mathsf{q} \simeq_{(\Gamma\times\Gamma)} \mathsf{q}(\mathrm{P}\sigma \times \mathrm{P}\sigma)$*.*

Proof. 1. A witness for associativity up to fibrewise homotopy may be obtained locally for any $(\omega, \eta, \xi) \in (\mathrm{P}\Gamma \times_\Gamma \mathrm{P}\Gamma \times_\Gamma \mathrm{P}\Gamma)(I)$ as follows, where $\omega : \alpha \to \beta$, $\eta : \beta \to \gamma$ and $\xi : \gamma \to \delta$. Let $\vec{u}^{\omega\eta\xi}$ be the $\ddot{S}^I_+$-open box in Γ given by $u^{\omega\eta\xi}_{x_I 0} = \Gamma(x_I = z_I)(\mathrm{P}rr)_I(\alpha)$, $u^{\omega\eta\xi}_{x_I 1} = \Gamma(x_I = z_I)\Gamma(x_I\ y_I)\check{\mathfrak{q}}_I(\eta, \xi)$, $u^{\omega\eta\xi}_{y_I 0} = \Gamma(y_I = z_I)\hat{\mathfrak{q}}_I(\omega, \mathsf{q}_I(\eta, \xi))$, $u^{\omega\eta\xi}_{y_I 1} = \Gamma(y_I = z_I)\hat{\mathfrak{q}}_I(\mathsf{q}_I(\omega, \eta), \xi)$ and lastly $u^{\omega\eta\xi}_{z_I 0} = \hat{\mathfrak{q}}_I(\omega, \eta)$. Then we obtain $\Gamma(z_I = 1)(\Gamma \uparrow_{\ddot{S}^I_+} \vec{u}^{\omega\eta\xi})$, which is the top face of the hypercube below.

$$\begin{tikzcd}
 & \alpha \arrow[rr, "\mathsf{q}(\mathsf{q}(\omega,\eta),\xi)"] & & \delta \\
\alpha \arrow[ur, "r\alpha"] \arrow[rr, "\mathsf{q}(\omega,\mathsf{q}(\eta,\xi))"'] & & \delta \arrow[ur, "r\gamma"'] & \\
 & \alpha \arrow[uu, "r\alpha"'] \arrow[rr, "\mathsf{q}(\omega,\eta)"] & & \gamma \arrow[uu, "\xi"'] \\
\alpha \arrow[uu, "r\alpha"] \arrow[ur, "r\alpha"'] \arrow[rr, "\omega"'] & & \beta \arrow[uu, "\mathsf{q}(\eta,\xi)"] \arrow[ur, "\eta"'] &
\end{tikzcd}$$

Combining these paths to a natural transformation of the proper form, we have found our desired homotopy $\mathsf{q}(\mathsf{id} \times \mathsf{q}) \simeq_{(\Gamma\times\Gamma)} \mathsf{q}(\mathsf{q} \times \mathsf{id})$.
2. Observe that $\overline{(x\ y)}\check{\mathfrak{q}}(rs, \mathsf{id}) : \mathsf{q}(rs, \mathsf{id}) \simeq_{(\Gamma\times\Gamma)} \mathsf{id}$, whereas for the other part we have simply $\hat{\mathfrak{q}}(\mathsf{id}, rt) : \mathsf{id} \simeq_{(\Gamma\times\Gamma)} \mathsf{q}(\mathsf{id}, rt)$.

3. We know that $rs \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}(rs, rs)$ by construction, and applying Proposition 2.18 to $\hat{\mathfrak{p}}$ gives $\mathfrak{q}(rs, rs) \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}(\mathsf{id}, \mathfrak{p})$. Applying it instead to $\check{\mathfrak{p}}$ gives $\mathfrak{q}(rt, rt) \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}(\mathfrak{p}, \mathsf{id})$, and $rt \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}(rt, rt)$ also holds by construction.
4. We see that $rs\mathrm{P}\sigma \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}(\mathrm{P}\sigma, \mathrm{P}\sigma\mathfrak{p})$ by applying Proposition 2.18 to $\mathrm{PP}\sigma\hat{\mathfrak{p}}$. This implies that $p\mathrm{P}\sigma \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}(\mathfrak{p}\mathrm{P}\sigma, \mathfrak{q}(\mathrm{P}\sigma, \mathrm{P}\sigma\mathfrak{p}))$ by identity up to fibrewise homotopy. In turn, we have $\mathfrak{q}(\mathfrak{p}\mathrm{P}\sigma, \mathfrak{q}(\mathrm{P}\sigma, \mathrm{P}\sigma\mathfrak{p})) \simeq_{(\Gamma\times\Gamma)} \mathrm{P}\sigma\mathfrak{p}$ by associativity, inverse and identity up to fibrewise homotopy, thus $\mathfrak{p}\mathrm{P}\sigma \simeq_{(\Gamma\times\Gamma)} \mathrm{P}\sigma\mathfrak{p}$ as required. Finally, we find $\mathfrak{q}(\mathrm{P}\sigma \times \mathrm{P}\sigma) \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}(rs\mathrm{P}\sigma\mathfrak{q}, \mathrm{P}\sigma\mathfrak{q}) \simeq_{(\Gamma\times\Gamma)} \mathrm{P}\sigma\mathfrak{q}$ by applying Proposition 2.18 to $\mathrm{P}\sigma\hat{\mathfrak{q}}$. □

We may now finally come to proving the characterisation of acyclic fibrations in $\mathcal{K}$, which has been the motivating result for this subsection.

Proposition 2.20. *A Kan fibration $\sigma : \Delta \to \Gamma$ is acyclic precisely when it has a section $\tau : \Gamma \to \Delta$ such that $\tau\sigma \simeq_\Gamma \mathsf{id}_\Delta$.*

Proof. As it is immediate that Kan fibrations satisfying the latter are acyclic, suppose $\sigma : \Delta \to \Gamma$ is an acyclic fibration, with $\tau : \Gamma \to \Delta$, $\ell : \sigma\tau \simeq \mathsf{id}_\Gamma$ and $\ell' : \tau\sigma \simeq \mathsf{id}_\Delta$. For any I we may consider the +-shape $S^I = ((x_I, 1); \emptyset; I, x_I)$, so that for any $\alpha \in \Gamma(I)$ we have that $\tau_I(\alpha)$ is an S^I-open box for σ over $\ell_I(\alpha)$. Since σ is a Kan fibration, we find a Kan filler $\sigma^{-1}(\ell_I(\alpha)) \uparrow_{S^I} \tau_I(\alpha)$ which we use to define our section $\psi : \Gamma \to \Delta$ by $\hat{\psi}_I(\alpha) = \sigma^{-1}(\ell_I(\alpha)) \uparrow_{S^I} \tau_I(\alpha)$ and $\psi_I(\alpha) = t_I\hat{\psi}_I(\alpha)$. We have $\sigma_I\psi_I(\alpha) = \alpha$ as required, and this construction is natural by the familiar uniformity argument, hence ψ is well-defined.
Thus we are left to show that $\psi\sigma \simeq_\Gamma \mathsf{id}_\Delta$, for which we note first that $\mathfrak{p}\ell' : \mathsf{id}_\Delta \simeq \tau\sigma$, $\mathrm{P}\tau\mathfrak{p}\ell\sigma : \tau\sigma \simeq \tau\sigma\tau\sigma$, $\ell'\tau\sigma : \tau\sigma\tau\sigma \simeq \tau\sigma$ and $\hat{\psi}\sigma : \tau\sigma \simeq \psi\sigma$. This means we may compose these and take the inverse in order to obtain $\mathfrak{pq}(\mathfrak{p}\ell', \mathrm{P}\tau\mathfrak{p}\ell\sigma, \ell'\tau\sigma, \hat{\psi}\sigma) : \mathsf{id}_\Delta \simeq \psi\sigma$. The precise order of this composition will be irrelevant for our purposes due to the associativity up to fibrewise homotopy of composition, which was established in Proposition 2.19.
Applying Proposition 2.18 to $\mathrm{P}\ell\ell\sigma$, $\mathrm{P}(\mathrm{P}\sigma\ell')\ell'$ and $\mathrm{P}(\ell\sigma)\ell'$ respectively, we find $\mathfrak{q}(\mathrm{P}(\sigma\tau)\ell\sigma, \ell\sigma) \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}(\ell\sigma\tau\sigma, \ell\sigma)$ and so $\mathrm{P}(\sigma\tau)\ell\sigma \simeq_{(\Gamma\times\Gamma)} \ell\sigma\tau\sigma$; secondly $\mathfrak{q}(\mathrm{P}(\sigma\tau\sigma)\ell', \mathrm{P}\sigma\ell') \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}(\mathrm{P}\sigma\ell'\tau\sigma, \mathrm{P}\sigma\ell')$ and thus $\mathrm{P}(\sigma\tau\sigma)\ell' \simeq_{(\Gamma\times\Gamma)} \mathrm{P}\sigma\ell'\tau\sigma$; finally $\mathfrak{q}(\ell\sigma\tau\sigma, \mathrm{P}\sigma\ell') \simeq_{(\Gamma\times\Gamma)} \mathfrak{q}(\mathrm{P}(\sigma\tau\sigma)\ell', \ell\sigma)$. This enables us to carry out the following derivation.

$$\begin{aligned}
\mathrm{P}\sigma\mathfrak{pq}(\mathfrak{p}\ell', \mathrm{P}\tau\mathfrak{p}\ell\sigma, \ell'\tau\sigma, \hat{\psi}\sigma) &\simeq_{(\Gamma\times\Gamma)} \mathfrak{pq}(\mathfrak{p}\mathrm{P}\sigma\ell', \mathfrak{p}\mathrm{P}(\sigma\tau)\ell\sigma, \mathrm{P}\sigma\ell'\tau\sigma, \ell\sigma) && (2.19,\ 4)\\
&\simeq_{(\Gamma\times\Gamma)} \mathfrak{pq}(\mathfrak{p}\mathrm{P}\sigma\ell', \mathfrak{p}\ell\sigma\tau\sigma, \mathrm{P}(\sigma\tau\sigma)\ell', \ell\sigma) && (\textit{see above})\\
&\simeq_{(\Gamma\times\Gamma)} \mathfrak{pq}(\mathfrak{p}\mathrm{P}\sigma\ell', \mathfrak{p}\ell\sigma\tau\sigma, \ell\sigma\tau\sigma, \mathrm{P}\sigma\ell') && (\textit{see above})\\
&\simeq_{(\Gamma\times\Gamma)} \mathfrak{pq}(\mathfrak{p}\mathrm{P}\sigma\ell', r\sigma\tau\sigma, \mathrm{P}\sigma\ell') && (\textit{inverse})\\
&\simeq_{(\Gamma\times\Gamma)} \mathfrak{pq}(\mathfrak{p}\mathrm{P}\sigma\ell', \mathrm{P}\sigma\ell') && (\textit{identity})\\
&\simeq_{(\Gamma\times\Gamma)} \mathfrak{p}\mathrm{P}\sigma && (\textit{inverse})
\end{aligned}$$

Now let $\lambda : \mathrm{P}\sigma\mathfrak{p}\mathfrak{q}(\mathfrak{p}\ell', \mathrm{P}\tau\mathfrak{p}\ell\sigma, \ell'\tau\sigma, \hat{\psi}\sigma) \simeq_{(\Gamma\times\Gamma)} \mathrm{P}\sigma r$ be such a homotopy. Then for any $\kappa \in \Delta(I)$ we may consider $\vec{v}^{\kappa}$, the $\dot{S}^I_+$-open box for σ over $\lambda_I(\kappa)$ given by $v^{\kappa}_{x_I 0} = \Gamma(x_I = y_I)\mathrm{P}(\psi\sigma)_I(\kappa)$, $v^{\kappa}_{x_I 1} = \Gamma(x_I = y_I)r_I(\kappa)$ and $v^{\kappa}_{y_I 0} = (\mathrm{P}\sigma)_I\mathfrak{p}_I\mathfrak{q}_I(\mathfrak{p}_I\ell'_I, (\mathrm{P}\tau)_I\mathfrak{p}_I\ell_I\sigma_I, \ell'_I\tau_I\sigma_I, \hat{\psi}_I\sigma_I)(\kappa)$. This gives rise to a Kan filler $\lambda'_I(\kappa) = \sigma^{-1}(\lambda_I(\kappa)) \uparrow_{\dot{S}^I} \vec{v}^{\kappa}$, from which we are able to bring forth $\mathrm{P}t_I\lambda'_I(\kappa) \in P_\Gamma(\Delta)(I)$ as a suitable path for which $\mathrm{P}t_I\lambda'_I(\kappa) : \psi_I\sigma_I(\kappa) \to \kappa$. Thus $\mathrm{P}t\lambda' : \psi\sigma \simeq_\Gamma \mathsf{id}_\Delta$, which shows that any acyclic fibration σ has a section ψ as described. □

This was the last result required in order to prove that the axioms of a path category are satisfied in $\mathcal{K}$, which we shall do in the next subsection. Had we been able to do this without Proposition 2.20, then we could have simply noted it to be the instance of Proposition 1.9 for $\mathcal{K}$. We are led to believe that our current proof strategy is nevertheless the most efficient.

