Representation Theory and the McKay Correspondence

Bachelorthesis

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Contents

1 introduction

We start by giving an introduction to representation theory and then apply it to the symmetry groups belonging to the 5 platonic solids; the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron. These 5 solids constitute 3 important symmetry groups, namely the tetrahedral group, octahedral group and the icosahedral group. Each symmetry group is isomorphic to some permutation group. For these 3 permutation groups we can find the character table, which we shall learn is a matrix that contains all the important information about a group. A character of a group G is an important concept in representation theory and is a function from G to $\mathbb{C}$. The most interesting characters of a group are the irreducible characters. The number of irreducible characters of each group equals the number of conjugacy classes of that group.

The symmetry groups of platonic solids consist of the rotations which leave the solid invariant and hence are subgroups of $SO(3)$. We shall see that there exist a surjective grouphomomorphism between $SU(2)$ and $SO(3)$;

$$\phi : SU(2) \to SO(3)$$

The kernel of this homomorphism is $\{-1, 1\}$. When we take the pre-images of subgroups of $SO(3)$ under this homomorphism ϕ, we get subgroups of $SU(2)$ which we shall call binary groups (since they contain twice as many elements as the original subgroup of $SO(3)$). Hence to each symmetry group of a platonic solid we can associate a binary group. For a binary group G^* of a group G, we have that;

$$G^*/\{-1, 1\} \cong G$$

Using the known character tables of the symmetry groups of the platonic solids we can construct the character tables for the corresponding binary groups. Each of the 3 groups has a special "natural" character. This character is fully determined by the rotation angles of the corresponding group elements. Using this character we can construct a new inproduct on the space of all functions from G to $\mathbb{C}$. We use this new inproduct to make a Gram-matrix, from which we can construct a so called Dynkin-Coxeter diagram of a group. These diagrams turn out to be important graphs in the theory of Lie groups. John McKay (1939) was a mathematician that discovered a link between the Dynkin graphs belonging to the representations of the binary groups associated with the platonic solids (representation theory) and geometrical structures of these groups (singularity theory). This link is called the McKay correspondence.

2 Representation Theory

2.1 Representations

Recall that $\mathrm{GL}(n, \mathbb{C})$ is the group of all invertible $n \times n$ matrices with entries in $\mathbb{C}$. Now we define a representation of a group G over $\mathbb{C}$.

Definition 2.1. *A representation of a group G over $\mathbb{C}$ is a homomorphism ρ from G to $GL(n, \mathbb{C})$, for some n (which is the degree of the representation).*

We recall that a homomorphism between two groups $(G, \bullet)$ and $(H, \circ)$ is a function: $\phi \colon G \longrightarrow H$ which satisfies $\phi(a \bullet b) = \phi(a) \circ \phi(b)$ for all $a, b \in G$.

For a representation ρ of G then it must hold that:

1. $\rho(g \bullet h) = \rho(g)\,\rho(h)$ for all $g, h \in G$
2. $\rho(e) = I_n$ (identity matrix)
3. $\rho(g^{-1}) = \rho(g)^{-1}$ for all $g \in G$

When we are given a representation of a group we can simply transform this representation into another one. Let S be an arbitrary invertible $n \times n$ matrix and let ρ be a representation of a group G of degree n. Now we define $\phi : G \longrightarrow \mathrm{GL}(n, \mathbb{C})$, $g \mapsto S^{-1}(\rho(g))S$. This clearly is another representation of G of degree n.

Definition 2.2. *Let ρ and ϕ be two representations of G of degree n, then we say that ρ equivalent to ϕ if there exist an invertible $n \times n$ S such that:*

$$\rho(g) = S^{-1}(\phi(g))S, \ \forall\ g \in G$$

Equivalence of representations is easilly seen to be an equivalence relation on the set of representations of G. Let ρ, ϕ and θ be representations of a group G:

- Every representation is equivalent to itself (take $S = I_n$)
- If ρ is equivalent to ϕ, then ϕ is equivalent to ρ (take S^{-1})
- If ρ is equivalent to ϕ and ϕ is equivalent to θ, then ρ is equivalent to θ, (take $S_1 S_2$)

Like every other function a representation of a group G has a kernel defined as $Ker\ \rho = \{g \in G \mid \rho(g) = I_n\}$. We call a representation faithful when $Ker\ \rho = \{e\}$.

2.2 The group algebra

Let G be a finite group with elements $g_1, \ldots, g_n$. Using the elements of G as a basis we can construct a vectorspace over $\mathbb{C}$. The elements of this vectorspace $\mathbb{C}G$ are of the form $\lambda_1 g_1 + \ldots + \lambda_n g_n$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{C}\}$. Now let

$$u = \sum_{i=1}^{n} \lambda_i g_i$$

and

$$v = \sum_{i=1}^{n} \mu_i g_i$$

be elements of $\mathbb{C}G$ and $\lambda \in \mathbb{C}$. We then define addition and scalar multiplication on the elements of $\mathbb{C}G$ as follows:

$$u + v = \sum_{i=1}^{n} (\lambda_i + \mu_i) g_i$$

$$\lambda u = \sum_{i=1}^{n} (\lambda \lambda_i) g_i$$

We now easily check that $\mathbb{C}G$ is indeed a vectorspace over $\mathbb{C}$ with dimension equal to n (the order of our group G). The basis $g_1, g_2, \ldots, g_n$ that we used is called the natural basis of $\mathbb{C}G$.

Definition 2.3 (The group algebra)**.** *Let*

$$a = \sum_{g \in G} \lambda_g g$$

$$b = \sum_{h \in G} \mu_h h$$

be elements of the vectorspace $\mathbb{C}G$, then the vectorspace $\mathbb{C}G$ together with the multiplication defined by;

$$ab = \left(\sum_{g \in G} \lambda_g g \right) \left(\sum_{h \in G} \mu_h h \right) = \sum_{g,h \in G} \lambda_g \mu_h (gh)$$

is named the group algebra of G over $\mathbb{C}$.

As one can check the group algebra $\mathbb{C}G$ satisfies the following properties, for all $r,\ s,\ t \in \mathbb{C}G$ *and* $\lambda \in \mathbb{C}$:

1. $rs \in \mathbb{C}G$;

2. $r(st) = (rs)t$;
3. $re = er = r$;
4. $(\lambda r)s = \lambda(rs) = r(\lambda s)$;
5. $(r + s)t = rt + st$;
6. $r(s + t) = rs + rt$;
7. $r0 = 0r = 0$

2.3 $\mathbb{C}G$-modules

One of the most used concepts in representation theory is that of the $\mathbb{C}G$-module.

Definition 2.4. *Let V be a vector space over $\mathbb{C}$ and let G be a group. We call V a $\mathbb{C}G$-module if we can multiply vectors of V with group elements of G (gv, $v \in V$ and $g \in G$). Such that the following conditions are satisfied for all $u, v \in V, \lambda \in \mathbb{C}$ and $g, h \in G$;*

1. $gv \in V$;
2. $(hg)v = h(gv)$;
3. $ev = v$;
4. $g(\lambda v) = \lambda(gv)$;
5. $g(u + v) = gu + gv$.

Now let $\rho : G \longrightarrow GL(n, \mathbb{C})$ be a representation of a group G. Then for every $g \in G$ we find an invertible $n \times n$ matrix $\rho(g)$. Let $V = \mathbb{C}^n$ be the vectorspace that contains all the vectors $(c_1, \ldots, c_n)^T$, with $c_i \in \mathbb{C}$ for $1 \leq i \leq n$. Since $\rho(g) \in GL(n, \mathbb{C})$ we can multiply with elements of V using matrix multiplication ($\rho(g)v$, $v\ inV$ and $g \in G$). Now let $u, v \in V$, $\lambda \in \mathbb{C}$ and $g, h \in G$, we easily check that this multiplication satisfies the following properties;

- $\rho(g)v$ lies in V.
- Since ρ is a homomorphism, it holds that $\rho(gh)v = \rho(g)\rho(h)v$.
- Because $\rho(e) = I_n$, we have $\rho(e)v = v$.
- $\rho(g)(\lambda v) = \lambda(\rho(g)v)$ (due to the known properties of matrixmultiplication).
- $\rho(g)(v + u) = \rho(g)(v) + \rho(g)(u)$ (again from matrix multiplication).

These are precisely the conditions for $\mathbb{C}^n$ to be a $\mathbb{C}G$-module. So representations of a group G correspond to $\mathbb{C}G$-modules.

Definition 2.5. *Let V be a $\mathbb{C}G$-module and let $\mathcal{B}$ be a bases of V. For each $g \in G$ we denote $[g]_{\mathcal{B}}$ for the matrix of the endomorphism $\epsilon_g : V \to V, v \mapsto gv$ of V, relative to the bases $\mathcal{B}$.*

We can now construct a representation of a group G from a given $\mathbb{C}G$-module.

Theorem 2.6. *Let V be a $\mathbb{C}G$-module and let $\mathcal{B}$ be a bases of V. Then the function*

$$\rho_{V\mathcal{B}} \colon G \to GL(n, \mathbb{C}),\ g \mapsto [g]_{\mathcal{B}}$$

is a representation of G over $\mathbb{C}$.

Definition 2.7. *The 1-dimensional $\mathbb{C}G$-module V with multiplication defined as:*

$$gv = v \quad \textit{for all} \quad v \in V,\ g \in G$$

is defined as the trivial $\mathbb{C}G$-module.

Definition 2.8. *A $\mathbb{C}G$-module V is faithful if the only element of G which satisfies:*

$$gv = v \quad \textit{for all} \quad v \in V$$

is the identity element of G $(g = e)$.

2.3.1 $\mathbb{C}G$-submodules

Definition 2.9. *A $\mathbb{C}G$-submodule W of a $\mathbb{C}G$-module V is a subspace of V which satisfies;*

$$gw \in W$$

for all $w \in W$ and $g \in G$

Thus a $\mathbb{C}G$-submodule is itself a $\mathbb{C}G$-module. Remark that for any $\mathbb{C}G$-module V the zero subspace $\{0\}$ and V itself are $\mathbb{C}G$-submodules of V.

2.3.2 Irreducible $\mathbb{C}G$-modules

Definition 2.10. *A $\mathbb{C}G$-module V is called irreducible if it is a non-zero vectorspace and it doesn't have $\mathbb{C}G$-submodules different from $\{0\}$ and V itself. We call V reducible if there does exist a $\mathbb{C}G$-submodule W unequal to $\{0\}$ or V.*

We call a representation ρ reducible/irreducible if the $\mathbb{C}G$-module $\mathbb{C}^n$ defined by $gv = \rho(g)v$, where $v \in \mathbb{C}^n$ and $g \in G$ is reducible/irreducible.

2.3.3 The regular $\mathbb{C}G$-module

We can now construct a $\mathbb{C}G$-module using the group algebra $\mathbb{C}G$ just defined. Here we multiply a vector of the group algebra;

$$v = \sum_{i=1}^{n} \mu_i g_i \in \mathbb{C}G$$

with an element of the group $g \in (G, \circ)$. Hence we get;

$$gv = \sum_{i=1}^{n} \mu_i (g \circ g_i)$$

.
Now let $V = \mathbb{C}G$, then for all $u,\ v \in V$, $\lambda \in \mathbb{C}$ and $g,\ h \in G$ the following properties hold:

- $gv \in V$,
- $(gh)v = g(hv)$,
- $ev = v$,
- $g(\lambda v) = \lambda(gv)$,
- $g(u + v) = gu + gv$

Definition 2.11. *The $\mathbb{C}G$-module that we just defined is called the regular $\mathbb{C}G$-module. Also, the representation of G given by $\rho : G \to GL(|G|\ , \mathbb{C})$, $g \mapsto [g]_{\mathcal{B}}$ (with $\mathcal{B}$ the natural basis of $\mathbb{C}G$) is called the regular representation of G over $\mathbb{C}$.*

Remark that the regular representation of a group G is faithful.

