Floer Homology and Rabinowitz-Floer homology

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Abstract

Let (M,ω) be a closed symplectic $2n$-dimensional manifold that is symplectically aspherical with vanishing first Chern class. The (weak) Arnold conjecture states that the number of contractible periodic orbits $\mathcal{P}_0(H)$ of the Hamiltonian vector field X_H associated to a Hamiltonian H is bounded from below by the Betti numbers of the manifold:

$$\mathcal{P}(H) \geq \sum_{k=0}^{2n} \dim H_k(M;\mathbb{Z}_2).$$

These contractible orbits can also be viewed as the fixed points of a Hamiltonian diffeomorphism. The first goal of this thesis is to prove the Arnold conjecture using Floer homology.

Floer homology is an infinite-dimensional type of Morse homology where the periodic orbits are described as critical points of the symplectic action functional $\mathcal{A}_H$ on loop space. We prove that this homology is well-defined and does not depend on the choice of the Hamiltonian H and almost complex structure J used to define it. To prove the Arnold conjecture, we show that the Floer homology $\mathrm{HF}_*(M)$ is isomorphic to the Morse homology $\mathrm{HM}_*(M)$ of M.

In the second part we explore several recent papers on Rabinowitz-Floer homology, a Floer homology associated to the Rabinowitz action functional. The Rabinowitz-Floer homology $\mathrm{RFH}_*(\Sigma,W)$ is defined for an exact embedding of a contact manifold (Σ,ξ) into a symplectic manifold (W,ω). We look at two applications.

The first one is the existence of leaf-wise fixed points. These are generalizations of fixed points, associated to a coisotropic submanifold. We prove an existence result for leaf-wise fixed points for a hypersurface of contact type (Σ,ξ) in a symplectic manifold.

The second application is orderability of contact manifolds. A contact manifold is orderable when there exists a partial order on $\widetilde{\mathrm{Cont}}_0(\Sigma,\xi)$, the universal cover of the group of contactomorphisms. We establish conditions in terms of the Rabinowitz-Floer homology $\mathrm{RFH}_*(\Sigma,W)$ under which a Liouville fillable closed coorientable contact manifold (Σ,ξ) with Liouville filling (W,ω) is orderable.

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Contents

CHAPTER 1

Introduction

1.1. Motivation and main result

In this section we give some motivation about fixed point theory, after which we state the main result proven in the thesis.

The motivation comes from physics. There, a mechanical system is described as a Hamiltonian system on phase space, which is the cotangent bundle of a configuration space. An example is a particle whose dynamics are governed by time-dependent Hamiltonian function. The configuration space is $\mathbb{R}^3$. The phase space is $T^*\mathbb{R}^3$, consisting of pairs (x,p) where p denotes the momentum of the particle. The Hamiltonian is a time-dependent function $H : \mathbb{R} \times T^*\mathbb{R}^3 \to \mathbb{R}$. This phase space has a symplectic structure. This is a non-degenerate and closed 2-form $\omega \in \Omega^2(T^*\mathbb{R}^3)$. For $T^*\mathbb{R}^3$, the sympletic form is given by

$$\omega = \sum_{i=0}^{3} dx_i \wedge dp_i.$$

Using the non-degeneracy of ω, there is a unique vector field X_H defined by

$$\iota_{X_H}\omega = dH$$

called the Hamiltonian vector field. A particle in the Hamiltonian system $(T^*\mathbb{R}^3, H)$ moves according to the differential equation

$$\dot{\xi}(t) = X_H(\xi(t)) \tag{1.1}$$

where $\xi : \mathbb{R} \to T^*\mathbb{R}^3$. It is conceivable that the particle returns to its original position after time 1, which means $\xi(t) = \xi(t+1)$ for all $t \in \mathbb{R}$. This is called a periodic orbit of period 1. A natural question about a given Hamiltonian system is the following.

QUESTION 1.1 (Periodic orbits). *For a given Hamiltonian system, do any periodic orbits exist. If so, how many at least?*

Consider now a symplectic manifold (M,ω) with a 1-periodic Hamiltonian $H : S^1 \times M \to \mathbb{R}$. Define

$$\mathcal{P}(H) := \{x : S^1 \to M \mid x \text{ solves equation (1.1)}\}.$$

When giving an answer to Question 1.1 for contractible orbits, we look at the number of elements in the set

$$\mathcal{P}_0(H) := \{x \in \mathcal{P}(H) \mid x \text{ contractible in } M\}.$$

Elements $x \in \mathcal{P}(H)$ correspond to fixed points of the time-1 flow of X_H as $\varphi_H^1(x(0)) = x(0)$. We discuss an example.

EXAMPLE 1.2. Consider $S^2 \subset \mathbb{R}^3$ be the unit sphere with symplectic structure $\omega = dz \wedge d\theta$ in cylindrical coordinates. Consider $H = \alpha z$. Then φ_H^1 is a rotation through an angle α. In the case $\alpha < 2\pi$ the we have the following picture.

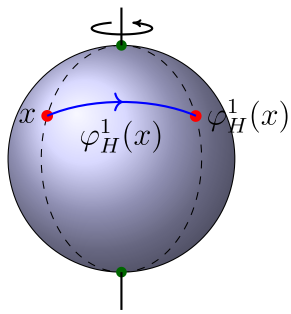

Figure 1. The fixed points of a rotation of the sphere S^2.

We see that $\varphi^1(H)$ has precisely two fixed points: the north and south pole.

We look at periodic orbits that are non-degenerate.

Definition 1.3. Let $H : S^1 \times M \to \mathbb{R}$ be a 1-periodic Hamiltonian and let $x \in \mathcal{P}(H)$. We say x is non-degenerate if for the time-1 flow φ_H^1 we have

$$\det(\mathrm{Id} - T_{x(0)}\varphi_H^1) \neq 0.$$

We say a Hamiltonian H is non-degenerate if all $x \in \mathcal{P}(H)$ are non-degenerate.

The following famous conjecture was first stated in 1965 by Arnold in [**Arn65**] (formulated differently, we formulate a homological version). It gives an answer to Question 1.1.

Theorem 1.4 (Arnold Conjecture)**.** *Let (M, ω) be a compact symplectic manifold of dimension $2n$. Let $H : S^1 \times M \to \mathbb{R}$ be a 1-periodic time-dependent Hamiltonian. Suppose H is non-degenerate. Then*

$$\#\mathcal{P}(H) \geq \sum_{i=0}^{2n} H_i(M, \mathbb{Z}_2)$$

This non-degenerate version of the Arnold conjecture has been proven in full generality by Fukaya-Ono in [**FO99**] and Liu-Tian in [**LT98**]. We refer the reader to [**MS12**] page 278 for a historical overview of the proof of the Arnold conjecture. In the first part of this thesis we prove the following version of the Arnold conjecture.

Theorem 1.5. *Let (M, ω) be a compact symplectic $2n$-dimensional manifold such that $\partial M = \emptyset$. Assume furthermore that*

- *For all $\alpha \in C^\infty(S^2, M)$ we have $\int_{S^2} \alpha^*\omega = 0$.*

- *For all $\alpha \in C^\infty(S^2, M)$, there exists a symplectic trivialization of the bundle α^*TM.*

Then

$$\#\mathcal{P}_0(H) \geq \sum_{i=0}^{2n} \dim H_i(M; \mathbb{Z}_2).$$

1.2. Overview of the topics covered

We prove Theorem 1.5 using Floer homology. This is an infinite dimensional Morse homology for the symplectic action functional. The proof is explained throughout the first 6 chapters. First, we introduce the necessary symplectic topology in chapter 2. In chapter 3 we discuss the properties of the symplectic action functional. Critical points of this functional generate the Floer chain groups. We then discuss the grading of the critical points by the Conley-Zehnder index in chapter 4, which is based on the Maslov index for paths of symplectic matrices. In chapter 5, we give a rigorous definition of Floer homology, with all technical machinery involved. In the following chapter, we establish that Floer homology does not depend on the choices made and is isomorphic to the Morse homology of M. We use these facts to prove Theorem 1.5.

In the final chapter of this thesis, we give an overview of Rabinowitz-Floer homology, a Floer homology associated to the Rabinowitz action functional. We discuss two applications of Rabinowitz-Floer homology. The first application is an existence result for leaf-wise fixed points in contact manifolds of restricted contact type. Leaf-wise fixed points are a generalization of the fixed points studied in Floer homology. The second application is a result about the orderability of contact manifolds. A contact manifold is orderable if there exists a partial order on $\widetilde{\mathrm{Cont}}_0$, the universal cover of the identity component of the group of diffeomorphisms that preserve the contact structure.

Throughout the text, we will reference to the appendix containing auxiliary results and background theory.

CHAPTER 2

Symplectic geometry and the Arnold conjecture

2.1. Introduction to symplectic geometry

We give a short introduction to symplectic geometry to introduce the concepts mentioned in Theorem (1.4). We first define symplectic vector spaces and symplectic manifolds. We then discuss Riemannian metrics and almost complex structures and the notion of ω-compatibility. These concepts arise in the Floer equation (3.6) which is derived in Chapter 3.

Definition 2.1 (Symplectic vector space)**.** A symplectic vector space (V, ω) is an $\mathbb{R}$-vector space V equipped with a bilinear form

$$\omega : V \times V \to \mathbb{R}$$

such that the following conditions are satisfied

(i) ω is skew-symmetric, which means it satisfies $\omega(u, v) = -\omega(v, u)$.

(ii) ω is non-degenerate, which means that, for $u, v \in V$, if $\omega(u, v) = 0$ for all $u \in V$, then $v = 0$.

In the following example we equip $\mathbb{R}^{2n}$ with the structure of a symplectic vector space.

EXAMPLE 2.2. Denote $v = (v_1, \ldots, v_{2n})$ and $w = (w_1, \ldots, w_{2n})$ two vectors in $\mathbb{R}^{2n}$. Let $\omega_0 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ be defined by

$$\omega_0(v, w) = \sum_{i=1}^{n} v_{2i-1} w_{2i} - v_{2i} w_{2i-1}.$$

The pair $(\mathbb{R}^{2n}, \omega_0)$ is a symplectic vector space. The form ω_0 is known as the standard symplectic form on $\mathbb{R}^{2n}$.

REMARK. Any symplectic vector space (V, ω) must be even dimensional. This follows from the fact that any real skew symmetric matrix of odd dimension must have a kernel, which contradicts the conditions of Definition 2.1.

For any linear subspace of a symplectic vector space, we can define its symplectic complement.

Definition 2.3. Let (V, ω) be a symplectic vector space and $W \subseteq V$ a linear subspace. Then the symplectic complement of W with respect to ω is defined as

$$W^\omega := \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}$$

By linearity of ω, the subset W^ω is a linear subspace. Unlike the orthogonal complement with respect to an inner product, where $W^\perp \cap W = \{0\}$, this is not the case for $W^\omega \cap W$. There are several possibilities.

Definition 2.4. Let (V, ω) be a symplectic vector space and $W \subseteq V$ a linear subspace with the associated symplectic complement $W^\omega \subseteq V$.

- We call W isotropic if $W \subseteq W^\omega$.
- We call W coisotropic if $W^\omega \subseteq W$.
- We call W Lagrangian if it is both isotropic and coisotropic, i.e. $W^\omega = W$.
- We call W a symplectic subspace if $W^\omega \cap W = \{0\}$.

Linear endomorphisms of a vector space can preserve the symplectic structure.

Definition 2.5. Let (V, ω) be a symplectic vector space and $\Psi \in \mathrm{End}(V)$. We say Ψ is a linear symplectomorphism if

$$\Psi^*\omega = \omega.$$

Here, by the pull-back we mean that for $v, w \in V$, $\Psi^*\omega(v, w) = \omega(\Psi v, \Psi w)$. All linear symplectomorphisms of a symplectic vector space (V, ω) form a group denoted

$$(V, \omega).$$

We can transfer the definition of a symplectic vector space to smooth manifolds by considering a 2-form ω that equips every tangent space with the structure of an symplectic vector space. Let M be a manifold, then denote

$$\Omega^k(M) = \{\text{Differential } k\text{-forms on the manifold } M\}.$$

We have the following definition for a symplectic manifold.

Definition 2.6 (Symplectic manifold)**.** A symplectic manifold (M, ω) is a pair where M is a manifold and $\omega \in \Omega^2(M)$ such that the following holds

(i) For every $x \in M$, the bilinear map $\omega_x : T_xM \times T_xM \to \mathbb{R}$ is non-degenerate.

(ii) The 2-form ω is closed, i.e. $d\omega = 0$.

The above definition ensures that for every point $x \in M$, the tangent space T_xM has the structure of a symplectic vector space, as ω_x is non-degenerate, bilinear and skew-symmetric.
Extending the definitions of a isotropic, coisotropic and Lagrangian subspaces to the manifold setting yields the following definition.

Definition 2.7. Let (M, ω) be a symplectic manifold and $N \hookrightarrow M$ be a submanifold.

- We call N an isotropic submanifold if for every $x \in N$, the vector space $T_xN \subseteq T_xM$ is isotropic.

- We call N a coisotropic submanifold if for every $x \in N$, the vector space $T_xN \subseteq T_xM$ is coisotropic.

- We call N a Lagrangian submanifold if for every $x \in N$, the vector space $T_xN \subseteq T_xM$ is Lagrangian.

- We call N a symplectic submanifold if for every $x \in N$, the vector space $T_xN \subseteq T_xM$ is symplectic.

REMARK. Note that the Lagrangian condition can be rephrased. A submanifold $L \subseteq M$ is Lagrangian if and only if $\omega|_L = 0$ and $\dim L = \frac{1}{2} \dim M$.

A diffeomorphism $\psi : M \to M$ can preserve the symplectic form.

Definition 2.8. Let $\psi \in \mathrm{Diff}(M)$. We say ψ is a symplectomorphism if

$$\psi^*\omega = \omega.$$

All symplectomorphisms of a symplectic manifold (M, ω) form a group denoted

$$\mathrm{Symp}(M, \omega).$$

We define a Riemannian metric on a manifold M. In order to define Floer homology we need a Riemannian metric on $C^\infty(S^1, M)$.

Definition 2.9 (Riemannian metric)**.** A Riemannian metric (M, g) is a pair where M is a manifold with an inner product g_x on every vector space T_xM which varies smoothly with x. That is, a bilinear symmetric positive definite form

$$g_x : T_xM \times T_xM \to \mathbb{R}$$

which depends smoothly on x.

Another concept that shows up in the Floer equation is that of an almost complex structure. Of special interest is ω-compatibility which constitutes a relation between g, J and ω.

Recall that a complex structure J on a vector space V is an automorphism $J : V \to V$ such that $J^2 = -\operatorname{Id}$.

Definition 2.10 (Almost complex structure)**.** Let M be a manifold. Then an almost complex structure J is a complex structure on the tangent bundle TM. This is equivalent to saying that $J : TM \to TM$ is a fiber-preserving map such that $J^2 = -\operatorname{Id}$.

If M is a symplectic manifold, there is a concept of compatibility between the symplectic form ω on M and an almost complex structure J on M. We use this to construct a Riemannian metric on $C^\infty(S^1, M)$ using an almost complex structure and a given ω on M.

Definition 2.11 (ω-compatibility)**.** Let (M, ω) be a symplectic manifold and J an almost complex structure on M. Then J is said to be ω-compatible if the form

$$\langle \cdot , \cdot \rangle : TM \times TM \to \mathbb{R}$$

defined by

$$(v, w) \mapsto \langle v, w\rangle = \omega(v, Jw)$$

defines a Riemannian metric on M.

We denote the set of ω-compatible almost complex structures by

$$\mathcal{J}(M, \omega)$$

Remark. By proposition from [**MS98**][Proposition 4.1 (i)], we have $\mathcal{J}(M, \omega) \neq \emptyset$. Hence, given ω, we can always find an ω-compatible almost complex structure, which then induces a metric $\omega(\cdot , J\cdot)$ by definition of ω-compatibility. We use this to find a metric on $C^1(S^1, M)$.

In the introduction we say that the fact that ω is non-degenerate allows us to define the Hamiltonian vector field X_H associated to a Hamiltonian function $H : [0, 1] \times M \to \mathbb{R}$ defined by

$$\iota_{X_H} = dH.$$

We expand on this and define the group of Hamiltonian diffeomorphisms.

We can also integrate the vector field X_H and consider its flow. Suppose that M is closed [1], then the flow of X_H is complete. Then X_H generates a 1-parameter family of diffeomorphisms $\varphi_H^t : M \to M$ by

$$\frac{d\varphi_H^t}{dt} \circ \varphi_H^t = X_H$$

with initial condition $\varphi_H^0 = \operatorname{Id}_M$.

[1]From now on, we will say a manifold M is closed when M is compact and $\partial M = \emptyset$

Definition 2.12. Suppose $H \in C^\infty([0,1] \times M, \mathbb{R})$ is a Hamiltonian with associated Hamiltonian vector field X_H. Denote the flow of X_H by φ_H^t. Any map $\varphi : M \to M$ such that there exists H such that $\varphi = \varphi_H^1$ is a called a Hamiltonian diffeormorphism. The set of Hamiltonian diffeomorphisms forms a group denoted $\mathrm{Ham}(M, \omega)$.

Remark. There are several remarks to be made about the this definition.

- Note that this definition works also for non-compact manifolds. However, some properties of $\mathrm{Ham}(M, \omega)$ are not valid in this case. In the non-compact case therefore, often the restriction is made to $\mathrm{Ham}_c(M, \omega)$, the subset of Hamiltonian diffeomorphisms generated by compactly supported Hamiltonians. See Remark 10.5 in [**MS98**]for more details.

- A small computation shows that actually $\mathrm{Ham}(M, \omega) \subset \mathrm{Symp}(M, \omega)$ is a subgroup. In the case where M is closed, this subgroup is normal and path-connected. See [**MS98**] Proposition 10.2 for both the proof that it is a group and normal and path-connected.

In the introduction we already mentioned that there is a bijection between $x \in \mathcal{P}(H)$ and the fixed points of φ_H^1.
For any map $\varphi : M \to M$ denote

$$\mathrm{Fix}(\varphi) := \{x \in M \mid \varphi(x) = x\}$$

the set of fixed points. Then the bijection between $\mathcal{P}(H)$ and $\mathrm{Fix}(\varphi_H^1)$ is given by evaluation in zero, $ev_0 : \mathcal{P}(H) \to \mathrm{Fix}(\varphi_H^1)$ where $ev_0(x) = x(0)$. In this way, the Arnold conjecture can be interpreted as a statement about fixed points of Hamiltonian diffeomorphisms. However, we prove the weak Arnold conjecture about $\mathcal{P}_0(H)$, the subset of contractable periodic orbits. In this case, the relation to fixed points of Hamiltonian diffeomorphisms becomes more obscure. It may not be the case that for any loop of Hamiltonian diffeomorphisms φ^t such that $\varphi^0 = \mathrm{Id}$, that for $p \in M$ the path $x_p(t) := \varphi^t(p)$ is contractible. It is true that this is the case, but the proof requires Floer homology. This is the reason we state the Arnold conjecture as a statement about $\mathcal{P}_0(H)$ instead.

CHAPTER 3

The action functional

3.1. The loop space

The following section is largely based on Section 6 of [**AD14**], a very detailed and thorough book on Floer homology. Some inspiration is also taken from the excellent lecture notes on Floer homology by W.J. Merry which can be found on his website.

The action functional $\mathcal{A}_H$ defined in the next section is defined on an infinite dimensional path space which we define in this section. We define the space of contractible loops.

Definition 3.1. Let M be a smooth manifold. Then we denote the space of loops of class $W1,p$

$$\mathcal{L}_0 M := \{x \in W^{1,p}(S^1, M) \mid x \text{ is contractible}\}.$$

By contractible here, we mean that x is homotopic to the constant map. All loops and homotopies are with free endpoints. The choice for contractible loops is the choice of a connected component of the space of all loops. In this case the natural choice is the component containing the constant loops, which means contractible loops.
We need to restrict to this special class of functions instead of C^∞ functions, as such spaces would merely be Fréchet manifolds. By restricting to functions of class $W^{1,p}(S^1, M)$ however, one can give $\mathcal{L}_0 M$ the structure of a Banach manifold, by application of the inverse function theorem. The following remark describes the structure of a Banach manifold on $\mathcal{L}_0 M$.

Remark. Let $x \in \mathcal{L}_0 M$. Then there is a symplectic trivialization of the pullback bundle x^*TM, as this is a bundle over S^1. For the definition of pullback bundle see Definition C.8. This is a map $\Phi : x^*TM \to S^1 \times \mathbb{R}^{2n}$. Then we set

$$W^{1,p}(x^*TM) := \{Y \in \Gamma(S^1, x^*TM) \mid \Phi Y \in W^{1,p}(S^1, \mathbb{R}^{2n})\}.$$

One can show that the Banach manifold structure induced in this way does not depend on the choice of trivialization Φ. Let exp denote the exponential of a fixed Riemannian metric g. The atlas on the loop space is given by the pairs

$$(W^{1,p}(x^*TM), \exp_x)$$

where $W^{1,p}(x^*TM)$ is a Banach space as above and $\exp_x$ is the diffeomorphism. This construction is independent of the choice of Riemannian metric g. See Section 6.8 in [**AD14**] for more details, with Theorem 6.8.1 in particular.

Let us describe tangent vectors at some $x \in \mathcal{L}_0 M$. Recall that a tangent vector X at x is an equivalence class of paths $u(s)$ through x with $u(0) = x$ and $\dot{u}(0) = X$. As $u(s)$ is a loop for every $s \in \mathbb{R}$ we view it as a map

$$\begin{aligned} u : \mathbb{R} \times S^1 &\to M \\ (s,t) &\mapsto u(s,t) \end{aligned}$$

Then by the above definition we have $X(t) \in T_{x(t)}M$ for every $t \in S^1$. Hence, philosophically, a tangent vector X is a section of the tangent bundle along x. That is, $X \in \Gamma(S^1, x^*TM)$. One readily sees that a section of the bundle x^*TM is precisely what we describe above.
Using this notion of a tangent space, one can also define a metric on $\mathcal{L}_0 M$ using that M comes equipped with a symplectic form ω. In the definition of Floer homology, we will consider flow lines of the gradient $\nabla_J \mathcal{A}_H$ with respect to the L^2-metric. Let $J \in \mathcal{J}(M, \omega)$. Then by definition

$$g_J = \omega(\,\cdot\,, J\,\cdot\,)$$

is a Riemannian metric. Then define the L^2-metric.

Definition 3.2. Let $x \in \mathcal{L}_0 M$ and $X, Y \in T_x M$. Define an metric on $\mathcal{L}_0 M$ by

$$\langle X, Y \rangle_J = \int_{S^1} g_J(X(t), Y(t))\, dt$$

One readily checks that this is indeed a metric.

3.2. The action functional $\mathcal{A}_H$

To define the action functional $\mathcal{A}_H$ we will need two assumptions already mentioned in the introduction. We provide a few more details here. Let (M, ω) be closed symplectic.

Assumption 3.3 (Symplectic Aspherical)**.** For every smooth map $\alpha : S^2 \to M$ we have $\int_{S^2} \alpha^* \omega = 0$.

Symplectic manifolds with this property are called symplectic aspherical manifolds. This can be rephrased in the language of algebraic topology as follows. Let

$$\Delta(M) := im\{h : \pi_2(M) \to H_2(M)\} \otimes \mathbb{R}$$

where h denotes the Hurewicz map. Denote $\langle \cdot, \cdot \rangle$ the Kronecker pairing then Assumption 3.3 is equivalent to

$$\langle \omega, \sigma \rangle = 0$$

for all $\sigma \in \Delta(M)$.

Assumption 3.4 (Vanishing of c_1)**.** For every smooth map $\alpha : S^2 \to M$, there exists a symplectic trivialization of the fiber bundle $\alpha^* TM$.

Again, in more algebraic terms this means that

$$\langle c_1(TM, \omega), \sigma \rangle = 0$$

for all $\sigma \in \Delta(M)$. For details on the Chern class, see the Appendix A.3.

Remark. By the algebraic descriptions of both assumptions, it is immediatly clear that any manifold with $\pi_2(M) = 0$ satisfies both. Hence, an example of a suitable symplectic manifold M in our setting would be T^{2n} with its standard symplectic structure.

In order to define the symplectic action functional $\mathcal{A}_H$ we need the following lemma.

Lemma 3.5. *Let $x \in \mathcal{L}_0 M$ be a a loop and $u, v : D \to M$ such that $u(\partial D) = v(\partial D) = x(t)$. Then*

$$\int_D u^* \omega = \int_D v^* \omega$$

Proof.: We can define a function $u \# v : S^2 \to M$ by gluing the two disks on which u and v are defined along their boundaries, where we have $u(\partial \mathbb{D}) = v(\partial \mathbb{D})$ to define a map on S^2. Then by assumption 3.3, we get that $\int_{S^2} (u \# v)^* \omega = 0$. However, we have $\int_{\mathbb{D}} u^* \omega - \int_{\mathbb{D}} v^* \omega = \int_{S^2} (u \# v)^* \omega$, so the integrals agree.

Remark. Strictly speaking, $u \# v$ as defined above may not be smooth. It is defined as a map $u \# v : \mathbb{D} \# \mathbb{D} \to M$, where $\mathbb{D} \# \mathbb{D}$ is diffeomorphic to S^2 with $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$ is the unit disc in $\mathbb{C}$. We need a small argument by smoothening such that there exists $\varepsilon > 0$ such that $u(z) = u(\frac{z}{|z|})$ and $v(z) = v(\frac{z}{|z|})$ for all $z \in \mathbb{D}$ such that $1 - \varepsilon \leq z \leq 1$. Then we can glue these two discs to a sphere and the result follows.

We can then define the action functional associated to the Hamiltonian H.

Definition 3.6 (Symplectic action functional)**.** Let $H : S^1 \times M \to \mathbb{R}$ be a time-dependent 1-periodic Hamiltonian. The *symplectic action functional*

$$\mathcal{A}_H : \mathcal{L}_0 M \to \mathbb{R}$$

is defined by

$$\mathcal{A}_H(x) = \int_{\mathbb{D}} u^* \omega + \int_0^t H(t, x(t))\, dt \tag{3.1}$$

Here, $u : \mathbb{D} \to M$ is an arbitrary extension of $x : S^1 \to M$ to the disk. More explicitly, if $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$, then u is a map such that $u(e^{2i\pi t}) = x(t)$ for all $t \in S^1$. By the above lemma 3.5, the right hand side of equation 3.6 is indepedent of the choice of u. Hence $\mathcal{A}_H$ is well-defined. The functional $\mathcal{A}_H$ has the following interesting property.

Proposition 3.7. *A loop $x \in \mathcal{L}_0 M$ is a critical point of $\mathcal{A}_H$ if and only if x satisfies*

$$\dot{x}(t) = X_H(x(t)) \tag{3.2}$$

(i.e. $x \in \mathcal{P}_0(H)$.

PROOF. Let $x \in \mathcal{L}_0 M$. By critical points we mean critical in the variational sense. Therefore, we define a smooth family $\widetilde{x}_s(t) := \widetilde{x}(s,t)$ around $x(t)$ in the following way. Extend x to $\widetilde{x}(s,t)$ such that

$$\widetilde{x}(0,t) = x(t) \qquad \text{and} \qquad \frac{\partial \widetilde{x}}{\partial s}|_{(0,t)} = X(t).$$

Then we can compute the differential of $\mathcal{A}_H$ at x applied to X. Formally, we have

$$d\mathcal{A}_H(x)X = \frac{d\mathcal{A}_H(\widetilde{x})(s,\cdot)}{ds}(0) \tag{3.3}$$

To compute the right term, we need to slightly extend u to $\widetilde{u}$ such that

$$\widetilde{u}(0,z) = u(z) \qquad \text{and} \qquad \widetilde{u}(s, e^{2i\pi t}) = \widetilde{x}(s,t).$$

This allows us to extend X by setting $X(z) = \frac{\partial \widetilde{u}}{\partial s}|_{(0,z)}$. We can now compute using Lebesgue dominated convergence

$$\begin{aligned}
-\frac{d}{ds}\Big(\int_{\mathbb{D}} \widetilde{u}^*\omega\Big)|_{s=0} &= -\int_{\mathbb{D}} \Big(\frac{\partial}{\partial s}\widetilde{u}^*\omega\Big)|_{s=0} \\
&= -\int_{\mathbb{D}} u^*(\mathcal{L}_{X(z)}\omega) \\
&= -\int_{\mathbb{D}} u^*(di_{X(z)}\omega) \\
&= -\int_{S^1} x^*(i_{X(z)}\omega) \\
&= -\int_0^1 \omega(X(z), \dot{x}(t))\, dt \\
&= \int_0^1 \omega(\dot{x}(t), X(t))\, dt
\end{aligned}$$

Here in the third step we use Cartan's formula

$$\mathcal{L}_X \alpha = \iota_X d\alpha + d\iota_X \alpha$$

for any $\alpha \in \Omega^k(M)$ for $k \geq 0$ and the fact that $d\omega = 0$. We compute the second term.

$$\begin{aligned}
\frac{d}{ds}\int_0^1 H_t(\widetilde{x}(s,t))\, dt|_{s=0} &= \int_0^1 (dH_t)_{\widetilde{x}(0,t)}(X(t))\, dt \\
&= \int_0^1 \omega_{x(t)}(X(t), X_{H_t}(x(t)))\, dt
\end{aligned}$$

We have

$$\mathcal{A}_H(\widetilde{x}) = -\int_{\mathbb{D}} \widetilde{u}^*\omega + \int_0^1 H_t(\widetilde{x}(s,t))\, dt.$$

Then by the above calculation and equation 3.3, we have

$$d\mathcal{A}_H(x)X = \int_0^1 \omega(\dot{x}(t) - X_{H_t}(x), X)\, dt.$$

Now a critical point x is such that $d\mathcal{A}_H)(x)X = 0$ for all $X \in T_{x(t)}M$. By non-degeneracy of ω, this happens if and only if

$$\dot{x}(t) = X_{H_t}(x(t))$$

which is precicely equation 3.2. This proves Proposition 3.7. □

Using Proposition 3.7 we see that critical points of $\mathcal{A}_H$ correspond to solutions of equation (3.2). We now compute the gradient of $\mathcal{A}_H$ with respect to the metric $\langle \cdot, \cdot \rangle_J$ defined above.

Recall the definition of the gradient. Let (N, g) be a smooth manifold equipped with a Riemannian metric. Let $f \in C^1(N, \mathbb{R})$. Then the gradient of f is a vector field $\nabla f \in \Gamma(N, TN)$ such that for any vector field $X \in \Gamma(N, TN)$ we have

$$g_x(\nabla_x f, X(x)) = df(x)X(x)$$

for all $x \in N$. In our case for $\mathcal{A}_H : \mathcal{L}_0 M \to \mathbb{R}$ with metric $\langle \cdot, \cdot \rangle_J$ the gradient $\nabla_J \mathcal{A}_H$ is then defined by

$$\langle \nabla_J \mathcal{A}_H, X \rangle_J = d\mathcal{A}_H(\cdot)X$$

for any vector field $X \in \Gamma(\mathcal{L}_0 M, T\mathcal{L}_0 M)$. We have the following proposition.

Proposition 3.8. *For the action function $\mathcal{A}_H$,with respect to the metric $\langle \cdot, \cdot \rangle_J$, we have at $x \in \mathcal{L}_0 M$ that*

$$\nabla_J \mathcal{A}_H(t) = J(x(t))\dot{x}(t) + \nabla_{x(t)} H_t$$

Proof. This is a direct computation using that $g_J(\cdot, \cdot) = \omega(\cdot, J\cdot)$. □

One can readily compute that the gradient of H_t is the Hamiltonian vector field X_{H_t} composed with $J(x)$ by definition, so that the above can also be rewritten to read

$$\nabla_J \mathcal{A}_H(t) = J(x)\big(\dot{x}(t) - X_{H_t}(x(t))\big) \tag{3.4}$$

We look at negative gradient flow lines, which are maps

$$u : \mathbb{R} \to \mathcal{L}_0 M.$$

such that

$$\frac{d}{ds}u(s) = -\nabla_J \mathcal{A}_H(u(s)) \tag{3.5}$$

where $\nabla_J \mathcal{A}_H$ is give by equation (3.4). The gradient flow equation (3.5) is an ordinary differential equation on the loop space $\mathcal{L}_0 M$. However, $\mathcal{L}_0 M$ is an infinite dimensional space. Floer's idea was to instead regard the map u as a map

$$u : \mathbb{R} \times S^1 \to M.$$

Then, equation (3.5) can be rewritten to a partial differential equation on the finite dimensional space M. This equation is known as Floer's equation

$$\frac{\partial u}{\partial s} + J(u)(\frac{\partial u}{\partial t} - X_{H_t}(u)) = 0 \tag{3.6}$$

This is the central equation in this thesis. We will analyze solutions to this equation, which will be essential to defining Floer homology.

CHAPTER 4

The Conley-Zehnder index

The Floer chain complex used to define Floer homology is generated by critical points of the action functional $\mathcal{A}_H$. We saw in Chapter 3 that these critical points correspond to periodic orbits $x \in \mathcal{P}_0(H)$. In order to grade the Floer chain complex, we index the critical points using the Conley-Zehnder index, henceforth abbreviated as CZ-index. This index is based on the Maslov index for a path of symplectic matrices. In this chapter we give a description of this index. Furthermore, we work out one example where we compute the index.

We define the CZ-index and calculate this index for a path of rotations in the plane. This will be used in Chapter 5, where it is shown that the CZ-index equals the Fredholm index of the vertical differential of the Floer operator.

4.1. Maslov index for paths $\Phi : [0, 1] \to \mathrm{Sp}(2n)$.

We define the Maslov index for a path of symplectic matrices. In the following, let J_0 and ω_0 be the standard almost complex structure and standard symplectic form on $\mathbb{R}^{2n}$. One can also view this as $\mathbb{C}^n$ where J_0 denotes multiplication by the complex variable i.

Definition 4.1. We call the matrices

$$Aut(\mathbb{R}^{2n}, \omega_0) = \mathrm{Sp}(2n) = \{A \in \mathbb{R}^{2n\times 2n} \mid A^* J_0 A = J_0\}$$

the symplectic group.

Note that this is the same as $\mathrm{Sp}(V, \omega)$ defined in Chapter 2, where now (V, ω) is taken to be $(\mathbb{R}^{2n}, \omega_0)$, meaning $\mathrm{Sp}(2n) = \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0)$.

Define the subspace of symplectic matrices A that do not have 1 as an eigenvalue as follows

$$\mathrm{Sp}^*(2n; \mathbb{R}) = \{A \in \mathrm{Sp}(2n; \mathbb{R}) \mid \det(A - \mathrm{Id}) \neq 0\}.$$

Then the path space we look at are continuous paths that start at Id and end in $\mathrm{Sp}^*(2n; \mathbb{R})$. This space of paths is defined by

$$\mathcal{S} = \{\Phi \in C^0([0, 1], \mathrm{Sp}(2n)) \mid \Phi(0) = \mathrm{Id}, \Phi(1) \in \mathrm{Sp}^*(2n)\}.$$

We equip this space with the compact-open topology.

There are several possible ways to define the Conley-Zehnder index. We follow the exposition in [**SZ92**] and define a map ρ that we use to define the Conley-Zehnder index. The determinant map

$$\det : U(n) \to S^1$$

induces an isomorphism of fundamental groups by using the homotopy exact sequence of the fibration $SU(n) \to U(n) \to S^1$. Now $\mathrm{Sp}(2n;\mathbb{R})/U(n)$ is contractible, so $\pi_1(\mathrm{Sp}(2n;\mathbb{R})) \simeq \mathbb{Z}$. (Some details here) This isomorphism yields a natural continuous map

$$\rho : \mathrm{Sp}(2n;\mathbb{R}) \to S^1$$

which extends the determinant map.

We will use the following notation. Suppose (V_1, ω_1) and (V_2, ω_2) are two symplectic vector spaces. Define the product space

$$(V, \omega) = (V_1 \times V_2, \omega_1 \times \omega_2).$$

Let $(W, \widetilde{\omega}) = (W_1 \times W_2, \widetilde{\omega_1} \times \widetilde{\omega}_2)$ be another product space. Then any $A \in \mathrm{Lin}(V, W)$ can be decomposed as $A = A_1 \times A_2$ where for $v = (v_1, v_2) \in V$ we have $A_i : V_i \to W_i$ such that

$$A(v_1, v_2) = (A_1 v_1, A_2 v_2) \tag{4.1}$$

Equivalently, A_i is given by

$$A_i = (p_{W_i} \circ A)|_{V_i} : V_i \to W_i$$

Furthermore, we use Lemma 2.19 from [**MS98**] which says that any $\Psi \in \mathrm{Sp}(2n;\mathbb{R}) \cap \mathrm{O}(2n)$ is of the form

$$\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \tag{4.2}$$

such that $X^T Y = Y^T X$ and $X^T X + Y^T Y = \mathrm{Id}$. Note that this is equivalent to $X + iY \in U(n)$.

The formal definition of ρ is described by the following theorem.

Theorem 4.2. *Let (V, ω) be a finite dimensional symplectic vector space. Denote*

$$\mathrm{Aut}(V, \omega) = \{\Phi \in \mathrm{Aut}(V) \mid \Phi^* \omega = \omega\}.$$

There is a unique collection of maps

$$\rho : \mathrm{Sp}(V, \omega) \to S^1$$

one for each (V, ω), satisfying the following conditions.

Naturality *Let (V_1, ω_1) and (V_2, ω_2) be symplectic vector spaces. If $T : V_1 \to V_2$ is an isomorphism such that $T^*\omega_2 = \omega_1$ then*

$$\rho(TAT^{-1}) = \rho(A)$$

for any $A \in \mathrm{Sp}(V_1, \omega_1)$.

Product *Let (V_1, ω_1) and (V_2, ω_2) be symplectic vector spaces and (V, ω) the product*

$$(V, \omega) = (V_1 \times V_2, \omega_1 \times \omega_2).$$

Then for the decomposition $A = A_1 \times A_2$ as in (4.1) the following holds

$$\rho(A) = \rho(A_1)\rho(A_2).$$

Determinant *Suppose $A \in \mathrm{Sp}(2n; \mathbb{R}) \cap O(2n)$ is of the form (4.2). Then*

$$\rho(A) = \det(X + iY).$$

Normalization *Let $A \in \mathrm{Sp}(V, \omega)$. Suppose that A has no eigenvalues on $S^1 \subset \mathbb{C}$. Then*

$$\rho(A) = \pm 1.$$

We now construct the Conley-Zehnder index. We need a lemma.

Lemma 4.3. *The set $\mathrm{Sp}^*(2n)$ has two connected components*

$$\mathrm{Sp}(2n; \mathbb{R})^{\pm} = \{A \in \mathrm{Sp}(2n; \mathbb{R}) \mid \pm \det(\mathrm{Id} - A) > 0\}.$$

Furthermore, any loop in $\mathrm{Sp}(2n; \mathbb{R})^*$ is contractible in $\mathrm{Sp}(2n; \mathbb{R})$.

Proof. We refer to the proof of Lemma 3.2 in [**SZ92**]. ☐

We now define the Maslov index of a path $\Phi \in C^0([0,1], \mathrm{Sp}(2n))$. Note that this works for any path, meaning we do not require that $\Phi(1) \in \mathrm{Sp}^*(2n)$.

Definition 4.4 (Maslov index)**.** Suppose $\Phi \in C^0([0,1], \mathrm{Sp}(2n))$. Choose a continuous function

$$\alpha : [0,1] \to \mathbb{R} \qquad \text{such that} \qquad \rho(\Phi(t)) = e^{i\alpha(t)}.$$

Then define the Maslov index of Φ

$$\Delta(\Phi) = \frac{\alpha(1) - \alpha(0)}{\pi}.$$

This assigns to any $\Phi \in C^0([0,1], \mathrm{Sp}(2n))$ some number.

Remark. The author was not sure what to call this index. The definition relies on picking a lift α of $\rho(\Phi(t))$, meaning exactly that $\rho(\Phi(t)) = e^{i\alpha(t)}$. Then the difference $\alpha(1) - \alpha(0)$ is independent of the particular choice of a lift: any two lifts differ only by a translation by $2k\pi$ in $\mathbb{R}$ which is irrelevant for the difference $\alpha(0) - \alpha(1)$.

The Maslov index usually refers to the above index in the case that $\Phi \in C^0(S^1, \mathrm{Sp}(2n))$. Then the above index can be described as the degree of the path composed with the rotation map ρ. We have

$$\Delta(\Phi) = \deg(t \mapsto \rho(\Phi(t))).$$

Note that this index is independent of the choice of α.

4.2. The Conley-Zehnder index of a path $\Phi \in \mathcal{S}$.

Let now $\Phi \in \mathcal{S}$. By definition, $\Phi(1) \in \mathrm{Sp}^*(2n)$. Now choose any path $\Phi' \in C^\infty([0,1], \mathrm{Sp}(2n;\mathbb{R})^*)$ such that $\Phi'(0) = \Phi(1)$ and $\Phi'(1) \in \{W^+, W^-\}$ where

$$W^+ = -\,\mathrm{Id}$$

and

$$W^- = \mathrm{diag}(2, -1, \ldots, -1, \frac{1}{2}, -1, \ldots, -1).$$

in the following way.

Note that either $\Phi(1) \in \mathrm{Sp}(2n;\mathbb{R})^+$ or $\Phi(1) \in \mathrm{Sp}(2n;\mathbb{R})^-$ by definition of $\mathcal{S}$. We choose $\Phi'(1) = W^\pm$ whenever $\Phi(1) \in \mathrm{Sp}(2n;\mathbb{R})^\pm$. Figure 1 depicts the construction of Φ and Φ' as just outlined.

We can compute the Maslov index of Φ' as in Definition 4.4. From Lemma 3.2 in [**SZ92**], it follows that $\Delta(\Phi')$ is independent of the specific choice of path Φ' and depends only on the particular starting point $\Phi'(0) = \Phi(1)$. We are now ready to define the Conley-Zehnder index of a path $\Phi \in \mathcal{S}$.

Definition 4.5 (The Conley-Zehnder index)**.** Let $\Phi \in \mathcal{S}$. Define a path Φ' as described above. The Conley-Zehnder index of the path Φ is defined by

$$\mu_{\mathrm{CZ}}(\Phi) := \Delta(\Phi) + \Delta(\Phi').$$

The following Theorem 4.6 is Theorem 3.3 in [**SZ92**]. It lists several properties of the Conley-Zehnder index defined in Definition 4.5.

