Existence of contact structures in all dimensions

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Chapter 1

Introduction

A contact structure on a manifold M is a completely non-integrable hyperplane field $\xi \subset TM$. These structures originate from the work of Sophus Lie who introduced them in 1872 in his work on partial differential equations. Since then, these structures have been applied in many areas of physics and mathematics including Hamiltonian dynamics, geometric optics, fluid mechanics, knot theory, low dimensional topology, Riemannian geometry, and symplectic geometry.

Surprisingly one of the basic questions about them, their existence and classification was only recently answered in full generality in a celebrated paper by Borman, Eliashberg and Murphy [2], where they establish a multi-parametric h-principle for contact structures on any manifold. In this thesis we present a proof of the existence part of this h-principle showing the existence of contact structures on manifolds of all dimensions.

Theorem. *Any almost contact structure η on a manifold M is homotopic to a contact structure ξ.*

The almost contact structures referred to in this theorem are the formal homotopy counter part of a contact structure similar to for example almost symplectic and almost complex structures.

To better understand the statement of this theorem and the reason for almost contact structures showing up, we take a step back and start by giving a more detailed definition of contact structures.

Let M be a smooth manifold of dimension $(2n + 1)$ and $\xi \subset TM$ a smooth hyperplane field. More precisely, ξ consists of a collection hyperplanes

$$\xi_p \subset T_pM, \quad p \in M,$$

in the tangent bundle and varying smoothly in p. Such a hyperplane field is called coorientable if there exists a 1-form $\alpha \in \Omega(M)$ satisfying

$$\xi = \ker \alpha,$$

and we will always assume this is the case. Note that the defining 1-form α is not unique. Indeed, if $\lambda : M \to \mathbb{R} \setminus \{0\}$ is any non-vanishing function then $\xi = \ker \alpha = \ker \lambda\alpha$.

An important class of hyperplane fields are the integrable ones meaning that through each point $p \in M$ we can find a submanifold N satisfying $T_qN = \xi_q$ for all $q \in N$. The famous Frobenius theorem tells us that a hyperplane field $\xi = \ker \alpha$ is integrable if and only if α satisfies the condition

$$\alpha \wedge d\alpha = 0.$$

As we stated above, a contact structure is the exact opposite of an integrable hyperplane field i.e. it is maximally non-integrable. Hence, a contact structure consists of a hyperplane field $\xi = \ker \alpha$ satisfying the equation

$$\alpha \wedge (d\alpha)^n \neq 0,$$

called the contact condition, and which is as far away from the integrability condition as possible. It is easy to check that this condition is independent of the choice of contact form α. Furthermore, the contact condition is equivalent to $d\alpha$ being non-degenerate on $\xi = \ker \alpha$, implying that the pair $(\xi, \omega = d\alpha)$ is a symplectic vector bundle.

The simplest example of a contact manifold is $\mathbb{R}^3$ with Cartesian coordinates (x, y, z) and

$$\alpha := dz - xdy.$$

We claimed that contact structures naturally show up in the study of differential equations. To see how, consider a time-dependent differential equation

$$F(t, x(t), \dot{x}(t)) = 0,$$

and identify solutions with curves $\gamma : \mathbb{R} \to \mathbb{R}^3$. Trying to solve the differential equation we notice that there are two types of solutions:

- Formal: $\gamma(t) = (t, x(t), y(t))$ satisfying $F \circ \gamma = 0$,
- Genuine: $\gamma(t) = (t, x(t), \dot{x}(t))$ satisfying $F \circ \gamma = 0$.

The genuine solutions are the ones we are interested in, because they solve the differential problem. On the other hand, the formal solutions, where we replaced the derivative of x with an independent function y, do not see any of the differential information and only solve the underlying algebraic problem.

We observe that a formal solution $\gamma(t) = (t, x(t), y(t))$ is genuine precisely when $y = \frac{dx}{dt}$. This condition is equivalent to the condition

$$dx - ydt = 0$$

which says that the formal solutions are precisely the integral curves[1] of the standard contact structure on $\mathbb{R}^3$.

[1] Geometrically the contact conditions says that any integral manifold N of ξ satisfies $\dim N \leq n$. So, the existence of integral curves does not contradict the maximal non-integrability of the contact structure

It turns out that the relation between formal and genuine solutions is a lot stronger than it looks at first sight. Although the algebraic problem is much simpler, it is clear that the existence of formal solutions is a necessary condition for the existence of genuine solutions, so this problem needs to be solved first. A priori it is unreasonable to expect this also to be a sufficient condition. Surprisingly, it turns out that there are large classes for which this is true meaning that the solvability of the differential equation reduces to a homotopy theoretic problem. In this case we say that the differential equation satisfies an h-principle. Furthermore, it is often the case that the genuine solutions are also classified by the formal solutions. This means that any formal solution is homotopic to a genuine one and that two genuine solutions are homotopic if and only if their underlying formal solutions are homotopic.

The relevance of these observations for the existence of contact structures is that the definition of a contact structure is basically a differential equation given by the contact condition. Contact structures are precisely the genuine solutions of this equation, while the corresponding formal solutions are called almost contact structures.

It turns out that trying to show existence of contact structures and classify them using the h-principle approach is very successful. The first important result in this direction was proved by Gromov. Recall that a manifold is called open if each component is non-compact or has non-empty boundary.

Theorem (Gromov, 1969)**.** *Let M be an open manifold. Then, there is a one-to-one correspondence between isotopy classes of contact structures and homotopy classes of almost contact structures.*

For closed manifolds the situation turned out to be more subtle. The existence of contact structures on closed 3-manifolds was proved by Martinet based on the surgery description of 3-manifolds due to Lickorish and Wallace.

Theorem (Martinet, 1971)**.** *Any closed orientable 3-manifold admits a contact structure.*

However, Bennequin showed that the 1-parametric h-principle fails for contact structures on S^3, by constructing a contact structure which was non-isotopic but formally homotopic to the standard contact structure. This led Eliashberg to introduce a dichotomy of 'tight' and 'overtwisted' contact structures on 3-manifold. A contact structure is called overtwisted if it contains a disk with a specific germ of contact structure and tight otherwise. Furthermore, he established a h-principle for overtwisted contact structures.

Theorem (Eliashberg, 1989)**.** *Any almost contact homotopy class on a closed 3-dimensional manifold contains a unique up to isotopy overtwisted contact structure.*

This result proved to be rather hard to generalize to higher dimensions. Although there was some progress in specific cases, for example on 5-manifolds there were no known general results for all dimensions. Finally, the parametric h-principle for (overtwisted) contact structures in all dimensions was proved by Borman, Eliashberg and Murphy in [2]. We give a precise statement of this theorem together with an overview of the main ideas used in its proof in Chapter 3.

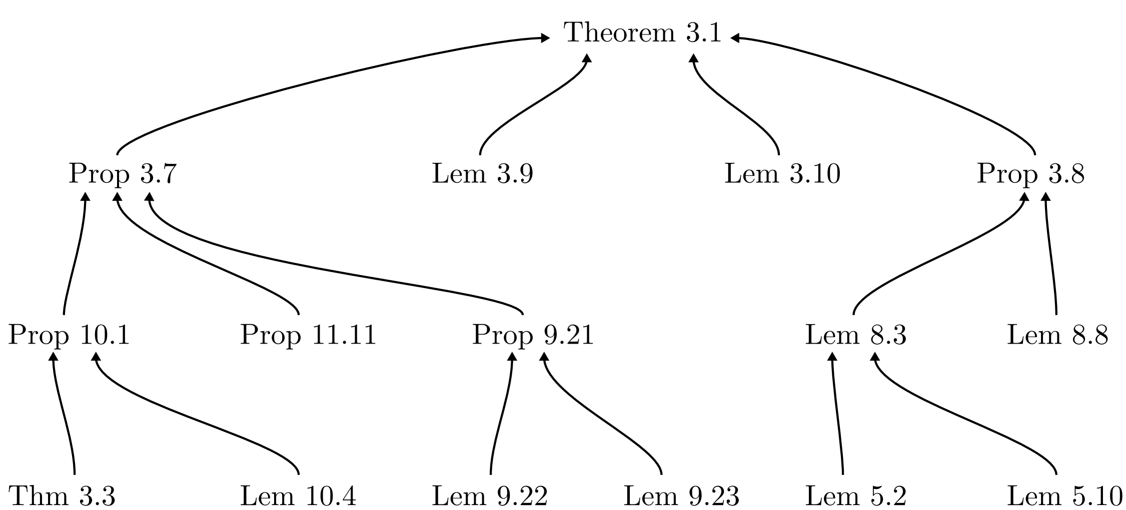
The text is structured as follows. We begin by introducing the basic definitions and fixing the notation in Chapter 2.

In Chapter 3 we give a precise statement of the main theorem and an overview of the main idea's in it's proof. The proof is based on two main results: Proposition 3.7 gives a reduction to a unique local problem in each dimension and Proposition 3.8 which is used to solve this local problem. To aid the presentation we prove Proposition 3.8 before Proposition 3.7.

In Chapter 4 up to Chapter 7 we prove all the results needed for the proof of Proposition 3.8. In Chapter 4 we define circle models, which describe the local extension problem. The essential property of these models is that all the important information about them is encoded in a function, called a contact Hamiltonian. Next, in Chapter 5 we study the relation between the circle models and their defining contact Hamiltonians. In Chapter 6 we introduce a connected sum operation for circle models and show that it is compatible with the defining contact Hamiltonians. Lastly, we introduce a distinguished class of contact Hamiltonians, called special, in Chapter 7. The circle models associated to special contact Hamiltonians are used to define overtwisted disks in higher dimensions which play an essential part in the proof proof of Proposition 3.8. In Chapter 8 the results from the previous chapters are combined to proof Proposition 3.8.

Chapters 9 up to 11 together form the proof of Proposition 3.7. In Chapter 9 we introduce another type of local model, called semi-contact saucers which are easier to obtain but as a trade-off are less detailed than circle models. We also show how to reduce the filling problem for a saucer to a filling problem for a circle model. In chapter 10 we start from Gromov's theorem to show that an almost contact structure on a manifold can be homotoped to a contact structure up to finitely many saucers. The argument in this chapter forms the core of the proof of Proposition 3.7. Although this chapter shows it is enough to solve the filling problem for finitely many saucers, and hence for finitely many circle models, there is a priori no restriction on what these models look like. In Chapter 11 we use an equivariant covering argument to show that these circle models can be chosen from a finite list of possibilities. This allows us to find a unique circle model for each dimension.

The diagram below outlines the logical dependency of the major propositions and lemma's used in the proof of the main theorem.



Chapter 2

Basic definitions

The goal of this chapter is to fix the notation and introduce the basic definitions. In the first two sections we define contact and almost contact structures and recall some well known results that hold for them. In the third section we introduce coordinates and contact structures on the manifolds we will consider most. Next, we recall some facts about contact vector fields and introduce star-shaped domains in section four. Finally, in section five, we define contact shells together with their equivalence and domination relation.

2.1 (Almost) contact structures

Let M be a manifold with tangent bundle TM, and $\xi \subset TM$ a smooth hyperplane field, which by definition is a smooth subbundle of codimension 1. If $\dim M = n$, the smoothness condition equivalent to requiring that for each point $p \in M$ there is an open neighborhood $U \subset M$ of p and smooth vector fields $X_1, \ldots X_{n-1}$ such that for al $q \in U$ we have

$$\xi_q = \mathrm{Span}\{\, X_1(q), \ldots, X_{n-1}(q) \,\} \subset T_q M.$$

In order to work with such hyperplane fields, it is useful to describe them in terms of differential forms.

Definition/Lemma 2.1. *Locally, ξ can be written as the kernel of some 1-form α. This can be done globally if and only if ξ is coorientable, which by definition means that the quotient line bundle TM/ξ is trivial. In this case TM/ξ has two components and a* coorientation *is a choice of one of these. In particular α induces a natural coorientation by picking the component on which it is positive.*

Proof. See Lemma 1.1.1 in [5]. □

The central objects of study are contact and almost contact structures which we now define. There are various (equivalent) definitions of these concepts and for completeness we will state a number of them and explain how they are related.

The most descriptive definition of a contact structure is the following.

Definition 2.2. *A* contact structure *a smooth manifold M is a maximally non-integrable hyperplane field $\xi \subset TM$. A manifold M equipped with such a structure is called a* contact manifold *and denoted by a pair (M, ξ).*

The maximally non-integrability means that there cannot be a smooth hypersurface in M whose tangent space is equal to ξ on an open subset. That is, contact structures are *completely non-integrable*. The maximality expresses the fact that if L is a isotropic submanifold of a $2n+1$-dimensional contact manifold $(M\xi)$, then $\dim L \leq n$. For a proof of these facts we refer the reader to Definition 1.5.11 and Proposition 1.5.12 in [5].

In practise the above definition is hard to work with and some of the immediate consequences, for example that a contact manifold needs to be of odd dimension, are not immediately clear. If we require the hyperplane field ξ to be coorientable we can write $\xi = \ker \alpha$ for a 1-form α on M, as in Lemma 2.1. The maximally non-integrability condition on ξ translates to a condition on α and we have the following equivalent definition of a (coorientable) contact structure.

Definition 2.3. *Let M be a smooth manifold of dimension $2n+1$. A (coorientable)* contact structure *is a hyperplane field $\xi = \ker \alpha \subset TM$ such that α satisfies the condition*

$$\alpha \wedge d\alpha^n \neq 0.$$

It follows immediately from this definition that M should be orientable. Furthermore, it is easy to see that the contact condition $\alpha \wedge d\alpha^n \neq 0$ can only be satisfied on manifolds of odd dimension so contact structures do not exist on even dimensional manifolds.
If $\lambda : M \to \mathbb{R} \setminus \{0\}$ is nowhere vanishing function then $\ker \alpha = \ker \lambda\alpha$. Furthermore $(\lambda\alpha) \wedge d(\lambda\alpha)^n = \lambda^{n+1}\alpha \wedge d\alpha$ from which we see that the choice of α defining ξ only matters up to multiplication by a non-vanishing function. This motivates us to define two differential p-forms α, α' to be equivalent if $\alpha = \lambda\alpha'$ for some $\lambda : M \to \mathbb{R} \setminus \{0\}$. The equivalence class $[\alpha]$ is called the *conformal* class of α. By considering these conformal classes we can define a contact structure without making any (explicit) reference to a hyperplane field as follows.

Definition 2.4. *A (coorientable)* contact structure *on a $2n+1$-dimensional manifold M is a conformal class of 1-forms $[\alpha]$, such that one (and hence any) representative α satisfies*

$$\alpha \wedge d\alpha^n \neq 0.$$

The contact condition $\alpha \wedge d\alpha^n \neq 0$ is equivalent to saying that $d\alpha$ is non-degenerate on $\ker \alpha = \xi$. Hence, the contact form induces on each hyperplane $\xi_p \subset T_pM$ a linear symplectic structure. However, since a contact structure is defined by a conformal class of a contact form it only induces a conformal symplectic structure on each hyperplane ξ_p, $p \in M$. Indeed, on $\ker \alpha$ we have

$$d(\lambda\alpha) = d\lambda \wedge \alpha + \lambda d\alpha = \lambda d\alpha,$$

using that $\xi = \ker \alpha$.

Remark 2.5. *Unless stated otherwise, we will always assume that the (almost) contact structures we consider are coorientable and given the natural coorientation induced by α, meaning we select the component of TM/ξ on which α is positive. This implies that if α and α' are two contact forms defining the same hyperplane field $\xi = \ker\alpha = \ker\alpha'$ then $\alpha = \lambda\alpha'$ for some $\lambda : M \to \mathbb{R}_+$. That is, the scalar function must take positive values.*
This also implies that instead of a conformal symplectic structure, we have an induced positive conformal symplectic structure on each hyperplane ξ_p. Hence, ξ is a positive conformal symplectic vector bundle over M.

To fix the notation we list here the contact manifolds that we will consider most often. On $\mathbb{R}^2$ we will use Cartesian coordinates or coordinates (u, θ) where $u := r^2$ and (r, θ) denotes standard polar coordinates. We use the coordinate u instead of r because this simplifies the notation in many situations. With this notation the contact structures on the following manifolds will be called standard, denoted by ξ_{st}.

Space	Coordinates	ξ_{st}
$\mathbb{R}^{2n-1} = \mathbb{R} \times (\mathbb{R}^2)^{n-1}$	$(z, u_1, \theta_1, \dots, u_{n-1}, \theta_{n-1})$	$\ker(\alpha_{st} := dz + \sum_{i=1}^{n-1} u_i d\theta_i)$
$\mathbb{R}^{2n+1} = \mathbb{R}^{2n-1} \times \mathbb{R}^2$	(x, v, θ) with $(v, \theta) = (u_n, \theta_n)$ and $x = (z, u_1, \theta_1, \dots, u_{n-1}, \theta_{n-1})$	$\ker(\alpha_{st} + v d\theta)$
$\mathbb{R}^{2n-1} \times T^*\mathbb{R}$	(x, q, p) with (q, p) standard coordinates on $T^*\mathbb{R}$	$\ker(\alpha_{st} + p dq)$
$R^{2n-1} \times T^*S^1$	(x, q, p) with (q, p) standard coordinates on T^*S^1	$\ker(\alpha_{st} + p dq)$

Observe that although all the contact structures above are denoted by ξ_{st}, only in the case of $\mathbb{R}^{2n-1}$ we have $\xi_{st} = \ker\alpha_{st}$.

To express when two contact structures are the same we introduce the following notion of a contactomorphism.

Definition 2.6. *A* contactomorphism *between contact manifolds (M, ξ) and (M', ξ') is a diffeomorphism $f : M \to M'$ sending ξ to ξ', that is, $T_pF(\xi_p) = \xi_{f(p)}$ for all $p \in M$. If $\xi = \ker\alpha$ and $\xi' = \ker\alpha'$ this is equivalent to*

$$f^*\alpha' = \lambda\alpha, \qquad \textit{for } \lambda : M \to \mathbb{R} \setminus \{0\}. \tag{2.1}$$

It is easy to see that condition 2.1 is well-defined on conformal classes. Hence, in the language of Definition 2.4, a contactomorphism between contact structures $(M, [\alpha])$ and $(M', [\alpha'])$ is a diffeomorphism $f : M \to M'$ such that $f^*[\alpha'] = [\alpha]$.
Similarly, we have a notion of contact embedding.

Definition 2.7. *A contact embedding is a smooth embedding which is a contactomorphism onto its image.*

We now want to define almost contact structures. Again there are various equivalent ways to define these.
Similar to almost symplectic, complex and Kähler (and more) structures we want to define an almost contact structure to be "a contact structure without the differential information". The differential information contained in a contact structure consists of the relation between the contact and the symplectic form given by $\omega = d\alpha$. Removing this relation gives the following definition for an almost contact structure.

Definition 2.8. *An (coorientable)* almost contact structure *on a manifold M is a (coorientable) hyperplane field $\eta \subset TM$ together with a conformal symplectic structure ω_p on each hyperplane $\xi_p \subset T_pM$, depending smoothly on p.*

Again, to work with almost contact structures it is convenient to formulate them in terms of differential forms. We use again the notation $[\alpha]$ to denote the conformal class of the differential form α.

Definition 2.9. *A (coorientable)* almost contact structure *on a manifold M of dimension $(2n+1)$ is an pair of equivalence classes $\eta := ([\alpha], [\omega])$, where α is a 1-form on M and ω a 2-form on $\ker \alpha$, such that $\alpha \wedge \omega^n \neq 0$.*

As before, note that the condition $\alpha \wedge \omega^n \neq 0$ is independent of the representatives α and ω. Indeed, if $\alpha' = \lambda\alpha$ and $\omega' = \gamma\omega$ for $\lambda, \omega : M \to \mathbb{R} \setminus \{0\}$ then

$$\alpha' \wedge \omega'^n = \lambda\gamma^n \alpha \wedge \omega^n \neq 0.$$

This situation is slightly different from the contact case where α and $\omega := d\alpha$ scale with the same factor. Hence, we might be tempted impose this as a condition on an almost structure and define it as an equivalence class of pairs, $\eta = [(\alpha, \omega)]$, so that they scale with the same factor.

To see this is too restrictive we consider a different definition for an almost contact structure which is equivalent to the above one. Observe that a contact structure on a $2n+1$-dimensional manifold M an isomorphism

$$TM \cong \mathrm{Span}(R_\alpha) \oplus \xi,$$

where $\alpha(R_\alpha) = 1$[1]. Since ξ is a conformal symplectic vector bundle this means that we get a reduction of the structure group of TM to $1 \times U(n)$, by choosing a compatible complex structure on ξ. From this perspective, the definition of an almost contact structure should encode this reduction of the structure group which gives rise to the following definition.

Definition 2.10. *A (cooriented)* almost contact structure *on a manifold M is a triple (η, J, ϵ) consisting of a (cooriented) hyperplane field $\eta \subset TM$, an almost complex structure J on η and a oriented line bundle $\epsilon \subset TM$ complementary to η and defining the coorientation.*

[1] Here, R_α denotes the unique section of TM/ξ satisfying $\alpha(R_\alpha) = 1$, which is called the Reeb vector field of α.

An almost contact structure as in the previous definition is the same thing as a stable almost contact structure on M, which by definition is an almost complex structure on $TM \oplus \epsilon^1$ where ϵ^1 is the oriented line bundle over M. To see this let J be a complex bundle structure on $TM \oplus \epsilon^1$ and assume $\epsilon^1 = \text{Span}\langle X \rangle$ for some non-vanishing vector field X. Define $Y := J(X)$ and take $\epsilon = \text{Span}\langle Y \rangle$, $\eta := TM/\epsilon$ and note that the restriction $J|_\eta$ is well-defined.
Conversely, given an almost contact structure (η, J, ϵ) we can extend J to $TM \oplus \epsilon^1$ as follows. Assume, $\eta^1 = \text{Span}\langle X \rangle$ and $\epsilon = \text{Span}\langle Y \rangle$. Then, define $J(Y) := X$ and $J(X) := -Y$.

The following lemma shows that Definition 2.9 and Definition 2.10 are indeed equivalent.

Lemma 2.11. *There is a one-to-one correspondence between isomorphism classes of symplectic vector bundles (ξ, ω) and isomorphism classes of complex vector bundles (ξ, J).*

Proof. This is an immediate consequence of Theorem 2.62 and Proposition 2.63 in [13] and we omit the details. □

Note that if $[\omega] = [\omega']$ as conformal classes and J is ω compatible then it is also ω' compatible. Hence, the above lemma is also true when we replace isomorphism classes of symplectic vector bundles by isomorphism classes of almost contact structures as in Definition 2.9, where we mean isomorphism as in Definition 2.12 below. In particular we see that the existence of an almost contact structure as in Definition 2.9 is equivalent to the existence of a stable almost complex structure. This confirms that scaling α and ω with the same scalar is too restrictive. For our purposes Definition 2.9 is easier to work with so we will use this as "the" definition of an almost contact structure.

The notion of an isomorphism of almost contact structures is defined as follows.

Definition 2.12. *An* isomorphism *between almost contact manifolds $(M, \eta := ([\alpha], [\omega]))$ and $(M', \eta' := ([\alpha'], [\omega'])$ is a diffeomorphism $f : M \to M'$ such that $f^*[\alpha'] = [\alpha]$ and $f^*[\omega'] = [\omega]$.*

Note that this definition says that f pulls back conformal classes to conformal classes. This means that $f^*\alpha' = \lambda\alpha$ and $f^*\omega' = \rho\omega$ for $\lambda, \rho : M \to \mathbb{R} \setminus \{0\}$ but λ and ρ can be different functions.
We also have the notion of an embedding of almost contact structures.

Definition 2.13. *An* embedding of almost contact structures *is a smooth embedding which is an isomorphism of almost contact structures onto it's image.*

Remark 2.14. *As we stated before the description hyperplane fields in terms of differential forms is not unique. This is the reason that in the above definitions of (almost) contact structures and maps we consider conformal classes of differential forms. Sometimes we want to work with fixed representatives of these classes. We then talk about* strict *(almost) contact structures and maps. More precisely, a* strict *contact manifold is a pair (M, α) where $\alpha \wedge (d\alpha)^n \neq 0$, and a strict almost contact structure is a triple (M, α, ω) where $\alpha \wedge \omega^n \neq 0$. Similarly, a strict contactomorphism between strict contact manifolds (M, α) and (M', α') is a diffeomorphism $f : M \to M'$ such that $f^*\alpha' = \alpha$ and a* strict *isomorphism between strict almost contact manifolds (M, α, ω) and (M', α, ω') is a diffeomorphism $f : M \to M'$ such that $f^*\alpha' = \alpha$ and $f^*\omega' = \omega$. Strict (almost) contact embeddings are defined analogously.*

2.2 Transverse contact structures & Giroux's theorem

Recall that a hypersurface $\Sigma \subset (M, \xi = \ker \alpha)$ in a contact manifold has a singular 1-dimensional foliation $\mathcal{F}$ defined by the distribution

$$(T\Sigma \cap \xi|_{\Sigma})^{\perp},$$

where $\perp$ denotes the symplectic complement with respect to $d\alpha$. It is well known, see for example Theorem 2.5.22 in [5], that in the 3-dimensional case this characteristic foliation determines the contact germ on Σ.

It turns out that a slightly weaker version of this theorem, given by Theorem 2.18 below, still holds in higher dimensions. To prove this we first introduce the notion of transverse contact structures.

Consider a manifold M of dimension $2m+2$ and α a 1-form on M. We say that α is *non-degenerate* if $\alpha \wedge d\alpha^{2m} \neq 0$.

Definition 2.15. *A one-dimensional foliation $\mathcal{F}$ on a manifold M of dimension $2m+2$ is said to have a* transverse contact structure *if there is a smooth codimension 1 distribution $\mathcal{H} = \ker \alpha$ for a smooth 1-form α on M (i.e. a smooth hyperplane field) such that*

(i) the tangent planes to $\mathcal{F}$ are contained in $\mathcal{H}$, i.e. $T\mathcal{F} \subset \mathcal{H}$,

(ii) $\mathcal{H}$ restricts to a contact structure on any $2m+1$ dimensional manifold N transverse to $\mathcal{F}$, i.e. $\mathcal{H} \cap TN$ is contact for any manifold N transverse to $\mathcal{F}$,

(iii) $\mathcal{H}$ is invariant under flowing along $\mathcal{F}$, i.e. if $\mathcal{F}$ is represented by a vector field X, then $\mathcal{L}_X\alpha = \rho\alpha$ for some smooth function ρ.

The main result of this section is the following lemma.

Lemma 2.16. *There is a bijective correspondence between equivalence classes of non-degenerate 1- forms α on a $2m+2$ dimensional manifold M and 1 dimensional foliations $\mathcal{F}$ with transverse contact structure $\mathcal{H} := \ker \alpha$.*

Proof. We will show that given a non-degenerate 1-form α on M there exists a unique 1 dimensional foliation $\mathcal{F}$ such that $\mathcal{H} := \ker \alpha$ is a transverse contact structure.

Let Ω be a volume form on M and define a vector field X by $\iota_X\Omega = \alpha \wedge d\alpha^m$. Since, α is non-degenerate X is non-vanishing and defines a foliation $\mathcal{F}$ on M. We now show that $\mathcal{H}$ is a transverse contact structure for $\mathcal{F}$. From our definition of X and since Ω is a volume form on M it is clear that α satisfies the contact condition when restricted to any manifold N transverse to $\mathcal{F}$. Next, observe that

$$0 = \iota_X(\iota_X\Omega) = (\iota_X\alpha) \wedge d\alpha^m - m\alpha \wedge (\iota_X d\alpha) \wedge d\alpha^{m-1}.$$

Wedging this equation with α, we conclude $\iota_X\alpha = 0$, implying $T\mathcal{F} \subset \mathcal{H}$. It remains to show that $\mathcal{H}$ is invariant under monodromy along $\mathcal{F}$. We claim that $\mathcal{L}_X\alpha = 0$. It follows from Darboux's theorem that near every point $p \in M$ we can find coordinates $(y, x_0, \ldots, x_{2m})$ such that $\mathcal{F}$ is given by $(x_1, \ldots, x_{2m})$ is constant and

$$\alpha|_{TN} = dx_0 + x_1 dx_2 + \cdots + x_{2m-1} dx_{2m}.$$

This implies that $\alpha = f dy + x_0 dx_1 + \cdots + x_{2m-1} dx_{2m}$ for a smooth function $f : M \to \mathbb{R}$. Moreover, since $T\mathcal{F} \subset \ker \alpha$ we have $\iota_X \alpha = f = 0$ so that $\alpha = x_0 dx_1 + \cdots + x_{2m-1} dx_{2m}$. Now compute $\mathcal{L}_X = \iota_X d\alpha + d(\iota_X \alpha) = 0$.

To show that $\mathcal{F}$ is unique it suffices to show that if X is any vector field tangent to $\mathcal{F}$ and $\mathcal{H} = \ker \alpha$ a transverse contact structure for $\mathcal{F}$, then X satisfies $\alpha \wedge d\alpha^m = \iota_X \Omega$ where Ω is a again a volume form.

To see this condition is satisfied observe that $\iota_X \alpha = 0$ since $\mathcal{F} \subset \mathcal{H}$. Furthermore, $\mathcal{L}_X \alpha = \rho\alpha$ so we can (locally) find a non vanishing function λ such that $\mathcal{L}_X(f\alpha) = 0$. Together this implies that $\iota_X d(\lambda\alpha) = 0$ which is equivalent to

$$\iota_X d\alpha = \lambda(X) \wedge \alpha.$$

In turn this gives that $\iota_X(\alpha \wedge (d\alpha)^m) = 0$ implying $\alpha \wedge (d\alpha)^m = \iota_X \Omega$ as desired. □

We get the following corollary for hypersurfaces in a contact manifold.

Corollary 2.17. *Let $S \subset M$ be a hypersurface in a contact manifold $(M, \xi = \ker \alpha)$ of dimension $2n+1$. Then, outside the singular locus $\Sigma \subset S$, the characteristic foliation $\mathcal{F}$ of S has a transverse contact structure invariant with respect to monodromy along the leaves of $\mathcal{F}$.*

Proof. Recall that the characteristic foliation is represented by a vector field X satisfying

$$\iota_X \Omega = \beta \wedge d\beta^{n-1},$$

where Ω is a volume form on S and $\beta := \alpha|_{TS}$. Outside the singular locus this means $\beta \wedge d\beta^{n-1} \neq 0$, hence non-singular on S which has dimension $2n$. Now apply the previous Lemma. □

The main use of these transverse contact structures is that they can be used to state a slightly weaker version of Giroux's theorem in higher dimensions. Recall that for a three-dimensional contact manifold this theorem says that the contact germ of a hypersurface is completely determined by its characteristic foliation. If we also keep track of the induced transverse contact structure this is still true in higher dimensions.

Theorem 2.18. *Let S_i be closed hypersurfaces in $(2n+1)$-dimensional contact manifolds $(M_i, \xi_i = \ker \alpha_i)$, with induced characteristic foliations $\mathcal{F}_i$ and transverse contact structures $\mathcal{H}_i$, $i = 0, 1$. Suppose there is a diffeomorphism $\phi : S_0 \to S_1$ preserving both the characteristic foliation and the transverse contact structure, then there exists a contactomorphism $\psi : \mathcal{O}p\, S_0 \to \mathcal{O}p\, S_1$ such that $\psi|_{S_0} = \phi$.*

Proof. Recall that for a hypersurface $S \subset (M^{2n+1}, \xi = \ker \alpha)$ the characteristic foliation is defined by the vector field X satisfying

$$\iota_X \Omega = \beta_0 \wedge (d\beta_0)^{n-1},$$

where Ω is a volume form and β_0 is the restriction of α to TS, see Lemma 2.5.20 in [5]. In the three dimensional case ($n = 1$) this implies that if $\phi : S_0 \to S_1$ preserves the characteristic foliation then the restrictions of α_0 and $\phi^*\alpha_1$ to S_0 agree up to a scaling factor. If $n \geq 2$ this need not be true. However, using Lemma 2.16 we see that if ϕ also preserves the transverse contact structure then it is still true that the restriction of α_0 and $\phi^*\alpha_1$ to S_0 agree up to scaling. A close inspection of the proofs of Theorem 2.5.22 and Theorem 2.5.23 in [5] shows that with the previous remark the proofs go through in all dimensions. □

2.3 Star-shaped domains

Given a strict contact manifold (M, α), there is a one-to-one correspondence between smooth function $H : M \to \mathbb{R}$, called contact Hamiltonians, and contact vector fields X_H on M. The correspondence is given by

$$\begin{aligned} &X \mapsto H_X := \alpha(X) \\ &H \mapsto X_H, \text{ defined uniquely by } \alpha(X_H) = H \text{ and } i_{X_H} d\alpha = dH(R_\alpha)\alpha - dH. \end{aligned} \tag{2.2}$$

Furthermore, this correspondence also holds for time dependent contact Hamiltonians $H : M \times I \to \mathbb{R}$ (or 1-periodic contact Hamiltonians $H : M \times S^1 \to \mathbb{R}$) by applying the construction from Equation 2.2 for each $t \in I$. Note that if the contact Hamiltonian is time dependent then the corresponding contact vector field will also be time dependent. Since there is also a one-to-one correspondence between contact vector fields and contact isotopies ϕ^t starting at $\phi^0 = Id$, defined by integrating the vector field, we see that contact Hamiltonians $H : M \times I \to \mathbb{R}$ correspond to contact isotopies $\phi_H^t : M \times I \to M$ by $\phi^0 = Id$ and

$$\alpha(\partial_t \phi_H^t(x)) = \alpha(X_H(\phi_H^t(x))) = H(\phi_H^t(x), t). \tag{2.3}$$

A good reference for these facts is Section 2.3 in [5].

On the strict contact manifold $(\mathbb{R}^{2n-1}, \alpha_{st})$ we can use this correspondence to consider X_z, the contact vector field associated to the z coordinate function. In coordinates we have $X_z = z\frac{\partial}{\partial z} + \sum_{i=1}^{n-1} u_i \frac{\partial}{\partial u_i}$. Denote by $X_z^t : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}$ it's flow over time t which is complete and fixes the origin.

Definition 2.19. *A compact domain $\Delta \subset (\mathbb{R}^{2n-1}, \alpha_{st})$ is called* star-shaped *if it contains the origin and its boundary is transverse to the vector field X_z.*

A usefull property of star-shaped domains is that given two such domains Δ and Δ' we can use X_z to flow one into the other while fixing the origin.

Lemma 2.20. *Given any two compact star-shaped domains $\Delta, \Delta' \subset (\mathbb{R}^{2n-1}, \xi_{st})$, there exists a $\Phi \in \mathrm{Cont}_0(\mathbb{R}^{2n-1}, \xi_{st})$ such that $\Phi(\Delta) \subset \mathrm{Int}\,\Delta'$.*

Proof. Consider $\Phi := X_z^t$ for appropriate $t \in \mathbb{R}$. □

Furthermore, star-shaped domains come with a useful description of open neighborhoods of the boundary.

Lemma 2.21. *For any compact star-shaped domain Δ, an open neighborhood of $\partial\Delta$ can be identified with $\partial\Delta \times (-1,1)$. Under this identification X_z corresponds to the constant vector field $\frac{\partial}{\partial\partial}\partial t$ where t denotes the coordinate on $(-1,1)$.*

Proof. Consider the map $\Phi : \partial\Delta \times (-1,1) \to \mathbb{R}^{2n-1}$ defined by $(x,t) \mapsto X_z^t(x)$. □

We will often encounter the following example of a compact, star-shaped domain called the *cylindrical domain*

$$\Delta_{cyl} := \{(z, u_1, \theta_1, \dots, u_{n-1}, \theta_{n-1}) \in \mathbb{R}^{2n-1} : |z| \leq 1, \sum_{i=1}^{n-1} u_i \leq 1\} \subset (\mathbb{R}^{2n-1}, \xi_{st}).$$

2.4 Contact shells

Suppose we have a manifold with an almost contact structure which is contact everywhere except for some closed set as in the following definition.

Definition 2.22. *A* contact shell *is a pair (B, η) consisting of*

(i) a $(2n+1)$ dimensional ball[2] *$B \subset \operatorname{Int} M$, contained in the interior of an ambient manifold M,*

(ii) an almost contact structure η on $\mathcal{O}p\,B$ which is a contact structure on an open set U containing ∂B.

A contact shell is called solid *if η is contact everywhere, i.e. if $B \subset U$.*

We think of these contact shells as holes in the contact structure and to show existence of a contact structure on the entire manifold we want to fill these holes. That is, we want to homotope such a shell to a solid one without changing the ambient contact structure.

We define an equivalence relation on the set of all contact shells encoding when two contact shells are the same for our filling problem. Being the same, means that if we can fill a contact shell then we can fill any shell equivalent to it.

Definition 2.23. *An* equivalence *between two contact shells (B, η), (B', η') is a diffeomorphism $f : \mathcal{O}p\,B \to \mathcal{O}p\,B'$, such that there exist an open set $U \supset \partial B$ satisfying*

1. *$f(B) = B'$ and $f^*(\eta'|_{f(U)}) = \eta|_U$,*
2. *$f^*\eta'$ is homotopic to η through almost contact structures, fixed on U.*

In this case (B, η) and (B', η') are said to be equivalent, denoted $(B, \eta) \sim (B', \eta')$.

[2] We will always allow balls and disks to have piecewise smooth (i.e. stratified by smooth submanifolds) boundary. For the sake of clarity we will use the convention $\dim B = 2n+1$, $\dim D = 2n$ and $\dim \Delta = 2n-1$ when defining balls/disks.

It is easy to see that this defines an equivalence relation. Note that since the homotopy is fixed on $U \supset \partial B$ we do not change the (almost) contact structure on the ambient manifold outside of B.

Given a contact shell (B, η), it can happen that U is large relative to B. In this case filling (B, η) reduces to filling a smaller contact shell contained in B. This situation gives rise to the notion of domination of contact shells.

Definition 2.24. *A contact shell (B_+, η_+) is said to* dominate *a contact shell (B_-, η_-), denoted by $(B_-, \eta_-) \prec (B_+, \eta_+)$, if there exist*

- *equivalences of contact shells $g_- : (B_-, \eta_-) \to (\widetilde{B}_-, \tilde{\eta}_-)$ and $g_+ : (B_+, \eta_+) \to (\widetilde{B}_+, \tilde{\eta}_+)$.*
- *an almost contact embedding $h : (\widetilde{B}_-, \tilde{\eta}_-) \hookrightarrow (\widetilde{B}_+, \tilde{\eta}_+)$,*

such that $h^*(\tilde{\eta}_+) = \tilde{\eta}_-$ and $\tilde{\eta}_+$ is contact on $\widetilde{B}_+ \setminus \operatorname{Int} h(\widetilde{B}_-)$. The composition $g_+^{-1} \circ h \circ g_- : B_- \to B_+$ is called a subordination *map.*

In [2] the following, slightly different, definition of domination is given:

Definition 2.25. *Given two shells (B_+, η_+) and (B_-, η_-) we say that (B_+, η_+)* dominates *(B_-, η_-) if there exists both*

- *a shell (B, η) with an equivalence $g : (B, \eta) \to (B_+, \eta_+)$ of contact shells,*
- *an embedding $h : B_- \to B$ such that $h^*\eta = \eta_-$ and η is a contact structure on $B \setminus \operatorname{Int} h(B_-)$.*

So, in our definition we allow for an extra equivalence. The next lemma says these two definitions are equivalent.

Lemma 2.26. *Definition 2.24 and Definition 2.25 are equivalent.*

Proof. Assume (B_-, η_-) is dominated by (B_+, η_+) according to Definition 2.25. Then, taking $g_- = \mathrm{Id}$ and $g_+ = g^{-1}$, it follows that $(B_-, \eta_-) \prec (B_+, \eta_+)$ according to Definition 2.24.

To show the converse, assume $(B_-, \eta_-) \prec (B_+, \eta_+)$ according to Definition 2.24. Consider the composition $h \circ g_- : B_- \to B_+$ this is an embedding and hence a diffeomorphism onto it's image.
Since g_- is an equivalence, we have $(g_-^{-1})^*\eta_- \simeq \tilde{\eta}_-$ by a homotopy of almost contact structures fixed on $U := \mathcal{O}p\, \partial \widetilde{B}_-$. Furthermore, since h is an almost contact embedding $(h^{-1})^*\tilde{\eta}_- = \tilde{\eta}_+$. Hence,

$$(h \circ g)^{-1*}\eta_- = h^{-1*} g^{-1*}\eta_- \simeq h^{-1*}\tilde{\eta}_- = \tilde{\eta}_+,$$

and this homotopy is fixed on $\partial \widetilde{B}_+ \subset h(U)$. We extend this homotopy to a homotopy on $\widetilde{B}_+$ fixed outside $\operatorname{Int} h(\widetilde{B}_-)$ to get an equivalence $(B, \eta) \sim (\widetilde{B}_+, \tilde{\eta}_+)$. Here $B := \widetilde{B}_+$ and η is the almost contact structure on $\mathcal{O}p\, B$ which agrees with $\tilde{\eta}_+$ outside $\operatorname{Int} h(\widetilde{B}_-)$ and with $(h \circ g)^{-1*}\eta_-$ everywhere else. Then, we have an equivalence $(B_+, \eta_+) \sim (B, \eta)$. Furthermore, $h \circ g_- : (B_-, \eta_-) \to (B, \eta)$ is an embedding such that $(h \circ g_-)^*\eta = \eta_-$ and η is a contact structure on $B \setminus \operatorname{Int} h \circ g_-(B_-) = \widetilde{B}_+ \setminus \operatorname{Int} h(\widetilde{B}_-)$ since it agrees with $\tilde{\eta}_+$. □

The domination relation is weaker than the equivalence relation. More precisely, if $(B_-, \eta_-) \sim (B_+, \eta_+)$ then $(B_-, \eta_-) \prec (B_+, \eta_+)$. This follows immediately by taking $g_- = g_+ = Id$ and $h = f$ where $f : (B_-, \eta_-) \to (B_+, \eta_+)$ is the equivalence.
Furthermore, the domination relation is reflexive, transitive but not symmetric or antisymmetric. This means it is a pre-order on the set of contact shells but not a partial order or equivalence relation. It is easy to see that $(B, \eta) \prec (B, \eta)$ and that $(B, \eta) \prec (B', \eta')$ does not imply $(B', \eta') \prec (B, \eta)$. Transitivity is proved in the following Lemma and the fact that domination is not antisymmetric follows from Example 5.6.

Lemma 2.27. *If we have contact shells $(B_-, \eta_-), (B, \eta)$ and (B_+, η_+) such that (B_-, η_-) is dominated by (B, η) and (B, η) is dominated by (B_+, η_+), then (B_-, η_-) is dominated by (B_+, η_+). Here domination refers to Definition 2.25.*

Proof. By definition of the domination relation we have equivalences $(B_+, \eta_+) \sim (\widetilde{B}_+, \tilde{\eta}_+)$, $(B, \eta) \sim (\widetilde{B}, \tilde{\eta})$ and isomorphisms

$$h_+ : (B, \eta) \to \Big(h_+(B), \tilde{\eta}_+|_{h_+(B)}\Big) \subset (\widetilde{B}_+, \tilde{\eta}_+)$$
$$h_- : (B_-, \eta_-) \to \Big(h_-(B_-), \tilde{\eta}|_{h_-(B_-)}\Big) \subset (\widetilde{B}, \tilde{\eta})$$

Let $\phi : \widetilde{B} \to B$ denote the diffeomorphism from the equivalence $(B, \eta) \sim (\widetilde{B}, \tilde{\eta})$. Then, $\phi^{-1*}(\tilde{\eta}) \cong \eta$ rel ∂B. This implies,

$$(\phi^{-1} \circ h_+^{-1})^*(\tilde{\eta}) \cong h^{-1*}\eta = \tilde{\eta}_+|_{h_+(B)}\,,\ \text{rel}\ \partial h_+(B).$$

Extending this to a homotopy on $\widetilde{B}_+$, fixed outside $h_+(B)$ gives an equivalence

$$(\widetilde{B}_+, \tilde{\eta}_+) \sim (\widetilde{B}_+, \hat{\eta}),$$

where $\hat{\eta}$ is defined by $\hat{\eta}|_{h_+(B)} = (h_+^{-1} \circ \phi^{-1})^*\tilde{\eta}$ and $\hat{\eta}|_{\widetilde{B}_+ \setminus h_+(B)} = \tilde{\eta}_+|_{\widetilde{B}_+ \setminus h_+(B)}$. By transitivity of the equivalence relation we conclude $(B_+, \eta_+) \sim (\widetilde{B}_+, \hat{\eta})$. The composition $h_+ \circ \phi \circ h_- : B_+ \to \widetilde{B}_+$ is an embedding satisfying $(h_+ \circ \phi \circ h_-)^*\hat{\eta} = \eta_-$. Moreover, outside $\operatorname{Int} h_+ \circ \phi \circ h_-(B_-) = h_+(B)$ we have that $\hat{\eta}$ is contact since h_+ comes from the assumption that B_+ dominates B. We conclude that (B_-, η_-) is dominated by (B_+, η_+). □

Finally,the following lemma says that the domination relation does indeed encode when it is possible to reduce the filling problem:

Lemma 2.28. *Let (B_-, η_-) be a contact shell which is equivalent to a solid shell. If (B_-, η_-) is dominated by a shell (B_+, η_+), then (B_+, η_+) is equivalent to a solid shell.*

Proof. We use again Definition 2.25. Hence, by assumption so we have an equivalence $g : (B, \eta) \to (B_+, \eta_+)$ and an embedding $h : B_- \to B$ satisfying $h^*\eta = \eta_-$.
The assumption that (B_-, η_-) is equivalent to a solid shell implies that we can find a homotopy of almost contact structure η_t, $t \in [0, 1]$ satisfying $\eta_0 = \eta_-$, η_1 is contact and $\eta_t|_{\mathcal{O}p\partial B_-} = \eta_0$ for all $t \in [0, 1]$.
Using $h^{-1} : B \to B_-$ we can pullback η_t to a homotopy $(h^{-1})^*\eta_t$ on B between η and a contact structure ξ extending $(h^{-1})^*\eta_1$, which makes (B, ξ) into a solid shell. By transitivity of the equivalence relation this means that (B_+, η_+) is equivalent to a solid shell. □

Chapter 3

Main theorem

With the basic definitions in place we are now ready to give the precise statement of the main theorem saying that any almost contact structure is homotopic to a contact structure. In this chapter we present the main ideas of this proof without going in to much details. There are two reasons for doing this. Firstly, the order in which the parts of the proof will be presented in subsequent chapters is different from the order in which they are used in the proof. Therefore, it is useful to see the logical structure of the proof in advance. Secondly, we hope that the elegant main ideas serve as motivation for the sometimes technical and involved details.

The core of the proof consists of two arguments, resulting in Proposition 3.7 and Proposition 3.8 below. The first argument, which we consider in Section 3.2, shows that we can find a contact structure on the complement of finitely many contact shells. In Section 3.3 we explain the second demonstrating how to extend the contact structures over these contact shells. In Section 3.4 we show how the core propositions combine to give the proof of the main theorem.

3.1 Main theorem

The statement of the main theorem is as follows.

Theorem 3.1. *Let M be a manifold and $A \subset M$ a closed subset, possibly the emptyset. Furthermore, let η be an almost contact structure on M which is contact on $\mathcal{O}p\, A$. Then, η is homotopic, through almost contact structures, relative to A to a contact structure ξ on M.*

We make no assumptions on the manifold M, other than it being of odd dimension which is necessary for the existence of an (almost) contact structure. Hence, taking $A = \emptyset$, this shows in particular that any closed manifold which admits an almost contact structure admits a contact structure.

By making minor modifications to the proof it is possible to obtain a parametric version of the above theorem. For the most part these modifications consist of adding parameters in the notation and in a few places the argument changes slightly. We will not give the details of this proof and refer the interested reader to [2]. For completeness we do give the statement of the theorem for which we first need to introduce some notation.

Let $\mathfrak{Cont}_{\mathrm{ot}}(M, A, \xi_0)$ denote the space of overtwisted contact structures ξ on M which coincide with some fixed contact structure ξ_0 on $\mathcal{O}p\, A$, with $A \subset M$ closed. Similarly, let $\mathfrak{cont}_{\mathrm{ot}}(M, A, \xi_0)$ denote the space of almost contact structures η on M which coincide with a fixed contact structure ξ_0 on $\mathcal{O}p\, A$. Given a contact embedding of an overtwisted disk $\phi : (D_{\mathrm{ot}}, \zeta_{\mathrm{ot}}) \to (M \setminus A, \xi)$ we denote by $\mathfrak{Cont}_{\mathrm{ot}}(M, A, \xi_0, \phi)$ and $\mathfrak{cont}_{\mathrm{ot}}(M, A, \xi_0, \phi)$ the subspaces of (almost) contact structures containing this specific overtwisted disk.