2.3.4 $\mathbb{C}G$-homomorphisms

For a group homomorphism ϕ between two groups $(G, \bullet)$ and $(H, \circ)$, we have that $\phi(a \bullet b) = \phi(a) \circ \phi(b)$. So we get the same thing if we first multiply two elements of G in G and sent the result to H using ϕ, as when we send each element separately to H and then multiply them in H. We say that the group homomorphism ϕ preserves the structure of G. A similar thing can be said about linear maps between vector spaces.
We shall now define a map between two $\mathbb{C}G$-modules V and W, called an $\mathbb{C}G$-homomorphism.

Definition 2.12 (**$\mathbb{C}G$-homomorphism**). *Let V and W be two $\mathbb{C}G$-modules. A $\mathbb{C}G$-homomorphism θ between V and W is a linear transformation that satisfies:*

$$\theta(gv) = g\theta(v) \quad \text{for all} \quad v \in V,\ g \in G$$

If θ is also a bijection we call it a $\mathbb{C}G$-isomorphism and two $\mathbb{C}G$-modules V and W are isomorphic if there exist a $\mathbb{C}G$-isomorphism between them (we then write $V \cong W$).

So a $\mathbb{C}G$-homomorphism has the property that it does't matter if we first multiply an element $v \in V$ with a group element $g \in G$ and send the result to W using θ or if we first send v to W and then muliply with g in W. Just as group homomorphisms and linear maps a $\mathbb{C}G$-homomorphism preserves the structure of $\mathbb{C}G$-modules.

Theorem 2.13. *For any $\mathbb{C}G$-homomorphism $\phi : V \to W$ between two $\mathbb{C}G$-modules V and W, we have that $Ker\ \phi$ is a $\mathbb{C}G$-submodule of V and that $Im\ \phi$ is a $\mathbb{C}G$-submodule of W.*

Proposition 2.14. *Suppose that two $\mathbb{C}G$-modules V and W are isomorphic, then*

1. *dim V = dim W*
2. *V is irreducible if and only if W is irreducible*

Proof of 2. Let θ be an $\mathbb{C}G$ isomorphism between V and W. Assume that V is reducible, so there exist a $\mathbb{C}G$-submodule X of V unequal to $\{0\}$ or V. We show that $\theta(X)$ is an $\mathbb{C}G$-submodule of W unequal to $\{0\}$ or W. Let $w \in \theta(X)$, so there exist $x \in X$ such that $w = \theta(x)$. Now $gw = g\theta(x) = \theta(gx)$ and since $gx \in X$ we have that $gw \in \theta(X)$. Let $x \in X$ such that $x \neq 0$ (this is possible because $X \neq \{0\}$). Since θ is injective we have that $\theta(x) \neq 0$ and hence $\theta(X) \neq \{0\}$. Since $X \neq V$, we can take $v \in V$ but $v \notin X$. It then follows that $\theta(v) \notin \theta(X)$ (because θ is injective). Hence $\theta(X) \neq W$. We have now showed that $\theta(X)$ is an $\mathbb{C}G$-submodule of W (and hence that W is reducible). □

Theorem 2.15. *Let V and W be two $\mathbb{C}G$-modules with basis $\mathcal{B}$, $\mathcal{B}'$ respectively. Then V and W are isomorpic if and only if the representations*

$$\rho : \quad g \mapsto [g]_{\mathcal{B}}$$

and

$$\sigma : \quad g \mapsto [g]_{\mathcal{B}'}$$

are equivalent.

Proposition 2.16. *Let V be a $\mathbb{C}G$-module such that $V = U_1 \oplus U_2 \oplus \ldots \oplus U_n$, where U_i is a $\mathbb{C}G$-submodules of V, for each $1 \leq i \leq n$. Now for each $i \in$ brace1, $\ldots$, $n\}$ we let $\mathcal{B}_i$ be a basis of U_i. Since V is a direct sum of the U_i's, it follows that $\mathcal{B}_1$, $\ldots$, $\mathcal{B}_n$ together constitute a basis $\mathcal{B}$ of V. Now let $g \in G$, we then have that:*

$$[g]_{\mathcal{B}} = \begin{pmatrix} [g]_{\mathcal{B}_1} & & 0 \\ & \ddots & \\ 0 & & [g]_{\mathcal{B}_n} \end{pmatrix}$$

Theorem 2.17 (Mashke's Theorem)**.** *Let G be a finite group end let V be a $\mathbb{C}G$-module. Now if U is a $\mathbb{C}G$-submodule of V, then there exist another $\mathbb{C}G$-submodule W of V such that:*

$$V = U \oplus W$$

Now since U and W are both $\mathbb{C}G$-modules, they could be reducible and hence we can apply Maschke's theorem again to U and W. Using induction we can write every $\mathbb{C}G$-module of a finite group G as a direct sum of irreducible $\mathbb{C}G$-submodules.

Theorem 2.18. *If G is a finite group and V a non-zero $\mathbb{C}G$-module, then V can be written as;*

$$V = U_1 \oplus \ldots \oplus U_n$$

where U_i is an irreducible $\mathbb{C}G$-submodule of V for each $1 \leq i \leq n$.

Let G be a finite group and V be a $\mathbb{C}G$-module that has a $\mathbb{C}G$-submodule U. By Maschke's theorem, there exist another $\mathbb{C}G$-submodule W of V such that $V = U \oplus W$. So for every $v \in V$, we have unique vectors $u \in U$ and $w \in W$ such that $v = u + w$. Now consider the function $\rho : V \to U,\ v = u + w \mapsto u$. It's eassily checked that ρ is a linear transformation. Now let $g \in G$, then we have $\rho(gv) = \rho(g(u + w)) = \rho(gu + gw) = gu = g\rho(v)$. So ρ is an $\mathbb{C}G$-homomorphism. We also have that $\rho^2(v) = \rho^2(u + w) = \rho(u) = u = \rho(v)$, hence ρ is a projection of V onto U.

Proposition 2.19. *Let G be a finite group and V be a $\mathbb{C}G$-module. Suppose that V is reducible and that U is an $\mathbb{C}G$-submodule of V. Then there exist a surjective $\mathbb{C}G$-homomorphism from V onto U.*

Theorem 2.20 (Schur's Lemma)**.** *Let V and W be irreducible $\mathbb{C}G$-modules.*

1. *If $\theta : V \to W$ is a $\mathbb{C}G$-homomorphism, then either $\theta = 0$ ($\theta(v) = 0$ for all $v \in V$) or θ is a $\mathbb{C}G$-isomorphism.*

2. *If $\theta : V \to V$ is a $\mathbb{C}G$-isomorphism, then θ is a scalar multiple of the identity endomorphism e_V ($e_V(v) = v$ for all $v \in V$).*

Theorem 2.21 (The decomposition of the regular $\mathbb{C}G$-module)**.** *Let $\mathbb{C}G$ be the regular $\mathbb{C}G$-module, then by Maschke's Theorem we can write:*

$$\mathbb{C}G \ = \ U_1 \oplus \ \dots \ \oplus U_k$$

with U_i irreducible for each $1 \leq i \leq k$. Now let U be any irreducible $\mathbb{C}G$-module, then there exist U_i such that $U \cong U_i$. The number of U_i's (in the decomposition of $\mathbb{C}G$) that are isomorphic to U equals $dim(U)$.

Definition 2.22. *We call a set $U_1, \dots, U_k$ of non-isomorphic irreducible $\mathbb{C}G$-modules complete if every arbitrary irreducible $\mathbb{C}G$-module U is isomorphic to U_i for some $i \in \{1, \dots, k\}$.*

Theorem 2.23. *Let $U_1, \dots, U_k$ be a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules. Then;*

$$\sum_{i=1}^{k} (dim U_i)^2 = |G|$$

2.4 Characters

Definition 2.24. *Let $A = (a)_{ij}$ be an $n \times n$ matrix. The number;*

$$trA \ = \ \sum_{i=1}^{n} a_{ii}$$

is defined as the trace of the matrix A.

Let A and B be $n \times n$ matrices and T an invertible $n \times n$ matrix. Using the definition of the trace of a matrix, we can derive the following;

- $tr(AB) = tr(BA)$
- $tr(T^{-1}AT) = tr(A)$

Definition 2.25 (Character)**.** *Let V be a $\mathbb{C}G$-module with basis $\mathcal{B}$. Let $g \in G$, then the function $\chi : G \to \mathbb{C}, g \mapsto tr[g]_{\mathcal{B}}$ is called the character of the $\mathbb{C}G$-module V.*

The character does not depend on the basis we choose for V. To see this, let $\mathcal{B}_0$ be another basis of V. Since $\mathcal{B}$ and $\mathcal{B}_0$ are both bases of V, there exist an invertible change of basis matrix T such that $[g]_{\mathcal{B}_0} = T^{-1}[g]_{\mathcal{B}}T$. Now using the properties of the trace of a matrix, we find that $tr\ [g]_{\mathcal{B}_0} = tr\ ((T^{-1}[g]_{\mathcal{B}})T) = tr[g]_{\mathcal{B}}$.

Definition 2.26. *We call a character χ of a group G irreducible if it is the character of an irreducible $\mathbb{C}G$-module.*

When we have two isomorphic $\mathbb{C}G$-modules V and W we know that for each $g \in G$ we can find a basis $\mathcal{B}_1$ of V and a basis $\mathcal{B}_2$ of W such that $[g]_{\mathcal{B}_1} = [g]_{\mathcal{B}_2}$. From this it follows that every two isomorphic $\mathbb{C}G$-modules have the same character.

Let χ be a character of a $\mathbb{C}G$-module V and let a and b be two conjugate elements of G (so there exist an element $g \in G$ such that $a = g^{-1}bg$). Now

$$\chi(a) = tr\ [a]_{\mathcal{B}} = tr\ ([g]_{\mathcal{B}}{}^{-1}[b]_{\mathcal{B}}[g]_{\mathcal{B}}) = tr\ [b]_{\mathcal{B}} = \chi(b)$$

We say that the character of a group G is constant on the conjugacy classes of G.

Definition 2.27 (Degree of a character)**.** *Let V be a $\mathbb{C}G$-module with character χ, then the degree of χ equals the dimension of V.*

Theorem 2.28. *Let V be a $\mathbb{C}G$-module with character χ. Now let $g \in G$ with order m. We then have that:*

1. *$\chi(e) = dim(V)$;*
2. *$\chi(g)$ is a sum of mth roots of unity;*
3. *$\chi(g^{-1}) = \overline{\chi(g)}$;*
4. *$\chi(g)$ is a real number if g is conjugate to g^{-1}.*

Theorem 2.29. *Let $\rho : G \to GL(n, \mathbb{C})$ be a representation with character χ. We then have the following;*

1. *Let $g \in G$, $|\chi(g)| = \chi(e) \Leftrightarrow \rho(g) = \lambda I_n$ (for some $\lambda \in \mathbb{C}$);*

2. $Ker\rho = \{g \in G : \chi(g) \;=\; \chi(e)\}$

Definition 2.30. *We define the kernel of a character χ as: $Ker\chi \;=\; \{g \in G :\; \chi(g) = \chi(e)\}$.*

Given a $\mathbb{C}G$-module V of a finite group G we can write it as a direct sum of irreducible $\mathbb{C}G$-submodules (Maschke's Theorem). Hence we have,

$$V = V_1 \oplus \ldots \oplus V_n$$

where V_i is an irreducible $\mathbb{C}G$-module for each $1 \leq i \leq n$. Now for each $i \in \{1, \ldots, n\}$ let $\mathcal{B}_i$ be a basis of the irreducible $\mathbb{C}G$-submodule V_i. We can use the bases $\mathcal{B}_1, \ \ldots, \ \mathcal{B}_n$ to form a basis $\mathcal{B}$ of V so that for each $g \in G$ we have;

$$[g]_{\mathcal{B}} = \begin{pmatrix} [g]_{\mathcal{B}_1} & & 0 \\ & \ddots & \\ 0 & & [g]_{\mathcal{B}_n} \end{pmatrix}$$

Let χ be the character of V and let χ_i be the character of the $\mathbb{C}G$-submodule V_i. Now for $g \in G$ we than have that $\chi(g) = tr[g]_{\mathcal{B}} = tr[g]_{\mathcal{B}_1} + \ldots + tr[g]_{\mathcal{B}_n} = \chi_1(g) + \ldots \chi_n(g)$.