Theorem 4.6. *Let μ_{CZ} be defined as in Definition 4.5. Then the following properties hold.*

(i) *Let $\Phi \in \mathcal{S}$. The Conley-Zehnder index $\mu_{\mathrm{CZ}}(\Phi)$ is an integer.*

(ii) *Two paths Φ and Ψ are homotopic in $\mathcal{S}$ if and only if*

$$\mu_{\mathrm{CZ}}(\Phi) = \mu_{\mathrm{CZ}}(\Psi).$$

(iii) *Let $\Phi \in \mathcal{S}$. Then*

$$\mathrm{sign}\det(\mathrm{Id} - \Phi(1)) = (-1)^{\mu_{\mathrm{CZ}}(\Phi) - n}.$$

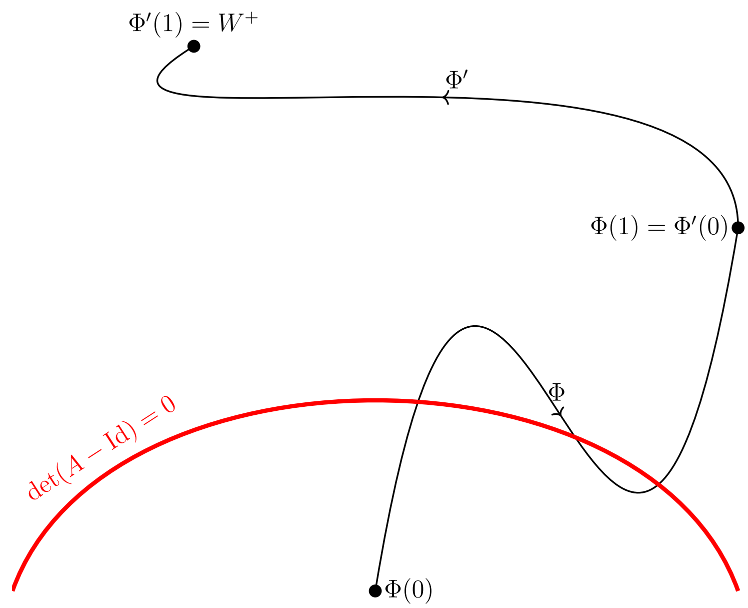

Figure 1. The path used to compute $\mu_{\mathrm{CZ}}(\Phi)$ for $\Phi \in \mathcal{S}$

(iv) *Let $S \in \mathrm{Sym}(\mathbb{R}^{2n\times 2n})$ be a non-singular matrix such that $\|S\| < 2\pi$. Define a path $\Psi(t) = \exp(J_0 t S)$. Then $\Psi \in \mathcal{S}$ and*

$$\mu_{\mathrm{CZ}}(\Psi) = \mathrm{ind}(S) - n.$$

Here by $\mathrm{ind}(S)$ *we mean the number of negative eigenvalues of S, counted with multiplicity.*

In this way, we have defined a map

$$\mu_{\mathrm{CZ}} : \mathcal{S} \to \mathbb{Z}.$$

Remark. There are two remarks to make here.
First of all, again this construction can be described as the degree of a map. Note that $(\rho(W^{\pm}))^2 = 1$. Given $\Phi \in \mathcal{S}$, let Φ' be constructed as above a path such that $\Phi'(1) = W^{\pm}$, depending on $\Phi(1)$. Denote $\widetilde{\Phi} : [0,2] \to \mathrm{Sp}(2n)$ the path

$$\widetilde{\Phi}(t) := \begin{cases} \Phi(t) & 0 \le t \le 1 \\ \Phi'(t-1) & 1 \le t \le 2 \end{cases}$$

Then

$$\mu_{\mathrm{CZ}}(\Phi) = \deg(t \mapsto (\rho(\widetilde{\Phi}(t)))^2).$$

The second remark concerns computability. The above construction is the easiest to explain, but actually computing μ_{CZ} of a path using this definition is hard. There is another equivalent definition by Robbin-Salamon defined in [**RS93**]. Here, philosophically speaking, it is counted how often a path $\Phi \in \mathcal{S}$ "crosses through" $\det(\Phi(t) - \mathrm{Id}) = 0$ with some sign, which is the thick red line in Figure 1. For this alternative definition, we refer the interested reader to [**RS93**] for details.

4.3. The Conley-Zehnder index for periodic orbits.

We have defined an index $\mu_{\mathrm{CZ}} : \mathcal{S} \to \mathbb{Z}$ for paths of symplectic matrices. Our goal is to define an index

$$\mathrm{CZ} : \mathcal{P}_0(H) \to \mathbb{Z}.$$

In order to do this, we associate to $x \in \mathcal{P}_0(H)$ a path of symplectic matrices $\Psi \in \mathcal{S}$, in the following way.

Recall that a symplectic trivialization of the bundle $x^*TM \to S^1$ is given by a map

$$\Phi : S^1 \times \mathbb{R}^{2n} \to x^*TM$$

which intertwines the standard symplectic form on $\mathbb{R}^{2n}$ and pulls back ω to the standard ω on $\mathbb{R}^{2n}$. Writing $\Phi_t = \Phi(t, \cdot)$ this means

$$\Phi_t^* \omega = \omega_0.$$

The following lemma tells us such trivializations exist.

Lemma 4.7. *For any smooth map $\varphi : \mathbb{D} \to M$, there exists a symplectic trivialization*

$$\Phi : \mathbb{D} \times \mathbb{R}^{2n} \to \varphi^*TM$$

Furthermore, any two such trivializations are homotopic via symplectic trivializations.

Proof. For the proof of existence we refer to [**MS98**] Lemma (2.65). The proof is basically an application of Gramm-Schmidt.
To prove that any two such trivializations are homotopic, let $\varphi : \mathbb{D} \to M$ be given, and let $\Phi, \Phi' : \mathbb{D} \times \mathbb{R}^{2n} \to x^*TM$ be two symplectic trivializations.
Then by definition of symplectic trivialization, for every $z \in \mathbb{D}$,

$$\Phi'(z)^{-1}\Phi(z)$$

is a unitary matrix. However, any smooth map $\mathbb{D} \to U(n)$ is smoothly homotopic to the constant map $z \mapsto \mathrm{Id}$. This proves that Φ and Φ' are homotopic. $\square$

Let $x \in \mathcal{P}_0(H)$. Choose any extension

$$\varphi : \mathbb{D} \to M$$

such that φ extends x to the disk $\mathbb{D}$, meaning

$$\varphi(e^{2\pi it}) = x(t).$$

where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. By Lemma (4.7) we can trivialize φ^*TM. By restriction to $\partial\mathbb{D}$ this gives a trivialization of x^*TM defined by

$$\Phi_x(t) := \Phi(e^{2\pi it}) : \mathbb{R}^{2n} \to x^*TM \tag{4.3}$$

that is periodic.

We prove that this construction does not depend on the particular choice of extension under Assumption (3.4).

Lemma 4.8. *The homotopy class of Φ_x does not depend on the particular choice of extension $\varphi : \mathbb{D} \to M$ if M satisfies Assumption 3.4.*

Proof. Suppose we have two extensions, $\varphi : \mathbb{D} \to M$ and $\varphi' : \mathbb{D} \to M$ such that $x(t) = \varphi(e^{2\pi it}) = \varphi'(e^{2\pi it})$, which the associated symplectic trivializations Φ and Φ'. We will construct a map $u : S^2 \to M$ in the following way.
Assume without loss of generality, by shrinking a bit, that there exists $\epsilon > 0$ such that

$$\varphi(z) = \varphi(\frac{z}{|z|})$$

and

$$\Phi(z) = \Phi(\frac{z}{|z|})$$

for $1 - \epsilon \leq |z| \leq 1$ and similarly for φ' and Φ'.

Define $u : S^2 \to M$ as follows, under the diffeomorphism $S^2 \simeq \mathbb{C} \cup \{\infty\}$

$$u(z) = \begin{cases} \varphi(z) & |z| \leq 1 \\ \varphi'(\frac{1}{z}) & |z| > 1 \end{cases}$$

Under Assumption (3.4), the bundle u^*TM is trivial. This yields a trivialization

$$\Theta : S^2 \times \mathbb{R}^{2n} \to u^*TM.$$

Now use the second part of Lemma (4.7). Restrict Θ to the upper hemisphere $\mathbb{D}^+$ and lower hemisphere $\mathbb{D}^-$. This yields two new trivializations of x^*TM which are homotopic as they agree along the equator $\mathbb{D}^+ \cap \mathbb{D}^-$, by Lemma (4.7). Hence $\Phi_x(t)$ and $\Psi'_x(t)$ are both homotopic to $\Theta(e^{2\pi it})$. This proves that the trivializations are homotopic. $\square$

We are now ready to define the index of a solution. Let $x \in \mathcal{P}_0(H)$, and choose any extension $\varphi : \mathbb{D} \to M$ with the associated restriction Φ_x, which is a trivialization of the bundle x^*TM as in equation (4.3). Define a path of symplectic matrices associated to x.

$$\Psi_x(t) := \Phi_x(t)^{-1} \circ T_{x(0)}\varphi_H^t \circ \Phi_x(0) : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \tag{4.4}$$

for $t \in [0,1]$. Then $\Phi_x(t) \in \mathrm{Sp}(2n;\mathbb{R})$ for every t, as Φ is a symplectic trivialization. Note that the solution $x \in \mathcal{P}_0(H)$ is non-degenerate if and only if $\Psi_x \in \mathcal{S}$. Then define the index of x.

Definition 4.9 (CZ-index of a periodic solution x)**.** Let $x \in \mathcal{P}(H)$ with the associated path $\Phi_x(t)$ as described by equation (4.4). Then define the Conley-Zehnder index of x by

$$\mathrm{CZ}(x) := \mu_{\mathrm{CZ}}(\Phi_x) \tag{4.5}$$

where $\mu(\Phi_x)$ is the Maslov index defined in Definition 4.5.

This index is determined uniquely by the condition that Φ_x extends to a disc bounded by the periodic solution x. To see this, let Φ_x' be any other extension. Then by Lemma 4.8, Φ_x and Φ_x' have the same homotopy class. Then the same is true for the associated Ψ_x and Ψ_x'. By Proposition 4.6, $\mathrm{CZ}(\Psi_x) = \mathrm{CZ}(\Psi_x')$. Hence, this way to define the index is independent of the particular choice of trivialization Φ.

4.4. Paths with a prescribed Conley-Zehnder index.

In the computation of the Fredholm index of the vertical differential of the Floer operator, it will be important to use the existence of paths of matrices with a specific index. We prove the following theorem.

Theorem 4.10. *For every $k \in \mathbb{Z}$ there exists a diagonal matrix $S_k \in M(2n;\mathbb{R})$ such that the associated path given by*

$$\Psi(t) = e^{tJ_0S_k}$$

is in $\mathcal{S}$ and satisfies $\mu_{\mathrm{CZ}}(\Psi) = k$.

The theorem will result from a reduction to the case $n = 1$. This is also our example of a rotation in $\mathbb{R}^2$. We consider matrices of the form

$$S = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}.$$

Direct computation shows that

$$\Psi(t) = \exp(tJ_0S) = \begin{pmatrix} \cos\theta t & -\sin\theta t \\ \sin\theta t & \cos\theta t \end{pmatrix} \tag{4.6}$$

Hence, we are computing $\mu_{\mathrm{CZ}}(\Psi)$ where Ψ is a path of rotations in $\mathbb{R}^2$.

Proof of Theorem 4.10. In order to prove this we generalize the case $n = 1$, that is $S \in M(2;\mathbb{R})$, to general n. We have the following claim in this case.

Claim 1. Let $S \in M(2n;\mathbb{R})$ invertible and symmetric such that $\|S\| \leq 2\pi$. Then for $\Psi(t) = e^{tJ_0S}$ we have

$$\mu_{\mathrm{CZ}}(\Psi) = \mathrm{ind}^-(S) - 1.$$

Here $\mathrm{ind}^-(S)$ denotes the number of negative eigenvalues of S.

Note that this Claim is (iv) of Theorem 4.6. For the sake of completeness, we prove this Claim.

Proof.: Proof of Claim 1 It is a small calculation to verify that $\Psi(t) \in \mathcal{S}$. As S is symmetric, it is diagonizable by orthogonal matrices. This means that there exists a path

$$s \mapsto \Lambda(s) \in O^+(2n)$$

such that $\Lambda(0) = \mathrm{Id}$ and $S(1) = (\Lambda(1))^T S \Lambda(1)$ is a diagonal matrix.

Let now

$$S(\lambda) = (\Lambda(\lambda))^T S \Lambda(\lambda)$$

and define

$$\Psi_\lambda(t) = e^{tJ_0 S(\lambda)}.$$

Note that $S(\lambda)$ is a path of symmetric matrices that connects $S = S(0)$ to a diagonal matrix $S(1)$. Then the number of negative eigenvalues of S_λ, which we denoted $\mathrm{ind}^-(S(\lambda))$, does not depend on λ. Also, $\|S\| < 2\pi$ means that $\Psi_\lambda(1)$ will never have eigenvalue 1. This reduces the proof to checking that the Claim is true for diagonal matrices. We can scale so that without loss of generality, S is of the form

$$S = \mathrm{diag}(\varepsilon, \dots, \varepsilon, -\varepsilon, \dots, -\varepsilon).$$

where $0 < \varepsilon < 2\pi$ and we denote the number of negative terms $k = \mathrm{ind}^-(S)$.

We now reduce to the case $n = 1$ by decomposing $\mathbb{R}^{2n}$ into n symplectic planes. Then there are three fundamental choices of diagonal matrices.

$$D_1 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad D_2 = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \quad D_3 = \begin{pmatrix} -\varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$$

It is now a matter of computing Conley-Zehnder indices for the associated matrices $\Psi_i(t) = \exp(tJ_0 D_i)$.

Let us do D_1 as an example. We have

$$e^{tJ_0D_1} = \begin{pmatrix} \cos\varepsilon t & \sin\varepsilon t \\ -\sin\varepsilon t & \cos\varepsilon t \end{pmatrix}.$$

Use now the determinant property of the rotation map ρ in Theorem 4.2. Then

$$\rho(\exp tJ_0D_1) = \exp(-i\varepsilon t).$$

As $0 < \varepsilon < 2\pi$, we have $\mu_{\mathrm{CZ}}(\exp(tJ_0D_1)) = -1$. To see this, follow the definition of μ_{CZ}. It is immediate that $\Delta(\Psi) = \frac{-\varepsilon}{\pi}$, as $\rho(\Psi(t)) = e^{-i\varepsilon t}$. Complete the path to the matrix $W^+ = -\operatorname{Id}$ as $\det(\operatorname{Id} - \Psi(1)) > 0$ by

$$\Psi'(t) = \begin{pmatrix} \cos{(1-t)\varepsilon + t\pi} & \sin{(1-t)\varepsilon + t\pi} \\ -\sin{(1-t)\varepsilon + t\pi} & \cos{(1-t)\varepsilon + t\pi} \end{pmatrix}.$$

Then $\rho(\Psi'(t)) = e^{-i((1-t)\varepsilon + t\pi)}$. Then $\Delta(\Psi') = \frac{\epsilon - \pi}{\pi}$. Then indeed $\mu_{\mathrm{CZ}}(\exp tJ_0D_1) = -1$. Note that D_1 has indeed 2 negative eigenvalues, such that $\mu_{\mathrm{CZ}}(\exp tJ_0D_1) = \operatorname{ind}^-(D_1) - 1$ is satisfied. Similar calculations for D_2 and D_3 show that indeed $\mu_{\mathrm{CZ}}(\exp(tJ_0S)) = \operatorname{ind}^-(S) - 1$ whenever S is a diagonal 2 by 2 matrix.

The general statement for a diagonal matrix S now follows from the product property of ρ, which translates to multiplicativity of the Conley-Zehnder indices. This proves the Claim.

We now go on to construct S_k from the building blocks we saw in the above claim. Choose some odd integer $l \in \mathbb{Z}$ and consider $S = \begin{pmatrix} l\pi & 0 \\ 0 & l\pi \end{pmatrix}$. Then the associated symplectic matrix is

$$\Psi(t) = e^{tJ_0S} = \begin{pmatrix} \cos l\pi t & -\sin l\pi t \\ \sin l\pi - t & \cos l\pi t \end{pmatrix}.$$

This path ends at $\Psi(1) = W^+$. Then $\mu_{\mathrm{CZ}}(e^{tJ_0S}) = -l$.

For $k = n \mod 2$, consider the matrix

$$S_k = \operatorname{diag}(-\pi, -\pi, \dots, -\pi, -\pi, (n-k-1)\pi, (n-k-1)\pi).$$

This matrix is precisely such that $\mu_{\mathrm{CZ}}(\exp(tJ_0S_k)) = k$.

For $k = n-1 \mod 2$, consider the matrix

$$S_k = \operatorname{diag}(-pi, -\pi, \dots, 1, -1, (n-k-2)\pi, (n-k-2)\pi).$$

This matrix is also such that $\mu_{\mathrm{CZ}}(\exp(tJ_0S_k) = k$.

$\square$

CHAPTER 5

Floer homology and transversality

5.1. Definition of Floer homology

In this section we define the Floer chain groups $\mathrm{CF}_*(H)$ and the boundary operator ∂_J. The homology associated to this chain complex is the Floer homology $HF_*(H,J)$.

We construct a chain complex from the periodic solutions $x : S^1 \to M$ of the Hamiltonian equation

$$\dot{x}(t) = X_H(x(t)) \tag{5.1}$$

and flow lines of $\nabla_J \mathcal{A}_H$ which were solutions $u : \mathbb{R} \times S^1 \to M$ of the Floer equation

$$\frac{\partial u}{\partial s} + J(u)\Big(\frac{\partial u}{\partial t} - X_H(u)\Big) = 0 \tag{5.2}$$

Reminiscent of Morse homology, we want these flow lines to run between critical points. Hence, we impose

$$\lim_{s\to\pm\infty} u(s,t) = x^{\pm}(t), \quad \text{uniformly in } t \tag{5.3}$$

with $x^{\pm} \in \mathcal{P}_0(H)$ and $\lim_{s\to\pm\infty} \frac{\partial u}{\partial s}(s,t) = 0$ uniformly in t. This condition can be rephrased in terms of the energy.

Definition 5.1. Let $u : \mathbb{R} \times S^1 \to M$ be a solution to the Floer equation (3.6). Then its energy is defined to be

$$E(u) := \int_{\mathbb{R}\times S^1} \Big\| \frac{\partial u}{\partial s} \Big\|_J^2 \, dsdt \tag{5.4}$$

The energy of a solution has the following properties.

Proposition 5.2. *Let $u : \mathbb{R} \times S^1 \to M$ be a solution to the Floer equation (3.6). Then*

(i) *$E(u) \geq 0$ and $E(u) = 0$ if and only if $u(s,t) = x(t)$ for some $x \in \mathcal{P}_0(H)$.*

(ii) *If $E(u) < \infty$, then there exist $x^{\pm} \in \mathcal{P}_0(H)$ such that*

$$\lim_{s\to\pm\infty} u(s,t) = x^{\pm}(t)$$

uniformly in t. Furthermore, $E(u) = \mathcal{A}_H(x^-) - \mathcal{A}_H(x^+)$.

Let $H : S^1 \times M \to \mathbb{R}$ be a Hamiltonian and $J \in \mathcal{J}(M, \omega)$. We define the moduli space of solutions running from x^- to x^+.

Definition 5.3.

$$\mathcal{M}(x^-, x^+; J, H) := \{u : \mathbb{R} \times S^1 \to M \mid u \text{ satisfies equations (5.2) and (5.3)}\}$$

In Section 5.4 of this chapter, we prove that for generic Hamiltonians, this space is a finite dimensional submanifold of a particular Banach manifold. This follows from application of the implicit function theorem. To prove that these spaces are manifolds, we will need to prove that a particular operator (the Floer operator $d^V \overline{\partial}_{H,J}$ defined in Section 5.2) is Fredholm with index $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+)$. This will be done throughout this chapter. We assume that $\mathcal{M}(x^-, x^+; J, H)$ has some manifold structure for the time being, and define Floer homology.

For $u \in \mathcal{M}(x^-, x^+; H, J)$ we have the Conley-Zehnder index for the associated critical points $x^{\pm}$ defined in Chapter 4. The chain groups are generated by the critical points and graded by the Conley-Zehnder index.

Definition 5.4 (Floer chain groups)**.** Let

$$\mathcal{P}_k(H) := \{x \in \mathcal{P}(H) \mid CZ(x) = k\}.$$

Then

$$\mathrm{CF}_k(H) := \bigoplus_{x \in \mathcal{P}_k(H)} \mathbb{Z}_2$$

In general any $\xi \in \bigoplus_{x \in \mathcal{P}_k(H)} \mathbb{Z}_2$ can be viewed as a function

$$\xi : \mathcal{P}_k(H) \to \mathbb{Z}_2$$

such that $\xi(x) \neq 0$ for only finitely many $x \in \mathcal{P}_k(H)$. For any $x \in \mathcal{P}_k(H)$, there is a particular $\delta_x \in \mathrm{CF}_k(H)$ given by

$$\delta_x(y) := \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases}.$$

In this case, we can write

$$\xi = \sum_{x \in \mathcal{P}_k(H)} \xi(x) \delta_x$$

meaning that these maps generate the complex. We will abbreviate the function δ_x by $\langle x \rangle$. By the above description of an element ξ, it is enough to know what a map does on the generators $\langle x \rangle$.

Associated to the complex $\mathrm{CF}_*(H)$ is the boundary operator ∂_J. It is defined on the generators by

Definition 5.5 (Floer boundary operator)**.**

$$\partial_J : \mathrm{CF}_{k+1}(H) \to \mathrm{CF}_k(H)$$

$$\partial(\langle x\rangle) = \sum_{\substack{y\in\mathcal{P}_0(H)\\ \mathrm{CZ}(y)=CZ(x)-1}} \eta(x,y)\langle y\rangle$$

Here

$$\eta(x,y) = \#\Big(\mathcal{M}(x,y;H,J)/\mathbb{R}\Big) \tag{5.5}$$

Philosophically, it counts the number of solutions u running from x to y modulo translations in the s variable; we only count unparametrized trajectories.

To be precise, note that there is an $\mathbb{R}$-action on $\mathcal{M}(x,y;J,H)$. Let $\sigma \in \mathbb{R}$ and $u \in \mathcal{M}(x,y;J,H)$. Then

$$(\sigma \cdot u)(s,t) = u(\sigma + s, t)$$

defines and $\mathbb{R}$-action. Assuming, for the moment, that $\mathcal{M}(x,y;J,H)$ is a finite dimensional manifold, we can define

$$\widehat{\mathcal{M}}(x,y;J,H) := \mathcal{M}(x,y,;J,H)/\mathbb{R}$$

as the quotient manifold equipped with the quotient topology. We will prove in Section 5.5 that $\widehat{\mathcal{M}}(x,y;J,H)$ is a compact 0-dimensional manifold whenever $\mathrm{CZ}(y)-\mathrm{CZ}(x)=1$. Hence, $\eta(x,y) = \#\widehat{\mathcal{M}}(x,y;J,H)$ as defined by equation (5.5) makes sense, as a compact 0-dimensional manifold is a finite number of points.

We define the Floer homology of (H,J).

Definition 5.6 (Floer Homology)**.** Let the Floer chain complex $(\mathrm{CF}_*(H),\partial_J)$ be as in Definitions 5.4 and 5.5. The homology of this complex is called the Floer homology of (H,J):

$$\mathrm{HF}_*(H,J) := H_*(CF_*(H),\partial_J).$$

We use this homology to prove the Arnold conjecture. To do this, we will show that the above homology is well-defined and canonically independent of the choice of (H,J) used to define it. We will see that the pair (H,J) will have to satisfy some conditions to make the spaces $\mathcal{M}(x,y;J,H)$ into finite dimensional manifolds. However, it will turn out these choices are generic.

Then we prove that for a particular Hamiltonian, the Floer homology and the Morse homology of M coincide. The Arnold conjecture is then a straightforward consequence of this.

REMARK. Note that to properly define Floer homology as in Definition 5.6, we need $\partial_J^2 = 0$. This is a consequence of the topology of the manifold $\widehat{\mathcal{M}}(x, z; J, H)$ where $\mathrm{CZ}(x) = \mathrm{CZ}(z) - 2$. It will be a consequence of Theorem 5.38, which involves complicated machinery like elliptic regularity and gluing. It is then only after understanding this Theorem that we can properly prove that $\partial_J^2 = 0$. Using Theorem 5.38, proving that $\partial_J^2 = 0$ will turn out to be a simple consequence of the fact that a compact 1-dimensional manifold has an even number of boundary points. This is done in Corollary 5.40.

5.2. Solutions to Floer's equation as a section

In this section, we consider solutions $u : \mathbb{R} \times S^1 \to M$ to equation (3.6). We describe solutions to this equation as zeros of a section of a Banach bundle over a Banach space. The goal is to endow the spaces $\mathcal{M}(x, y; J, H)$ for $x, y \in \mathcal{P}_0(H)$ with a useful manifold structure. When we describe $\mathcal{M}(x, y; J, H)$ as the zeros of a section, we can use the implicit function theorem in infinite dimensions to accomplish this.

Define the Banach bundle

$$\mathcal{E}^p \to W^{1,p}(\mathbb{R} \times S^1, M)$$

whose fiber over u is

$$\mathcal{E}^p_u := L^p(\mathbb{R} \times S^1, u^*TM).$$

We define the Floer section.

Definition 5.7. Let $\mathcal{E}^p \to W^{1,p}(\mathbb{R} \times S^1, M)$ be as above. Then the Floer section

$$\overline{\partial}_{H,J} : W^{1,p}(\mathbb{R} \times S^1, M) \to \mathcal{E}^p$$

is given by

$$\overline{\partial}_{H,J}(u) = \frac{\partial u}{\partial s} + J(u)\Big(\frac{\partial u}{\partial t} - X_H(u)\Big).$$

As said, we want to use the implicit function theorem. See Appendix Theorem C.13 for the full theorem. Therefore, we need to compute the vertical derivative of the Floer section $\overline{\partial}_{H,J}$. For the definition of the vertical derivative, see Definition C.9.

Lemma 5.8. *The vertical derivative*

$$d^V\overline{\partial}_{H,J}(u) : W^{1,p}(\mathbb{R} \times S^1, u^*TM) \to \mathcal{E}_u = L^p(\mathbb{R} \times S^1, u^*TM)$$

is given by

$$d^V\overline{\partial}_{H,J}(u)Y = \nabla_s Y + J(u)\nabla_s Y + \nabla_Y J(u)\frac{\partial u}{\partial t} - \nabla_Y \nabla H(u)$$

PROOF. This is a small computation. Let u_λ with $\lambda \in (-\varepsilon, \varepsilon)$ denote a smooth path in $W^{1,p}(\mathbb{R} \times S^1, M)$ such that $u_0 = u$ and $\frac{\partial u_\lambda}{\partial \lambda}(t)|_{\lambda=0} = Y(t)$. Then

$$\begin{aligned} d^V \overline{\partial}_{H,J}(u)Y &= \frac{\partial}{\partial \lambda}|_{\lambda=0}\Big(\overline{\partial}_{H,J}(u_\lambda)\Big) \\ &= \frac{\partial}{\partial \lambda}|_{\lambda=0}\Big(\frac{\partial u_\lambda}{\partial s} + J(u_\lambda)\frac{\partial u_\lambda}{\partial t} - \nabla H_t(u_\lambda)\Big) \\ &= \nabla_s\Big(\frac{\partial u_\lambda}{\partial \lambda}|_{\lambda=0}\Big) + J(u_0)\nabla_t\Big(\frac{\partial u_\lambda}{\partial \lambda}|_{\lambda=0}\Big) + \nabla_Y J(u_0)\frac{\partial u_0}{\partial t} - \nabla_Y \nabla H_t(u_0) \\ &= \nabla_s Y + J(u)\nabla_t Y + \nabla_Y J(u)\frac{\partial u}{\partial t} - \nabla_Y \nabla H_t(u) \end{aligned}$$

Here we used that ∇ is torsion free in the third line. $\square$

In order to describe $\mathcal{M}(x^-, x^+; J, H)$ as the zeros of the Floer section we need a Fredholm section. Therefore, we need to restrict the Floer section to an appropriate space

$$\mathcal{B}^{1,p}(x^-, x^+) \subset W^{1,p}(\mathbb{R} \times S^1, M).$$

These spaces are defined as follows.

Let $u : \mathbb{R} \times S^1 \to M$ and $x^-, x^+ \in \mathcal{P}_0(H)$ with $\lim_{s \to \pm\infty} u(s,t) = x^\pm(t)$. We say u decays suitably if there exist positive constants $K = K(u)$ and $\delta = \delta(u)$ such that

$$\Big\|\frac{\partial u}{\partial s}\Big\| \leq K e^{-\delta|s|}, \quad \Big\|\frac{\partial u}{\partial s} - X_H(u)\Big\| \leq K e^{-\delta|s|}.$$

Definition 5.9. Let $x^-, x^+ \in \mathcal{P}_0(H)$. Then

$$\mathcal{B}^{1,p}(x^-, x^+) := \{(s,t) \mapsto \exp_{u(s,t)} Y(s,t) \mid u : \mathbb{R} \times S^1 \text{ decays suitably}, Y \in W^{1,p}(\mathbb{R} \times S^1, u^*TM)\}.$$

Let us fix $x^-, x^+ \in \mathcal{P}_0(H)$ and denote the Floer section

$$\overline{\partial}_{H,J} : \mathcal{B}^{1,p}(x^-, x^+) \to \mathcal{E}^p.$$

Then its vertical derivative is a map

$$d^V \overline{\partial}_{H,J}(u) : W^{1,p}(\mathbb{R} \times S^1, u^*TM) \to \mathcal{E}_u.$$

There are several small things to verify.

First note that for $u \in \mathcal{B}^{1,p}(x^-, x^+)$ we wish to know that $\overline{\partial}_{H,J}(u) \in L^p(\mathbb{R} \times S^1, M)$. This is indeed the case. This follows immediately from Lemma 13.3.1 in [**AD14**].

Secondly, we wish to know whether $\mathcal{M}(x^+, x^-) \subset \mathcal{B}^{1,p}(x^-, x^+)$. This is Proposition 8.2.3 in [**AD14**].

Furthermore, it is important to note that $\mathcal{B}^{1,p}$ is a Banach space with tangent space $T_u\mathcal{B}^{1,p}(x^-, x^+) = W^{1,p}(\mathbb{R} \times S^1, u^*TM)$.

The atlas is defined similarly to the one for $\mathcal{L}_0 M$. For full details we refer the reader to [**AD14**] Section 8.2.d. We define charts in $W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$. Let w decay sufficiently. Note that we can identify $W^{1,p}(R\times S^1,\mathbb{R}^{2n})$ with $W^{1,p}(\mathbb{R}\times S^1, w^*TM$ for some sufficiently decaying w. Furthermore, the injection $W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})\hookrightarrow L^\infty(\mathbb{R}\times S^1,\mathbb{R}^{2n})$ is continuous by the Sobolev embedding theorem, hence

$$\|Y\|_{L^\infty}\leq K\|Y\|_{W^{1,p}}.$$

Let $R<\rho$, where ρ is the injectivity-radius of the metric we use on M. For every sufficiently decaying w, define

$$\Phi_w : \{Y\in W^{1,p}(w^*TM)mid\|Y\|_{W^{1,p}}\leq R/K\}\to\mathcal{B}^{1,p}(x^-,x^+)$$

given by

$$Y\mapsto\exp_w Y.$$

This is a smooth bijection on the image, as it is smaller than R. This forms an atlas.

5.3. The operator $d^V\overline{\partial}_{H,J}(u)$ is Fredholm of index $\mathrm{CZ}(x^-)-\mathrm{CZ}(x^+)$

Let $u\in\mathcal{B}^{1,p}(x^-,x^+)$ such that $\overline{\partial}_{H,J}(u)=0$. In the previous section we defined $d^V\overline{\partial}_{H,J}(u)$. The goal of this section is to prove the following theorem.

Theorem 5.10. *The operator $d^V\overline{\partial}_{H,J}(u)$ defined in Lemma 5.8 is Fredholm of index $\mathrm{CZ}(x^-)-\mathrm{CZ}(x^+)$.*

To prove this we will transfer everything to a linear setting, using symplectic trivializations. The operator in the linear setting will turn out to be a perturbed Cauchy-Riemann operator. We will prove that this operator is Fredholm in the next subsection. After this, we compute its Fredholm index separately in subsection 5.3.2. Recall that the operator $d^V\overline{\partial}_{H,J}(u)$ is given as in Lemma 5.8.

Let $\Phi:\mathbb{R}\times S^1\times\mathbb{R}^{2n}\to u^*TM$ be a symplectic trivialization, with limits $\Phi^\pm:=\Phi|_{\pm\infty\times S^1}$ trivializations of the limits $(x^\pm)^*TM\to S^1$. Then in the coordinates $\xi=\Phi^{-1}Y$, we have that $d^V\overline{\partial}_{H,J}(u)$ takes the form

$$D_S\xi=\frac{\partial\xi}{\partial s}+J_0\frac{\partial\xi}{\partial t}+S\xi \tag{5.6}$$

where

$$S(s,t):=\Phi^{-1}(s,t)\circ(\nabla_s\Phi+J(u)\nabla_t\Phi+\nabla_\Phi J(u)\frac{\partial u}{\partial t}-\nabla_\Phi\nabla H_t(u)). \tag{5.7}$$

This is a direct computation, where $D_S\xi=\Phi^{-1}(d^V\overline{\partial}_{H,J}(u))(\Phi\xi)$ Here, we use that $\nabla_s(\Phi\xi)=\Phi\frac{\partial\xi}{\partial s}+(\nabla_s\Phi)\xi$ and likewise for t combined with the fact that $J\Phi=\Phi J_0$. We separate the terms to find the operator of equation 5.6 with $S(s,t)$ as in equation (5.7).

For this $S \in C^\infty(\mathbb{R} \times S^1, \mathrm{End}(\mathbb{R}^{2n})$, the following is true.

Lemma 5.11. *The limit matrices*

$$S^\pm(t) := \lim_{s\to\pm\infty} S(s,t)$$

given by

$$S^\pm(t) := (\Phi_t^\pm)^{-1} \circ J(\nabla_t \Phi^\pm - \nabla_{\Phi^\pm} X_H(x^\pm)).$$

are symmetric

Proof. Computing the limits is straightforward. To see that $S^\pm$ are symmetric, we look at the matrix elements $S_{ij}^\pm$. Let us omit the $\pm$-superscript for the moment to avoid clutter. Let e_j denote the j-th basis vector of $\mathbb{R}^{2n}$ and define the symplectic basis $(\xi_j)_{1\le j\le 2n}$ by

$$\xi_j(t) = \Phi(t) e_j.$$

Then the matrix elements of $S(t)$ are given by

$$S_{ij} = g_J(\xi_i, J(\nabla_t \xi_j - \nabla_{\xi_j} X_H) = g_J(\xi_i, J[X_H, \xi_j]).$$

Note that by definition

$$g_J(\xi_i, J[X_H, \xi_j]) = \omega(\xi_i, [\xi_j, X_H]).$$

To get a term involving this Lie bracket, consider the cyclic identity

$$\begin{aligned} d\omega(\xi_i, \xi_j, X_H) &= \xi_i \omega(\xi_j, X_H) + \xi_j \omega(X_H, \xi_i) + X_H \omega(\xi_i, \xi_j) + \omega(X_H, [\xi_i, \xi_j]) + \omega(\xi_i, [\xi_j, X_H]) + \omega(\xi_j, [X_H, \xi_i]) \\ &= -\xi_i dH(\xi_j) + \xi_j dH(\xi_i) + X_H \omega(\xi_i, \xi_j) + dH([\xi_i, \xi_j]) + \omega(\xi_i, [\xi_j, X_H]) + \omega(\xi_j, [X_H, \xi_i]) \end{aligned}$$

Now note that

$$d^2 H(\xi_i, \xi_j) = \xi_i dH(\xi_j) - \xi_j dH(\xi_i) - dH([\xi_i, \xi_j])$$

so that the above equation becomes

$$d\omega(\xi_i, \xi_j, X_H) = +X_H \omega(\xi_i, \xi_j) + \omega(\xi_i, [\xi_j, X_H]) + \omega(\xi_j, [X_H, \xi_i]) = \omega(\xi_i, [xi_j, X_H]) + \omega(\xi_j, [X_H, \xi_i]).$$

As ω is closed, we are left with

$$\omega(\xi_i, [\xi_j, X_H]) = \omega(\xi_j, [\xi_i, X_H])$$

so that

$$S_{ij} = g_J(\xi_i, J[\xi_j, X_H]) = \omega(\xi_i, [\xi_j, X_H]) = \omega(\xi_j, [\xi_i, X_H]) = g_J(\xi_j, J[\xi_i, X_H]) = S_{ji}.$$

Hence S is indeed symmetric. This proves Lemma 5.11. □

Let $S : S^1 \to \mathrm{End}(\mathbb{R}^{2n})$. We define such an operator to be non-degenerate whenever the symplectic matrices associated to it are.

Definition 5.12. Let $S : S^1 \to \mathrm{End}(\mathbb{R}^{2n})$. Let $R : [0,1] \to \mathrm{Sp}(2n)$ be given as the solution of

$$\frac{dR(t)}{dt} = J_0 S(t) R(t), \quad R(0) = \mathrm{Id} \tag{5.8}$$

We say S is non-degenerate when

$$\det(R - \mathrm{Id}) \neq 0.$$

Remark. When given a loop $S : S^1 \to \mathrm{End}(\mathbb{R}^{2n})$ by the path of symplectic matrices associated to S we mean $R : [0,1] \to \mathrm{Sp}(2n)$ as defined by equation (5.8).

Note that the trivializations are isomorphisms. Hence, the following is true.

Theorem 5.13. *The operator $d^V\overline{\partial}_{H,J}(u)$ is Fredholm if and only if D_S is and*

$$\mathrm{ind}(d^V\overline{\partial}_{H,J}(u)) = \mathrm{ind}(D_S).$$

Hence, proving that $d^V\overline{\partial}_{H,J}(u)$ is Fredholm is equivalent to proving that D_S is Fredholm, which is what we will do. We prove the linear analogue of Theorem 5.10. We state it for any S with the properties that S defined by equation (5.7) has.

Theorem 5.14. *Let*

$$D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

with

$$D_S\xi = \frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t} + S\xi$$

where $S(s,t)$ is a symmetric matrix with non-degenerate limits $S^{\pm}(t)$. Suppose furthermore that $\Psi^{\pm}(t) \in \mathcal{S}$ are the symplectic matrices associated to $S^{\pm}$ as in equation (5.8). Then D_S is a Fredholm operator with

$$\mathrm{ind}(D_S) = \mu_{\mathrm{CZ}}(\Psi^-) - \mu_{\mathrm{CZ}}(\Psi^+).$$

Remark. Compare this to Theorem 5.10, where the index is given in terms of $\mathrm{CZ}(x^{\pm})$. A small computations shows that $\mathrm{CZ}(x^{\pm}) = \mu_{\mathrm{CZ}}(\Psi^{\pm})$ by the definition of $\mathrm{CZ}(x^{\pm})$ in Definition 4.9.
To compute the Conley-Zehnder index in Definition 4.9, we defined from $\Phi^{\pm} : S^1 \times \mathbb{R}^{2n} \to (x^{\pm})^*TM$ the matrices in equation (4.4) by

$$\Psi_{x^{\pm}}(t) := (\Phi_{x^{\pm}}(t))^{-1} \circ T_{x(0))}\varphi_H^t \circ \Phi_{x^{\pm}}(0).$$

We show that

$$\frac{d\Psi_{x^{\pm}}(t)}{dt} = J_0 S^{\pm}(t)\Psi_{x^{\pm}}(t) \tag{5.9}$$

so that the matrices associated to $S^{\pm}$ are $\Psi_{x^{\pm}}$ (i.e. $\Psi^{\pm} = \Psi_{x^{\pm}}$).
Note that

$$\Phi_{x^{\pm}}(t) \circ \Psi_{x^{\pm}}(t) = T_{x(0)}\varphi_H^t \circ \Phi_{x^{\pm}}(0)$$

by equation (4.4). Differentiate both sides of the equation to t which yields

$$\Phi^{\pm}(t)\frac{d\Psi_{x^{\pm}}(t)}{dt} + (\nabla_t \Phi_{x^{\pm}})\Psi_{x^{\pm}}(t) = (\nabla_{\Phi_{x^{\pm}}} X_H)\Psi_{x^{\pm}}.$$

Use that Φ is a symplectic trivialization, meaning $\Phi_{x^{\pm}}(t) \circ J_0 = J \circ \Phi_{x^{\pm}}(t)$. Therefore

$$\begin{aligned}
\Phi_{x^{\pm}}(t) J_0 \frac{d\Psi_{x^{\pm}}(t)}{dt} &= J\Psi_{x^{\pm}}(t)\frac{d\Psi_{x^{\pm}}(t)}{dt} \\
&= J(\nabla_{\Phi_{x^{\pm}}} X_H - \nabla_t \Phi_{x^{\pm}})\Psi_{x^{\pm}}(t) \\
&= -\Phi_{x^{\pm}}(t) S^{\pm}(t)\Psi_{x^{\pm}}(t)
\end{aligned}$$

Using that $\Phi_{x^\pm}(t)$ is an isomorphism and $J_0^2 = -\mathrm{Id}$ we find that equation (5.9) holds. Therefore, the matrices $\Psi^\pm$ associated to $S^\pm$ are valid to compute the indices $\mathrm{CZ}(x^\pm)$.

5.3.1. D_S is Fredholm.
The aim of this subsection is to prove that D_S is Fredholm. This is the statement of the following proposition. We will compute the Fredholm index of D_S in the next subsection.

Proposition 5.15. *Let D_S be the perturbed Cauchy-Riemann operator*

$$D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

given by

$$D_S Y = \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + SY. \tag{5.10}$$

Assume that $S^\pm(t) := \lim_{s \to \pm\infty} S(s,t)$ exist, where the convergence is uniform in t, and that these limit matrices are symmetric for all $t \in S^1$ and $S^\pm$ are non-degenerate. Then D_S is a Fredholm operator.

By the conditions on $S(s,t)$, also Ψ and $\frac{\partial \Psi}{\partial t}$ converge, uniformly in t, as $s \to \pm\infty$. Hence, we define the limits

$$\lim_{s \to \pm\infty} \Psi(s,t) = \Psi^\pm(t). \tag{5.11}$$

To prove Proposition 5.15 we first consider the perturbed Cauchy-Riemann operator D_Σ, where $\Sigma : S^1 \to \mathrm{Sym}(\mathbb{R}^{2n})$ and

$$D_\Sigma : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

is defined by

$$D_\Sigma Y = \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + \Sigma Y.$$

Suppose furthermore that Σ is non-degenerate. We then prove the following lemma.

Lemma 5.16. *For every $1 < p < \infty$, the operator D_Σ is bijective.*

Then using the semi-Fredholm Lemma, we prove Proposition 5.15 using Lemma 5.16.

Remark. By D_A we will always mean the perturbed Cauchy-Riemann operator associated to the operator A. However, we are dealing with two possible domains for A. To avoid confusion, by S we will always mean $S : \mathbb{R} \times S^1 \to \mathrm{End}(\mathbb{R}^{2n})$. Whenever S is s-independent, we will denote it $\Sigma : S^1 \to \mathrm{End}(\mathbb{R}^{2n})$. In this way, we have the two operators D_S and D_Σ.

Proof of Lemma 5.16. We prove Lemma 5.16 in two steps. First we treat the case $p = 2$. We prove this by defining a bounded inverse. After this, we tackle the general case $p > 1$ using a duality argument.

Proof of Lemma 5.16 in the case $p = 2$:

Consider the operator

$$A : W^{1,2}(S^1, \mathbb{R}^{2n}) \to L^2(S^1, \mathbb{R}^{2n})$$

defined by

$$AY = J_0 \frac{\partial Y}{\partial t} + \Sigma Y.$$

In fact, denote $\mathcal{H} = L^2(S^1, \mathbb{R}^{2n})$. We will regard A as an operator $A : \mathcal{H} \to \mathcal{H}$ an unbounded operator defined on the dense domain $W^{1,2}(S^1, \mathbb{R}^{2n}) \subset \mathcal{H}$. We have the following claim about the operator A.

Claim 2. Suppose Σ is non-degenerate (in the sense of Definition 5.12). Then the operator A is invertible.

Proof of Claim 2: Suppose $v \in \mathcal{H}$. We wish to find a unique $u \in W^{1,2}(S^1, \mathbb{R}^{2n})$ such that

$$Au = v.$$

Let Ψ be given by equation (5.8). The equation $Au = v$ implies

$$\dot{u} = J_0(Su - v).$$

We can then construct a solution u_x, for any $x \in \mathbb{R}^{2n}$, satisfying $Au_x = v$ and $u_x(0) = x$.

$$u_x(t) = \Psi(t)(x - \int_0^t \Psi(\tau)^{-1} J_0 v(\tau)\, d\tau)$$

for any $x \in \mathbb{R}^{2n}$. We wish now to impose periodicity on u_x. This is the case if and only if

$$x = \Psi(1)\Big(x - \int_0^1 \Psi(\tau)^{-1} J_0 v(\tau)\, d\tau\Big).$$

Rewriting this yields

$$(\Psi(1) - \mathrm{Id})x = \Psi(1) \int_0^1 \Psi(\tau)^{-1} J_0 v(\tau)\, d\tau.$$

By the non-degeneracy condition, this can be solved as the operator $\Psi(1) - \mathrm{Id}$ is invertible. Thus, we have shown that A is invertible. This proves Claim 2.

