Universiteit Utrecht

Bachelor Thesis

Applications of sheaves to intuitionistic logic

Author:
Thomas Blom
3915387

Supervisor:
Dr. J. van Oosten

January 19, 2016

Acknowledgements

First and foremost I would like to thank dr. J. van Oosten for his patience and his guidance.

Secondly I would like to thank my girlfriend for distracting me whenever I thought I didn't need it, but I actually did need it.

Contents

1 Introduction

Intuitionism is a philosophy of mathematics that originates with the work of the mathematician L.E.J. Brouwer. It is a constructive approach to approach to mathematics, which means that it is, in a sense, weaker than classical mathematics. For example, the intermediate value theorem from analysis does not hold in intuitionistic mathematics (see [8]).

A possible way to study intuitionism, is to construct models for intuitionistic logic inside classical mathematics. An example of such models are the Kripke models defined in section 2.3. The study of models is called *semantics*. Such models can also be defined constructively, as is done for Kripke models in [12, chapter 2, section 5] by Troelstra and Van Dalen. This thesis will only focus on constructing models inside classical mathematics, since this allows the use of classical reasoning when researching intuitionistic mathematics.

Sheaves are structures that have their origin in the study of analytic continuation of functions. Henri Cartan gave a definition of a sheaf in terms of opens of a topological space at the end of the forties. These sheaves also found their applications in algebraic geometry, where Grothendieck defined the notion of sheaves over sites, and the topos of sheaves over a given site, also called a Grothendieck topos. During the sixties, Lawvere and Tierney gave an axiomatization of *elementary toposes*, of which Grothendieck toposes are an important example. The basic idea of Lawvere was that a topos is a ‘universe of sets’. In a topos, one can interpret (higher-order) logic. [5, Prologue]

The main aim of this thesis is to introduce categories of sheaves over topological spaces, which we will call *sheaf toposes*, show how one can interpret logic in such a sheaf topos, and show some applications of this to the study of intuitionistic logic. We won’t discuss the most general notion of sheaves, sheaves over sites, since this requires a lot of preliminary category theory. A treatment of such sheaves can be found in [5].

In chapter 2, we will discuss a bit of intuitionistic logic and briefly discuss Kripke semantics. In chapter 3, we will study sheaf toposes and show how one can interpret logic in these toposes, which is similar to the interpretation of logic in Kripke models. The advantage of sheaves, is that a sheaf topos can be seen as a ‘mathematical universe’. There are operations which construct new sheaves out of old ones, similar to the way new sets can be constructed out of existing ones. We can, for example, mimick the construction of the continuum using Dedekind cuts in a sheaf topos, providing us with an intuitionistic model of the continuum. In a sheaf topos, it is also possible to interpret higher-order logic, which makes it possible to study, for example, real-valued functions. In chapter 4, we will look at some applications of the theory developed in chapter 3. We will construct the Dedekind reals in a sheaf topos and look at Brouwer’s continuity ‘theorem’, which states that every total real-valued function is continuous. We will also look at the construction of the continuum using Cauchy sequences and compare this to the Dedekind reals. In the appendix we discuss the exponential sheaf, which is defined in section 3.2.5. This discussion has been put in the appendix to avoid cluttering up the text, though some of the lemmas proved in the appendix are necessary for the application in chapter 4.

The reader is assumed to have some basic knowledge of topology and (classical) first-order predicate logic. In particular, the reader is assumed to be familiar with classical model theory and natural deduction in order to understand chapter 2.

2 Intuitionistic logic

Intuitionism can be seen as a constructive approach to mathematics. It is a philosophy according to which mathematics deals with mental constructions. That is to say, mathematics is not about formal manipulation of symbols, neither it is about objects that exist independently of us. This means that a mathematical object only exists when a construction of this object is known. A consequence of this philosophy is that mathematical statements cannot be assumed to simply be true or false. A statement is true if we know a proof of it, a statement is false if we can show that assuming the existence of such a proof leads to a contradiction. [12, chapter 1, section 1.6]

Such a philosophy of mathematics implies that one cannot use ordinary mathematical logic. For example, in classical mathematics, the law of excluded middle, which is the statement that a proposition P is either true or false, is taken for granted. In intuitionistic mathematics, such a principle cannot be accepted, since this would imply that for every statement we either have a proof that this statement is true, or have a proof that it is false (i.e. we can show that the assumption that there is a proof for P leads to a contradiction). For example, if our proposition P is the Goldbach conjecture, then we cannot assume that P is either true or false, since it has not yet been proved, yet it also has not been disproved yet. When expressed in propositional logic, this means that the statement

$$P \vee \neg P$$

is not valid. This also means we cannot use the law of excluded middle in a proof, hence a lot of statements that are valid in classical logic, are not valid in intuitionistic logic.
In the next section, we will look at two so-called 'weak counterexamples', which are the kind of counterexamples L.E.J. Brouwer used to show that the law of excluded middle should not be valid [8, p.2]. They are called weak counterexamples because they don't actually derive a contradiction from the assumption that the law of excluded middle is valid, but they do show why it's not valid in intuitionistic mathematics [12, p.11]. We will then take a look at the intuitionistic version of first-order predicate logic and describe a kind of semantics for this, called Kripke semantics.

2.1 Weak counterexamples

Define, as on page 3 of [8], the sequence $(a_k)_{k\in\mathbb{N}}$ by

$$a_k = \begin{cases} 2^{-k_0} & \text{if } k > k_0 \text{ and } k_0 \text{ is the first} \\ & \text{counterexample to Goldbach's conjecture;} \\ 2^{-k} & \text{else.} \end{cases}$$

Note that any a_k can be calculated in a finite amount of time, so this a constructive defnition. It is also a Cauchy sequence, so it defines a real number α.
We cannot however (yet) say $\alpha = 0$ or $\alpha > 0$, since this would imply either a proof of, or a counterexample to, the Goldbach conjecture. We also cannot claim $\alpha = 0 \vee \alpha \neq 0$ for the same reason, so we see again that the law of excluded middle is not valid. In the future, however, we might be able to claim $\alpha = 0$ or $\alpha > 0$, if the Goldbach conjecture is either

proved or disproved. This counterexample also teaches something about the structure of the (intuitionistic) continuum. We see that $\alpha < 0 \vee \alpha = 0 \vee \alpha > 0$ is not valid, hence $\forall x \in \mathbb{R}\ (x < 0 \vee x = 0 \vee x > 0)$ is not valid in the intuitionistic continuum.
In [12, p.13], Troelstra and Van Dalen give a similar example, using Fermat's last theorem. They define the formula $A(n)$ by

$$\forall m_1, m_2, m_3, k\ (m_1 + m_2 + m_3 + k \leq n \rightarrow (m_1+1)^{k+3} + (m_2+1)^{k+3} \neq (m_3+1)^{k+3})$$

and define the real number x via the Cauchy sequence

$$x_k = \begin{cases} 2^{-k_0} & \text{if } k > k_0 \text{ and } k_0 \text{ is the first} \\ & \text{natural number such that } A(k) \text{ does not hold;} \\ 2^{-k} & \text{else.} \end{cases}$$

When *Constructivism in mathematics* [12] was published in 1988, a proof of Fermat's last theorem was not known, so from an intuitionistic standpoint, one could not claim $x = 0 \vee x > 0$. These days, a proof for Fermat's last theorem is known, so there is a proof for $x = 0$, in particular there is a proof for $x > 0 \vee x = 0$. This illustrates that the information available at a certain moment can influence whether a statement is true. This idea can be seen as a motivation for the Kripke semantics which we will define in section 2.3.

2.2 Intuitionistic first-order predicate logic

In (classical) first-order predicate logic, a deductive system consists of a first-order language $\mathcal{L}$, a collection of sentences in this language called *axioms*, and *rules of inference* (also called *natural deduction rules*). Using these rules of inference, one formalizes the notion of proof in mathematics, which can be done through the use of *proof trees*. In intuitionistic first-order predicate logic, this is done in the same way. The only difference is that $\neg\phi$ is considered an abbreviation for $\phi \rightarrow \bot$ and that, when $\bot$ is inferred from $\neg\phi$, one may not conclude ϕ. A treatment of this deductive system can be found in [12, chapter 2, section 1]. One can also find a motivation there on why this system is used to describe intuitionistic reasoning. If we are given an $\mathcal{L}$-sentence ϕ and a collection of $\mathcal{L}$-sentences Γ, then we write $\Gamma \vdash \phi$ if there exists an intuitionistic proof of ϕ having only sentences from Γ as assumptions.