Theorem 3.2. *The inclusion map induces an isomorphism*

$$j_* : \pi_0(\mathfrak{Cont}_{\mathrm{ot}}(M, A, \xi_0)) \to \pi_0(\mathfrak{cont}(M, A, \xi_0)),$$

and moreover the map

$$j : \mathfrak{Cont}_{\mathrm{ot}}(M, A, \xi_0, \phi) \to \mathfrak{cont}_{\mathrm{ot}}(M, A, \xi_0, \phi),$$

is a (weak) homotopy equivalence.

The surjectivity on π_0 means that any almost contact structure is homotopic to a contact structure. The injectivity on π_0 says that two overtwisted contact structures which are homotopic as almost contact structures are homotopic as contact structures. Hence by Gray stability, Theorem 3.5, this implies that they are isotopic. So, an isomorphism on π_0 means that each homotopy class of almost contact structures contains a unique, up to isotopy, contact structure.
For higher homotopy groups we get similar statements. For example, the surjectivity on π_1 says that every 1-parameter family of almost contact structures $\{\xi_t\}_{t\in[0,1]}$ joining two contact structures ξ_0 and ξ_1 is homotopic, while keeping ξ_0 and ξ_1 fixed, to a family of contact structures $\{\tilde{\xi}_t\}_{t\in[0,1]}$.

3.2 Reduction to a local problem

The starting point of the proof is the famous result by Gromov stating that contact structures on open manifolds admit a (multi) parametric h-principle. With the same notation as before the statement is as follows:

Theorem 3.3. *Let M be a manifold, $A \subset M$ a closed subset (possibly the emptyset) and ξ_0 a contact structure on $\mathcal{O}p\, A$. If (M, A) is relatively open*[1]*, then the inclusion*

$$j : \mathfrak{Cont}(M, A, \xi_0) \to \mathfrak{cont}(M, A, \xi_0)$$

is a (weak) homotopy equivalence.

We will not provide a proof of this theorem but instead refer the reader to [9] or [4].

[1] A pair (M,A) is called *relatively open* if for any $x \in M \setminus A$ there either exists a path in $M \setminus A$ connecting x with a boundary point of M, or a proper path $\gamma : [0, \infty) \to M \setminus A$ with $\gamma(0) = x$. This condition is satisfied if $M \setminus A$ is an open manifold, i.e. non-compact or with boundary.

For open manifolds this already proves the main theorem so let us assume we have a closed almost contact manifold (M, η) of dimension $2n+1$ and that for some closed set $A \subset M$, possibly empty, the restriction $\eta|_A$ is contact.
Choose an embedded annulus $B := S^{2n} \times [0,1] \subset M \setminus A$, which can always be done using a local coordinate chart of M. Since B is closed we can apply Theorem 3.3 to the complement $M \setminus B$. Hence, we can homotope $\eta|_{M \setminus B}$ to a contact structure ξ on $M \setminus B$ and the homotopy can be assumed to be relative to $A \subset M$. This homotopy can be extended to a homotopy on M, starting at η and ending at an almost contact structure η' satisfying $\eta|_{M \setminus B} = \xi$. By slightly enlarging B we can assume that $\eta'|_{\mathcal{O}p\,\partial B}$ is contact making $(B, \eta'|_B)$ into a contact shell.

The key observation is that since B is fibered by spheres S^{2n} we obtain a 1-parameter family of almost contact structures to which we can apply the 1-parametric part of Theorem 3.3. More precisely, pick an open neighborhood $U_t := S^{2n} \times (t-\epsilon, t+\epsilon)$ around each sphere $S^{2n} \times \{t\}$, $t \in [0,1]$ and consider the restriction $\eta'_t := \eta'|_{U_t}$. By picking ϵ small enough we can assume that η'_0 and η'_1 are contact. Hence, identifying all the neighborhoods U_t with $U := S^{2n} \times (-\epsilon, \epsilon)$ we obtain a one parameter family of almost contact structures η'_t, $t \in [0,1]$ such that η_0 and η_1 are contact structures.
The 1-parameter case of Theorem 3.3 tells us that we can find a homotopy from η'_t to a family of contact structures ζ_t, $t \in [0,1]$ satisfying $\zeta_t = \eta'_t$ for $t = 0, 1$. Using again the identification $U_t \cong U$ we obtain a contact structure ζ_t on an open neighborhood U_t of each slice $S^{2n} \times \{t\} \subset B$. We call such a (smooth) family a semi-contact structure on B and denote it by $\zeta := \{\zeta_t\}_{t \in [0,1]}$.

Remark 3.4. *(i) Restricting ζ_t to $S^{2n} \times \{t\}$ for each $t \in [0,1]$ induces on B an almost contact structure η_ζ. Unlike η' this almost contact structures agrees with a contact structure in all $2n$ directions along the fiber S^{2n} and is an almost contact structure only in the t coordinate direction. Morally speaking we have reduced a $2n+1$ dimensional problem to a one dimensional problem.*

(ii) The contact germs ζ_0 and ζ_1 are just the restriction of the contact structure on $\mathcal{O}p\,\partial B$. Hence, we can forget about the ambient manifold and just consider (B, ζ) because as long as we keep ζ_0 and ζ_1 (which corresponds to keeping η_ζ fixed on ∂B) fixed we can always glue back B into the ambient manifold.

The next step is to relate the contact germs on different fibers to each other. For this, we use the Gray stability theorem.

Theorem 3.5. *Let $\{\xi_t\}_{t \in I}$ be a smooth family of contact structures on a closed manifold M. Then there is an isotopy ψ_t, $t \in I$, such that*

$$T\psi_t(\xi_0) = \xi_t \text{ for all } t \in I.$$

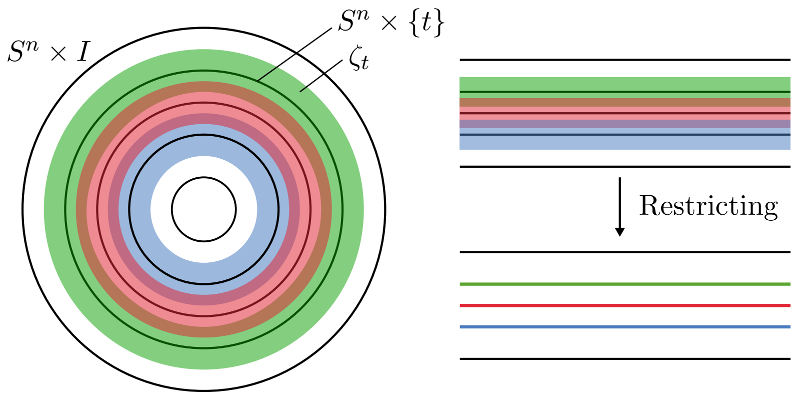


Figure 3.1: The annulus $S^n \times I \subset M$ together with the semi-contact structure $\zeta = \{\zeta_t\}_{t \in I}$. Note that the ζ_t overlap but restricting ζ_t to $S^{2n} \times \{t\}$ induces an almost contact structure η_ζ as in Lemma 9.2.

Apply this theorem to the family ζ_t, $t \in [0,1]$ on $U = S^{2n} \times (-\epsilon, \epsilon)$ and assume that for all $t \in [0,1]$ the hypersurface $\psi_t(S^{2n} \times \{0\})$ is the graph of a function $\phi_t : S^{2n} \to (-\epsilon, \epsilon)$. This realizes the germs of contact structures ζ_t as the restriction of ζ_0 to the graphs of the functions ϕ_t. Observe that since $\psi_0 = \text{Id}$ we always have $\phi_0 = 0$. Furthermore, the functions ϕ_t for $t \in (0,1)$ do not matter so much since we can always change them using a homotopy fixing η_ζ relative to the boundary ∂B. So, the information we need to remember is just the contact structure ζ_0 and one function $\phi := \phi_1$. To finish the proof we have to homotope η_ζ to a contact structure "connecting" the germs on the graphs of ϕ_0 and ϕ_1.

Remark 3.6. *Note that up to this point we have not used any properties unique to contact structures. Indeed, the only things we have used so far are the h-principle theorem by Gromov and a Moser type argument in the form of the Gray stability theorem and there are many other types of structures for which these statements hold. In fact, the above argument is a common idea in h-principle type proofs where it is usually the case that to proof even existence of some structure in n dimensions we need a 1-parametric h-principle in $n-1$ dimensions.*

We have an immersion $F : S^{2n} \times [0,1] \to S^{2n} \times (-\epsilon, \epsilon)$ defined by

$$(x,t) \mapsto (x, t\phi(x)).$$

Around the points where ϕ is positive this becomes an embedding and we can easily obtain the required contact structure by pulling back ζ_0 to B. Therefore, the interesting region is where ϕ becomes negative.

Consider the (singular) characteristic foliation $\mathcal{F}$ induced on $S^{2n} \times \{0\}$ by the contact structure ζ_0. Recall that associated to a contact form α defining ζ_0 we have a contact vector field R_α called the Reeb vector field which is transverse to the contact hyperplanes. This implies that around the singular points of $\mathcal{F}$ the Reeb vector field is transverse to $S^{2n} \times \{0\}$ and we can use this to "push up" the graph of ϕ until it is positive. As before this gives us the required contact structure around the singular points.

Away from the singular points any vector field representing the characteristic foliation is non-vanishing, so locally just a constant vector field. Using Giroux's theorem this allows us to locally identify $S^{2n} \times \{0\}$ with the hyperplane

$$\Pi := \{(x,q,p) \in \mathbb{R}^{2n-1} \times T^*\mathbb{R} \mid p = 0\} \subset (\mathbb{R}^{2n-1} \times T^*\mathbb{R}, \ker(\alpha_{st} + pdq),$$

whose characteristic foliation is given by straight lines in the q-direction.

Hence, locally the contact germ ζ_0 equals the restriction of ξ_{st} to $\mathcal{O}p\,\Pi$ while ζ_1 equals the restriction of ξ_{st} to the graph of a function $\tilde{\phi} : \Pi \to \mathbb{R}$ corresponding to ϕ.
Using a partition of unity argument and various local deformations, the part of B where ϕ is still negative can be cut up into finitely many contact shells, called circle models.
Intuitively, each circle model is obtained by restricting the function $\tilde{\phi}$ to a small subset of the form $\Delta \times I$, where $\Delta \subset \mathbb{R}^{2n-1}$ is star-shaped, obtaining a function $K : \Delta \times I \to \mathbb{R}$. Taking a quotient in the q direction we can replace I by S^1 obtaining a function $K : \Delta \times S^1 \to \mathbb{R}$. The pair (Δ, K) of such a function and its star-shaped domain is called a contact Hamiltonian.
We view S^1 as the angular coordinate on $\mathbb{R}^2$ and by picking a constant so that $K + C$ is strictly positive we can define a ball B_K as in Figure 3.2. Furthermore we endow B_K with a contact shell structure η_K coming from the region between Π and the graph of K as above. Again, this implies that the contact germ on ∂B_K is determined by $K : \Delta \times S^1 \to \mathbb{R}$.

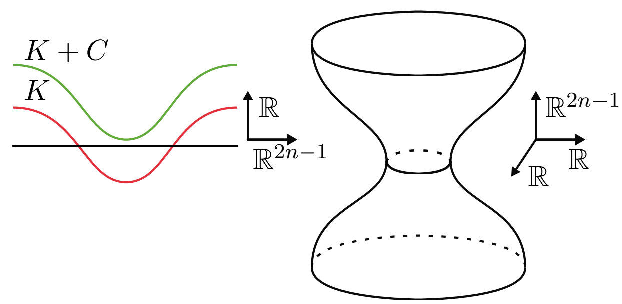


Figure 3.2: Circle model associated to the Hamiltonian K.

We have now reduced the problem to filling finitely many circle models (B_K, η_K) determined by contact Hamiltonians (Δ, K). A priori there are still infinitely many possible choices for these contact Hamiltonians but it turns out that using an equivariant covering argument it is possible to further reduce to the case that in a fixed dimension all circle models are modelled by one universal contact Hamiltonian. Together, these observations add up to the first half of the proof of Theorem 3.1 expressed by the following proposition.

Proposition 3.7. *For each dimension* $2n+1$ *there exists a contact Hamiltonian* $(K_{univ}, \Delta_{univ})$ *such that the following is true.*
Let M *be a* $(2n+1)$*-dimensional manifold,* $A \subset M$ *a closed subset, and* η *an almost contact structure on* M *which is contact on* $\mathcal{O}p\,A \subset M$*. Then, there exists (finitely many) disjoint balls* $B_i \subset M$*, for* $i = 1, \ldots, L$ *such that* η *is homotopic relative to* A*, through almost contact structures, to an almost contact structure* η' *satisfying*

(i) η' *is a contact structure on* $M \setminus \bigcup_{i=1}^{N} B_i$*,*

(ii) the contact shells $(B_i, \eta'|_{B_i})$ *are equivalent to* $(B_{K_{univ}}, \eta_{K_{univ}})$ *for* $i = 1, \ldots, L$*.*

3.3 Solving the local problem

The second half of the proof consists of showing that the circle model $(B_{K_{univ}}, \eta_{K_{univ}})$ appearing in the previous proposition is equivalent to a solid shell. This boils down to studying the contact Hamiltonians (K, Δ) defining the circle models.
The set of contact Hamiltonians (Δ, K) has three important properties which combine to give the proof of Proposition 3.8. We briefly discuss these and then sketch the proof of the proposition.

Firstly, there exists a preorder $\leq$ on the set of all contact Hamiltonians, which we refer to as the domination relation for contact Hamiltonians. Morally speaking, $(K, \Delta) \leq (K', \Delta')$ means that K is smaller than K' in the sense that $\Delta \subset \Delta'$ and $K \leq K'$, as functions on Δ. The essential property of this relation is that it is compatible with the domination relation on circle models $\prec$, in the sense that if $(K, \Delta) \leq (K', \Delta')$ then the associated circle models satisfy $(B_K, \eta_K) \prec (B_{K'}, \eta_{K'})$. An important consequence of the compatibility is that if we can find a contact Hamiltonian (Δ, K) producing a solid circle model (B_K, η_K) then any circle model $(B_{K'}, \eta_{K'})$ modelled by a Hamiltonian $(\Delta', K') \geq (\Delta, K)$ is equivalent to a solid shell.

Secondly, given a contactomorphism $\phi : \Delta \to \Delta'$ and a contact Hamiltonian $K : \Delta \to \mathbb{R}$ we can construct a new contact Hamiltonian $\phi_* K : \Delta' \to \mathbb{R}$, called the push-forward Hamiltonian, defined by $\phi_* K(\phi(x)) := c_\phi(x) K(x)$. Here $c_\phi : \Delta \to \mathbb{R}_+$ is the positive function satisfying $\phi^* \alpha_{st} = c_\phi \alpha_{st}$. Intuitively, the contactomorphism $\phi : \Delta \to \Delta'$ can be viewed as a contact coordinate change and $\phi_* K$ is just the Hamiltonian K in the new coordinates.
What makes this construction useful is that the circle models defined by K and $\phi_* K$ are equivalent. Hence, we can model the same circle model by different Hamiltonians and we can choose the Hamiltonian which is the easiest to work with or satisfies some useful properties.

Thirdly, given two contact shells (B, η) and (B', η) in an almost contact manifold M we can connect them by a thin tube obtaining a new contact shell $(B \# B', \eta \# \eta')$, the connected sum. Furthermore, if we take the connected sum of two circle models (B_K, η_K) and $(B_{K'}, \eta_{K'})$, the resulting shell is again a circle model. The contact Hamiltonian describing it, is denoted by $(K \# K', \Delta \# \Delta')$.
This construction is basically just the topological connected sum construction taking into account the smoothness of the contact structures along the gluing and the requirement that the connecting tube has to be (contact) embedded into the ambient manifold

To see how these observations can be used to proof Proposition 3.8 recall that the almost contact structure on a circle model (B_K, η_K) corresponds to the region between the zero section of $\mathbb{R}^{2n-1} \times T^*S^1$ and the graph of $K : \Delta \times S^1 \to \mathbb{R}$ in the same space. In the same way as before this implies that (B_K, η_K) is solid if and only if K is strictly positive. In terms of the domination relation we can express this by saying that (B_K, η_K) is solid if and only if $(\Delta, K) \geq (\Delta, K_0)$, where K_0 denotes the constant function with value zero. Indeed, if we define the empty set to be the circle model defined by (Δ, K) for any Δ, then this follows from the compatibility of the domination relations on Hamiltonians and circle models.
From this point of view the filling problem for a ball is just a special case of the filling problem for an annulus. Indeed, we can view the ball (B_K, η_K) as an annulus obtained as the difference of two circle models (B_K, η_K) and (B_{K_0}, η_{K_0}) which is just the empty set.
In general given a contact embedding $\iota : (B_{K'}, \eta_{K'}) \to (B_K, \eta_K)$ we can define an annulus $(B_K \setminus \operatorname{Int} B_{K'}, \eta_K|_{B_K \setminus \operatorname{Int} B_{K'}})$. It is important to note that this embedding need not be a subordination map so it is not necessary that $K' \leq K$. The filling problem for such an annulus translates into finding a contactomorphism $\phi \in \operatorname{Cont}(\mathbb{R}^{2n-1}, \xi_{st})$ such that $\phi_* K' \leq K$.
The biggest obstruction in finding such a contactomorphism is that since the push-forward construction is basically multiplication by a positive function it can never change the sign of a Hamiltonian. In particular this implies that if K is somewhere negative and K' is everywhere non-negative then we cannot find the required ϕ. This is also the situation when trying to fill a ball since here $K' = K_0$ while K can be somewhere negative. Hence, we need a trick to resolve this.

Using the connected sum construction we can obtain (B_K, η_K) as part of an annulus whose inner boundary is described by a somewhere negative function in the following way.
Assume somewhere in our manifold we have a solid contact shell $(B_{K'}, \eta_{K'})$ dominating a solid circle model modelled by a slightly smaller Hamiltonian $K'_\epsilon := K' - \epsilon$. Consider the connected sum $B_K \# B_{K'}$ and remove from it the circle model B_{K_ϵ}, see Figure 3.3. This way we obtain an almost contact annulus with a contact germ on the outer sphere modelled by $K \# K'$ and a contact germ on the inner sphere modelled by K'_ϵ.

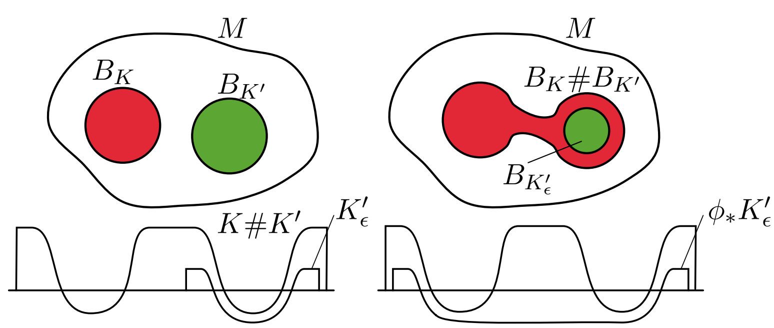


Figure 3.3: Top: Turning the filling problem for a ball into a filling problem for an annulus using the connected sum construction. Bottom: Using the push-forward construction to obtain $\phi_* K'_\epsilon \leq K \# K'$.

There is a large class of Hamiltonians called special which have very nice properties including being somewhere negative. If K' is such a special contact Hamiltonian then it is possible to find the required contactomorphism $\phi \in \mathrm{Cont}(\mathbb{R}^{2n-1}, \xi_{st})$ such that $\phi_* K'_\epsilon \leq K \# K'$. Basically this is done by defnining ϕ in such a way that it stretches the region where K' and hence K_ϵ, which may be very small, to cover the region where $K \# K'$ is negative. Furthermore, we make sure that c_ϕ is very large on the region where K_ϵ is negative so that in the end we have $\phi_* K_\epsilon \leq K \# K'$. In turn this ϕ induces a homotopy of almost contact structures (relative to the boundary of the annulus) between $\eta_K \# \eta_{K'}$ and a contact structure.

Recall that we assumed the existence of a circle model $(B_{K'}, \eta_{K'})$ modelled by a special contact Hamiltonian. It turns out that it is enough to require the existence of a contact ball (not necessarily a circle model) (B, ξ) containing a so called overtwisted disk (D_{ot}, xi_{ot}) in its boundary. Morally, an overtwisted disk is defined to be the lower hemisphere (together the induced contact germ) of a circle model associated to a special contact Hamiltonian which is smaller than K_{univ}. These disks are also used to define overtwisted contact structures in higher dimensions.

The above observations amount to the second half of the proof of Theorem 3.1 expressed by the following proposition.

Proposition 3.8. *Let (Δ_{cyl}, K) be a special contact Hamiltonian and (B, ξ) a contact ball (not necessarily a circle model) such that there is a contact embedding $(D_K, \eta_K) \subset (\partial B, \xi)$. Then, given any contact Hamiltonian $K' \geq K$ the connected sum*

$$(B_{K'} \# B, \eta_{K'} \# \xi)$$

connecting the north pole of $B_{K'}$ and the south pole of $D_K \subset \partial B$, is equivalent to a solid contact shell.

There is one point we still need to address. Namely, in order to apply Proposition 3.8 to fill the finitely many circle models equivalent to $(B_{K_{univ}}, \eta_{K_{univ}})$ obtained by Proposition 3.7, we need as many balls containing an overtwisted disk in their boundary.
The following lemma whose proof essentially boils down to the path-connectedness of the linear symplectic group, says that we can always create an overtwisted disk in an almost contact manifold.

Lemma 3.9. *Let (M, η) be an almost contact manifold and $B \subset M$ a ball. Then η is homotopic relative to $M \setminus \mathcal{O}p\, B$ to an almost contact structure η' containing an overtwisted disk in ∂B.*

Note that the homotopy does not change the almost contact structure outside $\mathcal{O}p\, B$. Therefore, it is easy to see that we can apply this homotopy without changing the contact structure on $\mathcal{O}p\, A$, where $A \subset M$ is as in Theorem 3.1.
Furthermore, once we have one overtwisted disk, we have as many as we need.

Lemma 3.10. *Every neighborhood of an overtwisted disk in an (almost) contact manifold contains a foliation by overtwisted disks.*

The basic idea of the proof is that if K is a special contact Hamiltonian, then a slightly perturbed Hamiltonian K_ϵ will still be special. The disks $(D_{K_\epsilon}, \eta_{K_\epsilon})$ foliate a neighborhood of the disk (D_K, η_K).

3.4 Proof of the main theorem

We now show how Proposition 3.7 and Proposition 3.8 imply the proof of Theorem 3.1. Unlike the previous sections the argument here is presented in full detail.

Proof of Theorem 3.1. Choose a ball $B \subset M \setminus A$, and apply Lemma 3.9 to homotope $\eta|_B$ to a contact structure containing an overtwisted disk. As we noted before, this changes η on $\mathcal{O}p\, B$ but the result will stil be an almost contact structure. For the rest of the proof we want to keep the contact structure on $A \cup B$ fixed.

Using Proposition 3.7 we deform η relative to $A \cup B$ to an almost contact structure η' which is contact in the complement of finitely many balls $B_1, \dots, B_L \subset M \setminus A \cup B$. Furthermore, each $(B_i, \eta'|_{B_i})$ is equivalent to $(B_{K_{univ}}, \eta_{K_{univ}})$.

By Lemma 3.10 we can find disjoint balls $\widehat{B}_1, \dots, \widehat{B}_L \subset \operatorname{Int} B$ such that the boundary of each ball contains an overtwisted disk (D_{K_i}, η_{K_i}) for special $K_i \leq K_{univ}$, $i = 1, \dots, L$. The balls B_i and $\widehat{B}_i$ can be connected inside $M \setminus A$ and without intersecting each other, by taking their (ambient) connected sums. This yields contact shells $(B_i \# \widehat{B}_i, \eta'|_{B_i \# \widehat{B}_i})$, $i = 1, \dots, L$. From Proposition 3.7 and properties of the connected sum construction we get isomorphisms

$$(B_i \# \widehat{B}_i, \eta'\big|_{B_i \# \widehat{B}_i}) \cong (B_i \# \widehat{B}_i, \eta'\big|_{B_i} \# \eta'\big|_{\widehat{B}_i}) \cong (B_{K_{univ}} \# \widehat{B}_i, \eta_{K_{univ}} \# \eta'\big|_{\widehat{B}_i}), \quad i = 1, \dots, L.$$

These shells satisfy the conditions for Proposition 3.8 and hence we can homotope $\eta_{K_{univ}} \# \eta'|_{\widehat{B}_i}$ relative to the boundary of $B_{K_{univ}} \# \widehat{B}_i$ to a contact structure. Going back under the isomorphisms this gives an equivalence of $(B_i \# \widehat{B}_i, \eta'|_{B_i \# \widehat{B}_i})$ to a solid shell.

Since all the homotopies we used are relative to A this concludes the proof. □

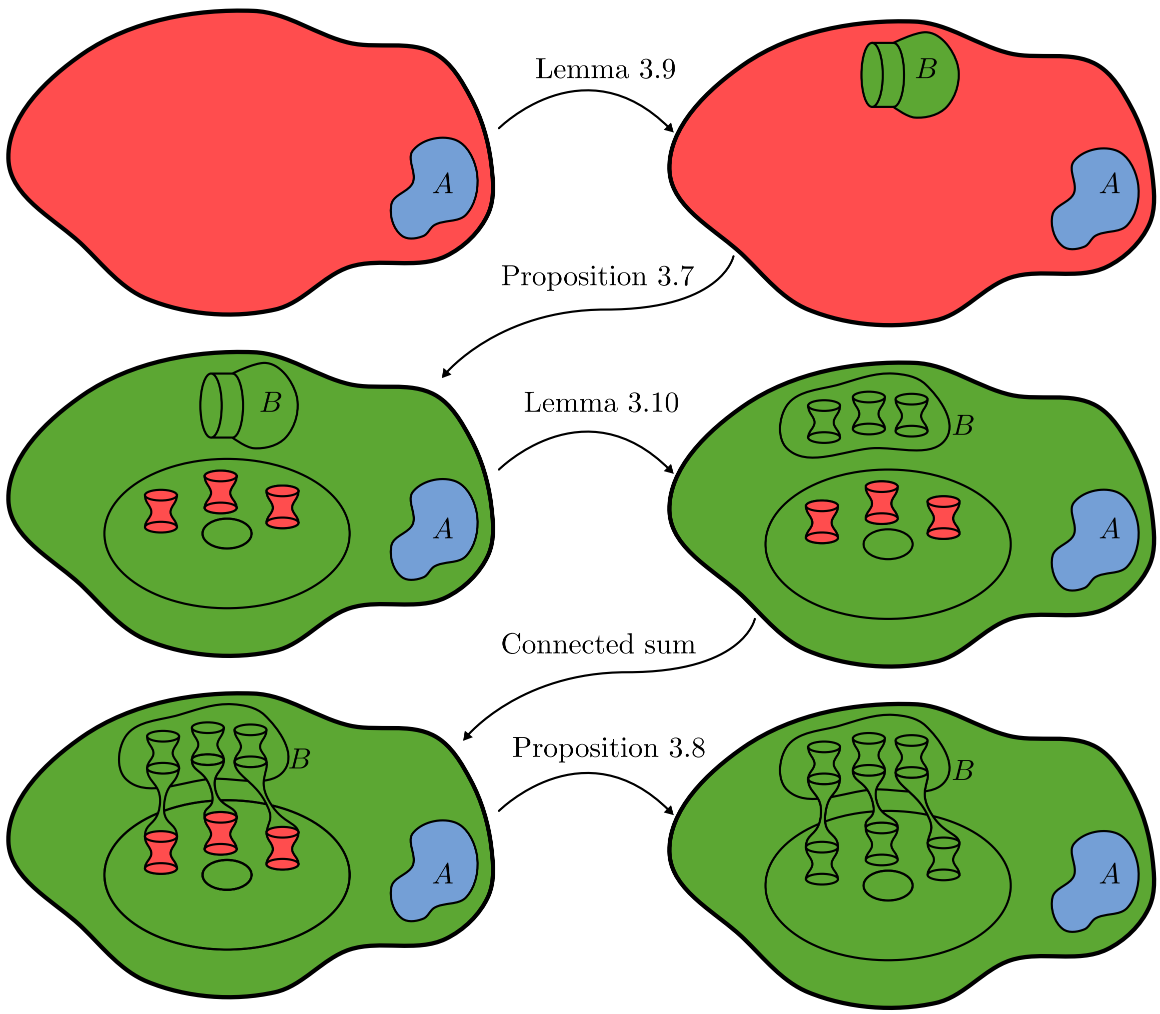


Figure 3.4: Overview of the steps in the main theorem and their result on the manifold M. Note that the ball B changes appearance but this is just to make the picture more readable.

Chapter 4

Circle models

As we have seen in Chapter 3 we can use topological methods to obtain a contact structure on a large part of the manifold, as is summarized in the statement of Proposition 3.7. This leaves us with finitely many contact shells of a specific form, called *circle models*. In this chapter we give a precise definition of these circle models and study some of their immediate properties.
In short, a circle model (B_K, η_K) is a contact shell with the special property that the contact structure near the boundary ∂B_K is encoded in a function K called a contact Hamiltonian. In the next chapters we will see that the equivalence and domination relations for circle models translate into relations on their defining functions. This allows us to formulate the filling problem in terms of functions, which are relatively easy to understand, leading to a proof of Proposition 3.7.

The precise definition of a circle model is rather involved. To clarify some parts of the definition we start by giving some motivation in the first section. The motivation is given by the proof of Proposition 3.7. We have seen in Chapter 3 that the essential ingredients in this proof are the push forward construction and the fact that the contact structure near the boundary of a circle model is encoded by a function. It turns out that if we want circle models to satisfy these conditions there is only one possible definition.
In the second definition we explain the precise definition. The construction of a circle model from its defining contact Hamiltonian involves several choices of auxiliary functions and constants. In the third section we show that suitable functions and constants exist and that the specific choice does not matter so that circle models are well defined.
In the last section we focus on the contact structure near the boundary of the circle model and calculate the precise form of the induced characteristic foliation on the boundary.

4.1 Intuition for circle models

Before considering the general case we illustrate the motivating properties in the following example, the simplest case of a 3-dimensional circle model.

Consider the (strict) contact manifold $(\mathbb{R}^3, \alpha_0 := dz + v d\theta)$, take $\Delta := I$ and let $K : \mathcal{O}p\, I \to \mathbb{R}_+$ be a smooth, positive function. We view K as a positive function on $(\mathcal{O}p\, I) \times S^1$ by defining it to be independent of the S^1 coordinate. Using K we define a (piecewise smooth) ball as follows,

$$B_K := \{ (x, v, \theta) \in \mathbb{R}^3 \mid 0 \le v \le K(x, \theta) \}.$$

Restricting the standard contact structure $\xi_{st} = \ker \alpha_0$ to an open neighborhood of B_K gives it the structure of a solid contact shell. This is the simplest example of a circle model.

Observe that the ball B_K is a (trivial) fiber bundle over D^2. It is trivialized by the map

$$\psi : I \times D^2 \to B_K, (x, r, \phi) \mapsto (x, rK(x, \phi), \phi),$$

where $(r, \phi) \in D^2$ denote standard polar coordinates. In the trivialization the projection map given by $\pi(x, r, \phi) := (r, \phi)$ the projection onto D^2.
Since, K is defined on an open neighborhood of I we can slightly extend ψ, to a diffeomorphism between $\mathcal{O}p\,(I \times D^2)$ and $\mathcal{O}p\,B_K$,also be denoted by ψ. Furthermore, equipping $\mathcal{O}p\,(I \times D^2)$ with the contact structure $\xi := \ker(\psi^* \alpha_{st})$, making ψ into a contactomorphism.

The coordinate expression for $\alpha := \psi^* \alpha_{st}$ is given by

$$dx + rK(x, \theta) d\phi. \tag{4.1}$$

This coordinate description of the contact structure on B_K allows us to read of its special properties:

(i) The hyperplane field $\xi = \ker \alpha$ is fiberwise contact in the sense that

$$\xi \cap T(I_p) = \xi^1_{st}, \text{ for all } p \in D^2,$$

where $\xi^1_{st} = \ker dx$ is the standard (trivial) contact structure on $I \subset \mathbb{R}$. This follows immediately from the fact that $\alpha|_{T(I_p)} = dx$.

(ii) The contact structure near the boundary $\partial(I \times D^2)$ is encoded by K. Indeed, an open neighborhood of this boundary is contactomorphic to the hypersurface in $(\mathbb{R}^3, \xi_{st})$ described by K.

The relevance of the first property is that it ensures that the push-forward construction, which we introduce in Chapter 5, produces equivalent circle models. We illustrate this for our simple example above.
Let $F : I \to [a, b]$ be a contactomorphism between I and some compact interval $[a, b] \subset (\mathbb{R}, \xi^1_{st})$. As usual this means that $F^*(\alpha^1_{st}) = c_F \alpha^1_{st}$ for some positive function c_F, which in this case is just $\frac{dF}{dx}$.
This contactomorphism induces a new contact Hamiltonian $F_*K : [a, b] \times S^1 \to \mathbb{R}_+$ which we call the push-forward contact Hamiltonian. It is defined by

$$F_*K(F(x), \theta) = c_F(x) K(x, \theta).$$

We can use this new Hamiltonian to define a circle model B_{F_*K} in exactly the same way as above. It turns out that B_K and B_{F_*K} are contactomorphic by the map $G : B_K \to B_{F_*K}$ defined by

$$G(x, v, \theta) := (F(x), c_F(x) v, \theta).$$

Observe that the fact that G is a contactomorphism is a consequence of the fact that ξ is fiberwise contact. Conversely, for any fiberwise contact structure a map of the above form is automatically a contactomorphism. We will see later that this is still true in the general case when the fiber I is replaced by a compact star-shaped domain $\Delta \subset \mathbb{R}^{2n-1}$ and $K : \Delta \times S^1 \to \mathbb{R}$ is allowed to become negative.

Now let us return to the main story. The point of the previous discussion is that (almost) contact structures of the form given by Equation 4.1 have exactly those properties needed for the proof of Proposition 3.8. It turns out that the converse is also true.

To prove this claim we start in the situation that we have a hyperplane field $\xi = \ker \alpha$ on a piecewise smooth ball $B \cong \Delta \times D^2 \subset \mathbb{R}^{2n-1} \times \mathbb{R}^2$. The following lemma and it's corollary say that if ξ is fiberwise contact then

$$\alpha = \alpha_{st} + \rho d\theta,$$

where α_{st} is the contact form on the fiber Δ and $\rho : B \to \mathbb{R}$ a smooth function. Furthermore, we will see that if we impose that ξ is part of a contact shell structure on B whose contact germ near ∂B is described by a function then we get extra conditions on ρ and α essentially has to be of the same form as in Equation 4.1.

Recall that a connection on a fiber bundle $M \xrightarrow{\pi} B$ is the choice of a complement $\mathcal{H} \subset TM$ to the vertical bundle $\mathcal{V} := \ker d\pi \subset TM$ so that $TM = \mathcal{H} \oplus \mathcal{V}$.

Lemma 4.1. *Consider a fiber bundle $F \to M \xrightarrow{\pi} B$ with compact fiber F. Let $\xi = \ker \alpha \subset TM$ be a (co-oriented) hyperplane field on M such that for each fiber F_b the intersection $\xi \cap T(F_b)$ is a contact structure on F_b. Then,*

- *(i) there exists a (natural) connection $\mathcal{H}$ on the fiber bundle $M \xrightarrow{\pi} B$;*
- *(ii) the parallel transport with respect to $\mathcal{H}$ preserves the contact structure on the fibers and the coorientation of ξ.*

Proof. We denote $\xi_b^\nu := \xi \cap T(F_b)$ and let

$$\xi^\nu = \bigcup_{b \in B} \xi_b^\nu,$$

which is a codimension one subbundle of the vertical bundle $\mathcal{V}$. Since ξ is cooriented we can write $\xi = \ker \alpha$ for $\alpha \in \Omega^1(M)$. Let $\omega := d\alpha|_\xi$ and define

$$\mathcal{H} := (\xi^\nu)^\omega,$$

the subspace ω-orthogonal to ξ^ν inside (ξ, ω). Note that $\mathcal{H}$ only depends on ξ. Indeed, if α' is another 1-form with $\ker \alpha' = \xi$, then $\alpha' = \alpha_{st}\alpha$ for a positive function $\alpha_{st} \in C^\infty(M)$. This immediately implies $d\alpha'|_\xi = \alpha_{st} d\alpha|_\xi$ so that they define the same $\mathcal{H}$. To see that $\mathcal{H}$ is a connection note that $\omega|_{\xi^\nu}$ is non-degenerate since $\alpha|_{TF_b}$ defines a contact form. This implies that

$$\xi = \xi^\nu \oplus \mathcal{H}.$$

Combining this with $\xi^\nu = \xi \cap \mathcal{V}$ gives

$$TM = \mathcal{V} \oplus \mathcal{H},$$

proving $\mathcal{H}$ is a connection.

Now consider for each path $\gamma : I \to B$ the parallel transport diffeomorphism $\mathcal{P}_\gamma : F_{\gamma(0)} \xrightarrow{\cong} F_{\gamma(1)}$, with respect to $\mathcal{H}$, which exists since F is compact. It follows from Theorem 3.1 in [1] that to show $\mathcal{P}_\gamma$ is a contactomorphism it suffices to show that for each horizontal vector field $X \in \Gamma(\mathcal{H})$ and each section $Y \in \Gamma(\xi^\nu)$ we have $[X, Y] \in \Gamma(\xi^\nu)$. By naturality of the Lie bracket and since $\Gamma(\xi^\nu) \subset \gamma(\mathcal{V})$, we have $d\pi[X, Y] = [d\pi(X), 0] = 0$ which implies $[X, Y] \in \Gamma(\mathcal{V})$. From the definition of $\mathcal{H}$ we know $0 = \omega(X, Y) = d\alpha(X, Y)$. Furthermore, $X, Y \in \Gamma(\xi)$ implies $0 = \alpha(X) = \alpha(Y)$. Consequently,

$$\alpha([X, Y]) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y]) = d\alpha(X, Y) = 0,$$

meaning $[X, Y] \in \Gamma(\xi)$. We conclude, $[X, Y] \in \Gamma(\xi) \cap \Gamma(\mathcal{V}) = \Gamma(\mathcal{V} \cap \xi) = \Gamma(\xi^\nu)$ and so $\mathcal{P}_\gamma$ preserves ξ^ν for each γ.

Now let $\gamma : I \to B$ be a path. Since, $\mathcal{P}_{\gamma(t)} : F_{\gamma(0)} \to F_{\gamma(t)}$ is a contactomorphism for each $t \in I$ we have

$$(\mathcal{P}_{\gamma(t)})^*(\alpha|_{TF_{\gamma(t)}}) = \lambda_t(\alpha|_{F_{\gamma(0)}}),$$

for a continuous family of non-vanishing functions $\lambda_t \in C^\infty(F_{\gamma(0)})$. Since $\lambda_0 = 1$ which is positive we have that λ_t is positive for all $t \in I$. □

The normal form of α follows as a corollary of the previous lemma.

Corollary 4.2. *Let $F \to M \xrightarrow{\pi} B$ be a fiber bundle with compact fiber $F = \Delta \subset \mathbb{R}^{2n-1}$ and base $B = D^2 \subset \mathbb{R}^2$. Assume $\xi \subset TM$ is a cooriented hyperplane field such that*

$$\xi \cap T\Delta_b = \xi_{st}, \text{ for all } b \in \Delta,$$

where $\xi_{st} = \ker \alpha_{st}$ is the standard contact structure on $\mathbb{R}^{2n-1}$. Then there exists a trivialization $\phi : \Delta \times D^2 \to M$, such that

$$(T\phi)^{-1}(\xi) = \ker\left(\alpha_{st}(x) + \rho(x, v, \theta)d\theta\right), \tag{4.2}$$

for a smooth function $\rho : \Delta \times D^2 \to \mathbb{R}$. Here (v, θ) are our usual scaled polar coordinates on D^2.

Proof. First note that since B is contractible it is immediate that $M \cong D^2 \times \Delta$. However, since we will construct a different trivialization $\phi : D^2 \times \Delta \xrightarrow{\cong} M$ we denote the total space by M to avoid confusion.

Define a diffeomorphism $\phi : \Delta \times D^2 \to M$ by

$$\phi(x, v, \theta) \mapsto \mathcal{P}_\gamma(x),$$

where $\gamma : I \to \Delta^2$, defined by $\gamma(t) := (tv, \theta)$, is the ray through (v, θ) and we identify Δ with $\Delta_0 = \pi^{-1}(0)$.

By the previous lemma we know the parallel transport preserves the contact distribution on F. Let $\alpha \in \Omega^1(M)$ by any 1-form such that $\xi = \ker \alpha$, then $\mathcal{P}^*_\gamma(\alpha|_{T(F_{\gamma(1)})}) = \alpha_{st}(\alpha|_{T(F_{\gamma(0)})})$, for a positive function $\alpha_{st} \in C^\infty(F_{\gamma(0)})$. By rescaling α we can assume without loss of generality that

$$\mathcal{P}_\gamma(\alpha|_{T(F_{\gamma(1)}}) = \alpha_{st}.$$

Hence,

$$\phi^*(\alpha) = \alpha_{st} + \eta dv + \rho d\theta,$$

for $\eta, \rho \in C^\infty(\Delta \times D^2)$. The upshot of choosing the special paths for our trivialization is that it implies that dv cannot appear. To see this consider the horizontal lift $\frac{\partial}{\partial u}^{\#} \in \mathfrak{X}(M)$ of the coordinate vectorfield $\frac{\partial}{\partial u} \in \mathfrak{X}(B)$. Since $\frac{\partial}{\partial u}$ is tangent to the paths used in the construction of ϕ we have that

$$(T\phi)^{-1}(\frac{\partial}{\partial u}^{\#}) = \frac{\partial}{\partial u}.$$

Moreover, by the definition of $\mathcal{H}$ and the fact that $\frac{\partial}{\partial u}^{\#}$ is horizontal we have $\alpha(\frac{\partial}{\partial u}^{\#}) = 0$. We conclude,

$$\phi^*(\alpha) = \alpha_{st} + \rho d\theta.$$

□

Remark 4.3. *The geometric interpretation of the function ρ is that it describes the rotation of the hyperplanes. Indeed, the kernel of the form $\alpha_{st} + \rho d\theta$ is given by*

$$\xi_{st} \oplus \text{Span}\{\frac{\partial}{\partial v}, \frac{\partial}{\partial \theta} - \rho R_{\alpha_{st}}\},$$

where $R_{\alpha_{st}}$ is the Reeb vector field of α_{st} which is equal to $\frac{\partial}{\partial z}$. In the three dimensional case, when $n = 1$, this becomes $\text{Span}\{\frac{\partial}{\partial v}, \frac{\partial}{\partial \theta} - \rho\frac{\partial}{\partial z}\}$. *Hence, along a ray $\gamma_{(x,\theta)} : \mathbb{R}_{\geq 0} \to \mathbb{R}^3$ given by $t \mapsto (x, t, \theta)$ for fixed (x, θ), the function $\rho_{(x,\theta)} := \rho \circ \gamma_{(x,\theta)}$ encodes the rotation of the hyperplanes.*

For $n > 1$ a visualization becomes impossible due to a lack of dimensions. However, viewing $\mathbb{R}^{2n-1}$ as the z-axis in the above picture and note that ξ_{st} is constant along the ray $\gamma_{(x,\theta)}$ we interpret ρ in the same way.

In the previous lemma we considered arbitrary hyperplane fields $\xi = \ker\alpha$ and so in the normal form $\alpha = \alpha_{st} + \rho d\theta$ there is no restriction on $\rho : \Delta \times D^2 \to \mathbb{R}$ other than being smooth. However, if we assume that α is part of an almost contact structure of a contact shell $(B := \Delta \times D^2, \eta := (\alpha, \omega))$, we get extra conditions on ρ. The first condition on ρ is imposed by the requirement that ξ is a contact structure on $\mathcal{O}p\,\partial B$.
Writing out the contact condition gives

$$\begin{aligned}\alpha \wedge d\alpha^n &= (\alpha_{st} + \rho d\theta) \wedge (d\alpha_{st} + \frac{\partial\rho}{\partial x} dx \wedge d\theta + \frac{\partial\rho}{\partial v}\rho dv \wedge d\theta)^n \\ &= n\frac{\partial\rho}{\partial v}\alpha_{st} \wedge \alpha_{st}^{n-1} \wedge dv \wedge d\theta.\end{aligned}$$

Hence, $\xi = \ker\alpha$ is contact if and only if $\frac{\partial\rho}{\partial v} > 0$ which geometrically means that a hyperplane is contact if and only if it twists enough. So, the first condition on ρ is that we require $\frac{\partial\rho}{\partial v} > 0$ on $\mathcal{O}p\,\partial B$.

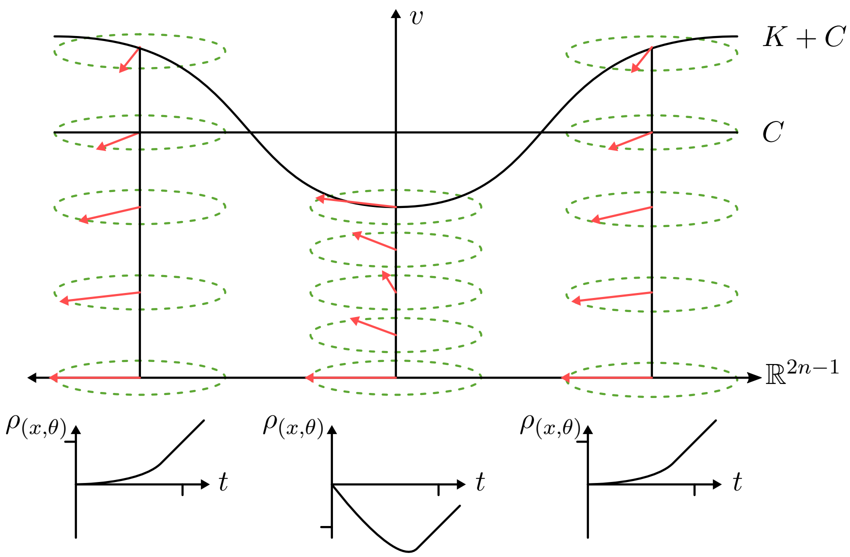


Figure 4.1: The rotation of the hyperplanes described by ρ. The red arrows are the normal vectors to the hyperplanes. The hash mark on the horizontal axis is at $t = K(x,\theta) + C$ and the hash mark on the vertical axis is at $\rho_{(x,\theta)} = K(x,\theta)$.

Remark 4.4. *In the example above we have $\rho = rK(x,\phi)$ on $I \times D^2$. Note that $\frac{\partial \rho}{\partial r} = K$ is positive everywhere, which is the reason that we have a contact structure on the entire ball.*

We now consider the description of the contact structure near the boundary of a contact shell $(B := \Delta \times D^2, \eta := (\alpha, \omega))$ with $\alpha = \alpha_{st} + \rho d\theta$ and $\frac{\partial \rho}{\partial v} > 0$ on $\mathcal{O}p\,\partial B$. As we stated before requiring this contact germ to be encoded in a function K imposes further restrictions on ρ. The boundary of B splits in two (smooth) parts,

$$\partial B = \Sigma^1_B \cup \Sigma^2_B,$$

where $\Sigma^1_B := (\partial\Delta) \times D^2$ and $\Sigma^2_B := \Delta \times S^1$. We claim that, if we impose the extra conditions $\rho|_{\Delta\times\{0\}} = 0$ and $\left.\frac{\partial \rho}{\partial v}\right|_{\mathcal{O}p\,\partial B} > 0$, the contact structure on $\mathcal{O}p\,\partial B$ is determined by the restriction of ρ to the boundary component Σ^2_B, denoted by

$$K := \rho|_{\Sigma^2_B}.$$

Of course, in practise we want to prescribe K which then imposes another restriction on the possible choices for ρ.
To prove the claim we give a more direct description of the contact germ in terms of K.
Given $K : \Delta \times S^1 \to \mathbb{R}$, the requirements $\rho|_{\Delta\times\{0\}} = 0$ together with $\left.\frac{\partial \rho}{\partial v}\right|_{\mathcal{O}p\,B} > 0$ imply that $K|_{\mathcal{O}p\,\Delta\times S^1} > 0$. This allows us to define submanifolds

$$\Sigma^1_K := \{\, (x,v,\theta) \in \mathbb{R}^{2n+1} \mid x \in \partial\Delta, 0 \le v \le K(x,\theta) \,\} \subset (\mathbb{R}^{2n+1}, \xi_{st}),$$

and

$$\Sigma^2_K := \{\, (x,p,q) \in \mathbb{R}^{2n-1} \times T^*S^1 \mid x \in \Delta, v = K(x,q) \,\} \subset (\mathbb{R}^{2n-1} \times T^*S^1, \xi_{st}),$$

endowed with the germ of a contact structure by restricting the contact structure on the ambient manifold.
The condition $\frac{\partial\rho}{\partial v}\Big|_{\mathcal{O}p\,\partial B} > 0$ also implies that we have diffeomorphisms

$$\psi_1 : \mathcal{O}p\,\Sigma_B^1 \to \mathcal{O}p\,\Sigma_K^1 \subset \mathbb{R}^{2n+1} \text{ defined by } (x, v, \theta) \mapsto (x, \rho(x, \theta), \theta),$$

and

$$\psi_2 : \mathcal{O}p\,\Sigma_B^2 \to \mathcal{O}p\,\Sigma_K^2 \subset \mathbb{R}^{2n-1} \times T^*S^1 \text{ defined by } (x, v, \theta) \mapsto (x, \rho(x, \theta), \theta).$$

In fact, it is easy to see that ψ_1 and ψ_2 are strict contactomorphisms . The reason Σ_B^2 cannot be embedded in $(\mathbb{R}^{2n+1}, \xi_{st})$ is that in general K is allowed to be negative on $\operatorname{Int}\Delta \times S^1$.