We now decompose the Hilbert space $\mathcal{H}$. We defined A as an unbounded operator on $\mathcal{H}$. First note that A is a closed operator with dense domain in $\mathcal{H}$. Furthermore, A is symmetric as Σ is symmetric. Note that by symmetric we mean that

$$\langle Au, v\rangle_{\mathcal{H}} = \langle u, Av\rangle_{\mathcal{H}}.$$

This is in the language of unbounded operators, where symmetric means that A agrees with its adjoint A^* on the domain of A. This is easily verified, using $J_0^* = -J_0$ and $\Sigma^T = \Sigma$. Hence, we can decompose $\mathcal{H}$ into positive and negative eigenspaces of A. Write

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-.$$

Define

$$A^\pm = A|_{\mathcal{H}^\pm \cap W^{1,2}(S^1, \mathbb{R}2n)}.$$

Then $A^\pm$ are self-adjoint unbounded linear operators with dense domain. Now apply the Hille-Yosida theorem, 13.37 from [**Rud73**] to $A^\pm$.

Hence the operators A^- and $-A^+$ generate families $E_{-A^+}(s) \in B(\mathcal{H}^+)$ and $E_{A^-}(s) \in B(\mathcal{H}^-)$. The decomposition of $\mathcal{H}$ comes with two orthogonal projections

$$p^\pm : \mathcal{H} \to \mathcal{H}^\pm.$$

We define a path of linear operators.

$$K(s) = \begin{cases} E_{-A^+}(s)p^+ & s \geq 0 \\ -E_{A^-}(-s)p^- & s < 0 \end{cases}$$

This path is continuous for $s \neq 0$ but discontinious at $s = 0$. Furthermore, we have the following estimate.

$$\|K(s)x\|_{\mathcal{H}} = \|E_{-A^+}(s)p^+x\|_{\mathcal{H}} \leq e^{-\mu s}\|x\|_{\mathcal{H}}$$

for μ the smallest element of $\operatorname{Spec} A^+$. Together with the same calculation for $s < 0$ yields, for the operator norm

$$\|K(s)\| \leq e^{-\delta|s|}. \tag{5.12}$$

for some $\delta > 0$.

Define an operator

$$Q : L^2(\mathbb{R}, \mathcal{H}) \to W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

by integration

$$Qv(s,t) = \int_{-\infty}^{\infty} K(-\sigma)v(s+\sigma, t)d\sigma.$$

This operator is well defined, as it converges in $L^2(\mathbb{R}\times S^1,\mathbb{R}^{2n})$ because of the estimate of equation (5.12) used in the following calculation.

$$\begin{aligned}\int_{\mathbb{R}}\|K(-\sigma)v(s+\sigma,t)\|_{L^2(S^1\times\mathbb{R})}\,d\sigma &= \int_R(\int_{\mathbb{R}}\|K(-\sigma)v(s+\sigma)\|^2_{L^2(S^1)}\,ds)^{\frac{1}{2}}\,d\sigma\\ &\leq \int_{\mathbb{R}}(\int_{\mathbb{R}}e^{-2\delta|s|}\|v(s+\sigma)\|^2_{L^2(S^1)}\,ds)^{\frac{1}{2}}\,d\sigma\\ &= \int_{\mathbb{R}}e^{-2\delta|\sigma|}\|v\|_{L^2(S^1\times\mathbb{R})}\,d\sigma\\ &< +\infty\end{aligned}$$

Now we claim that the operator Q we defined is the inverse of D_Σ. Suppose $v\in L^2(\mathbb{R},\mathcal{H})$ and let w such that $Qv=w$. We can decompose

$$w=w^++w^-.$$

By definition

$$w^+(s)=\int_{-\infty}^{s}E_{-A^+}(s-\tau)p^+v(\tau)\,d\tau$$

and

$$w^-(s)=-\int_{s}^{\infty}E_{A^-}(\tau-s)p^-v(\tau)\,d\tau.$$

Direct computation yields

$$\begin{aligned}\frac{d}{ds}w^+(s)&=p^+v(s)-A^+\int_{-\infty}^{s}E_{-A^+}(s-\tau)p^+v(\tau)\,d\tau\\ &=p^+v(s)-A^+w^+(s)\end{aligned}$$

The same computation shows

$$\frac{d}{ds}w^-(s)=p^-v(s)-A^-w^-(s).$$

Therefore, in general we have

$$\frac{d}{ds}w=v-Aw$$

which means $D_\Sigma\circ Q=\mathrm{Id}$.

A similar calculation shows that $Q\circ D_\Sigma=\mathrm{Id}$. Let $v\in W^{1,2}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$. We decompose $v=v^++v^-$ such that

$$Av=A^+v^++A^-v^-.$$

We have

$$Q(Av)=\int_{-\infty}^{s}E_{-A^+}(s-\sigma)A^+v^+(\sigma)\,d\sigma-\int_{s}^{+\infty}E_{A^-}(\sigma-s)A^-v^-(\sigma)\,d\sigma.$$

We compute the first term.

$$\begin{aligned}\int_{-\infty}^{s} A^+E_{-A^+}(s-\sigma)v^+(\sigma)\,d\sigma &= \int_{-\infty}^{s} -\frac{\partial}{\partial s}(E_{-A^+}(s-\sigma)v^+(\sigma))\,d\sigma \\ &= -\frac{\partial}{\partial s}\int_{-\infty}^{s} E_{-A^+}(s-\sigma)v+(\sigma)\,d\sigma + v^+(s) \\ &= -\frac{\partial}{\partial s}\int_{-\infty}^{0} E_{-A^+}(-\sigma)v^+(s+\sigma)\,d\sigma + v^+(s) \\ &= -\int_{-\infty}^{0} E_{-A^+}(-\sigma)\frac{\partial v^+}{\partial s}(s+\sigma)\,d\sigma + v^+(s)\end{aligned}$$

An analogous calculation shows that the second term equals

$$-\int_{s}^{+\infty} A^-E_{A^-}(\sigma-s)v^-(\sigma)\,d\sigma = \int_{0}^{+\infty} E_{A^-}(-\sigma)\frac{\partial v^-}{\partial s}(s+\sigma)\,d\sigma + v^-(s).$$

Taking the sum we get

$$Q(Av) = -Q(\frac{\partial v}{\partial s}) + Y.$$

This precisely means $Q \circ D_\Sigma = \mathrm{Id}$. Hence we have proven that for $p = 2$, the operator D_Σ is invertible. This concludes the proof of Lemma 5.16 in the case $p = 2$.
We now tackle the general case $p > 1$.

Proof of Lemma 5.16 in the case $p > 1$: The proof will consist of a series of inequalities that are listed as lemmas. Proving all these inequalities would be tedious, so we refer to the relevant lemmas in [**AD14**].

We remind the reader we are still in the case where $\Sigma = \Sigma(t)$ does not depend on s. The following two lemmas are technical and should be regarded as preludes to Lemma 5.19.

Lemma 5.17. *Suppose $p > 1$. There exists a constant $c > 0$ such that for every $k \in \mathbb{R}$ and every $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ the following inequality holds.*

$$\|u\|_{W^{1,p}([k,k+1]\times S^1)} \leq c(\|D_\Sigma u\|_{L^p([k-\frac{1}{2},k+\frac{2}{3}]\times S^1)} + \|u\|_{L^p([k-\frac{1}{2},k+\frac{3}{2}]\times S^1)}).$$

Proof.: This is Lemma 8.7.11 in [**AD14**]. Consider first $k = 0$, in which case the inequality follows from the elliptic regularity results for the Cauchy Riemann operator. This is Theorem B.12. By translating the inequality is satisfied for all k. This lemma leads to another inequality, this time for $p > 2$.

Lemma 5.18. *Suppose $p > 2$. There exists a constant $c_1 > 0$ such that for every $k \in \mathbb{R}$ and every $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ the following inequality holds.*

$$\|u\|_{W^{1,p}([k,k+1]\times S^1)} \leq c_1(\|D_\Sigma u\|_{L^p([k-1,k+2]\times S^1)} + \|u\|_{L^2([k-1,k+2]\times S^1)}).$$

Proof.: This is Lemma 8.7.12 in [**AD14**]. It is sufficient to prove for $k = 0$. We replace the $\|u\|_{L^p([-\frac{1}{2},\frac{3}{2}])}$ by a $W^{1,2}$-term using Theorem B.7. The previous Lemma replaces this with L^2-terms. We require $p > 2$ to use Hölder's inequality to replace one of the L^2-terms by an L^p term which gives the resulting inequality. The above lemma gives an estimate of $\|u\|$ for $u \in W^{1,p}(\mathbb{R} \times S^1)$ in terms of $\|D_\Sigma u\|_{Lp(S^1)}$. This lemma is the central inequality we will use.

Lemma 5.19. *Suppose $p > 2$. There exists a constant $c > 0$ such that if $u \in W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ and $D_\Sigma u \in L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ Then the following two conditions are satisfied.*

(i) $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$

(ii) $\|u\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq c\|D_\Sigma u\|_{L^p(\mathbb{R}\times S^1)}$

Proof.: We refer the reader to the proof of Lemma 8.7.13 in [**AD14**]. Here, the inverse Q constructed in the previous part of the proof (the $p = 2$ case) is used together with Young's and Hölder's inequalities. We can now first prove the following claim.

Claim 3. D_Σ is bijective for $p > 2$.

Proof of Claim 3: Let $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \cap W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Then by Lemma 5.19 (ii), we have

$$\|u\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq c\|D_\Sigma u\|_{L^p(\mathbb{R}\times S^1)} \tag{5.13}$$

Then in particular, this inequality holds for $u \in C_0^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. However, the space $C_0^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is dense in $W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$, see [**Eva98**] Section 5.3.2 or [**MS12**] Appendix B. Hence the inequality (5.13) is true for any $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. This implies that D_Σ is injective.

Also, $W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is complete, so its image under D_Σ in $L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is closed. It remains to show that D_Σ is surjective. By closedness of the image, it is sufficient to prove the image is dense inside $L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.

Suppose $v \in L^p(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \cap L^2(\mathbb{R}\times S^1, \mathbb{R}^{2n})$. Note that the space $L^p(\mathbb{R}\times S^1, \mathbb{R}^{2n}) \cap L^2(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is dense in $L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Now we use that surjectivity of D_Σ for $p = 2$ has been established, which gives us an $u \in W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ such that $D_\Sigma u = v$. Then by Lemma 5.19 (i), we have $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. This establishes that D_Σ is surjective and hence that D_Σ is bijective for $p > 2$. This proves Claim 3.

For the general case $p > 1$, it remains to show that D_Σ is bijective for $1 < p < 2$. This is a duality argument via the dual D_Σ^* and an application of Riesz's theorem.

Claim 4. The operator D_Σ is bijective for $1 < p < 2$.

Proof of Claim 4: The idea of the proof is the same as the case $p > 2$ in the sense that we will derive again inequality (5.13) in the case $1 < p < 2$. Because of the density argument (approximation by smooth functions), it is enough to show this inequality for compactly supported smooth functions $u \in C_0^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. To this end, introduce the adjoint operator.

Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Denote the adjoint

$$D_\Sigma^* : W^{1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

defined by

$$f \mapsto -\frac{\partial f}{\partial s} + J_0 \frac{\partial f}{\partial t} + \Sigma f.$$

A small calculation shows that the above operator is indeed the adjoint, in the sense that for $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ and $v \in W^{1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ we have

$$\int_{\mathbb{R}\times S^1} \langle D_\Sigma u, v\rangle = \int_{\mathbb{R}\times S^1} \langle u, {D_\Sigma}^* v\rangle.$$

We use here that Σ is symmetric. Note that $q > 2$ (as $1 < p < 2$) and D_Σ^* is of the same form as D_Σ. Hence, the results we proved for D_Σ also apply to D_Σ^*. In particular, also D_Σ^* is bijective. This can be used to derive inequality (5.13) for the case $1 < p < 2$. We refer the reader to [**AD14**] pages 282-283. The inequality results from some computations using D_Σ^*. By inequality (5.13), injectivity is immediate.

For surjectivity, again one first shows that D_Σ has closed image. Suppose that its image is not closed. Then by Riesz's theorem, there would be a nonzero $w \in L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ such that $\langle w, D_\Sigma v\rangle = 0$ for all $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Then $D_\Sigma^* w = 0$. This is a direct computation, preformed in [**AD14**] Lemma 8.5.2. As w solves a perturbed Cauchy-Riemann equation, by elliptic regularity (Theorem B.13), $w \in W^{1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Therefore, $w \in \ker(D_\Sigma^*)$ which means $w = 0$ as D_Σ was injective. This is a contradiction, which proves Claim 4. Hence, we have shown that the operator D_Σ is bijective for the general case $p > 1$ as required. The above claims therefore prove Lemma 5.16.

□

We now know that the operator D_Σ is bijective in the case where $\Sigma = \Sigma(t)$ and is symmetric. We now proceed to prove that it is Fredholm in the more general case of theorem, using that D_Σ is invertible. Let us return now to the case where $S = S(s,t)$ with limits $\lim_{s\to\pm\infty} S(s,t) = S^\pm(t)$.

We start with a Lemma.

Lemma 5.20. *Let $Y \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Then there exists $c > 0$ and $M > 0$ sufficiently large such that*

$$\|Y\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq c(\|D_S Y\|_{L^p(\mathbb{R}\times S^1)} + \|Y\|_{L^p([-M,M]\times S^1)}) \tag{5.14}$$

PROOF. From Proposition 5.16, we know D_Σ is bijective. By the Banach open mapping theorem, its inverse is continuous, hence there exists $K > 0$ such that

$$\|Y\|_{W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})} \leq K\|Y\|_{L^p(\mathbb{R}\times S^1,\mathbb{R}^{2n}}.$$

Recall that there exists uniform limits $S^{\pm}(t) = \lim_{s\to\pm} S(s,t)$. For a constant $M > 0$, truncate $Y \in W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$ by setting $Y(s,t) = 0$ for $|s| \leq M-1$. By uniform convergence

$$\|D_S Y - D_\Sigma Y\|_{L^p(\mathbb{R}\times S^1)} \leq \sup_{\substack{t\in S^1\\ |s|\geq M}} |S(s,t) - S^{\pm}(t)| \|Y\|_{L^p(\mathbb{R}\times S^1)} \leq \varepsilon\|Y\|_{W^{1,p}(\mathbb{R}\times S^1)}.$$

Therefore, there exists a constant $C > 0$ such that

$$\|Y\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq C\|D_S Y\|_{L^p(\mathbb{R}\times S^1)}.$$

Choose a bump function $\beta \in C^\infty(\mathbb{R},[0,1])$ such that

$$\beta(s) = \begin{cases} 1 & |s| \leq M-1 \\ 0 & |s| \geq M \end{cases}$$

This way, we can write

$$Y(s,t) = \beta(s)Y(s,t) + (1-\beta(s))Y(s,t).$$

Recall the estimate of Lemma 8.7.2 in [**AD14**], which is Appendix Lemma B.11 which says that $Z \in W^{1,p}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$ satisfies

$$\|Z\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq C(\|D_S Z\|_{L^p(\mathbb{R}\times S^1)} + \|Z\|_{L^p(\mathbb{R}\times S^1)}).$$

Hence,

$$\begin{aligned} \|Y\|_{W^{1,p}(\mathbb{R}\times S^1)} &\leq C'(\|D_S(\beta Y + (1-\beta)Y)\|_{L^p(\mathbb{R}\times S^1)} + \|\beta Y + (1-\beta)Y\|_{L^p(\mathbb{R}\times S^1)} \\ &\leq C'(\|D_S\beta Y\|_{L^p(\mathbb{R}\times S^1)} + \|D_S(1-\beta Y)\|_{L^p(\mathbb{R}\times S^1)} + \|\beta Y\|_{L^p(\mathbb{R}\times S^1)} + \|(1-\beta)Y\|_{L^p(\mathbb{R}\times S^1)} \end{aligned}$$

Note that $(\beta(s)-1)Y(s,t) = 0$ for $|s| \leq M-1$. Therefore, there exists $K > 0$ such that

$$\|Y\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq K\|D_S Y\|_{L^p(\mathbb{R}\times S^1)}.$$

Then the above inequality gives

$$\|Y\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq C'(\|D_S\beta Y\|_{L^p(\mathbb{R}\times S^1)} + (1+K)\|D_S(1-\beta Y)\|_{L^p(\mathbb{R}\times S^1)} + \|\beta Y\|_{L^p(\mathbb{R}\times S^1)}).$$

Adding $K+1(\|D_S\beta Y\|_{L^p(\mathbb{R}\times S^1)} + \|\beta Y\|_{L^p(\mathbb{R}\times S^1)})$ and choosing $C = C'(1+K)$ we get

$$\|Y\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq C(\|D_S\beta Y\|_{L^p(\mathbb{R}\times S^1)} + \|D_S(1-\beta)Y)\|_{L^p(\mathbb{R}\times S^1)} + \|\beta Y\|_{L^p(\mathbb{R}\times S^1)}) \tag{5.15}$$

Note that $D_S(\beta Y) = \beta D_S Y + \dot\beta(s)Y$. For $s \in [-M,M]$, we have $|\dot\beta(s)| \leq R$ whereas $\dot\beta(s) = 0$ for $|s| \geq M$. Hence

$$\|D_S\beta Y\|_{L^p(\mathbb{R}\times S^1)}\| \leq \|D_S Y\|_{L^p(\mathbb{R}\times S^1} + R\|Y\|_{L^p([-M,M]\times S^1)},$$
$$\|D_S(1-\beta)Y\|_{L^p(\mathbb{R}\times S^1)}\| \leq \|D_S Y\|_{L^p(\mathbb{R}\times S^1} + R\|Y\|_{L^p([-M,M]\times S^1)}$$

. Plugging this into equation (5.15), we find a constant C_2 such that

$$\|Y\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq C_2(\|D_S Y\|_{L^p(\mathbb{R}\times S^1)} + \|Y\|_{L^p([-M,M]\times S^1)}) \tag{5.16}$$

This proves Lemma 5.20. □

We can now prove the Fredholm property of D_S, which was Proposition 5.15.

Proof of Proposition 5.15. Let D_S be as in the hypothesis. We prove that $\ker(D_S)$ is finite-dimensional, $\operatorname{im}(D_S)$ is closed and $\operatorname{coker}(D_S)$ is finite dimensional.
The first two follow from application of the semi-Fredholm Lemma C.10. Lemma 5.20 gives a $C > 0$ such that equation (5.14) holds. By Rellich's theorem, the restriction operator to the truncated cylinder

$$K : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p([-M, M] \times S^1, \mathbb{R}^{2n})$$

is compact. Then by Lemma 5.20, we see the hypothesis of the semi-Fredholm Lemma C.10 are satisfied for $X = W^{1,p}(\mathbb{R}\times S^1, \mathbb{R}^{2n})$, $Y = L^p(\mathbb{R}\times S^1, \mathbb{R}^{2n})$ and $Z = L^p([-M, M]\times S^1, \mathbb{R}^{2n})$. Therefore, $\ker(D_S)$ is finite dimensional and $\operatorname{im}(D_S)$ is closed.
To prove that $\operatorname{coker}(D_S)$ is finite dimensional, we use a duality argument. Let $q \in \mathbb{R}$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. We consider the adjoint operator

$$D_S^* : W^{1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

given by

$$D_S^* Y = -\frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S^T Y.$$

Let $Z \in L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ such that $\langle \operatorname{im}(D_S), Z\rangle = 0$. Suppose that $D_S^* Z = 0$, then by elliptic regularity $Z \in W^{1,q}(()\mathbb{R} \times S^1, \mathbb{R}^{2n})$, so $Z \in \ker(D_S^*)$. Denote

$$\Lambda := \{Z \in L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \mid \langle \operatorname{im}(D_S), Z\rangle = 0\}.$$

We conclude

$$\Lambda \subset \ker(D_S^*).$$

Note that D_S^* is of the same form as D_S. Hence, we can repeat the procedure of the first part of the proof (applying the semi-Fredholm Lemma together with Rellich's theorem and Lemma 5.20) to conclude that $\ker(D_S^*)$ is finite dimensional. We now use the Hahn-Banach theorem to conclude that $\operatorname{coker}(D_S)$ is finite dimensional.

By Hahn-Banach, we find linear $\varphi : L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to \mathbb{R}$ such that $\varphi(Z) = 0$ for all $Z \in \operatorname{im}(D_S)$. By the Reisz representation theorem, φ is of the form $\varphi = \varphi_X$ for some $X \in L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ with $\varphi_X(Y) = \langle X, Y\rangle$. As all φ vanish on $\operatorname{im}(D_S)$, we have $X \in \Lambda$. However, Λ is finite-dimensional, so then $\operatorname{coker}(D_S)$ is also finite-dimensional. This proves Proposition 5.15. $\square$

5.3.2. The Fredholm index of D_S.

In this subsection we compute the index of the operator D_S, where $S = S(s,t)$ with limits uniform in t given by

$$\lim_{s\to\pm\infty} S(s,t) = S^{\pm}(t).$$

This we do as follows.
The first step is to replace $S(s,t)$ by the matrix S_σ which is $S^{\pm}$ on either end. The index of D_{S_σ} will be the same as that of D_S, provided that σ is large enough, as the index is invariant under small peturbations. This means

$$\mathrm{ind}(D_S) = \mathrm{ind}(D_{S_\sigma}).$$

In the second step we perturb D_{S_σ} again. This time, S_σ is replaced by a diagonal matrix $\Sigma_\sigma = \Sigma_\sigma(s)$, such that it is a constant matrices $\Sigma_\sigma^{\pm}$ for s outside $[-\sigma,\sigma]$. Then another Fredholm invariance theorem (under homotopy) implies that

$$\mathrm{ind}(D_{S_\sigma}) = \mathrm{ind}(D_{\Sigma_\sigma}).$$

We have then reduced the problem to the case of computing the index of D_{Σ_σ} which we can do by hand using Section 4.4.

Lemma 5.21. *Let $\sigma \in \mathbb{R}_{>0}$. Suppose the matrices $S(s,t)$ are given with the uniform limits $S^{\pm}(t)$. Define*

$$S_\sigma(s,t) = \begin{cases} S^-(t) & s \le -\sigma \\ S(s,t) & -\sigma < s < \sigma \\ S^+(t) & s \ge \sigma \end{cases}$$

Then

$$\mathrm{ind}(D_S) = \mathrm{ind}(D_{S_\sigma}).$$

Proof. Let $\varepsilon > 0$. As $S^{\pm}(t)$ are the limits of $S(s,t)$ uniformly in t, there exists a $\tau \in \mathbb{R}$ such that

$$\|S(s,t) - S^-(t)\| < \varepsilon \tag{5.17}$$

and

$$\|S(s,t) - S^+(t)\| < \varepsilon \tag{5.18}$$

for all $|s| > \tau$. Consider a smooth bump-function $\rho : \mathbb{R} \to [0,1]$ such that

$$\rho(s) = \begin{cases} 1 & |z| \le \tau \\ 0 & |z| \ge 2\tau \end{cases}$$

We define a perturbation of D_S in the following way. Let

$$S^\rho_\tau = \begin{cases} \rho(s)S(s,t) + (1-\rho(s))S^-(t) & s \le -\tau \\ S(s,t) & |s| \le \tau \\ \rho(s)S(s,t) + (1-\rho(s))S^+(t) & s \ge \tau \end{cases}$$

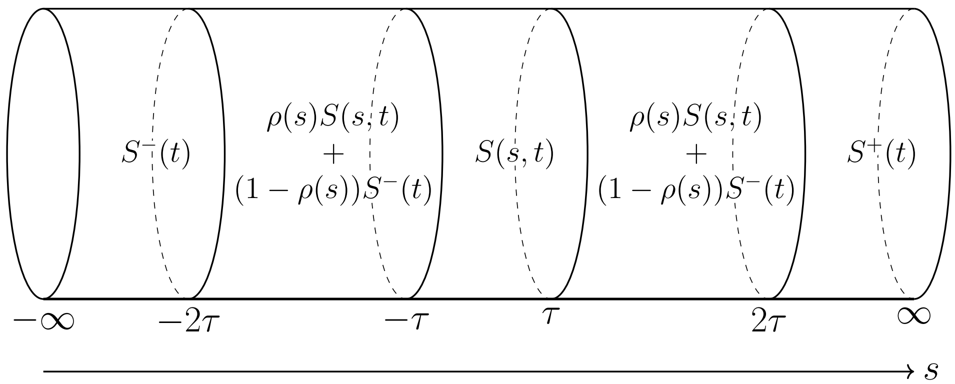

Figure 1. The cylinder of matrices $S^\rho_\tau(s,t)$.

We rewrite D_S as a perturbation of $D_{S^\rho_\tau}$ by some $A(s,t)$, by noting that

$$D_S = D_{S^\rho_\tau} + A(s,t)$$

where

$$A(s,t) = \begin{cases} (1-\rho(s))(S(s,t) - S^-(t)) & s \leq -\tau \\ 0 & |s| \leq \tau \\ (1-\rho(s))(S(s,t) - S^+(t)) & s \geq \tau \end{cases}$$

By equation (5.17) and (5.18), and the fact that $|\rho(s)| \leq 1$ for all $s \in \mathbb{R}$, it is immediate that

$$\|A(s,t)\| < \varepsilon.$$

Heuristically speaking, τ was chosen to be big enough such that the matrix $A(s,t)$ is sufficiently small. Then by Theorem C.11 (ii) in the appendix, we have $\mathrm{ind}(D_S) = \mathrm{ind}(D_{S^\rho_\sigma})$ by choosing ε as in the theorem. Note that this means that τ depends on the choice of ε.

We use that S^ρ_τ looks like Figure 1 as $\rho(s) = 0$ for $|s| \geq 2\tau$. That is, it coincides with $S^\pm$ for $\pm s \geq 2\tau$. However, τ depends on the choice of ε made. The goal was to relate the Fredholm index $\mathrm{ind}(D_S)$ to $\mathrm{ind}(D_{S_\sigma})$ for some fixed σ. Recall that the Fredholm index is a continuous map on the set of Fredholm operators, which is an open subset of the space $\mathrm{Lin}(W^{1,p}, L^p)$. Therefore, it suffices to connect S^ρ_τ to S_σ defined by

$$S_{s_0}(s,t) = \begin{cases} S^-(t) & s \leq -s_0 \\ S(s,t) & -so < s < s_0 \\ S^+(t) & s \geq s_0 \end{cases}$$

The Fredholm index is constant throughout this path if we pick it constant for s large. This is the case as the Fredholm index is a continuous map on the set of Fredholm operators, which is an open subset of the space of all operators. $\square$

We now modify S again. As $S^{\pm}(t)$ are non-degenerate and symplectic, the symplectic matrices $\Psi^{\pm}(t)$ given by equation (5.8) have well-defined Conley-Zehnder indices. We denote

$$\mu_{\mathrm{CZ}}(\Psi^{\pm}) = k^{\pm}.$$

The following lemma relates the index of D_S to the index of an operator of which we can compute the Fredholm index by hand.

Lemma 5.22. *Let $\Sigma_\sigma : \mathbb{R} \to M(2n, \mathbb{R})$ be a path of diagonal matrices such that*

$$\Sigma_\sigma(s) = S_{k^+} \quad s \geq \sigma$$

and

$$\Sigma_\sigma(s) = S_{k^-} \quad s \leq -\sigma.$$

Then

$$\mathrm{ind}(D_{\Sigma_\sigma}) = \mathrm{ind}(D_S).$$

Remark. Here, by the matrices $S_{k^\pm}$ we mean the ones defined in Section 4.4. They are diagonal and defined such that for the associated paths of symplectic matrices $\Psi_{k^\pm}$ defined by equation (5.8) we have $\mu_{\mathrm{CZ}}(\Psi_{k^\pm}) = k^{\pm}$.

Proof. We prove that there exists a path of symmetric matrices that interpolates between $\Psi^{\pm}$ and $\Psi_{k^\pm}$ in the following way.

Claim 5. There exist maps

$$S^{\pm} : [0,1] \times S^1 \to \mathrm{Sym}(2n, \mathbb{R})$$

such that

(i) The path $S^{\pm}(s, \cdot) : [0,1] \to \mathrm{Sym}(2n, \mathbb{R})$ is such that the path of symplectic matrices associated to it (by equation (5.8) is in $\mathcal{S}$ for all $s \in [0,1]$.

(ii) The symplectic matrices associated to $S^{\pm}(0,t)$ are $\Psi^{\pm}(t)$.

(iii) The symplectic matrices associated to $S^{\pm}(1,t)$ are $\Psi_{k^\pm}(t)$.

Proof.: Note that

$$\mu_{\mathrm{CZ}}(\Psi_{k^\pm}) = k^{\pm} := \mu_{\mathrm{CZ}}(\Psi^{\pm}).$$

Then by Theorem 4.6 (ii), these paths of matrices must be homotopic. Hence there exist C^1 homotopies

$$H^{\pm} : [0,1] \times [0,1] \to \mathrm{Sp}(2n)$$

such that $H^{\pm}(s, \cdot) \in \mathcal{S}$ for all $t \in [0,1]$ and $H^{\pm}(0,t) = \Psi^{\pm}(t)$ and $H^{\pm}(1,t) = \Psi_{k^\pm}(t)$.

Let $\rho \in C^\infty([0,1],[0,1])$ be such that $\dot\rho > 0$ and for a small $\varepsilon > 0$ we have $\rho(t) = 0$ if $t < \varepsilon$ and $\rho(t) = 1$ if $t > 1-\varepsilon$.

There exists a homotopy

$$\Lambda : [0,1] \times [0,1] \to [0,1]$$

between ρ and Id (i.e. $\Lambda(0,t) = \rho(t)$ and $\Lambda(1,t) = t$).
Note that

$$\frac{d\Psi^\pm(\Lambda_s(t))}{dt} = J_0\dot\Lambda_s(t)S^\pm(t)\Psi_{k^\pm}(\Lambda_s(t))$$

using the chain rule and the fact that $\frac{d\Psi^\pm(t)}{dt} = J_0S^\pm(t)\Psi^\pm(t)$. Hence associated to the composition $\Psi^\pm(\Lambda_s(t))$ are the matrices

$$\dot\Lambda_s(t)S^\pm.$$

We want these to be 1-periodic, which happens precisely when we choose the homotopy Λ such that for every s

$$\frac{d\Lambda_s}{dt}(0) = \frac{d\Lambda_s}{dt}(1) \tag{5.19}$$

The condition in equation (5.19) ensures that when we view that homotopy $\Psi^\pm(\Lambda_s(t))$ between $\Psi(t)$ and $\Psi(\rho(t))$, all associated matrices will be periodic. The same holds for $\Psi_{k^\pm}(\Lambda_s(t))$.

The second step is to homotope from $\Psi^\pm(\rho(t))$ to $\Psi_{k^\pm}(t)$ using the homotopy $H^\pm$ by precomposing with ρ.
Define

$$G^\pm : [0,1] \times [0,1] \to \mathrm{Sym}(2n,\mathbb{R})$$

by

$$G(s,t) := H^\pm(s,\rho(t)).$$

Now note that we picked $\rho(t)$ such that $\rho(t)$ is constant around $t = 0$ and $t = 1$. This implies that the matrices associated to $G^\pm(\,\cdot\,,t)$ will be periodic for every $s \in [0,1]$ as these associated matrices will be zero around $t = 0$ and $t = 1$.

The third and final step is to homotope from $\Psi_{k^\pm}$ back to $\Psi_{k^\pm}(t)$ like in the first step. The condition in equation (5.19) ensures that all associated symmetric matrices will be 1-periodic.

Concatenating these three steps yields a map $S^\pm : [0,1] \times S^1 \to \mathrm{Sym}(2n,\mathbb{R})$ that satisfies all the required properties. This proves Claim 5.

We use this path to interpolate by introducing a parameter $\lambda \in [0,1]$ in the following way. We can define a map

$$S : [0,1] \times \mathbb{R} \times S^1 \to \mathrm{Sym}(2n;\mathbb{R})$$

that satisfies

$$S(\lambda, s, t) = \begin{cases} S(s,t) & \lambda = 0 \\ S^+(\lambda, t) & s \geq \sigma \\ S^-(\lambda, t) & s \leq -\sigma_0 \\ S(s) & \lambda = 1 \end{cases}$$

This can readily be done. By Lemma 5.21, we know we can assume S to be independent of s for $|s|$ large. We can use the map that we obtained from Claim 5 to modify $S(\lambda, s, t)$ at both ends (for $|s|$ large). Modifying the middle part is easy; as long we make sure the ends extend, we can modify S on this compact subset of $\mathbb{R} \times S^1$ without changing the Fredholm index. This is due to Theorem C.14, which implies that changing on this compact set only, we change D_S by a compact operator. Then Theorem C.11 states that this operator has the same index.

Write $S_\lambda(s,t) := S(\lambda, s, t)$. Then the family of operators D_{S_λ} are all Fredholm operators and all have the same index by the above explanation. Note that $S_0 = S_\sigma$ and $S_1 = \Sigma_\sigma$. Therefore,

$$\mathrm{ind}(D_S) = \mathrm{ind}(D_{S_\sigma}) = \mathrm{ind}(D_{\Sigma_\sigma})$$

where we used Lemma 5.21 for the first equality. This proves Lemma 5.22. □

Our aim is to compute the index of D_{Σ_σ}, so that by Lemma 5.22, we know $\mathrm{ind}(D_S)$. We prove the following lemma.

Lemma 5.23. *Let $\Sigma : \mathbb{R} \to M(2n, \mathbb{R})$ be as in Lemma 5.22. Then*

$$\mathrm{ind}(D_{\Sigma_\sigma}) = k^- - k^+.$$

In order to prove this lemma, we will reduce to the case $\mathbb{R}^2$. Therefore, we mention the following lemma.

Lemma 5.24. *Let $p > 2$ and*

$$F_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^2) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^2)$$

defined by

$$\xi \mapsto \frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t} + S(s)\xi.$$

Suppose

$$S(s) = \begin{pmatrix} \alpha_1(s) & 0 \\ 0 & \alpha_2(s) \end{pmatrix}.$$

where

$$\alpha_i(s) = \begin{cases} \alpha_i^- & s \leq -s_0 \\ \alpha_i^+ & s \geq s_0 \end{cases}$$

to impose the limit behaviour of S and furthermore suppose that $\alpha_i^\pm \notin 2\pi\mathbb{Z}$. Then

(i) *If $\alpha_1(s) = \alpha_2(s) = \alpha(s)$. Then*

$$\dim\ker F_S = 2\#\{l \in \mathbb{Z} \mid a^- < 2\pi l < a^+\}$$

and

$$\dim\ker F_S^* = 2\#\{l \in \mathbb{Z} \mid a^+ < 2\pi l < a^+\}.$$

(ii) *If $\sup_{s\in\mathbb{R}} \|S(s)\| < 1$ then*

$$\dim\ker F_S = \#\{i \in \{1,2\} | a_i^- < 0, a_i^+ > 0\}$$

and

$$\dim\ker F_S^* = \#\{i \in \{1,2\} \mid a_i^+ < 0, a_i^- > 0\}.$$

The proof of this lemma is a computation involving power series. We refer to [**AD14**] page 293 for the proof. We now prove Lemma 5.23.

Proof. To compute the Fredholm index of D_{S_1} we use that $\operatorname{coker} D_{S_1}$ is isomorphic to $\ker D_{S_1}^*$, where the adjoint is defined as

$$D_{S_1}^* : W^{1,q}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^q(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

by

$$\xi \mapsto -\frac{\partial \xi}{\partial s} + J_0 \frac{\partial \xi}{\partial t} + S^t(s)\xi$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Hence, it is enough to only compute kernels of operators of this type. Recall $J_0 = \begin{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}$ As J_0 only has 2×2-blocks, we can reduce to the $\mathbb{R}^2$ case. As above, we have a diagonal matrix defined by

$$S(s) = \begin{pmatrix} \alpha_1(s) & 0 \\ 0 & \alpha_2(s) \end{pmatrix}.$$

where

$$\alpha_i(s) = \begin{cases} \alpha_i^- & s \leq -s_0 \\ \alpha_i^+ & s \geq s_0 \end{cases}$$

to impose the limit behavious of S.

We can now explicitely compute the dimensions of the kernel. This is a case distinction in the parity of $k^+ - n$ and $k^- - n$, which leads us to four cases. As the computations are totally analogous, we only treat the case $k^- = n \mod 2$ and $k^+ = n \mod 2$. The diagonal matrices $S_{k^\pm}$ are in this case

$$S_{k^-} = \mathrm{diag}(-\pi, \dots, -\pi, (n-1-k^-)\pi, (n-1-k^-)\pi)$$

and

$$S_{k+} = \mathrm{diag}(-\pi, \dots, -\pi, (n-1-k^+)\pi, (n-1-k^+)\pi).$$

Then we are in case (i) of Lemma 5.24. This lemma then gives

$$\dim\ker D_{\Sigma_\sigma} = 2\#\{l \in \mathbb{Z} \mid n-1-k^- < 2l < n-1-k^+\}$$

. This means

$$\dim\ker D_{S_{\Sigma_\sigma}} = \begin{cases} k^- - k^+ & k^- > k^+ \\ 0 & k^- \leq k^+ \end{cases}.$$

Similarly, by case (i) of Lemma 5.24, we find

$$\dim\ker D^*_{\Sigma_\sigma} = \begin{cases} k^+ - k^- & k^+ > k^- \\ 0 & k^+ \leq k^- \end{cases}.$$

Then

$$\mathrm{ind}(D_\Sigma) = \dim\ker D_{S_\Sigma} - \dim\operatorname{coker} D^*_{S_\Sigma}$$

hence by the above

$$\mathrm{ind}(D_{S_1}) = k^- - k^+.$$

This proves the lemma in one parity case. The other three follow by applying Lemma 5.24 appropriately and doing the same computation. This proves Lemma 5.23. □

5.4. A manifold structure on $\mathcal{M}(x^-, x^+; J, H)$ using transversality

In this section we prove that for a generic choice of non-degenerate H, the spaces $\mathcal{M}(x^-, x^+; J, H)$ are manifolds of dimension $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+)$. We primarily follow [**FHS94**].

We consider a non-degenerate Hamiltonian $H_0 \in C^\infty(S^1 \times M, \mathbb{R})$. Define the space of perturbations of H_0 that have the same periodic orbits.

Definition 5.25. Let H_0 be non-degenerate. Then for $l \geq 2$ define

$$C^l(H_0) := \{H \in C^l(M \times S^1, \mathbb{R}) \mid H \text{ agrees up to second order with } H_0 \text{ around every } x \in \mathcal{P}(H_0)\}.$$

The space $C^l(H_0)$ is a Banach space. We are interested in pairs (H, J) that are regular in the following sense.

Definition 5.26. We say $(H,J) \in C^\infty(S^1 \times M, \mathbb{R}) \times \mathcal{J}(M,\omega)$ is a regular pair if H is non-degenerate and the Floer operator $d^V\overline{\partial}_{H,J}(u)$ is surjective for all $u \in \mathcal{M}(x^-,x^+;H,J)$ for all $x^-,x^+ \in \mathcal{P}(H)$. We denote the collection of regular pairs by

$$(\mathcal{H} \times \mathcal{J})_{\text{reg}} := \{(H,J) \in C^\infty(S^1 \times M, \mathbb{R}) \times \mathcal{J}(M,\omega) \mid \ (H,J) \text{ is a regular pair}\}.$$

By application of the implicit function theorem, this means that for a pair $(H,J) \in (\mathcal{H} \times \mathcal{J})_{\text{reg}}$, the spaces $\mathcal{M}(x^-,x^+;J,H)$ are all finite dimensional submanifolds of the Banach spaces $\mathcal{B}^{1,p}(x^-,x^+)$ of dimension $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+)$. We prove that after picking a $J \in \mathcal{J}(M,\omega)$, this happens for generic Hamiltonians H.

Theorem 5.27. *Let $H_0 \in C^\infty(S^1 \times M, \mathbb{R})$ be non-degenerate and $J \in \mathcal{J}(M,\omega)$. Then the set*

$$\mathcal{H}_{reg}(H_0) := \{H \in C^\infty(H_0) \mid (H,J) \in (\mathcal{H},\mathcal{J})_{reg}\}$$

is of the second category in $C^\infty(H_0)$.

This says that for a generic choice of H, we may assume that the spaces $\mathcal{M}(x^-,x^+;H,J)$ are manifolds of dimension $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+)$, by application of the implicit function theorem.

The proof of Theorem 5.27 reasons by the Sard-Smale transversality theorem of a projection $(u,H) \mapsto u$. We follow [**FHS94**], taking several technical results in this paper for granted. The first step is defining a space $\mathcal{Z}(x,y;H)$ of pairs (u,H) that satisfy the Floer equation $\overline{\partial}_{H,J}(u) = 0$ for H with $u \in \mathcal{B}^{1,p}(x,y)$. We will prove that the space $\mathcal{Z}(x,y,J)$ is a Banach manifold for $x \neq y$. The strategy is again to describe $\mathcal{Z}(x,y,J)$ as the zeros of a section of a Banach bundle over a Banach manifold. We then apply the Sard-Smale Theorem C.12 to the projection $(u,H) \mapsto H$ and show that the regular values of this projection are precisely the H such that $d^V\overline{\partial}_{H,J}$ are surjective. By the Sard-Smale theorem, such regular values are of the second category.

To prove Theorem 5.27, we need two additional lemma's. We need that for solutions $u \in \mathcal{M}(x,y)$, the set of $(s,t) \in \mathbb{R} \times S^1$ where u is injective is dense. We make the following definitions.

Definition 5.28 (Critical points)**.** Let $u \in \mathcal{M}(x,y)$. A point $(s,t) \in \mathbb{R} \times S^1$ is called a critical point of u if either $\frac{\partial u}{\partial s}(s,t) = 0$ or $u(s,t) = x(t)$ or $u(s,t) = y(t)$. We denote the set of critical points of u inside $\mathbb{R} \times S^1$ by $C(u)$.

Definition 5.29 (Regular points)**.** Let $u \in \mathcal{M}(x,y)$. A point $(s,t) \in \mathbb{R} \times S^1$ is called a regular point of u if it is not a critical point and $u(s,t) \neq u(s',t')$ for every $s \neq s_0$. We denote the set of regular points of u inside $\mathbb{R} \times S^1$ by $R(u)$.

The critical and regular points defined in Definitions 5.28 and 5.29 have the following property. This is Theorem 8.2 in [**SZ92**] or a combination of Theorem 4.3 and Lemma 4.1 in [**FHS94**].

Lemma 5.30. *Assume $x, y \in \mathcal{P}_0(H)$ and $x \neq y$. Let $u \in \mathcal{M}(x, y)$. The set $C(u)$ is discrete and the set $R(u)$ is open and dense in $\mathbb{R} \times S^1$.*

Note that Lemma 5.30 is highly non-trivial; it is the main content of the paper [**FHS94**]. The following explaination should be interpreted as philosophical in nature.

The fact that $C(u)$ is discrete follows (in [**FHS94**]) from the Carleman Similary Principle (Theorem 2.2 in [**FHS94**], Theorem 2.3.5 in [**MS12**]). An immediate Corollary says that, roughly speaking, if u satisfies a perturbed Cauchy-Riemann equation, then its derivative must be non-vanishing on some small ball. If u satisfies the Floer equation, we look at the local flow of X_H denoted φ_H^t. We then compare u to $v = (\varphi_H^t)^{-1}u$. Heuristically this is a reduction to the case $X_H = 0$. Then v satisfies a perturbed Cauchy-Riemann equation. Using the Carleman Similarity Principle, dv is non-vanishing on a ball, which implies that $\frac{\partial u}{\partial s}$ vanishes on a discrete set only.
The fact that $R(u)$ is dense is even more involved and uses a unique continuation lemma for solutions of a perturbed Cauchy-Riemann equation that follows from the Carleman Similarity Principle. For more details, we refer the reader to the relevant Lemmas and Theorems in [**FHS94**].

We now give a proof of Theorem 5.27 using these Lemma 5.30.

Proof (*of Theorem 5.27*). Let $H_0 : S^1 \times M \to \mathbb{R}$ be a Hamiltonian non-degenerate Hamiltonian and let $J \in \mathcal{J}(M, \omega)$. Let $C^l(H_0)$ be as above. Let $x, y \in \mathcal{P}(H_0)$ and $\mathcal{Z}(x, y; J)$ be the space of pairs (u, H) such that u satisfies $\overline{\partial}_{H,J}(u) = 0$. We prove that this space is an infinite dimensional manifold, so that we can apply the Sard-Smale theorem to the projection $(u, H) \mapsto H$.