2.3 Kripke semantics

If one uses the classical notion of a structure for a first-order language, then classical logic is valid in this structure. In this section, we will define what a Kripke model is for a first-order language. They form an interpretation for intuitionistic first-order predicate logic and can be defined in classical mathematics. They can be motivated by the idea that, whether a statement P is true at a certain moment, depends on the information available at that time. The following definitions are based on the ones given in [12, chapter 2, section 5], although slightly different.

Definition 2.3.1. Let $\mathcal{L}$ be a first-order language. A Kripke model for this language $\mathcal{L}$ is a triple $\mathcal{K} = (K, \leq, D)$, where $(K, \leq)$ is a non-empty partially ordered set and D is a function assigning an $\mathcal{L}$-structure $D(k)$ to every element k of K, in such a way that:

1. $D(k) \subseteq D(k')$ if $k' \leq k$, for all $k, k' \in K$.
2. For every n-ary function symbol f, the interpretations $f_k : D(k)^n \to D(k)$ coincide, so $f_k(x) = f_{k'}(x)$ if $k' \leq k$ and $x \in D(k)$.
3. For every n-ary relation symbol R, the interpretations $R_k \subseteq D(k)^n$ satisfy $R_k \subseteq R_{k'}$ if $k' \leq k$.
4. For every constant c, the interpretations $c_k \in D(k)$ satisfy $c_k = c_{k'}$ if $k' \leq k$.

Note that this slightly differs from the definition usually given, where $D(k') \subseteq D(k)$ if $k' \leq k$ (in particular the one given in [12]). The reason we dualized this, is that the similarities between Kripke semantics and sheaves will then be more visible in section 3.3. We can think of the $k \in K$ as 'possible moments in time', where k' is a later moment than k if $k' < k$, and where $D(k)$ is the 'information' available at moment k.
We now give an inductive interpretation for terms:

Definition 2.3.2. Let t be a term with free variables $x_1, \ldots, x_n$, let $k \in K$ and let $m_1, \ldots, m_n \in D(k)$. Then define the interpretation $t^M(m_1, \ldots, m_n)$ in the following way:

1. If t is a constant symbol c, then $t^M(m_1, \ldots, m_n) = c_k$.
2. If t is of the form $f(t_1, \ldots, t_l)$, where f is a l-ary function symbol and $t_1, \ldots, t_l$ are terms, then $t^M(m_1, \ldots, m_n) = f_k(t_1^M(m_1, \ldots, m_n), \ldots, t_l^M(m_1, \ldots, m_n))$.

We now inductively define a relation $\Vdash$, called 'forcing', between elements of K and formulas of $\mathcal{L}$. If $k \Vdash \phi$ holds, then we say that k forces ϕ.

Definition 2.3.3. Let $k \in K$ and let ϕ be an $\mathcal{L}$-formula with free variables $x_1, \ldots x_n$ and let $m_1, \ldots, m_n \in D(k)$. We then define:

1. For atomic ϕ, so ϕ is of the form $\bot$, $(t_1 = t_2)$ or $R(t_1, \ldots, t_l)$, we set

$$\begin{array}{rl} k \Vdash \bot & \text{never holds} \\ k \Vdash (t_1 = t_2) & \text{iff} \quad t_1(m_1, \ldots, m_n) = t_2(m_1, \ldots, m_n) \\ k \Vdash R(t_1, \ldots, t_l)(m_1, \ldots, m_n) & \text{iff} \quad (t_1(m_1, \ldots, m_n), \ldots, t_l(m_1, \ldots m_n)) \in R_k \end{array}$$

2. If ϕ is of the form $\phi_1 \wedge \phi_2$, then

$$k \Vdash \phi(m_1, \ldots, m_n) \quad \text{iff} \quad k \Vdash \phi_1(m_1, \ldots, m_n) \quad \text{and} \quad k \Vdash \phi_2(m_1, \ldots, m_n)$$

3. if ϕ is of the form $\phi_1 \vee \phi_2$, then

$$k \Vdash \phi(m_1, \ldots, m_n) \quad \text{iff} \quad k \Vdash \phi_1(m_1, \ldots, m_n) \quad \text{or} \quad k \Vdash \phi_2(m_1, \ldots, m_n)$$

4. If ϕ is of the form $\phi_1 \to \phi_2$, then $k \Vdash \phi(m_1, \ldots, m_n)$ if and only if for every $k' \leq k$,

$$\text{if} \quad k' \Vdash \phi_1(m_1, \ldots, m_n), \quad \text{then} \quad k' \Vdash \phi(m_1, \ldots, m_n)$$

5. If ϕ is of the form $\neg\psi$, then $k \Vdash \phi(m_1, \ldots, m_n)$ if and only if for no $k' \leq k$, $k' \Vdash \psi(m_1, \ldots, m_n)$
6. If ϕ is of the form $\exists x\psi(x)$, then $k \Vdash \phi(m_1, \ldots, m_n)$ if and only if for some $m \in D(k)$, $k \Vdash \psi(m, m_1, \ldots, m_n)$
7. If ϕ is of the form $\forall x\psi(x)$, then $k \Vdash \phi(m_1, \ldots, m_n)$ if and only if for all $k' \leq k$ and all $m \in D(k')$, $k' \Vdash \psi(m, m_1, \ldots, m_n)$.

Note that the definition of $k \Vdash \neg\psi$ can also be deduced from the definition of $k \Vdash (\phi_1 \to \phi_2)$ and the fact that $k \Vdash \perp$ never holds.
Most parts of this definition are quite intuitive, although part 4, 5 and 7 might seem somewhat strange. If we see elements $k \in K$ as moments in time, we can interpret $k \Vdash \phi$ as 'ϕ holds at moment k' or 'there exists a proof of ϕ at moment k'. We can then make sense of part 4,5 and 7. For $\phi_1 \to \phi_2$ to hold, we also want it to hold at later moments, so we don't just want that ϕ_1 implies ϕ_2 at moment k, but also at later moments. Similarly, if $\neg\phi$ means that a proof of ϕ cannot exist, then also at later moments one should not be able to prove ϕ. For $\forall x\psi$, we similarly don't want this to hold at just at moment k, but also at later moments. An easy consequence of the definition for $\Vdash$, is that if $k \Vdash \phi$, then $k' \Vdash \phi$ for all $k' \leq k$. If for every $k \in K$, we have $k \Vdash \phi$, then we write $\mathcal{K} \Vdash \phi$.
If we are given a set of $\mathcal{L}$-sentences Γ, we write $\mathcal{K} \Vdash \Gamma$ if $\mathcal{K} \Vdash \psi$ for every $\psi \in \Gamma$, and we write $\Gamma \Vdash \phi$ for an $\mathcal{L}$-sentence ϕ if for every Kripke model $\mathcal{K}$ such that $\mathcal{K} \Vdash \Gamma$, we have $\mathcal{K} \Vdash \phi$. We have the following theorem, which allows the use of Kripke models to study intuitionistic logic:

Theorem 2.3.4 (Soundness theorem). *Let Γ be a set of $\mathcal{L}$-sentences. Then $\Gamma \Vdash \phi$ if $\Gamma \vdash \phi$.*

A proof of this can be found in [12, chapter 2, section 5]. One can in fact also prove the converse if the interpretation of equality in a Kripke model is slightly changed, a proof of this can be found in section 6 of chapter 2 of [12].

3 Interpreting logic in toposes of sheaves

In this section we will take a quick look at what a category is. We will then study a special kind of category, the category of sheaves over a topological space. Categories of this kind are an example of toposes, categories with a certain structure that allows for the interpretation of higher-order intuitionistic logic. We will only study these kind of toposes, because a full treatment of topos theory is far beyond the scope of this thesis. We only study toposes of sheaves over a topological space, since this makes the theory a lot easier and we don't need other kinds of toposes for our applications in chapter 4. From now on, we will simply refer to a *topos of sheaves over a topological space* as a *sheaf topos*. Section 3.1 is based on the first three chapters of [7]. Section 3.2 is based on the first two chapters of [6] and §7 of [3], where I have tried to rewrite and simplify definitions to the context of sheaves of topological spaces, instead of sheaves over sites or complete Heyting algebra's. Section 3.3 is based on chapters 4 and 7 of [7], chapters 1 and 2 of [6] and §5 and §7 of [3].