Define a diffeomorphism $F : \mathcal{O}p\,\partial\Sigma_K^1 \to \mathcal{O}p\,\partial\Sigma_K^2$ by

$$(x, v, \theta) \mapsto (x, p, q).$$

Observe that $F(\partial\Sigma_K^1) = \partial\Sigma_K^2$ and $F^*(\alpha_{st} + pdq) = \alpha_{st} + vd\theta$, so F is a strict contactomorphism.
Define an equivalence relation $\sim$ on $\mathbb{R}^{2n+1} \sqcup \mathbb{R}^{2n-1} \times T^*S^1$ where $(x, v, \theta) \sim (x, p, q)$ if $(x, p, q) = F(x, v, \theta)$. This equivalence relation respects the contact structure and glues Σ_K^1 and Σ_K^2 along their boundary. The resulting quotient is given by

$$\Sigma_K := \Sigma_K^1 \cup \Sigma_K^2 \subset \left(\mathbb{R}^{2n+1}, \xi_{st}\right) \sqcup (\mathbb{R}^{2n-1} \times T^*S^1, \xi_{st})/\sim . \tag{4.3}$$

The contactomorphisms ψ_1 and ψ_2 glue to a contactomorphism $\psi : \mathcal{O}p\,\partial B \to \mathcal{O}p\,\Sigma_K$. Hence, the contact germ on ∂B is the same as the contact germ on Σ_K which by the above description is easily seen to depend only on K.

Let us briefly summarize the results of this section. Given a contact shell $(B := \Delta \times D^2, \eta = (\alpha, \omega))$ such that ξ is fiberwise contact, there is a contact form $\alpha = \alpha_{st} + \rho d\theta$ such that $\xi = \ker\alpha$. Furthermore, $\frac{\partial\rho}{\partial v} > 0$ on $\mathcal{O}p\,\partial B$ and if we assume that $\rho|_{\Delta\times\{0\}} = 0$ we get a function $K : \Delta \times S^1 \to \mathbb{R}$, satisfying $K|_{\mathcal{O}p\,\Delta\times S^1} > 0$, which encodes the contact structure near the boundary of B.

In the next section we define circle models by showing that there is a map in the other direction. That is, given a pair (Δ, K) we can construct a contact shell (B_K, η_K) such that the contact structure on $\mathcal{O}p\,\partial B_K$ depends only on K.

4.2 Definition of circle models

In this section we give a precise definition of circle models. Let $\Delta \subset R^{2n-1}$ be a compact, star-shaped domain, and $K : \mathcal{O}p\,(\Delta) \times S^1 \to \mathbb{R}$ a smooth function, with

$$K|_{\partial\Delta\times S^1} > 0. \tag{4.4}$$

Such a function is called a *contact Hamiltonian* and denoted by a pair (K, Δ). We will define a map

$$\begin{Bmatrix}\text{Contact Hamiltonians}\\ (\Delta, K)\end{Bmatrix} \to \begin{Bmatrix}\text{Circle models}\\ (B_K, \eta_K)\end{Bmatrix} \subset \begin{Bmatrix}\text{Equivalence classes}\\ \text{of contact shells}\end{Bmatrix}$$

mapping a contact Hamiltonian (Δ, K) to an equivalence class of contact shells (B_K, η_K) called circle model.

Definition 4.5. *Let (Δ, K) be any contact Hamiltonian. Pick a constant $C > 0$ satisfying $\min K + C|_{\Delta\times S^1} > 0$. Use this to define a (piecewise smooth) ball,*

$$B_{K,C} := \{ (x, v, \theta) \in \mathbb{R}^{2n+1} \mid x \in \Delta, 0 \le v \le K(x, \theta) + C \}.$$

The boundary $\partial B_{K,C}$ is the union of two (smooth) manifolds,

$$\Sigma^1_{K,C} := \{ (x, v, \theta) \in \mathbb{R}^{2n+1} \mid x \in \partial\Delta, 0 \le v, \le K(x, \theta) + C \},$$

and

$$\Sigma^2_{K,C} := \{ (x, v, \theta) \in \mathbb{R}^{2n+1} \mid x \in \Delta, v = K(x, \theta) + C \}.$$

To define a suitable almost contact structure on $\mathcal{O}p\, B_{K,C}$ we need to make two more choices:

- *Pick a smooth function $\rho : \mathcal{O}p\, B_{K,C} \to \mathbb{R}$ satisfying the following conditions*

$$\begin{aligned} &(i)\ \rho(x, 0, \theta) = 0 && \text{for } x \in \mathcal{O}p\,\Delta, \theta \in S^1, \\ &(ii)\ \partial_v \rho(x, v, \theta) > 0 && \text{on } \mathcal{O}p\,\Sigma^1_{K,C} \\ &(iii)\ \rho(x, v, \theta) = v - C && \text{on } \mathcal{O}p\,\Sigma^2_{K,C} \end{aligned} \tag{4.5}$$

- *Pick a closed subset W such that $\mathcal{O}p\,\Sigma^1_K \subset W \subset \{ (x, v, \theta) \in \mathbb{R}^{2n+1} \mid \frac{\partial\rho}{\partial v}(x, v, \theta) > 0 \}$ and a smooth function $g : \mathbb{R}^{2n+1} \to [0, 1]$ satisfying*

  (i) $g|_{\mathcal{O}p\,\Sigma^1_{K,C}} = 1$

  (ii) $\operatorname{supp}(g) \subset W$

Define an almost contact structure $\eta_{K,\rho,g}$ on $\mathcal{O}p\, B_{K,C}$ by $\eta_{K,\rho,g} := (\alpha_\rho, \omega_\rho)$, where

$$\alpha_\rho := \alpha_{st} + \rho d\theta, \quad \text{and} \quad \omega_{\rho,g} := d\alpha_{st} + ((1 - g)dv + g d\rho) \wedge d\theta. \tag{4.6}$$

The circle model (B_K, η_K) is defined to be the equivalence class of the contact shell $(B_{K,C}, \eta_{K,\rho,g})$ is a circle model associated to (Δ, K). We will often identify (B_K, η_K) with any of it's representatives. In particular we will also call $(B_{K,C}, \eta_{K,\rho,g})$ a circle model for (Δ, K).

Remark 4.6. *Most of the definition of the circle model is motivated by our observations in the previous section. However, there are two points in the definition we want to clarify. Firstly, condition (iii) in the definition of ρ has two functions. It ensures that $\rho|_{\Sigma^2_{K,C}} = K$ and that $\frac{\partial\rho}{\partial v} = 1$. While the reason for the first property is clear, we could expect $\frac{\partial\rho}{\partial v} > 0$ instead of the second condition. The reason we require $\frac{\partial\rho}{\partial v} = 1$ is that this ensures that the contact structure on $\mathcal{O}p\,\partial B_K$ embeds smoothly, and not just continuously into the ambient manifold.*

Secondly, the definition of ω_ρ looks rather unnatural at first sight. Instead we could expect the definition $\omega_\rho := d\alpha_{st} + d\rho \wedge d\theta$ which equals $d\alpha_\rho$. Following the proof of Lemma 4.7 below, it is easy to see that this does not define an almost contact structure, since the condition $\alpha \wedge \omega^n > 0$ might fail to be true. For this reason we need the extra factor $dv \wedge d\theta$. Unfortunately, defining $\omega_\rho := d\alpha_{st} + (dv + d\rho) \wedge d\theta$ has the problem that $\omega_\rho \neq d\alpha_\rho$ near the boundary. Hence, we have to interpolate between the two definitions using the function g so that both conditions are satisfied.

At this point there are various things we need to prove to show that the map $(\Delta, K) \mapsto (B_K, \eta_K)$ is well-defined.

- We need to show that $(B_{K,C}, \eta_{K,\rho,g})$ defines a contact shell.
- During the construction we assume existence of a triple (C, ρ, g) satisfying suitable conditions. (The choice of W is implicit in the choice of g.) We need to show that such a quadruple exists.
- We need to show that different choices (C, ρ, g) and $(\widetilde{C}, \tilde{\rho}, \tilde{g})$ yield equivalent contact shells

We prove these statements in the next section.

4.3 Circle models are well-defined

Lemma 4.7. *Every circle model $(B_{K,C}, \eta_{K,\rho,g})$, defined in Definition 4.5, is a contact shell.*

Proof. We need to show that $\eta_{K,\rho,g} := (\alpha, \omega)$, defined on $\mathcal{O}p B_{K,C}$, is an almost contact structure which is contact near $\partial B_{K,C}$. By Definition 2.9 this means we need to check that $\alpha \wedge \omega^n > 0$ everywhere and $\omega = d\alpha$ near $\partial B_{K,C}$.

We have

$$\begin{aligned}\omega^n &= (d\alpha_{st} + ((1-g)dv + gd\rho) \wedge d\theta)^n \\ &= (d\alpha_{st})^n + n(d\alpha_{st})^{n-1} \wedge ((1-g)dv + gd\rho) \wedge d\theta \\ &= n(d\alpha_{st})^{n-1} \wedge ((1-g)dv + gd\rho) \wedge d\theta\end{aligned}$$

since $d\alpha_{st}$ is a 2-form on $\mathbb{R}^{2n-1}$ which implies $(d\alpha_{st})^n = 0$. Using this we find

$$\begin{aligned}\alpha \wedge \omega^n &= n\,(\alpha_{st} + \rho d\theta) \wedge (d\alpha_{st})^{n-1} \wedge ((1-g)dv + gd\rho) \wedge d\theta \\ &= n\alpha_{st} \wedge (d\alpha_{st})^{n-1} \wedge ((1-g)dv + gd\rho) \wedge d\theta \\ &= n\,(1 - g + g\partial_v \rho)\,\alpha_{st} \wedge (d\alpha_{st})^{n-1} \wedge dv \wedge d\theta\end{aligned}$$

where we used in the last line that $\alpha_{st} \wedge (d\alpha_{st})^{n-1}$ is a volume form on $\mathbb{R}^{2n-1}$, implying that all terms coming from $d\rho$ cancel except for $\partial_v \rho\, dv$. Hence, $\eta_{K,\rho,g}$ is almost contact if and only if $1 - g + g\,\partial_v \rho > 0$. To see this is the case, recall that $\mathrm{supp}(g) \subset \{\, (x, v, \theta) \in \mathcal{O}p\, B_{K,C} \mid \frac{\partial \rho}{\partial v} > 0 \,\}$. Hence, $1 - g + g\partial_v \rho > 0$ everywhere.

To see that $\omega = d\alpha$ on $\mathcal{O}p\,\partial B_{K,C}$ note that $g|_{\mathcal{O}p\,\Sigma^1_{K,C}} = 1$. Hence, on an open neighborhood of $\partial B_{K,C}$ we have

$$\omega = d\alpha_{st} + d\rho \wedge d\theta = d\alpha,$$

concluding the proof. □

Next we show that there exists triples (C, ρ, g). The existence of C follows from the fact that $\Delta \times S^1$ is compact and the existence of the bump function g is also easy to see. The interesting part is the existence of ρ which is given by the following lemma.

Lemma 4.8. *Let (K, Δ) be a contact Hamiltonian and $C > 0$ a constant satisfying $K+C > 0$. Then there exists a function $\rho : \mathcal{O}p\, B_{K,C} \to \mathbb{R}$, satisfying the properties in Equation 4.5.*

Proof. Recall that $K|_{\partial\Delta\times S^1} > 0$ and by assumption we have $K + C > 0$. Hence, there exists an $\epsilon > 0$ such that

(i) $K(x, \theta) + C > 2\epsilon > 0$ for all $(x, \theta) \in \mathcal{O}p\, \Delta \times S^1$;

(ii) There exists an open $\partial\Delta \subset U_\epsilon \subset \mathbb{R}^{2n-1}$ such $K|_{U_\epsilon \times S^1} > \epsilon$.

Cover $\mathcal{O}p\, B_{K,C}$ with two open sets

$$U_1 := \{\, (x, v, \theta) \in \mathcal{O}p\, B_{K,C} \mid 0 \le v < K(x, \theta) + C - \epsilon \,\},$$

$$U_2 := \{\, (x, v, \theta) \in \mathcal{O}p\, B_{K,C} \mid v \ge K(x, \theta) + C - 2\epsilon \,\},$$

and let $g : \mathcal{O}p\, B_{K,C} \to [0, 1]$ be a smooth bump function satisfying

(i) $g|_{U_1} = 1$ and $g|_{U_2} = 0$,

(ii) $\frac{\partial g}{\partial v} < 0$ on $\operatorname{Int} U_2 \setminus U_1$.

Note that this function is smooth since the boundaries ∂U_1 and ∂U_2 are smooth which in turn follows from the smoothness of K.

Define smooth functions $\rho_1, \rho_2 : \mathcal{O}p\, B_{K,C} \to \mathbb{R}$ by

$$\rho_1(x, v, \theta) := \frac{K(x, \theta) - 2\epsilon}{K(x, \theta) + C - \epsilon} v, \qquad \rho_2(x, v, \theta) := v - C,$$

and take

$$\rho := g\rho_1 + (1 - g)\rho_2.$$

We claim that this function satisfies the conditions from Equation 4.5. To see this observe the following,

(i) Since $\rho_1(x, 0, \theta) = 0$ and $g(x, 0, \theta) = 1$ we find $\rho(x, 0, \theta) = \rho_1(x, 0, \theta) = 0$.

(ii) Compute

$$\begin{aligned}\partial_v \rho &= \partial_v g\,(\rho_1 - \rho_2) + g\partial_v \rho_1 + (1 - g)\partial_v \rho_2 \\ &\ge \partial_v\,(\rho_1 - \rho_2) + \min(\partial_v \rho_1, \partial_v \rho_2).\end{aligned}$$

The set $\{\, (x, v, \theta) \in \mathcal{O}p\, B_{K,C} \mid x \in U_\epsilon \,\}$ is an open neighborhood of $\Sigma^1_{K,C}$ on which $\min(\partial_v \rho_1, \partial_v \rho_2) > 0$. Furthermore, whenever $\partial_v g < 0$ we have $\rho_2 \ge \rho_1$. Hence, $\partial_v \rho > 0$ on $\mathcal{O}p\, \Sigma^1_{K,C}$.

(iii) The set U_2 contains an open neighborhood $\mathcal{O}p\, \Sigma^2_{K,C}$. Since, $1 - g|_{U_2} = 0$ this immediately implies $\rho(x, v, \theta) = v - C$ on $\mathcal{O}p\, \Sigma^2_{K,C}$.

□

Remark 4.9. *In the special case that $K : \mathcal{O}p\,\Delta \times S^1 \to \mathbb{R}_+$ is a strictly positive function we don't need the above lemma to construct a suitable ρ. In fact, we can take $C = 0$ and $\rho(x, v, \theta) = v$. It is easy to see that this defines a solid circle model independent of the choice of (W, g).*

On the other hand, if K is negative somewhere then any ρ satisfying Equation 4.5 must have $\partial_v \rho < 0$ somewhere. The proof of Lemma 4.7 shows that in this case the circle model $(B_{K,C}, \eta_{K,\rho,g})$ cannot be solid. So, we can produce solid circle models if and only if K is strictly positive. (Ofcourse, by choosing ρ the right way it is still posible to produce a non-solid circle model even if K is strictly positive.)

It remains to be shown that the circle model construction is independent of the choices for C, ρ and g.

Lemma 4.10. *Different choices (C, ρ, g) and $(\widetilde{C}, \tilde{\rho}, \tilde{g})$ in the construction of $(B_{K,C}, \eta_{K,\rho,g})$, yield equivalent shells.*

Proof. Consider the special case of two choices (C, ρ, g) and $(C, \rho, \tilde{g})$. By assumption we have two sets $\operatorname{supp}(g) \subset W$ and $\operatorname{supp}(\tilde{g}) \subset \widetilde{W}$ both contained in the set $\{ (x, v, \theta) \in \mathcal{O}p\, B_{K,C} \mid \frac{\partial \rho}{\partial v}(x, v, \theta) > 0 \}$. Define a 1-parameter family of functions $h_t : \mathcal{O}p\, B_{K,C} \to [0, 1]$, $t \in I$ by

$$h_t := (1 - t)g + t\tilde{g}.$$

Note that $\Sigma^1_{K,C} \subset \operatorname{supp}(h_t) \subset W \cup \widetilde{W}$ for all $t \in I$. Hence we can define $\eta_t := \eta_{K,\rho,h_t}$ and it is easy to see that this defines a homotopy between $\eta_{K,\rho,g}$ and $\eta_{K,\rho,\tilde{g}}$ relative to $\mathcal{O}p\, \partial B_{K,C}$.

Next, consider the special case of two choices (C, ρ, g) and $(C, \tilde{\rho}, g)$. Define a smooth 2-parameter family of functions

$$\rho_{(x,\theta)}(v) := \rho(x, v, \theta) : \mathbb{R}_{\geq 0} \to \mathbb{R},$$

for $(x, \theta) \in \mathcal{O}p\, \Delta \times S^1$. Technically, this is an extension of $\rho_{(x,\theta)}(v)$ by defining $\rho_{(x,\theta)}(v) := v - C$ whenever $v \geq K(x, \theta) + C$. Define $\tilde{\rho}_{(x,\theta)} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ in the same way.

The idea is that near $\partial B_{K,C}$ the map $\phi := (x, v, \theta) \mapsto (x, \tilde{\rho}^{-1}_{(x,\theta)} \circ \rho_{(x,\theta)}(v), \theta)$ is defined and satisfied $\phi^* \eta_{K,r\tilde{h}o,g} = \eta_{K,\rho,g}$. However, this map might not well-defined (let alone a diffeomorphism) on the interior of $B_{K,C}$ since $\tilde{\rho}^{-1}_{(x,\theta)} \circ \rho_{(x,\theta)}(v)$ might not be well-defined. This is fixed as follows.

Pick a small neighborhood $\partial\Delta \subset U \subset \mathbb{R}^{2n-1}$ and a bump function $\lambda : \mathbb{R}^{2n-1} \to [0, 1]$ satisfying

(i) $\{ (x, v, \theta) \in \mathbb{R}^{2n+1} \mid x \in U \} \subset \{ (x, v, \theta) \in \mathbb{R}^{2n+1} \mid \partial_v \rho(x, v, \theta) > 0, \partial_v \tilde{\rho}(x, v, \theta) > 0 \}$

(ii) $\lambda|_{\mathcal{O}p\, \partial\Delta} = 1$ and $\operatorname{supp}(\lambda) \subset U$.

With this information we can construct a diffeomorphism $\phi : \mathcal{O}p\, B_{K,C} \to \mathcal{O}p\, B_{K,C}$ by

$$\phi(x, v, \theta) := (x, \lambda(x) \left(\tilde{\rho}^{-1}_{(x,\theta)} \circ \rho_{(x,\theta)}(v) \right) + (1 - \lambda(x))\, v, \theta).$$

This map is smooth, as can be seen from the coordinate expression. Furthermore, $\phi(B_{K,C}) = B_{K,C}$ and $\phi(\Delta \times \{0\}) = \Delta \times \{0\}$. Note that away from $\Sigma^1_{K,C}$ this is just $\phi(x, v, \theta) = (x, v, \theta)$ and on $\mathcal{O}p\, \partial B_{K,C}$ we have $\phi(x, v, \theta) = (x, \tilde{\rho}^{-1}_{(x,\theta)} \circ \rho_{(x,\theta)}(v), \theta)$.

Consider $\phi^*\eta_{K,\tilde\rho,g} = \eta_{K,\tilde\rho\circ\phi,g\circ\phi}$. From the previous special case we $\eta_{K,\tilde\rho\circ\phi,g\circ\phi}$ is equivalent to $\eta_{K,\tilde\rho\circ\phi,g}$. On $\mathcal{O}p\,\partial B_{K,C}$ we have $\tilde\rho\circ\phi = \rho$. This means that the straight line homotopy $h_t : \mathcal{O}p\,B_{K,C} \to \mathbb{R}$, $t \in I$ defined by

$$h_t := (1-t)\tilde\rho\circ\phi + t\rho,$$

is relative to $\mathcal{O}p\,B_{K,C}$. It is easy to see that h_t satisfied the condition in Equation 4.5 for all $t \in I$. Hence $\eta_t := \eta_{K,h_t,g}$ is a homotopy between $\eta_{K,\tilde\rho\circ\phi,g}$ and $\eta_{K,\rho,g}$, relative to $\mathcal{O}p\,B_{K,C}$. We conclude that the choices (C,ρ,g) and $(C,\tilde\rho,g)$ produce equivalent circle models.

Lastly, consider the special case of two choices (C,ρ,g) and $(\widetilde{C},\rho,g)$. Pick a family of diffeomorphisms $\psi_{(x,\theta)} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, depending smoothly on (x,θ) and satisfying

$$\psi_{(x,\theta)}(v) = v + (\widetilde{C} - C), \text{ on } \mathcal{O}p\,\Sigma^1_{K,C}.$$

One possible way to do this is to define $U_\epsilon := \{\,(x,v,\theta) \in \mathbb{R}^{2n+1} \mid K(x,\theta) + C - \epsilon \leq v \leq K(x,\theta) + C + \epsilon\,\}$. For $\epsilon > 0$ small enough U_ϵ is contained in the neighborhood $\mathcal{O}p\,\Sigma^2_{K,C}$ where $\rho(x,v,\theta) = v - C$. Pick a smooth bump function $\lambda : \mathbb{R}^{2n+1} \to [0,1]$ satisfying $\lambda = 1$ on $\mathcal{O}p\,\Sigma^2_{K,C}$ and $\operatorname{supp}(\lambda) \subset U_\epsilon$. Define

$$\psi_{(x,\theta)}(v) := (1-g)\frac{\widetilde{C}}{C}v + g\left(v + (\widetilde{C} - C)\right).$$

As before, ψ induces a diffeomorphism $\Psi : \mathcal{O}p\,B_{K,C} \to \mathcal{O}p\,B_{K,\widetilde{C}}$, mapping $B_{K,C}$ to $B_{K,\widetilde{C}}$, defined by

$$\Psi(x,v,\theta) = (x, \psi_{(x,\theta)}(v), \theta).$$

Note that $\Psi : (B_{K,C}, \eta_{K,\rho\circ\psi,g\circ\psi}) \to (B_{K,\widetilde{C}}, \eta_{K,\rho,g})$ is an isomorphism of almost contact structures, so in particular an equivalence of contact shells. By our previous results we know that $(B_{K,C}, \eta_{K,\rho\circ\psi,g\circ\psi})$ is equivalent to $(B_{K,C}, \eta_{K,\rho,g})$ finishing the proof.

The equivalences constructed for the three special cases above can be composed to give an equivalence in the general case. □

4.4 The characteristic foliation on ∂B_K.

We now use the fact that the contact structure near the boundary of a circle model is encoded by (K,Δ) to describe the characteristic foliation on ∂B_K, which is also encoded by K. Recall that for a hypersurface $\Sigma \subset (M^{2n+1}, \xi = \ker\alpha)$ the characteristic foliation Σ_ξ is the singular 1-dimensional foliation found by integrating the singular 1-dimensional distribution $\mathcal{F} := \ker d\alpha|_{T\Sigma\cap\xi} \subset T\Sigma \cap \xi$. By Equation 4.3 we know that $\mathcal{O}p\,\partial B_K$ is contactomorphic to

$$\Sigma_K := \Sigma^1_K \cup \Sigma^2_K \subset \left(\mathbb{R}^{2n+1}, \xi_{st}\right) \sqcup (\mathbb{R}^{2n-1} \times T^*S^1, \xi_{st})/\sim. \tag{4.7}$$

First consider the characterstic foliation on $\Sigma^1_K \subset (\mathbb{R}^{2n-1} \times R^2, \alpha_{st} + v d\theta)$. By definition $\Sigma^1_K = \{\,(x,v,\theta) \in \mathbb{R}^{2n+1} \mid x \in \partial\Delta, 0 \leq v \leq K(x,\theta) + \theta\,\}$. The diffeomorphism $\phi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ given by

$$(x,v,\theta) \mapsto (x, (K(x,\theta) + C)\,v, \theta),$$

is a contactomorphism between $\mathcal{O}p\left(\partial\Delta \times D^2\right) \subset (\mathbb{R}^{2n+1}, \xi = \ker\alpha_{st} + (K(x,\theta)+C)vd\theta)$ and $\mathcal{O}p\,\Sigma^1_K \subset (\mathbb{R}^{2n+1}, \alpha_{st} + vd\theta)$. Hence, it is enough to describe the characteristic foliation on $\partial\Delta \times D^2$. We claim this is given by $\mathcal{F} = \mathrm{Span}\{\, v\frac{\partial}{\partial v}\,\}$, under the identification

$$T(\partial\Delta \times D^2) \cong T\partial\Delta \oplus TD^2.$$

To see this observe that $\alpha_{st}|_{TD^2} = 0$ implying $v\frac{\partial}{\partial v} \in \xi \cap T(\partial\Delta \times D^2)$, and $\frac{\partial}{\partial\theta} \notin \xi \cap T(\partial\Delta \times D^2)$ whenever $v \neq 0$. Similarly,

$$d\alpha = d\alpha_{st} + v\partial_x K(x,\theta)dx \wedge d\theta + (K(x,\theta) + C))\,dv \wedge d\theta$$

and $d\alpha_{st}|_{TD^2} = 0$ which gives $\iota_{\frac{\partial}{\partial v}} d\alpha = (K(x,\theta) + C)\,d\theta$. Since $K(x,\theta) + C > 0$ and $\frac{\partial}{\partial\theta} \notin \xi \cap T(\partial\Delta \times D^2)$ we conclude $v\frac{\partial}{\partial v} \in \ker d\alpha|_{\xi\cap T(\partial\Delta\times D^2)}$. Hence, $v\frac{\partial}{\partial v} \subset \mathcal{F}$ and the claim follows for dimensional reasons. Since $d\phi : T\mathbb{R}^{2n+1} \to T\mathbb{R}^{2n+1}$ only scales $v\frac{\partial}{\partial v}$ the characteristic foliation on Σ^1_K is spanned by $v\frac{\partial}{\partial v}$.

The situation for the characteristic foliation on $\Sigma^2_K \subset (\mathbb{R}^{2n-1} \times T^*S^1, \alpha_{st} + pdq)$ is more complicated. To see what it looks like we view Σ^2_K as the graph of $K : \Delta \times S^1 \to \mathbb{R}$ where we identify $\Delta \times S^1$ with a subset of $\Pi := \{\, (x,p,q) \in \mathbb{R}^{2n-1} \times T^*S^1 \mid p = 0\,\}$. The following lemma gives the description and explains why we call K a contact Hamiltonian. Note, that this lemma also shows that the characteristic foliation on Σ^2_K does not have any singular points.

Lemma 4.11. *Given a smooth function $K : \Delta \times S^1 \to \mathbb{R}$ the characteristic foliation on $\Gamma_K := \{\, (x,p,q,) \in \mathbb{R}^{2n-1} \times T^*S^1 \mid x \in \Delta, p = K(x,q)\,\}$ induced by the contact structure $\xi = \ker\alpha_{st} + pdq$ is represented by the vector field $\frac{\partial}{\partial q} + X_{K_t}$. Here X_{K_t} is the vector field on $\mathbb{R}^{2n-1}$ induced by viewing $K_t := K(\cdot,t)$ as a 1-parameter family of contact Hamiltonians on $(\mathbb{R}^{2n-1}, \alpha_{st})$.*

Proof. Recall that the contact vector field associated to a contact Hamiltonian K on $(\mathbb{R}^{2n-1}, \alpha_{st})$ is uniquely defined by

$$\alpha_{st}(X_K) = K, \quad d\alpha(X_K, -) = dK(R_\alpha)\alpha - dK.$$

If Y is some vector field representing the characteristic foliation on a hypersurface Σ then it must satisfy

$$\iota_Y\Omega = \alpha \wedge (d\alpha)^{n-1}\big|_{T\Sigma},$$

where Ω is a volume form on Σ. Hence, we can compute both $\frac{\partial}{\partial q} - X_K$ and Y and check they agree up to scalar. Suppose $X_K = a\frac{\partial}{\partial z} + \sum_i b_i\frac{\partial}{\partial u_i} + \sum_i c_i\frac{\partial}{\partial\theta_i}$, for functions a, b_i, c_i, $i = 1, \ldots, n-1$. Then we find,

$$\alpha_{st}(X_K) = a + \sum_i u_i c_i, \qquad d\alpha_{st}(X_K, -) = \sum_i b_i d\theta_i - c_i du_i.$$

On the other hand

$$\alpha_{st}(X_K) = K,$$

$$\begin{aligned}
d\alpha_{st}(X_K, -) &= dK(R_{\alpha_{st}})\alpha_{st} - dK \\
&= \partial_z K dz + \sum_i u_i \partial_z K d\theta_i - \left(\partial_z K dz + \sum_i \partial_{u_i} K du_i + \partial_{\theta_i} K d\theta_i \right) \\
&= \sum_i \left(u_i \partial_z K - \partial_{\theta_i} K \right) d\theta_i - \sum_i \partial_{u_i} K du_i
\end{aligned}$$

From this we conclude

$$a = K - \sum_i u_i \partial_{u_i} K, \quad b_i = u_i \partial_z K - \partial_{\theta_i} K, \quad c_i = \partial_{u_i} K.$$

Suppose $Y = d\frac{\partial}{\partial z} + \sum_i e_i \frac{\partial}{\partial u_i} + \sum_i f_i \frac{\partial}{\partial \theta_i} + g\frac{\partial}{\partial q}$. Consider the volume form $\Omega = dz \wedge (\bigwedge_i du_i \wedge d\theta_i) \wedge dx$ on Γ_K. Furthermore, $\alpha|_{\Gamma_K} = \alpha_{st} - Kdx$. Which implies

$$\begin{aligned}
\alpha \wedge d\alpha^{n-1} &= (\alpha_{st} - Kdx) \wedge (d\alpha_{st} - dK \wedge dx)^{n-1} \\
&= (\alpha_{st} - Kdx) \wedge \left(d\alpha_{st}{}^{n-1} - (n-1) d\alpha_{st}{}^{n-2} \wedge dK \wedge dx \right) \\
&= (\alpha_{st} - Kdx) \wedge d\alpha_{st}{}^{n-1} - (n-1)\alpha_{st} \wedge d\alpha_{st}{}^{n-2} \wedge dK \wedge dx.
\end{aligned}$$

We calculate the first and second term in this sum seperately.

$$(\alpha_{st} - Kdx) \wedge d\alpha_{st}{}^{n-1} = \alpha \wedge \left((n-1)! \bigwedge_i du_i \wedge d\theta_i \right)$$
$$= (n-1)! dz \wedge \left(\bigwedge_i du_i \wedge d\theta_i \right) - (n-1)! K dx \wedge \left(\bigwedge_i du_i \wedge d\theta_i \right)$$

$$\alpha_{st} \wedge d\alpha_{st}{}^{n-2} \wedge dK \wedge dx = \left(dz + \sum_i u_i d\theta_i \right) \wedge \left(\sum_i du_i \wedge d\theta_i \right)^{n-2} \wedge \left(\partial_z K dz + \sum_i (\partial_{u_i} du_i + \partial_{\theta_i} d\theta_i) \right) \wedge dx$$
$$= \sum_i dz \wedge \left((n-2)! \bigwedge_{j \neq i} (du_j \wedge d\theta_j) \right) \wedge (\partial_{u_i} K du_i + \partial_{\theta_i} K d\theta_i) \wedge dx$$
$$+ \sum_i (u_i d\theta_i) \wedge \left((n-2)! \bigwedge_{j \neq i} (du_j \wedge d\theta_j) \right) \wedge (\partial_z K dz + \partial_{u_i} K du_i) \wedge dx$$
$$= \sum_i (n-2)! \partial_{u_i} K dz \wedge \left(\bigwedge_{j \neq i} du_i \wedge d\theta_i \right) \wedge du_i \wedge dx$$
$$+ \sum_i (n-2)! \partial_{\theta_i} K dz \wedge \left(\bigwedge_{j \neq i} du_i \wedge d\theta_i \right) \wedge d\theta_i \wedge dx$$
$$- \sum_i (n-2)! u_i \partial_z K dz \wedge d\theta_i \left(\bigwedge_{j \neq i} du_i \wedge d\theta_i \right) \wedge dx$$
$$- \sum_i (n-2)! u_i \partial_{u_i} K \left(\bigwedge_i du_i \wedge d\theta_i \right) \wedge dx$$

So, in total we have

$$\alpha \wedge d\alpha^{n-1} = (n-1)! dz \wedge \left(\bigwedge_i du_i \wedge d\theta_i \right)$$
$$- (n-1)! \left(K - \sum_i u_i \partial_{u_i} K \right) \left(\bigwedge_i du_i \wedge d\theta_i \right) \wedge dx$$
$$- (n-1)! \sum_i \partial_{\theta_i} K dz \wedge \left(\bigwedge_{j \neq i} du_i \wedge d\theta_i \right) \wedge d\theta_i \wedge dx$$
$$+ (n-1)! \left(\sum_i u_i \partial_z K - \partial_{\theta_i} K \right) dz \wedge d\theta_i \left(\bigwedge_{j \neq i} du_i \wedge d\theta_i \right) \wedge dx$$

Hence we conclude

$$d = -(n-1)! \left(K - \sum_i u_i \partial_{u_i} K \right), \quad e_i = -(n-1)!(u_i \partial_z K - \partial_{\theta_i} K), \quad f_i = -(n-1)! \partial_{u_i} K, \quad g = -(n-1)!$$

Comparing this with our earlier computed

$$a = K - \sum_i u_i \partial_{u_i} K, \quad b_i = u_i \partial_z K - \partial_{\theta_i} K, \quad c_i = \partial_{u_i} K,$$

we see they agree up to a factor $-(n-1)!$. □

Chapter 5

Domination and conjugation for circular model shells

In the previous chapter we have seen that a large class of contact shells, the circle models, are described by a single function (K, Δ). The upshot of this is that the study of these contact shells is reduced to the study of the contact Hamiltonians describing them.

In the first section we show that we can use contactomorphisms of $(\mathbb{R}^{2n-1}, \xi_{st})$ to manipulate the contact Hamiltonians. To illustrate this idea consider two functions $K_0, K_1 : [0, 4] \to \mathbb{R}$ as in Figure 5.1.
At some points we have $K_1 > K_0$ and at some points $K_0 > K_1$. We can reparametrize the interval using a diffeomorphism ϕ and define a new Hamiltonian $\widetilde{K}_1 := K_1 \circ \phi$. Choosing ϕ the right way we have that $\widetilde{K}_1 > K_0$ everywhere. Hence, we can change the qualitive behaviour of the Hamiltonians by reparametrizing.

The main result of the first section is that such a reparametrization does not change the equivalence class of the circle model associated to it. More precisely, if we construct circle models (B_K, η_K) and $(B_{\widetilde{K}}, \eta_{\widetilde{K}})$ where $\widetilde{K}$ is a reparametrization of K, then these models are equivalent. In particular, since equivalent shells have the same contact germ near the boundary, it follows that one contact germ can be modelled by different Hamiltonians.

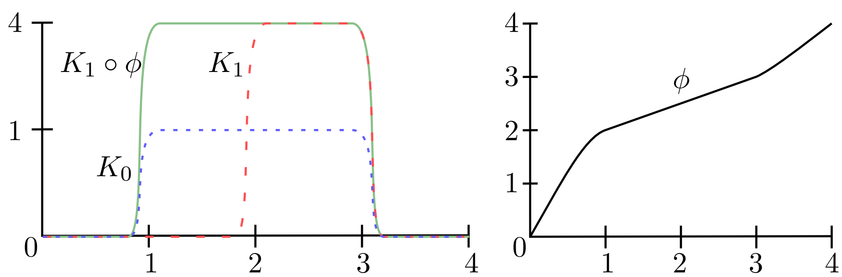


Figure 5.1: The graphs of the functions $K_0, K_1, K_1 \circ \phi$ and ϕ.

In the second section we define a pre-order on the set of contact Hamiltonians encoding the domination relation for the associated circle models. Essentially, this pre-order is given by $(\Delta, K) \leq (\Delta', K')$ if $K \leq K'$ as functions.
In the third section we show that this relation encodes what we want in the sense that $(\Delta, K) \leq (\Delta', K')$ implies $(B_K, \eta_K) \prec (B_{K'}, \eta_{K'})$. We also proof that given any circle model (B_K, η_K) and contact Hamiltonian (Δ', K') it is possible to construct a circle model $(B_{K'}, \eta_{K'})$ containing (B_K, η_K).

5.1 Conjugation of contact Hamiltonians

Recall a contact Hamiltonian is a smooth function $K : \mathcal{O}p\,\Delta \times S^1 \to \mathbb{R}$. Consider a contactomorphism $\Phi : (\mathbb{R}^{2n-1}, \xi_{st} = \ker \alpha_{st}) \to (\mathbb{R}^{2n-1}, \xi_{st} = \ker \alpha_{st})$. We have

$$\Phi^* \alpha_{st} = \lambda_\Phi \alpha_{st}$$

for a smooth positive function $\lambda_\Phi : \mathbb{R}^{2n-1} \to \mathbb{R}_+$, since we fixed the coorientation in Remark 2.5. If $\Delta \subset \mathbb{R}^{2n-1}$ is a compact star shaped domain it is easy to see that $\Phi(\Delta)$ is also compact and star-shaped. Furthermore, if $K|_{\partial\Delta\times S^1} > 0$ then $\Phi_* K|_{\partial f(\Delta)\times S^1} > 0$. Hence, given a contact hamiltonian (K, Δ) we get, using f, a new contact Hamiltonian $(\Phi_* K, \Phi(\Delta))$ called the push-forward contact Hamiltonian defined by

$$\Phi_* K : \Phi(\mathcal{O}p\,\Delta) \times S^1 \to \mathbb{R} \quad \text{where} \quad \Phi_* K(\Phi(x), \theta) := \lambda_\Phi(x) K(x, \theta). \tag{5.1}$$

The reason for scaling the Hamiltonian with a factor $\lambda_\Phi(x)$ will become clear from the proof of Lemma 5.2 below.
Observe that this only allows us reparametrizes $\mathcal{O}p\,\Delta \subset (\mathbb{R}^{2n-1}, \xi_{st})$ and leaves the S^1 coordinate unchanged. In view of Lemma 5.12, where we show that any contact Hamiltonian can be replaced by a time independent one, i.e. not depending on S^1, this does not restrict the usefulness of the operation.

This construction defines an action of the group of contactomorphism of $(\mathbb{R}^{2n-1}, \xi_{st})$ on the set of all contact Hamiltonians (K, Δ) given by $\Phi \cdot (K, \Delta) = (\Phi_* K, \Phi(\Delta))$. This action is called the *conjugation* action, for the following reason. Let ϕ_K^t for $t \in I$ be the unique contact isotopy with $\phi_K^0 = Id$ and

$$\alpha(\partial_t \phi_K^t(x)) = K(\phi_K^t(x), t),$$

as in Section 2.3. Then the following Lemma says $\phi_{\Phi_* K}^t$ comes from ϕ_K^t by conjugating with Φ.

Lemma 5.1. *Let $\Phi : (\mathbb{R}^{2n-1}, \xi_s t) \to (\mathbb{R}^{2n-1}, \xi_{st})$ be a contactomorphism and (K, Δ), $(\Phi_* K, \Phi(\Delta))$ defined as above. Then,*

$$\phi_{\Phi_* K}^t = \Phi \circ \phi_K^t \circ \Phi^{-1}, \text{ for all } t \in I.$$

Proof. Using the notation $y := \Phi(x)$ we want to show that for all $t \in I$ we have $\phi_{\Phi_* K}^t(y) =$

$\Phi \circ \phi^t_K(x)$. From the defining relations for ϕ^t_K it follows that

$$\begin{aligned}\alpha_{st}\left(\partial_t(\Phi\circ\phi^t_K\circ\Phi^{-1})(y)\right) &= \alpha_{st}(d\Phi(\partial_t\phi^t_K(x)))\\ &= \Phi^*\alpha_{st}(\partial_t\phi^t_K(x))\\ &= \lambda_\Phi(x)K(\phi^t_K(x),t)\\ &= \Phi_*K(\Phi\circ\phi^t_K\circ\Phi^{-1}(y),t)\end{aligned}$$

On the other hand we have

$$\alpha_{st}(\partial_t\phi^t_{\Phi_*K}(y)) = \Phi_*K(\phi^t_{\Phi_*K}(y),t).$$

We conclude $\phi^t_{\Phi_*K} = \Phi\circ\phi^t_K\circ\Phi^{-1}$ for $t\in I$. □

The interesting property of this conjugation action is that, up to isomorphism, all the Hamiltonians in the same orbit give the same circle model.

Lemma 5.2. *A contactomorphism Φ of $(\mathbb{R}^{2n-1},\xi_{st})$, restricting to a contactomorphism between star-shaped domains $\Phi:\Delta\to\Delta'$, induces an equivalence of the contact shells*

$$\widehat{\Phi}:(B_K,\eta_K)\to(B_{\Phi_*K},\eta_{\Phi_*K})$$

defined by (K,Δ) and (Φ_*K,Δ').
In particular, for a given model $(B_{\Phi_*K,\widetilde{C}},\eta_{\Phi_*K,\tilde\rho,\tilde g})$ we will construct a model $(B_{K,C},\eta_{K,\rho,g})$ such that the two models are isomorphic as almost contact structures.

Proof. Assume we are given a circle model $(B_{\Phi_*K,\widetilde{C}},\eta_{\Phi_*\widetilde{K},\tilde\rho,\tilde g})$, we will first construct a circle model $(B_{K,C},\eta_{K,\rho,g})$ and a diffeomorphism between these models.
Choose a $C\geq 0$ such that $\min(K)+C>$ and pick a family of diffeomorphisms $\psi_{(x,\theta)}:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ for $(x,\theta)\in\Delta\times S^1$, restricting to diffeomorphisms

$$\psi_{(x,\theta)}:[0,K(x,\theta)+C]\to[0,\lambda_\Phi(x)K(x,\theta)+\widetilde{C}],$$

satisfying

$$\psi_{(x,\theta)}(v)=\lambda_\Phi(x)(v-C)+\widetilde{C}\text{ for }(x,\theta,v)\in\mathcal{O}p\,\{v=K(x,\theta)+C\}. \tag{5.2}$$

Note that by definition $\lambda_\Phi(x)K(x,\theta)+\widetilde{C}=\Phi_*K(\Phi(x),\theta)+\widetilde{C}>0$ so that such a family of diffeomorphisms $\psi_{x,\theta}$ does indeed exist. From this data we get a diffeomorphism

$$\widehat{\Phi}:B_{K,C}\to B_{\Phi_*K,\widetilde{C}}\text{ defined by }\widehat{\Phi}(x,v,\theta)=(\Phi(x),\psi_{x,\theta}(v),\theta).$$

Next, we construct ρ and g such that $\widehat{\Phi}:(B_{K,C},\eta_{K,\rho,g})\to(B_{\Phi_*K,\widetilde{C}},\eta_{\Phi_*K,\tilde\rho,\tilde g})$ is an almost contact isomorphism as follows.
Define

$$\rho(x,\theta):[0,K(x,\theta)+C]\to\mathbb{R},\text{ by }\rho_{x,\theta}(v):=\frac{1}{\lambda_\Phi(x)}\tilde\rho_{\Phi(x),\theta)}(\psi_{x,\theta}(v)),$$

and $g := \tilde{g} \circ \widehat{\Phi}$. Observe that Equation 5.2 ensures that ρ satisfies the conditions in Equation 4.5. Write $\eta_{\Phi_* K, \tilde{\rho}, \tilde{g}} = ([\alpha_{\tilde{\rho}}], [\tilde{\omega}])$ and define $\eta_{K,\rho,g} := ([\alpha_\rho], [\omega])$ where

$$\alpha_\rho := \alpha_{st} + \rho d\theta, \qquad \omega := \widehat{\Phi}^* \tilde{\omega}.$$

To show that this is a well-defined circle model we need to check that $[d\alpha|_{\ker \alpha_\rho}] = [\omega]$ on $\mathcal{O}p\, \partial B_{K,C}$ and that ω is homotopic to $\omega_{\rho,g}$, through almost contact structures and relative to $\mathcal{O}p\, \partial B_{K,C}$. Note that

$$\begin{aligned} \widehat{\Phi}^* \alpha_{\tilde{\rho}} &= \widehat{\Phi}^* \left(\alpha_{st} + \tilde{\rho} d\theta\right) \\ &= \lambda_\Phi \alpha_{st} + \lambda_\Phi \rho d\theta = \lambda_\Phi \alpha_\rho. \end{aligned}$$

So, from $[d\alpha_{\tilde{\rho}}|_{\ker \alpha_{\tilde{\rho}}}] = [\tilde{\omega}]$ on $\mathcal{O}p\, \partial B_{\Phi_* K, \widetilde{C}}$ it follows that on $\mathcal{O}p\, \partial B_{K,C}$ we have

$$\omega = \widehat{\Phi}^* \omega = f d\Phi^* \alpha_{\tilde{\rho}} = f d(\lambda_\Phi \alpha_\rho) = f \lambda_\Phi d\alpha_\rho,$$

for some smooth function $f : M \to \mathbb{R}_{>0}$, proving the first claim.
For the second claim it is enough to show that we can homotope $\widehat{\Phi}^* \omega_{\tilde{\rho},\tilde{g}}$ to $\omega_{\rho,g}$, since by definition $\tilde{\omega}$ is homotopic to $\omega_{\tilde{\rho},\tilde{g}}$, and pulling back this homotopy shows ω is homotopic to $\widehat{\Phi}^* \omega_{\rho,g}$. A homotopy between $\widehat{\Phi}^* \omega_{\tilde{\rho},\tilde{g}}$ and $\omega_{\rho,g}$ is given by

$$H_t := (1-t)\Big(\lambda_\Phi + t\Big) d\alpha_{st} + \Big(g d\big((1-t)\lambda_\Phi \rho + t\rho\big) + (1-g) d\big((1-t)\psi + tv\big)\Big) \wedge d\theta,$$

since

$$H_0 = \lambda_\Phi d\alpha_{st} + \big(g d(\lambda_\Phi \rho) + (1-g) d\psi\big) \wedge d\theta = \widehat{\Phi}^* \omega_{\tilde{\rho},\tilde{g}}$$

and

$$H_1 = d\alpha_{st} + \big(g d\rho + (1-g) dv\big) \wedge \theta = \omega_{\rho,g}.$$

To show that $\alpha_\rho \wedge H_t^n \neq 0$ for all $t \in [0,1]$ we first prove the following identity.
Taking $\omega := f_4 d\alpha_{st} + (g df_1 + (1-g) df_2) \wedge d\theta$, and $\alpha := \alpha_{st} + f_3 d\theta$, for arbitrary functions $f_1, f_2, f_3, f_4 : \mathcal{O}p\, B_{K,C} \to \mathbb{R}$, we have

$$\alpha \wedge \omega^n \neq 0 \leftrightarrow f_4\big(g \partial_v f_1 + (1-g) \partial_v f_2\big) \neq 0. \tag{5.3}$$

To see this we compute

$$\begin{aligned} \alpha \wedge \omega^n &= (\alpha_{st} + f_3 d\theta) \wedge (f_4 d\alpha_{st} + (g df_1 + (1-g) df_2) \wedge d\theta)^n \\ &= n f_4 \left(g \partial_v f_1 + (1-g) \partial_v f_2\right) \alpha_{st} \wedge d\alpha_{st}^{n-1} \wedge dv \wedge d\theta. \end{aligned}$$

For α_ρ and H_t as above we have

$$f_1^t = (1-t)\lambda_\Phi \rho, \qquad f_2 = (1-t)\psi + tv, \qquad f_3^t = \rho, \qquad f_4 = (1-t)\lambda_\Phi + t.$$

So, by Equation 5.3 it suffices to show

$$f_4\Big(g \partial_v f_1 + (1-g) \partial_v f_2\Big) > 0, \text{ for all } t \in [0,1].$$

We compute

$$\partial_v f_1^t = (1-t)\lambda_\Phi \partial_v \rho + t\partial_v \rho, \qquad \partial_v f_2^t = (1-t)\partial_v \psi + t.$$

Hence,

$$f_4\Big(g\partial_v f_1 + (1-g)\partial_v f_2\Big) = \Big((1-t)\lambda_\Phi + t\Big)\Big(g\big((1-t)\lambda_\Phi \partial_v \rho + t\partial_v \rho\big) + (1-g)\big((1-t)\partial_v \psi + t\big)\Big)$$

First observe that $\big((1-t)\lambda_\Phi + t\big) > 0$ so it is enough to show

$$\Big(g\big((1-t)\lambda_\Phi \partial_v \rho + t\partial_v \rho\big) + (1-g)\big((1-t)\partial_v \psi + t\big)\Big) > 0.$$

Since ψ is a diffeomorphism we have $\partial_v \psi > 0$ hence, $\big((1-t)\partial_v \psi + t\big) > 0$. This implies that whenever $g = 0$ we have the correct inequality. Whenever $g > 0$ we have $\partial_v \rho > 0$ since we required $g = 0$ on $\mathcal{O}p\,\{x \in \Delta \mid \partial_v \rho \leq 0\}$. This gives the required inequality showing $\alpha_\rho \wedge H_t^n \neq 0$ for all $t \in [0,1]$. □

5.2 Domination of contact Hamiltonians

In this section we will introduce a preorder $\leq$ on the set of contact Hamiltonians (Δ, K). We show in the next section that this preorder is natural in the sense that the association $(\Delta, K) \to (B_K, \eta_K)$ is a map between preordered sets. That is, $(\Delta, K) \leq (\Delta', K')$ implies $(B_K, \eta_K) \prec (B'_K, \eta'_K)$.