Theorem 2.31. *The character χ of a $\mathbb{C}G$-module V is equal to a sum of irreducible characters χ_i of G.*

Definition 2.32. *The character of the regular $\mathbb{C}G$-module is called the regular character and is denoted by χ_{reg}.*

Decomposing the regular $\mathbb{C}G$-module as a direct sum of $\mathbb{C}G$-submodules we get that,

$$\mathbb{C}G = (U_1 \oplus \ldots \oplus U_1) \oplus \ldots \oplus (U_n \oplus \ldots \oplus U_n)$$

Where each U_i is an irreducible $\mathbb{C}G$-module and the number of times that U_i appears in the decomposition of $\mathbb{C}G$ equals $d_i = dim(U_i)$.

Theorem 2.33. *The regular character χ_{reg} can be written as,*

$$\chi_{reg} = d_1\chi_1 + \ldots + d_n\chi_n$$

where χ_i is the character of the irreducible $\mathcal{C}G$-module U_i.

Given a group G, let $\mathcal{B}$ $(\{g_1, \ldots, g_n\})$ be the natural basis of the regular $\mathbb{C}\mathbb{G}$-module. For the regular character χ_{reg} it now holds that,

1. $\chi_{reg}(1) = |G|$, since $dim(\mathbb{C}G) = |G|$.
2. $\chi_{reg}(g) = 0$ if $g \neq e$. To see this let $g \in G$ and $g \neq e$, we then have that $g_i g = g_j$ with $j \neq i$. Hence if we look at the matrix $[g]_{\mathcal{B}}$ all the diagonal elements are zero (column i only contains a 1 at position $j \neq i$). Hence the sum of the diagonal elements, $\chi(g)$, equals 0.

Characters of $\mathbb{C}G$-modules are functions from G to $\mathbb{C}$. The set of all functions from G to $\mathbb{C}$ forms a vector space X over $\mathbb{C}$. We can define an inner product on this vector space which has to satisfy the following conditions for all $\theta,\ \phi,\ \theta_1,\ \theta_2 \in X$ and $\lambda,\ \lambda_1,\ \lambda_2 \in \mathbb{C}$;

1. $\langle \theta,\ \phi \rangle = \overline{\langle \phi,\ \theta \rangle}$
2. $\langle \lambda_1\theta_1 + \lambda_2\theta_2,\ \phi \rangle = \lambda_1 \langle \theta_1,\ \phi \rangle + \lambda_2 \langle \theta_2,\ \phi \rangle$
3. $\langle \theta,\ \theta \rangle > 0$ if $\theta \neq 0$

One such an inner product on X is;

$$\langle \theta,\ \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \theta(g)\overline{\phi(g)}$$

For this inner product on X it holds that;

1. $$\langle \theta,\ \phi \rangle \ = \ \langle \phi,\ \theta \rangle = \frac{1}{|G|} \sum_{g \in G} \theta(g)\phi(g^{-1})$$

2. Let G have l conjugacy classes with representatives $g_1, \ldots, g_l$, then

$$\langle \theta,\ \phi \rangle \ = \ \sum_{i=1}^{l} \frac{\theta(g_i)\overline{\phi(g_i)}}{|C_G(g_i)|}$$

where $C_G(g_i)$ is the centralizer of g_i.

Theorem 2.34. *The irreducible characters of a group G form an orthonormal set of vectors in X. So let U and V be non-isomorphic irreducible $\mathbb{C}G$-modules with characters χ and ϕ respectively, then:*

$$\langle \chi, \phi \rangle = 0$$
$$\langle \chi, \chi \rangle = 1$$

Let G be a group and let $U_1, \ldots, U_n$ be a complete set of non-isomorphic irreducible $\mathbb{C}G$-modules. Now let V be a $\mathbb{C}G$-module with character χ. We can decompose U as;

$$U = (U_1 \oplus \ldots U_1) \oplus \ldots \oplus (U_n \oplus \ldots \oplus U_n)$$

and the number of U_i that appears in the decomposition of U is some integer k_i. Let χ_i be the character of the irreducible $\mathbb{C}G$-module U_i. Using the decomposition of V we can now write the character χ of U as; $\chi \ = \ k_1\chi_1 + \ldots + k_n\chi_n$.

Now using the linearity of the innerproduct and the orthogonality of the irreducible characters, we have for each irreducible character χ_i of G;

$$\langle \chi, \chi_i \rangle = \langle k_1\chi_1 + \ldots + k_n\chi_n, \chi_i \rangle = k_1\langle \chi_1, \chi_i \rangle + \ldots + k_i\langle \chi_i, \chi_i \rangle + \ldots + k_n\langle \chi_n, \chi_i \rangle = k_i$$

and k_i equals the number of times that the irreducible $\mathbb{C}G$-module U_i appeared in the decomposition of U. Hence when we are given a $\mathbb{C}G$-module U with character χ, we can find the number of times that an irreducible $\mathbb{C}G$-module U_i with character χ_i appears in the decomposition of U by taking the innerproduct of χ with χ_i.

When we take the inner product of χ with itself we find that;

$$\begin{aligned} \langle \chi, \chi \rangle &= \langle k_1\chi_1 + \ldots + k_n\chi_n,\ k_1\chi_1 + \ldots + k_n\chi_n \rangle \\ &= {k_1}^2\langle \chi_1, \chi_1 \rangle + \ldots + {k_n}^2\langle \chi_n, \chi_n \rangle \\ &= \sum_{i=1}^{n} {k_i}^2 \end{aligned}$$

As we have already seen is that when two $\mathbb{C}G$-modules are isomorphic, then their characters are equal. The following theorem states that the converse of this is also true.

Theorem 2.35. *Let V and W be two $\mathbb{C}G$-modules that have characters χ and ϕ respectively. Then V and W are isomorphic if and only if $\chi = \phi$.*

Theorem 2.36. *Let G be a group, and let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G. Then $\chi_1, \ldots, \chi_k$ are linearly independent vectors in X (the vectorspace of functions from G to $\mathbb{C}$).*

Proof. Assume, to the contrary, that $\chi_1, \ldots, \chi_k$ are linearly dependent. In that case there exist a linear relation between $\chi_1, \ldots, \chi_k$. So there exist complex numbers λ_i (where $\lambda_i \neq 0$ for at least one $i \in \{1, \ldots, k\}$) such that;

$$\lambda_i\chi_i + \ldots + \lambda_k\chi_k = 0$$

where 0 is the zero-function (defined by $0 : G \to \mathbb{C}, g \mapsto 0$).
If we now take the innerproduct of this zero-function with χ_i we have:

$$0 = \langle 0, \chi_i \rangle = \langle \lambda_i\chi_i + \ldots \lambda_k\chi_k, \chi_i \rangle = \lambda_i$$

since this must hold for all $i \in \{1, \ldots, k\}$ we have a contradiction. Hence the irreducible characters $\chi_1, \ldots, \chi_k$ of G are linearly independent in X. □

Functions that are constant on conjugacy classes are called class functions. For instance we see that characters are class functions. The set of all class functions from G to $\mathbb{C}$ is a vector subspace of the vector space X.

Theorem 2.37. *The number of irreducible characters of G equals the number of conjugacy classes of G.*

We now introduce an important matrix of a group G that can be constructed using the irreducible characters of G and representatives of its conjugacy classes. This matrix is called the character table and it contains all of the important information about a group G.

Definition 2.38. *Let $\chi_1, \ldots, \chi_n$ be the irreducible characters of a group G and let $g_1, \ldots, g_n$ be the representatives of the conjugacy classes of G. The character table of G is defined as the $n \times n$ matrix whose ij-entry equals $\chi_1(g_j)$.*

Because the irreducible characters are linearly independent in X, the rows in the character table are linealry independent. Since the character table is a square matrix it follows that it is invertible.

Let G be a group and let N be a normal subgroup of G.

Theorem 2.39. *Let $\tilde{\chi}$ be a character of the group G/N. Then the function $\chi : G \to \mathbb{C}$ defined by*

$$\chi(g) = \tilde{\chi}(Ng)\ (g \in G)$$

is a character of G and has the same degree as $\tilde{\chi}$.

2.5 Conjugacy classes

Let G be a group and let $x \in G$. The conjugacy class of x is denoted by x^G.

Theorem 2.40. *Let G be a group and let $x \in G$. The size of the conjugacy class x^G satisfies:*

$$|x^G| = |G|/|C_G(x)|$$

3 Unitary matrices

A unitary matrix U is a complex square matrix (let's say $n \times n$) which satisfies:

$$UU^* = U^*U = I_n$$

Here U^* is the complex transpose of U, so $U^* = \overline{U^T}$ and hence it holds that $U^{-1} = U^*$. The row and column vectors of $U \in U(n)$ form an orthonormal set in $\mathbb{C}^n$. We call an element $U \in U(n)$ hermitian if $U = U^*$.
Let U be a unitary matrix, then the following hold:

1. U is normal (this follows immediately from the definition of a normal matrix (trivial))

2. U preserves the inner product between two complex vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$:

$$\langle U\mathbf{u}, U\mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$$

3. U is diagonalizable, so $U = VDV^*$, where V is unitary and D is unitary and diagonal

4. $|det(U)| = 1$

5. Its eigenspaces are orthogonal

Theorem 3.1. *Let $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{C}$. Then the collection of all unitary matrices in $M_n(\mathbb{C})$ is a group and we call this group the unitary group and denote it by U_n.*

For all unitary matrices we have that $det(U) = \pm 1$. Now one can easily check that;

$$SU(n) := \{U \in U(n) : det(U) = 1\}$$

is a normal subgroup of $U(n)$, called the special unitary group.

4 Complex numbers

We can represent every complex number by a 2×2 real matrix. Namely let $a + bi \in \mathbb{C}$, then

$$\beta : \mathbb{C} \to M_2(\mathbb{R}) : \quad a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Now it can eassily be shown that the following statements hold for all $z_1,\ z_2 \in \mathbb{C}$:

- $\theta(z_1 + z_2) = \theta(z_1) + \theta(z_2)$
- $\theta(z_1 z_2) = \theta(z_1)\theta(z_2)$
- $\theta\left(\frac{z_1}{z_2}\right) = [\theta(z_2)]^{-1}\theta(z_1),\ if z_2 \neq 0$

Let $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we then have;

$$\beta(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = aI_2 + bQ$$

I_n and Q are both orthogonal matrices ($Q^T = Q^{-1}$)

4.1 isomorphism $SO(2)$ and U(1)

The group of complex numbers of unit length $U(1) = \{e^{i\theta} : 0 \leq i < 2\pi\}$, is isomorphic to the group of rotations $SO(2)$. This follows immediately by checking that the map:

$$\phi : U(1) \to SO(2),\ e^{i\theta} \mapsto \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

is a group isomorphism. The plane $\mathbb{R}^2$ and the complex plane $\mathbb{C}$ can be identified with each other if we let $z = x + iy \in \mathbb{C}$ represent $(x, y) \in \mathbb{R}^2$. Due to the isomorphism just discussed, we can represent every plane rotation ρ_θ (through an angle θ), with muliplication by the complex number $e^{i\theta} \in U(1)$.

5 Quaternions

Just as the complex numbers $\mathbb{C}$ are an extension of the real numbers $\mathbb{R}$, the quaternions $\mathbb{H}$ are an extension of the complex numbers. The quaternions were introduced by Hamilton in 1843, who used them for rotations in three dimensional space. The quaternions form a four-dimensional algebra over the real numbers. The basis elements of $\mathbb{H}$ are 1, i, j and k and satisfy

$$i^2 \ = \ j^2 \ = k^2 \ = \ ijk \ = -1$$

Furthermore multiplication of these basis elements satisfy;

$$ij = -ji, \qquad ik = -ki, \qquad jk = -kj$$

5.1 $\mathbb{H}$ as a unital ring

All quaternions are of the form: $a + bi + cj + dk$, where a, b, c and d are real numbers. So

$$\mathbb{H} \ = \ \{a + bi + cj + dk \ : \ a, b, c, d \in \mathbb{R}\}$$

Let $a, b \in \mathbb{H}$, so $a = a_0 + a_1 i + a_2 j + a_3 k$ and $b = b_0 + b_1 i + b_2 j + b_3 k$. Now addition in $\mathcal{H}$ is defined as:

$$a + b = (a_0 + b_0) + (a_1 + b_1)i + (a_2 + b_2)j + (a_3 + b_3)k$$

We see that addition is well-defined in $\mathbb{H}$.