3.1 Some category theory

In order to understand something about toposes of sheaves over a topological space, we will first need a little category theory, which we will cover in this section.

3.1.1 Definition of a category

A *category* $\mathcal{C}$ consists of two classes $\mathcal{C}_0$ and $\mathcal{C}_1$ of *objects* and *arrows* respectively, having the following structure:

1. Every arrow has a 'domain' and a 'codomain'. This means that there are two operations dom and cod, which assign an object to every arrow. If we are given an $f \in \mathcal{C}_1$, this is often written as $f : X \to Y$, where $X, Y \in \mathcal{C}_0$ are the domain and the codomain of f respectively.

2. Given two arrows f, g where $\text{cod}(f) = \text{dom}(g)$, there exists a composition $gf : \text{dom}(f) \to \text{cod}(g)$. So if $f : X \to Y$ and $g : Y \to Z$, there exists a composition $gf : X \to Z$.

3. This composition is associative. So if we are given f, g, h which can be composed, then $h(gf) = (hg)f$.

4. For every object X, there exists an *identity arrow* $\text{id}_X : X \to X$, such that for any f with domain X, we have $f\text{id}_X = f$ and for any g with codomain X, we have $\text{id}_X g = g$.

Categories exist everywhere in mathematics. One can, for example, consider the category of sets, whose objects are sets and whose arrows are functions. Note that for almost any kind of mathematical structure, we get objects and certain morphisms between these objects. Examples are groups and homomorphisms, topological spaces and continuous maps and posets and monotone maps. All these structures give rise to a category. We can, for example, consider the category of groups, where the objects are groups and the arrows are homomorphisms. Later, we will consider the category of sheaves over a topological space, where the arrows are sheaf morphisms.

There is one other type of category that will be important later, the category of opens of a topological space. Let T be a topological space. We define the category of opens $\mathcal{T}$ by setting $\mathcal{T}_0$ to be the opens of T. There is a unique arrow $U \to V$ if and only if $U \subseteq V$. The composition of two arrows $U \to V$ and $V \to W$ is simply defined to be the arrow $U \to W$.

If we are given an arrow $f : C \to C'$ in a category $\mathcal{C}$, then we say that f is an *isomorphism* is there exists an arrow $g : C' \to C$ such that $gf = \mathrm{id}_C$ and $fg = \mathrm{id}_{C'}$. In the category of sets, these are simply bijections between sets. In the category of groups, these are the isomorphism as they are usually defined in group theory. In the category of topological spaces, these are homeomorphisms. If there exists an isomorphism $f : C \to C'$ in $\mathcal{C}$, then we call C and C' isomorphic. Isomorphic objects are, in a sense, the same: they satisfy the same categorical properties.

3.1.2 Functors and natural transformations

Definition 3.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ is an operation which assigns an object $F(X) \in \mathcal{D}_0$ to every $X \in \mathcal{C}_0$ and an arrow $F(f) \in \mathcal{D}_1$ to every arrow $f \in \mathcal{C}_1$. For every arrow $f : X \to Y$, it has to satisfy $F(f) : F(X) \to F(Y)$, and if we are given two arrows $f : X \to Y$ and $g : Y \to Z$, then it has to satisfy $F(gf) = F(g)F(f)$. Furthermore, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for each $X \in \mathcal{C}_0$.

An example of a functor is the forgetful functor U : Grp $\to$ Set, where Grp is the category of groups and Set is the category of sets. It sends a group $(G, \cdot)$ to its underlying set G and homomorphisms are sent to the corresponding function. It forgets, in a sense, the group structure. A similar functor can be defined for any mathematical structure with an underlying set structure and arrows which are functions between these sets, so there also is a forgetful functor on the category of topological spaces or posets.

If we are given two functors $F, G : \mathcal{C} \to \mathcal{D}$, we can define what a natural transformation $\mu : F \to G$ is.

Definition 3.1.2. A *natural transformation* $\mu : F \to G$ is a collection of arrows $(\mu_C : F(C) \to G(C))_{C \in \mathcal{C}_0}$ in $\mathcal{D}$, indexed by the objects in $\mathcal{C}$, such that the following diagram commutes

$$\begin{array}{ccc} F(C) & \xrightarrow{\mu_C} & G(C) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(C') & \xrightarrow[\mu_{C'}]{} & G(C') \end{array}$$

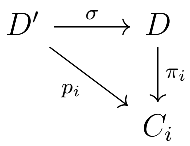
for every arrow $f : C \to C'$ in $\mathcal{C}$.

We won't look at examples of natural transformations here, since it will only complicate things at this point, but we will see examples when we start discussing sheaves.
It is easy to see that if we have two natural transformations $\mu : F \to G$ and $\nu : G \to H$, we can define a natural transformation $\nu\mu$ which consists of the arrows $(\nu_C \mu_C)_{C \in \mathcal{C}_0}$. If we are given two categories $\mathcal{C}$ and $\mathcal{D}$, we can define the category $\mathcal{D}^{\mathcal{C}}$ whose objects are functors $F : \mathcal{C} \to \mathcal{D}$ and whose arrows are natural transformations between functors. When we start discussing presheaves, we will see an example of such a category.

3.1.3 Products in a category

Let $\mathcal{C}$ be a category. If we are given a collection of objects $(C_i)_{i\in I}$, we can define what a product of $(C_i)_{i\in I}$ in the category $\mathcal{C}$ is.

Definition 3.1.3. We define a *product* of $(C_i)_{i\in I}$ to be an object D together with a collection of arrows $\pi_i : D \to C_i$ for all $i \in I$ satisfying the following property: if we are given an object D' and a collection of arrows $p_i : D' \to C_i$ for all $i \in I$, then there exists a unique arrow $\sigma : D' \to D$ such that $p_i = \pi_i \sigma$ for all $i \in I$. So σ is the unique arrow such that the diagram



commutes for every $i \in I$. The maps π_i are usually called projections and the map σ is called the product of the maps p_i.

A collection $(C_i)_{i\in I}$ does not need to have a product and if a product exists, there usually exist several (isomorphic) products. We have the following lemma:

Lemma 3.1.4. *Let $(C_i)_{i\in I}$ be a collection of objects in $\mathcal{C}$ and assume D together with the projections π_i is a product of $(C_i)_{i\in I}$. Then $D' \in \mathcal{C}_0$ is a product of $(C_i)_{i\in I}$ if and only if D' is isomorphic to D.*

Proof. First assume we are given a $D' \in \mathcal{C}$ and an isomorphism $\phi : D' \to D$. Then the object D' together with the maps $\pi_i \phi$ is a product of $(C_i)_{i\in I}$.
Now assume we are given a product $D' \in \mathcal{C}$ with projections π'_i. Since both D and D' with the projections π_i and π'_i are a product, we get two unique maps $\sigma : D' \to D$ and $\sigma' : D \to D'$ such that $\pi'_i = \pi_i \sigma$ and $\pi_i = \pi'_i \sigma'$ for all $i \in I$. We claim that these maps are inverse to each other. Note that $\pi_i = \pi_i \sigma \sigma'$ and $\pi'_i = \pi'_i \sigma' \sigma$ for all i. By definition of a product, the maps $\sigma\sigma'$ and $\sigma\sigma'$ are the unique maps having this property, hence $\sigma\sigma' = \mathrm{id}_D$ and $\sigma'\sigma = \mathrm{id}_{D'}$. ■

In Set, a product of a collection $(C_i)_{i\in I}$, where I is a set, always exists. We can take the usual product of sets $\Pi_{i\in I} C_i$ with the usual projection maps as our product. It is easy to check that this satisfies the definition of a product in a category: assume an object D' with maps $p_i : D' \to C_i$ is given. Then define $\sigma : D' \to D$ to be the product of the maps p_i, so if $x \in D'$, then $\sigma(x) = (p_i(x))_{i\in I}$. If I is a finite set $\{1, ..., n\}$, then this is simply the map $x \mapsto (p_1(x), ..., p_n(x))$.
There is a special kind of product, the 'empty' product, where the index set I is simply empty. If we read the definition carefully, it is clear that an 'empty' product is an object $D \in \mathcal{C}$ such that there exists exactly one arrow $f : C \to D$ for every object C. An object satisfying this property is called a terminal object. There can also be similar object, which we call the initial object:

Definition 3.1.5. Let $\mathcal{C}$ be a category. An object C in $\mathcal{C}$ is called a *terminal object* if for every object C', there exists exactly one arrow $f : C' \to C$. An object C is called an *initial object* if for every object C', there exists exactly one arrow $f : C \to C'$.