The definition of the preorder is as follows.

Definition 5.3. *A contact Hamiltonian* (Δ, K) *is said to be* dominated *by* (Δ', K')*, denoted by* $(\Delta, K) \leq (\Delta', K')$ *if*

(i) $\Delta \subset \Delta'$

(ii) $K'(x,\theta) > 0$ *for* $x \in \Delta' \setminus \Delta$

(iii) There exists a compact star-shaped domain $\widetilde{\Delta} \subset \operatorname{Int}\Delta$ *(possibly* $\tilde{\Delta} = \emptyset$*) such that*

- *(a)* $K(x,\theta) \leq K'(x,\theta)$ *for* $x \in \mathcal{O}p\,\Delta \setminus \operatorname{Int}\widetilde{\Delta}$
- *(b)* $K'(x,\theta) \geq 0$ *for* $x \in \mathcal{O}p\,\partial\widetilde{\Delta}$
- *(c)* $K(x,\theta) \leq 0$ *for* $x \in \mathcal{O}p\,\widetilde{\Delta}$
- *(d)* $K|_{\operatorname{Int}\widetilde{\Delta}} \not\equiv 0$

Taking $\tilde{\Delta} = \emptyset$ the domination relation says that K' is greater than (or equal to) K where they are both defined and K' is positive on $\Delta \setminus \Delta'$. Condition (iii) states that whenever K and K' are both negative we can also allow K' to be smaller than K.

Remark 5.4. *Note that if we only allow for* $\tilde{\Delta} = \emptyset$ *then Definition 5.3 is the same as the partial order introduced in Section 4.1 in [2]. There, the more general definition follows from Proposition 4.9.*

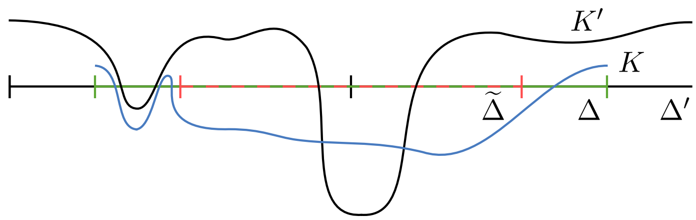


Figure 5.2: Illustration of the domination relation for Hamiltonians $K \leq K'$.

Lemma 5.5. *The domination relation defined a preorder on the set of contact Hamiltonians. More precisely,*

(i) for all (Δ, K) we have $(\Delta, K) \leq (\Delta, K)$ (reflexive)

(ii) $(\Delta_0, K_0) \leq (\Delta_1, K_1)$ and $(\Delta_1, K_1) \leq (\Delta_2, K_2)$ implies $(\Delta_0, K_0) \leq (\Delta_2, K_2)$ (transitive)

Proof. Since none of the conditions in Definition 5.3 depend on the $\theta \in S^1$ coordinate of K, we drop this in our notation to make the proof more readable. To show $\leq$ is reflexive it is enough to observe that $\Delta \subset \Delta$ and taking $\widetilde{\Delta} = \emptyset$ we have $K \leq K$ on Δ.
To show $\leq$ is transitive denote by $\widetilde{\Delta}_0^1$ and $\widetilde{\Delta}_1^2$ the sets used in the relations $(\Delta_0, K_0) \leq (\Delta_1, K_1)$ and $(\Delta_1, K_1) \leq (\Delta_2, K_2)$. We check that taking $\widetilde{\Delta} := \widetilde{\Delta}_0^1 \cup \widetilde{\Delta}_1^2$ the conditions of Definition 5.3 are satisfied so that $(\Delta_0, K_0) \leq (\Delta_2, K_2)$.

(i) By assumption $\Delta_0 \subset \Delta_1$ and $\Delta_1 \subset \Delta_2$, hence $\Delta_0 \subset \Delta_2$.

(ii) By assumption $K_1|_{\Delta_1 \setminus \Delta_0} > 0$ and $K_1|_{\mathcal{O}p\,\widetilde{\Delta}_1^2}$. This implies that $\widetilde{\Delta}_1^2 \subset \operatorname{Int} \Delta_0$ so that $K_2 \geq K_1$ on $\Delta_1 \setminus \Delta_0$. Since by assumption $K_1|_{\Delta_1 \setminus \Delta 0} > 0$ and $K_2|_{\Delta_2 \setminus \Delta_1} > 0$ we conclude $K_2|_{\Delta_2 \setminus \Delta_0} > 0$.

(iii) (a) By assumption we have $K_1 \geq K_0$ on $\mathcal{O}p\,\Delta_0 \setminus \operatorname{Int} \widetilde{\Delta}_0^1$ and $K_2 \geq K_1$ on $\mathcal{O}p\,\Delta_1 \setminus \operatorname{Int} \widetilde{\Delta}_1^2$ hence also on $\Delta_0 \setminus \operatorname{Int} \widetilde{\Delta}_1^2$. Combining this gives $K_2 \geq K_1 \geq K_0$ on $\mathcal{O}p\,\Delta_0 \setminus \operatorname{Int} \widetilde{\Delta}$.

(b) By assumption $K_2 \geq 0$ on $\mathcal{O}p\,\partial \widetilde{\Delta}_1^2$ and $K_1 \geq 0$ on $\mathcal{O}p\,\partial \widetilde{\Delta}_0^1$. Since $\mathcal{O}p\,\partial \widetilde{\Delta}_0^1 \subset \Delta_0$ we have $K_2 \geq K_1 \geq 0$ on $\mathcal{O}p\,\partial \widetilde{\Delta}_0^1$. Combining this gives $K_2 \geq 0$ on $\mathcal{O}p \partial \widetilde{\Delta}$.

(c) By assumption $K_0 \leq 0$ on $\mathcal{O}p\,\widetilde{\Delta}_0^1$. Since $K_1 \leq 0$ on $\mathcal{O}p \widetilde{\Delta}_1^2$ and, as before, $\widetilde{\Delta}_1^2 \subset \operatorname{Int} \Delta_0$ we have $0 \leq K_1 \leq K_2$ on $\mathcal{O}p \widetilde{\Delta}_1^2$. Again, combining this gives $K_2 \leq 0$ on $\mathcal{O}p \widetilde{\Delta}$.

(d) This follows immediately since by assumption $K_0 \not\cong 0$ on $\operatorname{Int} \widetilde{\Delta}_0^1 \subset \operatorname{Int} \widetilde{\Delta}$.

□

The following example shows that $\leq$ is not antisymmetric in the sense that $(\Delta, K) \leq (\Delta', K')$ and $(\Delta', K') \leq (\Delta, K)$ implies $(\Delta, K) = (\Delta', K')$. Furthermore, using Proposition 5.9 it shows that the same is true for $\prec$ on the set of circle models.

Example 5.6. *Let $\Delta := [0, 1]$ and $\widetilde{\Delta} := [\frac{1}{4}, \frac{3}{4}]$, viewed as compact star-shaped domains in $(\mathbb{R}, \xi_{st})$. Define two contact Hamiltonians K, K' satisfying the following conditions*

1. $K = K'$ *on* $\mathcal{O}p\,\Delta \setminus \operatorname{Int}\widetilde{\Delta}$

2. $K = K' \cong 0$ *on* $U := \mathcal{O}p\,\partial\widetilde{\Delta}$

3. $K < 0$ *on* $\widetilde{\Delta} \setminus U$

4. $K'(\frac{1}{2}) = 0$

see Figure 5.3 below.

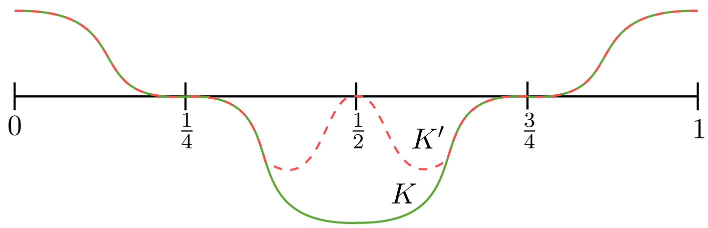


Figure 5.3: The graphs of K and K' used to show that the domination relation is not anti-symmetric

It follows immediately that $(\Delta, K) \leq (\Delta, K')$ *and* $(\Delta, K') \leq (\Delta, K)$ *while* $K \neq K'$. *According to Proposition 5.9 below this implies* $(B_K, \eta_K) \prec (B_{K'}, \eta_{K'})$ *and* $(B_K, \eta_K) \succ (B_{K'}, \eta_{K'})$. *However, in the three dimensional case zero's of the contact Hamiltonian correspond closed curves of the characteristic foliation and we see that these circle models have non diffeomorphic characteristic foliation on their boundaries. Furthermore, by Giroux's theorem the characteristic foliation determines the germ of contact structure near the boundary implying that these circle models can never be equivalent. This shows that the domination relation* $\prec$ *on contact shells is not antisymmetric.*

We end this section with the observation in the following lemma saying that in the three dimensional case the domination relation on the set of contact Hamiltonians is essentially trivial.
In the case $n = 1$ (corresponding to $\dim M = 3$), we have that any compact star-shaped domain $\Delta \subset \mathbb{R}$ is a compact interval. It turns out that this implies that up to conjugation any somewhere negative contact Hamiltonian (Δ, K) is minimal with respect to the domination relation.

Lemma 5.7. *Let* (Δ, K) *be a somewhere negative Hamiltonian with* $\Delta \subset \mathbb{R}$ *compact and star-shaped. Then, for any other contact Hamiltonian* $(\widetilde{\Delta}, \widetilde{K})$ *there exists a contactomorphism* $\Phi : \mathcal{O}p\,\Delta \to \mathcal{O}p\,\widetilde{\Delta}$ *satisfying* $(\Phi(\Delta), \Phi_*(K)) \leq (\widetilde{\Delta}, \widetilde{K})$.

Proof. Without loss of generality we can assume that $\Delta = \tilde{\Delta} = [-1, 1]$ and that $K(0) < 0$. This means that there exist $\delta > 0$ and $0 < \sigma \ll 1$ satisfying

$$K(z) < 0 \quad \text{if } z \in [-\sigma, \sigma], \text{ and}$$

$$\widetilde{K}(z) > \delta \quad \text{if } z \in [-1, -1+\sigma] \cup [1-\sigma, 1].$$

Pick a diffeomorphism $\Phi : [-1,1] \to [-1,1]$ such that the following subsets get mapped linearly onto each other,

$$[-1,-\sigma] \text{ maps onto } [-1,-1+\sigma], \text{ and}$$

$$[\sigma,1] \text{ maps onto} [1-\sigma,1].$$

Recall that $(\Phi_* K)(\Phi(z)) := \Phi'(z)K(z)$, so that choosing σ small enough we obtain

$$\begin{aligned} (\Phi_* K)(z) &< 0 & &\text{if } z \in [-1+\sigma, 1-\sigma] \\ (\Phi_* K)(z) &\leq \frac{\sigma}{1-\sigma}\max(K) < \delta < \widetilde{K}(z) & &\text{if } z \in [-1,-1+\sigma] \cup [1-\sigma,1]. \end{aligned}$$

Hence, $\Phi_* K < \widetilde{K}$. □

Remark 5.8. *In higher dimensions the previous proof does not hold. The problem is that there does not necessarily exist a contactomorphism* $\Phi \in \mathrm{Cont}(\mathbb{R}^{2n-1}, \xi_{st})$ *such that* $\widetilde{\Delta} \setminus \Phi(U) \subset V$*, where*

$$U := \{\, x \in \Delta \mid K(x) < 0 \,\}, V := \{\, x \in \widetilde{\Delta} \mid \widetilde{K}(x) > \delta \,\}.$$

The reason for this is that in general Δ *and* $\widetilde{\Delta}$ *can have different shapes while all compact star-shaped domains in* $\mathbb{R}$ *are an interval.*

It might still be true (although it seems unlikely) that a result similar Lemma 5.7 is true in higher dimensions, but the proof of such a result would use a different argument.

5.3 Circle model map respects domination

The main result we want to show that the domination relation on contact Hamiltonians is compatible with the domination relation on contact shells.

Proposition 5.9. *If* $(\Delta, K) \leq (\Delta', K')$ *then* $(B_K, \eta_K) \prec (B_{K'}, \eta_{K'})$*. More precisely, given a circle model shell* $(B_{K,C}, \eta_{K,\rho,g})$ *there exists a circle model shell* $(B_{K',C'}, \eta_{K',\rho',g'})$ *such that* $B_{K,C} \subset B_{K',C'}$ *and the inclusion is a subordination map.*

To prove this we need several lemma's which happen to be interesting results on their own.

Lemma 5.10. *Let* $(B_{K,C}, \eta_{K,\rho,g})$ *be a circle model for* (Δ, K)*. For any other contact Hamiltonian* (Δ', K') *there exists a circle model* $(B_{K',C'}, \eta_{K',\rho',g'})$ *and an embedding of almost contact structures*

$$(B_{K,C}, \eta_{K,\rho,g}) \hookrightarrow (B_{K',C'}, \eta_{K',\rho',g'}).$$

Furthermore,

(i) if $\Delta \subset \mathrm{Int}\,\Delta'$ *then the embedding can be taken to be an inclusion map;*

(ii) if $(\Delta, K) \leq (\Delta', K')$ *with* $\widetilde{\Delta} = \emptyset$ *then we can take* $C' = C$, $\operatorname{supp} g \subset \operatorname{supp} g'$, $\rho'|_{B_{K,C}} = \rho$ *and the inclusion to be a subordination map.*

Proof. Suppose we are given a circle model $(B_{K,C}, \eta_{K,\rho,g})$ for (Δ, K). By Lemma 2.20 we can find a contactomorphism $\Phi \in \mathrm{Cont}_0^c(\mathbb{R}^{2n-1})$ such that $\Delta \subset \Phi(\Delta')$. By Lemma 5.2, any circle model associated to $(\Phi(\Delta'), \Phi_* K')$ is isomorphic to a circle model associated to (Δ', K'). Therefore, it is enough to show that, assuming $\Delta \subset \mathrm{Int}\,\Delta'$, we can find a circle model $(B_{K',C'}, \eta_{K',\rho',g'})$ such that the inclusion

$$(B_{K,C}, \eta_{K,\rho,g}) \hookrightarrow (B_{K',C'}, \eta_{K',\rho',g'}) \tag{5.4}$$

is an embedding of almost contact structures. So, assume $\Delta \subset \mathrm{Int}\,\Delta'$ and pick $C' \geq 0$ satisfying

$$K'(x,\theta) + C' > K(x,\theta) + C \text{ for all } (x,\theta) \in \Delta \times S^1.$$

That such a C' exists follows from compactness of Δ, Δ' and S^1. This choice of C' ensures that, as sets, $B_{K,C} \subset \mathrm{Int}\, B_{K',C'}$. Recall that the defining conditions of ρ in Equation 4.5 only restrict the behaviour on $\mathcal{O}p\,\Delta \cong \{\, (x,v,\theta) \in \mathbb{R}^{2n+1} \mid x \in \mathcal{O}p\,\Delta, v = 0 \,\}$ and on $\mathcal{O}p\,\partial B_{K,C}$.
Since $B_{K,C} \subset \mathrm{Int}\, B_{K',C'}$ we can find disjoint opens $\mathcal{O}p\,\partial B_{K,C}$ and $\mathcal{O}p\,\partial B_{K',C'}$ and it follows that any $\rho : \mathcal{O}p\, B_{K,C} \to \mathbb{R}$ used to define $\eta_{K,\rho,g}$ can be extended to a $\rho' : \mathcal{O}p\, B_{K',C'} \to \mathbb{R}$ satisfying the conditions in Equation 4.5. Similarly, we can extend the bump function g to a bump function g' suitable for the construction of $\eta_{K',\rho',g'}$. It is clear that choosing C', ρ' and g', produces a circle model such that the inclusion in Equation 5.4 is an embedding of almost contact structures.
It remains to be shown that if $(\Delta, K) \leq (\Delta', K')$ with $\widetilde{\Delta} = \emptyset$ the inclusion is a subordination map. In this case since $K \leq K'$ we can choose $C' = C$ and still have $B_{K,C} \subset B_{K',C'}$. As before, we extend ρ and g to suitable functions ρ' and g'. Moreover, since $K \leq K'$ we can ensure that $\partial_v \rho'|_{B_{K',C'} \setminus B_{K,C}} > 0$, so that $\eta_{K',\rho',g'}$ is a contact structure outside $B_{K,C}$. Indeed, for $(x,\theta, K(x,\theta) + C) \in \partial B_{K,C}$ and $(x, \theta, K'(x,\theta) + C') \in \partial B_{K',C'}$, observe

$$\rho'(x,\theta,K'(x,\theta) + C') - \rho(x,\theta,K(x,\theta) + C) = K'(x,\theta) - K(x,\theta) \geq 0$$

and if $K'(x,\theta) = K(x,\theta)$ then we can take $\rho'(x,\theta,v) = \rho(x,\theta,v)$. □

For the next lemma we need to introduce some notation. Given a domain (not necessarily star-shaped) $\Delta \subset (\mathbb{R}^{2n-1}, \xi_{st})$ we define

$$F_+(\Delta) := \{\, F \in C^\infty(\mathcal{O}p\,\Delta) \mid \mathrm{supp}(F) \subset \mathrm{Int}\,\Delta, F \geq 0, F \not\equiv 0 \,\},$$

and we consider the action of $\mathrm{Cont}^c(\mathrm{Int}\,\Delta)$, the set of contactomorphisms of $\mathrm{Int}\,\Delta$ with compact support inside $\mathrm{Int}\,\Delta$, on $F_+(\Delta)$ given by

$$\Phi_* F := (\lambda_\Phi F) \circ \Phi^{-1},$$

where $F \in F_+(\Delta)$ and $\Phi \in \mathrm{Cont}^c(\mathrm{Int}\,\Delta)$. Note that the elements of $F_+(\Delta)$ are not contact Hamiltonians because they are not positive near the boundary of Δ.

Lemma 5.11. *If $\Delta \subset (\mathbb{R}^{2n-1}, \xi_{st})$ is star-shaped and $K, H \in F_+(\Delta)$, then there exists a $\Phi \in \mathrm{Cont}^c(\mathrm{Int}\,\Delta)$ such that $\Phi_* K \geq K'$.*

Proof. By a linear change of coordinates assume that $K(0) > 0$. Consider the closed set $V := \operatorname{supp}(H) \subset \operatorname{Int}\Delta$. For $T > 0$ very large define $U := X_z^{-T}(V)$ where $X_z^t : \mathbb{R}^{2n-1} \to \mathbb{R}^{2n-1}$ denotes the flow of the contact vector field X_z defined by viewing the z-coordinate as a function on $\mathbb{R}^{2n-1}$. By taking T large enough we can make sure that

$$\min_U(K) > 0, \quad \text{and} \quad e^T \min_U(K) > \max(H),$$

since $(X_z^t)^*\alpha_{st} = e^t\alpha_{st}$. Define $\widetilde{X} := gX_z$ where $g : \Delta \to I$ is a smooth bump function separating V from $\partial\Delta$. Then $\widetilde{X}$ is supported in $\operatorname{Int}\Delta$ and equals X_z on V.
Clearly $\Phi := \widetilde{X}^T \in \operatorname{Cont}^c(\operatorname{Int}\Delta)$ and we claim that $\Phi_*K \geq H$. Indeed,

$$(\Phi_*K)(x) = (\lambda_\Phi K)(\Phi^{-1}(x)) \geq e^T \min_U(K) \geq H(x), \text{ for all } x \in \Delta,$$

completing the proof. □

Next, we give the proof of Proposition 5.9.

Proof of Proposition 5.9. Since $(\Delta, K) \leq (\Delta', K')$ we can pick contact Hamiltonians $K_i : \Delta \to \mathbb{R}$, $i = 0, 1$ satisfying

(i) $K \leq K_0$ and $K_1 \leq K'$

(ii) $K_0 \leq K_1$ on $\Delta \setminus \widetilde{\Delta}$

(iii) $-K_i|_{\widetilde{\Delta}} \in F_+(\widetilde{\Delta})$ for $i = 0, 1$.

By Lemma 5.10 it is enough to show that (B_{K_0}, η_{K_0}) is dominated by (B_{K_1}, η_{K_1}). Combining Lemma 5.11 and condition (iii) gives a $\Phi \in \operatorname{Cont}^c(\operatorname{Int}\widetilde{\Delta})$ such that

$$\Phi_*(K_0|_{\widetilde{\Delta}}) \leq K_1|_{\widetilde{\Delta}}.$$

Moreover, since $\operatorname{supp}(\Phi) \operatorname{Int}\widetilde{\Delta}$ we can extend Φ to a contactomorphism of Δ, also denoted by Φ, which satisfies the above equation and $\Phi(\Delta) = \Delta$. By Lemma 5.2 the circle models defined by (Δ, K_0) and (Δ, Φ_*K_0) are isomorphic and by Lemma 5.10 we see that $(B_{\Phi_*K_0}, \eta_{\Phi_*K_0})$ is dominated by (B_{K_1}, η_{K_1}). □

An immediate consequence of the previous proposition is that we can restrict our attention to filling a class of particularly nice circle models.

Lemma 5.12. *Given any circle model (B_K, η_K) modelled by a time dependent contact Hamiltonian (Δ, K) there exists a (time-independent) special contact Hamiltonian (Δ_{cyl}, K') such that $(B_{K'}, \eta_{K'}) \prec (B_K, \eta_K)$*

Proof. By combining Lemma 2.20 and Lemma 5.2 we can assume that $\Delta_{cyl}, \subset \Delta$. If $K|_{\mathcal{O}p\,(\Delta\setminus\operatorname{Int}\Delta_{cyl})} > 0$ then $K' := K|_{\mathcal{O}p\,\Delta_{cyl}}$ satisfies the conditions to be a contact Hamiltonian. Replacing K' by a smaller special contact Hamiltonian using Example 7.4 and applying again Proposition 5.9 completes the proof.

In general it does not need to be true that $K|_{\mathcal{O}p\,(\Delta \setminus \operatorname{Int}\Delta_{cyl})} > 0$ and we need to do a little extra work to achieve this.

Choose an open set $U \supset \partial\Delta$ such that $K|_U > 0$ and a bump function $g : \Delta \to [0,1]$ satisfying $g|_{\mathcal{O}p\,\partial\Delta} = 0$ and $\operatorname{supp} g \subset U$. Use this to define a contact Hamiltonian $f : \mathbb{R}^{2n-1} \to \mathbb{R}$ by

$$(u_i, \phi_i, z) \mapsto (1 - g(u_i, \phi_i, z))z.$$

Away from $\mathcal{O}p\,\partial\Delta$ the associated contact vector field is just X_z so that for large enough T the set $X_f^T(U)$ covers $\Delta \setminus \operatorname{Int}\Delta_{cyl}$. Applying Lemma 5.2 we are in the same situation as above and the proof concludes as before. $\square$

Recall, from Chapter 4, that we have a map

$$\left\{\begin{matrix}\text{Contact Hamiltonians}\\ (\Delta, K)\end{matrix}\right\} \to \left\{\begin{matrix}\text{Circle models}\\ (B_K, \eta_K)\end{matrix}\right\}.$$

The results of this chapter can be summarized as saying that this map descents to a map from the orbit space of orbits of the conjugation action (on the set of contact Hamiltonian) to the set of equivalence classes of contact shells.

$$\left\{\begin{matrix}\text{Orbits of the conjugation action}\\ \operatorname{Cont}(\mathbb{R}^{2n-1}, \xi_{st}) \cdot \{(\Delta, K)\}\end{matrix}\right\} \to \left\{\begin{matrix}\text{Equivalence classes}\\ \text{of contact shells}\end{matrix}\right\}.$$

By definition this map is surjective and it is not hard to see that it is not injective.

Furthermore, the map is compatible with the domination relations in the sense that $(\Delta, K) \leq (\widetilde{\Delta}, \widetilde{K})$ implies $(B_K, \eta_K) \prec (B_{\widetilde{K}}, \eta_{\widetilde{K}})$. Note that although $\prec$ descents to a preorder on the set of equivalence classes of circle models, $\leq$ does not give a well defined preorder on the orbit space of the conjugation action.

In the three dimensional case any orbit containing a somewhere negative Hamiltonian is minimal with respect to the domination relation in the sense that any contact Hamiltonian dominates some element in the orbit. This means that it suffices to prove that a circle model associated to a somewhere negative contact Hamiltonian can be filled.

In higher dimensions we do not know if such an orbit exists. However, the family of orbits containing a special contact Hamiltonian are minimal in the sense that any Hamiltonian dominates an element of one of these orbits. By Proposition 3.8 this is enough to solve the filling problem.

Chapter 6

Connected sums

Recall that the topological connected sum operation allows us to construct, from two manifolds M and N with boundary, a new manifold $M\#N$ by identifying a disk D^n in the boundary of each manifold and gluing these to the boundary of a tube $D^n \times I$. It is well known that this can be done in a smooth way. That is, starting with two smooth manifolds, the connected sum is again a smooth manifold.

The goal of this chapter is to show that this construction is also compatible with (almost) contact structures in the sense that if M and N are (almost) contact manifolds then so is the connected sum. Since we only use the connected sum in Proposition 3.8 where we apply it to circle models, we only consider connected sums of contact shells.

In the first section we define abstract boundary connected sums. These are connected sums of contact shells where we view them as almost contact manifolds and don't take into account their ambient manifolds. In particular, the connecting tube is not contained in an ambient (contact) manifold.

In the second section we apply the abstract connected sum to circle model shells. It turns out that starting with two circle models, their (abstract) connected sum is equivalent to a circle model. To see this, we define the contact Hamiltonian $(\Delta_-\#\Delta_+, K_-\#K_+)$ modelling the connected sum circle model.

In the third section we show that the connected sum construction in the case of circle models can actually be performed inside an ambient (contact) manifold. This boils down to showing that the connecting tube can be embedded in the ambient manifold containing the circle models.

6.1 Abstract boundary connected sum

Consider $\mathbb{R}^{2n}$ with (polar) coordinates $(u_1, \theta_1, \dots, u_{n-1}, \theta_{n-1}, v, \theta)$ and equipped with the Liouville 1-form and Liouville vector field

$$\lambda := \sum_{i=1}^{n-1} u_i d\theta_i + v d\theta, \qquad L := \sum_{i=1}^{n-1} u_i \frac{\partial}{\partial u_i} + v \frac{\partial}{\partial v},$$

which we view as the hyperplane $(\mathbb{R}^{2n}, \lambda) \subset (\mathbb{R}^{2n+1}, \alpha_{st})$ defined by $z = 0$. Note in particular that λ is just the restrictin of α_{st} to the tangent bundle $T\mathbb{R}^{2n}$ and similarly L is the projection of X_z to $T\mathbb{R}^{2n}$. As usual we will denote by $L^t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the time t flow of L.

In order to glue the connecting tube we need to identify embedded disks in the boundary of the contact shells we want to connect. Moreover, to ensure that the gluing is compatible with the contact structure we need some restriction on which disks we allow. These restrictions are expressed in the definition of a gluing disk.

Definition 6.1. *A* gluing disk *for a contact shell* $\big(B^{2n+1}, \eta = ([\alpha], [\omega])\big)$ *is a smooth embedding* $\iota : D \to \partial B$*, of a star-shaped (with respect to L) disk*[1] $D \subset \mathbb{R}$*, satisfying* $\iota^*\alpha = \lambda$*.*

Note that since the disk embeds into the boundary of B where η is contact we also have $\iota^*\omega = d\alpha$.
A generic disk in the boundary of the a contact shell will not be a gluing disk. Luckily, there is an easy criterion to check if a point is contained in a gluing disk, given in the following definition.

Definition 6.2. *A* gluing place *for a contact shell* (B, η) *is a point* $p \in \partial B$ *such that* $T_p\partial B = \xi_p$*, where* $\xi = \ker\alpha$ *for some* α *representing* η *near* $\mathcal{O}p\,\partial B$*.*

These points allow us to identify gluing disks using the following lemma.

Lemma 6.3. *Given a gluing disk* $\iota : D \to \partial B$ *for a contact shell* (B, η) *the point* $\iota(0) \in \partial B$ *is a gluing place. Conversely, every gluing place* $p \in \partial B$ *is contained is some gluing disk.*

Proof. The first claim of the lemma follows immediately from the definitions. Indeed, under the embedding $\iota : D \to \partial B$ the tangent space $T_p\partial B$ corresponds to the tangent space $T_0\mathbb{R}^{2n}$ on which λ vanishes.

Conversely, assume $p \in \partial B$ such that $T_p\partial B = \ker\alpha_p$. This implies that at the point p the Reeb vector field is transverse to ∂B. By continuity we can find an open neighborhood U of p such that R_α is transverse to ∂B on $U \cap \partial B$ In turn we can find an open $V \subset (\mathbb{R}^{2n+1}, \alpha_{st})$ and a diffeomorphism $\phi : V \to U$, where $\mathbb{R}^{2n+1}$ has coordinates $(x_1, y_1, \dots, x_n, y_n, z)$, such that

$$T\phi(\frac{\partial}{\partial z}) = R_\alpha \text{ and } \phi^{-1}(U \cap \partial B) = \{\, q \in V \mid z(q) = 0\,\}.$$

This means that $\phi^*\alpha = dz + \beta$ for some 1 form β on $\mathbb{R}^{2n}$ satisfying $\beta_0 = 0$. Consider the family of 1-forms defined by

$$\alpha_t := dz + (1-t)\beta + t\lambda \text{ for } t \in [0, 1].$$

[1] Recall that we allowed disks to have piecewise smooth boundary. So, $D \subset \mathbb{R}^{2n}$ is a compact domain, containing the origin with piecewise smooth boundary and star-shaped with respect to L.

Now, following the proof of the Gray stability theorem, see for example Theorem 2.2.2 in [5], we note that $\alpha_t = \alpha$ at the origin for all $t \in [0,1]$ so that we can find a small open neighborhood V' of the origin where α_t is a contact form for all t. Hence, we can find an isotopy ψ_t, fixing the origin, such that $\psi_1^* \alpha_1 = \alpha_0 := \phi^* \alpha$. The isotopy ψ_t constructed in the proof of the Gray stability theorem integrates a vector field tangent to $\ker \alpha_t$, hence preserves transversality to the Reeb vectorfield. Let D be a small disk containing the origin in $\psi_1(\{ z = 0 \})$ then $\phi \circ \psi_1^{-1} : D \to \partial B$ is a gluing disk. □

Consider a contact shell (B, η) with a gluing disk $\iota : D \to \partial B$. We claim that the Reeb vectorfield R_α is transverse to $\iota(D)$. Indeed, suppose it is not, then we have $R_\alpha = T\iota(X)$ for some smooth vectorfield X on D. This implies that for all vector fields $Y = T\iota(X)$ tangent to $\iota(D)$ we have

$$0 = d\alpha(R_\alpha, Y) = d\alpha(T\iota(V), T\iota(X)) = d\iota^*\alpha(V, X) = d\lambda(V, X).$$

However, $d\lambda$ is non-degenerate implying $V = 0$ which is a contradiction because then $R_\alpha = 0$. Using transversality, the flow of R_α allows us to extend the embedding $\iota : D \to \partial B$ to a contact embedding

$$\Phi : D \times (-\epsilon, 0] \to B, \text{ with } \Phi^*\alpha = dz + \lambda,$$

and such that $\Phi|_{D\times\{0\}} = \iota$. This extension allows us to make the gluing to the connecting tube compatible with the contact structure near the boundary of the contact shell.

Consider two contact shells $(B_\pm, \eta_\pm)$ with gluing disks $\iota_\pm : D \to \partial B_\pm$ such that ι_+ preserves and ι_- reverses orientation. Extend them to contact embeddings

$$\begin{aligned} \Phi_+ : D \times (-\epsilon, 0] \to B_+ &\text{ with } \Phi_+^*\alpha_+ = dz + \theta, \\ \Phi_- : D \times [0, \epsilon) \to B_- &\text{ with } \Phi_-^*\alpha_+ = dz + \theta, \end{aligned} \tag{6.1}$$

satisfying $\Phi_\pm|_{D\times\{0\}} = \iota_\pm$.

To define the connecting tube, let $\ell > 0$ and pick a smooth function $\tau : [-\ell, \ell] \to \mathbb{R}_{\geq 0}$ such that $\tau(z) = 0$ for z near $\pm\ell$. Define

$$T := \{ (p, z) \in \mathbb{R}^{2n} \times [-\ell, \ell] \mid p \in L^{-\tau(z)}(D) \}.$$

The *abstract boundary connected sum* is defined to be the almost contact manifold

$$(B_+ \#_T B_-, \eta_+ \# \eta_-) := ((B_+, \eta_+) \sqcup (T, \ker(dz + \lambda)) \sqcup (B_-, \eta_-)) / \sim \tag{6.2}$$

identifying $\Phi_+(p, 0) \sim (p - \ell) \in T$ and $\Phi_-(p, 0) \sim (p, \ell) \in T$. Note that since $\Phi_\pm^*\alpha_\pm = dz + \lambda$ the gluing is compatible with the (almost) contact structures.

6.2 Abstract connected sum of circle model contact shells

A priori, taking the abstract connected sum of two circle models produces a contact shell. We show here that this contact shell is actually (equivalent to) a circle model. To do this we construct a contact Hamiltonians describing the connected sum.

The connected sum construction will only be used in the proof of Proposition 3.8. There, all the Hamiltonians involved will be special as in Definition 7.2 in the next chapter. In particular, these Hamiltonians are all defined over Δ_{cyl} and are spherically symmetric meaning that they are invariant under rotations in $\mathbb{R}^2$. The precise definition is as follows.

Definition 6.4. *A contact Hamiltonian* $K : \mathcal{O}p\,\Delta \times S^1 \to \mathbb{R}$ *is called* spherically symmetric *if it only depends on the coordinates* $u := \sum_{i=1}^{n-1} u_i$ *and* z*, where* $(z, u_1, \theta_1, \dots, u_{n-1}, \theta_{n-1})$ *denote the coordinates on* $\Delta \subset \mathbb{R}^{2n-1}$.

When we are considering a spherically symmetric contact Hamiltonian we will often commit a slight abuse of notation and write it as $K(u, z)$, instead of $K(z, u_1, \theta_1, \dots, u_{n-1}, \theta_{n-1})$. Since this simplifies the notation we will also make this assumption in this the rest of this chapter, i.e. Δ will always mean Δ_{cyl} and all the Hamiltonians are spherically symmetric. We note that with a little extra work all the results also hold for general contact Hamiltonians.

Consider a circle model shell $(B_{K,C}, \eta_{K,\rho,g})$ associated to a contact Hamiltonian (K, Δ). To take their connected sum we need to find suitable gluing disks. Luckily, every circle model has two canonical gluing disks. Let $w = (u_1, \theta_1, \dots, u_{n-1}, \theta_{n-1})$, $u = \sum_{i=1}^{n-1} u_i$, and let (w, z, v, θ) be coordinates on $\mathbb{R}^{2n+1}$, then the gluing disks are given by:

$$D_\pm := \{(w, v, \theta) \in \mathbb{R}^{2n} \mid u \le 1, v \le K(u, \pm 1)\} \subset \mathbb{R}^{2n}, \tag{6.3}$$

with embeddings $\iota_\pm : D_\pm \to (\partial B_{K,C}, \eta_{K,\rho,g})$ defined by

$$\iota_\pm(w, v, \theta) = (w, \pm 1, \rho^{-1}_{(w,\pm 1,\theta)}(v), \theta) \in \mathbb{R}^{2n+1}.$$

Observe that $K(u, \pm 1)$ is positive so that $D_\pm$ is well-defined. Furthermore, since $(w, \pm 1, \theta) \in \partial B_{K,C}$ we have $\partial_v \rho_{(w,\pm 1,\theta)} > 0$ and hence

$$\rho_{(w,\pm 1,\theta)} : [0, K(w, \pm 1, \theta) + C] \to [0, K(w, \pm 1, \theta)]$$

is a diffeomorphism and the definition of $\iota_\pm$ is well-defined. Lastly, observe that $\iota_\pm(0, 0, 0) = (0, \pm 1, 0, 0)$ which are the north and south pole of $B_{K,C}$ respectively.

Suppose we have two circle model shells $(B_{K_\pm}, \eta_\pm)$ associated to contact Hamiltonians $(K_\pm, \Delta)$. Assume that $E(u) := K_\pm(u, \pm 1)$ is well-defined and that all derivatives of $K_\pm$ in the z direction vanish at ± 1. That is, $K_+(u, +1) = K_-(u, -1)$ for $u \le 1$ and $\partial_z K(u, \pm 1) = 0$. It is easy to see that given any contact Hamiltonians (Δ, K) we can pick a Hamiltonian (Δ, K'), slightly smaller than K with respect to the domination relation these conditions.

Pick $\ell > 0$ and define a smooth function $\tau : [-\ell, \ell] \to \mathbb{R}_{\ge 0}$ such that $\tau(z) = 0$ for z near $\pm\ell$. Define the domain

$$\Delta \#_{\tau,\ell} \Delta := Z^{-1}_{1+\ell}(\Delta) \cup T_{\tau,\ell} \cup Z^{1}_{1+\ell}(\Delta) \subset \mathbb{R}^{2n-1}, \tag{6.4}$$

where

$$T_{\tau,\ell} := \{ (w, z) \in \mathbb{R}^{2n-1} \mid u \le e^{-\tau(z)}, |z| \le \ell \} \subset \mathbb{R}^{2n-1}. \tag{6.5}$$

We will drop τ from the notation whenever we take $\tau := 0$.

It follows from Lemma 8.5 below that $\Delta \#_{\tau,\ell} \Delta$ is contactomorphic to $\Delta \#_\ell \Delta$ which is easily seen to be star-shaped so we can use this as the domain for a contact Hamiltonian. Define the connected sum contact Hamiltonian $K_+ \#_\tau K_- : \Delta \#_{\tau,\ell} \Delta \to \mathbb{R}$ by

$$K_+ \#_\tau K_-(u, z) = \begin{cases} K_+ \circ Z_{1+\ell}(u, z) & \text{on } Z^{-1}_{1+\ell}(\Delta) \\ e^{-\tau(z)} E(u) & \text{for } (z, q) \in T_{\tau,\ell} \\ K_- \circ Z^{-1}_{1+\ell}(u, z) & \text{on } Z_{1+\ell}(\Delta) \end{cases} \tag{6.6}$$

which is smooth since we assumed the derivates of $K_\pm$ in the z direction to vanish at ± 1. It is easy to see that $K_+ \#_\tau K_-|_{\partial \Delta \#_{\tau,\ell} \Delta} > 0$ so that it is a well-defined contact Hamiltonian. Hence we get an associated circle model

$$(B_{K_+ \#_\tau K_-}, \eta_{K_+ \#_\tau K_-}).$$

The following Lemma is the key result in this section saying that this circle model is the result of applying the abstract boundary connected sum to the circle models (B_{K_+}, η_{K_+}) and (B_{K_-}, η_{K_-}).

Lemma 6.5. *The contact shell $(B_{K_+ \#_\tau K_-}, \eta_{K_+ \#_\tau K_-})$ is equivalent to the abstract connected sum $(B_{K_+} \#_T B_{K_-}, \eta_{K_+} \#_T \eta_{K_-})$ with tube*

$$T := \{ (w, v, \theta) \in \mathbb{R}^{2n+1} \mid u \leq e^{-\tau(z)}, v \leq e^{-\tau(z)} E(u) \}$$

and the connected sum is done at the north pole of B_{K_+} and the south pole of B_{K_-}.

The proof follows immediately from the definitions and Lemma 8.5.

6.3 Ambient boundary connected sum

In practise we always want to apply the connected sum construction to contact shells inside an ambient (almost) contact manifold. In order to do this we need to make sure that the connecting tube embeds in an (almost) contact way into the manifold containing the contact shells. We prove that this can be done for all contact shells so that it holds in particular for the circle model connected sum from the previous section.

Let $(M, \eta = (\alpha, \omega))$ be an almost contact manifold. Assume that we have contact shells $(B_\pm, \eta_\pm) \subset (M, \eta)$ equipped with gluing discs $\iota_\pm : \Delta \to \partial B_\pm$ where $\iota_\pm^* \alpha = \lambda$ and such that ι_+ preserves and ι_- reverses the orientation.

Definition 6.6. *A* connecting path γ *for contact shells $(B_\pm, \eta_\pm) \subset (M, \eta)$ with gluing disks $\iota_\pm : D \to \partial B_\pm$ is a smooth embedding $\gamma : [0, 1] \to \operatorname{Int} M$ satisfying*

(i) $\gamma(0) = \iota_+(0)$, $\gamma(1) = \iota_-(0)$, *and* $\gamma(t) \notin B_+ \cup B_-$ *for* $t \in (0, 1)$,

(ii) η *is a contact structure on* $\mathcal{O}p\,\Gamma$, *where* $\Gamma := \gamma([0, 1])$.

(iii) γ *is transverse to* $\ker \alpha$.

The next lemma shows that such paths can always be constructed.

Lemma 6.7. *Let $(B_\pm, \eta_\pm) \subset (M, \eta)$ be contact shells equipped with gluing discs $\iota_\pm : D \to \partial B_\pm$. If η is contact on $M \setminus (\operatorname{Int} B_- \cup \operatorname{Int} B_+)$, then there exists a connecting path γ.*

Proof. Pick any path $\gamma : [0,1] \to \operatorname{Int} M$ satisfying (i). Condition (ii) follows from the assumption that η is contact outside $B_- \cup B_+$.

The third condition can by satisfied since any path in a contact manifold is C^0-approximated by a transverse path. In the three dimensional case this follows from Theorem 3.3.1 in [5]. For arbitrary dimension this is a consequence of Theorem 14.2.2 in [4]. □

For our purpose it is not a problem to assume that η is contact on $M \setminus (\operatorname{Int} B_- \cup \operatorname{Int} B_+)$. Indeed, in the proof of Theorem 3.1 we apply Proposition 3.7 before Proposition 3.8, so that this holds whenever we want to take an ambient connected sum. However, if η is only almost contact, then it follows from Lemma 7.8 and the fact that the connected sum of two contractible manifolds is again contractible that we can homotope η to be contact on $\mathcal{O}p\,\Gamma$ relative to $\mathcal{O}p\,B_- \cup B_+$.

The following lemma shows that connecting paths allow us to take ambient connected sums.

Lemma 6.8. *Every neighbourhood $\left(B_+ \cup \mathcal{O}p\,\Gamma \cup B_-, \eta|_{B_+ \cup \mathcal{O}p\,\Gamma \cup B_-}\right)$ contains the image of an almost contact embedding of an abstract connected sum $(B_+ \#_T B_-, \eta_+ \#_T \eta_-)$.*

Proof. We need to show that for a suitable choice of $\tau : [-\ell, \ell] \to \mathbb{R}_{\geq 0}$, the connecting tube $T_{\tau,\ell}$ is contained in $\mathcal{O}p\,\gamma$.

As in Equation 6.1 we can use the flow of the Reeb vector field to extend the gluing disks $\iota_\pm : D \to \partial B_\pm$ to contact embeddings

$$\Phi_\pm : D \times (\mp\ell - \epsilon, \mp\ell + \epsilon) \to \mathcal{O}p\,\iota_\pm(D), \text{ satisfying } \Phi_\pm^*\alpha = dz + \lambda \text{ and } \Phi_\pm|_{D \times \mp\{\ell\}} = \iota_\pm.$$

Furthermore, we can assume that $\Phi_+^{-1}(\Gamma) = \{0\} \times [-\ell, -\ell + \epsilon)$ and $\Phi_-^{-1}(\Gamma) = \{0\} \times (\ell - \epsilon, \ell]$.

Recall we have a normal form for contact structures in the neighbourhood of a transverse curve, see for example Theorem 2.5.15 and Example 2.5.16 in [5]. Using this result we can, after picking some constant $C > 0$ large enough and possibly decreasing ϵ, extend the embeddings $\Phi_\pm$ to a contact embedding

$$\Phi : (D \times (-\ell - \epsilon, \ell + \epsilon)) \cup \left(L^{-C}(D) \times [\ell, \ell]\right) \cup (D \times (\ell - \epsilon, \ell + \epsilon)) \to M,$$

such that the image is contained in $\mathcal{O}p\,(\iota_+(D) \cup \Gamma \iota_-(D))$ and $\Phi(0 \times [-\ell, \ell]) = \Gamma$.

To find the required contact embedding to finish the proof, it suffices to pick $\tau : [-\ell, \ell] \to \mathbb{R}_{\geq 0}$ so that the tube T is contained in the domain of Φ. □

Chapter 7

Overtwisted disks

In this chapter we define overtwisted disks, overtwisted (almost) contact structures and show they have properties which are very useful for the proof of Theorem 3.1. The chapter is divided into two parts.
The goal of the first part, consisting of the first three sections, is to give a precise definition of overtwisted disks. Essentially, an overtwisted disk is defined to be a certain part of the boundary of a circle model shell (B_K, η_K), where the Hamiltonian K is "special" and smaller than a universal Hamiltonian K_{univ}.
In the first section we explain the significance of the universal contact Hamiltonian and it's meaning in the definition of overtwisted disks.
The second section deals with the precise definition of special contact Hamiltonians and the definition of overtwisted disks and overtwisted (almost) contact manifolds.
In the third section we compare this definition (for all dimensions) to the definition given in [3], for the three dimensional case.
The aim of the second part of the chapter, consisting of the last two sections, is to prove that in any (almost) contact manifold we can create as many overtwisted disks as we want.
In the fourth section we prove that on a contractible manifold any two almost contact structures are homotopic. This allows us to create an overtwisted disk in any almost contact manifold. Furthermore, we show that this procedure can be done inside an ambient manifold.
In the last section we show that an open neighborhood of an overtwisted disk contains a foliation by overtwisted disks, meaning that we have as many overtwisted disks as we want.

7.1 Intuition about overtwisted disks & universal contact Hamiltonians

Just as in the three dimensional case, the definition of overtwisted contact structures for all dimensions, which we give in the next section, is defined by specifying a model disk with a specific germ of a contact structure. Any disk with a contact germ contactomorphic to the model is then called overtwisted. The model overtwisted disk is defined to be a certain part of the boundary of a circular model shell (B_K, η_K), whose contact Hamiltonian K has to satisfy two properties. It is required to be special as defined in Definition 7.2 below and it has to be smaller than K_{univ} with respect to the domination relation. The aim of this section is to explain how the definition of an overtwisted disk arises and the role of K_{univ} in this definition.

In the previous chapter we have seen that there is a partial order $\leq$ on the set of all contact Hamiltonians compatible with the notion of domination of contact shells. As we explained there this means that if we are trying to solve the filling problem for a circle model (B_K, η_K), then this problem can be reduced to solving the filling problem for a smaller circle model $(B_{K'}, \eta_{K'})$ if we can find a $K' \leq K$. In other words, if K is large then we can find many different Hamiltonians K_i such that $K_i \leq K$ and solving the filling problem for any of these solves the filling problem for K. Hence we may expect that for large K our problem becomes easier.