2.2.3 Verifying the axioms

In this final subsection of the current chapter, we verify that the axioms of a path category for the Kan fibrations and homotopy equivalences in $\mathcal{K}$. More than once we are merely required to check that the constructions which mostly suggest themselves meet the requirements. The only axiom for which Proposition 2.20 is strictly required is Axiom 3, though some other results are used elsewhere. We shall now treat the axioms in their given order.

Proposition 2.21. *Kan fibrations are closed under composition, which means Axiom 1 holds in $\mathcal{K}$.*

Proof. Let $\sigma : \Delta \to \Gamma$ and $\tau : \Theta \to \Delta$ be Kan fibrations. Let S be any open box shape on some arbitrary I, and suppose $\vec{u}$ is an S-open box for $\sigma\tau$ over some $\alpha \in \Gamma(I)$. We shall assume that S is a $+$-shape, as the proof for $-$-shapes differs only in notation. We write $\tau\vec{u}$ for the S-open box for σ over α given by $(\tau\vec{u})_{yb} = \tau_{I-y}(u_{yb})$: that this is indeed an open box follows by the naturality of τ from the fact that $\vec{u}$ is. Because σ is a Kan fibration, we obtain a Kan filler $\beta = \sigma^{-1}(\alpha) \uparrow_S (\tau\vec{u}) \in \sigma_I^{-1}(\alpha)$. We find now that $\vec{u}$ is an S-open box for τ over β, since $u_{yb} \in \tau^{-1}_{I-y}(\tau_{I-y}(u_{yb})) = \tau^{-1}_{I-y}(\Delta(y = b)(\beta))$ as required. Thus by τ being a Kan fibration as well, we obtain another Kan filler $\gamma = \tau^{-1}(\beta) \uparrow_S \vec{u} \in \Theta(I)$. We define $(\sigma\tau)^{-1}(\alpha) \uparrow_S \vec{u}$ to be this filler γ, and note that $\Theta(y = b)(\gamma) = u_{yb}$ is immediate, whereas uniformity follows from the uniformity of the two filler operations and the naturality of τ. That is, for any $f : I \to K$ with suitable domain we have that $\Theta(f)(\gamma) = \tau^{-1}(\Delta(f)(\beta)) \uparrow_{fS} f\vec{u} = \tau^{-1}(\sigma^{-1}(\Gamma(f)(\alpha)) \uparrow_{fS} \tau f\vec{u}) \uparrow_{fS} f\vec{u}$ as required, hence this definition yields a Kan structure on $\sigma\tau$ as intended. □

For Axiom 2 (and likewise for Axiom 3) we note that $\mathcal{C}\mathsf{Set}$ has pullbacks as it is a presheaf category, hence for the existence part we only need to show that this pullback also lies in $\mathcal{K}$. This follows once we have shown that the pullback of a Kan fibration is again Kan, by virtue of the following lemma.

Lemma 2.22. *If $\sigma : \Delta \to \Gamma$ is a Kan fibration, then Δ is Kan if Γ is.*

Proof. By definition Δ is Kan whenever $1^\Delta : \Delta \to 1$ is a Kan fibration. But $1^\Delta = 1^\Gamma\sigma$, hence if Γ is Kan the composition $1^\Gamma\sigma$ is again Kan by Proposition 2.21, so that Δ is Kan whenever Γ is. □

That Kan fibrations are closed under pullback along any map can be found (in a somewhat different formulation) in [BCH14] as Theorem 6.1, but we will present a more detailed argument here.

Proposition 2.23. *Kan fibrations are closed under pullback along any map, hence Axiom 2 holds in $\mathcal{K}$.*

Proof. Let $\tau : \Theta \to \Gamma$ be a Kan fibration and $\sigma : \Delta \to \Gamma$ any map. Consider the pullback $\Delta \times_\Gamma \Theta$ with projections $\tau' : \Delta \times_\Gamma \Theta \to \Delta$ and $\sigma' : \Delta \times_\Gamma \Theta \to \Theta$. Then for any I the elements of $(\Delta \times_\Gamma \Theta)(I)$ are pairs $(\alpha, \kappa) \in (\Delta(I) \times \Theta(I))$ such that $\sigma_I(\alpha) = \tau_I(\kappa)$, with $\tau'_I(\alpha, \kappa) = \alpha$ and $\sigma'_I(\alpha, \kappa) = \kappa$. Now let S be any open box shape on I (we again only treat the $+$-shape case), and suppose $\vec{u}$ is an S-open box for τ' over some $\alpha \in \Delta(I)$. Then $\sigma'\vec{u}$ (cf. the proof of Proposition 2.21) is an S-open box for τ over $\sigma_I(\alpha)$, hence since τ is a Kan fibration we have a filler $\tau^{-1}(\sigma_I(\alpha)) \uparrow_S \sigma'\vec{u} \in \Theta(I)$. We can use this to define $\tau'^{-1}(\alpha) \uparrow_S \vec{u} = (\alpha, \tau^{-1}(\sigma_I(\alpha)) \uparrow_S \sigma'\vec{u})$, which is well-defined since $\tau_I(\tau^{-1}(\sigma_I(\alpha)) \uparrow_S \sigma'\vec{u}) = \sigma_I(\alpha)$ as required. That it is of the right form with respect to τ' is immediate; uniformity follows by componentwise naturality (and uniformity of the filler belonging to τ). Thus τ' with these fillers is a Kan fibration as desired. □

By combining this result with Proposition 2.20, we are rewarded for our efforts in establishing the latter with the following concise proof that acyclic fibrations are preserved under pullback along any map as well.

Proposition 2.24. *Acyclic fibrations are preserved under pullback along any map, hence Axiom 3 holds in $\mathcal{K}$.*

Proof. Consider the situation as in Proposition 2.23 where we take the pullback $\Delta \times_\Gamma \Theta$ of $\sigma : \Delta \to \Gamma$ and $\tau : \Theta \to \Gamma$, except now τ is an acyclic fibration. Let $\psi : \Gamma \to \Theta$ be a section of τ such that $\ell : \psi\tau \simeq_\Gamma \mathsf{id}_\Theta$, which exists by Proposition 2.20. This gives rise to a map $\psi' : \Delta \to \Delta \times_\Gamma \Theta$ such that $\psi'_I(\alpha) = (\alpha, \psi_I\sigma_I(\alpha))$, which we immediately recognise to be a section of τ'. Moreover, the map $\ell' = (r, \ell) : \Delta \times_\Gamma \Theta \to P(\Delta \times_\Gamma \Theta)$ is well-defined since $(P\sigma)(P\tau')\ell' = r\sigma\tau' = r\tau\sigma' = (P\tau)\ell\sigma' = (P\tau)(P\sigma')\ell'$ as required, and consequently $\ell' : \psi'\tau' \simeq_\Delta \mathsf{id}$. Since we know that τ' is a Kan fibration because τ is, this shows that τ' is in fact an acyclic fibration as well. $\square$

Now for Axiom 4 we know that the isomorphisms of any category satisfy 2-out-of-6. Since we have already shown that the homotopy equivalence relation is a congruence relation, it is possible to take the homotopy category $\mathrm{Ho}(\mathcal{K})$ of $\mathcal{K}$ in which we quotient the morphisms of $\mathcal{K}$ up to this homotopy equivalence. The isomorphisms of this homotopy category are then precisely (the homotopy classes of) those maps which are homotopy equivalences, so this construction reflects the property of 2-out-of-6 to the class of homotopy equivalences. As a result, we may conclude the following.

Proposition 2.25. *Homotopy equivalences satisfy 2-out-of-6, which means Axiom 4 holds in $\mathcal{K}$.*

Since we already established in Proposition 2.20 that every acyclic fibration has a section, we only need to show that isomorphisms are acyclic fibrations to complete the proof for Axiom 5. Since this fact is rather evident, we shall provide only partial details.

Proposition 2.26. *Isomorphisms are acyclic fibrations, hence Axiom 5 holds in $\mathcal{K}$.*

Proof. Let $\sigma : \Delta \to \Gamma$ be an isomorphism with its inverse $\tau : \Gamma \to \Delta$. That it is a homotopy equivalence is immediate, as we may simply take the respective reflexivity maps. It is not much more difficult to see that σ is also a Kan fibration. For arbitrary I, let S be any open box shape on I, and take $\vec{u}$ to be an S-open box for σ over some $\alpha \in \Gamma(I)$: we claim that it suffices to define the desired Kan filler as $\tau_I(\alpha)$. Since $\sigma_I\tau_I(\alpha) = \alpha$, we have that $\tau_I(\alpha) \in \sigma_I^{-1}(\alpha)$. Furthermore $u_{yb} = \tau_{I-y}(\Gamma(y = b)(\alpha)) = \Delta(y = b)(\tau_I(\alpha))$ as well as $\Delta(f)(\tau_I(\alpha)) = \tau_K(\Gamma(f)(\alpha))$ for any $f : I \to K$ by naturality, hence $\tau_I(\alpha)$ is of the appropriate form. $\square$

Finally we establish that the path object as given in Definition 2.8 is a factorisation of the diagonal as a weak equivalence followed by a uniform Kan fibration, thereby checking the validity of Axiom 6. Once this is done, we will have proven that $\mathcal{K}$ is indeed a path category, as Axiom 7 has already been ensured to hold.