- From the property that $a_i + (b_i + c_i) \ = (a_i + b_i) + c_i$ for all $a_i, b_i, c_i \in \mathbb{R}$, it follows that for all $a, b, c \in \mathbb{H}$, $a + (b + c) \ = \ (a + b) + c$. Hence addition in $\mathbb{H}$ is associative.

- The quaternion $0 = 0 + 0i + 0j + 0k$ is an additive identity in $\mathbb{H}$.

- Let $a \in \mathbb{H}$, so $a = a_0 + a_1 i + a_2 j + a_3 k$. Now $-a = -a_0 - a_1 i - a_2 j - a_3 k \in \mathbb{H}$ and $a + (-a) = 0$. Hence every element of $\mathbb{H}$ has an additive inverse.

- Let $a, b \in \mathbb{H}$, then $a + b = b + a$ (since addition in $\mathbb{R}$ is commutative we have that $a_i + b_i = b_i + a_i$, for all $0 \leq i \leq 3$). So addition in $\mathbb{H}$ is commutative.

Hence the quaternions $\mathbb{H}$ form an abelian group with respect to addition.

Using the relations between i, j, k, one can verify that for two quaternions $a, b \in \mathbb{H}$ ($a = a_0 + a_1 i + a_2 j + a_3 k$ and $b = b_0 + b_1 i + b_2 j + b_3 k$). it holds that;

$$
\begin{aligned}
ab &= (a_0 + a_1 i + a_2 j + a_3 k)(b_0 + b_1 i + b_2 j + b_3 k) \\
& (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) + \\
& (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) i + \\
& (a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1) j + \\
& (a_0 b_3 + a_1 b_2 - a_2 b_1 + a_3 b_0) k +
\end{aligned}
$$

Given a quaternion $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, we cal q_0 the scalar part and $\mathbf{q} = (q_1, q_2, q_3)$ the vector part. With this notation we can also write the product of two quaternions a, b as;

$$
\begin{aligned}
ab &= a_0 b_0 - (a_1 b_1 + a_2 b_2 + a_3 b_3) + \\
& a_0 (b_1 i + b_2 j + b_3 k) + b_0 (a_1 i + a_2 j + a_3 k) + \\
& (a_2 b_3 - a_3 b_2) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k \\
&= a_0 b_0 - \mathbf{a} \cdot \mathbf{b} + a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}
\end{aligned}
$$

Hence the product of two quaternions has scalar part equal to $a_0 b_0 - \mathbf{a} \cdot \mathbf{b}$ and vector part $a_0 \mathbf{b} + b_0 \mathbf{a} + \mathbf{a} \times \mathbf{b}$.

Multiplication of quaternions is not commutative, as is evident from the cross-product term of ab. For each pair of quaternions $a, b \in \mathbb{H}$ there is a unique product $ab \in \mathbb{H}$, hence multiplication is well-defined in $\mathbb{H}$. Regarding multiplication in $\mathbb{H}$ the following properties hold;

- For all $a, b, c \in \mathbb{H}$ we have that $a(bc) = (ab)c$ (this follows from the associative and distributive laws that hold in $\mathbb{R}$). Hence multiplication in $\mathbb{H}$ is associative.
- The quaternion $e = 1 + 0i + 0j + 0k$ has the property that for all $q \in \mathbb{H}$, $qe = eq = q$. Hence $\mathbb{H}$ has a multiplicative identity.
- The left and right distributive laws hold in $\mathbb{H}$.

We have now showed that $\mathbb{H}$ is a unital ring (with multiplicative identity).

Definition 5.1 (Pure quaternion)**.** *A quaternion $q \in \mathbb{H}$ of the form $q = q_1 i + q_2 j + q_3 k$ is called a pure quaternion. The collection of all pure quaternions is denoted by $\mathbb{H}_p$. Hence;*

$$
\mathbb{H}_p = \{q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H} : \; q_0 = 0\}
$$

Definition 5.2. *Let $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, the number $N(q) = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2$ is called the norm of q (the norm of a quaternion is similar to the modulus of a complex number).*

Definition 5.3. *A quaternion $q \in \mathbb{H}$ for which $N(q) = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2 = 1$ is called a unit quaternion.*

Definition 5.4. *Similar as with the complex numbers we can take the conjugate of a quaternion. Let $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, then the conjugate of q is $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k \in \mathbb{H}$.*

We can eassily check:

$$\begin{aligned} q\overline{q} &= {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2 \\ &= \overline{q}q \\ &= N(q) \end{aligned}$$

When $q \neq 0$, then $N(q) \neq 0$ it follows that $q\frac{\overline{q}}{N(q)} = \frac{\overline{q}}{N(q)}q = 1$. Hence for the inverse of a quaternion q it holds that;

$$q^{-1} = \frac{\overline{q}}{N(q)}$$

5.2 Representation of $\mathbb{H}$

Just like there is a bijection from $\mathbb{C}$ to $\mathbb{R}^2$, there is a bijection from $\mathbb{H}$ to $\mathbb{R}^4$. This bijection is given by:

$$\rho : \mathbb{H} \to \mathbb{R}^4, \quad a + bi + cj + dk \mapsto (a, b, c, d)^T$$

Now in a similar way as complex numbers can be represented by 2×2 real matrices, a quaternion can be represented by a 2×2 complex matrix. Let $a + bi + cj + dk \in \mathbb{H}$

$$\alpha : \mathbb{H} \to M_2(\mathbb{C}), \quad a + bi + cj + dk \mapsto \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix}$$

This is a representation of $\mathbb{H}$ over $\mathbb{C}$.
Now:

$$\alpha(a+bi+cj+dk) = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Hence we let,

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$
$$\mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \mathbf{j} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

We now let $\mathbb{H}'$ be the set of all matrices of the form $a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ with $a, b, c, d \in \mathbb{R}$. We shall call the matrices of $\mathbb{H}'$ quaternions. From the definitions of the matrices $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ we now have that every matrix $A \in \mathbb{H}'$ is of the form:

$$A = \begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix}$$

where $x = a + bi,\ y = c + di \in \mathbb{C}$.

Theorem 5.5. *The set of all unit quaternions is a group. The group of all unit quaternions is isomorphic to the group of 2×2 special unitary matices, $SU(2)$.*

The conjugate of $A = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H}'$ is $\overline{A} = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$. For a pure quaternion $A \in \mathbb{H}'$ it holds that $\overline{A} = -A$ and $tr(A) = 0$.

5.3 Rotations

A pure quaternion can be seen as a vecor in $\mathbb{R}^3$, using the map;

$$\rho : \mathbb{H}_p \to \mathbb{R}^3,\ ai + bj + ck \mapsto (a, b, c)$$

Now let us consider an arbitrary unit quaternion $q = q_0 + \mathbf{q}$. Since it is unit we have that ${q_0}^2 + \|\mathbf{q}\|^2 = 1$. We now search for some angle θ such that;

$${q_0}^2 = cos^2(\theta)$$

$$\|\mathbf{q}\|^2 = sin^2(\theta)$$

It turns out that there exist a unique $\theta \in [0, \pi)$ such that $q_0 = cos(\theta)$ and $\|\mathbf{q}\| = sin(\theta)$. We can now write our unit quaternion as $q = cos(\theta) + \mathbf{u}sin(\theta)$, where $\mathbf{u} = \mathbf{q}/\|\mathbf{q}\|$.
Let's consider the map;

$$L_q : \mathbb{H}_p \to \mathbb{H}_p, \mathbf{v} \mapsto q\mathbf{v}q^*$$

where q is a unit quaternion. We will show that this map can be seen as a rotation of $\mathbb{R}^3$.

We have;

$$\begin{aligned}
L_q(\mathbf{v}) = q\mathbf{v}q^* &= (q\mathbf{v})q^* \\
&= (-\mathbf{q}\cdot\mathbf{v} + q_0\mathbf{v} + \mathbf{q}\times\mathbf{v})q^* \\
&= -(\mathbf{q}\cdot\mathbf{v})q_0 - (q_0\mathbf{v} + \mathbf{q}\times\mathbf{v})\cdot(-\mathbf{q}) - (-\mathbf{q}\cdot\mathbf{v})\mathbf{q} + \\
&\quad q_0(q_0\mathbf{v} + \mathbf{q}\times\mathbf{v}) - (q_0\mathbf{v} + \mathbf{q}\times\mathbf{v})\times\mathbf{q} \\
&= -(\mathbf{q}\cdot\mathbf{v})q_0 + (q_0\mathbf{v} + \mathbf{q}\times\mathbf{v})\cdot\mathbf{q} + (\mathbf{q}\cdot\mathbf{v})\mathbf{q} + \\
&\quad q_0(q_0\mathbf{v} + \mathbf{q}\times\mathbf{v}) - (q_0\mathbf{v} + \mathbf{q}\times\mathbf{v})\times\mathbf{q} \\
&= -q_0(\mathbf{q}\cdot\mathbf{v}) + q_0(\mathbf{v}\cdot\mathbf{q}) + (\mathbf{q}\times\mathbf{v})\cdot\mathbf{q} + (\mathbf{q}\cdot\mathbf{v})\mathbf{q} + \\
&\quad {q_0}^2\mathbf{v} + q_0(\mathbf{q}\times\mathbf{v}) - q_0(\mathbf{v}\times\mathbf{q}) - (\mathbf{q}\times\mathbf{v})\times\mathbf{q} \\
&= (\mathbf{q}\cdot\mathbf{v})\mathbf{q} + {q_0}^2\mathbf{v} + 2q_0(\mathbf{q}\times\mathbf{v}) + (\mathbf{q}\times(\mathbf{q}\times\mathbf{v})) \\
&= (\mathbf{q}\cdot\mathbf{v})\mathbf{q} + {q_0}^2\mathbf{v} + 2q_0(\mathbf{q}\times\mathbf{v}) + \mathbf{q}(\mathbf{q}\cdot\mathbf{v}) - \|\mathbf{q}\|^2\mathbf{v} \\
&= ({q_0}^2 - \|\mathbf{q}\|^2)\mathbf{v} + 2(\mathbf{q}\cdot\mathbf{v})\mathbf{q} + 2q_0(\mathbf{q}\times\mathbf{v})
\end{aligned}$$

The map L_q preserves length, since;

$$\begin{aligned}
\|L_q(\mathbf{v})\| &= \|q\mathbf{v}q^*\| \\
&= |q|\cdot\|\mathbf{v}\|\cdot|q^*| \\
&= \|\mathbf{v}\|
\end{aligned}$$

Now let $\mathbf{v}$ be a pure quaternion along the direction of $\mathbf{q}$ (the vector part of q), so we have $\mathbf{v} = k\mathbf{q}$ for ome $k \in \mathbb{R}$. For the image of $\mathbf{v}$ under the map L_q it holds that;

$$\begin{aligned}
L_q(\mathbf{v}) &= q\mathbf{v}q^* \\
&= q(k\mathbf{q})q^* \\
&= ({q_0}^2 - \|\mathbf{q}\|^2)(k\mathbf{q}) + 2(\mathbf{q}\cdot(k\mathbf{q}))\mathbf{q} + 2q_0(\mathbf{q}\times(k\mathbf{q})) \\
&= k{q_0}^2\mathbf{q} - k\|\mathbf{q}\|^2\mathbf{q} + 2k\|\mathbf{q}\|^2\mathbf{q} + 2kq_0(\mathbf{q}\times\mathbf{q}) \\
&= k{q_0}^2\mathbf{q} + k\|\mathbf{q}\|^2\mathbf{q} \\
&= k({q_0}^2 + \|\mathbf{q}\|^2)\mathbf{q} \\
&= k\mathbf{q} \\
&= \mathbf{v}
\end{aligned}$$

Hence $\mathbf{v}$ is left fixed by L_q.