Let $p > 2$ and $l \geq 2$ and let $\mathcal{B}^{1,p}(x, y)$ be as described in Definition 5.9. We construct a bundle over $\mathcal{B}^{1,p} \times C^l(H_0)$. This is the bundle $\mathcal{E}$ whose fiber over (u, H) is

$$\mathcal{E}_{(u,H)} := L^p(u^*TM).$$

Define a smooth section of this bundle

$$\sigma : \mathcal{B}^{1,p}(x, y) \times C^l(H_0) \to \mathcal{E}$$

by setting

$$\sigma(u, H) = \overline{\partial}_{H,J}(u).$$

We prove that the vertical differential of this section is surjective, which is equivalent to saying the section is transversal to the zero section.

Note that

$$T_H C^l(H_0) = \{h \in C^l(S^1 \times M, \mathbb{R}) \mid \ h \text{ vanishes up to second order around points } x(t) \text{ for } x \in \mathcal{P}_0(H_0) \ \}.$$

By direct computation the vertical derivative of the section σ

$$d^V\sigma(u,H) : W^{1,p}(u^*TM) \times T_H C^l(H_0) \to L^p(u^*TM)$$

given by

$$d^V\sigma(u,H)(\xi,h) = d^V\overline{\partial}_{H,J}(u)\xi - \nabla h.$$

We prove that this map is surjective whenever $\sigma(u,h) = 0$.

Let $\frac{1}{p} + \frac{1}{q} = 1$ and suppose $\eta \in L^q(u^*TM)$ is non-zero and orthogonal to the range of $d^V\sigma(u,H)$, meaning

$$\int_{\mathbb{R}\times S^1} \langle d^V\sigma(u,H)(\xi,h), \eta\rangle_J \, dsdt = 0$$

for all (ξ, h). That is, suppose $d^V\sigma(u,H)$ is not surjective. This happens if and only if

$$d^V\overline{\partial}_{H,J}(u)\eta = 0 \tag{5.20}$$

and

$$\int_{\mathbb{R}\times S^1} dh(u)\eta \, dsdt = 0 \tag{5.21}$$

for all $h \in T_H C^l(H_0)$.

By elliptic regularity and duality, η is smooth. We will show that for any $\eta \in L^q(u^*TM)$ such that equation (5.21) is satisfied for all h, we have $\eta = 0$. This is then a contradiction with the fact that $d^V\sigma(u,H)$ is not surjective. We prove the following claim.

Claim 6. There exists a smooth function $\lambda : \mathbb{R} \times S^1 \setminus C(u) \to \mathbb{R}$ such that for all $(s,t) \in \mathbb{R} \times S^1$ we have

$$\eta(s,t) = \lambda(s,t)\frac{\partial u}{\partial s}(s,t).$$

Proof. Suppose that $\frac{\partial u}{\partial s}$ and η are linearly independent at some point $(s_0, t_0) \in \mathbb{R}\times S^1$. By Lemma 5.30, we may asusme $(s_0, t_0) \in R(u)$. We can construct a neighborhood $U_0 \subset M \times S^1$ of the point $(u(s_0, t_0), t_0)$ with the property that the open set

$$V_0 = \{(s,t) \mid (u(s,t),t) \in U_0\}$$

is a small neighborhood of (s_0, t_0).

Suppose this is not the case, then there would exist a sequence (s_n, t_n) such that $t_n \to t_0$ and $u(s_n, t_n) \to u(s_0, t_0)$ and $|s_n - s_0| > \delta$ (i.e. it converges in U_0 but not in V_0). The sequence s_n must be bounded because $u(s_0, t_0)$ is regular, or $u(s_0, t_0)$ would be one of the end-points. Assume it converges to some s'. Then $|s' - s_0| > \delta$ for some $\delta > 0$ and $u(s', t_0) = u(s_0, t_0)$ so $u \notin R(u)$ by definition of $R(u)$ in Definition 5.29. This contradiction proves the existence of V_0 for some small U_0.

We can embed a sufficiently small ball inside U_0. Let $\varepsilon > 0$ be sufficiently small and t close to t_0 such that $g_t : B_\varepsilon(0, s_0) \to U_0$ given by

$$g_t(r, s) := \exp_{u(s,t)}(r\eta(s, t))$$

is an embedding. Then $g_t(0, s) = u(s, t)$ and $\frac{\partial g_t(0,s)}{\partial r} = \eta(s, t)$. We use this function g_t to generate a Hamiltonian function h_t.

Let $\beta : \mathbb{R} \to [0, 1]$ be a cut-off function equal to 1 near 0. As g_t is an embedding, we can construct $h_t : M \to \mathbb{R}$ supported in U_0 that is of the following form

$$h_t(g_t(r, s)) = r\beta(r)\beta(s - s_0)\beta(t - t_0).$$

The Hamiltonian h_t has the property that it vanishes outside U_0 and $h_t(u(s, t)) = 0$. Furthermore, by differentiating we see that

$$dh_t(u(s, t), t)\eta(s, t) = \rho(s - s_0)\rho(t - t_0).$$

In particular, we have that equation (5.21) is not satisfied for this particular h. This is a contradiction, which proves Claim 6. $\square$

We claim that $\lambda(s, t)$, which exists by virtue of Claim 6 has the following property.

Claim 7. The function $\lambda(s, t)$ is such that $\frac{\partial \lambda}{\partial s} \equiv 0$.

Proof. Assume that $\frac{\partial \lambda}{\partial s}(s_0, t_0) \neq 0$ for some $(s_0, t_0) \in \mathbb{R} \times S^1 \setminus C(u)$. As $\eta(s, t) \neq 0$ everywhere but a discrete set, we may assume $\lambda(s_0, t_0) \neq 0$. Furthermore, by Lemma 5.30 we may assume $(s_0, t_0) \in R(u)$. Choose some small neighborhood V_0 of (s_0, t_0) such that on $\lambda(s, t) \neq 0$ and $\frac{\partial \lambda}{\partial s}(s, t) \neq 0$ for all $(s, t) \in V_0$. Construct a compactly supported function $\alpha : V_0 \to \mathbb{R}$ such that

$$\int_{V_0} \lambda(s, t) \frac{\partial \alpha}{\partial s}(s, t)\, dsst \neq 0 \tag{5.22}$$

As above, we can construct a Hamiltonian $h : M \times S^1 \to \mathbb{R}$ such that

$$h_t(u(s, t)) = \alpha(s, t)$$

for all $(s, t) \in V_0$. This yields

$$dh_t(u(s, t), t)\eta(s, t) = \lambda(s, t)\frac{\partial \alpha}{\partial s}(s, t)$$

for all $(s, t) \in V_0$. Again, together with equation (5.22) this implies that the integral in equation (5.21) is non-zero. This contradiction proves Claim 7. $\square$

Since $C(u)$ is discrete, an immediate consequence is that $\lambda(s,t) = \lambda(t)$ such that

$$\eta(s,t) = \lambda(t)\frac{\partial u}{\partial s}(s,t).$$

Assume $\eta \neq 0$. Then $\eta(s,t)$ vanishes on a discrete set only as it satisfies $(\overline{\partial}_{H,J}(u))^*\eta = 0$ together with Corollary 2.3 in [**FHS94**]. As $\eta \neq 0$, we have $\lambda(t) \neq 0$ for all $t \in S^1$. Assume that $\lambda(t) > 0$. We can then compute

$$\int_0^1 \Big\langle \frac{\partial u}{\partial s}(s,t), \eta(s,t) \Big\rangle_J dt = \int_0^t \lambda(t) \Big|\frac{\partial u}{\partial s}(s,t)\Big|_J^2 dt > 0. \tag{5.23}$$

On the other hand we have

$$d^V\overline{\partial}_{H,J}(u)\frac{\partial u}{\partial s} = 0 \qquad \text{and} \qquad (d^V\overline{\partial}_{H,J}(u))^*\eta = 0. \tag{5.24}$$

We claim that

$$\frac{d}{ds}\int_0^1 \Big\langle \frac{\partial u}{\partial s}(s,t), \eta(s,t) \Big\rangle_J dt = 0 \tag{5.25}$$

This requires a small computation. Note that

$$(d^V\overline{\partial}_{H,J}(u))^*\eta = -\nabla_s\eta + J(u)\nabla_t\eta + \nabla_\eta J\frac{\partial u}{\partial t} - \nabla_\eta \nabla H(u).$$

Hence, the part except for ∇_s is self-adjoint. Write $d^V\overline{\partial}_{H,J}(u) = \nabla_s + A_u$. Therefore, we can compute

$$\begin{aligned}
\frac{d}{ds}\int_0^1 \Big\langle \frac{\partial u}{\partial s}(s,t), \eta(s,t) \Big\rangle_J dt &= \int_0^1 \Big(\Big\langle \nabla_s \frac{\partial u}{\partial s}(s,t), \eta(s,t) \Big\rangle_J + \Big\langle \frac{\partial u}{\partial s}(s,t), \nabla_s\eta(s,t) \Big\rangle_J \Big) dt \\
&= \int_0^1 \Big\langle \eta, d^V\overline{\partial}_{H,J}(u)\frac{\partial u}{\partial s} - A_u\frac{\partial u}{\partial s} \Big\rangle_J + \Big\langle A_u\eta + (d^V\overline{\partial}_{H,J}(u))^*\eta, \frac{\partial u}{\partial s} \Big\rangle_J dt \\
&= \int_0^1 \Big\langle \eta, d^V\overline{\partial}_{H,J}(u)\frac{\partial u}{\partial s} \Big\rangle_J - \Big\langle (d^V\overline{\partial}_{H,J}(u))^*\eta, \frac{\partial u}{\partial s} \Big\rangle_J dt \\
&= 0
\end{aligned}$$

Combining equation (5.23) and equation (5.25) we find by integrating over t that

$$\int_{-\infty}^{+\infty}\int_0^1 \Big\langle \frac{\partial u}{\partial s}(s,t), \eta(s,t) \Big\rangle_J dsdt = +\infty \tag{5.26}$$

However, this contradicts the assumption that $\frac{\partial u}{\partial s} \in L^p(u^*TM)$ and $\eta \in L^q(u^*TM)$. Hence, this contradicts the assumption that η is non-zero meaning that for any η such that

$$\int_{-\infty}^{+\infty}\int_0^1 \Big\langle d^V\sigma(u,H)(\xi,h), \eta \Big\rangle_J dtds = 0$$

for all $(\xi, h) \in W^{1,p}(\xi^*TM) \times T_H C^l(H_0)$ we must have $\eta \equiv 0$.

Therefore, $d^V\sigma(u,h)$ is indeed surjective whenever $\sigma(u,h)=0$.

Let
$$\mathcal{Z}(x,y;H) := \{(u,H) \in \mathcal{B}^{1,p}(x,y) \times C^l(H_0) \mid \sigma(u,H)=0\}.$$
By the implicit function theorem and the above conclusion, $\mathcal{Z}(x,y;H)$ is a Banach submanifold of $\mathcal{B}^{1,p}(x^-,x^+)$. Hence, we can apply the Sard-Smale theorem to the following projection. Consider the projection
$$\pi : \mathcal{Z}(x,y;H) \to C^l(H_0)$$
defined by
$$(u,H) \mapsto H.$$

This map is a Fredholm map between two Banach manifolds of the same index as $d^V\overline{\partial}_{H,J}$. By the Sard-Smale transversality theorem, Theorem C.12, it follows that the regular values of π are of the second category in $C^l(H_0)$. We show that the regular values of π are precisely $H \in C^l(H_0)$ such that (H,J) is a regular pair.

Suppose H is a regular value of π and let $u \in \mathcal{M}(x,y;J,H)$. Suppose $d^V\overline{\partial}_{H,J}(u)$ is not surjective. Then there exists a non-zero $\eta \in L^q(u^*TM)$ such that
$$\int_{\mathbb{R}\times S^1} \langle d^V\overline{\partial}_{H,J}(u)\xi, \eta\rangle_J \, dsdt = 0 \tag{5.27}$$
for all $\xi \in W^{1,p}(u^*TM)$. Surjectivity of $d\pi(u,H)$ implies that for every $h \in C^l(H_0)$ there exists an $\xi \in W^{1,p}(u * TM)$ such that $d^V\sigma(u,H)(\xi,h)=0$. This is the case as
$$T_{(u,H)}\mathcal{Z}(x,y;H) = \{(\xi,h) \in W^{1,p}(u^*TM) \times C^l(H_0) \mid d^V\sigma(u,H)(\xi,h)=0\}.$$
In the proof above, we already showed that for this ξ we cannot have that equation (5.27) is satisfied for a non-zero η. This contradiction shows that $d^V\overline{\partial}_{H,J}(u)$ is surjective.

Conversely, suppose that $d^V\overline{\partial}_{H,J}(u)$ is surjective for all $u \in \mathcal{M}(x,y;J,H)$. Suppose $h \in C^l(H_0)$ is given. By surjectivity we can pick ξ such that $d^V\overline{\partial}_{H,J}(u)\xi = -\nabla h$. This means that $(\xi,h) \in T_{(u,H)}\mathcal{Z}(x,y;H)$ and $d\pi(u,H)(\xi,h) = h$ which means $d\pi(u,H)$ is surjective.

This proves the statement of the theorem in C^l. For completeness we still need to extend to C^∞. This is done via an argument due to Taubes (according to [**MS12**] from personal communications, hence no reference given) and explained more thoroughly in [**MS95**] in the proof of Theorem 3.1.2 (ii). Here however, a result is proven for a general J-holomorphic curve instead. For a discussion in the Floer setting, we refer the reader to the proof of Theorem 5.1 (i) in [**FHS94**] where the extension to C^∞ is made in the case of a perturbation of J on page 18. The argument in the case of a perturbation of H is analogous. This proves Theorem 5.27. $\square$

REMARK. As stated, the above argument to extend to the C^∞ case is due to Taubes. One can prove Theorem 5.27 without this argument, which is done in [**AD14**] Section 8.5 and 8.3. This uses an argument originally due to Floer in [**Flo88**].
Here one considers a Banach space $C^\infty_\epsilon(H_0)$ of perturbations h of H_0 such that the norm $\|h\|_\epsilon = \sum_{k\geq 0} \epsilon_k \sup |d^k h(x,t)|$ is finite. The sequence ϵ_k is chosen such that C^∞_ϵ is dense in C^∞. Then one considers the above argument where the Banach space $C^l(H_0)$ is replaced by the Banach space $C^\infty_\epsilon(H_0)$ of Hamiltonians $H = H_0 + h$ where $\|h\|_\varepsilon < +\infty$.

5.5. Broken trajectories and gluing

Suppose from now on we are in the case where we have fixed the pair $(H, J) \in (\mathcal{H} \times \mathcal{J})_{\text{reg}})$. By the previous sections, the moduli spaces $\mathcal{M}(x, y)$ are manifolds. In this section we discuss the compactness of $\mathcal{M}(x, y)$. A priori, this manifold can not be compact as we can translate in the s-parameter. That is, let $u \in \mathcal{M}(x, y)$. Then there is an action

$$\mathbb{R} \times \mathcal{M}(x, y) \to \mathcal{M}(x, y)$$

given by

$$(\sigma, u(s,t)) \mapsto \sigma \cdot u(s,t) = u(s+\sigma, t).$$

We can quotient by this action to define the space $\widehat{\mathcal{M}}(x, y)$.

Definition 5.31. Let $x, y \in \mathcal{P}_0(H)$. Define

$$\widehat{\mathcal{M}}(x, y) := \mathcal{M}(x, y)/\mathbb{R}$$

equipped with the quotient topology.

The space $\widehat{\mathcal{M}}(x, y)$ is the space of unparametrized trajectories of finite energy running between x and y. We saw its topology is instrumental in defining the Floer boundary operator $\overline{\partial}_J$ in Definition 5.5. The space $\widehat{\mathcal{M}}(x, y)$ has the following unique limit property.

Lemma 5.32. *Let $x, y \in \mathcal{P}_0(H)$. Let $(u_n) \subset \mathcal{M}(x, y)$ be a sequence with two sequences $(s_n), (\sigma_n) \subset \mathbb{R}$ such that*

$$\lim_{n\to\infty} u_n(s_n + s, \cdot) = u_s \in \mathcal{M}(x, z_s)$$

and

$$\lim_{n\to\infty} u_n(\sigma_n + s, \cdot) = u_\sigma \in \mathcal{M}(x, z_\sigma)$$

for two $u_s, u_\sigma \in \mathcal{P}_0(H)$ distinct from x. Then

(i) $z_s = z_\sigma$.

(ii) *Denote $z := z_s = z_\sigma$. Let $\pi : \mathcal{M}(x, y) \to \widehat{\mathcal{M}}(x, y)$, then $\pi(u_s) = \pi(u_\sigma)$. This is equivalent to the existence of $\tau \in \mathbb{R}$ such that*

$$u_s(\tau + s, t) = u_\sigma(s, t).$$

For a proof we refer to [**AD14**] Proposition 9.1.2. Note that this immediately implies that $\widehat{\mathcal{M}}(x,y)$ equipped with the quotient topology is a Hausdorff space.

We denote the space of all flow lines by

$\mathcal{M}(J,H) = \mathcal{M} := \{u \in C^\infty(\mathbb{R}\times S^1, M) \mid u$ is a contractible solution to equation 3.6 and $E(u) < +\infty\}$.

This is the space of all possible solutions to Floer's equation with finite energy. By Theorem 6.5.6 in [**AD14**], if u is such that $E(u) < +\infty$, then there exist $x^\pm \in \mathcal{P}_0(H)$ such that

$$\lim_{s\to\pm\infty} u(s,\cdot) = x^\pm$$

in $C^\infty(S^1, M)$. This implies that $\mathcal{M}$ is the union of all the moduli spaces.

$$\mathcal{M}(J,H) = \bigcup_{x^-,x^+\in\mathcal{P}_0(H)} \mathcal{M}(x^-,x^+;J,H).$$

To study the topology of $\widehat{\mathcal{M}}(x,y)$ we will want to prove a compactness theorem about $\mathcal{M}$. It is this theorem that is the central reason why we assume the manifold M to satisfy Assumption 3.3.

Theorem 5.33. *The space $\mathcal{M}$ is compact in the C^∞_{loc}-topology.*

Theorem 5.33 can be interpreted as a "No bubbling" result. It asserts that no bubbles are formed. Such bubbles form an obstruction to the compactness of the space $\mathcal{M}$. Assumption 3.4 will prove crucial in proving this result.

Remark. We give a brief explanation of what is meant by "bubbling" in the more general sense. We follow the introduction of Section 4.2 in [**MS12**]. Let (M,ω) be a compact symplectic manifold and $S(\Sigma, j_\Sigma)$ be a closed Riemann surface. In our case this is the surface $\mathbb{R}\times S^1$, which is of course not closed. However, our additional assumptions on M fixes this. LEt $J \in \mathcal{J}(M,\omega)$. Suppose we are given a sequence of J-holomorphic maps $u^\nu : \Sigma \to M$ such that their derivatives are unbounded, i.e. $\sup_\nu \|du^\nu\|_{L^\infty} = \infty$. Suppose we look at sequence with uniformly bounded energy (this is the case in $\mathcal{M}$). Then we can rescale the sequence u^ν to a sequence $v^\nu : U \to M$ where U is some ball around 0 in $\mathbb{C}$ by passing through a coordinate chart. Using the limit v of the sequence v^ν we obtain a map that extends to the Riemann-sphere $v : \mathbb{C}\cup\{\infty\} \to M$ by removing a singularity around 0. This map is called a bubble. It is a non-constant J-holomorphic map with positive energy.

The proof of Theorem 5.33 requires the following technical result.

Theorem 5.34. *On the space $\mathcal{M}$, the C^0_{loc} and the C^∞_{loc}-topologies coincide.*

Proof. The proof is results from elliptic regularity combined with a bound on the gradient of $u \in \mathcal{M}$ (Lemma 5.35 below). We refer the reader to the proof of [**AD14**] Proposition 6.5.3. $\square$

The central point in proving Theorem 5.33 is proving that the derivatives of $u \in \mathcal{M}$ have a bound. This is described by the following Lemma. Note that Assumption 3.3 is required in the hypothesis.

Lemma 5.35. *Suppose M is closed and satisfies Assumption 3.4. Then there exists $C > 0$ such that for all $u \in \mathcal{M}$ we have*

$$\|\nabla u(s,t)\|_J \leq C.$$

Proof. For a proof we refer the reader to [**AD14**] Proposition 6.6.2. The proof reasons by contradiction by considering a sequence (s_k, t_k) and a sequence $(u_k) \subset \mathcal{M}$ such that

$$\lim_{k\to\infty} \|\nabla u_k(s_k, t_k)\|_J = +\infty.$$

Then define a rescaling of u_k denoted v_k using the half maximum lemma. This v_k, by application Arzela-Ascoli, using compactness of M and elliptic regularity, converges to some smooth J-holomorphic v (meaning it satisfies $\frac{\partial v}{\partial s} + J(v)\frac{\partial v}{\partial t} = 0$). One can compute that v has finite energy (Lemma 6.6.4 in [**AD14**]). The contradiction is then obtained by proving that whenever Assumption 3.3 is satisfied, there exist no non-constant J-holomorphic $w : \mathbb{C} \to M$ with finite energy. $\square$

Let us give an outline of the proof of Theorem 5.33.

Proof of Theorem 5.33. Consider a sequence $(u_n) \subset \mathcal{M}$. Note that the bound in Lemma 5.35 is independent of u, hence this inequality asserts the equicontinuity of $u \in \mathcal{M}$. By Arzela-Ascoli, the closure $\overline{\mathcal{M}} \subset C^0(\mathbb{R} \times S^1, M)$ is compact. Hence, (u_n) has some subsequence that converges to some u_0 in $C^0_{\mathrm{loc}}(\mathbb{R} \times S^1, M)$.

By elliptic regularity, u_0 is of class C^∞. $\square$

Recall that our goal is to define the boundary operator $\partial : CF_k \to CF_{k+1}$ such that $\partial \circ \partial = 0$. To do this we first consider $\widehat{\mathcal{M}}(x,y)$ for which the index difference is 1: $CZ(x) - CZ(y) = 1$. To prove $\partial \circ \partial = 0$ we consider $\widehat{\mathcal{M}}(x,y)$ where the index difference is 2.

5.5.1. $\widehat{\mathcal{M}}(x,y)$ is compact 0-dimensional if $\mathrm{CZ}(x) - \mathrm{CZ}(y) = 1$.

We start by proving compactness of $\widehat{\mathcal{M}}(x,y)$ in the case where the index difference is 1.

Theorem 5.36. *Suppose $CZ(y) = CZ(x) + 1$. Then the space $\widehat{\mathcal{M}}(x,y)$ is compact.*

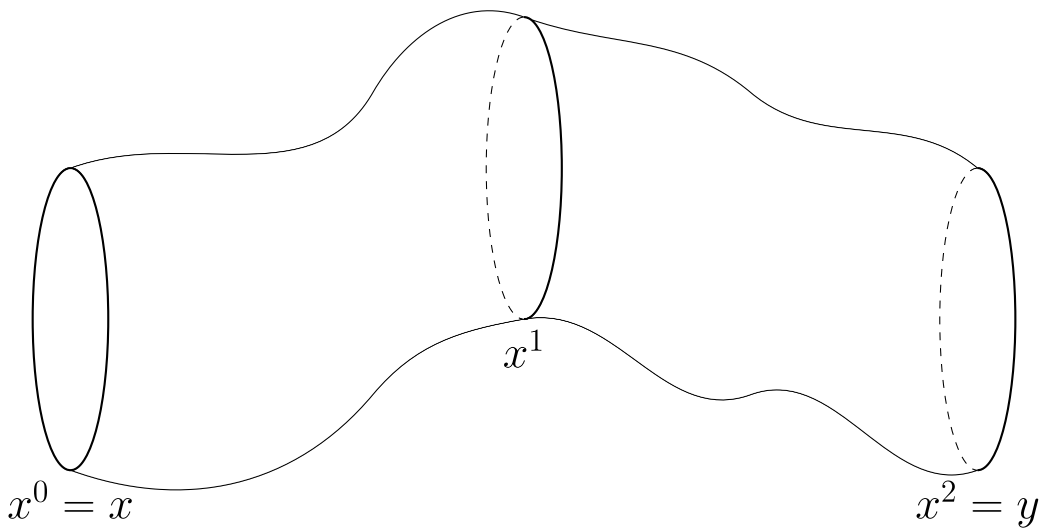

Figure 2. An picture of Theorem 5.37 in the case $l = 1$.

This theorem implies that this space is a finite set, as it is compact and zero dimensional. Then the number $\eta(x, y)$ defined in Definition 5.5 is a well defined finite number.

We attack this problem in the following way. Consider $\widehat{\mathcal{M}}(x, y)$ for an arbitrary index difference. Suppose $(u_n) \subset \widehat{\mathcal{M}}(x, y)$ is a sequence. It might be that (u_n) has no convergent subsequence. However, we will show that there always exists a subsequence that converges to a broken flow line from x to y. This is an collection of intermediate critical points $x = x_0, \ldots, x_k = y$ together with solutions $u_k \in \mathcal{M}(x_k, x_{k+1})$. The following theorem gives a precise description of this behavior.

Theorem 5.37. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}(x, y)$. Then there exists a subsequence of (u_n) and with*

- *Critical points $x^0 = x, x^1, \ldots, x^l, x^{l+1} = y$*
- *Sequences $(s_n^k) \subset \mathbb{R}$ for $1 \leq k \leq l$*
- *Elements $u^k \in \mathcal{M}(x^k, x^{k+1})$ such that for every $k = 0, \ldots, l$, we have*
$$\lim_{n\to\infty} u_n \cdot s_n^k = u^k$$
in the C^∞_{loc} topology.

Figure 2 depicts the situation. We will prove now prove this theorem and then explain how it solves Theorem 5.36.

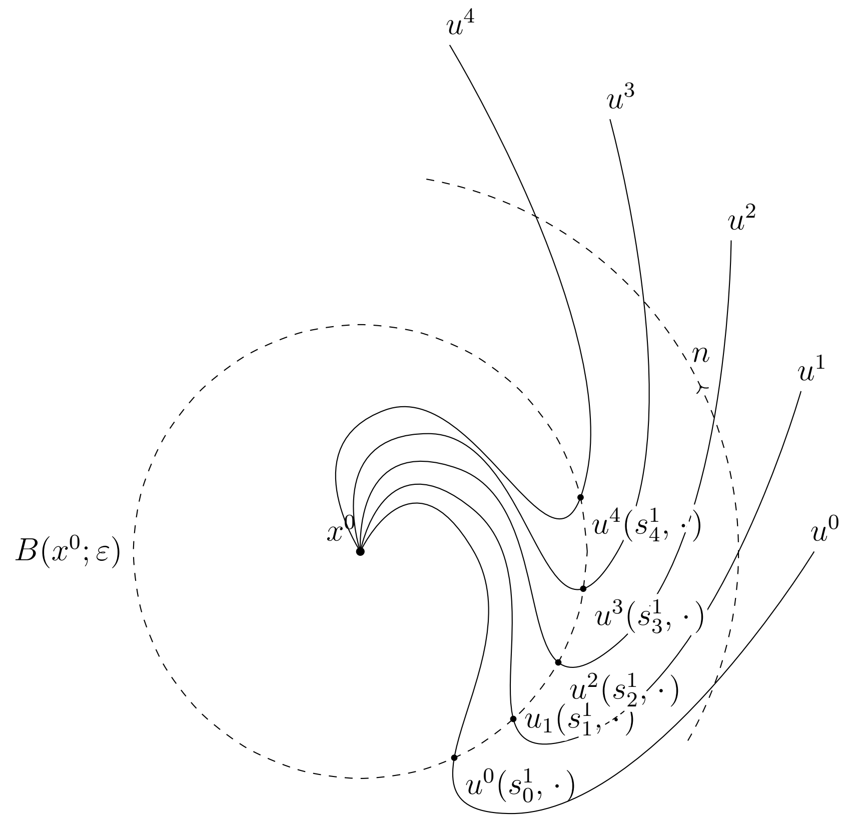

Figure 3. The definition of s_n^1 as the sequence of numbers for which u_n leaves the ball $B(x^0;\varepsilon)$.

Proof. Let $\varepsilon > 0$ be sufficiently small such that balls with radius ε centered around the critical points of $\mathcal{A}_H$ are disjoint. This is possible as there are only finitely many critical points. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ be a sequence.

If we view u_n as a path in $\mathcal{L}M$, then it must leave $B(x,\varepsilon)$ at some point, for it goes to $y \neq x$. Denote $x^0 := x$. Define

$$s_n^1 = \inf\{s \in \mathbb{R} \mid d_\infty(u_n(s,\cdot), x^0) > \varepsilon\} \tag{5.28}$$

Here d_∞ is the distance function inducing the C^∞ topology. Loosely speaking, $s_n^1 \in \mathbb{R}$ such that u_n leaves the ball $B(x^0;\varepsilon)$ for the first time. Figure 3 depicts the situation. By compactness of $\mathcal{M}$, we can extract a subsequence such that the sequence $(u_n \cdot s_n^1)_{n\in\mathbb{N}}$ converges to some $u^1 \in \mathcal{M}$.
By definition of s_n^1 (5.28), we must have

$$u^1(s,\cdot) \in \overline{B}(x^0;\varepsilon)$$

for all $s \leq 0$ and

$$u^1(0,\cdot) \in \partial B(x^0;\varepsilon).$$

To see this note that if $s \leq 0$, then $s + s_n^1 \leq s_n^1$ for all $n \in \mathbb{N}$. Therefore

$$\begin{aligned} d_\infty(u^1(s,\cdot),x^0) &= d_\infty(\lim_{n\to\infty} u_n(s+s_n^1,\cdot),x^0) \\ &= \lim_{n\to\infty} d_\infty(u_n(s+s_n^1,\cdot),x^0) \\ &\leq \varepsilon \end{aligned}$$

where we use $s + s_n^1 \leq s_n^1$ and the definition of s_n^1 (5.28).
This means that u^1 is a trajectory from $x^0 = x$ to some other critical point, as it exits the ball $B(x^0,\varepsilon)$. Hence, $u^1 \in \mathcal{M}(x^0,x^1)$ for some $x^1 \in \operatorname{Crit}\mathcal{A}_H$. We have two cases.

If $x^1 = y$ we are done. This is the case where $l = 0$.

If $x^1 \neq y$ we proceed by induction. This means that we have found

- Sequences $(s_n^0)_{n\in\mathbb{N}},\dots(s_n^k)_{n\in\mathbb{N}}$
- Trajectories $u^j \in \mathcal{M}(x^{j-1},x^j)$ such that $x^j \neq y$ for $0 \leq j \leq k$ with the property that
$$\lim_{n\to\infty} u_n \cdot s_n^j = u^j$$
for all $0 \leq j \leq k$.

We consider the last trajectory $u^k in(x^{k-1},x^k)$ which tends to x^k as $s \to \infty$. This implies the existence of an $s' \in \mathbb{R}$ such that

$$u^k(s,\cdot) \in B(x^k;\varepsilon)$$

for all $s \geq s'$.
Note that $u_n(s'+s_n^k,\cdot)$ converges to $u^k(s',\cdot)$ by the induction hypothesis. Therefore, for n sufficiently large,

$$u_n(s'+s_n^k,\cdot) \in B(x^k;\varepsilon).$$

However, the trajectory (u_n) tends to the orbit y, where $y \neq x^k$. Hence u_n must leave $B(x_k;\varepsilon)$ for some $s > s_n^k + s'$. We define the first point $s \in \mathbb{R}$ at which this happens.

$$s_n^{k+1} = \sup\{s \geq s_n^k + s' \mid u_n(r) \in B(x_k;\varepsilon), s_n^k + s' \leq r \leq s\} \tag{5.29}$$

This is the point at which u_n leaves the ball $B(x^k;\varepsilon)$ for the first time, as illustrated in figure 4. By compactness of $\mathcal{M}$, extract a subsequence such that $u_n \cdot s_n^{k+1}$ converges to some $u^{k+1} \in \mathcal{M}$. It remains to show that $u^{k+1} \in \mathcal{M}(x^k,x^{k+1})$ for some $x^{k+1} \neq x^k$.

Claim 8. We have $s_n^{k+1} - s_n^k \to \infty$ as $n \to \infty$.

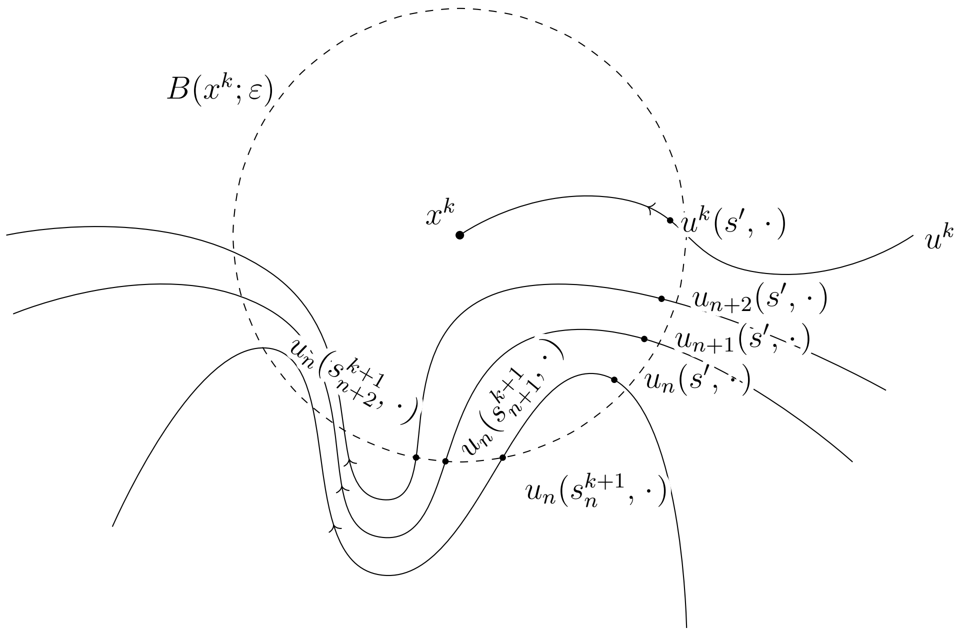

Figure 4. The definition of s_n^k as the sequence of numbers for which u_n leaves $B(x^k;\varepsilon)$ for the first time.

Proof.: Assume $s_n^{k+1} - s_n^k < C$ for all n for some $C \in \mathbb{R}$. The interval $[s', s^{k+1} - s_n^k]$ is contained in some compact interval K. On this compact interval, $u_n \cdot s_n^k$ converges to u^k uniformely. Then for any $s \in [s', s^{k+1} - s_n^k]$ we have $u_n(s_n^k + s, \cdot) \in B(x^k, \varepsilon)$. However, $u_n(s_n^{k+1}) \in \partial B(x^k, \varepsilon)$, whih is a contradiction. This proves Claim 8.

Let $s \in \mathbb{R}_{<0}$. If n is sufficiently large, by Claim 8 we have

$$s_n^k + s' < s_n^{k+1} + s < s_n^{k+1}.$$

By definition of s_n^{k+1} (5.29) and $s < 0$ we have

$$u_n(s_n^{k+1} + s, \cdot) \in B(x^k; \varepsilon).$$

This holds for any $s < 0$. Therefore,

$$u_n(s_n^{k+1}, \cdot) \in B(x^k, \varepsilon),$$

which implies

$$u^{k+1}((-\infty, 0), \cdot) \subset \overline{B}(x^k, \varepsilon).$$

Note that for $s = 0$,

$$u_n(s_n^{k+1}, \cdot) \in \partial B(x^k, \varepsilon).$$

Taking the limit we have

$$u^{k+1}(0) \in \partial B(x^k, \varepsilon).$$

This means that u^{k+1} leaves the ball $B(x^k, \varepsilon)$ as required. This proves Theorem 5.37. $\square$

The proof of Theorem 5.36 is a straightforward consequence of Theorem 5.37.

Proof. Proof of Theorem 5.36 Let x, y be as in the hypothesis and et $(u_n) \subset \mathcal{M}(x, y)$ be a sequence. Then Theorem 5.37 ensures the existence of a subsequence with properties as in the hypothesis of this theorem. However, the index difference is 1, so that there can be no intermediate critical points ($\mathcal{M}(x, y) = \emptyset$ whenever $x \neq y$ and $\mathrm{CZ}(x) = \mathrm{CZ}(y)$) ensuring that the sequence (u_n) has a convergent subsequence. $\square$

We use this same idea to study the structure of the space $\widehat{\mathcal{M}}(x, z)$ whenever $\mathrm{CZ}(x) - \mathrm{CZ}(z) = 2$. In this case, a sequence $(u_n) \subset \widehat{\mathcal{M}}(x, z)$ either has a convergent subsequence or converges to some broken flow line in $\mathcal{M}(x, y) \times \mathcal{M}(y, z)$ for $\mathrm{CZ}(x) < \mathrm{CZ}(y) < \mathrm{CZ}(z)$. The idea is to "compactify" $\widehat{\mathcal{M}}(x, z)$ be adding the limits of sequences converging to broken trajectories. This is done in the next subsection.

5.5.2. The compactification of $\widehat{\mathcal{M}}(x, z)$ is compact 1-dimensional.

We add the broken trajectories to $\widehat{\mathcal{M}}(x, z)$. We have the following theorem about the resulting space

Theorem 5.38. *Suppose $(H, J) \in (\mathcal{H} \times \mathcal{J})_{reg}$. Let $x, z \in \mathcal{P}_0(H)$ such that*
$$\mathrm{CZ}(x) - \mathrm{CZ}(z) = 2.$$
Define the compactified space of trajectories by
$$\overline{\mathcal{M}}(x, z) := \underline{\mathcal{M}}(x, z) \cup \Big(\bigcup_{\substack{y \in \mathcal{P}_0(H) \\ \mathrm{CZ}(x) < \mathrm{CZ}(y) < \mathrm{CZ}(z)}} \underline{\mathcal{M}}(x, y) \times \underline{\mathcal{M}}(y, z) \Big) \tag{5.30}$$
Then $\overline{\mathcal{M}}(x, z)$ is a compact 1-dimensional manifold with boundary. The boundary is given by
$$\partial \overline{\mathcal{M}}(x, z) = \bigcup_{\substack{y \in \mathcal{P}_0(H) \\ \mathrm{CZ}(x) < \mathrm{CZ}(y) < \mathrm{CZ}(z)}} \underline{\mathcal{M}}(x, y) \times \underline{\mathcal{M}}(y, z).$$

We already know that the space $\overline{\mathcal{M}}(x, z)$ is compact, by Theorem 5.37. We also know that $\underline{\mathcal{M}}(x, z)$ is a manifold of dimension 1. Therefore, we only need to study what happens around the boundary points. Hence, Theorem 5.38 is a result of the following gluing statement.

Theorem 5.39. *Let $x, y, z \in \mathcal{P}(H)$ such that*
$$CZ(x) = CZ(y) + 1 = CZ(z) + 2.$$
Let $(\hat{u}, \hat{v}) \in \widehat{\mathcal{M}}(x, y) \times \widehat{\mathcal{M}}(y, z)$. Denote $\pi : \mathcal{M}(x, z) \to \widehat{\mathcal{M}}(x, z)$ the projection. Then for some $\rho_0 > 0$ there exists a differentiable map
$$\psi : [\rho_0, +\infty) \to \mathcal{M}(x, z)$$

such that for the induced map

$$\overline{\psi} = \pi \circ \psi$$

we have that

(i) *The map $\hat{\psi}$ is an embedding*

$$\hat{\psi} : [\rho_0, +\infty) \to \widehat{\mathcal{M}}(x, z)$$

such that

$$\lim_{\rho \to \infty} \hat{\psi}(\rho) = (\hat{u}, \hat{v}) \in \overline{\mathcal{M}}(x, z).$$

(ii) *Suppose $(\hat{w}_n) \subset \widehat{\mathcal{M}}(x, z)$ is a sequence that converges to $(\hat{u}, \hat{v})$. Then for n sufficiently large, there exist $\rho_n \in \mathbb{R}$ such that $\hat{\psi}(\rho_n) = \hat{w}_n$.*

Proof. The proof of Theorem 5.39 is too long to give a full proof here. The proof relies on using a parametrized version of the Floer equation to interpolate between two solutions u and v of the Floer equation using a parameter ρ. We then exponentiate along these interpolations w_ρ in order to find ψ. However, elliptic regularity is required to prove that such solutions indeed converge to true solutions of the Floer equation. We refer the reader to Sections 9.3,9.4,9.5 and 9.6 in [**AD14**] for a thourough proof. $\square$

Using Theorem 5.39, Theorem 5.38 can be proven. We now have the tools to prove that the Floer boundary operator as defined in Definition 5.5 has the required property $\partial_J \circ \partial_J = 0$, thereby proving that Floer homology is well defined. This fact is now a straightforward Corollary using Theorem 5.38.

Corollary 5.40. *Let $\partial_J : \mathrm{CF}_{k+1}(H) \to \mathrm{CF}_k(H)$. Then $\partial_J \circ \partial_J = 0$.*

Proof. We compute ∂_J^2 of some generator $x \in \mathrm{CF}_{k+1}(H)$. Recall that ∂_J was defined on the generators and extended by linearity. Some notation is ommited.

$$\begin{aligned}
\partial(\partial(x)) &= \partial\Big(\sum_{\mathrm{CZ}(y)=k} \eta(x,y)y \Big) \\
&= \sum_{\mathrm{CZ}(y)=k} \eta(x,y)\partial y \\
&= \sum_{\mathrm{CZ}(y)=k} \eta(x,y)\Big(\sum_{\mathrm{CZ}(z)=k-1} z \Big) \\
&= \sum_{\mathrm{CZ}(z)=k-1} \Big(\sum_{\mathrm{CZ}(y)=k} \eta(x,z)\eta(z,y) \Big) y
\end{aligned}$$

By definition, $\eta(a,b) = \#\widehat{\mathcal{M}}(a,b)$. Note that

$$\begin{aligned}\sum_{\mathrm{CZ}(y)=k} \eta(x,z)\eta(z,y) &= \sum_{\mathrm{CZ}(y)=k} \#\widehat{\mathcal{M}}(x,y) \times \widehat{\mathcal{M}}(y,z) \\ &= \#\Big(\bigcup_{\mathrm{CZ}(y)=k} \widehat{\mathcal{M}}(x,y) \times \widehat{\mathcal{M}}(y,z)\Big) \\ &= 0 \mod 2\end{aligned}$$

Here we used that $\partial\overline{\mathcal{M}}(x,z) = \bigcup_{\mathrm{CZ}(y)=k} \widehat{\mathcal{M}}(x,y) \times \widehat{\mathcal{M}}(y,z)$. Note that $\overline{\mathcal{M}}(x,z)$ is a compact 1-dimensional manifold. Therefore, $\partial\overline{\mathcal{M}}(x,z)$ contains an even number of points. We conclude that indeed $\partial_J^2 = 0$ which proves Corollary 5.40. $\square$

CHAPTER 6

$HF_*(M)$ is isomorphic to $HM_*(M)$

In this chapter we prove that there exists an isomorphism between Floer homology and Morse homology. This is the final step in proving the Arnold conjecture. This can be done in several ways. Here, we construct a particular Hamiltonian H, small in some sense, such that the Floer complex $\mathrm{CF}_*(H, J)$ and the Morse complex $\mathrm{CM}_*(H, J)$ agree. This is what we do here, so that we can disregard the invariance of Floer homology with respect to the choices made.

6.1. Invariance of Floer homology

In this section we give a brief summary of the proof of the following theorem.

Theorem 6.1. *Let* $(H, J), (H', J') \in (\mathcal{H} \times \mathcal{J})_{reg}$. *Then*

$$\mathrm{HF}_*(M; H, J) \simeq \mathrm{HF}_*(M; H', J').$$

Many of the techniques required to prove this theorem have already been used throughout the text. We give a brief summary of the proof.

Let $(H^\alpha, J^\alpha), (H^\beta, J^\beta) \in (\mathcal{H} \times \mathcal{J})_{\mathrm{reg}}$. Consider a smooth homotopy

$$\Gamma^{\alpha,\beta} : (H^\alpha, J^\alpha) \simeq (H^\beta, J^\beta)$$

by which we mean a map

$$\Gamma^{\alpha,\beta} : \mathbb{R} \to C^\infty(S^1 \times M, \mathbb{R}) \times \mathcal{J}(M, \omega)$$

such that

$$\Gamma^{\alpha,\beta}(s) = \begin{cases} (H^\alpha, J^\alpha) & s \leq -R \\ (H^\beta, J^\beta) & s \geq R \end{cases} \tag{6.1}$$

The main ingredient of the proof of Theorem 6.1 is associating to such homotopies a chain map that induces an isomorphism in homology.