Proposition 2.27. *The map $r : \Gamma \to \mathrm{P}\Gamma$ is a weak equivalence and the map $(s, t) : \mathrm{P}\Gamma \to \Gamma$ is a Kan fibration for any Γ, hence Axiom 6 holds in $\mathcal{K}$.*

Proof. To see that $r : \Gamma \to \mathrm{P}\Gamma$ is a homotopy equivalence, we may take $s : \mathrm{P}\Gamma \to \Gamma$ and observe that we have $\acute{\mathsf{c}} : rs \simeq \mathsf{id}$ and $sr = \mathsf{id}_\Gamma$. Thus it remains to show that the map $(s,t) : \mathrm{P}\Gamma \to \Gamma \times \Gamma$ which is given by $s_I = \Gamma(x_I = 0)$, $t_I = \Gamma(x_I = 1)$ is indeed a Kan fibration. To this end, let S be an arbitrary open box shape on a given I, and let $\vec{u}$ be an S-open box for (s,t) over some $(\alpha, \beta) \in \Gamma(I) \times \Gamma(I)$. This means we have a family of adjacent elements $u_{yb} \in \Gamma(I, x_{I-y} - y)$ such that $X(x_{I-y} = 0)(u_{yb}) = X(y = b)(\alpha)$ and $X(x_{I-y} = 1)(u_{yb}) = X(y = b)(\beta)$. Taking $S^* = ((x,a); J, x_I; I, x_I)$, we may therefore instead consider the S^*-open box $\vec{u}^*$ in Γ which is defined by $u^*_{yb} = X(x_{I-y} = x_I)(u_{yb}) \in X(I, x_I - y)$ for all $y \in J$, along with $u^*_{x_I 0} = \alpha$ and $u^*_{x_I 1} = \beta$. Now if Γ is Kan, then we find a Kan filler of $\vec{u}^*$ in $\mathrm{P}\Gamma(I)$ which has the right properties to serve as a Kan filler of $\vec{u}$ for (s,t) over (α, β). $\square$

This leaves us to conclude this chapter by summarising our efforts in the following theorem.

Theorem 2.28. *The category $\mathcal{K}$ with its additional structure as presented in Definition 2.13 is a path category. That is, the full subcategory $\mathcal{K}$ of $\mathcal{C}\mathsf{Set}$ consisting of the uniform Kan cubical sets, where the fibrations are the uniform Kan fibrations as defined in Definition 2.6, the weak equivalences are the homotopy equivalences as defined in Definition 2.12, and the path objects are as defined in Definition 2.8, is a path category.*

As discussed during the Introduction, the importance of this result lies in the fact that it allows us to understand the propositional character of the identity types belonging to the cubical set model of type theory in terms of the latter being an instance of a path category. Although this provides us with a different perspective on the more type-theoretically motivated approach of [BCH14], at the same time we should keep in mind that this identification is not a perfect one. For instance, our path category $\mathcal{K}$ is only a full submodel of the original one. Furthermore, we have seen at times that stronger statements are true than those which were required to show that $\mathcal{K}$ is a path category, which is unsurprising given that the category of cubical sets possess a far richer model structure than a path category demands. We shall see similar disparities in the next chapter, in which we first treat exponentials and Π-types in the context of a path category, after which we study those encountered in cubical sets along the same lines.

Chapter 3

Exponentials and product types

So far we have only considered the bare structure of a path category which is sufficient for interpreting the identity types. Although these make up what is perhaps the most interesting part of dependent type theory, this does not mean that there is nothing worthwhile to be said about other kinds of types. For instance, one may study the Π-types and the related topic of function extensionality, which is what we will do in this final chapter. We shall follow the structure of the previous two chapters in that we begin by working in the abstract setting of a path category, after which we observe how the results there translate to the specific case of our category $\mathcal{K}$ of Kan cubical sets. In particular we examine what function extensionality looks like in path categories, after which we identify well-behaved witnesses to function extensionality in cubical sets. In this way we expand both on the results treated in [vdBM16] and those in [Hub15].

3.1 ...in path categories

In this section we shall look at exponentials and Π-types in the context of a path category, for which [vdBM16] is again our primary reference. We then come to the notions of function extensionality for these objects, which will feature in a pair of equivalence results. We begin by providing the definition of a (weak) homotopy exponential.

Definition 3.1. Whenever X and Y are objects of a path category $\mathcal{C}$, a *weak homotopy exponential* for X and Y is an object X^Y along with a map $\mathsf{ev} : X^Y \times Y \to X$ such that for any map $h : A \times Y \to X$ there is a map $H : A \to X^Y$ such that $\mathsf{ev}(H \times \mathsf{id}_Y) \simeq h$. If H is unique up to homotopy with this property, we say that X^Y is a *homotopy exponential*.

We say that $\mathcal{C}$ has (weak) homotopy exponentials if it has a (weak) homotopy exponential for every two objects X and Y. Similarly, we say that a path category $\mathcal{C}$ has (weak) homotopy Π-types if it has a (weak) homotopy Π-type $\Pi_a X$ for every pair of fibrations $b : X \to Y$ and $a : Y \to Z$, where (weak) homotopy Π-types are defined as follows.

Definition 3.2. Whenever $b : X \to Y$ and $a : Y \to Z$ are fibrations in a path category $\mathcal{C}$, a *weak homotopy* Π-*type* for a and b is an object $\Pi_a X$ with a fibration $\Pi_a b : \Pi_a X \to Z$ along with a map $\mathsf{ev} : a^*\Pi_a X \to X$ such that $b\mathsf{ev} = a^*\Pi_a b$, which satisfies the following: if $c : W \to Z$ and $m : a^*W \to X$ are such that $bm = a^*c$, then there is a map $n : W \to \Pi_a X$ such that $\Pi_a bn = c$ and $\mathsf{ev}a^*n \simeq_Y m$. If n is unique up to fibrewise homotopy with these properties, we say that $\Pi_a X$ is a *homotopy* Π-*type*.

There is an alternative way of distinguishing between the weak and non-weak homotopy exponentials and Π-types of a path category $\mathcal{C}$ which is already hinted at in Remark 6.2 in [vdBM16], although the statement there is (in its current state) incomplete. We formulate and prove this characterisation for homotopy exponentials first, and afterwards for homotopy Π-types.

Proposition 3.3. *A weak homotopy exponential X^Y in $\mathcal{C}$ is a homotopy exponential precisely when there is a weak homotopy exponential $(PX)^Y$ along with a map $e : (PX)^Y \to P(X^Y)$ such that the square*

$$\begin{tikzcd}
(PX)^Y \times Y \arrow[r, "(s',t')e\times\mathsf{id}_Y"] \arrow[d, "\mathsf{ev}'"'] & (X^Y \times X^Y) \times Y \arrow[d, "(\mathsf{ev}\times\mathsf{ev})(\pi_1\times\mathsf{id}_Y,\pi_2\times\mathsf{id}_Y)"] \\
PX \arrow[r, "(s,t)"'] & X \times X
\end{tikzcd}$$

commutes and induces the following quasi-pullback square for any object A:

$$\begin{tikzcd}
\mathrm{Ho}(\mathcal{C})(A,(PX)^Y) \arrow[r, "(s',t')e\circ_"] \arrow[d, "\mathsf{ev}'(_\times\mathsf{id}_Y)"'] & \mathrm{Ho}(\mathcal{C})(A, X^Y \times X^Y) \arrow[d, "(\mathsf{ev}\times\mathsf{ev})(\pi_1\times\mathsf{id}_Y,\pi_2\times\mathsf{id}_Y)(_\times\mathsf{id}_Y)"] \\
\mathrm{Ho}(\mathcal{C})(A\times Y, PX) \arrow[r, "(s,t)\circ_"'] & \mathrm{Ho}(\mathcal{C})(A\times Y, X\times X)
\end{tikzcd}$$

Proof. Suppose X^Y is a homotopy exponential. We show that $P(X^Y)$ is a weak homotopy exponential $(PX)^Y$ and choose $\mathsf{id} : P(X^Y) \to P(X^Y)$ as our map e. This requires us to define an evaluation map $\mathsf{ev}' : P(X^Y) \times Y \to PX$ as follows. We see that $(s,t)r\mathsf{ev}(s' \times \mathsf{id}) \simeq (\mathsf{ev}\times\mathsf{ev})(s'\times\mathsf{id}_Y, t'\times\mathsf{id}_Y)$, hence by Proposition 1.8 we obtain a map $\mathsf{ev}' \simeq r\mathsf{ev}(s'\times\mathsf{id})$ such that the desired identity $(s,t)\mathsf{ev}' = (\mathsf{ev}\times\mathsf{ev})(s'\times\mathsf{id}_Y, t'\times\mathsf{id}_Y)$ holds. Therefore $P(X^Y)$ is indeed a weak homotopy exponential $(PX)^Y$, as for any $h : A \times Y \to PX$ there is an $H : A \to X^Y$ such that $\mathsf{ev}(H \times \mathsf{id}_Y) \simeq sh$, so $r'H : A \to P(X^Y)$ satisfies $\mathsf{ev}'(r'H \times \mathsf{id}_Y) \simeq r\mathsf{ev}(H \times \mathsf{id}_Y) \simeq rsh \simeq h$. This leaves us to show that

$$\begin{tikzcd}
\mathrm{Ho}(\mathcal{C})(A, P(X^Y)) \arrow[r, "(s',t')\circ_"] \arrow[d, "\mathsf{ev}'(_\times\mathsf{id}_Y)"'] & \mathrm{Ho}(\mathcal{C})(A, X^Y \times X^Y) \arrow[d, "(\mathsf{ev}\times\mathsf{ev})(\pi_1\times\mathsf{id}_Y,\pi_2\times\mathsf{id}_Y)(_\times\mathsf{id}_Y)"] \\
\mathrm{Ho}(\mathcal{C})(A \times Y, PX) \arrow[r, "(s,t)\circ_"'] & \mathrm{Ho}(\mathcal{C})(A \times Y, X \times X)
\end{tikzcd}$$

is a quasi-pullback square, which is that the induced map to the pullback is an epimorphism in Set, i.e. a surjection. Thus we must demonstrate that for any two maps $(f : A \times Y \to PX, g : A \to X^Y \times X^Y)$ which satisfy $(s,t)f \simeq (\mathsf{ev} \times \mathsf{ev})(\pi_1 \times \mathsf{id}_Y, \pi_2 \times \mathsf{id}_Y)(g \times \mathsf{id}_Y)$ there is some $h : A \to P(X^Y)$ such that $(f, g) \simeq (\mathsf{ev}'(h \times \mathsf{id}_Y), (s', t')h)$. Since X^Y is a homotopy exponent, the fact that $\mathsf{ev}(\pi_1 g \times \mathsf{id}_Y) \simeq sf \simeq tf \simeq \mathsf{ev}(\pi_2 g \times \mathsf{id}_Y)$ implies $\pi_1 g \simeq \pi_2 g$. Hence there is a map $h : A \to P(X^Y)$ with $(s', t')h = g$ which then moreover satisfies $\mathsf{ev}'(h \times \mathsf{id}_Y) \simeq r\mathsf{ev}(s'h \times \mathsf{id}_Y) \simeq rsf \simeq f$ as required.
For the converse implication, suppose X^Y and $(PX)^Y$ are weak homotopy exponentials, and let $e : (PX)^Y \to P(X^Y)$ be a map such that the two given squares are commutative and a quasi-pullback respectively. In order to show that X^Y is now a homotopy exponential, consider any maps $h : A \times Y \to X$ and $H_1, H_2 : A \to X^Y$ such that $\mathsf{ev}(H_1 \times \mathsf{id}_Y) \simeq h \simeq \mathsf{ev}(H_2 \times \mathsf{id}_Y)$. Then we find that $(s,t)rh \simeq (\mathsf{ev} \times \mathsf{ev})(\pi_1 \times \mathsf{id}_Y, \pi_2 \times \mathsf{id}_Y)((H_1, H_2) \times \mathsf{id}_Y)$, so by quasi-pullback there is a map $K : A \to (PX)^Y$ satisfying $(s', t')eK \simeq (H_1, H_2)$. By Proposition 1.8 there is then a map $K' \simeq eK$ satisfying the identity $(s', t')K' = (H_1, H_2)$, which means X^Y is a homotopy exponential. □

Regarding the map $e : (PX)^Y \to P(X^Y)$ as being a type-theoretic proof term, this result tells us that the homotopy exponentials are precisely those which satisfy a form of function extensionality. The homotopy Π-types are similarly characterised by the following proposition, the proof of which proceeds along the same lines as the previous one.