In Set, the empty set is the (unique) initial object and the terminal objects are the singletons.

3.2 Toposes of sheaves over a topological space

In this section, we will define, for a given topological space T, the topos of sheaves over this space, which we denote by $\mathrm{Shv}(T)$. We will then study the structure of this topos a bit, enough to be able to interpret logic in this topos in section 3.3.

3.2.1 Presheaves over a topological space

As seen above, if we have a topological space T, we can consider the category $\mathcal{T}$ of opens of this space, where there is an arrow $U \to V$ if and only if $U \subseteq V$. We can also dualize this category by saying that there is an arrow $U \to V$ if and only if $V \subseteq U$. This category will be denoted by $\mathcal{T}^{\mathrm{op}}$. We will now define what a presheaf over a topological space is.

Definition 3.2.1. Let T be a topological space. A *presheaf* F over T is a functor $F : \mathcal{T}^{\mathrm{op}} \to \mathrm{Set}$. The category of presheaves over T is the category $\mathrm{Set}^{\mathcal{T}^{\mathrm{op}}}$, whose objects are presheaves and whose arrows are natural transformations. We will call these arrows *presheaf morphisms*, or simply *morphisms*, from now on.

When discussing presheaves, we usually assume a topological space T is given.
This definition might seem somewhat abstract, so we will take a look at what presheaves are and then consider some examples. A presheaf F over T consists of a set $F(U)$ for every open $U \subseteq T$ and a map $F(U \subseteq V) : F(V) \to F(U)$ for every pair $U \subseteq V$ of opens. The elements of $F(U)$ are called *sections* (over U) and the elements of $F(T)$ are sometimes called *global sections*. If we are given $x \in F(U)$ and $V \subseteq U$ open, then we will denote $F(U \subseteq V)(x)$ by $x \restriction V$. These maps are called *restrictions*. Since a functor $F : \mathcal{T}^{\mathrm{op}} \to \mathrm{Set}$ has to preserve composition of arrows in $\mathcal{T}^{\mathrm{op}}$, we see that for any opens $W \subseteq V \subseteq U$ and any $x \in U$, the following holds:

$$(x \restriction V) \restriction W = x \restriction W. \tag{1}$$

Since a functor has to preserve the identity, we see that

$$x \restriction U = x \quad \text{if} \quad x \in F(U) \tag{2}$$

for any open $U \subseteq T$. We therefore see that a presheaf is actually a collection of sets $F(U)$ indexed by the opens $U \subseteq T$ together with a collection of functions $(\cdot) \restriction V : F(U) \to F(V)$, for every pair of opens $V \subseteq U$, which satisfy equalities (1) and (2).
A presheaf morphism ϕ is a natural transformation between two presheaves F and G. This means that a presheaf morphism is a collection of maps $\phi_U : F(U) \to G(U)$ such that, for $V \subseteq U$, the equation $\phi_U(x) \restriction V = \phi_V(x \restriction V)$ holds for all $x \in F(U)$. We will often omit subscripts for presheaf morphisms, since it is usually clear from the context what the subscript should be.

Two examples of presheaves are the empty presheaf $\mathbf{0}$, where $\mathbf{0}(U) = \emptyset$ for all U and the presheaf $\mathbf{1}$ where every $\mathbf{1}(U) = \{*\}$ is a singleton set. It is easy to see that the first presheaf is an initial object and the second one a terminal object. A morphism from $\mathbf{0}$ to a presheaf F always consists of empty maps $\emptyset \to F(U)$, while a morphism from F to $\mathbf{1}$ always consists of maps that send every element of $F(U)$ to the element $*$. If $\phi : \mathbf{1} \to F$ is a morphism, then it simply picks a global section, since if $\phi_T(*) = x$, then $\phi_U(*) = x \restriction U$

is fully determined by x. So there is a 1-1 correspondence between morphisms $\phi : \mathbf{1} \to F$ and global sections of F.
Another example of a presheaf is Ω, which is defined by $\Omega(U) = \{V \subseteq U \mid V \text{ is open}\}$ and where, for $V, W \subseteq U$ open, $V \upharpoonright W = V \cap W$.
For any topological space X, we get the presheaf $\widetilde{X}$ over T of continuous maps to X. The sections are defined by

$$\widetilde{X}(U) = \{f : U \to X \mid f \text{ is continuous}\}$$

and the restrictions are defined by $f \upharpoonright V = f|_V$, so ordinary function restriction. If we are given another topological space Y and a continuous map $f : X \to Y$, then we get a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ defined by

$$\widetilde{f}_U(g) = f \circ g.$$

It is quite easy to see that $\widetilde{(\cdot)}$ is in fact a functor from the category Top of topological spaces to the category of presheaves over T.
We can make this a little more general. If we are given a map $f : X \times T \to Y$, we can also define a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$, by setting

$$\widetilde{f}(g)(x) = f(g(x), x).$$

In chapter 4 and the appendix, these presheaves and morphisms are important.
Lastly, if we are given a presheaf F, we can define, for any open $U \subseteq T$, the presheaf $F \upharpoonright U$ by

$$(F \upharpoonright U)(V) = \begin{cases} F(V) & \text{if } \quad V \subseteq U \\ \emptyset & \text{otherwise.} \end{cases}$$

This can be seen as a restriction of the presheaf F to the open $U \subseteq T$. The reason we use this notation will become clear when we start to discuss power objects in section 3.2.5. Note that this 'restriction' to $U \subseteq T$ is in fact a functor from the category of presheaves over T to the category of presheaves over U. If we are given a morphism $f : F \to G$, then we can define a morphism $f \upharpoonright U : F \upharpoonright U \to G \upharpoonright U$ by defining $(f \upharpoonright U)_V = f_V$ if $V \subseteq U$, and $(f \upharpoonright U)_V$ is the empty function otherwise.

3.2.2 Sheaves over a topological space

A sheaf is a special kind of presheaf. Before we can define what a sheaf is, we need some extra definitions.

Definition 3.2.2. Let F be a presheaf. Let $U \subseteq T$ be open and assume $\mathcal{U} = \{U_i\}_{i \in I}$ is a cover of U, i.e. $\cup_i U_i = U$. A *compatible family* $(x_i)_{i \in I}$ with respect to $\mathcal{U}$ is a collection of sections $x_i \in F(U_i)$ which are 'pairwise compatible', i.e. $x_i \upharpoonright (U_i \cap U_j) = x_j \upharpoonright (U_i \cap U_j)$ for all $i, j \in I$.

Definition 3.2.3. Let F be a presheaf, $U \subseteq T$ open and let $(x_i)_{i \in I}$ be a compatible family with respect to a cover $\{U_i\}_{i \in I}$ of U. We call a section $x \in F(U)$ an *amalgamation* of $(x_i)_{i \in I}$ if, for any $i \in I$, we have $x \upharpoonright U_i = x_i$.

If we consider the presheaf $\widetilde{X}$ for some topological space X, then a compatible family with respect to some cover $\mathcal{U}$ of $U \subseteq T$ is a collection of continuous maps $f_i : U_i \to X$. It is clear that a compatible family, in this case, has exactly one amalgamation, namely the function f defined by

$$f(x) = f_i(x) \text{ if } x \in U_i.$$

The existence of such a unique amalgamation is exactly what defines a sheaf.

Definition 3.2.4. A presheaf F is a *sheaf* if, for any $U \subseteq T$, every compatible family has exactly one amalgamation. If we are given a compatible family $(x_i)_{i\in I}$, we denote its amalgamation by $\bigvee x_i$. The category of sheaves, denoted by $\mathrm{Shv}(T)$, is the category which has sheaves over T as objects and presheaf morphisms as arrows, which from now on will be called *sheaf morphisms* or simply *morphisms*.

