Switching from codimension 2 bifurcations of equilibria in delay differential equations

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June 30, 2016

Maikel Bosschaert: *Switching from codimension 2 bifurcations of equilibria in Delay Differential Equations, June 30, 2016*

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1. Introduction

Many phenomena in physics, biology and chemistry are modeled by Ordinary Differential Equations (ODEs). A vital tool in gaining insight in such systems is analyzing stability of the equilibria. This is done by analyzing the eigenvalues of the linearization of the ODE at an equilibrium. When ODEs depend on certain (control) parameters the location of the equilibrium and eigenvalues may change. In the event of one or more eigenvalues crossing the imaginary axis, stability of the equilibrium changes and a bifurcation may occur. For example, when two complex conjugate eigenvalues with non-zero imaginary part cross the imaginary axis, an Andronov-Hopf bifurcation occurs, in which a limit cycle spawns off the equilibrium. It is possible to reduce the ODE to a simple form called the normal form by applying certain coordinate transformations. By inspecting the normal form coefficients at criticality, i.e. at the parameter value where the bifurcation occurs, one can predict the nature of a particular bifurcation occurring. In the example of the Hopf bifurcation, this means that one can predict whether the periodic orbit is stable or unstable and thus give information about the nature of the Hopf bifurcation (sub- or supercritical).

The Hopf bifurcation is an example of a codimension one bifurcation, i.e. a bifurcation that can be encountered in generic ODEs by varying one parameter. In bifurcations with codimension two more interesting phenomena can arise. In particular the Bogdanov-Takens bifurcation is of great interest. It happens when at criticality we encounter a double eigenvalue zero. One typically needs two parameters in order for this bifurcation to occur. For example, by varying one parameter we may encounter a Hopf point with two complex conjugate eigenvalues having zero real part. Then, by also varying a second parameter, an entire branch of Hopf points can be obtained. The two purely imaginary eigenvalues in the imaginary axis can meet at the origin. Under certain genericity conditions we expect to find, for nearby parameter values, a fold, a Hopf, and a homoclinic bifurcation curve. One of these conditions depends on the normal form coefficients at the criticality. The normal form coefficients at the criticality give information about the type of Hopf and homoclinic bifurcation curves occurring. However, no information about where to expect these bifurcation curves is given. This has recently been done for the Bogdanov-Takens bifurcation in the finite dimensional case, i.e. for ODEs, using parameter-dependent normal forms [32, 1, 36].

A general form of a first order autonomous ODE, for $x(t) \in \mathbb{R}^n$ is

$$\frac{d}{dt}x(t) = F(x(t), \alpha), \tag{ODE}$$

where $F : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a function of the state variable $x \in \mathbb{R}^n$ and the parameter

$\alpha \in \mathbb{R}^p$. Here all occurrences $x(\cdot)$ are assumed to appear simultaneously. This may not always be the case. For example, when the birth rate of predators is affected by prior levels of predators or prey rather than by only the current levels in a predator-prey model. In these type of systems, called delay differential equations (DDEs), the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. A general form of an autonomous DDE for $x(t) \in \mathbb{R}^n$ is

$$\frac{d}{dt}x(t) = f(x_t, \alpha), \tag{DDE}$$

where

$$x_t(\theta) = x(t+\theta), \qquad \theta \in [-h, 0],$$

represents the solution in the past and $\alpha \in \mathbb{R}^p$ is the parameter. In this equation, f is a functional operator from $C(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R}^p \to \mathbb{R}^n$ and $h > 0$ is assumed to be finite.

The stability of a steady-state, i.e. a constant solution φ_0 at the parameter α_0 such that $f(\varphi_0, \alpha_0) = 0$, is now given by the eigenvalues of the generator of the semigroup generated by the linear part of the DDE. These eigenvalues can be obtained from a finite dimensional *characteristic matrix* leading to the *characteristic equation* which is of exponential polynomial nature, giving rise to infinitely many eigenvalues. It follows that numerical methods are in general necessary to analyze the characteristic equation, see for example [42, Chapter 2] or [14, Chapter XI].

In contrast to ODEs, the state space of DDEs is infinite dimensional. To deal with this situation rigorously a new mathematical framework has been developed which is called the *perturbation theory of dual semigroups*, also known as *sun-star calculus*. Using this framework the existence of a finite dimensional smooth center manifold can be established, making it is possible to 'lift' the normalization method for local bifurcations of ODEs presented in [33] to the infinite dimensional setting of DDEs. One of the advantages using this normalization technique is that the reduction to the center manifold, and calculation of the critical normal form coefficients can be done simultaneously, using the so-called *homological equation*. This 'lifting' has been done first by Sebastiaan Janssens in his Master Thesis [29] for all five codimensional two bifurcations:

- Cusp
- Bogdanov-Takens
- Bautin (Generalized Hopf)
- Fold-Hopf
- Hopf-Hopf

The obtained critical normal form expressions are remarkably similar to those occurring in ODEs.

The derived critical normal form coefficients have been numerically evaluated in [29] for two models: a Van der Pol oscillator with delayed feedback [31], in which a transcritical Bogdanov-Takens bifurcation is encountered, and a neural mass model from [48] and [47], in which Hopf-Hopf, fold-Hopf and Bautin bifurcation points occur. It has been shown how to compute the critical normal forms coefficients systematically using the symbolic and numerical computer algebra system Maple.

This lays a foundation for the implementation of automatic calculation of critical normal form coefficients into a numerical continuation software package for DDEs. This has been done for Hopf points on steady-state and generalized-Hopf, fold-Hopf and Hopf-Hopf points on Hopf curves, in the Master Thesis of Bram Wage [49]. Instead of writing a standing alone, this functionality was added to `DDE-BifTool`, a `Matlab` package for numerical bifurcation and stability analysis of delay differential equations with several fixed discrete and/or state-dependent delays developed at the University of Leuven [17]. As a separate part of this thesis, the functionality to detect, locate and compute critical normal form coefficients has been added in the following situations:

- Fold encountered along steady-state curves
- Cusp, Bogdanov-Takens and fold-Hopf encountered along fold curves
- Bogdanov-Takens encountered along Hopf curves,

thereby capturing all cases in which local codimension one and two bifurcations generally occur.

It turns out that detection and location of Bogdanov-Takens bifurcation involve more work compared with the other codimension one and two bifurcations, see Chapter 5. For location of Bogdanov-Takens points detected on fold branches we cannot use the bisection method, but instead have to apply Newton to a special defining system. This defining system is derived following the methods developed in [3] and [21] for locating Bogdanov-Takens points in ODEs. An additional advantage is that the resulting defining system allows the continuation of Bogdanov-Takens points and detection of triple zero singularities.

The next step is to perform a parameter-dependent center manifold reduction near codimension 2 bifurcations. Such reduction is necessary for deriving asymptotics of codimension 1 non-equilibrium (e.g. saddle homoclinic orbits and non-hyperbolic cycles), emanating from some codimension 2 local bifurcations. For this we need a generalization of the parameter-dependent center manifold currently available for DDEs, see [14, Chapter IX.9]. This generalization should not impose the constraint that the steady-state remains fixed under variation of parameters. Then the Bogdanov-Takens and fold-Hopf bifurcations can be treated as well.

We will perform the parameter-dependent manifold reduction and normalization near the generic and transcritical Bogdanov-Takens, generalized Hopf, fold-Hopf, Hopf-transcritical and Hopf-Hopf bifurcations. This will allow us to initialize the continuation of the saddle homoclinic orbits emanating from a (transcritical) Bogdanov-Takens point and codimension 1 cycle bifurcations emanating from the generalized Hopf, zero-Hopf, Hopf-transcritical and Hopf-Hopf bifurcations. The homoclinic orbits near the Bogdanov-Takens bifurcation are approximated with the second-order homoclinic predictor derived in [32, 1, 36]. The codimension 1 cycle bifurcations are approximated using the predictors from [37].

1.1. Structure of this thesis

In Chapters 2, 3 and 4 we will review the sun-star calculus needed for the rest of the thesis. It contains three key elements:

- The functional analytic framework
- A center manifold
- Normal form calculation

Those already familiar with the theory can safely skip Chapters 2 and 3, except for Section 3.6, where a generalization of the parameter-dependent center manifold Theorem for DDEs is given. For those totally unfamiliar with sun-star calculus, this review might be too concise for the first read. Therefore, the reader may consult [14] on which Chapters 2 and 3 mostly rely.

Using the results obtained in Section 3.6 we describe the computation of the normal form coefficients on the parameter-dependent center manifold in the infinite dimensional setting in Chapter 4.

In Chapter 5 we turn our attention to detecting and locating Bogdanov-Takens points in DDEs with multiple delays. Here we will derive test functions for Bogdanov-Takens points on fold and Hopf curves. Contrary to all other codimension 2 bifurcations, we cannot use the bisection method to locate Bogdanov-Takens points on fold curves. Therefore, we derive a defining system, for which we prove regularity. Then, using Newton's iteration method, we can accurately locate Bogdanov-Takens points detected on either fold or Hopf curves.

In Chapter 6 we will apply the method described in Chapter 4 to the generic and transcritical Bogdanov-Takens, generalized Hopf, fold-Hopf, Hopf-transcritical and Hopf-Hopf bifurcations. We will explicitly calculate the normal form coefficients necessary for the predictors.

Chapter 7 describes how to initialize the continuation of the homoclinic orbits near the generic and transcritical Bogdanov-Takens bifurcations and the limit cycles near the generalized Hopf.

In Chapter 8 of this thesis we will illustrate the homoclinic predictors for the generic and transcritical Bogdanov-Takens and the nonhyperbolic cycle predictors on various models.

After some final comments are made there are four appendices. In the first Appendix A we perform the center manifold reduction combined with normalization for generic and transcritical Bogdanov-Takens bifurcation in ODE. Although the generic Bogdanov-Takens bifurcation has already been treated in [32, 1, 36] an alternative derivation is presented, one which is more suitable for the DDE case.

Appendices B and C describe the normal forms and predictors. In the first Section of Appendix C we summarize the method used in [32, 1, 36] to obtain a second-order predictor for the homoclinic orbit near a generic Bogdanov-Takens bifurcation. In the second Section we will show that the same procedure can be applied to obtain the second-order predictor for the homoclinic orbits near a transcritical Bogdanov-Takens bifurcation. In the remaining sections the asymptotics for the codimension 1 cycle bifurcations emanating from generalized Hopf, fold-Hopf, Hopf-transcritical and Hopf-Hopf bifurcations are presented.

Lastly, In Appendix D we discuss a subtle nonuniqueness problem with the second-order predictor for the generic Bogdanov-Takens bifurcation.

1.2. New results

Here is a short list of novelties this thesis brings

- Robust methods for detection of Bogdanov-Takens points on fold and Hopf curves in DDEs.
- Special defining system to locate and continue Bogdanov-Takens point in DDEs with multiple delays.
- General method to derive normal form coefficients on the parameter-dependent center manifold in DDEs. This makes it possible to obtain asymptotics of codimension 1 global bifurcations involving cycles, emanating from codimension 2 local bifurcations.
- Derivation of the normal form coefficients on the parameter-dependent center manifold for the generic and transcritical Bogdanov-Takens, generalized Hopf, fold-Hopf, Hopf-transcritical and Hopf-Hopf bifurcations, using the functional analytic framework of sun-star calculus and systematically tracing all freedom in solutions of singular linear equations.
- A second-order approximation of the homoclinic solutions near the generic and transcritical Bogdanov-Takens bifurcations in DDEs.
- Predictors for the codimension 1 cycle bifurcations emanating from generalized Hopf, fold-Hopf, Hopf-transcritical and Hopf-Hopf bifurcations in DDEs.
- Improved derivation of coefficients of the smooth normal form for the generic Bogdanov-Takens bifurcation in ODEs.
- Actual implementation of all developed methods into the standard public software `DDE-BifTool`.

1.3. Acknowledgments

I would first like to thank my supervisor Prof. Dr. Yuri Kuznetsov. Without the weekly appointments (which could easily extend up to two hours) and many email correspondences, this thesis would not have been completed. Every time when I was working on problems that seemed to lead to a dead end, Yuri could point me into new directions in which solutions could be found.

I am thankful to Sebastiaan Janssens, Hil Meijer, Sid Visser and Jan Sieber for comments, suggestions, and discussions that led to text improvements.

I would also like to thank Prof. Dr. Odo Diekmann for helping with proving a general center manifold theorem for parameterized delay differential equations.

Finally, I must express my profound gratitude to my mother Marjon van Uden and to my girlfriend Veerle van Harten for providing me with unfailing support and continuous encouragement throughout my years of study and through the process of researching and writing this thesis. This accomplishment would not have been possible without them.

2. Sun-star calculus

In this Chapter we will review the functional analytic framework of sun-star calculus following [14]. We will start in the first two sections to show the need of the framework through the variation-of-constants formula and bounded perturbations of the semigroup generator. In Section 2.3 we describe the framework for a general (non-reflexive) Banach space X on which a C_0-semigroup is defined. Then in the next Section 2.4, we show how this framework will look like for $X = C([-h, 0], \mathbb{R})$, $h > 0$ and the shift semigroup

$$(T_0(t)\varphi)(\theta) = \begin{cases} \varphi(t+\theta), & -h \leq t+\theta \leq 0, \\ \varphi(0), & t+\theta \geq 0, \end{cases}$$

generated by the trivial DDE

$$\begin{cases} \dot{x}(t) = 0, & t > 0, \\ x(\theta) = \varphi(\theta), & -h \leq \theta \leq 0. \end{cases}$$

In Section 2.5 we turn our attention to general linear DDEs. There we will see how the framework enables us to construct a semigroup for such DDE using the shift semigroup.

2.1. Variation-of-constant formula

The variation-of-constants formula plays an important role in the study of the stability, existence of bounded solutions and the asymptotic behavior of non-linear ODEs and partial differential equations PDEs. In particular it can be used in proving a center manifold theorem, which plays a key role in the description and understanding of the dynamics of nonlinear systems and their bifurcations. The variation-of-constants formula is well known for the finite dimensional semi-linear ordinary differential equation

$$\begin{cases} \dot{u}(t) = Au(t) + G(u), & u \in \mathbb{R}^n, \\ u(0) = u_0, \end{cases}$$

and gives the integral equation

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}G(u(s))\,ds,$$

where e^{tA} is the solution operator for the homogeneous system

$$\dot{u}(t) = Au(t)$$

and A a linear operator on $\mathbb{R}^n$. For PDEs or ODEs defined on infinite dimensional Banach spaces we need the notion of a C_0-semigroup.

Definition 2.1. Let X be a complex Banach space and let, for each $t \geq 0$, $T(t) : X \to X$ be a bounded linear operator. Then the family $\{T(t)\}_{t\geq 0}$ is called a *strongly continuous semigroup*, or a C_0-semigroup, if the following three properties hold:

(i) $T(0) = I$ (the identity),

(ii) $T(t)T(s) = T(t+s)$for $t, s \geq 0$,

(iii) for any $\varphi \in X$, $\|T(t)\varphi - \varphi\| \to 0$ as $t \downarrow 0$.

The first two properties are algebraic, and state that T is a representation of the semigroup $(\mathbb{R}_+, +)$; the last is topological, and means that the map T is continuous in the strong operator topology.

One can associate with such a semigroup the abstract differential equation

$$\frac{d}{dt}(T(t)\varphi) = A(T(t)\varphi) \tag{2.1}$$

where the infinitesimal generator A is defined by

$$A\,\varphi = \lim_{t\downarrow 0} \frac{1}{t}\,(T(t)\varphi - \varphi) \tag{2.2}$$

whenever the limit exists. Note that the infinitesimal generator A only makes sense if T is strongly continuous.

Definition 2.2. The domain of $A, \mathcal{D}(A)$, is the set of $\varphi \in X$ for which this limit does exist; $\mathcal{D}(A)$ is a linear subspace and A is linear on this domain. The operator A is closed, although not necessarily bounded, and the domain is dense in X.

Obviously, the above e^{tA} is a C_0-semigroup.

The variation-of-constants formula for DDEs has for some time been a puzzling part of the theory. Let L be a continuous linear operator from the state space $X = C([-h, 0], \mathbb{R}^n)$ into $\mathbb{R}^n$ and $h > 0$ some constant. Consider the DDE

$$\begin{cases} \dot{x}(t) = Lx_t + g(x_t), & t \geq 0, \\ x(\theta) = \varphi, & -h \leq \theta \leq 0, \end{cases} \tag{2.3}$$

where the non-linearity $g : X \to \mathbb{R}^n$ satisfies $g(0) = 0$ and has a continuous Fréchet derivative such that $Dg(0) = 0$. Then according to [25] $x = x(\cdot;\varphi)$ is a solution to (2.3) if and only if x satisfies

$$x_t = T(t)\varphi + \int_0^t T(t-s)X_0 g(x_s)\,ds \tag{2.4}$$

in which X_0 is the matrix-valued function defined by

$$X_0(\theta) = \begin{cases} 0, & h \leq \theta < 0, \\ I, & \theta = 0, \end{cases}$$

where I is the identity matrix in $\mathbb{R}^n$ and $T(t)$ is the semigroup associated with the homogeneous equation

$$\dot{x}(t) = Lx_t \tag{2.5}$$

on the state space X. The variation-of-constants formula (2.4) indicates that $T(t)$ is evaluated at X_0 although this function is not continuous and so is not in the state space X where the semigroup is defined. We refer to equation (2.5) as the unperturbed system and equation (2.3) as the perturbed system. Then $g(x_t)$ perturbs the generator of the semigroup and causes the right hand side of (2.4) to leave the state space X. In [8] the sun-star calculus framework was constructed framework a very natural generalization of the notion of a bounded perturbation of the generator and lead to a new version of the variation-of-constants formula, namely the formula (3.2) in Chapter 3.

2.2. Shift semigroup I

Another reason to introduce the sun-star calculus is a problem with the domain of the semigroup generator. First consider the simple DDE

$$\dot{x}(t) = \alpha x(t - \tau) \quad \text{for } t > 0, \tag{2.6}$$

where α and τ are parameters with $\tau > 0$ and $x \in \mathbb{R}$. In order for (2.6) to make sense we have to define a history function φ defined on the interval $[-\tau, 0]$. Then we can solve the equation for the interval $[0, \tau]$, i.e.

$$x(t) = x(0) + \int_0^t \frac{d}{dt}x(s)\, ds = \varphi(0) + \int_0^t \varphi(s - \tau)\, ds, \qquad 0 \leq t \leq \tau.$$

By shifting (translating) the result back to the interval $[-\tau, 0]$ we can once more integrate the equation. Repeating this process n times and 'gluing' the solutions together gives a solution on the interval $[0, n\tau]$. In this abstract view of solving the differential equation (2.6) we can separate two different processes: *extending* and *translating*. The first ingredient is specific for a particular equation, but the second is the same for all delay equations.

Motivated by above we will consider the trivial DDE

$$\begin{cases} \dot{x}(t) = 0, & t > 0, \\ x(\theta) = \varphi(\theta), & -h \leq \theta \leq 0, \end{cases} \tag{2.7}$$

where the extension is as simple as possible. We assume that the initial condition given by the function φ is an element of state-space $X = C([-h, 0], \mathbb{R}^n)$. The solution to the

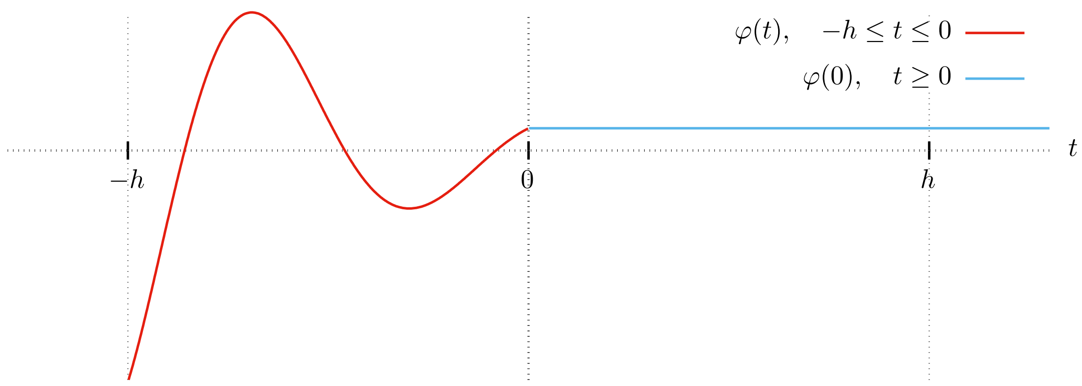


Figure 2.1.: Graph of the solution $x(t) \in \mathbb{R}^n$ to the trivial DDE defined in (2.7).

trivial DDE is given by

$$x(t) = \begin{cases} \varphi(t), & -h \leq t \leq 0, \\ \varphi(0), & t \geq 0, \end{cases}$$

see Figure 2.1.

In order to distinguish the time at which we inspect the state from the variable passing through the interval $[-h, 0]$ we shall write

$$x_t(\theta) = x(t+\theta), \qquad t \geq 0, \quad \text{and} \quad -h \leq \theta \leq 0. \tag{2.8}$$

With this notation, $x_t \in X$ is the state at time t. For each $t \geq 0$

$$(T_0(t)\varphi)(\theta) = \begin{cases} \varphi(t+\theta), & -h \leq t+\theta \leq 0, \\ \varphi(0), & t+\theta \geq 0 \end{cases} \tag{2.9}$$

defines a bounded linear operator $T_0(t) : X \longrightarrow X$. The operator $T_0(t)$ maps the initial state φ at time zero onto the state x_t at time t.

The family $\{T_0(t)\}_{t\geq 0}$ of operators defined in equation (2.9) clearly satisfies the conditions of Definition 2.1. The infinitesimal generator of $\{T_0(t)\}_{t\geq 0}$ can be explicitly calculated and is given in the following Lemma.

Lemma 2.3. [14, Chapter II, Lemma 2.1]. *The infinitesimal generator of T_0 is given by*

$$\mathcal{D}(A_0) = \{\varphi \in X : \dot{\varphi} \in C([-h;0], \mathbb{C}), \dot{\varphi}(0) = 0\}, \qquad A_0\varphi = \dot{\varphi}.$$

Defining $u(t,\theta) = (T(t)\varphi)(\theta)$, we can write (2.1) as the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \theta},$$

which describes translation with unit speed, and thus incorporates the shifting part. However the extension rule is incorporated in $\mathcal{D}(A_0)$ in the form of the condition $\dot{\varphi}(0) = 0$.

If we perturb the trivial equation (2.7), this rule will change together with the domain of definition of the generator. This will lead to unpleasant technical complications if we want to relate solutions of linear and nonlinear equations to each other by means of the variation-of-constants formula.

The sun-star calculus described in the next Section will overcome these difficulties. It will allow us to derive a homoclinic predictor near a Bogdanov-Takens bifurcation as in [32, 1, 36].

The main idea is to embed the space X into a bigger space $X^{\odot\star}$. It will turn out that one has a notion of generator on the space $X^{\odot\star}$ as well, for which shifting and extension are both described by the action of the operator, whereas the domain is determined by the translation only. In the next Section we will briefly review the process of constructing the space $X^{\odot\star}$.

2.3. Sun-star calculus: The abstract setting

In the following there will be five different spaces involved, therefore it can be useful to consult the diagram in Figure 2.2, where the relations between these spaces are illustrated.

Definition 2.4. Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. A linear operator from X into $\mathbb{F}$ is called a *linear functional*. Then the collection on all continuous (bounded) linear functionals on X is called the *dual* space of X. The notation $X^\star$ is used to denote the dual space of X, so that $X^\star$ is shorthand for $B(X, \mathbb{F})$. We denote the pairing of a functional $x^\star$ from the dual space $X^\star$ and an element x of X with the bracket: $x^\star(x) = \langle x^\star, x\rangle$.

Definition 2.5. The *adjoint operator* $T^\star$ of an operator T in X is an operator such that

$$\langle x^\star, Tx\rangle = \langle T^\star x^\star, x\rangle$$

for all $x^\star \in X^\star$ and $x \in X$.

Let A be the infinitesimal generator of the C_0-semigroup $\{T(t)\}_{t\geq 0}$ defined on a Banach space X. The adjoint semigroup family $\{T^\star(t)\}_{t\geq 0}$ consisting of operators on the dual

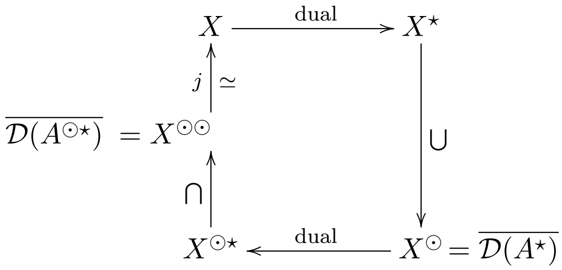


Figure 2.2.: Diagram illustrating the relations between the spaces $X, X^\star, X^\odot, X^{\odot\star}, X^{\odot\odot}$ for the $\odot$-reflective case.

space $X^\star$ defined by $T^\star(t) := (T(t))^\star$. If we equip $X^\star$ with its norm topology then in general the family $\{T^\star(t)\}_{t\geq 0}$ need not be a C_0-semigroup, i.e. it need not be strongly continuous anymore. Since we want to embed X into a bigger space $X^{\odot\star}$, where we also have the notion of a generator, we cannot lose strong continuity when constructing the spaces in between. We are led to the following definition:

$$X^\odot := \left\{ x^\star \in X^\star : \lim_{t\downarrow 0} \|T^\star(t)x^\star - x^\star\| = 0 \right\}.$$

Thus, $X^\odot$ (pronunciation X-sun) is precisely the subspace of $X^\star$ on which the action of $\{T^\star(t)\}_{t\geq 0}$ is strongly continuous. In order to see that this space is of any relevance, i.e. for example not empty, and what the generator looks like, we need the notion of the adjoint operator of a densely defined operator.

Definition 2.6. The *adjoint* $A^\star$ of a densely defined unbounded operator A is defined by $x^\star \in \mathcal{D}(A^\star)$ if and only if $y^* \in X^\star$ exists such that

$$\langle x^\star, Ax \rangle = \langle y^\star, x \rangle$$

for all $x \in \mathcal{D}(A)$, and in that case,

$$A^\star x^\star = y^\star.$$

Let A be the infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$ as in (2.2). The adjoint $A^\star$ is the generator of the adjoint semigroup $\{T^\star(t)\}_{t\geq 0}$ in the weak* sense, i.e.,

$$\frac{1}{t} \langle T^\star(t)x^\star - x^\star, x \rangle \qquad \text{converges or all } x \in X \text{ as } t \downarrow 0$$

if and only if $x^\star \in \mathcal{D}(A^\star)$, and in that case, the limit equals $\langle A^\star x^\star, x \rangle$.

One can show, see [14, Appendix II], that in the norm closure

$$X^\odot = \overline{\mathcal{D}(A^\star)}. \tag{2.10}$$

This is the closure with respect to the strong topology of X^* and shows that $\mathcal{D}(A^\star) \subseteq X^\odot$. The restriction of $\{T^\star(t)\}_{t\geq 0}$ to $X^\odot$, denoted $\{T^\odot(t)\}_{t\geq 0}$, is a strongly continuous semigroup of linear bounded operators on the space $X^\odot$. The infinitesimal generator of $\{T^\odot(t)\}_{t\geq 0}$ is the operator denoted $A^\odot$, which is the restriction of $A^\star$, to the domain

$$\mathcal{D}(A^\odot) = \left\{ x^\star \in \mathcal{D}(A^\star) : A^\star x^\star \in X^\odot \right\}.$$

The domain $\mathcal{D}(A^\odot)$ is weak* dense in $X^\star$. Starting from the C_0-semigroup $\{T^\odot(t)\}_{t\geq 0}$, we can repeat the same procedure once more and define $X^{\odot\star}, X^{\odot\odot}$, and $T^{\odot\star}$ and $T^{\odot\odot}$. The natural map $j : X \to X^{\odot\star}$, defined by

$$\left\langle jx, x^\odot \right\rangle := \left\langle x^\odot, x \right\rangle, \quad \forall x \in X \, \forall x^\odot \in X^\odot,$$

can be shown to be an embedding. Thus one can identify $X^{\odot\star}$ isomorphically with a closed subspace of X. If j is onto then X is called $\odot$-*reflexive* with respect to T.

2.4. Shift semigroup II

We will calculate $\{T_0^\star(t)\}_{t\geq 0}$ for the shift semigroup explicitly and show that it fails to be strongly continuous. In order to do so we first need to introduce two definitions and one theorem.

Definition 2.7. A function $f : [a,b] \to \mathbb{F}$ is said to be of *bounded variation* (BV) if there exists $M_f > 0$ such that for every partition $P : a = \sigma_0 < \sigma_1 < \cdots < \sigma_n = b$ of $[a,b]$,

$$V(f,P) = \sum_{i=0}^{n} |f(\sigma_{i+1}) - f(\sigma_i)| \leq M_f,$$

The quantity $V(f,P)$ is called the variation of f over P, and

$$V(f) := \sup_P V(f,P).$$

An important subspace of BV is the space of all normalized functions of bounded variation, defined by

$$NBV := \{f \in [a,b] : f(a) = 0, \text{ and } f \text{ is right continuous on } (a,b)\}.$$

Lastly, we extend the domain of definition of $f \in NBV$ to whole $\mathbb{R}$, by putting $f(\theta) = 0$ for $\theta \leq 0$ and $f(\theta) = f(h)$ for $\theta \geq h$.

Definition 2.8. Let $f : [a,b] \to \mathbb{C}^{n\times n}$ and $\varphi : [a,b] \to \mathbb{C}^n$ be given. For any partition P of $[a,b]$ and any choice $\tau_j \in [\sigma_{j-1}, \sigma_j]$ we introduce the sum

$$S(f,\varphi,P) = \sum_{j=1}^{n} \left(f(\sigma_j) - f(\sigma_{j-1})\right) \varphi(\tau_j).$$

Suppose $A \in \mathbb{C}^n$ exists such that

$$\forall \varepsilon > 0 \qquad \exists \delta = \delta(\varepsilon) > 0 \qquad \text{such that} \qquad |A - S(f,\varphi,P)| < \varepsilon$$

for all partitions P with width $\mu(P) < \delta$ and any choice of points $\tau_j \in [\sigma_{j-1}, \sigma_j]$. We then say that φ is *Riemann-Stieltjes integrable* with respect to f over $[a,b]$ and we shall write

$$A = \int_a^b df(\tau)\varphi(\tau).$$

The Riemann–Stieltjes integral appears in the original formulation of F. Riesz's theorem which represents the dual space of the Banach space $X = C([a,b],\mathbb{R}^n)$ as Riemann–Stieltjes integrals against functions of bounded variation.

Theorem 2.9. *(A corollary of the Riesz representation theorem) Let L be a continuous linear mapping from $C([-h,0],\mathbb{F}^n)$ into $\mathbb{F}^n$. There exists an unique NBV (normalized* *bounded variation) function ζ defined on $[0,h]$ with values in $\mathbb{F}^{n\times n}$ such that for all $\varphi \in C([-h,0],\mathbb{F}^n)$*

$$L\varphi = \int_0^h d\zeta(\theta)\varphi(-\theta),$$

where the integral is an n vector whose i^{th} component is equal to

$$\sum_{j=1}^{n}\int_0^h \varphi_j(-\theta)d\zeta_{ij}(\theta).$$

Example. Consider the DDE

$$\dot{x}(t) = -x(t) + \beta\, x(t-1).$$

Let $\eta : [-1,0] \to \mathbb{R}$ be given such that

$$\eta(\theta) = \begin{cases} \eta(0) = -1, & \\ \eta(\theta) = 0, & 0 < \theta < 1, \\ \eta(1) = -\beta. & \end{cases}$$

Then $\int_0^1 d\eta(\theta)\varphi(-\theta) = -\varphi(0) + \beta\varphi(-1)$ for all $\varphi \in C([-1,0],\mathbb{R})$.

We are now ready to define the adjoint of the semigroup $\{T_0(t)\}_{t\geq 0}$ in (2.9). Let $f \in X^\star = C([-h,0],\mathbb{C}^n)^\star$ then by Theorem 2.9 there exists an unique $\bar{N}BV$ function ζ such that the pairing between f and $T_0(t)\varphi \in X = C([-h,0],\mathbb{C}^n)$ becomes

$$\begin{aligned} \langle f, T_0(t)\varphi\rangle &= \int_0^\infty d\zeta(\theta)T_0(t)\varphi(-\theta) \\ &= \int_0^t d\zeta(\theta)\varphi(0) + \int_t^\infty d\zeta(\theta)\varphi(t-\theta) \\ &= f(t)\varphi(0) + \int_0^\infty d_\sigma\zeta(t+\sigma)\varphi(-\sigma). \end{aligned}$$

It follows that

$$(T_0^\star(t)f)(\theta) = f(t+\theta), \qquad \text{for } \theta > 0.$$

Let $f(\theta) = 0$ for $\theta < h$ and $f(h) \neq 0$, the

$$\|T_0^\star(t)f - f\| = 2|f(h)|, \qquad \theta > 0.$$

Thus $T_0^\star$ fails to be strongly continuous.
Using the equality in (2.10) for the shift semigroup $\{T_0(t)\}_{t\geq 0}$ we have

$$X^\odot = \Big\{ f \in NBV | f(t) = c + \int_0^t g(\theta)\,d\theta \text{ for } t > 0, \text{ where } c \in \mathbb{C} \text{ and } g \in L^1 \text{ with } g(\theta) = 0, \text{ for (almost all) } \theta \geq h \Big\}.$$

Elements of the space $X^\odot$ for the shift semigroup $\{T_0(t)\}_{t\geq 0}$ are completely specified by the pair $(c, g) \in \mathbb{C} \times L^1([0, h], \mathbb{C})$. In these coordinates we have

$$\begin{aligned} T_0^\odot(t)(c, g) &= \left(c + \int_0^t g(\theta)\, d\theta, g(t + \cdot)\right), \\ A_0^\odot(c, g) &= (g(0), g'), \end{aligned} \tag{2.11}$$

where the domain of the generator $A_0^\odot$ is given by

$$\mathcal{D}(A_0^\odot) = \{(c, g) : c \in \mathbb{C} \text{ and } g \in AC \text{ with } g(\theta) = 0 \text{ for } \theta \geq h\}$$

For the space $X^{\odot\star}$ of the shift semigroup we take the space

$$X^{\odot\star} = \mathbb{C} \times L^\infty([-h, 0], \mathbb{C})$$

with pairing

$$\langle(\alpha, \varphi), (c, g)\rangle = \alpha c + \int_0^h \varphi(-\theta)g(\theta)\, d\theta.$$

Using this pairing with equation (2.11) we can calculate

$$T_0^{\odot\star}(t)(\alpha, \varphi) = (\alpha, \varphi_t^\alpha),$$

where

$$\varphi_t^\alpha = \begin{cases} \varphi(t + \theta), & t + \theta \leq 0, \\ \alpha, & t + \theta > 0 \end{cases}$$

and the generator is given by

$$A_0^{\odot\star}(\alpha, \varphi) = (0, \dot{\varphi}), \qquad \mathcal{D}(A_0^{\odot\star}) = \{(\alpha, \varphi) | \varphi \in \text{Lip}(\alpha)\}, \tag{2.12}$$

where $\text{Lip}(\alpha)$ denotes the subset of $L^\infty([-h, 0], \mathbb{C})$ whose elements contain a Lipschitz continuous function which assumes the value α at $\theta = 0$.

Taking the closure of this space gives

$$X^{\odot\odot} = \overline{\mathcal{D}(A_0^{\odot\star})} = \{(\alpha, \varphi) | \varphi \in C(\alpha)\},$$

where $C(\alpha)$ denotes the closed subspace of $L^\infty([-h, 0], \mathbb{C})$ whose elements contain a continuous function with the value α at zero. Thus the function $\varphi \in X = C([-h, 0], \mathbb{R}^n)$ gets assigned to the couple $(\varphi(0), \varphi)$ by the embedding j, we see that $X^{\odot\odot} = j(X)$. From now on, we shall omit the embedding operator j in our notation and identify X and $X^{\odot\odot}$. In other words, we shall go back and forth between

$$\varphi \in X \text{ and } (\varphi(0), \varphi) \in X^{\odot\odot}.$$

As explained in [14, II.6] all results obtain upward of Lemma 2.3 holds when replacing $\mathbb{C}$ with $\mathbb{C}^n$ or $\mathbb{R}^n$ everywhere. A list of the representations of the spaces and the dual pairings between the spaces is given in Table 2.1 (due to [29]).

Space	Representation	Pairing
X X^*	$\phi \in C([-h,0],\mathbb{R}^n)$ $f \in \mathrm{NBV}([0,h],\mathbb{R}^n)$	$\langle f,\phi\rangle = \int_0^h df(\theta)\,\phi(-\theta)$
$X^{\odot}$ $X^{\odot *}$	$(c,g) \in \mathbb{R}^n \times L^1([0,h],\mathbb{R}^n)$ $(a,\phi) \in \mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$	$\langle(\alpha,\phi),(c,g)\rangle$ $= c^T\alpha + \int_0^h g(\theta)\,\phi(-\theta)\,d\theta$
X $X^{\odot}$	$\phi \in C([-h,0],\mathbb{R}^n)$ $(c,g) \in \mathbb{R}^n \times L^1([0,h],\mathbb{R}^n)$	$\langle(c,g),\phi\rangle$ $= c^T\phi(0) + \int_0^h g(\theta)\,\phi(-\theta)\,d\theta$

Table 2.1.: Representations for the abstract spaces $X, X^\star, X^\odot$ and $X^{\odot\star}$ for the case of the semigroup $\{T(t)\}_{t\geq 0}$ associated with the linear equation (2.13). Also indicated are the dual pairings that we will encounter in this thesis.

2.5. Bounded Linear Perturbations

We have calculated the $\odot$-star space for the trivial DDE from (2.7). Now consider the linear DDE

$$\begin{cases} \dot{x}(t) = \int_0^h d\zeta(\theta)x(t-\theta), & t>0, \\ x(\theta) = \varphi(\theta), & -h \leq \theta \leq 0. \end{cases} \tag{2.13}$$

We would like to see this equation as bounded perturbation of the trivial DDE. Motivated by the $\odot$-star framework for the shift semigroup above (in particular by equation (2.12)) we write the delay equation in the space $X^{\odot\star}$

$$\frac{d}{dt}x_t = A_0^{\odot\star}x_t + Bx_t, \tag{2.14}$$

where $B : X \to X^{\odot\star}$ is defined by

$$B\varphi = (\langle\zeta,\varphi\rangle_n\,,0) = \langle\zeta,\varphi\rangle\, r^{\odot\star},$$

with $r^{\odot\star} = (I_n, 0)$. The semigroup $\{T(t)\}_{t\geq 0}$ corresponding to (2.14) and the shift semigroup are related by the abstract integral equation

$$T(t)x = T_0(t)x + \int_0^t T_0^{\odot\star}(t-\tau)BT(\tau)x\,d\tau. \tag{2.15}$$

The integral has to be understood in the weak $\star$ sense, i.e.

$$\left\langle \int_0^t T_0^{\odot\star}(t-\tau)BT(\tau)x\,d\tau, x^\odot \right\rangle := \int_0^t \left\langle BT(\tau)x, T_0^{\odot}(t-\tau)x^\odot \right\rangle\,d\tau$$

for arbitrary $x^\odot \in X^\odot$. So in principle the integral takes values in $X^{\odot\star}$ but one can show that in fact it takes values in the closed subspace $X^{\odot\odot} = j(X)$. It can be shown that (2.15) defines an unique strongly continuous semigroup $T(t)$ with $\mathcal{D}(A) = \left\{x \in \mathcal{D}(A_0^{\odot\star}) : A_0^{\odot\star}x + Bx \in X\right\}$ and $Ax = A_0^{\odot\star}x + Bx$. Then, by taking duality and restriction we obtain semigroups $\{T(t)\}^\star_{t\geq 0}$, $\{T(t)\}^\odot_{t\geq 0}$ and $\{T(t)\}^{\odot\star}_{t\geq 0}$ with

- $\mathcal{D}(A^\star) = \mathcal{D}(A_0^\star)$ and for $x^\odot \in \mathcal{D}(A^\star)$ we have $A^\star x^\odot = A_0^\star x^\odot + B^\star x^\odot$
- $\mathcal{D}(A^\odot) = \{x^\odot \in \mathcal{D}(A_0^\star) : A_0^\star x^\odot + B^\star x^\odot \in X^\odot\}$ and $A^\odot x^\odot = A_0^\star x^\odot + B^\star x^\odot$
- $\mathcal{D}(A^{\odot\star}) = \mathcal{D}(A_0^{\odot\star})$ and $A^{\odot\star} x = A_0^{\odot\star} x + Bx$

In other words the domains are not affected by the perturbation B. Since the perturbation B is defined only in the finite dimensional span, we see that

$$A^{\odot\star}(\alpha, \varphi) = (\langle \zeta, \varphi \rangle, \dot{\varphi}). \tag{2.16}$$

We end this Section with a theorem which relates solutions of (2.13) with the semigroup defined by the abstract integral equation (2.15).

Theorem 2.10. [14, Theorem III.4.1] *Let, with $\{T_0(t)\}$ and B as defined above, $\{T(t)\}$ be the semigroup defined by the abstract integral equation (2.15). If $x(\cdot;\varphi)$ is a solution of (2.13) then*

$$T(t)\varphi = x_t(\cdot;\varphi).$$

3. Bifurcations of DDEs

3.1. Semiflows for nonlinear DDEs

Consider the nonlinear differential equation

$$\dot{u} = A_0^{\odot\star} u + R(u,p), \tag{3.1}$$

where $R : \mathcal{O} \to X^{\odot\star}, \mathcal{O} \subset X \times \mathbb{R}^m$ is assumed to be smooth. However, the results in this Section can be weakened to R being locally Lipschitz.

Formal integration of (3.1) leads to the abstract integral equation

$$u(t) = T_0(t)\varphi + \int_0^t T_0^{\odot\star}(t-s)R(u(s),p)ds. \tag{3.2}$$

As for linear DDEs we needed the notion of a semigroup, here we need the notion of a non-linear semigroup, called a semiflow.

Definition 3.1. [14, Definition VII.2.1] A *semiflow* on M is a map $S : D \to M$ on an open subset

$$D \subset [0,\infty) \times M$$

with the following properties:

1. For every $x \in M$ there exists an interval I_x, either $I_x = [0,\infty)$ or $I_x = [0,t_x)$ with some $t_x > 0$, so that
$$\{(t,x) \in [0,\infty) \times M | t \in I_x\} = D;$$
2. $S(0,x) = x$ on M;
3. $x \in M, s \in I_x$ and $t \in I_{S(s,x)}$ imply $t + s \in I_x$ and
$$S(t,S(s,x)) = S(t+s,x);$$
4. all maps
$$I_x \ni t \mapsto S(t,x) \in M, \qquad x \in M,$$
are continuous
5. all maps
$$\{y \in M : ((t,y) \in D\} \ni x \mapsto S(t,x) \in M, \qquad t \geq 0,$$
are continuous.

Definition 3.2. [14, Definition VII.2.9] Let P be a topological space and let $\Delta \subset [0,\infty) \times M \times P$ be open. A *parameterized semiflow* $\Sigma : \Delta \to M$ is a map such that for every $p \in P$, the map $S_p : D_p \to M$, where

$$D_p = \{(t,x) \in [0,\infty) \times M : (t,x,p) \in \Delta\}$$

and

$$S_p(t,x) = \Sigma(t,x,p)$$

is a semiflow on M.

It can be shown that [14, Theorem VII.3.4] there exists for each initial data φ_0 and parameter value p an unique solution $u_{\varphi,p}(t)$ of (3.2) on some maximal interval $I_{\varphi,p}$. Set

$$\Delta := \{(t,\varphi,p) \in [0,\infty) \times X \times P | t \in I_{\varphi,p}\},$$

and

$$\Sigma(t,\varphi,p) := u_{\varphi,p}(t),$$

then Σ defines a parameterized semiflow. The first two properties of Definition 3.1 are easy to prove [14, Corollary VII.3.6]. The remaining properties requires more work, which the reader can verify in [14, VII.4]. Now consider the parameter-dependent DDE

$$\begin{cases} \dot{x}(t) = g(x_t,\alpha), & t > 0, \\ x(\theta) = \varphi(\theta), & -h \leq \theta \leq 0, \end{cases} \tag{3.3}$$

where $g : C([-h,0],\mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^n$ is as smooth a necessary. Let $(e_1,\ldots,e_n)$ be the canonical basis for $\mathbb{R}^n$, then the n elements

$$r_i^{\odot\star} := (e_i, 0) \in \mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n),$$

for $i = 1,\ldots,n$ are linearly independent. Define a map $R : \mathcal{O} \to X^{\odot\star}, \mathcal{O} \subset X \times P$ by

$$R(\varphi,p) := \sum_{i=1}^{n} g_i(\varphi,p) r_i^{\odot\star}.$$

The next Proposition states that the maximal solutions of (3.2) are in one-to-one correspondence with the $\mathbb{R}^n$-valued functions of (3.3) .

Proposition 3.3. [14, Proposition VII.6.1] *Let* $(\varphi,p) \in \mathcal{O}$.

1. Suppose $x : [-h,t_+) \to \mathbb{R}^n$ is a solution to equation (3.3) and $x_0 = \varphi \in X$. Then $t_+ \leq \sup I_{\varphi,p}$ and for $0 \leq t \leq t_+$,

$$x_t = u_{\varphi,p}(t) = \Sigma(t,\varphi,p).$$

2. Consider the function $\tilde{x} : [-h,0] \cup I_{\varphi,p} \to \mathbb{R}^n$, given by $\tilde{x}_0 = \varphi$ and

$$\tilde{x}(t) = \Sigma(t,\varphi,p)(0), \qquad \forall t \in I_{\varphi,p}.$$

Then $\tilde{x}$ is a solution of (2.13) and

$$\tilde{x}_t = \Sigma(t,\varphi,p), \qquad \forall t \in I_{\varphi,p}.$$

3.2. Spectrum of the Generator

The stability of a steady-state solution of a nonlinear DDE results in analyzing the spectrum of the generator of the semigroup corresponding to the linear part of the DDE. Therefore we will concentrate on the spectrum of the generator in this Section. In general, the spectrum of an operator may consist of three different types of points, namely, the residual spectrum, the continuous spectrum, and the point spectrum. Moreover, points of the point spectrum are called eigenvalues of this operator. Consider the linear DDE

$$\begin{cases} \dot{x}(t) = \int_0^h d\zeta(\theta)x(t-\theta), & t \geq 0, \\ x_0 = \varphi, & -h \leq \theta \leq 0, \end{cases} \tag{3.4}$$

with corresponding generator A and semigroup $\{T(t)\}_{t\geq 0}$. The spectrum $\sigma(A)$ of A contains of eigenvalues finite type only, i.e. all eigenvalues are isolated points of $\sigma(A)$ and the corresponding (generalized) eigenspaces are finite.

Theorem 3.4.

1. [14, Corollary IV.3.3] *The spectrum of A consists of point spectrum only and is given by*

$$\sigma(A) = \{\lambda \in \mathbb{C} |\, \det \Delta(\lambda) = 0\},$$

where

$$\Delta(\lambda) = \lambda I - \int_0^h e^{-\lambda\theta} d\zeta(\theta) \tag{3.5}$$

is called the characteristic matrix and

$$\det \Delta(\lambda) = 0$$

the characteristic equation.

2. $$\sigma(A) = \sigma(A^\star) = \sigma(A^\odot) = \sigma(A^{\odot\star})$$

3. *The algebraic multiplicity of the eigenvalue λ equals the order of λ as a zero of* $\det \Delta$

Example. For the simple DDE (2.6) we can try, as in ODEs, the solution of the form $x(t) = e^{\lambda t}$. This leads to the equation

$$\lambda - \alpha e^{-\lambda\tau} = 0 \tag{3.6}$$

which has a countable set of complex conjugate solutions, with real parts accumulating at $-\infty$ and imaginary parts growing fast [51]. Non-real roots $z = u + iv$, $u = \text{Re } z$ and $v = \text{Im } z$, of equation (3.6) yield oscillatory solutions

$$t \to e^{ut} \left(c \cos(v\,t) + d \sin(v\,t)\right),$$

of the DDE (2.6), which is in obvious contrast to scalar autonomous ODEs, with all solutions monotone.

Remark. In `DDE-BifTool` we will exploit the fact the set

$$\Lambda(\beta) = \{\lambda \in \sigma(A) | \text{Re } \lambda > \beta\}$$

is a finite set of isolated eigenvalues of A when detecting codim-2 bifurcation.

3.3. Eigenspaces

For the normal form computation of the Bogdanov-Takens bifurcation we will need a representation for the (generalized) eigenfunctions and adjoint (generalized) eigenfunctions of the generator A and $A^\star$ respectively. These can be constructed using characteristic matrix (3.5). First we need the notion of a *Jordan chain.*

Definition 3.5. [14, Chapter IV.4] An ordered set $(x_0, x_1, \ldots, x_{k-1})$ of vectors in X is called a *Jordan chain* for Δ at λ if $x_0 \neq 0$ and

$$\Delta(z)\Big[x_0 + (z-\lambda)x_1 + \cdots + (z-\lambda)^{k-1}x_{k-1}\Big] = \mathcal{O}((z-\lambda)^k)$$

for $|z - \lambda| \to 0$. The number k is called the length of the chain and the maximal length of a chain starting with x_0 is called the rank of x_0.

Lemma 3.6. [14, Chapter IV.5.12] *Let λ be a simple eigenvalue of the generator A, then there is an eigenfunction ϕ such that*

$$A\phi = \lambda\phi, \tag{3.7}$$

and an adjoint eigenfunction $\phi^\odot$ such that

$$A^\star \phi^\odot = \lambda \phi^\odot. \tag{3.8}$$

Let (q) and (p) be the Jordan chain for $\Delta(\lambda)$ and $\Delta(\lambda)^T$ respectively, i.e. q and p are null vectors. Then the corresponding eigenfunction and adjoint eigenfunction are given by

$$\begin{aligned} \phi(\theta) &= e^{\lambda\theta} q, \\ \phi^\odot(\theta) &= p^T + p^T \int_0^\theta \Big(\int_\sigma^h e^{\lambda(\sigma-\tau)} d\zeta(\tau)\Big)\, d\sigma, \end{aligned}$$

Furthermore the following identity hold

$$\langle \phi^\odot, \phi \rangle = p^T \Delta'(\lambda) q, \tag{3.9}$$

which can be normalized to satisfy $\langle \phi^\odot, \phi \rangle = 1$.

Lemma 3.7. [29, Lemma 2.5] *Let $\lambda_1 \neq \lambda_2$ be simple eigenvalues of A. Let ϕ_{λ_1} be an eigenfunction of A corresponding to λ_1 eigenfunction and $\phi^\odot_{\lambda_2}$ an eigenfunction of $A^\star$ corresponding to λ_2. Then $\left\langle \phi^\odot_{\lambda_2}, \phi_{\lambda_1} \right\rangle = 0$.*

Lemma 3.8. [29, Lemma 2.7] *Let λ be eigenvalue of the generator A with algebraic multiplicity two and geometric multiplicity one, then there are (generalized) eigenfunctions $\phi_{1,2}$ such that*

$$A\phi_0 = \lambda\phi_0, \qquad A\phi_1 = \lambda\phi_1 + \phi_0 \tag{3.10}$$

and adjoint (generalized) eigenfunctions $\phi_{1,2}^{\odot}$ such that

$$A^{\star}\phi_1^{\odot} = \lambda\phi_1^{\odot}, \qquad A^{\star}\phi_0^{\odot} = \lambda\phi_0^{\odot} + \phi_1^{\odot}. \tag{3.11}$$

Let (q_0, q_1) and (p_1, p_0) be the Jordan chain for $\Delta(\lambda)$ and $\Delta(\lambda)^T$ respectively. Then the (generalized) eigenfunctions and adjoint (generalized) eigenfunctions are given by

$$\begin{aligned}
&\phi_0(\theta) = e^{\lambda\theta}q_0, \\
&\phi_1(\theta) = e^{\lambda\theta}(\theta q_0 + q_1), \\
&\phi_1^{\odot}(\theta) = p_1^T + p_1^T\int_0^{\theta}(\int_{\sigma}^{h} e^{\lambda(\sigma-\tau)}d\zeta(\tau))\,d\sigma, \\
&\phi_0^{\odot}(\theta) = p_0^T + p_0^T\int_0^{\theta}(\int_{\sigma}^{h} e^{\lambda(\sigma-\tau)}d\zeta(\tau))\,d\sigma \\
&\qquad + p_1^T\int_0^{\theta}(\int_{\sigma}^{h} e^{\lambda(\sigma-\tau)}(\sigma-\tau)\,d\zeta(\tau))\,d\sigma.
\end{aligned} \tag{3.12}$$

Furthermore the following identities hold

$$\langle\phi_0^{\odot}, \phi_0\rangle = p_0^T\Delta'(\lambda)q_0 + \frac{1}{2}p_1^T\Delta''(\lambda)q_0, \tag{3.13}$$

$$\langle\phi_1^{\odot}, \phi_1\rangle = \langle\phi_0^{\odot}, \phi_0\rangle, \tag{3.14}$$

$$\langle\phi_1^{\odot}, \phi_0\rangle = 0, \tag{3.15}$$

$$\langle\phi_0^{\odot}, \phi_1\rangle = p_0^T\Delta'(\lambda)q_1 + \frac{1}{2}p_0^T\Delta''(\lambda)q_0 + \frac{1}{2}p_1^T\Delta''(\lambda)q_1 + \frac{1}{6}p_1^T\Delta'''(\lambda)q_{0.} \tag{3.16}$$

which can be normalized to satisfy $\langle\phi_0^{\odot}, \phi_0\rangle = \delta_{ij}$.

Proof. The (generalized) eigenspace at the double eigenvalue λ of A is given by

$$\mathcal{N}((A-\lambda)^2)$$

which leads to the expressions in (3.10) and similarly for (3.11). The representations of the (generalized) eigenfunctions and adjoint (generalized) eigenfunctions can be in Theorem IV.5.5 and IV.5.9 in [14].

To prove (3.13) observe that

$$\begin{aligned}
\langle \phi_0^\odot, \phi_0 \rangle &= \int_0^h d\phi_0^\odot(\theta)\phi_0(-\theta) \\
&= p_0^T q_0 + \int_0^h d\phi_0^{\odot\prime}(\theta)\phi_0(-\theta)d\theta \\
&= p_0^T q_0 + p_0^T \int_0^h (\int_\theta^h e^{\lambda(\theta-\tau)} d\zeta(\tau) e^{-\lambda\theta})\, d\theta\, q_0 \\
&\quad + p_1^T \int_0^h (\int_\theta^h e^{\lambda(\theta-\tau)}(\theta-\tau)\, d\zeta(\tau) e^{-\lambda\theta})\, d\theta\, q_0 \\
&= p_0^T q_0 + p_0^T \int_0^h \int_0^\tau e^{-\lambda\tau}\, d\theta\, d\zeta(\tau)\, q_0 \\
&\quad + p_1^T \int_0^h \int_0^\tau e^{-\lambda\tau}(\theta-\tau)\, d\theta\, d\zeta(\tau)\, q_0 \\
&= p_0^T q_0 + p_0^T \int_0^h \tau e^{-\lambda\tau}\, d\zeta(\tau)\, q_0 \\
&\quad + \frac{1}{2} p_1^T \int_0^h \tau^2 e^{-\lambda\tau}\, d\theta\, d\zeta(\tau)\, q_0 \\
&= p_0^T \Delta'(\lambda) q_0 + \frac{1}{2} p_1^T \Delta''(\lambda) q_0,
\end{aligned}$$

where we used Fubini's theorem to change the order of integration.

Equations (3.14) and (3.15) follow from

$$\begin{aligned}
\langle \phi_1^\odot, \phi_1 \rangle &= \langle A^\star \phi_0^\odot - \lambda\phi_0^\odot, \phi_1 \rangle \\
&= \langle \phi_0^\odot, A\phi_1 \rangle - \langle \lambda\phi_0^\odot, \phi_1 \rangle \\
&= \langle \phi_0^\odot, \lambda\phi_1 + \phi_0 \rangle - \langle \lambda\phi_0^\odot, \phi_1 \rangle \\
&= \langle \phi_0^\odot, \phi_0 \rangle
\end{aligned}$$

and

$$\langle \phi_1^\odot, \phi_0 \rangle = \langle \phi_1^\odot, A\phi_1 - \lambda\phi_1 \rangle = \langle A^\star \phi_1^\odot, \phi_1 \rangle - \lambda\langle \phi_1^\odot \lambda\phi_1 \rangle = 0.$$

To prove the normalization conditions $\langle \phi_0^\odot, \phi_0 \rangle = \delta_{ij}$ we start by showing that $\langle \phi_0^\odot, \phi_0 \rangle$ is non-vanishing. Consider the direct sum decomposition

$$\begin{aligned}
X &= \mathcal{N}((\lambda - A)^2) \oplus \mathcal{R}((\lambda - A)^2) \\
&= \mathcal{N}((\lambda - A)^2) \oplus \overline{\mathcal{R}((\lambda - A)^2)} \\
&= \mathcal{N}((\lambda - A)^2) \oplus {}^\perp\mathcal{N}((\lambda - A^\star)^2),
\end{aligned}$$

Theorem IV.2.5 in [14]. Since $\phi_0 \in \mathcal{N}((\lambda - A)^2)$ and $\mathcal{N}((\lambda - A^\star)^2)$ is spanned by $\phi_0^\odot$ and $\phi_1^\odot$ it follows that $\langle \phi_0^\odot, \phi_0 \rangle \neq 0$. Now a choice has to be made, we can either scale ϕ_0 or $\phi_1^\odot$ with $\alpha = \langle \phi_1^\odot, \phi_1 \rangle^{-1}$ in order to achieve the normalization

$$\langle \phi_1^\odot, \phi_1 \rangle = \langle \phi_0^\odot, \phi_0 \rangle = 1.$$

Note that scaling ϕ_0 results also in scaling ϕ_1 in order for ϕ_1 to stay a generalized eigenfunction and similarly for $\phi_1^{\odot}$ and $\phi_0^{\odot}$. It remains to normalize $\langle \phi_0^{\odot}, \phi_1 \rangle$. Suppose first we have scaled the function ϕ_0. Then we note that for any scalar δ the function $\phi_0^{\odot} \mapsto \phi_0^{\odot} + \delta \phi_1^{\odot}$ is a generalized eigenvector of $A^\star$. By setting $\delta = -\langle \phi_0^{\odot}, \phi_1 \rangle$ we obtain $\langle \phi_0^{\odot}, \phi_1 \rangle = 0$. In the case we scaled the function $\phi_1^{\odot}$, we note that for any scalar δ the function $\phi_1 \mapsto \phi_1 + \delta \phi_0$ is a generalized eigenvector of A. Then again taking $\delta = -\langle \phi_0^{\odot}, \phi_1 \rangle$ yields $\langle \phi_0^{\odot}, \phi_1 \rangle = 0$.

□

3.4. Stability around a steady-state

In this Section we keep parameter α fixed and, therefore, we also suppress it in the notation. Let φ_0 be a stationary point the semiflow Σ generator by (3.2), that is,

$$\Sigma(t, \varphi_0) = \varphi_0 \qquad \forall t \geq 0.$$

Equivalently one can show that $\bar{\varphi}$ is a constant function

$$\varphi_0(\theta) = \bar{x}, \qquad \theta \in [-h, 0],$$

and satisfies

$$\bar{x} = f(\varphi_0),$$

see [13, Theorem 3.9].

Definition 3.9. [14, VII.5.7] We say that φ_0 is (locally) stable whenever for every $\varepsilon > 0$ we can find $\delta > 0$ such that $\|\varphi - \varphi_0\| \leq \delta$ guarantees that $[0, \infty) \times \{\varphi\} \subset \Delta$ and

$$\|\Sigma(t, \varphi) - \varphi_0\| \leq \varepsilon \qquad \forall t \geq 0.$$

When φ_0 is not stable, we say it is unstable. When we can find $\varepsilon > 0, K > 0$ and $\omega < 0$ such that

$$\Sigma(t, \varphi_0) = K e^{\omega k} \qquad t \geq 0,$$

for all φ with $\|\varphi - \varphi_0\| \leq \varepsilon$, we say that φ_0 is (locally) exponentially stable.

The next Theorem states the "principle of linearized stability" for DDEs. It relates the stability of a stationary point to the stability of the semigroup obtained from differentiation of the semiflow.

Theorem 3.10. *The stationary point φ_0 is*

(i) unstable if Re $\lambda > 0$ for some root λ of the characteristic equation,

(ii) (locally) exponentially stable if Re $\lambda < 0$ for all roots λ of the characteristic equation.

When A does have eigenvalues on the imaginary axis then there exists, as in ODEs, a center manifold which we will describe in the next Section.

3.5. Center Manifold

If the spectrum of A contains eigenvalues on the imaginary axis in the complex plane then there exists, as in ODEs, a invariant local center manifold $\mathcal{W}_c$. Since the (generalized) eigenspaces are finite dimensional and also the number of eigenvalues is any vertical stripe in the complex plane are finite, the center manifold $\mathcal{W}_c$ is finite dimensional. Consider the AIE

$$u(t) = T(t-s)u(s) + \int_s^t T^{\odot\star}(t-\tau)R(u(\tau))\,d\tau, \tag{3.17}$$

where $R : X \to X^{\odot\star}$ is defined by $R(\phi) = r^{\odot\star}g(\phi)$. To prove the existence of a smooth global center manifold, one needs to define cut-off functions $\xi : \mathbb{R}_+ \to \mathbb{R}$ be C^∞-smooth and such that

(i) $\xi(y) = 1$ for $0 \leq y \leq 1$,

(ii) $0 \leq \xi(y) \leq 1$ for $1 \leq y \leq 2$,

(iii) $\xi(y) = 0$ for $y \geq 1$.

Then the non-linearity R is modified in the center and the hyperbolic directions separately; for δ positive we let

$$R_\delta(u) = R(u)\xi\left(\frac{\|P_0^{\odot\star}u\|}{\delta}\right)\xi\left(\frac{\|(I-P_0^{\odot\star})\,u\|}{\delta}\right).$$

This is used to prove the existence of a Lipschitz continuous (global) center manifold. Using results of contractions on scales of Banach spaces a smooth center manifold is constructed. Either from Theorem IX.5.3 and Corollary IX.7.10 in [14] or Theorem 6.13 in [15] we obtain

Theorem 3.11. *(Center manifold: invariance and relation to bounded orbits) Assume that $g \in C^k, k \geq 1, g(0) = 0, Dg(0) = 0$ and let $\Lambda_0 \neq \emptyset$. There exists a C^k-mapping $\phi \to \mathcal{C}(\phi)$ of a neighborhood of the origin in X_0 into X and a positive constant δ such that*

(i) Im $(\mathcal{C})$ is locally invariant in the sense that $u^\star(\phi)(t)$ satisfies the equation $\mathcal{C}(P_0(u^\star(\phi)(t))) = u^\star(\phi)(t)$ and $u^\star(\phi)(t)$ is a solution of (3.17) on the interval $I = [S,T], S < 0 < T$, provided for t in this interval $\|u^\star(\phi)(t)\| \leq \delta$,

(ii) Im $(\mathcal{C})$ is tangent to X_0 at zero: $\mathcal{C}(0) = 0$ and $D\mathcal{C}(0)\varphi = \varphi$ for all $\varphi \in X_0$,

(iii) Im $(\mathcal{C})$ contains all solutions of (3.17) which are defined on $\mathbb{R}$ and bounded above by δ in the supreme norm.

Here P_0 is the projection of X onto the center subspace X_0.

Let $y(t) = P_0(u^\star(\phi)(t))$ then $y(t)$ satisfies the equation

$$y(t) = T(t)y(0) + \int_0^t T^{\odot\star}(t-\tau)P_0^{\odot\star}R_\delta(\mathcal{C}(y(\tau)))\,d\tau,$$

and, consequently the ODE in X_0 yields

$$\dot{y} = Ay + P_0^{\odot\star} R_\delta(\mathcal{C}(y(\tau))), \tag{3.18}$$

where $P_0^{\odot\star}$ is the projection of $X^{\odot\star}$ onto the center subspace X_0. Equivalently, we write the ODE (3.18) in $X^{\odot\star}$ as

$$\dot{u}(t) = A^{\odot\star} u(t) + R(u_\delta(t)), \tag{3.19}$$

see Theorem VI.2.2 in [14].

In the absence of unstable directions (i.e no positive eigenvalues) the center manifold is attractive, see Theorem [14, Theorem IX.8.1]. An immediate consequence is that all solutions which remain bounded and sufficiently small for all time lie on the center manifold. Since our interest here lies with local analysis only (the global bifurcation predicted by a Bogdanov-Takens point comes from local analysis) we dropped the dependence δ in equations (3.18) and (3.19) and write

$$\dot{y} = Ay + P_0^{\odot\star} R(\mathcal{C}(y(\tau))) \tag{3.20}$$

and

$$\dot{u}(t) = A^{\odot\star} u(t) + R(u(t)). \tag{3.21}$$

3.6. Parameter-Dependent Center Manifold

Suppose that the steady-state $\varphi_0 \in X = C([-h, 0], \mathbb{R}^n)$ at the critical parameter value $\alpha_0 \in \mathbb{R}^m$ is a stationary solution of (3.3), i.e.

$$f(\varphi_0, \alpha_0) = 0.$$

By a change of coordinates it can always be arranged that $(\varphi_0, \alpha_0) = (0, 0)$. Furthermore, suppose that the linearization has eigenvalues on the imaginary axis.

We follow the method used in [14, Chapter IX.9] to modify the center manifold theorem in order to include parameter dependence. However, there the assumption that the equilibrium is fixed for all parameter values α is made. This assumption is commonly applied in the literature when considering Delay Differential Equations, see for example [19, 18]. In general this may not always be the case. In particular the equilibrium at the generic Bogdanov-Takens and fold-Hopf bifurcation can disappear when parameters are varied.

One way to consider the parameter-dependent center manifold is to extend the DDE (3.3) as

$$\begin{cases} \dot{x}(t) &= f(x_t, \alpha), \\ \dot{\alpha}(t) &= 0. \end{cases} \tag{3.22}$$

That is, we treat parameters as variables, which have trivial dynamics. This approach was suggested in [14, VII.7] but never elaborated. We need to write the extended system as an abstract integral equation in order to apply the center manifold theorem. Therefore, we need to derive the C_0-semigroups $\{\mathbf{T}(t)\}$ and $\{\mathbf{T}^{\odot\star}(t)\}$ corresponding to the extended system. To do so we expand (3.22) as

$$\begin{cases} \dot{x}(t) &= Lx_t + M\alpha + g(x_t, \alpha), \\ \dot{\alpha}(t) &= 0, \end{cases} \tag{3.23}$$

where

$$L = D_1 f(\varphi_0, \alpha_0),$$
$$M = D_2 f(\varphi_0, \alpha_0),$$

and g contains only nonlinear terms with respect to x_t and α. The linearization of (3.23), i.e. when $g = 0$, is equivalent to the system

$$\begin{cases} \dot{u} &= A_0^{\odot\star} u + Bu + J_1\alpha, \\ \dot{\alpha} &= 0, \end{cases}$$

where

$$Bu = (Lu)\, r^{\odot\star}, \qquad J_1\alpha = (M\alpha)\, r^{\odot\star}, \tag{3.24}$$

and the first equation is an inhomogeneous differential equation. Let $\left\{T_0^{\odot\star}(t)\right\}$ be the C_0-semigroup on X generated by $A_0^{\odot\star}$, then

$$\begin{cases} u(t) &= T_0(t-s)u(s) + \int_s^t T_0^{\odot\star}(t-\tau)\left[Bu(\tau) + J_1\alpha_0\right] d\tau \\ \alpha(t) &= \alpha_0. \end{cases} \tag{3.25}$$

From [14, Lemma III.2.23] it follows that (3.25) is equivalent to the abstract integral equation

$$\begin{cases} u(t) &= T(t-s)u(s) + \int_s^t T^{\odot\star}(t-\tau) J_1\alpha_0 \, d\tau, \\ \alpha(t) &= \alpha_0, \end{cases} \tag{3.26}$$

where $\{T(t)\}$ is the C_0-semigroup on X generated by $A_0^{\odot\star} + B$ in X. We thus obtain

$$\mathbf{T}(t) = \begin{pmatrix} T(t) & \int_0^t T^{\odot\star}(t-\tau) J_1 \, d\tau \\ 0 & I_{m\times m} \end{pmatrix}. \tag{3.27}$$

Lemma 3.12. *The generator* $\mathbf{A}$ *of the* C_0*-semigroup* $\{\mathbf{T}(t)\}$ *is given by*

$$\mathbf{A} = \begin{pmatrix} A & M \\ 0 & 0 \end{pmatrix},$$

where A *is the generator of the* C_0*-semigroup* $\{T(t)\}$.