Proposition 3.4. *A weak homotopy Π-type $\Pi_a X$ in $\mathcal{C}$ for a pair of fibrations $b : X \to Y$ and $a : Y \to Z$ is a homotopy Π-type precisely when there is a weak homotopy Π-type $\Pi_a P_Y(X)$ along with a map $e : \Pi_a P_Y(X) \to P_Z(\Pi_a X)$ such that the square*

$$\begin{tikzcd}
a^*\Pi_a P_Y(X) \arrow[r, "a^*((s',t')e)"] \arrow[d, "\mathsf{ev}'"'] & a^*(\Pi_a X \times_Z \Pi_a X) \arrow[d, "(\mathsf{ev}\times\mathsf{ev})(a^*\pi_1,a^*\pi_2)"] \\
P_Y(X) \arrow[r, "(s,t)"'] & X \times_Y X
\end{tikzcd}$$

commutes and induces the following quasi-pullback square for any object W:

$$\begin{tikzcd}
\mathrm{Ho}(\mathcal{C}(Z))(W,\Pi_a P_Y(X)) \arrow[r, "(s',t')e\circ"] \arrow[d, "\mathsf{ev}'a^*"'] & \mathrm{Ho}(\mathcal{C}(Z))(W,\Pi_a X\times_Z \Pi_a X) \arrow[d, "(\mathsf{ev}\times\mathsf{ev})(a^*\pi_1,a^*\pi_2)a^*"] \\
\mathrm{Ho}(\mathcal{C}(Y))(a^*W,P_Y(X)) \arrow[r, "(s,t)\circ_"'] & \mathrm{Ho}(\mathcal{C}(Y))(a^*W,X\times_Y X)
\end{tikzcd}$$

Proof. Suppose $\Pi_a X$ is a homotopy Π-type for $b : X \to Y$ and $a : Y \to Z$. We show that $P_Z(\Pi_a X)$ is a weak homotopy Π-type $\Pi_a P_Y(X)$, with fibration $\Pi_a bs : P_Z(\Pi_a X) \to Z$. As $(s,t)r\mathsf{ev}a^*s' \simeq_Y (\mathsf{ev}a^*s', \mathsf{ev}a^*t')$, by Proposition 1.8 there is $\mathsf{ev}' \simeq_Y r\mathsf{ev}a^*s'$ such that $(s,t)\mathsf{ev}' = (\mathsf{ev}a^*s', \mathsf{ev}a^*t')$. Now if $c : W \to Z$ and $m : a^*W \to P_Y(X)$ are such that $bsm = a^*c$, then there is $n : W \to \Pi_a X$ such that $\Pi_a bn = c$ and $\mathsf{ev}a^*n \simeq_Y sm$. Thus $r'n : W \to P_Z(\Pi_a X)$ is such that $\Pi_a bs'r'n = \Pi_a bn = c$ and $\mathsf{ev}a^*(r'n) \simeq_Y r\mathsf{ev}a^*n \simeq_y rsm \simeq_Y m$ as required. Now let $e : P_Z(\Pi_a X) \to P_Z(\Pi_a X)$ be the identity map, and consider the diagram given below.

$$\begin{tikzcd}
\mathrm{Ho}(\mathcal{C}(Z))(W,P_Z(\Pi_a X)) \arrow[r, "(s',t')\circ_"] \arrow[d, "\mathsf{ev}'a^*"'] & \mathrm{Ho}(\mathcal{C}(Z))(W,\Pi_a X\times_Z \Pi_a X) \arrow[d, "(\mathsf{ev}\times\mathsf{ev})(a^*\pi_1,a^*\pi_2)a^*"] \\
\mathrm{Ho}(\mathcal{C}(Y))(a^*W,P_Y(X)) \arrow[r, "(s,t)\circ_"'] & \mathrm{Ho}(\mathcal{C}(Y))(a^*W,X\times_Y X)
\end{tikzcd}$$

To see that this is a quasi-pullback, suppose we have $f : a^*W \to P_Y(X)$ and $g : W \to \Pi_a X \times_Z \Pi_a X$ such that $(s,t)f \simeq_Y (\mathsf{ev}\times\mathsf{ev})(a^*\pi_1, a^*\pi_2)a^*g$. Then $\mathsf{ev}a^*(\pi_1 g) \simeq_Y sf \simeq_Y tf \simeq_Y \mathsf{ev}a^*(\pi_2 g)$ implies that $\pi_1 g \simeq_Z \pi_2 g$ since $\Pi_a X$ is a homotopy Π-type. Thus we have our required map $h : W \to P_Z(\Pi_a X)$ such that $(s',t')h = g$ and moreover $\mathsf{ev}'a^*h \simeq_Y r\mathsf{ev}a^*(s'h) \simeq_Y rsf \simeq_Y f$.
For the converse implication, suppose that $\Pi_a X$ and $\Pi_a P_Y(X)$ are weak homotopy Π-types and $e : \Pi_a P_Y(X) \to P_Z(\Pi_a X)$ is as described. To see that $\Pi_a X$ is now a homotopy Π-type, let $c : W \to Z$ and $m : a^*W \to X$ be such that $bm = a^*c$, and take any $(n_1,n_2) : W \to \Pi_a X \times_Z \Pi_a X$ satisfying $\mathsf{ev}a^*n_1 \simeq_Y m \simeq_Y \mathsf{ev}a^*n_2$. Then $(s,t)rm \simeq_Y (\mathsf{ev}\times\mathsf{ev})(a^*\pi_1,a^*\pi_2)a^*(n_1,n_2)$, hence by quasi-pullback there is some $k : W \to \Pi_a P_Y(X)$ for which we find $(s',t')ek \simeq_Z (n_1,n_2)$. By Proposition 1.8 there is now some $k' \simeq_Z ek$ with $(s',t')k' = (n_1,n_2)$, which shows that $n_1 \simeq_Z n_2$ as required. $\square$

These two propositions enable us to talk about function extensionality for homotopy exponentials and Π-types in a way which avoids having to specify a choice of path object and exponential or Π-type. This is rather useful as these are at best unique up to homotopy equivalence. Conversely, this tells us that the homotopy exponentials and Π-types for which function extensionality does not hold are very weak indeed. In the next section we shall find that the exponentials and Π-types in cubical sets are more well-behaved, as we are able to explicitly construct a suitable proof term for both.

3.2 ...in cubical sets

In this section we shall concern ourselves with the way in which the exponentials and Π-types are instantiated in our path category $\mathcal{K}$ of Kan cubical sets. We begin by treating the exponentials by giving a definition and explicitly constructing a proof term witnessing function extensionality for these. In order to express path objects as exponentials as well, we consider a slight variation on the definition of an exponential which is also found in [Hub15]: we shall call these *path exponentials* and prove a few additional results about them. We then continue by treating the Π-types as we did the exponentials, which is to say that we define them and exhibit a proof term for function extensionality for Π-types. Though such a witness to function extensionality is already considered in [Hub15] as well, our discussion extends the one provided there as we consider this proof term in a broader context.

3.2.1 Exponentials and path exponentials

The exponentials in $\mathcal{K}$ are given in the following way.

Definition 3.5. Whenever Γ and Δ are two cubical sets, their *exponential* Γ^Δ is the cubical set such that $\Gamma^\Delta(I) = \mathcal{C}\mathsf{Set}(\mathbf{y}(I) \times \Delta, \Gamma)$.

What is important to note about these exponentials is that they do not behave in the same way as they would in say simplicial sets. Taking $\mathbf{y}(x_\emptyset)$ as the obvious candidate for an interval object in cubical sets, one is able to give a Kan cubical set Γ for which $\Gamma^{\mathbf{y}(x_\emptyset)}$ is not Kan, as is done in [BCH14]. Since exponentials are Kan whenever both cubical sets are (which derives from the analogous result for Π-types), this reflects the fact that representable cubical sets are not Kan, hence in particular $\mathbf{y}(x_\emptyset)$ is not in $\mathcal{K}$. Moreover, this shows that $\Gamma^{\mathbf{y}(x_\emptyset)}$ cannot be isomorphic to $\mathrm{P}\Gamma$, which is a cubical set if Γ is.
We therefore require an alternative kind of exponential for which this is the case, to which must correspond an alternative notion of product as well, hence we first introduce the latter.

Definition 3.6. For any two cubical sets Γ and Δ we define the *degeneracy product* $\Gamma \otimes \Delta$ to be the cubical set where $(\Gamma \otimes \Delta)(I)$ is the subset of those $(u, v) \in \Gamma(I) \times \Delta(I)$ which are co-degenerate. That is, we have $(u, v) \in (\Gamma \otimes \Delta)(I)$ precisely when there are two disjoint subsets $I_1, I_2 \subseteq I$ along with $u' \in \Gamma(I_1)$ and $v' \in \Delta(I_2)$ such that $u = \Gamma(\imath_{I-I_1})(u')$ and $v = \Delta(\imath_{I-I_2})(v')$.

Hence the degeneracy product is what is called the *separated product* in [Hub15]. While one readily sees that the degeneracy product is indeed a cubical set, $\Gamma \otimes \Delta$ is Kan only if at least one of Γ and Δ is trivial.

However, this will not be an issue as we intend to use the degeneracy product only indirectly in defining the notion of a path exponential in which the degeneracy product takes the place of the usual product, which is what we do next.

Definition 3.7. The *path exponential* of two cubical sets Γ and Δ is the cubical set $[\Delta, \Gamma]$ given by $[\Delta, \Gamma](I) = \mathcal{C}\mathsf{Set}(\mathbf{y}(I) \otimes \Delta, \Gamma)$.

We claim that $[\mathbf{y}(x_\emptyset), \Gamma]$ is isomorphic to $\mathrm{P}\Gamma$, or more generally $[\mathbf{y}(I), \Gamma]$ is isomorphic to $\mathrm{P}^n\Gamma$ where $n = |I|$ and P^n indicates n iterations of taking the path object. For clearly $\mathbf{y}(I) \cong \mathbf{y}(J)$ whenever $|I| = |J|$, and furthermore $\mathbf{y}(I) \otimes \mathbf{y}(J) \cong \mathbf{y}(I \cup J)$ whenever I and J are disjoint, since pairs of functions $f : I \to H$ and $g : J \to H$ co-degenerate on H correspond bijectively with functions $h : I \cup J \to H$. Thus $\mathbf{y}(I) \otimes \mathbf{y}(x_\emptyset) \cong \mathbf{y}(I) \otimes \mathbf{y}(x_I) \cong \mathbf{y}(I, x_I)$, hence we obtain $[\mathbf{y}(x_\emptyset), \Gamma](I) \cong \mathcal{C}\mathsf{Set}(\mathbf{y}(I, x_I), \Gamma) \cong \Gamma(I, x_I) = \mathrm{P}\Gamma(I)$ by the Yoneda lemma, where the general case follows along the same lines. Thus $[\mathbf{y}(I), \Gamma]$ is Kan whenever Γ is, despite that fact that $\mathbf{y}(I)$ is not Kan. We are able to show that for $[\Delta, \Gamma]$ to be Kan it is indeed sufficient that Γ is Kan.