We will from now on call a category of sheaves a *sheaf topos*. As we just saw, all presheaves of the form $\widetilde{X}$ for some topological space X are sheaves. In fact, all examples of presheaves we saw above are sheaves, except for $\mathbf{0}$.
There are two possible reasons why a presheaf might not be a sheaf. The first is that some amalgamations might not exist and the second reason is that they might not be unique (or, of course, a combination of both). In the case of $\mathbf{0}$, the first is what goes wrong. This might sound strange at first, since it looks like there are no compatible families. The empty set is a cover of $\emptyset \subseteq T$ though, so there exists exactly one compatible family, namely the empty compatible family (with respect to the empty cover of $\emptyset$). This implies that the set $\mathbf{0}(\emptyset)$ should have exactly one element. This is something that holds for any sheaf F, the set $F(\emptyset)$ should always have exactly one element. A consequencence of this is that the sheaf F defined by $F(\emptyset) = \{*\}$ and $F(U) = \emptyset$ otherwise, is an initial object in the category of sheaves.
It might sound strange that sheaf morphisms are the same as presheaf morphisms. A sheaf can be seen as a presheaf with an additional property (see definition 3.2.4) which induces an operation $\bigvee$ on compatible families. One would expect a sheaf morphism to commute with this operation. Luckily, we have the following lemma.

Lemma 3.2.5. *Let F, G be sheaves and let $\phi : F \to G$ be a presheaf morphism. If $(x_i)_{i\in I}$ is a compatible family in F for $U \subseteq T$, then $\phi_U(\bigvee x_i) = \bigvee \phi_{U_i}(x_i)$.*

Proof. If $(x_i)_{i\in I}$ is a compatible family, then so is $(\phi_{U_i}(x_i))_{i\in I}$. This is a direct consequence of the naturality of ϕ. Hence, there exists a unique amalgamation $\bigvee \phi_{U_i}(x_i)$. All we need to show is that $\phi_U(\bigvee x_i)$ is an amalgamation of $(\phi_{U_i}(x_i))_{i\in I}$. This follows from the naturality of ϕ, since for all $j \in I$ we have

$$\phi_U\left(\bigvee x_i\right) \upharpoonright U_j = \phi_{U_j}\left(\bigvee x_i \upharpoonright U_j\right) = \phi_{U_j}(x_j).$$

■

Before we go on, we will prove the following lemma:

Lemma 3.2.6. *Let F and G be (pre)sheaves and $\phi : F \to G$ a morphism. Then ϕ is an isomorphism if and only if all the maps $\phi_U : F(U) \to G(U)$ are bijections.*

Proof. First assume ϕ is an isomorphism and let ψ be its inverse. Then $\psi\phi = \mathrm{id}_F$ and $\phi\psi = \mathrm{id}_G$, in particular $\psi_U\phi_U = \mathrm{id}_{F(U)}$ and $\phi_U\psi_U = \mathrm{id}_{G(U)}$, hence all maps ϕ_U are bijections.

Now assume all maps ϕ_U are bijections. We define $\psi : G \to F$ by $\psi_U = \phi_U^{-1}$. To show that ψ is a morphism, we need to show that it's natural. Let $V \subseteq U \subseteq T$ open and let $x \in G(U)$. Note that $x = \phi(y)$ for some $y \in F(U)$. We see that

$$\psi_U(x) \upharpoonright V = \phi_U^{-1}(\phi_U(y)) \upharpoonright V = y \upharpoonright V = \phi_V^{-1}(\phi_V(y \upharpoonright V)) = \phi_V^{-1}(\phi_U(y) \upharpoonright V) = \psi_V(x \upharpoonright V),$$

so the naturality of ψ follows from the naturality of ϕ. Since they are clearly inverse to each other, we conclude that ϕ is an isomorphism. ■

3.2.3 Products of sheaves

In the following sections, we will study the structure of a sheaf topos, so that we will be able to interpret intuitionistic logic in a sheaf topos in section 3.3.
If we are given two sheaves F and G, we define their product $F \times G$ by

$$(F \times G)(U) = F(U) \times G(U) \quad \text{and} \quad (x, y) \upharpoonright V = (x \upharpoonright V, y \upharpoonright V).$$

We also get two projection morphisms $\pi_1 : F \times G \to F$ and $\pi_2 : F \times G \to G$, defined by

$$\pi_1((x, y)) = x \quad \text{and} \quad \pi_2((x, y)) = y$$

Lemma 3.2.7. *The product of two sheaves $F \times G$ is also a sheaf, and it is a product of F and G in Shv(T) in the categorical sense (see definition 3.1.3).*

Proof. It is easy to see that $F \times G$ is a sheaf if F and G are sheaves. If $((x_i, y_i))_{i\in I}$ is a compatible family, then so are $(x_i)_{i\in I}$ in F and $(y_i)_{i\in I}$ in G. The unique amalgamation of $((x_i, y_i))_{i\in I}$ in $F \times G$ is $(\bigvee x_i, \bigvee y_i)$.
Assume we are given a sheaf H and two morphisms $p_1 : H \to F$ and $p_2 : H \to G$. We define the morphism $\sigma : H \to F \times G$ by $\sigma(x) = (p_1(x), p_2(x))$. It is easy to see that this is a morphism and in fact the only morphism $H \to F \times G$ such that $\pi_i\sigma = p_i$ for $i = 1, 2$. ■

This proof can be easily extended to arbitrary products $\Pi_{i\in I}F_i$, the details are left to the reader.

3.2.4 Subsheaves and the subobject classifier

Just like we can define subsets of sets, we can also give a definition of subsheaves of a sheaf.

Definition 3.2.8. If we are given a presheaf F, then we call G a *subpresheaf* of F if $G(U) \subseteq F(U)$ for all open $U \subseteq T$. If F and G are both sheaves, then we call F a *subsheaf* of G.

An interesting question to ask, is which subpresheaves of a sheaf F are in fact a subsheaf. Assume we are given a sheaf F and a subpresheaf G. If we are given a compatible family $(x_i)_{i\in I}$ in G, then this is also compatible family in F, hence it has a unique amalgamation. Therefore, if $(x_i)_{i\in I}$ has an amalgamation in G, it is necessarily unique. Therefore, a subpresheaf of a sheaf F is a subsheaf if and only if it contains all amalgamations of its compatible families. This leads us to the following lemma.

Lemma 3.2.9. *If we are given a sheaf F and a subpresheaf G, then there exists a smallest subsheaf $\overline{G}$ of F containing G. Furthermore, if we are given a presheaf morphism $\phi : G \to H$ for some sheaf H, then there is a unique extension $\overline{\phi} : \overline{G} \to H$.*

Proof. We define $\overline{G}$ in the following way:

$$\overline{G}(U) = \{x \in F(U) \mid \text{There exists a compatible family } (x_i)_{i\in I} \text{ in } G \text{ such that } \bigvee x_i = x\}.$$

This can be seen as 'adding the amalagamations to G.' It is clear that any subsheaf G' of F containing G, also contains $\overline{G}$. Therefore we only need to show that $\overline{G}$ is a sheaf. Assume we are given a compatible family $(x_i)_{i\in I}$ in $\overline{G}$ and let x be the section of F such that $x = \bigvee x_i$. We need to show that $x \in \overline{G}$. For any $i \in I$, we can write $x_i = \bigvee_{j\in J_i} x_{i,j}$ for some compatible family $(x_{i,j})_{j\in J_i}$ in G. Define the index set $K = \{(i,j) \mid i \in I, j \in J_i\}$. Then $(x_{i,j})_{(i,j)\in K}$ is a compatible family in G such that $\bigvee_{(i,j)\in K} x_{i,j} = x$, so $x \in \overline{G}$. We conclude that $\overline{G}$ is a sheaf, hence the smallest subsheaf of F containing G.