Proof. Let $w = (u, \alpha_0) \in X \times \mathbb{R}^m$. Using definition (2.2) it follows that

$$\begin{aligned}
\mathbf{A}w &= \lim_{t\downarrow 0} \frac{1}{t} \left(\mathbf{T}(t)w - w\right) \\
&= \lim_{t\downarrow 0} \frac{1}{t} \left(\begin{pmatrix} T(t) & \int_0^t T^{\odot\star}(t-\tau)\, d\tau M \\ 0 & I_{m\times m} \end{pmatrix} \begin{pmatrix} u \\ \alpha_0 \end{pmatrix} - \begin{pmatrix} u \\ \alpha_0 \end{pmatrix} \right) \\
&= \lim_{t\downarrow 0} \frac{1}{t} \left(\begin{pmatrix} T(t)u + \int_0^t T^{\odot\star}(t-\tau)\, d\tau M\alpha_0 \\ \alpha_0 \end{pmatrix} - \begin{pmatrix} u \\ \alpha_0 \end{pmatrix} \right) \\
&= \lim_{t\downarrow 0} \frac{1}{t} \left(\begin{pmatrix} T(t)u - u + \int_0^t T^{\odot\star}(t-\tau)\, d\tau M\alpha_0 \\ 0 \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} Au + M\alpha_0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} A & M \\ 0 & 0 \end{pmatrix} w.
\end{aligned}$$

□

For the C_0-semigroup $\{\mathbf{T}^{\odot\star}(t)\}$ more work needs to be done. We have the following theorem.

Theorem 3.13. *The C_0-semigroup $\{\mathbf{T}^{\odot\star}(t)\}$ on $(X \times \mathbb{R}^m)^{\odot\star} \cong X^{\odot\star} \times \mathbb{R}^m$ is given by*

$$\mathbf{T}^{\odot\star}(t) = \begin{pmatrix} T^{\odot\star}(t) & \int_0^t T^{\odot\star}(t-\tau)J_1\, d\tau \\ 0 & I_{n\times m} \end{pmatrix} \tag{3.28}$$

when we consider $X^{\odot\star}$ as a subspace of $X^{\odot\star\star\star}$ for the component $\int_0^t T^{\odot\star}(t-\tau)J_1\, d\tau$.

Proof. First notice that since

$$\begin{aligned}
\langle \mathbf{T}w, w^\star \rangle &= \left\langle \begin{pmatrix} T(t) & \int_0^t T^{\odot\star}(t-\tau)J_1\, d\tau \\ 0 & I_{n\times m} \end{pmatrix} \begin{pmatrix} u \\ \alpha_0 \end{pmatrix}, \begin{pmatrix} u^\star \\ \alpha_0^\star \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} T(t)u \\ 0 \end{pmatrix}, \begin{pmatrix} u^\star \\ \alpha_0^\star \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} \int_0^t T(t-\tau)\, d\tau M\alpha_0 \\ \alpha_0 \end{pmatrix}, \begin{pmatrix} u^\star \\ \alpha_0^\star \end{pmatrix} \right\rangle,
\end{aligned}$$

we can treat each component in the block operator matrix (3.27) separately. Therefore, we only need to focus on the term $\int_0^t T^{\odot\star}(t-\tau)C\, d\tau$. Define the operators

$$\begin{aligned}
\tilde{L}_1\alpha &= \begin{pmatrix} M\alpha \\ 0 \end{pmatrix}, \qquad &:& \qquad \mathbb{R}^m \to X^{\odot\star} \cong \mathbb{R}^n \times L_\infty, \\
\tilde{L}_2(t)x^{\odot\star} &= \int_0^t T^{\odot\star}(\sigma)x^{\odot\star}\, d\sigma, \qquad &:& \qquad X^{\odot\star} \to X.
\end{aligned}$$

It follows that

$$\tilde{L}\alpha := \tilde{L}_2(t) \circ \tilde{L}_1\alpha = \int_0^t T^{\odot\star}(\sigma)(M\alpha, 0)\, d\sigma, \qquad : \qquad \mathbb{R}^m \to X.$$

The operator $\tilde{L}_2(t)^\star$ maps $X^\star$ into $X^\odot$ and $\tilde{L}_2(t)^\star = \tilde{L}_2^\odot(t)$, where $\tilde{L}_2^\odot(t) : X^\star \to X^\odot$ is defined by

$$\tilde{L}_2^\odot(t)x^\star = \int_0^t T^\star(\sigma)x^\star \, d\sigma$$

where we consider $X^\odot$ as a subspace of $X^{\odot\star\star}$, see [14, Corollary III.2.18]. Let $x^\odot = (c, g) \in \mathbb{R}^n \times L_1 \cong X^\odot \subset X^{\odot\star\star}$, then

$$\left\langle \tilde{L}_1\alpha, x^\odot \right\rangle = \left\langle \begin{pmatrix} M\alpha \\ 0 \end{pmatrix}, x^\odot \right\rangle = (M\alpha, c) = \left(\alpha, M^T c\right),$$

where $(\cdot,\cdot)$ is the standard inner product on $\mathbb{R}^n$. It follows that the adjoint $\tilde{L}_1^\star : X^\odot \to \mathbb{R}^m$ is given by $\tilde{L}_1^\star x^\odot = M^T c$. We obtain that

$$\tilde{L}^\star x^\star = \left(\tilde{L}_1^\star(t) \circ \tilde{L}_2^\odot\right) x^\star = M^T \left(\int_0^t T^\star(\sigma)\, d\sigma x^\star\right), \qquad : \qquad X^\star \to \mathbb{R}^m.$$

The map $\left(\tilde{L}_2^{\odot\star}(t)\right) = \left(\tilde{L}_2^\odot(t)\right)^\star : X^{\odot\star} \to X^{\odot\star}$ is given by

$$\tilde{L}_2^{\odot\star}(t)x^{\odot\star} = \int_0^t T^{\odot\star}(\sigma)x^{\odot\star} \, d\sigma.$$

Indeed, we have

$$\begin{aligned}
\left\langle \tilde{L}_2^\odot(t)x^\star, x^{\odot\star} \right\rangle &= \left\langle \int_0^t T^\star(\sigma)x^\star \, d\sigma, x^{\odot\star} \right\rangle \\
&= \int_0^t \left\langle T^\star(\sigma)x^\star, x^{\odot\star} \right\rangle d\sigma \\
&= \int_0^t \left\langle x^\star, T^{\odot\star}(\sigma)x^{\odot\star} \right\rangle d\sigma \\
&= \left\langle x^\star, \int_0^t T^{\odot\star}(\sigma)x^{\odot\star} \, d\sigma \right\rangle.
\end{aligned}$$

Let $x^\odot = (c, 0) \in \mathbb{R}^n \times L_1$ and $x^{\odot\star} = (M\alpha, 0) \in \mathbb{R}^n \times L_\infty \cong X^{\odot\star}$, then

$$\left(\tilde{L}_1^\star x^\odot, \alpha\right) = \left(M^T c, \alpha\right) = (c, M\alpha) = \left\langle \begin{pmatrix} c \\ 0 \end{pmatrix}, \begin{pmatrix} M\alpha \\ 0 \end{pmatrix} \right\rangle = \left\langle x^\odot, x^{\odot\star} \right\rangle.$$

Thus $\tilde{L}_1^{\star\star} : \mathbb{R}^m \to X^{\odot\star}$ is given by

$$\tilde{L}_1^{\star\star}\alpha = \begin{pmatrix} M\alpha \\ 0 \end{pmatrix}.$$

It follows that

$$\begin{aligned}\tilde{L}^{\star\star}\alpha &= \left(\tilde{L}_2^{\odot\star}(t)\circ\tilde{L}_1^{\star\star}\right)\alpha\\ &= \tilde{L}_2^{\odot\star}(t)(M\alpha,0)\\ &= \int_0^t T^{\odot\star}(\sigma)(M\alpha,0)\,d\sigma\\ &= \int_0^t T^{\odot\star}(\sigma)\,d\sigma M\alpha.\end{aligned}$$

and we obtain (3.28). □

Lemma 3.14. *The generator of* $\{\mathbf{T}^{\odot\star}(t)\}$ *is given by*

$$\mathbf{A}^{\odot\star} = \begin{pmatrix} A^{\odot\star} & M \\ 0 & 0 \end{pmatrix},$$

where $A^{\odot\star}$ *is the generator of the* C_0*-semigroup* $T^{\odot\star}$*.*

Proof. The proof is similar to Lemma 3.12. □

The projection $\mathbf{P}^{\odot\star}$ from $(X\times\mathbb{R}^m)^{\odot\star}$ onto the center subspace of $\mathbf{A}$ is given by $\mathbf{P}^{\odot\star}w = (P_0^{\odot\star}u,\alpha)$. Next, define the nonlinearity $\mathbf{R}: X\times\mathbb{R}^m \to X^{\odot\star}\times\mathbb{R}^m$ by

$$\mathbf{R}(w) = (R(u,\alpha),0),$$

where

$$R(\varphi,\alpha) = r^{\odot\star}\left(g(\varphi,\alpha)\right) \tag{3.29}$$

and $w=(u,\alpha_0)$.

It follows that the abstract integral equation for extended system (3.23) in given by

$$(u(t),\alpha) = \mathbf{T}(t-s)(u(s),\alpha) + \int_s^t \mathbf{T}^{\odot\star}(t-\tau)\mathbf{R}(u(\tau),\alpha)\,d\tau. \tag{3.30}$$

To this equation one can apply the Center Manifold Theorem 3.11. By projecting the center manifold flow on the center subspace $X_0\times\mathbb{R}^m$ we obtain the ODE

$$\frac{dz}{dt} = \mathbf{A}z + \mathbf{P}^{\odot\star}(\mathbf{R}(\mathcal{C}(z))), \tag{3.31}$$

where $z=(y,\alpha)$. From the abstract integral equation (3.30) we see that the mapping $\mathcal{C}$ from $X_0\times\mathbb{R}^m$ into $X\times\mathbb{R}^m$ is given by

$$\mathcal{C}(\varphi,\alpha) = (\mathcal{C}_1(\varphi,\alpha),\alpha). \tag{3.32}$$

We conclude that the ODE (3.31) on the center subspace $X_0 \times \mathbb{R}^m$ can be written as

$$\begin{cases} \dot{y}(t) = Ay + M\alpha + P^{\odot\star}(R(\mathcal{C}_1(y,\alpha),\alpha)), \\ \dot{\alpha}(t) = 0, \end{cases} \tag{3.33}$$

and on the center manifold in $X^{\odot\star} \times \mathbb{R}^m$ as

$$\begin{cases} \dot{u}(t) = A^{\odot\star}u(t) + J_1\alpha + R(u(t),\alpha), \\ \dot{\alpha}(t) = 0. \end{cases} \tag{3.34}$$

Remark 3.15. We implicitly assumed that the mapping $(u,\alpha) \mapsto R(u,\alpha)$ from $X \times \mathbb{R}^m$ into $X^{\odot\star}$ is C^k-smooth. This condition is violated whenever one takes one of the delays as parameters, see [14, Remark IX.9.2].

4. Computation of normal forms on the parameter-dependent center manifold

4.1. The method

In this Section we will extend the normalization method from [29] to include parameters. There are two different approaches to choose from. In the first approach the variables and parameters are treated separately. Then there are two maps relating the original system and the normal form on the center subspace. In the second approach the parameters are treated as variables as well. Then there will only one map relating the two systems. Although the first approach is more natural, there are two advantages to the second approach. The first advantage is that since there is only one map between the systems, the expansion is more easy to implement. The second advantage is that it may give more insight into solving the coefficients. In [32, 1, 36] there is a 'big' system to solve to obtain some coefficients simultaneously. From the second approach it is clearer to see that these coefficients can be obtained without using this 'big' system, making it more adaptable to the DDE case. Despite these advantages we have chosen to use the first approach in this thesis to better mimic the ODE case.

Suppose that the steady-steady $\varphi_0 = 0$ at the parameter value $\alpha_0 = 0 \in \mathbb{R}^p$ is a solution of (3.3),

$$\dot{x} = f(\varphi_0, \alpha_0) = 0.$$

Let the generator $A^{\odot\star}$ of the linearization of the DDE at $\varphi_0 = 0$ and the critical parameter value $\alpha_0 = 0$ have $0 < n_c < \infty$ eigenvalues on the imaginary axis. Then there exists a finite dimensional local center manifold $\mathcal{W}_{loc}$ of dimension $n \geq n_c$, depending on the multiplicities of the eigenvalues, tangent to the center subspace X_0. We consider bounded solutions $u(t)$ that are in $\mathcal{W}_{loc}$ for all time. Let $y(t)$ be the projection of $u(t)$ onto X_0, i.e. $y(t) = P_0^{\odot\star}(u(t))$. Since X_0 is spanned by some basis Φ of (generalized) eigenvectors , we can express $y(t)$ uniquely relative to Φ. The corresponding coordinate vector $z(t)$ of $y(t)$ satisfies some ODE admitting an expansion of the form

$$\dot{z} = G(z, \beta) = \sum_{|\nu|=1}^{N} \sum_{|\mu|=0}^{M} \frac{1}{\nu!\mu!} g_{\nu\mu} z^{\nu} \beta^{\mu} + \mathcal{O}(\|z\|^{N+1} \|\beta\|^{M+1}), \qquad \forall t \in \mathbb{R}, \tag{4.1}$$

with unknown normal form coefficients $g_{\nu\mu} \in \mathbb{R}^{n_c}$ and parameters β. Here ν and μ are multi-indices of length n and p respectively. For a multi-index ν one has

$$\begin{aligned}\nu &= (\nu_1, \nu_2, \dots, \nu_n),\\ \nu! &= \nu_1!\nu_2!\dots\nu_n!,\\ |\nu| &= \nu_1 + \nu_2 + \dots + \nu_n,\\ z^\nu &= z_1^{\nu_1} z_2^{\nu_2} \dots z_n^{\nu_n}.\end{aligned}$$

The series is supposed to be truncated after some sufficiently high order N and M.
On $\mathcal{W}_{loc}$ itself $u(t)$ satisfies the ODE

$$\dot{u}(t) = A^{\odot\star}u(t) + J_1\alpha + R(u(t), \alpha),$$

where J_1 is the derivative with respect to the parameters $D_2 f(\varphi_0, \alpha_0) \in \mathbb{R}^{n\times p}$ and R is given by (3.29). The nonlinearity can expanded by

$$R(u,\alpha) = \sum_{\substack{j=1\dots N,\, k=1\dots M,\\ (j,k)\notin\{(0,0),(1,0),(0,1)\}}} \frac{1}{j!k!} D_2^k D_1^j f(0,0)(\overbrace{u,\dots,u}^{j\text{ times}}, \overbrace{\alpha,\dots\alpha}^{k\text{ times}}) r^{\odot\star},$$

where $D_2^k D_1^j f(0,0)(\overbrace{u,\dots,u}^{j\text{ times}}, \overbrace{\alpha,\dots\alpha}^{k\text{ times}})$ is the jth order Fréchet derivative of f with respect to its first argument and the kth order derivative of f with respect to its second argument evaluated at the point $(0,0) \in (C([-h,0),\mathbb{R}^n), \mathbb{R}^p)$.

In order to define a coordinate-version of the parameter-dependent center manifold mapping $\mathcal{C} : V \times \mathbb{R}^p \subset X_0 \to X$ defined in (3.32) relative to Φ, we introduce the mapping $\mathcal{H} : V \subset \mathbb{R}^{n_c} \times \mathbb{R}^p \to X$ defined by

$$\mathcal{H}(z,\beta) = \mathcal{C}(\xi(z), \beta).$$

Here the coordinate mapping $\xi \to z(\xi)$ is a smooth C^k injection onto some neighbourhood $V \subset \mathbb{R}^{n_c}$. Then $\mathcal{H}$ admits the expansion

$$\mathcal{H}(z,\beta) = \sum_{|\nu|=1}^{N} \sum_{|\mu|=0}^{M} \frac{1}{\nu!\mu!} H_{\nu\mu} z^\nu \beta^\mu + \mathcal{O}(\|z\|^{N+1}\|\beta\|^{M+1}). \tag{4.2}$$

As before, let $y(t)$ be the projection of the solution $u(t)$ onto the center subspace X_0 and $z(t)$ is its coordinate with respect to Φ. It then follows that

$$u(t) = \mathcal{H}(z(t), \beta), \qquad \forall t \in \mathbb{R}.$$

Differentiating both sides of this relation yields the *homological equation*

$$A^{\odot\star}\mathcal{H}(z,\beta) + J_1\alpha + R(\mathcal{H}(z,\beta), \alpha) = D_z\mathcal{H}(z,\beta)\dot{z}, \tag{4.3}$$

due to $\dot{\beta} = 0$. To relate the parameters α to the parameters β, we define the mapping

$$\alpha = K(\beta), \qquad K : \mathbb{R}^p \to \mathbb{R}^p.$$

We expand K as

$$K(\beta) = \sum_{|\mu|=1}^{N} \frac{1}{\mu!} K_\mu \beta^\mu. \tag{4.4}$$

Substituting (4.1), (4.2) and (4.4) into (4.3) and equating coefficients of the same order in z and β, one can solve recursively for the unknown coefficients g_ν, $H_{\nu\mu}$ and K_μ by applying the Fredholm alternative, and taking inverses or bordered inverses as described in the next Section.

4.2. Solvability

In this Section we are interested in solving the system

$$\left(\lambda I - A^{\odot\star}\right) \begin{pmatrix} \psi_0 \\ \psi(\theta) \end{pmatrix} = \begin{pmatrix} c \\ \varphi \end{pmatrix}, \tag{4.5}$$

where $\psi_0 = \psi(0)$. Here $(c, \varphi) \in \mathbb{R}^n \times C([-h, 0], \mathbb{R}^n)$ is given. There are two situations two consider depending on weather λ is an eigenvalues or not. First we suppose that λ is not an eigenvalue of the generator A, then by Corollary 1 it belongs to the resolvent $\rho(A)$ of A. In this case there is an unique solution, which can be found by the variations-of-constants formula.

Corollary 4.1. [14, Corollary IV.5.4] *The resolvent* $(\lambda I - A^{\odot\star})^{-1}$ *has the following representation:*

$$(\lambda I - A^{\odot\star})^{-1} \begin{pmatrix} c \\ \varphi \end{pmatrix} = \begin{pmatrix} \psi(0) \\ \psi \end{pmatrix},$$

where

$$\psi(\theta) = e^{\lambda\theta}\psi_0 + \int_\theta^0 e^{-\lambda\sigma}\varphi(\sigma)\, d\sigma, \qquad (-h \le \theta \le 0),$$

with

$$\psi_0 = \Delta(\lambda)^{-1} \left[c + \int_0^h d\zeta(\tau) \int_0^\tau e^{-\lambda\sigma}\varphi(\sigma - \tau)\, d\sigma \right]$$

Now suppose that λ is an eigenvalue, then there may not be a solution, and if there is, it may not be unique. The *Fredholm alternative* gives sufficient and necessary conditions for the system (4.5) to be solvable.

Lemma 4.2. [46, Lemma 33] *(*Fredholm solvability*) Let λ be arbitrary. Then* (4.5) *has a solution* $(\psi_0, \psi(\theta)) \in \mathcal{D}(A^{\odot\star})$ *if and only of* (c, φ) *annihilates* $N(\lambda I - A^{\odot})$*, i.e. if and only if*

$$\langle \phi^{\odot}, (c, \varphi) \rangle = 0, \qquad \forall \phi^{\odot}(\lambda I - A^{\odot}).$$

Proposition 4.3. *Let λ be a simple eigenvalue of Δ. Let $\{q_0\}$ and $\{p_1\}$ be Jordan chains for $\Delta(\lambda)$, $\Delta(\lambda)^T$ respectively, i.e.*

$$\Delta(\lambda)q_0 = 0, \qquad p_1\Delta(\lambda) = 0. \tag{4.6}$$

Then the product $\langle p_1, q_0\rangle \neq 0$ and the bordered system

$$\begin{pmatrix} \Delta(\lambda) & q_0 \\ p_1^T & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}, \tag{4.7}$$

is nonsingular. It defines the unique solution x such that $\Delta(\lambda)x = y$ and $\langle p_1, x\rangle = 0$ if and only if $p_1^T y = 0$. We denote this solution by $x = \Delta^{INV}(\lambda)y$.

Proof. Equation (4.6) follows from the definition of a Jordan chain. We have the following decomposition

$$\begin{aligned}\mathbb{R}^n &= \mathcal{R}(\Delta(\lambda)) \oplus \mathcal{N}(\Delta(\lambda)) \\ &= \mathcal{N}(\Delta^T(\lambda))^\perp \oplus \mathcal{N}(\Delta(\lambda)).\end{aligned}$$

Since $\mathcal{N}(\Delta(\lambda)) = q_0$ and $\mathcal{N}(\Delta^T(\lambda))^\perp = \{x \in \mathbb{R}^n : \langle p_1, x\rangle = 0\}$ it follows that $\langle p_1, q_0\rangle$ is nonvanishing. Inspecting the null-space of the bordered system in (4.7) yields the system

$$\begin{cases} \Delta(\lambda)x + aq_0 &= 0, \\ p_1^T x &= 0. \end{cases}$$

Multiplying the first equation with p_1 gives that $a = 0$. Since x cannot be in the span of q_0 by the second equation, it follows that $x = 0$. Lastly consider the system

$$\begin{cases} \Delta(\lambda)x + aq_0 &= y, \\ p_1^T x &= 0. \end{cases}$$

By the Fredholm alternative we have $\Delta(0)x = y$ if and only if $p_1^T y = 0$. Assuming that $p_1^T y = 0$ is satisfied, it follows that

$$p_1^T \Delta(\lambda)x + ap_1^T p_1 = a\|p_1\|^2 = p_1^T y = 0.$$

Therefore, $a = 0$ and $\Delta(\lambda)x = y$. □

Proposition 4.4. *Let λ be a double eigenvalue of Δ. Let $\{q_0, q_1\}$ and $\{p_1, p_0\}$ be Jordan chains for $\Delta(\lambda)$, $\Delta(\lambda)^T$ respectively, then*

$$\begin{pmatrix} \Delta(\lambda) & 0_n \\ \Delta'(\lambda) & \Delta(\lambda) \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = 0, \qquad \begin{pmatrix} p_0 & p_1 \end{pmatrix}\begin{pmatrix} \Delta(\lambda) & 0_n \\ \Delta'(\lambda) & \Delta(\lambda) \end{pmatrix} = 0$$

holds.

Proof. By definition

$$\Delta(z)\,[q_0 + (z-\lambda)q_1] = \mathcal{O}((z-\lambda))$$

holds at $z = \lambda$. Thus $\Delta(\lambda)q_0 = 0$. Differentiation with respect to z yields

$$\Delta'(z)\,[q_0 + (z-\lambda)q_1] + \Delta(z)q_1 = 0.$$

Evaluating at $z = \lambda$ proves the statement for the chain $\{q_0, q_1\}$. The statement for the chain $\{p_1, p_0\}$ follows similarly. □

Remark 4.5. This result can be generalized to Jordan chain of length $n \in \mathbb{N}$, see [14, Chapter IV Exercise 5.11].

Proposition 4.6. *Suppose that λ is a double eigenvalue of A. Let (q_0, q_1) and (p_1, p_0) be Jordan chains for $\Delta(\lambda)$, $\Delta(\lambda)^T$ respectively, then the augmented system*

$$\begin{pmatrix} \Delta(\lambda) & p_1 \\ q_0^T & 0 \end{pmatrix} \begin{pmatrix} x \\ a \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}, \tag{4.8}$$

is nonsingular. It defines the unique solution x such that $\Delta(\lambda)x = y$ and $\langle q_0, x\rangle = 0$ if and only if $p_1^T y = 0$. We denote this solution by $x = \Delta^{INV}(\lambda)y$.

Remark. Note that the same symbol Δ^{INV} is used as in Proposition 4.3. It will be clear from the contents which bordered system should used.

Proof. Consider the equation

$$\begin{pmatrix} \Delta(\lambda) & p_1 \\ q_0^T & 0 \end{pmatrix} \begin{pmatrix} x \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which is equivalent to the system

$$\begin{cases} \Delta(\lambda)x + ap_1 &= 0, \\ q_0^T x &= 0. \end{cases} \tag{4.9}$$

Since

$$p_1^T \Delta(\lambda)x + ap_1^T p_1 = ap_1^T p_1 = 0,$$

we have $a = 0$. Then $x = cq_0$, for some constant c. The second equation in (4.9) then implies that $c = 0$, so that $x = 0$. Suppose now that $(x, a)^T$ is the solution to (4.8). Lastly consider the system

$$\begin{cases} \Delta(\lambda)x + ap_1 &= y, \\ q_0^T x &= 0. \end{cases}$$

By the Fredholm alternative we have $\Delta(0)x = y$ if and only if $p_1^T y = 0$. Assuming that $p_1^T y = 0$ is satisfied, it follows that

$$p_1^T \Delta(\lambda)x + ap_1^T p_1 = a\|p_1\|^2 = p_1^T y = 0.$$

Therefore, $a = 0$ and $\Delta(\lambda)x = y$. □

Proposition 4.7. [29, Proposition 3.6] *Suppose λ is simple eigenvalue of A and assume that (4.5) is consistent for a given $(c,\varphi) \in X^{\odot\star}$. Let the vectors $q, p \in \mathbb{R}^n$ and eigenfunctions $\phi, \phi^{\odot}$ be as in Lemma 3.6. Assume that the eigenfunctions are normalized to $\langle \phi, \phi^{\odot} \rangle = 1$, then*

$$\psi(\theta) = e^{\lambda\theta}\psi_0 + \int_\theta^0 e^{\lambda(\theta-s)}\varphi(s)\,ds \qquad (\theta \in [-h,0])$$

with

$$\psi_0 = \xi + \gamma q, \qquad \xi = \Delta(\lambda)^{INV}\left[c + \int_0^h d\eta(\tau)\int_0^\tau e^{-\lambda s}\varphi(s-\tau)\,ds\right].$$

For the constant γ given by

$$\gamma = -p^T\Delta'(\lambda)\xi - p^T\int_0^h\int_\tau^h e^{-\lambda s}d\eta(s)\int_{-\tau}^0 e^{\lambda\sigma}\varphi(\sigma)\,d\sigma\,d\tau.$$

the pairing $\langle \phi_0^{\odot}, \psi \rangle = 0$.

Corollary 4.8. [29, Corollary 3.7] *Suppose in addition that $(c,\varphi) = (\zeta, 0) + \kappa\phi$, where $\zeta \in \mathbb{R}^n$ is an arbitrary vector and κ is a scalar. Then*

$$\psi_0 = \xi + \gamma q, \qquad \psi(\theta) = e^{\lambda\theta}\left(\psi_0 - \kappa\theta q\right) \qquad (\theta \in [-h,0])$$

with

$$\xi = \Delta(\lambda)^{INV}\left[\zeta + \kappa\Delta'(\lambda)q\right], \qquad \text{and} \qquad \gamma = -p^T\Delta'(\lambda)\xi + \frac{1}{2}\kappa p^T\Delta''(\lambda)q.$$

For readability purposes we will use the notation $\psi(\theta) = B_\lambda^{INV}(\zeta,\kappa)$ whenever convenient.

Proposition 4.9. *Suppose λ is a double eigenvalue of A and assume that (4.5) is consistent for a given $(c,\varphi) \in X^{\odot\star}$. Let $q_0, q_1, p_0 p_1 \in \mathbb{R}^n$ be as in Lemma 3.8, then*

$$\psi(\theta) = e^{\lambda\theta}\psi_0 + \int_\theta^0 e^{\lambda(\theta-s)}\varphi(s)\,ds \qquad (\theta \in [-h,0])$$

with

$$\psi_0 = \xi + \gamma q_0, \qquad \xi = \Delta(\lambda)^{INV}\left[c + \int_0^h d\eta(\tau)\int_0^\tau e^{-\lambda\theta}\varphi(s-\tau)\,ds\right].$$

The constant γ may be arbitrary, however when

$$\begin{aligned}\gamma = &-p_0^T\Delta'(\lambda)\xi - \frac{1}{2}p_1^T\Delta''(\lambda)\xi \\ &- p_0^T\int_0^h\int_0^s e^{-\lambda s}\int_{-\tau}^0 e^{-\lambda\sigma}\varphi(\sigma)\,d\sigma\,d\tau\,d\zeta(s) \\ &- p_1^T\int_0^h\int_0^s e^{-\lambda s}(\tau - s)\int_{-\tau}^0 e^{-\lambda\sigma}\varphi(\sigma)\,d\sigma\,d\tau\,d\zeta(s),\end{aligned}$$

then the pairing $\langle \phi_0^{\odot}, \psi \rangle = 0$.

Proof. From formula (2.16) we have seen that

$$A^{\odot\star}(\alpha,\varphi)=(\int_0^h d\zeta(\theta)\psi(t-\theta),\dot{\psi}),$$

it follows that

$$(\lambda I-A^{\odot\star})\begin{pmatrix}\psi_0\\ \psi\end{pmatrix}=\begin{pmatrix}\lambda\psi_0-\int_0^h d\zeta(\theta)\psi(-\theta)\\ \lambda\psi-\dot{\psi}\end{pmatrix}=\begin{pmatrix}c\\ \varphi\end{pmatrix}. \tag{4.10}$$

By the variations-of-constants formula we have

$$\psi(\theta)=e^{\lambda\theta}\psi_0+\int_\theta^0 e^{\lambda(\theta-s)}\varphi(s)\,ds \qquad (\theta\in[-h,0])\,.$$

Substituting into the first equation of (4.10) yields

$$\Delta(\lambda)\psi_0=\left\{c+\int_0^h d\zeta(\tau)\int_0^\tau e^{-\lambda s}\varphi(s-\tau)\,ds\right\}.$$

It follows that

$$\psi_0=\Delta(\lambda)^{INV}\left\{c+\int_0^h d\zeta(\tau)\int_0^\tau e^{-\lambda s}\varphi(s-\tau)\,ds\right\}+\gamma q_0,$$

where γ is some constant. Now define

$$\tilde{\psi}(\theta)=e^{\lambda\theta}\xi+\int_\theta^0 e^{\lambda(\theta-s)}\varphi(s)\,ds,$$

so that

$$\psi(\theta)=\tilde{\psi}(\theta)+\gamma\phi_0(\theta).$$

Pairing with $\phi_0^\odot$ yields

$$\langle\phi_0^\odot,\psi\rangle=\left\langle\phi_0^\odot,\tilde{\psi}+\gamma\phi_0\right\rangle=\left\langle\phi_0^\odot,\tilde{\psi}\right\rangle+\gamma,$$

thus for $\gamma=-\left\langle\phi_0^\odot,\tilde{\psi}(\theta)\right\rangle$ the pairing $\langle\phi_0^\odot,\psi\rangle$ vanishes. A straightforward calculation yields

$$
\begin{aligned}
\left\langle \phi_0^{\odot}, \tilde{\psi}(\theta) \right\rangle &= \int_0^h d\phi_0^{\odot}(\tau)\tilde{\psi}(-\tau) \\
&= p_0^T\xi + \int_0^h \left[\phi_0^{\odot}\right]'(\tau)\tilde{\psi}(-\tau)d\tau \\
&= p_0^T\xi + p_0^T \int_0^h \int_\tau^h e^{\lambda(\tau-s)} d\zeta(s)\tilde{\psi}(-\tau)d\tau \\
&\quad + p_1^T \int_0^h \int_\tau^h e^{\lambda(\tau-s)}(\tau-s)d\zeta(s)\tilde{\psi}(-\tau)d\tau \\
&= p_0^T\xi + p_0^T \int_0^h \int_\tau^h e^{\lambda(\tau-s)} d\zeta(s) \left(e^{-\lambda\tau}\xi + \int_{-\tau}^0 e^{-\lambda(\tau+\sigma)}\varphi(\sigma)\, d\sigma \right) d\tau \\
&\quad + p_1^T \int_0^h \int_\tau^h e^{\lambda(\tau-s)}(\tau-s)d\zeta(s) \left(e^{-\lambda\tau}\xi + \int_{-\tau}^0 e^{-\lambda(\tau+\sigma)}\varphi(\sigma)\, d\sigma \right) d\tau \\
&= p_0^T\xi + p_0^T \int_0^h \int_\tau^h e^{-\lambda s} d\zeta(s) \left(\xi + \int_{-\tau}^0 e^{-\lambda\sigma}\varphi(\sigma)\, d\sigma \right) d\tau \\
&\quad + p_1^T \int_0^h \int_\tau^h e^{-\lambda s}(\tau-s)d\zeta(s) \left(\xi + \int_{-\tau}^0 e^{-\lambda\sigma}\varphi(\sigma)\, d\sigma \right) d\tau \\
&= p_0^T\xi + \frac{1}{2}p_0^T \int_0^h \int_\tau^h e^{-\lambda s} \left(\xi + \int_{-\tau}^0 e^{-\lambda\sigma}\varphi(\sigma)\, d\sigma \right) d\tau d\zeta(s) \\
&\quad + p_1^T \int_0^h \int_0^s e^{-\lambda s}(\tau-s) \left(\xi + \int_{-\tau}^0 e^{-\lambda\sigma}\varphi(\sigma)\, d\sigma \right) d\tau d\zeta(s) \\
&= p_0^T\Delta'(\lambda)\xi + p_1^T\Delta''(\lambda)\xi + p_0^T \int_0^h \int_\tau^h e^{-\lambda s} \left(\int_{-\tau}^0 e^{-\lambda\sigma}\varphi(\sigma)\, d\sigma \right) d\tau d\zeta(s) \\
&\quad + p_1^T \int_0^h \int_0^s e^{-\lambda s}(\tau-s) \left(\int_{-\tau}^0 e^{-\lambda\sigma}\varphi(\sigma)\, d\sigma \right) d\tau d\zeta(s)
\end{aligned}
$$

□

Corollary 4.10. *Suppose that in addition that $\lambda = 0$, then*

$$\psi(\theta) = \xi + \gamma q_0 - \int_\theta^0 \varphi(s)\, ds \qquad (\theta \in [-h, 0])$$

with

$$\xi = \Delta(0)^{INV} \left[-c - \int_0^h d\eta(\tau) \int_0^\tau \varphi(s-\tau)\, ds \right]$$

is the solution to the system

$$A^{\odot\star} \begin{pmatrix} \psi(0) \\ \psi(\theta) \end{pmatrix} = \begin{pmatrix} c \\ \varphi(\theta) \end{pmatrix}.$$

In order for the pairing $\langle \phi_0^{\odot}, \psi \rangle$ *to vanish the constant* γ *must satisfy*

$$\begin{aligned}\gamma = &-p_0^T \Delta'(0)\xi - \frac{1}{2} p_1^T \Delta''(0)\xi \\ &- p_0^T \int_0^h \int_0^s \int_{-\tau}^0 \varphi(\sigma)\, d\sigma\, d\tau\, d\zeta(s) \\ &- p_1^T \int_0^h \int_0^s (\tau - s) \int_{-\tau}^0 \varphi(\sigma)\, d\sigma\, d\tau\, d\zeta(s).\end{aligned}$$

5. Detection and location of BT-points in DDEs

In general a Bogdanov-Takens point is encountered while continuing either a Hopf or a fold bifurcation. In Section 5.1, we first describe the method used for detecting a Bogdanov-Takens point while continuing a Hopf curve in `DDE-BifTool` [17], and compare this with the detection method used in `MatCont` [11, 12]. In this Section we will also describe the method to detect a Bogdanov-Takens point while continuing a fold bifurcation curve.

After detecting a BT-point we need to locate the point accurately in order to calculate the normal form coefficients. This will be done in Section 5.2 by applying Newton-Raphson to a special defining system based on the Jordan chain of the characteristic matrix.

5.1. Detection

5.1.1. Detection while continuing fold

In `DDE-BifTool` a fold bifurcation curve is continued using the defining system

$$\begin{cases} f(x^0, x^0, \dots, x^0, \alpha) &= 0, \\ \Delta(x^0, \alpha, 0) q_0 &= 0, \\ c^T q_0 - 1 &= 0, \end{cases} \tag{5.1}$$

where $\alpha \in \mathbb{R}^2$. At a fold point the characteristic matrix $\Delta(x^0, \alpha, 0)$ has a one simple zero eigenvalue.

From [14, Corollary IV.5.12] we have to following result

Lemma 5.1. *Let λ be a simple zero of $\det \Delta$. The Jordan chain for Δ at λ has rank one and is given by q_0 with $\Delta(\lambda) q_0 = 0$. The corresponding eigenvector of A is given by*

$$\phi_\lambda(\theta) = p_1 e^{\lambda\theta}, \qquad -h \le \theta \le 0.$$

The Jordan chain for Δ^T at λ has rank one and is given by p_1 with $p_1^T \Delta(\lambda) = 0$. The corresponding eigenfunction of $A^\star$ is given by

$$\phi_\lambda^\odot(\theta) = p_1 + \int_0^\theta \left(\int_\sigma^h e^{\lambda(\sigma-\tau)} d\zeta^T(\tau) \right) d\sigma \, p_1, \qquad 0 < \theta \le h.$$

Furthermore,

$$\left\langle \phi_\lambda^\odot, \phi_\lambda \right\rangle = p_1 \Delta'(\lambda) q_0 \neq 0, \tag{5.2}$$

where Δ' *denotes the derivative of* Δ.

When continuing a fold bifurcation point a Bogdanov-Takens bifurcation can be encountered in the event of an additional zero eigenvalue. In this situation, we have from Proposition 4.6, vectors $q_0, q_1, p_1, p_0 \in \mathbb{R}^n$ such that

$$\begin{aligned} \Delta(x^0, \alpha, 0) q_0 &= 0, \\ \Delta'(x^0, \alpha, 0) q_0 &= -\Delta(x^0, \alpha, 0) q_1, \\ p_1^T \Delta(x^0, \alpha, 0) &= 0, \\ p_1^T \Delta(x^0, \alpha, 0)' &= -p_0^T \Delta(x^0, \alpha, 0). \end{aligned} \tag{5.3}$$

holds. Therefor, the identity in (5.2) vanishes

$$p_1 \Delta'(\lambda) q_0 = p_0^T \Delta(0) q_0 = 0$$

at a Bogdanov-Takens point.

Corollary 5.2. *To test for a Bogdanov-Takens point while continuing a fold point we monitor sign of the function* $\psi_{BT}^f = p_1^T \Delta'(\lambda) q_0$ *and the derivative of* ψ_{BT}^f *with respect to the parameters.*

The monitoring of the derivative is necessary since the functions depends on the vectors p_1^T and q_0.

5.1.2. Detection while continuing Hopf

A Bogdanov-Takens point can be detected while continuing either a Hopf or a fold bifurcation curve. In `DDE-BifTool` a Hopf bifurcation point is continued using the defining system

$$\begin{cases} f(x^0, x^0, \ldots, x^0, \alpha) &= 0 \\ \Delta(x^0, \alpha, i\omega) v &= 0 \\ c^T v - 1 &= 0. \end{cases} \tag{5.4}$$

The vector $c \in \mathbb{R}^n$ is chosen as $c = v^{(0)} / ({v^{(0)}}^T v^{(0)})$ where $v^{(0)}$ is the initial value of v.

The defining system (5.4) differs from the defining system used in `MatCont`

$$\begin{cases} f(x^0, \alpha) &= 0 \\ \left(A^2(x^0, \alpha) + \kappa I_n\right) v &= 0 \\ v^T v - 1 &= 0 \\ v^T l_0 &= 0, \end{cases} \tag{5.5}$$

where $A = D_x f(x, \alpha)$. The reference vector $l_0 \in \mathbb{R}^n$ is not orthogonal to the real two dimensional eigenspace of A corresponding to the eigenvalues $\lambda_1 + \lambda_2 = 0, \lambda_1 \lambda_2 = \kappa$. A solution to (5.5) with $\kappa > 0$ corresponds to the Hopf bifurcation point with $\omega^2 = \kappa$, while $\kappa < 0$ specifies a neutral saddle with two real eigenvalues $\lambda_{1,2} = \pm - \kappa$. A Bogdanov-Takens point is thus detected by monitoring the sign of κ.

To analyze the sign of ω in the defining system (5.4) while continuing a Hopf point and passing a Bogdanov-Takens point we consider the normal form of a Bogdanov-Takens point

$$\dot{w} = (\dot{w}_0, \dot{w}_1) = \begin{pmatrix} w_1 \\ \beta_1 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 \end{pmatrix}. \tag{5.6}$$

on the center manifold. Without loss of generality we assume that $a = 1$ and $b = -1$. Then there is a Hopf bifurcation curve at $\beta_2^H = -\sqrt{-\beta_1}$ for the equilibrium $E_1 = \left(i\sqrt{\beta_1}, 0\right)$, where $\beta_1 < 0$. On the curve β_2^H the eigenvalues and eigenvectors of E_1 are given by

$$\pm i\omega_0$$

and

$$v_{\mp}^H = \begin{pmatrix} \mp \dfrac{i}{\omega_0} \\ 1 \end{pmatrix},$$

respectively, where $\omega_0 = \sqrt{2}(-\beta_1)^{1/4}$. The characteristic matrix becomes

$$\Delta(x^0, \alpha, i\omega) = i\omega I_n - A,$$

where A is the Jacobian of (5.6). A simple calculation gives

$$\Delta(x^0, \alpha, i\omega) = \begin{pmatrix} i\omega & -1 \\ w_1 - 2w_0 & w_0 - \beta_2 + i\omega \end{pmatrix}.$$

Taking $c = (1, 0)^T$ the defining system (5.4) becomes

$$S(w, \beta, v, \omega) = \begin{pmatrix} w_1 \\ \beta_1 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 \\ i\omega v_0 - v_1 \\ v_0(w_1 - 2w_0) + v_1(w_0 - \beta_2 + i\omega) \\ v_0 - 1 \end{pmatrix}. \tag{5.7}$$

The Jacobian of S is given by

$$DS(w, \beta, v, \omega) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2aw_0 + bw_1 & bw_0 + \beta_2 & 1 & w_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i\omega & -1 & iv_0 \\ v_1 - 2v_0 & v_0 & 0 & -v_1 & w_1 - 2w_0 & i\omega + w_0 - \beta_2 & iv_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The kernel ker $DS(E_1, \beta_1, \beta_2^H, v_\mp^H, \pm\omega_0)$ on the Hopf curve is given by

$$\text{span}\left\{\begin{pmatrix} \pm\frac{2\beta_1}{\sqrt{2}(-\beta_1)^{3/4}\mp i\beta_1} \\ 0 \\ -\frac{4a\beta_1^{3/2}}{i\sqrt{2}(-\beta_1)^{3/4}+\beta_1} \\ 0 \\ 0 \\ \mp\frac{(-\beta_1)^{3/4}}{\sqrt{2}\beta_1} \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{\beta_1}{\mp i\sqrt{2}(-\beta_1)^{3/4}+\beta_1} \\ 0 \\ -\frac{2ia\beta_1^{3/2}}{\pm i\sqrt{2}(-\beta_1)^{3/4}+\beta_1} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}.$$

It follows that the tangent vector to the curve $(E_1(s), \beta_1(s), \beta_2^H(s), v_\mp^H(s), \pm\omega_0(s))$ has a nonvanishing component $\dot{\omega}(s) = 1$ for all s. Since at a Bogdanov-Takens, where we have $\omega_0 = 0$, we can use ω as a test function as in the ODE case. In this situation one can thus use the bisection method to locate the Bogdanov-Takens point, which only relays on the continuity of the curve $\omega(s)$. The transcritical Bogdanov-Takens bifurcation can be treated similar. We obtain the following Lemma.

Lemma 5.3. *A regular test function for a generic or transcritical Bogdanov-Takens point encountered on a Hopf curve is*

$$\psi_{BT}^H = \omega_0.$$

5.2. Defining systems for BT-points in DDEs

Suppose the generator A of the C_0-semigroup associated to (3.3) has a double zero eigenvalue at $(x(t), x(t-\tau_1), \ldots, x(t-\tau_m), \alpha) = (x^0, x^0, \ldots, x^0, \alpha_0)$. Then by Lemma 3.8 there are nonzero vectors $v, w \in \mathbb{R}^n$ such that

$$\Delta(x^0, \alpha_0, 0)v = 0 \tag{5.8}$$

and

$$\Delta'(x^0, \alpha_0, 0)v = -\Delta(x^0, \alpha_0, 0)w. \tag{5.9}$$

A possible defining system could thus contain $\tilde{f}(x) = f(x, x, \ldots, x, \alpha) = 0$, $\Delta(x, \alpha, 0)v = 0$ and $\Delta'(x, \alpha, 0)v + \Delta(x, \alpha, 0)w = 0$. Since $\tilde{f} : \mathbb{R}^{n+2} \to \mathbb{R}^n$ and $\Delta(x, \alpha, 0) : \mathbb{R}^n \to \mathbb{R}^n$ we need 2 more constraints. Any multiple of v in (5.8) still satisfies (5.8). As a first constraint we can therefore demand that $(v, v) = 1$. Notice that we need to scale w with the same factor $1/(v, v)$ such that (5.9) is still satisfied. We now notice that q_1 in (5.9) can be replaced by $w + cv$, for some constant c. It follows that $(w + cv, v) = (w, v) + c$. Thus for $c = -(w, v)$ we have $(\tilde{w}, q_0) = 0$, where $\tilde{w} = w + cv$. For the second constraint we thus demand that $(w, v) = 0$. We therefore calculate a BT-point by solving the defining system

$$\bar{S}(x,v,w,\alpha) = \begin{pmatrix} f(x,x,\ldots,x,\alpha) \\ \Delta(x,\alpha,0)v \\ \Delta'(x,\alpha,0)v + \Delta(x,\alpha,0)w \\ (v,v)-1 \\ (w,v) \end{pmatrix}. \tag{5.10}$$

In order to apply the standard Newton iteration procedure we need (5.10) to be regular at a BT-point. Before given regularity conditions we will first look what has been done in the ODE case in the article [3]. Then we will look at a defining system in the DDE case in which only one time delay is assumed in the article [54].

5.2.1. Defining systems with implicitly defined functions

In [21] a defining system for computing a BT-point is given for which the algebraic multiplicity of the zero eigenvalue at the critical value may be greater or equal to 2. In [3] the same defining system is used when the algebraic multiplicity is equal to 2. Since this is the case of interests for us now we will describe the defining system given there. For a moment we therefore consider the ODE

$$\dot{u}(t) = f(u(t),\alpha), \tag{5.11}$$

where $u(t) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $f \in C^k(\mathbb{R}^{n+2}, \mathbb{R}^n), k \geq 3$. We assume that (u^0, α^0) is a BT-point of (5.11), that is (u^0, α^0) is a stationary point of (5.11) and

$$J_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{5.12}$$

is the only entry in the Jordan normal form of

$$f_u^0 = \frac{\partial f}{\partial u}(u^0, \alpha^0), \tag{5.13}$$

which belongs to the zero eigenvalue. Let us assume that we are given vectors $b_0, c_0 \in \mathbb{R}^n$ such that the $(n+1)\times(n+1)$ matrix (where ^{T} represents the transpose)

$$\tilde{A}(u,\alpha) = \begin{pmatrix} f_u(u,\alpha) & b_0 \\ c_0^T & 0 \end{pmatrix} \tag{5.14}$$

is non-singular for (u,α) in some domain $\Omega \in \mathbb{R}^{n+2}$. Then we calculate a BT-point by solving the following defining system

$$\tilde{S}(u,\alpha) = \begin{pmatrix} f(u,\alpha) \\ g(u,\alpha) \\ h(u,\alpha) \end{pmatrix} = 0. \tag{5.15}$$

Here the functions $g, h \in C^{k-1}(\Omega, \mathbb{R})$ are implicitly defined through

$$\tilde{A}\begin{pmatrix} v \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \tilde{A}\begin{pmatrix} w \\ h \end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix} \qquad \text{for } (u,\alpha)\in\Omega. \tag{5.16}$$

The vanishing of g and h in (5.16) implies that

$$\tilde{A}v = 0 \qquad \tilde{A}w = v. \tag{5.17}$$

It follows that v and w are in span of the generalized eigenspace of the zero eigenvalue. However, it does not imply that the generalized eigenspace is 2-dimensional. Thus care should be taken when finding a solution to the defining system (5.15).

For the Newton process we need to calculate the Jacobian of $\tilde{S}(u, \alpha)$.

Lemma 5.4. [3] *In addition to (5.16) define the function* $\bar{g}, \bar{h} \in C^{k-1}(\Omega, \mathbb{R})$ *and* $\Psi, \zeta \in C^{k-1}(\Omega, \mathbb{R}^n)$ *by the adjoint equations*

$$\begin{pmatrix}\Psi^T & \bar{g}\end{pmatrix} \tilde{A} = \begin{pmatrix}0 & 1\end{pmatrix}, \qquad \begin{pmatrix}\zeta^T & \bar{h}\end{pmatrix} \tilde{A} = \begin{pmatrix}\Psi^T & 0\end{pmatrix}. \tag{5.18}$$

Then the following relations hold for all $z \in (u, \alpha) \in \Omega$

$$g = \bar{g} = -\Psi^T f_u v, \qquad h = \bar{h} = \Psi^T v \tag{5.19}$$

and

$$g_z = -\Psi^T f_{uz} v, \qquad h_z = -\Psi^T f_{uz} w - \zeta^T f_{uz} v. \tag{5.20}$$

It follows that the Jacobian of (5.15) can be calculated by

$$\tilde{S}' = \begin{pmatrix} f_u & f_\alpha \\ g_u & g_\alpha \\ h_u & h_\alpha \end{pmatrix} = \begin{pmatrix} f_u & f_\alpha \\ -\Psi^T f_{uu} v & -\Psi^T f_{u\alpha} v \\ -\Psi^T f_{uu} w - \zeta^T f_{uu} v & -\Psi^T f_{u\alpha} w - \zeta^T f_{u\alpha} v \end{pmatrix}. \tag{5.21}$$

For the Newton process we need the Jacobian $\tilde{S}'$ to be non-singular at a BT-point (u^0, α^0). Let the superscript '0' always denote evaluation (u^0, α^0).

Lemma 5.5. [3] *Let* (u^0, α^0) *be a BT-point of (5.11) in some domain* $\Omega \in \mathbb{R}^{n+2}$ *where the matrices (5.14) are non-singular. Then* (u^0, α^0) *is a regular solution of the defining system (5.15) if and only the following transversality conditions are satisfied*

$$0 \neq \Psi^{0T} f^0_\alpha, \tag{T1}$$

$$\begin{aligned} 0 \neq\ & \Psi^{0T} f^0_{uu} v^0 v^0 \left(\zeta^{0T} f^0_{uz} v^0 \delta_2 + \Psi^{0T} f^0_{uz} w^0 \delta_2\right) \\ & - \Psi^{0T} f^0_{uz} v^0 \delta_2 \left(\zeta^{0T} f^0_{uu} v^0 v^0 + \Psi^{0T} f^0_{uu} v^0 w^0\right). \end{aligned} \tag{T2}$$

The vector δ_2 is constructed in such a way that (5.11) is put into linear normal form (including parameters). For more information on the vector δ_2 and a proof of Lemma 5.5 we refer to [3]. The first transversality condition (T1) is equivalent to rank $(f_u, f_\alpha) = n$ and hence a necessary condition for the non-singularity of S'^0. In the second transversality condition (T2) we see the non-degeneracy condition $ab \neq 0$ for the co-dim 2 Bogdanov-Takens bifurcation.

We now consider a DDE of the form (3.3). By using Proposition 4.4 we can determine the analog defining system of (5.16) for DDEs. Let A denote the generator of the C_0-semigroup associated with the DDE. Suppose that $\alpha = 0$ is eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1. Let

$$\Delta^0 := \Delta(x^0, \alpha^0, 0) \tag{5.22}$$

and

$$\Delta := \Delta(x, \alpha, 0) \tag{5.23}$$

be the characteristic matrix at a BT-point and the characteristic matrix at a point (x, α) nearby the BT-point respectively. Then there are vectors v^0 and w^0 such that

$$\Delta^0 v^0 = 0, \tag{5.24}$$

$$\Delta^0 w^0 = -\Delta'^0 v^0. \tag{5.25}$$

Since the matrix $\Delta(x^0, \alpha^0, 0)$ has rank $n-1$ we can assume, as in (5.14), that there are vectors $b_0, c_0 \in \mathbb{R}^n$ such that

$$A(x, \alpha) = \begin{pmatrix} \Delta(x, x, \ldots, x, \alpha, 0) & b_0 \\ c_0^T & 0 \end{pmatrix} \tag{5.26}$$

is non-singular for some neighborhood $\Omega \in \mathbb{R}^{n+2}$ of (x^0, α^0). Then we calculate a BT-point by solving the following defining system

$$S(x, \alpha) = \begin{pmatrix} f(x, x, \ldots, x, \alpha) \\ g(x, \alpha) \\ h(x, \alpha) \end{pmatrix} = 0. \tag{5.27}$$

Here the functions $g, h \in C^{k-1}(\Omega, \mathbb{R})$ are implicitly defined through

$$A \begin{pmatrix} v \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad A \begin{pmatrix} w \\ h \end{pmatrix} = \begin{pmatrix} -\Delta'(x, \alpha, 0) v \\ 0 \end{pmatrix} \qquad \text{for } (x, \alpha) \in \Omega. \tag{5.28}$$

The equivalent of Lemma 5.4 becomes

Lemma 5.6. *In addition to (5.28) define the function* $\bar{g}, \bar{h} \in C^{k-1}(\Omega, \mathbb{R})$ *and* $\Psi, \zeta \in C^{k-1}(\Omega, \mathbb{R}^n)$ *by the adjoint equations*

$$\begin{pmatrix} \Psi^T & \bar{g} \end{pmatrix} A = \begin{pmatrix} 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} \zeta^T & \bar{h} \end{pmatrix} A = \begin{pmatrix} -\Psi^T \Delta'(x, \alpha, 0) & 0 \end{pmatrix}. \tag{5.29}$$

Then the following relations hold for all $z \in (x, \alpha) \in \Omega$

$$g = \bar{g} = -\Psi^T \Delta v, \tag{5.30}$$

$$h = \bar{h} = -\Psi^T \Delta' v \tag{5.31}$$

and

$$g_z = -\Psi^T \Delta_z v, \tag{5.32}$$

$$h_z = -\Psi^T \Delta_z w - \zeta^T \Delta_z v - \Psi^T \Delta'_z v \tag{5.33}$$

Proof. Equations (5.28) and (5.29) give the following 8 identities

$$\Delta v + b_0 g = 0, \tag{5.34}$$
$$c_0 v = 1, \tag{5.35}$$
$$\Delta w + b_0 h = -\Delta' v, \tag{5.36}$$
$$c_0 w = 0, \tag{5.37}$$
$$\Psi^T \Delta + c_0 \bar{g} = 0, \tag{5.38}$$
$$\Psi^T b_0 = 1, \tag{5.39}$$
$$\zeta^T \Delta + c_0 \bar{h} = -\Psi^T \Delta', \tag{5.40}$$
$$\zeta^T b_0 = 0. \tag{5.41}$$

Equations (5.34) and (5.39) gives $\Psi^T \Delta v + g = 0$. Equations (5.35) and (5.38) gives $\Psi^T \Delta v + \bar{g} = 0$. It follows that $g = \bar{g} = -\Psi^T \Delta v$. For the identity of the function h we first notice that equations (5.37) and (5.38) gives $\Psi^T \Delta w = 0$ and equations (5.34) and (5.41) gives $\zeta \Delta v = 0$. Then equations (5.36) and (5.39) gives $\Psi^T \Delta w + \Psi^T b_0 h = h = -\Psi^T \Delta' v$ and equations (5.35) and (5.40) gives $\zeta^T \Delta v + c_0^T \bar{h} v = \bar{h} = -\Psi^T \Delta' v$. Thus, we have $h = -\bar{h} = \Psi^T \Delta' v$. For the identity of g_z we differentiate (5.30), (5.34) and (5.38) with respect to $z = (x, \alpha)$

$$g_z = -\Psi_z^T \Delta v - \Psi^T \Delta_z v - \Psi^T \Delta v_z, \tag{5.42}$$
$$\Delta_z v + \Delta v_z + b_0 g_z = 0, \tag{5.43}$$
$$\Psi_z^T \Delta + \Psi^T \Delta_z + c_0 g_z = 0. \tag{5.44}$$

Multiplying (5.43) and (5.44) with Ψ^T from the left and v from the right respectively gives The last 2 equations together with (5.42) implies that $g_z = -\Psi^T \Delta_z v$. Lastly, for the identity of h_z, we differentiate (5.31), (5.36) and (5.40) with respect to $z = (x, \alpha)$

$$h_z = -\Psi_z^T \Delta' v - \Psi^T \Delta_z' v - \Psi^T \Delta' v_z, \tag{5.45}$$
$$\Delta_z w + \Delta w_z + b_0 h_z = -\Delta_z' v - \Delta' v_z, \tag{5.46}$$
$$\zeta_z^T \Delta + \zeta^T \Delta_z + c_0 h_z = -\Psi_z^T \Delta' - \Psi^T \Delta_z'. \tag{5.47}$$

Multiplying (5.46) and (5.47) with Ψ from the left and v from the right respectively gives

$$\Psi^T \Delta_z w + \Psi^T \Delta w_z + h_z = -\Psi^T \Delta_z' v - \Psi^T \Delta' v_z, \tag{5.48}$$
$$\zeta_z^T \Delta v + \zeta^T \Delta_z v + h_z = -\Psi_z^T \Delta' v - \Psi^T \Delta_z' v. \tag{5.49}$$

Using equations (5.37) and (5.41) we notice that $\Psi^T \Delta w_z = -g c_0 w_z = 0$ and $\zeta_z^T \Delta v = -\zeta_z^T b_0 g = 0$. Thus equations (5.48) and (5.49) can be simplified to

$$\Psi^T \Delta_z w + h_z = -\Psi^T \Delta_z' v - \Psi^T \Delta' v_z, \tag{5.50}$$
$$\zeta^T \Delta_z v + h_z = -\Psi_z^T \Delta' v - \Psi^T \Delta_z' v. \tag{5.51}$$

Solving (5.50) for $\Psi^T \Delta' v_z$ and (5.51) for $\Psi_z^T \Delta' v$ and substituting the result into (5.45) gives, after rearranging terms,

$$h_z = -\Psi^T \Delta_z w - \zeta^T \Delta_z v - \Psi^T \Delta_z' v. \tag{5.52}$$

□

The Jacobian of the defining system (5.27) is now given by

$$S'(x,\alpha) = \begin{pmatrix} f_x & f_\alpha \\ g_x & g_\alpha \\ h_x & h_\alpha \end{pmatrix} \tag{5.53}$$

$$= \begin{pmatrix} -\Delta & f_\alpha \\ -\Psi^T \Delta_x v & -\Psi^T \Delta_\alpha v \\ -\Psi^T \Delta_x w - \zeta^T \Delta_x v & -\Psi^T \Delta_\alpha w - \zeta^T \Delta_\alpha v \\ -\Psi^T \Delta_x' v & -\Psi^T \Delta_\alpha' v \end{pmatrix}. \tag{5.54}$$

Here we cannot mimic the proof of Lemma 5.5 for the non-singularity of S'. This is a result of replacing the Jacobian $f_u(u,\alpha)$ in (5.14) with the characteristic matrix at $(x,\alpha,0)$ in (5.26). Indeed at the zero eigenvector the Jordan normal form of $f_u(u,\alpha)$ has a Jordan block at a BT-point whereas the characteristic matrix at a BT-point in general does not.

To derive transversality condition for the non-singularity of S' at a BT-point we start by investigating when S' is injective. By expanding $S'(x^0,\alpha^0)\vartheta = 0$ with $\vartheta = (\vartheta_1, c_1, c_2) \in \Omega$, we obtain

$$-\Delta\vartheta_1 + c_1 f_{\alpha_1} + c_2 f_{\alpha_2} = 0, \tag{5.55}$$

$$-\Psi^T \Delta_x v \vartheta_1 - c_1 \Psi^T \Delta_{\alpha_1} v - c_2 \Psi^T \Delta_{\alpha_2} v = 0 \tag{5.56}$$

and

$$\begin{aligned} &-\Psi^T \Delta_x w \vartheta_1 - \zeta^T \Delta_x v \vartheta_1 - \Psi^T \Delta_x' v \vartheta_1 \\ &\quad - c_1 \Psi^T \Delta_{\alpha_1} w - c_2 \Psi^T \Delta_{\alpha_2} w - c_1 \zeta^T \Delta_{\alpha_1} v \\ &\quad\quad - c_2 \zeta^T \Delta_{\alpha_2} v - c_1 \Psi^T \Delta_{\alpha_1} v - c_2 \Psi^T \Delta_{\alpha_2} v = 0 \end{aligned} \tag{5.57}$$

Multiplying (5.55) by Ψ^T from the left gives

$$c_1 \Psi^T f_{\alpha_1} + c_2 \Psi^T f_{\alpha_2} = 0.$$

Assuming the transversality condition

$$\Psi f_{\alpha_1} \neq 0, \tag{5.58}$$

this yields

$$c_1 = -\frac{\Psi^T f_{\alpha_2}}{\Psi^T f_{\alpha_1}}\, c_2 := c_\alpha c_2 \tag{5.59}$$

If $\vartheta_1 \neq 0$ there is a vector $v_\alpha$such that $\vartheta_1 = c_2 v_\alpha$. It follows that from (5.55)

$$\vartheta_1 = c_2 v_\alpha + cv, \tag{5.60}$$

for some constant c.
Substituting (5.59) and (5.60) into (5.56) gives

$$-\Psi^T \Delta_x v(c_2 v_\alpha + cv) - c_\alpha c_2 \Psi^T \Delta_{\alpha_1} v - c_2 \Psi^T \Delta_{\alpha_2} v = 0 \tag{5.61}$$

Collecting terms in c_2 and c yields

$$-c_2 \Psi^T (\Delta_x v_\alpha + c_\alpha \Delta_{\alpha_1} + \Delta_{\alpha_2}) v - c\Psi^T \Delta_x vv = 0 \tag{5.62}$$

Substituting (5.59) and (5.60) into (5.57) gives

$$\begin{aligned} &-\Psi^T \Delta_x w(c_2 v_\alpha + cv) - \zeta^T \Delta_x v(c_2 v_\alpha + cv) - \Psi^T \Delta'_x v(c_2 v_\alpha + cv) \\ &\qquad - c_\alpha c_2 \Psi^T \Delta_{\alpha_1} w - c_2 \Psi^T \Delta_{\alpha_2} w - c_\alpha c_2 \zeta^T \Delta_{\alpha_1} v \\ &\qquad\qquad - c_2 \zeta^T \Delta_{\alpha_2} v - c_\alpha c_2 \Psi^T \Delta_{\alpha_1} v - c_2 \Psi^T \Delta_{\alpha_2} v = 0 \end{aligned}$$

Collecting terms in c_2 and c yields

$$\begin{aligned} &-c_2 \Psi^T (\Delta'_x v_\alpha + c_\alpha \Delta_{\alpha_1} + \Delta_{\alpha_2}) v \\ &\qquad - c_2 \Psi^T (\Delta_x v_\alpha + c_\alpha \Delta_{\alpha_1} + \Delta_{\alpha_2}) w - c_2 \zeta^T (\Delta_x v_\alpha + c_\alpha \Delta_{\alpha_1} + \Delta_{\alpha_2}) v \\ &\qquad\qquad - c(\Psi^T \Delta_x w + \zeta^T \Delta_x v + \Psi^T \Delta'_x v) v = 0. \end{aligned}$$

We impose a second transversality condition

$$d_0 := \det \begin{pmatrix} \Psi^T A_1 v & \Psi^T B_1 v \\ \Psi^T A_1 w + \zeta^T A_1 v + \Psi^T A_2 v & \Psi^T B_1 w + \zeta^T B_1 v + \Psi^T B_2 v \end{pmatrix} \neq 0 \tag{5.63}$$

where

$$A_1 := \Delta_x v, \qquad A_2 := \Delta'_x v \tag{5.64}$$

and

$$B_1 := \Delta_x v_\alpha + c_\alpha \Delta_{\alpha_1} + \Delta_{\alpha_2}, \qquad B_2 := \Delta'_x v_\alpha + c_\alpha \Delta_{\alpha_1} + \Delta_{\alpha_2}. \tag{5.65}$$

Lemma 5.7. *Suppose $\Psi^T f_{\alpha_1} \neq 0$ and $d_0 \neq 0$ are both satisfied. Then the defining system S is regular at a BT-point.*

Proof. Using the definitions of A_1, A_2, B_1 and B_2 in (5.62) and (5.62) yields

$$-c_2 \Psi^T B_1 v - c\Psi^T A_1 v = 0 \tag{5.66}$$

and

$$-c_2(\Psi^T B_1 w + \zeta^T B_1 v + \Psi^T B_2 v) - c(\Psi^T A_1 w + \zeta^T A_1 v + \Psi^T A_2 v) = 0.$$

respectively. Since $d \neq 0$ it follows that $c_2 = c = 0$. From (5.60) that $\vartheta = 0$. Surjectivity is proved in a similar way. □

5.2.2. Defining systems without implicitly defined functions

We are now also able to proof the regularity of the defining system (5.10).

Lemma 5.8. *Suppose $\Psi^T f_{\alpha_1} \neq 0$ and $d_0 \neq 0$ are both satisfied. Then the defining system (5.10) is regular at a BT-point.*

Proof. The Jacobian of $\bar{S}$ reads

$$\bar{S}'(w) = \begin{pmatrix} -\Delta & 0 & 0 & f_{\alpha_1} & f_{\alpha_2} \\ \Delta_x q_0 & \Delta & 0 & \Delta_{\alpha_1} q_0 & \Delta_{\alpha_2} q_0 \\ \Delta'_x q_0 + \Delta_x q_1 & \Delta' & \Delta & \Delta'_{\alpha_1} q_0 + \Delta_{\alpha_1} q_1 & \Delta'_{\alpha_2} q_0 + \Delta_{\alpha_2} q_1 \\ 0 & 2q_0 & 0 & 0 & 0 \\ 0 & q_1 & q_0 & 0 & 0 \end{pmatrix}, \tag{5.67}$$

where $w = (x, q_0, q_1, \alpha_1, \alpha_2)$. By expanding $\bar{S}'(x^0, q_0^0, q_1^0, \alpha_1^0, \alpha_2^0)\vartheta = 0$ with $\vartheta = (\vartheta_1, \vartheta_2, \vartheta_3, c_1, c_2) \in \Omega$, we obtain

$$-\Delta\vartheta_1 + c_1 f_{\alpha_1} + c_2 f_{\alpha_2} = 0, \tag{5.68}$$

$$\Delta_x v\vartheta_1 + \Delta\vartheta_2 + c_1\Delta_{\alpha_1} v + c_2\Delta_{\alpha_2} v = 0, \tag{5.69}$$

$$\begin{aligned}(\Delta'_x v + \Delta_x w)\vartheta_1 + \Delta'\vartheta_2 + \Delta\vartheta_3 \\ + c_1(\Delta'_{\alpha_1} v + \Delta_{\alpha_1} w) + c_2(\Delta'_{\alpha_2} v + \Delta_{\alpha_2} w) = 0,\end{aligned} \tag{5.70}$$

$$2v\vartheta_2 = 0 \tag{5.71}$$

and

$$w\vartheta_2 + v\vartheta_3 = 0. \tag{5.72}$$

We see that (5.68) and (5.55) are the equivalent. By multiplying (5.69) by Ψ^T from the left we get

$$\Psi^T\Delta_x v\vartheta_1 + c_1\Psi^T\Delta_{\alpha_1} v + c_2\Psi^T\Delta_{\alpha_2} v = 0, \tag{5.73}$$

which is equivalent to (5.56). Multiplying (5.69) and (5.70) from the left by ζ and Ψ^T respectively yields

$$\zeta\Delta_x v\vartheta_1 + \zeta\Delta\vartheta_2 + c_1\zeta\Delta_{\alpha_1} v + c_2\zeta\Delta_{\alpha_2} v = 0, \tag{5.74}$$

$$\begin{aligned}\Psi^T(\Delta'_x v + \Delta_x w)\vartheta_1 + \Psi^T\Delta'\vartheta_2 + \Psi^T\Delta\vartheta_3 \\ + c_1\Psi^T(\Delta'_{\alpha_1} v + \Delta_{\alpha_1} w) + c_2\Psi^T(\Delta'_{\alpha_2} v + \Delta_{\alpha_2} w) = 0.\end{aligned} \tag{5.75}$$

Adding the last two equations gives (5.57). Since the equations (5.55), (5.56) and (5.57) are equivalent to the equations (5.68), (5.69) and (5.70) if that, under the assumption $d_0 \neq 0$ that

$$c = c_2 = c_1 = \vartheta_1 = 0. \tag{5.76}$$

Substituting into (5.69) yields

$$\Delta\vartheta_2 = 0. \tag{5.77}$$

It follows that $\vartheta_2 = \tilde{c}v$, for some constant . By substituting into $\vartheta_2 = \tilde{c}v$ into (5.71) we obtain that $\tilde{c} = 0$ so that ϑ_2. In a similar way we substitute (5.76) into (5.70) to obtain

$$\Delta\vartheta_3 = 0. \tag{5.78}$$

Together with (5.72) gives that $\vartheta_3 = 0$. □

6. Coefficients of parameter-dependent normal forms

Using the method outlined in Section 4.1, will derive those coefficients needed for the predictors of the nonhyperbolic cycles emanating from generalized Hopf, fold-Hopf, Hopf-transcritical and Hopf-Hopf bifurcations, and the predictors of the homoclinic orbits emanating from generic and transcritical Bogdanov-Takens bifurcation, see Appendix C. We assume in all situations that $\varphi_0 = 0$ is a steady-state of the DDE (3.3) at the parameter value $\alpha_0 = 0 \in \mathbb{R}^2$.