We know that the proof of Theorem 3.1 splits in two parts. By the first part, Proposition 3.7, it is enough to solve the filling problem for finitely many circle models equivalent to $(B_{K_{univ}}, \eta_{K_{univ}})$ for some universal Hamiltonian K_{univ}. Considering our previous observation we should try to make this K_{univ} as large as possible with respect to the domination relation to make the problem of filling the shell $(B_{K_{univ}}, \eta_{K_{univ}})$ as easy as possible.

We will see later that the proof of Proposition 3.7 actually shows that we can homotope any almost contact structure to a genuine one up to finitely many circle model shells each described by a contact Hamiltonian K_i, $i = 1, \dots, L$, from a finite list. The universal Hamiltonian K_{univ} is then defined to be any Hamiltonian smaller than any of the K_i, $i = 1, \dots, L$. This shows two things. Firstly, there are many possible choices for K_{univ}, since any Hamiltonian smaller than the K_i, $i = 1, \dots, L$ will do. However, by our previous observation the smaller we choose K_{univ} the weaker the statement of Proposition 3.7 becomes.

Secondly, we can interpret K_{univ} as a border between the Hamiltonians which are bigger than K_{univ} for which we expect the filling problem to be solvable, and the ones which are smaller than K_{univ} and for which we expect the filling problem to be hard or impossible to solve. The statement of Proposition 3.7 can then be interpreted as saying that the Hamiltonians, describing the circle models where the almost contact structure is not contact, can always be chosen so big (i.e. greater or equal to K_{univ}) that we can solve the filling problem for them.

The second part of the proof of Theorem 3.1, which is Proposition 3.8, shows that the filling problem for a circular model (B_K, η_K) can be solved if somewhere in the manifold there is a circle model shell $(B_{K'}, \eta_{K'})$ modelled by a contact Hamiltonian K' which is special and smaller than K. By Giroux's theorem this can be relaxed to requiring only part of the boundary of the circle model $(B_{K'}, \eta_{K'})$ to be present.
We want to choose the definition of an overtwisted disk in such a way that if a manifold contains an overtwisted disk then we can homotope the almost contact structure to a contact structure. So the question is, what object should be present in the manifold in order to solve the filling problem. The answer follows immediately from the above considerations. Namely, it should be part of the boundary of a circle model (B_K, η_K) where K is special and smaller than K_{univ}. We will see in the next section that this is precisely how overtwisted disks are defined.

7.2 Special contact Hamiltonians & overtwisted disks

In this section we give a precise definition of overtwisted disks. As we stated in the previous section an overtwisted disk is a subset of the boundary ∂B_K of a circular model shell (B_K, η_K) associated to a special contact Hamiltonian. To define these special contact Hamiltonians we need the notions of a special function and a spherically symmetric contact Hamiltonian.

Special functions are defined as follows.

Definition 7.1. *A smooth function $k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is called* special *if $k(1) > 0$ and*

$$ak(\frac{x}{a}) < k(x), \text{ for all scalars } a > 1 \text{ and } x \in \mathbb{R}_{\geq 0}. \tag{7.1}$$

This condition implies that k has a zero in $(0, 1)$ and the y-intercepts of all tangent lines to the graph of k are negative. To see the first statement rewrite 7.1 as $k(\frac{x}{a}) < \frac{k(x)}{a}$. Taking the limit for $a \to \infty$ gives $k(0) < 0$ and since $k(1) > 0$ this means k has a zero in $(0, 1)$.
Differentiating 7.1 with respect to a and substituting $u := \frac{x}{a}$ gives

$$k(u) - uk'(u) < 0 \quad \text{for all } u \in \mathbb{R}_{\geq 0}. \tag{7.2}$$

Rewriting this as $\frac{k(u)}{u} < k'(u)$ gives the second claim.

Recall that a contact Hamiltonian $K : \mathcal{O}p\,\Delta \times S^1 \to \mathbb{R}$ is called spherically symmetric if it only depends on the coordinates (u, z) where $u = \sum_{i=1}^{n-1} u_i$. Using these notions we define special contact Hamiltonians as follows.

Definition 7.2. *A contact Hamiltonian $K : \mathcal{O}p\,\Delta_{cyl} \times S^1 \to \mathbb{R}$ is called* special *if K is spherically symmetric and there exists a $z_D \in (-1, 1)$ together with a special function $k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that:*

(i) $K(-z, u) = K(z, u)$ on $\{(z, u) \in \Delta_{cyl} : |z| \in \mathcal{O}p\{1\}\} \subset \mathbb{R}^{2n-1}$,

(ii) For any $\delta \in \mathcal{O}p\{1\}$ we have $K(u, z) \leq K(u, \delta)$ for all $u \in [0, 1]$ and $|z| \leq \delta$,

(iii) K *is non-increasing in the positive* z *direction for* $z \in \mathcal{O}p[-1, z_D] \subset [-1, 1]$,

(iv) $\begin{cases} \text{if } n = 1 : K(z_D) < 0 \\ \text{if } n > 1 : K(u, z) \geq k(u) \text{ with equality if } z \in \mathcal{O}p\{z_D\}. \end{cases}$

We denote such a special contact Hamiltonian by a pair (K, z_D).

Remark 7.3. *Note that a special contact Hamiltonian is in particular a contact Hamiltonian so it also satisfies* $K|_{\partial\Delta_{cyl} \times S^1} > 0$.

The first condition will be used when taking connected sums during the proof of Proposition 3.8, and has no geometric meaning. On the other hand, conditions (ii), (iii) and (iv) describe the way the characteristic foliation on overtwisted disks behaves, see Figure 7.2.

For the case $n = 1$ the conditions (i) and (iv) are need for the proof of Proposition 3.8. Conditions (ii) and (iii) are not needed for this proof, however they ensure that Definition 7.5 and Definition 7.6 are equivalent.

For the case $n > 1$ the situation is a little different. Here, all conditions are needed to prove the parametric analogue of Proposition 3.8. We will not do this and refer the interested reader to Proposition 5.3 in [2]. To prove Proposition 3.8 we use conditions (i), (ii) and (iv). That is, for our purpose we could remove condition (iii) from the above definition but we include it so our definition does not deviate from the one in [2].

The following example shows that special contact Hamiltonians exist, and that there are many of them.

Example 7.4. *For positive constants* $a, b, c \in \mathbb{R}_{>0}$, *satisfying* $b < 1$ *and* $c > \frac{a}{1-b}$, *define the (piecewise smooth) function* $k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *by*

$$k(x) := \begin{cases} c(x-b) - a & \text{if } x \geq b \\ -a & \text{if } x \leq b \end{cases}$$

see Figure 7.1.

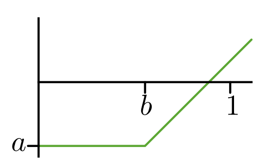


Figure 7.1: The graph of $k : \mathbb{R}_{\geq 0} \to \mathbb{R}$.

Since $c > \frac{a}{1-b}$ *we have* $k(1) > 0$ *and from the graph of* k *it is easy to see that it satisfies Equation 7.1. Hence, except from smoothness,* k *is a special function. Use this to define the (piecewise smooth) spherically symmetric contact Hamiltonian*

$$K(u, z) := \max(k(u), k(|z|)).$$

It is easy to see that $K(u, z)$ *satisfies conditions (i) and (ii) from Definition 7.2. Taking any* $z_D \in [-1, 0)$, *conditions (iii) and (iv) are also satisfied.*

Given any contact Hamiltonian $K : \mathcal{O}p\,\Delta_{cyl} \times S^1 \to \mathbb{R}$ *we can choose* a, b *and* c *in such a way that the above construction produces a special contact Hamiltonian smaller than* K*. Hence, there are plenty of special contact Hamiltonians.*
It remains to solve the smoothness problems of the above construction. However,this is easy since by a small perturbation one can always find a smooth functions C^0*-close to* k *and* K *which are special.*
One way to do this explicitly is to take an $\epsilon > 0$ *and let* $g : \mathbb{R}_{\geq 0} \to [0,1]$ *be a smooth function satisfying*

$$g(x) = \begin{cases} 1 & \text{if } x \geq b \\ 0 & \text{if } x \leq b - \epsilon \end{cases}$$

Define, $k_\epsilon(x) := g\left(c(x-b) - a + \epsilon\right) - (1-g)a$ *and*

$$K(u,z) := \frac{k_\epsilon(u)e^{Nk_\epsilon(u)} + k_\epsilon(|z|)e^{Nk_\epsilon(|z|)}}{e^{Nk_\epsilon(u)} + e^{Nk_\epsilon(|z|)}}.$$

For $\epsilon \to 0$ *and* $N \to \infty$ *these tend to the non-smooth* k *and* K *defined previously.*

To define overtwisted contact manifolds we first define model overtwisted disks. An overtwisted disk is then defined to be any disk with a germ of a contact structure contactomorphic to some model and a contact manifold is overtwisted if it contains such an overtwisted disk. More precisely, let (K, z_D) be a special contact Hamiltonian. Define a $2n$-dimensional disk

$$D_{K,C,z_D} := \{\, (w,v,\theta) \in \partial B_{K,C} \mid z(w) \in [-1, z_D] \,\} \subset B_{K,C}, \tag{7.3}$$

which is part of the boundary of the circle model shell $(B_{K,C}, \eta_{K,\rho,g})$ associated to K. Restrict $\eta_{K,\rho,g}$ to $\mathcal{O}p\,D_{K,C}$ and denote the resulting pair, consisting of a disk with a contact germ, by $(D_{K,z_D}, \eta_{K,\rho,g})$.

Definition 7.5. *An* overtwisted disk (D_{ot}, ξ_{ot}) *is a* $2n$*-dimensional disk with a germ of a contact structure such that there is a contactomorphism*

$$(D_{ot}, \xi_{ot}) \cong (D_{K,C,z_D}, \eta_{K,\rho,g}),$$

where K *is a special contact Hamiltonian satisfying* $K < K_{univ}$*.*
A $2n+1$*-dimensional contact manifold* (M, ξ) *is said to be* overtwisted *if it admits a contact embedding* $(D_{ot}, \xi_{ot}) \hookrightarrow (M, \xi)$ *of some overtwisted disk.*

7.3 The three dimensional case

The three dimensional case is of special interest since here we can draw pictures of the overtwisted disks. Furthermore we compare our definition to the usual one for three dimensions as given in [3].

First we want to see what special contact Hamiltonians look like for $n = 1$. In this case $\Delta_{cyl} = [-1, 1]$ and any spherically symmetric contact Hamiltonian depends only on z. Also, condition (iv) in Definition 7.2 states that any special contact Hamiltonian is somewhere negative. Using Lemma 5.7 this implies that the condition that the Hamiltonian K modelling an overtwisted disk should satisfy $K < K_{univ}$ is void since any somewhere negative Hamiltonian is minimal. So, consider a special contact Hamiltonian $K : [-1, 1] \to \mathbb{R}$ and a point $z_D \in [-1, 1]$ such that $K(z_D) < 0$. Using the results from Section 4.4 and noting that in the 3-dimensional case closed orbits of the characteristic foliation correspond to zero's of the Hamiltonian, we can visualize a three dimensional overtwisted disk as in Figure 7.2 below.

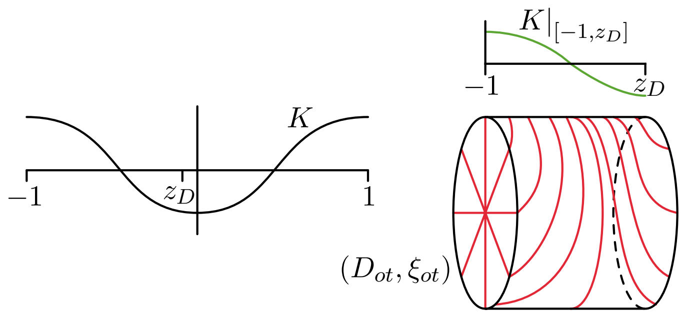


Figure 7.2: Illustration of a three dimensional overtwisted disk together with its characteristic foliation.

Recall that in three dimensions an overtwisted disk is usually defined to be the disk

$$D_\pi := \{ (z, u := r^2, \theta) \in \mathbb{R}^3 \mid z = 0, r \leq \pi \},$$

inside $\mathbb{R}^3$ with the standard overtwisted contact structure $\xi_{ot} := \ker(\cos r dz + r \sin r d\theta)$. Denote by (D_π, ξ_{ot}) this disk with the germ of the standard overtwisted contact structure. The characteristic foliation on D is given by straight lines in the radial direction. This gives the well known definition of an overtwisted contact manifold in three dimensions.

Definition 7.6. *A manifold (M, ξ), with $\dim M = 3$, is called overtwisted if it contains admits a contact embedding $(D_\pi, \xi_{ot}) \hookrightarrow M$.*

The following lemma shows that this definition is equivalent to Definition 7.5 with $n = 1$.

Lemma 7.7. *A three dimensional contact manifold (M, ξ) is overtwisted in the sense of Definition 7.5 if and only if it is overtwisted in the sense of Definition 7.6.*

Proof. The obvious problem comparing Definition 7.5 and Definition 7.6 is that the first is defined using a piecewise smooth disk while the latter is defined using a smooth disk. As in Equation 4.3 an open neighborhood of an overtwisted disk is contactomorphic to an open neighborhood of

$$\Sigma_{K,C,z_D} := \Sigma^1_{K,C} \cup \Sigma^2_{K,C,z_D} \subset \big(\mathbb{R}^3, \xi_{st}\big) \sqcup (\mathbb{R} \times T^*S^1, \xi_{st})/\sim,$$

where $\Sigma^{z_D}_{K,C} := \{ (z, p, q) \in \mathbb{R} \times T^*S^1 \mid z \in [-1, z_D], v = K(z, \theta) \} \subset \Sigma^2_{K,C}$, and $\Sigma^1_{K,C} := \{ (z, v, \theta) \in \mathbb{R}^3 \mid z = -1, 0 \leq v \leq K(z, \theta) \} \subset (\mathbb{R}^3, \xi_{st})$

For $\epsilon > 0$ consider the disk

$$\Sigma^1_{K,C,\epsilon} := \{ (z, v, \theta) \in \mathbb{R}^3 \mid 0 \le v \le K(x, \theta), z = \sqrt{\epsilon - \epsilon v} \} \subset (\mathbb{R}^3, \xi_{st}),$$

and define

$$\Sigma_{K,C,z_D,\epsilon} := \sigma^1_{K,C,\epsilon} \cup \Sigma^2_{K,C,z_D} \subset \left(\mathbb{R}^3, \xi_{st}\right) \sqcup (\mathbb{R} \times T^*S^1, \xi_{st})/\sim .$$

It is easy to see that $\Sigma_{K,C,z_D,\epsilon}$ is a smooth surface whose characteristic foliation is diffeomorphic to that of an overtwisted disk D_π as in Definition 7.6.

From Giroux's theorem we know that in the three dimensional case the germ of contact structure on a smooth surface is determined by the characteristic foliation. Furthermore, choosing $\epsilon > 0$ small enough we see that we can find a disk $\Sigma_{K,C,z_D,\epsilon}$ is an arbitrary small open neighborhood of Σ_{K,C,Z_D}. Combining these two observations it is clear that Definition 7.5 and Definition 7.6 are equivalent. □

7.4 Creating overtwisted disks

To show that we can create overtwisted disks in any almost contact manifold we start by proving that on a contractible manifold any two almost contact structures are homotopic. The idea is that using the contractibility the problem reduces to finding a homotopy between linear almost contact structures. The proof then boils down to the fact that the space of linear symplectic structures, which can be identified with the space of symplectic matrices, is path connected.

Lemma 7.8. *On a contractible manifold any two almost contact structures are homotopic.*

Proof. Let $\eta := ([\alpha], [\omega])$ and $\eta' := ([\alpha'], [\omega'])$ be two almost contact structures on a contractible manifold M of dimension $2n+1$. It is enough to find a homotopy between fixed representatives (α, ω) and (α', ω') of η and η' respectively.
View α as a map $\alpha : M \to T^*M$. Since M is contractible there exists a homotopy $H_t : M \to M$, $t \in I$, between the identity map and the constant map c_p for some $p \in M$. Looking at the composition $\alpha \circ H_t : M \to T^*M$ gives a homotopy between α and $\alpha_p \in T^*_pM$. Similarly, denoting as usual $\xi := \ker \alpha \subset TM$, we get a homotopy from $\omega : M \to \Lambda^2\xi$ to $\omega_p \in \Lambda^2\xi_p$. Moreover, since

$$(\alpha \circ H_t) \wedge (\omega \circ H_t)^n = (\alpha \wedge \omega^n) \circ H_t \ne 0,$$

the homotopy from (α, ω) to (α_p, ω_p) is through almost contact structures. Repeating the same argument for (α, ω') we have reduced to the case where we want to find a homotopy between two linear almost contact structures (α_p, ω_p) and (α'_p, ω'_p) on T_pM.

Viewing α_p and α'_p as elements of T^*_pM it is clear we can find a homotopy $F_t \in T^*M$, $t \in I$, between them. Indeed, any two non-zero elements of a vector space can be connected by a path not going through the origin. Consider the trivial vector bundle $T_pM \times I \to I$ and the subbundle ξ with fiber $\xi_t := \ker F_t \subset T_pM \cong T_pM \times \{t\}$. By definition ξ_0 and ξ_1 are endowed with linear symplectic structures defined by ω_p and ω'_p, respectively.
Now, ξ is a bundle over a contractible base, hence it is trivial. Under the isomorphism $\xi \cong \mathbb{R}^{2n} \times I$ we can represent a linear symplectic structure on $\xi_t \cong \mathbb{R}^{2n} \times \{t\}$ by a matrix in $\mathrm{Sp}(2n, \mathbb{R})$. Hence, finding a homotopy between ω_p and ω'_p reduces to finding a path in $\mathrm{Sp}(2n, \mathbb{R})$ connecting the matrices representing ω_p and ω'_p. By path connectedness of $\mathrm{Sp}(2n, \mathbb{R})$ this can be done. $\square$

To create overtwisted an overtwisted disk in an almost contact manifold (M, η) we want to find a ball $B \subset M$ and use the previous lemma to homotope $\eta|_B$ to an almost contact structure on B containing an overtwisted disk. However, the homotopy constructed in the previous lemma will also change the almost contact structure on ∂B. The next lemma shows that using a smooth bump function we can extend the homotopy to the entire manifold.

Lemma 7.9. *Any almost contact structure η on a manifold M is homotopic to an almost contact structure η' containing an overtwisted disk. Moreover, the homotopy can be taken to be fixed outside an arbitrary small ball.*

Proof. Let $D_{ot} \subset \mathbb{R}^{2n+1}$ with contact structure ξ_{ot} on $U := \mathcal{O}p\, D_{ot}$, be any overtwisted disk. Since the contact structure ξ_{ot} is defined on an open neighborhood of D_{ot} we can find closed balls $B_\pm$ such that $D_{ot} \subset \operatorname{Int} B_-$, $B_- \subset \operatorname{Int} B_+ \subset U$ and a smooth bump function $g : U \to I$ such that $g|_{B_-} = 1$ and $\operatorname{supp}(g) \subset \operatorname{Int} B_+$. If $p \in M$ is an arbitrary point then it is easy to see that we can embed B_+ into an (arbitrary small) neighborhood of p, and we identify B_+ with its image under this embedding. Applying lemma 7.8 we obtain a homotopy H_t on B_+ between $\eta|_{B_+}$ and $\xi_{ot}|_{B_+}$. Consider the homotopy H_{gt} and observe that this keeps the almost contact structure on ∂B_+ fixed. Hence, it can be extended (smoothly) to a homotopy on M between η and an almost contact structure containing (D_{ot}, ξ_{ot}). $\square$

7.5 Foliations by overtwisted disks

In the proof of Theorem 3.1 we use the connected sum construction from the next chapter to connected each circle model shell from Proposition 3.7 to a contact shell containing an overtwisted disk. Thus, we might need (finitely) many overtwisted disks while by definition an overtwisted contact manifold only needs to contain one overtwisted disk. In this section we show that this does not pose a problem since any neighborhood of an overtwisted disk contains a foliation by overtwisted disks. Thus, the existence of one overtwisted disk in a manifold implies the existence of infinitely many overtwisted disks in this manifold.
Intuitively, the idea of the proof is the following. Suppose the overtwisted disk is modelled by a special contact Hamiltonian K. We show that for $\delta > 0$ small enough $K - \delta$, will also be special. Since, the Hamiltonian basically describes the radius of the associated circle model shell we have that for varying δ the circle model associated to $K - \delta$ foliate an open neighborhood of (B_K, η_K). Hence, since the $K-\delta$ are special we get a foliation by overtwisted disks. Of course, there are some details to be taken care of and the results about conjugation of contact Hamiltonians from Chapter 5 will play an important role.

Remark 7.10. *We point out the difference of this argument with the one for the three dimensional case given in [3]. Here, instead of connecting each contact shell to an overtwisted disk the shells are instead connected summed to each other and afterwards connected to an overtwisted disk. This means that only one overtwisted disk is needed and we do not need to make use of foliations by overtwisted disks.*

This is possible since in the three dimensional case any somewhere negative contact Hamiltonian is special. This implies that the connected sum of two circle models associated to special contact Hamiltonians is again special. It follows from Definition 7.2 that is no longer true in higher dimensions. Hence, Proposition 3.8 can only applied if we connect each circle model modelled by a special contact Hamiltonian directly to an overtwisted disk.

For $\delta \neq 0$ consider the scaling contactomorphism $S_\delta : (\mathbb{R}^{2n-1}, \xi_{st}) \to (\mathbb{R}^{2n-1}, \xi_{st})$ given by

$$S_\delta(z, u_1, \dots, u_{n-1}, \theta_1, \dots, \theta_{n-1}) := \left(\frac{z}{\delta}, \frac{u_1}{\delta}, \dots, \frac{u_{n-1}}{\delta}, \theta_1, \dots, \theta_n\right).$$

We use this contactomorphism to construct from any special contact Hamiltonian $K : \Delta_{cyl} \times S^1 \to \mathbb{R}$ a family $K_\delta : \Delta_{cyl} \times S^1 \to \mathbb{R}$ of special contact Hamiltonians which are slightly smaller than K. To do this define a scaled version of the cylindrical domain by

$$\Delta_\delta := \{(z, u_1, \theta_1, \dots, u_{n-1}, \theta_{n-1}) \in \mathbb{R}^{2n-1} : |z| \leq \delta, u := \sum_{i=1}^{n-1} u_i \leq \delta\} \subset (\mathbb{R}^{2n-1}, \xi_{st}).$$

Observe that $S_\delta(\Delta_\delta) = \Delta_{cyl}$. Next, consider the family of functions $K_\delta : \Delta_\delta \times S^1 \to \mathbb{R}$ defined by

$$K_\delta := K - (\delta - 1). \tag{7.4}$$

For $\delta <$ these functions satisfy $K_\delta < K$. Consider a circle model $(B_{K,C}, \eta_{K,\rho,g})$ associated to K. Recall that C is chosen big enough so that $K + C > 0$. This means that for $\epsilon > 0$ small enough $K_\delta + C > 0$ for all $\delta \in [1-\epsilon, 1]$. Moreover, the circle models $(B_{K_\delta,C}, \eta_{K_\delta,\rho_\delta,g_\delta})$, for $\delta \in [1-\epsilon, 1]$, foliate part of the interior of $B_{K,C}$, see Figure 7.3.

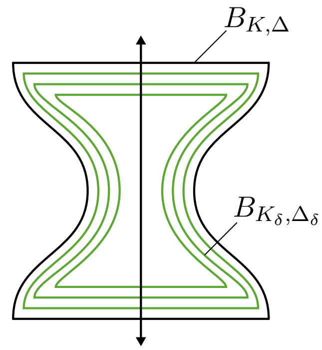


Figure 7.3: Foliation of (part) of the interior of $B_{K,C}$ by boundaries $\partial B_{K_\delta,\Delta_\delta}$, $\delta \in [1-\epsilon, 1]$, inducing a foliation by overtwisted disks.

The contact Hamiltonians K_δ are not special. This follows immediately from the fact that $\Delta_\delta \neq \Delta_{cyl}$, which is a bit silly, but also condition (iv) from Definition 7.2 does not need to be satisfied. However, we claim that the $(B_{K_\delta}, \eta_{K_\delta})$ are equivalent to circle models associated to special contact Hamiltonians $(\widetilde{K}_\delta, \Delta_\delta)$.To see this define $\widetilde{K}_\delta : \Delta_\delta \times S^1 \to \mathbb{R}$ by $\widetilde{K}_\delta := (S_\delta)_* K_\delta$ using the push forward construction from Chapter 5. Unfolding the definition of push forward we see

$$\widetilde{K}_\delta = (S_\delta)_* \left(K - (\delta - 1)\right) = \frac{K \circ S_\delta^{-1}}{\delta} + \frac{\delta - 1}{\delta}. \tag{7.5}$$

With $\epsilon > 0$ as above we claim that the family $\widetilde{K}_\delta$, $\delta \in [1-\epsilon, 1]$ consists of special contact Hamiltonians provided K is special.

Lemma 7.11. *Let $K : \Delta_{cyl} \times S^1 \to \mathbb{R}$ be a special contact Hamiltonian. For $\epsilon > 0$ small enough, the family $K_\delta : \Delta_{cyl} \times S^1 \to \mathbb{R}$, $\delta \in [1-\epsilon, 1]$, defined as above consists of special contact Hamiltonians.*

Proof. From Equation 7.5 it is clear that $\widetilde{K}_\delta$ satisfies condition $(i)-(iii)$ in Definition 7.2 if we take $\tilde{z}_D := \frac{z_D}{\delta}$. The interesting part is to find the correct special function $\tilde{k}_\delta$ associated to $\widetilde{K}_\delta$ and show it satisfies condition (iv). Let $k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be special function for K. It follows from condition (iv), equation 7.5 and the definition of $\tilde{z}_D$ that for $x \in \mathcal{O}p\tilde{z}_D$ we have

$$\tilde{k}_\delta(x) = \frac{k(\delta x)}{\delta} + \frac{\delta - 1}{\delta}.$$

We claim that this functions is special and therefore we can define $\tilde{k}_\delta$ by the previous equation for all $x \in \mathbb{R}_{\geq 0}$. To see this, let $a > 1$ and $x \in \mathbb{R}_{\geq 0}$ then,

$$a\tilde{k}_\delta(\frac{x}{a}) - \tilde{k}_\delta(x) = a\frac{k(\frac{x\delta}{a})}{\delta} + a\frac{\delta - 1}{\delta} - \frac{k(\delta x)}{\delta} - \frac{\delta - 1}{\delta} < (a-1)\frac{\delta - 1}{\delta} < 0,$$

where we used that k is special and hence $a\frac{k(\frac{x\delta}{a})}{\delta} < \frac{k(x\delta)}{\delta}$. Hence, $\tilde{k}$ satisfies Equation 7.1. Moreover, if ϵ is small enough we have $\tilde{k}_\delta(1) > 0$ so that $\tilde{k}_\delta$ is special and condition (iv) in Definition 7.2 is also satisfied. $\square$

With this we come to the main result of this section.

Proposition 7.12. *Every neighborhood of an overtwisted disk in a contact manifold contains a foliation by overtwisted disks.*

Proof. Consider an overtwisted disk $(D_{K,C,z_D}, \eta_{K,\rho,g})$ defined by a special contact Hamiltonian $K : \mathcal{O}p\,\Delta_{cyl} \times S^1 \to \mathbb{R}$. Let $U := \mathcal{O}p\,D_{K,C,z_D}$ be an open neighborhood of this overtwisted disk. Pick $\epsilon > 0$ small enough so that the disks

$$(D_{K_\delta,C,z_{D,\delta}}, \eta_{K_\delta,\rho_\delta,g_\delta}), \qquad \text{for } \delta \in [1-\epsilon, 1], \tag{7.6}$$

where K_δ is defined as in Equation 7.4 and $z_{D,\delta} := \frac{z_D}{\delta}$, is contained in U. These disks are defined as in Definition 7.3, and hence each disk $D_{K_\delta,C,z_{D,\delta}}$ is a subset of $(\partial B_{K_\delta,C}, \eta_{K_\delta,\rho_\delta,g_\delta})$. By Lemma 5.2 we get equivalences

$$(B_{K_\delta,C,}, \eta_{K_\delta,\rho_\delta,g_\delta}) \cong (B_{\widetilde{K}_\delta,\tilde{C}}, \eta_{\widetilde{K}_\delta,\tilde{\rho}_\delta,\tilde{g}_\delta}),$$

where the family $\{\widetilde{K}_\delta : \mathcal{O}p\,\Delta_{cyl} \times S^1 \to \mathbb{R}\}_{\delta\in[1-\epsilon,1]}$ is defined as in Equation 7.5. Moreover, by Lemma 7.11 we know these contact Hamiltonians are special and since they are smaller than K they are smaller than K_{univ}. Recall that by the definition of an equivalence we get contactomorphisms between the boundaries. Hence, the disks from Equation 7.6 are overtwisted since they are contactomorphic to model overtwisted disks. $\square$

Chapter 8

Filling circle models

Combining the results of the previous chapters we are now in a position to show that circle models associated to special contact Hamiltonians can be filled. The main goal of this chapter is to present the proof of this statement, which is Proposition 3.8. As we stated in Chapter 7 the definition of special contact Hamiltonians, and thus of overtwisted disks, is motivated by the proof of this proposition. Our second goal is to see how these conditions show up in the proof and interpret their meaning in a more geometric way.

In the first section we present the main idea's of the proof in the three dimensional case. As we have seen in Lemma 5.7 the domination relation is essentially trivial in this case some of the technical parts of the proof simplify.
The idea is to use a connected sum construction to reduce the filling problem for a ball to a filling problem for an annulus. If the boundary spheres of the annulus are modelled as the boundary spheres of contact shells the problem essentially reduces to conjugating the Hamiltonian of the inner sphere to be smaller than the Hamiltonian of the outer sphere. In the three dimensional case this can be done by stretching the domain of K which is just an interval.

There are two parts of this proof that need to be generalized for the higher dimensional case. Firstly, we need to find a higher dimensional analogue of scaling along the z-direction. Since in the higher dimensional case the contact structure on Δ is no longer trivial this requires some work. More precisely, scaling along the z direction in $(\mathbb{R}^{2n-1}, \xi_{st})$ (using a contactomorphism) can only be done at the cost of a scaling in the u direction. We can correct this scaling in the u direction at the cost of a twisting along the z-axis, which does not pose a problem due to rotational symmetry. We explicitly construct these contactomorphisms in the second section.

Secondly, since the domination relation is no longer trivial in the higher dimensional case, we need to do some extra work to make sure that after applying the conjugation, the Hamiltonian modelling the inner sphere is smaller than the one modelling the outer sphere. We will see in the third section that this will be true if we require the Hamiltonians to have some extra symmetries. The precise conditions basically follow from the description of the scaling contactomorphism and will explain the definition of special contact Hamiltonians we saw in Section 7.2.

In the last section we combine these ideas to give a precise proof of Proposition 3.8

8.1 Idea of the proof of Proposition 3.8

Assume we are in the case $\dim M = 3$ and we have a contact Hamiltonian (Δ, K) (assumed to be time independent) modelling a circle model $(B_K, \eta_K) \subset M$ which we want to fill.

Depending on the Hamiltonian K this problem is easy or hard. It could happen that K is everywhere positive. In this case we know from Remark 4.9 that (B_K, η_K) is equivalent to a solid shell and we are done. In general (hard) K might be somewhere negative and the previous argument does not hold. Before considering this case we first look at a different problem of trying to fill an annulus bounded by two circle models.

Suppose we have another contact Hamiltonian (Δ', K') satisfying $\Delta' \subset \operatorname{Int}\Delta$. By Lemma 5.10 we can define an almost contact annulus

$$(A := B_K \setminus B_{K'}, \eta_K|_A),$$

which is contact near the boundary $\partial A = \partial B_K \cup \partial B_{K'}$. The annulus is contact if and only if the inclusion map $i : B_{K'} \hookrightarrow B_K$ is a subordination map. Again by Lemma 5.10 we know that this can be achieved if $K' \leq K$. Hence, the filling problem for the annulus reduces to conjugating K' to be smaller than K.

Note that the filling problem for an annulus is lot easier than for a ball. Indeed, we see from Equation 5.1 that it is impossible to conjugate a somewhere negative Hamiltonian in a strictly positive one. By Remark 4.9, saying that a circle model is solid if and only if the defining Hamiltonian is strictly positive, this tells us that we cannot use conjugation to solve the filling problem for a ball. On the other hand it is possible to change the order of Hamiltonians with respect to the domination relation using conjugation. By the above discussion we know this is enough to solve the filling problem for an annulus.

The key observation is that by taking a connected sum with a contact ball (not necessarily a circle model), any filling problem for a circle model can be turned into a filling problem for an annulus. Indeed, assume there is some contact ball $(B, \xi) \subset (M, \eta)$ somewhere in our manifold and consider the connected sum $(B_K \# B, \eta_K \# \xi)$. Take a slightly smaller ball $\widetilde{B} \subset B$ and forget that we already have a contact structure on $B \setminus \widetilde{B}$. Consider the annulus $B_K \# B \setminus \widetilde{B}$ with the restriction of the almost contact structure, see Figure 8.1. The upshot is that to solve the (hard) filling problem for the circle model (B_K, η_K) it suffices to solve the (easier) filling problem for the annulus $B_K \# B \setminus \widetilde{B}$ with its induced almost contact structure.

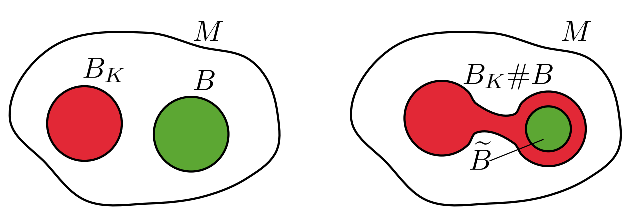


Figure 8.1: Reducing the filling problem for the circle model (B_K, η_K) to the filling problem for the annulus $B_K \# B \setminus \widetilde{B}$ with its induced almost contact structure.

Before continuing there are some technicalities we need to address. In order to solve the filling problem for the annulus we want the ball B (and hence also $\widetilde{B}$) to be a circle model for some Hamiltonian (Δ', K'). For the rest of the proof it will be convenient to assume that $(\Delta, K) = (\Delta', K')$ which we will do from now on. Under the hypothesis of Proposition 3.8 we can do this since we assume that $(\Delta', K') \leq (\Delta, K)$ so before taking the connected sum can first replace (B_K, η_K) by $(B_{K'}, \eta_{K'})$ using domination.
To take a connected sum $(B_K \# B_K, \eta_K \# \eta_K)$ we know from Chapter 6 that we need $K(p_-) = K(p_+)$, where $p_\pm$ are the boundary components of Δ which is just a closed interval in dimension three, i.e. $\partial\Delta = \{p_-\} \cup \{p_+\}$.
Furthermore, to solve the filling problem we need this Hamiltonian need to be somewhere negative. Indeed, from Equation 5.1 it is easy to see that conjugation can never change the sign of a Hamiltonian. Hence, if K is somewhere negative and K' everywhere positive we can never conjugate it to be smaller than K.

Remark 8.1. *Observe that so far we put three conditions on K. We want it to be time-independent, somewhere negative and satisfy $K(p_-) = K(p_+)$. These are precisely conditions (i) and (iv) from Definition 7.2, for the three dimensional case. This explains why we require the Hamiltonian to be special in the statement of Proposition 3.8. As we pointed out in the discussion following Definition 7.2 these are the only relevant conditions for the three dimensional case.*

Observe that these requirements do not make the statement of the proposition less general since given any Hamiltonian K we can find a somewhere negative Hamiltonian which is smaller with respect to the domination relation. In fact, we can find a contact Hamiltonian K satisfying all the conditions of Definition 7.2. Hence, we assume from now on that K is special. In the higher dimensional case the same statement holds for special contact Hamiltonians by Example 7.4.

For our argument it is important that $\widetilde{B}$ is a contact ball. Indeed, if this is not the case then, after filling the annulus, we still need to fill a ball $\widetilde{B}$ inside an ambient contact manifold. This was exactly our starting point so it would mean that we did not make any progress. On the other hand we said we want B and $\widetilde{B}$ to be circle models. But then, Remark 4.9 says that because of our assumptions on K (and hence also on K_ϵ) these circle models cannot be contact. At first sight it looks like we have conflicting conditions but it turns out that this can be resolved as follows.

Without loss of generality assume $\Delta = [-1, 1]$ and $K(z_D) < 0$ for $z_D \in (-1, 1)$. Define the Hamiltonian $(\Delta_\epsilon, K_\epsilon)$ by

$$\Delta_\epsilon := \{\, |z| \leq 1 - \epsilon \,\} \quad \text{and} \quad K_\epsilon := K - \epsilon,$$

where we assume that $\epsilon > 0$ is so small that $K|_{\mathcal{O}p\,\partial\Delta_\epsilon} > 0$. By Proposition 5.10 we can assume that, for $\epsilon > 0$ small enough, the inclusion

$$(B_{K_\epsilon,C}, \eta_{K_\epsilon,\rho_\epsilon,g_\epsilon}) \to (B_{K,C}, \eta_{K,\rho,g}),$$

is a subordination map so that we can define a $(2n+1)$-dimensional contact annulus (A, ξ_A) by

$$A := B_{K,C} \setminus B_{K_\epsilon,C}, \qquad \xi_A := \eta_{K,\rho,g}|_A\,.$$

Inside this contact annulus sits another contact ball $(B_\epsilon, \xi_\epsilon)$ given by

$$B_\epsilon := \{ (z, v, \theta) \in B_{K'} \mid z \in [-1, z_D], K_\epsilon(z) \le v \le K(z) \} \subset A \quad \text{and} \quad \xi_\epsilon := \xi_A|_{B_\epsilon}, \tag{8.1}$$

which is basically "half" the annulus, see Figure 8.1.

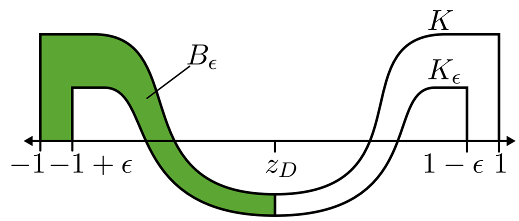


Figure 8.2: The ball B_ϵ (in green) contained in the annulus $A = B_{K,C} \setminus B_{K_\epsilon,C}$.

Since K is assumed to be special the intersection $\partial B_\epsilon \cap \partial B_K$ with its induced contact germ is an overtwisted disk. Conversely, by Theorem 2.18, any contact ball containing a disk with a contact germ as in Equation 7.3 contains a contact ball $(B_\epsilon, \xi_\epsilon)$ as in Equation 8.1.
The upshot is that this allows us to work with (B_K, η_K) instead of a general contact ball (B, ξ). To see this recall that an equivalence of contact shells is always relative to the boundary. Also, when taking a connected sum of a circle model (B_K, η_K) with a contact ball (B, ξ) containing an overtwisted disk $D_{ot} \subset \partial B$ the connecting tube intersects ∂B only at points inside D_{ot}. This follows immediately from the fact that gluing disk D_- from Equation 6.3 is contained in D_{ot}. Together, this implies that if we homotope the almost contact structure $\eta_K \# \xi_\epsilon$ on $B_K \# B_\epsilon$ relative to the boundary we can forget about the ambient manifold and its almost contact structure. In particular we can replace $(\widetilde{B}, \xi)$ by (B_K, η_K), because they are both ambient manifolds for $(B_\epsilon, \xi_\epsilon)$, and solve the filling problem for

$$(B_K \# B_\epsilon, \eta_K \# \xi_\eta) \subset (B_K \# B_K, \eta_K \# \eta_K),$$

as long as we keep everything fixed on $B_K \setminus \operatorname{Int} B_\epsilon$.
This also explains why, in the statement of Proposition 3.8, we only require B to contain an (overtwisted disk in its boundary instead of being a circle model.

Now let us continue with the filling problem. By the above considerations we can assume that we need to fill a connected sum contact shell $(B_K \# B_\epsilon, \eta_K \# \xi_\epsilon)$ embedded inside $(B_K \# B_K, \eta_K \# \eta_K)$, where $K : I \to \mathbb{R}$ is special hence in particular $K(z_D) < 0$ for $z_D \in \Delta$. As we described the idea is to "stretch" K using conjugation so it's domain covers points where $K \# K$ is negative.

To define the "stretching" contactomorphism used for the conjugation recall that we assumed that $\Delta = [-1, 1]$ and that the connecting tube of the connected sum $\Delta \# \Delta$ is given by $[-\ell, \ell]$. Then, $\Delta \# \Delta = [-2 - \ell, 2 + \ell]$. We can find a small interval $[z_-, z_+ := z_D] \subset \Delta$ containing z_D such that $K|_{[z_-, z_+]} < 0$. Define a (smooth family) of contactomorphisms $\phi_t : \mathbb{R} \to \mathbb{R}$ by

$$\phi_t(z) := \begin{cases} z + \ell + 1 & \text{for } z \in \mathcal{O}p\,[z_+, \infty) \\ \phi_t'(z) \ge 1 & \text{for } z \in [z_-, z_+] \\ z + (1 - 2t)(\ell + 1) & \text{for } z \in \mathcal{O}p\,(-\infty, z_-] \end{cases} \tag{8.2}$$

which restricts to a family of contact embeddings $\phi_t : \Delta \to \Delta\#\Delta$. It is easy to see that $(\phi_1)_* K_\epsilon \leq K\#K$, as desired.
Note that the family is fixed for $z \in [z_+, \infty)$ which is related to the idea of keeping the homotopy of almost contact structures fixed on parts of the annulus A not contained in $B_K\#B_\epsilon$. In terms of Hamiltonians the situation is depicted in Figure 8.3.

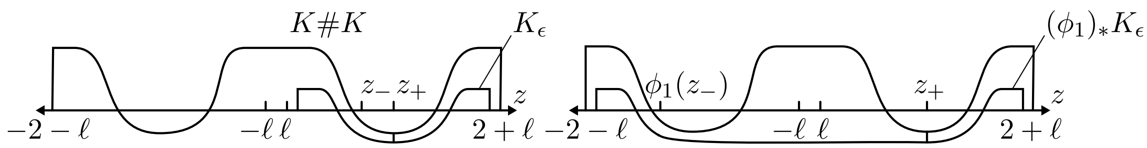


Figure 8.3: The effect of the 'streching' homotopy on K_ϵ using the push-forward construction.

Remark 8.2. *We assume here that over connecting tube of $\Delta\#\Delta$ we have that the connected sum contact Hamiltonian $K\#K$ is constant with value $K|_{\partial\Delta}$. That is, in the notation of Chapter 6 we assume $\tau = 0$. We know this is not true in general but we show in Lemma 8.5 below that we can always reduce to this case so we gloss over this technical detail here.*

To complete the proof we still need to show how the existence of a homotopy of conjugations as in Equation 8.2 implies the existence of an equivalence between $(B_K\#B_\epsilon, \eta_K\#\xi_\epsilon)$ and a solid shell. The idea is that by the results of Chapter 5 we can construct, for each $t \in [0,1]$, a contact embedding of circle models

$$\Phi_t : (B_{(\phi_t)_* K_\epsilon}, \eta_{(\phi_t)_* K_\epsilon}) \hookrightarrow (B_K\#B_K, \eta_K\#\eta_K).$$

The fact that $(\phi_1)_* K_\epsilon \leq K\#K$ implies that for $t = 1$ the embedding is a subordination map. The precise statement is given by the following lemma. We immediately proof this lemma in its full generality (not just for the three dimensional case) since the proof does not simplify for the thee dimensional case.

Lemma 8.3. *Let (Δ, K) and $(\widetilde{\Delta}, \widetilde{K})$ be two contact Hamiltonians and $\phi_t : \Delta \to \operatorname{Int}\widetilde{\Delta}$, $t \in I$ a smooth family of contact embeddings. Then, there exists a smooth family of almost contact embeddings*

$$\Phi_t : (B_{K,C}, \eta_{K,\rho,g}) \to \left(B_{\widetilde{K},\widetilde{C}}, \eta_{\widetilde{K},\tilde{\rho}_t,\tilde{g}}\right), \quad t \in I,$$

for suitable choices in the construction of the circle models. Furthermore, these embeddings satisfy the following properties:

(i) For $t \in [0,1]$ satisfying $(\phi_t)_* K \leq \widetilde{K}$ we can choose $\eta_{\widetilde{K},\tilde{\rho}_t,\tilde{g}}$ to be contact on $B_{\widetilde{K},\widetilde{C}} \setminus \Phi_t(B_{K,C})$.

(ii) If $\phi_t|_A = \phi_0|_A$ for all $t \in [0,1]$, then $\eta_{\widetilde{K},\tilde{\rho}_t,\tilde{g}} = \eta_{\widetilde{K},\tilde{\rho}_0,\tilde{g}}$ on the set

$$\{ (x, v, \theta) \in B_{\widetilde{K},\widetilde{C}} \mid x \in \phi_0(\Delta) \}.$$

Proof. We first construct $(B_{K,C}, \eta_{K,\rho,g})$ using any choice of C, ρ, g. Since $[0,1]$ and Δ are compact we can find C' such that $(\phi_t)_* K + C' > 0$ for all $t \in I$. Together with Lemma 5.2 this implies that we get family of almost contact embeddings

$$\Psi_t : (B_{K,C}, \eta_{K,\rho,g}) \to \left(B_{(\phi_t)_*K,C'}, \eta_{(\phi_t)_*K,\rho'_t,g'_t}\right), \quad t \in [0,1].$$

Furthermore, inspecting the proof of Lemma 5.2 we see that we can make sure the construction is smooth in the t parameter.

Using compactness again we can pick a $\widetilde{C} > 0$ satisfying $(\phi_t)_*K + C' < \widetilde{K} + \widetilde{C}$. Applying Lemma 5.10 with this $\widetilde{C}$ we get a family of almost contact embeddings

$$\iota_t : \left(B_{(\phi_t)_*K,C'}, \eta_{(\phi_t)_*K,\rho'_t,g'_t}\right) \to \left(B_{\widetilde{K},\widetilde{C}}, \eta_{\widetilde{K},\tilde{\rho}_t,\tilde{g}}\right), \quad t \in [0,1],$$

which as maps between sets are just inclusion maps. Note that $\tilde{g}$ can be chosen independent of t since the $\tilde{g}_t$ with smallest support can be used in the construction for all $t \in [0,1]$. Furthermore, note that if $(\phi_t)_*K \leq \widetilde{K}$ then we can choose ι_t to be a subordination map.

Composing these maps we get a smooth family of contact embeddings

$$\Phi_t : (B_{K,C}, \eta_{K,\rho,g}) \to \left(B_{\widetilde{K},\widetilde{C}}, \eta_{\widetilde{K},\tilde{\rho}_t,\tilde{g}}\right), \quad t \in [0,1]$$

which is a subordination map whenever $(\phi_t)_*K \leq \widetilde{K}$.

To see that condition (ii) can be satisfied, note that if $\phi_t = \phi_0$ then $(\phi_t)_*K = (\phi_0)_*K$. Hence, we can choose $\tilde{\rho}_t$ such that it agrees with $\tilde{\rho}_0$ on

$$\{\, (x, v, \theta) \in B_{\widetilde{K},\widetilde{C}} \mid x \in \phi_0(\Delta) \,\}.$$

$\square$

Apply this lemma to the family $\phi_t : \Delta \to \Delta \# \Delta$ defined in Equation 8.2 and for notational convenience denote by $\eta_t = \eta_{K\#K,\rho_t,g}$ the family of almost contact structures obtained from the lemma. Observe that Φ_0 is just the inclusion $\iota : (B_{K_\epsilon}, \eta_{K_\epsilon}) \to (B_K \# B_K, \eta_K \# \eta_K)$, of $(B_{K_\epsilon}, \eta_{K_\epsilon})$ into the right-hand factor of $(B_K \# B_K, \eta_K \# \eta_K)$. Furthermore, $\Phi_t = \Phi_0$ on $\mathcal{O}p\,\{\, (z, v, \theta) \in B_{K_\epsilon} \mid z \geq z_+ \,\}$, and Φ_1 is a subordination map. Equivalently, in terms of η_t this means that $\eta_0 = \eta_K \# \eta_K$,

$$\eta_t = \eta_K \# \eta_K, \qquad \text{on } \mathcal{O}p\,\iota\,(\{\, (z, v, \theta) \in B_K \mid z \geq z_+ \,\})\,,$$

and η_1 is contact on $B_K \# B_K \setminus \Phi_1(B_{K_\epsilon})$.

The family of almost contact structures η_t satisfies all the properties we want except one. That is, this family is not fixed on $\mathcal{O}p\,\partial\,(B_K \# B_\epsilon)$. In order to obtain this we need to use one more technical trick.

The family of contact embeddings induces a contact isotopy

$$\Psi_t : \left(\iota(B_{K_\epsilon}), \eta_K \# \eta_K|_{\iota(B_{K_\epsilon}}\right) \to (B_K \# B_K, \eta_K \# \eta_K),$$

as in Diagram 8.4.