Proposition 3.8. *The path exponential* $[\Delta, \Gamma]$ *is Kan whenever* Γ *is.*

Proof. Let Γ be Kan and Δ be any cubical set, and take S to be any given $+$-shape on I, with $\vec{F}$ an S-open box in $[\Delta, \Gamma]$. This means we have $F_{yb} \in [\Delta, \Gamma](I - y)$ for all $y \in J$ and $F_{x0} \in [\Delta, \Gamma](I - x)$ satisfying adjacency conditions, which come down to $F_{yb_H}((k, z = c), v) = F_{zcH}((k, y = b), v)$ for all H and all $k : I - y, z \to H$. We wish to construct a Kan filler $F = [\Delta, \Gamma] \uparrow_S \vec{F} \in [\Delta, \Gamma](I)$. Before we can do so, we first determine a suitable $F_{x1} \in [\Delta, \Gamma](I - x)$, which we define as follows. If for any $y \in J$ we have $g(y) = b$ then $F_{x1H}(g, v)$ is taken to be $F_{yb_H}((g - y, x = 1), v)$, which is well-defined by and guarantees adjacency. Now assume g is defined on J. Consider $g' = (g, x = x_H) : I \to H, x_H$, which we may use to define a $g'S$-open box $\vec{u}^{Fgv}$ in Γ as follows. For $y \in J$ we let $u^{Fgv}_{g(y)b} = F_{yb_{H,x_H - g(y)}}(g' - y, r_H\Delta((g(y) = b))(v))$, and $u^{Fgv}_{x_H0} = F_{x0H}(g, v)$. We check that the adjacency conditions are satisfied: we have

$$\Gamma(g(z) = c)u^{Fgv}_{g(y)b} = F_{yb_{H,x_H - g(y,z)}}((g' - (y, z), z = c), r_H\Delta(g(y) = b, g(z) = c)(v))$$
$$= F_{zcH,x_H - g(y,z)}((g' - (y, z), y = b), r_H\Delta(g(y) = b, g(z) = c)(v)) = \Gamma(g(y) = b)u_{g(z)c}$$

whenever both $y, z \in J$. If only $y \in J$, then again as required we find

$$\begin{aligned} s_H u_{g(y)b} &= F_{yb_{H-g(y)}}((g - y, x = 0), \Delta(g(y) = b)(v)) \\ &= F_{x0H-g(y)}((g(y) = b)g, \Delta(g(y) = b)(v)) = \Gamma(g(y) = b)u_{x_H0}. \end{aligned}$$

Thus we obtain a Kan filler $\Gamma \uparrow_{g'S} \vec{u}^{Fgv} \in \Gamma(H, x_H)$, which allows us to take $F_{x1H}(g, v) = t_H(\Gamma \uparrow_{g'S} \vec{u}^{Fgv})$. This is represented by the following diagram (though note that only one of the $y \in J$ is shown here).

$$\begin{tikzcd}
F_{y0}(tg' - y, sv) \arrow[r, dotted, "F_{x1H}(g,v)"] & F_{y1}(tg' - y, tv) \\
F_{y0}(sg' - y, sv) \arrow[r, "F_{x0}(g,v)"'] \arrow[u, "F_{y0}(g'-y,rsv)"] \arrow[ur, phantom, "\Gamma\uparrow_{g'S}\vec{u}^{Fgv}"] & F_{y1}(sg' - y, tv) \arrow[u, "F_{y1}(g'-y,rtv)"']
\end{tikzcd}$$

We check that F_{x1} thus defined is natural. Let $h : H \to K$. If $g(y) = b$ for some $y \in J$, then we use the naturality of F_{yb} to calculate as required that

$$\begin{aligned}
\Gamma(h)F_{x1H}(g, v) &= \Gamma(h)F_{yb_H}((g - y, x = 1), v) \\
&= F_{yb_K}((hg - y, x = 1), \Delta(h)(v)) = F_{x1K}(hg, \Delta(h)(v)),
\end{aligned}$$

so suppose g is defined on J. If $h(g(y)) = b$ for some $y \in J$, then

$$\begin{aligned}
\Gamma(h)F_{x1H}(g, v) &= \Gamma(k - g(y))F_{yb_{H-g(y)}}((g - y, x = 1), \Delta(g(y) = b)(v)) \\
&= F_{yb_K}((hg - y, x = 1), \Delta(h)(v)) = F_{x1K}(hg, \Delta(h)(v)).
\end{aligned}$$

Finally, if h is defined on $g(J)$, then naturality follows from the uniformity of the Kan filler involved.

Now to define F from this. If $f(y) = b$ for any $y \in J, x$, then by taking $F_H(f, v) = F_{yb_H}(f - y, v)$ we ensure that F is a Kan filler of $\vec{F}$ as desired. Hence we only need to deal with the case where f is defined on J, x. Consider the $+$-shape $S^f = ((x_H, 1); f(J, x); H, x_H)$. We define an S^f-open box $\vec{w}^{Ffv}$ in Γ by $w^{Ffv}_{f(y)b} = r_H F_{yb_{H-f(y)}}(f - y, \Delta(f(y) = b)(v))$ for $y \in J, x$ and $w^{Ffv}_{x_H 0} = \Gamma(x_{H-f(x)} = f(x))(\Gamma \uparrow_{(f-x)'S} \vec{u}^{\digamma})$, where $\digamma$ is the string $F(f - x)\Delta(f(x) = 0)(v)$. We check that adjacency holds for $y, z \in J, x$:

$$\begin{aligned}
&\Gamma(f(z) = c)w_{f(y)b} = r_H F_{yb_{H-f(y,z)}}((f-y, z, z = c), \Delta(f(y) = b, f(z) = c)(v)) \\
&= r_H F_{zc_{H-f(y,z)}}((f-y, z, y = b)), \Delta(f(y) = b, f(z) = c)(v) = \Gamma(f(y) = b)w_{f(z)c}.
\end{aligned}$$

Furthermore $s_H w_{f(y)b} = F_{yb_{H-f(y)}}(f - y, \Delta(f(y) = b)(v))$, which has to be $\Gamma(f(y) = b)w_{x_H 0} = F_{yb_{H-f(y)}}(f - y, \Delta(\imath_{f(x)}(f(x) = 0, f(y) = b))(v))$. At first glance, these two seem to be different. However, v must be degenerate in $f(x)$, which means $\Delta(\imath_{f(x)}(f(x) = 0))(v) = v$ and so adjacency holds after all. It is here that the crucial difference with the usual exponent lies, for then we would have to involve v in some filler construction in Δ at this point in the argument, hence we would require that Δ is Kan as well.

In this case, we now have $\Gamma \uparrow_{S^F} \vec{w}^{Ffv} \in \Gamma(H, x_H)$, so that we may take $F_H(f,v) = t_H(\Gamma \uparrow_{S^f} \vec{w}^{Ffv})$. This construction is captured in the diagram below, where we only show the variable x.

$$\begin{tikzcd}
F_{x0}(f-x, sv) \arrow[r, dotted, "F_H(f,v)"] & F_{y1}(tg'-y, tv) \\
F_{x0}(f-x, sv) \arrow[u, "rF_{x0}(f-x,sv)"] \arrow[r, "\Gamma\uparrow_{(f-x)'S}\vec{u}^F"'] \arrow[ur, phantom, "\Gamma\uparrow_{S^f}\vec{w}^{Ffv}"] & F_{x0}(f-x, tv) \arrow[u, "rF_{x1}(f-x,tv)"']
\end{tikzcd}$$

In order to complete the proof, we need to verify that F thus defined is indeed natural, and moreover that it meets the uniformity requirements. Let $h : H \to K$. If $f(y) = b$ for some $y \in J, x$, then we find as required that $\Gamma(h)F_H(f,v) = \Gamma(h)F_{yb_H}(f-y,v) = F_{yb_K}(hf-y,v) = F_K(hf,v)$. Suppose therefore that f is defined on J, x. If $h(f(y)) = b$ for some $y \in J, x$, then

$$\begin{aligned}
\Gamma(h)F_H(f,v) &= \Gamma(h - f(y))F_{yb_{H-f(y)}}(f-y, \Delta(f(y) = b)(v)) \\
&= F_{yb_K}(hf-y, \Delta(h)(v)) = F_K(hf, \Delta(h)(v)).
\end{aligned}$$

When h is instead defined on J, x naturality follows from the uniformity of the Kan filler involved. Finally, to prove uniformity of the filler F, suppose $k : I \to K$ is defined on J, x. If $f(k(y)) = b$ for some $y \in J, x$, then $F_H(fk, v) = F_{yb_H}((fk-y), v) = F_{yb_H}((f-k(y))(k-y), v)$ as required; note that we do not need to treat the constructed case of $f(k(x)) = 1$ separately. If instead f is defined on $k(J, x)$, then the open box used in defining $F_H(fk, v)$ is precisely the same as the one which is obtained when carrying out the construction using $k\vec{F}$, hence our construction ensures uniformity as well. $\square$

Thus we may use the path exponential $[\mathbf{y}(x_\emptyset), \Gamma]$ as the path object $\mathrm{P}\Gamma$ when constructing a witness to function extensionality for exponentials in $\mathcal{K}$, which is what we will do next. Supposing Γ and Δ to be Kan cubical sets, we are after maps $\mu : \mathrm{P}(\Gamma^\Delta) \to (\mathrm{P}\Gamma)^\Delta$ and $\nu : (\mathrm{P}\Gamma)^\Delta \to \mathrm{P}(\Gamma^\Delta)$ which act as lower fillers in the diagram below.

$$\begin{tikzcd}
\Gamma^\Delta \arrow[r, "r^\Delta"] \arrow[d, "r"'] & (\mathrm{P}\Gamma)^\Delta \arrow[d, "(s^\Delta, t^\Delta)"] \\
\mathrm{P}(\Gamma^\Delta) \arrow[r, "(s,t)"'] & \Gamma^\Delta \times \Gamma^\Delta
\end{tikzcd}$$

First we determine how these maps are actually defined. For $M \in \Gamma^\Delta(I)$ we find $r_I(M)_H(f,g)_K(h,v) = M_K(hf,v)$, while for $F \in \mathrm{P}(\Gamma^\Delta)(I)$ (considered as path exponential) we have $s_I(F)_H(f,v) = F_H(f, x_\emptyset = 0)_H(\mathsf{id}_H, v)$ and similarly $t_I(F)_H(f,v) = F_H(f, x_\emptyset = 1)_H(\mathsf{id}_H, v)$.

On the other hand we take $r^{\Delta}_I(M)_H(f,v)_K(h,g) = M_K(hf, \Delta(h)(v))$, while for $G \in (\mathrm{P}\Gamma)^{\Delta}$ we use $s^{\Delta}_I(G)_H(f,v) = G_H(f,v)_H(\mathsf{id}_H, x_\emptyset = 0)$ and similarly $t^{\Delta}_I(G)_H(f,v) = G_H(f,v)_H(\mathsf{id}_H, x_\emptyset = 1)$ (again in terms of a path exponential). Now we have function extensionality for exponentials as follows.