6.1. Generic Bogdanov-Takens bifurcation

Let the eigenvectors ϕ_0, ϕ_1 and adjoint eigenvectors $\phi_1^{\odot}, \phi_0^{\odot}$ be as in Lemma 3.8, with $\lambda = 0$. The smooth normal form on the parameter-dependent center manifold takes the form

$$\begin{aligned}\dot{w} &= G(w,\beta)\\ &= \begin{pmatrix} w_1 \\ \beta_1 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 + g_1(w,\beta) \end{pmatrix}\\ &\quad + \mathcal{O}(\|\beta\|\, w_1^2) + \mathcal{O}(\|\beta\|^2 \|w\|^2 + \|\beta\|\, \|w\|^3 + \|w\|^4),\end{aligned} \tag{6.1}$$

where $w = (w_0, w_1)$, $\beta = (\beta_1, \beta_2)$ and

$$g_1(w,\beta_2) = (a_1\beta_2 + d w_0)\, w_0^2 + (b_1\beta_2 + e w_0)\, w_1 w_0.$$

This normal form can be derived from [7]. The functions $\mathcal{H}, K$ and R defined in Section 4.1 can be expanded as

$$
\begin{aligned}
\mathcal{H}(w,\beta) = &[\phi_0,\phi_1]\, w + [H_{0010}, H_{0001}]\,\beta + \frac{1}{2}H_{2000}w_0^2 + H_{1100}w_0w_1 + \frac{1}{2}H_{0200}w_1^2 \\
&+ H_{1010}\beta_1 w_0 + H_{1001}\beta_2 w_0 + H_{0110}\beta_1 w_1 + H_{0101}\beta_2 w_1 + \frac{1}{2}H_{0002}\beta_2^2 \\
&+ \frac{1}{6}H_{3000}w_0^3 + \frac{1}{2}H_{2100}w_0^2 w_1 + H_{1101}\beta_2 w_1 w_0 + \frac{1}{2}H_{2001}\beta_2 w_0^2 \\
&+ \mathcal{O}(|w_1|^3 + |w_0 w_1^2| + |\beta_2 w_1^2| + |\beta_1|\|w\|^2 + \|\beta\|^2\|w\| + \|\beta\|^3) \\
&+ \mathcal{O}(\beta_1^2 + |\beta_1\beta_2|) + \mathcal{O}(\|(w,\beta)\|^4), \qquad (6.2)\\
K(\beta) = &[K_{10}, K_{01}]\,\beta + \frac{1}{2}K_{02}\beta_2^2 + \mathcal{O}(\beta_1^2 + |\beta_1\beta_2|) + \mathcal{O}(\|\beta\|^3) \qquad (6.3)
\end{aligned}
$$

and

$$
\begin{aligned}
R(u,\alpha) = \Big(&\frac{1}{2}D_1^2 f(0,0)(u,u) + \frac{1}{6}D_1^3 f(0,0)(u,u,u) + D_2D_1 f(0,0)(u,\alpha) \\
&+ D_2 f(0,0)(\alpha) + \frac{1}{2}D_1^2 f(0,0)(\alpha,\alpha) + \frac{1}{2}D_2 D_1^2(u,u,\alpha) \\
&+ \mathcal{O}\left(\|u\|\,\|\alpha\|^2 + \|\alpha\|^3\right) + \mathcal{O}(\|u,\alpha\|^4)\Big) r^{\odot\star}.
\end{aligned}
$$

Using the notation used in [32, 1, 36], we write that last expression as

$$
\begin{aligned}
R(u,\alpha) = &\frac{1}{2}B(u,u) + \frac{1}{6}C(u,u,u) + A_1(u,\alpha) + \frac{1}{2}J_2(\alpha,\alpha) \\
&+ \frac{1}{2}B_2(u,u,\alpha) + \mathcal{O}\left(\|u\|\,\|\alpha\|^2 + \|\alpha\|^3\right) + \mathcal{O}(\|u,\alpha\|^4), \qquad (6.4)
\end{aligned}
$$

where

$$B(u,u) = D_1^2 f(0,0)(u,u) r^{\odot\star}$$

and similar for the other multilinear forms.

We insert the Taylor expansions (6.4), (6.2) and (6.3) into the homological equation

$$A^{\odot\star}\mathcal{H}(w,\beta) + J_1 K(\beta) + R(\mathcal{H}(w,\beta), K(\beta)) = D_w\mathcal{H}(w,\beta)\dot{w}, \qquad (6.5)$$

see (4.3). Collecting the w_0^2, w_0w_1 and w_1^2 terms in the homological equations lead to the systems

$$
\begin{aligned}
A^{\odot\star}H_{2000} &= 2a\phi_1 - B(\phi_0,\phi_0), \qquad (6.6)\\
A^{\odot\star}H_{1100} &= H_{2000} + b\phi_1 - B(\phi_0,\phi_1), \qquad (6.7)\\
A^{\odot\star}H_{0200} &= 2H_{1100} - B(\phi_1,\phi_1). \qquad (6.8)
\end{aligned}
$$

By pairing equations (6.6) and (6.7) with $\phi_1^\odot$ yields

$$a = \frac{1}{2} p_1^T B(\phi_0, \phi_0),$$
$$b = -\left\langle \phi_1^\odot, H_{2000} \right\rangle + p_1^T B(\phi_0, \phi_1).$$

Pairing equation (6.8) with $\phi_0^\odot$ yields

$$\left\langle \phi_1^\odot, H_{2000} \right\rangle = -p_0^T B(\phi_0, \phi_0),$$

so that

$$b = p_0^T B(\phi_0, \phi_0) + p_1^T B(\phi_0, \phi_1).$$

In order to solve for H_{2000}, H_{1100} and H_{0200} we write (6.8) as

$$\begin{aligned} A^{\odot\star} H_{2000} &= 2a \begin{pmatrix} q_1 \\ \theta q_0 + q_1 \end{pmatrix} - \begin{pmatrix} B(\phi_0, \phi_0) \\ 0 \end{pmatrix}, \\ A^{\odot\star} H_{1100} &= \begin{pmatrix} H_{2000}(0) \\ H_{2000}(\theta) \end{pmatrix} + b \begin{pmatrix} q_1 \\ \theta q_0 + q_1 \end{pmatrix} - \begin{pmatrix} B(\phi_0, \phi_1) \\ 0 \end{pmatrix}, \\ A^{\odot\star} H_{0200} &= 2 \begin{pmatrix} H_{1100}(0) \\ H_{1100}(\theta) \end{pmatrix} - \begin{pmatrix} B(\phi_1, \phi_1) \\ 0 \end{pmatrix}. \end{aligned} \tag{6.9}$$

Using Corollary 4.10 it follows that

$$H_{2000}(\theta) = \xi_1 + \left(\gamma_1 + a\theta^2\right) q_0 + 2a\theta q_1,$$

where

$$\xi_1 = \Delta(0)^{INV} \left[B(\phi_0, \phi_0) - 2a\Delta'(0) q_1 - a\Delta''(0) q_0 \right].$$

Here $\Delta(0)^{INV}$ is defined as in Proposition 4.6. In order for equation (6.8) to be solvable, we need

$$2 \left\langle \phi_1^\odot, H_{1100} \right\rangle - p_1^T B(\phi_1, \phi_1) = 0.$$

Let

$$\tilde{H}_{2000}(\theta) = \xi_1 + a\theta^2 q_0 + 2a\theta q_1,$$

so that

$$H_{2000}(\theta) = \tilde{H}_{2000}(\theta) + \gamma_1 \phi_0,$$

then

$$
\begin{aligned}
2\left\langle \phi_1^{\odot}, H_{1100}\right\rangle - p_1^T B(\phi_1,\phi_1) &= 2\left\langle \phi_1^{\odot}, H_{1100}(\theta)\right\rangle - p_1^T B(\phi_1,\phi_1),\\
&= 2\left\langle A^{\odot}\phi_0^{\odot}, H_{1100}(\theta)\right\rangle - p_1^T B(\phi_1,\phi_1),\\
&= 2\left\langle \phi_0^{\odot}, A^{\odot\star}H_{1100}(\theta)\right\rangle - p_1^T B(\phi_1,\phi_1),\\
&= 2\left\langle \phi_0^{\odot}, H_{2000} + b\phi_1 - B(\phi_0,\phi_1)\right\rangle\\
&\quad - p_1^T B(\phi_1,\phi_1),\\
&= 2\left\langle \phi_0^{\odot}, \tilde{H}_{2000}(\theta) + \gamma_1\phi_0\right\rangle - 2p_0^T B(\phi_0,\phi_1)\\
&\quad - p_1^T B(\phi_1,\phi_1),\\
&= 2\left\langle \phi_0^{\odot}, \tilde{H}_{2000}(\theta)\right\rangle + 2\gamma_1 - 2p_0^T B(\phi_0,\phi_1)\\
&\quad - p_1^T B(\phi_1,\phi_1).
\end{aligned}
$$

Since

$$
\begin{aligned}
\left\langle \phi_0^{\odot}, \tilde{H}_{2000}(\theta)\right\rangle &= p_0\left(\frac{a}{3}\Delta^{(3)}(0)q_0 + a\Delta''(0)q_1 + \Delta'(0)\xi_1\right)\\
&\quad + p_1\left(\frac{a}{12}\Delta^{(4)}(0)q_0 + \frac{a}{3}\Delta^{(3)}(0)q_1 + \frac{1}{2}\Delta''(0)\xi_1\right),
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\gamma_1 &= -p_0\left(\frac{a}{3}\Delta^{(3)}(0)q_0 + a\Delta''(0)q_1 + \Delta'(0)\xi_1\right)\\
&\quad - p_1\left(\frac{a}{12}\Delta^{(4)}(0)q_0 + \frac{a}{3}\Delta^{(3)}(0)q_1 + \frac{1}{2}\Delta''(0)\xi_1\right)\\
&\quad + p_0^T B(\phi_0,\phi_1) + \frac{1}{2}p_1^T B(\phi_1,\phi_1).
\end{aligned}
$$

Using the expression for H_{2000} we can calculate

$$
H_{1100}(\theta) = \xi_2 + \theta\xi_1 + \left(\gamma_1\theta + \frac{b}{2}\theta^2 + \frac{a}{3}\theta^3\right)q_0 + \left(b\theta + a\theta^2\right)q_1,
$$

where

$$
\begin{aligned}
\xi_2 = \Delta(0)^{INV}\Bigg[B(\phi_0,\phi_1) - \Delta'(0)\xi_1 - \left(\gamma_1\Delta'(0) + \frac{1}{2}b\Delta''(0) + \frac{a}{3}\Delta'''(0)\right)q_0\\
- \left(b\Delta'(0) + a\Delta''(0)\right)q_1\Bigg].
\end{aligned}
$$

The H_{0200} term is given by

$$
\begin{aligned}
H_{0200}(\theta) &= \xi_3 + 2\theta\xi_2 + \theta^2\xi_1\\
&\quad + \left(\gamma_1\theta^2 + \frac{b}{3}\theta^3 + \frac{a}{6}\theta^4\right)q_0 + \left(\frac{2a}{3}\theta^3 + b\theta^2\right)q_1,
\end{aligned}
$$

where

$$\xi_3 = \Delta(0)^{INV}\Big[B(\phi_1,\phi_1) - 2\Delta'(0)\xi_2 - \Delta''(0)\xi_1 + \left(-\gamma_1\Delta''(0) - \frac{b}{3}\Delta^{(3)}(0) - \frac{a}{6}\Delta^{(4)}(0)\right) q_0 + \left(-\frac{2a}{3}\Delta^{(3)}(0) - b\Delta''(0)\right) q_1\Big].$$

Collecting the linear $\beta_{1,2}$ terms yields the systems

$$\begin{aligned} A^{\odot\star}H_{0010} &= \phi_1 - J_1K_{10}, \\ A^{\odot\star}H_{0001} &= -J_1K_{01}, \end{aligned} \tag{6.10}$$

which we write as

$$\begin{aligned} A^{\odot\star}H_{0010} &= \begin{pmatrix} q_1 \\ \theta q_0 + q_1 \end{pmatrix} - \begin{pmatrix} J_1K_{10} \\ 0 \end{pmatrix}, \\ A^{\odot\star}H_{0001} &= -\begin{pmatrix} J_1K_{01} \\ 0 \end{pmatrix}. \end{aligned} \tag{6.11}$$

Here we made the implicit identification between J_1 and M, see (3.24). We have two alternatives to solve $H_{0010}, H_{0001}, K_{10}$ and K_{01}, either we follow [32, 1, 36], in which a 'big' system is used, or we follow Appendix A.1, where only scaling and translation is used. In the next two subSection we will derive both alternatives.

6.1.1. Solving $H_{0010}, H_{0001}, K_{10}$ and K_{01} with big system

Using Corollary 4.10 it follows that

$$H_{0010}(\theta) = \xi_4 + \frac{1}{2}\theta^2 q_0 + \theta q_1, \tag{6.12}$$

where

$$\Delta(0)\xi_4 = J_1K_{10} - \frac{1}{2}\Delta''(0)q_0 - \Delta'(0)q_1. \tag{6.13}$$

Similarly, we find that

$$H_{0001}(\theta) = \xi_5, \tag{6.14}$$

where

$$\Delta(0)\xi_5 = J_1K_{01}. \tag{6.15}$$

Combining equations (6.13) and (6.15) yields the $n \times (n+2)$ dimensional system

$$\begin{pmatrix} -\Delta(0) & J_1 \end{pmatrix} \begin{pmatrix} \xi_4 & \xi_5 \\ K_{10} & K_{01} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\Delta''(0)q_0 + \Delta'(0)q_1 & 0 \end{pmatrix}.$$

In order to determine (ξ_4, K_{10}) and (ξ_5, K_{01}) we extent the system with 2 more equations. To this end we start by collecting the $w_0\beta_1, w_0\beta_2, w_1\beta_1$ and $w_1\beta_2$-terms in the homological equation

$$A^{\odot\star}H_{1010} + A_1(\phi_0, K_{10}) = -B(\phi_0, H_{0010}) + H_{1100}, \tag{6.16}$$
$$A^{\odot\star}H_{1001} + A_1(\phi_0, K_{01}) = -B(\phi_0, H_{0001}), \tag{6.17}$$
$$A^{\odot\star}H_{0110} + A_1(\phi_1, K_{10}) = -B(\phi_1, H_{0010}) + H_{0200} + H_{1010},$$
$$A^{\odot\star}H_{0101} + A_1(\phi_1, K_{01}) = -B(\phi_1, H_{0001}) + H_{1001} + \phi_1.$$

Pairing these equations with $\phi_1^\odot$ yields

$$p_1^T A_1(\phi_0, K_{10}) + p_1^T B(\phi_0, H_{0010}) = \langle \phi_1^\odot, H_{1100} \rangle, \tag{6.18}$$
$$p_1^T A_1(\phi_0, K_{01}) + p_1^T B(\phi_0, H_{0001}) = 0, \tag{6.19}$$
$$p_1^T A_1(\phi_1, K_{10}) + p_1^T B(\phi_1, H_{0010}) = \langle \phi_1^\odot, H_{0200} \rangle + \langle \phi_1^\odot, H_{1010} \rangle, \tag{6.20}$$
$$p_1^T A_1(\phi_1, K_{01}) + p_1^T B(\phi_1, H_{0001}) = \langle \phi_1^\odot, H_{1001} \rangle + 1. \tag{6.21}$$

Pairing equations (6.16) and(6.17) with $\phi_0^\odot$ yields

$$p_0^T A_1(\phi_0, K_{10}) + p_0^T B(\phi_0, H_{0010}) = \langle \phi_0^\odot, H_{1100} \rangle - \langle \phi_1^\odot, H_{1010} \rangle, \tag{6.22}$$
$$p_0^T A_1(\phi_0, K_{01}) + p_0^T B(\phi_0, H_{0001}) = -\langle \phi_1^\odot, H_{1001} \rangle. \tag{6.23}$$

Adding (6.21) to (6.23) and (6.20) to (6.22) gives

$$p_0^T A_1(\phi_0, K_{01}) + p_1^T A_1(\phi_1, K_{01}) + p_0^T B(\phi_0, H_{0001}) + p_1^T B(\phi_1, H_{0001}) = 1, \tag{6.24}$$
$$p_0^T A_1(\phi_0, K_{10}) + p_1^T A_1(\phi_1, K_{10}) + p_0^T B(\phi_0, H_{0010}) + p_1^T B(\phi_1, H_{0010}) = \langle \phi_1^\odot, H_{0200} \rangle + \langle \phi_0^\odot, H_{1100} \rangle. \tag{6.25}$$

From the w_1^2-terms in the homological equations we obtain

$$A^{\odot\star}H_{0200} = 2H_{1100} - B(\phi_1, \phi_1),$$

from which we derive that

$$\langle \phi_1^\odot, H_{0200} \rangle = 2\langle \phi_0^\odot, H_{1100} \rangle - p_0^T B(\phi_1, \phi_1), \tag{6.26}$$
$$\langle \phi_1^\odot, H_{1100} \rangle = \frac{1}{2} p_1^T B(\phi_1, \phi_1). \tag{6.27}$$

Substituting (6.26) and (6.27) into (6.25) and (6.18) respectively, yields

$$p_1^T A_1(\phi_0, K_{10}) + p_1^T B(\phi_0, H_{0010}) = \frac{1}{2} p_1^T B(\phi_1, \phi_1), \tag{6.28}$$
$$p_0^T A_1(\phi_0, K_{10}) + p_1^T A_1(\phi_1, K_{10}) + p_0^T B(\phi_0, H_{0010}) + p_1^T B(\phi_1, H_{0010}) = 3\langle \phi_0^\odot, H_{1100} \rangle - p_0^T B(\phi_1, \phi_1). \tag{6.29}$$

Combining the last two equations with (6.19) and (6.24) gives the so-called "big" system

$$\begin{pmatrix} p_1^T B\phi_0 & p_1^T A_1\phi_0 \\ p_0^T B\phi_0 + p_1^T B\phi_1 & p_0^T A_1\phi_0 + p_1^T A_1\phi_1 \end{pmatrix} \begin{pmatrix} H_{0010} & H_{0001} \\ K_{10} & K_{01} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} p_1^T B(\phi_1,\phi_1) & 0 \\ 3\left\langle \phi_0^{\odot}, H_{1100}\right\rangle - p_0^T B(\phi_1,\phi_1) & 1 \end{pmatrix}. \tag{6.30}$$

From now on we assume that there are finitely many delays

$$0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_m = h.$$

This allows us to write the multilinear form B as

$$B_i = \begin{pmatrix} B_i^0 & B_i^1 & \cdots & B_i^m \end{pmatrix},$$

where

$$B_i^j = \begin{pmatrix} \dfrac{\partial f_i}{\partial x_1^j \partial x_1^j} & \cdots & \dfrac{\partial f_i}{\partial x_1^j \partial x_n^j} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_i}{\partial x_n^j \partial x_1^j} & \cdots & \dfrac{\partial f_i}{\partial x_n^j \partial x_n^j} \end{pmatrix}.$$

Using the formula given in (6.12) we see that

$$\begin{pmatrix} p_1^T B\phi_0 & p_1^T A_1\phi_0 \end{pmatrix} \begin{pmatrix} H_{0010} \\ K_{10} \end{pmatrix} = p_1 \begin{pmatrix} B^0\phi_0(0) & B^1\phi_0(-\tau_1) & \cdots & B^m\phi_0(-\tau_m) & p_1 A_1\phi_0 \end{pmatrix} \times \begin{pmatrix} \xi_4 \\ \xi_4 - \tau_1 q_1 + \frac{1}{2}\tau_1^2 q_0 \\ \vdots \\ \xi_4 - \tau_m q_1 + \frac{1}{2}\tau_m^2 q_0 \\ K_{10} \end{pmatrix} = \frac{1}{2} p_1^T B(\phi_1,\phi_1),$$

which we can write as

$$\begin{pmatrix} p_1^T \left(\Sigma_{j=0}^m B^j \phi_0(-\tau_j)\right) & p_1^T A_1\phi_0 \end{pmatrix} \begin{pmatrix} \xi_4 \\ K_{10} \end{pmatrix} = \frac{1}{2} p_1^T B(\phi_1,\phi_1) - p_1^T \left(\Sigma_{j=1}^m B^j \phi_0(-\tau_j) \left(-\tau_j q_1 + \frac{1}{2}\tau_j^2 q_0\right)\right).$$

Using the same method for the entry $p_0 B\phi_0 + p_1 B\phi_1$ yields the system

$$\begin{pmatrix} -\Delta(0) & J_1 \\ p_1^T\left(\Sigma_{j=0}^m B^j\phi_0(\tau_j)\right) & p_1^T A_1\phi_0 \\ p_0^T\left(\Sigma_{j=0}^m B^j\phi_0(\tau_j)\right) + p_1^T\left(\Sigma_{j=0}^m B^j\phi_1(\tau_j)\right) & p_0^T A_1\phi_0 + p_1^T A_1\phi_1 \end{pmatrix} \left(\begin{bmatrix} \xi_4 & \xi_5 \\ K_{10} & K_{01} \end{bmatrix}\right) = \begin{pmatrix} E_1 & 0 \\ E_2 & 0 \\ E_3 & 1 \end{pmatrix}, \quad (6.31)$$

where

$$\begin{aligned} E_1 =& \Delta'(0)q_1 + \frac{1}{2}\Delta''(0)q_0, \\ E_2 =& \frac{1}{2}p_1^T B(\phi_1,\phi_1) + p_1^T\left(\Sigma_{j=1}^m B^j\phi_0(-\tau_j)\left(-\tau_j q_1 + \frac{1}{2}\tau_j^2 q_0\right)\right), \\ E_3 =& 3\left\langle \phi_0^{\odot}, H_{1100}\right\rangle - p_0^T B(\phi_1,\phi_1) \\ & - p_0^T\left(\Sigma_{j=1}^m B^j\phi_0(-\tau_j)\left(-\tau_j q_1 + \frac{1}{2}\tau_j^2 q_0\right)\right) \\ & - p_1^T\left(\Sigma_{j=1}^m B^j\phi_1(-\tau_j)\left(-\tau_j q_1 + \frac{1}{2}\tau_j^2 q_0\right)\right). \end{aligned}$$

from which we can determine $(\xi_4, \xi_5, K_{10}, K_{01})$. Here $\left\langle \phi_0^{\odot}, H_{1100}\right\rangle$ is given by

$$\begin{aligned} \left\langle \phi_0^{\odot}, H_{1100}\right\rangle = p_0^T & \left(\frac{1}{12}a\Delta^{(4)}(0)q_0 + \frac{1}{6}b\Delta^{(3)}(0)q_0 + \frac{1}{2}\gamma_1\Delta''(0)\right) q_0 \\ & + p_0^T\left(\frac{1}{3}a\Delta^{(3)}(0) + \frac{1}{2}b\Delta''(0)\right) q_1 \\ & + p_0^T\left(\frac{1}{2}\Delta''(0)\xi_1 + \Delta'(0)\xi_2\right) \\ & + p_1^T\left(-\frac{1}{60}a\Delta^{(5)}(0) + \frac{1}{24}b\Delta^{(4)}(0) + \frac{1}{6}\gamma_1\Delta^{(3)}(0)\right) q_0 \\ & + p_1^T\left(\frac{1}{12}a\Delta^{(4)}(0) + \frac{1}{6}b\Delta^{(3)}(0)\right) q_1 \\ & + p_1^T\left(\frac{1}{6}\Delta^{(3)}(0)\xi_1 + \frac{1}{2}\Delta''(0)\xi_2\right). \end{aligned} \quad (6.32)$$

6.1.2. Solving $H_{0010}, H_{0001}, K_{10}$ and K_{01} without a big system

Let $\gamma = (\gamma_1, \gamma_2) = p_1^T J_1$, then by applying the Fredholm alternative to (6.11) it follows that

$$\begin{aligned} K_{10} &= s_1 + \delta_1 s_2, \\ K_{01} &= \delta_2 s_2, \end{aligned} \quad (6.33)$$

where $s_1 = \frac{1}{\gamma_1^2+\gamma_2^2}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}, s_2 = \begin{pmatrix}-\gamma_2\\ \gamma_1\end{pmatrix}$ and $\delta_{1,2}$ are some constants to be determined below. Using Corollary 4.10 it follows that

$$H_{0010}(\theta) = \Delta^{INV}(0)J_1 s_1 + \delta_1 \Delta^{INV}(0)J_1 s_2 \tag{6.34}$$

$$+ \Delta^{INV}(0)\left[J_1(s_1+\delta_1 s_2) - \frac{1}{2}\Delta''(0)q_0 - \Delta'(0)q_1\right] \tag{6.35}$$

$$+ \frac{1}{2}\theta^2 q_0 + \theta q_1 + \zeta_1\phi_0(\theta), \tag{6.36}$$

$$H_{0001}(\theta) = \delta_2\Delta^{INV}(0)J_1 s_2 + \zeta_2\phi_0(\theta). \tag{6.37}$$

For the moment we fix the values $\delta_1 = 0, \delta_2 = 1$ and $\xi_{1,2} = 0$ to obtain

$$\begin{cases} K_{10} &= s_1,\\ K_{01} &= s_2,\\ H_{0010}(\theta) &= \Delta^{INV}(0)J_1 s_1 + \Delta^{INV}(0)\left[J_1 s_1 - \frac{1}{2}\Delta''(0)q_0 - \Delta'(0)q_1\right]\\ & \quad +\frac{1}{2}\theta^2 q_0 + \theta q_1 + \zeta_1\phi_0(\theta),\\ H_{0001} &= \Delta^{INV}(0)J_1 s_2 + \zeta_2\phi_0(\theta). \end{cases} \tag{6.38}$$

Evaluating these vectors on the equations (6.19) and (6.24) gives

$$\begin{aligned} v_1 &= p_1^T A_1(\phi_0, K_{01}) + p_1^T B(\phi_0, H_{0001}),\\ v_2 &= p_0^T A_1(\phi_0, K_{01}) + p_0^T B(\phi_0, H_{0001}) + p_1^T A_1(\phi_1, K_{01}) + p_1^T B(\phi_1, H_{0001}). \end{aligned}$$

To make $(v_1, v_2) = (0, 1)$ we first use the freedom $H_{0001} \to H_{0001} + \zeta_2\phi_0$, so that

$$p_1^T A_1(\phi_0, K_{01}) + p_1^T B(\phi_0, H_{0001}) \to p_1^T A_1(\phi_0, K_{01}) + p_1^T B(\phi_0, H_{0001}) + 2a\zeta_2.$$

Thus, for

$$\zeta_2 = -\frac{p_1^T A_1(\phi_0, K_{01}) + p_1^T B(\phi_0, H_{0001})}{2a}$$

we have $v_1 = 0$. Then we can scale $(H_{0001}, K_{01}) \to \delta_2(H_{0001}, K_{01})$ to make $v_2 = 1$ without affecting v_0. This gives

$$\delta_2 = \frac{1}{p_0^T A_1(\phi_0, K_{01}) + p_0^T B(\phi_0, H_{0001}) + p_1^T A_1(\phi_1, K_{01}) + p_1^T B(\phi_1, H_{0001})}.$$

Rearranging equations (6.28) and (6.29) yields

$$\begin{aligned} \frac{1}{2}p_1^T B(\phi_1, \phi_1) &= p_1^T B(\phi_0, H_{0010}) + p_1^T A_1(\phi_0, K_{10}),\\ 3\left\langle \phi_0^{\odot}, H_{1100}\right\rangle &= p_0^T B(\phi_1, \phi_1) + p_0^T B(\phi_0, H_{0010}) + p_0^T A_1(\phi_0, K_{10})\\ &\quad + p_1^T B(\phi_1, H_{0010}) + p_1^T A_1(\phi_1, K_{10}). \end{aligned}$$

Note that the first equation looks different from the ODE case in Appendix A.1, but is identical. Evaluating (6.38) on the right hand side of these equations gives

$$v_3 = p_0^T B(\phi_1,\phi_1) + p_0^T B(\phi_0, H_{0010}) + p_0^T A_1(\phi_0, K_{10}) + p_1^T B(\phi_1, H_{0010}) + p_1^T A_1(\phi_1, K_{10}),$$
$$v_4 = p_1^T B(\phi_0, H_{0010}) + p_1^T A_1(\phi_0, K_{10}).$$

To make $v_4 = \frac{1}{2} p_1^T B(\phi_1,\phi_1)$ we use the freedom $H_{0010} \to H_{0010} + \zeta_1\phi_0$, so that

$$v_4 \to v_4 + 2a\xi_1$$

Thus, for

$$\zeta_1 = \frac{\frac{1}{2} p_1^T B(\phi_1,\phi_1) - p_1^T B(\phi_0, H_{0010}) - p_1^T A_1(\phi_0, K_{10})}{2a}$$

we have $v_4 = \frac{1}{2} p_1^T B(\phi_1,\phi_1)$. Then, after reevaluating v_3, we can translate $(H_{0010}, K_{10}) \to (H_{0010} + \delta_1 H_{0001}, K_{10} + \delta_1 K_{01})$ to make $v_3 = 3\left\langle \phi_0^\odot, H_{1100}\right\rangle$ without affecting v_4, see equation (6.19). This gives, using equation (6.24),

$$v_3 \to v_3 + \delta_1.$$

It follows that for

$$\begin{aligned}\delta_1 = 3\left\langle \phi_0^\odot, H_{1100}\right\rangle - p_0^T B(\phi_1,\phi_1) - p_0^T B(\phi_0, H_{0010})\\ - p_0^T A_1(\phi_0, K_{10}) - p_1^T B(\phi_1, H_{0010}) - p_1^T A_1(\phi_1, K_{10})\end{aligned}$$

we obtain $v_3 = 3\left\langle \phi_0^\odot, H_{1100}\right\rangle$. Here $\left\langle \phi_0^\odot, H_{1100}\right\rangle$ is given by (6.32).

6.1.3. Determining coefficients $K_{02}, H_{0002}, H_{1001}, H_{0101}$

We now determine the coefficients K_{02} and H_{0002} by collecting the β_2^2 term

$$\begin{aligned}A^{\odot\star} H_{0002} = -\big(2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001})\\ + J_2(K_{01}, K_{01}) + J_1^{\odot\star} K_{02}\big).\end{aligned}$$

The solvability condition implies that

$$\begin{aligned}p_1^T J_1 K_{02} = -p_1^T\big(2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001})\\ + J_2(K_{01}, K_{01})\big).\end{aligned}$$

From equation (6.10) we derive that

$$1 = p_1^T J_1 K_{10},$$

therefore

$$K_{02} = -p_1^T\big(2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01})\big)K_{10}.$$

Having calculated K_{02}, its easy to see that

$$H_{0002}(\theta) = \Delta(0)^{INV}\Big[2A_1(H_{0001}, K_{01}) + B\left(H_{0001}, H_{0001}\right) + J_2\left(K_{01}, K_{01}\right) + J_1K_{02}\Big] + \gamma_6 q_0.$$

Collecting the $w_0\beta_2$ and $w_1\beta_2$-terms in the homological equations yields

$$\begin{aligned} A^{\odot\star}H_{1001} &= -\left(A_1(\phi_0, K_{01}) + B(\phi_0, H_{0001})\right), \\ A^{\odot\star}H_{0101} &= -\left(A_1(\phi_1, K_{01}) + B(\phi_1, H_{0001})\right) + H_{1001} + \phi_1. \end{aligned}$$

It follows that

$$\begin{aligned} H_{1001}(\theta) &= \xi_7, \\ H_{0101}(\theta) &= \xi_8 + \frac{\theta^2}{2}q_0 + \theta q_1, \end{aligned}$$

where

$$\begin{aligned} \xi_7 &= \Delta(0)^{INV}\left[A_1(\phi_0, K_{01}) + B(\phi_0, H_{0001})\right], \\ \xi_8 &= \Delta(0)^{INV}\left[A_1(\phi_1, K_{01}) + B(\phi_1, H_{0001}) - \xi_7 - \frac{\Delta''(0)}{2}q_0 - \Delta'(0)q_1\right]. \end{aligned}$$

6.1.4. Determining coefficients d, e, a_1, b_1

The $w_0^3, w_0^2w_1, w_0^2\beta_2$ and $w_0w_1\beta_2$ terms in the homological equation lead to the systems

$$\begin{aligned} A^{\odot\star}H_{3000} =& 6aH_{1100} - \left(3B(\phi_0, H_{2000}) + C(\phi_0, \phi_0, \phi_0)\right) + 6d\phi_1 \\ A^{\odot\star}H_{2100} =& 2aH_{0200} + 2bH_{1100} + H_{3000} + 2e\phi_1 \\ & - \left(2B(\phi_0, H_{1100}) + B(\phi_1, H_{2000}) + C(\phi_1, \phi_0, \phi_0)\right), \\ A^{\odot\star}H_{2001} =& 2aH_{0101} + 2a_2\phi_1 - \Big(A_1(H_{2000}, K_{01}) \\ & + 2B(\phi_0, H_{1001}) + B(H_{0001}, H_{2000}) \\ & + B_2(\phi_0, \phi_0, K_{01}) + C(\phi_0, \phi_0, H_{0001})\Big), \\ A^{\odot\star}H_{1101} =& bH_{0101} + H_{1100} + H_{2001} + b_2\phi_1 - \Big(A_1(H_{1100}, K_{01}) \\ & + B(\phi_0, H_{0101}) + B(\phi_1, H_{1001}) \\ & + B(H_{0001}, H_{1100}) + B_2(\phi_0, \phi_1, K_{01}) \\ & + C(\phi_0, \phi_1, H_{0001})\Big). \end{aligned}$$

The solvability condition gives the following expressions for the cubic coefficients:

$$
\begin{aligned}
d &= -\frac{1}{2}ap_1^T B(\phi_1,\phi_1) + \frac{1}{2}p_1^T\left(B(\phi_0,H_{2000}) + \frac{1}{6}C(\phi_0,\phi_0,\phi_0)\right),\\
H_{3000}(\theta) &= \xi_9 + \left(\frac{a^2\theta^4}{2} + ab\theta^3 + 3a\gamma_1\theta^2 + 3d\theta^2\right)q_0\\
&\quad + \left(2a^2\theta^3 + 3ab\theta^2 + 6d\theta\right)q_1 + 3a\theta\left(\theta\xi_1 + 2\xi_2\right),\\
e &= p_1^T\left(B(\phi_0,H_{1100}) + \frac{1}{2}B(\phi_1,H_{2000}) + \frac{1}{2}C(\phi_1,\phi_0,\phi_0)\right)\\
&\quad - p_1^T\Bigg(b\Delta'(0)\xi_2 + a\Delta'(0)\xi_3 + \frac{1}{2}\Delta'(0)\xi_9\\
&\quad + \frac{1}{2}b^2\Delta''(0)q_1 + \frac{3}{2}d\Delta''(0)q_1 + \frac{1}{2}b\gamma_1\Delta''(0)q_0\\
&\quad + \frac{1}{2}b\Delta''(0)\xi_1 + \frac{5}{2}a\Delta''(0)\xi_2\\
&\quad + \frac{1}{6}b^2\Delta^{(3)}(0)q_0 + \frac{1}{2}d\Delta^{(3)}(0)q_0 - \frac{1}{12}a^2\Delta^{(5)}(0)q_0\\
&\quad + \frac{5}{12}a^2\Delta^{(4)}(0)q_1 + \frac{7}{24}ab\Delta^{(4)}(0)q_0 + \frac{7}{6}ab\Delta^{(3)}(0)q_1\\
&\quad + \frac{5}{6}a\Delta^{(3)}(0)\xi_1 + \frac{5}{6}a\gamma_1\Delta^{(3)}(0)q_0\Bigg),\\
a_2 &= \frac{1}{2}p_1^T\Bigg(A_1(H_{2000},K_{01}) + 2B(\phi_0,H_{1001}) + B(H_{0001},H_{2000})\\
&\quad + B_2(\phi_0,\phi_0,K_{01}) + C(\phi_0,\phi_0,H_{0001})\Bigg)\\
&\quad ap_0^T\Bigg(A_1(\phi_1,K_{01}) + B(\phi_1,H_{0001}) + A_1(\phi_0,K_{01})\\
&\quad + B(\phi_0,H_{0001}),\\
H_{2001}(\theta) &= \xi_{10} + \left(\frac{a\theta^3}{3} + a_2\theta^2\right)q_0 + \left(a\theta^2 + 2a_2\theta\right)q_1,\\
b_2 &= -p_1^T\Bigg(-A_1(K_{01},H_{1100}) - B(H_{0001},H_{1100}) - B(\phi_0,H_{0101})\\
&\quad - B(\phi_1,H_{1001}) - B_2(\phi_0,\phi_1,K_{01}) - C(\phi_0,\phi_1,H_{0001})\\
&\quad + \Delta'(0)\xi_2 + b\Delta'(0)\xi_8 + \Delta'(0)\xi_{11} + b\Delta''(0)q_1\\
&\quad + a_2\Delta''(0)q_1 + \frac{1}{2}\gamma_1\Delta''(0)q_0 + a\Delta''(0)\xi_8\\
&\quad + \frac{1}{6}a\Delta^{(4)}(0)q_0 + \frac{1}{3}a_2\Delta^{(3)}(0)q_0 + \frac{2}{3}a\Delta^{(3)}(0)q_1\\
&\quad + \frac{1}{3}b\Delta^{(3)}(0)q_0 + \frac{1}{2}\Delta''(0)\xi_1\Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_9 = \Delta(0)^{INV} \Bigg[& 3B(\phi_0, H^{2000}) + C(\phi_0, \phi_0, \phi_0) \\
& + \left(-\frac{1}{2} a^2 \Delta^{(4)}(0) - ab\Delta^{(3)}(0) - 3a\gamma_1 \Delta''(0) - 3d\Delta''(0) \right) q_0 \\
& + \left(-2a^2 \Delta^3(0) - 3ab\Delta''(0) - 6d\Delta'(0) \right) q_1 - 3a \left(\Delta''(0)\xi_1 + 2\Delta'(0)\xi_2 \right) \Bigg], \qquad (6.39)
\end{aligned}
$$

$$
\begin{aligned}
\xi_{10} = \Delta(0)^{INV} \Bigg[& A_1(K_{01}, H_{2000}) + B(H_{0001}, H_{2000}) + 2B(H_{1001}, \phi_0) \\
& + B_2(K_{01}, \phi_0, \phi_0) + C(H_{0001}, \phi_0, \phi_0) \\
& - \left(-a\Delta''(0) - 2a_2 \Delta'(0) \right) q_1 + 2a\Delta'(0)\xi_8 \\
& + \left(-\frac{1}{3} a\Delta^{(3)}(0) - a_2 \Delta''(0) \right) q_0 \Bigg]. \qquad (6.40)
\end{aligned}
$$

6.2. Transcritical Bogdanov-Takens bifurcation

Here we consider the case in which the origin remains a steady-state under variation of the parameters. For ODEs this hasn't been done yet. Therefore, we derived the homoclinic predictor and center manifold reduction combined with normalization in the Appendices.

The eigenvectors ϕ_0, ϕ_1 and adjoint eigenvectors $\phi_1^\odot, \phi_0^\odot$ are the same as in the previous Section. The ODE on the parameter-dependent center manifold takes the form

$$
\begin{aligned}
\dot{w} &= G(w, \beta) \\
&= \begin{pmatrix} w_1 \\ \beta_1 w_0 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 + g_2(w, \beta) \end{pmatrix} \qquad (6.41) \\
&\quad + \mathcal{O}(\|\beta\| \, w_1^2) + \mathcal{O}(\|\beta\|^2 \, \|w\|^2 + \|\beta\| \, \|w\|^3 + \|w\|^4),
\end{aligned}
$$

where $w = (w_0, w_1)$ and $\beta = (\beta_1, \beta_2)$ and where

$$
g_2(w, \beta) = (a_1 \beta_2 + a_2 \beta_1 + d w_0) \, w_0^2 + (b_1 \beta_2 + b_2 \beta_1 + e w_0) \, w_1 w_0.
$$

The function $\mathcal{H}, K$ and R can be expanded as

$$
\begin{aligned}
\mathcal{H}(w,\beta) =& [\phi_0,\phi_1]w + \frac{1}{2}H_{2000}w_0^2 + H_{1100}w_0w_1 + \frac{1}{2}H_{0200}w_1^2 + H_{1010}\beta_1 w_0 \\
&+ H_{1001}\beta_2 w_0 + H_{0110}\beta_1 w_1 + H_{0101}\beta_2 w_1 + \frac{1}{2}H_{0102}\beta_2^2 w_1 \\
&+ H_{0111}\beta_1\beta_2 w_1 + \frac{1}{2}H_{0120}\beta_1^2 w_1 + \frac{1}{2}H_{1002}\beta_2^2 w_0 + H_{1011}\beta_1\beta_2 w_0 \\
&+ \frac{1}{2}H_{1020}\beta_1^2 w_0 + H_{1101}\beta_2 w_1 w_0 + H_{1110}\beta_1 w_1 w_0 \\
&+ \frac{1}{2}H_{2001}\beta_2 w_0^2 + \frac{1}{2}H_{2010}\beta_1 w_0^2 + \frac{1}{2}H_{2100}w_1 w_0^2 + \frac{1}{6}H_{3000}w_0^3 \\
&+ \mathcal{O}(|\beta_2 w_1^2| + |\beta_1 w_1^2| + |w_1^3| + |w_0 w_1^2|) + \mathcal{O}(\|(w,\beta)\|^4), \qquad (6.42)\\
K(\beta) =& [K_{10},K_{01}]\,\beta + \frac{1}{2}K_{20}\beta_1^2 + K_{11}\beta_1\beta_2 + \frac{1}{2}K_{02}\beta_2^2 + \mathcal{O}(\|\beta\|^3). \qquad (6.43)
\end{aligned}
$$

and

$$
\begin{aligned}
R(u,\alpha) = \Bigg(&\frac{1}{2}B(u,u) + \frac{1}{6}C(u,u,u) + A_1(u,\alpha) \\
&+ \frac{1}{2}B_2(u,u,\alpha) + \mathcal{O}\left(\|u\|\,\|\alpha\|^2\right) + \mathcal{O}(\|u,\alpha\|^4)\Bigg). \qquad (6.44)
\end{aligned}
$$

Note that, since the steady-state φ_0 remains fixed under variations of parameters, we left out all coefficients in the expansion of $\mathcal{H}$ which solely depend on the parameters.

We insert the Taylor expansions (6.4), (6.42) and (6.3) into the homological equation

$$
A^{\odot\star}\mathcal{H}(z,\alpha) + R(\mathcal{H}(z,\alpha),\alpha) = D_z\mathcal{H}(z,\alpha)\dot{z}, \qquad (6.45)
$$

see (4.3).

6.2.1. Linear and quadratic terms

Collecting the coefficients of the linear and quadratic terms in the homological equation lead to the systems

$$
\begin{aligned}
w_0: &\quad A^{\odot\star}\phi_0 = 0,\\
w_1: &\quad A^{\odot\star}\phi_1 = \phi_0,\\
w_0^2: &\quad A^{\odot\star}H_{2000} = 2a\phi_1 - B(\phi_0,\phi_0), && (6.46)\\
w_0w_1: &\quad A^{\odot\star}H_{1100} = b\phi_1 + H_{2000} - B(\phi_0,\phi_1), && (6.47)\\
w_1^2: &\quad A^{\odot\star}H_{0200} = 2H_{1100} - B(\phi_1,\phi_1),\\
w_0\beta_1: &\quad A^{\odot\star}H_{1010} = \phi_1 - A_1(\phi_0,K_{10}), && (6.48)\\
w_0\beta_2: &\quad A^{\odot\star}H_{1001} = -A_1(\phi_0,K_{01}), && (6.49)\\
w_1\beta_1: &\quad A^{\odot\star}H_{0110} = -A_1(\phi_1,K_{10}) + H_{1010}, && (6.50)\\
w_1\beta_2: &\quad A^{\odot\star}H_{0101} = -A_1(\phi_1,K_{01}) + H_{1001} + \phi_1. && (6.51)
\end{aligned}
$$

The solvability condition implies that

$$
\begin{aligned}
0 &= 2a - p_1^T B(\phi_0,\phi_0), && (6.52)\\
0 &= b + \left\langle \phi_1^{\odot}, H_{2000}\right\rangle - p_1^T B(\phi_0,\phi_1), && (6.53)\\
0 &= 2\left\langle \phi_1^{\odot}, H_{1100}\right\rangle - p_1^T B(\phi_1,\phi_1), && (6.54)\\
0 &= 1 - p_1^T A_1(\phi_0,K_{10}), && (6.55)\\
0 &= -p_1^T (A(\phi_0,K_{01}), && (6.56)\\
0 &= -p_1^T A_1(\phi_1,K_{10}) + \left\langle \phi_1^{\odot}, H_{1010}\right\rangle, && (6.57)\\
0 &= -p_1^T A_1(\phi_1,K_{01}) + \left\langle \phi_1^{\odot}, H_{1001}\right\rangle + 1. && (6.58)
\end{aligned}
$$

Pairing equations (6.48) and (6.49) with $\phi_0^{\odot}$ yields

$$
\begin{aligned}
\left\langle \phi_1^{\odot}, H_{1010}\right\rangle &= -p_1^T A_1(\phi_0,K_{10}),\\
\left\langle \phi_1^{\odot}, H_{1001}\right\rangle &= -p_1^T A_1(q_0,K_{01}).
\end{aligned}
$$

Substituting into equations (6.57) and (6.58) gives

$$
\begin{aligned}
0 &= -p_1^T A_1(\phi_1,K_{10}) - p_1^T A_1(\phi_0,K_{10}),\\
0 &= -p_1^T A_1(\phi_1,K_{01}) - p_1^T A_1(q_0,K_{01}) + 1.
\end{aligned}
$$

Together with equations (6.55), and (6.56), one computes K_{10} and K_{01} by solving the 2-dimensional system

$$
\begin{pmatrix} p_1^T A_1\phi_0 \\ p_0^T A_1\phi_0 + p_1^T A_1\phi_1 \end{pmatrix} \left(\begin{bmatrix} K_{10} & K_{01} \end{bmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Since the quadratic w terms remain unchanged compared with generic case we have

$$
\begin{aligned}
a &= \frac{1}{2} p_1^T B(\phi_0,\phi_0),\\
b &= p_0^T B(\phi_0,\phi_0) + p_1^T B(\phi_0,\phi_1)
\end{aligned}
$$

and

$$
\begin{aligned}
H_{2000}(\theta) &= \xi_1 + \left(\gamma_1 + a\theta^2\right) q_0 + 2a\theta q_1, \\
H_{1100}(\theta) &= \xi_2 + \theta\xi_1 + \left(\gamma_1\theta + \frac{b}{2}\theta^2 + \frac{a}{3}\theta^3\right) q_0 + \left(b\theta + a\theta^2\right) q_1, \\
H_{0200}(\theta) &= \xi_3 + 2\theta\xi_2 + \theta^2\xi_1 + \left(\gamma_1\theta^2 + \frac{b}{3}\theta^3 + \frac{a}{6}\theta^4\right) q_0 \\
&\quad + \left(\frac{2a}{3}\theta^3 + b\theta^2\right) q_1,
\end{aligned} \tag{6.59}
$$

where

$$
\begin{aligned}
\Delta(0)\xi_1 &= B(\phi_0, \phi_0) - 2a\Delta'(0)q_1 - a\Delta''(0)q_0, \\
\Delta(0)\xi_2 &= B(\phi_0, \phi_1) - \Delta'(0)\xi_1 - \left(\gamma_1\Delta'(0) + \frac{1}{2}b\Delta''(0) + \frac{a}{3}\Delta'''(0)\right) q_0, \\
&\quad - \left(b\Delta'(0) + a\Delta''(0)\right) q_1, \\
\Delta(0)\xi_3 &= B(\phi_1, \phi_1) - 2\Delta'(0)\xi_2 - \Delta''(0)\xi_1 \\
&\quad + \left(-\gamma_1\Delta''(0) - \frac{b}{3}\Delta^{(3)}(0) - \frac{a}{6}\Delta^{(4)}(0)\right) q_0 \\
&\quad + \left(-\frac{2a}{3}\Delta^{(3)}(0) - b\Delta''(0)\right) q_1.
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma_1 = -p_0 &\left(\frac{a}{3}\Delta^{(3)}(0)q_0 + a\Delta''(0)q_1 + \Delta'(0)\xi_1\right) \\
&- p_1\left(\frac{a}{12}\Delta^{(4)}(0)q_0 + \frac{a}{3}\Delta^{(3)}(0)q_1 + \frac{1}{2}\Delta''(0)\xi_1\right) \\
&+ p_0^T B(\phi_0, \phi_1) + \frac{1}{2}p_1^T B(\phi_1, \phi_1).
\end{aligned}
$$

The remaining four unknowns in equations (6.48)-(6.51) can now be solved using Corollary 4.10

$$
\begin{aligned}
H_{1010} &= \xi_4 + \frac{\theta^2}{2}q_0 + \theta q_1, && (6.60) \\
H_{1001} &= \xi_5, \\
H_{0110} &= \xi_6 + \theta\xi_4 + \frac{\theta^3}{6}q_0 + \frac{\theta^2}{2}q_1, \\
H_{0101} &= \xi_7 + \theta\xi_5 + \frac{\theta^2}{2}q_0 + \theta q_1, && (6.61)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta(0)\xi_4 &= A_1(\phi_0, K_{10}) - \frac{1}{2}\Delta''(0)q_0 - \Delta'(0)q_1,\\
\Delta(0)\xi_5 &= A_1(\phi_0, K_{01}),\\
\Delta(0)\xi_6 &= A_1(\phi_1, K_{10}) - \Delta'(0)\xi_4 - \frac{1}{6}\Delta^{(3)}(0)q_0 - \frac{1}{2}q_1\Delta''(0),\\
\Delta(0)\xi_7 &= A_1(\phi_1, K_{01}) - \Delta'(0)\xi_5 - \frac{1}{2}\Delta''(0)q_0 - \Delta'(0)q_1.
\end{aligned}
$$

6.2.2. Cubic terms

6.2.2.1. Coefficients K_{20}, K_{02} and K_{11}

To solve the coefficients K_{20}, K_{02} and K_{11} we collect the $w_0\beta_1^2$,$w_0\beta_2^2$,$w_1\beta_1^2$,$w_1\beta_2^2$,$w_0\beta_1\beta_2$ and $w_1\beta_1\beta_2$ terms in the homological equation

$$
\begin{aligned}
w_0\beta_1^2: &\quad A^{\odot\star}H_{1020} = 2H_{0110} - 2A_1(H_{1010}, K_{10}) - A_1(\phi_0, K_{20}), && (6.62)\\
w_1\beta_1^2: &\quad A^{\odot\star}H_{0120} = H_{1020} - 2A_1(H_{0110}, K_{10}) - A_1(\phi_1, K_{20}), && (6.63)\\
w_0\beta_2^2: &\quad A^{\odot\star}H_{1002} = -2A_1(H_{1001}, K_{01}) - A_1(\phi_0, K_{02}), && (6.64)\\
w_1\beta_2^2: &\quad A^{\odot\star}H_{0102} = 2H_{0101} + H_{1002} - 2A_1(H_{0101}, K_{01}) - A_1(\phi_1, K_{02}), && (6.65)\\
w_0\beta_1\beta_2: &\quad A^{\odot\star}H_{1011} = H_{0101} - A_1(H_{1001}, K_{10}) - A_1(H_{1010}, K_{01})\\
&\qquad - A_1(\phi_0, K_{11}), && (6.66)\\
w_1\beta_1\beta_2: &\quad A^{\odot\star}H_{0111} = H_{0110} + H_{1011} - A_1(H_{0101}, K_{10}) - A_1(H_{0110}, K_{01})\\
&\qquad - A_1(\phi_1, K_{11}). && (6.67)
\end{aligned}
$$

Pairing with $\phi_1^\odot$ yields

$$
\begin{aligned}
0 &= 2\left\langle \phi_1^\odot, H_{0110}\right\rangle - p_1^T\left(2A_1(H_{1010}, K_{10}) + A_1(\phi_0, K_{20})\right), && (6.68)\\
0 &= \left\langle \phi_1^\odot, H_{1020}\right\rangle - p_1^T\left(2A_1(H_{0110}, K_{10}) + A_1(\phi_1, K_{20})\right), && (6.69)\\
0 &= -p_1^T\left(2A_1(H_{1001}, K_{01}) + A_1(\phi_0, K_{02})\right), && (6.70)\\
0 &= 2\left\langle \phi_1^\odot, H_{0101}\right\rangle + \left\langle \phi_1^\odot, H_{1002}\right\rangle - p_1^T\left(2A_1(H_{0101}, K_{01}) + A_1(\phi_1, K_{02})\right), && (6.71)\\
0 &= \left\langle \phi_1^\odot, H_{0101}\right\rangle - p_1^T\left(A_1(H_{1001}, K_{10}) + A_1(H_{1010}, K_{01}) + A_1(\phi_0, K_{11})\right), && (6.72)\\
0 &= \left\langle \phi_1^\odot, H_{0110}\right\rangle + \left\langle \phi_1^\odot, H_{1011}\right\rangle - p_1^T\big(A_1(H_{0101}, K_{10}) + A_1(H_{0110}, K_{01})\\
&\quad + A_1(\phi_1, K_{11})\big). && (6.73)
\end{aligned}
$$

Pairing equations (6.63),(6.64) and (6.66) with $\phi_0^\odot$ yields

$$
\begin{aligned}
\left\langle \phi_1^\odot, H_{1020}\right\rangle &= 2\left\langle \phi_0^\odot, H_{0110}\right\rangle - p_0^T\left(2A_1(H_{1010}, K_{10}) + p_0^T A_1(\phi_0, K_{20})\right),\\
\left\langle \phi_1^\odot, H_{1002}\right\rangle &= -p_0^T\left(2A_1(H_{1001}, K_{01}) + A_1(\phi_0, K_{02})\right),\\
\left\langle \phi_1^\odot, H_{1011}\right\rangle &= \left\langle \phi_0^\odot, H_{0101}\right\rangle - p_0^T\left(A_1(H_{1001}, K_{10}) + A_1(H_{1010}, K_{01}) + A_1(\phi_0, K_{11})\right).
\end{aligned}
$$

By substituting these equations into (6.69),(6.71) and (6.73) we obtain

$$\begin{aligned}
0 &= 2\left\langle \phi_0^{\odot}, H_{0110}\right\rangle - p_0^T\left(2A_1(H_{1010},K_{10}) + p_0^T A_1(\phi_0,K_{20})\right)\\
&\quad - p_1^T\left(2A_1(H_{0110},K_{10}) + A_1(\phi_1,K_{20})\right),\\
0 &= 2\left\langle \phi_1^{\odot}, H_{0101}\right\rangle - p_0^T\left(2A_1(H_{1001},K_{01}) + A_1(\phi_0,K_{02})\right)\\
&\quad - p_1^T\left(2A_1(H_{0101},K_{01}) + A_1(\phi_1,K_{02})\right),\\
0 &= \left\langle \phi_1^{\odot}, H_{0110}\right\rangle + \left\langle \phi_0^{\odot}, H_{0101}\right\rangle - p_0^T\left(A_1(H_{1001},K_{10}) + A_1(H_{1010},K_{01})\right)\\
&\quad - p_0^T A_1(\phi_0,K_{11}) - p_1^T\left(A_1(H_{0101},K_{10}) + A_1(H_{0110},K_{01}) + A_1(\phi_1,K_{11})\right).
\end{aligned}$$

Together with equations (6.68), (6.70) and (6.72), one computes K_{20}, K_{02} and K_{11} by solving the 2-dimensional system

$$\begin{pmatrix} p_1^T A_1\phi_0 \\ p_0^T A_1\phi_0 + p_1^T A_1\phi_1 \end{pmatrix}\left(\begin{bmatrix} K_{20} & K_{02} & K_{11}\end{bmatrix}\right) = \begin{pmatrix} E_1 & E_3 & E_5 \\ E_2 & E_4 & E_6 \end{pmatrix},$$

where

$$\begin{aligned}
E_1 &= 2\left\langle \phi_1^{\odot}, H_{0110}\right\rangle - 2p_1^T A_1(H_{1010},K_{10}),\\
E_2 &= 2\left\langle \phi_0^{\odot}, H_{0110}\right\rangle - 2p_0^T A_1(H_{1010},K_{10}) - 2p_1^T A_1(H_{0110},K_{10}),\\
E_3 &= -2p_1^T A_1(H_{1001},K_{01}),\\
E_4 &= 2\left\langle \phi_1^{\odot}, H_{0101}\right\rangle - 2p_0^T A_1(H_{1001},K_{01}) - 2p_1^T A_1(H_{0101},K_{01})\\
E_5 &= \left\langle \phi_1^{\odot}, H_{0101}\right\rangle - p_1^T\left(A_1(H_{1001},K_{10}) + A_1(H_{1010},K_{01})\right),\\
E_6 &= \left\langle \phi_1^{\odot}, H_{0110}\right\rangle + \left\langle \phi_0^{\odot}, H_{0101}\right\rangle - p_0^T\left(A_1(H_{1001},K_{10}) + A_1(H_{1010},K_{01})\right)\\
&\quad - p_1^T\left(A_1(H_{0101},K_{10}) + A_1(H_{0110},K_{01})\right).
\end{aligned}$$

Using the expression in equations (6.60)-(6.61) one calculates the pairings

$$\begin{aligned}
\left\langle \phi_1^{\odot}, H_{0110}\right\rangle &= p_1^T\left(\frac{1}{2}\Delta''(0)\xi_4 + \Delta'(0)\xi_6 + \frac{1}{24}\Delta^{(4)}(0)q_0 + \frac{1}{6}\Delta^{(3)}(0)q_1\right),\\
\left\langle \phi_0^{\odot}, H_{0110}\right\rangle &= p_1^T\left(\frac{1}{6}\Delta^{(3)}(0)\xi_4 + \frac{1}{2}\Delta''(0)\xi_6 - \frac{1}{120}\Delta^{(5)}(0)q_0 + \frac{1}{24}\Delta^{(4)}(0)q_1\right)\\
&\quad + p_0^T\left(\frac{1}{2}\Delta''(0)\xi_4 + \Delta'(0)\xi_6 + \frac{1}{24}\Delta^{(4)}(0)q_0 + \frac{1}{6}\Delta^{(3)}(0)q_1\right),\\
\left\langle \phi_1^{\odot}, H_{0101}\right\rangle &= p_1^T\left(\frac{1}{2}\Delta''(0)\xi_5 + \Delta'(0)\xi_7 + \frac{1}{6}\Delta^{(3)}(0)q_0 + \frac{1}{2}q_1\Delta''(0)\right),\\
\left\langle \phi_0^{\odot}, H_{0101}\right\rangle &= p_0^T\left(\frac{1}{2}\Delta''(0)\xi_5 + \Delta'(0)\xi_7 + \frac{1}{6}\Delta^{(3)}(0)q_0 + \frac{1}{2}q_1\Delta''(0)\right)\\
&\quad + p_1^T\left(\frac{1}{6}\Delta^{(3)}(0)\xi_5 + \frac{1}{2}\Delta''(0)\xi_7 + \frac{1}{24}\Delta^{(4)}(0)q_0 + \frac{1}{6}\Delta^{(3)}(0)q_1\right).
\end{aligned}$$

6.2.2.2. Coefficients a, a_1, a_2, b_1, b_2, d and e

To solve the coefficients a, a_1, a_2, b_1, b_2, d and e we collect the w_0^3, $w_0^2 w_1$, $w_0^2\beta_1$, $w_0^2\beta_2$, $w_0 w_1 \beta_1$ and $w_0 w_1\beta_2$ terms in the homological equation

$$
\begin{aligned}
w_0^3: &\quad A^{\odot\star} H_{3000} = 6d\phi_1 + 6aH_{1100} - 3B(\phi_0, H_{2000}) - C(\phi_0,\phi_0,\phi_0), &(6.74)\\
w_0^2 w_1: &\quad A^{\odot\star} H_{2100} = 2e\phi_1 + 2aH_{0200} + 2bH_{1100} + H_{3000} - C(\phi_0,\phi_0,\phi_1)\\
&\qquad - 2B(\phi_0, H_{1100}) - B(\phi_1, H_{2000}), &(6.75)\\
w_0^2\beta_2: &\quad A^{\odot\star} H_{2001} = 2a_1\phi_1 + 2aH_{0101} - A_1(H_{2000}, K_{01}) - 2B(\phi_0, H_{1001})\\
&\qquad - B_2(\phi_0,\phi_0,K_{01}), &(6.76)\\
w_0^2\beta_1: &\quad A^{\odot\star} H_{2010} = 2a_2\phi_1 + 2aH_{0110} + 2H_{1100} - A_1(H_{2000}, K_{10})\\
&\qquad - 2B(\phi_0, H_{1010}) - B_2(\phi_0,\phi_0,K_{10}) &(6.77)\\
w_0 w_1\beta_2: &\quad A^{\odot\star} H_{1101} = b_1\phi_1 + H_{1100} + H_{2001} + bH_{0101} - A_1(H_{1100}, K_{01})\\
&\qquad - B(\phi_0, H_{0101}) - B(\phi_1, H_{1001})\\
&\qquad - B_2(\phi_0,\phi_1,K_{01}). &(6.78)\\
w_0 w_1\beta_1: &\quad A^{\odot\star} H_{1110} = b_2\phi_1 + H_{0200} + H_{2010} + bH_{0110} - A_1(H_{1100}, K_{10})\\
&\qquad - B(\phi_0, H_{0110}) - B(\phi_1, H_{1010})\\
&\qquad - B_2(\phi_0,\phi_1,K_{10}), &(6.79)
\end{aligned}
$$

Pairing with $\phi_1^{\odot}$ yields and rearranging yields

$$
\begin{aligned}
d &= -a\left\langle \phi_1^{\odot}, H_{1100}\right\rangle + \frac{1}{6}p_1^T\left(3B(\phi_0, H_{2000}) + C(\phi_0,\phi_0,\phi_0)\right),\\
e &= -a\left\langle \phi_1^{\odot}, H_{0200}\right\rangle - b\left\langle \phi_1^{\odot}, H_{1100}\right\rangle - \frac{1}{2}\left\langle \phi_1^{\odot}, H_{3000}\right\rangle\\
&\quad + \frac{1}{2}p_1^T\left(C(\phi_0,\phi_0,\phi_1) + 2B(\phi_0, H_{1100}) + B(\phi_1, H_{2000})\right),\\
a_1 &= -a\left\langle \phi_1^{\odot}, H_{0101}\right\rangle + \frac{1}{2}p_1^T\big(A_1(H_{2000}, K_{01}) + 2B(\phi_0, H_{1001})\\
&\quad + B_2(\phi_0,\phi_0,K_{01})\big),\\
a_2 &= -a\left\langle \phi_1^{\odot}, H_{0110}\right\rangle - \left\langle \phi_1^{\odot}, H_{1100}\right\rangle + \frac{1}{2}p_1^T\big(A_1(H_{2000}, K_{10})\\
&\quad + 2B(\phi_0, H_{1010}) + B_2(\phi_0,\phi_0,K_{10})\big),\\
b_1 &= -\left\langle \phi_1^{\odot}, H_{1100}\right\rangle - \left\langle \phi_1^{\odot}, H_{2001}\right\rangle - b\left\langle \phi_1^{\odot}, H_{0101}\right\rangle + p_1^T\big(A_1(H_{1100}, K_{01})\\
&\quad + B(\phi_0, H_{0101}) + B(\phi_1, H_{1001}) + B_2(\phi_0,\phi_1,K_{01})\big),\\
b_2 &= -\left\langle \phi_1^{\odot}, H_{0200}\right\rangle - \left\langle \phi_1^{\odot}, H_{2010}\right\rangle - b\left\langle \phi_1^{\odot}, H_{0110}\right\rangle + p_1^T\big(A_1(H_{1100}, K_{10})\\
&\quad + B(\phi_0, H_{0110}) + B(\phi_1, H_{1010}) + B_2(\phi_0,\phi_1,K_{10})\big).
\end{aligned}
$$

Pairing equations (6.74), (6.76) and (6.77) with $\phi_0^\odot$ yields

$$\begin{aligned}
\langle \phi_1^\odot, H_{3000} \rangle &= 6a \langle \phi_0^\odot, H_{1100} \rangle - p_0^T \left(3B(\phi_0, H_{2000}) + C(\phi_0, \phi_0, \phi_0)\right), \\
\langle \phi_1^\odot, H_{2001} \rangle &= 2a \langle \phi_0^\odot, H_{0101} \rangle - p_0^T \big(A_1(H_{2000}, K_{01}) + 2B(\phi_0, H_{1001}) \\
&\quad + B_2(\phi_0, \phi_0, K_{01})\big), \\
\langle \phi_1^\odot, H_{2010} \rangle &= 2a \langle \phi_0^\odot, H_{0110} \rangle + 2 \langle \phi_0^\odot, H_{1100} \rangle - p_0^T \big(A_1(H_{2000}, K_{10}) \\
&\quad + 2B(\phi_0, H_{1010}) + B_2(\phi_0, \phi_0, K_{10})\big).
\end{aligned}$$

It thus remains to calculate $\langle \phi_1^\odot, H_{1100} \rangle$, $\langle \phi_1^\odot, H_{0200} \rangle$ and $\langle \phi_0^\odot, H_{1100} \rangle$. From equation (6.54) it follows that

$$\langle \phi_1^\odot, H_{1100} \rangle = \frac{1}{2} p_1^T B(\phi_1, \phi_1).$$

From equation (6.59) and the representation of $\phi_1^\odot$ in (3.12) we obtain

$$\begin{aligned}
\langle \phi_1^\odot, H_{0200} \rangle = p_1^T \Big(&-\frac{1}{30} a\Delta^{(5)}(0) q_0 + \frac{1}{6} a\Delta^{(4)}(0) q_1 + \frac{1}{12} b\Delta^{(4)}(0) q_0 \\
&+ \frac{1}{3} b\Delta^{(3)}(0) q_1 + \frac{1}{3}\Delta^{(3)}(0)\xi_1 + \Delta''(0)\xi_2 \\
&+ \Delta'(0)\xi_3 + \frac{1}{3}\gamma_1 \Delta^{(3)}(0) q_0.
\end{aligned}$$

Then, pairing equation (6.47) with $\phi_0^\odot$ we obtain

$$\langle \phi_1^\odot, H_{0200} \rangle = 2 \langle \phi_1^\odot, H_{1100} \rangle - p_1^T B(\phi_1, \phi_1).$$

Remark 6.1. One can easily check that the transformations

$$\begin{aligned}
H_{1100} &\to H_{1100} + \gamma\phi_0, \\
H_{0200} &\to H_{1100} + \gamma\phi_0, \\
H_{1010} &\to H_{1010} + \gamma\phi_0, \\
H_{1001} &\to H_{1001} + \gamma\phi_0, \\
H_{0110} &\to H_{0110} + \gamma\phi_0, \\
H_{0101} &\to H_{0101} + \gamma\phi_0,
\end{aligned}$$

leave the coefficients $K_{10}, K_{01}, K_{20}, K_{11}, K_{02}, a, b, d, e, a_1, a_2, b_1, b_2$ invariant.

6.3. Generalized Hopf bifurcation

Since the eigenvalues (B.3) are simple, Lemma 3.6 gives an eigenfunction ϕ and an adjoint eigenfunction $\phi^\odot$ such that

$$A\phi = i\omega_0\phi, , \qquad A^\star \phi^\odot = i\omega_0 \phi^\odot. \tag{6.80}$$

Let the vectors $q, p \in \mathbb{R}^n$ satisfy

$$\Delta(i\omega_0)q = 0, \qquad p^T\Delta(i\omega_0) = 0,$$

then the eigenfunctions are given by

$$\begin{aligned}\phi(\theta) &= e^{i\omega_0\theta}q,\\ \phi^\odot(\theta) &= p^T + p^T\int_0^\theta (\int_\sigma^h e^{i\omega_0(\sigma-\tau)}d\zeta(\tau))\,d\sigma.\end{aligned}$$

Furthermore, using Lemma 3.6 we normalize the eigenfunctions such that

$$\langle \phi^\odot, \phi \rangle = 1$$

holds.

Any point $y \in X_0$ from the critical eigenspace can be represented as

$$y = z\phi + \bar{z}\bar{\phi}, \qquad z \in \mathbb{C},$$

where $z = \langle \phi^\odot, y\rangle$. Therefore, the homological equation (4.3) can be written as

$$A^{\odot\star}\mathcal{H}(z,\beta) + J_1K(\beta) + R(\mathcal{H}(z,\beta), K(\beta)) = D_z\mathcal{H}(z,\beta)\dot{z} + D_{\bar{z}}\mathcal{H}(z,\beta)\dot{\bar{z}}. \tag{6.81}$$

Then $\mathcal{H}$, K and R admits the expansions

$$\begin{aligned}\mathcal{H}(z,\bar{z},\beta) =& z\phi + \bar{z}\bar{\phi}\\ &+ \sum_{j+k=2}^{3}\sum_{|\mu|=0}^{1}\frac{1}{j!k!\mu!}H_{jk\mu}z^j\bar{z}^k\beta^\mu + \mathcal{O}(\|z\|^4\|\beta\|^2), &(6.82)\\ K(\beta) =& K_{10}\beta_1 + K_{01}\beta_2 + \mathcal{O}(\|\beta\|^2), &(6.83)\\ R(u,\beta) =& \frac{1}{2}B(u,u) + A_1(u,\beta) + \mathcal{O}\left(\|u\|^3 + \|u\|\,\|\beta\|^2 + \|\beta\|^3\right),\end{aligned}$$

where $\beta = (\beta_1, \beta_2)$.