Proposition 6.2. *Let* $\Gamma^{\alpha,\beta} = \Gamma : (H^\alpha, J^\alpha) \simeq (H^\beta, J^\beta)$ *as above. There exist a chain map*

$$\Phi^{\Gamma^{\alpha,\beta}} \mathrm{CF}(H^\alpha) \to \mathrm{CF}(H^\beta)$$

with the following properties.

(i) *Let* $(H^\alpha, J^\alpha) \in (\mathcal{H} \times \mathcal{J})_{reg}$. *For the constant homotopy* $\Gamma^{\alpha,\alpha}(s) = (H^\alpha, J^\alpha)$ *we have*

$$\Phi^{\Gamma^{\alpha,\alpha}} = \mathrm{Id}\,.$$

(ii) *Let* $(H^\alpha, J^\alpha), (H^\beta, J^\beta), (H^\gamma, J^\gamma) \in (\mathcal{H}\times\mathcal{J})_{reg}$, *and let* $\Gamma^{\alpha,\beta} : (H^\alpha, J^\alpha) \simeq (H^\beta, J^\beta)$, $\Gamma^{\beta,\gamma} : (H^\beta, J^\beta) \simeq (H^\gamma, J^\gamma)$ *and* $\Gamma^{\alpha,\gamma} : (H^\alpha, J^\alpha) \simeq (H^\gamma, J^\gamma)$. *Then*

$$\Phi^{\Gamma^{\alpha,\gamma}} \text{ is homotopic to} \Phi^{\Gamma^{\beta,\gamma}} \circ \Phi^{\Gamma^{\alpha,\beta}}.$$

Remark. By a chain map, we require that in every degree

$$\partial_{J^\beta} \circ \Phi^{\Gamma^{\alpha,\beta}} = \Phi^{\Gamma^{\alpha,\beta}} \circ \partial_{J^\alpha}.$$

This means that the chain maps induce maps in homology

$$\varphi^{\Gamma^{\alpha,\beta}} : \mathrm{HF}_*(H^\alpha, J^\alpha) \to \mathrm{HF}_*(H^\beta, J^\beta).$$

Then property (ii) implies that

$$\varphi^{\Gamma^{\alpha,\gamma}} = \varphi^{\Gamma^{\beta,\gamma}} \circ \varphi^{\Gamma^{\alpha,\beta}}.$$

From Proposition 6.2, it follows immediatly that $\varphi^{\Gamma^{\alpha,\beta}}$ depends only on the end points (H^α, J^α) and (H^β, J^β) and that $\varphi^{\Gamma^{\alpha,\beta}}$ is an isomorphism.

We will give a brief description of the proof. We will see that many techniques used in previous chapters make a reappearance, hence we can be concise. For more details we refer the reader to Chapter 11 in [**AD14**]

Let $\Gamma(s) = (H_s, J_s)$ be a given homotopy between some given regular (H^+, J^+) and (H^-, J^-). Associated to the s-dependent Hamiltonian is the Hamiltonian vector field $X_{s,t}$ given by

$$\iota_{X_{s,t}}\omega = dH_{s,t}.$$

We look at solutions of the parametrized Floer equation.

$$\frac{\partial u}{\partial s} + J_s(u)\Big(\frac{\partial u}{\partial t} - X_{s,t}(u)\Big) = 0 \tag{6.2}$$

The energy of a solution to equation (6.2) can be defined just as in Defintion 5.4, where we now use the metric

$$g_J = g_{J_s} := \omega(\cdot\,, J_s\,\cdot).$$

We look at moduli spaces for Γ defined by

$$\mathcal{M}^\Gamma := \{u \in C^\infty(S^1 \times \mathbb{R}, M) \mid u \text{ solves equation (6.2), is contractible and } E(u) < +\infty\}.$$

Let now $x^- \in \mathcal{P}_0(H^-)$ and $x^+ \in \mathcal{P}_0(H^+)$. We can consider the component of $\mathcal{M}^\Gamma$ of solutions running from x^- to x^+ defined by

$$\mathcal{M}^\Gamma(x^-, x^+) := \{u \in \mathcal{M}^\Gamma \mid \lim_{s\to\pm\infty} u(s,t) = x^\pm(t)\}.$$

We can describe these spaces as finite dimensional submanifolds of a $W^{1,p}$-Banach manifold in the same way as $\mathcal{M}(x^-, x^+; J, H)$.

Let
$$\overline{\partial}_\Gamma : \mathcal{B}^{1,p}(x^-, x^+) \to \mathcal{E}^p$$
be a section given by
$$\overline{\partial}_\Gamma(u) := \frac{\partial u}{\partial s} + J_s(u)\Big(\frac{\partial u}{\partial t} - X_{s,t}(u)\Big)$$
where the fiber over u is given by $\mathcal{E}^p_u := L^p(\mathbb{R} \times S^1, u^*TM)$ as before. We have the following analogue of Theorem 5.10.

Theorem 6.3. *Let $(H^+, J^+), (H^-, J^-) \in (\mathcal{H} \times \mathcal{J})_{reg}$ and $x^\pm \in \mathcal{P}_0(H^\pm)$. Let $\Gamma : (H^-, J^-) \simeq (H^+, J^+)$. Then at every zero $u \in \mathcal{M}^\Gamma(x^-, x^+)$, the vertical derivative $d^V\overline{\partial}_\Gamma(u)$ is a Fredholm operator of index $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+)$.*

The proof of Theorem 6.3 is very similar to that of Theorem 5.10. Recall that in order to prove Theorem 5.10, we transfered to a linear setting, in which the vertical derivative was a perturbed Cauchy-Riemann operator, with some peturbation S. We showed that the Fredholm index of this operator only depends on the assymptotic ends $S^\pm$. In this case, for s large, we are in this situation as well, as the homotopy Γ is constant for $|s| > R$. We refer the reader to Theorem 11.1.7 in [**AD14**].

If the vertical derivative $d^V\overline{\partial}_\Gamma(u)$ is surjective, we know by the implicit function theorem that $\overline{\partial}_\Gamma^{-1}(0) = \mathcal{M}^\Gamma(x^-, x^+)$ is a smooth submanifold of $\mathcal{B}^{1,p}(x^-, x^+)$ of dimension $\mathrm{CZ}(x^-) - \mathrm{CZ}(x^+)$. The following transversality result shows that for generic Γ, surjectivity is satisfied. It is the analogue of Theorem 5.27.

Consider $h : \mathbb{R} \times S^1 \times M \to \mathbb{R}$ with compact support such that $\|h\|_varepsilon < \infty$.

Theorem 6.4. *There exists a countable intersection of dense open subsets of $\mathcal{H}_{reg}$ inside a neighborhood of 0 in C^∞_ε, such that for $h \in \mathcal{H}_{reg}$ we have that for $\Gamma_h := (H + h, J)$, the vertical differential $d^V\overline{\partial}_{\Gamma_h}(u)$ is surjective for every $u \in \mathcal{M}^{\Gamma_h}(x^-, x^+)$ for every $x^\pm\mathcal{P}_0(H^\pm)$.*

Remark. Note that for every h, we have that Γ_h connects (H^-, J^-) and (H^+, J^+), as h has compact support.

The proof of this statement is easier than that of the analogous Theorem 5.27. We refer the reader to Theorem 11.1.6 in [**AD14**]. The proof again proceeds by looking at a section of $\mathcal{E}^p \to W^{1,}(\mathbb{R} \times S^1, u^*TM) \times C^\infty_\varepsilon(H_0)$. Again, we want to prove that if $\langle \hat{u}, \nabla h\rangle = 0$ for some $\hat{u} \in L^q(u^*TM)$, then $\hat{u} = 0$. Suppose that $\hat{u}$ is non-zero somewhere. We can now let h depend on s. Picking an h supported around this point immediatly proves the theorem.

By the above transversality result, the moduli spaces $\mathcal{M}^\Gamma(x^-, x^+)$ have a nice manifold structure for generic Γ, via the implicit function theorem.

We now sketch the definition of $\Phi^{\Gamma^{\alpha,\beta}}$. Let $x \in \mathcal{P}_0(H^\alpha)$. Then for $x \in \mathrm{CF}_k(H^\alpha)$, we have

$$\Phi_k^{\Gamma^{\alpha,\beta}}(x) = \bigoplus_{\substack{y \in \mathcal{P}_0(H^\beta) \\ \mathrm{CZ}(y)=k}} \eta^\Gamma(x,y)y$$

where

$$\eta^\Gamma(x,y) := \#\mathcal{M}^\Gamma(x,y).$$

Here, two important properties need to be verified.

First, we need to prove that η^Γ is well defined, meaning that $\mathcal{M}^\Gamma(x,y)$ is finite. This will again be done by proving it is a compact 0-dimensional manifold.

Secondly, we want $\Phi^{\Gamma^{\alpha,\beta}}$ to be a chain map, so that it induces a map in homology. This means we need to check that

$$\Phi^{\Gamma^{\alpha,\beta}} \circ \overline{\partial}_{J^\alpha} = \overline{\partial}_{J^\beta} \circ \Phi^{\Gamma^{\alpha,\beta}} \tag{6.3}$$

This will be done in a way similar to proving that for the Floer boundary we have $\partial_J^2 = 0$. We will describe terms that arise when computing equation (6.3) as boundary points of some 1-dimensional compact manifold.

Both these properties rely on compactness results for the spaces $\mathcal{M}^\Gamma(x,y)$. These are more difficult to prove than compactness results for trajectory spaces of the Floer equation $\widehat{\mathcal{M}}(x,y)$. Like in the unparametrized Floer case, the main tool is to compactify $\mathcal{M}^\Gamma(x,y)$ by adding the limits of sequences that converge to broken trajectories. This is similar to the construction of the compactified $\overline{\mathcal{M}}(x,y)$ of Theorem 5.38. However, the definition of a broken trajectory is more involved, as the intermediate critical points belong to $\mathcal{P}_0(H^\alpha)$ at one end and to $\mathcal{P}_0(H^\beta)$ at the other end. Roughly speaking, the trajectory consists of two broken trajectories; one inside $\mathcal{M}(H^\alpha, J^\alpha)$ that ends at some critical point x_k and one in $\mathcal{M}(H^\beta, J^\beta)$ that starts at some critical point y_0. These two are connected by a cylinder $w \in \mathcal{M}^\Gamma(x_k, y_0)$. We will only state the compactness result and the existence of the gluing map.

Theorem 6.5. *Suppose $(u_n) \subset \mathcal{M}^\Gamma(x,y)$ is a sequence.*
Then there exists a subsequence of (u_n) together with critical points $x = x_0, x_1, \ldots, x_k$ of $\mathcal{A}_{H^\alpha}$ and critical points $y_0, y_1, \ldots, y_l = y$ of $\mathcal{A}_{H^\beta}$.
These critical points are accompanied by sequences $(s_n^i)_n \subset \mathbb{R}$ for $0 \le i \le k-1$ such that $s_n^i \to -\infty$ and $(\sigma_n^j)_n \subset \mathbb{R}$ for $0 \le j \le l-1$ such that $\sigma_n^j \to +\infty$.
There exist elements $u^i \in \mathcal{M}(x_i, x_{i+1}; H^\alpha, J^\alpha)$ for $0 \le i \le k-1$ and elements $v^j \in$

$\mathcal{M}(y_j, y_{j+1}; H^\beta, J^\beta)$ for $0leqj \le l-1$ and a $w \in \mathcal{M}^\Gamma(x_k, y_0)$ such that for $0 \le i \le k-1$ and $0 \le j \le l-1$ we have

$$\lim_{n\to\infty} u_n \cdot s_n^i = u^i,$$

$$\lim_{n\to+\infty} u_n \cdot \sigma_n^j = v^j$$

and

$$\lim_{n\to\infty} u_n = w.$$

For a proof we refer to [**AD14**], Theorem 11.1.10. A key result in understanding $\eta^\Gamma(x,y)$ is the following lemma. It gives a bound on the number of broken flowlines between two critical points.

Lemma 6.6. *Let k and l be as in Theorem 6.5. Then*

$$\mathrm{CZ}(x) - \mathrm{CZ}(y) \ge k + l.$$

This is an immediate result from the manifold structure of $\mathcal{M}(x, y; H, J)$. From the conditions of Theorem 6.5, it is immediate that if $\mathrm{CZ}(x) = \mathrm{CZ}(y)$, then $\mathcal{M}^\Gamma(x,y)$ is compact of dimension 0 (by Theorem 6.4) hence $\eta^\Gamma(x,y)$ is well defined.

Furthermore, property (i) of Theorem 6.2 is also satisfied immediately. Let $k = \mathrm{CZ}(x) = \mathrm{CZ}(y)$ and $\Gamma = \mathrm{Id}$. Then

$$\mathcal{M}^\Gamma(x,y) = \mathcal{M}(x,y;H,J)$$

so that

$$\eta^\Gamma(x,y) = \begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases}.$$

Therefore, $\Phi_k^\Gamma = \mathrm{Id}$ for every k.

We still want to show that Φ^Γ is a chain map. To do this, we need to compactify the space $\mathcal{M}^\Gamma(x,z)$ where $CZ(x) - CZ(z) = 1$. This is the statement of the following theorem, which is an analogue of Theorem 5.38.

Theorem 6.7. *Let $x \in \mathcal{P}_0(H^\alpha)$ and $z \in \mathcal{P}_0(H^\beta)$ with $\mathrm{CZ}(x) - \mathrm{CZ}(y) = 1$. Then the space*

$$\overline{\mathcal{M}}^\Gamma(x,z) := \mathcal{M}^\Gamma(x,z) \cup \widetilde{\mathcal{M}}^\Gamma(x,z)$$

is a compact 1-dimensional manifold with boundary $\partial\overline{\mathcal{M}}^\Gamma(x,z) = \widetilde{\mathcal{M}}^\Gamma$ given by

$$\widetilde{\mathcal{M}}^\Gamma := \Bigg(\bigcup_{\substack{y \in \mathcal{P}_0(H^\alpha) \\ \mathrm{CZ}(x) - \mathrm{CZ}(y) = 1}} \widehat{\mathcal{M}}(x, y; H^\alpha, J^\alpha) \times \mathcal{M}^\Gamma(y,z) \Bigg) \cup \Bigg(\bigcup_{\substack{y' \in \mathcal{P}_0(H^\beta) \\ \mathrm{CZ}(y') = \mathrm{CZ}(x)}} \mathcal{M}^\Gamma(x, y') \times \widehat{\mathcal{M}}(y', z; H^\beta, J^\beta) \Bigg).$$

This is Theorem 11.1.15 in [**AD14**]. Now, one can check that Φ^Γ is a chain map, as by the above Theorem 6.7, we have

$$\sum_{\substack{y\in\mathcal{P}_0(H^\alpha)\\ \mathrm{CZ}(x)-\mathrm{CZ}(y)=1}} \eta(x,y)\eta^\Gamma(y,z) = \sum_{\substack{y'\in\mathcal{P}_0(H^\beta)\\ \mathrm{CZ}(x)=\mathrm{CZ}(y')}} \eta^\Gamma(x,y')\eta(y,z)$$

is satisfied modulo 2. These terms arise when computing equation (6.3). However, these are precisely terms that count the boundary components of $\overline{\mathcal{M}}^\Gamma(x,z)$ which is a compact 1-dimensional manifold, by Theorem 6.7, which has an even number of boundary points.

Hence, the main point is to prove Theorem 6.7. This is done by the existence of a gluing map, as we already know the components of $\overline{\mathcal{M}}^\Gamma(x,z)$ are manifolds of the right dimension. This is an analogue of Theorem 5.39.

Theorem 6.8. *Let $x\in\mathcal{P}_0(H^\alpha)$ and $y,z\in\mathcal{P}_0(H^\beta)$ such that*

$$\mathrm{CZ}(x)=\mathrm{CZ}(y)=\mathrm{CZ}(z)+1.$$

Let $u\in\mathcal{M}^\Gamma(x,y)$ and $\hat{v}\in\widehat{\mathcal{M}}(y,z;H^\beta,J^\beta)$. Then, there exists for $\rho_0>0$ an embedding

$$\psi:[\rho_0,+\infty)\to\mathcal{M}^\Gamma(x,z)$$

such that

$$\lim_{\rho\to\infty}\psi(\rho)=(u,\hat{v})$$

and if $(w_n)_n\subset\mathcal{M}^\Gamma(x,z)$ is a sequence that tends to $(u,\hat{v})$, then there exist ρ_n such that $w_n=\psi(\rho_n)$ for n sufficiently large.

Note that by $\varphi(\rho)$ converging to $(u,\hat{v})$ we mean that it converges to the broken flow line as in Theorem 6.5. In this way, the gluing statement Theorem 6.8 shows that the compactified space $\overline{\mathcal{M}}^\Gamma(x,z)$ is a 1-dimensional compact manifold with the right boundary, as claimed in Theorem 6.7.

This concludes the sketch of the proof of Theorem 6.1. We have defined the required chain maps that induce isomorphisms in homology, and shown that they are well-defined.

6.2. Small Hamiltonians

We have defined what the chain groups $\mathrm{CF}_*(H,J)$ are. Here, we denote $\mathrm{CM}_*(H,J)$ the Morse complex associated with the Morse function H and the vector field $\nabla_J H$ defined with respect to the metric g_J. The goal of this section is to prove the following theorem.

Theorem 6.9. *Let (M,ω) be a compact symplectic manifold satisfying Assumptions 3.3 and 3.4. There exists a non-degenerate Hamiltonian $H:M\to\mathbb{R}$ for which we can find a $J\in\mathcal{J}(M,\omega)$ such that both the Morse and Floer complexes exists and*

$$CF_*(H,J)=CM_{*+n}(H,J).$$

By Section 6.1, we know that if the above choice is such that $(H, J) \in (\mathcal{H} \times \mathcal{J})_{\text{reg}}$ then we can prove the Arnold conjecture. For all regular pairs (H, J), the Floer homology groups are isomorphic, so it is sufficient to prove the Arnold conjecture for this particular choice, which is straightforward using Theorem 6.9. We will say a few words about the proof of this theorem.

We accomplish this by starting with a non-degenerate Hamiltonian H_0. Note that we have two notions of non-degeneracy here. On the Floer side, H_0 is non-degenerate when its periodic trajectories are non-degenerate. On the Morse side, H_0 is non-degenerate if all of its critical points are non-degenerate. In this case H_0 is a Morse function. In order to prove Theorem 6.9 our aim is to find H with the following properties.

(i) $\text{Crit}(\mathcal{A}_H) = \text{Crit}(H)$

(ii) H is nondegenerate both in the Floer and Morse sense.

(iii) For $x \in \text{Crit}(H)$, we have the index formula $\text{ind}_H(x) = \mu_{CZ}(x) + n$.

(iv) The differentials of both complexes exist and coincide.

We start with H_0 a non-degenerate Hamiltonian in the Floer sense, and consider

$$H = H_0/k$$

for k sufficiently large. The first three items (i),(ii) and (ii) are relatively easy. Item (iv) is the hard part. Hence, for the first three items we refer the reader to page 360 in [**AD14**], which refers to the relevant results. These are Proposition 6.1.5, Proposition 5.4.5 and Proposition 7.2.1 in [**AD14**]. The first three items imply that as vector spaces, $\text{CF}_k(H, J)$ and $\text{CM}_{k+n}(H, J)$ are indeed the same. We will say a bit more about item (iv).

To define the trajectories used in Morse theory, we need a Morse function H together with a vector field X satisfying the Morse-Smale condition. We therefore want X to be the gradient of H with respect to the metric g_J for $J \in \mathcal{J}(M, \omega)$. Hence, we need to pick a J such that this is satisfied. We need the following transversality result, which is Theorem 10.1.2 in [**AD14**].

Theorem 6.10. *Let H be a Morse function on a symplectic manifold M. There exists a dense subset $\mathcal{J}_{reg}(H) \subset \mathcal{J}(M, \omega)$ of almost complex structures such that the pair $(H, -JX_H)$ is Morse-Smale (i.e. H is a Morse function and $-JX_H$ is a gradient-like vector field satisfying the Smale condition).*

Using Theorem 6.10, we choose a J such that $(H, -JX_H)$ is Morse-Smale such that we can define the Morse complex, as in the Appendix Section A.1. We compare solutions to the Floer equation to trajectories of the vector field $-JX_H$.

Note that if u solves the Floer equation, but does not depend on t, then u is a trajectory of X_H.

Let $H_k = H/k$ with k sufficiently large. Consider the following proposition.

Proposition 6.11. *For k sufficiently large, then any $u \in C^\infty(\mathbb{R} \times S^1, M)$ such that $\overline{\partial}_{J,H_k}(u) = 0$ connecting two critical points x and y such that*

$$\mathrm{ind}_{H_k}(x) - \mathrm{ind}_{H_k}(y) \leq 2$$

is independent of t.

By the above Proposition 6.11, trajectories of the Floer equation associated to (H, J) connecting x and y coincide with the trajectories of the vector field $-JX_H$ which was a Smale vector field (by choice of J). We need to verify that the spaces $\mathcal{M}(x, y; H, J)$ can still be constructed as before. This is done by the following Proposition, which is Corollary 10.1.8 in [**AD14**].

Proposition 6.12. *Suppose H is a sufficiently small Hamiltonian. Then the Fredholm operator $d^V\overline{\partial}_{J,H}(u)$ is surjective along every trajectory of $-JX_H$.*

Therefore, the spaces $\mathcal{M}(x, y; H, J)$ are manifolds by the implicit function theorem as before. In this way we can define the Floer complex $\mathrm{CF}_*(H, J)$. The above two propositions then show that the differentials of both complexes agree, hence Theorem 6.9 holds.

6.3. The Arnold conjecture

We can now prove the Arnold conjecture. In Section 4.9 of [**AD14**], it is proven that the Morse homology of a manifold coincides with the cellular homology in Theorem 4.9.3. We will exploit this now. The cellular homology of the manifold is isomorphic to the singular homology $\mathrm{H}_*(M; \mathbb{Z}_2)$.

Proof of Theorem 1.5. Suppose M is as in the hypothesis of the Arnold conjecture. This means M is closed, symplectically aspherical and with vanishing first Chern class. We have for a regular pair (H, J) there is a well defined Floer complex $CF_*(H, J)$ and associated to it the Floer homology $HF_*(H, J)$. Recall that $CF_k(H, J; \mathbb{Z}_2)$ is generated by the periodic orbits of the Hamiltonian equation with coefficients in $\mathbb{Z}_2$. Hence,

$$\#\mathcal{P}_0(H) = \sum_{k=0}^{2n} \dim \mathrm{CF}_k(H, J).$$

It follows that

$$\#\mathcal{P}_0(H) \geq \sum_{k=0}^{2n} \dim \mathrm{HF}_k(H, J; \mathbb{Z}_2).$$

By Theorem 6.9 and invariance of Floer homology, we have

$$\#\mathcal{P}_0(H) \geq \sum_{k=0}^{2n} \dim H_k(M; \mathbb{Z}_2).$$

This proves Theorem 1.5, the Arnold conjecture. $\square$

CHAPTER 7

Rabinowitz-Floer homology

In this chapter we will describe a flavour of Floer homology defined relatively recently. The Rabinowitz-Floer homology is a Floer homology associated to a hypersurface of contact type in an exact symplectic manifold.

We will first give an overview of the contact topology required to define Rabinowitz-Floer homology.

After this, we give the definition of the Rabinowitz-Floer homology groups $\mathrm{RFH}_*(M, \Sigma)$. We sketch the definition given in [**CF09**] and point out the central points in the definition.

Secondly, we consider a perturbation of the Rabinowitz action functional. We show that it gives rise to leaf-wise fixed points, which are a generalization of the fixed points encountered in Hamiltonian Floer homology as described in the main body of the thesis. We give the proof of an existence result of leaf-wise fixed points due to P. Albers and U. Frauenfelder in [**AF**].

In the final section, we describe another application of Rabinowitz-Floer homology. We show that non-vanishing of the Rabinowitz-Floer homology of a manifold implies that it is orderable. Orderability means that there is a partial order on $\widetilde{\mathrm{Cont}_0(\Sigma, \xi)}$, the universal cover of the group of contactomorphisms of a contact manifold (Σ, ξ). This result is due to P. Albers and W. J. Merry in [**AM14**].

The notation and conditions on our manifolds will vary throughout this chapter, as every paper requires its own conditions. We will try to make clear what the set-up is in each separate case.

7.1. Preliminaries on Contact Topology

Rabinowitz-Floer homology is defined for a coisotropic submanifold of contact type. We follow the exposition in [**Gei08**] to introduce contact manifolds.
Let M be a differential manifold of dimension $2n+1$.

Definition 7.1. A contact structure on M is a maximally non-integrable hyperplane field $\xi \subset TM$.

Recall that non-integrable has two interpretations. Geometrically, it means that there is no hypersurface in M that is tangent to ξ along an open subset of the hypersurface.

Non-integrable can also be described in terms of 1-forms. For every neighborhood $U \subset M$ there is a form $\alpha \in \Omega^1(U)$ such that $\xi|_U = \ker \alpha$. We refer the reader to [**Gei08**] Lemma 1.1.1. The hyperplane field ξ is non-integrable if for every locally defining 1-form α we have

$$\alpha \wedge (d\alpha)^n \neq 0.$$

Remark. There exists a global $\alpha \in \Omega(M)$ such that $\xi = \ker \alpha$ if and only if ξ is coorientable. Recall that coorientability means that the bundle $(TM/\xi)^*$ admits a global section. Pulling this section back to T^*M via the projection $\pi : TM \to TM/\xi$ yields the form α. Conversely, α such that $\ker \alpha = \xi$ yields a non-zero section of TM/ξ.
Whenever we assume ξ to be coorientable we call this global α the contact form. Note that this is slight abuse of language, as any α' such that $\alpha = f\alpha'$ with $f : M \to \mathbb{R} \setminus \{0\}$ has the property that $\xi = \ker \alpha'$

When defining these structures it is natural to give a name to maps that preserve such structures.

Definition 7.2. Suppose (M_1, ξ_1) and (M_2, ξ_2) are two contact manifolds. A contactomorphism φ is a diffeomorphism $\varphi : M_1 \to M_2$ such that for $d\varphi : TM_1 \to TM_2$ we have $d\varphi(\xi_1) = \xi_2$. If a contactomorphism $\varphi : M_1 \to M_2$ exists, M_1 and M_2 are said to be contactomorphic.

Remark. In the case that (M_1, ξ_1) and (M_2, ξ_2) are coorientable with contact forms $\xi_i = \ker \alpha_i$ for $i = 1, 2$ we can rephrase what it means to be contactomorphic in terms of the contact forms. M_1 and M_2 are contactomorphic if and only if there exists a nowhere vanishing function $f : M_1 \to \mathbb{R} \setminus \{0\}$ such that

$$\varphi^* \alpha_2 = f \alpha_1.$$

Note that this is sufficient, as it is sufficient for $\varphi^*\alpha_2$ and α_2 to define the same hyperplane field, so they differ a nonwhere vanishing function at most. In the special case that $f \equiv 1$, φ is sometimes called a strict contactomorphism.

When considering a contact manifold (M, ξ) the contactomorphisms $\varphi : M \to M$ form a group when equipped with the composition of functions, which we denote

$$\mathrm{Cont}(M, \xi)$$

Associated to a contact form α is the Reeb vector field.

Definition 7.3. Let α be a contact form on M. The Reeb vector field R_α is the unique vector field defined by

(i) $\iota_{R_\alpha} d\alpha = 0$

(ii) $\alpha(R_\alpha) = 1$

REMARK. The fact that the Reeb vector field defined as such is well-defined and unique follows from linear algebra. Condition (i) determines the direction of R_α while condition (ii) provides a normalization. We refer the reader to Lemma/Definition 1.1.9 in [**Gei08**].

Suppose that (W, ω) is a symplectic manifold.

Definition 7.4. A Liouville vector field Y on (W, ω) is a vector field satisfying the equation

$$\mathcal{L}_Y \omega = \omega$$

Such a Liouville vector field connects contact geometry and symplectic geometry in the following way.

Lemma 7.5. *Suppose (W, ω) is a symplectic manifold and let Y be a Liouville vector field. Define $\alpha \in \Omega^1(W)$ by*

$$\alpha := \iota_Y \omega.$$

Then the form α is a contact form on any hypersurface $\Sigma \subset W$ transverse to Y. A hypersurface with this property is called a hypersurface of contact type.

PROOF. The proof follows from the Cartan formula, $d\omega = 0$ and the fact that ω^n is a volume form when restricted to the tangent bundle of Σ. We refer the reader to Lemma/Definition 1.4.5 in [**Gei08**]. $\square$

This yields a particular class of hypersurfaces inside a symplectic manifold that we will study in the upcoming sections. Another required concept relates Hamiltonians on a contact manifold to vector fields fixing the contact hyperplanes.

Definition 7.6. Let (M, ξ) be a cooriented contact manifold with associated contact form α and let X be a vector field on M. Denote φ_X^t the flow of X [1]. Then X is called a contact vector field if

$$d\varphi_X^t(\xi) = \xi \text{ for all } t \in \mathbb{R}.$$

If the contact form is preserved, i.e. $(\varphi_X^t)^* \alpha = \alpha$ for all $t \in \mathbb{R}$, then X is called a strict contact vector field.

[1] Note that the domain may not be the full manifold.

Remark. The above definition can also be characterized in terms of the Lie derivative. A vector field X is a contact vector field of ξ if and only if $\mathcal{L}_X\alpha = \lambda\alpha$ for some function $\lambda : M \to \mathbb{R}$. This condition is independent of the choice of contact form α.

The following lemma describes a relation between Hamiltonians and contact vector fields.

Lemma 7.7. *Let $(M, \xi = \ker\alpha)$ be a contact manifold. For a fixed α there is a bijective correspondence between contact vector fields X on M and Hamiltonians $H : M \to \mathbb{R}$. This correspondence*

$$\{\textit{contact vector fields } X \textit{ on } M\ \} \longleftrightarrow \{\textit{Hamiltonians } H : M \to \mathbb{R}\}$$

is given by the two maps

$$X \longmapsto H_X := \alpha(X)$$

and

$$H \longmapsto X_H$$

where X_H is defined uniquely by $\alpha(X_H) = H$ and $\iota_{X_H} d\alpha = dH(R_\alpha)\alpha - dH$

Using this lemma, we can speak of contact Hamiltonians as Hamiltonians that are associated to a contact vector field. That is, a Hamiltonian H with the property that there exists a contact vector field X such that $H = H_X = \alpha(X)$. This means that the above gives a bijection between the set of contact vector fields and the set of contact Hamiltonians. Note that X_H defined as such is unique as $d\alpha$ is non-degenerate on ξ, whereas $R_\alpha \in \ker(dH(R_\alpha)\alpha - dH)$.

Lemma 7.7 is useful in the following setting. Let (M, α) be a closed contact manifold, and $H : [0,1] \times M \to \mathbb{R}$ a time-dependent family of Hamiltonians. We have an associated time-dependent contact vector field X_t. Then the flow of X_t, denoted φ_t is globally defined (by closedness) and is such that

$$\varphi_t^*\alpha = \lambda_t\alpha$$

for $\lambda : [0,1] \times M \to \mathbb{R}_+$. Hence, we can construct from a Hamiltonian a contact isotopy.

7.2. The Rabinowitz action functional and Rabinowitz-Floer homology

We define the Rabinowitz action functional. The Floer homology associated to this functional gives rise to Rabinowitz-Floer homology. We follow the exposition in [**CF09**] where the Rabinowitz-Floer homology was first defined. We first describe the setting.

Let (V, λ) be an exact convex symplectic manifold that is connected and without boundary. By exactness we mean that the two form $\omega := d\lambda$ is symplectic. By convex we mean the strong version of convex (at infinity) as outlined in [**MS12**]. This is Definition 9.2.6 combined with Remark 9.2.7.

Definition 7.8. Let (M, ω) be a symplectic manifold. Then it is called convex (at infinity) if there exists a pair (f, J) with $J \in \mathcal{J}(M, \omega)$ and $f : M \to [0, \infty)$ a smooth proper function such that for

$$\omega_f := -d(df \circ J)$$

we have that $\omega = \omega_f$ globally.

Note that this is the stronger definition of Remark 9.2.7. In [**CF09**], convexity is defined differently. Here (V, ω) is convex if there exists an exhaustion $\bigcup_k V_k = V$ of compact sets $V_k \subset V_{k+1}$ such that $\lambda|_{\partial V_k}$ is a contact form. We will however adhere to Definition 7.8. This Definition implies the one in [**CF09**]. To see this, we refer to the computation in [**FS03**] on pages 2 and 3. Using f, we can construct an exhaustion as the inverse image of an exhaustion of compact sets in $[0, \infty)$ (which are compact as f is proper), if we avoid singular values, which has the required properties. This is the case as the gradient of f is a Liouville vector field that defines the same contact structure as λ on the boundaries (see Exercise 9.2.9 (ii) in [**MS12**]).

Define a vector field Y_λ by $\iota_{Y_\lambda}\omega = \lambda$. We say (V, λ) is complete if Y_λ is complete. We say that (V, λ) has bounded topology if $Y_\lambda \neq 0$ outside some compact set.

Remark. It is important to note that (V, λ) is complete and of bounded topology if and only if "it looks like a contact manifold with a cylindrical end attached". To be precise, there exists and embedding $\varphi : M \times \mathbb{R}_+ \to V$ such that $\varphi^*\lambda = e^r \alpha_M$ with contact form $\alpha_M = \varphi^*\lambda|_{M \times \{0\}}$ such that $V \setminus \varphi(M \times \mathbb{R}_+)$ is compact.

We will consider a hypersurface (without boundary) inside (V, λ). We say $\Sigma \subset V$ is exact convex if there exists a contact form on α such that $\alpha - \lambda|_\Sigma$ is exact and $V \setminus \Sigma$ consists of one compact and one noncompact component. Note that we can modify λ to $\widetilde{\lambda} = \lambda + dh$ such that $\omega = d\widetilde{\lambda}$ and $\alpha = \widetilde{\lambda}|_\Sigma$. Hence, we can consider the following general setting.

From now on, let (V, λ) be a complete exact convex symplectic manifold with bounded topology. Suppose $\Sigma \subset V$ is an exact convex hypersurface with contact form α so that $\lambda|_\Sigma = \alpha$.

Remark. By Lemma 1.4 in [**CF09**], the hypothesis that V be complete and of bounded topology are superfluous as this can always be arranged. However, we mention these conditions as they are needed to define the Rabinowitz-Floer homology.

Let $F : V \to \mathbb{R}$ be a smooth time-independent Hamiltonian with associated Hamiltonian vector field X_F by $\iota_{X_F}\omega = -dF$.

Recall the definition of the loop space $\mathcal{L}M := C^\infty(S^1, M)$.

Definition 7.9. The Rabinowitz action functional is defined as

$$\mathcal{A}^F : \mathcal{L}M \times \mathbb{R} \longrightarrow \mathbb{R}$$

by

$$(x, \eta) \longmapsto \mathcal{A}^F(x, \eta) := \int_{S^1} x^*\lambda - \eta \int_0^1 F(x(t))\, dt$$

Similarly to Floer homology, we look at critical points of the functional $\mathcal{A}^F$. A computation shows the following.

Proposition 7.10. *Let $(x, \eta) \in \operatorname{Crit} \mathcal{A}^F$. Then (x, η) satisfies*

$$\dot{x}(t) = \eta X_F(x(t)), \forall t \in S^1 \tag{7.1}$$

and

$$\int_0^1 F(x(t))\, dt = 0 \tag{7.2}$$

Proof. To get the first equation (7.1), we compute the differential of $\mathcal{A}^F$ analogous to the computation as in the proof of Proposition 3.7. The second equation (7.2 arises from the variation in η, which is just linear in the term $\eta \int_0^1 F(x(t))\, dt$. This yields equation (7.2. □

An immediate corollary is the following.

Corollary 7.11. *Let $(x, \eta) \in \operatorname{Crit} \mathcal{A}^F$. Then (x, η) satisfies*

$$\dot{x}(t) = \eta X_F(x(t)), \forall t \in S^1 \tag{7.3}$$

and

$$F(x(t)) = 0, \forall t \in S^1 \tag{7.4}$$

Proof. We have equation (7.1) which implies that $x(t)$ is the reparametrized flow of the vector field X_F. Denote the flow by φ_F^t. Then equation (7.1) implies that

$$x(t) = \varphi_F^{\eta t}(x(0)).$$

By definition of the Hamiltonian vector field X_F, its flow $\varphi_F^t : M \to M$ leaves level sets of F invariant. Therefore equation (7.2) implies that

$$F(x(t)) = 0, \forall t \in S^1.$$

This proves Corollary 7.11. □

Hence, critical points of the Rabinowitz action function $\mathcal{A}^F$ are periodic orbits of the Hamiltonian vector field X_F with period η that lie on the hypersurface $\Sigma := F^{-1}(0)$.

Consider a Hamiltonian $F : W \to \mathbb{R}$ such that $\Sigma = F^{-1}(0)$ such that X_F has compact support and agrees with $R = R_\alpha$, the Reeb flow of α, on Σ.

Consider $\mathcal{A}^F$ for such a Hamiltonian. Then critical points are precisely Reeb orbits

$$\dot{x}(t) = \eta R(x(t)).$$

We will give a consise explanation of the following Theorem, which is Theorem 1.1 in [**CF09**].

Theorem 7.12. *Let (V, λ) with hypersurface (Σ, α) be as above. Let F define Σ as above. Then we can define the Floer homology of $\mathcal{A}^F$ denoted* $\mathrm{RFH}(\mathcal{A}^F)$*. The Floer homology* $\mathrm{RFH}(\mathcal{A}^F)$ *is independent of the choice of F used to define Σ, so that we can speak of the Rabinowitz-Floer homology* $\mathrm{RFH}(\Sigma, V)$*.*

We will look into some details of the proof of the above theorem. Most important are the conditions on Σ and V, as this ensures that flow lines of $\mathcal{A}^F$ will be compact modulo breaking, so that the definition of the Floer homology is standard (i.e. as in the Hamiltonian Floer homology in the main body of the thesis). The only difference is that $\mathcal{A}^F$ is Morse-Bott so that we define the homology using flow lines with cascades. This requires a non-degeneracy assumption on the Reeb orbits of the contact manifold (Σ, α).

Assumption 7.13. Let $T \in \mathbb{R}$ and denote θ_t the Reeb-flow of R_α. Then the set of closed T-periodic Reeb orbits $\mathcal{R}_T \subset \Sigma$ is a closed submanifold with the rank of $d\alpha$ on $\mathcal{R}_T$ is locally constant and $T_p\mathcal{R}_T = \ker(T_p\theta_T - \mathrm{Id})$ for all $p \in \mathcal{R}_T$.

This assumption is generically satisfied, meaning we may assume $\mathcal{A}^F$ to be Morse-Bott. See page 11 of [**CF09**] for more details.

Consider the metric g_J on $\mathcal{L}V \times \mathbb{R}$ given by

$$g_{(x,\eta)}((\hat{x}_1, \hat{\eta}_1), (\hat{x}_2, \hat{\eta}_2)) := \int_0^1 \omega(\hat{x}_1(t), J_t(x(t))\hat{x}_2(t))\, dt + \hat{\eta}_1\hat{\eta}_2 \tag{7.5}$$

for $(x, \eta) \in \mathcal{L}V \times \mathbb{R}$ and $(\hat{x}_1, \hat{\eta}_1), (\hat{x}_2, \hat{\eta}_2) \in T_{(x,\eta)}(\mathcal{L}V \times \mathbb{R})$. The gradient of $\mathcal{A}^F$ with respect to this metric gives rise to the following PDE for the gradient flow lines. They are maps $(x, \eta) \in C^\infty(\mathbb{R} \times S^1, V) \times C^\infty(\mathbb{R}, \mathbb{R})$ that solve

$$\begin{cases} \dfrac{\partial x}{\partial s} + J_t(x)\Big(\dfrac{\partial x}{\partial t} - \eta X_F(x)\Big) = 0 \\ \dfrac{\partial \eta}{\partial s} + \displaystyle\int_0^1 H(x(t))\, dt = 0 \end{cases} \tag{7.6}$$

In the Floer case, to prove that the moduli spaces of flow lines were compact up to breaking, we showed bounds on solutions u of the gradient flow equation. The same can be done here. Note that in the Floer case, we assumed our manifold to be compact. Here, V is generally not compact, so we need to additional condition that V is convex to find these bounds. Furthermore, we need bounds on derivatives of u. Here we use that V is exact so that no bubbling occurs. However, we also need a bound on $\eta \in \mathbb{R}$. This a central Proposition in [**CF09**].

Proposition 7.14. *There exists $\varepsilon > 0$ such that for every $M > 0$ there exists a constant c_M such that if*

$$\|\nabla_g \mathcal{A}^H(x,\eta)\|_J \leq \varepsilon$$

and

$$|\mathcal{A}^H(x,\eta)| \leq M,$$

then

$$|\eta| \leq c_M.$$

This is a Proposition that should be regarded as an analogue of Theorem 5.35 for η. A direct consequence of this Proposition is that the L^∞-norm of η is uniformely bounded. This is Corollary 3.3 in [**CF09**].

We define the Rabinowitz-Floer homology as the Floer homology of $\mathcal{A}F$ in the following way. Assume that $\mathcal{A}^F$ is Morse-Bott and let $h : \mathrm{Crit}(\mathcal{A}^F) \to \mathbb{R}$ be a Morse function. Then let

$$\mathrm{CF}(\mathcal{A}^F, h) = \bigoplus_{c \in \mathrm{Crit}(h)} \xi_c c$$

with coefficients $\xi_c \in \mathbb{Z}_2$ such that

$$\#\{c \in \mathrm{Crit}(h) \mid \xi_c \neq 0, \mathcal{A}^F(c) \leq \kappa\} < +\infty$$

for all $\kappa \in \mathbb{R}$.

The boundary operator is defined using J_t. We require some conditions so that cascades remain inside some compact subset of V. These are three conditions. We refer the reader to page 17 of [**CF09**]. They are phrased in terms of the symplectically embedded $\Sigma \times \mathbb{R}_+$ which was possible as V is convex and of bounded topology.

Choosing a Riemannian metric g_c on $\mathrm{Crit}(\mathcal{A}^F)$ we can consider for $c^-, c^+ \in \mathrm{Crit}(h)$ the moduli space of gradient flow lines with cascades running from c^- to c^+ denoted

$$\mathcal{M}_{c^-,c^+}(\mathcal{A}^F, h, J, g_c).$$

We refer the reader to Appendix A.2 for details. Here, the choice of J together with the convexity at ∞ guarantees compactness up to breaking. By exactness of ω, no bubbling will occur.

For generic choices in J and g_c, this is a smooth manifold with a compact 0-dimensional component $\mathcal{M}^0_{c^-,c^+}(\mathcal{A}^F, h, J, g_c)$. Therefore, we can set

$$\eta(c^-, c^+) := \#\mathcal{M}^0_{c^-,c^+}(\mathcal{A}^F, h, J, g_c) \mod 2.$$

Then we can define the boundary operator $\partial : \mathrm{CF}(\mathcal{A}^H, h) \to \mathrm{CF}(\mathcal{A}^H, h)$ by setting

$$\partial c = \sum_{c' \in \mathrm{Crit}(h)} \eta(c, c')c'.$$

By compactness modulo breaking, we have $\partial^2 = 0$. Then define $\mathrm{RFH}_*(\mathcal{A}^F, h, J, g_c) := H_*(\mathrm{CF}(\mathcal{A}^F, h), \partial)$. One can prove that this homology is independent of the choice of particular h, J and g_c. Furthermore, $\mathrm{HF}(\mathcal{A}^F)$ is independent of the particular choice of H used to define it (for a hypersurface Σ). Hence we can speak of the Rabinowitz-Floer homology

$$\mathrm{RFH}(\Sigma, V)$$

.