Proposition 3.9. *Let Γ and Δ be Kan cubical sets. Then function extensionality for Γ^{Δ} is witnessed by a pair of maps $\mu : \mathrm{P}(\Gamma^{\Delta}) \to (\mathrm{P}\Gamma)^{\Delta}$ and $\nu : (\mathrm{P}\Gamma)^{\Delta} \to \mathrm{P}(\Gamma^{\Delta})$ which commute with the respective source and target maps. Moreover μ commutes strictly and ν commutes up to fibrewise homotopy over $\Gamma^{\Delta} \times \Gamma^{\Delta}$ with the reflexivity maps.*

Proof. We begin by defining the map $\mu : \mathrm{P}(\Gamma^{\Delta}) \to (\mathrm{P}\Gamma)^{\Delta}$ using the identity $\mu_I(F)_H(f,v)_K(h,g) = F_K(hf,g)_K(\mathsf{id}_K, \Delta(h)(v))$, which we may straightforwardly show to be natural. This has to be done on three different levels: μ must be natural, but in order to be well-defined, $\mu_I(F)$ is also required to be natural for any $F \in \mathrm{P}(\Gamma^{\Delta})(I)$, and in turn $\mu_I(F)_H(f,v)$ ought to be natural for any $(f,v) \in (\mathbf{y}(I) \otimes \Delta)(H)$. For the first, if $k : I \to J$, then

$$\begin{aligned}\mu_I(F)_H(fk,v)_K(h,g) &= F_K(hfk,g)_K(\mathsf{id}_K, \Delta(h)(v)) \\ &= \mu_J(\mathrm{P}(\Gamma^{\Delta})(k)(F))_H(f,v)_K(h,g)\end{aligned}$$

as required. Secondly, if $k : H \to H'$, then we find in turn that

$$\begin{aligned}\mu_I(F)_H(f,v)_K(hk,g) &= F_K(hkf,g)_K(\mathsf{id}_K, \Delta(hk)(v)) \\ &= \mu_I(F)_{H'}(kf, \Delta(k)(v))_K(h,g)\end{aligned}$$

again as required. For the last of these, if $k : K \to K'$, then we conclude

$$\begin{aligned}\Gamma(k)\mu_I(F)_H(f,v)_K(h,g) &= \Gamma(k)F_K(hf,g)_K(\mathsf{id}_K, \Delta(h)(v)) \\ &= F_K(hf,g)_{K'}(k, \Delta(kh)(v)) = F_{K'}(khf,kg)_{K'}(\mathsf{id}_{K'}, \Delta(kh)(v)) \\ &= \mu_I(F)_H(f,v)_{K'}(kh,kg).\end{aligned}$$

Now to define $\nu : (\mathrm{P}\Gamma)^{\Delta} \to \mathrm{P}(\Gamma^{\Delta})$ we need to reason in terms of a case distinction. We take $\nu_I(G)_H(f,g)_K(h,v) = G_K(hf,v)_K(\mathsf{id}_K, hg)$ if $hg(x_\emptyset)$ is undefined. If $hg(x_\emptyset)$ is defined, let $\vec{u}$ be the $\mathrm{P}(hg)\dot{S}^{\emptyset}$-open box in Γ given by the data $u_{hg(x_\emptyset)0} = G_K(hf,v)_{K,x_K - hg(x_\emptyset)}(\imath_{x_K}(hg(x_\emptyset) = 0), x_\emptyset = 0)$, along with $u_{hg(x_\emptyset)1} = G_K(hf,v)_{K,x_K - hg(x_\emptyset)}(\imath_{x_K}(hg(x_\emptyset) = 1), x_\emptyset = x_K)$ and finally $u_{x_K 0} = G_K(hf,v)_K(\mathsf{id}_K, x_\emptyset = 0)$. We now take $\nu_I(G)_H(f,g)_K(h,v)$ to be $t_K(\Gamma \uparrow_{\mathrm{P}(hg)\dot{S}^{\emptyset}} \vec{u})$, which corresponds to the diagram below.

$$\begin{tikzcd}
G(hf,v)(s,s) \arrow[r, dotted, "\nu(G)(f,g)(h,v)"] & G(hf,v)(t,t) \\
G(hf,v)(s,s) \arrow[u, "rG(hf,v)(s,s)"] \arrow[r, "G(hf,v)(\mathsf{id},s)"'] \arrow[ur, phantom, "\Gamma\uparrow_{\mathrm{P}(hg)\dot{S}^{\emptyset}}\vec{u}"] & G(hf,v)(t,s) \arrow[u, "G(hf,v)(rs,x)"']
\end{tikzcd}$$

We verify that ν is also a natural transformation in the same way as we did for μ, though now we have to distinguish between certain cases, hence we shall proceed in the reverse order. Let $k : K \to K'$ be any map, for which we must show that $\Gamma(k)\nu_I(G)_H(f,g)_K(h,v) = \nu_I(G)_H(f,g)'_K(kh, \Delta(k)(v))$. If $hg(x_\emptyset)$ is undefined, then we have

$$\begin{aligned}\Gamma(k)G_K(hf,v)_K(\mathsf{id}_K, hg) = G_K(hf,v)_{K'}(k, khg)& \\ = G_{K'}(khf, \Delta(k)(v))_{K'}(\mathsf{id}_{K'}, khg)&\end{aligned}$$

as required. If on the other hand $hg(x_\emptyset)$ is defined yet $khg(x_\emptyset) = b$, then the left hand side is $\Gamma(k - hg(x_\emptyset))G_K(hf,v)_{K-hg(x_\emptyset)}(hg(x_\emptyset = b), x_\emptyset = b)$ by uniformity, which is $G_K(hf,v)_{K'}(k, khg) = G_{K'}(khf, \Delta(k)(v))_{K'}(\mathsf{id}_K, khg)$, i.e. the right hand side. Lastly, if $khg(x_\emptyset)$ is defined, then the equality again holds due to uniformity of the Kan filler. So let k instead be any map $k : H \to H'$, which means we must have $\nu_I(G)_{H'}(kf, kg)_K(h,v) = \nu_I(G)_H(f,g)_K(hk, v)$. We are in either of the two cases depending on whether $hkg(x_\emptyset)$ is defined, and in each of these we can use the naturality of G to prove the desired equalities as we did for μ. The same remark applies to maps $k : I \to J$ and showing that $\nu_I(G)_H(fk, g) = \nu_J((\mathrm{P}\Gamma)^\Delta(k)(G))_H(f,g)$, which concludes this part of the proof.

Having established that μ and ν are well-defined, we now have to demonstrate that they satisfy all the requirements. First, we check that they commute with the source and target maps. For the source maps we find that

$$\begin{aligned}s_I^\Delta \mu_I(F)_H(f,v) = \mu_I(F)_H(f,v)_H(\mathsf{id}_H, x_\emptyset = 0)& \\ = F_H(f, x_\emptyset = 0)_H(\mathsf{id}_H, v) = s_I(F)_H(f,v)&\end{aligned}$$

as well as that

$$\begin{aligned}s_I \nu_I(G)_H(f,v) = \nu_I(G)_H(f, x_\emptyset = 0)_H(\mathsf{id}_H, v)& \\ = G_H(f,v)_H(\mathsf{id}_H, x_\emptyset = 0) = s_I^\Delta(G)_H(f,v).&\end{aligned}$$

The cases for t and t^Δ differ only in the value assigned to $x_\emptyset$, hence we find that μ and ν indeed commute with the source and target map. All that remains then is to show that they behave as described with respect to r and r^Δ. We have indeed that $\mu_I r_I = r_I^\Delta$, since

$$\begin{aligned}\mu_I r_I(M)_H(f,v)_K(h,g) = r_I(M)_K(hf,g)_K(\mathsf{id}_K, \Delta(h)(v))& \\ = M_K(hf, \Delta(h)(v)) = r_I^\Delta(M)_H(f,v)_K(h,g).&\end{aligned}$$

Finally, we can define a homotopy $\ell : r \simeq_{(\Gamma^\Delta \times \Gamma^\Delta)} \nu r^\Delta$, by doing so in each of several cases. Here the crucial observation is that νr^Δ is strictly identical to r in those instances where it is not given by means of a uniform Kan filler, and that when it is given as such the uniform Kan filler involved in fact constitutes part of the required fibrewise homotopy. Whenever $hg(x_\emptyset)$ is undefined we have that

$$\begin{aligned}\nu_I r_I^\Delta(M)_H(f,g)_K(h,v) = r_I^\Delta(M)_K(hf,v)_K(\mathsf{id}_K, hg)& \\ = M_K(hf,v) = r_I(M)_H(f,g)_K(h,v).&\end{aligned}$$

On the other hand, when $hg(x_\emptyset)$ is defined, the open box in Γ which is being filled has $r_I^\Delta(M)_K(hf,v)_K(\mathsf{id}_K, x_\emptyset = 0) = M_K(hf,v)$ as side opposite the part being filled, and the two other sides are reflexivity paths of the form

$$\begin{aligned}&r_I^\Delta(M)(hf,v)_{K,x_K - hg(x_\emptyset)}(\imath_{x_K}(hg(x_\emptyset) = b), k) \\ &\qquad = M_{K,x_K - hg(x_\emptyset)}(\imath_{x_K}(hg(x_\emptyset) = b)hf, r_K\Delta(hg(x_\emptyset) = b)(v)) \\ &\qquad\qquad = r_K M_{K - hg(x_\emptyset)}(hf, \Delta(hg(x_\emptyset) = b)(v)).\end{aligned}$$

Now let us see how we can use this to define $\ell_I(M)_H(f,g)_K(h,v)$. First, we distinguish between whether $f(x_I)$ is defined or not. If $f(x_I) = 0$, we take it to be $r_I(M)_H(f - x_I, g)_K(h,v)$, whereas if $f(x_I) = 1$ we take it to be $\nu_I r_I^\Delta(M)_H(f - x_I, g)_K(h,v)$. Now if $f(x_I)$ is defined, we look at whether $hg(x_\emptyset)$ and $hf(x_I)$ are defined. If neither of these is defined, we take it to be $r_I(M)_{H-f(x_I)}(f - x_I, g)_K(h - f(x_I), v)$, which we also do when $hg(x_\emptyset)$ is defined and $hf(x_I) = 0$; if instead $hf(x_I) = 1$ we take it to be $\nu_I r_I^\Delta(M)_{H-f(x_I)}(f - x_I, g)_K(h - f(x_I), v)$. When it is rather $hg(x_\emptyset)$ which is undefined and $hf(x_I)$ which is defined, we shall take it to be $r_I(M)_{H-f(x_I)}(f - x_I, g)_K(\imath_{hf(x_I)}(h - f(x_I)), v)$. Finally, if both of these are defined, we shall take it to be the preserving composition of the two paths $r_I(M)_{H-f(x_I)}(f - x_I, g)_K(\imath_{hf(x_I)}(h - f(x_I)), v)$ and (notice the adjustment) $\Gamma(x_{K-hf(x_I)} = hf(x_I))(\Gamma\uparrow_{\mathrm{P}((h-f(x_I))g)\dot{S}^\emptyset} \vec{u})$, where $\vec{u}$ is the open box used to obtain $\nu_I r_I^\Delta(M)_{H-f(x_I)}(f - x_I, g)_{K-hf(x_I)}(h - f(x_I), \Delta(hf(x_I) = 1)(v))$. By preserving composition we mean here the composition such that applying $hg(x_\emptyset) = b$ to it yields the same terms which are found by applying it to $r_I(M)_{H-f(x_I)}(f - x_I, g)_K(\imath_{hf(x_I)}(h - f(x_I)), v)$. This composition exists since we have determined above that applying $hg(x_\emptyset) = b$ to the Kan filler $\Gamma(x_{K-hf(x_I)} = hf(x_I))(\Gamma\uparrow_{\mathrm{P}((h-f(x_I))g)\dot{S}^\emptyset} \vec{u})$ yields the right reflexivity paths.