For the predictors derived in Section C.3 we need the parameter-dependent normal form

$$\dot{z} = (i\omega + \beta_1 + ib_{11}\beta_1 + ib_{12}\beta_2)\,z + (\beta_2 + \operatorname{Im}(c_1(0))i)\,z|z|^2 + (d_2(0) + \operatorname{Im}(c_2(0))i)\,z|z|^4.$$

Critical normal form coefficients We start by calculating the critical normal form coefficients. Collecting the coefficients of the terms $z^2, z\bar{z}, z^3$ and $z^2\bar{z}$ in the homological equation yields the systems

$$\begin{aligned}\left(A^{\odot\star} - 2i\omega_0\right)H_{2000} &= -B\left(\phi,\phi\right),\\ A^{\odot\star}H_{1100} &= -B\left(\phi,\bar{\phi}\right),\\ \left(A^{\odot\star} - 3i\omega_0\right)H_{3000} &= -3B\left(\phi,H_{2000}\right) - C\left(\phi,\phi,\phi\right),\\ \left(A^{\odot\star} - i\omega_0\right)H_{2100} &= -B\left(\bar{\phi},H_{2000}\right) - 2B\left(\phi,H_{1100}\right) - C\left(\phi,\phi,\bar{\phi}\right).\end{aligned}$$

The first three solutions can be solved using Corollary 4.1, we have

$$\begin{aligned}
H_{2000} &= e^{2i\omega_0\theta}\Delta^{-1}(2i\omega)B(\phi,\phi),\\
H_{1100} &= \Delta^{-1}(0)B(\phi,\bar{\phi}),\\
H_{3000} &= e^{3i\omega_0\theta}\Delta^{-1}(3i\omega)\left(3B\left(\phi,H_{2000}\right)+C\left(\phi,\phi,\phi\right)\right).
\end{aligned}$$

For the fourth equation Corollary 4.8 gives the solution

$$H_{2100} = B^{INV}_{i\omega_0}\left(\left(\bar{\phi},H_{2000}\right)+2B\left(\phi,H_{1100}\right)+C\left(\phi,\phi,\bar{\phi}\right)\right).$$

Notice that the Fredholm alternative implies that the expression

$$p^T\left(B\left(\bar{\phi},H_{2000}\right)+2B\left(\phi,H_{1100}\right)+C\left(\phi,\phi,\bar{\phi}\right)\right)$$

vanishes, which implies that the first Lyapunov coefficient also vanishes. Continuing to compute the second Lyapunov coefficient we collect the coefficients corresponding to the $z^3\bar{z}$ and $z^2\bar{z}^2$ terms in the homological equation. We obtain the systems

$$\begin{aligned}
\left(A^{\odot\star}-2i\omega_0\right)H_{3100} &= 6c_1H_{2000}-B\left(\bar{\phi},H_{3000}\right)-3B\left(\phi,H_{2100}\right)-3B\left(H_{1100},H_{2000}\right)\\
&\quad -3C\left(\phi,\bar{\phi},H_{2000}\right)-3C\left(\phi,\phi,H_{1100}\right)-D\left(\phi,\phi,\phi,\bar{\phi}\right),\\
A^{\odot\star}H_{2200} &= -2B\left(\bar{\phi},H_{2100}\right)-2B\left(\phi,H_{1200}\right)-B\left(H_{0200},H_{2000}\right)\\
&\quad -2B\left(H_{1100},H_{1100}\right)-C\left(\phi,\phi,H_{0200}\right)-4C\left(\phi,\bar{\phi},H_{1100}\right)\\
&\quad -C\left(\bar{\phi},\bar{\phi},H_{2000}\right)-D\left(\phi,\phi,\bar{\phi},\bar{\phi}\right).
\end{aligned}$$

Both system are non-singular and can be solved with Corollary (4.1). The critical normal form coefficient $c_2(0)$ is calculated by applying the Fredholm alternative to the system obtained from the coefficient corresponding to the $z^3\bar{z}^2$ term in the homological equation. This gives

$$\begin{aligned}
c_2(0) = \frac{1}{12}p^T\Bigg(&2B\left(\bar{\phi},H_{3100}\right)+3B\left(\phi,H_{2200}\right)+B\left(\overline{H}_{2000},H_{3000}\right)\\
&+6B\left(H_{1100},H_{2100}\right)+3B\left(\overline{H}_{2100},H_{2000}\right)\\
&+6C\left(\bar{\phi},H_{1100},H_{2000}\right)+6C\left(\phi,\bar{\phi},H_{2100}\right)+C\left(\bar{\phi},\bar{\phi},H_{3000}\right)\\
&+3C\left(\phi,\phi,\overline{H}_{2100}\right)+3C\left(\phi,\overline{H}_{2000},H_{2000}\right)+6C\left(\phi,H_{1100},H_{1100}\right)\\
&+D\left(\phi,\phi,\phi,\overline{H}_{2000}\right)+6D\left(\phi,\phi,\bar{\phi},H_{1100}\right)+3D\left(\phi,\bar{\phi},\bar{\phi},H_{2000}\right)\\
&+E\left(\phi,\phi,\phi,\bar{\phi},\bar{\phi}\right)\Bigg).
\end{aligned}$$

Parameter-related coefficients Next we derive the parameter-related coefficients that provide a linear approximation to the parameter transformation. Following [37] and [4] first expand the eigenvalue $\lambda(\alpha)$ and $c_1(\alpha)$ in the normal form (B.4) in the original parameters α and truncated to the fifth order

$$\dot{z} = (i\omega_0 + \gamma_{110}\alpha_1 + \gamma_{101}\alpha_2)\, z + (c_1(0) + \gamma_{210}\alpha_1 + \gamma_{201}\alpha_2)\, z|z|^2 + c_2(\alpha)z|z|^4. \tag{6.84}$$

The parameters β are given through the relation

$$\alpha = \left(\mathrm{Re}\begin{pmatrix} \gamma_{110} & \gamma_{101} \\ \gamma_{210} & \gamma_{201} \end{pmatrix}\right)^{-1} \beta$$

and $\dfrac{\partial}{\partial \beta_1} b_1(\beta) = \mathrm{Im}(\gamma_{110}\alpha_1 + \gamma_{101}\alpha_2)$, compare with (C.70).

The homological equation (4.3) becomes

$$A^{\odot\star}\mathcal{H}(z,\bar{z},\alpha) + J_1\alpha + R(\mathcal{H}(z,\bar{z},\alpha),\alpha) = D_z\mathcal{H}(z,\bar{z},\alpha)\dot{z} + D_{\bar{z}}\mathcal{H}(z,\bar{z},\alpha)\dot{\bar{z}}, \tag{6.85}$$

where $\mathcal{H}$ and R admits the expansions

$$\begin{aligned} \mathcal{H}(z,\bar{z},\alpha_1,\alpha_2) =& z\phi + \bar{z}\bar{\phi} \\ &+ \sum_{j+k=2}^{3} \sum_{|\mu|=0}^{1} \frac{1}{j!k!\mu!} H_{jk\mu} z^j \bar{z}^k \alpha + \mathcal{O}(\|z\|^4\|\alpha\|^2), \\ R(u,\alpha) =& \frac{1}{2}B(u,u) + A_1(u,\alpha) + \mathcal{O}\left(\|u\|^3 + \|u\|\,\|\alpha\|^2 + \|u\|^2\,\|\alpha\| + \|\alpha\|^2\right). \end{aligned} \tag{6.86}$$

Collecting the coefficients of the terms α and $z\alpha$ in the homological equation yields the systems

$$\begin{aligned} A^{\odot\star}H_{00\mu} &= -J_1 v_\mu, \\ \left(A^{\odot\star} - i\omega_0\right) H_{10\mu} &= \gamma_{1\mu}\phi - A\left(\phi, v_\mu\right) - B\left(\phi, H_{00\mu}\right), \end{aligned}$$

where $\mu = (10),(01)$ and $v_{10} = (1,0)^T, v_{01} = (0,1)^T$. The first equation has the solution

$$H_{00\mu} = \Delta(0)^{-1} J_1 v_\mu$$

and the Fredholm alternative gives

$$\gamma_{1\mu} = p^T\left(A_1\left(\phi, v_\mu\right) + B\left(\phi, H_{00\mu}\right)\right).$$

This leads to the solutions

$$H_{10\mu} = B_{i\omega_0}^{INV}\left(A_1\left(\phi, v_\mu\right) + B\left(\phi, H_{00\mu}\right), -\gamma_{1\mu}\right)$$

for the second equation.

To determine $\gamma_{2\mu}$ we collect the coefficients corresponding to the $z^2\beta, z\bar{z}\alpha$ and $z^2\bar{z}\alpha$ terms in the homological equation. We obtain the systems

$$\begin{aligned}
\left(A^{\odot\star} - 2i\omega_0\right) H_{20\mu} &= 2\gamma_{1\mu} H_{2000} - A_1\left(H_{2000}, v_\mu\right) - 2B\left(\phi, H_{10\mu}\right) \\
&\quad - B\left(H_{2000}, H_{00\mu}\right) - B_2\left(\phi, \phi, v_\mu\right) - C\left(\phi, \phi, H_{00\mu}\right), \\
A^{\odot\star} H_{11\mu} &= 2\operatorname{Re}(\gamma_{1\mu}) H_{1100} - B\left(\bar{\phi}, H_{10\mu}\right) - B_2\left(\phi, \bar{\phi}, v_\mu\right) - C\left(\phi, \bar{\phi}, H_{0\mu}\right) \\
&\quad - A_1\left(H_{1100}, v_\mu\right) - B\left(\phi, H_{01\mu}\right) - B\left(H_{1100}, H_{00\mu}\right), \\
\left(A^{\odot\star} - i\omega_0\right) H_{21\mu} &= 2\gamma_{2\mu}\phi + \left(2\gamma_{1\mu} + \bar{\gamma}_{1\mu}\right) H_{2100} + 2c_1(0) H_{10\mu} - A_1\left(H_{2100}, v_\mu\right) \\
&\quad - B\left(\bar{\phi}, H_{20\mu}\right) - 2B\left(\phi, H_{11\mu}\right) - B\left(H_{2100}, H_{00\mu}\right) \\
&\quad - B\left(H_{2000}, H_{01\mu}\right) - 2B\left(H_{1100}, H_{10\mu}\right) - B_2\left(H_{2000}, \bar{\phi}, v_\mu\right) \\
&\quad - 2B_2\left(\phi, H_{1100}, v_\mu\right) - 2C\left(\phi, \bar{\phi}, H_{10\mu}\right) - C\left(H_{2000}, \bar{\phi}, H_{00\mu}\right) \\
&\quad - C\left(\phi, \phi, H_{01\mu}\right) - 2C\left(\phi, H_{1100}, H_{00\mu}\right) - C_1\left(\phi, \phi, \bar{\phi}, v_\mu\right) \\
&\quad - D\left(\phi, \phi, \bar{\phi}, H_{00\mu}\right).
\end{aligned}$$

Using Corollary 4.1 we find the solutions

$$\begin{aligned}
H_{20\mu}(\theta) &= e^{2i\omega_0\theta}\Delta(2i\omega_0)^{-1}\Big[- 2\gamma_{1\mu} H_{2000}(0) + A_1\left(H_{2000}, v_\mu\right) + 2B\left(\phi, H_{10\mu}\right) \\
&\quad + B\left(H_{2000}, H_{00\mu}\right) + B_2\left(\phi, \phi, v_\mu\right) + C\left(\phi, \phi, H_{00\mu}\right)\Big] \\
&\quad - 2\gamma_{1\mu} \int_0^h d\zeta(\tau) \int_0^\tau e^{-2i\omega_0\sigma} H_{2000}(\sigma - \tau)\, d\sigma \\
&\quad - 2\gamma_{1\mu} \int_\theta^0 e^{-2i\omega_0\sigma} H_{2000}(\sigma)\, d\sigma \\
&= e^{2i\omega_0\theta}\Delta(2i\omega_0)^{-1}\Big[- 2\gamma_{1\mu} H_{2000}(0) + A_1\left(H_{2000}, v_\mu\right) + 2B\left(\phi, H_{10\mu}\right) \\
&\quad + B\left(H_{2000}, H_{00\mu}\right) + B_2\left(\phi, \phi, v_\mu\right) + C\left(\phi, \phi, H_{00\mu}\right)\Big] \\
&\quad - 2\gamma_{1\mu}\left(\Delta'(2i\omega) - I\right)\, \Delta^{-1}(2i\omega) B(\phi, \phi), \\
&\quad + 2\gamma_{1\mu}\Delta^{-1}(2i\omega) B(\phi, \phi)\, \theta, \\
H_{11\mu}(\theta) &= \Delta(0)^{-1}\Big[- 2\operatorname{Re}(\gamma_{1\mu}) H_{1100}(0) + B\left(\bar{\phi}, H_{10\mu}\right) + B_2\left(\phi, \bar{\phi}, v_\mu\right) \\
&\quad + C\left(\phi, \bar{\phi}, H_{0\mu}\right) + A_1\left(H_{1100}, v_\mu\right) + B\left(\phi, H_{01\mu}\right) + B\left(H_{1100}, H_{00\mu}\right)\Big] \\
&\quad - 2\operatorname{Re}(\gamma_{1\mu})\left(\Delta'(0) - I\right)\, \Delta^{-1}(0) B(\phi, \bar{\phi}) \\
&\quad + 2\operatorname{Re}(\gamma_{1\mu})\, \Delta^{-1}(0) B(\phi, \bar{\phi})\, \theta.
\end{aligned}$$

Applying the Fredholm alternative gives

$$\begin{aligned}\gamma_{2\mu} = \frac{1}{2}p^T \Big[& A_1\left(H_{2100}, v_\mu\right) + B\left(\bar{\phi}, H_{20\mu}\right) + 2B\left(\phi, H_{11\mu}\right) \\ & + B\left(H_{2100}, H_{00\mu}\right) + B\left(H_{2000}, H_{01\mu}\right) + 2B\left(H_{1100}, H_{10\mu}\right) \\ & + B_2\left(H_{2000}, \bar{\phi}, v_\mu\right) + 2B_2\left(\phi, H_{1100}, v_\mu\right) + 2C\left(\phi, \bar{\phi}, H_{10\mu}\right) \\ & + C\left(H_{2000}, \bar{\phi}, H_{00\mu}\right) + C\left(\phi, \phi, H_{01\mu}\right) + 2C\left(\phi, H_{1100}, H_{00\mu}\right) \\ & + C_1\left(\phi, \phi, \bar{\phi}, v_\mu\right) + D\left(\phi, \phi, \bar{\phi}, H_{00\mu}\right).\end{aligned}$$

6.4. Fold-Hopf bifurcation

Since the eigenvalues (B.5) are simple Lemma 3.6 gives eigenfunctions $\phi_{0,1}$ and adjoint eigenfunctions $\phi_{0,1}^{\odot}$ such that

$$A\phi_0 = 0, \qquad A\phi_1 = i\omega_0\phi_1, \qquad A^\star\phi_0^{\odot} = 0, \qquad A^\star\phi_1^{\odot} = i\omega_0\phi_1^{\odot}. \tag{6.87}$$

Let the vectors $q_0, q_1, p_0, p_1 \in \mathbb{R}^n$ satisfy

$$\Delta(0)q_0 = 0, \qquad \Delta(i\omega_0)q_1 = 0, \qquad p_0^T\Delta(0) = 0, \qquad p_1^T\Delta(i\omega_0) = 0,$$

then the eigenfunctions are given by

$$\begin{aligned}\phi_0(\theta) &= q_0, \\ \phi_0^{\odot}(\theta) &= p_0^T + p_0^T\int_0^\theta (\int_\sigma^h d\zeta(\tau))\, d\sigma, \\ \phi_1(\theta) &= e^{i\omega_0\theta}q_1, \\ \phi_1^{\odot}(\theta) &= p_1^T + p_1^T\int_0^\theta (\int_\sigma^h e^{i\omega_0(\sigma-\tau)}d\zeta(\tau))\, d\sigma.\end{aligned}$$

Furthermore, using Lemma 3.6 and 3.7 we normalize the eigenfunctions such that the ‘bi-orthogonality’ relation

$$\left\langle \phi_i^{\odot}, \phi_j \right\rangle = \delta_{ij}, \qquad 1 \le i, j \le 2,$$

holds.

Following [33], any point $y \in X_0$ from the critical eigenspace can be represented as

$$y = z_0\phi_0 + z_1\phi_1 + \bar{z}_1\bar{\phi}_1, \qquad z_{1,2} \in \mathbb{C},$$

where $z_0 = \left\langle \phi_0^{\odot}, y, \right\rangle$ and $z_1 = \left\langle \phi_1^{\odot}, y, \right\rangle$. Therefore, the homological equation (4.3) can be written as

$$\begin{aligned}A^{\odot\star}\mathcal{H}(z,\beta) + J_1(\beta) + R(\mathcal{H}(z,\beta), K(\beta)) \\ = D_{z_0}\mathcal{H}(z,\beta)\dot{z}_0 + D_{z_1}\mathcal{H}(z,\beta)\dot{z}_1 + D_{\bar{z}_1}\mathcal{H}(z,\beta)\dot{\bar{z}}_1,\end{aligned} \tag{6.88}$$

where $z = (z_1, z_2)$. Here, the functions $\mathcal{H}$, K and R admits the expansions

$$\begin{aligned} \mathcal{H}(z_0, z_1, \bar{z}_1, \beta_1, \beta_2) =& z_0\phi_0 + z_1\phi_1 + \bar{z}_1\bar{\phi}_1 \\ &+ \sum_{j+k+l=2}^{3} \sum_{|\mu|=0}^{1} \frac{1}{j!k!l!\mu!} H_{jkl\mu} z_0^j z_1^k \bar{z}_1^l \beta^\mu + \mathcal{O}(\|z\|^4 \|\beta\|^2), \qquad (6.89) \\ K(\beta) =& K_{10}\beta_1 + K_{01}\beta_2 + \mathcal{O}(\|\beta\|^2), \\ R(u, \beta) =& \frac{1}{2} B(u, u) + A_1(u, \beta) + \mathcal{O}\left(\|u\|^3 + \|u\| \, \|\beta\|^2 + \|\beta\|^3\right). \end{aligned}$$

Critical normal form coefficients We start by solving the critical normal form coefficients. Collecting the $z_0^2, z_1^2, z_0 z_1, z_1 \bar{z}_1$ we obtain the systems

$$\begin{aligned} A^{\odot\star} H_{20000} =& g_{200}\phi_0 - B(\phi_0, \phi_0), \\ \left(A^{\odot\star} - 2i\omega_0\right) H_{02000} =& - B(\phi_1, \phi_1), \\ \left(A^{\odot\star} - i\omega_0\right) H_{11000} =& g_{110}\phi_1 - B(\phi_0, \phi_1), \\ A^{\odot\star} H_{01100} =& g_{011}\phi_0 - B(\phi_1, \bar{\phi}_1). \end{aligned}$$

By the Fredholm alternative we obtain the quadratic coefficients

$$g_{200} = \frac{1}{2} p_0^T B(\phi_0, \phi_0), \qquad g_{110} = p_1^T B(\phi_0, \phi_1), \qquad g_{011} = p_0^T B(\phi_1, \bar{\phi}_1).$$

Then, using Corollary 4.1 and 4.8 we obtain

$$\begin{aligned} H_{20000}(\theta) =& B_0^{INV}(B(\phi_0, \phi_0), -g_{200}), \\ H_{02000}(\theta) =& e^{2i\omega_0\theta} \Delta(2i\omega_0)^{-1} B(\phi_0, \phi_0), \\ H_{11000}(\theta) =& B_{i\omega_0}^{INV}(B(\phi_0, \phi_1), -g_{110}), \\ H_{01100}(\theta) =& B_0^{INV}(B(\phi_1, \bar{\phi}_1), -g_{011}). \end{aligned}$$

Collecting the resonant $z_0^j z_1^k \bar{z}_1^l$ term in (6.88) with $j + k + l = 3$ yield the systems

$$\begin{aligned} A^{\odot\star} H_{30000} =& 6g_{300}\phi_0 + 3g_{200}H_{20000} - 3B(\phi_0, H_{20000}) - C(\phi_0, \phi_0, \phi_0), \\ A^{\odot\star} H_{11100} =& g_{111}\phi_0 - B(\phi_0, H_{01100}) - B(\phi_1, H_{10100}) - B(\bar{\phi}_1, H_{11000}) \\ &- C(\phi_0, \phi_1, \bar{\phi}_1) + (g_{101} + g_{110}) H_{01100} + g_{011} H_{20000}, \\ \left(A^{\odot\star} - i\omega_0\right) H_{21000} =& 2g_{210}\phi_1 - 2B(\phi_0, H_{11000}) - B(\phi_1, H_{20000}) - C(\phi_0, \phi_0, \phi_1) \\ &+ (2g_{110} + g_{200}) H_{11000}, \\ \left(A^{\odot\star} - i\omega_0\right) H_{02100} =& 2g_{021}\phi_1 - 2B(\phi_1, H_{01100}) - B(\bar{\phi}_1, H_{02000}) - C(\phi_1, \phi_1, \bar{\phi}_1) \\ &+ 2g_{011} H_{11000}. \end{aligned}$$

The Fredholm alternative yields

$$\begin{aligned}
g_{300} &= \frac{1}{6}p_0^T\left(3B(\phi_0,H_{20000})+C(\phi_0,\phi_0,\phi_0)\right),\\
g_{111} &= p_0^T\left(B(\phi_0,H_{01100})+B(\phi_1,\bar{H}_{11000})+B(\bar{\phi}_1,H_{11000})+C(\phi_0,\phi_1,\bar{\phi}_1)\right),\\
g_{210} &= \frac{1}{2}p_1^T\left(2B(\phi_0,H_{11000})+B(\phi_1,H_{20000})+C(\phi_0,\phi_0,\phi_1)\right),\\
g_{021} &= \frac{1}{2}p_1^T\left(2B(\phi_1,H_{01100})+B(\bar{\phi}_1,H_{02000})+C(\phi_1,\phi_1,\bar{\phi}_1)\right).
\end{aligned}$$

Parameter-related coefficients The parameter-related linear terms in (6.88) give

$$\begin{aligned}
A^{\odot\star}H_{00010} =& \phi_0 - J_1K_{10},\\
A^{\odot\star}H_{00001} =& -J_1K_{01}.
\end{aligned}$$

Let $\gamma = (\gamma_1,\gamma_2) = p_1^T J_1$, then by the Fredholm alternative we obtain the orthogonal frame

$$K_{10} = s_1 + \delta_1 s_2, \qquad K_{01} = \delta_2 s_2, \tag{6.90}$$

where

$$s_1^T = \gamma/\|\gamma\|^2, \qquad s_2^T = (-\gamma_2,\gamma_1)$$

and $\delta_{1,2}\in\mathbb{R}$ are some constants. Using Corollary (4.8) we obtain

$$\begin{aligned}
H_{00010}(\theta) &= \Delta(0)^{INV}\left(J_1K_{10}-\Delta'(0)q_0\right)+\delta_3q_0+\theta q_0\\
&= r_1+\delta_1r_2+\delta_3q_0-r_3,\\
H_{00001}(\theta) &= \delta_2r_2+\delta_4q_0,
\end{aligned}\tag{6.91}$$

where

$$r_1 = \Delta(0)^{INV}\left(J_1s_1\right), \qquad r_2 = \Delta(0)^{INV}\left(J_1s_2\right), \qquad r_3 = \Delta(0)^{INV}\left(\Delta'(0)q_0\right)-\theta q_0,$$

and the constants δ_3 and δ_4 are not chosen such that $\langle\phi_0^{\odot},H_{00010}\rangle = 0$ and $\langle\phi_0^{\odot},H_{00001}\rangle = 0$, but will be determined below.

Collecting the $z_0\beta_1, z_0\beta_2, z_1\beta_1$ and $z_1\beta_2$ terms in the homological equation yields the systems

$$\begin{aligned}
A^{\odot\star}H_{10010} =& H_{20000}-B(\phi_0,H_{00010})-A_1(\phi_0,K_{10}),\\
A^{\odot\star}H_{10001} =& -B(\phi_0,H_{00001})-A_1(\phi_0,K_{01}),\\
\left(A^{\odot\star}-i\omega_0\right)H_{01010} =& H_{11000}-B(\phi_1,H_{00010})-A_1(\phi_1,K_{10}),\\
\left(A^{\odot\star}-i\omega_0\right)H_{01001} =& \phi_1-B(\phi_1,H_{00001})-A_1(\phi_1,K_{01}).
\end{aligned}$$

The Fredholm alternative yields

$$
\begin{aligned}
0 =& p_0^T B(\phi_0, H_{00010}) + p_0^T A_1(\phi_0, K_{10}),\\
0 =& p_0^T B(\phi_0, H_{00001}) + p_0^T A_1(\phi_0, K_{01}),\\
0 =& p_1^T B(\phi_1, H_{00010}) + p_1^T A_1(\phi_1, K_{10}),\\
1 =& p_1^T B(\phi_1, H_{00001}) + p_1^T A_1(\phi_1, K_{01}).
\end{aligned}
$$

Substituting (6.90) and (6.91) into the above equations yields

$$
\begin{aligned}
0 =& p_0^T B(\phi_0, r_1) + \delta_1 p_0^T B(\phi_0, r_2) + \delta_3 p_0^T B(\phi_0, q_0) - p_0^T B(\phi_0, r_3)\\
&+ p_0^T A_1(\phi_0, s_1) + \delta_1 p_0^T A_1(\phi_0, s_2),\\
0 =& \delta_2 p_0^T B(\phi_0, r_2) + \delta_4 p_0^T B(\phi_0, q_0) + \delta_2 p_0^T A_1(\phi_0, s_2),\\
0 =& p_1^T B(\phi_1, r_1) + \delta_1 p_1^T B(\phi_1, r_2) + \delta_3 p_1^T B(\phi_1, q_0) - p_1^T B(\phi_1, r_3)\\
&+ p_1^T A_1(\phi_1, s_1) + \delta_1 p_1^T A_1(\phi_1, s_2),\\
1 =& \delta_2 p_1^T B(\phi_1, r_2) + \delta_4 p_1^T B(\phi_1, q_0) + \delta_2 p_1^T A_1(\phi_1, s_2).
\end{aligned}
$$

Which we can solve for $\delta_1, \delta_2, \delta_3$ and δ_4 by solving the systems

$$
\begin{aligned}
LL \begin{pmatrix} \delta_1 \\ \delta_3 \end{pmatrix} &= - \begin{pmatrix} p_0^T A_1(\phi_0, s_1) + p_0^T B(\phi_0, r_1) - p_0^T B(\phi_0, r_3) \\ p_1^T A_1(\phi_1, s_1) + p_1^T B(\phi_1, r_1) - p_1^T B(\phi_1, r_3) \end{pmatrix},\\
\mathrm{Re}(LL) \begin{pmatrix} \delta_2 \\ \delta_4 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{aligned}
$$

where

$$
LL = \begin{pmatrix} p_0^T B(\phi_0, r_2) + p_0^T A_1(\phi_0, s_2) & p_0^T B(\phi_0, \phi_0) \\ p_1^T B(\phi_1, r_2) + p_1^T A_1(\phi_1, s_2) & p_1^T B(\phi_1, \phi_0) \end{pmatrix}.
$$

6.5. Hopf-transcritical bifurcation

The critical normal form coefficients for the Hopf-transcritical bifurcation remain the same as for the fold-Hopf bifurcation. Therefore, we proceed with the parameter-related equations. The homological equation (4.3) can be written as

$$
\begin{aligned}
&A^{\odot\star}\mathcal{H}(z,\beta) + R(\mathcal{H}(z,\beta), K(\beta))\\
&\qquad = D_{z_0}\mathcal{H}(z,\beta)\dot{z}_0 + D_{z_1}\mathcal{H}(z,\beta)\dot{z}_1 + D_{\bar{z}_1}\mathcal{H}(z,\beta)\dot{\bar{z}}_1, \quad (6.92)
\end{aligned}
$$

where $z = (z_1, z_2)$. Here, the functions $\mathcal{H}$, K and R admits the expansions

$$\begin{aligned}\mathcal{H}(z_0, z_1, \bar{z}_1, \beta_1, \beta_2) =& z_0\phi_0 + z_1\phi_1 + \bar{z}_1\bar{\phi}_1 \\ &+ \sum_{j+k+l=2}^{3} \sum_{|\mu|=0}^{1} \frac{1}{j!k!l!\mu!} H_{jkl\mu} z_0^j z_1^k \bar{z}_1^l \beta^\mu + \mathcal{O}(\|z\|^4 \|\beta\|^2), \qquad (6.93)\\ K(\beta) =& K_{10}\beta_1 + K_{01}\beta_2 + \mathcal{O}(\|\beta\|^2), \\ R(u, \beta) =& \frac{1}{2} B(u, u) + A_1(u, \beta) + \mathcal{O}\left(\|u\|^3 + \|u\| \, \|\beta\|^2\right).\end{aligned}$$

Collecting the coefficients of the $z_0\beta_1, z_0\beta_2, z_1\beta_1$ and $z_1\beta_2$ terms in the homological equation we obtain the systems

$$A^{\odot\star} H_{10010} = \phi_0 - A_1(\phi_0, K_{10}), \qquad (6.94)$$
$$A^{\odot\star} H_{10001} = - A_1(\phi_0, K_{01}),$$
$$\left(A^{\odot\star} - i\omega_0\right) H_{01010} = - A_1(\phi_1, K_{10}),$$
$$\left(A^{\odot\star} - i\omega_0\right) H_{01001} = \phi_1 - A_1(\phi_1, K_{01}). \qquad (6.95)$$

By the Fredholm alternative we have

$$0 = 1 - p_0^T A_1(\phi_0, K_{10}), \qquad (6.96)$$
$$0 = - p_0^T A_1(\phi_0, K_{01}),$$
$$0 = - p_1^T A_1(\phi_1, K_{10}),$$
$$0 = 1 - p_1^T A_1(\phi_1, K_{01}). \qquad (6.97)$$

Let

$$K_{10} = \delta_1 e_1 + \delta_2 e_2,$$
$$K_{01} = \delta_3 e_1 + \delta_4 e_2,$$

where $e_1 = (1, 0), e_2 = (0, 1)$ and $\delta_i \, (i = 1, \ldots, 4) \in \mathbb{R}$ are to be determined. Substituting into equations (6.96)-(6.97), give the systems

$$\begin{pmatrix} p_0^T A_1(\phi_0, e_1) & p_0^T A_1(\phi_0, e_2) \\ p_1^T A_1(\phi_1, e_1) & p_1^T A_1(\phi_1, e_2) \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} p_0^T A_1(\phi_0, e_1) & p_0^T A_1(\phi_0, e_2) \\ p_1^T A_1(\phi_1, e_1) & p_0^T A_1(\phi_1, e_2) \end{pmatrix} \begin{pmatrix} \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Having determined K_{10} and K_{01} equations (6.94)-(6.95) are consistent, we obtain

$$H_{10010}(\theta) = B_0^{INV}(A_1(\phi_0, K_{10}), -1),$$
$$H_{10001}(\theta) = B_0^{INV}(A_1(\phi_0, K_{01}), 0),$$
$$H_{01010}(\theta) = B_{i\omega_0}^{INV}(A_1(\phi_1, K_{10}), 0),$$
$$H_{01001}(\theta) = B_{i\omega_0}^{INV}(A_1(\phi_1, K_{01}), -1).$$

Collecting the coefficient of the $\bar{z}_1^2$ term in the homological equation gives the systems

$$\left(A^{\odot\star} + 2i\omega_0\right) H_{00200} = -B(\bar{\phi}_1, \bar{\phi}_1).$$

Using using Corollary 4.1 we obtain

$$H_{00200} = e^{-2i\omega_0\theta}\Delta(-2i\omega_0)^{-1}B(\bar{\phi}_1, \bar{\phi}_1).$$

6.6. Hopf-Hopf bifurcation

Since the eigenvalues (B.12) are simple Lemma 3.6 gives eigenfunctions $\phi_{1,2}$ and adjoint eigenfunctions $\phi_{1,2}^{\odot}$ such that

$$A\phi_1 = i\omega_1\phi_1, \qquad A\phi_2 = i\omega_2\phi_2, \qquad A^\star\phi_1^{\odot} = i\omega_1\phi_1^{\odot}, \qquad A^\star\phi_2^{\odot} = i\omega_2\phi_2^{\odot}. \tag{6.98}$$

Let the vectors $q_1, q_2, p_1, p_2 \in \mathbb{R}^n$ satisfy

$$\Delta(i\omega_1)q_1 = 0, \qquad \Delta(i\omega_2)q_2 = 0, \qquad p_1\Delta(i\omega_1) = 0, \qquad p_2\Delta(i\omega_2) = 0,$$

then the eigenfunctions are given by

$$\begin{aligned}
\phi_1(\theta) &= e^{i\omega_1\theta}q_1,\\
\phi_1^{\odot}(\theta) &= p_1^T + p_1^T\int_0^\theta \Big(\int_\sigma^h e^{i\omega_1(\sigma-\tau)}d\zeta(\tau)\Big)\, d\sigma,\\
\phi_2(\theta) &= e^{i\omega_2\theta}q_2,\\
\phi_2^{\odot}(\theta) &= p_2^T + p_2^T\int_0^\theta \Big(\int_\sigma^h e^{i\omega_2(\sigma-\tau)}d\zeta(\tau)\Big)\, d\sigma.
\end{aligned}$$

Furthermore, using Lemma 3.6 and 3.7 we normalize the eigenfunctions such that the 'bi-orthogonality' relation

$$\left\langle \phi_i^{\odot}, \phi_j \right\rangle = \delta_{ij}, \qquad 1 \le i, j \le 2,$$

holds.

Following [33], any point $y \in X_0$ from the critical eigenspace can be represented as

$$y = z_1\phi_1 + \bar{z}_1\bar{\phi}_1 + z_2\phi_2 + \bar{z}_2\bar{\phi}_2, \qquad z_{1,2} \in \mathbb{C},$$

where $z_1 = \left\langle \phi_1^{\odot}, y, \right\rangle$ and $z_2 = \left\langle \phi_2^{\odot}, y, \right\rangle$. Therefore, the homological equation (4.3) can be written as

$$\begin{aligned}
&A^{\odot\star}\mathcal{H}(z,\beta) + J_1(\beta) + R(\mathcal{H}(z,\beta), K(\beta))\\
&\qquad = D_{z_1}\mathcal{H}(z,\beta)\dot{z}_1 + D_{\bar{z}_1}\mathcal{H}(z,\beta)\dot{\bar{z}}_1 + D_{z_2}\mathcal{H}(z,\beta)\dot{z}_2 + D_{\bar{z}_2}\mathcal{H}(z,\beta)\dot{\bar{z}}_2,
\end{aligned} \tag{6.99}$$

where $z = (z_1, z_2)$. Then $\mathcal{H}$, K and R admits the expansions

$$\begin{aligned}\mathcal{H}(z_1, \bar{z}_1, z_2, \bar{z}_2, \beta_1, \beta_2) =& z_1\phi_1 + \bar{z}_1\bar{\phi}_1 + z_2\phi_2 + \bar{z}_2\bar{\phi}_2 + \\ &+ \sum_{j+k+l+m=2}^{3} \sum_{|\mu|=0}^{1} \frac{1}{j!k!l!m!\mu!} H_{jklm\mu} z_1^j \bar{z}_1^k z_2^l \bar{z}_2^m \beta^\mu + \mathcal{O}(\|z\|^6 \|\beta\|^2),\end{aligned} \tag{6.100}$$

$$K(\beta) = K_{10}\beta_1 + K_{01}\beta_2 + \mathcal{O}(\|\beta\|^2), \tag{6.101}$$

$$R(u, \beta) = \frac{1}{2}B(u, u) + A_1(u, \beta) + \mathcal{O}\left(\|u\|^3 + \|u\| \, \|\beta\|^2 + \|\beta\|^3\right).$$

The linear terms in (6.99) give back the eigenfunctions (6.98) and the parameter-related equations

$$A^{\odot\star} H_{0000\mu} = -J_1 K_\mu,$$

where $\mu = (10), (01)$. Let

$$K_\mu = \gamma_{1\mu} e_1 + \gamma_{2\mu} e_2, \tag{6.102}$$

where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$H_{0000\mu}(\theta) = -\gamma_{1\mu}\Delta(0)^{-1} J_1 e_1 - \gamma_{2\mu}\Delta(0)^{-1} J_1 e_2 \tag{6.103}$$

Collecting the $z_i\beta_j, 1 \le i, j \le 2$ terms yields the systems

$$\begin{aligned}(A^{\odot\star} - i\omega_1) H_{100010} &= \phi_1 - A_1(\phi_1, K_{10}) - B(\phi_1, H_{000010}), \\ (A^{\odot\star} - i\omega_1) H_{100001} &= -A_1(\phi_1, K_{01}) - B(\phi_1, H_{000001}), \\ (A^{\odot\star} - i\omega_2) H_{001010} &= -A_1(\phi_2, K_{10}) - B(\phi_2, H_{000010}), \\ (A^{\odot\star} - i\omega_2) H_{001001} &= \phi_2 - A_1(\phi_2, K_{01}) - B(\phi_2, H_{000001}).\end{aligned}$$

By the Fredholm alternative we obtain

$$\begin{aligned} 1 &= p_1^T \left(A_1(\phi_1, K_{10}) + B(\phi_1, H_{000010})\right), \\ 0 &= p_1^T \left(A_1(\phi_1, K_{01}) + B(\phi_1, H_{000001})\right), \\ 0 &= p_2^T \left(A_1(\phi_2, K_{10}) + B(\phi_2, H_{000010})\right), \\ 1 &= p_2^T \left(A_1(\phi_2, K_{01}) + B(\phi_2, H_{000001})\right). \end{aligned} \tag{6.104}$$

Substituting (6.102) and (6.103) into (6.104) yields

$$
\begin{aligned}
1 &= p_1^T\Big(\gamma_{110}A_1(\phi_1,e_1)+\gamma_{210}A_1(\phi_1,e_2)\\
&\qquad -\gamma_{110}B(\phi_1,\Delta(0)^{-1}J_1e_1)-\gamma_{210}B(\phi_1,\Delta(0)^{-1}J_1e_2)\Big),\\
0 &= p_1^T\Big(\gamma_{101}A_1(\phi_1,e_1)+\gamma_{201}A_1(\phi_1,e_2)\\
&\qquad -\gamma_{101}B(\phi_1,\Delta(0)^{-1}J_1e_1)-\gamma_{201}B(\phi_1,\Delta(0)^{-1}J_1e_2)\Big),\\
0 &= p_2^T\Big(\gamma_{110}A_1(\phi_2,e_1)+\gamma_{210}A_1(\phi_2,e_2)\\
&\qquad -\gamma_{110}B(\phi_2,\Delta(0)^{-1}J_1e_1)-\gamma_{210}B(\phi_2,\Delta(0)^{-1}J_1e_2)\Big),\\
1 &= p_2^T\Big(\gamma_{101}A_1(\phi_2,e_1)+\gamma_{201}A_1(\phi_2,e_2)\\
&\qquad -\gamma_{101}B(\phi_2,\Delta(0)^{-1}J_1e_1)-\gamma_{201}B(\phi_2,\Delta(0)^{-1}J_1e_2)\Big).
\end{aligned}
\tag{6.105}
$$

Note that $\Delta(0)^{-1}J_1e_i$ is a constant function of θ. We can solve (6.105) for $(\gamma_{1\mu},\gamma_{2\mu})$ by solving the two 2×2-dimensional systems

$$
\mathrm{Re}(\begin{pmatrix} M_{11} & M_{12}\\ M_{31} & M_{32}\end{pmatrix})\begin{pmatrix}\gamma_{110}\\ \gamma_{101}\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\qquad \mathrm{Re}(\begin{pmatrix} M_{23} & M_{24}\\ M_{43} & M_{44}\end{pmatrix})\begin{pmatrix}\gamma_{210}\\ \gamma_{201}\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix},
$$

where

$$
\begin{aligned}
M_{11} &= p_1^T\left(A_1(\phi_1,e_1)-B(\phi_1,\Delta(0)^{-1}J_1e_1)\right),\\
M_{12} &= p_1^T\left(A_1(\phi_1,e_2)-B(\phi_1,\Delta(0)^{-1}J_1e_2)\right),\\
M_{23} &= p_1^T\left(A_1(\phi_1,e_1)-B(\phi_1,\Delta(0)^{-1}J_1e_1)\right),\\
M_{24} &= p_1^T\left(A_1(\phi_1,e_2)-B(\phi_1,\Delta(0)^{-1}J_1e_2)\right),\\
M_{31} &= p_2^T\left(A_1(\phi_2,e_1)-B(\phi_2,\Delta(0)^{-1}J_1e_1)\right),\\
M_{32} &= p_2^T\left(A_1(\phi_2,e_2)-B(\phi_2,\Delta(0)^{-1}J_1e_2)\right),\\
M_{43} &= p_2^T\left(A_1(\phi_2,e_1)-B(\phi_2,\Delta(0)^{-1}J_1e_1)\right),\\
M_{44} &= p_2^T\left(A_1(\phi_2,e_2)-B(\phi_2,\Delta(0)^{-1}J_1e_2)\right).
\end{aligned}
$$

Lastly, for initialization of the Neimark-Sacker curves (C.87), we need the resonant cubic critical normal form coefficients $g_{2100}, g_{1011}, g_{1110}$ and g_{0021}. Collecting the $z_1|z_1|^2, z_1|z_2|^2$,

$z_1|z_2|^2$ and $|z_1|^2 z_2$ terms in the homological equations leads to the systems

$$\begin{aligned}\left(A^{\odot\star} - i\omega_1\right) H_{210000} =& 2g_{2100}\phi_1 - 2B(\phi_1, H_{110000}) - B(\bar{\phi}_1, H_{200000}) - C(\phi_1,\phi_1,\bar{\phi}_1),\\ \left(A^{\odot\star} - i\omega_1\right) H_{101100} =& g_{1011}\phi_1 - B(\bar{\phi}_2, H_{101000}) - B(\phi_1, H_{001100}) - B(\phi_2, H_{100100})\\ &- C(\phi_1,\phi_2,\bar{\phi}_2),\\ \left(A^{\odot\star} - i\omega_2\right) H_{111000} =& g_{1110}\phi_2 - B(\bar{\phi}_1, H_{101000}) - B(\phi_1, H_{011000}) - B(\phi_2, H_{110000})\\ &- C(\phi_1,\bar{\phi}_1,\phi_2),\\ \left(A^{\odot\star} - i\omega_2\right) H_{002100} =& 2g_{0021}\phi_2 - 2B(\phi_2, H_{001100}) - B(\bar{\phi}_2, H_{002000}) - C(\phi_2,\phi_2,\bar{\phi}_2).\end{aligned}$$

Using the Fredholm alternatives yields

$$\begin{aligned}g_{2100} &= \frac{1}{2}p_1^T\left(2B(\phi_1, H_{110000}) + B(\bar{\phi}_1, H_{200000}) + C(\phi_1,\phi_1,\bar{\phi}_1)\right),\\ g_{1011} &= p_1^T\left(B(\bar{\phi}_2, H_{101000}) + B(\phi_1, H_{001100}) + B(\phi_2, H_{100100}) + C(\phi_1,\phi_2,\bar{\phi}_2)\right),\\ g_{1110} &= p_2^T\left(B(\bar{\phi}_1, H_{101000}) - B(\phi_1, \bar{H}_{100100}) - B(\phi_2, H_{110000}) + C(\phi_1,\bar{\phi}_1,\phi_2)\right),\\ g_{0021} &= \frac{1}{2}p_2^T\left(2B(\phi_2, H_{001100}) + B(\bar{\phi}_2, H_{002000}) + C(\phi_2,\phi_2,\bar{\phi}_2)\right).\end{aligned}$$

It thus remains to obtain expression for the functions H_{110000}, H_{200000}, H_{101000}, H_{001100}, H_{100100} and H_{002000}. For this we collect the $|z_1|^2, z_1^2, z_1z_2, |z_2|^2, z_1\bar{z}_2$ and $z_2\bar{z}_1$ terms in the homological equations, yielding to the systems

$$\begin{aligned}A^{\odot\star}H_{110000} =& -B(\phi_1,\bar{\phi}_1),\\ \left(A^{\odot\star} - 2i\omega_1\right) H_{200000} =& -B(\phi_1,\phi_1),\\ \left(A^{\odot\star} - i\omega_1 - i\omega_2\right) H_{101000} =& -B(\phi_1,\phi_2),\\ A^{\odot\star}H_{001100} =& -B(\phi_2,\bar{\phi}_2),\\ \left(A^{\odot\star} - i\omega_1 + i\omega_2\right) H_{100100} =& -B(\phi_1,\bar{\phi}_2),\\ \left(A^{\odot\star} - 2i\omega_2\right) H_{002000} =& -B(\phi_2,\phi_2).\end{aligned}$$

Using Corollary 4.1 we obtain the solutions

$$\begin{aligned}H_{110000}(\theta) =& \Delta(0)^{-1}B(\phi_1,\bar{\phi}_1),\\ H_{200000}(\theta) =& e^{2i\omega_1\theta}\Delta(2i\omega_1)^{-1}B(\phi_1,\phi_1),\\ H_{101000}(\theta) =& e^{i(\omega_1+\omega_2)\theta}\Delta(i\left(\omega_1+\omega_2\right))^{-1}B(\phi_1,\phi_2),\\ H_{001100}(\theta) =& \Delta(0)^{-1}B(\phi_2,\bar{\phi}_2),\\ H_{100100}(\theta) =& e^{i(\omega_1-\omega_2)\theta}\Delta(i\left(\omega_1-\omega_2\right))^{-1}B(\phi_1,\bar{\phi}_2),\\ H_{011000}(\theta) =& e^{2i\omega_2\theta}\Delta(2i\omega_2)^{-1}B(\phi_2,\phi_2).\end{aligned}$$

7. Implementation in DDE-BifTool

In this Chapter we briefly describe the implementation of the predictors in `DDE-BifTool` and it should be read with the source code at hand, which can be downloaded from sourceforge.net[1]. The procedure is similar in all cases. First convert the bifurcation point to its type, i.e. a Bogdanov-Takens, a fold-Hopf, etc. Then calculate the coefficients as derived in Chapter 6. Call the predictor file to setup a branch with two initial points using the predictors from Appendix C. Then the branch of homoclinic orbits or non-hyperbolic limit cycles can be continued as usual.

In the first Section we mainly follow the method used in [32, 1, 36] to initialize the continuation of the homoclinic orbit emanating from the generic and transcritical Bogdanov-Takens bifurcation. In the second Section we only describe the method to initialize the continuation of limit cycles emanating from generalized Hopf points. The remaining cases, i.e. the fold-Hopf, Hopf-transcritical and Hopf-Hopf bifurcations, are treated similarly.

7.1. Bogdanov-Takens

We suppose that the parameter-dependent DDE (3.3) is a Bogdanov-Takens point at the steady-state φ_0 and parameter α_0. Furthermore, we assume that one has successfully located a Bogdanov-Takens point denoted by `bt` with `DDE-BifTool`. For both the generic and transcritical case there are vectors q_0, q_1, p_1, p_0 such that the normalizations in Lemma 3.8 hold. These are calculated in the file `p_tobt(funcs,point)`. Depending on weather a generic or transcritical Bogdanov-Takens bifurcation is considered either the function `nmfm_bt_pm(funcs,bt)` or `p_totbt(funcs,point)` should be called. In these files the coefficients derived in Section 6.1 and 6.2, respectively.

7.1.1. Initialize continuation homoclinic orbit in the generic case

Step 1: Compute coefficients a, b, H_{2000}, H_{1100}, H_{0200}, K_{10}, K_{01}, H_{0001}, H_{0010}, K_{02}, H_{0002}, H_{1001}, H_{0101}, d, e, a_2, and b_2.

Step 2: Let $A := \|x(\pm\infty, \varepsilon) - x(0, \varepsilon)\|$ be the amplitude of the initial homoclinic orbit. Using (C.9) we approximate A for small ε by

$$A = \left\| \varepsilon^2 \left(\frac{2}{a} \right) \phi_0 - \varepsilon^2 \left(\frac{-4}{a} \right) \phi_0 \right\| = \varepsilon^2 \frac{6}{|a|} \|\phi_0\| = \varepsilon^2 \frac{6\sqrt{m}}{|a|}.$$

[1] `https://sourceforge.net/projects/ddebiftool/files/latest/download`

Here we used that for finitely many delays m we have

$$\|\phi_0\| = \|\Phi_0\| = \|\underbrace{(q_0, \dots q_0)}_{m \text{ times}}\| = \sqrt{m}.$$

The amplitude is chosen by the user, so that we get

$$\varepsilon = \sqrt{\frac{A|a|}{6\sqrt{m}}}.$$

Step 3: We choose the initial half-return time T such that, at the end points, the distance,

$$k := \|x(\pm\infty, \varepsilon) - x(\pm T, \varepsilon)\|$$

is sufficiently small. For ε small we approximate k using (C.9) as

$$k = \varepsilon^2 \frac{6\text{sech}^2(\pm\varepsilon T)}{|a|} \|\phi_0\|.$$

Hence the initial half-return time is given by solving

$$k = A\text{sech}^2(\varepsilon T) \|\phi_0\|,$$

or, equivalently,

$$\text{sech}(\varepsilon T) = \sqrt{l_1},$$

where $l_1 = \sqrt{\frac{k}{A\|\phi_0\|}} > 0$. We conclude that

$$T = \frac{1}{\varepsilon}\text{sech}^{-1}(l_1) = \frac{1}{\varepsilon}\log\left(\frac{1 + \sqrt{1 - l_1^2}}{l_1}\right).$$

Step 4: Compute the initial homoclinic orbit by discretizing the interval $[0, 1]$ into equidistant point f_i (the fine mesh) and the evaluate (C.9) at each t where t is given by

$$t = (2f_i - 1)T, \qquad f_i \in [0, 1].$$

By taking the limit of t in (C.9) to infinity the saddle point is approximated by

$$x_0 = \epsilon^2 \left(\frac{10b}{7a} H_{0001} + \frac{2}{a}\phi_0\right).$$

Note that x_0 is uniquely defined, since $H_{0001}(\theta)$ and $\phi(\theta)$ are constant functions.

Step 5: Compute a second homoclinic orbit with slightly bigger amplitude and add the two orbits to a homoclinic branch. Then the branch of homoclinic orbits can be continued.

7.1.2. Initialize continuation homoclinic orbits in the transcritical case

Step1: Compute coefficients a, b, H_{2000}, H_{1100}, H_{0200}, K_{10}, K_{01}, H_{1010}, H_{1001}, H_{0110}, H_{0101}, K_{20}, K_{02}, d, e, a_1, a_2, b_1 and b_2.

Step2: Let $A := \|x(\pm\infty, \varepsilon) - x(0, \varepsilon)\|$ be the amplitude of the initial homoclinic orbit. Using (C.42) we approximate A for small ε by

$$A = \left\| \varepsilon^2 \left(-\frac{1}{2a} \right) \phi_0 - \varepsilon^2 \left(\frac{1}{a} \right) \phi_0 \right\| = \varepsilon^2 \frac{3}{2|a|} \|\phi_0\| = \varepsilon^2 \frac{3\sqrt{m}}{2|a|}.$$

Here we used that for finitely many delays m we have

$$\|\phi_0\| = \|\Phi_0\| = \| \underbrace{(q_0, \dots q_0)}_{m \, \text{times}} \| = \sqrt{m}.$$

The amplitude is chosen by the user, so that we get

$$\varepsilon = \sqrt{A \frac{2|a|}{3\sqrt{m}}}$$

Step3: We choose the initial half-return time T such that, at the end points, the distance,

$$k := \|x(\pm\infty, \varepsilon) - x(\pm T, \varepsilon)\|$$

is sufficiently small. For ε small we approximate k using (C.42) as

$$k = \varepsilon^2 \frac{3\text{sech}^2(\varepsilon T)}{2|a|} \|\phi_0\| .$$

Hence the initial half-return time is given by solving

$$k = A\text{sech}^2(\varepsilon T) \|\phi_0\| ,$$

or equivalently

$$\text{sech}(\varepsilon T) = \sqrt{l_1},$$

where $l_1 = \sqrt{\dfrac{k}{A \|\phi_0\|}} > 0$. We conclude that

$$T = \frac{1}{\varepsilon}\text{sech}^{-1}(l_1) = \frac{1}{\varepsilon} \log \left(\frac{1 + \sqrt{1 - l_1^2}}{l_1} \right).$$

Step 4: Compute the initial homoclinic orbits by discretizing the interval $[0, 1]$ into equidistant point f_i (the fine mesh) and the evaluate (C.42) and(C.69) at each t where t is given by

$$t = (2f_i - 1)T, \qquad f_i \in [0, 1].$$

By taking the limit of t in (C.42) and (C.69) to infinity the saddle points are approximated by

$$s_1 = 0,$$
$$s_2 = \frac{1}{a}\epsilon^2\phi_0,$$

respectively.

Step 5: Compute for both homoclinic orbits a second homoclinic orbit with slightly bigger amplitude and add the two orbits to a homoclinic branch. Then the branches of homoclinic orbits can be continued.

7.2. Generalized Hopf

Step 1: Compute the coefficients H_{1001}, H_{0001}, H_{1100}, H_{2000}, H_{3000}, H_{2100} and the second Lyapunov coefficient $c_2(0)$.

Step 2: Compute the initial periodic orbit by discretizing the interval $[0,1]$ into equidistant point f_i (the fine mesh) and then evaluate (C.74) at each t where t is given by

$$t = (2f_i - 1)T, \qquad f_i \in [0,1]$$

and $\epsilon \ll 1$ is provided by the user. Here the period T is given by (C.73). Substituting, (C.72) into equation 6.83 we obtain an approximation for the parameters.

Step 3: Compute a second period orbit as in **step 2** with slightly bigger ϵ

Step 4: From the data collected in **step 2** and **step 3** setup a branch of non-hyperbolic limit cycles with the same structure used in the `DDE-BifTool` extension `ddebiftool_extra_psol`. Now the branch can be continued as usual.

8. Examples

In this Chapter we demonstrate the predictors of the nonhyperbolic cycles emanating from generalized Hopf, fold-Hopf, Hopf-transcritical and Hopf-Hopf bifurcations, and the predictors of the homoclinic orbits emanating from generic and transcritical Bogdanov-Takens bifurcations, in various models. In the first example we include the minimal `Matlab/Octave` code to reproduce the main results. This code can then be used as a starting point to investigate other models, by simple modifications. The code for the other examples can be found online on sourceforge.net[1].

8.1. Delayed feedback on the dynamical model of a financial system

In [45] the delayed financial system

$$\begin{cases} \dot{x} &= z + (y-a)x + k_1(x - x(t-\tau_1)), \\ \dot{y} &= 1 - by - x^2 + k_2(y - y(t-\tau_2)), \\ \dot{z} &= -x - cz + k_3(z - z(t-\tau_3)) \end{cases} \tag{8.1}$$

is considered. The variables and parameters have the following meaning:

- The variables x, y and z describes the interest rate, the investment demand, and the price index respectively.
- $a > 0$ is the saving amount.
- $b > 0$ is the cost per investment.
- $c > 0$ is the elasticity of demand of commercial markets.
- k_i $(i = 1, 2, 3)$ are the feedback strength.

We fix the parameters

$$b = 0.1, \quad c = 1, \quad k_1 = k_2 = 0, \quad k_3 = 1$$

and take a and τ_3 as control parameters. It can analytically verified that the steady-state

$$H_0 = (x_0, y_0, z_0) = (0, \frac{1}{b}, 0)$$

[1] https://sourceforge.net/projects/ddebiftool/files/latest/download

undergoes a Hopf bifurcation at parameter values $(a, \tau_3) \approx (10, 0.9708)$, see [45, Theorem 3 (iii)]. To treat τ_3 as an ordinary parameter we rescale the time by

$$t \to \frac{t}{\tau_3}.$$

Then the system (8.1) becomes

$$\begin{cases} \dot{x} &= \tau_3 \left(z + (y-a)x\right), \\ \dot{y} &= \tau_3 \left(1 - by - x^2\right), \\ \dot{z} &= \tau_3 \left(-x - cz + (z - z(t-1))\right), \end{cases} \tag{8.2}$$

where we have taken into account that $k_1 = k_2 = 0, \quad k_3 = 1$. Continuing the Hopf point H_0 at $(a, \tau_3) \approx (10, 0.9708)$, a generalized Hopf point is detected at parameter values $(a, \tau_3) \approx (13.0581, 1.7659)$, with the second Lyapunov coefficient $\ell_2 \approx 0.0227$. We continue the curve **LPC** of fold bifurcation of limit cycles emanating from the generalized Hopf point using the predictor from Section C.3. In Figure 8.1 we have plotted the Hopf curve, the continued curve **LPC** and the predictor in parameter space. There we see that for nearby parameter values the continued curve **LPC** and the predictor are nearly identical. In Figure 8.2 we compare the limit cycles along the curve **LPC** with the predicted limit cycle given by C.74.

```
%% initialize system
clear   % clear variables
close all   % close figures
addpath('../../ddebiftool',...
    '../../ddebiftool_extra_psol',...
    '../../ddebiftool_extra_nmfm',...
    '../../ddebiftool_utilities');

% fixed constants
b=0.1; c=1; k1=0; k2=0; k3=1;

% DDE
sys_sonpark=@(xx,par)[...
 par(2)*(xx(3,1,:)+(xx(2,1,:)-par(1))*xx(1,1,:)+k1*(xx(1,1,:)-xx(1,2,:)));
 par(2)*(1-b*xx(2,1,:)-xx(1,1,:)^2+k2*(xx(2,1,:)-xx(2,3,:)));
 par(2)*(-xx(1,1,:)-c*xx(3,1,:)+k3*(xx(3,1,:)-xx(3,4,:)))];

funcs=set_funcs(...
    'sys_rhs', @(xx,par) sys_sonpark(xx,par),...
    'sys_tau', @() [3 4 5],...
    'x_vectorized',1);

%% setup hopf point
```

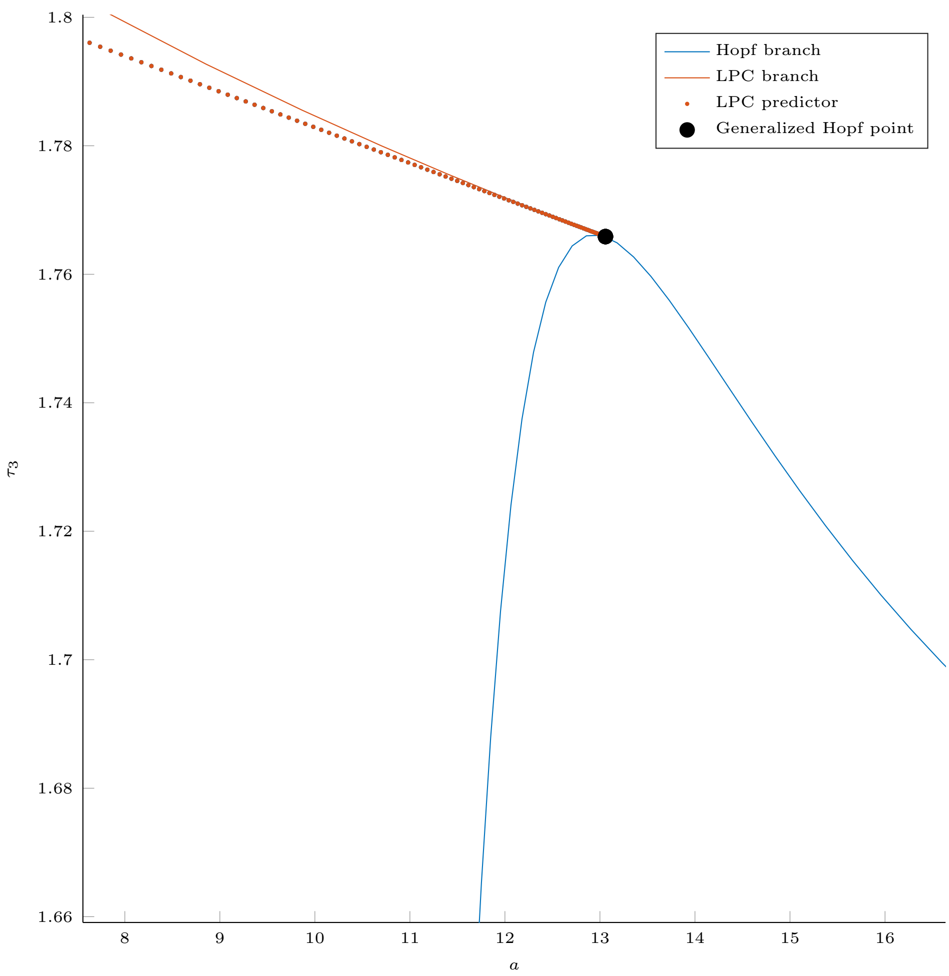


Figure 8.1.: Bifurcation diagram near a generalized Hopf bifurcation in the delayed financial system (8.1). We see that the predictor gives a good approximation to the continued curve **LPC** for nearby parameter values in parameter space.

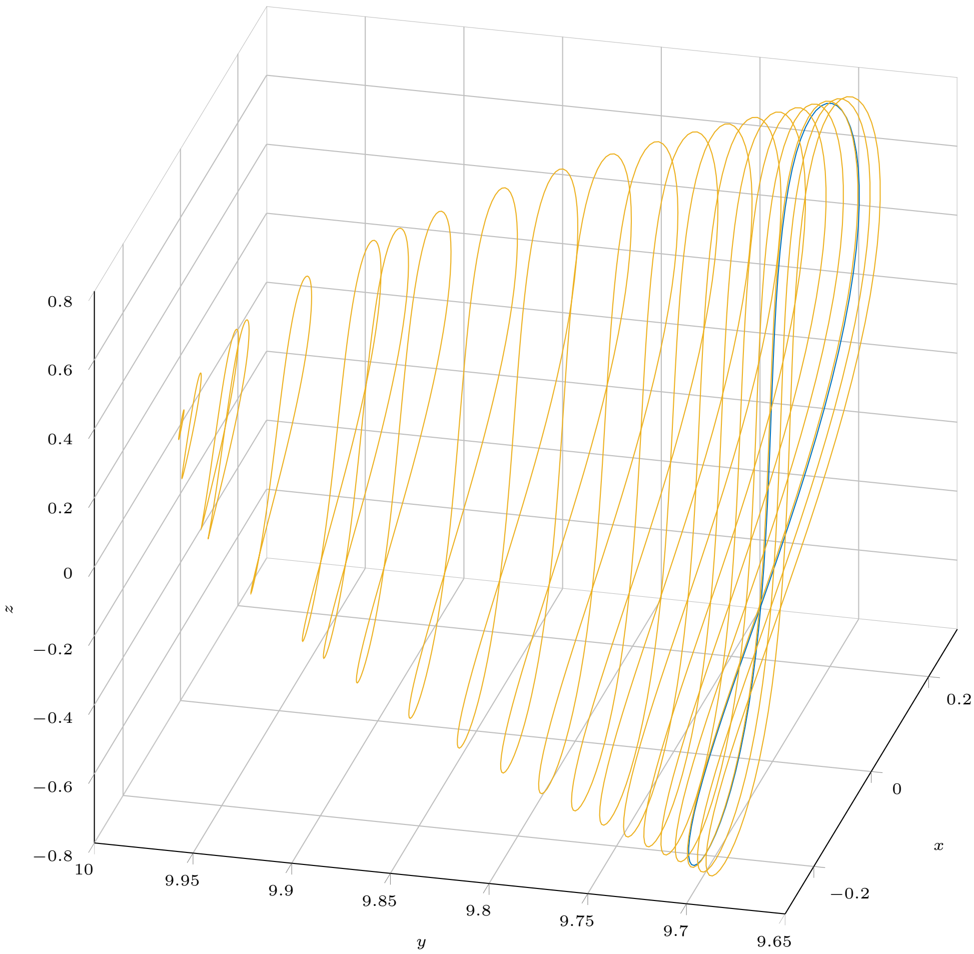


Figure 8.2.: Plot of nonhyperbolic limit cycles along the curve **LPC** in phase-space emanating from the generalized Hopf bifurcation in the delayed financial system (8.1). The yellow orbits are the computed limit cycles. The blue orbit is the predicted limit cycle.

```
ind_a=1;
ind_tau3=2;

stst.kind='stst';
xstar=0;
ystar=1/b;
zstar=0;
a=10;
tau3=0.9708;
stst.x=[xstar; ystar; zstar];
stst.parameter=[a tau3 0 0 1];

method=df_mthod(funcs,'stst');
[stst,s]=p_correc(funcs,stst,[],[],method.point);
stst.stability=p_stabil(funcs,stst,method.stability);
stst.stability.l1

%% continue hopf point
hopf1=p_tohopf(funcs,stst);
hopf2=hopf1;
hopf2.parameter(ind_a)=hopf2.parameter(ind_a)+0.001;

method=df_mthod(funcs,'hopf');
[hopf2,s]=p_correc(funcs,hopf2,ind_tau3,[],method.point)

hopf_br=df_brnch(funcs,[ind_a ind_tau3],'hopf');
hopf_br.point=hopf1;
hopf_br.point(2)=hopf2;

figure(1);clf;
hopf_br.parameter.max_bound=[[ind_a 17];[ind_tau3 3]];
hopf_br.method.continuation.steplength_growth_factor=1.05;
hopf_br=br_contn(funcs,hopf_br,200);

%% detect codim-2 bifurcations
hopf_br=br_stabl(funcs,hopf_br,0,0);
[hopf_br,hopf_testfuncs]=LocateSpecialPoints(funcs,hopf_br);

%% extract generalized Hopf point
genh_ind=br_getflags(hopf_br,'genh');
genh=hopf_br.point(genh_ind);

%% continue LPC using the predictor
[LPCfuncs,LPCbranch,psol]=init_GH_LPC(funcs,genh,0.5);

% figure(2);
LPCbranch=br_contn(LPCfuncs,LPCbranch,50);

%% plot bifurcation diagram
figure(2);clf; hold on

cm=colormap('lines');
figure(2);clf;hold on
getpars=@(br,ind) arrayfun(@(p)p.parameter(ind),br);
```

```
getx=@(br,ind) arrayfun(@(p)p.x(ind),br);
hopf_br_pl=plot(getpars(hopf_br.point,ind_a),getpars(hopf_br.point,ind_tau3
    ),'Color',cm(1,:));
LPCbranch_pl=plot(getpars(LPCbranch.point,ind_a),getpars(LPCbranch.point,
    ind_tau3),'Color',cm(2,:));

% add predictor
eps=linspace(0,2);
betas=[0*eps; -2*genh.nmfm.L2*eps.^2];
gamma110=genh.nmfm.gamma110;
gamma101=genh.nmfm.gamma101;
gamma210=genh.nmfm.gamma210;
gamma201=genh.nmfm.gamma201;
KK=inv(real([[gamma110 gamma101; gamma210 gamma201]]))*betas;

preditor_pl=plot(genh.parameter(ind_a)+KK(1,:),genh.parameter(ind_tau3)+KK
    (2,:),'.');
genh_pl=plot(genh.parameter(ind_a),genh.parameter(ind_tau3),'k.','
    MarkerSize',16);

legend([hopf_br_pl LPCbranch_pl preditor_pl genh_pl],...
    {'Hopf branch','LPC branch','LPC predictor','Generalized Hopf point'});
axis([7.5586    16.6371    1.6591    1.8004]);
xlabel('$a$','Interpreter','LaTeX');
ylabel('$\tau_3$','Interpreter','LaTeX');

%% plot cycles on the curve LPCbranch
figure(4); clf; hold on
plot3(psol.profile(1,:),psol.profile(2,:),psol.profile(3,:),'Color',cm(1,:)
    );
% plot LPC in phase-space
for i=10:30
    plot3(LPCbranch.point(i).profile(1,:),LPCbranch.point(i).profile(2,:)
        ,...
        LPCbranch.point(i).profile(3,:),'Color',cm(3,:));
end

xlabel('$x$','Interpreter','LaTeX');
ylabel('$y$','Interpreter','LaTeX');
zlabel('$z$','Interpreter','LaTeX');

view(-76,28)
grid on
```

8.2. A neural mass model

In [48] and [46] the following model of two interacting layers of neurons is considered

$$\begin{cases} \dot{x}_1(t) &= -x_1(t) - ag(bx_1(t-\tau_1)) + cg(dx_2(t-\tau_2)), \\ \dot{x}_2(t) &= -x_2(t) - ag(bx_2(t-\tau_1)) + cg(dx_1(t-\tau_2)). \end{cases} \tag{8.3}$$

The variables $x_1(t)$ and $x_2(t)$ represent the population-averaged neural activity at time t in layers one and two, respectively. The parameter $a > 0$ is a measure of the strength of inhibitory feedback, while $c > 0$ measures the strength of the excitatory effect of one layer on the other. The parameters $b > 0$ and $d > 0$ are saturation rates and the delays $\tau_{1,2}$ represent time lags in the inhibitory feedback loop and excitatory inter-layer connection. Note that the system is symmetric with respect to interchanging the labels 1 and 2, so equilibria are necessarily of the form (x_0, x_0). The function g is smooth, strictly increasing and satisfies $g(0) = 0$ and $g'(0) = 1$. We fix the numerical parameter values

$$b = 2.0, \qquad d = 1.2, \qquad \tau_1 = 12.7, \qquad \tau_2 = 20.2,$$

and take for $g : \mathbb{R} \to \mathbb{R}$ the sigmoidal form

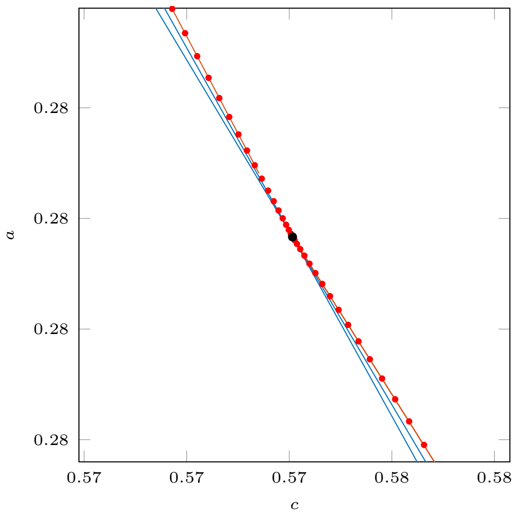
$$g(z) = [\tanh(z-1) + \tanh(1)] \cosh(1)^2.$$

The trivial equilibrium $(x_1, x_2) = (0, 0)$ undergoes a Hopf-Hopf at parameter values $(c, a) \approx (0.5741, 0.2798)$. The critical normal form coefficients are given by

$$\begin{aligned} g_{2100} &\approx -0.0007 + 0.0020i, \\ g_{1011} &\approx -0.0020 + 0.0040i, \\ g_{1110} &\approx -0.0016 - 0.0045i, \\ g_{0021} &\approx -0.0008 - 0.0022i, \\ \theta &\approx 2.4716, \\ \delta &\approx 2.2107. \end{aligned}$$

From [34] we conclude that we are in simple case I, i.e. nearby the Hopf-Hopf point there will be two stable periodic orbits present and an unstable torus. By using the predictors two Neimark-Sacker curves are obtained. In Figure 8.3 the predictors for the parameter values are compared with continued Neimark-Sacker curves.