One can check that for $\mathbf{v}, \mathbf{w} \in \mathbb{H}_p$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ we have that $L_q(\lambda_1\mathbf{v} + \lambda_2\mathbf{w}) = \lambda_1 L_q(\mathbf{v}) + \lambda_2 L_q(\mathbf{w})$. So L_q is a linear map.

Theorem 5.6. *For every unit quaternion $q = q_0 + \boldsymbol{q} = cos(\theta/2) + \boldsymbol{u}sin(\theta/2)$ (with $\boldsymbol{u} = \boldsymbol{q}/\|\boldsymbol{q}\|$), the linear map L_q corresponds to rotation about $\boldsymbol{u}$ through an angle θ. So for every $\boldsymbol{v} \in \mathbb{H}_p$, $L_q(\boldsymbol{v})$ is the vector we get when we rotate $\boldsymbol{v}$ through an angle θ around the vector $\boldsymbol{u}$.*

Proof. Write $\mathbf{v}$ as $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel$, where $\mathbf{v}_\perp$ is the component of $\mathbf{v}$ perpendicular to $\mathbf{q}$ and $\mathbf{v}_\parallel$ is the component of $\mathbf{v}$ parallel to $\mathbf{q}$ (so $\mathbf{v}_\parallel = \lambda\mathbf{q}$ for some $\lambda \in \mathbb{R}$). Now applying the operator L_q to $\mathbf{v}$ we get;

$$L_q(\mathbf{v}) = L_q(\mathbf{v}_\perp + \mathbf{v}_\parallel) = L_q(\mathbf{v}_\perp) + L_q(\lambda\mathbf{q}) = L_q(\mathbf{v}_\perp) + \lambda\mathbf{q}$$

and;

$$\begin{aligned} L_q(\mathbf{v}_\perp) &= ({q_0}^2 - \|\mathbf{q}\|^2)\mathbf{v}_\perp + 2(\mathbf{q}\cdot\mathbf{v}_\perp)\mathbf{q} + 2q_0(\mathbf{q}\times\mathbf{v}_\perp) \\ &= ({q_0}^2 - \|\mathbf{q}\|^2)\mathbf{v}_\perp + 2q_0(\mathbf{q}\times\mathbf{v}_\perp) \\ &= ({q_0}^2 - \|\mathbf{q}\|^2)\mathbf{v}_\perp + 2q_0\|\mathbf{q}\|(\mathbf{u}\times\mathbf{v}_\perp) \end{aligned}$$

where $\mathbf{u} = \mathbf{q}/\|\mathbf{q}\|$. Now let $\mathbf{a} = \mathbf{u}\times\mathbf{v}_\perp$. We then get;

$$L_q(\mathbf{v}_\perp) = ({q_0}^2 - \|\mathbf{q}\|^2)\mathbf{v}_\perp + 2q_0\|\mathbf{q}\|\mathbf{a}$$

Since $\|\mathbf{a}\| = \|\mathbf{u}\times\mathbf{v}_\perp\| = \|\mathbf{u}\|\|\mathbf{v}_\perp\|sin(\pi/2) = \|\mathbf{v}_\perp\|$, $\mathbf{a}$ and $\mathbf{v}_\perp$ have the same length. Now using $q_0 = cos(\theta/2)$ and $\|\mathbf{q}\| = sin(\theta/2)$ we have that;

$$\begin{aligned} L_q(\mathbf{v}_\perp) &= \left(cos\left(\frac{\theta}{2}\right)^2 - sin\left(\frac{\theta}{2}\right)^2\right)\mathbf{v}_\perp + 2cos\left(\frac{\theta}{2}\right)sin\left(\frac{\theta}{2}\right)\mathbf{a} \\ &= cos(\theta)\mathbf{v}_\perp + sin(\theta)\mathbf{a} \end{aligned}$$

Let α be the angle between $L_q(\mathbf{v}_\perp)$ and $\mathbf{v}_\perp$. Taking the inner product between them we get;

$$L_q(\mathbf{v}_\perp)\cdot\mathbf{v}_\perp = \|L_q(\mathbf{v}_\perp)\|\|\mathbf{v}_\perp\|cos(\alpha) = \|\mathbf{v}_\perp\|^2 cos(\alpha)$$

So;

$$cos(\alpha) = \frac{L_q(\mathbf{v}_\perp)\cdot\mathbf{v}_\perp}{\|\mathbf{v}_\perp\|^2} = \frac{(cos(\theta)\mathbf{v}_\perp + sin(\theta)\mathbf{a})\cdot\mathbf{v}_\perp}{\|\mathbf{v}_\perp\|^2} = cos(\theta)$$

So the angle between $L_q(\mathbf{v}_\perp)$ and $\mathbf{v}_\perp$ is θ. Now taking the inner product of the vector $L_q(\mathbf{v}_\perp)$ with $\mathbf{q}$ we get;

$$L_q(\mathbf{v}_\perp)\cdot\mathbf{q} = cos(\theta)(\mathbf{v}_\perp\cdot\mathbf{q}) + sin(\theta)(\mathbf{a}\cdot\mathbf{q}) = 0$$

Hence $L_q(\mathbf{v}_\perp)$ is orthogonal to $\mathbf{q}$. □

For a unit quaternion q of the form $q = q_0 + \mathbf{q} = cos\left(\frac{\theta}{2}\right) + \mathbf{u}sin\left(\frac{\theta}{2}\right)$., we can write;

$$\begin{aligned} L_q(\mathbf{v}) &= \left(cos^2\left(\frac{\theta}{2}\right) - sin^2\left(\frac{\theta}{2}\right)\right)\mathbf{v} + 2\left(sin\left(\frac{\theta}{2}\right)\mathbf{u}\cdot\mathbf{v}\right)sin\left(\frac{\theta}{2}\right)\mathbf{u} \\ &\quad + 2cos\left(\frac{\theta}{2}\right)\left(sin\left(\frac{\theta}{2}\right)\mathbf{u}\times\mathbf{v}\right) \\ &= cos(\theta)\mathbf{v} + (1 - cos(\theta))(\mathbf{u}\cdot\mathbf{v})\mathbf{u} + sin(\theta)(\mathbf{u}\times\mathbf{v}) \end{aligned}$$

5.4 Composition of rotations

Assume we have two unit quaternions p, q. As we have just seen L_p and L_q are rotations. Let $\mathbf{u} \in \mathbb{H}_p$. As pq is also a unit quaternion, we now apply the rotation L_{pq} to $\mathbf{u}$. We then get.

$$\begin{aligned} L_{qp}(\mathbf{u}) &= (qp)\mathbf{u}(qp)* \\ &= q(p\mathbf{u}p*)q* \\ &= q(L_p(\mathbf{u}))q* \\ &= L_q \circ L_p(\mathbf{u}) \end{aligned}$$

So the rotation corresponding to the unit quaternion pq equals the composite of the rotations corresponding to p and q.
We shall now prove the following lemma.

Lemma 5.7. *For any two non zero quaternions $a, b \in \mathbb{H}$ we have that; $ab = -ba$ if and only if a and b are pure quaternions and are perpendicular to each other (with respect to the dot product in $\mathbb{R}^3$).*

Proof. First let a and b be non zero quaternions and assume that $ab = -ba$. We have;

$$\begin{aligned} ab &= a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 + \\ &= (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)i + \\ &= (a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1)j + \\ &= (a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0)k \end{aligned}$$

and;

$$\begin{aligned} -ba &= -a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3 + \\ &= (-a_1b_0 - a_0b_1 - a_3b_2 + a_2b_3)i + \\ &= (-a_2b_0 + a_3b_1 - a_0b_2 - a_1b_3)j + \\ &= (-a_3b_0 - a_2b_1 + a_1b_2 - a_0b_3)k \end{aligned}$$

Since $ab = -ba$ we find the following array of equations:

$$\begin{aligned} a_0b_0 &= a_1b_1 + a_2b_2 + a_3b_3 \\ a_1b_0 + a_0b_1 &= 0 \\ a_0b_2 + a_2b_0 &= 0 \\ a_0b_3 + a_3b_0 &= 0 \end{aligned}$$

Assume that $a_0 \neq 0$, we then have that;

$$\begin{aligned} b_1 &= -\frac{b_0}{a_0}a_1 = -ca_1 \\ b_2 &= -\frac{b_0}{a_0}a_2 = -ca_2 \\ b_3 &= -\frac{b_0}{a_0}a_3 = -ca_3 \\ b_0 &= \frac{b_0}{a_0}a_0 = ca_0 \end{aligned}$$

If we now substitute these values in the equation $a_0b_0 = a_1b_1 + a_2b_2 + a_3b_3$ we get, $a_0b_0 = -\frac{b_0}{a_0}({a_1}^2 + {a_2}^2 + {a_3}^2)$ or equivalently: ${a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2 = 0$. This is only possible when a is the zero quaternion. Hence $a_0 = 0$. The same argument for b shows that $b_0 = 0$ and hence that a and b are pure quaternions. Since $a_1b_1 + a_2b_2 + a_3b_3 = a_0b_0 = 0$, a and b are perpendicular.
Now assume that a and b are pure quaternions that are perpendicular to each other; so $a = a_1i + a_2j + a_3k$ and $b = b_1i + b_2j + b_3k$; and $a_1b_1 + a_2b_2 + a_3b_3 = 0$. In this case;

$$\begin{aligned} ab &= (a_2b_3 - a_3b_2)i + \\ &= (-a_1b_3 + a_3b_1)j + \\ &= (a_1b_2 - a_2b_1)k \\ &= -ba \end{aligned}$$

□

6 The Platonic Solids

Definition 6.1 (**Platonic Solid**). *A platonic solid is a solid in three-dimensions with flat faces, straight edges and sharp vertices (in other words a polyhedron) which is convex and satisfies the following properties;*

- *Each face is the same regular polygon*
- *The same number of polygons meet at each vertex*

In total there are five platonic solids, namely the tetrahedron, cube (hexahedron), octahedron, dodecahedron, icosahedron and are named after the number of faces they possess. For instance take the icosahedron that has 20 triangular faces (icosa is Greek for twenty). The 5 platonic solids are depicted in figure 1.

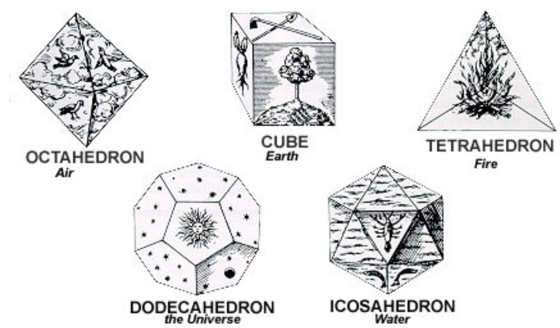

Figure 1: The five platonic solids [8]

The platonic solids were named after The Greek philosopher Plato. He described each of the classical elements with one of these solids based on the physical properties they possess. He associated air with the octahedron, water with the icosahedron, fire with the tetrahedron and earth with the cube. Plato thaught that the fifth platonic solid, the dodecahedron, was used by god for arranging the constellations of heaven (see again figure 1).

6.1 Symmetry groups of the solids

Each of the five platonic solids has an associated symmetry group. This is the group of transformations which leave the solid invariant. These rotations form a subgroup of $SO(3)$, which is the group of all rotations about the origin of $\mathbb{R}^3$ under the operation of composition.

6.2 Dual solids

For each polyhedron there exist a dual polyhedron. The dual polyhedron is the polyhedron we get from the original polyhedron when we interchange the faces and vertices. To show what is meant by this we refer to figure 2. When we are given a polyhedron and we take the dual of its dual, we get the original polyhedron back again. When the dual of a polyhedron is the polyhedron itself, we call it a self-dual polyhedron.