Assume we are given another sheaf H and a presheaf morphism $\phi : G \to H$. It is easy to see that an extension $\overline{\phi} : \overline{G} \to H$ has to be unique. If $x \in \overline{G}$, write $x = \bigvee x_i$ for sections x_i of G. Then $\overline{\phi}$ has to satisfy

$$\overline{\phi}(x) = \bigvee \overline{\phi}(x_i) = \bigvee \phi(x_i), \tag{3}$$

hence it is fully determined by ϕ. We therefore define $\overline{\phi} : \overline{G} \to H$ by (3). We need to show that this is well-defined, so we need to show that the definition does not depend on our choice of compatible family $(x_i)_{i\in I}$. Let $x \in \overline{G}(U)$ and let $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ be compatible families in G with respect to the covers $\{U_i\}_{i\in I}$ and $\{U_j\}_{j\in J}$ of U, such that $\bigvee x_i = \bigvee y_j = x$. Define the cover $\{V_{ij}\}_{(i,j)\in I\times J}$ by $V_{ij} = U_i \cap U_j$. This is a cover over U and a refinement of both $\{U_i\}_{i\in I}$ and $\{U_j\}_{j\in J}$. Define $x_{ij} = x_i \restriction V_{ij}$ and $y_{ij} = y_j \restriction V_{ij}$. Then $(x_{ij})_{(i,j)\in I\times J}$ and $(y_{ij})_{(i,j)\in I\times J}$ are both compatible families. It is easy to see that $\bigvee \phi(x_{ij}) = \bigvee \phi(x_i)$ and $\bigvee \phi(y_{ij}) = \bigvee \phi(y_j)$, since

$$\left(\bigvee \phi(x_i)\right) \restriction V_{ij} = \left(\left(\bigvee \phi(x_i)\right) \restriction U_i\right) \restriction V_{ij} = \phi(x_i) \restriction V_{ij} = \phi(x_{ij})$$

and similarly for y_{ij}. We also see that $x_{ij} = x \restriction V_{ij} = y_{ij}$, hence

$$\bigvee \phi(x_i) = \bigvee \phi(x_{ij}) = \bigvee \phi(y_{ij}) = \bigvee \phi(y_j).$$

We therefore see that $\overline{\phi} : \overline{G} \to H$ is well-defined, so there is a unique extension of ϕ to $\overline{G}$. ■

If we are given a sheaf morphism $f : F \to G$ and a subsheaf H of G, then we can define a subsheaf $f^{\#}(H)$ of F by setting $f^{\#}(H)(U) = f^{-1}(H(U))$. It is easy to see that this is indeed a sheaf.

The sheaf Ω defined in section 3.2.1 (recall that $\Omega(U)$ is the set of all opens $V \subseteq U$) has a special property with respect to subsheaves. If we are given a sheaf F, then there exists a *natural* 1-1 correspondence between morphisms $\phi : F \to \Omega$ and subsheaves of F. By natural, we mean that it satisfies the following property: if $\phi : G \to \Omega$ corresponds to the subsheaf H, then ϕf corresponds to the subsheaf $f^{\#}(H)$.

Lemma 3.2.10. *If we are given a sheaf F, then there is a natural 1-1 correspondence between morphisms $\phi : F \to \Omega$ and subsheaves of F. For this reason, Ω is called the subobject classifier.*

Proof. Assume we are given a subsheaf G of F. If we are given a section $x \in F(U)$, then there is a greatest open $V_x \subseteq U$ such that $x \restriction V_x \in G(V_x)$. To see this, let $\{V_i\}$ be the collection of opens such that $x \restriction V_i \in G(V_i)$ for all i and let $V_x = \cup V_i$. Then $\{V_i\}$ covers V_x and $(x \restriction V_i)$ is a compatible family such that $\bigvee(x \restriction V_i) = x \restriction V_x$. Since G is a sheaf, we conclude that $x \restriction V_x \in G(V_x)$. We now define a morphism $\phi_G : F \to \Omega$ by setting

$$\phi_G(x) = V_x.$$

Now assume we are given a morphism $\phi : F \to \Omega$. We define a subsheaf G_ϕ of F by setting

$$G_\phi(U) = \{x \in F(U) \mid \phi(x) = U\}.$$

To see that this is a sheaf, note that for a compatible family (x_i) in G with respect to a cover $\{U_i\}$ of U, we have

$$\phi(\bigvee x_i) = \bigvee \phi(x_i) = \bigcup U_i = U.$$

It is easy to see that these operations are inverse to each other. If G is a subsheaf of F, then

$$G_{\phi_G}(U) = \{x \in F(U) \mid \phi_G(x) = U\} = \{x \in F(U) \mid x \in G(U)\} = G(U).$$

If $\phi : F \to \Omega$ is a morphism, note that if $x \in F(U)$ and $\phi(x) = V$, then V is the largest open such that $x \restriction V \in G_\phi(V)$. We therefore see that $\phi_{G_\phi}(x) = \phi(x)$ by definition.
To see that this correspondence is natural, let $\phi : G \to \Omega$ correspond the subsheaf H of G and let $f : F \to G$. We then get a morphism $\phi f : F \to \Omega$. Let $x \in F(U)$. Then $\phi f(x) = U$ if and only if $f(x) \in H(U)$, which is by definition the case if and only if $x \in f^{\#}(H)(U)$. ■

3.2.5 Exponential and power objects

If we are given two sets X and Y, then we can define the set Y^X of functions $f : X \to Y$. A similar thing can be done with sheaves.

Definition 3.2.11. Let F and G be sheaves. We define the *exponential sheaf* G^F by setting

$$G^F(U) = \{\text{all morphisms } \phi : F \restriction U \to G\}$$

and by letting $\phi \restriction V$ simply be the restriction of the morphism $\phi : F \restriction U \to G$ to $F \restriction V$, as defined at the end of section 3.2.1.

We of course need to show that this is indeed a sheaf.

Lemma 3.2.12. *The (pre)sheaf G^F from definition 3.2.11 is a sheaf.*

Proof. This proof is actually an application of lemma 3.2.9. Assume we are given a compatible family $(\phi_i)_{i\in I}$ in G^F with respect to a cover $\{U_i\}$ of U. Then we are given morphisms $\phi_i : F \upharpoonright U_i \to G$. We can define a subpresheaf H of F by $H(V) = F(V)$ if $V \subseteq U_i$ for some i, and $H(V) = \emptyset$ otherwise. We then get a presheaf morphism $\phi : H \to G$ defined by $\phi(x) = \phi_i(x)$ if $x \in F(V)$ for some $V \subseteq U_i$. This is well defined since $(\phi_i)_{i\in I}$ is a compatible family. It is easy to check that the smallest subsheaf of F containing H is $F \upharpoonright U$, so by lemma 3.2.9, there is a unique map $\overline{\phi} : F \upharpoonright U \to G$ extending ϕ. Since this map is unique, we see that $(\phi_i)_{i\in I}$ has a unique amalgamation $\overline{\phi} = \bigvee \phi_i$. ■

The sheaf G^F also comes with a morphism $\mathrm{ev}_{F,G} : G^F \times F \to G$ which we call *evaluation*, defined by $\mathrm{ev}_{F,G}(\phi, x) = \phi(x)$. It has the following property:

Lemma 3.2.13. *Given a sheaf H and a map $\phi : H \times F \to G$, there is a unique map $\widetilde{\phi} : H \to G^F$ making*

$$\begin{array}{ccc} H \times F & \xrightarrow{\widetilde{\phi}\times id_F} & G^F \times F \\ & \searrow^{\phi} & \downarrow ev_{F,G} \\ & & G \end{array} \tag{4}$$

commute.

Proof. Assume a map $\phi : H \times F \to G$ is given. We define, for $x \in H(U)$, a morphism $\phi_x : F \upharpoonright U \to G$ by $\phi_x(y) = \phi(x \upharpoonright V, y)$ for all sections $y \in F(V)$ for every $V \subseteq U$. We then set $\widetilde{\phi}(x) = \phi_x$.
It is clear that $\widetilde{\phi}$ makes diagram (4) commute. To see that it is the unique map having this property, assume $\psi : H \to G^F$ is a map also making (4) commute. Then let $x \in H(U)$ and write ψ_x for $\psi(x)$. To see that $\psi_x = \phi_x$, let $y \in F(V)$ for some $V \subseteq U$ open. Then

$$\begin{aligned} \psi_x(y) &= \psi(x)(y) = (\psi(x) \upharpoonright V)(y) = \psi(x \upharpoonright V)(y) \\ &= \mathrm{ev}_{F,G}(\psi \times \mathrm{id}_F)(x \upharpoonright V, y) = \phi(x \upharpoonright V, y) = \phi_x(y) \end{aligned}$$

for all $y \in F(V)$, so $\widetilde{\phi}(x) = \phi_x = \psi(x)$ for all $x \in H(U)$, so $\widetilde{\phi} = \psi$.

■

This property is often used as a definition for the exponential object when discussing category theory in general, giving a definition that is unique up to isomorphism. Since we are only looking at sheaf toposes, it is easier to give the above direct definition.
The above definition is sometimes hard to work with. In the appendix, we look at the sheaf $\widetilde{B}^{\widetilde{A}}$, where A and B are given topological spaces, and try to find an alternative description that is easier to work with.

If we are given a set X, then we can define the power set $\mathcal{P}(X)$ to be the set of subsets of X. It is well-known that we can see this set as the set 2^X of functions $X \to \{0, 1\}$. Given a sheaf F, we can define the power sheaf $\mathcal{P}(F)$ in a similar way.