Remark 8.1. As mentioned in the introduction, the expressions for the critical normal form coefficients for DDEs have first been derived in [29]. There the model (8.3) has been used to verify the expressions. However, due to a mistake in the expressions for the zero-Hopf and Hopf-Hopf bifurcations the coefficients obtained are wrong. In particular, the prediction of a stable torus in Section 4.2.9 in [29] is incorrect. Numerical simulation over a longer time of integration show that the torus slowly shrinks to a stable cycle.



(a)

Figure 8.3.: Bifurcation diagram near the Hopf-Hopf point. The two Neimark-Sacker curves emanate from the Hopf-Hopf point. In (a) the predicted parameter values (red dots) are located to a high precision at the continued Neimark-Sacker curves (orange). The blue curves are the Hopf branches, which meet transversely at the Hopf-Hopf point.

8.3. Delayed Ratio-Dependent Holling-Tanner Predator Prey System

In [39] the Bogdanov-Takens bifurcation of the following delayed predator prey system

$$\begin{cases} \dot{x} &= rx\left(1 - \frac{x}{K}\right) - \frac{\alpha y(x-\bar{m})}{Ay + x - \bar{m}} - \bar{h}, \\ \dot{y} &= sy\left(1 - \frac{dy(t-\bar{\tau})}{x(t-\bar{\tau}) - \bar{m}}\right), \end{cases} \tag{8.4}$$

is considered. Here x and y stand for prey and predator population (or densities) at time t, respectively. The predator growth is of logistic type with growth rate r and carrying capacity K in the absence of predation; α and A stand for the predator capturing rate and half saturation constant, respectively; s is the intrinsic growth rate of predator; however, carrying capacity x/b (b is the conversion rate of prey into predators) is the function on the population size of prey. The parameters $\alpha, A, \bar{m}, \bar{h}, s, b$, and $\bar{\tau}$ are all positive constants. $\bar{m}$ is a constant number of prey using refuges; h is the rate of prey harvesting. System (8.4) can be transformed into

$$\begin{cases} \dot{x} &= (x+m)(1-x-m) - \frac{xy}{ay+x} - h, \\ \dot{y} &= \delta y\left(\beta - \frac{y(t-\tau)}{x(t-\tau)}\right), \end{cases} \tag{8.5}$$

see [39] for the transformation and the meaning of the new parameters. Let

$$\begin{aligned} 0 < m &< \frac{1}{2}\left(1 - \frac{\beta}{a\beta+1}\right), \\ h &= \frac{1}{4}\left(\frac{\beta}{a\beta+1} - 1\right)^2 + \frac{m\beta}{a\beta+1}. \end{aligned} \tag{8.6}$$

Then $P_\star = (x_\star, y_\star)$ is an interior positive equilibrium point of systems (8.5), where

$$x_\star = -\frac{1}{2}\left(\frac{\beta}{a\beta+1} + 2m - 1\right), \qquad y_\star = \beta x_\star. \tag{8.7}$$

We fix the parameter values $(\beta, \tau, a, m, h, \delta) = (0.5, 0.7812, 0.5, 0.02, 0.098, 0.64)$. Then there is a codimension 2 Bogdanov-Takens point at $(x_\star, y_\star) = (0.28, 0.14)$ with critical normal form coefficients $(a, b) = (-0.3816222682, -1.6894735830)$. We start the continuation with (β, δ) free, `amplitude=0.02` and `TTolerance=1e-03`. The dependence of $P_\star$ on the parameter β yields a generic Bogdanov-Takens bifurcation. In Figure 8.4 we have plotted the predicted and corrected orbit in phase space and the homoclinic predictor in parameter space. In Figure 8.4b one can see the similarity with the unfolding of the normal form for the generic Bogdanov-Takens bifurcation in Figure B.1. However the phase portraits are different since the sign of the product ab is positive. This leads to unstable periodic orbits near the Bogdanov-Takens curve as seen in the simulation in Figure 8.4d.

Remark 8.2. The parameter-dependent normal form reduction in [39] is incorrect because the results from [18] were used based on the assumption that steady-state remains fixed under variation of parameters.

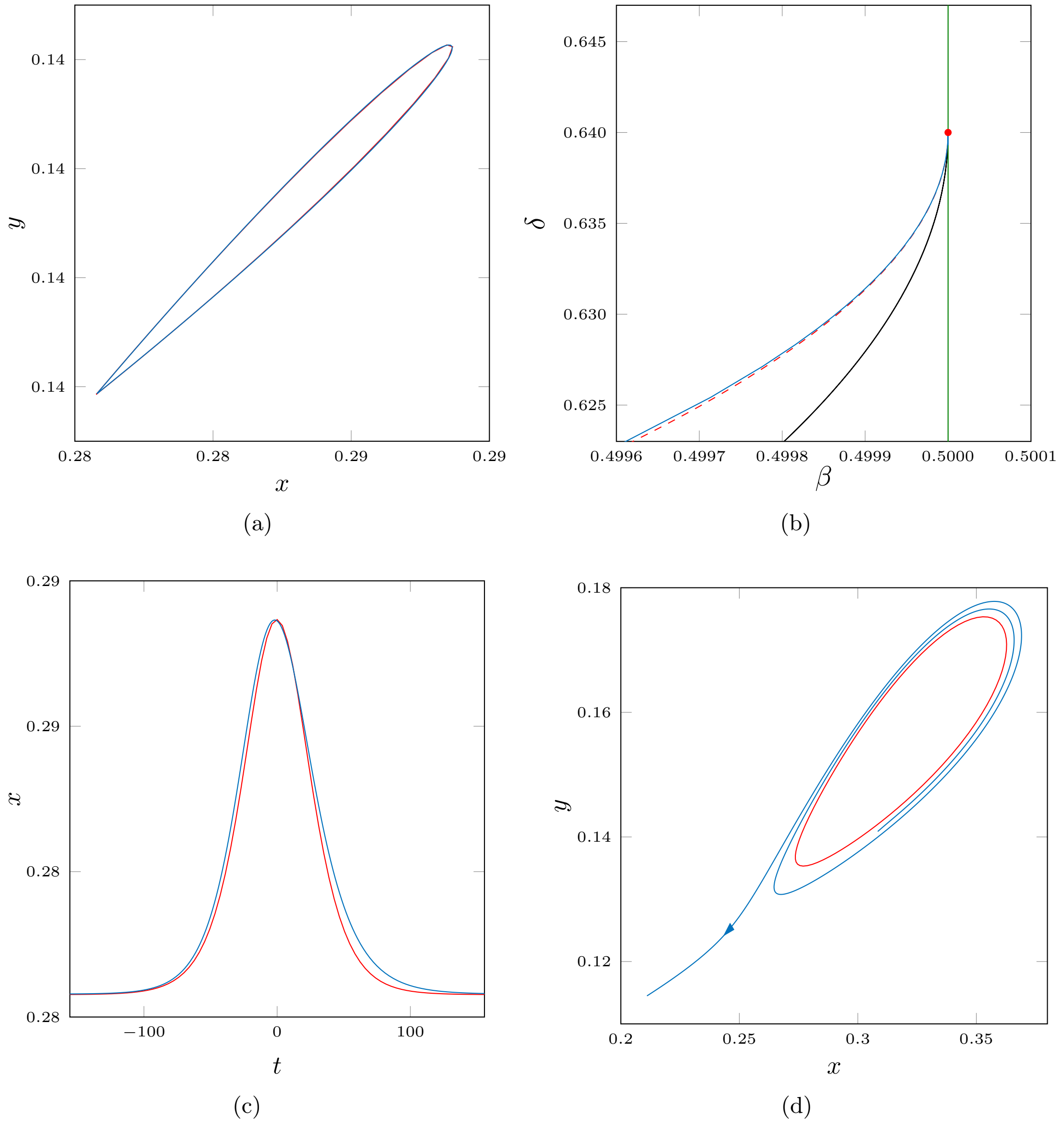


Figure 8.4.: (a) The comparison of (nearly identical) homoclinic orbits in phase space between computed (blue) and predicted (red) using the second-orbit corrector for `amplitude=0.02` and `TTolerance=1e-03`. (b) Predicted (red) and computed (blue) homoclinic bifurcation curves in parameter space. The black curve is the Hopf curve. The green curve is the fold curve. The red dot is the Bogdanov-Takens point. (c) Time plot of the homoclinic orbits from (a). In the simulation in (d) at $(\beta, \delta) \approx (0.4912, 0.5396)$ we see that the periodic orbit (red) near the Hopf curve is unstable, as predicted by the sign of the normal form coefficients ab.

8.4. A neural network

In this example we will consider the model

$$\begin{cases} \mu\dot{u}_1(t) = -u_1(t) + q_{11}\alpha(u_1(t\text{-}T)) - q_{12}u_2(t\text{-}T) + e_1, \\ \mu\dot{u}_2(t) = -u_2(t) + q_{21}\alpha(u_1(t\text{-}T)) - q_{22}u_2(t\text{-}T) + e_2, \end{cases} \tag{8.8}$$

which describes the dynamics of a neural network consisting of an excitatory and inhibitory neurons [20]. The variables and parameters occurring in (8.8) have the following neurophysiological meaning:

- $u_1, u_2 : \mathbb{R} \to \mathbb{R}$ denote the total post-synaptic potential of the excitatory and inhibitory neuron, respectively.
- $\mu > 0$ is a time constant characterizing the dynamical properties of cell membrane.
- $q_{ik} \geq 0$ represents the strength of the connection line from the kth neuron to the ith neuron.
- $\alpha : \mathbb{R} \to \mathbb{R}$ is the transfer function which describes the activity generation of the excitatory neuron as a function of its total potential u_1. The function α is smooth, increasing and has an unique turning point at $u_1 = \theta$. The transfer function corresponding to the inhibitory neuron is assumed to be the identity.
- $T \geq 0$ is a time delay reflecting synaptic delay, axonal and dendritic propagation time.
- e_1 and e_2 are external stimuli acting on the excitatory and inhibitory neuron, respectively.

We consider equation (8.8) with

$$\alpha(u_1) = \frac{1}{1+e^{-4u_1}} - \frac{1}{2}, \qquad q_{11} = 2.6, \qquad q_{21} = 1.0, \qquad q_{22} = 0.0,$$
$$\mu = 1.0, \qquad T = 1.0, \qquad e_2 = 0.0,$$

and $Q := q_{12}$, $E := e_1$ as bifurcation parameters. Substituting into (8.8) yields

$$\begin{cases} \dot{u}_1(t) = -u_1(t) + 2.6\alpha(u_1(t\text{-}T)) - Qu_2(t\text{-}T) + E, \\ \dot{u}_2(t) = -u_2(t) + \alpha(u_1(t\text{-}T)). \end{cases} \tag{8.9}$$

Notice that for any steady-state we have the symmetry $(u_1, u_2, E) \to (-u_1, -u_2, -E)$. There a two generic codimension 2 Bogdanov-Takens bifurcation in this system. One is located at $P_0 = (u_1, u_2, Q, E) \approx (-0.2617, -0.2402, 2.6000, 0.0505)$. The second follows from the symmetry. We start the continuation with (Q, E) free, `amplitude=0.04` and `TTolerance=1e-03`. The dependence of P_0 on the parameters (Q, E) yields a generic Bogdanov-Takens bifurcation, see [20]. Notice that the normal form reduction in [20] is wrong, which leads to the normal form for a transcritical Bogdanov-Takens bifurcation. In Figure 8.5 we have plotted the predicted and corrected orbit in phase space and the homoclinic predictor in parameter space.

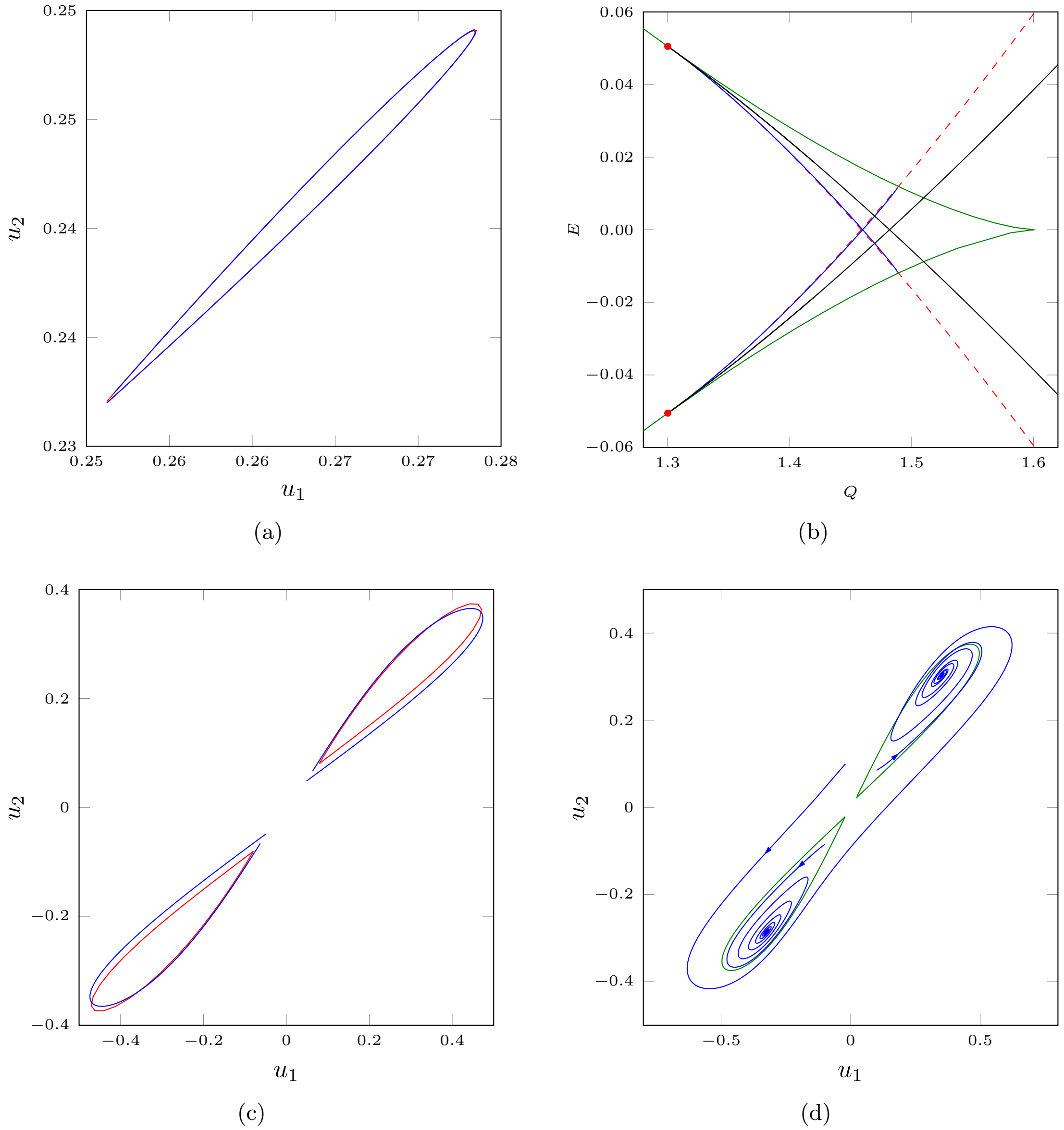


Figure 8.5.: (a) The comparison of (nearly identical) homoclinic orbits in phase space between computed (blue) and predicted (red) using the second-orbit corrector for `amplitude=0.04` and `TTolerance=1e-03`. (b) Predicted (red) and computed (blue) homoclinic bifurcation curves in parameter space. The black curves are the Hopf curves. The green curve is the fold curve. (c) The comparison of both homoclinic orbits in phase space between computed (blue) and predicted (dashed red) using the second-orbit corrector for `amplitude=0.6` and `TTolerance=1e-04`. (d) Simulation with the `Matlab` integrator `dde23` of the dynamics in phase space at the point of intersection of the homoclinic curves in Figure 8.5b.

8.5. Van der Pol oscillator with delayed feedback

We consider the Van der Pol oscillator with delay feedback [31] given by

$$\ddot{x}(t) + \varepsilon(x^2(t) - 1)\dot{x}(t) + x(t) = \varepsilon g(x(t - \tau)) \tag{8.10}$$

where $\varepsilon > 0$ is a parameter, $\tau > 0$ is a delay and $g : \mathbb{R} \to \mathbb{R}$ is a smooth function with $g(0) = 0$ and $g'(0) \neq 0$. We rewrite the Van der Pol equation (8.10) as

$$\begin{cases} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \varepsilon g(x_1(t-\tau)) - \varepsilon(x_1^2 - 1)x_2 - x_1. \end{cases} \tag{8.11}$$

Rescaling time with $t \to \dfrac{t}{\tau}$ to normalize the delay yields

$$\begin{cases} \dot{x}_1 &= \tau x_2, \\ \dot{x}_2 &= \tau \left(\varepsilon g(x_1(t-\tau)) - \varepsilon(x_1^2 - 1)x_2 - x_1\right). \end{cases} \tag{8.12}$$

As in [31], we consider (8.10) with

$$g(x) = \frac{e^x - 1}{c_1 e^x + c_2},$$

with $c_1 = \dfrac{1}{4}$ and $c_2 = \dfrac{1}{2}$. Then the trivial equilibrium undergoes a Bogdanov-Takens bifurcation at parameter values $(\tau, \varepsilon) = (0.75, 0.75)$. In Figure 8.6 we have plotted the predicted and corrected orbits in phase space and the homoclinic predictors in parameter space.

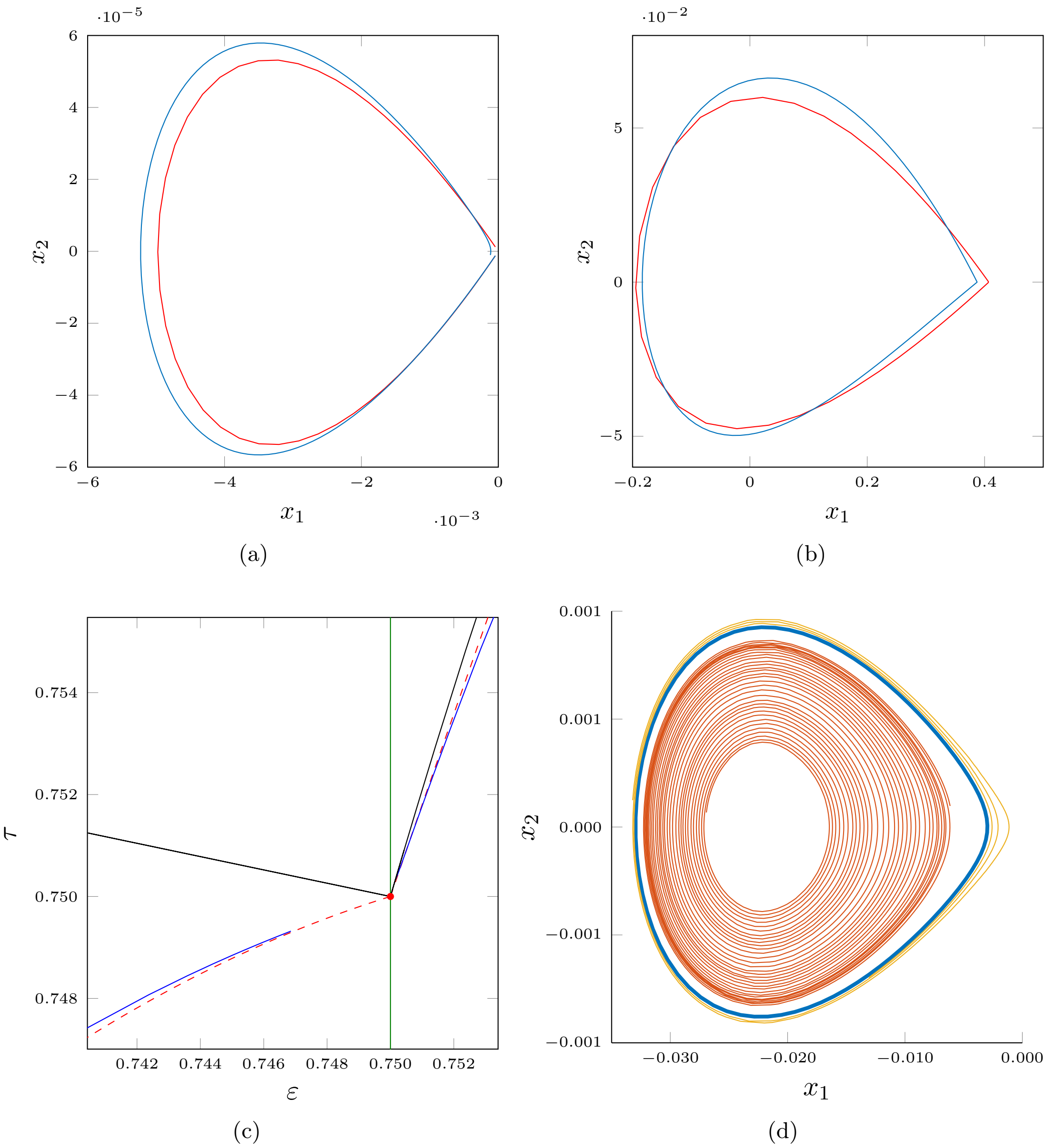


Figure 8.6.: (a) The comparison of homoclinic orbits= to the **trivial equilibrium** between computed (blue) and predicted (red) using the second-orbit corrector for `amplitude=0.05` and `TTolerance=1e-08`. (b) The comparison of homoclinic orbits in phase space to the **nontrivial equilibrium** between computed (blue) and predicted (red) using the second-orbit corrector for `amplitude=0.5` and `TTolerance=1e-08`. (c) Predicted (dashed red) and computed (blue) homoclinic bifurcation curves in parameter space. The black curves are the Hopf curves. The green curve is the fold curve. (d) Simulation with the `Matlab` integrator `dde23` of the dynamics in phase space at $(\epsilon, \tau) \approx (0.7528, 0.0.7549)$, between the Hopf and homoclinic bifurcation curves. The yellow and red curves converge to the blue cycle as predicted.

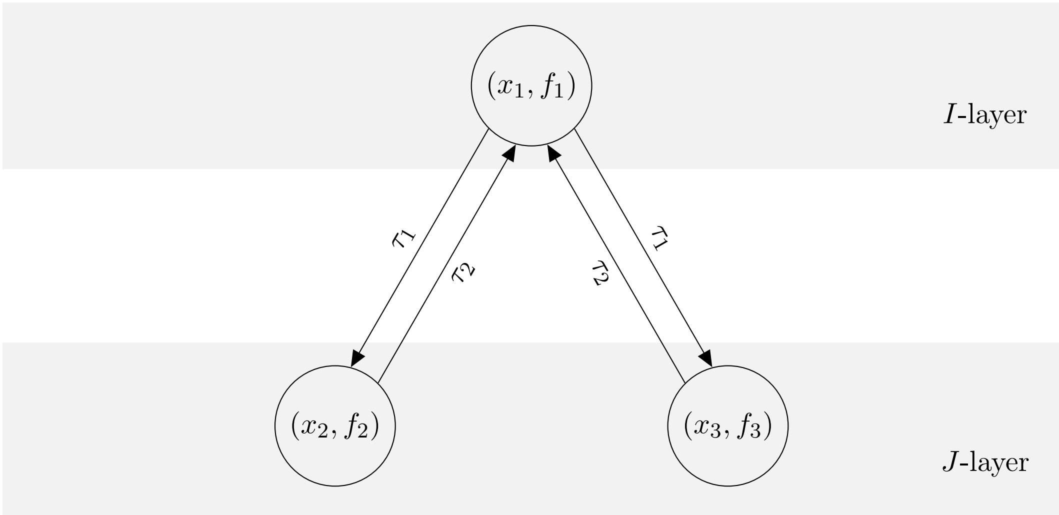


Figure 8.7.: The graph of architecture for model (8.13)

8.6. Tri-neuron BAM neural network model

We consider a three-component system of a tri-neuron BAM (bidirectional associative memory) neural network model with multiple delays [16]. The architecture of this BAM model is illustrated in Figure 8.7.

In this model, there is only one neuron with the activation function f_1 on the I-layer and there are two neurons with respective activation functions f_2 and f_3 on the J-layer. We assume that the time delay from the I-layer to the J-layer is τ_1, while the time delay from the J-layer to the I-layer is τ_2. Then the network can be described by the following DDE:

$$\begin{cases} \dot{x}_1(t) &= -\mu_1 x_1(t) + c_{21} f_1(x_2(t-\tau_2)) + c_{31} f_1(x_3(t-\tau_2)), \\ \dot{x}_2(t) &= -\mu_2 x_2(t) + c_{12} f_2(x_1(t-\tau_1)), \\ \dot{x}_3(t) &= -\mu_3 x_3(t) + c_{13} f_3(x_1(t-\tau_1)), \end{cases} \tag{8.13}$$

where:

- $x_i(t)\,(i = 1, 2, 3)$ denote the state of the neuron at time t;
- $\mu_i(i = 1, 2, 3)$ describe the attenuation rate of internal neurons processing on the I-layer and the J-layer and $\mu_i > 0$;
- the real constants $c_{i1}$and c_{1i} $(2, 3)$ denote the neurons in two layers: the I-layer and the J-layer.

Letting $u_1(t) = x_1(t - \tau_1), u_2(t) = x_2(t), u_3(t) = x_3(t)$ and $\tau = \tau_1 + \tau_2$, then system (8.13) is equivalent to the following system:

$$\begin{cases} \dot{u}_1(t) &= -\mu_1 u_1(t) + c_{21} f_1(u_2(t-\tau)) + c_{31} f_1(u_3(t-\tau)), \\ \dot{u}_2(t) &= -\mu_2 u_2(t) + c_{12} f_2(u_1(t)), \\ \dot{u}_3(t) &= -\mu_3 u_3(t) + c_{13} f_3(u_1(t)). \end{cases} \tag{8.14}$$

Lemma 8.3. *Assume that $f_i(0) = 0\,(i = 1,2,3)$, $f_i'(0) \neq 0\,(i = 1,2,3)$ and $\mu_2 \neq \mu_3$, then the steady-state $(u_1,u_2,u_3) = (0,0,0)$ has a double zero eigenvalue at*

$$c_{21} = c_{21}^0 = \frac{\mu_2^2\left(\mu_1\left(\mu_3\tau+1\right)+\mu_3\right)}{c_{12}\left(\mu_2-\mu_3\right)f_1'(0)f_2'(0)},$$
$$c_{31} = c_{31}^0 = \frac{\mu_3^2\left(\mu_1\left(\mu_2\tau+1\right)+\mu_2\right)}{c_{13}\left(\mu_3-\mu_2\right)f_1'(0)f_3'(0)}.$$

Proof. The characteristic matrix of (8.14) is given by

$$\Delta(\lambda) = \begin{pmatrix} \lambda+\mu_1 & -e^{-\lambda\tau}c_{21}f_1'(0) & -e^{-\lambda\tau}c_{31}f_1'(0) \\ -c_{12}f_2'(0) & \lambda+\mu_2 & 0 \\ -c_{13}f_3'(0) & 0 & \lambda+\mu_3 \end{pmatrix}.$$

Thus the characteristic equation becomes

$$\begin{aligned} \det\Delta(\lambda) = \lambda^3 + (\mu_1+\mu_2+\mu_3)\lambda^2 + \big(&-c_{12}c_{21}f_1'(0)f_2'(0)e^{-\lambda\tau} \\ &- c_{13}c_{31}f_1'(0)f_3'(0)e^{-\lambda\tau} + \mu_1\mu_2 + \mu_3\mu_2 + \mu_1\mu_3\big)\lambda \\ &+ \mu_1\mu_2\mu_3 - e^{-\lambda\tau}\left(c_{12}c_{21}\mu_3 f_2'(0) + c_{13}c_{31}\mu_2 f_3'(0)\right)f_1'(0) = 0. \end{aligned} \tag{8.15}$$

Clearly, $\lambda = 0$ is a root if and only if

$$\mu_1\mu_2\mu_3 = \left(c_{12}c_{21}\mu_3 f_2'(0) + c_{13}c_{31}\mu_2 f_3'(0)\right)f_1'(0).$$

Taking the derivative of (8.15) with respect to λ gives

$$\begin{aligned} \frac{d}{d\lambda}\det\Delta(\lambda) = 3\lambda^2 + 2(\mu_1+\mu_2+\mu_3)\lambda + \big(&-c_{12}c_{21}f_1'(0)f_2'(0)e^{-\lambda\tau} \\ &- c_{13}c_{31}f_1'(0)f_3'(0)e^{-\lambda\tau} + \mu_1\mu_2 + \mu_3\mu_2 + \mu_1\mu_3\big) \\ &+ \tau\left(c_{12}c_{21}f_2'(0)e^{-\lambda\tau} + c_{13}c_{31}f_3'(0)e^{-\lambda\tau}\right)f_1'(0)\lambda \\ &+ \tau e^{-\lambda\tau}\left(c_{12}c_{21}\mu_3 f_2'(0) + c_{13}c_{31}\mu_2 f_3'(0)\right)f_1'(0) = 0. \end{aligned} \tag{8.16}$$

Therefore, we have

$$\frac{d}{d\lambda}\det\Delta(0) = \big(-c_{12}c_{21}f_1'(0)f_2'(0) - c_{13}c_{31}f_1'(0)f_3'(0) + \mu_1\mu_2 + \mu_3\mu_2 + \mu_1\mu_3\big) = 0.$$

For any $\tau > 0$, it is easy to see that $\det\Delta(\lambda) = \dfrac{d}{d\lambda}\det\Delta(\lambda) = 0$ if and only if the following conditions are satisfied

$$\begin{cases} \left((1-\tau\mu_3)c_{12}c_{21}f_2'(0) + (1-\tau\mu_2)c_{13}c_{31}f_3'(0)\right)f_1'(0) = \mu_1\mu_2 + \mu_3\mu_2 + \mu_1\mu_3, \\ \\ \left(c_{12}c_{21}\mu_3 f_2'(0) + c_{13}c_{31}\mu_2 f_3'(0)\right)f_1'(0) = \mu_1\mu_2\mu_3. \end{cases} \tag{8.17}$$

By solving (8.17) for (c_{21}, c_{31}) we get $(c_{21}, c_{31}) = (c_{21}^0, c_{31}^0)$.

Taking the derivative of (8.16) yields

$$\begin{aligned}\frac{d^2}{d\lambda^2}\det\Delta(\lambda) = 6\lambda + 2\left(\mu_1+\mu_2+\mu_3\right) + \tau f_1'(0)\big(c_{12}c_{21}f_2'(0)e^{-\lambda\tau} + c_{13}c_{31}f_3'(0)e^{-\lambda\tau}\big)\\ + \tau\left(c_{12}c_{21}f_2'(0)e^{-\lambda\tau} + c_{13}c_{31}f_3'(0)e^{-\lambda\tau}\right)f_1'(0)\\ - \tau^2\left(c_{12}c_{21}f_2'(0)e^{-\lambda\tau} + c_{13}c_{31}f_3'(0)e^{-\lambda\tau}\right)f_1'(0)\lambda\\ - \tau^2 e^{-\lambda\tau}\left(c_{12}c_{21}\mu_3 f_2'(0) + c_{13}c_{31}\mu_2 f_3'(0)\right)f_1'(0) = 0.\end{aligned} \tag{8.18}$$

Then we can obtain

$$\begin{aligned}&\frac{d^2}{d\lambda^2}\det\Delta(0)|_{(c_{21},c_{31})=(c_{21}^0,c_{31}^0)}\\ &= 2\left(\mu_1+\mu_2+\mu_3\right) + 2\tau f_1'(0)\big(c_{12}c_{21}^0 f_2'(0) + c_{13}c_{31}^0 f_3'(0)\big)\\ &\quad - \tau^2 f_1'(0)\left(c_{12}c_{21}^0\mu_3 f_2'(0) + c_{13}c_{31}^0\mu_2 f_3'(0)\right)\\ &= 2\left(\mu_1+\mu_2+\mu_3\right) + \tau\left(\frac{\mu_2^2\left(\mu_1\left(\mu_3\tau+1\right)+\mu_3\right)}{\left(\mu_2-\mu_3\right)} + \frac{\mu_3^2\left(\mu_1\left(\mu_2\tau+1\right)+\mu_2\right)}{\left(\mu_3-\mu_2\right)}\right)\\ &\quad - \tau^2\left(\frac{\mu_2^2\left(\mu_1\left(\mu_3\tau+1\right)+\mu_3\right)}{\left(\mu_2-\mu_3\right)}\mu_3 + \frac{\mu_3^2\left(\mu_1\left(\mu_2\tau+1\right)+\mu_2\right)}{\left(\mu_3-\mu_2\right)}\mu_2\right)\\ &= 2\left(\mu_1+\mu_2+\mu_3\right) + 2\tau\left(\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3\right) + \tau^2\mu_1\mu_2\mu_3.\end{aligned}$$

Since $\tau > 0$ and $\mu_i > 0 (i = 1, 2, 3)$ the second derivative of the characteristic equations at $(\lambda, c_{21}, c_{31}) = (0, c_{21}^0, c_{31}^0)$ doesn't vanish and we obtain a double zero eigenvalue. □

Consider that there are eigenvalues $\lambda \neq 0$ on the imaginary axis for (c_{21}^0, c_{31}^0). Substituting $\lambda = i\omega, (\omega > 0)$ and (c_{21}^0, c_{31}^0) into (8.15), and rearranging terms according to its real and imaginary part yields

$$\begin{cases} a_2\omega^2 - a_0 = b_0\cos\tau\omega + b_1\omega\sin\tau\omega, \\ -\omega^3 + a_1\omega = b_0\sin\tau\omega - b_1\omega\cos\tau\omega, \end{cases} \tag{8.19}$$

where

$$\begin{aligned} a_2 &= \mu_1+\mu_2+\mu_3,\\ a_1 &= \mu_1\mu_2+\mu_2\mu_3+\mu_1\mu_3,\\ a_0 &= \mu_1\mu_2\mu_3\\ b_0 &= -f_1'(0)\left(c_{12}c_{21}^0\mu_3 f_2'(0) + c_{13}c_{31}^0\mu_2 f_3'(0)\right),\\ b_1 &= -f_1'(0)\left(c_{12}c_{21}^0 f_2'(0) + c_{13}c_{31}^0 f_3'(0)\right).\end{aligned}$$

By squaring and adding the above equations, it follows that

$$\left(\omega^3 - a_1\omega\right)^2 + \left(a_0 - a_2\omega^2\right)^2 = b_1^2\omega^2 + b_0^2.$$

Simplifying yields

$$\omega^6 + d_1\omega^4 + d_2\omega^2 = 0, \tag{8.20}$$

where

$$d_1 = \mu_1^2 + \mu_2^2 + \mu_3^2$$
$$d_2 = \mu_1^2\mu_2^2 + \mu_3^2\mu_2^2 + \mu_1^2\mu_3^2 - (\tau\mu_1\mu_2\mu_3 + \mu_1\mu_2 + \mu_3\mu_2 + \mu_1\mu_3)^2 < 0.$$

Substituting $z = \omega^2$ into (8.20) and dividing the resulting equation by $z \neq 0$ yields

$$z^2 + d_1 z + d_2 = 0.$$

Solving for positive z gives

$$z_0 = \frac{1}{2}\left(-d_1 + \sqrt{d_1^2 - 4d_2}\right).$$

By letting $\omega_0 = \sqrt{z_0}$, we obtain from (8.19)

$$\cos\tau\omega_0 = \frac{a_2b_0\omega_0^2 - a_1b_1\omega_0^2 - a_0b_0 + b_1\omega_0^4}{b_1^2\omega_0^2 + b_0^2}.$$

Lemma 8.4. [16, Lemma 3] *Let* $(c_{21}, c_{31}) = (c_{21}^0, c_{31}^0)$ *and* $0 < \tau < \tau_0$*, where*

$$\tau_0 = \frac{1}{\omega_0}\arccos\frac{a_2b_0\omega_0^2 - a_1b_1\omega_0^2 - a_0b_0 + b_1\omega_0^4}{b_1^2\omega_0^2 + b_0^2},$$

them all roots of the characteristic equation (8.15), except the double zero roots, have negative real parts.

Remark 8.5. Note that the obtained expressions for (c_{21}^0, c_{31}^0) are different from the expressions obtained in [16]. All though numerical simulation was used, the error there was not discovered. Here we see that the predictor is of great value, which gives an extra verification of the analysis.

Remark 8.6. We rederived the expression for τ_0 since the derivation in [16] contained an error.

As in the simulations in [16, Example 1] we consider (8.14) with the activation functions

$$f_1(x) = \tanh(x) + 0.1x^2, \quad f_2(x) = f_3(x) = \tanh(x),$$

and parameters

$$\mu_1 = 0.1, \mu_2 = 0.3, \mu_3 = 0.2, c_{12} = c_{13} = 1, \tau = 5.$$

Then from Lemma 8.3 we obtain two critical values

$$(c_{21}^0, c_{31}^0) = (0.36, -0.22).$$

Furthermore, since $\tau < \tau_0 = 5.46336$ the center manifold is attractive. We write the system (8.14) as

$$\begin{cases} \dot{u}_1(t) &= -\mu_1 u_1(t) + \left(c_{21}^0 + \alpha_1\right) f_1(u_2(t-\tau)) + \left(c_{31}^0 + \alpha_2\right) f_1(u_3(t-\tau)), \\ \dot{u}_2(t) &= -\mu_2 u_2(t) + c_{12} f_2(u_1(t)), \\ \dot{u}_3(t) &= -\mu_3 u_3(t) + c_{13} f_3(u_1(t)), \end{cases} \tag{8.21}$$

where (α_1, α_2) are the new parameter values such that at $(\alpha_1, \alpha_2) = (0, 0)$ we have a Bogdanov-Takens bifurcation. The critical normal form coefficients

$$(a, b) \approx (0.0012060198, -0.0135096097),$$

indicate stable cycles. In Figure 8.8 we have plotted the predicted and corrected orbits in phase space and the homoclinic predictors in parameter space. Note that in Figure 8.8b the continued homoclinic orbit, corresponding to the curve in the upper half plane, only exists for a very short time. Without the predictor this homoclinic orbit would be difficult to find. In Figure 8.9 simulations are made to confirm the dynamics.

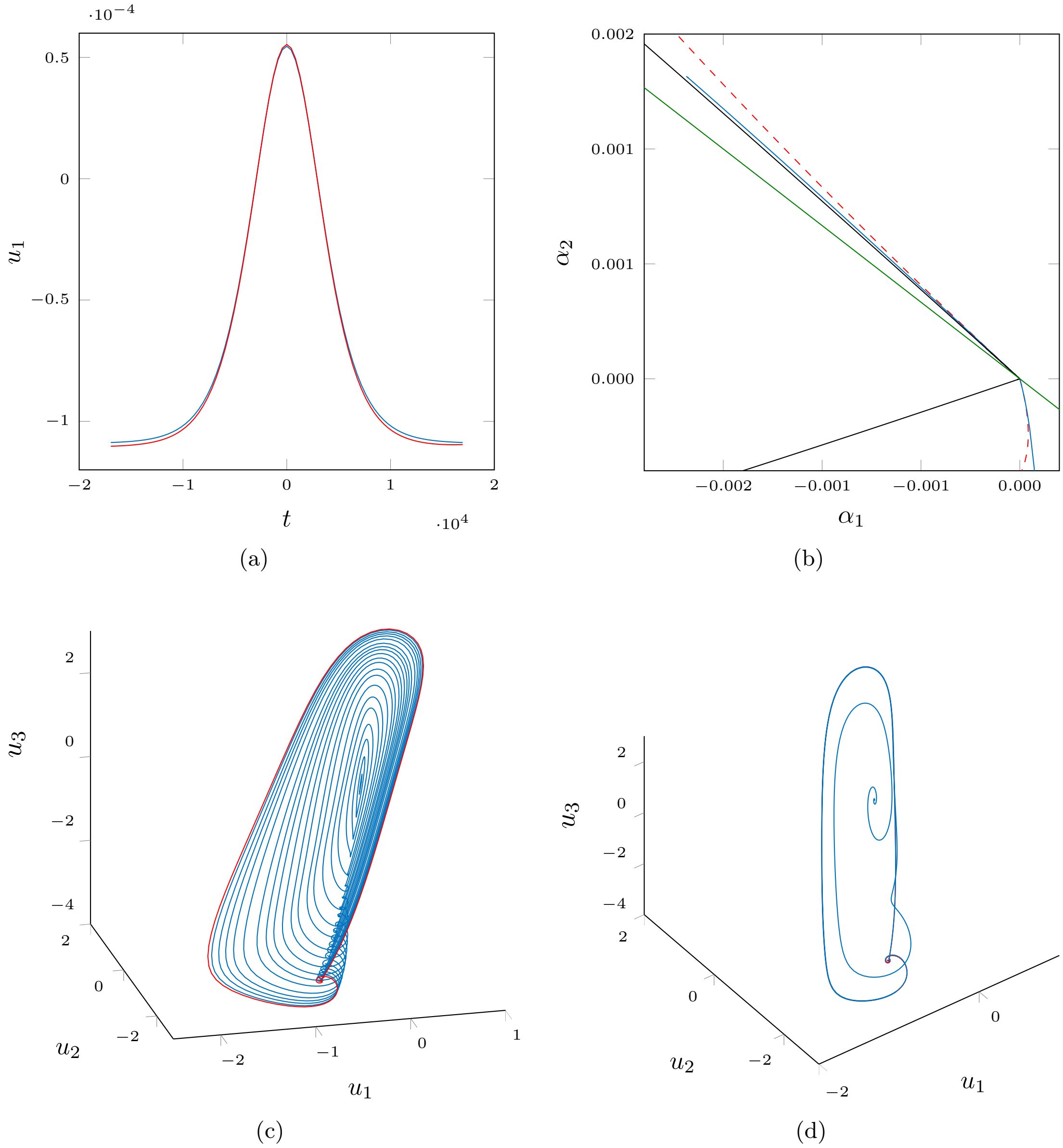


Figure 8.8.: (a) The comparison of homoclinic orbits in phase space to the **nontrivial equilibrium** between computed (blue) and predicted (red) using the second-orbit corrector for `amplitude=1e-3` and `TTolerance=1e-9`. (b) Predicted (red) and computed (blue) homoclinic bifurcation curves in parameter space. The black curves are the Hopf curves. The green curve is the fold curve. (c) Continued homoclinic orbit to the nontrivial equilibrium in phase space. (d) Simulation of orbit starting near the origin and converging to the homoclinic orbit for parameter values $(\alpha_1, \alpha_2) \approx (-0.1701, -0.1598)$.

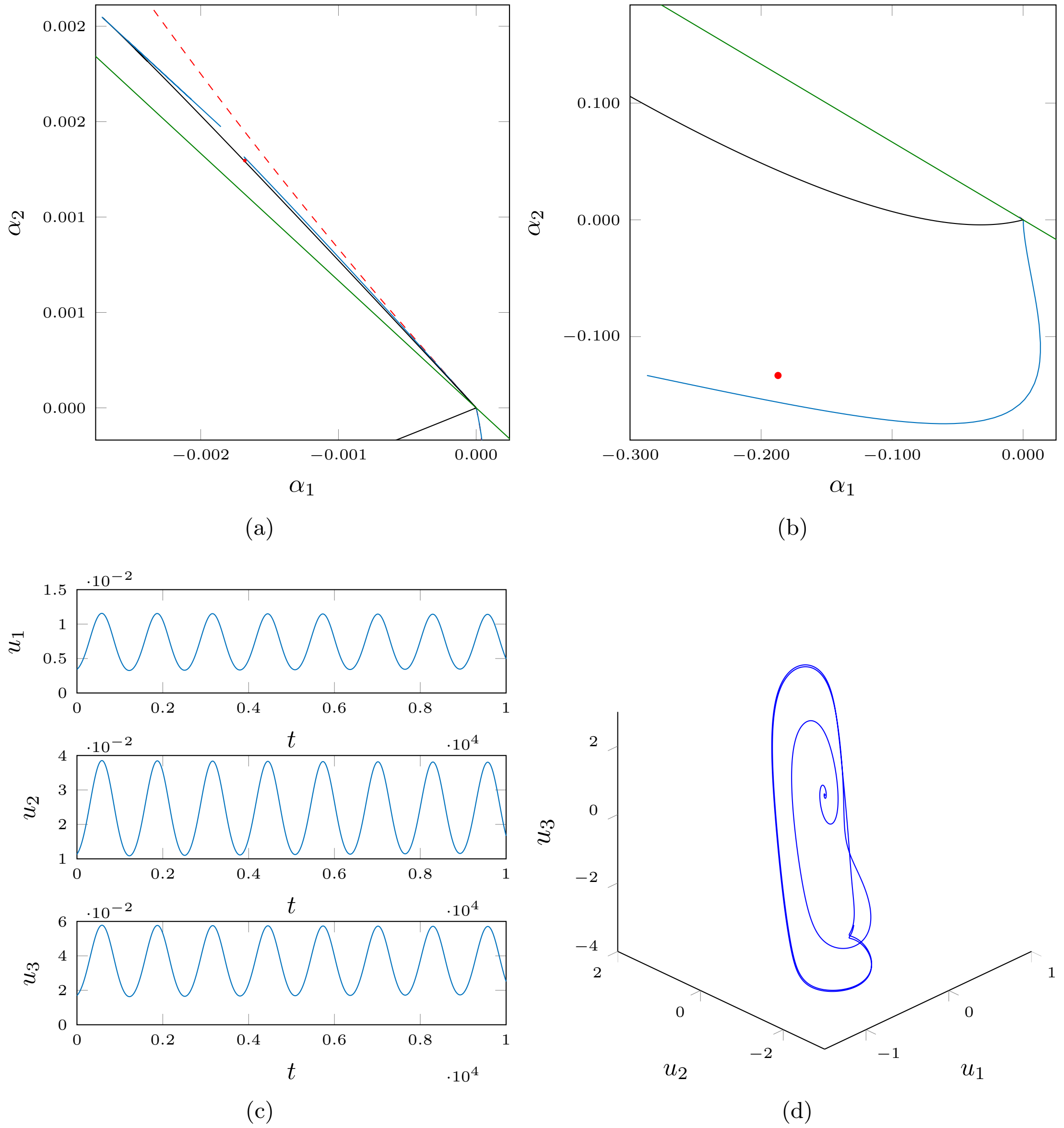


Figure 8.9.: (a) The Hopf curve in the upper half plane connects the Bogdanov-Takens point at the origin to a generic Bogdanov-Takens point. The red dot corresponds to the parameter value of the simulation in (c). (b) Plot of the homoclinic curve in the lower half plane. The red dot corresponds to the simulation in (d). (c) Simulation of orbit started at $(u_0, u_1, u_2) = (0.003471872461226, 0.011366542840623, 0.016898360549174)$ shows a periodic orbit at parameter value $(\alpha_1, \alpha_2) \approx (-0.00168, 0.001295)$ between the Hopf and homoclinic bifurcation curves in the upper half plane. (d) Simulation of orbit starting near the origin, $(u_0, u_1, u_2) = (0.0001, 0, 0)$ for parameter value $(\alpha_1, \alpha_2) \approx (-0.1871 - 0.1333)$, converging to a periodic orbit.

8.7. Neimark-Sacker curves emanating from a Hopf-Hopf point

In [28] the state-dependent DDE

$$\dot{u}(t) = -\gamma u(t) - \kappa_1 u(t - a_1 - cu(t)) - \kappa_2 u(t - a_2 - cu(t)) \tag{8.22}$$

is considered. For the parameter values

$$\kappa_1 = 5.95, \qquad \kappa_2 = 2.3, \qquad c = 1, \qquad \gamma = 4.75, \qquad a_1 = 1.3, \qquad a_2 = 6$$

a stable torus is present near a Hopf-Hopf bifurcation. We will keep the parameters c, γ, a_1 and a_2 fixed for the rest of this example. Since there are no normal form coefficients for state-dependent DDEs available, the state-dependent DDE must first be approximated with constant delays. Expanding the delays about their steady-state $u = 0$ values yields

$$u(t - a_i - cu(t)) = u(t - a_i) + \dot{u}(t - a_i)(-cu(t)) + \frac{1}{2}\ddot{u}(t - a_i)(-cu(t))^2 + \dots.$$

Then, using the state-dependent DDE (8.22) we remove the $\dot{u}, \ddot{u}$ terms etc.

$$\begin{aligned}\dot{u}(t) &= -\gamma u(t) - \kappa_1 u(t - a_1 - cu(t)) - \kappa_2 u(t - a_2 - cu(t)) \\ &= -\gamma u(t) - \kappa_1 u(t - a_1) - \kappa_2 u(t - a_2) + h.o.t\end{aligned}$$

Therefore

$$\dot{u}(t - a_i) = \gamma u(t - a_i) - \kappa_1 u(t - a_1 - a_i) - \kappa_2 u(t - a_2 - a_i) + h.o.t.$$

Expanding up to order three yields

$$\begin{aligned}\dot{u}(t) = &-\gamma u(t) - \kappa_1 u(t - a_1) - \kappa_2 u(t - a_2) \\ &- cu(t)\sum_{i=1}^{2}\kappa_i\left[\gamma u(t - a_i) + \sum_{j=1}^{2}\kappa_j u(t - a_i - a_j)\right] \\ &- \sum_{i,j=1}^{2}\kappa_i\kappa_j cu(t)u(t - a_i)\left[\gamma u(t - a_i - a_j) + \sum_{m=1}^{2}\kappa_m u(t - a_i - a_j - a_m)\right. \\ &- \frac{1}{2}(c(u(t))^2\sum_{i=1}^{2}\kappa_i\left[\gamma^2 u(t - a_i) + 2\gamma\sum_{j=1}^{2}\kappa_j u(t - a_i - a_j)\right. \\ &\quad \left. + \sum_{j,m=1}^{2}\kappa_j\kappa_m u(t - a_i - a_j - a_m)\right] + h.o.t.\end{aligned}$$

Truncating the higher order terms gives the DDE

$$\begin{aligned}
\dot{u}(t) =& -\gamma u - \kappa_1 u(t-\tau_1) - \kappa_2 u(t-\tau_2) \\
& - \kappa_1 c u(\gamma u(t-\tau_1) + \kappa_1 u(t-\tau_3) + \kappa_2 u(t-\tau_4)) \\
& - \kappa_2 c u(\gamma u(t-\tau_2) + \kappa_1 u(t-\tau_4) + \kappa_2 u(t-\tau_5)) \\
& - \kappa_1 \kappa_1 c^2 u(t) u(t-\tau_1)(\gamma u(t-\tau_3) + \kappa_1 u(t-\tau_6) + \kappa_2 u(t-\tau_7)) \\
& - \kappa_1 \kappa_2 c^2 u u(t-\tau_1)(\gamma u(t-\tau_4) + \kappa_1 u(t-\tau_7) + \kappa_2 u(t-\tau_8)) \\
& - \kappa_2 \kappa_1 c^2 u(t) u(t-\tau_2)(\gamma u(t-\tau_4) + \kappa_1 u(t-\tau_7) + \kappa_2 u(t-\tau_8)) \\
& - \kappa_2 \kappa_2 c^2 u u(t-\tau_2)(\gamma u(t-\tau_5) + \kappa_1 u(t-\tau_8) + \kappa_2 u(t-\tau_9)) \\
& - \frac{1}{2}(c u(t))^2 \kappa_1 (\gamma^2 u(t-\tau_1) + 2\gamma(\kappa_1 u(t-\tau_3) + \kappa_2 u(t-\tau_4)) \\
& \quad + \kappa_1 \kappa_1 u(t-\tau_6) + 2\kappa_1 \kappa_2 u(t-\tau_7) + \kappa_2 \kappa_2 u(t-\tau_8)) \\
& - \frac{1}{2}(c u(t))^2 \kappa_2 (\gamma^2 u(t-\tau_2) + 2\gamma(\kappa_1 u(t-\tau_4) + \kappa_2 u(t-\tau_5)) \\
& \quad + \kappa_1 \kappa_1 u(t-\tau_7) + 2\kappa_1 \kappa_2 u(t-\tau_8) + \kappa_2 \kappa_2 u(t-\tau_9))
\end{aligned}$$

with 9 constant delays, where

$$\begin{aligned}
\tau_1 &= a_1, \\
\tau_2 &= a_2, \\
\tau_3 &= 2a_1, \\
\tau_4 &= a_1 + a_2, \\
\tau_5 &= 2a_2, \\
\tau_6 &= 3a_1, \\
\tau_7 &= 2a_1 + a_2, \\
\tau_8 &= a_1 + 2a_2, \\
\tau_9 &= 3a_2.
\end{aligned}$$

Using `DDE-BifTool` we find a Hopf-Hopf point at parameter values

$$(\kappa_1, \kappa_2) \approx (2.0809, 3.7868)$$

with critical normal form coefficients

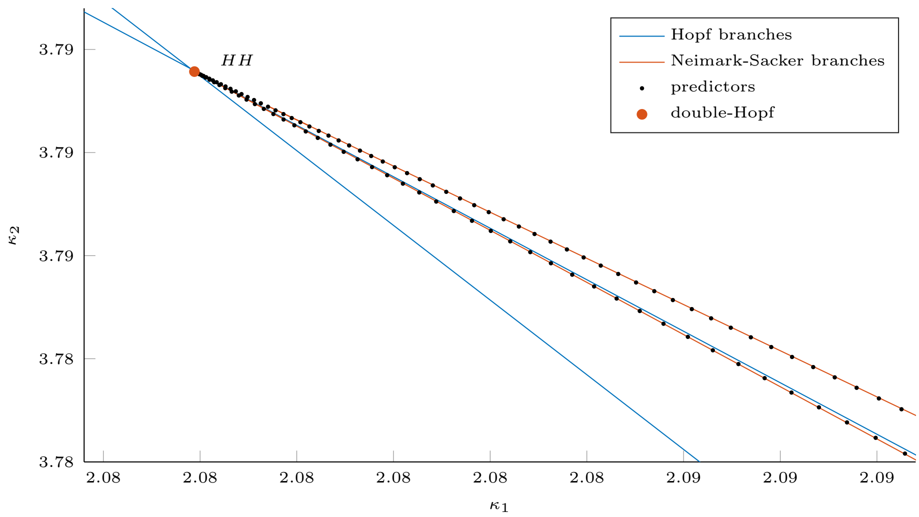
$$\begin{aligned}
g_{2100} =& -0.027604441169896 - 0.017897738883975i, \\
g_{1011} =& -0.020673889394881 - 0.015338623328501i, \\
g_{1110} =& 0.000615291160368 - 0.032074905848412i, \\
g_{0021} =& -0.003907332082037 - 0.011857396811775i, \\
\theta =& 5.291049995449255, \\
\delta =& -0.022289571325884.
\end{aligned}$$

We conclude that we are in the 'simple' case III, see [34]. Thus, a stable torus is indeed predicted. Furthermore, two Neimark-Sacker bifurcations curves should emanate from the Hopf-Hopf point. Using the predictors from Section C.6, we initialize and continue these curves, see Figure 8.10.

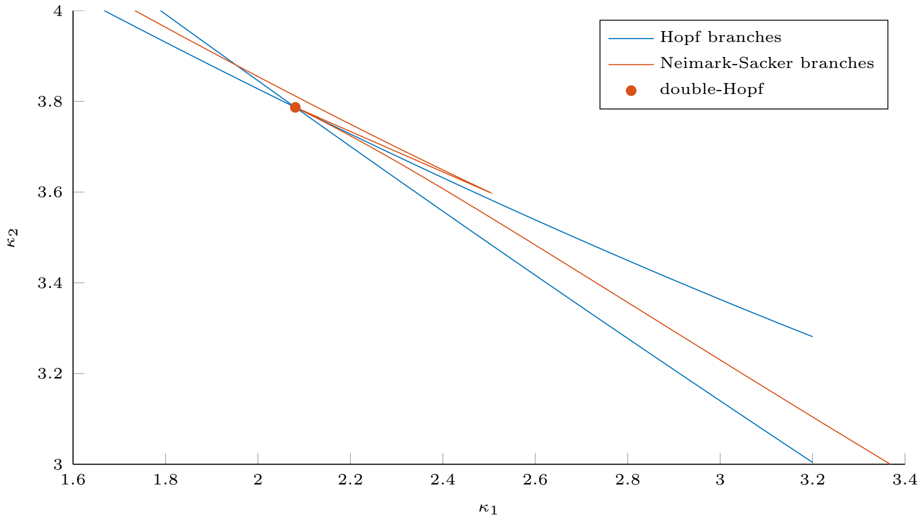
To simulate the torus we fixed the parameters

$$\kappa_1 = 2.757858545579159, \qquad \kappa_2 = 3.383471633934356.$$

and take the history function $u(t) = 0.036964714041287$, see Figure 8.10.



(a)



(b)

Figure 8.10.: Bifurcation diagram near the Hopf-Hopf point. The two Neimark-Sacker curves emanate from the Hopf-Hopf point. In (a) the predicted parameter values overlap the continued Neimark-Sacker curves. In (b) a larger part of the bifurcation diagram in shown.

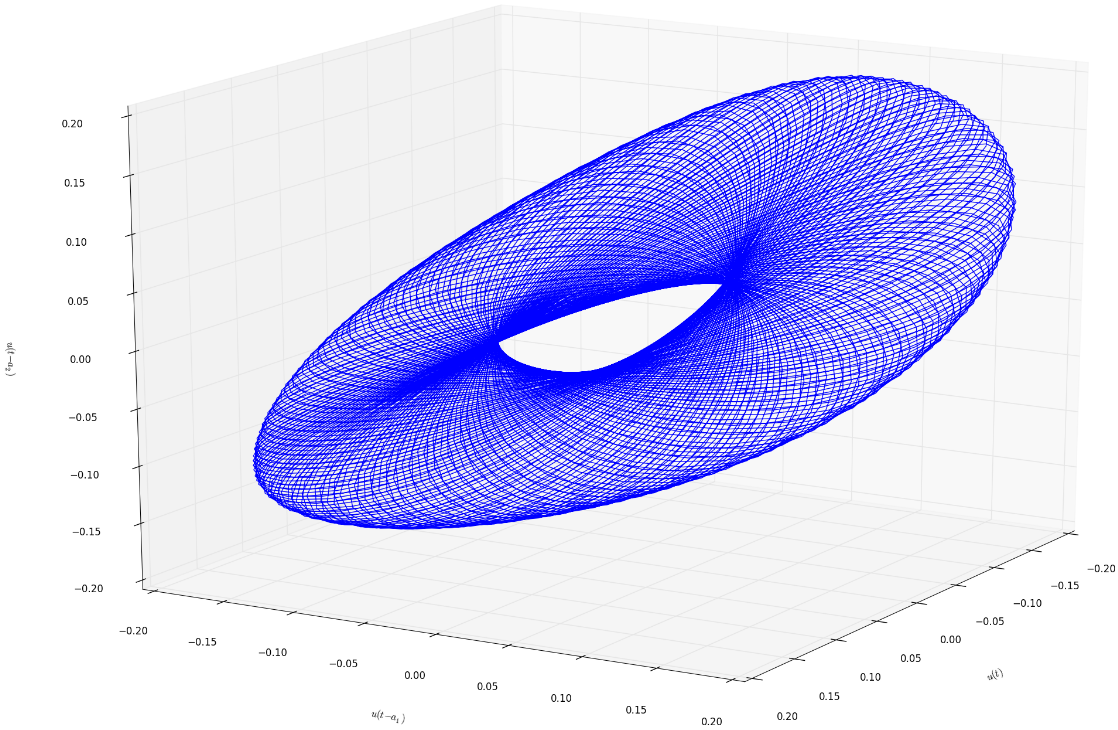


Figure 8.11.: A stable torus near a Neimark-Sacker bifurcation curve at $(\kappa_1, \kappa_2) = 2.757858545579159, 3.3834716339343560$. For the simulation 3000 time steps were used, with only the last 400 time steps plotted. Furthermore, the torus remains stable after increasing the time steps to 300.000.

8.8. Van der Pol's oscillator with delayed position and velocity feedback

In [6] a generalization of the Van der Pol's oscillator treated in Section 8.5 is considered. In the Van der Pol's oscillator (8.10) the nonlinear forcing g only depends on the delayed position. In [30] it is shown that under certain conditions a zero-Hopf bifurcation is present. However, only two of the four possible bifurcation diagrams of the zero-Hopf bifurcation were possible. By letting the forcing g also depend on the delayed velocity $\dot{x}(t-\tau)$ a full realization of all four generic unfoldings can be obtained. Therefore, the generalization of Van der Pol's oscillator with delayed feedback

$$\ddot{x}(t) + \varepsilon(x^2(t) - 1)\dot{x}(t) + x(t) = g(\dot{x}(t-\tau), x(t-\tau)), \tag{8.23}$$

is considered in [6], where $g \in C^3, g(0,0) = 0, g_{\dot{x}}(0,0) = a$ and $g_x(0,0) = b$. One immediately sees that the trivial equilibrium $(\dot{x}, x) = (0,0)$ is an equilibrium for all parameter values $(\mu_1, \mu_2) = (b-1, \tau - \tau_0)$. Then the normal form (B.6) cannot be used here. Instead the normal form for the Transcritical-Hopf bifurcation must be used, see for example [23]. Since we are in the first place interested in verifying the predictors for the generic fold-Hopf bifurcation we artificially modify the DDE (8.23) to

$$\ddot{x}(t) + \varepsilon(x^2(t) - 1)\dot{x}(t) + x(t) = 0.05\mu_1 + g(\dot{x}(t-\tau), x(t-\tau)), \tag{8.24}$$

i.e. we remove the fixed equilibrium at the origin.

Linearization of equation (8.23) around the trivial solution $x = 0$ gives

$$\ddot{x}(t) - \varepsilon\dot{x}(t) + x(t) = a\dot{x}(t-\tau) + bx(t-\tau).$$

From which we obtain the characteristic equation

$$\Delta(\lambda, \tau) = \lambda^2 - \epsilon\lambda + 1 - (a\lambda + b)e^{-\lambda\tau} = 0.$$

Let

$$b = 1, \qquad \tau = \tau_0 \neq \epsilon + a, \qquad \epsilon^2 - a^2 < 2, \tag{8.25}$$

then the characteristic equation has a simple zero and a pair of purely imaginary roots $\lambda = \pm i\omega_0$. Here ω_0 and τ_0 are defined by

$$\omega_0 = \sqrt{2 - \epsilon^2 + a^2}, \qquad \tau_0 = \frac{1}{\omega_0}\arccos\left(\frac{1 - (1 + \epsilon a)\omega_0^2}{a^2\omega_0^2 + 1}\right),$$

see [6, Proposition 2.1]. We set the function g to

$$g(x_2, x_1) = (1 + \mu_1)x_1 - 0.2x_2 - 0.2x_1^2 - 0.2x_1x_2 - 0.2x_2^2 + 0.5x_1^3 \tag{8.26}$$

and $\epsilon = 0.3$. Then the conditions (8.25) are satisfied and

$$\omega_0 \approx 1.396424004376894, \qquad \tau_0 \approx 1.757290761249588.$$

This leads to the critical normal form coefficients

$$
\begin{aligned}
&e - 0.147637371155013,\\
&s = 1,\\
&\theta = -0.630334154175547.
\end{aligned}
$$

We conclude that we are in unfolding case III. Furthermore, since the DDE (8.24) with g as in (8.26) does not contain terms of order four or higher the dynamics should be described by the truncated normal form (B.9). We thus, in particular expect a stable limit cycle, a stable torus, and a spherelike surface to be present for nearby parameter values (μ_1, μ_2). In Figure 8.12 the bifurcation diagram of the fold-Hopf point is shown. We see the resemblance with the theoretical unfolding in Figure B.3 by a reflection in the μ_1 axis. In Figures 8.13 and 8.14 we simulated the dynamics below and above the Neimark-Sacker curve. The obtained plot correspond with the predictions made.

Hopf-transcritical bifurcation Setting $\mu_1 = 0$ in (8.24) yields a Hopf-transcritical at the origin. Since the critical normal form coefficients remain the same, there will be two Neimark-Sacker bifurcation curves. Furthermore, the fold and Hopf curve will meet transversely. This is illustrated in the bifurcation diagram in Figure 8.15.

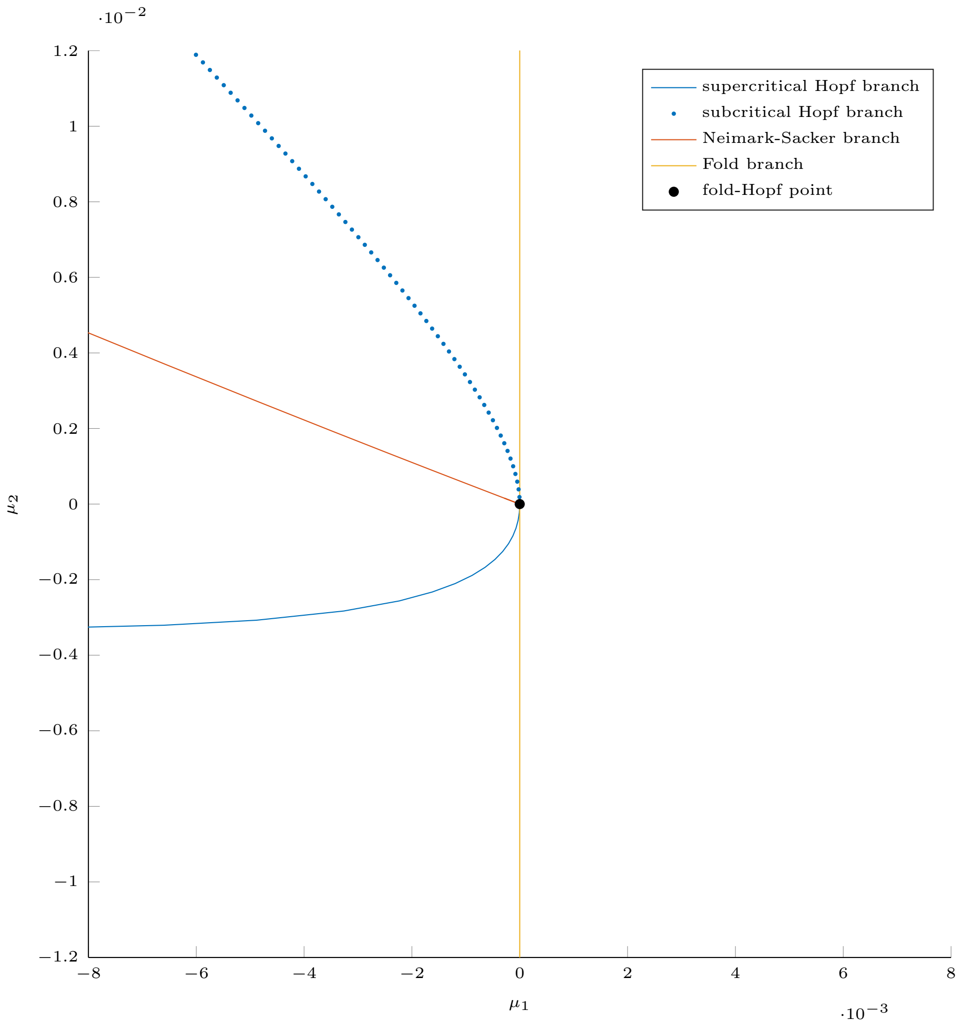


Figure 8.12.: Unfolding of the fold-Hopf point in (8.24) with `DDE-BifTool`. The Neimark-Sacker curve was initiated using the predictor described in Section C.4.