Remark. In [**CFO09**], it is proven that the above definition can also be made by considering the Floer homology of $\mathcal{A}^F$ for $\mathcal{A}^F(x, \eta) \in (a, b)$. Then one defines truncated Rabinowitz-Floer homology groups

$$\mathrm{RFH}^{(a,b)}(\Sigma, V) := \mathrm{HF}^{(a,b)}(\mathcal{A}^F).$$

The full Rabinowitz-Floer homology is then defined by taking the limits

$$\mathrm{RFH}_*(\Sigma, V) := \varinjlim_b \varprojlim_a \mathrm{RFH}_*^{(-a,b)}(\Sigma, V).$$

It is proven here that these two definitions coincide. The definition in this Remark is the one used in [**AM14**]. We will discuss the results of this paper when discussing orderability.

7.3. The peturbed Rabinowitz-Floer action functional and leaf-wise fixed points

We first discuss the definition of a leaf-wise fixed point in the general setting of a coisotropic manifold inside a symplectic manifold. We then prove an existence theorem for these leaf-wise fixed points using a perturbed version of Rabinowitz-Floer homology.

Let (M, ω) be a symplectic manifold and $N \subseteq M$ be a submanifold. Recall that N is called coisotropic if

$$(T_xN)^\omega \subseteq T_xN \text{ for all } x \in N.$$

as in 2.7. We have an inclusion map

$$\iota_N : N \to M.$$

In this case $(TN)^{\iota_N^*\omega}$ is a subbundle of TM of rank $\dim N - \operatorname{corank} \iota_N^*\omega$. Note that $d\iota_N^*\omega = \iota_N^* d\omega = 0$, hence the distribution is involutive. By the Frobenius theorem, this gives a foliation of N. We refer the reader to any standard book on differentiable manifolds, for example [**War00**] Theorem 1.60. We call these integral manifolds the leaves of the foliation and denote

$$N_x^\omega := \{\text{The isotropic leaf in } N \text{ passing through } x\}.$$

Denote the collection of leaves by

$$N^\omega.$$

We define what it means for a point $x \in N$ to be a leaf-wise fixed point.

Definition 7.15. Let (M, ω) be a symplectic manifold and $N \subseteq M$ a coisotropic submanifold. Let $\varphi : M \to M$ be a map. We call $x \in N$ a leaf-wise fixed point for φ with respect to N if and only if

$$\varphi(x) \in N_x.$$

We denote the set of all leaf-wise fixed points of φ with respect to N by

$$\operatorname{Fix}(\varphi, N) := \{x \in N \mid \varphi(x) \in N_x\}.$$

In the case where (M, ω) is a symplectic manifold with some coisotropic submanifold $N \subseteq M$, we can define the minimal symplectic action

$$A(M, \omega, N) := \inf\left(\left\{ \int_{\mathbb{D}} u^*\omega \mid u \in C^\infty(\mathbb{D}, M), \exists F \in N^\omega \text{ such that } u(S^1) \subseteq F\right\} \cap (0, \infty)\right)$$

We will consider fixed points of Hamiltonian diffeomorphisms $\varphi \in \operatorname{Ham}_c(M, \omega)$, where $\operatorname{Ham}_c(M, \omega) \subset \operatorname{Ham}(M, \omega)$ denotes the subgroup of compactly supported Hamiltonian diffeomorphisms. On $\operatorname{Ham}(M, \omega)$ a norm can be defined, called the Hofer norm. Recall the set $\mathcal{H}(M, \omega)$ which contains all $H \in C^\infty([0, 1] \times M, \mathbb{R})$ such that the Hamiltonian flow $\varphi_H^t : M \to M$ exists and is surjective for every $t \in [0, 1]$.
Let $\varphi \in \operatorname{Ham}(M, \omega)$. Define

$$\|H\| := \int_0^1 \Big(\sup_M H^t - \inf_M H^t\Big)\, dt.$$

The Hofer norm of φ is given by

$$\|\varphi\|_\omega := \inf\{\|H\| \mid H \in \mathcal{H}(M, \omega) \text{ with } \varphi_H^1 = \varphi\}.$$

We will prove the following theorem on the existence of leaf-wise fixed points, using Rabinowitz-Floer homology.

Theorem 7.16. *Let M be an exact symplectic manifold with symplectic form $\omega = d\lambda$. Suppose furthermore that M is convex at infinity and let $\Sigma \subset M$ be a closed hypersurface of contact type, such that Σ bounds a compact region in M. Let $\varphi \in \operatorname{Ham}_c(M, \omega)$. If $\|\varphi\|_\omega < A(M, \omega, \Sigma)$ then*

$$\operatorname{Fix}(\varphi, \Sigma) \neq \emptyset.$$

This is a result by P. Albers and U. Frauenfelder from [**AF**]. In this paper, the above theorem is Theorem A, except for some minor rephrasing. The proof relies on the analysis of flow lines of the perturbed Rabinowitz action functional.

Note that a compact hypersurface Σ of contact type such that its Liouville vector field is globally defined is said to be of restricted contact type. However, if ω is exact, every connected hypersurface of contact type is of restricted contact type. All our symplectic manifolds are exact, so the distinction is somewhat superfluous.

Remark. In [**AF**], the condition on $\varphi \in \mathrm{Ham}_c(M,\omega)$ reads $\|\varphi\|_\omega < \wp(\Sigma,\alpha)$. Here $\alpha = \lambda|_\Sigma$ is the contact form (Σ is of contact type). Furthermore, $\wp(\Sigma,\alpha) > 0$ is the minimal period of a Reeb orbit of (Σ,α) that is contractible in M with the added condition that if no contractible Reeb orbit exists we set $\wp(\Sigma,\alpha) = +\infty$.
We have $A(M,\omega,\Sigma) = \wp(\Sigma,\alpha)$. To see this, note that by exactness of ω, we can apply Stokes' theorem. For a $u \in C^\infty(\mathbb{D}, M)$ such that $u(S^1)$ is on a leaf of the Reeb foliation, we have

$$\int_{\mathbb{D}} u^*\omega = \int_{S^1} u^*\alpha.$$

Because of the normalization $\alpha(R_\alpha) = 1$, this integral is precisely the period of the Reeb orbit. Furthermore, if no contractible Reeb orbit exists, then also $A(M,\omega,N) = +\infty$ as $\inf\{\emptyset\} = +\infty$.

We recall the notion of convexity at infinity, which is Definition 7.8. In general, the manifold M will not be compact. To ensure that the J-holomorphic curves used to define the Rabinowitz-Floer homology stay inside compact sets, we need some regularity. Therefore, we require M to be convex (at ∞). The conditions on Σ will give a bound on the Lagrange multiplier η.

We now consider a perturbation of the Rabinowitz action functional we discussed in Section 7.2 and see how it gives rise to leaf-wise fixed points.

The flow φ_F^t preserves the hypersurface $\Sigma = F^{-1}(0)$. Hence, Σ is foliated by leaves

$$L_x := \{\varphi_F^t(x) \mid t \in \mathbb{R}\}$$

for $x \in \Sigma$.
The strategy to prove Theorem 7.16 is to choose F cleverly and peturb $\mathcal{A}^F$ in a way such that critical points of the peturbed functional give rise to leaf-wise fixed points. We begin by describing the Hamiltonians we will use.

Definition 7.17. Let $F, H : M \times S^1 \to \mathbb{R}$ be a pair of Hamiltonians. A pair (F,H) is called a good pair if it satisfies

(i) $F(\cdot, t) = 0$ for all $t \in [\frac{1}{2}, 1]$

(ii) $H(\,\cdot\,,t) = 0$ for all $t \in [0, \frac{1}{2}]$.

(iii) F is of the form
$$F(x,t) = \rho(t)f(x)$$
where $\rho : S^1 \to [0,1]$ is a smooth map such that $\int_0^1 \rho(t)\,dt = 1$ and $f : M \to \mathbb{R}$.

The definition has many conditions, hence might seem restrictive. However, such good pairs are abundant in the following sense.

Lemma 7.18. *Suppose $\widetilde{F} : M \to \mathbb{R}$ is an autonomous Hamiltonian and let $\widetilde{H} : M \times S^1 \to \mathbb{R}$ be arbitrary. then there exist $F, H : M \times S^1 \to \mathbb{R}$ such that*

- *The pair (F,H) is a good pair.*
- *The Hamiltonian flows of (F,H) denoted $(\varphi_F^t, \varphi_H^t)$ are time reparametrizations of the flows of $\varphi_{\widetilde{F}}^t, \varphi_{\widetilde{H}}^t)$*

We define the perturbed Rabinowitz action functional. Recall that we assume our manifold M to be exact ($\omega = d\lambda$). Define the following.

Definition 7.19 (Peturbed Rabinowitz action functional)**.** Let M be an exact symplectic manifold and let (F,H) be good pair. The perturbed Rabinowitz action functional is defined by
$$\mathcal{A}_H^F : \mathcal{L}M \times \mathbb{R} \longrightarrow \mathbb{R}$$
$$(x,\eta) \longmapsto -\int_0^1 x^*\lambda - \int_0^1 H(x,t)\,dt - \eta \int_0^1 F(x,t)\,dt.$$

We look at critical points of this functional. We have the following lemma.

Lemma 7.20. *The critical points $(x,\eta) \in \mathcal{L}M \times \mathbb{R}$ of the action functional $\mathcal{A}_H^F$ are the solutions of*
$$\begin{cases} \dot{x}(t) = \eta X_F(x,t) + X_H(x,t) \quad \text{for all } t \in S^1 \\ \displaystyle\int_0^1 F(x,t)\,dt = 0 \end{cases} \tag{7.7}$$

Choosing a particular F will make sure that (x,η) satisfying equation (7.7) will give rise to a leaf-wise fixed point. Let M be a exact symplectic manifold ($\omega = d\lambda$) which is convex at infinity and Σ a closed hypersurface such that $(\Sigma, \alpha = \lambda|_\Sigma)$ is a contact manifold and Σ bounds a compact region in M. In this case, Σ is of restricted contact type, and is foliated by the leaves of the characteristic line bundle spanned by the Reeb vector field of α. Hence, we can talk about leaf-wise fixed points in this setting. For $x \in \Sigma$, denote L_x the leaf through x.

To see that (Σ, α) is of constricted contact type, note the following. Define a vector field Y by

$$\iota_Y d\lambda = \lambda.$$

Then this vector field is a Liouville vector field for (Σ, α). We need to check that $\mathcal{L}_Y\omega = \omega$ and $Y \pitchfork \Sigma$. This follows directly from the Cartan formula. Note that $\lambda(Y) = d\lambda(Y, Y) = 0$. Hence

$$\begin{aligned}\mathcal{L}_Y\lambda &= d\iota_Y\lambda + \iota_Y d\lambda \\ &= d0 + \iota_Y d\lambda \\ &= \lambda\end{aligned}$$

using the definition of Y. Hence,

$$\begin{aligned}\mathcal{L}_Y\omega &= \mathcal{L}_Y d\lambda \\ &= d\mathcal{L}_Y\lambda \\ &= d\lambda \\ &= \omega\end{aligned}$$

To prove that $Y \pitchfork \Sigma$, suppose conversely that $Y \in T_x\Sigma$. This implies

$$d\alpha(Y, R) = 0.$$

However,

$$d\lambda(Y, R) = \lambda(R) = \alpha(R) = 1.$$

Therefore, $Y \pitchfork \Sigma$.

We define an F such that the critical points of $\mathcal{A}_H^F$ will give rise to leaf-wise fixed points, using the Liouville vector field Y.

The flow φ_Y^t is defined near Σ because of transversality. Reparametrize such that for a fixed $\delta_0 > 0$, the flow $\varphi_Y^t|_\Sigma$ is defined for all $|t| \leq \delta_0$. We can define

$$\hat{G} : \Sigma \to \mathbb{R}$$

by

$$\hat{G}(\varphi_Y^t(x)) = t.$$

Let $0 < \delta \leq \delta_0$ and set

$$U_\delta := \{x \in M \mid |\hat{G}(x)| < \delta\}.$$

Because Σ bounds a compact region, we can extend $\hat{G}$ to the whole of M to a function $G : M \to \mathbb{R}$ that is locally constant outside U_{δ_0}, coincides with $\hat{G}$ on $U_{\frac{\delta_0}{2}}$ and $G^{-1}(0) = \Sigma$.

These conditions imply that the Hamiltonian vector field X_G satisfies $X_G|_\Sigma = R_\alpha$, the Reeb vector field. Fix some smooth function $\rho : S^1 \to \mathbb{R}$ with $\int_0^1 \rho(t)\, dt = 1$ and $\operatorname{supp}(\rho) \subset (0, \frac{1}{2})$. Let

$$F : S^1 \times M \to \mathbb{R}$$

defined by

$$F(t,x) := \rho(t)G(x) \tag{7.8}$$

This F has Hamiltonian vector field

$$X_F(t,x) = \rho(t)X_G(x).$$

For the above F, the functional $\mathcal{A}_H^F$ has the following property.

Proposition 7.21. *Let $F : S^1 \times M \to \mathbb{R}$ be as in equation (7.8). Let $H : S^1 \times M \to \mathbb{R}$ be such that $H(t,\cdot) = 0$ for $t \in [0, \frac{1}{2}]$ so that (F, H) is a good pair. Suppose $(x, \eta) \in \operatorname{Crit} \mathcal{A}_H^F$. Then the point $p = x(\frac{1}{2})$ satisfies $\varphi_H(p) \in L_p$, so it is a leaf-wise fixed point.*

Proof. We use the conditions on F and H. Recall that $F(t,x) = \rho(t)G(x)$. We prove the following claim.

Claim 9. For $t \in [0, \frac{1}{2}]$, we have $x(t) \in \Sigma$.

Proof. We use the chain rule and the fact that $(x, \eta) \in \operatorname{Crit} \mathcal{A}_H^F$ for the good pair (F, H) to compute

$$\begin{aligned}\frac{d}{dt}G(x(t)) &= dG(x(t))\frac{\partial x}{\partial t}\\ &= dG(x(t))\big(X_H(t,x) + \eta X_F(t,x)\big)\end{aligned}$$

We assumed $H(t,\cdot) = 0$ for all $t \in [0, \frac{1}{2}]$. Hence $X_H = 0$ on this interval. Furthermore, $X_F(t,x) = \rho(t)X_G(x)$. Using the definition of the Hamiltonian vector field $\iota_{X_G}\omega = dG$, we have $dG(X_G) = 0$. We conclude

$$\frac{d}{dt}G(x(t)) = 0.$$

Therefore, $G(x(t)) = c$ for some constant c. Hence

$$\int_0^1 \rho(t)G(x(t))\,dt = c\int_0^1 \rho(t)\,dt = c.$$

However, using that $F(t,x) = \rho(t)G(x(t))$ and equation (7.7), we find $c = 0$. The construction of G is such that $G^{-1}(0) = \Sigma$, thus $x(t) \in \Sigma$ for $t \in [0, \frac{1}{2}]$. This proves Claim 9. □

We then proceed in two steps. Define $p := x(\frac{1}{2})$. First, we show that $x(1) \in L_p$. Then we show that $x(1) = \varphi_H(p)$, which completes the proof of Proposition 7.21.

As above, for $t \in [0, \frac{1}{2}]$, we have

$$\dot{x}(t) = \eta\rho(t)X_G(x).$$

because $F(t,\cdot) = 0$. By construction of G, we have $X_G|_\Sigma = R_\alpha$. Using Claim 9, we know that $x(t) \in \Sigma$. As the foliation is generated by the Reeb vector field R, we have

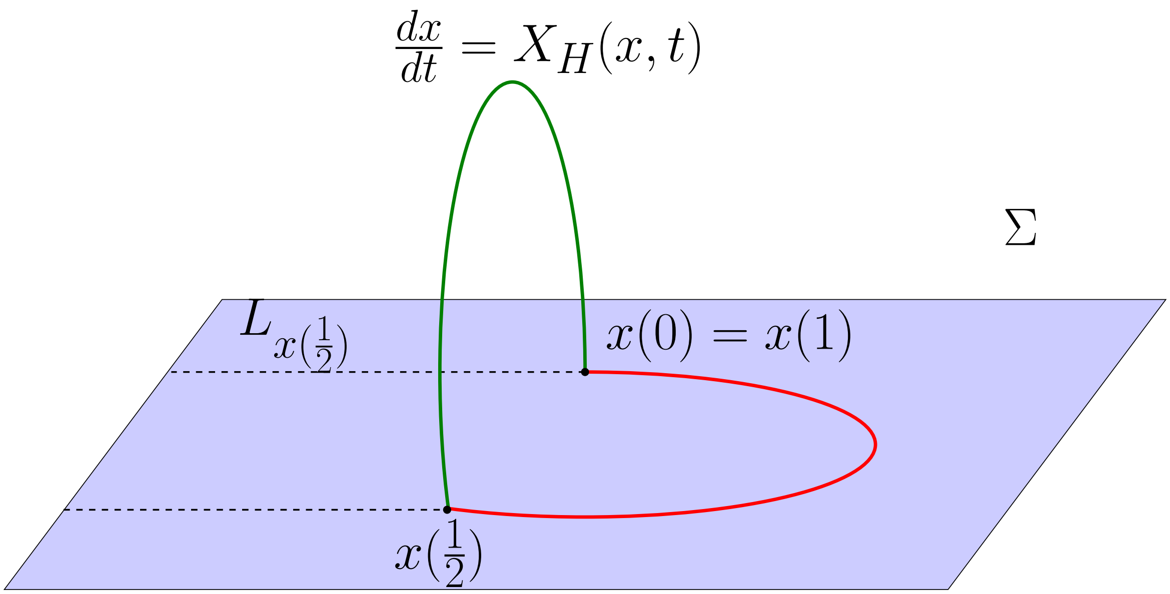

Figure 1. A critical point of $\mathcal{A}_H^F$.

$x(0) \in L_{x(\frac{1}{2})}$. As $x(0) = x(1)$, we have $x(1) \in L_p$.
For the second part, let $t \in [\frac{1}{2}, 1]$. By construction, $F(t, \cdot) = 0$ on this interval. Using equation (7.7), we find

$$\dot{x}(t) = X_H(t, x).$$

Note that this is precisely what it means to be the flow of H, hence

$$x(1) = \varphi_H(x(\frac{1}{2})) = \varphi_H(p).$$

This proves the second part.
We conclude that for $p := x(\frac{1}{2})$, we have $x(1) \in L_p$ and $x(1) = \varphi_H(p)$, so $\varphi_H(p) \in L_p$. This is the definition of a leaf-wise fixed point, hence this concludes the proof of Proposition 7.21. □

Figure 1 gives an illustration of the situation. For $t \in [0, \frac{1}{2}]$, u runs along the Reeb flow. For $t \in [\frac{1}{2}, 1]$ the path u flows along X_H.

Thus we see that critical points of the peturbed Rabinowitz action functional give rise to leaf-wise fixed points. To prove then the existence of these leaf-wise fixed points, we need to establish some analytical properties of the perturbed Rabinowitz action functional.

We repeat the steps we took in order to define Floer homology. Our approach is heuristic and some details are left out. For the functional $\mathcal{A}_H^F$ as defined above, we look at the gradient flow lines. Instead of a single H, we will now look at a parametrized family H_s of Hamiltonians with the following properties.

- For $s \geq 1$, $H_s(t,x) = H_+(t,x)$ for some fixed $H_+ : S^1 \times M \to \mathbb{R}$.
- For $s \leq -1$, $H_x(t,x) = H_+(t,x)$ for some fixed $H_- : S^1 \times M \to \mathbb{R}$.
- $H_s(t,\cdot) = 0$
- H_s has compact support uniformly in s

We look at gradient flow lines of $\mathcal{A}_H^F$. To define this, define the same metric we defined in Section 7.2 which was remniscent of Definition 3.2.

Choose a family $J(s,t) \in \mathcal{J}(M,\omega)$ of compatible almost complex structures, such that $J(s,t) = J_+(t)$ and $J(s,t) = J_-(t)$ for $s \geq +1$ and $s \leq -1$ respectively. Similarly to the definition of g_J, we define a metric by setting

$$g_{(s,t)}(\,\cdot\,,\,\cdot\,) := \omega(\,\cdot\,, J(s,t)\,\cdot\,).$$

Let $(x,\eta) \in \mathcal{L} \times \mathbb{R}$, we define the L^2-metric on $T_{(x,\eta)}(\mathcal{L}M \times \mathbb{R})$ similarly to Definition 3.2.

$$\mathfrak{g}_s : T_{(x,\eta)}(\mathcal{L}M \times \mathbb{R}) \times T_{(x,\eta)}(\mathcal{L}M \times \mathbb{R}) \longrightarrow \mathbb{R}$$

by setting for $(\xi_1, Y_1), (\xi_2, Y_2) \in T_{(x,\eta)}(\mathcal{L}M \times \mathbb{R})$

$$\mathfrak{g}_s((\xi_1, Y_1), (\xi_2, Y_2)) := \int_0^1 g_{(s,t)}(\xi_1, \xi_2)\, dt + Y_1 Y_2 \tag{7.9}$$

We can now define the gradient flow lines.

Definition 7.22. Let $\mathcal{A}_H^F$ be the peturbed Rabinowitz action functional. A gradient flow line is a map $u = (x,\eta) \in C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R})$ which satisfies

$$\frac{\partial u}{\partial s} = -\nabla_s \mathcal{A}_H^F(u(s)) \tag{7.10}$$

The gradient is taken with repect to the L^2-metric $\mathfrak{g}_s$.

Analogously to Floer homology, we switch view points. Instead of viewing x as a map $x : \mathbb{R} \to \mathcal{L}M$ we view it as a map $x : \mathbb{R} \times S^1 \to M$. Then we can rewrite equation (7.10) to read

$$\begin{cases} \dfrac{\partial x}{\partial s}(s,t) + J(s,t,x)(\dfrac{\partial x}{\partial t}(s,t) - X_{H_s}(t,x) - \eta X_F(t,x)) = 0 \\ \dfrac{\partial \eta}{\partial s}(s,t) - \displaystyle\int_0^1 F(t,x)\, dt = 0 \end{cases} \tag{7.11}$$

We define the energy of a gradient flow line.

Definition 7.23. Let $u \in C^\infty(\mathbb{R}, \mathcal{L}M \times \mathbb{R})$. The energy of u is defined as

$$E(u) := \int_{-\infty}^{\infty} \|\frac{\partial u}{\partial s}\|^2 \, ds.$$

Like in the Floer theory that is the main theory of thesis, the energy of a gradient flow line has a bound. In the Floer case, the energy was equal to the difference of the action values of the two limits of the flow line. However, in this case there is an additional term.

Lemma 7.24. *Suppose u is a gradient flow line. Denote $\lim_{s\to\pm\infty} u(s,\cdot) = u^\pm(\cdot)$. Then*

$$E(u) \leq \mathcal{A}_H^F(u^-) - \mathcal{A}_H^F(u^+) + \int_{-\infty}^{\infty} \Big\|\frac{\partial H_s}{\partial s}\Big\|_- \, ds.$$

From this it follows that $\mathcal{A}_H^F(u(s_0))$ is bounded for any $s_0 \in \mathbb{R}$ by

$$|\mathcal{A}_H^F(u(s_0))| \leq \max\{\mathcal{A}_{H_-}^F(u^-), -\mathcal{A}_{H_+}^F(u^+)\} + \int_{-\infty}^{\infty} \Big\|\frac{\partial H_s}{\partial s}\Big\|_- \, ds \tag{7.12}$$

We study the convergence of sequences of gradient flow lines. In the Floer case, then every reparametrization of a sequence of gradient flow lines has a C^∞_{loc}-convergent subsequence. This is also the case now. Let $u_n = (x_n, \eta_n)$ be a sequence of gradient flow lines. We can find bounds on the "loop parts" of (u_n) like in the Floer case. However, we also need an L^∞-bound on the "multiplier-part" η-part,

The central point is to bound $\|\eta\|_{L^\infty(\mathbb{R})}$ in some way. This is achieved by Proposition 2.10 in [**AF**].

Proposition 7.25. *Given two critical points $u^-, u^+ \in \operatorname{Crit} \mathcal{A}_H^F$ then there exists a constant K such that for every gradient flow line $u = (x, \eta)$ with $\lim\limits_{s\to\pm\infty} u = u^\pm$ we have*

$$\|\eta\|_{L^\infty(\mathbb{R})} \leq K.$$

The proof of the lemma uses the following inequality that is the content of the Fundamental Lemma (1.23) in [**AF10**] and Lemma 2.11 in [**AF**]. It is the perturbed version of Proposition 7.14. (TODO: CHANGE ALL THE SIGMAS HERE TO SOMETHING ELSE SUPER UGLY)

Lemma 7.26. *There exists a constant $C > 0$ such that for all $(x, \eta) \in C^\infty(S^1, M) \times \mathbb{R}$ we have*

$$\|\nabla_s \mathcal{A}_H^F(x, \eta)\| < \frac{1}{C} \implies |\eta| \leq C(\mathcal{A}_H^F(x, \eta) + 1) \tag{7.13}$$

Let us prove Proposition 7.25 using Lemma 7.26.

PROOF. We have

$$E(w) \leq \mathcal{A}^F_{H_-}(u_-) - \mathcal{A}^F_{H_+}(u^+) + \int_0^1 \left\| \frac{\partial H_s}{\partial s} \right\| ds.$$

Fix C as in Lemma 7.26. Let $\sigma \in \mathbb{R}$ and define

$$\tau(\Sigma) := \inf\{\tau \geq 0 \mid \|\nabla_s \mathcal{A}^F_H(u(\sigma+\tau)\| \leq \frac{1}{C}\} \tag{7.14}$$

We have that

$$\|\nabla_s \mathcal{A}^F_H(u(s))\|^2 \geq \frac{1}{C^2}.$$

Therefore,

$$E(u) \geq \int_\sigma^{\sigma+\tau(\sigma)} \|\nabla_s \mathcal{A}^F_H(u(s))\|^2 \geq \frac{\tau(\sigma)}{C^2} \tag{7.15}$$

We now use that u is a gradient flow line, hence solves equation (7.11). The second line implies that

$$\left\| \frac{\partial \eta}{\partial s} \right\|_{L^\infty} \leq \|F\|_{L^\infty} \tag{7.16}$$

Recall that $F(t,x) = \rho(t)G(x)$. Here G was defined to be locally constant outside a compact set. Therefore, $dF = 0$ outside some compact set. This implies $\|F\|_{L^\infty}$ is finite, which means $\left\|\frac{\partial \eta}{\partial s}\right\|$ is finite.
We also have a bound on $\mathcal{A}^F_H(u(s))$ for any $s \in \mathbb{R}$ given by equation (7.12), where we integrate over H_s now in the last term. We write the following inequality to express $|\eta(\sigma)|$ in terms of an integral over $\frac{\partial \eta}{\partial s}$

$$|\eta(\sigma)| \leq |\eta(\sigma + \tau(\sigma))| + \int_\sigma^{\sigma+\tau(\sigma)} \left| \frac{\partial \eta}{\partial s} \right| ds \tag{7.17}$$

By the definition of $\tau(s_0)$ (equation 7.14) and Lemma 7.26, we have

$$|\eta(s_0 + \tau(s_0))| \leq C\Big(|\mathcal{A}^F_H(u(s_0+\tau(s_0)))| + 1\Big).$$

By equation (7.12), we have

$$|\eta(s_0 + \tau(s_0))| \leq C\Big(\max\{\mathcal{A}^F_{H_-}(u^-), -\mathcal{A}^F_{H_+}(u^+)\} + \int_0^1 \left\| \frac{\partial H_s}{\partial s} \right\| ds + 1\Big) \tag{7.18}$$

On the other hand, equation (7.16) implies

$$\int_\sigma^{s_0+\tau(s_0)} \left| \frac{\partial \eta}{\partial s} \right| ds \leq \|F\|_{L^\infty} \tau(s_0).$$

Using equation (7.15, we find

$$\int_\sigma^{s_0+\tau(s_0)} \left| \frac{\partial \eta}{\partial s} \right| ds \leq C^2 \|F\|_{L^\infty} E(u) \tag{7.19}$$

Combining equations (7.18) and (7.19) implies

$$|\eta(s_0)| \le C\Big(\max\{\mathcal{A}^F_{H_-}(u^-), -\mathcal{A}^F_{H_+}(u^+)\} + \int_0^1 \Big\|\frac{\partial H_s}{\partial s}\Big\|\,ds + 1\Big) + C^2\|F\|_{L^\infty}E(u).$$

This holds for any $s_0 \in \mathbb{R}$. Hence this inequality proves Proposition 7.25. □

We have the tools required to prove Theorem 7.16, the existence of leaf-wise fixed points. We will need the following lemma. Recall the definition of a Morse-Bott function in Definition A.14 in the Appendix A.2. Let $\mathcal{A} : \mathcal{E} \to \mathbb{R}$ be a functional. We call $C \subset \operatorname{Crit}\mathcal{A}$ a Morse-Bott component if $\mathcal{A}$ is Morse-Bott on this set.

Lemma 7.27. *Let Σ be as in the hypothesis of Theorem 7.16. Then $\Sigma \subset \operatorname{Crit}\mathcal{A}^F_H$ is a Morse-Bott component.*

Proof. Let $p \in \Sigma$ so that $(p, 0)$ is a critical point. Then the Hessian of $\mathcal{A}^F$ at $(p, 0)$ is well-defined. It is given by the vertical derivative. A computation like (FLOER CASE) shows that the gradient of $\mathcal{A}^F$ satisfies

$$\nabla_s \mathcal{A}^F(v,\eta) = \begin{pmatrix} J_t(v)\Big(\frac{\partial v}{\partial t} - \eta X_F(v)\Big) \\ \int_0^1 F(v)\,dt \end{pmatrix} \tag{7.20}$$

Then the requirement for a tangent vector $(\hat v, \hat\eta) \in C^\infty(S^1, T_pM)\times\mathbb{R}$ to be in the kernel of the Hessian is equivalent to saying it solves

$$\begin{cases} \dfrac{\partial \hat v}{\partial t} = \hat\eta X_F(t,p) \\ \displaystyle\int_0^1 dF(t,p)\hat v\,dt = 0 \end{cases} \tag{7.21}$$

Here we used that $F(t,p) = \rho(t)G(p)$. We can integrate the first equation to find an expression for $\hat v(1)$. Using that $\int_0^1 \rho(t)\,dt = 1$, we find

$$\hat v(1) = \hat v(0) + \hat\eta X_G(p).$$

As $\hat v$ is a loop, we must have $\hat\eta = 0$, so $\hat v = \hat v_0 \in T_pM$ is a constant path. Then the second equation, using that $\int_0^1 \rho(t)\,dt = 1$ again, implies $dG(p)\hat v_0$ As $G^{-1}(0) = \Sigma$, we have

$$\hat v_0 \in T_p\Sigma.$$

This proves Lemma 7.27. □

Proof of Theorem 7.16. Let $H : S^1 \times M \to \mathbb{R}$ be as in the hypothesis of the Theorem ($\varphi = \varphi_H$ and $\|\varphi\|_\omega < A(M,\omega,\Sigma)$. Choose a parametrized family of functions $\beta_r \in C^\infty(\mathbb{R},[0,1])$ for $r \ge 0$ satisfying the following conditions

(i) For $r \ge 1$ we have

$$\begin{cases} \beta_r(s) = 1 & \text{for } |s| \le r-1 \\ \beta_r(s) = 0 & \text{for } |s| \ge r \end{cases} \tag{7.22}$$

and $\dot{\beta}_r(s) \cdot s \leq 0$ for all $s \in \mathbb{R}$

(ii) For $r \leq 1$ we have

$$\begin{cases} \beta_r(s) \leq r & \text{for all } s \in \mathbb{R} \\ \beta_r(s) = 0 & \text{for all } s \in (-\infty, -1) \cup (1, \infty) \end{cases} \tag{7.23}$$

(iii) The functions defined by

$$\beta^{\pm}_{\infty}(s) := \lim_{r \to \infty} \beta_r(s \mp r)$$

exists as limits in the C^∞-topology.

Define

$$K_r(s, t, x) = \beta_r(s) H(t, x) \tag{7.24}$$

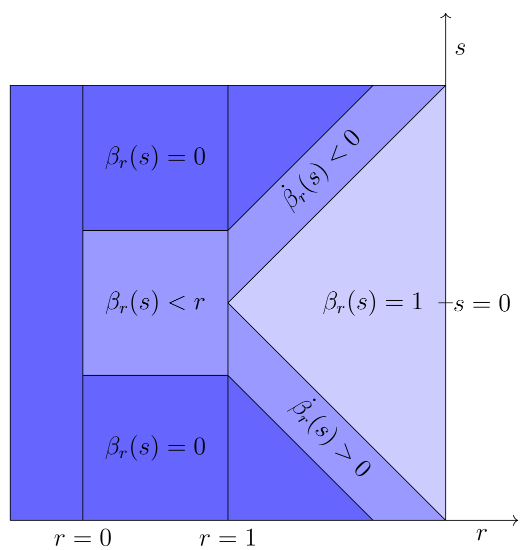

Figure 2. The function $\beta_r(s) \in C^\infty(\mathbb{R}, [0, 1])$.

To see how $\beta_r(s)$ interpolates between the perturbed and unperturbed functional, Figure 2 gives a schematic view of $\beta_r(s)$ as just defined.

Fix some $p \in \Sigma$. We consider the moduli space of solutions of equation (7.7) for $H = K_r$ that start at p and end somewhere in Σ. Concretely, consider

(7.25)

$$\mathcal{M}(\Sigma, p) := \left\{ (r, u) \in [0, \infty) \times C^\infty(\mathbb{R}, \mathcal{L} \times \mathbb{R}) \;\middle|\; \begin{array}{c} u \text{ satisfies equation (7.11) with } H = K_r \\ \lim\limits_{s \to -\infty} u(s) = (p, 0) \\ \lim\limits_{s \to +\infty} u(s) \in \Sigma \end{array} \right\}$$

To prove the theorem we need the following compactness result about $\mathcal{M}$.

Claim 10. Suppose there is no leaf-wise fixed point. Then the moduli space $\mathcal{M}$ is compact with $\partial\mathcal{M} = (0, p, 0)$.

Let us assume the Claim. The proof is now straightforward using the techniques demonstrated before. The moduli space $\mathcal{M}$ is a zero section of a Fredholm section of a Banach-bundle over a Banach manifold of index 1.

To see this, note that this the r-parameter interpolates between the unperturbed functional at the $r = 0$ end, and the perturbed functional at the $r = +\infty$ end. Consider the unperturbed functional. In this case, the point $(p, 0)$ is a critical point, hence the Fredholm index is 0. The Fredholm index is invariant under perturbations, so that the same is true for the perturbed functional. By introducing the r-parameter, we consider a one-parameter familiy of functionals. This is then a problem of Fredholm index 1.

By the Morse-Bott condition, the section is regular at the boundary. Peturb the section away from this boundary $(0, p, 0)$ to a transverse section, so that $\mathcal{M}$ is a smooth compact submanifold of the base space, with boundary $(0, p, 0)$. However, such a manifold cannot exist, therefore the assumption that there are no leaf-wise fixed points is false. This proves Theorem 7.15. The more difficult part is proving Claim 10. We refer the reader to page 10 of [**AF**], where several lemmas are used mentioned in the same paper. $\square$

For the actual definition of the perturbed Rabinowitz-Floer homology, we proceed as in Floer homology. Choose some s-independent $H : S^1 \times M \to \mathbb{R}$ with the additional condition that $H(t, \cdot) = 0$ for $t \in [0, \frac{1}{2}]$. Assume that the peturbed Rabinowitz action functional $\mathcal{A}_H^F$ from Definition 7.19 is Morse. For generic H with the above condition, this is true. We refer the reader to Theorem 2.14 in [**AF**]. Note that this is easy for any H but the vanishing of H for $t \in [0, \frac{1}{2}]$ is a complicating factor. Furthermore, it is interesting because the unperturbed Rabinowitz functional can never be Morse. The proof is in Appendix A of [**AF**].

As we may assume $\mathcal{A}_H^F$ to be Morse, we do not have to work with the Morse-Bott homology, as was the case for the unperturbed functional.
Define the chain complex. Let

$$\mathcal{C}_\kappa := \{\alpha \in \operatorname{Crit} \mathcal{A}_H^F(\alpha) \geq \kappa\}.$$

$$CRF(\mathcal{A}_H^F) := \{ \sum_{\alpha \in \operatorname{Crit} \mathcal{A}_H^F} \xi_\alpha \alpha \mid \xi_\alpha \in \mathbb{Z}_2, |\mathcal{C}_\kappa| < \infty \text{ for all } \kappa \in \mathbb{R}\} \tag{7.26}$$

Then define the moduli space of solutions running between critical points modulo reparametrization in the s-variable.

$$\widehat{\mathcal{M}}(\alpha_-, \alpha_+) := \{w : \mathbb{R} \times S^1 \to M \mid \ w \text{ solves (7.11) and } \lim_{s \to \pm\infty} w = c_\pm\}/\mathbb{R} \tag{7.27}$$

In the Floer case we saw that we need transversality of the moduli spaces $\overline{\mathcal{M}}(\alpha_-, \alpha_+)$. In this case, this is more difficult. Abstract peturbation theory developed in the theory of polyfolds by Hofer, Wysocki and Zehnder is required to prove transversality.

Let us for now assume that we have perturbed the gradient flow equation such that the resulting manifolds, also denoted $\overline{\mathcal{M}}(\alpha_-, \alpha_+)$ are smooth submanifolds of some Banach space. Like in Floer theory, denote $\eta(\alpha_-, \alpha_+)$ the number of elements in the zero-dimensional component of $\overline{\mathcal{M}}(\alpha_-, \alpha_+)$ modulo $\mathbb{Z}_2$. Define the boundary map on the generators

$$\partial : CRF(\mathcal{A}_H^F) \to CFH(\mathcal{A}_H^F)$$

by

$$\alpha \mapsto \sum_\beta \eta(\alpha, \beta)\beta.$$

The following theorem gives an interesting result about leaf-wise fixed points in terms of the Rabinowitz-Floer homology $\mathrm{RFH}_*(M, \Sigma)$.

Theorem 7.28. $HF(\mathcal{A}_H^F) \cong \mathrm{RFH}(M, \Sigma)$

Proof. The proof reasons by just taking a homotopy from H to 0. Because of compactness results (Theorem 2.9 in [**AF**]), the homology is independent of homotopies, so that we have $\mathrm{HF}(\mathcal{A}_H^F) \simeq \mathrm{HF}(\mathcal{A}_0^F)$. Note that $\mathrm{RFH}(M, \Sigma) = \mathrm{HF}(\mathcal{A}_0^F)$ to prove the theorem. □

An immediate Corollary is the following. It gives a condition for the existence of leaf-wise fixed points in terms of the vanishing of the Rabinowitz-Floer homology $\mathrm{RFH}(M, \Sigma)$.

Corollary 7.29. *Suppose* $\mathrm{RFH}(M, \Sigma) \neq 0$. *Then for any* $\varphi \in \mathrm{Ham}_c(M, \omega)$ *we have*

$$\mathrm{Fix}(\varphi, \Sigma) \neq \emptyset.$$

This Corollary is surprising as there is no assumption on $\|\varphi\|_\omega$, contrary to many other results. However, the condition that $\mathrm{RFH}_*(M, \Sigma) \neq 0$ may be hard to verify.

7.4. Orderability

We will first explain the setting by defining a particular class of contact manifolds. Then we define what is means for a general contact manifold to be orderable. After this we discuss an orderability result.

We introduce a class of manifolds that, heuristically speaking, are symplectic manifolds that have a contact manifold as their boundary. From the perspective of contact manifolds, we fill out our contact manifold to form a symplectic manifold. This will be the setting in which we consider orderability, as in [**AM14**].

We first introduce a Liouville domain.

Definition 7.30. Let W be a compact manifold with boundary and $\lambda \in \Omega^1(W)$ such that $d\lambda$ is non-degenerate. We call the pair (W, λ) a Liouville domain if the unique vector field Y_λ defined by $\iota_{Y_\lambda} d\lambda = \lambda$ is transverse to ∂W pointing outwards.

Remark. First of all, as $d\lambda$ is non-degenerate and closed, it is a symplectic form on W, meaning $(W, d\lambda)$ is symplectic. Furthermore, Y_λ is a Liouville vector field as defined in Definition 7.4. Note that the above definition is equivalent to saying that $\lambda|_{\partial W}$ is a positive contact form on ∂W, as ∂W is then a hypersurface transverse to Y_λ. Hence, by Lemma 7.5, ∂W is a contact manifold with contact form $\lambda|_{\partial W}$.

We can complete Liouville domains by attaching a cylindrical end to ∂W and extending the form λ appropriately. This is done by considering the Liouville vector field Y_λ and using that it is transverse to ∂W. Let $\psi^t = \psi^t_{Y_\lambda}$ denote the flow of Y_λ. We can define an embedding $\partial W \times (0, 1] \hookrightarrow W$ by $(x, r) \mapsto \psi^{\log r}(x)$. Using this identification, we can glue a cylindrical end to W.

Definition 7.31. Let (W, λ) be a Liouville domain. We define the completion of $(\widetilde{W}, \widetilde{\omega})$ by setting

$$\widetilde{W} := W \cup_{\partial W} (\partial W \times [1, \infty).$$

Associated to this is the form $\widetilde{\lambda} \in \Omega^1(\widetilde{W})$ defined by

$$\widetilde{\lambda} = \begin{cases} \lambda|_W & \text{on } W \subset \widetilde{W} \\ r\lambda|_{\partial W} & \text{on } \partial W \times [1, \infty) \subset \widetilde{W} \end{cases}$$

Then $d\widetilde{\lambda}$ is a symplectic form on $\widetilde{W}$. The manifold $(\widetilde{W}, d\widetilde{\lambda})$ is called a Liouville manifold.

Remark. We can also extend the vector field Y_λ to $\widetilde{W}$ by $Y_{\widetilde{\lambda}} = r\frac{\partial}{\partial r}$ on $\partial W \times [1, \infty)$.

We can approach this construction from the other point of view by starting with a closed connected coorientable contact manifold (Σ, ξ). We require that this contact manifold plays the role of ∂W. That is, we say (Σ, α) is Liouville fillable if there exists a Liouville domain (W, λ) such that $\partial W = \Sigma$ and $\alpha := \lambda_\Sigma$ is a positive contact form on Σ. The Liouville filling of (Σ, ξ) is then the completed Liouville manifold $(\widetilde{W}, d\widetilde{\lambda})$.

A way to construct a symplectic manifold from a contact manifold is the symplectization.

Definition 7.32. Let (Σ, ξ) be a coorientable contact manifold with associated contact form α. Then the symplectization of Σ is the sympletic manifold defined by

$$S\Sigma = \Sigma \times (0, \infty)$$

with symplectic form

$$\omega := d(r\alpha).$$

This definition is somewhat reminiscent of the Liouville filling. Indeed, suppose (Σ, ξ) is Liouville fillable, then we can embed $S\Sigma$ into the Liouville filling $(\widetilde{W}, \widetilde{\omega})$, using the flow of $Y_{\widetilde{\lambda}}$.

The last thing we discuss is how to lift contact isotopies $\varphi : [0,1] \to \mathrm{Cont}_0(\Sigma, \xi)$ to symplectic isotopies of $S\Sigma$.

Definition 7.33. Let $\{\varphi_t\} \in \mathrm{Cont}(\Sigma, \xi)$. Then there exists $\{\Phi_t\} \in \mathrm{Ham}(S\Sigma, d(r\alpha))$ which we call the symplectization of φ_t. This is defined in the following way. Let ρ_t be defined by $\varphi_t^*\alpha = \rho_t\varphi_t$. Then

$$\Phi_t : S\Sigma \to S\Sigma$$

is defined by

$$\Phi_t(x, r) = (\varphi_t(x), r\rho_t(x)^{-1}).$$

One can prove that indeed the path Φ_t is Hamiltonian, generated by the Hamiltonian function

$$H : S^1 \times S\Sigma \to \mathbb{R}$$

given by

$$H_t(x, r) = rh_t(x)$$

where h_t denotes the contact Hamiltonian associated to φ_t. See for example Proposition 2.3 in [**AF11**].

We define what it means for a contact manifold to be orderable. Recall that for any set, there is the notion of a partial order.

Definition 7.34. Let S be a set. A partial order is a binary relation $\leq$ on S such that the relation is reflexive, anti symmetric and transitive, i.e.