Definition 3.2.14. Given a sheaf F, we define the *power sheaf* $\mathcal{P}(F)$ to be the sheaf Ω^F. By the correspondence of lemma 3.2.10 and definition 3.2.11, we see that this corresponds to the sheaf G defined by

$$G(U) = \{\text{subsheaves of } F \upharpoonright U\},$$

where the restriction $H \restriction V$ for some $H \in G(U)$ and $V \subseteq U$ is simply the restriction of the sheaf H to V, as defined at the end of section 3.2.1.

We will usually use the second version of $\mathcal{P}(F)$. This also comes with a map $\mathrm{ev}_{F,\Omega} : \mathcal{P}(F) \times F \to \Omega$. From the 1-1 correspondence of lemma 3.2.10, we see that $\mathrm{ev}_{F,\Omega}(G, x)$ is the largest open $U \subseteq T$ such that $x \restriction U \in G(U)$. Since the codomain of $\mathrm{ev}_{F,\Omega}$ is Ω, it can also be seen as a subsheaf of $\mathcal{P}(F) \times F$. We will often, by abuse of notation, denote this map or subsheaf by $\in$.

3.3 Interpreting intuitionistic logic in a sheaf topos

In this section, we show how one can interpret intuitionistic logic in a sheaf topos. We will see that, using exponential and power objects, we can interpret higher-order logic. We will then show that we can construct quotients in a sheaf topos and we will study the so-called natural numbers object, which will both be important for the applications in chapter 4.

3.3.1 Many-sorted logic

We will interpret a slightly different kind of logic than the first-order predicate logic defined in section 2.2. We will look at many-sorted intuitionistic logic. Our language $\mathcal{L}$ consists of the logical symbols $=$, $\wedge$, $\vee$, $\rightarrow$, $\perp$, $\exists$ and $\forall$, and above that of:

1. A set of sorts $S, T, \ldots$.
2. A denumerable collection of variables $x_1^S, x_2^S, \ldots$ of sort S for each sort S.
3. A collection of function symbols $f : S_1, \ldots, S_n \rightarrow S$.
4. A collection of relation symbols $R \subseteq S_1, \ldots S_m$.

Note that $\neg\phi$ is an abbreviation of $\phi \rightarrow \perp$ and $\phi \leftrightarrow \psi$ of $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. Functions and relation symbols have a sequence of sorts $S_1, \ldots S_n$ as their domain, and each function has a certain sort S as its target. We treat constants as 0-placed function symbols. We will now define terms and formulas for this language.

Definition 3.3.1 ([7, p.32-33] and [6, p.11]). Terms of sort S are defined by

1. x^S is a term of sort S if x^S is a variable of sort S;
2. if $t_1, \ldots, t_n$ are terms of sorts $S_1, \ldots, S_n$ respectively and $f : S_1, \ldots S_n \rightarrow S$ is a function symbol, then $f(t_1, \ldots, t_n)$ is a term of sort S.

Formulas are defined by

1. $\perp$ is a formula.
2. if t and s are terms of the same sort, then $t = s$ is a formula.
3. if $R \subseteq S_1, \ldots, S_n$ is a relation symbol and $t_1, \ldots, t_n$ are terms of sorts $S_1, \ldots, S_n$ respectively, then $R(t_1, \ldots t_n)$ is a formula.
4. if ϕ and ψ are formulas, then $\phi \wedge \psi$, $\phi \vee \psi$, and $\phi \rightarrow \psi$ are formulas.
5. if ϕ is a formula and x a variable of some sort, then $\exists x\phi$ and $\forall x\phi$ are formulas.

A deductive system for this langauge is defined in the same way as for intuitionistic first-order logic, so simply natural deduction for classical logic, where one may not conclude ϕ when one infers $\perp$ from $\neg\phi$. If one can prove a sentence ϕ intuitionistically from a set of sentences Γ, we will write $\Gamma \vdash \phi$.

A language with multiple sorts can make it easier to describe certain mathematical objects, for example, vector spaces. Here, one has a two sorts, namely *scalars* S and *vectors* V.

There exists a function symbol $S \times S \to S$ multiplying scalars, and a function symbol $S \times V \to V$ multiplying vectors with a scalar, while (in general) there is no function symbol for multiplication of vectors. There also exists a function symbol for addition of scalars and one for addition of vectors, but not one adding scalars to vectors.

3.3.2 Interpreting intuitionistic logic

We will now show how one can interpret many-sorted logic in a sheaf topos $\mathrm{Shv}(T)$. We will first give an interpretation which looks a lot like the interpretation of first-order logic in a Kripke model. We will then show that we can also assign an open $U \subseteq T$ to every $\mathcal{L}$-sentence. The opens of T will then act as 'truth values' for sentences in the language $\mathcal{L}$. We first need to define what an interpretation for a many-sorted language $\mathcal{L}$ in a sheaf topos is.

Definition 3.3.2. Let T be a topological space and let $\mathcal{L}$ be a many-sorted langauge. An interpretation of $\mathcal{L}$ in $\mathrm{Shv}(T)$ consists of:

1. A sheaf $[\![S]\!]$ for every sort S.
2. A sheaf morphism $[\![f]\!] : [\![S_1]\!] \times \ldots \times [\![S_n]\!] \to [\![S]\!]$ for every function symbol $f : S_1, \ldots, S_n \to S$.
3. A subsheaf $[\![R]\!]$ of $[\![S_1]\!] \times \ldots \times [\![S_n]\!]$ for every relation symbol $R \subseteq S_1, \ldots, S_n$.

Note that the empty product is the terminal sheaf $\mathbf{1}$, so a 0-placed function symbol c of sort S is a sheaf morphism $[\![c]\!] : \mathbf{1} \to [\![S]\!]$, or equivalently, a global section of $[\![S]\!]$. A 0-placed relation symbol is a subsheaf of $\mathbf{1}$. It is easily shown that this is of the form $\mathbf{1} \upharpoonright U$ for some open $U \subseteq T$, so we can also view its interpretation as an open $U \subseteq T$. This follows from the fact that it corresponds to an arrow $\mathbf{1} \to \Omega$ by lemma 3.2.10, and the global sections of Ω are opens $U \subseteq T$. We will now give an interpretation for terms:

Definition 3.3.3. Let t be a term of sort S with free variables $x_1, \ldots, x_n$ of sorts $S_1, \ldots, S_n$ respectively. We define the sheaf morphism $[\![t]\!] : [\![S_1]\!] \times \ldots \times [\![S_n]\!] \to [\![S]\!]$ by:

1. If $t = x^S$ for a variable of sort S, then $[\![t]\!] = \mathrm{id}_S$.
2. If $t = f(t_1, \ldots, t_m)$, then we define $[\![t]\!] : [\![S_1]\!] \times \ldots \times [\![S_n]\!] \to [\![S]\!]$ by setting

   $$[\![t]\!](a_1, \ldots, a_n) = [\![f]\!]([\![t_1]\!](a_1, \ldots, a_n), \ldots, [\![t_m]\!](a_1, \ldots, a_n))$$

   for any $a_1, \ldots, a_n \in ([\![S_1]\!] \times \ldots \times [\![S_n]\!])(U)$ and any open $U \subseteq T$. If the variable x_i does not occur in term t_j for some $1 \leq i \leq n$ and $1 \leq j \leq m$, then we simply ignore it. So if only the free variables $x_2, \ldots, x_n$ occur in t_1, then by $[\![t_1]\!](a_1, \ldots, a_n)$, we mean $[\![t_1]\!](a_2, \ldots, a_n)$.

If $\phi(x_1, \ldots x_n)$ is a given formula, where $x_1, \ldots x_n$ are of sort $S_1, \ldots, S_n$ respectively, and $a_1, \ldots, a_n$ is a section of $[\![S_1]\!] \times \ldots \times [\![S_n]\!]$, we will define a forcing relation $V \Vdash \phi(a_1, \ldots, a_n)$ for opens $V \subseteq T$. This relation is similar to the forcing relation of Kripke semantics from section 2.3, but there are some subtle differences where the structure of the topological space T is used.

Definition 3.3.4 (**[6, p.13 and p.32]**). Let $a_1, \dots, a_n \in (\llbracket S_1 \rrbracket \times \dots \times \llbracket S_n \rrbracket)(U)$ and write $\mathbf{a}$ for $(a_1, \dots, a_n)$ and $\mathbf{a} \upharpoonright V$ for $(a_1 \upharpoonright V, \dots, a_n \upharpoonright V)$. We define the relation $U \Vdash \phi(\mathbf{a})$, by

1. $U \Vdash \perp$ if and only if $U = \emptyset$.