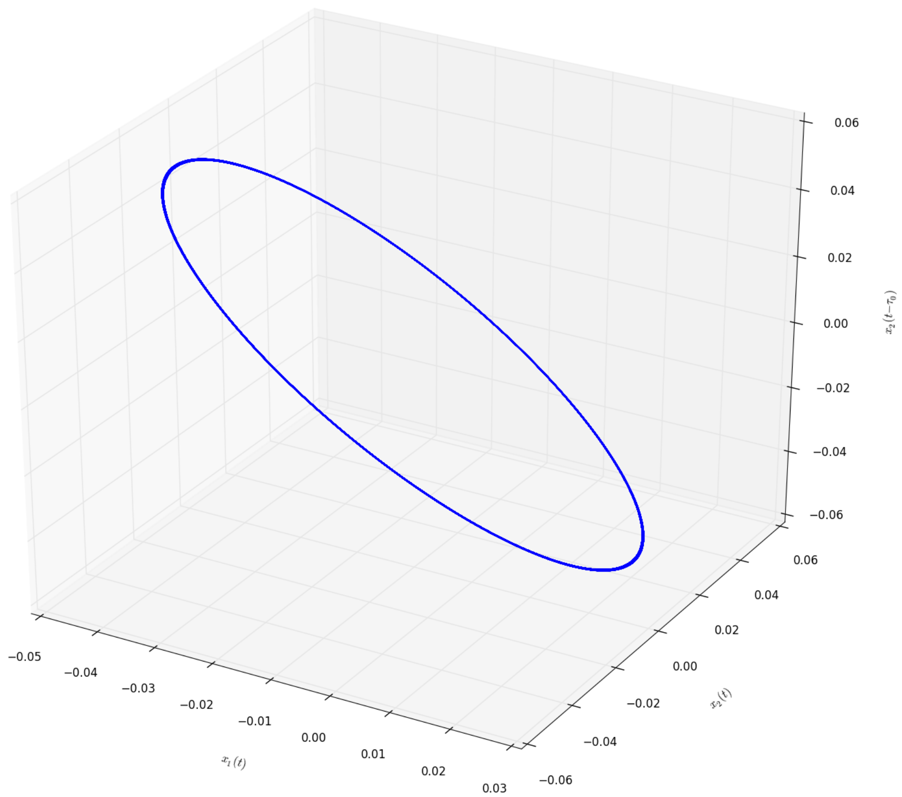
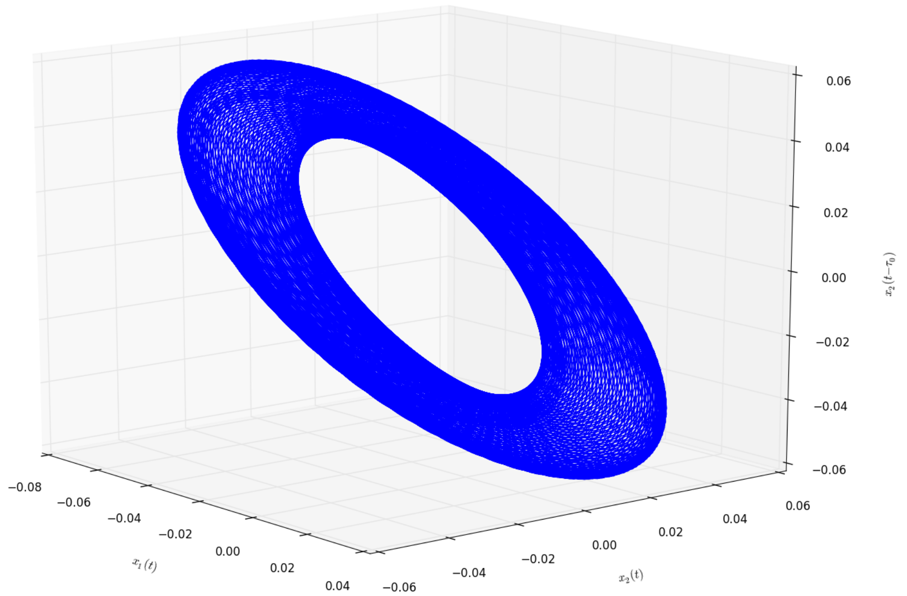
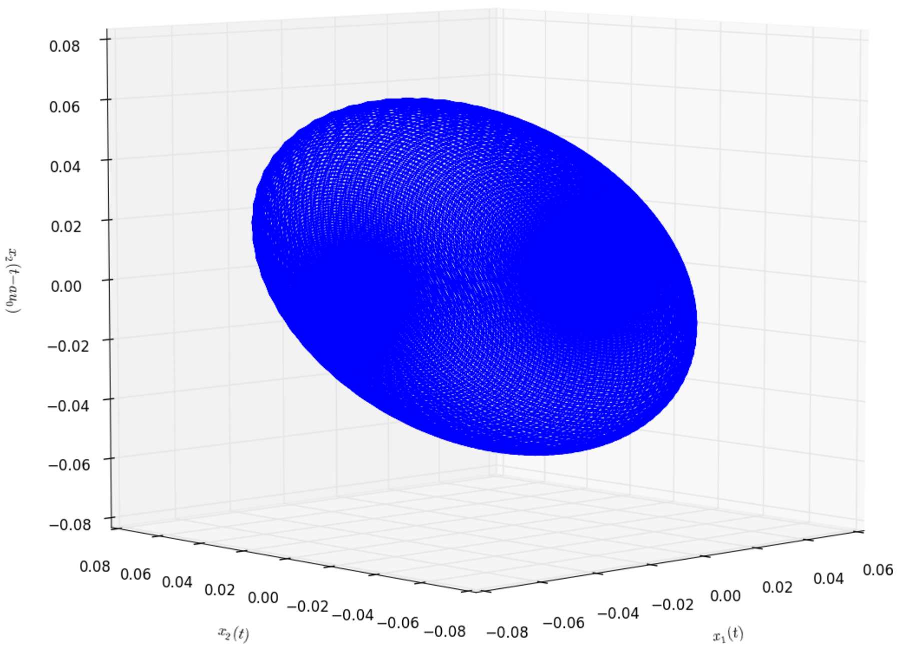


Figure 8.13.: Simulation over 12.000 time steps at parameter values $(\mu_1, \mu_2) = (-0.006871405962603, 0.003871232826592 - 0.0008)$ just below the Neimark-Sacker curve in the unfolding of the fold-Hopf point, see Figure 8.12. As predicted a stable cycle is present.



(a) Simulation over 300.000 time steps at parameter values $(\mu_1, \mu_2) = (-0.006871405962603, 0.003871232826592 + 0.00001)$ just above the Neimark-Sacker curve in the unfolding of the fold-Hopf point, see Figure 8.12 As predicted a stable torus is present.



(b) Simulation over 300.000 time steps at parameter values $(\mu_1, \mu_2) = (-0.006871405962603, 0.003871232826592 + 0.0000792541)$, where the torus becomes a spherelike surface.

Figure 8.14.: Simulation near the fold-Hopf point at the origin in (8.24).

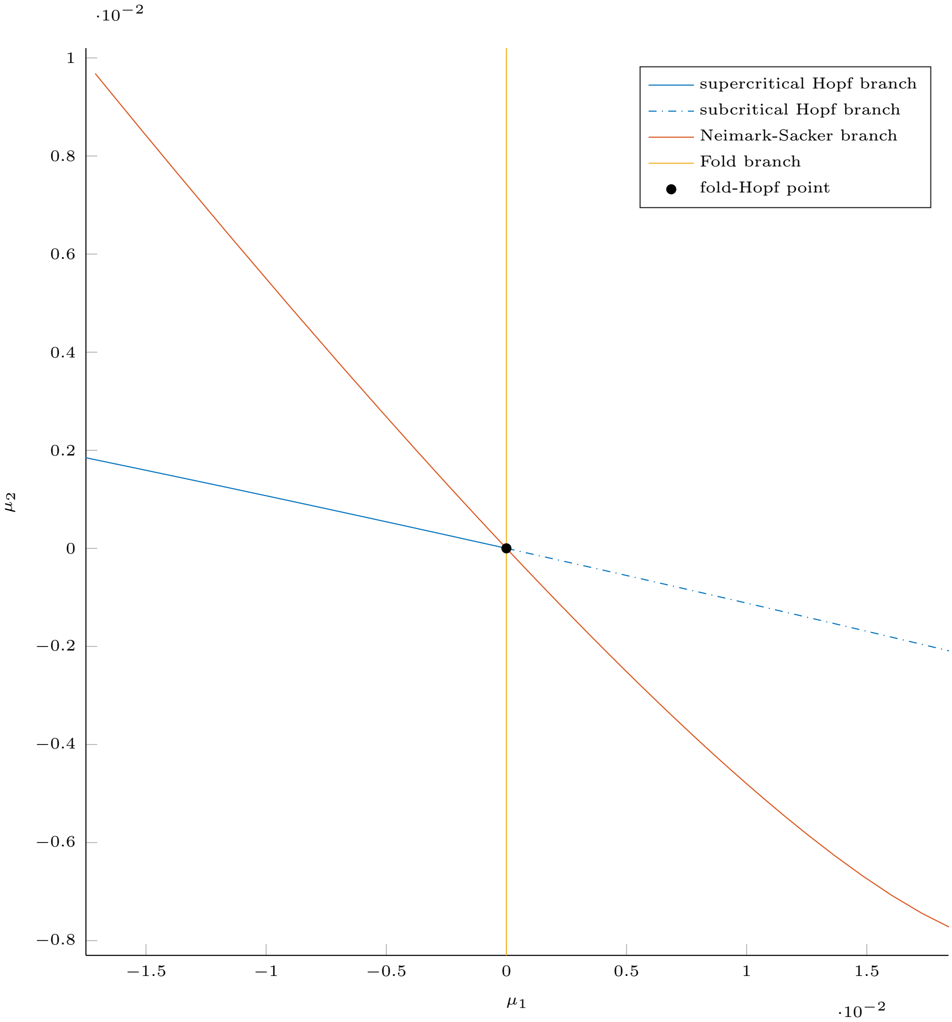


Figure 8.15.: Bifurcation diagram near the Hopf-transcritical point in (8.24) with `DDE-BifTool`. The Neimark-Sacker curves were initiated using the predictor described in Section C.5.

9. Final remarks

Unfortunately, there is a point at which one needs to stop working on a Master thesis. Even though there are so many things to further investigate. For example, the defining system derived in Chapter 5 could be used to continue Bogdanov-Takens points in three parameters and detect triple zero bifurcations, for which normal form coefficients can be computed. In Section 8.7 the state-dependent DDE is approximated with a DDE with discrete delays, since there is no normal forms for state-dependent DDEs available yet. Much research is possible here. Another subject not been touched on yet is the computation of normal form for cycles in DDE. With the framework of sun-star calculus and the method for deriving coefficients on the (parameterized) center manifold available, the next logical step to take is to 'lift' the normal forms for cycles, as done in [10] in the finite dimensional case, to the infinite dimensional setting. Another direction in which one may go is to optimize the algorithms used to continue cycles and their bifurcations, a subject not being touched upon in this thesis at all. While testing the models used for the examples in this thesis it could take a lot of time to continue homoclinic orbits and cycles. Lastly, one major disadvantage of `DDE-BifTool` is the lacking of a graphical user interface (GUI). This may discourage students and researchers from using it. Therefore, I think it is essential for adding this to the software.

A. Center manifold reduction with normalization for BT bifurcation in ODE

In this Appendix we derive the coefficients needed for the homoclinic predictor(s) emanating the generic and transcritical Bogdanov-Takens in an ODE

$$\dot{x} = f(x, \alpha), \tag{A.1}$$

where $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^m$ with $n \geq 2$, $m = 2$, and f as smooth as necessary, has an equilibrium at $(x_0, \alpha_0) = (0, 0)$. The Taylor expansion of (A.1) at the equilibrium is given by

$$\begin{aligned} f(x, \alpha) = Ax + \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + A_1(x, \alpha) + J_1(\alpha) + J_2(\alpha, \alpha) \\ + \frac{1}{2}B_2(x, x, \alpha) + \mathcal{O}\left(\|x\|^4 + \|x\| \, \|\alpha\|^2 + \|\alpha\|^3\right), \end{aligned} \tag{A.2}$$

where $A = f_x(x_0, \alpha_0)$, $J_1 = f_\alpha(x_0, \alpha_0)$, and B, J_2, C, A_1 and B_2 are the standard multilinear forms. Suppose furthermore that at the equilibrium (x_0, α_0) the Jacobian matrix A has a double (but not semi-simple) zero eigenvalue. Then, there exist two real linearly independent (generalized) eigenvectors, $q_{0,1} \in \mathbb{R}^n$, of A, such that

$$Aq_0 = 0, \qquad Aq_1 = q_0,$$

and two adjoint (generalized) eigenvectors $p_{0,1} \in \mathbb{R}^n$, of A, such that

$$p_1^T A = 0, \qquad p_0^T A = p_1.$$

Using the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ these vectors can be normalized to satisfy

$$p_i^T q_j = \delta_{ij}, \qquad i = 0, 1, \; j = 0, 1.$$

As in [35], we impose the condition

$$q_0^T q_0 = 1, \qquad q_1^T q_0 = 0,$$

to uniquely define the vectors $\{q_0, q_1, p_1, p_0\}$ up to a $\pm$ sign.

A.1. Generic Bogdanov-Takens bifurcation

We want to relate the system (A.1) near (x_0, α_0) to the smooth normal form

$$\begin{cases} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= \beta_1 + \beta_2 w_1 + (a + a_1\beta_2)\, w_0^2 \\ & \quad + (b + b_1\beta_2)\, w_0 w_1 + d w_0^3 + e w_0^2 w_1, \end{cases} \tag{A.3}$$

corresponding to the bifurcation on its center manifold following [1, 32]. In order to relate both systems to each other, we need a parameterization H of the center manifold in terms of the original variables x and a transformation K of the bifurcation branch to the original parameters α,

$$\begin{aligned} x &= H(w,\beta), \qquad H : \mathbb{R}^{n_c+2} \to \mathbb{R}^n, \\ \alpha &= K(\beta), \qquad K : \mathbb{R}^2 \to \mathbb{R}^n. \end{aligned}$$

We thus obtained the center manifold $(x, \alpha) = (H(w, \beta), K(\beta))$ for this system. The invariance of the center manifold implies the homological equation

$$H_{w_0}(w,\beta)\dot{w}_0 + H_{w_1}(w,\beta)\dot{w}_1 = f(H(w,\beta), K(\beta)). \tag{A.4}$$

We expand the mappings H and K as

$$\begin{aligned} H(w,\beta) = {} & [q_0, q_1]\, w + [H_{0010}, H_{0001}]\, \beta + \frac{1}{2} H_{2000} w_0^2 + H_{1100} w_0 w_1 + \frac{1}{2} H_{0200} w_1^2 \\ & + H_{1010}\beta_1 w_0 + H_{1001}\beta_2 w_0 + H_{0110}\beta_1 w_1 + H_{0101}\beta_2 w_1 + \frac{1}{2} H_{0002}\beta_2^2 \\ & + \frac{1}{6} H_{3000} w_0^3 + \frac{1}{2} H_{2100} w_0^2 w_1 + H_{1101}\beta_2 w_1 w_0 + \frac{1}{2} H_{2001}\beta_2 w_0^2 \\ & + \mathcal{O}(|w_1|^3 + |w_0 w_1^2| + |\beta_2 w_1^2| + |\beta_1| \|w\|^2 + \|\beta\|^2 \|w\| + \|\beta\|^3) \\ & + \mathcal{O}(\beta_1^2 + |\beta_1\beta_2|) + \mathcal{O}(\|(w,\beta)\|^4), \end{aligned} \tag{A.5}$$

$$K(\beta) = [K_{10}, K_{01}]\, \beta + \frac{1}{2} K_{02}\beta_2^2 + \mathcal{O}(\beta_1^2 + |\beta_1\beta_2|) + \mathcal{O}(\|\beta\|^3). \tag{A.6}$$

Below we will derive the coefficients needed to relate the homoclinic orbit in (A.3) to the homoclinic orbit on the center manifold of (A.1). The derivation in [1, 32] leads to a 'big' system to be solved. The derivation presented here does not involve a 'big' system, making more suitable to implement for the DDE case, cf. Section 6.1.1 and 6.1.2.

A.1.1. Linear terms

Collecting the coefficients of linear terms in the homological equation (A.4) yields the systems

$$\begin{aligned} w_0: &\quad Aq_0 = 0,\\ w_1: &\quad Aq_1 = q_0,\\ \beta_1: &\quad AH_{0010} = q_1 - J_1K_{10},\\ \beta_2: &\quad AH_{0001} = -J_1K_{01}. \end{aligned} \tag{A.7}$$

Let $\gamma = (\gamma_1, \gamma_2) = p_1^T J_1$, then it follows from the Fredholm alternative that

$$\begin{aligned} K_{10} &= s_1 + \delta_1 s_2,\\ K_{01} &= \delta_2 s_2,\\ H_{0010} &= q_0 - A^{INV} J_1 s_1 - \delta_1 A^{INV} J_1 s_2 + \xi_1 q_0,\\ H_{0001} &= -\delta_2 A^{INV} J_1 s_2 + \xi_2 q_0. \end{aligned} \tag{A.8}$$

where $s_1 = \frac{1}{\gamma_1^2+\gamma_2^2}\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}$, $s_2 = \begin{pmatrix}-\gamma_2\\ \gamma_1\end{pmatrix}$ and $\delta_{1,2}, \xi_{1,2}$ are some constants to be determined. The expression $x = A^{INV}$y is defined by solving the non-singular bordered system

$$\begin{pmatrix} A & p_1\\ q_0^T & 0\end{pmatrix}\begin{pmatrix} x\\ s\end{pmatrix} = \begin{pmatrix} y\\ 0\end{pmatrix}. \tag{A.9}$$

A.1.2. Coefficients $a, b, H_{2000}, H_{1100}, H_{0200}$

Collecting the quadratic terms $w\beta$ and ww in the homological equation yields the systems

$$w_0\beta_2 : AH_{1001} = -A_1(K_{01}, q_0) - B(H_{0001}, q_0), \tag{A.10}$$
$$w_1\beta_2 : AH_{0101} = q_1 + H_{1001} - A_1(K_{01}, q_1) - B(H_{0001}, q_1), \tag{A.11}$$
$$w_1\beta_1 : AH_{0110} = H_{0200} + H_{1010} - B(H_{0010}, q_1) - A_1(K_{10}, q_1), \tag{A.12}$$
$$w_1^2 : AH_{0200} = 2H_{1100} - B(q_1, q_1), \tag{A.13}$$
$$w_0\beta_1 : AH_{1010} = H_{1100} - B(H_{0010}, q_0) - A_1(K_{10}, q_0), \tag{A.14}$$
$$w_0w_1 : AH_{1100} = bq_1 + H_{2000} - B(q_0, q_1), \tag{A.15}$$
$$w_0^2 : AH_{2000} = 2aq_1 - B(q_0,q_0). \tag{A.16}$$

From the Fredholm alternative we obtain

$$0 = p_1^T A_1(K_{01}, q_0) + p_1^T B(H_{0001}, q_0), \tag{A.17}$$
$$1 = -p_1^T H_{1001} + p_1^T A_1(K_{01}, q_1) + p_1^T B(H_{0001}, q_1), \tag{A.18}$$
$$0 = p_1^T H_{0200} + p_1^T H_{1010} - p_1^T B(H_{0010}, q_1) - p_1^T A_1(K_{10}, q_1), \tag{A.19}$$
$$0 = 2p_1^T H_{1100} - p_1^T B(q_1, q_1), \tag{A.20}$$
$$0 = p_1^T H_{1100} - p_1^T B(H_{0010}, q_0) - p_1^T A_1(K_{10}, q_0), \tag{A.21}$$
$$0 = b + p_1^T H_{2000} - p_1^T B(q_0, q_1), \tag{A.22}$$
$$0 = 2a - p_1^T B(q_{0,}q_0), \tag{A.23}$$

Multiplying the equations (A.10)-(A.16) with p_0^T yields

$$p_1^T H_{1001} = -p_0^T A_1(K_{01}, q_0) - p_0^T B(H_{0001}, q_0), \tag{A.24}$$
$$p_1^T H_{0101} = p_0^T H_{1001} - p_0^T A_1(K_{01}, q_1) - p_0^T B(H_{0001}, q_1) \tag{A.25}$$
$$p_1^T H_{0110} = p_0^T H_{0200} + p_0^T H_{1010} - p_0^T B(H_{0010}, q_1) - p_0^T A_1(K_{10}, q_1) \tag{A.26}$$
$$p_1^T H_{0200} = 2p_0^T H_{1100} - p_0^T B(q_1, q_1), \tag{A.27}$$
$$p_1^T H_{1010} = p_0^T H_{1100} - p_0^T B(H_{0010}, q_0) - p_0^T A_1(K_{10}, q_0) \tag{A.28}$$
$$p_1^T H_{1100} = p_0^T H_{2000} - p_0^T B(q_0, q_1) \tag{A.29}$$
$$p_1^T H_{2000} = -p_0^T B(q_{0,}q_0). \tag{A.30}$$

Substituting these equation into equations (A.17)-(A.23) we obtain

$$0 = p_1^T A_1(K_{01}, q_0) + p_1^T B(H_{0001}, q_0), \tag{A.31}$$
$$1 = p_0^T A_1(K_{01}, q_0) + p_0^T B(H_{0001}, q_0) + p_1^T A_1(K_{01}, q_1) + p_1^T B(H_{0001}, q_1), \tag{A.32}$$
$$0 = 3p_0^T H_{1100} - p_0^T B(q_1, q_1) - p_0^T B(H_{0010}, q_0) - p_0^T A_1(K_{10}, q_0), \tag{A.33}$$
$$- p_1^T B(H_{0010}, q_1) - p_1^T A_1(K_{10}, q_1), \tag{A.34}$$
$$0 = 2p_0^T H_{2000} - 2p_0^T B(q_0, q_1) - p_1^T B(q_1, q_1), \tag{A.35}$$
$$0 = p_0^T H_{2000} - p_0^T B(q_0, q_1) - p_1^T B(H_{0010}, q_0) - p_1^T A_1(K_{10}, q_0), \tag{A.36}$$
$$0 = b - p_0^T B(q_{0,}q_0) - p_1^T B(q_0, q_1), \tag{A.37}$$
$$0 = 2a - p_1^T B(q_{0,}q_0). \tag{A.38}$$

From the last two equations we have

$$\begin{aligned} a &= \frac{1}{2} p_1^T B(q_0, q_0), \\ b &= p_1^T B(q_0, q_1) - p_1^T H_{2000} = p_1^T B(q_0, q_1) - p_0^T A H_{2000} \\ &= p_1^T B(q_0, q_1) + p_0^T B(q_{0,}q_0) \end{aligned}$$

and subsequently

$$
\begin{aligned}
H_{2000} &= A^{INV}\left(2aq_1 - B(q_0,q_0)\right) + \xi_3 q_0,\\
H_{1100} &= A^{INV}\left(bq_1 + H_{2000} - B(q_0,q_1)\right) + \xi_4 q_0,
\end{aligned} \tag{A.39}
$$

where $\xi_{3,4}$ are some constants. The constant ξ_3 is determined by equation (A.35) and gives

$$
\xi_3 = -p_0^T\left(H_{2000}\right) + p_0^T B(q_0,q_1) + \frac{1}{2}p_1^T B(q_1,q_1).
$$

Then we can solve

$$
H_{0200} = A^{INV}\left(2H_{1100} - B(q_1,q_1)\right) + 2\xi_4 q_1 + \xi_5 q_0.
$$

A.1.3. Determining $H_{0010}, K_{10}, H_{0001}$ and K_{01}

Take the equations (A.8) and fix the values $\delta_1 = 0, \delta_2 = 1$ and $\xi_{1,2} = 0$ to obtain

$$
\begin{cases}
K_{10} &= s_1,\\
K_{01} &= s_2,\\
H_{0010} &= q_0 - A^{INV} J_1 s_1,\\
H_{0001} &= -A^{INV} J_1 s_2.
\end{cases} \tag{A.40}
$$

Evaluating these vectors on the equations (A.31) and (A.32) gives

$$
\begin{aligned}
v_1 &= p_1^T A_1(K_{01},q_0) + p_1^T B(H_{0001},q_0),\\
v_2 &= p_0^T A_1(K_{01},q_0) + p_0^T B(H_{0001},q_0) + p_1^T A_1(K_{01},q_1) + p_1^T B(H_{0001},q_1).
\end{aligned}
$$

To make $(v_1, v_2) = (0,1)$ we first use the freedom $H_{0001} \to H_{0001} + \xi_2 q_0$, so that

$$
p_1^T A_1(K_{01},q_0) + p_1^T B(H_{0001},q_0) \to p_1^T A_1(K_{01},q_0) + p_1^T B(H_{0001},q_0) + 2a\xi_2.
$$

Thus, for

$$
\xi_2 = -\frac{p_1^T A_1(K_{01},q_0) + p_1^T B(H_{0001},q_0)}{2a}
$$

we have $v_1 = 0$. Then we can scale

$$
(H_{0001}, K_{01}) \to \delta_2(H_{0001}, K_{01})
$$

to make $v_2 = 1$ without affecting v_0. This gives

$$
\delta_2 = \frac{1}{p_0^T A_1(K_{01},q_0) + p_0^T B(H_{0001},q_0) + p_1^T A_1(K_{01},q_1) + p_1^T B(H_{0001},q_1)}.
$$

Rearranging equations (A.33) and (A.36) yields

$$\begin{aligned} 3p_0^T H_{1100} &= p_0^T B(q_1, q_1) + p_0^T B(H_{0010}, q_0) + p_0^T A_1(K_{10}, q_0) \\ &\quad + p_1^T B(H_{0010}, q_1) + p_1^T A_1(K_{10}, q_1), \\ p_0^T H_{2000} &= p_0^T B(q_0, q_1) + p_1^T B(H_{0010}, q_0) + p_1^T A_1(K_{10}, q_0). \end{aligned}$$

Evaluating (A.40) on these equations gives

$$\begin{aligned} v_3 &= p_0^T B(q_1, q_1) + p_0^T B(H_{0010}, q_0) + p_0^T A_1(K_{10}, q_0) + p_1^T B(H_{0010}, q_1) + p_1^T A_1(K_{10}, q_1), \\ v_4 &= p_0^T B(q_0, q_1) + p_1^T B(H_{0010}, q_0) + p_1^T A_1(K_{10}, q_0). \end{aligned}$$

To make $v_4 = p_0^T H_{2000}$ we use the freedom $H_{0010} \to H_{0010} + \xi_1 q_0$, so that

$$v_4 \to v_4 + 2a\xi_1$$

Thus, for

$$\xi_1 = \frac{p_0^T H_{2000} - p_0^T B(q_0, q_1) - p_1^T B(H_{0010}, q_0) - p_1^T A_1(K_{10}, q_0)}{2a}$$

we have $v_4 = p_0^T H_{2000}$. Then, after reevaluating v_3, we can translate $(H_{0010}, K_{10}) \to (H_{0010} + \delta_1 H_{0001}, K_{10} + \delta_1 K_{01})$ to make $v_3 = 3p_0^T H_{1100}$ without affecting v_4, see equation (A.31). This gives, using equation (A.32),

$$v_3 \to v_3 + \delta_1.$$

It follows that for

$$\begin{aligned} \delta_1 = 3p_0^T H_{1100} - p_0^T B(q_1, q_1) - p_0^T B(H_{0010}, q_0) - p_0^T A_1(K_{10}, q_0) \\ - p_1^T B(H_{0010}, q_1) - p_1^T A_1(K_{10}, q_1) \end{aligned}$$

we obtain $v_3 = 3p_0^T H_{1100}$. Notice that as before δ_1 still depends on ξ_4 due to the freedom $H_{1100} \to H_{1100} + \xi_4 q_0$ and translates

$$(H_{0010}, K_{10}) \to (H_{0010} + 3\xi_4 H_{0001}, K_{10} + 3\xi_4 K_{01}).$$

This way of determining $H_{0010}, K_{10}, H_{0001}$ and K_{01} is more suitable to implement in the DDE case. Also, it gives better inside on how the freedom affects the vectors.

A.1.4. Coefficients $K_{02}, H_{0002}, H_{1001}, H_{0101}$

The β_2^2 term in the homological equation yields the equation

$$AH_{0002} + J_1 K_{02} = -\left(2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01})\right). \quad \text{(A.41)}$$

The Fredholm alternative gives

$$p_1^T J_1 K_{02} = -p_1^T \left(2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01})\right). \tag{A.42}$$

Using that

$$p_1^T J_1 K_{10} = 1 \tag{A.43}$$

we see that

$$K_{02} = -\left[p_1^T \left(2A_1(K_{01}, H_{0001}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01})\right)\right] K_{10}$$

solves (A.42). Note that K_{02} is not uniquely defined since equation (A.41) still admits the freedom

$$(H_{0002}, K_{02}) \to (H_{0002}, K_{02}) + \delta_3 \left(H_{0001}, K_{01}\right).$$

Also, K_{02} is affected by the freedom in H_{1100}. Indeed, since $H_{1100} \to H_{1100} + \xi_4 q_0$ implies

$$(H_{0010}, K_{10}) \to (H_{0010} + 3\xi_4 H_{0001}, K_{10} + 3\xi_4 K_{01}),$$

we have

$$K_{02} \to K_{02} - 3\xi_4 z_1 K_{01},$$

where

$$z_1 = p_1^T \left(2A_1(H_{0001}, K_{01}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01})\right). \tag{A.44}$$

Now that (A.41) is consistent, we have

$$\begin{aligned} H_{0002} = -A^{INV} \left(J_1 K_{02} + 2A_1(K_{01}, H_{0001}) + B(H_{0001}, H_{0001}) + J_2(K_{01}, K_{01})\right) \\ + \xi_6 q_0 + \delta_3 H_{0001}, \end{aligned}$$

for some constant ξ_6.

Lastly, equations (A.10)-(A.11) give

$$H_{1001} = A^{INV} \left(-A_1(K_{01}, q_0) - B(H_{0001}, q_0)\right) + \xi_7 q_0, \tag{A.45}$$

$$H_{0101} = A^{INV} \left(q_1 + H_{1001} - A_1(K_{01}, q_1) - B(H_{0001}, q_1)\right) + \xi_7 q_1 + \xi_8 q_0, \tag{A.46}$$

for some constants $\xi_{7,8}$.

A.1.5. Coefficients d, e, a_1, b_1

Collecting the systems corresponding to the $w_0^3, w_0^2 w_1, w_0^2\beta_2$ and $w_0 w_1 \beta_2$ term in the homological equations yields

$$w_0^3 : AH_{3000} = 6dq_1 + 6aH_{1100} - 3B(q_0, H_{2000}) - C(q_0, q_0, q_0), \tag{A.47}$$

$$\begin{aligned} w_0^2 w_1 : AH_{2100} &= 2eq_1 + 2aH_{0200} + 2bH_{1100} + H_{3000} \\ &\quad - 2B(q_0, H_{1100}) - B(q_1, H_{2000}) - C(q_0, q_0, q_1), \end{aligned} \tag{A.48}$$

$$\begin{aligned} w_0^2\beta_2 : AH_{2001} &= 2a_1 q_1 + 2aH_{0101} - A_1(K_{01}, H_{2000}) - 2B(q_0, H_{1001}) \\ &\quad - B(H_{0001}, H_{2000}) - B_2(q_0, q_0, K_{01}) - C(q_0, q_0, H_{0001}), \end{aligned} \tag{A.49}$$

$$\begin{aligned} w_0 w_1 \beta_2 : AH_{1101} &= b_1 q_1 + bH_{0101} + H_{1100} + H_{2001} - A_1(K_{01}, H_{1100}) \\ &\quad - B(q_0, H_{0101}) - B(q_1, H_{1001}) - B(H_{0001}, H_{1100}) \\ &\quad - B_2(q_0, q_1, K_{01}) - C(q_0, q_1, H_{0001}). \end{aligned} \tag{A.50}$$

Multiplying the systems with p_1^T and solving for the coefficients d, e, a_1 and b_1 yields

$$\begin{aligned} d &= p_1^T \left(-aH_{1100} + \frac{1}{2}B(q_0, H_{2000}) + \frac{1}{6}C(q_0, q_0, q_0) \right), \\ e &= p_1^T \Big(- aH_{0200} - bH_{1100} - \frac{1}{2}H_{3000} + B(q_0, H_{1100}) \\ &\quad + \frac{1}{2}B(q_1, H_{2000}) + \frac{1}{2}C(q_0, q_0, q_1) \Big), \\ a_1 &= p_1^T \Big(- aH_{0101} + \frac{1}{2}A_1(K_{01}, H_{2000}) + B(q_0, H_{1001}) \\ &\quad + \frac{1}{2}B(H_{0001}, H_{2000}) + \frac{1}{2}B_2(q_0, q_0, K_{01}) + \frac{1}{2}C(q_0, q_0, H_{0001}) \Big), \\ b_1 &= p_1^T \Big(- bH_{0101} - H_{1100} - H_{2001} + B(q_0, H_{0101}) + A_1(K_{01}, H_{1100}) \\ &\quad + B(q_1, H_{1001}) + B(H_{0001}, H_{1100}) + B_2(q_0, q_1, K_{01}) + C(q_0, q_1, H_{0001}) \Big), \end{aligned}$$

where

$$\begin{aligned} p_1^T H_{3000} &= p_0^T A H_{3000} \\ &= p_0^T \left(6aH_{1100} - 3B(q_0, H_{2000}) - C(q_0, q_0, q_0) \right), \\ p_1^T H_{2001} &= p_0^T A H_{2001} \\ &= p_0^T \Big(2aH_{0101} - A_1(K_{01}, H_{2000}) - 2B(q_0, H_{1001}) \\ &\quad - B(H_{0001}, H_{2000}) - B_2(q_0, q_0, K_{01}) - C(q_0, q_0, H_{0001}) \Big). \end{aligned}$$

A.2. Transcritical Bogdanov-Takens bifurcation

In this Section we impose the constraint that the equilibrium at the origin remains fixed under parameter variation as in [27]. Then, the Taylor expansion (A.2) becomes

$$\begin{aligned} f(x,\alpha) = Ax + \frac{1}{2}B(x,x) + A_1(x,\alpha) + \frac{1}{6}C(x,x,x) + \frac{1}{2}B_2(x,x,\alpha) \\ \mathcal{O}\left(\|x\|^4 + \|x\|\,\|\alpha\|^2 + \|x\|^2\,\|\alpha\|\right). \end{aligned} \tag{A.51}$$

We want to relate this system to the smooth normal form

$$\begin{cases} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= \beta_1 w_0 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 \\ &\qquad + (a_1\beta_2 + a_2\beta_1 + d w_0)\, w_0^2 + (b_1\beta_2 + b_2\beta_1 + e w_0)\, w_1 w_0, \end{cases}$$

see (C.11), corresponding to this bifurcation on its center manifold. In order to relate both systems to each other, we need a parameterization H of the center manifold in terms of the original variables x and a transformation K of the bifurcation branch to the original parameters α,

$$\begin{aligned} x &= H(w,\beta), \qquad H : \mathbb{R}^{n_c+2} \to \mathbb{R}^n, \\ \alpha &= K(\beta), \qquad K : \mathbb{R}^2 \to \mathbb{R}^n. \end{aligned}$$

We thus obtained the center manifold $(x,\alpha) = (H(w,\beta), K(\beta))$ for this system. The invariance of the center manifold implies the homological equation

$$H_{w_0}(w,\beta)\dot{w}_0 + H_{w_1}(w,\beta)\dot{w}_1 = f(H(w,\beta), K(\beta)). \tag{A.52}$$

We can expand the mappings H and K as

$$\begin{aligned} H(w,\beta) =& q_0 w_0 + q_1 w_1 \\ &+ \frac{1}{2}H_{2000}w_0^2 + H_{1100}w_0w_1 + \frac{1}{2}H_{0200}w_1^2 + H_{1010}\beta_1 w_0 \\ &+ H_{1001}\beta_2 w_0 + H_{0110}\beta_1 w_1 + H_{0101}\beta_2 w_1 + \frac{1}{2}H_{0102}\beta_2^2 w_1 \\ &+ H_{0111}\beta_1\beta_2 w_1 + \frac{1}{2}H_{0120}\beta_1^2 w_1 + \frac{1}{2}H_{0201}\beta_2 w_1^2 + \frac{1}{2}H_{0210}\beta_1 w_1^2 \\ &+ \frac{1}{6}H_{0300}w_1^3 + \frac{1}{2}H_{1002}\beta_2^2 w_0 + H_{1011}\beta_1\beta_2 w_0 + \frac{1}{2}H_{1020}\beta_1^2 w_0 \\ &+ H_{1101}\beta_2 w_1 w_0 + H_{1110}\beta_1 w_1 w_0 + \frac{1}{2}H_{1200}w_1^2 w_0 + \frac{1}{2}H_{2001}\beta_2 w_0^2 \\ &+ \frac{1}{2}H_{2010}\beta_1 w_0^2 + \frac{1}{2}H_{2100}w_1 w_0^2 + \frac{1}{6}H_{3000}w_0^3 + \mathcal{O}(\|w\|^4), \end{aligned} \tag{A.53}$$

$$K(\beta) = K_{10}\beta_1 + K_{01}\beta_2 + \frac{1}{2}K_{20}\beta_1^2 + K_{11}\beta_1\beta_2 + \frac{1}{2}K_{02}\beta_2^2 + \mathcal{O}(\|\beta\|^3). \tag{A.54}$$

Note that, since the equilibrium x_0 remains fixed under variations of parameters, we left out all coefficients of which the terms (w, β) solely depend on the parameters β.

Then the left hand side of the homological equation (A.52) becomes

$$
\begin{aligned}
&H_{w_0}(w,\beta)\dot{w}_0 + H_{w_1}(w,\beta)\dot{w}_1 = \\
&(q_0 + H_{2000}w_0 + H_{1100}w_1 + H_{1010}\beta_1 + H_{1001}\beta_2)\, w_1 \\
&+ (q_1 + H_{1100}w_0 + H_{0200}w_1 + H_{0110}\beta_1 + H_{0101}\beta_2) \\
&\times \Big(\beta_1 w_0 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 \\
&\qquad + (a_1\beta_2 + a_2\beta_1 + d w_0)\, w_0^2 + (b_1\beta_2 + b_2\beta_1 + e w_0)\, w_1 w_0\Big) \\
&= q_0 w_1 \\
&+ a q_1 w_0^2 + H_{1100} w_1^2 + (H_{2000} + b q_1)\, w_0 w_1 \\
&+ H_{1010} w_1 \beta_1 + (H_{1001} + q_1)\, w_1 \beta_2 \\
&+ (6aH_{1100} + 6dq_1)\, w_0^3 + 2H_{1200} w_1^3 \\
&+ (2aH_{0200} + 2bH_{1100} + 2eq_1 + H_{3000})\, w_0^2 w_1 \\
&+ (2aH_{0110} + 2a_2 q_1 + 2H_{1100})\, w_0^2 \beta_1 \\
&+ (2aH_{0101} + 2a_1 q_1)\, w_0^2 \beta_2 + (2bH_{0200} + 2H_{2100})\, w_0 w_1^2 \\
&+ 2H_{0110}\beta_1^2 w_0 + 2H_{1110}\beta_1 w_1^2 + (2H_{0200} + 2H_{1101})\, \beta_2 w_1^2 \\
&+ H_{1020} w_1 \beta_1^2 + (2H_{0101} + H_{1002})\, w_1 \beta_2^2 \\
&+ (bH_{0110} + b_2 q_1 + H_{0200} + H_{2010})\, w_0 w_1 \beta_1 \\
&+ (bH_{0101} + b_1 q_1 + H_{1100} + H_{2001})\, w_0 w_1 \beta_2 \\
&+ H_{0101} w_0 \beta_1 \beta_2 + (H_{0110} + H_{1011})\, w_1 \beta_1 \beta_2.
\end{aligned} \tag{A.55}
$$

Inserting (A.53) and (A.54) into the right hand side of the homological equation (A.52) yields

$$
\begin{aligned}
&f(H(w,\beta),K(\beta)) = \\
&\quad AH(w,\beta) + \frac{1}{2}B(H(w,\beta),H(w,\beta)) + \frac{1}{6}C(H(w,\beta),H(w,\beta),H(w,\beta)) \\
&\qquad + A_1(H(w,\beta),K(\beta)) + \frac{1}{2}B_2(H(w,\beta),H(w,\beta),K(\beta)) + \dots \\
&\quad = A(q_0w_0 + q_1w_1 + \frac{1}{2}H_{2000}w_0^2 + H_{1100}w_0w_1 + \frac{1}{2}H_{0200}w_1^2 + H_{1010}\beta_1w_0 \\
&\qquad + H_{1001}\beta_2w_0 + H_{0110}\beta_1w_1 + H_{0101}\beta_2w_1 + \frac{1}{2}H_{0102}\beta_2^2w_1 \\
&\qquad + H_{0111}\beta_1\beta_2w_1 + \frac{1}{2}H_{0120}\beta_1^2w_1 + \frac{1}{2}H_{0201}\beta_2w_1^2 + \frac{1}{2}H_{0210}\beta_1w_1^2 \\
&\qquad + \frac{1}{6}H_{0300}w_1^3 + \frac{1}{2}H_{1002}\beta_2^2w_0 + H_{1011}\beta_1\beta_2w_0 + \frac{1}{2}H_{1020}\beta_1^2w_0 \\
&\qquad + H_{1101}\beta_2w_1w_0 + H_{1110}\beta_1w_1w_0 + \frac{1}{2}H_{1200}w_1^2w_0 + \frac{1}{2}H_{2001}\beta_2w_0^2 \\
&\qquad + \frac{1}{2}H_{2010}\beta_1w_0^2 + \frac{1}{2}H_{2100}w_1w_0^2 + \frac{1}{6}H_{3000}w_0^3) \\
&\quad + \frac{1}{2}B(q_0w_0 + q_1w_1 + \frac{1}{2}H_{2000}w_0^2 + H_{1100}w_0w_1 + \frac{1}{2}H_{0200}w_1^2 + H_{1010}\beta_1w_0 \\
&\qquad + H_{1001}\beta_2w_0 + H_{0110}\beta_1w_1 + H_{0101}\beta_2w_1, q_0w_0 + q_1w_1 \\
&\qquad + \frac{1}{2}H_{2000}w_0^2 + H_{1100}w_0w_1 + \frac{1}{2}H_{0200}w_1^2 + H_{1010}\beta_1w_0 + H_{1001}\beta_2w_0 \\
&\qquad + H_{0110}\beta_1w_1 + H_{0101}\beta_2w_1) \\
&\quad + \frac{1}{6}C(q_0w_0 + q_1w_1, q_0w_0 + q_1w_1, q_0w_0 + q_1w_1) \\
&\quad + A_1(q_0w_0 + q_1w_1 + \frac{1}{2}H_{2000}w_0^2 + H_{1100}w_0w_1 \\
&\qquad + \frac{1}{2}H_{0200}w_1^2 + H_{1010}\beta_1w_0 + H_{1001}\beta_2w_0 + H_{0110}\beta_1w_1 + H_{0101}\beta_2w_1, \\
&\qquad K_{10}\beta_1 + K_{01}\beta_2 + \frac{1}{2}K_{20}\beta_1^2 + K_{11}\beta_1\beta_2 + \frac{1}{2}K_{02}\beta_2^2) \\
&\quad + \frac{1}{2}B_2(q_0w_0 + q_1w_1, q_0w_0 + q_1w_1, K_{10}\beta_1 + K_{01}\beta_2) + \dots
\end{aligned}
$$

$$
\begin{aligned}
=&Aq_0w_0 + Aq_1w_1 \\
&+ \frac{1}{2}\left(AH_{2000} + B(q_0,q_0)\right)w_0^2 + \frac{1}{2}\left(AH_{0200} + B(q_1,q_1)\right)w_1^2 \\
&+ \left(AH_{1100} + B(q_0,q_1)\right)w_0w_1 + \left(AH_{1010} + A_1(q_0,K_{10})\right)w_0\beta_1 \\
&+ \left(AH_{0110} + A_1(q_1,K_{10})\right)w_1\beta_1 + \left(AH_{0101} + A_1(q_1,K_{01})\right)w_1\beta_2 \\
&+ (C(q_0,q_0,q_0) + AH_{3000} + 3B(q_0,H_{2000}))w_0^3 \\
&+ (C(q_1,q_1,q_1) + AH_{0300} + 3B(q_1,H_{0200}))w_1^3 \\
&+ (C(q_0,q_0,q_1) + AH_{2100} + 2B(q_0,H_{1100}) + 2B(q_1,H_{2000}))w_0^2w_1 \\
&+ (A_1(K_{10},H_{2000}) + AH_{2010} + 2B(q_0,H_{1010}) + B_2(q_0,q_0,K_{10}))\beta_1w_0^2 \\
&+ (A_1(K_{01},H_{2000}) + AH_{2001} + 2B(q_0,H_{1001}) + B_2(q_0,q_0,K_{01}))\beta_2w_0^2 \\
&+ (C(q_0,q_1,q_1) + AH_{1200} + B(q_0,H_{0200}) + 2B(q_1,H_{1100}))w_0w_1^2 \\
&+ (2A_1(K_{10},H_{1010}) + AH_{1020} + A_1(q_0,K_{20}))\beta_1^2w_0 \\
&+ (2A_1(K_{01},H_{1001}) + AH_{1002} + A_1(q_0,K_{02}))\beta_2^2w_0 \\
&+ (A_1(K_{10},H_{0200}) + AH_{0210} + 2B(q_1,H_{0110}) + B_2(q_1,q_1,K_{10}))\beta_1w_1^2 \\
&+ (A_1(K_{01},H_{0200}) + AH_{0201} + 2B(q_1,H_{0101}) + B_2(q_1,q_1,K_{01}))\beta_2w_1^2 \\
&+ (2A_1(K_{10},H_{0110}) + AH_{0120} + A_1(q_1,K_{20}))\beta_1^2w_1 \\
&+ (2A_1(K_{01},H_{0101}) + AH_{0102} + A_1(q_1,K_{02}))\beta_2^2w_1 \\
&+ (A_1(K_{10},H_{1100}) + AH_{1110} + B(q_0,H_{0110}) + B(q_1,H_{1010}) + B_2(q_0,q_1,K_{10}))\beta_1w_0w_1 \\
&+ (A_1(K_{01},H_{1100}) + AH_{1101} + B(q_0,H_{0101}) + B(q_1,H_{1001}) + B_2(q_0,q_1,K_{01}))\beta_2w_0w_1 \\
&+ (A_1(K_{01},H_{1010}) + A_1(K_{10},H_{1001}) + AH_{1011} + A_1(q_0,K_{11}))\beta_1\beta_2w_0 \\
&+ (A_1(K_{01},H_{0110}) + A_1(K_{10},H_{0101}) + AH_{0111} + A_1(q_1,K_{11}))\beta_1\beta_2w_1 + \ldots. \qquad \text{(A.56)}
\end{aligned}
$$

A.2.1. Linear and quadratic terms

Equating (A.55) with (A.56) and collecting the linear and quadratic terms in (w, β) by leads to the equations

$$
\begin{aligned}
w_0 &: \quad Aq_0 = 0 \\
w_1 &: \quad Aq_1 = q_0 \\
w_0^2 &: \quad AH_{2000} = 2aq_1 - B(q_0, q_0) \\
w_0 w_1 &: \quad AH_{1100} = bq_1 + H_{2000} - B(q_0, q_1) \\
w_1^2 &: \quad AH_{0200} = 2H_{1100} - B(q_1, q_1)
\end{aligned} \tag{A.57}
$$

$$w_0\beta_1 : \quad AH_{1010} = q_1 - A_1(q_0, K_{10}) \tag{A.58}$$

$$w_0\beta_2 : \quad AH_{1001} = -A_1(q_0, K_{01}) \tag{A.59}$$

$$w_1\beta_1 : \quad AH_{0110} = -A_1(q_1, K_{10}) + H_{1010} \tag{A.60}$$

$$w_1\beta_2 : \quad AH_{0101} = -A_1(q_1, K_{01}) + H_{1001} + q_1 \tag{A.61}$$

Left multiplying the equations corresponding to the quadratic terms with the adjoint vector p_1^T, yields,

$$
\begin{aligned}
0 &= 2a - p_1^T B(q_0, q_0), \\
0 &= b + p_1^T H_{2000} - p_1^T B(q_0, q_1), \\
0 &= 2p_1^T H_{1100} - p_1^T B(q_1, q_1),
\end{aligned}
$$

$$0 = 1 - p_1^T A_1(q_0, K_{10}), \tag{A.62}$$

$$0 = -p_1^T A_1(q_0, K_{01}), \tag{A.63}$$

$$0 = -p_1^T A_1(q_1, K_{10}) + p_1^T H_{1010}, \tag{A.64}$$

$$0 = -p_1^T A_1(q_1, K_{01}) + p_1^T H_{1001} + 1. \tag{A.65}$$

From the first two equations we have

$$
\begin{aligned}
a &= \frac{1}{2} p_1^T B(q_0, q_0), \\
b &= -p_1^T H_{2000} + p_1^T B(q_0, q_1).
\end{aligned}
$$

Multiplying (A.57) with p_0^T from the left yields

$$p_1^T H_{2000} = -p_0^T B(q_0, q_0).$$

It follows that

$$b = p_0^T B(q_0, q_0) + p_1^T B(q_0, q_1).$$

We thus recover the critical normal coefficients (a, b) for the generic Bogdanov-Takens bifurcation.

Multiplying (A.58) and (A.59) with p_0^T from the left yields

$$\begin{aligned} p_1^T H_{1010} &= -p_0^T A_1(q_0, K_{10}), \\ p_1^T H_{1001} &= -p_0^T A_1(q_0, K_{01}). \end{aligned}$$

Substituting into equations (A.64) and (A.65) gives

$$\begin{aligned} 0 &= -p_1^T A_1(q_1, K_{10}) - p_0^T A_1(q_0, K_{10}), \\ 0 &= -p_1^T A_1(q_1, K_{01}) - p_0^T A_1(q_0, K_{01}) + 1. \end{aligned}$$

Together with equations (A.62) and (A.63), one computes K_{10} and K_{01} by solving the 2-dimensional system

$$\begin{pmatrix} p_1^T A_1 q_0 \\ p_0^T A_1 q_0 + p_1^T A_1 q_1 \end{pmatrix} \left(\begin{bmatrix} K_{10} & K_{01} \end{bmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The operator A^{INV}, defined in Lemma 4.6, can serve to tackle the remaining unknowns in equations (A.57)-(A.61):

$$\begin{aligned} H_{2000} &= A^{INV}\left(2aq_1 - B(q_0, q_0)\right), \\ H_{1100} &= A^{INV}\left(bq_1 + H_{2000} - B(q_0, q_1)\right), \\ H_{0200} &= A^{INV}\left(2H_{1100} - B(q_1, q_1)\right), \\ H_{1010} &= A^{INV}\left(q_1 - A_1(q_0, K_{10})\right), \\ H_{1001} &= A^{INV}\left(-A_1(q_0, K_{01})\right), \\ H_{0110} &= A^{INV}\left(-A_1(q_1, K_{10}) + H_{1010}\right), \\ H_{0101} &= A^{INV}\left(-A_1(q_1, K_{01}) + H_{1001} + q_1\right), \end{aligned}$$

where A^{INV} is as in (A.9). To ensure that $2H_{1100} - B(q_1, q_1)$ is in the image of A we have to translate H_{2000} with a scalar multiple of q_0. The solvability condition implies that

$$2p_1^T H_{1100} = p_1^T B(q_1, q_1)$$

Let

$$H_{2000} = \tilde{H}_{2000} + \gamma q_0,$$

where $\tilde{H}_{2000}$ is given by $A^{INV}\left(2aq_1 - B(q_0, q_0)\right)$, then

$$\begin{aligned} p_1^T H_{1100} &= \langle A^T p_0, H_{1100} \rangle \\ &= \langle p_0, A H_{1100} \rangle \\ &= p_0^T \left(bq_1 + H_{2000} - B(q_0, q_1)\right) \\ &= p_0^T \tilde{H}_{2000} + \gamma - p_0^T B(q_0, q_1). \end{aligned}$$

Therefore

$$\gamma = \frac{1}{2} p_1^T B(q_1, q_1) - p_0^T \tilde{H}_{2000} + p_0^T B(q_0, q_1).$$

A.2.2. Cubic terms

Equating (A.55) with (A.56) and collecting the cubic terms in (w, β) by leads to the equations

$$
\begin{aligned}
w_0^3 :\quad & AH_{3000} = -C(q_0,q_0,q_0) + 6aH_{1100} - 3B(q_0,H_{2000}) + 6dq_1, && \text{(A.66)}\\
w_1^3 :\quad & AH_{0300} = -C(q_1,q_1,q_1) - 3B(q_1,H_{0200}) + 3H_{1200}, &&\\
w_0^2w_1 :\quad & AH_{2100} = -C(q_0,q_0,q_1) + 2aH_{0200} + 2bH_{1100} - 2B(q_0,H_{1100}) && \text{(A.67)}\\
& \qquad - B(q_1,H_{2000}) + H_{3000} + 2eq_1, &&\\
w_0^2\beta_1 :\quad & AH_{2010} = 2aH_{0110} - A_1(H_{2000},K_{10}) - 2B(q_0,H_{1010}) &&\\
& \qquad - B_2(q_0,q_0,K_{10}) + 2H_{1100} + 2a_2q_1, && \text{(A.68)}\\
w_0^2\beta_2 :\quad & AH_{2001} = 2aH_{0101} - A_1(H_{2000},K_{01}) - 2B(q_0,H_{1001}) &&\\
& \qquad - B_2(q_0,q_0,K_{01}) + 2a_1q_1, && \text{(A.69)}\\
w_0w_1^2 :\quad & AH_{1200} = -C(q_0,q_1,q_1) + 2bH_{0200} - B(q_0,H_{0200}) - 2B(q_1,H_{1100}) &&\\
& \qquad + 2H_{2100}, &&\\
w_0\beta_1^2 :\quad & AH_{1020} = -2A_1(H_{1010},K_{10}) - A_1(q_0,K_{20}) + 2H_{0110}, && \text{(A.70)}\\
w_0\beta_2^2 :\quad & AH_{1002} = -2A_1(H_{1001},K_{01}) - A_1(q_0,K_{02}), && \text{(A.71)}\\
w_1^2\beta_1 :\quad & AH_{0210} = -A_1(H_{0200},K_{10}) - 2B(q_1,H_{0110}) - B_2(q_1,q_1,K_{10}) + 2H_{1110}, &&\\
w_1^2\beta_2 :\quad & AH_{0201} = -A_1(H_{0200},K_{01}) - 2B(q_1,H_{0101}) - B_2(q_1,q_1,K_{01}) &&\\
& \qquad + 2H_{0200} + 2H_{1101}, &&\\
w_1\beta_1^2 :\quad & AH_{0120} = -2A_1(H_{0110},K_{10}) - A_1(q_1,K_{20}) + H_{1020}, && \text{(A.72)}\\
w_1\beta_2^2 :\quad & AH_{0102} = -2A_1(H_{0101},K_{01}) - A_1(q_1,K_{02}) + 2H_{0101} + H_{1002}, && \text{(A.73)}\\
w_0w_1\beta_1 :\quad & AH_{1110} = -A_1(H_{1100},K_{10}) + bH_{0110} - B(q_0,H_{0110}) - B(q_1,H_{1010}) &&\\
& \qquad - B_2(q_0,q_1,K_{10}) + H_{0200} + H_{2010} + b_2q_1, && \text{(A.74)}\\
w_0w_1\beta_2 :\quad & AH_{1101} = -A_1(H_{1100},K_{01}) + bH_{0101} - B(q_0,H_{0101}) - B(q_1,H_{1001}) &&\\
& \qquad - B_2(q_0,q_1,K_{01}) + H_{1100} + H_{2001} + b_1q_1, && \text{(A.75)}\\
w_0\beta_1\beta_2 :\quad & AH_{1011} = -A_1(H_{1001},K_{10}) - A_1(H_{1010},K_{01}) - A_1(q_0,K_{11}) + H_{0101}, && \text{(A.76)}\\
w_1\beta_1\beta_2 :\quad & AH_{0111} = -A_1(H_{0101},K_{10}) - A_1(H_{0110},K_{01}) - A_1(q_1,K_{11}) + H_{0110} &&\\
& \qquad + H_{1011}. && \text{(A.77)}
\end{aligned}
$$

A.2.3. Coefficients K_{20}, K_{11} and K_{02}

To solve the coefficients K_{20}, K_{02} and K_{11} we left multiply the systems in equations (A.70), (A.72), (A.71), (A.73), (A.76) and (A.77) with p_1^T

$$0 = p_1^T \left(-2A_1(H_{1010}, K_{10}) - A_1(q_0, K_{20}) + 2H_{0110}\right), \tag{A.78}$$
$$0 = p_1^T \left(-2A_1(H_{0110}, K_{10}) - A_1(q_1, K_{20}) + H_{1020}\right), \tag{A.79}$$
$$0 = p_1^T \left(-2A_1(H_{1001}, K_{01}) - A_1(q_0, K_{02})\right), \tag{A.80}$$
$$0 = p_1^T \left(-2A_1(H_{0101}, K_{01}) - A_1(q_1, K_{02}) + 2H_{0101} + H_{1002}\right), \tag{A.81}$$
$$0 = p_1^T \left(-A_1(H_{1001}, K_{10}) - A_1(H_{1010}, K_{01}) - A_1(q_0, K_{11}) + H_{0101}\right), \tag{A.82}$$
$$0 = p_1^T \left(-A_1(H_{0101}, K_{10}) - A_1(H_{0110}, K_{01}) - A_1(q_1, K_{11}) + H_{0110} + H_{1011}\right). \tag{A.83}$$

Left multiplying equations (A.70),(A.71) and (A.76) with p_0^T yields

$$\begin{aligned}
p_1^T H_{1020} &= p_0^T \left(-2A_1(H_{1010}, K_{10}) - A_1(q_0, K_{20}) + 2H_{0110}\right), \\
p_1^T H_{1002} &= p_0^T \left(-2A_1(H_{1001}, K_{01}) - A_1(q_0, K_{02})\right), \\
p_1^T H_{1011} &= p_0^T \left(-A_1(H_{1001}, K_{10}) - A_1(H_{1010}, K_{01}) - A_1(q_0, K_{11}) + H_{0101}\right).
\end{aligned}$$

By substituting these equations into (A.79), (A.81) and (A.83) we obtain

$$\begin{aligned}
0 =& p_1^T \left(-2A_1(H_{0110}, K_{10}) - A_1(q_1, K_{20})\right) \\
&+ p_0^T \left(-2A_1(H_{1010}, K_{10}) - A_1(q_0, K_{20}) + 2H_{0110}\right), \\
0 =& p_1^T \left(-2A_1(H_{0101}, K_{01}) - A_1(q_1, K_{02}) + 2H_{0101}\right) \\
& p_0^T \left(-2A_1(H_{1001}, K_{01}) - A_1(q_0, K_{02})\right), \\
0 =& p_1^T \left(-A_1(H_{0101}, K_{10}) - A_1(H_{0110}, K_{01}) - A_1(q_1, K_{11}) + H_{0110}\right) \\
& p_0^T \left(-A_1(H_{1001}, K_{10}) - A_1(H_{1010}, K_{01}) - A_1(q_0, K_{11}) + H_{0101}\right).
\end{aligned}$$

Together with equations (A.78), (A.80) and (A.82), one computes K_{20}, K_{02} and K_{11} by solving the 2-dimensional system

$$\begin{pmatrix} p_1^T A_1 q_0 \\ p_0^T A_1 q_0 + p_1^T A_1 q_1 \end{pmatrix} \left(\begin{bmatrix} K_{20} & K_{02} & K_{11} \end{bmatrix}\right) = \begin{pmatrix} E_1 & E_3 & E_5 \\ E_2 & E_4 & E_6 \end{pmatrix},$$

where

$$\begin{aligned}
E_1 &= p_1^T \left(-2A_1(H_{1010}, K_{10}) + 2H_{0110}\right) \\
E_2 &= p_1^T \left(-2A_1(H_{0110}, K_{10})\right) \\
&\quad + p_0^T \left(-2A_1(H_{1010}, K_{10}) + 2H_{0110}\right), \\
E_3 &= -2p_1^T A_1(H_{1001}, K_{01}) \\
E_4 &= p_1^T \left(-2A_1(H_{0101}, K_{01}) + 2H_{0101}\right) \\
&\quad p_0^T \left(-2A_1(H_{1001}, K_{01})\right), \\
E_5 &= p_1^T \left(-A_1(H_{1001}, K_{10}) - A_1(H_{1010}, K_{01}) + H_{0101}\right) \\
E_6 &= p_1^T \left(-A_1(H_{0101}, K_{10}) - A_1(H_{0110}, K_{01}) + H_{0110}\right) \\
&\quad p_0^T \left(-A_1(H_{1001}, K_{10}) - A_1(H_{1010}, K_{01}) + H_{0101}\right).
\end{aligned}$$

A.2.4. The coefficients a_1, a_2, b_1, b_2, d and e.

Lastly, we need to solve the coefficients a_1, a_2, b_1, b_2, d and e.
Left multiplying equations (A.66), (A.67), (A.69), (A.68), (A.75) and (A.74) with p_1^T and rearranging terms yields

$$
\begin{aligned}
d =& \frac{1}{6}p_1^T\Big(- 6aH_{1100} + 3B(q_0, H_{2000}) + C(q_0, q_0, q_0)\Big),\\
e =& \frac{1}{2}p_1^T\Big(- 2aH_{0200} - 2bH_{1100} + 2B(q_0, H_{1100})\\
&+ B(q_1, H_{2000}) - H_{3000} + C(q_0, q_0, q_1)\Big).\\
a_1 =& \frac{1}{2}p_1^T\Big(- 2aH_{0101} + A_1(H_{2000}, K_{01}) + 2B(q_0, H_{1001}) + B_2(q_0, q_0, K_{01})\Big),\\
a_2 =& \frac{1}{2}p_1^T\Big(- 2aH_{0110} + A_1(H_{2000}, K_{10}) + 2B(q_0, H_{1010}) + B_2(q_0, q_0, K_{10}) - 2H_{1100}\Big),\\
b_1 =& p_1^T\Big(A_1(H_{1100}, K_{01}) - bH_{0101} + B(q_1, H_{1001}) + B_2(q_0, q_1, K_{01}) - H_{1100} - H_{2001}\Big),\\
b_2 =& p_1^T\Big(A_1(H_{1100}, K_{10}) - bH_{0110} + B(q_0, H_{0110}) + B(q_1, H_{1010}) + B_2(q_0, q_1, K_{10})\\
&- H_{0200} - H_{2010}\Big),
\end{aligned}
$$

Where the expressions $p_1^T H_{2001}$, $p_1^T H_{2010}$ and $p_1^T H_{3000}$ can be found by left multiplying equations (A.66) (A.69) and (A.68) with p_1^T

$$
\begin{aligned}
p_1^T H_{3000} &= p_0^T\Big(- C(q_0, q_0, q_0) + 6aH_{1100} - 3B(q_0, H_{2000})\Big),\\
p_1^T H_{2001} &= p_0^T\Big(2aH_{0101} - A_1(H_{2000}, K_{01}) - 2B(q_0, H_{1001}) - B_2(q_0, q_0, K_{01})\Big),\\
p_1^T H_{2010} &= p_0^T\Big(2aH_{0110} - A_1(H_{2000}, K_{10}) - 2B(q_0, H_{1010}) - B_2(q_0, q_0, K_{10}) + 2H_{1100}\Big).
\end{aligned}
$$

B. Parameter-dependent normal forms for codim 2 equilibrium bifurcations

B.1. Bogdanov-Takens bifurcation

Suppose that the system (ODE) at the critical parameter value $\alpha_0 = (0,0) \in \mathbb{R}^2$ undergoes a Bogdanov-Takens bifurcation at the origin. Then the smooth normal form on the parameter-dependent center manifold takes the form

$$\begin{aligned} \dot{w} &= G(w,\beta) \\ &= \begin{pmatrix} w_1 \\ \beta_1 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 + g_1(w,\beta) \end{pmatrix} \\ &\quad + \mathcal{O}(\|\beta\|\, w_1^2) + \mathcal{O}(\|\beta\|^2 \|w\|^2 + \|\beta\| \|w\|^3 + \|w\|^4), \end{aligned}$$

where $w = (w_0, w_1)$, $\beta = (\beta_1, \beta_2)$ and

$$g_1(w,\beta_2) = (a_1\beta_2 + d w_0)\, w_0^2 + (b_1\beta_2 + e w_0)\, w_1 w_0.$$

This normal form can be derived from [7]. The restriction of (ODE) to the two-dimensional center manifold W^c at the critical parameter value α_0 can be transformed to the smooth normal form

$$\begin{cases} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= a w_0^2 + b w_0 w_1 + \mathcal{O}(\|\,(w_0,w_1)\,\|^3). \end{cases}$$

The following two-parameter family provides an universal unfolding of the codimension 2 Bogdanov-Takens bifurcation

$$\dot{w} = \begin{pmatrix} w_1 \\ \beta_1 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 \end{pmatrix}, \tag{B.1}$$

[22]. In Figure B.1 the bifurcation diagram for $a = 1$ and $b = -1$ is shown. A detailed analysis of this, or equivalent unfoldings, can be found in many textbooks and articles, see for example [5, 22, 34].

B.2. Transcritical Bogdanov-Takens bifurcation

Many articles in which Bogdanov-Takens bifurcations in DDEs are studied, deal with models in which the steady-state remains fixed under variation of parameters [53, 55, 50,

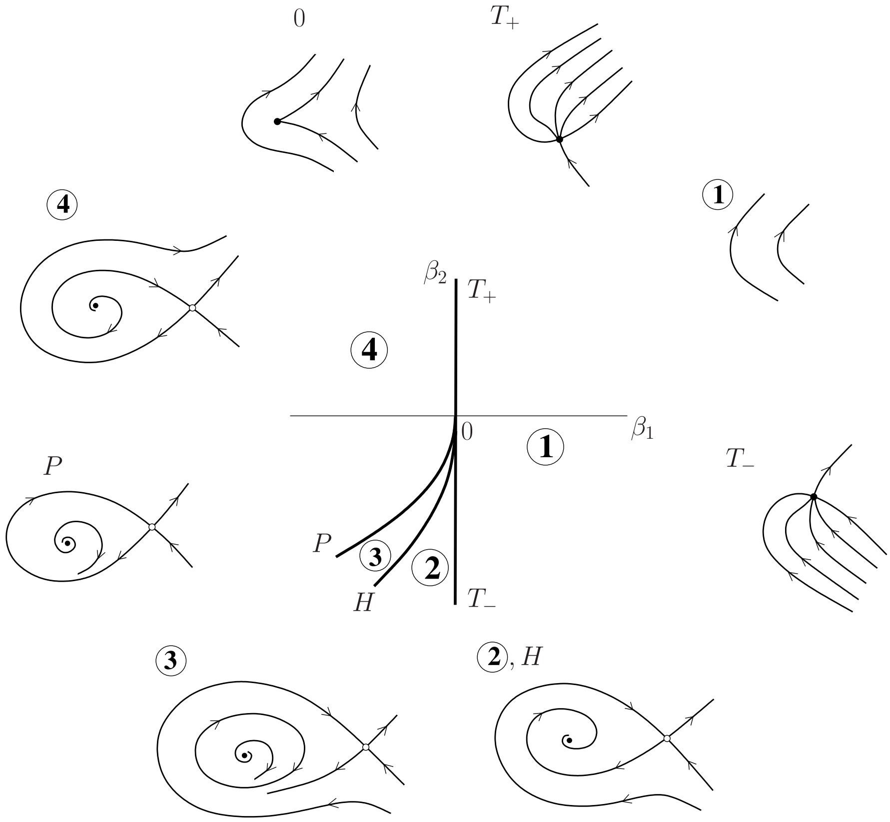


Figure B.1.: Phase portraits for the generic Bogdanov-Takens bifurcation given by (B.1) with $a = 1$ and $b = -1$. There is a supercritical Hopf bifurcation curve, i.e. the limit cycles in the Hopf bifurcations are stable; a homoclinic bifurcation curve and a fold bifurcation curve, which all meet each other tangentially at the origin. The Hopf curve is given by $\beta_2 = -\sqrt{-\beta_1}$ for $\beta_1 < 0$. The fold bifurcation curve is given by $\beta_1 = 0$. At $\beta_2 = \frac{10}{7}\beta_1 + \mathcal{O}(\beta_1^{3/2})$ for $\beta_1 < 0$, there exists a homoclinic orbit which can be found by using Melnikov's integral, see [22].