Note that $\Psi_0 = \mathrm{Id}$ and $\Psi_t = \mathrm{Id}$ on $\mathcal{O}p\,\iota\,(\{\, (z, v, \theta) \in B_K \mid z \geq z_+ \,\})$. By the isotopy extensions theorem, for example Theorem 8.13 in [10], we can extend Ψ_t to an isotopy

$$\widehat{\Psi}_t : B_K \# B_K \to B_K \# B_K,$$

satisfying $\widehat{\Psi}_0 = \mathrm{Id}$ and $\widehat{\Psi}_t = \Psi_t$ on $\iota(B_{K_\epsilon})$. satisfying the following properties

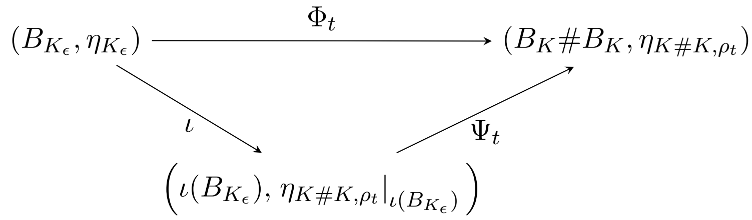


Figure 8.4: The defining relations for the contact isotopy Ψ_t.

(i) $\widehat{\Psi}_0 = \mathrm{Id}$,

(ii) $\widehat{\Psi}_t \circ \iota = \Phi_t : B_{K_\epsilon} \to B_K\#B_K$,

(iii) $\widehat{\Psi}_t = \mathrm{Id}$ on $\mathcal{O}p\, \iota\{\, z \in [z_+, 1]\,\} \cup \mathcal{O}p\, \partial B_K\#B_K$,

(iv) $\widehat{\Psi}_1\, (B_K\#B_K \setminus \mathrm{Int}\, \iota(B_{K_\epsilon})) = B_K\#B_K \setminus \mathrm{Int}\, \Phi_1(B_{K_\epsilon})$.

Define an almost contact structure on $B_K\#B_K$ by

$$\hat{\eta}_t := \widehat{\Psi}_t^*(\eta_t).$$

Note that it follows immediately from the definition that $\hat{\eta}_0 = \eta_K\#\xi$ and that $\hat{\eta}_1$ is contact on $B_K\#B_\epsilon$. Furthermore, we claim that $\hat{\eta}_t$ is fixed on $\mathcal{O}p\, \partial\, (B_K\#B_\epsilon)$. Indeed, a point in $\mathcal{O}p\, \partial\, (B_K\#B_\epsilon)$ is contained in one of the following regions

(i) $\mathcal{O}p\, \partial\, (B_K\#B_K)$ where $\widehat{\Psi}_t = \mathrm{Id}$ and $\eta_t = \eta_K\#\eta_K$ implying $\hat{\eta}_t = \hat{\eta}_0$ for all $t \in [0, 1]$,

(ii) $\mathcal{O}p\, \iota\, (\{\, z = z_+\,\})$ where $\widehat{\Psi}_t = \mathrm{Id}$ and $\eta_t = \eta_K\#\eta_K$ implying $\hat{\eta}_t = \hat{\eta}_0$ for all $t \in [0, 1]$,

(iii) $\mathcal{O}p\, \iota(\partial B_{K_\epsilon})$ where $\hat{\eta}_t = \Psi_t^*\eta_t = \iota_*\Phi_t^*\eta_t = \iota_*\eta_{K_\epsilon} = \eta_K\#\eta_K$ implying $\hat{\eta}_t = \hat{\eta}_0$ for all $t \in [0, 1]$.

We conclude that $\hat{\eta}_t$ gives an equivalence of contact shells between $(B_K\#B_\epsilon, \eta_K\#\xi_\epsilon)$ and $(B_K\#B_\epsilon,\, \hat{\eta}_1|_{B_K\#B_\epsilon})$ which is contact. This completes the argument for the three dimensional case.

We want to use the same ideas in the proof of the higher dimensional case. Both the proof of Lemma 8.3 and the isotopy extension argument do not depend on the dimension and so they immediately carry over. The only thing that we need to generalize is the stretching homotopy from Equation 8.2. As we explained in the introduction, there are two parts that need to be generalized.

1. In the three dimensional case Δ is just an interval inside $(\mathbb{R}, \xi_{st} = \ker dz)$. Since the contact structure is trivial any diffeomorphism is automatically a contactomorphism. In the higher dimensional case this is no longer true and we have to find a generalization of the stretching diffeomorphism which is also a contactomorphism.

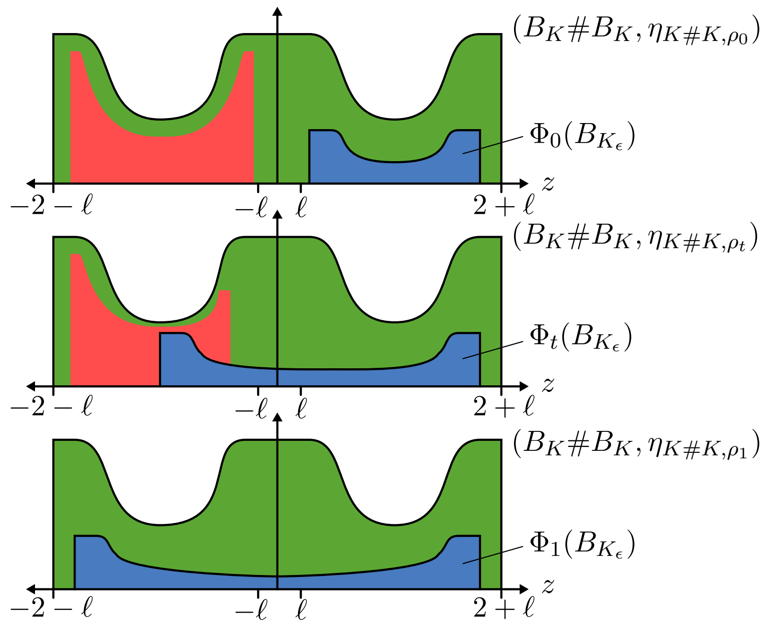


Figure 8.5: The images of the almost contact embeddings $\Phi_t(B_{K_\epsilon})$. The red and region regions indicate where (outside of $\Phi_t(B_{K_\epsilon})$) we have an almost contact and contact structure.

2. In the three dimensional case any somewhere negative function is minimal with respect to the domination relation. We used this in our proof to conclude that $(\phi_1)_* K \leq K \# K$. In the higher dimensional case the domination relation is more complicated and the contact Hamiltonian K needs to satisfy extra conditions (exactly the conditions for being a special function) so that after the conjugation it is small enough.

We carry out these generalizations in the next section.

8.2 Generalizing the stretching

Consider $(\mathbb{R}^{2n-1}, \xi_{st})$ and recall that the standard contact structure is given by the kernel of the 1-form

$$dz + \sum_{i=1}^{n-1} u_i d\phi_i.$$

As in the three dimensional case we want to change the z-direction using a diffeomorphism $h : \mathbb{R} \to \mathbb{R}$. It is easy to see that the naive guess for such a diffeomorphism of $\mathbb{R}^{2n-1}$, given by $(z, u_i, \phi_i) \mapsto (h(z), u_i, \phi_i)$, is not a contactomorphism. Instead, we need to add a scaling in the u_i directions.

Definition 8.4. *Let $h : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism and define a contactomorphism Φ_h of $(\mathbb{R}^{2n-1}, \xi_{st})$ by*

$$\Phi_h(z, u_i, \phi_i) := (h(z), h'(z) u_i, \phi_i),$$

called the transverse scaling *contactomorphism. It is easy to see that* $(\Phi_h)^{-1} = \Phi_{h^{-1}}$ *and* $\Phi_h^*\alpha_{st} = h'(z)\alpha_{st}$.

Geometrically this map should be thought of as manipulating the z-direction at the cost of a scaling in the u-direction. To visualize this, consider a function $f : \mathbb{R} \to \mathbb{R}_{>0}$ and observe that Φ_h gives a contactomorphism between compact domains,

$$\begin{aligned}\Phi_h : \{\, (z, u_i, \phi_i) \in \mathbb{R}^{2n-1} \mid u \leq f(z), z \in [a, b] \,\} &\xrightarrow{\cong} \\ &\{\, (z, u_i, \phi_i) \in \mathbb{R}^{2n-1} \mid u \leq (h' \cdot f)(h^{-1}(z)), z \in [h(a), h(b)] \,\}, \quad (8.3)\end{aligned}$$

where we use the usual notation $u = \sum_{i=1}^{n-1} u_i$.

From Equation 5.1 we can directly compute that the conjugation action of transverse scaling on a spherically symmetric contact Hamiltonian $K : \mathcal{O}p\,\Delta \times S^1 \to \mathbb{R}$, is given by

$$(\Phi_h)_* K(u, z) = h'(h^{-1}(z))K\left(h^{-1}(z), \frac{u}{h'(h^{-1}(z))}\right). \tag{8.4}$$

This contactomorphism allows us to show that for any circle model connected sum there is an embedding (as almost contact structures) into a circle model connected sum with a connecting tube of constant radius. That is, as we claimed in Remark 8.2, it suffices to proof Proposition 3.8 assuming that $\tau = 0$.

Lemma 8.5. *Let* $(\Delta = \Delta_{cyl}, K)$ *be a special contact Hamiltonian. Then for any choice of* τ *and* ℓ*, there exists an* $\ell' > \ell$ *and a contact embedding*

$$\Phi : \Delta \#_{\ell'} \Delta \to \Delta \#_{\tau,\ell} \Delta,$$

satisfying

(i) $\Phi = Z_{\pm(\ell-\ell')}$ *on* $\mathcal{O}p\,\{\, (z, u_i, \phi_i) \in \Delta\#_{\ell'}\Delta \mid \pm z \geq \ell' \,\}$,

(ii) $(\Phi_*(K\#K), \Phi(\Delta\#_{\ell'}\Delta)) \geq (K\#_\tau K, \Delta\#_{\tau,\ell}\Delta)$.

Proof. Recall that by definition $E(u) := K(u, \pm 1) > 0$ so that we can pick a constant $C > 0$ satisfying

$$0 < C < \frac{\min(E)}{\max(E)} \leq 1. \tag{8.5}$$

Choose $\ell' > 0$ satisfying

$$\ell' > \frac{1}{2C}\int_{-\ell}^{\ell} e^{\tau(z)} dz. \tag{8.6}$$

Since $\tau \geq 0$ and $C \leq 1$ we see $\frac{e^{\tau(z)}}{C} \geq 1$ implying $\ell' > \ell$. By Equation 8.6 we can pick a diffeomorphism $h : [-\ell', \ell'] \to [-\ell, \ell]$ satisfying

(i) $h'(z) = 1$ on $\mathcal{O}p\,\partial[-\ell', \ell']$,

(ii) $h'(z) \leq Ce^{-\tau(h(z))}$.

Condition (i) allows us to smoothly extend h as a translation outside $[-\ell', \ell]$ to a diffeomorphism $h : \mathbb{R} \to \mathbb{R}$. We claim that the associated transverse scaling contactomorphism $\Phi_h : (\mathbb{R}^{2n-1}, \xi_{st}) \to (\mathbb{R}^{2n-1}, \xi_{st})$ is the required contact embedding. To see this note that by Equation 8.3

$$\Phi_h(\Delta \#_{\ell'} \Delta) = \{ (z, u_i, \phi_i) \in \mathbb{R}^{2n-1} \mid u \le h'(h^{-1}(z)), z \in [-2-\ell, 2+\ell] \} \subset \Delta \#_{\tau,\ell} \Delta.$$

To check the order on the Hamiltonians it suffices to check them on the connecting tube since Φ_h is just a translation so that $\Phi_*(K\#K)$ and $K\#_\tau K$ agree everywhere else. It follows from Equation 8.4, Equation 8.5 and Condition (ii) on h that

$$(\Phi_h)_* E(u,z) = h'(h^{-1}(z)) E(\frac{u}{h'(h^{-1}(z))}) < e^{-\tau(z)} E(u) = (K\#_\tau K)(u,z),$$

finishing the proof. □

Let us continue with the main story. The problem with using Φ_h as the generalization of the stretching contactomorphism is that the scaling in the u-direction implies that $\Phi_h(\Delta) \not\subset \Delta \# \Delta$. To fix this we need to counteract the u-scaling which can be done at the cost of a twist in the ϕ_i direction.

Definition 8.6. *Let $g : \mathbb{R} \to R$ be a smooth function and $z_0 \in \mathbb{R}$, define a contactomorphism Ψ_{g,z_0} of $(\mathbb{R}^{2n-1}, \xi_{st})$ by*

$$\Psi_{g,z_0}(z, u_i, \phi_i) := \left(z, \frac{u_i}{1 + g(z)u}, \phi_i - \int_{z_0}^{z} g(s) ds \right),$$

called the twisting *contactomorphism. It is easy to see that $(\Phi_{g,z_0})^{-1} = \Psi_{-g,z_0}$ and $\Psi_{g,z_0}^* \alpha_{st} = \frac{1}{(1-g(z)u)} \alpha_{st}$.*

Each ϕ_i coordinate is $S^1 = \mathbb{R}/\mathbb{Z}$ valued. The choice of $z_0 \in \mathbb{R}$ is just the choice of a reference point which does not get twisted and the function g describes the amount of twisting in the ϕ_i-direction when moving in the z-direction. To visualize Ψ_{g,z_0} observe that it is a contactomorphism between compact domains,

$$\Psi_{g,z_0} : \{ (z, u_i, \phi_i) \in \mathbb{R}^{2n-1} \mid 1 + g(z)u > 0 \} \xrightarrow{\cong} \{ (z, u_i, \phi_i) \in \mathbb{R}^{2n-1} \mid 1 - g(z)u > 0 \}.$$

In particular, picking $f_j : \mathbb{R} \to \mathbb{R}_{>0}$, $j = 1, 2$ and defining $g(z) := \frac{1}{f_2(z)} - \frac{1}{f_1(z)}$ we get a contactomorphism between

$$\Psi_{g,z_0} : \{ (z, u_i, \phi_i) \in \mathbb{R}^{2n-1} \mid u \le f_1(z) \} \xrightarrow{\cong} \{ (z, u_i, \phi_i) \in \mathbb{R}^{2n-1} \mid u \le f_2(z) \} \tag{8.7}$$

The conjugation action of Ψ_{g,z_0} on a spherically symmetric contact Hamiltonian $K : \mathcal{O}p\,\Delta \times S^1 \to \mathbb{R}$ is given by

$$(\Psi_{g,z_0})_* K(u,z) = (1 - g(z)u)\, K \left(\frac{u}{1 - g(z)u}, z \right) \tag{8.8}$$

using Equation 5.1.

The generalization of the stretching contactomorphism is given by the composition of the transverse scaling and twisting contactomorphisms. For an orientation preserving diffeomorphism $h : \mathbb{R} \to \mathbb{R}$ and a point $z_0 \in \mathbb{R}$, define $g(z) := 1 - \frac{1}{h'(h^{-1}(z))}$. The composition $\Gamma_{h,z_0} := \Psi_{g,z_0} \circ \Phi_h$ is a contactomorphism of $(\mathbb{R}^{2n-1}, \xi_{st})$ which by Equation 8.3 and Equation 8.7 restricts to a contactomorphism of compact domains

$$\Gamma_{h,z_0} : \{ (z, u_i, \phi_i) \in \mathbb{R}^{2n-1} \mid u \leq 1, z \in [a,b] \} \xrightarrow{\cong} \{ (z, u_i, \phi_i) \in \mathbb{R}^{2n-1} \mid u \leq 1, z \in [h(a), h(b)] \}. \tag{8.9}$$

Remark 8.7. *In particular this shows why we defined Δ_{cyl} to be spherically symmetric since this implies $\Gamma_{h,z_0}(\Delta_{cyl}) \subset \Delta_{cyl} \# \Delta_{cyl}$ for $h : [-1,1] \to [-\ell-2, \ell+2]$.*
We also see that the length of Δ_{cyl} in the z-direction does not matter. That is, instead of taking $z \in [-1,1]$ any other interval $[a,b]$ would also work.
Similarly, the size of Δ_{cyl} in the u-direction, i.e. $u \leq 1$ seems arbitrary. However, we will use in the proof of Lemma 8.8 that $u \leq 1$ for all points in Δ_{cyl}. Therefore, the size of Δ_{cyl} in the u-direction can be any positive constant less or equal than 1.

The coordinate description of Γ_{h,z_0} is given by

$$\Gamma_{h,z_0}(z, u_i, \phi_i) = \left(h(z), \frac{h'(z)u_i}{1 + (h'(z)-1)u}, \phi_i - \int_{z_0}^{z} \left(1 - \frac{1}{h'(h^{-1}(s))} \right) ds \right).$$

Suppose we have a connected set $A \subset \mathbb{R}$ on which h is just a translation, i.e. $h(z) = z + \tau$ for $z \in A$. From the above description of Γ_{h,z_0} we see that if $z_0 h(A)$ then Γ_{h,z_0} is just a translation

$$\Gamma_{h,z_0} = Z_\tau \text{ on the set } \{ (z, u_i, \phi_i) \in \mathbb{R}^{2n-1} \mid z \in A \}. \tag{8.10}$$

The conjugation action of Γ_{h,z_0} on a spherically symmetric contact Hamiltonian $K : \mathcal{O}p\,\Delta \times S^1 \to \mathbb{R}$ is given by

$$(\Gamma_{h,z_0})_* K(u,z) = \tilde{h}(u,z) H \left(\frac{u}{\tilde{h}(u,z)}, h^{-1}(z) \right), \tag{8.11}$$

where we defined

$$\tilde{h}(u,z) := h'(h^{-1}(z)) - \big(h'(h^{-1}(z)) - 1 \big)\, u. \tag{8.12}$$

The main result of this section is the following lemma, giving the generalization of the stretching homotopy from Equation 8.2.

Lemma 8.8. *Let (Δ, K) be a special contact Hamiltonian and $\phi_t : \mathbb{R} \to \mathbb{R}$ the family of contactomorphisms from Equation 8.2. Then the family of contact embeddings*

$$\Gamma_t := \Gamma_{\phi_t, \ell+2} : \Delta \to \Delta \#_\ell \Delta, \quad t \in [0,1],$$

satisfies

(i) $\Gamma_0 = Z_{1+\ell}$,

(ii) $\Gamma_t = Z_{1+\ell}$ *on* $\mathcal{O}p\,\{\,(z, u_i, \phi_i) \in \Delta \mid z \in [z_+, 1]\,\}$ *for all* $t \in [0, 1]$,

(iii) $(\Gamma_1)_* K \leq K \# K$.

Proof. The first two conditions follow from the definition of ϕ_t and Equation 8.10.

Condition (iii) is the most important one. From Equation 8.2 we see that $\phi_t' := \frac{d\phi_t(z)}{dz} \geq 1$ for all $t \in [0, 1]$. Furthermore, for all $(z, u_i, \phi_i) \in \Delta$ we have $u \leq 1$. This implies, by Equation 8.12,

$$\tilde{\phi}_t(u, z) := \phi_t'(\phi_t^{-1}(z)) - \big(\phi_t'(\phi_t^{-1}(z)) - 1\big)\, u \geq 1.$$

Together with condition (iv) from Definition 7.2 this implies that for $z \in \phi_t([z_-, z_+])$ we have

$$(\Gamma_t)_* K(u, z) = (\Gamma_t)_* k(u, z) = \tilde{\phi}_t(u, z) k\left(\frac{u}{\tilde{\phi}_t(u, z)}\right) \leq k(u).$$

Note that this equation in the motivation for condition (iv) in Definition 7.2 and for the way we defined special functions in Definition 7.1. In fact it is the only time we use this condition.

Hence, using Equation 8.11 we compute

$$(\Gamma_1)_* K(u, z) = \begin{cases} K(u, z + (1 + \ell)) & \text{if } z \in \mathcal{O}p\,[-2 - \ell, z_- - 1 - \ell] \\ \leq k(u) & \text{if } z \in [z_- - 1 - \ell, z_+ + 1 + \ell] \\ K(u, z - (1 + \ell)) & \text{if } z \in \mathcal{O}p\,[z_+ + 1 + \ell, 2 + \ell] \end{cases}$$

On the other hand,

$$(K \# K)(u, z) = \begin{cases} K(u, z + (1 + \ell)) & \text{if } z \in [-2 - \ell, -\ell] \\ E(u) & \text{if } z \in [-\ell, \ell] \\ K(u, z - (1 + \ell)) & \text{if } z \in [\ell, 2 + \ell] \end{cases}$$

By condition (ii) from Definition 7.2 we have

$$k(u) \leq K(u, z) \leq E(u),$$

which we use to conclude that $(\Gamma_1)_* K \leq K \# K$.

Again, this is the motivation for condition (ii) from Definition 7.2 and the only time we use it. □

8.3 Proof of Proposition 3.8

At this point the proof of Proposition 3.8 is just a matter of collecting the results from the previous sections. For completeness we recall the statement of the proposition.

Proposition. *Let* (Δ_{cyl}, K) *be a special contact Hamiltonian and* (B, ξ) *a contact ball (not a circle model) such that there is a contact embedding* $(D_K, \eta_K) \subset (\partial B, \xi)$*. Then, given any contact Hamiltonian* $K' \gtrsim K$ *the connected sum*

$$(B_{K'} \# B, \eta_{K'} \# \xi)$$

connecting the north pole of $B_{K'}$ *and the south pole of* $D_K \subset \partial B$*, is equivalent to a solid contact shell.*

Proof. Define

$$K_\epsilon := K - \epsilon, \quad \Delta_\epsilon := \{ (z, u_i, \phi_i) \in \mathbb{R}^{2n-1} \mid u \leq 1 - \epsilon, |z| \leq 1 - \epsilon \},$$

for $\epsilon > 0$ small enough so that K_ϵ is a special contact Hamiltonian.
Define a contact ball $(B_\epsilon, \xi_\epsilon)$ by

$$B_\epsilon := \{ (x, v, \theta) \in B_{K,C} \mid z(x) \in [-1, z_+], K_\epsilon(x, \theta) \leq v \leq K(x, \theta) \} \subset B_{K,C} \setminus \operatorname{Int} B_{K_\epsilon, C},$$

and $\xi_\epsilon := \eta_{K,\rho,g}|_{B_\epsilon}$ which is contact for $\epsilon > 0$ small enough. Note that we suppress the dependence of $(B_\epsilon, \xi_\epsilon)$ on the choices C, ρ and g.
Since there is an embedding $(D_K, \eta_K) \subset (\partial B, \xi)$ we can assume by picking $\epsilon > 0$ small enough that there exists an embedding $(B_\epsilon, \xi_\epsilon) \subset (B, \xi)$. Together with the fact that $(K_\epsilon, \Delta_\epsilon) \leq (K, \Delta) \leq (K', \Delta')$ this implies that it suffices to show that the contact shell $(B_K \# B_\epsilon, \eta_K \# \eta_\epsilon)$ viewed as a subset of $(B_K \# B_K, \eta_K \# \eta_K)$ is equivalent to a solid shell.
Apply Lemma 8.3 to the smooth family of embeddings Γ_t obtained from Lemma 8.8 and let

$$\Phi_t : (B_{K_\epsilon}, \eta_{K_\epsilon}) \to (B_{K\#K}, \eta_{K\#K,\rho t}), \quad t\ in[0,1]$$

be the resulting family of contact embeddings.
We finish the proof with the same argument as in the three dimensional case by noting that Ψ_t is the composition of an inclusion and an isotopy as in Diagram 8.4. This isotopy can be extended to an isotopy $\widehat{\Psi}_t : B_K \# B_k \to B_K \# B_K$. The exact same argument as before gives that

$$\hat{\eta}_t := \widehat{\Psi}_t^*(\eta_{K\#K,\rho t}),$$

gives the required equivalence of contact shells. ◻

Chapter 9

Saucers and Interval models

The aim of the following chapters is to show that any almost contact structure on a manifold M can be homotoped to be contact everywhere except in finitely many circle models which we know how to fill by Proposition 3.8.
The first step in this direction is to introduce another type of model shell, called semi-semi contact saucers. These are fibered balls (saucers) equipped with a special kind of contact shell structure (semi-contact structure).
Saucers are the type of holes in the contact structure which show up more or less naturally when reducing the global filling problem to a local one while circle models although harder to obtain are easier to work with when solving the local problem. Both models share the property that they are fiberwise contact. The main difference is that for circle models the fibers Δ have codimension two and are all equipped with ξ_{st}^{2n-1} while for saucers the fibers have codimension one and we have a contact structure on an open neighborhood of the fiber (not on the fiber itself). Furthermore, these contact structures do not need to equal ξ_{st}^{2n+1} but can be different for different fiber.

The precise definition of semi-contact structures is given in the first section and that of saucers in the second section.
In the third section we specify a special subclass of semi-contact saucers called regular semi-contact saucers. These play an important role in the following chapter and share various properties with circle models. In particular they are also modelled by a single function and satisfy a similar domination relation.
The main result of this chapter is that each regular semi-contact saucer dominates a circle model, which is stated in Proposition 9.21. To prove this we define another type of model shell, called interval models in the fourth section. Morally speaking this definition is just a variation on the definition of circle models by replacing S^1 everywhere by $I = [0, 1]$. These interval models are only used in the proof of Proposition 9.21 which is given in the fifth section. This is done by showing that each regular semi-contact saucer dominates an interval model which in turn dominates a circle model.

In summary, in order to turn an almost contact structure into a contact structure we go through three types of models, see Figure 9.1.

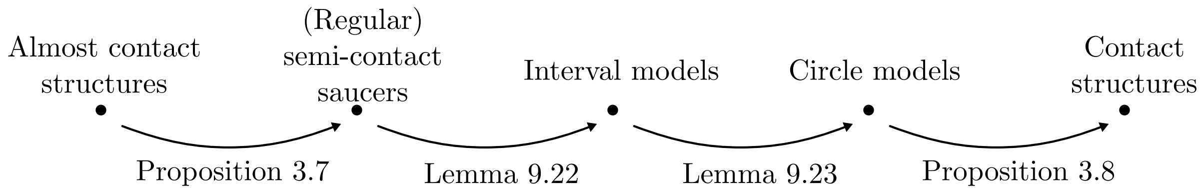


Figure 9.1: Note that Lemma 9.22 and Lemma 9.22 together combine into Proposition 9.21

9.1 Semi-contact structures

To interpolate between contact and almost contact structures we want to define a new type of structure which morally speaking is an almost contact which is contact in a fixed number (but not necessarily all) directions. To do this we first define so called semi-contact structures and we will see that these induce almost contact structures with the required properties.

Definition 9.1. *Consider a smooth fiber bundle* $F \to M \xrightarrow{\pi} B$ *with* $\dim M = 2n+1$*. A* semi-contact structure ζ *on* M *is a family* $\{\zeta_b\}_{b \in B}$*, depending smoothly on* b*, where* ζ_b *is a germ of a contact structure on* $F_b := \pi^{-1}(b)$*. More precisely, each* $\zeta_b = \ker \alpha_b$ *for a smooth* 1 *form* α_b *defined on* $\mathcal{O}p\, F_b \subset M$ *and we require that* α_b *depends smoothly on* b*.*

This notion is stronger than that of an almost contact structure and weaker than that of a contact structure.

Lemma 9.2. *A contact structure on* M *induces an semi-contact structure on* M *which in turn induces an almost contact structure on* M*.*

Proof. Suppose we have a contact structure ξ on M. Define a family of contact germs on F_b by $\zeta_b := \xi|_{\mathcal{O}p\, F_b}$. It is clear that this defines a semi-contact structure $\zeta = \{\zeta_b\}_{b \in B}$.

For the second statement assume we have an semi-contact structure ζ on M with $\zeta_b = \ker \alpha_b$ for a smooth family of 1-forms on $\mathcal{O}p\, F_b$. For $x \in \mathcal{O}pF_b$ denote by $\alpha_b(x)$ the differential form α_b at the point x. Define an almost contact structure $\eta_\zeta := [(\lambda, \omega)]$ on M by

$$\lambda(x) = \alpha_{\pi(x)}(x) \text{ and } \omega(x) = d\alpha_{\pi(x)}(x) \tag{9.1}$$

Note that α_b is a smooth 1-form on $\mathcal{O}p\, F_b$ so we can look at $d\alpha_b$, keeping b fixed giving us a family of 2-forms $\beta_b := d\alpha_b$. The above definition should be read as $\omega(x) = \beta_{\pi(x)}(x)$. Hence, in general $d\lambda \neq \omega$.

Since α_b is required to be smooth in b we see that λ and ω are smooth differential forms. Furthermore, it is easy to see that for each $x \in M$

$$(\lambda \wedge \omega^n)(x) = \alpha_{\pi(x)}(x) \wedge d\alpha^n_{\pi(x)}(x) \neq 0$$

so that (λ, ω) defines an almost contact structure. $\square$

Remark 9.3. *Observe that the restriction of $\lambda(x)$ to the fiber F_x equals $\alpha_{\pi(x)}(x)$. That is, a semi-contact structure looks like a contact structure in the fiber directions.*

We are mainly interested in a special case of this construction where the fiber is a closed manifold and the B is a closed interval with π the trivial projection.

Let Σ be a closed $2n$-dimensional manifold and let $\psi_s : \Sigma \to \mathbb{R}$, $s \in [0, 1]$ be a smooth family of (smooth) functions. Denote the graph of ψ_s by $\Gamma_s \subset \Sigma \times \mathbb{R}$ and $\Sigma_s := \Sigma \times \{s\} \subset \Sigma \times [0, 1]$. We can pick a smooth family of diffeomorphisms $\Psi_s : \mathcal{O}p\Sigma_s \to \mathcal{O}p\,\Gamma_s$ satisfying $\Psi_s|_{\Sigma_s} = \mathrm{Id} \times \psi_s$. Suppose that there exists a contact structure ξ on $\Sigma \times \mathbb{R}$. Then we can define a semi contact structure $\zeta = \{\zeta_s\}_{s\in[0,1]}$ on $\Sigma \times [0, 1]$ by

$$\zeta_s := \Psi_s^* \xi.$$

Definition 9.4. *A semi-contact structure of the above form is said to be of* immersion type *with defining functions ψ_s and contact structure ξ.*

The term immersion type is motivated by the fact that using Ψ_0 and Ψ_1 the boundary $\Sigma \times \{0\} \cup \Sigma \times \{1\} = \partial(\Sigma \times I)$ can be immersed inside $\Sigma \times \mathbb{R}$. Moreover, the contact germ on the boundary is just the restriction of ξ to the image of this immersion. We will see in Section 9.3 that regular semi-contact saucers are of immersion type. Hence, when viewing them as contact shells, the contact germ on their boundary can be entirely immersed in an ambient contact manifold. Recall that to describe the contact germ on the boundary of a circle model we have to take a quotient as in Equation 4.7. In particular, the contact germ cannot (entirely) be immersed in an ambient contact manifold.

9.2 Saucers

We want to define a semi-contact shell as a ball B in some ambient manifold M equipped with a semi-contact structure ζ on $\mathcal{O}p\,B$. However, since we can only define semi-contact structures on fibered manifold we need a more restrictive notion of a ball called a *saucer* which come equipped with a fibration.

Definition 9.5. *Let $D \subset M$ be a $2n$-dimensional embedded*[1] *disk in an ambient manifold M of dimension $2n+1$. Identify a neighborhood of D with $D \times \mathbb{R}$. Let $f_\pm : D \to \mathbb{R}$ be two smooth functions satisfying $f_- < f_+$ on $\mathrm{Int}\, D$ and whose ∞-jets coincide along ∂D. This defines a* saucer

$$B_{f_\pm,D} := \{\, (w, v) \in D \times \mathbb{R} \mid f_-(w) \le v \le f_+(w) \,\} \subset M \tag{9.2}$$

Every saucer $B_{f_\pm,D}$ comes with a family of disks

$$D_s := \{\, (w, p) \in D \times \mathbb{R} \mid p = (1 - s)f_-(w) + sf_+(w) \,\} \subset M \text{ for } s \in [0, 1].$$

The interiors $\mathrm{Int}\, D_s$ foliate $\mathrm{Int}\, B_{f_\pm,D}$ and the ∞-jets of the disks D_s all agree along their common boundary $S = \partial D_s$. Hence, we can view any saucer as a quotient space,

$$B_{f_\pm,D} = D \times I/S \times I.$$

[1]technically D is the image of the embedding

We use this identification to parametrize $B_{f_\pm,D}$ by coordinates $(w,s) := (1-s)f_-(w) + sf_+(w) \in B_{f_\pm,D}$. Note that in these coordinates $D_{s_0} = \{\,(w,s) \in B_{f_\pm,D} \mid s = s_0\,\}$. Furthermore, from this identification it is clear how to define a semi-contact structure on a saucer,

Definition 9.6. *A* semi-contact saucer *is a pair $(B_{f_\pm,D}, \zeta = \{\zeta_s\}_{s\in I})$ where $B_{f_\pm,D}$ is a saucer and $\{\zeta_s\}_{s\in I}$ a smooth family of contact structures ζ_s on $\mathcal{O}p\, D_s$ satisfying the following two conditions:*

(i) For all $s \in [0,1]$ the contact structures ζ_s coincide along $\mathcal{O}p\, S$;

(ii) Using ζ_0 and ζ_1 the almost contact structure η_ζ can be extended to a smooth almost contact structure on $\mathcal{O}p\, B_{f_\pm,D}$ as indicated below.

The second condition is a technicality needed to make sure that the induced almost contact structure η_ζ, as defined in Equation 9.1, fits smoothly in the ambient (almost) contact manifold.
To extend η_ζ to $\mathcal{O}p\, B_{f_\pm,D}$ note that the boundary of the saucer is given by $\partial B_{f_\pm,D} = D_0 \cup D_1$. It follows that ζ_0 and ζ_1 give us a contact structure on $\mathcal{O}p\, \partial B_{f_\pm,D}$. Hence, we can extend η_ζ by defining it to be equal to $\zeta_0 \cup \zeta_1$ outside $B_{f_\pm,D}$ as illustrated in Figure 9.2. Condition (ii) in the above definition ensures this extension is smooth.

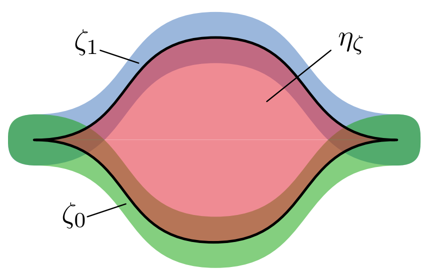


Figure 9.2: extension of the almost contact structure on a semi-contact saucer

If we extend $B_{f_\pm,D}$ to a slightly larger, closed ball B' then η_ζ is an almost contact structure on $\mathcal{O}pB'$ which is a contact structure near the boundary. That is, (B', η_ζ) is a contact shell. For notational reason we will usually just say that a semi-contact saucer $(B_{f_\pm,D}, \zeta)$ can be viewed as a contact shell. The reader should keep in mind that in this case we refer to the above extension construction.

Remark 9.7. *We might object that since a saucer is assumed to sit in an ambient manifold there is no room for the above extensions. For example, in the next chapter we will partition part of the ambient manifold by saucers, in this case the extensions would overlap. However, any equivalence of contact shells is assumed to fix the contact germ on the boundary. This means that if a semi-contact saucer viewed as a contact shell is equivalent to a solid one then we can also homotope the saucer (without its extension) to be contact.*

The upshot of viewing semi-contact saucers as contact shells enables us to talk about equivalence and domination, as in Definition 2.23 and Definition 2.24. The equivalence relation can already be defined for the semi-contact saucer without viewing it as a contact shell.

Definition 9.8. *An* equivalence of semi-contact saucers *between* (B, ζ) *and* (B', ζ') *is a diffeomorphism* $\phi : \mathcal{O}p\, B \to \mathcal{O}p\, B'$ *such that*

(i) $\phi^*\zeta'$ *is homotopic to* ζ *rel* ∂B*, where by definition* $(\phi^*\zeta')_s := \phi^*\zeta'_s$.

(ii) $\phi^*\zeta'_0 = \zeta_0$ *and* $\phi^*\zeta'_1 = \zeta_1$.

Observe that this notion of equivalence of semi-contact saucers is defined in such a way that when viewing them as contact shells (using the above extension) they are also equivalent as contact shells as in Definition 2.23.

9.3 Regular semi-contact saucers

We define here a distinguished subclass of semi-contact saucers called regular semi-contact saucers. To do this we first mimic the construction of circle models in the sense that we construct a semi-contact saucer (B_ϕ, ζ_ϕ) from a function $\phi : D \to \mathbb{R}$, called a saucer Hamiltonian. Then, requiring the defining saucer Hamiltonian to satisfy some extra conditions we obtain regular semi-contact saucers. The nice property of regular semi-contact saucers is that they dominate circle models, which we show in the last section.
We let $(q, p) \in \mathbb{R}^2$ denote the standard Cartesian coordinates and consider the strict contact manifold $(\mathbb{R}^{2n+1} = \mathbb{R}^{2n-1} \times \mathbb{R}^2, \xi_{st} = \ker \alpha_{st} + pdq)$. Define $\Pi := \{\, (x, q, p) \in \mathbb{R}^{2n+1} \mid p = 0 \,\} \subset \mathbb{R}^{2n+1}$. The construction of a Hamiltonian semi-contact structure in Definition 9.12 is a special case of the following construction.

Definition 9.9. *Consider any saucer* $B_{f_\pm, D}$ *as in Definition 9.5 and let* $\phi_\pm : D \to \mathbb{R}$ *be two functions satisfying*

(i) $J^\infty \phi_-|_{\partial D} = J^\infty \phi_+|_{\partial D}$,

(ii) $\phi_+ > \phi_-$ *on* $\mathcal{O}p\, \partial D \cap \operatorname{Int} D$.

As usual the saucer$B_{f_\pm, D}$ *comes with a foliation by disks* D_s*,* $s \in [0, 1]$ *giving coordinates* (w, s)*. By the assumptions on* $f_\pm$ *and* $\phi_\pm$ *we can find a smaller disk* $\widetilde{D} \subset D$ *such that* $f_+ > f_-$ *and* $\phi_+ > \phi_-$ *on* $D \setminus \widetilde{D}$.
Introduce the notation $D_{[a,b]} := \bigcup_{s \in [a,b]} D_s$ *for* $[a, b] \subset [0, 1]$ *and*

$$\widetilde{D}_s := \{\, (w, s) \in D_s \mid w \in \widetilde{D} \,\}.$$

Pick smooth bump functions $g_\pm : B_{f_\pm, D} \to [0, 1]$ *satisfying*

(i) $g_+ = 1$ *on* $\mathcal{O}p\, \widetilde{D}_1 \cap B_{f_\pm, D}$ *and* $\operatorname{supp} g_+ \subset D_{[1-\epsilon, 1]}$,

(ii) $g_- = 1$ *on* $\mathcal{O}p\, \widetilde{D}_0 \cap B_{f_\pm, D}$ *and* $\operatorname{supp} g_- \subset D_{[0, \epsilon]}$,

for $0 < \epsilon < \frac{1}{2}$ *very small.*

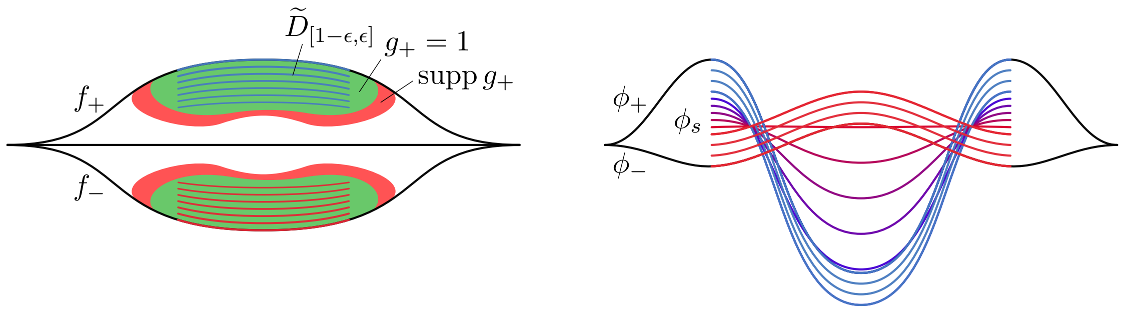


Figure 9.3: The foliation of $B_{f_\pm,D}$ by disks D_s and the graphs of ψ_s defining the immersion type semi-contact structure, $s \in [0,1]$.

Note that a point in $\mathcal{O}p\, D_s$ is of the form $(w, p+t)$ where $s = \frac{p - f_-(w)}{f_+(w) - f_-(w)}$ and $t \in \mathbb{R}$. With this notation, define a smooth family of diffeomorphisms $\Phi_s : \mathcal{O}p\, D_s \to \mathcal{O}p\, \Gamma_s$ by

$$\Phi_s(w, p+t) := (w, \phi_s(w,p) + t),$$

where $\phi_s : D_s \to \mathbb{R}$ is given by

$$\phi_s(w,p) := g_+(w,p)\Big(p - f_+(w) + \phi_+(w)\Big) + g_-(w,p)\Big(p - f_-(w) + \phi_-(w)\Big) + \Big(1 - g_-(w,p)\Big)\Big(1 - g_+(w,p)\Big)\Big((1-s)\phi_-(w) + s\phi(w)\Big) \quad (9.3)$$

and $\Gamma_s := \{ (w,p) \in D \times \mathbb{R} \mid w \in D, p = \phi_s(w) \}$ is the graph of ϕ_s. The semi-contact structure of immersion type $\zeta_{\phi_\pm, g_\pm} := \{\zeta_s\}_{s \in I}$ is defined by

$$\zeta_s := \Phi_s^*(\xi_{st}),$$

the pullback of the contact germ on Γ_s. We refer to $\zeta_{\phi_\pm, g_\pm}$ as the semi-contact structure defined by the functions ϕ_- and ϕ_+.

Remark 9.10. *The notation in Equation 9.3 is quite unwieldy. In practise when we want to describe a semi-contact structure of immersion type defined by two functions $\phi_\pm : D \to \mathbb{R}$ we say that ζ_s is obtained by restricting ξ on $D \times \mathbb{R}$ to an open neighborhood of the graph of the function*

$$\phi_s := (1-s)\phi_- + s\phi_+.$$

This is just to make the notation more readable and we really mean a family of functions defined as in Equation 9.3.

The analogue of a contact Hamiltonian for saucers is given by the following definition.

Definition 9.11. *Let $D \subset \Pi$ be a $2n$-dimensional disk, possibly with piecewise smooth boundary and $\phi : D \to \mathbb{R}$ a smooth function satisfying*

(i) ϕ is positive on $\operatorname{Int} D \cap \mathcal{O}p, \partial D$,

(ii) the ∞-jet of ϕ vanishes on ∂D, i.e. $J^\infty \phi|_{\partial D} = 0$.

The pair (ϕ, D) *is called a* saucer Hamiltonian*.*

The associated semi-contact saucer is defined as follows.

Definition 9.12. *Let* (ϕ, D) *be any saucer Hamiltonian. Pick a smooth function* $F : D \to \mathbb{R}$ *compactly supported in* $\operatorname{Int} D$ *such that* $\phi + F|_{\operatorname{Int} D} > 0$*.*
Define a saucer

$$B_{\phi,F} := \{ (w, p) \in D \times \mathbb{R} \mid 0 \leq p \leq \phi(w) + F(w) \} \subset \mathbb{R}^{2n+1}.$$

As before, the interior of this saucer is foliated by disks

$$D_s := \{ (w, p) \in D \times \mathbb{R} \mid p = s\,(\phi(w) + F(w)) \} \subset B_F$$

and we will parametrize $B_{\phi,F}$ *with coordinates* $(w, s) := s(\phi(w) + F(w))$*.*
The semi-contact structure of immersion type $\zeta_{\phi,g_\pm}$ *is defined by applying the construction in Definition 9.9 with* $f_- = 0$*,* $f_+ = \phi + F$*,* $\phi_- = 0$*,* $\phi_+ = \phi$ *and with* $\operatorname{supp} F \subset \widetilde{D}$*. The pair* $(B_{F,\phi}, \zeta_{\phi,g_\pm})$ *is called the* Hamiltonian semi-contact saucer *associated to* (ϕ, D)*.*

Up to a (canonical) almost contact isomorphism $(B_{\phi,F}, \zeta_{\phi,g_\pm})$ is independent of the choice of F, g and W, hence we can talk about the Hamiltonian semi-contact saucer (B_ϕ, ζ_ϕ).
The reason for using bump functions $g_\pm$ in the definition of ϕ_s is to make sure that ζ satisfies Condition (ii) of Definition 9.6. To see this condition holds with the above definition consider the immersion $\Phi : B_{\phi,F} \to \mathbb{R}^{2n+1}$ given by

$$\Phi(w, s) := (w, \phi_s(w)). \tag{9.4}$$

The fact that $\phi > 0$ on $\operatorname{Int} D \setminus \operatorname{supp} F$ together with the definition of ϕ_s implies that Φ is an embedding near the boundary of $B_{\phi,F}$. Hence, near the boundary $\eta_{\zeta_\phi} = \Phi^*\xi_{st}$ from which it is clear that Condition (ii) is satisfied. Furthermore, if $\phi|_{\operatorname{Int} D} > 0$ then Φ is an embedding everywhere so that $(B_\phi, \eta_{\zeta_\phi})$ viewed as a contact shell is solid.
In coordinates the almost contact structure $\eta_{\zeta_\phi} = ([\alpha], [\omega])$ is given by

$$\alpha(x, q, p) = \alpha_{st}(x) + \rho(x, q, p)dq, \qquad \omega(x, q, p) = d\alpha_{st}(x) + dp \wedge dq, \tag{9.5}$$

where we defined $\rho(x, q, p) := \phi_{s(x,q,p)}(x, q)$ using that $w = (x, q)$ and the coordinate s can be described as a function of (x, q, p). This expression follows immediately form the definition of ϕ_s and Equation 9.1.

Compare this with the coordinate expression of the almost contact structure on a circle model, which looks very similar. This comparison goes even further since the domination relation on Hamiltonian saucers viewed as shells is also encoded in a partial order on the set of saucer Hamiltonians.

Definition 9.13. *There is a partial order* $\leq$ *on the set of pairs* (D, ϕ) *where*

$$(D, \phi) \leq (D', \phi')$$

is defined to mean $D' \subset D$ *together with* *(i)* $\phi' \leq \phi|_{D'}$

(ii) $\phi|_{(\operatorname{Int} D)\setminus D'} > 0$.

This definition is similar to Definition 5.3 for the case $\widetilde{\Delta} = \emptyset$. The domination on saucer Hamiltonians is also compatible with domination of contact shells.

Lemma 9.14. *If $(\phi', D') \leq (\phi, D)$ then $(B_{\phi'}, \zeta_{\phi'})$ is dominated by (B_ϕ, ζ_ϕ). More specifically, given a saucer $(B_{\phi',F}, \zeta_{\phi'})$ there exists a saucer $(B_{\phi,F}, \tilde{\zeta})$ equivalent to $(B_{\phi,F}, \zeta_\phi)$ such that the inclusion*

$$(B_{\phi',F}, \zeta_{\phi'}) \subset (B_{\phi,F}, \tilde{\zeta})$$

is a subordination map.

Proof. Consider the semi-contact saucer $(B_{\phi',F}, \zeta_{\phi'})$, where $F : D' \to \mathbb{R}$ compactly supported in $\operatorname{Int} D'$ and such that $\phi' + F > 0$ on $\operatorname{Int} D'$. Extend F by zero to the whole of D denoting this new function also by F. Since $(\phi', D') \leq (\phi, D)$ we have $\phi + F > 0$ on $\operatorname{Int} D$. Hence $(B_{\phi,F}, \zeta_\phi)$ is a well defined saucer.