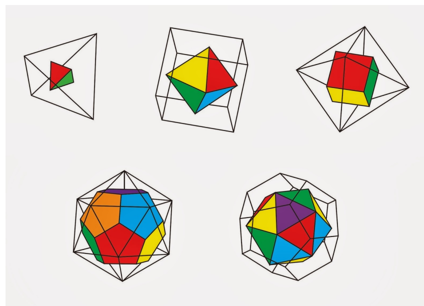

Figure 2: Dual polyhedra [9]

6.3 Tetrahedron

We shall now descibe the symmetry group of the tetrahedron. There are two main cathegories of rotation of the tetrahedron. We can rotate through $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ around an axis that runs from a vertex of the solid to the centroid of the opposing face, see figure 3.

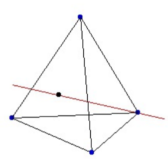

Figure 3: vertex-face axis [10]

Another possible rotation is through π around an axis that runs from midpoint to midpoint of opposing edges, see figure 4.

There are 4 possible vertex-face axex and 3 different edge-edge axes. In total they give 11 possible rotations of the tetrahedron and as one can check all of these rotations are unique. Together with the identity rotation we count 12 rotations of the tetrahedron. The symmetry group of the tetrahedron is called the tetrahedral group and is denoted by T.

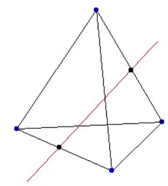

Figure 4: edge-edge axis [10]

We denote S_n for the set of all permutations of the numbers $1, 2, \dots, n$. The order of S_n is $n!$. The subset of all even permutations within S_n is called the alternating group and is denoted by A_n. In S_n there as many odd elements as even elements and from this it follows that the order of A_n is $\frac{n!}{2}$. Another fact (which will not be proven here) is that the 3-cycles generate A_n, so given an even permutation we can write it as a product of 3-cycles.
We now number the four vertices of the terahedron. Each of the rotations just discussed, then gives an unique permutation of the numbers $1, 2, 3, 4$. A rotation r of the terahedron will swap the location of 3 vertices and a rotation q will swap 4 vertices in pairs. Every possible rotation of the tetrahedron is now contained in the following set:

e	(12)(34)	(13)(24)	(14)(23)
(123)	(124)	(134)	(234)
(132)	(142)	(143)	(243)

But these are exactly the elements of A_4. It turns out that the rotational symmetry group of the tetrahedron and the alternating group of degree 4 are isomorphic. The tetrahedron is a self-dual polyhedron.

6.4 The Cube and the Octahedron

The octahedron and the cube are dual solids, hence they have the same rotational symmetry group. So any rotation of the cube is a rotation of the octahedron and any rotation of the octahedron is a rotation of the cube. The symmetry group of the octahedron and the cube is called the octahedral group and is denoted with O.

We shall discuss the octahedral group O using the rotations of the cube. The cube has two more faces and twice as many vertices and edges as the tetrahedron. The rotations of the cube are;

- r: a rotation of $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$ around an axis through the centroid of opposing faces (a face-face axis, see figure 5)
- q: a rotation through π around an axis through the midpoints of diagonally opposing edges (an edge-edge axis, see figure 6)

- s: a rotation of $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ around an axis through diagonally opposite vertices (a vertex-vertex axis, see figure 7)

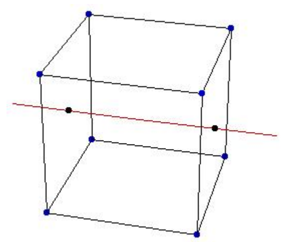

Figure 5: face-face axis [10]

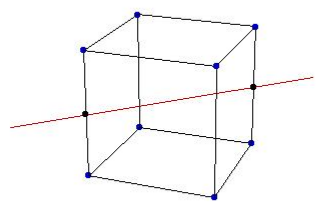

Figure 6: edge-edge axis [10]

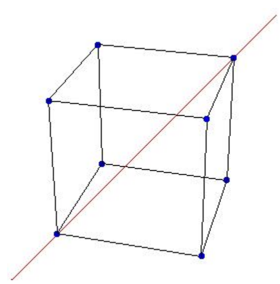

Figure 7: vertex-vertex axis [10]

In total there are 3 face-face axes, 6 edge-edge axes and 4 vertex-vertex axes. When we include the identity rotation, the total number of possible rotations of the cube is 24.

We now number the four main diagonals of the cube $1, 2, 3, 4$ (the diagonal is the same as a vertex-vertex axis just discussed). A rotation r will cyclicly permute these diagonals and hence result in a 4-cycle. A rotation s will give us a 3-cycle (it will fix the axis of rotation, which is a diagonal in this case). A rotation q will permute two diagonals and hence gives us a 2 cycle. Each of the elements of the symmetry group of the cube (octahedron group O) corresponds to an element of S_4, so whe have a mapping

$\phi : O \to S_4$. We know that O contains 24 elements, just like S_4. If we can now proof that ϕ is surjective, we know that O and S_4 are isomorphic. A result states that (12) and (1234) generate S_4 and we know that both these elements lie in the image of ϕ. The image of ϕ is a subgroup of S_4 and hence must equal S_4.

6.5 The Dodecahedron and the Icosahedron

The dodecahedron and the icosahedron are also dual solids, so their symmetry groups are identical. The symmetry group of the dodecahedron and the icosahedron is called the icosahedral group and is denoted by I. We shall discuss the icosahedral group using the rotations of the dodecahedron. There are three categories of rotations of the dodecahedron;

- r: a rotation of $\frac{2\pi}{5}$, $\frac{4\pi}{5}$, $\frac{6\pi}{5}$ and $\frac{8\pi}{5}$ around an axis through the centroid of opposing faces (a face-face axis)
- q: a rotation through π around an axis through the midpoints of diagonally opposing edges (an edge-edge axis)
- s: a rotation of $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ around an axis through diagonally opposite vertices (an vertex-vertex axis)

In total there are 6 different face-face axes, 15 edge-edge axes and 10 vertex-vertex axes. Including the identity rotation we count for 60 different rotations (which leave the dodecahedron and the icosahedron fixed).
The icosahedral group I is isomorphic to A_5. To see this we take the dodecahedron and look at the inscribed cubes whose volume is maximized inside the dodecahedron. There are 5 of these cubes and each edge of such a cube aligns with a diagonal of a pentagonal face of the dodecahedron (see figure 8). We will number these 5 cubes $1, 2, 3, 4$ and 5. Every rotation of the dodecahedron now corresponds to a permutation of the 5 cubes. One can show that the permutations of these cubes caused by rotations are generated by 3-cycles (which generate A_5) and that I is isomorphic to A_5.

6.6 Groups

The rotational symmetry groups of the platonic solids are all isomorphic to some permutation group. We have listed them in table 1.

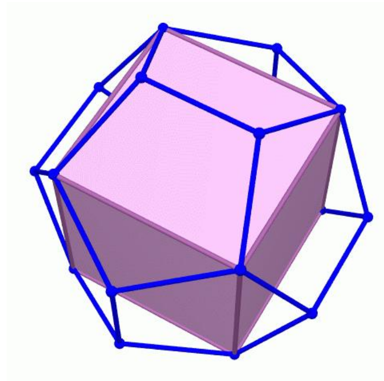

Figure 8: An inscribed cube of the dodecahedron [11]

group:	**Symbol**	**solids:**	**permutation group:**
tetrahedral	T	tetrahedron	A_4
octahedral	O	cube & octahedron	S_4
icosahedral	I	dodecahedron & icosahedron	A_5

Table 1: Isomorphic rotational symmetry groups and permutation groups

7 Binary groups

The symmetry groups of the platonic solids are subgroups of $SO(3)$. Every element of $SO(3)$ can be represented by a unit quaternion $q = q_0 + \mathbf{q} = cos(\theta/2) + \mathbf{u}sin(\theta/2)$ (with $\mathbf{u} = \mathbf{q}/\|\mathbf{q}\|$), corresponding to a rotation about $\mathbf{u}$ through an angle θ. We can represent every unit quaternion as a 2 special unitary matrix. using the map;

$$a + bi + cj + dk \mapsto \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}$$

The trace, in other words the character of this representation, for a unit quaternion is then easily checked to be $(a + bi) + (a - bi) = 2a$. Which is just 2 times the real part of a unit quaternion.

So given a rotation, we search for a rotation axis (and write is a a unit pure quaternion $\mathbf{u} = u_i\mathbf{i} + u_j\mathbf{j} + u_k\mathbf{k}$) and a rotation angle θ and write it as the unit quaternion $q = q_0 + \mathbf{q} = cos(\theta/2) + \mathbf{u}sin(\theta/2)$. The character of a rotation is then 2 times its real part, which is just $2cos\left(\frac{\theta}{2}\right)$. We shall call this character χ_0. Since characters of representations contain all the important information we don't need to look at the actual rotation axes, only the rotation angle is of importance when calculating the character.

We now consider the map:

$$\phi : SU(2) \to SO(3), q \mapsto L_q(: \mathbb{H}_p \to \mathbb{H}_p, \mathbf{v} \mapsto q\mathbf{v}q^*)$$

This is a surjective group homomorphism (with composition of rotations $L_{pq} = L_p \circ L_q$). The kernel of ϕ consist of all unit quaternions that are send to the identity rotation ($e : \mathbb{H}_p \to \mathbb{H}_p, \mathbf{v} \mapsto \mathbf{v}$) of $\mathbb{R}^3$. This kernel is equal to $\{1,\ -1\}$. We also have that for a unit quaternion q, $\phi(q) = \phi(-q)$ (since for every $\mathbf{v} \in \mathbb{H}_p$ we have $L_{-q}(\mathbf{v}) = (-q)\mathbf{v}(-q)^* = -q\mathbf{v}(-q^*) = q\mathbf{v}q^* = L_q(\mathbf{v})$). So each rotation can be represented by 2 different unit quaternions q and $-q$. However the character of these two quaternions are not the same but we have that $\chi_0(q) = -\chi_0(-q)$.

The pre-images of subgroups subgroup of $SO(3)$ under ϕ are subgroups of $SU(2)$. These groups are called binary groups. For a subgroup G of $SO(3)$ we denote G^* for the corresponding subgroup of $SU(2)$ and call it the binary G group. With these notations for a group G we have the exact sequence:

$$1 \to \{\pm 1\} \to G^* \to G \to 1$$

The group $\{\pm 1\}$ is a normal subgroup of the binary group G^* and we have that:

$$G^*/\{\pm 1\} \cong G$$

We say that G^* is a group extension of G by $\{\pm 1\}$.

Group	Symbol	Binary Group	Symbol
tetrahedal	T	binary tetrahedal	2T
octahedral	O	binary octahedral	2O
icosahedral	I	binary icosahedral	2I

The corresponding binary groups of the symmetry groups of the platonic solids are given in figure 7.
The map:

$$\phi : G^* \to G, g^* \mapsto g$$

is a group homomorphism. When two elements are conjugated in G^* they don't need to be conjugated in G. We know which elements are conjugated in the symmetry groups of the platonic solids, we know search for the conjugated elements of G^*.
Let $a, b \in G$ and assume that a and b are conjugated. So there exist $c \in G$ such that $a = cbc^{-1}$. Since the map ϕ is surjective, there exist $p, q, r \in G^*$ such that $\phi(p) = a,\ \phi(q)$ and $\phi(r) = c$. Now we have that;

$$\phi(p) = \phi(r)\phi(q)\phi(r^{-1}) = \phi(rqr^{-1})$$

So either we have that $rqr^{-1} = p$ or $rqr^{-1} = -p$.
When p and $-p$ are conjugated there is nothing to worry about (then q is conjugated to p and to $-p$). In that case we have that $\chi(p) = -\chi(-p) = 0$. And since $\chi(p) = 2cos\left(\frac{\theta}{2}\right)$ $(p = cos\left(\frac{\theta}{2}\right) + \mathbf{u}sin\left(\frac{\theta}{2}\right))$. We must have that $\theta = \pi$ (or 180 degrees). When p and $-p$ are not conjugated, a conjugacy class of G can split in different conjugacy classes of G^*.