- $a \leq a$.
- $a \leq b$ and $b \leq a$ implies $a = b$.
- $a \leq b$ and $b \leq c$ implies $a \leq c$.

We begin by defining the notion of orderability in the sense of Eliashberg, described in [**EKP06**]. Let (Σ, ξ) be a contact manifold and denote $\mathrm{Cont}_0(\Sigma, \xi)$ the identity component of the group of contactomorphisms. Let $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ be its universal cover equipped with the quotient topology.

Definition 7.35. Let $\widetilde{f}, \widetilde{g} \in \widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$. We say $\widetilde{f} \succeq \widetilde{g}$ if $\widetilde{f}\widetilde{g}^{-1}$ is represented by a path generated by a non-negative contact Hamiltonian. We call a contact manifold (Σ, ξ) orderable if the relation $\succeq$ on $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ defines a partial order.

Remark. Note that the definition of $\succeq$ immediatly implies that it is both reflexive and transitive, hence it is the antisymmetry that is of particular importance.

There is an equivalent definition of orderability that we will use. This is a rephrasing of Proposition 1.9 in [**EKP06**].

Proposition 7.36. *Let (Σ, ξ) be a closed contact manifold. Then (Σ, ξ) is orderable if and only if there does not exist a loop $\varphi : S^1 \to \mathrm{Cont}_0(\Sigma, \xi)$ with $\varphi(0) = \mathrm{Id}_\Sigma$, such that φ is generated by a contact Hamiltonian H_t with $H_t(x) \geq 0$ for all $x \in \Sigma$ and $t \in [0, 1]$.*

A question in contact geometry is which manifolds are orderable. This question is relevant in the search for a analogue of the Gromov non-squeezing theorem in contact geometry. Recall that Gromov's non-squeezing result states that the symplectic ball B^{2n} cannot be symplectically embedded into the symplectic cylinder $C^{2n} := B^2 \times \mathbb{R}^{2n-1}$. This result is related to the geometry of the group of symplectomorphisms. Similary, there is a correspondence between contact non-squeezing and the geometry of the group of contacomorphisms. We refer the reader to the paper [**EKP06**]. This provides motivation to answer the orderability problem. In a particular setting, this question can sometimes be answered using Rabinowitz-Floer homology. The following result from the paper [**AM14**] by P. Albers and W. J. Merry provides some answer to this question.

Theorem 7.37. *Let (Σ, ξ) be a closed coorientable contact manifold that admits a Liouville filling (W, λ) such that $\mathrm{RFH}_*(\Sigma, W)$ is non-zero. Then $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ is orderable.*

The proof results from the existence of a map for every $Z \in \mathrm{RFH}(\Sigma, W)$ called $c(\,\cdot\,, Z)$ that assigns to elements of $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ a number in such a way that $c(\,\cdot\,, Z)$ descends to a map on $\widetilde{\mathrm{Cont}}_0(\Sigma, \xi)$ that is order reversing when one looks at the associated contact Hamiltonians. The map $c(\,\cdot\,, Z)$ is described in the following Theorem 7.38 which is the content of Theorem 1.1 in [**AM14**].

Recall that by $\mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ we denote the space of paths $\varphi : [0, 1] \to \mathrm{Cont}_0(\Sigma, \xi)$ such that $phi_0 = \mathrm{Id}_\Sigma$. These are often called contact isotopies. Let θ denote the flow of the Reeb vector field.

Proposition 7.38. *Suppose (Σ, ξ) satisfies the hypothesis of Theorem 7.37. Then for any non-zero class $Z \in \mathrm{RFH}_*(\Sigma, W)$ there exists a map $c(\,\cdot\,, Z) : \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi) \to \mathbb{R}$ satisfying the following properties.*

(i) *The map $c(\,\cdot\,, Z)$ descends to a well defined map*
$$\widetilde{c}(\,\cdot\,, Z) : \widetilde{\mathrm{Cont}}_0(\Sigma, \xi) \to \mathbb{R}.$$

(ii) *Let $\lambda \in \mathbb{R}$. Then*
$$c(t \mapsto \theta^{\lambda t}, Z) = -\lambda + c(\mathrm{Id}_\Sigma, Z).$$

(iii) *The map $c(\,\cdot\,, Z)$ is continuous with respect to the C^2-norm on $\mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$.*

(iv) *Let $\varphi, \psi \in \mathcal{P}\operatorname{Cont}_0(\Sigma,\xi)$ be generated by contact Hamiltonians H_t and G_t respectively, such that $H_t(x) \geq K_t(x)$ for all $x \in \Sigma$ and all $t \in [0,1]$. Then*
$$c(\varphi, Z) \leq c(\psi, Z).$$

Using Proposition 7.38 we can immediately prove Theorem 7.37.

Proof of Theorem 7.37. We use the equivalent definition of orderability from Proposition 7.36. By contradiction, suppose (Σ, ξ) is not orderable. Then a loop φ as in Proposition 7.36 exists, generated by a contact Hamiltonian H_t with $H_t(x) \geq \varepsilon$ for all $x \in \Sigma$ and $t \in [0,1]$ for some $\varepsilon > 0$. We first prove the following claim.

Claim 11. For the loop φ as described above, we have $c(\varphi, Z) < c(\mathrm{Id}_\Sigma, Z)$.

Proof of Claim 11. Denote K_ε the constant function with value ε. This function generates the path $\kappa_\varepsilon : t \mapsto \theta^{t\varepsilon}$. Using Proposition 7.38 (iv) we have
$$c(\varphi, Z) \leq c(\kappa_\varepsilon, Z).$$
Then by Proposition 7.38 (ii) it follows that
$$c(\kappa_\varepsilon, Z) = -\varepsilon + c(\mathrm{Id}_\Sigma, Z).$$
As $\varepsilon > 0$ we conclude
$$c(\varphi, Z) < c(\mathrm{Id}_\Sigma, Z).$$
This proves Claim 11. □

We see that for φ, the conditions (ii) and (iv) of Proposition 7.38 imply that
$$c(\varphi, Z) < c(\mathrm{Id}_\Sigma, Z).$$
However, contractability of φ together with Proposition 7.38 (i) implies that
$$c(\varphi, Z) = c(\mathrm{Id}_\Sigma, Z).$$
This is a contradiction, proving Theorem 7.37. □

We still need to prove Proposition 7.38 which is much more involved. We need several definitions. We first define the notion of a translated point of a contactomorphism. This term was introduced in [**San12**] in a somewhat different context. We follow Definition 2.3 from [**AM14**]. We then define a functional whose critical points are translated points.

Definition 7.39. Let (Σ, ξ) be a closed connected coorientable contact manifold, which associated contact form $\alpha \in \Omega^1(\Sigma)$ such that $\ker \alpha = \xi$. Let $\varphi \in \operatorname{Cont}_0(\Sigma, \xi)$. Let $f : \Sigma \to \mathbb{R} \setminus \{0\}$ be such that $\varphi^*\alpha = f\alpha$.
A translated point of φ is a point $x \in \Sigma$ such that there exists $\eta \in \mathbb{R}$ such that
$$\varphi(x) = \theta^\eta(x) \qquad \text{and} \qquad f(x) = 1.$$
The number η is called the time-shift.

Remark. Note that the above $f : \Sigma \to \mathbb{R} \setminus \{0\}$ exists and is unique by Remark 7.1

We also have a notion of contractibility of translated points with respect to a Liouville filling $(\widetilde{W}, \widetilde{\omega})$.

Definition 7.40. Let (Σ, ξ) be a closed contact manifold and $(\widetilde{W}, d\widetilde{\lambda})$ is Liouville filling, with $\alpha = \lambda_{|}\Sigma$. Let $[\varphi] \in \widetilde{\text{Cont}_0}(\Sigma, \xi)$ where x is a translated point of $\varphi(1)$ with time-shift η. We call (x, η) contractible if and only if the loop $\gamma : S^1 \to \Sigma$ defined by

$$\gamma(t) = \begin{cases} \varphi_{2t}(x) & \text{for } 0 \leq t \leq \dfrac{1}{2} \\ \theta^{2\eta(1-t)} & \text{for } \dfrac{1}{2} \leq t \leq 1 \end{cases} \tag{7.28}$$

is contractible in W.

Remark. Note that this notion is independent of the particular representative of $[\varphi]$ chosen. That is, suppose γ is as above in equation (7.28). Choose another representative ψ of $[\varphi]$ and let γ' be the associated loop as in equation (7.28). Then γ is contractible in W if and only if γ' is. This is immediate from the fact that φ is homotopic to ψ.

7.5. Rabinowitz Floer homology and orderability

We define the Rabinowitz Floer homology required to prove Theorem 7.37. We first define the functional we use, and then prove that the critical points of this functional are translated points.

Definition 7.41. Let $\varphi \in \text{Cont}_0(\Sigma, \xi)$. Define a functional on the loop space of the symplectization

$$\mathcal{A}_\varphi : \mathcal{L}S\Sigma \times \mathbb{R} \longrightarrow \mathbb{R}$$

by

$$\mathcal{A}_\varphi(u, \eta) := \int_0^1 u^*\lambda - \eta \int_0^1 \beta(t)(r(t) - 1)\, dt - \int_0^1 \dot{\chi}(t) H_{\chi(t)}(u(t))\, dt$$

Here $\beta : S^1 \to \mathbb{R}$ is a smooth function such that $\beta(t) = 0$ for all $t \in [\frac{1}{2}, 1]$ and $\int_0^1 \beta(t)\, dt = 1$. $\chi : [0,1] \to [0,1]$ is a smooth function such that $\dot{\chi}(t) \geq 0$, $\chi(\frac{1}{2}) = 0$ and $\chi(1) = 1$.
By $r(t)$ we denote the second component of the map $u : S^1 \to S\Sigma$, so that $r : S^1 \to \mathbb{R}$.

Remark. Let us make a few remarks on the similarities between this definition and the one of Definition 7.19 and 7.17.
Recall that the part involving η was a function $F : M \times [0,1] \to \mathbb{R}$ of the form $F(x,t) = \rho(t)f(x)$. In this case, this role is played by $F_0 := \beta(t)(r(t) - 1)$. The role of $\rho(t)$ is played by $\beta(t)$, with $\int_0^1 \beta(t)\, dt = 1$ whereas $r(t) - 1$ depends on $u : S^1 \to S\Sigma$. Furthermore, $F_0(\,\cdot\,, t) = 0$ for all $t \in [\frac{1}{2}, 1]$ as $\beta(0) = 0$ on this interval.
For the part involving H, setting $H_0(u,t) = \dot{\chi}(t) H_{\chi(t)}(u(t))$ we see that indeed $H_0(\,\cdot\,, t) = 0$ for $t \in [0, \frac{1}{2}]$ as in Definition 7.17. This is the case as $\chi(t) = 0$ for $t \in [0, \frac{1}{2}]$.

In this case, the critical points turn out to be translated points of φ. We denote the the critical values by

$$\operatorname{Spec}(\varphi) := \mathcal{A}_\varphi(\operatorname{Crit}(\mathcal{A}_\varphi)) \tag{7.29}$$

Again, two equations describe the critical points, reminiscent of equation (7.7).

Lemma 7.42. *Let $(u,\eta) \in \mathcal{L}S\Sigma \times \mathbb{R}$ and write $u = (x(t), r(t))$. The critical points (u,η) of the functional $\mathcal{A}_\varphi$ are the solutions of*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \eta\beta(t)R((x(t)) + \dot{\chi}(t)X_{H_{\chi(t)}}(u(t)) \\ \displaystyle\int_0^1 \beta(t)(r(t)-1)\,dt = 0 \end{cases} \tag{7.30}$$

Here R denotes the Reeb vector field on Σ.

Proof. The proof is very similar to that in the Floer case and in the peturbed Rabinowitz Floer case. In the proof of Lemma 7.19 some details were provided. $\square$

These critical points have the following interpetation, which is the content of Lemma 2.7 in [**AM14**].

Theorem 7.43. *Let $(u,\eta) \in \mathcal{L}S\Sigma \times \mathbb{R}$. Write $u(t) = (x(t), r(t))$ as above. Then (u,η) is a critical point of $\mathcal{A}_\varphi$ only if $p := x(\frac{1}{2})$ is a translated point of φ with time-shift $-\eta$. Conversely, given a translated point p of φ with time-shift $-\eta$, then the pair (p,η) gives rise to a unique critical point of $\mathcal{A}_\varphi$.*

Proof. The proof is similar to the one of Theorem 7.21. Recall that we denote by θ^t the flow of the Reeb vector field. Denote $\Phi_t : S\Sigma \to S\Sigma$ the lift of φ as described in Definition 7.33. Suppose that (u,η) is a critical point, where we write $u(t) = (x(t), r(t))$. Then u satisfies equation (7.30).
We first consider $t \in [0, \frac{1}{2}]$. Note $\chi(t) = 0$ on this interval. Hence,

$$\frac{\partial u}{\partial t} = \eta\beta(t)R(x(t)).$$

Note that the function $r(t) - 1$ is constant on flow lines of $\eta\beta(t)R(x)$, meaning $r(t) - 1$ is some constant c. The second line of equation (7.30) tells us that $c = 0$, meaning $r(t) = 1$. Note that $R|_{\Sigma\times 1} = R_\alpha$ where R_α denotes the Reeb vector field on Σ. Therefore

$$u(\frac{1}{2}) = (\theta^\eta(x(0)), 1) \tag{7.31}$$

Consider $t \in [\frac{1}{2}, 1]$. In this case, $\beta(t) = 0$. Hence,

$$\frac{\partial u}{\partial t} = \dot{\chi}(t)X_{H_{\chi(t)}}(u(t))$$

This implies that

$$u(t) = \Phi_{\chi(t)}(u(\frac{1}{2}))$$

on this interval, as H was defined to be such that its Hamiltonian flow is precisely the symplectization $\Phi_t : S\Sigma \to S\Sigma$. In particular for $t = 1$ we have $\chi(1) = 1$ so

$$x(1) = \varphi(x(\frac{1}{2})).$$

Now note that

$$x(1) = x(0) = \theta^{-\eta}(x(\frac{1}{2}))$$

using equation (7.31) so that

$$\varphi(x(\frac{1}{2})) = \theta^{-\eta}(x(\frac{1}{2})).$$

Equation (7.31) also implies that $\rho = 1$, as this is the second component. We conclude that $x(\frac{1}{2})$ is a translated point with time shift $-\eta$. This completes the first part of the proof.

Note that for λ we have the following

$$\lambda(X_H(x,r)) = dH(x,r)(r\frac{\partial}{\partial r}) = H(x,r) \tag{7.32}$$

Now $\beta(t) = 0$ for $t \in [\frac{1}{2}, 1]$. On the other hand, $\dot{\chi}(t) = 0$ for $t \in [0, \frac{1}{2}]$. Therefore, collecting non-zero terms for a critical point (u, η) yields

$$\begin{aligned}\mathcal{A}_\varphi(u,\eta) &= \int_0^{\frac{1}{2}} (r\alpha)(\eta\beta(t)R(x(t)))\, dt + \int_{\frac{1}{2}}^1 \lambda(\dot{\chi}(t)X_{H_{\chi(t)}}(u)) - \dot{\chi}(t)H_{\chi(t)}(u)\, dt \\ &= \eta\end{aligned}$$

Here, we used that equation (7.32) to conclude that $\lambda(\dot{\chi}X_{H_{\chi(t)}}(u)) = \dot{\chi}(t)H_{\chi(t)}(u)$ as well as $\int_0^{\frac{1}{2}} \beta(t)\, dt = 1$. $\square$

We have now defined the functional $\mathcal{A}_\varphi$ on $\mathcal{L}S\Sigma \times \mathbb{R}$. We use the embedding of $S\Sigma$ into $\widetilde{W}$ to extend $\mathcal{A}_\varphi$ to the whole of $\mathcal{L}\widetilde{W} \times \mathbb{R}$. We do this by extending the functions $r(t) - 1$ and H by truncating them. This is done in section 2.3 in [**AM14**].

We replace F_0 by a function $F : \widetilde{W} \to \mathbb{R}$ such that

$$F(x,r) := r - 1 \text{ on } \Sigma \times (\frac{1}{2}, \infty)$$

$$F|_{\widetilde{W}\setminus S\Sigma} := -\frac{3}{4}$$

and

$$\frac{\partial F}{\partial r}(x,r) \geq 0 \text{ for } (x,r) \in S\Sigma.$$

Next we truncate H. For $\kappa > 0$, let $\epsilon_\kappa \in C^\infty([0,\infty),[0,1])$ be defined by

$$\epsilon_\kappa(r) := \begin{cases} 1 & \text{for } r \in [e^{-\kappa}, e^{\kappa}] \\ 0 & \text{for } r \in [0, e^{-2\kappa}] \cup [e^{\kappa}+1, \infty) \end{cases} \tag{7.33}$$

with the following decay behavior

$$0 \leq \epsilon_\kappa'(r) \leq 2e^{2\kappa} \text{ for } r \in [e^{-2\kappa}, e^{2\kappa}]$$

$$-2 \leq \epsilon_\kappa'(r) \leq 0 \text{ for } r \in [e^{\kappa}, e^{\kappa}+1]$$

Define $H_t^\kappa : \widetilde{W} \to \mathbb{R}$ by setting

$$H_t^\kappa := \begin{cases} -\dfrac{3}{4} & \text{on } \widetilde{W} \setminus S\Sigma \\ \epsilon_\kappa(r) H_t(x,r) & \text{for } (x,r) \in S\Sigma \end{cases} \tag{7.34}$$

Then define the Rabinowitz Floer action functional on $\mathcal{L}\widetilde{W}$.

Definition 7.44. Define

$$\mathcal{A}_\varphi^\kappa : \mathcal{L}\widetilde{W} \times \mathbb{R} \to \mathbb{R}$$

by

$$\mathcal{A}_\varphi^\kappa(u,\eta) := \int_0^1 u^*\lambda - \eta \int_0^1 \beta(t) F(u(t))\, dt - \int_0^1 \dot{\chi}(t) H_{\chi(t)}^\kappa(u(t))\, dt$$

Definition 7.45. Let $\varphi \in \mathcal{P}\operatorname{Cont}_0(\Sigma,\xi)$. Define $\rho_t : \Sigma \to (0,\infty)$ by $\varphi_t^*\alpha = \rho_t\alpha$. Define the constant $\kappa(\varphi)$ by

$$\kappa(\varphi) := \max_{t\in[0,1]} \left| \int_0^t \max_{x\in\Sigma} \frac{\dot{\rho}_\tau(x)}{\rho_\tau(x)^2}\, d\tau \right| \tag{7.35}$$

The following lemma is necessary to define the Rabinowitz Floer homology $\mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W)$ for $a,b \in [-\infty,\infty] \setminus \operatorname{Spec}(\varphi)$. It should be viewed as a technical compactness result. We will consider only flow lines with energy less than some constant depending on κ to guarantee compactness (i.e. we truncate the groups).

Lemma 7.46. *Suppose $\kappa > \kappa(\varphi)$. If $(u,\eta) \in \operatorname{Crit}(\mathcal{A}_\varphi^\kappa)$ then $u(S^1) \subseteq S\Sigma$ and for $u(t) = (x(t), r(t))$ we have $r(S^1) \subseteq (e^{-\frac{\kappa}{2}}, e^{\frac{\kappa}{2}})$.*

We can now define $\mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W)$ as the gradient flow lines when choosing a suitable almost complex structure. The following is a sketch of the precise definition.

As in the Floer case, we start by picking an almost complex structure. As the manifold $S\Sigma$ is non-compact, we need some regularity to ensure J-holomorphic curves do not leave compact sets. This regularity is captured by the almost complex structure. In [**AM14**], a stronger notion of convexity is used, where we demand $\omega_f = \omega$ outside a compact set. This means that when we consider the symplectic manifold $(S\Sigma, d(r\alpha))$ then J is convex whenever there exists $S_0 > 0$ such that

$$-d(dr \circ J_t) = d(r\alpha)$$

on $\Sigma \times [S_0, \infty)$. Here, $f = r$ is our function.

Recall that we can embed $S\Sigma \hookrightarrow W$. We denote $\mathcal{J}_{\text{conv}}(W)$ the set of time-dependent almost complex structures $J = \{J_t\}_{t \in S^1}$ such that $J|_{S\Sigma} \in \mathcal{J}_{\text{conv}}(S\Sigma)$. Let $J \in \mathcal{J}_{\text{conv}}(W)$. Define the L^2-inner product on $\mathcal{L}W \times \mathbb{R}$ as in equation (7.9). For $(u, \eta) \in \mathcal{L}W \times \mathbb{R}$ and $\xi, \xi' \in \Gamma(u^*TW)$ and $Y, Y' \in \mathbb{R}$ set

$$\langle\langle(\xi, Y), (\xi', Y'\rangle\rangle_J := \int_0^1 d\lambda(J_t \xi, \xi')\, dt + YY' \tag{7.36}$$

We are interested in the gradient of $\mathcal{A}^\kappa_\varphi$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle_J$. Let φ be non-degenerate, $\kappa > \kappa(\varphi)$ and $J \in \mathcal{J}_{\text{conv}}(W)$. By assumption $\mathcal{A}^\kappa_\varphi$ is Morse-Bott, not necessarily Morse. Therefore, we define the Morse-Bott homology with cascades as in section A.2 in the appendix, with several small modifications. Instead of counting flow lines, we count flow lines with cascades. A short overview can be found in the appendix in A.2. This approach was first used by U. Frauenfelder in [**?**]. In many interesting cases however, $\mathcal{A}_\varphi$ is Morse and we can take $g \equiv 0$, thus defining the normal Morse homology.

Choose a Morse function $g : \text{Crit}(\mathcal{A}^\kappa_\varphi) \to \mathbb{R}$ and a Riemannian metric ρ on $\text{Crit}(\mathcal{A}^\kappa_\varphi)$ such that the negative gradient flow of g with respect to ρ is Morse-Smale.
Let $w^-, w^+ \in \text{Crit}(g)$ denote $w^\pm = (u^\pm, \eta^\pm)$. We are in the situation where we can define the moduli space of gradient flow lines with cascades of the quadruple $(\mathcal{A}^\kappa_\varphi, g, \langle\langle \cdot \rangle\rangle_J, \rho)$, which we denote

$$\mathcal{M}_{w^-, w^+}(\mathcal{A}^\kappa_\varphi, g, J, \rho)$$

The grading on $\text{Crit}(g)$ is the following.

$$\mu(u, \eta) := \begin{cases} \mu_C Z(u) - \dfrac{1}{2} \dim_{(u,\eta)} \text{Crit}(\mathcal{A}^\kappa_\varphi) + \text{ind}_g(u, \eta) & \text{for } \eta > 0 \\ \mu_C Z(u) - \dfrac{1}{2}{}_{(u,\eta)} \text{Crit}(\mathcal{A}^\kappa_\varphi) + \text{ind}_g(u, \eta) & \text{for } \eta < 0 \\ 1 - n + \text{ind}_g(u, \eta) & \text{for } \eta = 0 \end{cases} \tag{7.37}$$

Here, $\mu_C Z$ denotes the Conley-Zehnder index of the loop $t \mapsto u(\frac{t}{\eta})$ which was defined in Chapter 4.The number $\dim_{(u,\eta)} \text{Crit}(\mathcal{A}\kappa_\varphi)$ the local dimension of $\text{Crit}(\mathcal{A}^\kappa_\varphi)$ at (u, η). Recall that if $\mathcal{A}_\varphi$ is Morse-Bott then this number is defined as $\text{Crit}\, A_\varphi$ is a submanifold of M. However, the dimension may vary in the components.

Let $-\infty < a < b < \infty$ such that $a, b \notin \text{Spec}(\varphi)$ and define chain groups by

$$\text{RFC}^{(a,b)}_*(\mathcal{A}^\kappa_\varphi, g) := \text{Crit}^{(a,b)}_*(g) \otimes \mathbb{Z}_2$$

where

$$\mathrm{Crit}_*^{(a,b)}(g) := \{w \in \mathrm{Crit}(g) \mid a < \mathcal{A}_\varphi < b\}.$$

Note we need $a, b \notin \mathrm{Spec}(\varphi)$, or this is not well defined because of the definition of $\mathrm{Spec}(\varphi)$ in equation 7.29. The moduli spaces $\widehat{\mathcal{M}}(\mathcal{A}_phi, g, J, \rho)$ are finite dimensional smooth manifolds. As before, the zero dimensional components are compact. The boundary operator is defined by counting the number of elements in these zero dimensional components. From compactness and gluing arguments it follows that

$$\partial^2 = 0$$

hence we can look at the homology associated to this complex. This way, we define

$$\mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W) := H_*(\mathrm{RFC}_*^{(a,b)}(\mathcal{A}_\varphi, g), \partial).$$

Then this homology is independent of the particular choices made.

Taking direct limits in a and b we define the following spaces.

Definition 7.47.

$$\mathrm{RFH}_*^{b}(\mathcal{A}_\varphi, W) := \varinjlim_{a\downarrow-\infty} \mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W)$$

$$\mathrm{RFH}_*^{(a,\infty)}(\mathcal{A}_\varphi, W) := \varprojlim_{b\uparrow\infty} \mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W)$$

$$\mathrm{RFH}_*(\mathcal{A}_\varphi, W) := \varinjlim_{a\downarrow-\infty} \varprojlim_{b\uparrow\infty} \mathrm{RFH}^{(a,b)}(\mathcal{A}_\varphi, W)$$

In the following part we list properties of the Rabinowitz-Floer homology groups. This part is technical. We will just state the properties and then use them to prove Theorem 7.37.

Proposition 7.48. *The Rabinowitz Floer homology* $\mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W)$ *has the following properties.*

(i) *There is a universal object* $\mathrm{RFH}_*(\Sigma, W)$ *that comes with canonial isomorphisms*

$$\zeta_\varphi : \mathrm{RFH}_*(\Sigma, W) \to \mathrm{RFH}_*(\mathcal{A}_\varphi, W).$$

Let $\varphi, \psi \in \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$. *Then there is a map*

$$\zeta_{\varphi,\psi} : \mathrm{RFH}_*(\mathcal{A}_\varphi, W) \to \mathrm{RFH}_*(\mathcal{A}_\psi, W)$$

such that

$$\zeta_\psi = \zeta_{\varphi,\psi} \circ \zeta_\psi.$$

These maps define for every non-zero class $Z \in \mathrm{RFH}_*(\Sigma, W)$ *a particular non-zero class* $Z_\varphi \in \mathrm{RFH}_*(\mathcal{A}_\varphi, W)$ *by two conditions:*

$$Z_{\mathrm{Id}_\Sigma} = Z \in \mathrm{RFH}_*(\Sigma, W)$$

and

$$\zeta_{\varphi,\psi}(Z_\varphi) = Z_\psi.$$

(ii) *Let $a \leq b \leq \infty$. There exists a natural map*

$$j_\varphi^{a,b} : \mathrm{RFH}_*^a(\mathcal{A}_\varphi, W) \to \mathrm{RFH}_*^b(\mathcal{A}_\varphi, W)$$

and a natural map

$$p_*^{a,b} : \mathrm{RFH}_*^b(\mathcal{A}_\varphi, W) \to \mathrm{RFH}_*^{(a,b)}(\mathcal{A}_\varphi, W).$$

Furthermore, if $\mathrm{Spec}(\varphi) \cap [a, b] = \emptyset$ *then* $j_\varphi^{a,b}$ *is an isomorphism and* $p_\varphi^{a,b}$ *is the zero map.*

(iii) *There exist maps*

$$\zeta_{\varphi,\psi}^a : \mathrm{RFH}_*^a(\mathcal{A}_\varphi, W) \to \mathrm{RFH}_*^{a+K(\varphi,\psi)}(\mathcal{A}_\psi, W)$$

for some constant $K(\varphi, \psi) \geq 0$. Denote H_t and K_t the contact Hamiltonians associated with φ and ψ respectively. Then $K(\varphi, \psi)$ satisfies

$$K(\varphi, \psi) \leq e^{\max(\kappa(\varphi),\kappa(\psi))} \max(\|H - K\|_+, 0) \tag{7.38}$$

Furthermore, for all $Z \in \mathrm{RFH}_*^a(\mathcal{A}_\varphi, W)$ *we have*

$$\zeta_{\varphi,\psi}(j_\varphi^a(Z)) = j_\psi^{a+K(\varphi,\psi)}(\zeta_{\varphi,\psi}^a(Z))$$

(iv) Σ *is contained in* $\mathrm{Crit}(\mathcal{A}_{\mathrm{Id}_\Sigma})$ *as a Morse-Bott component as the constants. Let $\epsilon > 0$ such that $\epsilon < \wp(\Sigma, \xi)$. Then there is a canonical isomorphism*

$$\mathrm{RFH}_*^{(-\epsilon,\epsilon)}(\Sigma, W) \simeq H_{*+n-1}(\Sigma, \mathbb{Z}_2).$$

Using the above proposition we can define the following.

Definition 7.49. Let $Z \in \mathrm{RFH}_*(\Sigma, W)$ be a non-zero class and let $\varphi \in \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ be non-degenerate. Define the spectral number

$$\widetilde{c}(\varphi, Z) := \inf\{a \in \mathbb{R} \mid Z_\varphi \in j_\varphi^a(RFH_*^a(\mathcal{A}_\varphi, W))\}$$

Note that this definition works for non-degenerate φ only. To extend $\widetilde{c}(\,\cdot\,, Z)$ to the whole of $\mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ we make the following definition.

Definition 7.50. Let $\varphi \in \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ be a non-degenerate path. Let φ_k be a sequence of non-degenerate paths such that $\varphi_k \to \varphi$ in C^2. Let $\widetilde{c}(\,\cdot\,, Z)$ be as above. We set

$$c(\varphi, Z) = \lim_k \widetilde{c}(\varphi_k, Z).$$

We prove that the map $c(\,\cdot\,, Z)$ has the required properties listed in Proposition 7.38. We need the following Lemma which is part of Lemma 3.3 in [**AM14**].

Lemma 7.51. *Let $\varphi \in \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$. Then $c(\varphi, Z) \in \mathrm{Spec}(\varphi)$.*

Proof of Proposition 7.38. We follow the order of properties stated in the proposition. We prove (iv) first, then use it to prove (iii). The continuity statement in (iii) is required to prove both (i) and (ii).

(iv) Let $\varphi, \psi \in \mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ be non-degenerate. Denote H_t and K_t the contact Hamiltonians associated to φ and ψ respectively. We first prove the following Claim which is a direct result of the properties listed in Proposition 7.48.

Claim 12. For two non-degenerate paths φ and ψ we have

$$c(\psi, Z) \leq c(\varphi, Z) + K(\varphi, \psi).$$

PROOF OF CLAIM 7.5. Let $a \in \mathbb{R}$ be such that $Z_\varphi \in j^a_\varphi(\mathrm{RFH}^a_*(\mathcal{A}_\varphi, W))$. Now note that Proposition 7.48 (i) says that

$$Z_\psi = \zeta_{\varphi,\psi}(Z_\varphi).$$

Furthermore, Proposition 7.48 (iii) says that

$$\zeta_{\varphi,\psi} \circ j^a_\varphi = j^{a+K(\varphi,\psi)}_\psi(\zeta^a_{\varphi,\psi}(Z)).$$

From this it is immediate that

$$Z_\psi \in j^{a+K(\varphi,\psi)}_\psi(\mathrm{RFH}^{a+K(\psi,\varphi)}_*(\mathcal{A}_\psi, W)).$$

Therefore, it follows that $c(\psi, Z) \leq c(\psi, Z) + K(\varphi, \psi)$ as required, as this still holds for the infimum. This proves Claim 7.5. $\square$

We now combine the inequality of Claim 7.5 with equation (7.38) in Proposition 7.48 (iii) which yields

$$c(\psi, Z) \leq c(\varphi, Z) + e^{\max(\kappa(\varphi),\kappa\psi)} \max(\|H_t - K_t\|_+, 0) \tag{7.39}$$

Suppose now that $H_t(x) \leq K_t(x)$ for all $x \in \Sigma$ and all $t \in [0, 1]$. Then

$$c(\psi, Z) \leq c(\varphi, Z)$$

is immediate. The inequality is still satisfied when taking the limits to extend $c(\,\cdot\,, Z)$ to the whole of $\mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$. This proves property (iv).

(iii) Note that if $\varphi_k \to \varphi$ in C^2, then also $\kappa(\varphi_k) \to \kappa(\varphi)$ by the definition of $\kappa(\varphi)$ in equation (7.35) as the integral. Denote H_k the contact Hamiltonian associated to φ_k and H the contact Hamiltonian associated to φ. Then by C^2 convergence of φ_k and the definition of H_k as

$$H_k \circ \varphi^t_k := \alpha(\frac{d}{dt}\varphi^t_k)$$

we have that $\|H - H_k\|_+ \to 0$. Then from equation (7.39) it follows that $c(\varphi_k, Z)$ converges. Hence when $\mathcal{P}\,\mathrm{Cont}_0(\Sigma, \xi)$ is equipped with the C^2-topology, the function $c(\,\cdot\,, Z)$ is continuous. This proves property (iii).

(i) The first property relies on a property of $\mathrm{Spec}(\varphi)$. We use Lemma 3.8 from [**Sch93**] by M. Schwarz.

Lemma 7.52. *The set* $\mathrm{Spec}(\varphi) \subset \mathbb{R}$ *is residual, meaning that it is nowhere dense and depends only on* φ_1.

By property (iii) we know $c(\,\cdot\,, Z)$ is continuous. Furthermore, we know that $c(\varphi, Z) \in \operatorname{Spec}(\varphi_1)$ for all $\varphi \in \mathcal{P}\operatorname{Cont}_0(\Sigma, \xi)$. Suppose $\varphi \sim \psi$ via a homotopy H with fixed endpoints. Then $c(H_t, Z)$ takes values in the nowhere dense set $\operatorname{Spec}(\varphi_1)$, hence is constant. In particular $c(\varphi, Z) = c(\psi, Z)$. This proves property (i).

(ii) For this property, note that $\operatorname{Spec}(\theta^\lambda) = -\lambda + \operatorname{Spec}(\mathrm{Id}_\Sigma)$. Again by Lemma 7.52, $\operatorname{Spec}(\mathrm{Id}_\Sigma)$ is nowhere dense and by property (iii) the map $c(\,\cdot\,, Z)$ is continuous. As above, this implies
$$c(t \mapsto \theta^{\lambda t}, Z) = -\lambda + c(\mathrm{Id}_\Sigma, Z).$$
This proves property (ii).

We see that $c(\,\cdot\,, Z)$ satisfies all properties listed in Proposition 7.38. Hence this concludes the proof of Proposition 7.38. $\square$

APPENDIX A

Collection of auxillary results

Also, Hamiltonian diffeomorphisms actually preserve the symplectic form. That is, if we define the symplectic group

$$\mathrm{Symp}(M,\omega) = \{\psi \in \mathcal{C}^\infty(M,M) \mid \psi^*\omega = \omega\}$$

of diffeomorphisms the preserve the symplectic form, then

$$\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(M,\omega).$$

This is easily checked by the following computation

Let $\varphi \in \mathrm{Ham}(M,\omega)$, which is generated by some $H : M \to \mathbb{R}$. Then

$$\begin{aligned}
\frac{d}{dt}(\varphi_H^t)^*\omega &= {\varphi_H^t}^*(\mathcal{L}_{X_H}\omega) \\
&= {\varphi_H^t}^*(d\iota_{X_H}\omega + \iota_{X_H}d\omega) \\
&= {\varphi_H^t}^*(ddH) \\
&= 0
\end{aligned}$$

Here we used the Cartan formula, the definition of X_H and the fact that ω is closed. Now note $\varphi_H^0 = \mathrm{Id}_M$ so that $(\varphi_H^0)^*\omega = \omega$, so the same holds for $\varphi = \varphi_H^1$ by the above calculation, so $\varphi \in \mathrm{Symp}(M,\omega)$.

A.1. Morse theory

We provide a short overview of Morse theory. This is used in Chapter 6. Because many concepts in Morse theory are similar to those in Floer theory, a brief overview will suffice. We inform the reader that Morse theory may be approached in two different way. The classical way is as in Milnor's standard text [**Mil63**]. A more modern viewpoint, similar to how we treated Floer homology (involving Fredholm theory) can be found in [**Sch93**]. We will mostly follow the classical approach as this is more accessible. We base this approach on [**Nic07**].

In this section, let (M,ω) be a symplectic manifold. Let $J \in \mathcal{J}(M,\omega)$ with $g_J = \omega(\cdot\,, J\,\cdot)$ the associated metric.

Let $f : M \to \mathbb{R}$ be a smooth function. A point $x \in M$ is a critical point if for $df(x) : T_xM \to \mathbb{R}$ we have $df(x) = 0$ (when viewed as an element of T_x^*M). At a critical point x, we can construct a bilinear form which is known as the Hessian of f at x. We need Lemma 1.6 from [**Nic07**].

Lemma A.1. *Let $f \in C^\infty(M, \mathbb{R})$ and $x \in \mathrm{Crit}(f)$. Let $X, X', Y, Y' \in \Gamma(M, TM)$ be smooth vector fields on M such that $X(x) = X'(x)$ and $Y(x) = Y'(x)$. Denote $X(f) = df(X)$. Then*

$$X(Y(f))(x) = X'(Y'(f))(x) = Y(X(f))(x).$$

This is a small computation that depends on the fact that $x \in \mathrm{Crit}(f)$. Using the lemma, we define the Hessian of f at x.

Definition A.2. Let $x \in \mathrm{Crit}(f)$. Then define

$$\mathrm{Hess}_f(x) : T_xM \times T_xM \to \mathbb{R}$$

by

$$\mathrm{Hess}_f(x)(v, w) := X(Y(f))(x)$$

where $X, Y \in \Gamma(M, TM)$ such that $X(x) = v$ and $Y(x) = w$.

Remark. In local coordinates the Hessian is probably familiar to the reader. Let $(x_1, \dots, x_n)$ be local coordinates around x with $x^i(x) = 0$. Then $\mathrm{Hess}_f(x)$ is a matrix which has as its i, j^th entry the function

$$\frac{\partial^2 f}{\partial x_i \partial x_j}\Big|_x.$$

Let us rephrase to avoid confusion. In local coordinates, $X = \sum_i X^i \frac{\partial}{\partial x_i}$. Then the Hessian of f at x is given by

$$\mathrm{Hess}_f(x) := \sum_{i,j} h_{ij} X^i Y^j$$

with

$$h_{ij} := \frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

By Lemma A.1 this Definition does not depend on the choice of X and Y. Furthermore, the map $\mathrm{Hess}_f(x)$ is bilinear and symmetric also by Lemma A.1. We define what it means for $x \in \mathrm{Crit}(f)$ to be non-degenerate, in terms of this Hessian.

Definition A.3. Let $f \in C^\infty(M, \mathbb{R})$ and $x \in \mathrm{Crit}(f)$. Then x is called non-degenerate if and only if $\mathrm{Hess}_f(x)$ is non-degenerate. A function $f \in C^\infty(M, \mathbb{R})$ is called Morse if all $x \in \mathrm{Crit}(f)$ are non-degenerate.

Remark. Recall that a bilinear form $b : V \times V \to \mathbb{R}$ is non-degenerate if $b(v, w) = 0$ for all $w \in V$ if and only if $v = 0$. In the case of local coordinates, a critical point is non-degenerate whenever its Hessian matrix is such that $\det(h_{ij}) \neq 0$.

Recall that for a general bilinear form, we have the notion of an index. Let V be a finite-dimensional vector space and $b : V \times V \to \mathbb{R}$ be a bilinear symmetric non-degenerate form. Then there exists a basis $(e_1, \ldots, e_n)$ of V such that for $v = \sum_i v^i e_i$ we have

$$b(v, v) = -|v^1|^2 - \ldots - |v^\lambda|^2 + |v^{\lambda+1}|^2 + \ldots |v^n|^2.$$

Then the number λ does not depend on the choice of basis. We define

$$\mathrm{ind}(b) = \lambda.$$

We extend this to the Hessian.

Definition A.4. Let $f \in C^\infty(M, \mathbb{R})$ be a Morse function and $x \in \mathrm{Crit}(f)$. Then we define the Morse index of x to be

$$\mathrm{ind}_f(x) := \mathrm{ind}(\mathrm{Hess}_f(x)).$$

The beauty of Morse functions is that around a critical point, there exists coordinates such that they look like a quadratic polynomial. This is the famous Morse lemma, which is Corollary 1.17 in [**?**].

Lemma A.5. *Let $f \in C^\infty(M, \mathbb{R})$ be Morse with $x \in \mathrm{Crit}(f)$ such that $\mathrm{ind}_f(x) = \lambda$. Then there exists coordinates $(x_1, \ldots, x_n)$ centered at x such that*

$$f = f(x) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{j=\lambda+1}^{n} x_j^2.$$

These local coordinates give a chart (U, φ) of some open set around x and a diffeomorphism $\varphi : U \to \mathbb{R}^n$. A chart such that the Morse lemma holds is known as a Morse chart.

The goal of this section is to define the Morse homology. This homology is generated by the critical points of a Morse function, and the boundary operator counts the number of flow lines associated to a particular vector field running between critical points. In the Floer case, we needed some conditions to make sure that the gradient of the action functional connects critical points and to ensure that $\partial_J^2 = 0$. For Morse homology, we want similar results; we want flows that connect critical points and we want the boundary operator to be well-defined. We discuss the conditions on our vector field.

Definition A.6. Let $f \in C^\infty(M, \mathbb{R})$ be a Morse function. We say a $X \in \Gamma(M, TM)$ is a gradient-like vector field with respect to f if

(i) We have $df(x)X < 0$ for all $x \in M \setminus \mathrm{Crit}(f)$.

(ii) For every $p \in \mathrm{Crit}(f)$, there is a Morse chart centered at p such that

$$X = -2\sum_{i=1}^{\lambda} x^i \frac{\partial}{\partial x^i} + 2\sum_{j=\lambda+1}^{n} x^j \frac{\partial}{\partial x^j}$$

for $\lambda = \mathrm{ind}_f(p)$.

Philosophically, in a Morse chart X looks like the standard gradient on $\mathbb{R}^n$. The number of dimensions flowing down from a critical point is then precisely its index. These gradient-like vector fields always exists. This follows from partition on unity (we assume our manifolds to be paracompact). We refer the reader to [**AD14**] 2.1.c. In particular we can pick a Riemannian metric g such that the gradient ∇f of f with respect to g is a gradient-like vector field.

These gradient-like vector fields have the following useful property we wanted.

Proposition A.7. *Let M be a compact manifold and let $\gamma : \mathbb{R} \to M$ be the trajectory of a gradient-like vector field X. Then there exist $x^-, x^+ \in \mathrm{Crit}(f)$ such that*

$$\lim_{x \to \pm} \gamma(s) = x^{\pm}.$$

We can observe the space formed by everything flowing away from a critical point by a gradient-like vector field and likewise for flowing towards. We make this precise.

Definition A.8. Let $f \in C^\infty(M, \mathbb{R})$ be Morse and $x \in \mathrm{Crit}(f)$. Denote φ^s the flow of a gradient-like vector field X. Define the stable manifold

$$W^s(x) := \left\{ p \in M \mid \lim_{s \to +\infty} \varphi^s(p) = x \right\} \tag{A.1}$$

and the unstable manifold

$$W^u(x) := \left\{ p \in M \mid \lim_{s \to -\infty} \varphi^s(p) = x \right\} \tag{A.2}$$

As the name suggests, these spaces are indeed submanifolds of M. They are diffeomorphic to open disks with the property that

$$\dim W^u(x) = \operatorname{codim} W^s(x) = \mathrm{ind}_f(x).$$

See Proposition 2.1.5 in [**AD14**].

We want the moduli space of solutions running between critical points to be a manifold of the right dimension. Therefore, we require an additional condition on the gradient-like vector fied X.

Definition A.9. Let $f \in C^\infty(M, \mathbb{R})$ be Morse and X a gradient-like vector field. Then we say the pair (f, X) satisfies the Smale condition if for all $x, y \in \mathrm{Crit}(f)$ we have

$$W^u(x) \pitchfork W^s(b).$$

Note that this implies that for any two $x, y \in \mathrm{Crit}(f)$, we have

$$\dim(W^u(x) \cap W^s(y)) = \mathrm{ind}_f(x) - \mathrm{ind}_f(y).$$

Remark. Such gradient-like vector fields satisfying the Smale condition are still generic. Roughly speaking, for any gradient like X there exists a gradient-like X' satisfying the Smale condition close to X. See Theorem 2.2.5 in [**AD14**].