26, 38, 16, 31, 44]. Under this constraint the unfolding cannot be given by (B.1) anymore and we have to consider the normal form

$$\dot{w} = \begin{pmatrix} w_1 \\ \beta_1 w_0 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 \end{pmatrix}. \tag{B.2}$$

The trivial equilibrium at the origin undergoes a transcritical bifurcation at $\beta_1 = 0$. Therefore, we will refer to this case as a transcritical Bogdanov-Takens bifurcation.

Without loss of generality, we assume that $a > 0$ and $b < 0$. This will lead to supercritical Hopf curves. Other possible values can be obtained by straightforward reflections (including time reversal). In particular, the reflection $\beta_1 \to -\beta_1$ together with the transformation $(w_0, \beta_2) \to (w_0 - \frac{\beta_1}{a}, \beta_2 + \frac{b}{a}\beta_1)$ leaves the normal form (B.2) invariant. This extra reflection symmetry compared with the generic Bogdanov-Takens bifurcation leads to one additional Hopf and one additional homoclinic bifurcation curve. In Figure B.2 we have plotted the bifurcation diagram. For a more detailed analysis we refer to [27].

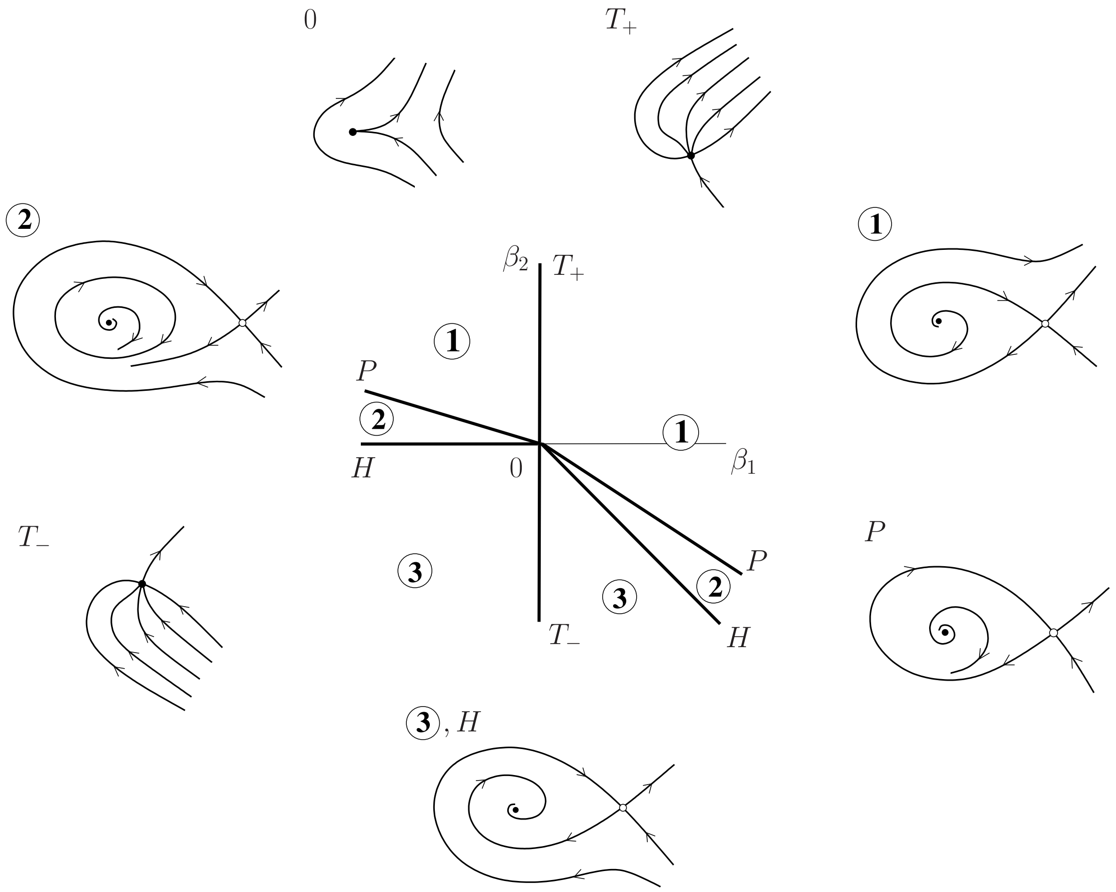


Figure B.2.: Phase portraits for the transcritical Bogdanov-Takens bifurcation given by (B.2) with $a = 1$ and $b = -1$. In contrast to the generic Bogdanov-Takens bifurcations there are now two Hopf and homoclinic bifurcation curves. One Hopf curve is given by $\beta_2 = -\beta_1$ for $\beta_1 > 0$. The other Hopf curve is given by $\beta_2 = 0$ for $\beta_1 < 0$. At $\beta_2 = -\frac{6}{7}\beta_1 + \mathcal{O}(\beta_1^{3/2})$ for $\beta_1 > 0$, there exists a homoclinic orbit which can be found by using Melnikov's integral, see [22]. The second homoclinic orbit is given by $\beta_2 = \frac{1}{7}\beta_1 + \mathcal{O}(\beta_1^{3/2})$ for $\beta_1 < 0$, which is derived using the extra reflection. Lastly there is a transcritical bifurcation curve for $\beta_1 = 0$.

B.3. Generalized Hopf bifurcation

Suppose that the system (3.3) has an equilibrium $x = 0$ at the critical parameter value $\alpha_0 = (0, 0) \in \mathbb{R}^2$ with the purely imaginary eigenvalues

$$\lambda_{1,2} = \pm i\omega_0, \qquad \omega_0 > 0. \tag{B.3}$$

Furthermore, suppose that the first Lyapunov coefficient $\ell_1(0) = 0$, then the restriction of (3.3) to the two-dimensional center manifold W^c can be transformed to the normal form

$$\dot{z} = \lambda(\alpha)z + c_1(\alpha)z|z|^2 + c_2(\alpha)z|z|^4 + \mathcal{O}(|z|^6), \tag{B.4}$$

where $\lambda(\alpha), c_1(\alpha), c_2(\alpha)$ are complex functions with $\omega_0\ell_1(0) = \text{Re } c_1(0) = 0$, $\lambda(0) = i\omega_0$ and $\omega_0\ell_2(0) = \text{Re } c_2(0) \neq 0$. Lastly, suppose that the map $\alpha \mapsto (\mu(\alpha), \ell_1(\alpha))$ is regular at $\alpha = 0$, Then, by the introduction of a complex variable, applying smooth invertible coordinate transformations that depend smoothly on the parameters, and performing smooth parameter and time changes, the system can be reduced to the complex form

$$\dot{z} = (\beta_1 + i)\, z + \beta_2 z|z|^2 \pm z|z|^4 + O(|z|^6).$$

B.4. Fold-Hopf bifurcation

Suppose that the system (3.3) has an equilibrium $x = 0$ at the critical parameter value $\alpha_0 = (0, 0) \in \mathbb{R}^2$ with the eigenvalues

$$\lambda_1 = 0, \qquad \lambda_{2,3} = \pm i\omega_0, \tag{B.5}$$

where $\omega_0 > 0$. The restriction of (3.3) to the three-dimensional center manifold W^c can be transformed to the normal form

$$\begin{cases} \dot{z}_0 &= \gamma(\alpha) + g_{200}(\alpha)z_0^2 + g_{011}(\alpha)|z_1|^2 + g_{300}(\alpha)z_0^3 + g_{111}(\alpha)z_0|z_1|^2 \\ &\quad + \mathcal{O}\left(\| (z_0, z_1, \overline{z}_1) \|^4\right), \\ \dot{z}_1 &= \Lambda(\alpha)z_1 + g_{110}(\alpha)z_0z_1 + g_{210}(\alpha)w^2z_1 + g_{021}(\alpha)z_1|z_1|^2 + \mathcal{O}\left(\| (z_0, z_1, \overline{z}_1) \|^4\right), \end{cases} \tag{B.6}$$

where $z_0 \in \mathbb{R}$, $z_1 \in \mathbb{C}$, $\gamma(0) = 0, \Lambda(0) = i\omega_0$ and the functions $g_{jkl}(\alpha)$ are real in the first equation and complex in the second. If $g_{110}(0)g_{011}(0) \neq 0$, then, generically, the restriction of (3.3) to the three-dimensional center manifold W^c can be reduced to the system

$$\begin{cases} \dot{z}_0 &= \delta(\alpha) + bz_0^2 + c|z_1|^2 + \mathcal{O}(\| (z_0, z_1, \overline{z}_1) \|^4), \\ \dot{z}_1 &= (\beta_2(\alpha) + i\omega_0(\alpha))z_1 + dz_0z_1 + ez_0^2z_1 + \mathcal{O}(\| (z_0, z_1, \overline{z}_1) \|^4), \end{cases} \tag{B.7}$$

where ω, b, c and e are real functions of α, while d is a complex function of α:

$$\omega(0) = 0, \quad b(0) = g_{200}, \quad c(0) = g_{011}, \quad d(0) = g_{110} - i\omega_0\frac{g_{300}}{g_{200}}$$

and

$$e(0) = \text{Re}\left(g_{210} + g_{110}\left(\frac{\text{Re}g_{021}}{g_{011}} - \frac{3}{2}\frac{g_{300}}{g_{200}} + \frac{g_{111}}{2g_{011}}\right) - \frac{g_{021}g_{200}}{g_{011}}\right).$$

The normal form given by (B.6) is referred to as the *Poincaré normal form*, while the normal form (B.7) is referred to as the *Gavrilov normal form.* As in the generalized Hopf bifurcation, a time reparametrization is needed to transfer the Poincaré normal form to the Gavrilov normal form. Therefore, we will use the Poincaré normal form to derive the coefficients. If we furthermore assume that $e(0)$ is non-zero and the map $\alpha \mapsto \beta$ is regular at $\alpha = \alpha_0$, the Gavrilov normal form (B.7) can be transformed into

$$\begin{cases} \dot{z}_0 &= \beta_1 + z_0^2 + s|z_1|^2 + \mathcal{O}(\|(z_0, z_1, \overline{z}_1)\|^4), \\ \dot{z}_1 &= (\beta_2 + i\omega_0)z_1 + (\theta + i\vartheta)\, z_0 z_1 + z_0^2 z_1 + \mathcal{O}(\|(z_0, z_1, \overline{z}_1)\|^4), \end{cases} \tag{B.8}$$

where

$$s = \text{sign}\,[b(0)c(0)] \qquad \text{and} \qquad \theta(0) = \text{Re}\,\frac{g_{100}(0)}{g_{200}(0)}.$$

Truncating the fourth order terms in (B.8) and making the substitution $z = \rho e^{i\varphi}$ we obtain the system

$$\begin{cases} \dot{z}_0 &= \beta_1 + z_0^2 + s\rho^2, \\ \dot{\rho} &= \rho\left(\beta_2 + \theta z_0 + z_0^2\right), \\ \dot{\varphi} &= \omega_0 + \vartheta z_0 \end{cases} \tag{B.9}$$

Removing the azimuthal term we obtain the *amplitude system*

$$\begin{cases} \dot{w} &= \beta_1 + z_0^2 + s\rho^2, \\ \dot{\rho} &= \rho\left(\beta_2 + \theta z_0 + z_0^2\right). \end{cases} \tag{B.10}$$

Equilibrium points of (B.10) with $\rho = 0$ correspond to equilibrium points for (B.8). Equilibrium points of (B.10) with $\rho > 0$ correspond to periodic solutions for (B.8). Limit cycles of (B.10) correspond to invariant tori for (B.8). Lastly, heteroclinic solutions of (B.10) correspond to spherelike surface for (B.8).

Depending on the signs of s and $\theta(0)$ in (B.10) 4 bifurcation diagrams for nearby parameter values can be distinguished:

I. $s = 1, \theta > 0$ subcritical Hopf bifurcations and no tori

II. $s = -1, \theta < 0$ subcritical Hopf bifurcations and no tori

III. $s = 1, \theta < 0$ sub- and supercritical Hopf bifurcations and torus "heteroclinic destruction"

IV. $s = -1, \theta > 0$ sub- and supercritical Hopf bifurcation and torus "blow-up"

The stability of the torus depend on the sign of $e(0)$. In Figure B.3 the bifurcation diagram of the amplitude system (B.10) with $s = 1, \theta < 0$ and $e < 0$ is shown. This unfolding will be seen in the example in Section 8.8. For the remaining unfoldings of (B.10) and a more detailed analysis we refer to [34].

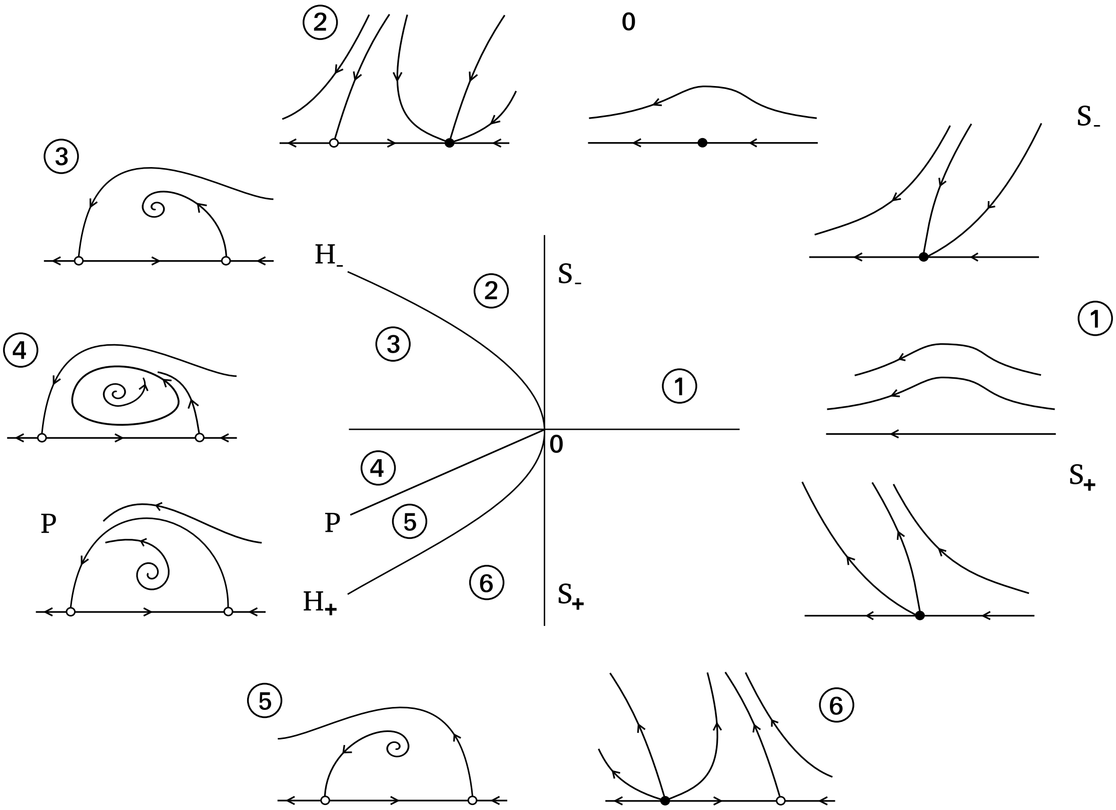


Figure B.3.: Unfolding of the fold-Hopf bifurcation when $s = 1, \theta(0) < 0$ and $e(0) < 0$.

Remark B.1. All results obtained for the truncated normal form (B.10) cannot be transferred back to the whole system (B.8). However, one can show using the Implicit Function Theorem, that the Neimark-Sacker, fold, and sub- and supercritical Hopf bifurcation curves survive adding higher order terms to system (B.9).

B.5. Hopf-transcritical

As for the Bogdanov-Takens bifurcation, many articles in which fold-Hopf bifurcations in DDEs are studied, deal with models in which the steady-state remains fixed under variation of parameters. Under this constraint the unfolding cannot be given by (B.6) anymore and we have to consider the normal form

$$\begin{cases} \dot{z}_0 &= \gamma(\alpha)z_0^2 + g_{200}(\alpha)z_0^2 + g_{011}(\alpha)|z_1|^2 + g_{300}(\alpha)z_0^3 + g_{111}(\alpha)z_0|z_1|^2 \\ & \quad + \mathcal{O}\left(\| (z_0, z_1, \overline{z}_1) \|^4\right), \\ \dot{z}_1 &= \Lambda(\alpha)z_1 + g_{110}(\alpha)z_0 z_1 + g_{210}(\alpha)w^2 z_1 + g_{021}(\alpha)z_1|z_1|^2 + \mathcal{O}\left(\| (z_0, z_1, \overline{z}_1) \|^4\right), \end{cases} \tag{B.11}$$

The bifurcation analysis can be carried out similar to the fold-Hopf case, [23]. An alternative approach is presented in [52]. There the transformation

$$z_1 \to z_1 + \delta$$

is made to transform the amplitude system of the Hopf-transcritical into the amplitude system (B.10). There are in general two solutions $\delta^{\pm}$, yielding to an additional Neimark-Sacker bifurcation curve in cases III and IV compared with the fold-Hopf bifurcation. Furthermore, the fold bifurcation curve becomes a transcritical bifurcation curve, which meets the Hopf bifurcation curve transversely.

B.6. Hopf-Hopf bifurcation

Suppose that the system (3.3) at the critical parameter value $\alpha_0 = (0, 0) \in \mathbb{R}^2$ undergoes two Hopf bifurcation simultaneously. Then the generator A contains two pairs of purely imaginary eigenvalues

$$\lambda_{1,4} = \pm\omega_1, \qquad \lambda_{2,3} = \pm\omega_2, \tag{B.12}$$

where we assume that $\omega_1 > \omega_2 > 0$. When no other eigenvalues on the imaginary axis exists this phenomenon is called the Hopf-Hopf bifurcation or double-Hopf bifurcation. Assume, furthermore that the non-resonance conditions

$$k\omega_1 \neq l\omega_2, \qquad k, l > 0, k + l \leq 5$$

are satisfied. The restriction of (3.3) to the four-dimensional center manifold W^c can be transformed to the normal form

$$\begin{cases} \dot{z}_1 & = (i\omega_1 + \beta_1) z_1 + g_{2100} z_1 |z_1|^2 + g_{1011} z_1 |z_2|^2 + g_{3200} z_1 |z_1|^4 \\ & \quad + g_{2111} z_1 |z_1|^2 |z_2|^2 + g_{1022} z_1 |z_2|^4 + \mathcal{O}\left(\|z_1, \overline{z_1}, z_2, \overline{z_2}\|^6\right), \\ \dot{z}_2 & = (i\omega_2 + \beta_2) z_2 + g_{1110} z_2 |z_1|^2 + g_{0021} z_2 |z_2|^2 + g_{2210} z_2 |z_1|^4 \\ & \quad + g_{1121} z_2 |z_1|^2 |z_2|^2 + g_{0032} z_2 |z_2|^4 + \mathcal{O}\left(\|z_1, \overline{z_1}, z_2, \overline{z_2}\|^6\right), \end{cases} \tag{B.13}$$

where $z_{1,}, z_2 \in \mathbb{C}^2$ and $g_{jklm} \in \mathbb{C}$. Moreover, if

$$(\text{Re } g_{2100})\,(\text{Re } g_{1011})\,(\text{Re } g_{1110})\,(\text{Re } g_{0021}) \neq 0$$

and the critical eigenpairs cross the imaginary axis with nonzero velocities, then (3.3) can be reduced to the system

$$\begin{cases} \dot{z}_1 & = (i\omega_1 + \beta_1) z_1 + \frac{1}{2} p_{11} z_1 |z_1|^2 + p_{12} z_1 |z_2|^2 + i r_1 z_1 |z_1|^4 \\ & \quad + \frac{1}{4} s_1 z_1 |z_2|^4 + \mathcal{O}\left(\|z_1, \overline{z_1}, z_2, \overline{z_2}\|^6\right), \\ \dot{z}_2 & = (i\omega_2 + \beta_2) z_2 + p_{21} z_2 |z_1|^2 + \frac{1}{2} p_{22} z_2 |z_2|^2 + \frac{1}{4} s_2 z_2 |z_1|^4 \\ & \quad + i r_2 z_2 |z_2|^4 + \mathcal{O}\left(\|z_1, \overline{z_1}, z_2, \overline{z_2}\|^6\right), \end{cases}$$

where the coefficients p_{jk} and s_k are complex, while the numbers r_k are real. Moreover, the real parts of the critical values are given by the expressions

$$\text{Re} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \text{Re} \begin{pmatrix} g_{2100} & g_{1011} \\ g_{1110} & g_{0021} \end{pmatrix}$$

and

$$\text{Re } s_1 = \text{Re } g_{1022} + \frac{1}{3} \text{Re } g_{1011} \left(6 \frac{\text{Re } g_{1121}}{\text{Re } g_{1110}} - 4 \frac{\text{Re } g_{0032}}{\text{Re } g_{0021}} - 6 \frac{(\text{Re } g_{3200})\,(\text{Re } g_{0021})}{(\text{Re } g_{2100})\,(\text{Re } g_{1110})} \right),$$

$$\text{Re } s_2 = \text{Re } g_{2210} + \frac{1}{3} \text{Re } g_{1110} \left(6 \frac{\text{Re } g_{2111}}{\text{Re } g_{1011}} - 4 \frac{\text{Re } g_{3200}}{\text{Re } g_{2100}} - 6 \frac{(\text{Re } g_{2100})\,(\text{Re } g_{0032})}{(\text{Re } g_{1011})\,(\text{Re } g_{0021})} \right).$$

Depending on the sign of

$$(\text{Re } p_{11})\,(\text{Re } p_{22}) = (\text{Re } g_{2100})\,(\text{Re } g_{0021}),$$

this bifurcation exhibits either 'simple' or 'difficult' dynamics for nearby parameter values. Each case includes many subcases depending on the signs of

$$\theta = \frac{\text{Re } g_{1011}}{\text{Re } g_{0021}}, \qquad \delta = \frac{\text{Re } g_{1110}}{\text{Re } g_{2100}},$$

see [34]. Generically, in all cases there are two half-lines along which there is a Neimark-Sacker bifurcation of limit cycles.

C. Predictors

In [1] a generalization of the Lindstedt-Poincaré method is used to approximate the homoclinic orbit emanating from a generic Bogdanov-Takens bifurcation. In the first Section we will shortly review the method and show the obtained results. In the second we will apply the same method to the transcritical Bogdanov-Takens bifurcation. However, in contrast to the generic Bogdanov-Takens bifurcation, there are two homoclinic orbits to approximate. One homoclinic solution to the trivial solution and another to the nontrivial equilibrium. These cases need to be treated separately. In the remaining sections we list known asymptotics for codimension 1 nonhyperbolic cycles emanating from generalized Hopf, fold-Hopf and Hopf-Hopf bifurcations obtained in [37]. Following the same method as in [37] we also derive asymptotics for codimension 1 nonhyperbolic cycles emanating from the Hopf-transcritical bifurcation.

C.1. Generic Bogdanov-Takens bifurcation

The smooth normal form for the restriction of a generic system (3.3) to its parameter-dependent two-dimensional center manifold near the Bogdanov-Takens bifurcation is

$$\begin{aligned} \dot{w} &= G_1(w,\beta) \\ &= \begin{pmatrix} w_1 \\ \beta_1 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 + g_1(w,\beta) \end{pmatrix} \\ &\quad + \mathcal{O}(|\beta_1| \|w\|^2 + |\beta_2| w_1^2) + \mathcal{O}(\|\beta\|^2 \|w\|^2 + \|\beta\| \|w\|^3 + \|w\|^4), \end{aligned} \tag{C.1}$$

where

$$g_1(w,\beta) = (a_1\beta_2 + d w_0)\, w_0^2 + (b_1 \beta_2 + e w_0)\, w_1 w_0.$$

To approximate homoclinic solution we take the normal form (C.1) and apply the singular rescaling

$$\beta_1 = -\frac{4}{a}\varepsilon^4, \qquad \beta_2 = \frac{b}{a}\varepsilon^2 \tau,$$
$$w_0 = \frac{\varepsilon^2}{a} u, \qquad w_1 = \frac{\varepsilon^3}{a} v, \qquad \varepsilon t = s.$$

This gives

$$\begin{cases} \dot{u} = & v, \\ \dot{v} = & -4 + u^2 + \varepsilon \dfrac{b}{a} v(u+\tau) + \varepsilon^2 \frac{1}{a^2} u^2 \,(a_1 b \tau + d u) \\ & + \varepsilon^3 \frac{1}{a^2} u v \,(b b_1 \tau + e u) + \mathcal{O}(\varepsilon^4), \end{cases} \tag{C.2}$$

or

$$\ddot{u} - u^2 + 4 = \varepsilon \frac{b}{a} \dot{u}(u+\tau) + \varepsilon^2 \frac{1}{a^2} u^2 \left(a_1 b \tau + du\right) + \varepsilon^3 \frac{1}{a^2} u\dot{u} \left(bb_1\tau + eu\right) + \mathcal{O}(\varepsilon^4). \quad \text{(C.3)}$$

The dot now indicates the derivative with respect to s. For $\varepsilon = 0$, (C.2) is a Hamiltonian system with the first integral

$$L(u,v) = \frac{1}{2}v^2 + 4u - \frac{1}{3}u^3 = h. \quad \text{(C.4)}$$

The Hamiltonian system has a well-known explicit homoclinic solution $(u_0(s), v_0(s))$ given by

$$\begin{cases} u_0(s) & = 2(1 - 3\,\mathrm{sech}^2(s)), \\ v_0(s) & = 12\,\mathrm{sech}^2(s)\tanh(s). \end{cases}$$

This solution defines a homoclinic orbit to the saddle $(2,0)$.

Introduce the non-linear transformation of time,

$$\frac{d\zeta}{ds} = \omega(\zeta), \quad \text{(C.5)}$$

where $\omega(\zeta)$ is a bounded function for all ζ. The new parameterization of time transforms (C.3) into

$$\omega \frac{d}{d\zeta}\left(\omega\dot{u}\right) - u^2 + 4 = \varepsilon \frac{b}{a}\omega\dot{u}(u+\tau) + \varepsilon^2 \frac{1}{a^2}u^2\left(a_1 b\tau + du\right) + \varepsilon^3 \frac{1}{a^2} u\omega\dot{u}\left(bb_1\tau + eu\right) + \mathcal{O}(\varepsilon^4). \quad \text{(C.6)}$$

The homoclinic solutions of (C.3) can be parameterized by ε and approximate by

$$\begin{pmatrix} u(\zeta) \\ v(\zeta) \\ \omega(\zeta) \\ \tau \end{pmatrix} = \begin{pmatrix} u_0(\zeta) \\ v_0(\zeta) \\ \omega_0(\zeta) \\ \tau_0 \end{pmatrix} + \varepsilon \begin{pmatrix} u_1(\zeta) \\ v_1(\zeta) \\ \omega_1(\zeta) \\ \tau_1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} u_2(\zeta) \\ v_2(\zeta) \\ \omega_2(\zeta) \\ \tau_2 \end{pmatrix} + \varepsilon^3 \begin{pmatrix} u_3(\zeta) \\ v_3(\zeta) \\ \omega_3(\zeta) \\ \tau_3 \end{pmatrix} + \mathcal{O}(\varepsilon^4). \quad \text{(C.7)}$$

By substituting the series expansion (C.7) into equation (C.6) and then successively collecting the terms with equal power in ε, the unknowns $u_i, v_i, \omega_i(\zeta)$ for $i = 0,1,2,3$ and τ_i for $i = 0,1,2$ have been solved. One obtains the following second-order correction

$$\begin{cases} \tau_0 = & \frac{10}{7}, \qquad \tau_1 = 0, \qquad \tau_2 = \frac{1}{a}\left(\frac{100}{49}b_1 - 4\frac{e}{b}\right) + \frac{1}{a^2}\left(\frac{288}{2401}b^2 - \frac{50ba_1}{49} + \frac{146}{49}d\right), \\ u_0(\zeta) = & -6\,\mathrm{sech}^2(\zeta) + 2, \\ v_0(\zeta) = & 12\,\mathrm{sech}^2(\zeta)\tanh(\zeta), \\ u_1(\zeta) = & 0, \\ v_1(\zeta) = & -\frac{6b}{7a}\tanh(\zeta)v_0(\zeta), \\ u_2(\zeta) = & \frac{1}{49a^2}(210a_1 b - 18b^2 - 147d)\,\mathrm{sech}^2(\zeta) - \frac{2}{7}\frac{5a_1b + 7d}{a^2}, \\ v_2(\zeta) = & \left(-\frac{5a_1b}{14a^2} + \frac{3b^2}{7a^2} + \frac{3d}{2a^2} - \left(\frac{27b^2}{98a^2} + \frac{9d}{4a^2}\right)\mathrm{sech}^2(\zeta)\right)v_0(\zeta). \end{cases}$$

We omit the third order approximation to the homoclinic orbit, which has also been derived. The second-order approximation to the homoclinic orbit of the smooth Bogdanov-Takens normal form system is given by

$$\begin{cases} w_0(t) = & \frac{\varepsilon^2}{a}\left(\sum_{i=0}^{2}\varepsilon^i u_i(\zeta)\right) + \mathcal{O}(\varepsilon^5), \\ w_1(t) = & \frac{\varepsilon^3}{a}\left(\sum_{i=0}^{2}\varepsilon^i v(\zeta)\right) + \mathcal{O}(\varepsilon^6), \\ \beta_1 = & -\frac{4}{a}\varepsilon^4, \\ \beta_2 = & \frac{b}{a}\varepsilon^2\left(\tau_0 + \varepsilon^2\tau_2\right) + \mathcal{O}(\varepsilon^5). \end{cases} \tag{C.8}$$

Substituting (C.8) into equations (6.2) and (6.3) gives the second-order homoclinic predictors

$$\begin{aligned} \alpha &= \frac{10}{7}\frac{b}{a}K_{01}\varepsilon^2 + \left(\frac{b}{a}\tau_2 K_{01} + \frac{50b^2}{49a^2}K_{02} - \frac{4}{a}K_{10}\right)\varepsilon^4, \\ x &= \varepsilon^2\left(\frac{10b}{7a}H_{0001} + \frac{1}{a}u_0(\zeta)\phi_0\right) + \varepsilon^3\left(\frac{1}{a}v_0(\zeta)\phi_1 + \frac{1}{a}u_1(\zeta)\phi_0\right) \\ &\quad + \varepsilon^4\Bigg(- \frac{4}{a}H_{0010} + \frac{50b^2}{49a^2}H_{0002} + \frac{b}{a}\tau_2 H_{0001} \\ &\quad + \frac{1}{a}u_2(\zeta)\phi_0 + \frac{1}{a}v_1(\zeta)\phi_1 + \frac{1}{2a^2}H_{2000}u_0^2(\zeta) + \frac{10b}{7a^2}H_{1001}u_0(\zeta)\Bigg) + \mathcal{O}(\varepsilon^5), \end{aligned} \tag{C.9}$$

for the original system (3.3).

C.2. Transcritical Bogdanov-Takens bifurcation

Suppose that a smooth autonomous 2D ODE has an equilibrium with a double (but not semi-simple) zero eigenvalue. It is well known (see, e.g. [22]) that such system is C^∞-equivalent near this equilibrium to the critical normal form

$$\dot{w} = \begin{pmatrix} w_1 \\ aw_0^2 + bw_0w_1 \end{pmatrix} + \mathcal{O}(\|w\|^3), \tag{C.10}$$

where $w = (w_0, w_1)$ and the w_0-component of the $\mathcal{O}(\|w\|^3)$-term is identically zero. We assume that $ab \neq 0$, i.e. that we consider a non-degenerate (codim 2) BT singularity. Any generic smooth two-parameter perturbation of (C.10), with the equilibrium at the origin is hold fixed, is topologically equivalent near the origin to the system

$$\dot{w} = \begin{pmatrix} w_1 \\ \beta_1 w_0 + \beta_2 w_1 + aw_0^2 + bw_0w_1 \end{pmatrix} + \mathcal{O}(\|w\|^3),$$

where $\beta = (\beta_1, \beta_2)$ are the unfolding parameters [27]. To accurately approximate homoclinic solutions in the two-parameter perturbation of (C.10), one has to consider the smooth normal form

$$\begin{aligned}\dot{w} &= G(w,\beta) \\ &= \begin{pmatrix} w_1 \\ \beta_1 w_0 + \beta_2 w_1 + a w_0^2 + b w_0 w_1 + g(w,\beta) \end{pmatrix} \\ &\quad + \mathcal{O}(\|\beta\|\, w_1^2) + \mathcal{O}(\|\beta\|^2 \|w\|^2 + \|\beta\| \|w\|^3 + \|w\|^4),\end{aligned} \tag{C.11}$$

where

$$g(w,\beta) = (a_1\beta_2 + a_2\beta_1 + d w_0)\, w_0^2 + (b_1\beta_2 + b_2\beta_1 + e w_0)\, w_1 w_0.$$

The next step is to study homoclinic solutions of the normal form (C.11). There are two homoclinic orbits to consider, see (B.2). One, in which the trivial equilibrium is the saddle to the homoclinic orbit, and the second, in which the nontrivial equilibrium, which coincides with the trivial equilibrium for $\beta = 0$, is the saddle to the homoclinic orbit. These two cases can be distinguished by the sign of β_1, since the coefficients in $g(w,\beta)$ do not effect the linear approximation to the homoclinic orbits.

C.2.1. Homoclinic to the trivial solution

To approximate homoclinic to the trivial solution we take the normal form (C.11) and apply the singular rescaling

$$\begin{gathered}\beta_1 = \varepsilon^2, \qquad \beta_2 = \frac{b}{a}\varepsilon^2\tau, \\ w_0 = \frac{\varepsilon^2}{a}u, \qquad w_1 = \frac{\varepsilon^3}{a}v, \qquad \varepsilon t = s.\end{gathered}$$

This gives

$$\begin{cases}\dot{u} = & v, \\ \dot{v} = & u\,(1+u) + \varepsilon\frac{b}{a}v(u+\tau) + \varepsilon^2\frac{1}{a^2}u^2\,(a_1 b\tau + a a_2 + du) \\ & + \varepsilon^3\frac{1}{a^2}uv\,(ab_2 + bb_1\tau + eu) + \mathcal{O}(\varepsilon^4),\end{cases} \tag{C.12}$$

or

$$\begin{aligned}\ddot{u} - u\,(1+u) =& \varepsilon\frac{b}{a}\dot{u}(u+\tau) + \varepsilon^2\frac{1}{a^2}u^2\,(a_1 b\tau + a a_2 + du) \\ &+ \varepsilon^3\frac{1}{a^2}u\dot{u}\,(ab_2 + bb_1\tau + eu) + \mathcal{O}(\varepsilon^4),\end{aligned} \tag{C.13}$$

where $0 < \varepsilon \ll 1$ and τ are the new parameters. The dot now indicates the derivative with respect to s. For $\varepsilon = 0$, (C.13) is a Hamiltonian system with the first integral

$$L(u,v) = \frac{1}{2}\left(v^2 - u^2\right) - \frac{1}{3}u^3 = h. \tag{C.14}$$

Every closed orbit of (C.14) surrounding $(-1,0)$ corresponds to a level curve

$$\Gamma_h = \left\{(u,v) : L(u,v) = h, -\frac{1}{6} < h < 0\right\}.$$

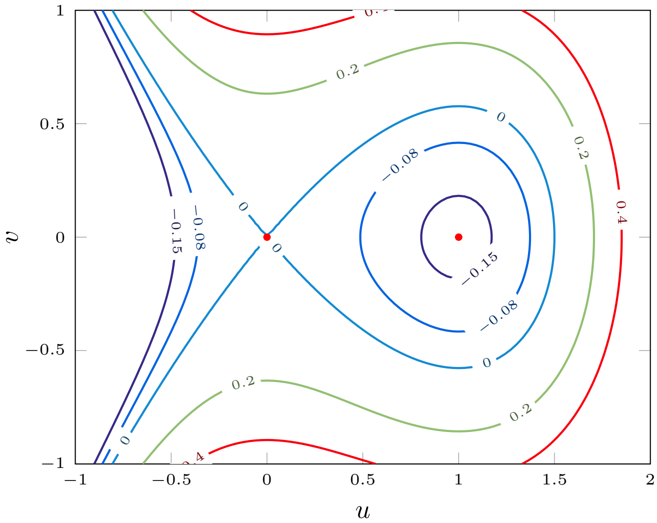


Figure C.1.: Contourplot of the Hamiltonian L in (C.14).

Γ_h shrinks to the equilibrium $(-1, 0)$ as $h \to -\dfrac{1}{6}$ and tends to a homoclinic orbit as $h \to 0$, see Figure C.1.

The Hamiltonian system L has a homoclinic orbit

$$\begin{pmatrix} u_0(s) \\ v_0(s) \end{pmatrix} = \frac{3}{2} \begin{pmatrix} -\operatorname{sech}^2(s/2) \\ \operatorname{sech}^2(s/2)\tanh(s/2) \end{pmatrix}. \tag{C.15}$$

to the saddle at the origin. Indeed, substituting homoclinic orbit (C.15) into the Hamiltonian L yields

$$\begin{aligned} L(u_0, v_0) &= \frac{9}{8}\operatorname{sech}^4(s/2)\tanh^2(s/2) - \frac{9}{8}\operatorname{sech}^4(s/2) + \frac{9}{8}\operatorname{sech}^6(s/2) \\ &= \frac{9}{8}\operatorname{sech}^4(s/2)\left(1 - \operatorname{sech}^2(s/2)\right) - \frac{9}{8}\operatorname{sech}^4(s/2) + \frac{9}{8}\operatorname{sech}^6(s/2). \\ &= 0. \end{aligned}$$

Furthermore,

$$\begin{cases} (u_0(\pm\infty), v_0(\pm\infty) = (0,0))\,, \\ (u_0(0), v_0(0)) = \left(-\frac{3}{2}, 0\right), \end{cases}$$

shows that the solution is indeed a homoclinic solution. Our next aim is to prove existence of homoclinic orbits for the perturbed system (C.12) for $\varepsilon \neq 0$, as done in [3] for the general case. We introduce the Banach spaces

$$X_0 = \left\{ z \in C(\mathbb{R}, \mathbb{R}^2) : \lim_{t\to\infty} z(t) \text{ and } \lim_{t\to-\infty} z(t) \text{ exists} \right\}$$

with norm

$$\|z\|_0 = \sup\left\{\|z(t)\| : t \in \mathbb{R}\right\}, \qquad \|\cdot\| \text{ some norm in } \mathbb{R}^m$$

and

$$X_1 = \left\{ z \in C^1(\mathbb{R}, \mathbb{R}^2) : z, \dot{z} \in X_0 \right\}, \qquad \|z\|_1 = \|z\|_0 + \|\dot{z}\|_0.$$

With $h = (z, \varepsilon) = (u, v, \varepsilon) \in X_0 \times \mathbb{R}$ let us write (C.12) as

$$F(h, \tau) = 0 \tag{C.16}$$

where $F : X_1 \times \mathbb{R} \times \mathbb{R} \to X_0 \times \mathbb{R}$ is defined by

$$F(h, \tau) = \begin{pmatrix} \dot{z} - g(z, \varepsilon, \tau) \\ v(0) \end{pmatrix},$$

with

$$g(z, \varepsilon, \tau) = \begin{pmatrix} v \\ u\,(1+u) \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ \frac{b}{a} v(u+\tau) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 \\ \frac{1}{a^2} u^2\,(a_1 b\tau + a a_2 + du) \end{pmatrix}$$
$$+ \varepsilon^3 \begin{pmatrix} 0 \\ \frac{1}{a^2} u\dot{u}\,(ab_2 + bb_1\tau + eu) \end{pmatrix} + \mathcal{O}(\varepsilon^4).$$

The condition $\dot{u}(0) = v(0) = 0$ is used to fix the phase of the homoclinic orbits.

Setting $h_0 = (z_0, 0) = (u_0, v_0, 0)$ we find

$$F(h_0, \tau) = 0 \qquad \text{for all } \tau.$$

Hence we have a trivial branch (h_0, τ) of homoclinic orbits and we look for values of τ at which bifurcation occurs.

Theorem C.1. *Consider the two-parameter system (C.12) and assume that $ab \neq 0$. Then equation (C.16) has an unique simple bifurcation point (in the sense of [9]) at (h_0, τ_0) with*

$$\tau_0 = \frac{6}{7}.$$

The emanating C^1-branch can be parameterized by ε

$$(h(\varepsilon), \tau(\varepsilon)) = (z(\varepsilon), \varepsilon, \tau(\varepsilon)) \in X_1 \times \mathbb{R}^2. \tag{C.17}$$

It has tangent

$$\left(z'(0), 1, \tau'(0)\right) = \left(z_1, 1, \tau'(0)\right)$$

where $z_1 = (u_1, v_1)$ is the unique solution in X_1 of the linear system

$$\begin{cases} \dot{u} - v = 0 \\ \dot{v} - u\,(2u_0 + 1) \quad = \dfrac{b}{a} v_0(u_0 + \tau_0) \end{cases} \qquad \text{and } v(0) = 0. \tag{C.18}$$

Proof. Consider the linearization of F with respect to $h \in X_1 \times \mathbb{R}$ about the trivial solution of (h_0, τ)

$$F_w(h_0,\tau)w = \begin{pmatrix} Lz - g_\varepsilon(z_0,0,\tau)\varepsilon \\ v(0) \end{pmatrix}, \qquad w = (z,\varepsilon) = (u,v,\varepsilon) \in X_1 \times \mathbb{R}.$$

where $L : X_1 \to X_0$ is given by (cf, (C.12))

$$Lz = \begin{pmatrix} \dot{u} - v \\ \dot{v} - u(2u_0+1) \end{pmatrix}, \qquad z = (u,v) \tag{C.19}$$

We need a few facts from the Fredholm theory of linear differential operator $Lz = \dot{z} - A(t)z$ which have the property that $\lim_{t\to\infty} A(t) = \lim_{t\to-\infty} A(t)$ exists and has no eigenvalues on the imaginary axis (see [2, Lemma 2.2], and also [43, Lemma 4.2]; [24]).

(i). $L : X_1 \to X_0$ is Fredholm of index 0;

(ii). dim $N(L)$= dim $N(L^*)$ where $L^*z = \dot{z} + A(t)^T z$;

(iii). $z \in R(L) \iff \int_{-\infty}^{\infty} \Psi^T(t)z(t)\,dt = 0 \qquad \forall \Psi \in N(L^*)$.

For the special case (C.19) we have

$$N(L) = \operatorname{span}\{\dot{z_0}\}, \qquad N(L^*) = \operatorname{span}\{(-\dot{v}_0, \dot{u}_0)\}. \tag{C.20}$$

Using (i) and the bordering Lemma [2] we find that $F_h(h_0,\tau) : X_1 \times \mathbb{R} \to X_0 \times \mathbb{R}$ also has Fredholm index 0. Since $\dot{v}(0) \neq 0$ the only way that $F_h(h_0,\tau)$ can have nontrivial null space is the case

$$g_\varepsilon(z_0, 0, \tau) \in R(L).$$

By (iii), (C.20) and (C.15) this is equivalent to a vanishing Melnikov integral

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} (-\dot{v}_0, \dot{u}_0) g_\varepsilon(z_0,0,\tau)\,dt \\ &= \int_{-\infty}^{\infty} \frac{3\tanh\left(\frac{t}{2}\right)\operatorname{sech}^2\left(\frac{t}{2}\right)\left(\frac{3}{2}b\tau\tanh\left(\frac{t}{2}\right)\operatorname{sech}^2\left(\frac{t}{2}\right) - \frac{9}{4}b\tanh\left(\frac{t}{2}\right)\operatorname{sech}^4\left(\frac{t}{2}\right)\right)}{2a}\,dt \\ &= \frac{6b(7\tau-6)}{35a} \end{aligned} \tag{C.21}$$

This is satisfied at $\tau = \tau_0$ and we obtain

$$N\left(F_h(h_0,\tau_0)\right) = \operatorname{span}\{(z_1, 1)\}, \qquad R\left(F_h(h_0,\tau_0)\right) = R(L) \times \mathbb{R} \tag{C.22}$$

where z_1 is the unique solution of (C.18).

The final condition for bifurcation from (h_0, τ_0) is (cf. [9])

$$F_{w\tau}(h_0,\tau_0)\begin{pmatrix} z_1 \\ 1 \end{pmatrix} \notin R\left(F_h(h_0,\tau_0)\right).$$

Using (C.22) and once more (iii) and (C.20) this turns out to be equivalent to

$$0 \neq \int_{-\infty}^{\infty} (-\dot{v}_0, \dot{u}_0) g_{\varepsilon\tau}(z_0, 0, \tau)\, dt = \int_{-\infty}^{\infty} \frac{9b \tanh^2\left(\frac{t}{2}\right) \operatorname{sech}^4\left(\frac{t}{2}\right)}{4a}\, dt = \frac{3b}{5a}$$

which is true by assumption. □

Actually, the homoclinic branch is as smooth in ε as (C.12). Introduce the non-linear transformation of time,

$$\frac{d\zeta}{ds} = \omega(\zeta), \tag{C.23}$$

where $\omega(\zeta)$ is a bounded function for all ζ. Since $\omega(\zeta)$ also depends on ε, we can expand $\omega(\zeta)$ in a power series of ε:

$$\omega(\zeta) = \omega_0(\zeta) + \varepsilon\omega_1(\zeta) + \varepsilon^2\omega_2(\zeta) + \varepsilon^3\omega_3(\zeta) + \dots, \tag{C.24}$$

Using that

$$\frac{d}{ds}u = \frac{d\zeta}{ds}\frac{d}{d\zeta}u = \omega(\zeta)\frac{d}{d\zeta}u = \omega(\zeta)\hat{u}'$$

and

$$\frac{d^2}{ds^2}u = \omega(\zeta)\frac{d}{d\zeta}\left(\omega(\zeta)\hat{u}'\right),$$

where the prime denotes the derivative of $\hat{u}$ with respect to the new independent variable ζ, the new parameterization of time transforms (C.13) into

$$\begin{aligned} \omega\frac{d}{d\zeta}\left(\omega\hat{u}'\right) - \hat{u}\left(1+\hat{u}\right) =& \varepsilon\frac{b}{a}\omega\hat{u}'(\hat{u}+\tau) + \varepsilon^2\frac{1}{a^2}\hat{u}^2\left(a_1 b\tau + aa_2 + d\hat{u}\right) \\ &+ \varepsilon^3\frac{1}{a^2}\omega\hat{u}'\left(ab_2 + bb_1\tau + e\hat{u}\right)\hat{u} + \mathcal{O}(\varepsilon^4). \end{aligned} \tag{C.25}$$

We approximate the branch (C.17) parameterized by ε

$$\begin{pmatrix} \hat{u}(\zeta) \\ \hat{v}(\zeta) \\ \omega(\zeta) \\ \tau \end{pmatrix} = \begin{pmatrix} \hat{u}_0(\zeta) \\ \hat{v}_0(\zeta) \\ \omega_0(\zeta) \\ \tau_0 \end{pmatrix} + \varepsilon \begin{pmatrix} \hat{u}_1(\zeta) \\ \hat{v}_1(\zeta) \\ \omega_1(\zeta) \\ \tau_1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \hat{u}_2(\zeta) \\ \hat{v}_2(\zeta) \\ \omega_2(\zeta) \\ \tau_2 \end{pmatrix} + \varepsilon^3 \begin{pmatrix} \hat{u}_3(\zeta) \\ \hat{v}_3(\zeta) \\ \omega_3(\zeta) \\ \tau_3 \end{pmatrix} + \mathcal{O}(\varepsilon^4). \tag{C.26}$$

Substituting (C.26) into (C.25) and collecting the terms of order ε^0 gives the system

$$\omega_0\frac{d}{d\zeta}\left(\omega_0\hat{u}_0'\right) - (1+\hat{u}_0)\hat{u}_0 = 0.$$

Then, for $\omega_0 = 1$ we obtain

$$\hat{u}_0'' - (1+\hat{u}_0)\hat{u}_0 = 0, \tag{C.27}$$

which is equivalent to (C.13) with $\epsilon = 0$ and thus has the exact homoclinic solution

$$\begin{pmatrix} u_0(s) \\ v_0(s) \end{pmatrix} = \frac{3}{2} \begin{pmatrix} -\operatorname{sech}^2(s/2) \\ \operatorname{sech}^2(s/2)\tanh(s/2) \end{pmatrix}. \tag{C.28}$$

Substituting $\omega_0 = 1$ and (C.26) into (C.25) and collecting the terms of order ε^1, ε^2 and ε^3 gives the systems:

$$\text{Order}(\varepsilon^1): \frac{d}{d\zeta}(\omega_1 \hat{u}_0') + \omega_1 \hat{u}_0'' + \hat{u}_1'' - (2\hat{u}_0 + 1)\,\hat{u}_1 = \frac{b}{a}\hat{u}_0'\,(\hat{u}_0 + \tau_0)\,, \tag{C.29}$$

$$\begin{aligned}\text{Order}(\varepsilon^2): &\frac{d}{d\zeta}\left(\omega_1 \hat{u}_1'\right) + \omega_1 \hat{u}_1'' + \frac{d}{d\zeta}\left(\omega_2 \hat{u}_0'\right) + \omega_2 \hat{u}_0'' \\ &+ \hat{u}_2'' - (2\hat{u}_0 + 1)\,\hat{u}_2 + \omega_1 \frac{d}{d\zeta}\left(\omega_1 \hat{u}_0'\right) - \hat{u}_1^2 \\ &= \frac{1}{a^2}\hat{u}_0^2\,(a_1 b\tau_0 + aa_2 + d\hat{u}_0) \\ &\quad + \frac{b}{a}\left(\hat{u}_0'(\hat{u}_1 + \tau_1) + \hat{u}_1'(\hat{u}_0 + \tau_0) + \omega_1 \hat{u}_0'(\hat{u}_0 + \tau_0)\right),\end{aligned} \tag{C.30}$$

$$\begin{aligned}\text{Order}(\varepsilon^3): &\frac{d}{d\zeta}\left(\omega_1 \hat{u}_2'\right) + \omega_1 \hat{u}_2'' + \frac{d}{d\zeta}\left(\omega_3 \hat{u}_0'\right) + \omega_3 \hat{u}_0'' \\ &+ \frac{d}{d\zeta}\left(\omega_2 \hat{u}_1'\right) + \omega_2 \hat{u}_1'' + \hat{u}_3'' - (2\hat{u}_0 + 1)\hat{u}_3 \\ &+ \omega_1 \frac{d}{d\zeta}\left(\omega_1 \hat{u}_1'\right) + \omega_1 \frac{d}{d\zeta}\left(\omega_2 \hat{u}_0'\right) + \omega_2 \frac{d}{d\zeta}\left(\omega_1 \hat{u}_0'\right) - 2\hat{u}_1 \hat{u}_2 \\ &= \frac{1}{a^2}\hat{u}_0'\,(ab_2 + bb_1\tau_0 + e\hat{u}_0)\,\hat{u}_0 \\ &+ \frac{b}{a}\Big(\hat{u}_0'(\hat{u}_2 + \tau_2) + \hat{u}_1'(\hat{u}_1 + \tau_1) + \hat{u}_2'(\hat{u}_0 + \tau_0) \\ &+ \omega_1\left[\hat{u}_0'(\hat{u}_1 + \tau_1) + \hat{u}_1'(\hat{u}_0 + \tau_0)\right] + \omega_2\left[\hat{u}_0'(\hat{u}_0 + \tau_0)\right]\Big) \\ &+ \frac{1}{a^2}\left(\hat{u}_0^2(a_1 b\tau_1 + d\hat{u}_0) + 2\hat{u}_0\hat{u}_1'(a_1 b\tau_0 + aa_2 + d\hat{u}_0)\right).\end{aligned} \tag{C.31}$$

We assume that for $\varepsilon \neq 0$ the homoclinic orbit of (C.25) is still given by

$$\begin{cases} \hat{u}(\zeta) &= \sigma\operatorname{sech}^2(\frac{\zeta}{2}), \\ \hat{v}(\zeta) &= \hat{u}'(\zeta) = -\sigma\omega(\zeta)\operatorname{sech}^2(\frac{\zeta}{2})\tanh(\frac{\zeta}{2}), \end{cases} \tag{C.32}$$

where σ is a parameter that depends on ε,

$$\sigma = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \varepsilon^3\sigma_3 + \dots. \tag{C.33}$$

For which it follows that $\sigma_0 = -\frac{3}{2}$ and

$$\hat{u}_i(\zeta) = \sigma_i \text{sech}^2(\frac{\zeta}{2}), \qquad i = 1, 2, 3, \tag{C.34}$$

$$\hat{v}_1(\zeta) = \left(\frac{3}{2}\omega_1 - \sigma_1\right) \text{sech}^2(\frac{\zeta}{2})\tanh(\frac{\zeta}{2}), \tag{C.35}$$

$$\hat{v}_2(\zeta) = \left(\frac{3}{2}\omega_2 - \sigma_1\omega_1 - \sigma_2\right) \text{sech}^2(\frac{\zeta}{2})\tanh(\frac{\zeta}{2}),$$

$$\hat{v}_3(\zeta) = \left(\frac{3}{2}\omega_3 - \sigma_1\omega_2 - \sigma_2\omega_1 - \sigma_3\right) \text{sech}^2(\frac{\zeta}{2})\tanh(\frac{\zeta}{2}). \tag{C.36}$$

Using assumptions (C.34)-(C.36) we solve the linear equations (C.29)-(C.31) for $\hat{u}(\zeta)$ one by one to determine τ_{i-1}, σ_i and $\omega_i(\zeta)$ for $i = 1, 2, 3$.

We multiply both sides of (C.29) with $\hat{u}_0'$ and integrate both sides from ζ_0 to ζ, and get

$$\int_{\zeta_0}^{\zeta} \hat{u}_0' \frac{d}{d\zeta}(\omega_1 \hat{u}_0')\, dx + \int_{\zeta_0}^{\zeta} \hat{u}_0' \omega_1 \hat{u}_0''\, dx \tag{C.37}$$

$$+ \int_{\zeta_0}^{\zeta} \hat{u}_0' \hat{u}_1''\, dx - \int_{\zeta_0}^{\zeta} \hat{u}_0' \left(2\hat{u}_0 + 1\right) \hat{u}_1\, dx \tag{C.38}$$

$$= \frac{b}{a} \int_{\zeta_0}^{\zeta} \left(\hat{u}_0'\right)^2 \left(\hat{u}_0 + \tau_0\right)\, dx$$

Differentiating (C.27) with respect to ζ yields

$$\hat{u}_0''' = \hat{u}_0'(1 + 2\hat{u}_0),$$

so that

$$\int_{\zeta_0}^{\zeta} \hat{u}_1 \hat{u}_0''' dx = \int_{\zeta_0}^{\zeta} \hat{u}_1 \hat{u}_0'(1 + 2\hat{u}_0) dx. \tag{C.39}$$

Using this expression and integration by parts we can simplify (C.37) and (C.38) to become

$$\begin{aligned}\int_{\zeta_0}^{\zeta} \hat{u}_0' \frac{d}{d\zeta}(\omega_1 \hat{u}_0')\, dx + \int_{\zeta_0}^{\zeta} \hat{u}_0' \omega_1 \hat{u}_0''\, dx &= \omega_1 \left(\hat{u}_0'\right)^2 \Big|_{\zeta_0}^{\zeta} - \int_{\zeta_0}^{\zeta} \hat{u}_0'' \omega_1 \hat{u}_0'\, dx \\ &\quad + \int_{\zeta_0}^{\zeta} \hat{u}_0' \omega_1 \hat{u}_0''\, dx \\ &= \omega_1 \left(\hat{u}_0'\right)^2 \Big|_{\zeta_0}^{\zeta}, \\ \int_{\zeta_0}^{\zeta} \hat{u}_0' \hat{u}_1''\, dx - \int_{\zeta_0}^{\zeta} \hat{u}_0' \left(2\hat{u}_0 + 1\right) \hat{u}_1\, dx &= \int_{\zeta_0}^{\zeta} \hat{u}_0' \hat{u}_1''\, dx - \int_{\zeta_0}^{\zeta} \hat{u}_1 \hat{u}_0''' dx \\ &= \left(\hat{u}_0' \hat{u}_1' - \hat{u}_1 \hat{u}_0''\right)\Big|_{\zeta_0}^{\zeta} - \int_{\zeta_0}^{\zeta} \hat{u}_0'' \hat{u}_1'\, dx \\ &\quad - \int \hat{u}_0'' \hat{u}_1'\, dx \\ &= \left(\hat{u}_0' \hat{u}_1' - \hat{u}_1 \hat{u}_0''\right)\Big|_{\zeta_0}^{\zeta}.\end{aligned}$$

Therefore, (C.29) can be written as

$$\omega_1 \left(\hat{u}_0'\right)^2 \Big|_{\zeta_0}^{\zeta} + \left(\hat{u}_0' \hat{u}_1' - \hat{u}_1 \hat{u}_0''\right)\Big|_{\zeta_0}^{\zeta} = \frac{b}{a} \int_{\zeta_0}^{\zeta} \left(\hat{u}_0'\right)^2 \left(\hat{u}_0 + \tau_0\right)\, dx. \tag{C.40}$$

Setting $\zeta_0 = -\infty$ or $\zeta = \infty$ gives

$$\begin{aligned}0 &= \frac{b}{a} \int_{\zeta_0}^{\zeta} \left(\hat{u}_0'\right)^2 \left(\hat{u}_0 + \tau_0\right)\, dx \\ &= \frac{6}{35} \frac{b}{a} \left(7\tau_0 - 6\right),\end{aligned}$$

from which we recover (C.21) and thus once more that $\tau_0 = \frac{6}{7}$. Taking the integration boundaries $\zeta_0 = 0$ and $\zeta = \infty$ in (C.40) we obtain

$$\frac{3\sigma_1}{4} = 0,$$

we which it follows that $\sigma_1 = 0$. To obtain ω_1 we set the integration boundaries in (C.40) from $\zeta_0 = 0$ to ζ, which yields

$$\begin{aligned}\frac{9}{4} \tanh^2 \left(\frac{\zeta}{2}\right) \operatorname{sech}^4 \left(\frac{\zeta}{2}\right) w_1(\zeta) = \frac{3}{280} &(6(10\cosh(\zeta) + \cosh(2\zeta) + 9) \\ &- 6(10\cosh(\zeta) \\ &+ \cosh(2\zeta) + 24)) \tanh^3 \left(\frac{\zeta}{2}\right) \operatorname{sech}^4 \left(\frac{\zeta}{2}\right).\end{aligned}$$

After simplifying we obtain

$$\omega_1(\zeta) = -\frac{b}{a}\frac{3}{7}\tanh\left(\frac{\zeta}{2}\right).$$

Substituting σ_1 and ω_1 into (C.34) and (C.35) gives the first-order correction to the initial homoclinic solution $(\hat{u}_0, \hat{v}_0)$

$$\begin{cases} \hat{u}_1(\zeta) & = 0, \\ \hat{v}_1(\zeta) & = -\frac{9}{14}\mathrm{sech}^2(\frac{\zeta}{2})\tanh^2(\frac{\zeta}{2}). \end{cases}$$

We can apply to same procedure to equation (C.30). First we multiply both sides with $\hat{u}_0'$

$$\begin{aligned} \mathrm{Order}(\varepsilon^2) :& \hat{u}_0' \frac{d}{d\zeta}\left(\omega_2 \hat{u}_0'\right) + \hat{u}_0'\omega_2\hat{u}_0'' + \hat{u}_0'\hat{u}_2'' - \hat{u}_0'\left(2\hat{u}_0 + 1\right)\hat{u}_2 + \hat{u}_0'\omega_1\frac{d}{d\zeta}\left(\omega_1\hat{u}_0'\right) \\ &= \hat{u}_0'\left[\frac{1}{a^2}\hat{u}_0^2\left(a_1 b\tau_0 + aa_2 + d\hat{u}_0\right) + \frac{b}{a}\hat{u}_0'\left(\tau_1 + \omega_1(\hat{u}_0 + \tau_0)\right)\right] \end{aligned}$$

Using the identity

$$\int_{\zeta_0}^{\zeta} \hat{u}_0'\omega_1\frac{d}{d\zeta}\left(\omega_1\hat{u}_0'\right)dx = \frac{1}{2}\left(\omega_1\hat{u}_0'\right)^2 = \frac{81}{392}\tanh^4\left(\frac{\zeta}{2}\right)\mathrm{sech}^4\left(\frac{\zeta}{2}\right)\Big|_{\zeta_0}^{\zeta}$$

we obtain the equality

$$\begin{aligned} &\omega_2\left(\hat{u}_0'\right)^2\Big|_{\zeta_0}^{\zeta} + \left(\hat{u}_0'\hat{u}_2' - \hat{u}_2\hat{u}_0''\right)\Big|_{\zeta_0}^{\zeta} + \tfrac{81}{392}\tanh^4\left(\tfrac{\zeta}{2}\right)\mathrm{sech}^4\left(\tfrac{\zeta}{2}\right)\Big|_{\zeta_0}^{\zeta} \\ = &\int_{\zeta_0}^{\zeta}\hat{u}_0'\left[\tfrac{1}{a^2}\hat{u}_0^2\left(a_1 b\tau_0 + aa_2 + d\hat{u}_0\right) + \frac{b}{a}\hat{u}_0'\left(\tau_1 + \omega_1(\hat{u}_0 + \tau_0)\right)\right]dx. \end{aligned}$$

To this equation we repeat the last procedure of changing the integration variables and obtain

$$\begin{cases} \tau_1 & = 0, \\ \sigma_2 & = \frac{3(6ab - 336a_1 b - 392aa_2 + 441d)}{784a^2}, \\ \omega_2(\zeta) & = -\frac{9\mathrm{sech}^2\left(\frac{\zeta}{2}\right)\left(-4a^2 + 4a(a-2b)\cosh(\zeta) + 6ab + 49d\right)}{784a^2}, \end{cases}$$

with

$$\begin{cases} \hat{u}_2(\zeta) = & -\frac{3\mathrm{sech}^2\left(\frac{\zeta}{2}\right)(6ab - 336a_1 b - 392aa_2 + 441d)}{784a^2}, \\ \hat{v}_2(\zeta) = & -\frac{3\tanh\left(\frac{\zeta}{2}\right)\mathrm{sech}^4\left(\frac{\zeta}{2}\right)}{1568a^2}\Big(\cosh(\zeta)\left(36a^2 - 78ab - 441d\right) + 336a_1 b(\cosh(\zeta) + 1) \\ & +12a(4b - 3a) + 392aa_2(\cosh(\zeta) + 1)\Big). \end{cases}$$

Lastly, we turn our attention to the third order terms. Multiplying both sides of equation (C.31) with $\hat{u}_0'$ yields

$$\begin{aligned}
&\hat{u}_0'\frac{d}{d\zeta}\left(\omega_1\hat{u}_2'\right)+\hat{u}_0'\omega_1\hat{u}_2''+\hat{u}_0'\frac{d}{d\zeta}\left(\omega_3\hat{u}_0'\right)+\hat{u}_0'\omega_3\hat{u}_0''\\
&+\hat{u}_0'\hat{u}_3''-\hat{u}_0'(2\hat{u}_0+1)\hat{u}_3+\hat{u}_0'\omega_1\frac{d}{d\zeta}\left(\omega_2\hat{u}_0'\right)+\hat{u}_0'\omega_2\frac{d}{d\zeta}\left(\omega_1\hat{u}_0'\right)\\
&=\hat{u}_0'\left[\frac{1}{a^2}\hat{u}_0'\left(ab_2+bb_1\tau_0+e\hat{u}_0\right)\hat{u}_0+\frac{b}{a}\Big(\hat{u}_0'(\hat{u}_2+\tau_2)+\hat{u}_2'(\hat{u}_0+\tau_0)\right.\\
&\left.+\omega_1\hat{u}_0'\tau_1+\omega_2\left[\hat{u}_0'(\hat{u}_0+\tau_0)\right]\Big)+\frac{1}{a^2}\left(\hat{u}_0^2(a_1b\tau_1+d\hat{u}_0)\right)\right].
\end{aligned}$$

Integrating both sides from ξ_0 to ξ and simplifying yields

$$\begin{aligned}
&\omega_1\left(\hat{u}_2'\right)^2\Big|_{\zeta_0}^{\zeta}+\omega_3\left(\hat{u}_0'\right)^2\Big|_{\zeta_0}^{\zeta}+\left(\hat{u}_0'\hat{u}_3'-\hat{u}_3\hat{u}_0''\right)\Big|_{\zeta_0}^{\zeta}\\
&+\frac{243\tanh^3\left(\frac{\zeta}{2}\right)\operatorname{sech}^6\left(\frac{\zeta}{2}\right)\left(-4a^2+4a(a-2b)\cosh(\zeta)+6ab+49d\right)}{21952a^2}=\\
&=\int_{\zeta_0}^{\zeta}\hat{u}_0'\left[\frac{1}{a^2}\hat{u}_0'\left(ab_2+bb_1\tau_0+e\hat{u}_0\right)\hat{u}_0+\frac{b}{a}\Big(\hat{u}_0'(\hat{u}_2+\tau_2)+\hat{u}_2'(\hat{u}_0+\tau_0)\right.\\
&\left.+\omega_1\hat{u}_0'\tau_1+\omega_2\left[\hat{u}_0'(\hat{u}_0+\tau_0)\right]\Big)+\frac{1}{a^2}\left(\hat{u}_0^2(a_1b\tau_1+d\hat{u}_0)\right)\right]dx.
\end{aligned}$$

Changing the integration variables as above we obtain

$$\begin{cases}
\tau_2 & =\frac{3\left(6a^2b+686a^2b_2-588a_1b^2-686a_2ab+588abb_1-686ae+735bd\right)}{2401a^2b},\\
\sigma_3 & =0,\\
\omega_3(\zeta) & =\tanh\left(\frac{\zeta}{2}\right)\left(\frac{c_8}{\cosh(\zeta)+1}+c_9\right),
\end{cases}$$

where

$$\begin{aligned}
c_8=&\frac{3bd}{8a^3}+\frac{3b^2}{28a^2}-\frac{27d}{56a^2}+\frac{e}{a^2}-\frac{39b}{196a}+\frac{27}{343},\\
c_9=&\frac{108a_1^2b^2}{343a^4}-\frac{81a_1bd}{98a^4}+\frac{243d^2}{448a^4}+\frac{1737a_1b^2}{2401a^3}-\frac{5085bd}{5488a^3}\\
&+\frac{36a_1a_2b}{49a^3}-\frac{27a_2d}{28a^3}-\frac{22905b^2}{268912a^2}+\frac{579a_2b}{686a^2}-\frac{18b_1b}{49a^2}\\
&+\frac{3e}{7a^2}+\frac{3a_2^2}{7a^2}+\frac{549b}{4802a}-\frac{3b_2}{7a}-\frac{27}{686},
\end{aligned}$$

with

$$\begin{cases} \hat{u}_3(\zeta) = & 0, \\ \hat{v}_3(\zeta) = & 3\text{sech}^2\left(\frac{\zeta}{2}\right)\tanh^2\left(\frac{\zeta}{2}\right)\left(\frac{c_{10}}{\cosh(\zeta)+1}+c_{11}\right), \end{cases}$$

where

$$\begin{aligned} c_{10} =& \frac{9bd}{16a^3}+\frac{9b^2}{56a^2}-\frac{81d}{112a^2}+\frac{3e}{2a^2}-\frac{117b}{392a}+\frac{81}{686}, \\ c_{11} =& \frac{162a_1^2b^2}{343a^4}-\frac{243a_1bd}{196a^4}+\frac{729d^2}{896a^4}+\frac{5211a_1b^2}{4802a^3} \\ & -\frac{15255bd}{10976a^3}+\frac{54a_1a_2b}{49a^3}-\frac{81a_2d}{56a^3}-\frac{68715b^2}{537824a^2} \\ & +\frac{27a_1b}{49a^2}+\frac{1737a_2b}{1372a^2}-\frac{27b_1b}{49a^2}-\frac{81d}{112a^2}+\frac{9e}{14a^2} \\ & +\frac{9a_2^2}{14a^2}+\frac{3105b}{19208a}-\frac{9b_2}{14a}+\frac{9a_2}{14a}-\frac{81}{1372}. \end{aligned}$$

The second-order approximation for the emanating homoclinic orbit to the trivial equilibrium for the normal form (C.11) is given by

$$\begin{aligned} w_0(t) &= \frac{\varepsilon^2}{a}\left(\hat{u}_0(t)+\varepsilon\hat{u}_1(t)+\varepsilon^2\hat{u}_2(t)\right)+\mathcal{O}(\varepsilon^5), \\ w_1(t) &= \frac{\varepsilon^3}{a}\left(\hat{v}_0(t)+\varepsilon\hat{v}_1(t)+\varepsilon^2\hat{v}_2(t)\right)+\mathcal{O}(\varepsilon^6), \\ \beta_1 &= \varepsilon^2, \\ \beta_2 &= \varepsilon^2(\tau_0+\varepsilon\tau_1+\varepsilon^2\tau_2)+\mathcal{O}(\varepsilon^5) \\ &= \frac{6}{7}\frac{b}{a}\varepsilon^2+\frac{3}{2401a^3}\left(-588a_1b^2+98a\left(-7a_2b+7ab_2+6bb_1\right)\right. \\ &\quad \left.-686ae+6b^3+735bd\right)\varepsilon^4+\mathcal{O}(\varepsilon^5). \end{aligned} \tag{C.41}$$

Substituting (C.41) into equations (6.42) and (6.43) gives the second-order homoclinic predictors

$$\begin{aligned} \alpha =& \left(\frac{b}{a}\varepsilon^2\left(\tau_0+\varepsilon^2\tau_2\right)\right)K_{01}, \\ x =& \left(\frac{10b}{7a}H_{0001}+\frac{1}{a}u_0(\zeta)\phi_0\right)\varepsilon^2+\frac{1}{a}v_0(\zeta)\phi_1\varepsilon^3+\left(-\frac{4}{a}H_{0010}+\frac{50b^2}{49a^2}H_{0002}\right. \\ & +\frac{b}{a}\left(\frac{1}{a}\left(\frac{100}{49}b_1-4\frac{e}{b}\right)+\frac{1}{a^2}\left(-\frac{50}{49}ba_1+\frac{288}{2401}b^2+\frac{146}{49}d\right)\right)H_{0001} \\ & \left.+\frac{1}{a}u_2(\zeta)\phi_0+\frac{1}{a}v_1(\zeta)\phi_1+\frac{1}{2a^2}H_{2000}u_0^2(\zeta)+\frac{10b}{7a^2}H_{1001}u_0(\zeta)\right)\varepsilon^4+\mathcal{O}(\varepsilon^5), \end{aligned} \tag{C.42}$$

for the original system (3.3).