Let p and q be two unit quaternions that are conjugated (in $\mathbb{H}'$), so there exist $a \in \mathbb{H}'$ such that $p = aqa^{-1} = aqa^*$. We can write p, q and a as:

$$p = cos\left(\frac{\alpha}{2}\right) + \mathbf{u}sin\left(\frac{\alpha}{2}\right)$$

$$q = cos\left(\frac{\beta}{2}\right) + \mathbf{v}sin\left(\frac{\beta}{2}\right)$$

$$a = cos\left(\frac{\gamma}{2}\right) + \mathbf{w}sin\left(\frac{\gamma}{2}\right)$$

so we have;

$$p = aqa^* = cos\left(\frac{\alpha}{2}\right) + \mathbf{u}sin\left(\frac{\alpha}{2}\right) = cos\left(\frac{\beta}{2}\right) + a\mathbf{v}a^*sin\left(\frac{\beta}{2}\right)$$

Hence we must have that $\alpha = \beta$ and that there is a rotation (corresponding to $a \in \mathbb{H}'$) that turns $\mathbf{v}$ to $\mathbf{u}$.

Theorem 7.1. *The conjugacy class of an element $g \in G^*$ does not split if q is a rotation of π rad and there exist another rotation p of π rad that is perpendicular to q.*

Proof. If q and $-q$ are conjugated there exist a $p \in G^*$ such that $-q = pqp^{-1}$ or equivalently $pq = -qp$. But since p and q are quaternions they then have to be pure and perpendicular to each other. We must also have that $\chi(p) = \chi(-p) = 2cos\left(\frac{\theta}{2}\right) = 0$. Thus $\theta = \pi$ rad. As we have already seen the angles of two conjugated elements in G^* must be equal and hence both are rotations of *pi* rad. □

In G^* -1 and 1 are not conjugated, since for any $g \in G^*$ we have that $g1g^{-1} = 1$.

8 The character tables of A_4, S_4 and A_5

The charactertables of A_4, S_4 and A_5 are listed in tables 2, 3 and 4 respectively ([1], chapter 18).

g_i	1	(12)(34)	(123)	(132)
$\|C_G(g_i)\|$	12	4	3	3
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

Table 2: The character table of A_4 ($\omega = e^{i\frac{2\pi}{3}}$)

g_i	1	(12)	(123)	(12)(34)	(1234)
$\|C_G(g_i)\|$	24	4	3	8	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	-1	2	0
χ_4	3	1	0	-1	-1
χ_5	3	-1	0	-1	1

Table 3: The character table of S_4

g_i	1	(12)(34)	(123)	(12345)	(12354)
$\|C_G(g_i)\|$	60	4	3	5	5
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	5	1	-1	0	0
χ_4	3	-1	0	$(\sqrt{5}+1)/2$	$(-\sqrt{5}+1)/2$
χ_5	3	-1	0	$(-\sqrt{5}+1)/2$	$(\sqrt{5}+1)/2$

Table 4: The character table of A_5

9 The symmetry groups

In the upcoming tables we use the following abbreviations.

- cc : conjugacy class
- x^G : representative element of a conjugacy class
- $|x^G|$: the order of a conjugacy class

9.1 Octahedral group

The octahedral group has 5 conjugacy classes. Since every rotation permutes the 4 diagonals of the cube, we can label each rotation as a permutation of $1, 2, 3, 4$. For each of the conjugacy classes we shall give a representative permutation. The conjugacy classes of the octahedral group are given in table 5.

cc	x^O	**description**	$\|x^O\|$
1	e	identity	1
2	(12)	180° through the midpoints of opposite edges	6
3	(123)	± 120° vertex rotations through diagonals	8
4	(1234)	± 90° through centers of opposite faces	6
5	(12)(34)	180° through centers of opposite faces	3

Table 5: The 5 conjugacy classes of the octahedral group

The binary symmetry group of the cube has 8 conjugacy classes. To find the number of elements in each of the conjugacy classes in the binary octahedral group, we simply look at the number of elements in the corresponding conjugacy classes of the octahedral group. If a conjugacy class in $2O$ doesn't split it contains twice as many elements as the corresponding conjugacy class in O and if it splits the two resulting conjugacy classes both contain the same number of elements as the corresponding conjugacy class in O. We now the order ot the group and the size of the conjugacy classes, hence we can calculate the size of the stabilizer of the representative elements. For this we use the formula;

$$|C_G(x)| = |G|/|x^G|$$

The conjugacy classes of the binary octahedral group ($2O$) are given in table 6.
The natural character χ_0 of the binary octahedral group is given in table 7;
As expected $\chi_0(e) = 2 = dim_{\mathbb{C}}(\mathbb{C}^2)$.

cc	x^{2O}	$\lvert x^{2O}\rvert$	$\lvert C_{2O}(x)\rvert$
1	e^+	1	48
2	e^-	1	48
3	(12)	12	4
4	$(123)^+$	8	6
5	$(123)^-$	8	6
6	$(1234)^+$	6	8
7	$(1234)^-$	6	8
8	(12)(34)	6	8

Table 6: The 8 conjugacy classes of the binary octahedral group

cc	e^+	e^-	(12)	$(123)^+$	$(123)^-$	$(1234)^+$	$(1234)^-$	(12)(34)
$\lvert C_{2O}(x)\rvert$	48	48	4	6	6	8	8	8
χ_0	2	-2	0	-1	1	$\sqrt{2}$	$-\sqrt{2}$	0

Table 7: The natural character of $2O$

If we take the inner product of the natural character with itself we find that:

$$\begin{aligned}\langle \chi_0,\ \chi_0\rangle &= \sum_{i=1}^{l} \frac{\chi_0(g_i)\overline{\chi_0(g_i)}}{|C_G(g_i)|} \\ &= \frac{2\cdot 2}{48} + \frac{-2\cdot -2}{48} + \frac{0\cdot 0}{4} + \frac{-1\cdot -1}{6} + \frac{1\cdot 1}{6} + \frac{\sqrt{2}\cdot\sqrt{2}}{8} + \frac{-\sqrt{2}\cdot -\sqrt{2}}{8} + \frac{0\cdot 0}{8} \\ &= 1\end{aligned}$$

and we conclude that the natural character is an irreducible character.

Now $\{\pm 1\}$ is a normal subgroup of the binary octahedral group $2O$ and we have that:

$$2O/\{\pm 1\} \cong O$$

We already know the 5 irreducible characters of O and we can lift these characters to find 5 irreducible characters of $2O$. These characters are different from the irreducible natural character χ_0 and we shall call them χ_1, χ_2, χ_3, χ_4 and χ_5. The binary octahedral group had 8 conjugacy classes, so we want to find 8 irreducible characters. Hence 2 more irreducible characters are required. Since the product of 2 characters is a character, we hope to get 2 more irreducible character by taking the product of the lifted characters χ_i (where $i \in \{1, 2, 3, 4, 5\}$) with the natural character χ_0. After taking the inner products of these resulting characters with themselves we find that $\chi_2\chi_0$ and $\chi_3\chi_0$ are irreducible and different from the ones we already had. We will call them χ_6 and χ_7 respectively. All of the irreducible characters just discussed are given in the charactertable of $2O$ 8.

cc	e^+	e^-	(12)	$(123)^+$	$(123)^-$	$(1234)^+$	$(1234)^-$	$(12)(34)$
$\|C_{2O}(g)\|$	48	48	4	6	6	8	8	8
χ_0	2	-2	0	-1	1	$\sqrt{2}$	$-\sqrt{2}$	0
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	-1	1	1	-1	-1	1
χ_3	2	2	0	-1	-1	0	0	2
χ_4	3	3	1	0	0	-1	-1	-1
χ_5	3	3	-1	0	0	1	1	-1
χ_6	2	-2	0	-1	1	$-\sqrt{2}$	$\sqrt{2}$	0
χ_7	4	-4	0	1	-1	0	0	0

Table 8: The irreducible characters of $2O$

As one eassily checks we find that

$$\langle\langle\theta,\phi\rangle\rangle = \langle\theta,\ \chi_0\phi\rangle = \frac{1}{|G|}\sum_{g\in G}\theta(g)\overline{\chi_0(g)\phi(g)}$$

is another inner product between characters.

We use this inner product to calculate the Gram-matrix L_{2O}. This is the matrix who's ij-th entry is $\langle\langle\chi_i,\chi_j\rangle\rangle$. It is a diagonal matrix since for the ij-th m_{ij} entry it holds that:

$$m_{ij} = \langle\chi_i,\chi_0\chi_j\rangle = \frac{1}{|G|}\sum_{g\in G}\chi_i(g)\overline{\chi_0(g)\chi_j(g)} = \frac{1}{|G|}\sum_{g\in G}\chi_j(g)\chi_0(g)\chi_i(g) = \langle\chi_j,\chi_0\chi_i\rangle = m_{ji}$$

where we used the fact that all of our irreducible characters are real.
We have;

$$L_{2O} = \begin{pmatrix} \langle\langle\chi_0,\chi_0\rangle\rangle & \cdots & \langle\langle\chi_0,\chi_7\rangle\rangle \\ \vdots & \ddots & \vdots \\ \langle\langle\chi_7,\chi_0\rangle\rangle & \cdots & \langle\langle\chi_7,\chi_7\rangle\rangle \end{pmatrix} = \begin{pmatrix} 0&1&0&0&0&1&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&1&1\\ 1&0&0&0&0&0&0&1\\ 0&0&1&0&1&0&0&0\\ 0&0&0&1&1&1&0&0 \end{pmatrix}$$

We shall now make a graph where the numbers i and j are connected if $\langle\langle\chi_i,\chi_j\rangle\rangle = 1$. This graph is called the Coxeter-Dynkin diagram. We can construct the Coxeter-Dynkin diagram for $2O$ by looking at the Gram-matrix L_{2O}. For the binary octahedral group the Coxeter-Dynkin diagram is given in figure 9.

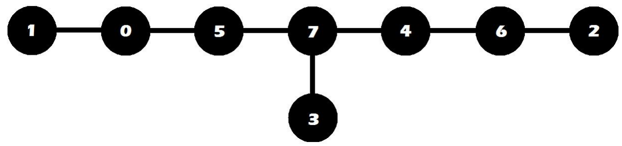

Figure 9: The Coxeter-Dynkin diagram for the binary octahedral group

9.2 Tetrahedral group

The tetrahedral group has 4 conjugacy classes. Since every rotation permutes the 4 vertices of the tetrahedron, we can label each rotation as a permutation of $1, 2, 3, 4$. For each of the conjugacy classes we shall denote a representative permutation. The conjugacy classes of the tetrahedral group are listed in table 9.

cc	x^T	**description**	$\|x^T\|$
1	e	identity	1
2	(123)	$+120°$ through vertex and center of opposite face	4
3	(132)	$-120°$ through vertex and center of opposite face	4
4	(12)(34)	180° through midpoints of opposite edges	3

Table 9: The 4 conjugacy classes of the tetrahedral group

The binary tetrahedral group has 7 conjugacy classes. To find the number of elements in each of the conjugacy classes in the binary tetrahedral group, we again look at the number of elements in the corresponding conjugacy classe of the tetrahedral group. The conjugacy classes of the binary tetrahedral group $2O$ are given in table 10.

cc	x^{2T}	$\|x^{2T}\|$	$\|C_{2T}(x)\|$
1	e^+	1	24
2	e^-	1	24
3	$(123)^+$	4	6
4	$(123)^-$	4	6
5	$(132)^+$	4	6
6	$(132)^-$	4	6
7	(12)(34)	6	4

Table 10: The 7 conjugacy classes of the binary tetrahedral group ($2T$)

The natural character χ_0 for the binary tetrahedral group is given in table 11.

cc	e^+	e^-	$(123)^+$	$(123)^-$	$(132)^+$	$(132)^-$	$(12)(34)$
$\lvert C_{2O}(x)\rvert$	24	24	6	6	6	6	4
χ_0	2	-2	1	-1	1	-1	0