Define the set

$$\widetilde{B} := B_{\phi,F} \setminus B_{\phi',F} = \{(v, w) \in D \times \mathbb{R} : w \in D, \phi'(w) + F(w) \leq v \leq \phi(w) + F(w)\}.$$

Again, $\widetilde{B}$ is foliated by disks

$$\widetilde{D}_s := \{\, (w, v) \in D \times \mathbb{R} \mid p = (1-s)\phi(w) + s\phi'(w) + F(w) \,\}.$$

which induces coordinates $(w, s) := (1-s)\phi(w) + s\phi'(w) + F(w)$. Moreover, since $(\phi, D) \leq (\phi', D')$ we have an embedding

$$\widetilde{\Phi} : \widetilde{B} \to \mathbb{R}^{2n+1} \text{ where } \widetilde{\Phi}(w, s) = (w, (1-s)\phi'(w) + s\phi(w)).$$

This gives us a genuine contact structure $\sigma := \widetilde{\Phi}^*(\xi_{st}^{2n+1})$ on $\widetilde{B}$. Clearly we have $B_{\phi,F} = B_{\phi',F} \cup \widetilde{B}$, as sets. We define a semi-contact structure $\tilde{\zeta}$ on $B_{\phi,F}$ by

$$\tilde{\zeta} = \begin{cases} \zeta_{\phi'} & \text{on } B_{\phi',F} \\ \sigma & \text{on } \widetilde{B}. \end{cases}$$

Too see that $(B_{\phi,F}, \zeta_\phi)$ is equivalent to $(B_{\phi,F}, \tilde{\zeta})$ it suffices to show that ζ is homotopic to semi-contact structures to ζ_ϕ relative to $\tilde{\zeta}_1$ and $\tilde{\zeta}_0$, the contact germ on $\partial B_{\phi,F} = D_0 \cup D_1$. Note that all the semi-contact structures involved are graphical and to describe such a semi-contact structure $\zeta = \{\zeta_s\}_{s \in I}$ it is enough to describe the graphs corresponding to ζ_0 and ζ_1 since the rest follows from interpolating. In particular, to describe a homotopy of graphical semi-contact structures it suffices to describe homotopies of the graphs corresponding to ζ_0 and ζ_1. Observe that $\tilde{\zeta}_0 = \zeta_{\phi,0}$ and $\tilde{\zeta}_1 = \zeta_{\phi,1}$. We homotope $\Gamma_{\frac{1}{2}\phi}$ to $\Gamma_{\phi'}$ and by interpolation this gives a homotopy between ζ_ϕ and $\tilde{\zeta}$, see Figure 9.4 below. $\square$

Choosing the pair (ϕ, D) of a special form the above construction produces regular semi-contact saucers. Recall from Section 2.2 that outside the singular locus of its singular locus, the characteristic foliation $\mathcal{F}$ on a hypersurface $\Sigma \subset (M, \xi)$ induces a contact structure on $\Sigma/\mathcal{F}$.

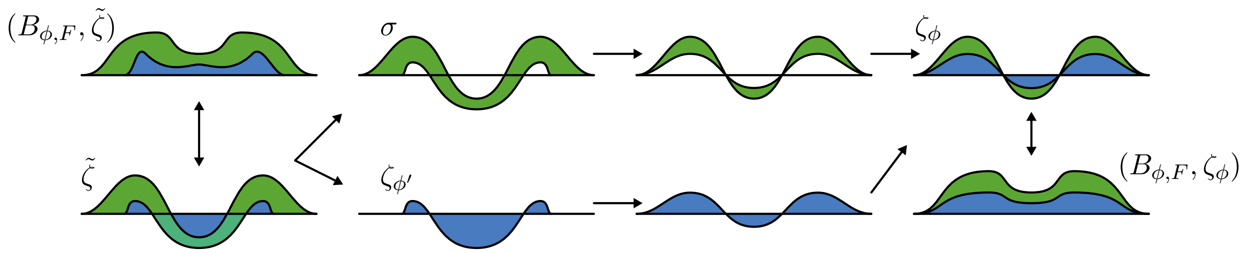


Figure 9.4: The equivalence between $(B_{\phi,F},\zeta_\phi)$ and $(B_{\phi,F},\tilde{\zeta})$ in steps.

Definition 9.15. *A saucer Hamiltonian (ϕ, D) is called* regular *if*

(i) the characteristic foliation $\mathcal{F}$ on D is diffeomorphic to the characteristic foliation on the standard round disk $D_{st} \subset \Pi$,

(ii) there exists a star-shaped disk $\Delta \subset D/\mathcal{F}$ with induced characteristic foliation by the transverse contact structure, and a "square" $S \subset D$ such that S is diffeomorphic to $\Delta \times I$ and the restriction of $\mathcal{F}$ to S is diffeomorphic to the product foliation on $\Delta \times I$,

(iii) we have that $\phi|_{\operatorname{Int} D \setminus S} > 0$.

Note that since $D \subset \Pi$ it also has the same induced transverse contact structure as the standard round disk in Π which by Giroux's theorem implies the existence of a contactomorphism $\psi : \mathcal{O}p\, D_{st} \to \mathcal{O}p\, D$.
The definition of a regular semi-contact saucer is now rather straightforward.

Definition 9.16. *A semi-contact saucers is* regular *if it is equivalent to a Hamiltonian semi-contact saucer defined by a regular saucer Hamiltonian (ϕ, D).*

Remark 9.17. *To avoid confusion we remark here that our definition for a regular semi-contact saucer is different from the one given in [2]. The upshot of our definition is that the proof of Proposition 9.21 simplifies. The trade off is that we have to do little more work in the next two chapters.*

In particular the class of semi-contact saucers satisfying the conditions in the following lemma are regular. We will encounter these in the following chapters.

Lemma 9.18. *Let $(B_{f_\pm,D},\zeta)$ be a saucer of immersion type where ζ comes from $(\mathbb{R}^{2n+1},\xi_{st})$ using a family of function $\phi_s : D \to \mathbb{R}$, $s \in [0,1]$. If $(\phi_1 - \phi_0, D)$ is a regular saucer Hamiltonian and ϕ_0, ϕ_1 are C^1-small, then $(B_{f_\pm,d},\zeta)$ is a regular semi-contact saucer.*

Proof. By assumption the characteristic foliation $\mathcal{F}$ is diffeomorphic to the characteristic foliation on the standard disk in Π. Define a disk

$$\widetilde{D} := \{(w,p) \in \mathbb{R}^{2n+1} \mid w \in D, p = \phi_0(w)\},$$

which is a subset of the graph of ϕ_0. The graph of ϕ_0 over D can also be described as the graph of a function $\tilde{h} : \widetilde{D} \to \mathbb{R}$ over $\widetilde{D}$.

If ϕ_0 is sufficiently C^1-small then the characteristic foliation $\widetilde{\mathcal{F}}$ and the induced transverse contact structure on $\widetilde{D}$ will also be diffeomorphic to the characteristic foliation and the induced transverse contact structure on the standard disk $D_{st} \subset \Pi$. Since this is also true for D we find contactomorphisms

$$\psi : \mathcal{O}p\, D_{st} \to \mathcal{O}p\, D \quad \text{and} \quad \widetilde{\psi} : \mathcal{O}p\, D_{st} \to \mathcal{O}p\, \widetilde{D}.$$

Composing gives us a contactomorphism $\Psi := \psi \circ \widetilde{\psi}^{-1} : \mathcal{O}p\, \widetilde{D} \to \mathcal{O}p\, D$ and my C^{-1}-smallness we can assume that the area between the graphs of ϕ_0 and ϕ_1 is contained in the domain of this map. Hence, under Ψ the area between $\widetilde{D}$ and the graph of $\tilde{h}$ gets mapped to the area between D and the graph of some function $h : D \to \mathbb{R}$.
Again, the C^1-smallness of ϕ_0 and ϕ_1 implies that h is C^1-close to $\phi_1 - \phi_0$. This implies that $h|_{\mathrm{Int}\, D \setminus S} > 0$. Hence, (D, h) is a regular saucer Hamiltonian and Ψ gives an equivalence between $(B_{f_\pm, D}, \zeta)$ and the regular semi-contact saucer defined by (D, h). □

9.4 Interval models

The definition of the interval model is very similar to the definition of a circle model. Let $\Delta \subset \mathbb{R}^{2n-1}$ be a compact star shaped domain and $K : \mathcal{O}p\, \Delta \times I \to \mathbb{R}$ be a smooth function satisfying

$$K|_{\partial\Delta \times I} > 0, \quad \text{and} \quad K|_{\Delta \times \{0\}} = K|_{\Delta \times \{1\}} > 0. \tag{9.6}$$

To emphasize the difference with a contact Hamiltonian we call a pair (K, Δ) as above an interval Hamiltonian. By the second condition we can also view K as a function $K : \mathcal{O}p\, \Delta \times S^1 \to \mathbb{R}$, where $S^1 = I/\partial I$. In particular any interval Hamiltonian is also a contact Hamiltonian.

Definition 9.19. *Let (Δ, K) be a contact Hamiltonian as above. Pick a constant $C > 0$ such that $\min K + C|_{\Delta \times I} > 0$. Use this to define a (piecewise smooth) ball*

$$B^I_{K,C} := \{\, (x, p, q) \in \mathbb{R}^{2n-1} \times T^*I \mid x \in \Delta, 0 \le p \le K(x, q) + C \,\}.$$

We divide the boundary $\partial B^I_{K,C}$ in three pieces:

$$\Sigma^I_{1,K,C} := \{\, (x, p, q) \in \mathbb{R}^{2n-1} \times T^*I \mid x \in \Delta, p = 0 \,\},$$
$$\Sigma^I_{2,K,C} := \{\, (x, p, q) \in \mathbb{R}^{2n-1} \times T^*I \mid x \in \Delta, p = K(x, q) + C \,\},$$
$$\Sigma^I_{3,K,C} := \{\, (x, p, q) \in \mathbb{R}^{2n-1} \times T^*I \mid x \in \partial(\Delta \times I), 0 \le p \le K(x, q) + C \,\},$$

Note that $\Sigma^I_{1,K,C}$ and $\Sigma^I_{2,K,C}$ are the "bottom" and "top" of the interval model while $\Sigma^I_{3,K,C}$ is the "side", see Figure 9.5 below.

To define an almost contact structure on this ball we need to make two more choices:

- *Pick a smooth function $\rho : \mathcal{O}p\, B^I_{K,C} \to \mathbb{R}$ satisfying the following conditions*

(i) $\rho(x,p,q) = p$ on $\mathcal{O}p\, \Sigma^I_{1,K,C}$,

(ii) $\rho(x,p,q) = p - C$ on $\mathcal{O}p\, \Sigma^I_{2,K,C}$,

(iii) $\partial_p \rho(x,p,q) > 0$ on $\mathcal{O}p\, \Sigma^I_{3,K,C}$.

- *Pick a closed subset* W^I *such that* $\mathcal{O}p\, \Sigma^I_{3,K,C} \subset W \subset \{\, (x,p,q) \in \mathbb{R}^{2n-1} \times T^*I \mid \partial_p \rho(x,p,q) > 0 \,\}$ *and a smooth function* $g : \mathbb{R}^{2n-1} \times T^*I \to [0,1]$ *satisfying*

  (i) $g|_{\mathcal{O}p\, \Sigma^I_{3,K,C}=1}$,

  (ii) $\operatorname{supp} g \subset W^I$.

Define an almost contact structure $\eta^I_{K,\rho,g} = (\alpha_\rho, \omega_{\rho,g})$ *on* $\mathcal{O}p\, B^I_{K,C}$ *by*

$$\alpha_\rho = \alpha_{st} + \rho dq, \qquad \omega_{\rho,g} := d\alpha_{st} + ((1-g)dp + g d\rho) \wedge dq.$$

The equivalence class (B^I_K, η^I_K) *of the pair* $(B^I_{K,C}, \eta^I_{K,\rho,g})$ *is called the interval model associated to* (K, Δ).

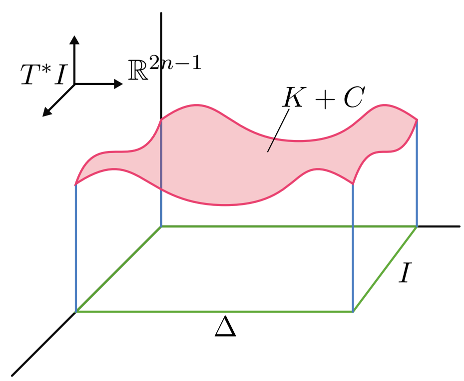


Figure 9.5: Illustration of B^I_K, the interval model shell associated to K.

As with the definition of circle models we have to check that the above definition is well-defined. This amounts to showing that the above definition actually defines a contact shell, that suitable choices for C, ρ, g and W^I exist and that up to equivalence the construction is independent of these choices. The proofs of these statements are similar to their circle model counterparts, so we state the following lemma without its proof.

Lemma 9.20. *The following statements are true:*

(i) *For any contact Hamiltonian* (K, Δ) *satisfying Equation 9.6 there exist* C, ρ, g *and* W^I *satisfying the conditions in Definition* 9.19.

(ii) *Every interval model* $(B^I_{K,C}, \eta^I_{K,\rho,g})$ *defined in Definition 9.19 is a contact shell.*

(iii) *For different choices of* C, ρ, g *and* W^I *the interval models are equivalent as contact shells.*

It is clear from the definition that the interval model shares many properties with the circle model. However, there is one important difference. Taking $C = 0$ the ball $B^I_{K,C}$ is not well-defined since K might be negative at some points. However, in this case the boundary $\partial B^I_{K,C}$ is an immersed submanifold in $\mathbb{R}^{2n-1} \times T^*I$. In fact, it is not hard to see that the restriction of η^I_K to $\mathcal{O}p\,\partial B^I_{K,C}$ is just the restriction of $\xi_{st} = \ker \alpha_{st} + pdq$. Hence, the contact germ on the boundary of an interval model can be immersed in an ambient contact manifold. Recall that regular semi-contact saucers also have this property.

We will see in the next section that the interval models form a bridge between regular contact saucers and circle models.

9.5 Domination of model shells

We now come to the main result of this chapter given by the following proposition.

Proposition 9.21. *For any regular semi-contact saucer (B, ζ), viewed as a contact shell, there exists a time-independent contact Hamiltonian (K, Δ) such that (B, ζ) dominates the circle model (B_K, η_K).*

To proof this proposition we need the following two lemma's.

Lemma 9.22. *For any regular semi contact saucer (B, ζ), viewed as a contact shell, there exists a Hamiltonian $K : \Delta \times I \to \mathbb{R}$ such that (B, ζ) dominates the interval model (B^I_K, η^I_K).*

Proof. This lemma is a direct consequence of the way we defined regular semi-contact saucers. Indeed, by the definition of a regular semi-contact saucer we can assume that (B, ζ) is of the form $(B_{\phi,F}, \zeta_\phi = (\alpha, \omega))$ as in Definition 9.16, associated to a regular saucer Hamiltonian (ϕ, D). It is easy to see that we can find a $\phi' \leq \phi$, as in Definition 9.13 satisfying $\phi'|_{\Delta\times\{0\}} = \phi'|_{\Delta\times\{1\}}$ and consider the dominated saucer, so we can assume that ϕ already has this property.
Recall there is a square $S \cong \Delta \times I \subset D$ such that $\phi|_{\operatorname{Int} D \setminus S} > 0$. Hence, we can choose $F : D \to \mathbb{R}$ used in the construction of $(B_{\phi,F}, \zeta_\phi)$ in such a way that $F|_{\Delta\times I} = C > 0$. Furthermore, the restriction

$$K := \phi|_{\Delta\times I}$$

satisfies the conditions in Equation 9.6 and can be used to define an interval model shell

$$(B^I_{K,C}, \eta^I_{K,\rho,g} = (\alpha_\rho, \omega_\rho)),$$

where we pick $C = F|_{\Delta\times I}$, which clearly satisfies $K + C > 0$.
We observe that the function ρ as defined in Equation 9.5 satisfies all the conditions in Definition 9.19 and we can assume we used this specific choice in the construction of the interval model. This implies that as sets

$$B^I_{K,C} \subset B_{\phi,F},$$

and using Equation 9.5 that

$$\alpha|_{B_{K,C}} = \alpha_\rho.$$

It is easily seen that the straight line homotopy between ω and $\omega_{\rho,g}$ for any choice of bump function g satisfying the conditions in Definition 9.19 is fixed near the boundary $\partial B_{\phi,F}$. Hence, after applying this homotopy we also have

$$\omega|_{B^I_{K,C}} = \omega_{\rho,g}.$$

Since $\phi|_{\mathrm{Int}\, D\setminus S} > 0$ it follows that η_{ζ_ϕ}, the almost contact structure induced by ζ_ϕ, is contact on $B_{\phi,F} \setminus B^I_{K,C}$ showing that (B, η_ζ) dominates $(B^I_{K,C}, \eta^I_{K,\rho,g})$. □

For the next lemma, recall that any Hamiltonian $K : \Delta \times I \to \mathbb{R}$ satisfying Equation 9.6 induces a Hamiltonian $K : \Delta \times S^1 \to \mathbb{R}$ which can be used to define a circle model.

Lemma 9.23. *Let $\Delta' \subset \mathrm{Int}\,\Delta$ be star-shaped domains and let $K : \Delta \times S^1 \to \mathbb{R}$ be such that $K|_{\Delta\times\{0\}} > 0$ and $K|_{\Delta\setminus \mathrm{Int}\,\Delta'} > 0$. Then the interval model (B^I_K, η^I_K) dominates the circle model $(B_{K'}, \eta_{K'})$, where $K' := K|_{\Delta'\times S^1}$.*

Proof. The idea of the proof is the following. Since $K' \leq K$ we would be done if we show that $(B^I_{K'}, \eta^I_{K'})$ is isomorphic to $(B_{K'}, \eta_{K'})$. Unfortunately, since cylinder coordinates (v, θ) are degenerate at $v = 0$ there does not even exists a diffeomorphism since it would map $\Sigma^I_{1,K'}$ to $\Delta \times \{0\} \subset \mathbb{R}^{2n-1} \times \mathbb{R}^2$. However, there does exist a diffeomorphism between a subset of $B^I_{K'}$ and a subset of $B_{K'}$ called the key-hole model $B^\epsilon_{K'}$, depicted in Figure 9.6, which we define below.

By this observation the proof consists of two steps. First we show that (B^I_K, η^I_K) dominates $(B^\epsilon_K, \eta^\epsilon_K)$. Second we show that (for $\epsilon > 0$ small enough) $(B^\epsilon_K, \eta^\epsilon_K)$ dominates the circle model $(B_{K'}, \eta_{K'})$.

Fix a choice of C, ρ, g and W^I as in definition 9.19 and observe that such a choice also satisfies all the conditions of 4.5. Hence, we can use this to define contact shell models

$$(B_{K,C}, \eta_{K,\rho,g}), (B_{K',C}, \eta_{K',\rho,g}), \text{ and } (B^I_{K,C}, \eta^I_{K,\rho,g}),$$

which we keep fixed from now on.

Since $K|_{\Delta\times\{0\}} > 0$ we can find $\epsilon > 0$ satisfying $K|_{\Delta\times(-\epsilon,\epsilon)} > \epsilon$. Define the keyhole model by

$$\begin{aligned} B^\epsilon_{K,C} &:= \{\, (x, v, \theta) \in B_{K,C} \mid \epsilon \leq v \leq K(x, \theta) + C, \epsilon \leq \theta \leq 1 - \epsilon \,\} \\ &= B_{K,C} \setminus (\{v < \epsilon\} \cup \{\theta \in (-\epsilon, \epsilon)\}), \end{aligned}$$

and $\eta^I_{K,\rho,g} := \eta_{K,\rho,g}|_{B^\epsilon_{K,C}}$, as in Figure 9.6. Note, that since we have the extra condition $\rho(x, p, q) = p$ on $\mathcal{O}p\,\Sigma^I_{1,K,C}$ which translates in the condition $\rho(x, v, \theta) = v$ on $\mathcal{O}p\,\{(x, 0, \theta) \in B_{K,C}\}$, the keyhole model is indeed a contact shell.

By the same equations we can cut out a subset of the interval model.

$$B^{\epsilon,I}_{K,C} := \{\, (x, p, q) \in B^I_{K,C} \mid \epsilon \leq p \leq K(x, q) + C, q \in ([-1, \epsilon] \cup [\epsilon, 1]) \,\},$$

with the restriction $\eta^{\epsilon,I}_{K,\rho,g} := \eta^{I}_{K,\rho,g}\Big|_{B^{\epsilon,I}_{K,C}}$. This is again a well defined contact shell.

It is easy to see that $(B^I_{K,C}, \eta^I_{K,C}$ dominates $(B^{\epsilon,I}_{K,C}, \eta^{\epsilon,I}_{K,\rho,g}$ which in turn is isomorphic to $(B^{\epsilon}_{K,C}, \eta^{\epsilon}_{K,\rho,g})$.

It remains to be shown that for $\epsilon > 0$ small enough the shell $(B^\epsilon_K, \eta^\epsilon_K)$ dominates $(B_{K'}, \eta_{K'})$. To do this we construct a contact isotopy of (B_K, η_K) "pushing" $B_K \setminus B^\epsilon_K$ into B^ϵ_K, as in Figure 9.6.

Let (y, z) denote the standard Cartesian coordinates on $\mathbb{R}^2$, so that we have coordinates $(x, y, z) \in \mathbb{R}^{2n-1} \times \mathbb{R}^2$. Since the ρ we fixed to model (B_K, η_K) has the property $\rho(x, v, \theta) = v$ on $\mathcal{O}p\,\{\,(x, 0, \theta) \in B_{K,C}\,\}$ we can find a $\delta > \epsilon > 0$ such that $\eta_{K,C}$ is contact on $\Delta \times \mathcal{O}p\,\{\,(y, 0) \in \mathbb{R}^2 \mid y \geq -2\delta\,\}$.

Pick a smooth function $k : \Delta \to [-\delta, \infty)$ satisfying $k(x) = -\delta$ on $\mathcal{O}p\,\partial\Delta$ and $k(x) = K(x, 0)$ on $\mathcal{O}p\,\Delta' \subset \operatorname{Int}\Delta$. Such a function exists since we assumed that $\Delta' \subset \operatorname{Int}\Delta$. Define

$$\Gamma_k := \{\,(x, y, 0) \in \Delta \times \mathbb{R}^2 \mid -2\delta \leq y \leq k(x)\,\},$$
$$\Gamma := \{\,(x, y, 0) \in \Delta \times \mathbb{R}^2 \mid -2\delta \leq y\,\}.$$

It is easy to see that there exists an isotopy $\{\psi_s\}_{s\in[0,1]}$ of Γ , satisfying

(i) $\psi_s(x, y, 0) = (x, g_s(x, y, 0), 0)$ for a smooth family of functions $g_s : \Gamma \to \mathbb{R}_{\geq 0}$, $s \in [0, 1]$,

(ii) $\psi_1(\Gamma_k) = \{\,(x, y, 0) \in \Delta \times R^2 \mid -2\delta \leq y \leq -\delta\,\}$,

(iii) $\psi_s = \operatorname{Id}$ on $\mathcal{O}p\,\{(x, y, 0) \in \Delta \times \mathbb{R}^2 \mid x \in \mathcal{O}p\Delta\,\}$.

This isotopy has the property that it preserves $\eta_{K,\rho,g}|_\Gamma = (\alpha_\rho|_\Gamma\,, \omega_{\rho,g}|_\Gamma) = (\alpha_{st}, d\alpha_{st})$. Using Theorem 2.6.13 in [5] this implies that we can extend ψ_s to a contact isotopy Ψ_s, $s \in [0, 1]$ of (B_K, η_K) and supported in $\mathcal{O}p\,\Gamma$.

If $\epsilon > 0$ is small enough then $\Psi_1(B_{K'}) \subset B^\epsilon_K$ so that the keyhole model $(B^\epsilon_K, \eta^\epsilon_K)$ dominates the circle model $(B_{K'}, \eta_{K'})$. □

With these two lemma's the proof of Proposition 9.21 consists of combining results.

Proof of Proposition 9.21. By Lemma 9.22 we find an interval model $(B^I_{\widetilde{K}}, \eta^I_{\widetilde{K}})$ dominated by (B, ζ). In turn we apply Lemma 9.23 to find a circle model $(B_{K'}, \eta_{K'})$ dominated by $(B^I_{\widetilde{K}}, \eta^I_{\widetilde{K}})$. Choosing a time-independent contact Hamiltonian $K \leq K'$ and applying Proposition 5.9 we get the required circle model (B_K, η_K) dominated by (B, ζ). □

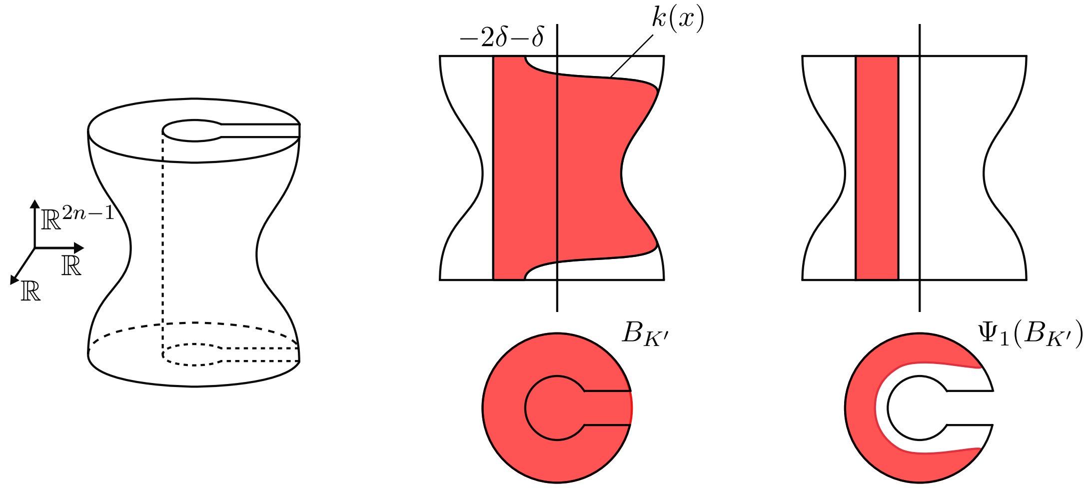


Figure 9.6: Figure 9.6: Left: keyhole model, Top right: The region $\{\, (x, y, 0)\Delta \times \mathbb{R}^2 \,\}$ together with the effect of ψ_s. Bottom right: Top view of the keyhole model and $B_{K'}$ together with (in red) the effect of Ψ_s.

Chapter 10

Reduction to saucers

This chapter contains the core argument in the proof of Proposition 3.7. We show that any almost contact structure is homotopic to an almost contact structure which is contact outside finitely many regular semi-contact saucers.

We split the argument in two parts. In the first section we use an h-principle argument to reduce the global problem to a local one. This is done by removing a closed set from the manifold to obtain an open manifold and using the fact that any almost contact structure on an open manifold is homotopic to a contact structure. We choose this closed set to be an annulus of the form $\Sigma \times [0,1]$, for $\Sigma \subset m$ a closed codimension one submanifold. The key observation is that this allows to use also the one parametric h-principle to obtain a semi-contact structure on the annulus which has a lot more structure than a general almost contact structure.

In the second section we study this semi-contact annulus. Using another well known property of contact structures, Gray stability, allows us to relate the contact structures on different fibers of the annulus and show that locally the semi-contact structure is of immersion type. This reduces the proof to studying the functions defining the semi-contact structure.

It turns out that on the parts of the annulus where the graphs of the functions defining the semi-contact structure are transverse to the Reeb vector field the induced almost contact structure can easily be made contact. The other parts we cut up in semi-contact saucers, which by compactness, are finite in number. Using a rather technical argument we can show that these semi-contact saucers are homotopic to regular ones, which we know how to fill using the results of the previous chapters.

10.1 Reduction to a local problem using h-principles

The goal of this chapter is to prove the following result.

Proposition 10.1. *Let M be a $(2n+1)$-dimensional manifold, $A \subset M$ a closed subset, and ξ_0 an almost contact structure on M which is contact on $\mathcal{O}p\, A \subset M$. Then, there exists finitely embedded saucers $B_i \subset M$, for $i = 1, \dots, N$ such that ξ_0 is homotopic relative to A to an almost contact structure ξ_1 satisfying* *(i)* ξ_1 *is a contact structure on* $M \setminus \bigcup_{i=1}^{N} B_i$*,*

(ii) the restriction of ξ_1 *to each saucer* B_i *is induced by a regular, semi-contact structure.*

Using Gromov's h-principle for contact structures on open manifold the proof of Proposition 10.1 immediately reduces to the proof of Lemma 10.4. We first state Gromov's h-principle and then show the reduction.

Let M be a $(2n+1)$-dimensional manifold, possibly with boundary, $A \subset M$ a closed subset and ξ_0 a contact structure on $\mathcal{O}p\, A \subset M$. Denote by $\mathfrak{Cont}(M, A, \xi_0)$ the space of contact structures on M that coincide with ξ_0 on $\mathcal{O}p\, A$. Similarly, denote by $\mathfrak{cont}(M, A, \xi_0)$ the space of almost contact structures on M that coincide with ξ_0 on $\mathcal{O}p\, A$. Observe that each $\xi \in \mathfrak{cont}(M, A, x_0)$ is contact on some open subset of M (containing $\mathcal{O}p\, A$) but might fail to be contact anywhere else. Since every contact structure is also an almost contact structure we have an inclusion map

$$j : \mathfrak{Cont}(M, A, \xi_0) \to \mathfrak{cont}(M, A, \xi_0).$$

Definition 10.2. *Let* M *be a manifold, possibly with boundary, and* $A \subset M$ *a closed set. Then, the pair* (M, A) *is called* relatively open *if for any point* $x \in M \setminus A$ *one of the following is true.*

(i) There exists a path, contained in $M \setminus A$*, connecting* x *with a boundary point of* M*.*

(ii) There exists a proper path $\gamma : [0, \infty) \to M \setminus A$ *with* $\gamma(0) = x$*.*

In particular, for any open manifold M the pair $(M, \emptyset)$ is relatively open.

Theorem 10.3. *Let* M *be a* $(2n+1)$*-dimensional manifold,* $A \subset M$ *a closed subset and* ξ_0 *a contact structre on* $\mathcal{O}p\, A \subset M$*. Suppose that* (M, A) *is relatively open. Then, the inclusion*

$$j : \mathfrak{Cont}(M, A, \xi_0) \to \mathfrak{cont}(M, A, \xi_0)$$

is a homotopy equivalence.

For a proof of this theorem we refer the reader to [8] and [9].

Using this theorem the proof of Proposition 10.1 follows from the following lemma.

Lemma 10.4. *Let* $\Sigma \subset M$ *be a closed submanifold of a manifold* M*, and* $\zeta^0 := \{\zeta_s^0\}_{s \in [0,1]}$ *a semi-contact structure on the annulus* $C := \Sigma \times [0, 1] \subset M$*. Then, there exist finitely many saucers* $B_i \subset C$*, for* $i = 1, \ldots, N$*, such that* ζ^0 *is homotopic relative to* $\{\zeta_s\}_{s \in \partial I}$ *to a semi-contact structure* ζ^1 *satisfying:*

(i) The almost contact structure induced by ζ^1 *is contact on* $C \setminus \bigcup_{i=1}^{N} B_i$*, and*

(ii) ζ^1 *is a regular semi-contact structure on* B_i*, for* $i = 1, \ldots, N$*.*

Proof of Proposition 10.1. Using a coordinate chart on M we can find an embedded annulus $C := S^{2n} \times I \subset M \setminus A$. Define the slightly smaller annulus $\widetilde{C} := S^{2n} \times [\frac{1}{4}, \frac{3}{4}] \subset C$. By the existence part of Theorem 10.3 we can find a homotopy ζ^t on $M \setminus \widetilde{C}$ interpolating between $\zeta|_{M\setminus\widetilde{C}}$ and a contact structure ζ' . To extend this homotopy to M, pick a function $g : M \to I$ such that $g = 1$ on $\mathcal{O}p\, M \setminus C$ and $g = 0$ on $\mathcal{O}p\, \widetilde{C}$. Then, the homotopy $\zeta^{g(x)t}$ is defined on M and interpolates between ζ and an almost contact structure $\tilde{\zeta}$ which is contact on $\mathcal{O}p\, M \setminus C$. Each sphere $S^{2n} \times s \subset C$, for $s \in [0, 1]$, has an open neighborhood diffeomorphic to $S^{2n} \times (-\delta, \delta)$ so by restricting $\tilde{\zeta}$ to these neighborhoods we get a 1-parameter family ζ^s, of almost contact structures on $S^{2n} \times (-\delta, \delta)$. Moreover, by picking δ small enough we have that ζ^0 and ζ^1 are contact. Apply the 1-parametric part of Theorem 10.3 to homotope ζ^s, fixing ζ^0 and ζ^1, to a family of contact structures.
To complete the proof apply Lemma 10.4. □

Remark 10.5. *Following the philosophy of the previous argument we might try to take a closed submanifold $\Lambda \subset M$ of codimension two and consider a closed set $\Lambda \times [0, 1] \times [0, 1] \subset M$ and apply the two parameter part of Theorem 10.3 with the idea that the resulting "codimension two semi-contact structure" is even nicer than a semi-contact structure.*

The reason that this does not work is that the proof of Lemma 10.7 and hence of Lemma 10.4 relies on the Gray stability theorem. This theorem is one parametric in nature and there is no two parametric analogue. Hence, one is exactly the right codimension.

10.2 From a semi-contact annulus to regular saucers

The technical part then consists of proving Lemma 10.4. Observe that since the homotopy in Lemma 10.4 fixes the semi-contact structure on ∂C, we can partition $[0, 1] = \bigcup_{i=0}^{N} [a_i, a_{i+1}]$ where $a_i < a_j$ for $i < j$, and proof the lemma for the restriction of Σ to each annulus $\Sigma \times [a_i, a_{i+1}]$. The first consequence of this is that if we make the partition fine enough the contact germs on all slices of the annulus will look like each other. This is made precise in Lemma 10.7. To prove this we first need another lemma.

Lemma 10.6. *Let $f : M \to N$ be a smooth map and $Z \subset M$ a submanifold satisfying*

(i) f maps Z diffeomorphically onto $f(Z)$,

(ii) $T_x f : T_x M \to T_{f(x)} N$ is an isomorphism for all $x \in Z$.

Then, f maps a neighborhood of Z diffeomorphically onto a neighborhood of $f(Z)$.

Proof. By (ii) we can cover $f(Z)$ we can find local inverses $g_i : U_i \to M$ such that the open sets U_i cover $f(Z)$. Moreover, we can assume that the cover $\{U_i\}$ is locally finite. Define $V_{ij} := \overline{\{\, y \in U_i \cap U_j \mid g_i(y) \neq g_j(y) \,\}}$ and $\tilde{U}_i := U_i \setminus \bigcup_j V_{ij}$. Since, $\{U_i\}$ is locally finite $\tilde{U}_i$ is open. As a union of open sets $W := \bigcup_i \tilde{U}_i$ is open. Observe that

$$W = \{\, y \in \bigcup_i U_i \mid g_i(y) = g_j(y) \text{ when } y \in U_i \cap U_j \,\}.$$

It follows from this description that the g_i patch together to a smooth inverse $g : W \to M$ of f. Moreover, it is clear that $f(Z) \subset W$ and that W is open. □

Lemma 10.7. *Let Σ be a closed manifold, $[a,b]$ an interval and consider $\{\zeta_s\}_{s\in[a,b]}$ a semi-contact structure on the annulus $\Sigma\times[a,b]$. Then, for each $s_0\in[a,b]$ there exists a $\sigma>0$ such that the restriction $\{\zeta\}_{s\in[s_0,s_0+\sigma]}$ is of immersion type with defining contact structure ζ_{s_0}.*

Proof. Since $[a,b]$ is compact there exists a $\delta>0$ such that each ζ_s is defined on $\Sigma\times[s-\delta,s+\delta]$. Hence, we can view $\{\zeta_s\}_{s\in[a,b]}$ as a smooth family of contact structures on $\Sigma\times[-\delta,\delta]$. For each $s_0\in[a,b]$ we can find, by Theorem 2.2.2 in [5], a $\sigma>0$ and an isotopy $\phi^s_{s_0}$, $s\in[s_0,s_0+\sigma]$, on $\Sigma\times[-\delta,\delta]$ satisfying $\phi^{s_0}_{s_0}=\mathrm{Id}$ and $(\phi^s_{s_0})^*\zeta_s=\zeta_{s_0}$.
We claim that by making σ smaller if necessary, the hypersurfaces $(\phi^s_{s_0})^{-1}(\Sigma\times 0)$ are graphical for all $s\in[s_0,s_0+\sigma]$. To see this look at the smooth map

$$F:\Sigma\times[s_0,s_0+\sigma]\to\Sigma\times[-\delta,\delta] \text{ defined by } F(x,s):=(\pi\circ(\phi^s_{s_0})^{-1}(x,0),s),$$

where $\pi:\Sigma\times[-\delta,\delta]\to\Sigma$ is the projection map. Observe that F satisfies the conditions of Lemma 10.6. This means that for σ small enough $\pi\circ(\phi^s_{s_0})^{-1}:\Sigma\to\Sigma$ is a diffeomorphism for all $s\in[s_0,s_0+\sigma]$. This proves $(\phi^s_{s_0})^{-1}(\Sigma\times 0)$ is graphical. □

Remark 10.8. *Note that in the proof of the previous Lemma we assign to each $s_0\in[a,b]$ a $\sigma>0$. Using compactness it is easy to see that we can pick one $\sigma>0$ such that the restriction $\{\zeta_s\}_{s\in[s_0,s_0+\sigma]}$ is of immersion type for all $s_0\in[a,b]$ at once.*

By the previous lemma and reparametrizing each interval in the partition we can assume we are considering an annulus $\Sigma\times[0,1]$ with an immersion type semi-contact structure $\{\zeta_s\}_{s\in[0,1]}$. More precisely, for some constant $R>0$ we have a contact structure $\mu=\zeta_0$ on $\Sigma\times[-R,R]$ and a family of functions $\psi_s:\Sigma\to[-R,R]$ such that the contact germs ζ_s are identified with the restriction of μ to a neighborhood of the graphs $\Gamma_s:=\mathrm{graph}(\psi_s)$.
In general, these functions do not need to be positive and the graphs Γ_s might intersect. The idea of the following lemma is that, using the Reeb vectorfield R_μ we can divide Σ into two parts. One part V where the angle between Σ and R_μ is large, and a part W where the angle is small. Over V, the Reeb vector field is transverse to Σ so we can flow Γ_1 along it to a strictly positive graph $\widetilde{\Gamma}=\mathrm{graph}(\tilde{\psi})$. The important point here is that since R_μ is a contact vector field, σ_1 can also be identified with a neighborhood of this new graph. Hence, we can homotope the graphs Γ_s to interpolate between Γ_0 and $\widetilde{\Gamma}$, while keeping ζ_0 and ζ_1 fixed. The result is that our new semi-contact structure is contact over V.
On W the function $\tilde{\psi}$ might still be negative. However, since here R_μ is almost tangent to Σ and the Reeb vector field is always transverse to the contact planes we can arrange that the contact planes will always be transverse to Σ over W. The upshot of this is that the induced characteristic foliation on W will be regular and hence over W the annulus $\Sigma\times[0,1]$ can be divided into regular semi-contact saucers.

Lemma 10.9. *Let $\zeta:=\{\zeta_s\}_{s\in[0,1]}$ a semi-contact structure on an annulus $\Sigma\times[0,1]\subset M$ where $\Sigma\subset M$ is a closed submanifold. Then, there is a partition $a_0<\cdots<a_N$ of $[0,1]$ such that the restriction of ζ to each $\Sigma\times[a_i,a_{i+1}]$ is equivalent to a semi contact structure $\tilde{\zeta}$ satisfying the following properties.*
There is a contact structure $\mu=\zeta_0$ on $\Sigma\times[-R,R]$ for a constant $R>0$ and a function $\psi:\Sigma\to[-\frac{R}{2},\frac{R}{2}]$ such that

(i) the germ of contct structure ζ_s is identified with the restriction of μ to a neighborhood of the graph $\Gamma_s := \mathrm{graph}(s\psi)$;

(ii) there are closed domains $V \subset \Sigma$ and $\widehat{V} \subset \mathrm{Int}\, V$ such that $\psi|_v > 0$, and the contact structure μ is transverse to all graphs Γ_s over $\hat{W} := \Sigma \setminus \mathrm{Int}\, \widehat{V}$ for all $s \in [a_i, a_{i+1}]$.

Proof. Equip $\Sigma \times [0,1]$ with a metric so we can measure angles between hyperplanes in the tangent bundle. By applying Lemma 10.7 we can partition $[0,1]$ and are free to assume that the semi-contact structure is of immersion type, and defined by a family of functions $\psi_s : \Sigma \to [-R, R]$ with $\psi_0 = 0$. Define θ to be the minimum over $\Sigma \times [-R, R]$ of the angle between R_μ and the contact distribution μ. By refining the partition from Lemma 10.7, we can assume that

$$||\Psi_s||_{C^0} \leq \tan\frac{\theta}{16} \qquad ||\Psi_s||_{C^1} \leq \epsilon \tag{10.1}$$

$$|\,\mathrm{angle}(\mu_{(x,0)}, \mu_{(x,t)})| \leq \frac{\theta}{16} \tag{10.2}$$

where we will pick the appropiate $\epsilon > 0$ later.

Define

$$\widehat{V} := \{\, (x,0) \in \Sigma \times 0 \mid \mathrm{angle}(\mu_{(x,0)} T_{(x,0)}\Sigma) \leq \frac{\theta}{4} \,\}^1.$$

It follows from Equation 10.2 and the definition of $\widehat{V}$ that $|\,\mathrm{angle}(R_\mu(x,t), T_{(x,t)}\Sigma)| \geq \frac{11\theta}{16}$ for all $(x,t) \in \Sigma \times [-R, R]$. Hence, $R_\mu(x,t)$ for $x \in \widehat{V}$ has a non-zero projection onto $\frac{\partial}{\partial u}$ so we can present $\widehat{V}$ as a disjoint union $\widehat{V} = \widehat{V}_+ \sqcup \widehat{V}_-$ defined by wether this component is positive or negative. Equivalently,

$$\widehat{V}_\pm := \{\, x \in \widehat{V} \mid du(\pm R_\mu(x)) > 0 \,\}.$$

The fact that the Reeb vector field is transverse to $\Sigma \times t$ can be used to homotope Γ_1 to the graph of a function ψ which is positive over $\mathcal{O}p\,\widehat{V}$ as follows.

Pick a smooth function $H : \Sigma \to [0,1]$ equal to ± 1 on $\widehat{V}_\pm$. Extend this to a function, still denoted by H, on $\Sigma \times [-R, R]$ as independent of the coordinate u. As in Equation 2.2 this function gives a contact vectorfield X_H on $\Sigma \times [-R, R]$ which is equal to $\pm R_\mu$ whenever H is equal to ± 1. Denote by $h_s : \Sigma \times [-R, R] \to \Sigma \times [-R, R]$ its flow over time s. We have to make sure that h_t does not rotate the contact planes to much so they stay transverse to the graphs. So, pick $\sigma > 0$ small enough so that $Th_s : T(\Sigma \times [-R, R]) \to T(\Sigma \times [-R, R])$ rotates any hyperplane by an angle $\leq \frac{\theta}{16}$ and $h_s(\Gamma_1)$ is graphical, for all $s \in [0, \sigma]$. Then, since $du(X_H) > 0$, we can assume by picking the $\epsilon > 0$ from Equation 10.1 small enough that the following holds:

(i) We have that $\widetilde{\Gamma} := h_\sigma(\Gamma_1)$ is graphical over Σ. More precisly, $\widetilde{\Gamma} := \{\, (x,t) \in \Sigma \times [-R, R] \mid t = \psi(x) \,\}$ for $\psi : \Sigma \to [-R, R]$.

(ii) For $x \in \widehat{V}$ we have $\psi(x) \geq \frac{\epsilon}{2}$.

[1] Note that both $\mu_{(x,0)}$ and $T_{(x,0)}\Sigma$ are hyperplanes in $T_{(x,0)}(\Sigma \times [0,1])$

(iii) For $s \in [0,1]$ defne $\widetilde{\Gamma}_s := \{\, z := (x,t) \in \Sigma \times [-R,R] \mid t = s\psi(x) \,\}$. Then, for $x \in \widehat{W} := \Sigma \setminus \widehat{V}$ and $s \in [0,1]$ we have $|\operatorname{angle}(T_z\widetilde{\Gamma}_s, \mu_z)| > \frac{\theta}{16}$. Hence, μ is transverse to all graphs $\widetilde{\Gamma}_s$.

Take $V := \{\, x \in \Sigma \mid \psi(x) \geq \frac{\epsilon}{4} \,\}$. Then, $\widehat{V} \subset \operatorname{Int} V$ and $\psi|_V > 0$.
Consider the 2-parameter family of functions, $F_{t,s} : \Sigma \to [-R,R]$ for $(t,s) \in I^2$, defined by

$$F_{t,s}(x) := (1-t)\psi_s(x) + ts\psi(x).$$

These functions interpolate between the 1-parameter families of functions $F_{0,s} = \psi_s$, $F_{1,s} = s\psi$. This means that the graphs induce a homotopy between ζ and an immersion type semi-contact structure $\tilde{\zeta}$ defined by the graphs of $s\psi$. Moreover, since $F_{t,0} = 0$ and $\operatorname{graph}(F_{t,1}) = h_t(\Gamma_1)$ and h_t is a contact isotopy, the homotopy between ζ and $\tilde{\zeta}$ fixes ζ_0 and ζ_1. □

With this lemma we are ready to finish the proof of Lemma 10.4. The idea of the proof is that by compactness of Σ we can cover the part where ψ is negative with finitely many disks and cut up the annulus in finitely many saucers using a partition of unit, see Figure 10.1.

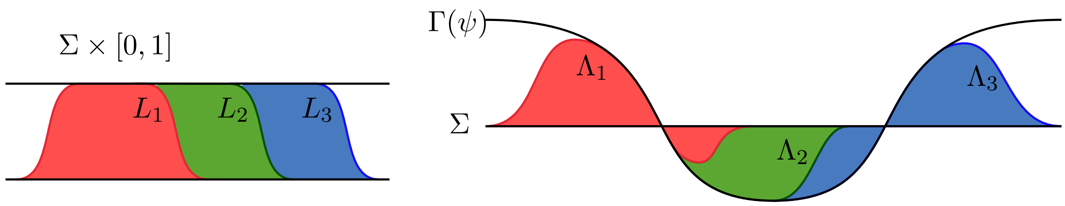


Figure 10.1: Take a finite partition of unity λ_i, $i = 1, \ldots, N$ (in the picture $N = 3$) and define $L_k := \sum_{i=1}^{k} \lambda_i$ and $\Lambda_k := L_k\psi$ to partition $\Sigma \times [0,1]$ into finitely many regions.

To ensure that the resulting saucers are regular we need to do some extra work. The essential ingredient of the proof is that whenever ψ is negative the graphs $\Gamma(s\psi)$, $s \in [0,1]$ are all transverse to the contact structure μ. This means that the induced characteristic foliation on any of the graphs (and in particular on Σ) are non-singular. Using Giroux's theorem this can be exploited to show that $\widehat{W}$ can in fact be covered by regular disks.

The last part of the proof consists of fixing the following problem which is also visible in Figure 10.1. Namely, a function $f : D \to \mathbb{R}$ describing a regular saucers is required to be positive on $D \setminus S$, which is not true in our above picture, caused by the negativity of ψ. To fix this we introduce a slightly perturbed version of ψ as in Figure 10.4. The proof is concluded by showing that the semi-contact structure induced by the perturbed ψ is homotopic to the original one.

Proof of Lemma 10.4. By applying Lemma 10.9 we can assume that we are in the following situation. We have a semi-contact structure of immersion type $\{\zeta_s\}_{s\in I}$ on an annulus $\Sigma \times I$ satisfying the following properties. There exists a contact structure $\mu = \zeta_0$ on $\Sigma \times [-R,R]$ for $R > 0$ and a function $\psi : \Sigma \to [-\frac{R}{2}, \frac{R}{2}]$ such that

(i) there exists a smooth family of diffeomorphisms $G_s : \mathcal{O}p\,\Sigma \times \{s\} \to \mathcal{O}p\,\Gamma(s\psi) \subset \Sigma \times [-R,R]$, such that $\zeta_s := G_s^*\mu$ and $G_s|_{\Sigma\times\{s\}} = \operatorname{Id} \times s\psi$, $s \in I$.

(ii) there exists closed domains $V \subset \Sigma$ and $\widehat{V} \subset \operatorname{Int} V$ such that $\psi|_V > 0$ and μ is transverse to the graphs $\Gamma(s\psi)$ over $\widehat{W} := \Sigma \setminus \operatorname{Int}(\widehat{V})$ for all $s \in I$.

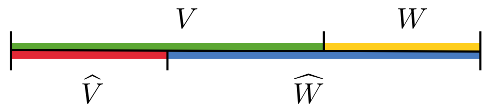


Figure 10.2: Schematic representaiton of the decomposition of Σ into V (green), $\widehat{V}$ (red), W (orange) and $\widehat{W}$ (blue).

Consider the following immersion $G : \Sigma \times [0,1] \to \Sigma \times [-R, R]$ defined by $G|_{\Sigma \times \{s\}} = G_s|_{\Sigma \times \{s\}}$. It is easy to see that the restriction of G to the set

$$\{ (x,s) \in \Sigma \times [0,1] \mid \psi(x) > 0 \},$$

is an embedding. Since the almost contact structure η on $\Sigma \times [0,1]$ equals $G^*\mu$ this implies that η is contact over the region $V \subset \Sigma$ where ψ is positive.