We can then consider the space of trajectories running from x^- to x^+ for a pair (f, X) where $f \in C^\infty(M, \mathbb{R})$ is Morse and X is gradient-like satisfying the Smale condition. Define

$$\mathcal{M}(x^-, x^+; f, X) := \{x \in M \mid x \in W^u(x^-) \cap W^s(x^+)\}.$$

Like in the Floer case, there is an action of $\mathbb{R}$ on $\mathcal{M}(x^-, x^+)$. Let $x \in \mathcal{M}(x^-, x^+)$ and $s \in \mathbb{R}$. Denote φ_X^t the flow of X. Then $s \cdot x = \varphi_X^s(x)$ is an action. Consider the quotient manifold

$$\widehat{\mathcal{M}}(x^-, x^+; f, X) := \widehat{\mathcal{M}}(x^-, x^+; f, X)/\mathbb{R}.$$

This is a manifold of dimension $\mathrm{ind}_f(x^-) - \mathrm{ind}_f(x^+) - 1$.

We can now define the Morse complex and Morse homology. Let M be a compact manifold, $f \in C^\infty(M, \mathbb{R})$ a Morse funtion and X a gradient-like vector field satisfying the Smale condition.

Definition A.10. The Morse chains are the vector spaces

$$\mathrm{CM}_k(f) := \bigoplus_{\substack{x \in \mathrm{Crit}(f) \\ \mathrm{ind}_f(x) = k}} \langle x \rangle \mathbb{Z}_2.$$

The chains are given like in Floer homology by counting the number of trajectories between the critical points.

Definition A.11. Define $\partial_X : \mathrm{CM}_k(f) \to \mathrm{CM}_{k-1}(f)$ by

$$\partial_X(x) := \sum_{\substack{y \in \mathrm{Crit}(f) \\ \mathrm{ind}_f(y) = k-1}} \eta(x, y) y$$

with $\eta(x, y) = \#\widehat{\mathcal{M}}(x, y) \mod 2$.

We need compactness results for the spaces $\widehat{\mathcal{M}}(x, y)$ in order to prove that $\eta(x, y)$ is well-defined and that $\partial_X^2 = 0$. We state these as Theorems here and refer the reader to [**AD14**].

The first theorem implies that $\eta(x, y)$ is well-defined and is Corollary 3.2.4 in [**AD14**].

Theorem A.12. *Suppose $x, y \in \mathrm{Crit}(f)$ such that $\mathrm{ind}_f(x) = \mathrm{ind}_f(y) + 1$. Then $\widehat{\mathcal{M}}(x, y)$ is a compact 0-dimensional manifold, hence a finite set.*

The following theorem implies that $\partial_X^2 = 0$ and is Theorem 3.2.7 in [**AD14**].

Theorem A.13. *Let $x, z \in \mathrm{Crit}(f)$ such that $\mathrm{ind}_f(x) = \mathrm{ind}_f(z) + 2$. Then*

$$\overline{\mathcal{M}}(x, z) := \widehat{\mathcal{M}}(x, z) \bigcup_{\substack{y \in \mathrm{Crit}(f) \\ \mathrm{ind}_f(y) = \mathrm{ind}_f(x) - 1}} \widehat{\mathcal{M}}(x, y) \times \widehat{\mathcal{M}}(y, z)$$

is a compact 1-dimensional manifold with boundary

$$\partial\overline{\mathcal{M}}(x,z) = \bigcup_{\substack{y\in\mathrm{Crit}(f)\\ \mathrm{ind}_f(y)=\mathrm{ind}_f(x)-1}} \widehat{\mathcal{M}}(x,y)\times\widehat{\mathcal{M}}(y,z).$$

Then the Morse homology of M for the pair (f,X) is given by

$$\mathrm{HM}_*(M;f,X) := H_*(\mathrm{CM}_*(f),\partial_X).$$

The homology thus defined depends on the choice of Morse function f and a gradient-like vector field X satisfying the Smale condition. However, it can be proven that the homology obtained is independent of the choice of (f,X). This is proven in Section 3.4 in [**AD14**].

For a particular manifold M, the pair (f,X) can be used to define a cellular decomposition of M. In this way, the Morse homology and the cellular homology coincide. Thus, the Morse homology of M is isomorphic to the integral homology of M. This is what we exploit in Chapter 6 to prove the Arnold conjecture. For a proof, we refer the reader to [**AD14**] Section 4.9.

A.2. Morse-Bott theory

Morse-Bott homology is defined as the homology associated to a chain complex generated by the critical points of a Morse-Bott function. The boundary operator is defined by counting gradient flow lines with cascades. This theory is required to define the Rabinowitz Floer homology $\mathrm{RFH}_*(\Sigma,W)$ of a hypersurface of contact type Σ in a Liouville manifold W, as the functional $\mathcal{A}_\varphi$ is not Morse in general but still Morse-Bott.

Remark. There are many ways to compute the homology of a smooth closed manifold using Morse-Bott functions. See the paper [**?**] by D.E. Hurtubise for an outline. The approach using cascades is what we explain in this section. Another path is provided by peturbing the Morse-Bott function to a Morse function. A additional ways are using a multicomplex, or using spectral sequences. We choose the cascade approach as this is the one used in [**AM14**] and [**AF**]. Flow lines with cascades where first introduced by U. Frauenfelder (one of the authors of [**AF**]) in his PhD thesis [**?**]. We follow the exposition in Appendix A of [**Fra08**].

We begin by defining a Morse-Bott function.

Definition A.14. Let (M,g) be a Riemannian manifold. A function $f\in C^\infty(M,\mathbb{R})$ is called Morse-Bott if the following conditions are satisfied.

(i) $\mathrm{Crit}(f)$ is a submanifold of M

(ii) For each $x \in \mathrm{Crit}(f)$ we have $T_x \mathrm{Crit}(f) = \ker(\mathrm{Hess}(f)(x))$.

The Morse-Bott condition implies the Morse condition of Definition **??**. Conversely, denote C_x the connected component of $\mathrm{Crit}(f)$ containing x. Then a Morse function such that $C_x = \{x\}$ for all $x \in \mathrm{Crit}(f)$ is a Morse-Bott function. We demonstrate a simple example of a Morse-Bott function and a function that is not Morse-Bott.

Example A.15. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be the functions $f(x_1, \dots, x_n) = x_1^2$ and $g(x_1, \dots, x_n) = x_1^3$. Then f is Morse-Bott whereas g is not. For f, the critical manifold $\mathrm{Crit}(f)$ is the hyperplane $x_1 = 0$. The condition $T_x \mathrm{Crit}(f) = \ker(\mathrm{Hess}(f)(x))$ is also satisfied for any $x \in \mathrm{Crit}(f)$. For g, the critical manifold $\mathrm{Crit}(g)$ is also the hyperplane $x_1 = 0$. However, the condition $T_x \mathrm{Crit}(g) = \ker(\mathrm{Hess}(g)(x))$ is not satisfied.

Morse-Bott functions have the property that their gradient flow lines are bounded and run between critical points (provided that M is compact). This is made precise in the following theorem.

Theorem A.16. *Let (M, g) be a compact Riemannian manifold and $f : M \to \mathbb{R}$ a Morse-Bott function. Suppose $u : \mathbb{R} \to M$ is a negative gradient flow line, that is, a solution of*

$$\dot{u}(t) = -\nabla f(u(t)).$$

Then there exist $x^{\pm} \in \mathrm{Crit}(f)$ and constants $\delta > 0$ and $K > 0$ such that

$$\lim_{s \to \pm\infty} u(s) = x^{\pm}$$

and

$$\left| \frac{\partial u}{\partial t}(t) \right| \leq K e^{-\delta t}.$$

Proof. This is a standard result in Morse-Bott theory. See for example Theorem A.3 in [**Fra08**]. □

We define flow lines with cascades. In the following, let (M, g) be a compact Riemannian manifold and $f : M \to \mathbb{R}$ a Morse-Bott function. Fix a Riemannian metric on (the submanifold) $\mathrm{Crit}(f)$ and a Morse function $h : \mathrm{Crit}(f) \to \mathbb{R}$ such that h satisfies the Morse-Smale condition from Definition **??**.
We look at critical points of h and define a grading on them. Let $x \in \mathrm{Crit}(h)$. Then the Morse index $\mathrm{ind}_h(x)$ as in Definition **??** makes sense, as h is Morse. Furthermore, the index as defined in Definition **??** still makes sense for $x \in \mathrm{Crit}(f)$ as the number of strictly negative eigenvalues of $d^V(df)(x)$. We define the Morse-Bott index as follows.

Definition A.17. Let $f : M \to \mathbb{R}$ be a Morse-Bott function and $h : \mathrm{Crit}(f) \to \mathbb{R}$ a Morse function. Let $p \in \mathrm{Crit}(h)$. We define the Morse-Bott index of p to be

$$\mathrm{ind}(p) = \mathrm{ind}_{f,h}(x) := \mathrm{ind}_f(x) + \mathrm{ind}_h(x).$$

We are now ready to define flow lines with cascades. The idea is to flow between critical points of h, flowing along negative gradient lines of h with respect to ρ when in $\mathrm{Crit}(f)$ and flowing between components of $\mathrm{Crit}(f)$ via the negative gradient flow lines of f with respect to g. The following definition makes this idea precise. Recall the definition of the stable and unstable manifolds of a Morse function at a critical point in equations (A.1) and (A.2).

Definition A.18. Let $p, p' \in \mathrm{Crit}(h)$ and $m \in \mathbb{N}$. A flow line from p to p' with m cascades consists of a collection of maps and a collection of times

$$((x_k)_{1\leq k\leq m}, (t_k)_{1\leq k\leq m-1})$$

where $x_k \in C^\infty(\mathbb{R}, M)$ and $t_k \in \mathbb{R}_{\geq 0}$ such that the following holds.

(i) The maps $x_k \in C^\infty(\mathbb{R}, M)$ are non-constant negative gradient flow lines of f with respect to g. They solve

$$\frac{\partial x_k}{\partial s} = -\nabla_g f(x_k).$$

(ii) There exist points $a \in W^u_h(p)$ and $b \in W^s_h(p')$ such that

$$\lim_{s\to-\infty} x_1(s) = a, \ \lim_{s\to\infty} x_m(s) = b.$$

(iii) For $1 \leq k \leq m-1$ there exist negative gradient flow lines of h with respect to ρ. That is, maps $u_k \in C^\infty(\mathbb{R}, \mathrm{Crit}(f))$ which solve

$$\frac{\partial u_k}{\partial s} = -\nabla_\rho h(u_k)$$

such that

$$\lim_{s\to\infty} x_k(s) = u_k(0), \ \lim_{s\to-\infty} x_{k+1}(s) = u_k(t_k).$$

The above definition of a flow line with cascades is visualized in Figure 1 for the case $m = 2$ for two critical points $p, p' \in \mathrm{Crit}(h)$.

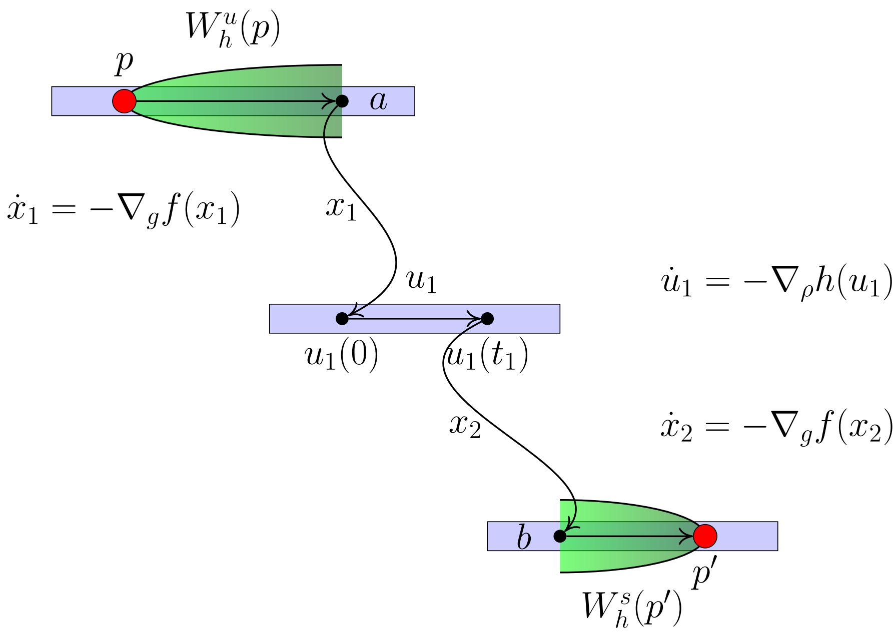

Figure 1. A gradient flow line from p to p' with two cascades

Here, the blue boxes are the submanifold $\mathrm{Crit}(f)$. From p the gradient flow of h meets a negative asymptotic end a of the gradient flow line x_1 of f. The positive asymptotic end of x_1 converges to some point in $\mathrm{Crit}(f)$. This is the first cascade. From this point, we flow via the gradient flow of h on $\mathrm{Crit}(f)$, which we call u_1 until we encounter some other negative asymptotic end of a gradient flow line x_2 of f after some time t_1. Figure 1 depicts the situation with two cascades.

We look at the moduli space of flow lines with m cascades running from p to p' denoted

$$\mathcal{M}_m(p, p').$$

We quotient out reparametrization. Note that there is a reparametrization action of $\mathbb{R}$ on gradient flow lines of h connecting two critical points on the same level (i.e. flow lines in $\mathcal{M}_0(p, p')$) and an action of $\mathbb{R}^m$ on $\mathcal{M}_m(p, p')$ by shifting time in each cascade. We denote the quotient by these actions by

$$\overline{\mathcal{M}}_m(p, p').$$

The space of all flow lines with cascades running from p to p' we denote

$$\overline{\mathcal{M}}(p, p') := \bigcup_{m \in \mathbb{N}} \overline{\mathcal{M}}_m(p, p').$$

One may wonder whether the above picture is accurate. The following lemma states that cascades can only "run down".

Lemma A.19. *Suppose $f(p) < f(p')$ then $\overline{\mathcal{M}}(p,p') = \emptyset$. If $f(p) = f(p')$ then $\widehat{\mathcal{M}}(p,p') = \widehat{\mathcal{M}}_0(p,p')$. If $f(p) > f(p')$ then $\widehat{\mathcal{M}}(p,p') \setminus \widehat{\mathcal{M}}_0(p,p') = \widehat{\mathcal{M}}(p,p')$.*

To consider homology, we need to take care of compactness. Like previous spaces of trajectories, there is a concept of broken flow lines with cascades.

The important result that follows from this is the following.

Theorem A.20. *Let $p, p' \in \mathrm{Crit}(h)$. Then for a generic choice of Riemannian metric g on M, the space $\widehat{\mathcal{M}}(p,p')$ is a smooth finite dimensional manifold of dimension*

$$\dim \widehat{\mathcal{M}}(p,p') = \mathrm{ind}(p) - \mathrm{ind}(p') - 1.$$

Furthermore, whenever $\mathrm{ind}(p) - \mathrm{ind}(p') = 1$, the manifold $\widehat{\mathcal{M}}(p,p')$ is a compact zero dimensional manifold, hence a finite set.

The proof is similar to the Floer case we consider in the main part of the thesis. We define a suitable Banach bundle $\mathcal{E} \to \mathcal{B}$ with a section $\mathcal{F} : \mathcal{B} \to \mathcal{E}$. Then prove that the vertical derivative of this section is a Fredholm operator of a suitable index. The proof is somewhat complicated because one needs to keep track of the m cascades. We refer the reader to the proof of Theorem A.11 in [**Fra08**].

We can now define the Morse-Bott homology. This is section A.3 in [**Fra08**]. Consider a pair (f,h) of a Morse-Bott function $f : M \to \mathbb{R}$ and a Morse function $h : \mathrm{Crit}(f) \to \mathbb{R}$. Define a chain complex

$$\mathrm{CMB}_k(M; f, h) := \bigoplus_{\substack{c \in \mathrm{Crit}(h) \\ \mathrm{ind}(c) = k}} \mathbb{Z}_2\langle c\rangle \tag{A.3}$$

Similarly to the Floer case, we define a boundary operator. Suppose $\mathrm{ind}(p) - \mathrm{ind}(p') = 1$. Set

$$\eta(p,p') := \#\widehat{\mathcal{M}}(p,p') \mod 2.$$

This is a finite number, as Theorem A.20 asserts that $\widehat{\mathcal{M}}(p,p')$ is a smooth compact manifold of dimension 0. Define the boundary operator

$$\partial_k : \mathrm{CMB}_k(M; f, h) \to \mathrm{CMB}_{k-1}(M; f, h)$$

by setting

$$\partial_k p = \sum_{\mathrm{ind}(p') = k-1} \eta(p,p') p'$$

on the generators $p \in \mathrm{Crit}(h)$ with $\mathrm{ind}(p) = k$ and linearly extending. Analogous to the Floer case, compactness and gluing implies that

$$\partial^2 = 0$$

so that we can define homology groups

$$\mathrm{HMB}_k(M; f, h, g, \rho) := \mathrm{H}_k(\mathrm{CMB}(M; f, h), \partial)$$

It is important that this definition does not depend on the choices made. In this case, the choice of (f, h) and the associated metrics g on M and ρ on $\mathrm{Crit}(f)$. This is indeed the case. The following theorem is Theorem A.17 in [**Fra08**].

Theorem A.21. *Let $(f^\alpha, h^\alpha, g^\alpha, \rho^\alpha)$ and $(f^\beta, h^\beta, g^\beta, \rho^\beta)$ be two regular quadrupels. Then $\mathrm{HMB}_*(M; f^\alpha, h^\alpha, g^\alpha, \rho^\alpha)$ and $\mathrm{HMB}_*(M; f^\beta, h^\beta, g^\beta, \rho^\beta)$ are naturally isomorphic.*

This gives rise to the Morse-Bott homology of M by setting

$$\mathrm{HMB}_*(M) := \mathrm{HMB}(M; f, h, g, \rho)$$

for a regular quadruple (f, h, g, ρ). Note that in the case that f is actually Morse, we can take $h = 0$ throughout this entire construction. This immediatly implies that $\mathrm{HMB}_*(M)$ is isomorphic to $\mathrm{HM}_*(M)$.

A.3. The Chern Class

In this section we establish the existence of the first Chern class c_1, which associates to a bundle $\pi : E \to B$ an element of $H^2(B; \mathbb{Z})$. If B is 2-dimensional, the Chern class is determined by the first Chern number. We first state the axiomatic definition of the first Chern class and then provide a theorem that is relevant to choosing a symplectic trivialization of the bundle x^*TM for $x \in \mathcal{P}(H)$.

Definition A.22 (The first Chern number)**.** There exists a unique functor c_1, the first Chern number, that assigns to every symplectic vector bundle $\pi : E \to \Sigma$ over a closed Riemann surface an integer $c_1(E) \in \mathbb{Z}$ satisfying the following conditions

(naturality) Two symplectic vector bundles E and E' over Σ are isomorphic if and only if they have the same dimension and the same Chern number.

(functoriality) For any smooth map $\varphi : \Sigma' \to \Sigma$ of oriented Riemann surfaces and any symplectic vector bundle $E \to \Sigma$, we have

$$c_1(\varphi^* E) = \deg(\varphi) c_1(E).$$

(additivity) For any two symplectic vector bundles $E_1 \to \Sigma$ and $E_2 \to \Sigma$, we have

$$c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2).$$

(normalization) The Chern number of $T\Sigma$ is

$$c_1(T\Sigma) = 2 - 2g$$

where g denotes the genus of the surface Σ.

The following lemma is what we use to find a trivialization of the bundle x^*TM.

Lemma A.23. *The first Chern number vanishes if and only if the bundle is trivial.*

Proof. This follows directly from the axioms. $\square$

Therefore, we see that the condition that for every smooth map $\alpha : S^2 \to M$ there exists a symplectic trivialization of the bundle $*^T M$ is equivalent to saying that the first Chern class vanishes over TM. The condition on smooth maps is however what we will use.

APPENDIX B

Distributions and Sobolev spaces and Elliptic regularity results

B.1. Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be an open subset. The goal is to define the Sobolev space $W^{k,p}(\Omega)$. The following is based on Chapter 5 of [**Eva98**] and Appendix B.1 of [**MS12**]. We define the weak derivative.

Definition B.1. Let $u : \Omega \to \mathbb{R}$ be locally integrable (so $u \in L^1_{\mathrm{loc}}(\Omega)$) and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index and denote $|\alpha| = \alpha_1 + \dots + \alpha_n$ the order of α. We call a locally integrable $u_\alpha : \Omega \to \mathbb{R}$ the weak derivative of u corresponding to α if and only if

$$\int_\Omega u(x)\partial^\alpha \varphi(x)\,dx = -1^{|\alpha|} \int_\Omega u_\alpha(x)\varphi(x)\,dx \tag{B.1}$$

for every $C_0^\infty(\Omega)$.

Such $\varphi \in C_0^\infty(\Omega)$ are often called test functions. If the weak derivative of u exists, it is unique almost everywhere. Suppose u_α and u'_α are both weak derivatives of u with respect to α. Then in particular

$$\int_\Omega u_\alpha(x)\varphi(x)\,dx = \int_\Omega u'_\alpha(x)\varphi(x)\,dx$$

for all $\varphi \in C_0^\infty(\Omega)$. Therefore

$$\int_\Omega (u_\alpha(x) - u'_\alpha(x))\varphi(x)\,dx = 0$$

for all $\varphi \in C_0^\infty(\Omega)$ which means that $u_\alpha - u'_\alpha = 0$ almost everywhere. By uniqueness (up to a set of measure 0) we can speak of the weak derivative of u with respect to α, which we denote

$$\partial^\alpha u := u_\alpha.$$

Remark. Suppose $u \in C^k(\Omega, \mathbb{R})$. Then u has weak derivatives up to order k and these agree with the "normal" strong derivates of u.

We can now define the Sobolev space $W^{k,p}(\Omega)$. Fix $k \in \mathbb{N}$ and $1 \le p < \infty$.

Definition B.2. The Sobolev space $W^{k,p}(U)$ is the space of equivalence classes of functions $u \in L^p(\Omega)$ such that for every multi-index α with $|\alpha| \le k$, the weak derivative $\partial^\alpha u$ exists and $\partial^\alpha u \in L^p(\Omega)$.

The space $W^{k,p}(U)$ has a norm associated to it.

Definition B.3. Let $u \in W^{k,p}(\Omega)$. Its $W^{k,p}$-norm is defined by

$$\|u\|_{W^{k,p}} := \Big(\int_\Omega \sum_{|\alpha|\le k} |\partial^\alpha u(x)|^p \, dx \Big)^{1/p}$$

Remark. The definition of $W^{k,p}(\Omega)$ does not exclude the case $p = \infty$. However, the $W^{k,\infty}$-norm is defined differently as

$$\|u\|_{W^{1,\infty}} := \sum_{|\alpha|\le k} \operatorname{ess\,sup} |\partial^\alpha u|.$$

Hence, we can define the Sobolev spaces for all $1 \le p \le \infty$.

One readily checks this is a norm. By defining this, we also have the notion of convergence in $W^{k,p}(U)$ given by norm convergence in the $W^{k,p}$-norm. Given this norm, it is natural to ask whether $W^{k,p}(\Omega)$ together with the $W^{k,p}$-norm is a Banach space. This is indeed the case.

Theorem B.4. *Let $k \in \mathbb{N}$ and $1 \le p \le \infty$. Then the Sobolev space $W^{k,p}(\Omega)$ is a Banach space. Furthermore, for $1 < p < \infty$, $W^{k,p}(\Omega)$ is reflexive.*

Proof. We refer to Theorem 2 on page 249 of [**Eva98**]. Here, the reflexivity statement is not present here, but we can view $W^{k,p}(\Omega)$ as a closed subspace of $L^p(\Omega, \mathbb{R}^N$ for suitable N and use that a closed subspace of a reflexive Banach space is reflexive combined with the fact that $L^p(\Omega, \mathbb{R}^N)$ is reflexive for $1 < p < \infty$. $\square$

Using this definition we can define the locally $W^{1,p}$ functions.

Definition B.5. Let $\Omega \subset \mathbb{R}^n$ be open. Then

$$W^{k,p}_{\text{loc}}(\Omega) := \{u : \Omega \to \mathbb{R} \mid \text{ For any } K \subset \Omega \text{ compactly contained, we have } u|_K \in W^{k,p}(K) \ \}.$$

We now collect several results about Sobolev spaces that we use, in particular in Chapter 5 when computing the index of the vertical derivative of the Floer operator. They give estimates of the $W^{k,p}$-norm of a function in terms of the Hölder norm and the L^p norm.

Theorem B.6 (Sobolev estimate)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and suppose $kp > n$ and $0 < \frac{k-n}{p} < 1$. Set $\mu := \frac{k-n}{p}$. Then there exists a constant $C = C(k, p, \Omega) > 0$ such that*

$$\|u\|_{C^{0,\mu}} \le C\|u\|_{W^{k,p}}.$$

for any $u \in C^\infty(\overline{\Omega})$. Furtheremore, the inclusion $W^{k,p}(\Omega) \hookrightarrow C^0(\Omega)$ is compact.

Theorem B.7 (Rellich's Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and suppose $kp < n$. Let $r := \frac{np}{n-kp}$ Then there exists a constant $C = C(k, p, \Omega)$ such that*

$$\|u\|_{L^r} \le C\|u\|_{W^{k,p}}$$

for $u \in C^\infty(\overline{\Omega})$. Furthermore, if $q < r$, then the inclusion $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.

B.2. Elliptic regularity

Recall that the Laplace operator Δ on $\mathbb{R}^n$ is given by

$$\Delta := \frac{\partial^2}{\partial {x_1}^2} + \ldots + \frac{\partial^2}{\partial {x_n}^2}.$$

Let $\Omega \subset \mathbb{R}^n$ be an open subset and $u \in C^2(\Omega, \mathbb{R})$. Then u is called harmonic iff $\Delta u = 0$. An important question is whether for a given $f : \mathbb{R}^n \to \mathbb{R}$ there exists a solution to $\Delta u = f$. It turns out there is a positive answer in the following setting of a weak solution. This definition is similar to the one of a weak derivative.

Definition B.8. Let $f \in L^1_{\mathrm{loc}}(\Omega)$. Then $u \in L^1_{\mathrm{loc}}(\Omega)$ is called a weak solution of $\Delta u = f$ if

$$\int_\Omega u(x)\Delta\varphi(x)\,dx = \int_\Omega f(x)\varphi(x)\,dx$$

for all test functions $\varphi \in C_0^\infty(\Omega)$.

Let ω_n be the volume form on the unit sphere in $\mathbb{R}^n$. Then define the fundamental solution of Laplace's equation

$$K(x) := \begin{cases} \dfrac{\log|x|}{2\pi}, & n = 2 \\ \dfrac{|x|^{2-n}}{(2-n)\omega_n}, & n \geq 3 \end{cases} \tag{B.2}$$

The following inequality is important.

Theorem B.9 (Calderon-Zygmund inequality)**.** *Let K be as in equation (B.2). Let $1 < p < \infty$ and let $f \in C_0^\infty(\mathbb{R}^n)$. Then there exists a constant $C = C(n,p) > 0$ such that*

$$\|\nabla(\partial_j K * f)\|_{L^p} \leq C\|f\|_{L^p}$$

for $j = 1, \ldots, n$.

This has the following consequence for the perturbed Cauchy-Riemann operator. We use this when proving the Fredholm property of the vertical derivative of the Floer operator.

Lemma B.10. *Let $p > 2$ and $D_S : W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ the perturbed Cauchy Riemann operator where $S \in C^\infty(\mathbb{R} \times S^1, \mathrm{End}(\mathbb{R}^{2n}))$ which converges to symmetric operators $S^\pm$ as $s \to \pm\infty$ with $\lim_{s\to\pm\infty} \frac{\partial S}{\partial s}(s,t) = 0$ uniformly in t.*
Let $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$. Then there exists $C > 0$ such that

$$\|u\|_{W^{1,p}(\mathbb{R}\times S^1)} \leq C(\|D_S u\|_{L^p(\mathbb{R}\times S^1)} + \|u\|_{L^p(\mathbb{R}\times S^1)}).$$

Recall the definition of the convolution product of functions. Let $f, g : \mathbb{R}^n \to \mathbb{R}$. Then

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy.$$

The following is Lemma B.2.2 in [**MS12**].

Lemma B.11. *Let $u, f \in L^1(\mathbb{R}^n)$ with compact support. Then u is a weak solution of $\Delta u = f$ if and only if $u = K * f$.*

We now concentrate on the elliptic regularity of solutions to the Floer equation. In particular, when proving transversality results, we had solutions of the Floer equation in a space $\mathcal{B}^{1,p}(x,y)$ modeled on a Sobolev space. We want to conclude that these solutions are of class C^∞. Using a symplectic trivialization we transfered everything to a linear setting. Here, the vertical derivative of the Floer equation took the form of a perturbed Cauchy-Riemann operator. This operator has several regularity results we used.

We start with two theorems on the elliptic regularity in the case where our manifold $M = \mathbb{R}^{2n}$. These results are easier to prove.

The central theorem on which this section is a regularity result for the Cauchy-Riemann operator $\overline{\partial}$. Compare this for example to Theorem B.3.1 in [**MS12**], which is very similar for the Laplace operator Δ (up to change in some degrees to take care of the extra derivatives). The following is Theorem 12.1.2 from [**AD14**].

Theorem B.12 (Elliptic regularity for the Cauchy-Riemann operator)**.** *Let $1 < p < \infty$ and $k \in \mathbb{N}$. Suppose $\Omega \subset \mathbb{C}$ is open. If $u \in L^p_{loc}(\Omega)$ is a weak solution of $\overline{\partial} u = f$ for $f \in W^{k,p}_{loc}(\Omega)$, then*

$$u \in W^{k+1,p}_{loc}(\Omega).$$

Moreover, for every relatively compact open set $\Omega' \subset \mathbb{C}$ such that $\overline{\Omega'} \subset \Omega$, there exists a constant $C = C(k, p, \Omega', \Omega) > 0$ such that for every $u \in C^\infty(\overline{\Omega})$ we have

$$\|u\|_{W^{k+1,p}(\Omega')} \leq C\Big(\|\overline{\partial} u\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)}\Big) \tag{B.3}$$

Remark. Note that we require $p > 2$.

From this theorem, we can prove a similar theorem for the peturbed Cauchy-Riemann operator, provided the peturbation is sufficiently well-behaved.

Theorem B.13 (Elliptic regularity for the peturbed Cauchy-Riemann operator)**.** *Let $1 < p < \infty$ and let $S \in C^\infty(\mathbb{R} \times S^1, \mathrm{End}(\mathbb{R}^{2n})$ such that the limits*

$$\lim_{s \to \pm\infty} S(s,t) = S^{\pm}(t)$$

exist with

$$\lim_{s \to \pm\infty} \frac{\partial S}{\partial s}(s,t) = 0$$

uniformly in t. Let D_S be the peturbed Cauchy-Riemann operator

$$D_S := \overline{\partial} + S.$$

If $u \in L^p(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ *is a weak solution to* $D_S u = 0$*, then* $u \in C^\infty(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \cap W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$.

Proof. For details, we refer to the proof of Theorem 12.1.3 in [**AD14**]. By induction, using Theorem B.12 we find that $u \in W^{k,p}_{\mathrm{loc}}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ for all $k \in \mathbb{N}$, which implies that u is smooth. Such an argument is sometimes called elliptic bootstrapping. The statement that $u \in W^{1,p}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ follows from a version of Theorem B.12 for the peturbed Cauchy-Riemann operator without conditions on S that we used. This is Theorem □

We now list regularity results for the non-linear case. The following is a non-linear version of Theorem B.13. It is Proposition 12.1.4 in [**AD14**].

Theorem B.14 (Elliptic regularity of the Floer equation)**.** *Let* $2 < p < \infty$*. Suppose* $u \in W^{1,p}_{loc}(\mathbb{R} \times S^1, M)$ *such that* $\overline{\partial}_{H,J}(u) = 0$*. Then* $u \in C^\infty(\mathbb{R} \times S^1, M)$.

When referring to elliptic regularity for a solution to Floer's equation to prove that it is smooth, we are referring to this theorem.

APPENDIX C

Banach manifolds and Banach bundles

Floer homology is Morse homology for the symplectic action functional. This functional is defined on the space of contractible loops, which can be given the structure of a Banach manifold. In this section we define such manifolds. Let us briefly recall the definitions and important theorems associated to Banach manifolds such as the inverse function theorem, Fredholm stability and the semi-Fredholm lemma. We follow the exposition of [**Lan85**]. We assume some familiarity with differentiable manifolds.

Let X be a vector space. We have the following definition of a *normed space*

Definition C.1 (Normed space)**.** Let X be a $\mathbb{F}$-vector space, where we will take $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, equiped with a map $\|\cdot\| : X \to \mathbb{R}$ with the following properties.

(i) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

(ii) $\|\alpha x\| = |\alpha|\|x\|$ if $x \in X$ and $\alpha \in \mathbb{F}$

(iii) $\|x\| > 0$ if $x \neq 0$

Then X is called a normed space.

Every normed space can then be regarded as a metric space, by definining the distance function $d^{\|\cdot\|} : X \times X \to R$ by $d(x, y) = \|x - y\|$, which has all properties associated to a metric (do we need to check?)

Using this metric, there is a special class of normed spaces.

Definition C.2 (Banach space)**.** A Banach space is a normed space for which the metric $d^{\|\cdot\|}$ is complete.

On these Banach spaces we can define a topology induced by the metric. Hence, we can speak of topological notions like continuity and openess in the context of Banach spaces.

Now we can define differentiable manifolds with these Banach spaces. First define atlasses.

Definition C.3 (Atlas)**.** Let X be a set. An atlass of class C^p on X is a collection of pairs (U_i, φ_i) satisfying

(i) Each U_i is a subset of X and the U_i together form a cover of X.

(ii) Each $\varphi_i : U_i \to E_i$ is a bijection unto an open subset $\varphi_i(U_i)$ of some Banach space E_i. Furthermore, for any i, j we have that $\varphi_i(U_i \cap U_j)$ is open in E_i.

(iii) The transitions $\varphi_j \circ {\varphi_i}^{-1} : \varphi(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a $\mathcal{C}^p$ isomorphism for each pair i, j.

Now one can construct a unique topology on X such that each U_i is open and the φ_i are homeomorphisms. Suppose we are given some homeomorphism $\varphi : U \to V$ from an open subset $U \subset X$ to an open subset V of a Banach space E. Then (U, φ) is *compatible* with the atlas $\{(U_i, \varphi_i)\}_i$ if each map $\varphi_i \varphi^{-1}$ is a $\mathcal{C}^p$ isomorphism. Two atlases are *compatible* if each chart of one is compatible with each the other atlas. Such compatibility is an equivalence relation, and thereby we can make the following definition.

Definition C.4 (C^p-manifold)**.** A C^p-manifold is a pair $(X, \mathcal{A})$ where $\mathcal{A}$ is an equivalence class of atlasses of class C^p on X.

If in some atlas the vector spaces E_i are isomorphic, we can find and equivalent atlas for which they are all equal to some vector space E. Note that for $E = \mathbb{R}^n$ we get the "regular" definition for an n-dimensional manifold. So, we extend the definition of a manifold to also take into account spaces that locally look like Banach spaces, instead of just $\mathbb{R}^n$ or $\mathbb{C}^n$.

Likewise, we can also define tangent vectors and tangent spaces. Automatically, these tangent spaces have the structure of a Banach space. Suppose X is a C^p manifold and let $x \in X$. We define a tangent vector at x.

Definition C.5. Consider a triple (U, φ, v) where (U, φ) is a chart of X and $v \in \varphi(U)$ is an element of a vector space E. Two triples (U, φ, v) and (V, ψ, w) are equivalent if for

$$T_{\varphi(x)}(\psi \circ \varphi^{-1}) : \varphi(U) \to \psi(U)$$

we have $T_{\varphi(x)}(\psi \circ \varphi^{-1})(v) = w$. An equivalence class of triples is called a tangent vector of X at x. The set of all tangent vectors is the tangent space $T_x X$ of X at x.

Note that a chart (U, φ) gives a bijection of the Banach space E and $T_x X$ by $(U, \varphi, v) \mapsto v$. In this way, $T_x X$ is endowed with the structure of a Banach space.

Suppose X and Y are manifolds and $f : X \to Y$ is a morphism of class C^p. We can define a tangent map $T_x f : T_x X \to T_x Y$ using charts. Let (U, φ) and (V, ψ) be charts at x and $f(x)$ respectively. Let v be a tangent vector at x, represented by $v \in \varphi(U)$. Define

$T_x f(v)$ as the tangent vector at $f(x)$ represented by $Df_{U,V}(x)v$, the derivative of the map $f_{U,V} := \varphi \circ f\psi^{-1}$.

We define bundles over Banach manifolds.

Definition C.6 (Banach vector bundle)**.** Let X be a manifold of class $\mathcal{C}^p$. A Banach vector bundle over X is a total space E together with a continuous map $\pi : E \to X$ such that the following holds

(i) For all $x \in X$ the fiber $\pi^{-1}(x) = E_x$ has the structure of a Banach space.

(ii) There exists a cover U_i of X and a collection of maps associated to it $\psi_i : \pi^{-1}(U_i) \to U_i \times E$ for some Banach space E such that each ψ_i is an isomorphism commuting with projection unto the first coordinate pr_2. That is, $pr_2 \circ \psi_i = \pi$ and for each $x \in U_i$ the map $\psi_i : \pi^{-1}(x) \to E$ is an isomorphism.

(iii) The transition maps are continuous C^p.

Really, the above defines a trivializing cover, and for a vector bundle one needs to have an equivalence class of such bundles over a space X.

For any manifold X there is the construction of the tangent bundle.

Definition C.7. At every point we have a tangent space $T_x X$. Let

$$TX = \bigsqcup_{x \in X} T_x X$$

be the disjoint union of tangent spaces. This comes with the projection $\pi : TX \to X$ by $T_x X \mapsto x$. We define a trivialization on this bundle. Let (U, φ) be a chart on X. Then the map $\Phi_U : \pi^{-1}(U) = TU \to U \times E$ is a bijection, using the definition of tangent vector as a triple (U, φ, v). This commutes with projection $U \times E \to U$. Similarly we can define transition maps

$$\Phi_{ij} = \Phi_j \circ \Phi_i^{-1} : \varphi_i(U_i \cap U_j) \times E \to \varphi_j(U_i \cap U_j) \times E$$

by

$$\Phi_{ij}(x, v) = (\varphi_{ij}(x), D\varphi_{ij}(x)v).$$

One can readily check that this satisfies the necessary properties of a vector bundle.

There are natural construction to construct from vector bundles new ones. One such construction that we will often use is the pullback bundle.

Definition C.8 (Pullback bundle)**.** Suppose $\pi : E \to X$ is a vector bundle and $f : X' \to X$ is a continuous map. Then we can define a vector bundle called the pullback bundle

$$f^*\pi : f^*(E) \to X.$$

Here $f^*(E)$ is defined by the fibered product

$$f^*(E) = X' \times_X E = \{(x', e) \in X' \times E \mid \pi(e) = f(x')\}$$

and

$$f^*\pi = \mathrm{pr}_1$$

projection unto the first coordinate.

One of the goals of Floer homology is to define moduli spaces of solutions to the Floer equation. To do this, we view the Floer operator as a section σ of a vector bundle $\pi : \mathcal{E} \to \mathcal{B}$ as described above. To establish this, one needs to prove that σ is transverse to the zero section. Let

$$D\sigma(u) : T_u\mathcal{B} \to T_{\sigma(u)}\mathcal{E}$$

denote the derivative, then we must have that it is complementary to the tangent space $T_u\mathcal{B}$ of the zero section.

A different way to phrase this is to look at the canonical splitting of the vector bundle $\mathcal{E}$ at the zero section. Denote by $\mathcal{E}_u$ the fiber at u. There is an canonical isomorphism $T_{(u,0)}\mathcal{E} \simeq T_u\mathcal{B} \oplus \mathcal{E}_u$. We make the following definition.

Definition C.9 (Vertical differential)**.** The vertical differential $d^V\sigma(u)$ of the section $\sigma \in \Gamma(\mathcal{B}, \mathcal{E})$ at $u \in \mathcal{B}$ is defined by $d^V\sigma(u) = pr_2 \circ D\sigma(u) : T_u\mathcal{B} \to \mathcal{E}_u$ where $pr_2 : T_{(u,0)}\mathcal{E} \simeq T_u\mathcal{B} \oplus \mathcal{E}_u \to \mathcal{E}_u$ denotes projection unto the second coordinate.

It is now immediate that transversality of σ to the zero section is the same as saying that $D^v_u\sigma$ is surjective for every $u \in \mathcal{B}$.

We now state several theorems concerning Banach spaces. These will be used throughout the text.

The statements are primarily from [**MS12**] and [**AD14**], collected throughout several chapters and the appendices. Recall the definition of a Fredholm operator and Fredholm index. Suppose X, Y are Banach spaces with $F : X \to Y$ a bounded linear operator. If F has finite-dimensional kernel and cokernel and has closed range, then we call F a Fredholm operator. In this case, its index is given by

$$\mathrm{ind}(F) = \dim\ker F - \dim\operatorname{coker} F.$$

Theorem C.10 (Semi-Fredholm lemma)**.** *Let X,Y,Z be Banach spaces. Assume $D : X \to Y$ is a bounded linear operator, and $K : X \to Z$ is a compact operator. Assume there is a $c > 0$ such that*

$$\|x\|_X \le c(\|Dx\|_Y + \|Kx\|_Z)$$

for $x \in X$. Then D has closed image and a finite dimensional kernel.

What we do to compute indices of Fredholm operators is perturb them, so that we get operators for which we can compute the index. Hence there is the following important theorem stating that the Fredholm property and index are stable.

Theorem C.11 (Fredholm stability)**.** *Suppose $D : X \to Y$ be a Fredholm operator. We have the following*

(i) *Let $K : X \to Y$ be a compact operator. Then $D + K : X \to Y$ is a Fredholm operator with* $\operatorname{ind}(D + K) = \operatorname{ind}(D)$.

(ii) *For any D there exists $\epsilon > 0$ such that for $A : X \to Y$ a bounded linear operator with $\|A\| < \epsilon$, then $D + A : X \to Y$ is a Fredholm operator with* $\operatorname{ind}(D + A) = \operatorname{ind}(D)$.

The next theorem is a theorem from functional analysis, called the *Sard-Smale Theorem*. It is used to prove that the Hamiltonians H such that the spaces of solutions to the Floer equation are finite dimensional manifolds are generic.

Theorem C.12 (Sard-Smale Theorem)**.** *Let X and Y be separable Banach spaces and $U \subset X$ be an open set. Suppose $F : U \to Y$ is a Fredholm map of class $\mathcal{C}^l$ where*

$$l \geq \max\{1, \operatorname{ind}(F) + 1\}.$$

Then the set

$$Y_{reg}(F) = \{y \in Y \mid x \in U, F(x) = y \implies imdf(x) = Y\}.$$

of regular values of F is residual in the sense of Baire.

This is also sometimes called of the second category in the sense of Baire. It is the same as saying that the set contains a countable intersection of open and dense sets. If A is a set with a subset B of the second category in the sense of Baire, we say that the choice of some $a \in B$ is generic.

Another important theorem is an infinite dimensional version of the implicit function theorem. This is the following statement. It is used to give the space of solutions to Floer's equation a manifold structure by describing them as regular values of a section of a Banach bundle over a Banach space.

Theorem C.13 (Implicit function theorem)**.** *Let X and Y be Banach spaces and let $U \subset X$ be an open set, and l a positive integer. If $F : Y \to Y$ is of class $\mathcal{C}^l$ and y is a regular value of F.*
Let

$$\mathcal{M} = F^{-1}(y) \subset X.$$

Then $\mathcal{M}$ is a $\mathcal{C}^l$ Banach manifold and $T_x\mathcal{M} = \ker df(x)$.
In the special case that F is a Fredholm map, then $\mathcal{M}$ is finite dimensional with

$$\dim \mathcal{M} = \operatorname{ind}(F).$$

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