Table 11: The natural character of $2T$

The inner product of the natural character with itself gives:

$$\begin{aligned}\langle \chi_0,\ \chi_0\rangle &= \sum_{i=1}^{l} \frac{\chi_0(g_i)\overline{\chi_0(g_i)}}{|C_G(g_i)|} \\ &= \frac{2\cdot 2}{24} + \frac{-2\cdot -2}{24} + \frac{1\cdot 1}{6} + \frac{-1\cdot -1}{6} + \frac{1\cdot 1}{6} + \frac{-1\cdot -1}{6} + \frac{0\cdot 0}{4} \\ &= 1\end{aligned}$$

and we conclude that the natural character is an irreducible character.
We already know the 4 irreducible characters of T and we can lift these characters to find 4 irreducible characters of $2T$. These characters are different than the irreducible natural character χ_0 and we shall call them χ_1, χ_2, χ_3 and χ_4. The binary tetrahedral group had 7 conjugacy classes, so we want to find 7 irreducible characters. Hence 2 more irreducible characters are required. Since the product of 2 characters is a character, we hope to get 2 more irreducible character by taking the product of the lifted characters χ_i (where $i \in \{1, 2, 3, 4\}$) with the natural character χ_0. After taking the inner products of these resulting product characters with themselves we find that $\chi_2\chi_0$ and $\chi_3\chi_0$ are irreducible and different from the already known irreducible characters. We will call them χ_5 and χ_6 respectively. All of the irreducible characters of the binary tetrahedral group are now given in the charactertable of $2T$ 12.

cc	e^+	e^-	$(123)^+$	$(123)^-$	$(132)^+$	$(132)^-$	$(12)(34)$
$\lvert C_{2T}(x)\rvert$	24	24	6	6	6	6	4
χ_0	2	-2	1	-1	1	-1	0
χ_1	1	1	1	1	1	1	1
χ_2	1	1	ω	ω	ω^2	ω^2	1
χ_3	1	1	ω^2	ω^2	ω	ω	1
χ_4	3	3	0	0	0	0	-1
χ_5	2	-2	ω	$-\omega$	ω^2	$-\omega^2$	0
χ_6	2	-2	ω^2	$-\omega^2$	ω	$-\omega$	0

Table 12: The irreducible characters of $2T$

We now use the inner product $\langle\langle \theta, \phi \rangle\rangle = \langle \theta,\ \chi_0\phi\rangle$ again to calculate the gram matrix L_{2T}.

$$L_{2T} = \begin{pmatrix} \langle\langle \chi_0, \chi_0 \rangle\rangle & \cdots & \langle\langle \chi_0, \chi_6 \rangle\rangle \\ \vdots & \ddots & \vdots \\ \langle\langle \chi_6, \chi_0 \rangle\rangle & \cdots & \langle\langle \chi_6, \chi_6 \rangle\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Once again we can construct the Coxeter-Dynkin diagram for the binary tetrahedral group, given in figure 10.

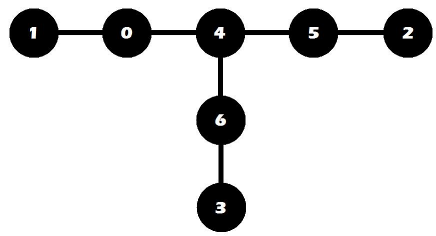

Figure 10: The Coxeter-Dynkin diagram for the binary tetrahedral group

9.3 Icosahedral group

The icosahedral group I has 5 conjugacy classes and order 60. Since every rotation permutes the 5 inscribed cubes of the dodecahedron, we can label each rotation as a permutation of $1, 2, 3, 4, 5$. For each of the conjugacy classes we shall give a representative permutation. The conjugacy classes of the icosahedral group are listed in table 13.

cc	x^I	**description**	$\lvert x^I \rvert$
1	e	identity	1
2	(12)	π rad through midpoints of opposite edges	15
3	(123)	$\pm\frac{\pi}{3}$ rad through opposite vertices	20
4	(12345)	$\frac{2\pi}{5}$ & $\frac{8\pi}{5}$ rad through centers of opposite faces	12
5	(12354)	$\frac{4\pi}{5}$ & $\frac{6\pi}{5}$ rad through centers of opposite faces	12

Table 13: The 5 conjugacy classes of the icosahedral group I

The binary icosahedral group $2I$ has 9 conjugacy classes. To find the number of elements in each of the conjugacy classes of the binary icosahedral group, we simply look at the number of elements in the corresponding conjugacy classes of the icosahedral group. Knowing the size of the conjugacy classes, we can calculate the size of the stabilizer

cc	x^{2I}	$\|x^{2i}\|$	$\|C_{2I}(x)\|$
1	e^+	1	120
2	e^-	1	120
3	(12)	30	4
4	$(123)^+$	20	6
5	$(123)^-$	20	6
6	$(12345)^+$	12	10
7	$(12345)^-$	12	10
8	$(12354)^+$	12	10
9	$(12354)^-$	12	10

Table 14: The 9 conjugacy classes of the binary icosahedral group ($2I$)

of the representative elements of the conjugacy classes. The conjugacy classes of the binary icosahedral group $2I$ are given in table 14.
The natural character χ_0 of the binary icosahedral group is given in table 15.

cc	e^+	e^-	(12)	$(123)^+$	$(123)^-$	$(12345)^+$	$(12345)^-$	$(12354)^+$	$(12354)^-$
$\|C_{2I}(x)\|$	120	120	4	6	6	10	10	10	10
χ_0	2	-2	0	1	-1	a	$-a$	a^{-1}	$-a^{-1}$

Table 15: The natural character of $2I$ ($a = 2cos\left(\frac{\pi}{5}\right) = (\sqrt{5}+1)/2$ and $a^{-1} = 2cos\left(\frac{2\pi}{5}\right) = (\sqrt{5}-1)/2$)

The inner product of the natural character with itself gives:

$$\begin{aligned}\langle \chi_0,\ \chi_0\rangle &= \sum_{i=1}^{9} \frac{\chi_0(g_i)\overline{\chi_0(g_i)}}{|C_G(g_i)|} \\ &= \frac{2\cdot 2}{120} + \frac{-2\cdot -2}{120} + \frac{0\cdot 0}{4} + \frac{1\cdot 1}{6} + \frac{-1\cdot -1}{6} + \frac{a\cdot a}{10} + \frac{-a\cdot -a}{10} \\ &\quad + \frac{b\cdot b}{10} + \frac{-b\cdot -b}{10} \\ &= 1\end{aligned}$$

and we conclude that the natural character is an irreducible character.

We know the 5 irreducible characters of I and we can lift these characters to find 5 more irreducible characters of $2I$. These characters are different from the irreducible natural character χ_0 and we shall call them χ_1, χ_2, χ_3, χ_4 and χ_5. The binary icosahedral group $2I$ had 9 conjugacy classes, so we want to find 9 irreducible characters. Hence 3 more irreducible characters are required. Since the product of 2 characters is a character, we hope to find more irreducible character by taking the product of the lifted characters χ_i (where $i \in \{1,2,3,4,5\}$) with the natural character χ_0. After taking the inner products of these resulting characters with themselves we find that $\chi_5\chi_0$ is an irreducible character and different from the already known irreducible characters of $2I$. We will call this character χ_6.

When we took the product of the lifted characters χ_i (where $i \in \{1,2,3,4,5\}$) with the natural character χ_0 we noticed that $\langle \chi_2\chi_0, \chi_2\chi_0 \rangle = 2$ and $\langle \chi_3\chi_0, \chi_3\chi_0 \rangle = 2$. From the theory of characters we know that for any character ϕ of the binary icosahedral group $2I$ we must have integers $d_0, d_1, d_2, d_3, d_4, d_5, d_6, d_8$ such that;

$$\phi = d_0\chi_0 + \ldots + d_8\chi_8$$

where $\chi_0, \ldots, \chi_8$ are the 9 irreducible characters. It follows that:

$$d_i = \langle \phi, \chi_i \rangle, \text{ for } 0 \leq i \leq 8$$

and

$$\langle \phi, \phi \rangle = \sum_{i=0}^{8} {d_i}^2$$

For $\phi \in \chi_2\chi_0,\ \chi_3\chi_0$ we have;

$$\langle \phi, \phi \rangle = \sum_{i=0}^{8} {d_i}^2 = 2$$

Hence ϕ must be the sum of two irreducible characters (since the only way when $\langle \phi, \phi \rangle = 2$ is when $d_i = 1$ for just two i in $1, \ldots, 8$). Since we already know 7 irreducible characters of $2I$ we hope that some of them are contained in $\chi_2\chi_0$ and $\chi_3\chi_0$.
For the case of $\chi_2\chi_0$ we find that $\langle \chi_2\chi_0, \chi_i \rangle = 1$ only for $i = 6$ and $\langle \chi_2\chi_0, \chi_i \rangle = 0$ for the rest. Since $\chi_2\chi_0$ is the sum of two irreducible characters this means that $\chi_2\chi_0 = \chi_6 + \chi_7$, where χ_7 is an irreducible character of $2I$ different from $\chi_0, \ldots, \chi_6$. Hence we have found another irreducible character $\chi_7 = \chi_2\chi_0 - \chi_6$ of $2I$.
In the same way we find that for $\chi_3\chi_0$, $\langle \chi_3\chi_0, \chi_i \rangle = 1$ only for $i = 6$ and $\langle \chi_2\chi_0, \chi_i \rangle = 0$ for the rest. Hence $\chi_8 = \chi_4\chi_0 - \chi_6$ is another irreducible character different from $\chi_0, \ldots, \chi_7$. We have now found the 9 irreducible characters of $2I$ and we can construct the character table, which is given in table 16.

Again we use the inner product defined by $\langle\langle \theta, \phi \rangle\rangle = \langle \theta,\ \chi_0\phi \rangle$ to calculate the gram matrix L_{2I}.

cc	e^+	e^-	(12)	$(123)^+$	$(123)^-$	$(12345)^+$	$(12345)^-$	$(12354)^+$	$(12354)^-$
$\|C_{2I}(x)\|$	120	120	4	6	6	10	10	10	10
χ_0	2	-2	0	1	-1	a	$-a$	a^{-1}	$-a^{-1}$
χ_1	1	1	1	1	1	1	1	1	1
χ_2	4	4	0	1	1	-1	-1	-1	-1
χ_3	5	5	1	-1	-1	0	0	0	0
χ_4	3	3	-1	0	0	a	a	$-a^{-1}$	$-a^{-1}$
χ_5	3	3	-1	0	0	$-a^{-1}$	$-a^{-1}$	a	a
χ_6	6	-6	0	0	0	-1	1	1	-1
χ_7	2	-2	0	1	-1	$-a^{-1}$	a^{-1}	$-a$	a
χ_8	4	-4	0	-1	1	1	-1	-1	1

Table 16: The irreducible characters of $2I$

$$L_{2I} = \begin{pmatrix} \langle\langle \chi_0, \chi_0 \rangle\rangle & \cdots & \langle\langle \chi_0, \chi_8 \rangle\rangle \\ \vdots & \ddots & \vdots \\ \langle\langle \chi_8, \chi_0 \rangle\rangle & \cdots & \langle\langle \chi_8, \chi_8 \rangle\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Once again we can construct the Coxeter-Dynkin diagram for the binary icosahedral group, given in figure 11.

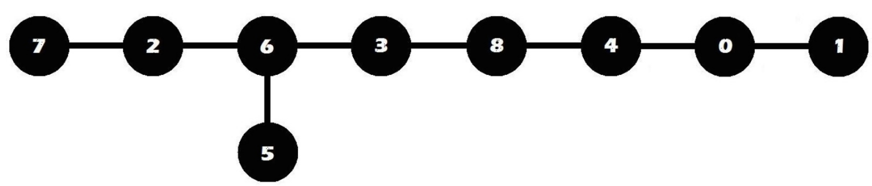

Figure 11: The Coxeter-Dynkin diagram for the binary icosahedral group

9.4 Note on the Coxeter-Dynkin diagrams

We have constructed the Coxeter-Dynkin diagrams for the groups $2T$, $2O$ and $2I$ (figures 10, 9, 11 respectively). These turn out to be famous graphs in the theory of Lie-groups and are named the affine (extended) Dynkin diagrams $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ respectively (they can be found in Bourbaki's book [3]).

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