Due to all these different cases, verifying naturality for ℓ is somewhat of a tedious exercise, hence we shall not provide fully explicit calculations at every step. Let $k : K \to K'$ be any map, so that we must have naturality of the form $\Gamma(k)\ell_I(M)_H(f,g)_K(h,v) = \ell_I(M)_H(f,g)_{K'}(kh,\Delta(k)(v))$. Suppose first that $hg(x_\emptyset)$ and $hf(x_I)$ are defined, so that the left hand side is $\Gamma(K)$ applied to the preserving composition. If both $khg(x_\emptyset)$ and $khf(x_I)$ are defined, then the equality immediately holds by the uniformity of the Kan fillers involved; we will not treat the similar cases in which k can be thought of as extending h. If $khg(x_\emptyset)$ is defined yet $khf(x_I) = b$, then the right hand side is $r_I(M)_{H-f(x_I)}(f-x_I,g)_{K'}(kh-f(x_I),\Delta(k)(v))$ or $\nu_I r_I^\Delta(M)_{H-f(x_I)}(f-x_I,g)_{K'}(kh-f(x_I),\Delta(k)(v))$ respectively, which as required are precisely the endpoints of the preserving composition with $\Gamma(k-hf(x_I))$ applied to them. If instead $khg(x_\emptyset) = b$ and $khf(x_I)$ is defined, then the right hand side is

$$
\begin{aligned}
&r_I(M)_{H-f(x_I)}(f-x_I,g)_{K'}(\imath_{khf(x_I)}(kh-f(x_I)),\Delta(k)(v)) = \\
&\Gamma(k-hf(x_I))r_I(M)_{H-f(x_I)}(f-x_I,g)_{K-hg(x_\emptyset)}(\imath_{hf(x_I)}(h-f(x_I)),\Delta(hg(x_\emptyset)=b)(v))
\end{aligned}
$$

which as required are the edges of the preserving composition with respect to $khg(x_\emptyset)$ with $\Gamma(k-hf(x_I))$ applied to them. Now suppose that $hg(x_\emptyset)$ is defined and $hf(x_I) = b$, with k undefined on $hg(x_\emptyset)$. The result is immediate for $b = 0$, hence we look at $b = 1$ to find as required that

$$
\begin{aligned}
&\Gamma(k)\nu_I r_I^\Delta(M)_{H-f(x_I)}(f-x_I,g)_K(h-f(x_I),v) \\
&\qquad = \Gamma(k-hg(x_\emptyset))M_{K-hg(x_\emptyset)}(hf-x_I,\Delta(hg(x_\emptyset)=1)(v)) \\
&= M_{K'}(kh(f-x_I),\Delta(k)(v)) = r_I(M)_{H-f(x_I)}(f-x_I,g)_{K'}(kh,\Delta(k)(v)).
\end{aligned}
$$

Finally, suppose $hg(x_\emptyset) = b$ and $hf(x_I)$ is defined, with k undefined on $hf(x_I)$. Here we find that

$$
\begin{aligned}
&\Gamma(k)r_I(M)_{H-f(x_I)}(f-x_I,g)_K(\imath_{hf(x_I)}(h-f(x_I)),v) \\
&\qquad\qquad = r_I(M)_{H-f(x_I)}(f-x_I,g)_{K'}(kh-f(x_I),\Delta(k)(v))
\end{aligned}
$$

again as required, which concludes this part of the verification. For the next part, let $k : H \to H'$ be any map, so that we must have naturality of the form $\ell_I(M)_{H'}(kf,kg)_K(h,v) = \ell_I(M)_H(f,g)_K(hk,v)$. If $f(x_I)$ is undefined, then the right hand side is as required (by naturality of r and νr^Δ) either $r_I(M)_H(f-x_I,g)_K(hk,v) = r_I(M)_{H'}(kf-x_I,kg)_K(h,v)$, or if $f(x_I = 1)$, $\nu_I r_I^\Delta(M)_H(f-x_I,g)_K(hk,v) = \nu_I r_I^\Delta(M)(kf-x_I,kg)_K(h,v)$. If $f(x_I)$ is defined yet $kf(x_I) = b$, then the right hand side will depend on $hkg(x_\emptyset)$. We find $r_I(M)_{H-f(x_I)}(f-x_I,g)_K(hk-f(x_I),v) = r_I(M)_{H'}(kf-x_I,kg)_K(h,v)$ as required if $hkg(x_\emptyset)$ is undefined; the cases where $hkg(x_\emptyset)$ is defined are (essentially) the same. Should both $f(x_I)$ and $kf(x_I)$ be defined, then the left and right hand sides do not lie in separate cases, and we can again simply appeal to the naturality of the maps involved.

To conclude, let $k : I \to J$ be any map, which means we must have $\ell_I(M)_H(f\mathrm{P}k, g) = \ell_J(\Gamma^\Delta(k)(M))_H(f, g)$. Loosely speaking, applying the map $\Gamma^\Delta(k)(M)$ also has the effect of precomposing k, and discarding x_I is the same as mapping it to x_J and then discarding the latter (which we do in every case). We therefore claim that we again have naturality here owing to the naturality of the constructions used. Thus ℓ is natural and of the required form $\ell : r \simeq_{(\Gamma^\Delta \times \Gamma^\Delta)} \nu r^\Delta$, which means we are done. $\square$

Had we shown instead that the map $(s^\Delta, t^\Delta) : (\mathrm{P}\Gamma)^\Delta \to \mathrm{P}(\Gamma^\Delta)$ is a Kan fibration and that r^Δ is a weak equivalence, then the existence of such μ and ν would have followed on general grounds from Theorem 1.7 since $\mathcal{K}$ is a path category. One of the main reasons for taking our current approach is that it yields an explicit formulation of the proof term ν. Furthermore it provides us with an illustration of one of the uses of path exponentials, which include a reformulation of the uniform Kan condition in terms of sections of open box inclusions, for which we refer to [Hub15]. Nevertheless the corresponding function extensionality statement for Π-types is undeniably more relevant, to which we come to in the final subsection.

3.2.2 Function extensionality for Π-types

Following [BCH14], we define the Π-types in $\mathcal{C}\mathsf{Set}$ as follows.

Definition 3.10. Let $\tau : \Theta \to \Delta$ and $\sigma : \Delta \to \Gamma$ be Kan fibrations of cubical sets. For any $\alpha \in \Gamma(I)$, define Δ_α to be the cubical set given by $\Delta_\alpha(H)$ being the subset of those $(h, v) \in \mathbf{y}(I)(H) \times \Delta(H)$ such that $\Gamma(h)(\alpha) = \tau_H(v)$. The Π-*type* $\Pi_\sigma\Theta$ is then the cubical set given by $(\Pi_\sigma\Theta)(I)_\alpha$ being the subset of those morphisms $F : \Delta_\alpha \to \Theta$ such that $\tau F = \pi_2$; the Kan fibration $\Pi_\sigma\tau : \Pi_\sigma\Theta \to \Gamma$ sends such F to the corresponding α.

One may verify that these Π-types are indeed cubical sets. More importantly, the fibration $\Pi_\sigma\tau : \Pi_\sigma\Theta \to \Gamma$ is Kan since σ and τ are, a result covered in [BCH14] and more extensively in [Hub15]. Thus by Lemma 2.22 we know that $\Pi_\sigma\Theta$ is Kan whenever Γ is, hence this is a valid construction within $\mathcal{K}$. We proceed by setting the stage for function extensionality for Π-types. Let $\tau : \Theta \to \Delta$ and $\sigma : \Delta \to \Gamma$ be Kan fibrations, so that we have the Π-type $\Pi_\sigma\Theta$, and consider the diagram:

$$\begin{tikzcd}
\Pi_\sigma\Theta \arrow[r, "r^\Delta"] \arrow[d, "r"'] & \Pi_\sigma\mathrm{P}_\Delta(\Theta) \arrow[d, "(s^\Delta, t^\Delta)"] \\
\mathrm{P}_\Gamma(\Pi_\sigma\Theta) \arrow[r, "(s,t)"'] & \Pi_\sigma\Theta \times_\Gamma \Pi_\sigma\Theta
\end{tikzcd}$$

Here the maps r, s, t are the usual ones, while for the others we shall define $r^{\Delta}_I(N)_H(f, v) = r_H N_H(f, v)$ for $N \in (\Pi_\sigma\Theta)(I)$; for $G \in (\Pi_\sigma \mathrm{P}_\Delta(\Theta))(I)$ we let $s^{\Delta}_I(G)_H(f, v) = s_H G_H(f, v)$ for $G \in (\Pi_\sigma \mathrm{P}_\Delta(\Theta))(I)$. The map t^Δ is defined similarly with t_H for s_H. We have function extensionality for Π-types in the following sense.

Theorem 3.11. *Let $\tau : \Theta \to \Delta$ and $\sigma : \Delta \to \Gamma$ be Kan fibrations. Then function extensionality for $\Pi_\sigma\Theta$ is witnessed by a pair of morphisms $\mu : \mathrm{P}_\Gamma(\Pi_\sigma\Theta) \to \Pi_\sigma \mathrm{P}_\Delta(\Theta)$ and $\nu : \Pi_\sigma \mathrm{P}_\Delta(\Theta) \to \mathrm{P}_\Gamma(\Pi_\sigma\Theta)$ which commute with the respective source and target maps. Moreover μ commutes strictly and ν commutes up to fibrewise homotopy over $\Pi_\sigma\Theta \times_\Gamma \Pi_\sigma\Theta$ with r and r^Δ.*

Proof. The first of these maps is easily defined in terms of μ_I for any I by $\mu_I(F)_H(f, v) = F_{H,x_H}(\mathrm{P}f, r_H(v))$. Observe first that this term exists since $\mathrm{P}\sigma_H r_H(v) = r_H \Gamma_H(v) = r_H \Gamma(f)(\Pi_\sigma\tau)_I(F) = \Gamma(\mathrm{P}f) r_H (\Pi_\sigma\tau)_I(F)$ and $\tau_{H,x_H}\mu_I(F)_H(f, v) = \tau_{H,x_H} F_{H,x_H}(\mathrm{P}f, r_H(v)) = r_H(v)$ as ought to be the case. Thus we are left to show naturality of $\mu_I(F)$ and of μ itself. The former entails that we have the identity $\Theta(\mathrm{P}h)\mu_I(F)_H(f, v) = \mu_I(F)_K(hf, \Delta(h)(v))$ for any map $h : H \to K$, which is the case since by naturality of F we find that $\Theta(\mathrm{P}k)F_{H,x_H}(\mathrm{P}f, r_H(v)) = F_{K,x_K}(\mathrm{P}(hf), r_K\Delta(h)(v))$. For the latter we must show $(\Pi_\sigma\Theta)(\mathrm{P}h)(F)_H(\mathrm{P}f, r_H(v)) = \mu_I(F)_H(fh, v)$ for any map $h : I \to J$, which is again immediate since $F_H(\mathrm{P}f)\mathrm{P}h, r_H(v)) = F_H(\mathrm{P}(fh), r_H(v))$.