Over the region $W \subset \Sigma$ where ψ becomes negative this is not the case. We want to cut up this part into finitely many regular saucers and solve the filling problem locally on each one of them.

To do this we look for a special family of opens covering W as in the next lemma. The opens of this cover will form the base disks over which the saucers are defined. The conditions we put on the cover ensure that the saucers are equivalent to regular ones.

Lemma 10.10. *Let $\Sigma \subset (M, \xi)$ be a closed hypersurface, transverse to the contact structure. Then, around each point $p \in \Sigma$ there exists an open neighborhood $U \ni p$ and a contactomorphism $\phi : U \to (\mathbb{R}^{2n+1}, \xi_{st})$ such that*

$$\phi(\Sigma \cap U) = \Pi \subset (\mathbb{R}^{2n+1}, \xi_{st})$$

.

Proof. Denote by $\mathcal{F}$ the characteristic foliation on Σ and suppose it is spanned by the non-vanishing vector field X. Locally any non-vanishing vector field can be written as a constant vector field which allows us to find a diffeomorphism $\psi : V \to R^{2n+1}$ such that $\psi(\Sigma \cap V) = \Pi$ and $\psi_* X = \frac{\partial}{\partial q}$. Here we use the usual coordinates $(x, q, p) \in \mathbb{R}^{2n+1} = \mathbb{R}^{2n-1} \times \mathbb{R}^2$ and $\Pi := \{ (x,q,p) \in \mathbb{R}^{2n+1} \mid p = 0 \}$.

If $\xi = \ker \alpha$ then $\tilde{\xi} := \ker \psi_* \alpha$ is a contact structure on $\mathbb{R}^{2n+1}$ transverse to Π which induces a characteristic foliation $\widetilde{\mathcal{F}} = \langle \psi_* X \rangle$ on Π. Hence, we have an induced contact structure on $\Pi / \widetilde{\mathcal{F}}$ which we can assume to be equal to ξ_{st}^{2n-1}. Indeed, using Darboux theorem we can (by possibly shrinking V) find a change of coordinates on Π such that the contact structure on $\Pi / \widetilde{\mathcal{F}}$ equals ξ_{st}^{2n-1}.

Hence, $\phi := \psi|_{\Sigma \cap U} \to \Pi \subset (\mathbb{R}^{2n+1}, \xi_{st})$ is a diffeomorphism both preserving the characteristic foliation and transverse contact structure. By Theorem 2.18 we find a contactomorphism $\phi : U \to (R^{2n+1}, \xi_{st})$ such that $\phi(U \cap \Sigma) = \Pi$. □

Using this lemma we construct a suitable cover of W.

Lemma 10.11. *Let $\Sigma \subset (M^{2n+1}, \xi)$ be a closed hypersurface, transversal to the contact structure. Then, there exists a finite cover $\{O_i\}_{i=1}^N$ of $\mathcal{O}p\,\Sigma$ inducing a cover $\{U\}_{i=1}^N$ of Σ. Furthermore there exist two partitions of unit $\{\lambda_i^\pm\}_{i=1}^N$ subordinate to $\{U_i\}_{i=1}^N$ such that*

(i) each U_i contains a "square" $S_i \subset U_i$ diffeomorphic to $\Delta \times I$ where $\Delta \subset \Sigma/\mathcal{F}$ is a star-shaped domain;

(ii) the restriction of $\mathcal{F}$ to S_i is diffeomorphic to the product foliation on $\Delta \times I$;

(iii) we have inclusions $\operatorname{supp} \lambda_i^- \subset S_i \subset \operatorname{supp} \lambda_i^+ \subset U_i$.

Proof. For each point $p \in \Sigma$ we can apply the previous lemma to find a contactomorphism $\phi : U \to (\mathbb{R}^{2n+1}, \xi_{st})$ with $\phi(\Sigma \cap U) = \Pi$ and we can assume that $\phi(p) = 0$. We denote by $\widetilde{\mathcal{F}}$ the characteristic foliation on Π. Pick a star-shaped domain $\Delta \subset \Pi/\widetilde{\mathcal{F}}$ and let $S := \phi^{-1}(\Delta \times I)$.

By compactness we can find finitely many $S_i \subset \Sigma$, $i = 1, \ldots, N$ covering Σ. For each $i = 1, \ldots, N$ let U_i be an open disk containing S_i, then $\{U_i\}_{i=1}^N$ is a finite cover of Σ.

Pick partitions of unit λ_i^- subordinate to S_i and λ_i^+ subordinate to U_i, $i = 1, \ldots, N$. Since both partitions of unit are finite we can modify them to satisfy the condition $S_i \operatorname{supp} \lambda_i^+$. □

Remark 10.12. *The conditions in the previous lemma have two immediate consequences which will be helpful later in the proof:*

(i) First, they imply that $\lambda_i^- < \lambda_i^+$ on $\operatorname{Int} D \setminus S$. Hence if $f_\pm : D \to \mathbb{R}_{>0}$ are two positive functions then the pair $(f_+\lambda_i^+ - f_-\lambda_i^-, U_i)$ is a regular saucer Hamiltonian as in Definition 9.15.

(ii) Second, $\operatorname{supp} \lambda_i^- \subset \operatorname{supp} \lambda_i^+$ tells us that $\lambda_i^+(x) = 1$ implies $\lambda_i^-(x) = 1$.

Next, we use the partition of unit λ_i^+ subordinate to $\{U_i\}_{i=1}^N$ to cut up the annulus into saucers. For $0 \leq k \leq N$, define

$$L_k := \sum_{i=1}^{k} \lambda_i^+ : W \to I,$$

and observe that the graphs $\Gamma(L_k)$ partition $\Sigma \times I$ into saucers. By assumption the almost contact structure η on $\Sigma \times i$, induced by $\{\zeta_s\}_{s \in I}$ is equal to $\eta = F^*(\mu)$ where $F : \Sigma \times I \to \Sigma \times [-R, R]$ is the immersion defined by

$$F(x,t) := (x, t\psi(x)).$$

Hence, looking at the graphs $F(\Gamma(L_k)) = \Gamma(L_k\psi)$ we see they in turn partition the area between $\Sigma \times \{0\}$ and $\Gamma(\psi)$ inside $\Sigma \times [-R, R]$.

We can foliate each saucer

$$B_i := \{(x,t) \in \Sigma \times I \mid L_{i-1}(x) \leq t \leq L_i\} \tag{10.3}$$

by disks

$$D_s^i := \{(x,t) \in B_i \mid t = (1-s)L_{i-1}(x) + sL_i(x)\},$$

and this defines coordinates (x,s) on B_i in the usual sense. Observe that the $F(D_s)$ foliate the area between $\Gamma(L_{i-1}\psi)$ and $\Gamma(L_i\psi)$. So we get a family of diffeomorphisms

$$G_i^s : D_s^i \to \Gamma\left((1-s)L_{i-1}\psi + sL_i\psi\right) \subset \Sigma \times [-R,R]$$

defined by

$$(x,s) \mapsto (x, (1-s)L_{i-1}\psi(x) + sL_i\psi(x)) \quad s \in [0,1],$$

which together give an immersion $G^i : B_i \to \Sigma \times [-R,R]$ defined by the same formula. The family G_s^i gives, in the usual way, a semi-contact structure of immersion type $\zeta_i := \{\zeta_{i,s}\}_{s\in[0,1]}$ on B_i. Note that ζ_i is equal to the restriction of ζ, the semi-contact structure on $\Sigma \times [0,1]$, to B_i and that the induced almost contact structure η_i satisfies

$$\eta_i = \eta|_{B_i} = G_i^*(\mu).$$

So, we have a (finite) partition of $(\Sigma \times [0,1], \zeta)$ in semi-contact saucers (B_i, ζ_i), $i = 1, \ldots, N$.

Unfortunately the above construction does not yield regular semi-contact saucers. Because ψ can be negative it can happen that $L_i\psi \leq L_{i-1}\psi$ on $U_i \setminus S_i$ which is not allowed. To remedy this we construct different semi-contact structures $\widetilde{\zeta_i}$ on the saucers B_i, $i = 1, \ldots, N$ which are regular. We will then show that the $\widetilde{\zeta_i}$ and ζ_i are homotopic for all i, and that the induced homotopy on $\Sigma \times [0,1]$ is relative to the boundary.

We can write $\psi = \psi^+ - \psi^-$ for positive functions $\psi^\pm$. One way to see this is to take $\epsilon > 0$ very small, and define the set

$$U_\epsilon := \{x \in \widehat{W} \mid \psi(x) \geq \epsilon\}.$$

Let $g : \widehat{W} \to [0,1]$ be a smooth bump function with $g|_{U_\epsilon} = 1$ and $g|_{\overline{\widehat{W}\setminus \operatorname{supp}\psi}} = 0$ which exists since $U_\epsilon \subset \operatorname{Int}\operatorname{supp}\psi$. Then,

$$\psi_+ := g\psi + \epsilon, \qquad \psi^- := (g-1)\psi + \epsilon,$$

are strictly positive and gives the desired decomposition $\psi = \psi_+ - \psi_-$, see Figure 10.3.

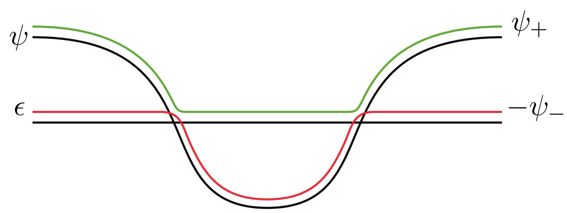


Figure 10.3: The decomposition of ψ into positive functions $\psi_\pm$.

Then define

$$\psi_i^\pm := \psi^\pm \lambda_i^\pm \quad \text{and} \quad \psi_i := \psi_i^+ - \psi_i^-, \quad i = 1, \ldots, N.$$

It follows from Remark 10.12 that $\lambda_i^+(x) = 1$ implies $\psi_i(x) = \psi(x)$ and $\psi_i|_{U_i \setminus S_i} > 0$. For $0 \leq k \leq N$, define

$$\Psi_k := \sum_{i=1}^{k} \psi_i.$$

When $L_k = 1$ we have $F(\Gamma(L_k)) = \Gamma(\Psi_k)$. Moreover, $\Psi_k - \Psi_{k-1} = \psi_i > 0$ on $\mathcal{O}p\, U_i \setminus S_i$. That is, the functions Ψ_k partition the area between $\Sigma \times \{0\}$ and $\Gamma(\psi)$ similarly to the $L_k\psi$ above.

Consider the family of diffeomorphisms

$$\widetilde{G}_i^s : D_s^i \to \Gamma\left((1-s)\Psi_{i-1} + s\Psi_i\right) \subset \Sigma \times [-R, R]$$

defined by

$$(x, s) \mapsto (x, (1-s)\Psi_{i-1}(x) + s\Psi_i(x)) \quad s \in [0, 1],$$

which together give an immersion $\widetilde{G}^i : B_i \to \Sigma \times [-R, R]$ defined by the same formula. This family induces in the usual way a semi-contact structure of immersion type on B_i which we denote by $\widetilde{\zeta}_i$.

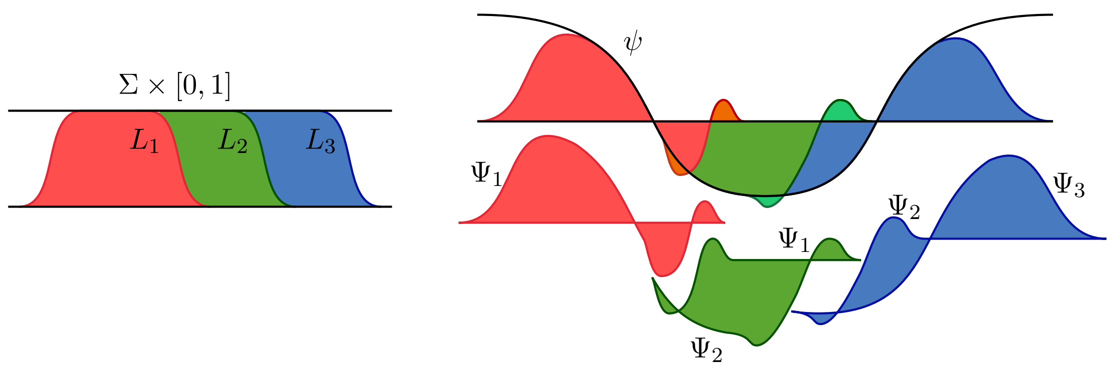


Figure 10.4: The partition of the annulus into saucers.

The upshot is that in contrast to the (B_i, ζ_i), the saucers $(B_i, \widetilde{\zeta}_i)$ are regular. To see this, first observe that $(\Psi_i - \Psi_{i-1}, U_i)$ satisfies the conditions in Definition 9.15.
By passing to a partition of $\Sigma \times [0, 1]$ we can make the $\Psi_k, 0 \leq k \leq N$, as C^1-small as we want. Furthermore, we can assume that the graphs $\Gamma(s\psi)$ are all contained in $\bigcup_{i=1}^N O_i$ and hence that each $(B_i, \widetilde{\zeta}_i) \subset O_i$, where $\{O_i\}_{i=1}^N$ is the cover of $\mathcal{O}p\, \Sigma \times \{0\}$ we constructed above. Hence, we can assume that the defining contact structure for each $\widetilde{\zeta}_i$ is just ξ_{st} on $\mathbb{R}^{2n+1}$.
This means that we are in the situation of Lemma 9.18 and we conclude that each $(B_i, \widetilde{\zeta}_i)$ is regular.

It remains to be shown that the semi-contact structures ζ_i and $\widetilde{\zeta}_i$ on B_i are homotopic. Consider the straight line homotopies

$$H_t^i := (1-t)L_i\psi + t\Psi_i, t \in [0, 1],$$

for $i = 1, \ldots, N$. Observe that

$$\mathcal{H}_t^i := \{\, \Gamma\left((1-s)H_t^{i-1} + sH_t^i\right) \}_{s \in [0,1]}, \quad t \in [0, 1]$$

is a family of foliations interpolating between the foliations defining ζ_i and $\widetilde{\zeta}_i$. Hence, this gives a homotopy between the semi-contact structures.

Clearly, this homotopy does not fix the contact germ on the boundary $\partial B_i = D_0^i \cup D_1^i$ so a priori it is not well-defined since applying the homotopy on one of the saucers might change the semi-contact structure on the neighbouring ones. Since

$$\Sigma \times [0,1] = \bigcup_{i=1}^{N} B_i,$$

we can apply the homotopy on all saucers B_i at once, giving a homotopy on $\Sigma \times [0,1]$ between ζ and $\widetilde{\zeta} := \bigcup_{i=1}^{N} \widetilde{\zeta}_i$. Moreover, we claim that this homotopy leaves the contact germs on $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ fixed so that it is well-defined. For $\Sigma \times \{0\}$ this follows immediately from the definitions. For $\sigma \times \{1\}$ note that if $1 \leq i \leq N$ and $x \in \Sigma$ are such that $L_i(x) = 1$ then it follows that $\Psi_i(x) = \psi(x)$. Hence, $H_t^i(x) = \psi(x)$ for all $i \in [0,1]$ proving the claim.

□

Chapter 11

Reduction to a universal model

In this chapter we prove the last ingredient to the proof of Proposition 3.7. In the previous chapters we proved that any almost contact structure on a manifold can be made contact up to finitely many circle models. However, there is no restriction on what these circle models can looks like. In particular, there are infinitely many possibilities for each of the defining contact Hamiltonians.

In this chapter we state and proof Proposition 11.11 below. This proposition says that any (time-independent) circle model can be made contact outside finitely many circle models inside it. Moreover, these circle models can be chosen from a finite list. That is, there are only finitely many possible choices for the contact Hamiltonians defining them.

This allows us to reduce to a universal model for each dimension, as in the statement of Proposition 3.7.

By Remark 11.12 there is an easier way to reduce to an universal in the three dimensional case, making this chapter unnecessary.

The idea of the proof in this chapter is the following. We construct a finite list of functions $\phi_i : U \to \mathbb{R}$, where $U \subset \mathbb{R}^{2n-1}$, for $i = 1, \ldots, N$.

By precomposing these functions with elements of the contactomorphism group on $(\mathbb{R}^{2n-1}, \xi_{st})$ and by taking finite sums of such functions we can construct new functions from this finite list. It turns out that for any contact Hamiltonian K we can construct in this way a contact Hamiltonian $K' \leq K$. Hence, the circle model associated to K dominates the circle model associated to K'. Moreover, since K' is constructed using finitely many building blocks we can show that the associated circle model can be made contact up to finitely many circle models from a finite list.

The basic idea of the function construction is illustrated by the following example.

Example 11.1. *Consider (non-smooth) function $f : [0,3] \subset \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) := \begin{cases} 1 & \text{for } x \in [0,1) \\ -3 & \text{for } x \in [1,2) \\ 1 & \text{for } x \in [2,3) \end{cases}$$

Let the additive group $\mathbb{Z}$ act on $\mathbb{R}$ by translation. That is, we identify $n \in \mathbb{Z}$ with the translation *map $\tau_n : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x + n$. Consider the interval $[0,3] \subset \mathbb{R}$ and the functions*

$$f_k := \sum_{i=0}^{k} f \circ \tau_{-i},$$

see Figure 11.1.

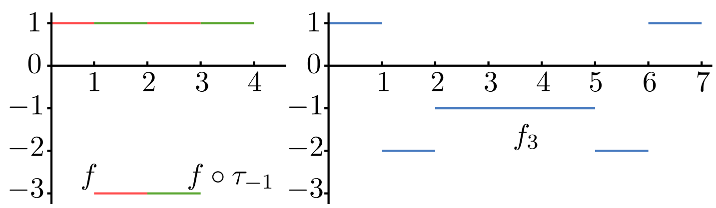


Figure 11.1: The graphs of f, $f \circ \tau_{-1}$ and $f_3 := \sum_{i=0}^{3} f \circ \tau_{-i}$

Note that as $k \to \infty$ the quotient $\frac{\text{are where } f_k \text{ is positive}}{\text{area where } f_k \text{ is negative}} \to 0$. So functions which are negative on an arbitrary large part of their domain can be written as sums of functions which are positive on most of their domain. Furthermore, these functions are just translations of a single function.
This is important since we know from Definition 5.3 that the domination relation is essentially encoded in the size of the set where a Hamiltonian is negative. A contact Hamiltonian $K : \Delta \to \mathbb{R}$ is only required to be positive on an arbitrary small neighbourhood of $\partial\Delta$ and so we would expect that it is impossible to find a single Hamiltonian which is dominated by all the possible Hamiltonians showing up in the proof of Proposition 3.7. The above example shows that this is possible.

This chapter basically consists of generalizing the above example. In the first section we define a suitable cover for the general case and in the second section we define a suitable function generalizing f from the example. In the last section we provide proofs of Proposition 11.11 and Proposition 3.7.

11.1 Constructing a suitable cover

The first thing we have to do is find a higher dimensional analogue of the translation action from Example 11.1. The essential property of the action in the example is that it allows us to construct a locally finite cover by compact sets. In general we also want that the group action is by contactomorphisms of $(\mathbb{R}^{2n+1}, \xi_{st})$ so the saucers constructed from these functions will be equivalent.

As usual define the hypersurface $\Pi := \{ (z, x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n+1} \mid y_n = 0 \} \subset (\mathbb{R}^{2n+1}, \xi_{st})$, where

$$\alpha = dz + \sum_{i=1}^{n-1}(x_i dy_i - y_i dx_i) + y_n dx_n = dz + \sum_{i=1}^{n-1} u_i d\theta_i + y_n dx_n$$

is the standard contact structure under the identification $\mathbb{R}^{2n+1} = \mathbb{R}^{2n-1} \times T^*\mathbb{R}^1$ and using Cartesian coordinates (x_n, y_n) on $T^*\mathbb{R}^1$.

Inside the group of contactomorphisms $\mathrm{Diff}(\mathbb{R}^{2n+1}, \xi_{st})$ there is a subgroup Θ defined to be the free group generated by the following contactomorphisms:

- Translations T in the z and the x_n coordinate:

$$T_z : (x, y, z) \mapsto (x, y, z + 1)$$

$$T_{x_n} : (x_1, \ldots, x_n, y, z) \mapsto (x_1, \ldots, x_n + \frac{1}{2}, y, z),$$

- Sheers S in each x_i and y_i coordinate for $i = 1, \ldots, n-1$ (not in the x_n and y_n coordinate):

$$S_{y_i} : (x, y_1, \ldots, y_i, \ldots, y_n, z) \mapsto (x, y_1, \ldots, y_i + 1, \ldots, y_n, z + x_i), i = 1, \ldots, n-1$$

$$S_{x_i} : (x_1, \ldots, x_i, \ldots, x_n, y, z) \mapsto (x_1, \ldots, x_i + 1, \ldots, x_n, y, z + y_i), i = 1, \ldots, n-1.$$

Note that Θ does not contain any translation or sheer in the y_n direction so that the action by Θ preserves Π. Also, the reason that we choose sheers instead of translation in the first $2n-2$ coordinates is that translations in those coordinates are not contactomorphisms of ξ_{st}, while sheers are.

One of the important properties of the translation action from Example 11.1 is that there are only finitely many $n \in \mathbb{Z}$ such that $\tau_n([0,3]) \cap [0,3] \neq \emptyset$ so that the functions f_k are well-defined. To see that Θ also has this property note that $[S_{y_j}, S_{x_j}] = S_{y_j} S_{x_j} S_{y_j}^{-1} S_{x_j}^{-1} = T_z^2$ while all other transformations commute. This means that we can write any element of Θ in the form

$$S_{x_1}^{k_1} \ldots S_{x_n-1}^{k_{n-1}} S_{y_1}^{l_1} \ldots S_{y_{n-1}}^{l_{n-1}} T_{x_n}^{k_n} T_z^{l_n},$$

for integers $k_1, \ldots, k_n$ and $l_1, \ldots, l_n$. The above identity implies that for any compact subset $Q \subset \Pi$ the set

$$S(Q) := \{\, g \in \Theta \mid g(Q) \cap Q \neq \emptyset \,\} \subset \Theta$$

is finite. We say that Θ acts *properly discontinuously* on Π.

The distances we can translate Q using Θ are bounded from below. To be able to translate over small distances pick an integer $N > 0$ and consider the scaling contactomorphism $C_N \in \mathrm{Diff}(\mathbb{R}^{2n+1}, \xi_{st})$ defined by

$$(x, y, z) \mapsto (Nx, Ny, N^2 z).$$

Define $\Theta_N := C_N^{-1} \Theta C_N$. To be precise, Θ_N is the free group generated by translations $T_{z,N} := C_N^{-1} \circ T_z \circ C_N$, $T_{x_j,N} := C_N^{-1} \circ T_{x_j,N} \circ C_N$ and sheers $S_{x_j,N} := C_N^{-1} \circ S_{x_j} \circ C_N$, $S_{y_j,N} := C_N^{-1} \circ S_{y_j} \circ C_N$. Clearly, by picking $N > 0$ large enough we can translate over arbitrary small distances. This allows us to construct covers of Π as in the following definition.

Definition 11.2. *A compact subset $Q \subset \Pi$ is said to generate a* Θ_N-equivariant cover *of* Π *if* $\Theta_N \cdot \mathrm{Int}(Q) = \Pi$. *In this case the θ_N-equivariant cover is given by* $\{g(Q)\}_{g \in \Theta_N}$.

The next example shows that such covers exist.

Example 11.3. *Let $a > \frac{1}{2}$ and consider the parallelepiped*

$$P := \{ (x, y, z) \in \mathbb{R}^{2n+1} \mid |z| \leq a, |x_j| \leq a, |y_j| \leq a, \text{ for } 1 \leq j \leq n-1, 0 < x_n \leq a, y_n = 0 \} \subset \Pi.$$

This is easily seen to generates a Θ-equivariant cover of Π. In turn, the scaled parallelepiped, $P_N := C_N^{-1}(P)$ generates a Θ_N-equivariant cover of Π.

The group Θ_N allows us to cover any compact domain in Π, however contact Hamiltonians are defined as functions on compact domains in $\mathbb{R}^{2n-1} \times S^1$. This space can be viewed as the zero-section of

$$\left(\mathbb{R}^{2n-1} \times T^*S^1, \xi_{st} = \ker(dz + \sum_{i=1}^{n-1} x_i dy_i - y_i dx_i + pdq) \right),$$

which in turn is the quotient of $(\mathbb{R}^{2n+1}, \xi_{st})$ by the contactomorphism $T_{x_n}^2$. Denote the quotient map by $\pi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n-1} \times T^*S^1$. Since we want to partition circle models we need to move our construction on $\mathbb{R}^{2n-1} \times T^*\mathbb{R}^1$ over to $\mathbb{R}^{2n-1} \times T^*S^1$. This can only be done if the set Q, generating the cover satisfies the condition of the following definition.

Definition 11.4. *A compact set $Q \subset \Pi$ generating a Θ-equivariant cover is called* sufficiently small *if $T_{x_n}(Q) \cap Q \neq \emptyset$.*

Remark 11.5. *Note that by taking N sufficiently large Q_N is sufficiently small for any compact Q. Furthermore, taking $\frac{1}{2} < a < 1$ in Example 11.3 the set P is sufficiently small, showing such sets exist.*

Assume Q is sufficiently small and consider the normal subgroup $\Upsilon \subset \Theta_N$ generated by $T_{x_n}^2$. Note that for any integer $N > 0$, we have $T_{x_n} = T_{x_n,N}^N$ so $T_{x_n}^2$ is indeed an element of Θ_N. Define the quotient group $\widehat{\Theta}_N := \Theta_N / \Upsilon \subset \mathrm{Diff}(\mathbb{R}^{2n-1} \times T^*S^1, \xi_{st})$ which preserves $\widehat{\Pi} := \pi(\Pi)$. Since Q is sufficiently small $\widehat{Q} := \pi(Q)$ is well-defined and generates a $\widehat{\Theta}_N$-equivariant cover of $\widehat{\Pi}$.

11.2 Constructing a suitable function

In this section construct functions ϕ and Φ_k analogous to the f and f_k in Example 11.1. In order to do this we first need a Θ-equivariant cover as in the previous section. Moreover, the set $Q \subset \Pi$ generating the cover and the function ϕ will determine a universal saucer. We want this saucer to be regular so we put some conditions on Q and ϕ.

Definition 11.6. *For fixed dimension $(2n+1)$, let $Q, Q', S \subset \Pi$ be closed sets satisfying the following conditions:*

(i) $Q' \subset \mathrm{Int}\, S$ and $S \subset \mathrm{Int}\, Q$ and S is diffeomorphic to $\Delta \times I$ for a compact star-shaped domain $\Delta \subset (\mathbb{R}^{2n-1}, \xi_{st})$;

(ii) Q', S and Q are sufficiently small and generating a Θ-equivariant cover of Π;

(iii) *the characteristic foliation $\mathcal{F}$ on Q is diffeomorphic to the characteristic foliation of the standard round disk in Π and the restriction of $\mathcal{F}$ to S is diffeomorphic to the product foliation on $\Delta \times I$.*

Furthermore, pick two smooth and non-negative functions $\phi_\pm : \Pi \to \mathbb{R}$ satisfying the following conditions:

(iv) $\operatorname{supp} \phi_- \subset S$ *and* $\operatorname{supp} \phi_+ \subset Q$;

(v) $\phi_+|_{\operatorname{Int} Q} > 0$, $J^\infty \phi_+|_{\partial Q} = 0$, *meaning its ∞-jet vanishes, and* $\phi_-|_{Q'} > 0$;

(vi) $\max \phi|_{Q'} < -(m+1)\mu$ *where* $\phi := \phi_+ - \phi_-$ *and* $\mu := \max(\phi)$ *(which is positive by condition v).*

(vii) *For a finite subset $\Lambda \subset \Theta$ define*

$$\Phi_\Lambda := \mu + \sum_{g \in \Lambda} \phi \circ g^{-1}.$$

Then for all $\Lambda \subset \Theta$ and $h \in \Theta$, $h \notin \Lambda$ we require Φ_Λ and $\Phi_{\Lambda \cup h}$ to be sufficiently C^1-small for Lemma 9.18.

The pair (Q, ϕ) is called the universal saucer Hamiltonian *in dimension* $(2n+1)$.

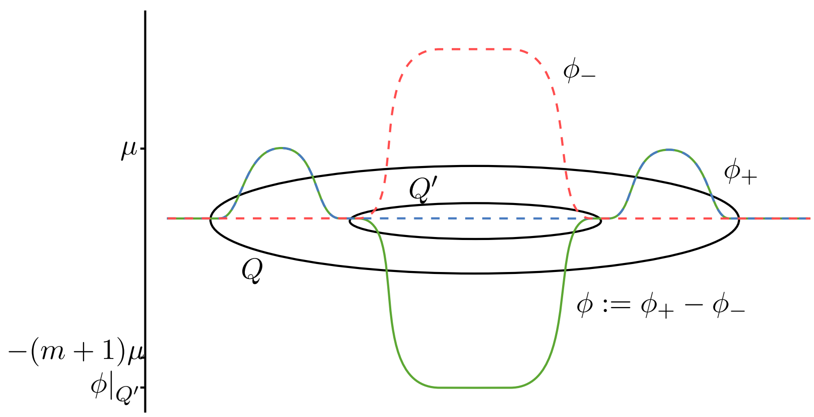


Figure 11.2: The graph of ϕ.

Condition (iv) is the analogue of picking $f|_{[1,2)} = -3$ in Example 11.1. That is, we define $\phi|_{Q'}$ to be so small (i.e. negative) that sums of translates of ϕ will always be negative on the union of translates of Q'. Conditions $(i) - (v)$ and (vii) ensure that the universal saucer Hamiltonian is regular as in Definition 9.15, so that the following is well-defined. Observe that in Condition (vii) all terms $g \in \Lambda$ with $g(Q) \cap Q = \emptyset$ are irrelevant, so it suffices to very (vii) only for subsets Λ of the finite set $S(Q)$. Hence, this gives finitely many conditions and can always be satisfied by taking ϕ_+ and ϕ_- small enough which can be achieved by replacing (ϕ_+, ϕ_-) by $(\epsilon\phi_+, \epsilon\phi_-)$ for $\epsilon > 0$ small enough. Lastly, the reason for adding the term $+\mu$ in the definition of Φ_Λ will become clear from the proof of Proposition 11.11.

Definition 11.7. *The regular Hamiltonian semi-contact saucer* (B_ϕ, ζ_ϕ) *associated to* (Q, ϕ)*, defined above, is called the* universal saucer *in dimension* $(2n+1)$.

The equivariant cover of Π allows us to translate the universal saucer models to obtain many different regular saucers all equivalent to the universal one. To do this, let $\Lambda \subset \Theta$ be a finite subgroup and $h \in \Theta$ such that $h \notin \Lambda$. Define functions $\Psi_\Lambda, \Psi_{\Lambda,h} : \Pi \to \mathbb{R}$ by

$$\Psi_\Lambda := \mu + \sum_{g \in \Lambda} \phi_+ \circ g^{-1}, \quad \text{and} \quad \Psi_{\Lambda,h} := \Psi_{\Lambda \cup h}.$$

Use these functions to define saucers $B_{\Lambda,h}$ over $h(Q)$ by

$$B_{\Lambda,h} := \{ (w, y_n) \in \mathbb{R}^{2n+1} \mid w \in h(Q), \Psi_\Lambda(w) \leq y_n \leq \Psi_{\Lambda,h}(w) \}.$$

Note that Condition v in Definition 11.6 implies that this is a well-defined saucer. To define a regular semi-contact structure on $B_{\Lambda,h}$ define functions $\Phi_\Lambda, \Phi_{\Lambda,h} : \Pi \to \mathbb{R}$ by

$$\Phi_\Lambda := \mu + \sum_{g \in \Lambda} \phi \circ g^{-1}, \quad \text{and} \quad \Phi_{\Lambda,h} := \Phi_{\Lambda \cup h}.$$

Then, define $\zeta_{\Lambda,h}$ to be the immersion type semi-contact structure with defining functions

$$\psi_s := (1-s)\Phi_\Lambda + s\Phi_{\Lambda,h}, \qquad s \in [0,1],$$

as in Definition 9.4. The following lemma shows that all these saucers are regular and that up to equivalence there are only finitely many different ones.

Lemma 11.8. *Up to equivalence the above construction builds at most* 2^m *regular semi-contact saucers* $(B_{\Lambda,h}, \zeta_{\Lambda,h})$*, where* $m = |S(Q)|$.

Proof. Consider a fixed semi-contact saucer $(B_{\Lambda,h}, \zeta_{\Lambda,h})$ so that $\zeta_{\Lambda,h}$ is defined over $h(Q)$ by the functions

$$(1-s)\Phi_\Lambda + s\Phi_{\Lambda,h}, \qquad s \in [0.1].$$

Since h is a contactomorphism of $(\mathbb{R}^{2n+1}, \xi_{st})$ it follows that this semi-contact structure is equivalent to the one defined by the functions

$$(1-s)\Phi_\Lambda \circ h + s\Phi_{\Lambda,h} \circ h, \qquad s \in [0,1],$$

over Q. Observe that $\Phi_{\Lambda,h} \circ h - \Phi_\Lambda \circ h = \phi$, showing that the number of different saucers we can make is bounded by the number of functions of the form Φ_Λ. In turn this is bounded by 2^m the number of finite subsets of $S(Q)$. Furthermore, the regularity of the saucers follows from Lemma 9.18. □

As in the previous section we can use the contactomorphism $C_N \in \mathrm{Diff}(\mathbb{R}^{2n+1}, \xi_{st})$ to define scaled versions of the saucers $(B_{\Lambda,h}, \zeta_{\Lambda,h})$. More precisely, pick N a positive integer. For any $g \in \Theta_N$ define

$$\phi_{g,N,+} := \frac{1}{N}\phi_+ \circ C_N \circ g^{-1}.$$

Since any $g \in \Theta_N$ is of the form $g = C_N^{-1} \circ g' \circ C_N$ for some $g' \in \Theta$ we see $\phi_{g,N} = \frac{1}{N}\phi \circ g' \circ C_N$ so it is the function describing the set obtained by applying C_N to the graph of $\phi \circ g'^{-1}$. Define scaled saucers by $B_{\Lambda,h,N} := C_N^{-1}(B_\Lambda, h)$. These can be described more directly by

$$B_{\Lambda,h,N} := \{ (w, y_n) \in \mathbb{R}^{2n+1} \mid w \in C_N^{-1} \circ g_k(Q), \Psi_{\Lambda,N}(w) \le y_n \le \Psi_{\Lambda,h,N}(w) \},$$

where

$$\Psi_{\Lambda,N} := \frac{\mu}{N} + \sum_{g \in \Lambda_N} \phi_{g,N,+} = \frac{\mu}{N} + \sum_{g' \in \Lambda} \frac{1}{N}\phi_+ \circ g'^{-1} \circ C_N,$$

with $\Lambda_N := C_N \circ \Lambda \circ C_N^{-1}$. The regular semi-contact structure $\zeta_{\Lambda,h,N}$ is defined in the same way as before by replacing Φ_Λ by

$$\Phi_{\Lambda,N} := \frac{\mu}{N} + \sum_{g \in \Lambda_N} \phi_{g,N} = \frac{\mu}{N} + \sum_{g' \in \Lambda} \frac{1}{N}\phi \circ g'^{-1} \circ C_N,$$

and replacing $\Phi_{\Lambda,h}$ by $\Phi_{\Lambda,h,N} := \Phi_{\Lambda \cup h,N}$.

The above saucers and functions are all defined on $\mathbb{R}^{2n+1}$. To define them on $\mathbb{R}^{2n-1} \times T^*S^1$ use the projection $\pi : \Pi \to \widehat{\Pi}$. Given a compactly supported function $f : \Pi \to \mathbb{R}$ we define a function

$$\sum_{g \in \upsilon} f \circ g^{-1}.$$

This function is 1-periodic in the x_n-variable and therefore defines a function $\hat{f} : \widehat{\Pi} \to \mathbb{R}$. We can apply this to any of the functions constructed above. In particular applying this construction to $\Phi_{\Lambda,N}$ and $\Psi_{\Lambda,N}$ yields functions $\widehat{\Phi}_{\Lambda,N}$ and $\widehat{\Psi}_{\Lambda,N}$ which we use to construct regular semi-contact saucers $(\widehat{B}_{\Lambda,h,N}, \hat{\zeta}_{\Lambda,h,N})$ as before.

11.3 Reducing to a universal model

The idea of the proof of Proposition 11.11 is that using the results from the previous sections we can construct for each (time independent) contact Hamiltonian K, a function $\Phi_{|\Lambda|}$, where Λ is a finite subset of $\widehat{\Theta}_N$, which is smaller than K with respect to the domination relation. This implies that the circular model associated to K dominates a contact shell encoded by $\Phi_{|\Lambda|}$. Moreover, since $\Phi_{|\Lambda|}$ is constructed by adding copies of ϕ, its associated contact shell decomposes in a part which is contact and finitely many saucers from the list of Lemma 11.8. This gives us the desired result.

Recall that we fixed in Definition 11.6 two regular and sufficiently small disks $Q, Q' \subset \Pi$ satisfying $Q' \subset \operatorname{Int} Q$ and a function $\phi = \phi_+ - \phi_-$. We keep these choices. This means that in order to construct the functions $\widehat{\Phi}_{\Lambda,N}$ we are only free to choose the integer $N > 0$ for the scaling and the subset $\Lambda_N \subset \widehat{\Theta}_N$. The following Lemma shows that we can always make the appropriate choices for N and Λ.

Remark 11.9. *Note that in the statement of the following lemma we talk about open sets $U, U' \subset \mathbb{R}^{2n-1}$ satisfying $U' \Subset U$, and a smooth function $K : U \to \mathbb{R}$. One should think of U as $\operatorname{Int} \Delta$ where $K : \Delta \times S^1 \to \mathbb{R}$ is a time-independent contact Hamiltonian. The set U' is such that K is positive on $U \setminus U'$.*

Lemma 11.10. *Fix sets $Q, Q' \subset \Pi$ and a function $\phi := \phi_+ - \phi_-$ as above. Then, for any bounded open domains $U, U' \subset \mathbb{R}^{2n-1}$ satisfying $\overline{U'} \subset \operatorname{Int} U$ and any smooth function $K : U \to \mathbb{R}$ which is positive on $U \setminus \overline{U}'$, there exists an integer $N > 0$ and a finite subset $\Lambda_N \subset \widehat{\Theta}_N$ such that*

(i) $U' \times S^1 \Subset \bigcup_{g\in\Lambda_N} g(\operatorname{Int} \widehat{Q}'_N) \subset \bigcup_{g\in\Lambda_N} g(\operatorname{Int}(\widehat{Q}_N)) \Subset U \times S^1$, *where* $\widehat{Q}'_N := \pi(Q'_N)$ *and* $\widehat{Q}_N := \pi(Q_N)$*;*

(ii) $\sum_{g\in\Lambda_N} \hat{\phi}_{g,N} < \begin{cases} -\frac{2\mu}{N} & \text{on } U' \times S^1; \\ K - \frac{\mu}{N} & \text{on } (U \setminus U') \times S^1. \end{cases}$

Proof. If $K : U \to \mathbb{R}$ is not only positive near ∂U but also on some set in the interior of U it can happen that $U \setminus U'$ has more than one connected component. However, enlarging U' so that $U \setminus U'$ becomes smaller, makes the statement of the lemma stronger. Indeed, enlarging U' will force our constructed function to be even smaller, see Figure 11.3. With this in mind

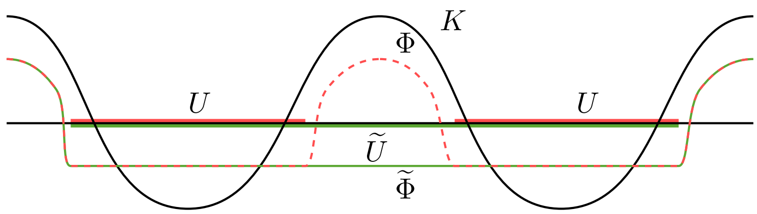


Figure 11.3: Enlarging the set U' (in red) to $\widetilde{U}'$ (in green), together with the corresponding Φ and $\widetilde{\Phi}$.

fix $\epsilon > 0$ with the property that the set

$$P := \{ (x, y, z) \in U \setminus U' \mid K(x, y, z) > \epsilon \}$$

separates ∂U from U' and redefine U' to include all the connected component of $U \setminus P$ which are disjoint from ∂U.

Now taking N big enough we can find a finite set $\Lambda \subset \widehat{\Theta}_N$ satisfying condition (i). This follows from the fact that $\widehat{Q}_N$ and $\widehat{Q}'_N$ become smaller as N increases but still produce $\widehat{\Theta}_N$-equivariant covers of $\mathbb{R}^{2n-1}$. By choosing N large enough we can further assume that

$$N > \frac{(m+1)\mu}{\epsilon}, \tag{11.1}$$

where $m = |S(Q)|$ and $\mu = \max(\phi)$ as before. Recall $\phi_{N,g} := \phi \circ C_N \circ g^{-1}$ implying that $\max(\hat{\phi}_{g,N}) = \frac{\mu}{N}$. Therefore, we find using Equation 11.1, that on $(U \setminus U') \times S^1$ we have

$$\sum_{g\in\Lambda_N} \hat{\phi}_{g,N} \leq \frac{m\mu}{N} = \frac{(m+1)\mu}{N} - \frac{\mu}{N} < \epsilon - \frac{\mu}{N} < K - \frac{\mu}{N}.$$

Recall that we required $\max(\phi|_{Q'}) < -m\mu$ in the definition of ϕ which means that $\max \hat{\phi}_{g,N}\Big|_{g(Q')} < -\frac{m\mu}{N}$. There are at most $m-1$ other elements $g' \in \widehat{\Theta}_N$ such that $g'^{-1}g(Q') \cap Q' \neq \emptyset$ and each such element satisfies $\max(\hat{\phi}_{g',N}\Big|_{g(Q')}) < \frac{\mu}{N}$. Hence, since every $x \in U'$ is contained in some $g(\widehat{Q}'_N)$ we have on $U' \times S^1$ the following inequality

$$\sum_{g\in\Lambda_N} \hat{\phi}_{g,N} < -\frac{(m+1)\mu}{N} + \frac{(m-1)\mu}{N} = -\frac{2\mu}{N}.$$

This proves the lemma. $\square$

The following Proposition is the last ingredient for the proof of Proposition 3.7.

Proposition 11.11. *For a fixed dimension, there is a finite list of saucers $\{(B_k, \zeta_k)\}$ for $k = 1, \dots, L$, with the following property.*
For any circle model contact shell (B_K, η_K) defined by a time-independent contact Hamiltonian $K : \Delta \times S^1 \to \mathbb{R}$, there exist finitely many disjoint balls $B_i \subset B_K$ for $i = 1, \dots, q$ so that the contact shell (B_K, η_K) is homotopic relative to $\mathcal{O}p\,\partial B_K$ to an almost contact structure ξ that is contact on $B_K \setminus \bigcup_{i=1}^q B_i$ and the restriction $\xi|_{B_i}$ is equivalent to one of the saucers (B_k, ζ_k) for $k = 1, \dots, L$.

Proof. Define $U := \mathcal{O}p\,\Delta$ and pick $U' \Subset U$ such that $K|_{U\setminus U'} > 0$. Applying Lemma 11.10 we find an integer $N > 0$, a finite set $\Lambda_N \subset \widehat{\Theta}_N$ and a function

$$\Phi := \frac{\mu}{N} + \sum_{g\in\Lambda} \hat{\phi}_{g,N} : U \times S^1 \to \mathbb{R}.$$

This function satisfies $\Phi < K$ on $(\mathcal{O}p\,\Delta \setminus U') \times S^1$, $\Phi < -\frac{\mu}{N}$ on $U' \times S^1$ and $\Phi = \frac{\mu}{N} > 0$ near $\mathcal{O}p\,\partial\Delta$. We linearly order in any way the elements of $\Theta : g_1, g_2, \dots$ this induces orderings of $\widehat{\Theta}_N$ and Λ. Using the ordering of Λ we define

$$\Phi_k := \frac{\mu}{N} + \sum_{i=1}^{k} \hat{\phi}_{g_i,N} : \mathcal{O}p\,\Delta \times S^1 \to \mathbb{R}, \quad k = 1, \dots, |\Lambda|,$$

and observe that $\Phi_0 = \frac{\mu}{N}$ and $\Phi = \Phi_{|\Lambda|}$. Consider the union

$$(B, \zeta) := \bigcup_{k=1}^{|\Lambda|} (\widehat{B}_{k,N}, \hat{\zeta}_{k,N}).$$

More precisely,

$$B := \{\, (w, q, p) \in \mathbb{R}^{2n-1} \times T^*S^1 \mid \frac{\mu}{N} \le p \le \widehat{\Psi}_{|\Lambda|,N} \,\} \subset \mathbb{R}^{2n+1}.$$

Here we have identified $\widehat{\Pi} \subset \mathbb{R}^{2n-1} \times T^*S^1$ with $\mathbb{R}^{2n-1} \times S^1$ so we can view $\widehat{\Psi}_{k,N}$ as functions on $\mathbb{R}^{2n+1}$ with polar coordinates. The almost contact structure ζ is defined to be $\hat{\zeta}_{k,N}$ on each $\hat{B}_{k,N}$.
As we saw in Equation 9.5 the almost contact structure η_ζ can be described by a function $\tilde{\rho} : B \to \mathbb{R}$. Indeed, this can be done for each $\hat{B}_{k,N}$ and since these descriptions agree on the overlap $\hat{B}_{k,N} \cap \hat{B}_{k',N}$ we can do this on the whole of B.

For a suitable choice of $C > 0$ in the construction of $B_{K,C}$ it follows that $B \subset \operatorname{Int} B_{K,C}$. Furthermore, as in the proof of Proposition 5.9 we can extend $\tilde{\rho}$ to a function $\rho : B_{K,C} \to \mathbb{R}$, used in the construction of $\eta_{K,\rho,g}$ such that we have an almost contact embedding

$$(B, \eta_\zeta) \subset (B_{K,C}, \eta_{K,\rho,g}).$$

It is easy to see that this extension can be chosen in such a way that $\eta_{K,\rho,g}|_{B_{K,C}\setminus B}$ is contact. Together with the fact that (B, ζ) decomposes as finitely many saucers from the finite list of Lemma 11.8 this completes the proof. □

The proof of Proposition 3.7 is now a matter of collecting results.

Proposition. *For each dimension $2n+1$ there exists a contact Hamiltonian $(K_{univ}, \Delta_{univ})$ such that the following is true.*
Let M be a $(2n+1)$-dimensional manifold, $A \subset M$ a closed subset, and η an almost contact structure on M which is contact on $\mathcal{O}p\, A \subset M$. Then, there exists (finitely many) disjoint balls $B_i \subset M$, for $i = 1, \dots, L$ such that η is homotopic relative to A, through almost contact structures, to an almost contact structure η' satisfying

- *(i) η' is a contact structure on $M \setminus \bigcup_{i=1}^N B_i$,*
- *(ii) the contact shells $(B_i, \eta'|_{B_i})$ are equivalent to $(B_{K_{univ}}, \eta_{K_{univ}})$ for $i = 1, \dots, L$.*

Proof. First, use Proposition 10.1 which allows us to assume that η is contact outside a finite number of disjoint saucers B_i, $i = 1, \dots, N$, and that the restriction $\eta|_{B_i}$ is a regular semi-contact structure. Next, using Proposition 9.21 we can replace the regular semi-contact saucers B_i by circular model shells defined by time-independent contact Hamiltonians. By Proposition 11.11, these circular model shells can in turn be replaced by finitely many regular semi-contact shells (B_p, ζ^p), $p = 1, \dots, L$ from the finite list of Lemma 11.8. Using Proposition 9.21 we can again replace these saucers by circular model shells (B_{K_p}, η_{K_p}) defined by time-independent contact Hamiltonians. However, at this point the contact Hamiltonians can be assumed to be from a finite list K_p, $p = 1, \dots L$, since we only need one Hamiltonian for each regular semi-contact saucer from the finite list. This allows us to choose any special Hamiltonian K_{univ} satisfying the property that $K_{univ} < \min_p K_p$. □

Remark 11.12. *Although the previous proof is correct in all dimensions, the three dimensional case (where $n = 1$) can be simplified. In fact, in the three dimensional case Proposition 11.11, and hence this chapter, is redundant. In this case the proof starts the same, invoking Proposition 10.1 and Proposition 9.21 to reduce to the case that η is contact everywhere except on finitely many circle model shells defined by time-independent contact Hamiltonians. However, now we do not use Proposition 3.7. Define K_{univ} to be any somewhere negative contact Hamiltonian. Then, by Lemma 5.7 and Proposition 5.9 we can replace all the circle model shells where η is not contact by $(B_{K_{univ}}, \eta_{K_{univ}})$. This concludes the proof.*

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