C.2.2. Homoclinic orbit to the nontrivial equilibrium

To approximate the second homoclinic orbit emanating from the transcritical Bogdanov-Takens bifurcation we use the transformation

$$\beta_1 = -\varepsilon^2, \qquad \beta_2 = \frac{b}{a}\varepsilon^2\tau,$$
$$w_0 = \frac{\varepsilon^2}{a}u, \qquad w_1 = \frac{\varepsilon^3}{a}v, \qquad \varepsilon t = s.$$

This transforms the normal form (C.11) into

$$\begin{cases} \dot{u} = & v, \\ \dot{v} = & u\,(u-1) + \varepsilon\frac{b}{a}v(u+\tau) + \varepsilon^2\frac{1}{a^2}u^2\,(a_1 b\tau - aa_2 + du) \\ & +\varepsilon^3\frac{1}{a^2}uv\,(-ab_2 + bb_1\tau + eu) + \mathcal{O}(\varepsilon^4). \end{cases} \tag{C.43}$$

or

$$\begin{aligned} \ddot{u} - u\,(u-1) =& \varepsilon\frac{b}{a}\dot{u}(u+\tau) + \varepsilon^2\frac{1}{a^2}u^2\,(a_1 b\tau - aa_2 + du) \\ &+ \varepsilon^3\frac{1}{a^2}u\dot{u}\,(-ab_2 + bb_1\tau + eu) + \mathcal{O}(\varepsilon^4), \end{aligned} \tag{C.44}$$

where $0 < \varepsilon \ll 1$ and τ are the new parameters. The dot now indicates the derivative with respect to s.

For $\varepsilon = 0$ this system is Hamiltonian with the first integral

$$L(u,v) = \frac{1}{2}\left(v^2 + u^2\right) - \frac{1}{3}u^3 = h, \tag{C.45}$$

which has the exact homoclinic solution

$$\begin{pmatrix} u_0(s) \\ v_0(s) \end{pmatrix} = \begin{pmatrix} 1 - \frac{3}{2}\operatorname{sech}^2(s/2) \\ \frac{3}{2}\operatorname{sech}^2(s/2)\tanh(s/2) \end{pmatrix}. \tag{C.46}$$

Every closed orbit of (C.45) surrounding $(0,0)$ corresponds to a level curve

$$\Gamma_h = \left\{(u,v) : L(u,v) = h, 0 < h < \frac{1}{6}\right\}.$$

Γ_h shrinks to the equilibrium $(0,0)$ as $h \to 0$ and tends to a homoclinic orbit as $h \to \frac{1}{6}$, see Figure C.2.

Theorem C.2. *Consider the two-parameter system* (C.43) *and assume that* $ab \neq 0$. *Then equation* (C.16) *has an unique simple bifurcation point (in the sense of* [9]*) at* (h_0, τ_0) *with*

$$\tau_0 = -\frac{1}{7}. \tag{C.47}$$

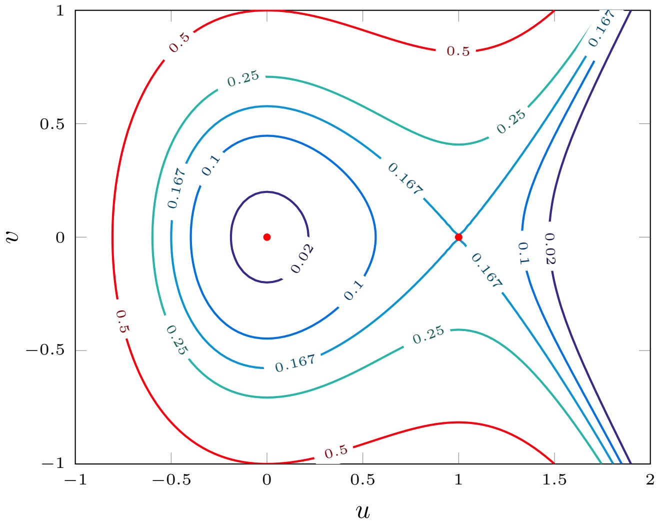


Figure C.2.: Contourplot of the Hamiltonian L in (C.45). There are periodic orbits surrounding the trivial equilibrium. At $h = \frac{1}{6} \approx 0.167$ there is a homoclinic orbit connecting to the equilibrium at $(1, 0)$.

The emanating C^1-branch can be parameterized by ε

$$(h(\varepsilon), \tau(\varepsilon)) = (z(\varepsilon), \varepsilon, \tau(\varepsilon)) \in X_1 \times \mathbb{R}^2. \tag{C.48}$$

It has tangent

$$\big(z'(0), 1, \tau'(0)\big) = \big(z_1, 1, \tau'(0)\big)$$

where $z_1 = (u_1, v_1)$ is the unique solution in X_1 of the linear system

$$\begin{cases} \dot{u} - v = 0 \\ \dot{v} - u\,(2u_0 - 1) \quad = \dfrac{b}{a} v_0(u_0 + \tau_0) \end{cases} \qquad \textit{and } v(0) = 0. \tag{C.49}$$

Proof. With $h = (z, \varepsilon) = (u, v, \varepsilon) \in X_0 \times \mathbb{R}$ let us write (C.43) as

$$F(h, \tau) = 0 \tag{C.50}$$

where $F : X_1 \times \mathbb{R} \times \mathbb{R} \to X_0 \times \mathbb{R}$ is defined by

$$F(h, \tau) = \begin{pmatrix} \dot{z} - g(z, \varepsilon, \tau) \\ v(0) \end{pmatrix},$$

with

$$g(z, \varepsilon, \tau) = \begin{pmatrix} v \\ u\,(u - 1) \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ \dfrac{b}{a} v(u + \tau) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 \\ \frac{1}{a^2} u^2\,(a_1 b\tau - a a_2 + du) \end{pmatrix}$$
$$+ \varepsilon^3 \begin{pmatrix} 0 \\ \frac{1}{a^2} u\dot{u}\,(-ab_2 + bb_1\tau + eu) \end{pmatrix} + \mathcal{O}(\varepsilon^4).$$

The condition $\dot{u}(0) = v(0) = 0$ is used to fix the phase of the homoclinic orbits.
Setting $h_0 = (z_0, 0) = (u_0, v_0, 0)$ we find

$$F(h_0, \tau) = 0 \qquad \text{for all } \tau.$$

Hence we have a trivial branch (h_0, τ) of homoclinic orbits and we look for values of τ at which bifurcation occurs. Following the proof of (C.1), we obtain τ_0 from the vanishing of the Melnikov integral

$$\begin{aligned}
0 &= \int_{-\infty}^{\infty} (-\dot{v}_0, \dot{u}_0) g_\varepsilon(z_0, 0, \tau)\, dt \\
&= \int_{-\infty}^{\infty} \frac{3}{2a} \tanh\left(\frac{t}{2}\right) \operatorname{sech}^2\left(\frac{t}{2}\right) \left[\frac{3}{2} b\tau \tanh\left(\frac{t}{2}\right) \operatorname{sech}^2\left(\frac{t}{2}\right)\right. \\
&\qquad \left. + \frac{3}{2} b \tanh\left(\frac{t}{2}\right)\left(1 - \frac{3}{2}\operatorname{sech}^2\left(\frac{t}{2}\right)\right) \operatorname{sech}^2\left(\frac{t}{2}\right)\right] dt \qquad \text{(C.51)} \\
&= \frac{6b(7\tau + 1)}{35a} \qquad \text{(C.52)}
\end{aligned}$$

This is satisfied at $\tau = \tau_0 = -\dfrac{1}{7}$. Furthermore, we need to verify that

$$0 \neq \int_{-\infty}^{\infty} (-\dot{v}_0, \dot{u}_0) g_{\varepsilon\tau}(z_0, 0, \tau)\, dt = \int_{-\infty}^{\infty} \frac{9b \tanh^2\left(\frac{t}{2}\right) \operatorname{sech}^4\left(\frac{t}{2}\right)}{4a}\, dt = \frac{3b}{5a}.$$

Which is true by assumption. □

Using the nonlinear time transformation from (C.23), we obtain the second-order differential equation

$$\begin{aligned}
\omega \frac{d}{d\zeta}\left(\omega \hat{u}'\right) - \hat{u}\left(\hat{u} - 1\right) =& \varepsilon \frac{b}{a} \omega \hat{u}'(\hat{u} + \tau) + \varepsilon^2 \frac{1}{a^2} \hat{u}^2 \left(a_1 b\tau - a a_2 + d\hat{u}\right) \\
&+ \varepsilon^3 \frac{1}{a^2} \omega \hat{u}' \left(-a b_2 + b b_1 \tau + e\hat{u}\right) \hat{u} + \mathcal{O}(\varepsilon^4).
\end{aligned} \qquad \text{(C.53)}$$

Substituting (C.26) into (C.25) and collecting the terms of order ε^0 gives the system

$$\omega_0 \frac{d}{d\zeta}\left(\omega_0 \hat{u}_0'\right) - \hat{u}_0(\hat{u}_0 - 1) = 0.$$

Then, for $\omega_0 = 1$ we obtain

$$\hat{u}_0'' - \hat{u}_0(\hat{u}_0 - 1) = 0, \qquad \text{(C.54)}$$

which has the exact homoclinic solution

$$\begin{pmatrix} u_0(s) \\ v_0(s) \end{pmatrix} = \begin{pmatrix} 1 - \frac{3}{2}\operatorname{sech}^2(s/2) \\ \frac{3}{2}\operatorname{sech}^2(s/2)\tanh(s/2) \end{pmatrix}. \qquad \text{(C.55)}$$

Substituting $\omega_0 = 1$ and (C.26) into (C.25) and collecting the terms of order ε^1, ε^2 and ε^3 gives the systems:

$$\text{Order}(\varepsilon^1): \frac{d}{d\zeta}(\omega_1 \hat{u}_0') + \omega_1 \hat{u}_0'' + \hat{u}_1'' - (2\hat{u}_0 - 1)\,\hat{u}_1 = \frac{b}{a}\hat{u}_0'\,(\hat{u}_0 + \tau_0)\,, \tag{C.56}$$

$$\begin{aligned}\text{Order}(\varepsilon^2): &\frac{d}{d\zeta}\left(\omega_1 \hat{u}_1'\right) + \omega_1 \hat{u}_1'' + \frac{d}{d\zeta}\left(\omega_2 \hat{u}_0'\right) + \omega_2 \hat{u}_0'' \\ &+ \hat{u}_2'' - (2\hat{u}_0 - 1)\,\hat{u}_2 + \omega_1 \frac{d}{d\zeta}\left(\omega_1 \hat{u}_0'\right) - \hat{u}_1^2 \\ &= \frac{1}{a^2}\hat{u}_0^2\,(a_1 b \tau_0 - a a_2 + d\hat{u}_0) \\ &\quad + \frac{b}{a}\left(\hat{u}_0'(\hat{u}_1 + \tau_1) + \hat{u}_1'(\hat{u}_0 + \tau_0) + \omega_1 \hat{u}_0'(\hat{u}_0 + \tau_0)\right)\end{aligned} \tag{C.57}$$

$$\begin{aligned}\text{Order}(\varepsilon^3): &\frac{d}{d\zeta}\left(\omega_1 \hat{u}_2'\right) + \omega_1 \hat{u}_2'' + \frac{d}{d\zeta}\left(\omega_3 \hat{u}_0'\right) + \omega_3 \hat{u}_0'' \\ &+ \frac{d}{d\zeta}\left(\omega_2 \hat{u}_1'\right) + \omega_2 \hat{u}_1'' + \hat{u}_3'' - (2\hat{u}_0 - 1)\hat{u}_3 \\ &+ \omega_1 \frac{d}{d\zeta}\left(\omega_1 \hat{u}_1'\right) + \omega_1 \frac{d}{d\zeta}\left(\omega_2 \hat{u}_0'\right) + \omega_2 \frac{d}{d\zeta}\left(\omega_1 \hat{u}_0'\right) - 2\hat{u}_1 \hat{u}_2 \\ &= \frac{1}{a^2}\hat{u}_0'\,(-a b_2 + b b_1 \tau_0 + e\hat{u}_0)\,\hat{u}_0 \\ &+ \frac{b}{a}\Big(\hat{u}_0'(\hat{u}_2 + \tau_2) + \hat{u}_1'(\hat{u}_1 + \tau_1) + \hat{u}_2'(\hat{u}_0 + \tau_0) \\ &+ \omega_1\left[\hat{u}_0'(\hat{u}_1 + \tau_1) + \hat{u}_1'(\hat{u}_0 + \tau_0)\right] + \omega_2\left[\hat{u}_0'(\hat{u}_0 + \tau_0)\right]\Big) \\ &+ \frac{1}{a^2}\left(\hat{u}_0^2(a_1 b \tau_1 + d\hat{u}_0) + 2\hat{u}_0 \hat{u}_1'(a_1 b \tau_0 - a a_2 + d\hat{u}_0)\right)\end{aligned} \tag{C.58}$$

We assume that for $\varepsilon \neq 0$ the homoclinic orbit of (C.53) is still given by

$$\begin{cases} \hat{u}(\zeta) & = \delta + \sigma \text{sech}^2(\frac{\zeta}{2}), \\ \hat{v}(\zeta) & = \hat{u}'(\zeta) = -\sigma\omega(\zeta)\text{sech}^2(\frac{\zeta}{2})\tanh(\frac{\zeta}{2}), \end{cases} \tag{C.59}$$

where σ and δ is a parameter that depends on ε,

$$\begin{cases} \sigma & = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \varepsilon^3\sigma_3 + \ldots \\ \delta & = \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \varepsilon^3\delta_3 + \ldots \end{cases} \tag{C.60}$$

For which it follows that $\sigma_0 = -\frac{3}{2}$ and

$$\hat{u}_i(\zeta) = \delta_i + \sigma_i \text{sech}^2(\frac{\zeta}{2}), \qquad i = 1, 2, 3, \tag{C.61}$$

$$\hat{v}_1(\zeta) = \left(\frac{3}{2}\omega_1 - \sigma_1\right) \text{sech}^2(\frac{\zeta}{2})\tanh(\frac{\zeta}{2}), \tag{C.62}$$

$$\hat{v}_2(\zeta) = \left(\frac{3}{2}\omega_2 - \sigma_1\omega_1 - \sigma_2\right) \text{sech}^2(\frac{\zeta}{2})\tanh(\frac{\zeta}{2}),$$

$$\hat{v}_3(\zeta) = \left(\frac{3}{2}\omega_3 - \sigma_1\omega_2 - \sigma_2\omega_1 - \sigma_3\right) \text{sech}^2(\frac{\zeta}{2})\tanh(\frac{\zeta}{2}). \tag{C.63}$$

Using assumptions (C.61)-(C.63) we solve the linear equations (C.56)-(C.58) for $\hat{u}(\zeta)$ one by one to determine τ_{i-1},δ_i σ_i and $\omega_i(\zeta)$ for $i = 1, 2, 3$.
We multiply both sides of (C.56) with $\hat{u}_0'$ and integrate both sides from ζ_0 to ζ, and get

$$\int_{\zeta_0}^{\zeta} \hat{u}_0' \frac{d}{d\zeta}(\omega_1 \hat{u}_0')\, dx + \int_{\zeta_0}^{\zeta} \hat{u}_0' \omega_1 \hat{u}_0''\, dx \tag{C.64}$$

$$+ \int_{\zeta_0}^{\zeta} \hat{u}_0' \hat{u}_1''\, dx - \int_{\zeta_0}^{\zeta} \hat{u}_0' \left(2\hat{u}_0 - 1\right) \hat{u}_1\, dx \tag{C.65}$$

$$= \frac{b}{a} \int_{\zeta_0}^{\zeta} \left(\hat{u}_0'\right)^2 \left(\hat{u}_0 + \tau_0\right)\, dx$$

Differentiating (C.54) with respect to ζ yields

$$\hat{u}_0''' = \hat{u}_0'(2\hat{u}_0 - 1),$$

so that

$$\int_{\zeta_0}^{\zeta} \hat{u}_1 \hat{u}_0''' dx = \int_{\zeta_0}^{\zeta} \hat{u}_1 \hat{u}_0'(2\hat{u}_0 - 1) dx. \tag{C.66}$$

Using this expression and integration by parts we can simplify (C.64) and (C.65) to become

$$\int_{\zeta_0}^{\zeta} \hat{u}_0' \frac{d}{d\zeta}(\omega_1 \hat{u}_0')\, dx + \int_{\zeta_0}^{\zeta} \hat{u}_0' \omega_1 \hat{u}_0''\, dx = \omega_1 \left(\hat{u}_0'\right)^2 \Big|_{\zeta_0}^{\zeta} - \int_{\zeta_0}^{\zeta} \hat{u}_0'' \omega_1 \hat{u}_0'\, dx + \int_{\zeta_0}^{\zeta} \hat{u}_0' \omega_1 \hat{u}_0''\, dx$$

$$= \omega_1 \left(\hat{u}_0'\right)^2 \Big|_{\zeta_0}^{\zeta},$$

$$\int_{\zeta_0}^{\zeta} \hat{u}_0' \hat{u}_1''\, dx - \int_{\zeta_0}^{\zeta} \hat{u}_0' \left(2\hat{u}_0 - 1\right) \hat{u}_1\, dx = \int_{\zeta_0}^{\zeta} \hat{u}_0' \hat{u}_1''\, dx - \int_{\zeta_0}^{\zeta} \hat{u}_1 \hat{u}_0''' dx$$

$$= \left(\hat{u}_0' \hat{u}_1' - \hat{u}_1 \hat{u}_0''\right) \Big|_{\zeta_0}^{\zeta} - \int_{\zeta_0}^{\zeta} \hat{u}_0'' \hat{u}_1'\, dx - \int \hat{u}_0'' \hat{u}_1'\, dx$$

$$= \left(\hat{u}_0' \hat{u}_1' - \hat{u}_1 \hat{u}_0''\right) \Big|_{\zeta_0}^{\zeta}.$$

Therefore, (C.29) can be written as

$$\omega_1 \left(\hat{u}_0'\right)^2 \Big|_{\zeta_0}^{\zeta} + \left(\hat{u}_0'\hat{u}_1' - \hat{u}_1\hat{u}_0''\right)\Big|_{\zeta_0}^{\zeta} = \frac{b}{a}\int_{\zeta_0}^{\zeta} \left(\hat{u}_0'\right)^2 \left(\hat{u}_0 + \tau_0\right)\, dx. \tag{C.67}$$

Setting $\zeta_0 = -\infty$ or $\zeta = \infty$ gives

$$0 = \frac{b}{a}\int_{\zeta_0}^{\zeta} \left(\hat{u}_0'\right)^2 \left(\hat{u}_0 + \tau_0\right)\, dx = \frac{6}{35}\frac{b}{a}\left(1 + 7\tau_0\right),$$

from which we recover (C.47) and thus once more that $\tau_0 = -\frac{1}{7}$. Taking the integration boundaries $\zeta_0 = 0$ and $\zeta = \infty$ in (C.67) we obtain

$$\frac{3}{4}\left(\delta_1 + \sigma_1\right) = 0,$$

from which it follows that $\sigma_1 = -\delta_1$. We set the integration boundaries in (C.67) from $\zeta_0 = 0$ to ζ, which yields

$$\begin{aligned}\frac{3}{16}\tanh^2\left(\frac{\zeta}{2}\right)\operatorname{sech}^4\left(\frac{\zeta}{2}\right)\left(\delta_1(12\cosh(\zeta) + \cosh(2\zeta) + 15) + 12\omega_1(\zeta)\right)\\ = \frac{864}{7}\frac{b}{a}\sinh^{10}\left(\frac{\zeta}{2}\right)\operatorname{csch}^7(\zeta).\end{aligned}$$

.

Taking the limit of $\zeta \to \infty$ in the above expression implies that This condition implies that

$$\delta_1 = 0.$$

Here we used the function ω_1 is a bounded function. Thus $\omega_1(\zeta)$ is given by

$$\omega_1(\zeta) = -\frac{3b\tanh\left(\frac{\zeta}{2}\right)}{7a}.$$

Substituting σ_1,δ_1 and ω_1 into (C.34) and (C.35) gives the first-order correction to the initial homoclinic solution $(\hat{u}_0, \hat{v}_0)$

$$\begin{cases}\hat{u}_1(\zeta) & = 0,\\ \hat{v}_1(\zeta) & = -\frac{9}{14}\operatorname{sech}^2(\frac{\zeta}{2})\tanh^2(\frac{\zeta}{2}).\end{cases}$$

We can apply to same procedure to equation (C.30). First we again multiply both sides with $\hat{u}_0'$

$$\begin{aligned}\hat{u}_0'\frac{d}{d\zeta}\left(\omega_2\hat{u}_0'\right) + \hat{u}_0'\omega_2\hat{u}_0'' + \hat{u}_0'\hat{u}_2'' - \hat{u}_0'\left(2\hat{u}_0 - 1\right)\hat{u}_2 + \hat{u}_0'\omega_1\frac{d}{d\zeta}\left(\omega_1\hat{u}_0'\right)\\ = \hat{u}_0'\left[\frac{1}{a^2}\hat{u}_0^2\left(a_1 b\tau_0 - aa_2 + d\hat{u}_0\right) + \frac{b}{a}\left(\hat{u}_0'\tau_1 + \omega_1\hat{u}_0'(\hat{u}_0 + \tau_0)\right)\right].\end{aligned}$$

Using the identity

$$\int_{\zeta_0}^{\zeta} \hat{u}_0'\omega_1 \frac{d}{d\zeta}\left(\omega_1\hat{u}_0'\right) dx = \frac{1}{2}\left(\omega_1\hat{u}_0'\right)^2\Big|_{\zeta_0}^{\zeta} = \frac{2592}{49}\frac{b^2}{a^2}\sinh^{12}\left(\frac{\zeta}{2}\right)\operatorname{csch}^8(\zeta)\Big|_{\zeta_0}^{\zeta}$$

we obtain the equality

$$\begin{aligned}
&\omega_2\left(\hat{u}_0'\right)^2\Big|_{\zeta_0}^{\zeta} + \left(\hat{u}_0'\hat{u}_2' - \hat{u}_2\hat{u}_0''\right)\Big|_{\zeta_0}^{\zeta} + \tfrac{2592}{49}\frac{b^2}{a^2}\sinh^{12}\left(\tfrac{\zeta}{2}\right)\operatorname{csch}^8(\zeta)\Big|_{\zeta_0}^{\zeta} \\
&= \textstyle\int_{\zeta_0}^{\zeta} \hat{u}_0'\Big[\tfrac{1}{a^2}\hat{u}_0^2\left(a_1b\tau_0 - aa_2 + d\hat{u}_0\right) + \dfrac{b}{a}\hat{u}_0'\left(\tau_1 + \omega_1(\hat{u}_0 + \tau_0)\right).
\end{aligned}$$

Taking the integration boundaries as before leads to

$$\begin{cases}
\tau_1 & = 0 \\
\delta_2 & = \frac{a_1b + 7aa_2 - 7d}{7a^2} \\
\sigma_2 & = -\frac{3\left(56a_1b + 392aa_2 + 6b^2 - 343d\right)}{784a^2} \\
\omega_2(\zeta) & = \frac{4\left(9b^2 + 49d\right)\cosh(\zeta) - 18b^2 - 245d}{392a^2(\cosh(\zeta)+1)}
\end{cases}$$

with

$$\begin{cases}
\hat{u}_2(\zeta) = & \frac{\operatorname{sech}^2\left(\frac{\zeta}{2}\right)}{784a^2}\bigg(56a_1b(\cosh(\zeta) - 2) + 392aa_2(\cosh(\zeta) - 2) \\
& \quad -18b^2 - 392d\cosh(\zeta) + 637d\bigg), \\
\hat{v}_2(\zeta) & = \frac{3\tanh\left(\frac{\zeta}{2}\right)\operatorname{sech}^2\left(\frac{\zeta}{2}\right)}{784a^2(\cosh(\zeta)+1)}\bigg(56a_1b(\cosh(\zeta) + 1) + 392aa_2(\cosh(\zeta) + 1) \\
& \quad +21\left(2b^2 - 7d\right)\cosh(\zeta) - 12\left(b^2 + 49d\right)\bigg).
\end{cases}$$

Lastly, we turn our attention to the third order terms. Multiplying both sides of equation (C.58) with $\hat{u}_0'$ yields

$$\begin{aligned}
&\hat{u}_0'\frac{d}{d\zeta}\left(\omega_3\hat{u}_0'\right) + \hat{u}_0'\omega_3\hat{u}_0'' + \hat{u}_0'\hat{u}_3'' - \hat{u}_0'(2\hat{u}_0 - 1)\hat{u}_3 \\
&+ \hat{u}_0'\omega_1\frac{d}{d\zeta}\left(\omega_2\hat{u}_0'\right) + \hat{u}_0'\omega_2\frac{d}{d\zeta}\left(\omega_1\hat{u}_0'\right) \\
&+ \hat{u}_0'\frac{d}{d\zeta}\left(\omega_1\hat{u}_2'\right) + \hat{u}_0'\omega_1\hat{u}_2'' \\
&= \frac{1}{a^2}\hat{u}_0'\left(-ab_2 + bb_1\tau_0 + e\hat{u}_0\right)\hat{u}_0 + \frac{b}{a}\Big(\hat{u}_0'(\hat{u}_2 + \tau_2) + \hat{u}_2'(\hat{u}_0 + \tau_0) \\
&+ \omega_1\left[\hat{u}_0'(\hat{u}_1 + \tau_1) + \hat{u}_1'(\hat{u}_0 + \tau_0)\right] + \omega_2\left[\hat{u}_0'(\hat{u}_0 + \tau_0)\right]\Big) + \frac{1}{a^2}\hat{u}_0^2(a_1b\tau_1 + d\hat{u}_0).
\end{aligned}$$

Integrating both sides from ξ_0 to ξ and simplifying yields

$$
\begin{aligned}
&\omega_3 \left(\hat{u}_0'\right)^2\Big|_{\zeta_0}^{\zeta} + \left(\hat{u}_0'\hat{u}_3' - \hat{u}_3\hat{u}_0''\right)\Big|_{\zeta_0}^{\zeta} \\
&+ \frac{27b\tanh^3\left(\frac{\zeta}{2}\right)\operatorname{sech}^6\left(\frac{\zeta}{2}\right)\left(-4\left(9b^2+49d\right)\cosh(\zeta)+18b^2+245d\right)}{21952a^3} \\
&- \frac{108b\sinh^{10}\left(\frac{\zeta}{2}\right)\operatorname{csch}^7(\zeta)\left(56a_1b+392aa_2+6b^2-343d\right)}{343a^3} \\
&= \int_{\zeta_0}^{\zeta} \frac{1}{a^2}\hat{u}_0'\left(-ab_2+bb_1\tau_0+e\hat{u}_0\right)\hat{u}_0 + \frac{b}{a}\Big(\hat{u}_0'(\hat{u}_2+\tau_2)+\hat{u}_2'(\hat{u}_0+\tau_0) \\
&+ \omega_1\left[\hat{u}_0'(\hat{u}_1+\tau_1)+\hat{u}_1'(\hat{u}_0+\tau_0)\right] + \omega_2\left[\hat{u}_0'(\hat{u}_0+\tau_0)\right]\Big) + \frac{1}{a^2}\hat{u}_0^2(a_1b\tau_1+d\hat{u}_0)\,dx.
\end{aligned}
$$

Changing the integration variables as above we obtain

$$
\begin{cases}
\tau_2 &= \frac{-49a_1b^2+49a(-7a_2b+7ab_2+bb_1)-343ae+18b^3+490bd}{2401a^2b}, \\
\delta_3 &= 0, \\
\sigma_3 &= 0, \\
\omega_3(\zeta) &= -\frac{1}{115248a^3}\Bigg[19208a^3\delta_3\cosh(\zeta)+3\tanh\left(\frac{\zeta}{2}\right) \\
&\quad\times\Bigg(7\operatorname{sech}^2\left(\frac{\zeta}{2}\right)\left(-1372ae+18b^3+147bd\right) \\
&\quad+12\left(98a_1b^2-98a\left(-7a_2b+7ab_2+bb_1\right)+686ae+6b^3-637bd\right)\Bigg)\Bigg].
\end{cases}
$$

with

$$
\begin{cases}
\hat{u}_3(\zeta) = & 0, \\
\hat{v}_3(\zeta) = & -\frac{1}{76832a^3}\Bigg[3\tanh^2\left(\frac{\zeta}{2}\right)\operatorname{sech}^2\left(\frac{\zeta}{2}\right) \\
& \quad +6\big(588a_1b^2-196a\left(-21a_2b+7ab_2+bb_1\right) \\
& \quad +1372ae+54b^3-3675bd\big)\Bigg].
\end{cases}
$$

The second-order approximation for the emanating homoclinic orbit to the trivial equilibrium for the normal form (C.11) is given by

$$
\begin{aligned}
w_0(t) &= \frac{\varepsilon^2}{a}\left(\hat{u}_0(t) + \varepsilon\hat{u}_1(t) + \varepsilon^2\hat{u}_2(t)\right) + \mathcal{O}(\varepsilon^5),\\
w_1(t) &= \frac{\varepsilon^3}{a}\left(\hat{v}_0(t) + \varepsilon\hat{v}_1(t) + \varepsilon^2\hat{v}_2(t)\right) + \mathcal{O}(\varepsilon^6),\\
\beta_1 &= -\varepsilon^2,\\
\beta_2 &= \varepsilon^2(\tau_0 + \varepsilon\tau_1 + \varepsilon^2\tau_2) + \mathcal{O}(\varepsilon^5)\\
&= \frac{1}{7}\frac{b}{a}\varepsilon^2 + \frac{1}{2401a^3}\big(343a^2b_2 - 49a_1b^2 - 343aa_2b + 49ab_1b\\
&\qquad -343ae + 18b^3 + 490bd\big)\,\varepsilon^4 + \mathcal{O}(\varepsilon^5).
\end{aligned}
\tag{C.68}
$$

Substituting (C.68) into equations (6.42) and (6.43) gives the second-order homoclinic predictors

$$
\begin{aligned}
\alpha &= \left(-\frac{b}{7a}K_{01} - K_{10}\right)\varepsilon^2 + \Bigg(\frac{b^2}{98a^2}K_{02} + \frac{b}{7a}K_{11} + \frac{1}{2}K_{20} +\\
&\qquad \frac{1}{2401a^3}\left(343a^2b_2 - 343aa_2b + 49abb1 - 343ae - 49a_1b^2 + 18b^3 + 490bd\right)K_{01}\Bigg)\varepsilon^4,\\
x &= \frac{1}{a}\phi_0u_0(\zeta)\epsilon^2 + \frac{1}{a}\left(\phi_0u_1(\zeta) + \phi_1v_0(\zeta)\right)\epsilon^3 + \frac{1}{2a^2}\Bigg(2b\tau_0H_{1001}u_0(\zeta)\\
&\qquad - 2aH_{1010}u_0(\zeta) + H_{2000}u_0(\zeta)^2 + 2a\phi_0u_2(\zeta) + 2a\phi_1v_1(\zeta)\Bigg) + \mathcal{O}(\varepsilon^5),
\end{aligned}
\tag{C.69}
$$

for the original system (3.3).

C.3. Generalized-Hopf bifurcation

The normal form is given by

$$
\dot{z} = \lambda(\beta)z + c_1(\beta)z|z|^2 + c_2(\beta)z|z|^4 + \mathcal{O}(|z|^6), \qquad z \in \mathbb{C}, \tag{C.70}
$$

where $\lambda(0) = i\omega$. This bifurcation is characterized by

$$
\ell_1 = \mathrm{Re}(c_1(0)) = 0, \qquad \ell_2 = \mathrm{Re}(c_2(0)) \neq 0.
$$

It is well known that a curve **LPC** of fold bifurcation of limit cycles emanate from this point. To approximate this curve we substitute $z = \rho e^{i\psi}$, $\lambda(\beta) = i\omega + \beta_1 + ib_1(\beta) + \mathcal{O}(|\beta|^2)$, with $b_1(0) = 0$ and $\mathrm{Re}(c_1(0)) = \beta_2 + \mathcal{O}(|\beta|^2)$ into (C.70) and truncate the normal form the fifth order in z

$$
\begin{aligned}
\dot{z} &= \dot{\rho}e^{i\psi} + \rho i\dot{\psi}e^{i\psi}\\
&= \left(i\omega + \beta_1 + ib_1(\beta)\right)\rho e^{i\psi} + \left(\beta_2 + \mathrm{Im}(c_1(0))i\right)\rho^3e^{i\psi} + \left(\ell_2 + \mathrm{Im}(c_2(0))i\right)\rho^5e^{i\psi}.
\end{aligned}
$$

Separating the real and imaginary parts yields

$$\begin{cases} \dot{\rho} &= \rho\left(\beta_1 + \beta_2\rho^2 + \ell_2\rho^4\right), \\ \dot{\psi} &= \omega + b_1(\beta) + \operatorname{Im}(c_1(0))\rho^2 + \operatorname{Im}(c_2(0))\rho^4. \end{cases} \tag{C.71}$$

The curve **LPC** occurs when

$$\begin{cases} 0 = & \beta_1 + \beta_2\rho^2 + \ell_2\rho^4, \\ 0 = & 2\beta_2\rho + 4\ell_2\rho^3. \end{cases}$$

Therefore, the curve **LPC** in (C.70) can be approximated by

$$\rho = \epsilon, \qquad \beta_1 = \ell_2\epsilon^4, \qquad \beta_2 = -2\ell_2\epsilon^2, \tag{C.72}$$

for $\epsilon > 0$. From the second equation in the amplitude system (C.71) we obtain an approximation for the period given by

$$\begin{aligned} T &= \frac{2\pi}{\omega + \frac{\partial}{\beta_1}b_1\beta_1 + \frac{\partial}{\beta_2}b_1\beta_2 + \operatorname{Im}(c_1(0))\epsilon^2} \\ &= \frac{2\pi}{\omega + \left(\operatorname{Im}(c_1(0)) - 2\ell_2\frac{\partial}{\beta_2}b_1\right)\epsilon^2}, \end{aligned} \tag{C.73}$$

using equitation (C.72). For a fourth order approximation in ϵ also the seventh order derivatives would be needed, see [40, Remark 3.3.2]. Lastly, to approximate the cycle we substitute $z = \epsilon e^{i\psi}$ and (C.72) in (6.82), this gives

$$\begin{aligned} x &= \mathcal{H}(\epsilon e^{i\psi}, \epsilon e^{-i\psi}, \ell_2\epsilon^4, -2\ell_2\epsilon^2) \\ &= 2\operatorname{Re}(e^{i\psi}\phi)\epsilon + \Big(H_{1100} - 2\ell_2 H_{0001} + \operatorname{Re}(e^{2i\psi}H_{2000})\Big)\epsilon^2 \\ &\quad + \left(-4\ell_2\operatorname{Re}(e^{i\psi}H_{1001}) + \frac{1}{3}\operatorname{Re}(e^{3i\psi}H_{3000}) + \operatorname{Re}(e^{i\psi}H_{2100})\right)\epsilon^3 + \mathcal{O}(|\epsilon|^4), \end{aligned} \tag{C.74}$$

with $\psi \in [0, 2\pi)$.

Since $\ell_1 = \operatorname{Re}(c_1(0)) = \beta_2 + \mathcal{O}(|\beta|^2)$, it is easy to see that the Hopf curve in the original system is related to the truncated normal form by

$$(\beta_1, \beta_2, z) = (0, \epsilon, 0)$$

for $\epsilon \neq 0$ small.

C.4. Fold-Hopf bifurcation

Following [37] we truncate the normal form (B.6) to obtain the system

$$\begin{cases} \dot{z}_0 &= \beta_1 + g_{200}z_0^2 + g_{011}|z_1|^2 + g_{111}z_0|z_1|^2, \\ \dot{z}_1 &= (i\omega + \beta_2 + ib_1(\beta))z_1 + g_{110}z_0z_1 + g_{210}z_0^2z_1 + g_{021}z_1|z_1|^2, \end{cases} \tag{C.75}$$

with $b_1(0) = 0$. Letting $z_1 = \rho e^{i\psi}$ we have

$$\begin{aligned} \dot{z}_1 &= \dot{\rho}e^{i\psi} + \rho i\dot{\psi}e^{i\psi} \\ &= (i\omega + \beta_2 + ib_1(\beta))\rho e^{i\psi} + g_{110}z_0\rho e^{i\psi} + g_{210}z_0^2\rho e^{i\psi} + g_{021}e^{i\psi}\rho^3. \end{aligned}$$

Separating the real and imaginary parts yields the three dimensional system

$$\begin{cases} \dot{z}_0 &= \beta_1 + g_{200}z_0^2 + g_{011}\rho^2 + g_{111}z_0\rho^2, \\ \dot{\rho} &= \rho\left(\beta_2 + \mathrm{Re}(g_{110})z_0 + \mathrm{Re}(g_{210})z_0^2 + \mathrm{Re}(g_{021})\rho^2\right), \\ \dot{\psi} &= \omega_0 + b_1(\beta) + \mathrm{Im}(g_{110})z_0 + \mathrm{Im}(g_{210})z_0^2 + \mathrm{Im}(g_{021})\rho^2. \end{cases} \tag{C.76}$$

The first two equations are independent of the third equation and can be studied separately. Therefore, we consider the system

$$\begin{cases} \dot{z}_0 &= \beta_1 + g_{200}z_0^2 + g_{011}\rho^2 + g_{111}z_0\rho^2, \\ \dot{\rho} &= \rho\left(\beta_2 + \mathrm{Re}(g_{110})z_0 + \mathrm{Re}(g_{210})z_0^2 + \mathrm{Re}(g_{021})\rho^2\right). \end{cases} \tag{C.77}$$

A Hopf bifurcation in the system (C.77) corresponds to a Neimark-Sacker bifurcation in the original system. The Jacobian of (C.77) is given by

$$J = \begin{pmatrix} 2g_{200}z_0 + g_{111}\rho^2 & 2g_{011}\rho + 2g_{111}z_0\rho \\ \rho\left(\mathrm{Re}(g_{110}) + 2\,\mathrm{Re}(g_{210})z_0 + \mathrm{Re}(g_{021})\right) & \beta_2 + \mathrm{Re}(g_{110})z_0 + \mathrm{Re}(g_{210})z_0^2 + 3\,\mathrm{Re}(g_{021})\rho^2 \end{pmatrix}.$$

Eigenvalues of J are purely imaginary when the trace of J vanishes, i.e.

$$2g_{200}z_0 + g_{111}\rho^2 + \beta_2 + \mathrm{Re}(g_{110})z_0 + \mathrm{Re}(g_{210})z_0^2 + 3\,\mathrm{Re}(g_{021})\rho^2 = 0$$

and the determinant of J is positive. Let $\rho = \epsilon$, then the approximation to the Neimark-Sacker curve can be obtained by solving the system

$$\begin{cases} 0 = & \beta_1 + g_{200}z_0^2 + g_{011}\epsilon^2 + g_{111}z_0\epsilon^2, \\ 0 = & \beta_2 + \mathrm{Re}(g_{110})z_0 + \mathrm{Re}(g_{210})z_0^2 + \mathrm{Re}(g_{021})\epsilon^2, \\ 0 = & 2g_{200}z_0 + g_{111}\epsilon^2 + \beta_2 + \mathrm{Re}(g_{110})z_0 + \mathrm{Re}(g_{210})z_0^2 + 3\,\mathrm{Re}(g_{021})\epsilon^2 \end{cases}$$

for (z_0, β_1, β_2). We obtain

$$\begin{cases} z_0 &= -\frac{(2\operatorname{Re}(g_{021})+g_{111})}{2g_{200}}\epsilon^2, \\ \beta_1 &= -g_{011}\epsilon^2, \\ \beta_2 &= \frac{\operatorname{Re}(g_{110})(2\operatorname{Re}(g_{021})+g_{111})-2\operatorname{Re}(g_{021})g_{200}}{2g_{200}}\epsilon^2. \end{cases} \tag{C.78}$$

Substituting (C.78) into the determinant of J and expending in ϵ yields

$$\det J = -2\left(g_{011}\operatorname{Re}(g_{110})\right)\epsilon^2 + \mathcal{O}(\epsilon^4).$$

We conclude that for ϵ small the determinant of J is positive when $g_{011}\operatorname{Re}(g_{110}) < 0$.

An approximation for the period of the cycle for the Neimark-Sacker predictor can be obtained from the third equation in the system (C.76), yielding

$$T = \frac{2\pi}{\omega + \frac{\partial}{\beta_1}b_1\beta_1 + \frac{\partial}{\beta_2}b_1\beta_2 + \operatorname{Im}(g_{110})z_0 + \operatorname{Im}(g_{210})z_0^2 + \operatorname{Im}(g_{021})\epsilon^2}.$$

Here (z_0, β_1, β_2) are as in (C.78). Lastly, to approximate the cycle itself we substitute $z_1 = \epsilon e^{i\psi}$ and (C.78) into (6.89), this gives

$$\begin{aligned} x &= \mathcal{H}\left(z_0, \epsilon e^{i\psi}, \epsilon e^{-i\psi}, \beta_1, \beta_2\right) \\ &= 2\operatorname{Re}\left(e^{i\psi}\phi_1\right)\epsilon + \left(\frac{\operatorname{Re}(g_{110})\left(2\operatorname{Re}(g_{021}) + g_{111}\right) - 2\operatorname{Re}(g_{021})g_{200}}{2g_{200}}H_{00001}\right. \\ &\qquad \left. - g_{011}H_{00010} + H_{01100} - \left(\frac{2\operatorname{Re}(g_{021}) + g_{111}}{2g_{200}}\right)\phi_0 + \operatorname{Re}\left(e^{2i\psi}\bar{H}_{02000}\right)\right)\epsilon^2, \end{aligned}$$

with $\psi \in [0, 2\pi)$.

C.4.1. Fold

The fold curve in the normal form is obtained by substituting $\rho = 0$ in the system (C.77). Then β_2 is unrestricted and

$$z_0 = \pm\sqrt{-\frac{\beta_1}{g_{200}}}.$$

The fold curve is therefore given by

$$(\beta_1, \beta_2) = (0, \beta_2).$$

C.5. Hopf-transcritical bifurcation

As in the fold-Hopf bifurcation in the previous Section we truncate the normal form (B.11) to obtain the system

$$\begin{cases} \dot{z}_0 &= \beta_1 z_0 + g_{200} z_0^2 + g_{011}|z_1|^2 + g_{111} z_0 |z_1|^2, \\ \dot{z}_1 &= (i\omega + \beta_2 + ib_1(\beta)) z_1 + g_{110} z_0 z_1 + g_{210} z_0^2 z + g_{021} z_1 |z_1|^2, \end{cases} \tag{C.79}$$

where $b_1(0) = 0$. Letting $z_1 = \rho e^{i\psi}$ we have

$$\begin{aligned} \dot{z}_1 &= \dot{\rho} e^{i\psi} + \rho i \dot{\psi} e^{i\psi} \\ &= (i\omega + \beta_2 + ib_1(\beta))\rho e^{i\psi} + g_{110} z_0 \rho e^{i\psi} + g_{210} z_0^2 \rho e^{i\psi} + g_{021} e^{i\psi} \rho^3. \end{aligned}$$

Separating the real and imaginary parts yields the three dimensional system

$$\begin{cases} \dot{z}_0 &= \beta_1 z_0 + g_{200} z_0^2 + g_{011} \rho^2 + g_{111} z_0 \rho^2, \\ \dot{\rho} &= \rho \left(\beta_2 + \mathrm{Re}(g_{110}) z_0 + \mathrm{Re}(g_{210}) z_0^2 + \mathrm{Re}(g_{021}) \rho^2\right), \\ \dot{\psi} &= \omega_0 + b_1(\beta) + \mathrm{Im}(g_{110}) z_0 + \mathrm{Im}(g_{210}) z_0^2 + \mathrm{Im}(g_{021}) \rho^2. \end{cases} \tag{C.80}$$

The first two equations are independent of the third equation and can be studied separately. Therefore, we consider the system

$$\begin{cases} \dot{z}_0 &= \beta_1 z_0 + g_{200} z_0^2 + g_{011} \rho^2 + g_{111} z_0 \rho^2, \\ \dot{\rho} &= \rho \left(\beta_2 + \mathrm{Re}(g_{110}) z_0 + \mathrm{Re}(g_{210}) z_0^2 + \mathrm{Re}(g_{021}) \rho^2\right). \end{cases} \tag{C.81}$$

A Hopf bifurcation in the system (C.81) corresponds to a Neimark-Sacker bifurcation in the original system. The Jacobian of (C.81) is given by

$$J = \begin{pmatrix} \beta_1 + 2g_{200} z_0 + g_{111} \rho^2 & 2g_{011}\rho + 2g_{111} z_0 \rho \\ \rho \left(\mathrm{Re}(g_{110}) + 2\,\mathrm{Re}(g_{210}) z_0 + \mathrm{Re}(g_{021})\right) & \beta_2 + \mathrm{Re}(g_{110}) z_0 + \mathrm{Re}(g_{210}) z_0^2 + 3\,\mathrm{Re}(g_{021}) \rho^2 \end{pmatrix}.$$

Eigenvalues of J are purely imaginary when the trace of J vanishes, i.e.

$$\beta_1 + 2g_{200} z_0 + g_{111} \rho^2 + \beta_2 + \mathrm{Re}(g_{110}) z_0 + \mathrm{Re}(g_{210}) z_0^2 + 3\,\mathrm{Re}(g_{021}) \rho^2 = 0$$

and the determinant of J is positive. Let $\rho = \epsilon$, then the approximation to the Neimark-Sacker curve can be obtained by solving the system

$$\begin{cases} 0 = & \beta_1 z_0 + g_{200} z_0^2 + g_{011} \epsilon^2 + g_{111} z_0 \epsilon^2, \\ 0 = & \beta_2 + \mathrm{Re}(g_{110}) z_0 + \mathrm{Re}(g_{210}) z_0^2 + \mathrm{Re}(g_{021}) \epsilon^2, \\ 0 = & \beta_1 + 2g_{200} z_0 + g_{111} \epsilon^2 + \beta_2 + \mathrm{Re}(g_{110}) z_0 + \mathrm{Re}(g_{210}) z_0^2 + 3\,\mathrm{Re}(g_{021}) \epsilon^2 \end{cases}$$

for (z_0, β_1, β_2). We obtain

$$\begin{cases} z_0 &= -\frac{\mathrm{Re}(g_{021})\epsilon^2 \pm L}{g_{200}}, \\ \beta_1 &= -g_{011}\epsilon^2 \pm 2L, \\ \beta_2 &= \frac{1}{g_{200}^2}\Bigg(\mathrm{Re}(g_{021})\epsilon^2 \left((\mathrm{Re}(g_{110}) - g_{200})\, g_{200} \pm 2\,\mathrm{Re}(g_{210})L \right) \\ & \quad + g_{200} \left(\pm \mathrm{Re}(g_{110})L - g_{011}\,\mathrm{Re}(g_{210})\epsilon^2 \right) - 2\,\mathrm{Re}(g_{021})^2\,\mathrm{Re}(g_{210})\epsilon^4 \Bigg), \end{cases} \tag{C.82}$$

where $L = \epsilon\sqrt{\mathrm{Re}(g_{021})^2\epsilon^2 + g_{011}g_{200}}$. Substituting (C.82) into the determinant of J and expending in ϵ yields

$$\det J = -2\left(g_{011}\,\mathrm{Re}(g_{110})\right)\epsilon^2 + \mathcal{O}(\epsilon^4).$$

We conclude that for ϵ small the determinant of J is positive when $g_{011}\,\mathrm{Re}(g_{110}) < 0$.

An approximation for the period of the cycle for the Neimark-Sacker predictor can be obtained from the third equation in the system (C.80), yielding

$$T = \frac{2\pi}{\omega + \dfrac{\partial}{\beta_1} b_1\beta_1 + \dfrac{\partial}{\beta_2} b_1\beta_2 + \mathrm{Im}(g_{110})z_0 + \mathrm{Im}(g_{210})z_0^2 + \mathrm{Im}(g_{021})\epsilon^2}.$$

Here (z_0, β_1, β_2) are as in (C.82). Lastly, to approximate the cycle itself we substitute $z_1 = \epsilon e^{i\psi}$ and (C.82) into (6.89), this gives

$$\begin{aligned} x &= \mathcal{H}\left(z_0, \epsilon e^{i\psi}, \epsilon e^{-i\psi}, \beta_1, \beta_2\right) \\ &= \left(\mp \frac{g_{011}}{\sqrt{g_{011}g_{200}}}\phi_0 + 2\,\mathrm{Re}\left(e^{i\psi}\phi_1\right)\right)\epsilon + \Bigg(\pm 2\sqrt{g_{011}g_{200}}\,\mathrm{Re}\left(e^{i\psi}H_{01010}\right) \\ &\quad + \mathrm{Re}\left(e^{2i\psi}H_{02000}\right) + H_{01100} - 2g_{011}H_{10010} \mp 2\frac{\sqrt{g_{011}g_{200}}}{g_{200}}\,\mathrm{Re}\left(e^{i\psi}H_{11000}\right) \\ &\quad + \frac{g_{011}}{2g_{200}}H_{20000} - \frac{\mathrm{Re}\left(g_{021}\right)}{g_{200}}q_0 \pm 2\frac{\mathrm{Re}\left(g_{110}\right)\sqrt{g_{011}g_{200}}}{g_{200}}\,\mathrm{Re}\left(e^{i\psi}H_{01001}\right) \\ &\quad - \frac{\mathrm{Re}\left(g_{110}\right)g_{011}}{g_{200}}H_{10001}\Bigg)\epsilon^2, \end{aligned}$$

with $\psi \in [0, 2\pi)$.

C.5.1. Transcritical bifurcation

The fold curve in the normal form is obtained by substituting $\rho = 0$ in the system (C.81). Then β_2 is unrestricted and

$$z_0 = -\frac{\beta_1}{g_{200}}.$$

The fold curve is therefore given by

$$(\beta_1, \beta_2) = (0, \beta_2)\,.$$

C.6. Hopf-Hopf bifurcation

We take the normal form (B.13) and truncate to the third order

$$\begin{cases} \dot{z}_1 &= (i\omega_1 + \beta_1 + ib_1(\beta))\, z_1 + g_{2100} z_1 |z_1|^2 + g_{1011} z_1 |z_2|^2, \\ \dot{z}_2 &= (i\omega_2 + \beta_2 + ib_2(\beta))\, z_2 + g_{1110} z_2 |z_1|^2 + g_{0021} z_2 |z_2|^2. \end{cases} \tag{C.83}$$

Letting $(z_1, z_2) = \left(\rho_1 e^{i\psi_1}, \rho_2 e^{i\psi_2}\right)$ we have

$$\begin{aligned} \dot{z_1} &= \dot{\rho}_1 e^{i\psi_1} + \rho_1 i\dot{\psi}_1 e^{i\psi_1} \\ &= (i\omega_1 + \beta_1 + ib_1(\beta))\, \rho_1 e^{i\psi_1} + g_{2100}\rho_1^3 e^{i\psi_1} + g_{1011}\rho_1 e^{i\psi_1}\rho_2^2, \\ \dot{z_2} &= \dot{\rho}_2 e^{i\psi_2} + \rho_2 i\dot{\psi}_2 e^{i\psi_2} \\ &= (i\omega_2 + \beta_2 + ib_2(\beta))\, \rho_2 e^{i\psi_2} + g_{1110}\rho_2 e^{i\psi_2}\rho_1^2 + g_{0021}\rho_2^3 e^{i\psi_2}. \end{aligned}$$

Separating the real and imaginary parts yields the four dimensional system

$$\begin{cases} \dot{\rho_1} = & \rho_1 \left(\beta_1 + \mathrm{Re}(g_{2100})\rho_1^2 + \mathrm{Re}(g_{1011})\rho_2^2\right), \\ \dot{\rho_2} = & \rho_2 \left(\beta_2 + \mathrm{Re}(g_{1110})\rho_1^2 + \mathrm{Re}(g_{0021})\rho_2^2\right), \\ \dot{\psi_1} = & \omega_1 + b_1(\beta) + \mathrm{Im}(g_{2100})\rho_1^2 + \mathrm{Im}(g_{1011})\rho_2^2, \\ \dot{\psi_2} = & \omega_2 + b_2(\beta) + \mathrm{Im}(g_{1110})\rho_1^2 + \mathrm{Im}(g_{0021})\rho_2^2. \end{cases} \tag{C.84}$$

There are two semi-trivial equilibria

$$(\rho_1, \rho_2) = \left(\sqrt{-\frac{\beta_1}{\mathrm{Re}(g_{2100})}}, 0\right), \qquad (\rho_1, \rho_2) = \left(0, \sqrt{-\frac{\beta_2}{\mathrm{Re}(g_{0021})}}\right)$$

of the amplitude system of (C.84). Translating to the original system provides the Hopf bifurcation curves

$$H_1 = \{(\beta_1, \beta_2) : \beta_1 = 0\}, \qquad \text{and} \qquad H_2 = \{(\beta_1, \beta_2) : \beta_2 = 0\}.$$

A nontrivial equilibrium to the amplitude system is given by

$$\begin{aligned} &(\rho_1, \rho_2) \\ &= \left(\frac{\sqrt{\beta_2 \,\mathrm{Re}\, g_{1011} - \beta_1 \,\mathrm{Re}\, g_{0021}}}{\sqrt{\mathrm{Re}\, g_{0021} \,\mathrm{Re}\, g_{2100} - \mathrm{Re}\, g_{1011} \,\mathrm{Re}\, g_{1110}}}, \frac{\sqrt{\beta_2 \,\mathrm{Re}\, g_{2100} - \beta_1 \,\mathrm{Re}\, g_{1110}}}{\sqrt{\mathrm{Re}\, g_{1011} \,\mathrm{Re}\, g_{1110} - \mathrm{Re}\, g_{0021} \,\mathrm{Re}\, g_{2100}}}\right) \end{aligned}$$

and corresponds to a torus of the original system. When

$$\mathrm{Re}(g_{1110})\beta_1 = \mathrm{Re}(g_{2100})\beta_2 \tag{C.85}$$

the nontrivial equilibrium coincides with the first semi-trivial equilibrium, and thus giving a predictor to a Neimark-Sacker bifurcation curve. Similarly, when

$$\mathrm{Re}(g_{0021})\beta_1 = \mathrm{Re}(g_{1011})\beta_2 \tag{C.86}$$

the nontrivial equilibrium coincides with the second semi-trivial equilibrium, and gives a predictor for a second Neimark-Sacker bifurcation curve. Therefore, we obtain two Neimark-Sacker bifurcation curves in (C.83), with approximation given by

$$\begin{aligned} (\rho_1,\rho_2,\beta_1,\beta_2) &= \left(\epsilon, 0, -\mathrm{Re}(g_{2100})\epsilon^2, -\mathrm{Re}(g_{1110})\epsilon^2\right), \\ (\rho_1,\rho_2,\beta_1,\beta_2) &= \left(0, \epsilon, -\mathrm{Re}(g_{1011})\epsilon^2, -\mathrm{Re}(g_{0021})\epsilon^2\right), \end{aligned} \tag{C.87}$$

where $\epsilon > 0$, which coincide with the predictors given in [37] and [41].

Period An approximation for the period of the cycle for the Neimark-Sacker predictors can be obtained from the third and fourth equation in the system (C.84), yielding

$$\begin{aligned} T_1 &= \frac{2\pi}{\omega_1 + \frac{\partial}{\beta_1} b_1(\beta)\beta_1 + \frac{\partial}{\beta_2} b_1(\beta)\beta_2 + \mathrm{Im}(g_{2100})\epsilon^2}, \\ T_2 &= \frac{2\pi}{\omega_2 + \frac{\partial}{\beta_1} b_2(\beta)\beta_1 + \frac{\partial}{\beta_2} b_2(\beta)\beta_2 + \mathrm{Im}(g_{0021})\epsilon^2}. \end{aligned}$$

Here (β_1, β_2) are as in C.85 and (C.86), respectively. Lastly, to approximate the cycles we substitute $z_1 = \epsilon e^{i\psi_1}$ and (C.85), and $z_2 = \epsilon e^{i\psi_2}$ and (C.86) into (6.100). We obtain

$$\begin{aligned} x &= \mathcal{H}\left(\epsilon e^{i\psi_1}, \epsilon e^{-i\psi_1}, 0, 0, \beta_1, \beta_2\right) \\ &= 2\,\mathrm{Re}\left(e^{i\psi_1}\phi_1\right)\epsilon + \Bigg(-\mathrm{Re}(g_{1110})H_{00001} - \mathrm{Re}(g_{2100})H_{00010} \\ &\qquad + H_{110000} + \mathrm{Re}\left(e^{2i\psi_1}H_{200000}\right)\Bigg)\epsilon^2 \end{aligned}$$

and

$$\begin{aligned} x &= \mathcal{H}\left(0, 0, \epsilon e^{i\psi_2}, \epsilon e^{-i\psi_2}, \beta_1, \beta_2\right) \\ &= 2\,\mathrm{Re}\left(e^{i\psi_2}\phi_2\right)\epsilon + \Bigg(-\mathrm{Re}(g_{0021})H_{00001} - \mathrm{Re}(g_{1011})H_{00010} \\ &\qquad + H_{001100} + \mathrm{Re}\left(e^{2i\psi_1}H_{002000}\right)\Bigg)\epsilon^2 \end{aligned}$$

with $\psi \in [0, 2\pi)$.

D. Nonuniqueness problem with the second-order predictor

Below we illustrate that the second-order homoclinic predictor derived in [1, 32] is not unique. We show that there is a transformation which does not affect the 'form' of the normal form, but does alter the predictor. Two possible directions of solving the nonuniqueness of the predictor are presented.

We take the smooth normal form for the generic Bogdanov-Takens bifurcation and truncate

$$\begin{cases} \dot{x}_0 &= x_1, \\ \dot{x}_1 &= \alpha_1 + \alpha_2 x_1 + (a + a_1\alpha_2)\, x_0^2 \\ & \qquad + (b + b_1\alpha_2)\, x_0 x_1 + d x_0^3 + e x_0^2 x_1. \end{cases} \tag{D.1}$$

The second order predictor is given by

$$\begin{cases} \alpha_1 = & -\dfrac{4}{a}\varepsilon^4, \\ \alpha_2 = & \dfrac{b}{a}\varepsilon^2 \left(\frac{10}{7} + \varepsilon^2 \left(\frac{1}{a} \left(\frac{100}{49} b_1 - 4\frac{e}{b} \right) + \frac{1}{a^2} \left(\frac{288}{2401} b^2 - \frac{50 b a_1}{49} + \frac{146}{49} d \right) \right) \right). \end{cases} \tag{D.2}$$

The transformation

$$\begin{cases} x_0 &= w_0 + \xi_7 w_0 \beta_2 = w_0\,(1 + \xi_7\beta_2)\,, \\ x_1 &= w_1 + \xi_7 w_1 \beta_2 = w_1\,(1 + \xi_7\beta_2)\,, \\ \alpha_1 &= \beta_1 + \xi_7 \beta_1 \beta_2 = \beta_1\,(1 + \xi_7\beta_2)\,, \\ \alpha_2 &= \beta_2. \end{cases} \tag{D.3}$$

gives the system

$$\begin{cases} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= \beta_1 + \beta_2 w_1 + a w_0^2 + b w_1 w_0 + (a_1 + a\xi_7 + a_1\beta_2\xi_7)\,\beta_2 w_0^2 \\ & \qquad + (b_1 + b\xi_7 + b_1\beta_2\xi_7)\,\beta_2 w_1 w_0 + \left(d + 2\beta_2 d\xi_7 + \beta_2^2 d\xi_7^2\right) w_0^3 \\ & \qquad + \left(e + 2\beta_2 e\xi_7 + \beta_2^2 e\xi_7^2\right) w_1 w_0^2, \end{cases}$$

which is included in the smooth normal form. Truncating to the third order gives

$$\begin{cases} \dot{w}_0 &= w_1, \\ \dot{w}_1 &= \beta_1 + \beta_2 w_1 + a w_0^2 + b w_1 w_0 + (a_1 + a\xi_7)\,\beta_2 w_0^2 + (b_1 + b\xi_7)\,\beta_2 w_1 w_0 \\ & \quad + d w_0^3 + e w_1 w_0^2, \end{cases} \tag{D.4}$$

which has the second order predictor

$$\begin{cases} \beta_1 = & -\frac{4}{a}\varepsilon^4, \\ \beta_2 = & \frac{b}{a}\varepsilon^2\left(\frac{10}{7} + \varepsilon^2\left(\frac{1}{a}\left(\frac{100}{49}\left(b_1 + b\xi_7\right) - 4\frac{e}{b}\right) + \frac{1}{a^2}\left(\frac{288}{2401}b^2 - \frac{50b(a_1+a\xi_7)}{49} + \frac{146}{49}d\right)\right)\right) \\ = & \frac{b}{a}\varepsilon^2\left(\frac{10}{7} + \varepsilon^2\left(\frac{1}{a}\left(\frac{100}{49}b_1 - 4\frac{e}{b}\right) + \frac{1}{a^2}\left(\frac{288}{2401}b^2 - \frac{50ba_1}{49} + \frac{146}{49}d\right) + \frac{50b\xi_7}{a49}\right)\right). \end{cases} \tag{D.5}$$

Substituting into the transformation (D.3) does not give back (D.5). However, this is to be expected since the predictors are not solutions to the systems, but approximate the solutions. The problem is that the predictor is invariant under the transformation in the K_{11} direction, which corresponds to the term $\beta_1\beta_2$ in the transformation (D.3). Indeed, the systems (D.1) and (D.4) are related by the transformation

$$\alpha = \frac{10}{7}\frac{b}{a}K_{01}\varepsilon^2 + \left(\frac{b}{a}\tau_2 K_{01} + \frac{50b^2}{49a^2}K_{02} - \frac{4}{a}K_{10}\right)\varepsilon^4,$$

in which no K_{11} term is present. Therefore, the transformation (D.3) gives a predictor for the system (D.4), which cannot be transformed back to the original system. One way to solve this problem is to use the freedom in $H_{1001} \to H_{1001} + \xi_7 q_0$ to make K_{11} vanish. This can be done since

$$K_{11} = z_1 K_{10},$$

where

$$z_1 = p_1^T\Big(H_{0101} - A_1(K_{10}, H_{0001}) - A_1(K_{01}, H_{0010}) - B(H_{0001}, H_{0010}) - J_2(K_{01}, K_{10})\Big)$$

and H_{0101} can be translated by $\tilde{H}_{0101} = H_{0101} + \xi_7 q_1$. Then for

$$\xi_7 = p_1^T\Big(H_{0101} - A_1(K_{10}, H_{0001}) - A_1(K_{01}, H_{0010}) - B(H_{0001}, H_{0010}) - J_2(K_{01}, K_{10})\Big)$$

we have $K_{11} = 0$. However, with this line of reasoning H_{0101} should be made zero as well, since the systems (D.1) and (D.4) are related by the transformation

$$\begin{aligned} x = \varepsilon^2 &\left(\frac{10b}{7a}H_{0001} + \frac{1}{a}u_0(\zeta)q_0\right) + \varepsilon^3\left(\frac{1}{a}v_0(\zeta)q_1 + \frac{1}{a}u_1(\zeta)q_0\right) + \varepsilon^4\Big(-\frac{4}{a}H_{0010} \\ &+ \frac{50b^2}{49a^2}H_{0002} + \frac{b}{a}\tau_2 H_{0001} + \frac{1}{a}u_2(\zeta)q_0 + \frac{1}{a}v_1(\zeta)q_1 + \frac{1}{2a^2}H_{2000}u_0^2(\zeta) \\ &+ \frac{10b}{7a^2}H_{1001}u_0(\zeta)\Big) + \mathcal{O}(\varepsilon^5), \end{aligned}$$

in state space. Although this can be done in this specific situation, this may not be true in general.

Another way to try to solve the problem is to include the K_{11} term into the predictor, i.e. we use the predictor

$$\alpha = \frac{10}{7}\frac{b}{a}K_{01}\varepsilon^2 + \left(\frac{b}{a}\tau_2 K_{01} + \frac{50b^2}{49a^2}K_{02} - \frac{4}{a}K_{10}\right)\varepsilon^4 - \frac{40b}{7a^2}K_{11}\epsilon^6. \tag{D.6}$$

In Figure D.1 we compare these two strategies for the system

$$\begin{cases} \dot{x}_0 & = x_1 + \omega_7 x_1 \alpha_2, \\ \dot{x}_1 & = \alpha_1 + \alpha_2 x_1 + a x_0^2 + b x_0 x_1. \end{cases} \tag{D.7}$$

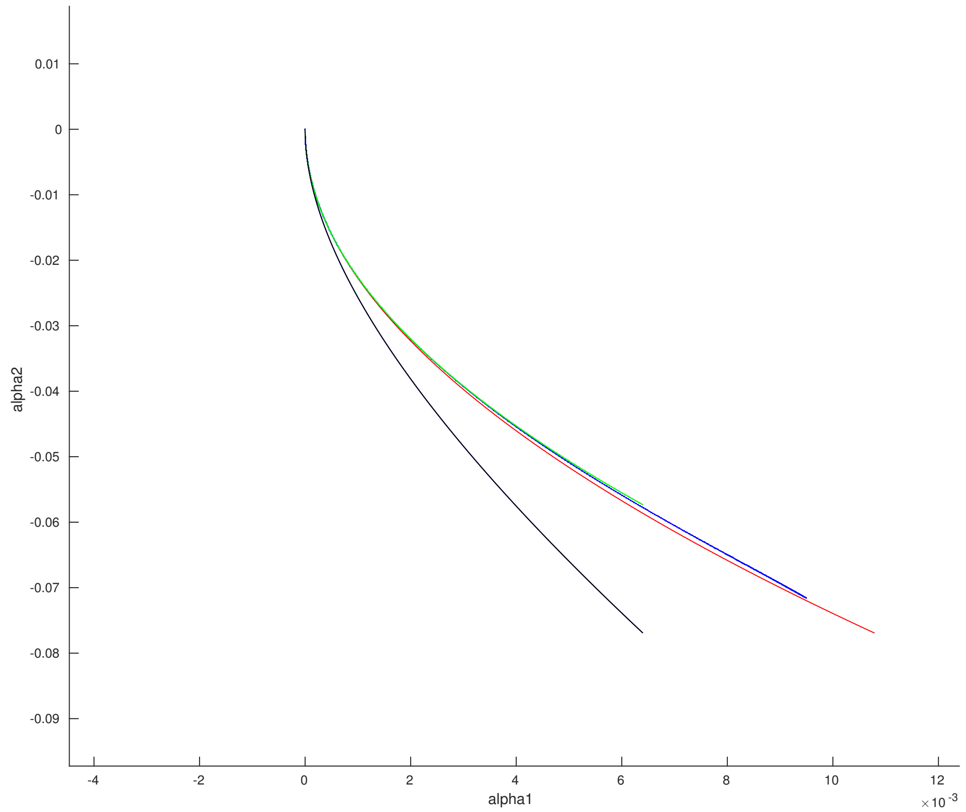


Figure D.1.: Comparison of the predictors using the freedom in $H_{1001} \to H_{1001} + \xi_7 q_0$, for the systems (D.7) with $\omega_7 = 12$ in parameter space. The dark blue curve is the computed homoclinic curve. The black curve is the predictor as it is implemented now. The red curve is the predictor with K_{11} made zero. Lastly, the green curve is created by using the extended predictor (D.6) without using any freedom.

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