As for the other map, we define $\nu_I(G)_H(f, v)$ by case distinction. Whenever $f(x_I) = b$ we take $\nu_I(G)_H(f, v)$ to be $\Theta(x_H = b)G_H(f - x_I, v)$. If instead $f(x_I)$ is defined, we let $\nu_I(G)_H(f, v)$ be given by $t'_H \tau^{-1}(r_H(v)) \uparrow_{\mathrm{P}f\dot{S}^I} \vec{w}$. Here $\vec{w}$ is a $\mathrm{P}f\dot{S}^I$-open box for τ over $r_H(v)$ consisting of the element $w_{f(x_I)0} = r_H\Theta(x_{H-f(x_I)} = 0)G_{H-f(x_I)}(f - x_I, \Delta(f(x_I) = 0)(v))$ together with $w_{f(x_I)1} = \Theta(x_{H-f(x_I)} = x_H)G_{H-f(x_I)}(f - x_I, \Delta(f(x_I) = 1)(v))$ and as last one $w_{x_H0} = s_H G_H(\imath_{f(x_I)}(f - x_I), v)$. This corresponds to the diagram below.

$$\begin{tikzcd}
sG(f - x_I, sv) \arrow[r, dotted, "\nu(G)(f{,}v)"] & tG(f - x_I, tv) \\
sG(f - x_I, sv) \arrow[u, "rsG(f - x_I{,}sv)"] \arrow[r, "sG(r(f - x_I){,}v)"'] \arrow[ur, phantom, "\tau^{-1}(rv)\uparrow_{\mathrm{P}f\dot{S}^I}\vec{w}"] & sG_(f - x_I, tv) \arrow[u, "G(f - x_I{,}tv)"']
\end{tikzcd}$$

We begin by showing that the terms involved are well-defined. If $f(x_I)$ is undefined then $\sigma_H(v) = \Gamma(f) r_I(\Pi_\sigma(\tau p))_I(G) = \Gamma(f - x_I)(\Pi_\sigma(\tau p))_I(G)$, hence $G_H(f - x_I, v)$ exists in these cases, and furthermore we have as required that $\tau_H\Theta(x_H = b)G_H(f - x_I, v) = (\mathrm{P}_\Delta(\Theta))_H G_H(f - x_I, v) = v$.

If instead $f(x_I) = b$, then we find again that

$$\begin{aligned}\sigma_{H-f(x_I)}\Delta(f(x_I) = b)(v) &= \Gamma(f(x_I) = b)\sigma_H(v) \\ &= \Gamma((f(x_I) = b)f)r_I(\Pi_\sigma(\tau p))_I(G) = \Gamma(f - x_I)(\Pi_\sigma(\tau p))_I(G)\end{aligned}$$

and $\tau_H \nu_I(G)_H(f, v) = v$ by construction, so everything is well-defined. Now we must show naturality of $\nu_I(G)$ and of ν, for which we consider some case distinctions. First we take any map $h : H \to K$ and show that $\Theta(h)\nu_I(G)_H(f, v) = \nu_I(G)_K(hf, \Delta(h)(v))$. If $f(x_I)$ is undefined we see that

$$\begin{aligned}\Theta(h)\Theta(x_H = b)G_H(f - x_I, v) &= \Theta(x_K = b)\Theta(\mathrm{P}h)G_H(f - x_I, v) \\ &= \Theta(x_K = b)G_K(hf - x_I, \Delta(h)(v)).\end{aligned}$$

When $f(x_I)$ is defined yet $hf(x_I) = b$, then by uniformity we have

$$\begin{aligned}&\Theta(h - f(x_I))\Theta(x_{H-f(x_I)} = b)G_{H-f(x_I)}(f - x_I, \Delta(f(x_I) = b)(v)) = \\ &\Theta(x_K = b)\Theta(h - f(x_I), x_{H-f(x_I)} = x_K)G_{H-f(x_I)}(f - x_I, \Delta(f(x_I) = b)(v)) \\ &\qquad = \Theta(x_K = b)G_K(hf - x_I, \Delta(h)(v)).\end{aligned}$$

If $hf(x_I)$ is defined as well, the equality follows once more by uniformity. Now let $h : I \to J$ be any map, so that we must show naturality of the form $\nu_I(G)_H(f\mathrm{P}h, v) = \nu_J((\Pi_\sigma \mathrm{P}_\Delta(\Theta))(h)(G))_H(f, v)$. As in earlier proofs, this can be done in each of the cases using the naturality of G in these arguments, but we will not carry out this verification explicitly.

Thus we have come to the part involving the respective path object maps. For completeness' sake, observe that $(\tau p)_H r_I^\Delta(N)_H(f, v) = \tau_H N_H(f, v) = v$ and $\tau_H s_H G_H(f, v) = (\tau p)_H G_H(f, v) = v$ as required. That μ and ν commute with the respective source and target maps follows from

$$\begin{aligned}s_I^\Delta \mu_I(F)_H(f, v) = s'_H \mu_I(F)_H(f, v) &= s'_H F_{H,x_H}(\mathrm{P}f, r_H(v)) \\ &= F_H((f, x_I = 0), v) = s_I(F)_H(f, v)\end{aligned}$$

and $s_I \nu_I(G)_H(f, v) = \nu_I(G)_H((f, x_I = 0), v) = s'_H G_H(f, v) = s_I^\Delta(G)_H(f, v)$. The case for t^Δ again only differs in that 0 is replaced with 1 throughout. As for the reflexivity maps, we find that

$$\begin{aligned}\mu_I r_I(N)_H(f, v) = r_I(N)_{H,x_H}(\mathrm{P}f), r_H(v)) &= N_{H,x_H}(\mathrm{P}f)\imath_{x_I}, r_H(v)) \\ &= N_{H,x_H}(\imath_{x_H} f, r_H(v)) = r_I^\Delta(N)_H(f, v)\end{aligned}$$

hence we have strict equality here. For the other part we wish to show that we have some homotopy $\ell : r \simeq_{(\Pi_\sigma \Theta \times_\Gamma \Pi_\sigma \Theta)} \nu r^\Delta$. We define $\ell_I(N)_H(f, v)$ as

follows. First, if $f(y_I) = 0$ then we take it to be $r_I(N)_H(f - y_I, v)$, whereas if $f(y_I) = 1$ we take it to be $\nu_I r_I^{\Delta}(N)_H(f - y_I, v)$. Now if $f(y_I)$ is defined, we must look at whether $f(x_I)$ is defined. In those cases where it is not, we define $\ell_I(N)_H(f, v)$ as $(\mathrm{P}rr)_I(N)_H(f, v)$. When $f(x_I)$ is also defined, it is rather the preserving composition (cf. the proof of Proposition 3.9) of $(\mathrm{P}rr)_I(N)_H(f, v)$ and $\Theta(x_{H-f(x_I),f(y_I)} = f(y_I))(\tau^{-1}(r_{H-f(y_I)}\Delta(f(y_I) = 1)(v)) \uparrow_{\mathrm{P}(f-y_I)\dot{S}^I} \vec{w})$. Here $\tau^{-1}(r_{H-f(y_I)}\Delta(f(y_I) = 1)(v)) \uparrow_{\mathrm{P}(f-y_I)\dot{S}^I} \vec{w}$ is the Kan filler involved in the definition of $\nu_I r_I^{\Delta}(N)_{H-f(y_I)}(f - y_I, \Delta(f(y_I) = 1)(v))$.
One last time we shall have to show that this is natural, namely in (f, v) and in N. For the former, let $h : H \to K$ be any map so that we must have $\Theta(h)\ell_I(N)_H(f, v) = \ell_I(N)_K(hf, \Delta(h)(v))$, and suppose first that $f(y_I) = b$ is undefined. Then $hf(y_I) = b$ as well, hence both sides are in the same case, and naturality obtains through that of r and νr^{Δ}. Suppose next that $f(y_I)$ is defined, yet $hf(y_I) = 0$. If $f(x_I)$ is undefined, then the left hand side is

$$\begin{aligned}\Theta(h)(\mathrm{P}rr)_I(N)_H(f, v) = r_I(N)_K((h - f(y_I))(f - y_I), \Delta(h)(v))\\ = r_I(N)_K(hf - y_I, \Delta(h)(v)).\end{aligned}$$

When $f(x_I)$ is also defined, the left hand side is

$$\begin{aligned}\Theta(h - f(y_I))r_I(N)_{H-f(y_I)}(f - y_I, \Delta(f(y_I) = 0)(v))\\ = r_I(N)_K(hf - y_I, \Delta(h)(v))\end{aligned}$$

as well. If instead $hf(y_I) = 1$ and f is undefined on x_I, then so is $hf - y_I$, hence by the definition of ν we again have

$$\begin{aligned}\nu_I r_I^{\Delta}(N)_K(hf - y_I, \Delta(h)(v)) = N_K(hf - x_I, y_I, \Delta(h)(v))\\ = (\mathrm{P}rr)_I(N)_K(hf, \Delta(h)(v)).\end{aligned}$$

Should $f(x_I)$ rather be defined, then we find that the left hand side is $\Theta(h - f(y_I))\nu_I r_I^{\Delta}(N)_{H-f(y_I)}(f - y_I, \Delta(f(y_I) = 1)(v))$, which by naturality is indeed $\nu_I r_I^{\Delta}(N)_K(hf - y_I, \Delta(h)(v))$. Now suppose $hf(y_I)$ is defined. If $f(x_I)$ is undefined both sides are once more in the same case, hence we may assume that $f(x_I)$ is defined yet $hf(x_I) = b$. We find as required that the left hand side is, by the preserving composition,

$$\begin{aligned}\Theta(h - f(x_I))N_{H-f(x_I)}((f - x_I)\imath_{y_I}, \Delta(f(x_I) = b)(v))\\ = N_K((hf - x_I)\imath_{y_I}, \Delta(h)(v)) = (\mathrm{P}rr)_I(N)_K(hf, \Delta(h)(v)).\end{aligned}$$

This leaves us to show naturality in N, that is, for any $h : I \to J$ we have $\ell_J((\Pi_\sigma\Theta)(h)(N))_H(f, v) = \ell_I(N)_H(f\mathrm{PP}h, v)$. As before, h cannot separate the left and right hand sides into different cases, hence for each of these we can simply resolve naturality using that of the maps involved, which concludes the proof that ℓ is our desired homotopy. □

As remarked before, the novelty of this theorem lies not so much in the fact that function extensionality holds for Π-types in cubical sets, as this follows from general considerations, such as from Theorem 1.7 if we show that r^{Δ} and (s^{Δ}, t^{Δ}) are a weak equivalence and a fibration respectively. Neither is the construction of the proof term ν itself a completely original one, as it is essentially a translation of the type-theoretic one in Theorem 3.20 of [Hub15]. The main contribution of this theorem is rather that we have shown this proof term to be very well-behaved with respect to the path structure involved and have sketched the background theory which explains why such a witness to function extensionality should exist at all. With this result we have come to the end of this chapter and of the present work.

Conclusions

In this work we set out to study the model of type theory in cubical sets from the perspective of a path category, thereby increasing the understanding of the former and adding to the results known for both. We have succeeded in this regard, with the establishing of Theorem 2.28 being our main achievement, along with other results such as Theorem 1.18 and Theorem 3.11 which are less original though not less worthwhile. Especially in Chapter 2 we have tread on previously uncovered grounds, as identifying a path category with the category of cubical sets forced us to consider the cubical set model in an alternative way.

Doubtlessly there are multiple interesting avenues left to explore. There is for instance the direction taken by [GS15], which explores the role of the uniformity condition in the construction of a model of type theory. Other possibilities for further research include working with variations of cubical sets, such as cubical sets with connections (as done by Thiery Coquand), or Steve Awodey's cartesian cubical sets. A topic which we would have liked but were unable to treat adequately is that of weak function extensionality in the context of path categories, specifically whether this is again in some way equivalent to function extensionality in the usual sense (as per the result from homotopy type theory). The author hopes that despite this remaining host of unanswered questions, his humble work has been a somewhat valuable contribution to this all.

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