2. $U \Vdash (t = s)(\mathbf{a})$ if and only if $\llbracket t \rrbracket(\mathbf{a}) = \llbracket s \rrbracket(\mathbf{a})$.

3. $U \Vdash R(t_1, \dots, t_m)(\mathbf{a})$ if and only if
$$(\llbracket t_1 \rrbracket(\mathbf{a}), \dots, \llbracket t_m \rrbracket(\mathbf{a})) \in \llbracket R \rrbracket(U).$$

4. $U \Vdash (\phi \wedge \psi)(\mathbf{a})$ if and only if
$$U \Vdash \phi(\mathbf{a}) \quad \text{and} \quad U \Vdash \psi(\mathbf{a}).$$

5. $U \Vdash (\phi \vee \psi)(\mathbf{a})$ if and only if
$$\{V \subseteq U \text{ open} \mid V \Vdash \phi(\mathbf{a} \upharpoonright V) \text{ or } V \Vdash \psi(\mathbf{a} \upharpoonright V)\}$$
is a cover of U.

6. $U \Vdash (\phi \to \psi)(a_1, \dots, a_n)$ if and only if for every open $V \subseteq U$,
$$\text{if} \quad V \Vdash \phi(\mathbf{a} \upharpoonright V), \quad \text{then} \quad V \Vdash \psi(\mathbf{a} \upharpoonright V).$$

7. $U \Vdash \neg\phi(\mathbf{a})$ if and only if for no nonempty $V \subseteq U$, $V \Vdash \phi(\mathbf{a} \upharpoonright V)$.

8. $U \Vdash \exists x^S \phi(x^S, \mathbf{a})$ if and only if the set
$$\{V \subseteq U \text{ open} \mid \text{there exists an } a \in \llbracket S \rrbracket(V) \text{ such that } V \Vdash \phi(a, \mathbf{a} \upharpoonright V)\}$$
is a cover of U.

9. $U \Vdash \forall x^S \phi(x^S, \mathbf{a})$ if and only if for all $V \subseteq U$ and every $a \in \llbracket S \rrbracket(V)$, $V \Vdash \phi(a, \mathbf{a} \upharpoonright V)$.

Note that the definition of $U \Vdash \neg\phi$ is a consequence of the definition of $U \Vdash \phi \to \psi$ and $U \Vdash \perp$. We have the following lemma:

Lemma 3.3.5 (**[6, p.33]**).

1. *If $U \Vdash \phi(a_1, \dots, a_n)$, then for any open $V \subseteq U$, we have $V \Vdash \phi(a_1 \upharpoonright V, \dots, a_n \upharpoonright V)$.*

2. *If $\{U_i\}_{i \in I}$ is a cover of U and for every $i \in I$, we have $U_i \Vdash \phi(a_1 \upharpoonright U_i, \dots, a_n \upharpoonright U_i)$, then $U \Vdash \phi(a_1, \dots, a_n)$. In particular, if $(a_i)_{i \in I}$ is a compatible family, then $U \Vdash \phi(\bigvee a)$ if and only if $U_i \Vdash \phi(a_i)$ for every $i \in I$.*

A proof of this lemma can be obtained by induction on $\mathcal{L}$-formulas and is left to the reader.
A consequence of lemma 3.3.5 is that we can define, for an $\mathcal{L}$-formula ϕ with free variables $x_1, \ldots, x_n$ of sorts $S_1, \ldots, S_n$, a subsheaf $[\![\phi]\!]$ of $[\![S_1]\!] \times \ldots \times [\![S_n]\!]$ by

$$[\![\phi]\!](U) := \{(a_1, \ldots, a_n) \in [\![S_1]\!] \times \ldots \times [\![S_n]\!] \mid U \Vdash \phi(a_1, \ldots, a_n)\}.$$

If ϕ has no free variables, then $[\![\phi]\!]$ is a subsheaf of $\mathbf{1}$, which, as we saw above, corresponds to an open $U \subseteq T$. We define $[\![\phi]\!]$, for a ϕ without free variables, to be the greatest open $U \subseteq T$ such that $U \Vdash \phi$. Using this interpretation, we can rewrite definition 3.3.4 to the following equivalent definition, which is based on definition 5.13 in [3, p.365-366]. Here int denotes the interior operator from topology.

Definition 3.3.6. Let $(a_1, \ldots, a_n) \in ([\![S_1]\!] \times \ldots \times [\![S_n]\!])(U)$ and write $\mathbf{a}$ for $(a_1, \ldots, a_n)$ and $\mathbf{a} \upharpoonright V$ for $(a_1 \upharpoonright V, \ldots, a_n \upharpoonright V)$. We define $[\![\phi(a_1, \ldots, a_n)]\!]$ by

1. $[\![\bot]\!] = \emptyset$.
2. $[\![(t = s)(\mathbf{a})]\!]$ is the greatest open $V \subseteq U$ such that $[\![t]\!](\mathbf{a}) = [\![s]\!](\mathbf{a})$.
3. $[\![R(t_1, \ldots, t_m)(\mathbf{a})]\!]$ is the greatest open $V \subseteq U$ such that
$$([\![t_1]\!](\mathbf{a} \upharpoonright V), \ldots, [\![t_m]\!](\mathbf{a} \upharpoonright V)) \in [\![R]\!](V).$$
4. $[\![(\phi \wedge \psi)(\mathbf{a})]\!] = [\![\phi(\mathbf{a})]\!] \cap [\![\psi(\mathbf{a})]\!]$.
5. $[\![(\phi \vee \psi)(\mathbf{a})]\!] = [\![\phi(\mathbf{a})]\!] \cup [\![\psi(\mathbf{a})]\!]$.
6. $[\![(\phi \to \psi)(\mathbf{a})]\!] = \mathrm{int}((U \setminus [\![\phi(\mathbf{a})]\!]) \cup [\![\psi(\mathbf{a})]\!])$.
7. $[\![\neg\phi(\mathbf{a})]\!] = \mathrm{int}(U \setminus [\![\phi(\mathbf{a})]\!])$.
8. $$[\![\exists x^S \phi(x^S, \mathbf{a})]\!] = \bigcup\{[\![\phi(a, \mathbf{a} \upharpoonright V)]\!] \mid V \subseteq U \text{ and } a \in F(V)\}.$$
9. $$[\![\forall x^S \phi(x^S, \mathbf{a})]\!] = \mathrm{int}\left(\bigcap\{[\![\phi(a, \mathbf{a} \upharpoonright V)]\!] \cup V^c \mid V \subseteq U \text{ and } a \in F(V)\}\right).$$

For the definition of $[\![\exists x^S \phi(x^S, \mathbf{a})]\!]$, note that $x \in [\![\exists x^S \phi(x^S, \mathbf{a})]\!]$ if and only if, for some neighborhood U_x of x, there is an $a \in [\![S]\!](U_x)$ such that $[\![\phi(a, \mathbf{a} \upharpoonright U_x)]\!] = U_x$. So we see that $[\![\exists x^S \phi(x^S, \mathbf{a})]\!] = U$ if and there is some cover $\{U_i\}_{i \in I}$ of U, such that for every $i \in I$, there is an $a \in [\![S]\!](U_i)$ for which $[\![\phi(a, \mathbf{a} \upharpoonright U_i)]\!] = U_i$.
For the definition of $[\![\forall x^S \phi(x^S, \mathbf{a})]\!]$, note that $[\![\forall x^S \phi(x^S, \mathbf{a})]\!] = U$ if and only if for every $V \subseteq U$ and every $a \in [\![S]\!](V)$, we have $[\![\phi(a, \mathbf{a} \upharpoonright V)]\!] = V$.
It is now an easy exercise to check that

$$U \Vdash \phi(\mathbf{a}) \quad \text{iff} \quad U \subseteq [\![\phi(\mathbf{a})]\!].$$

From now on, we will use the $[\![\cdot]\!]$-definition, since this is sometimes easier to work with. The advantage of this notation is that it has also has a meaning if ϕ contains free variables.

If we say that a certain sentence ϕ *holds* or if we say that it is *true*, we mean that $[\![\phi]\!] = T$ or, equivalently, $T \Vdash \phi$. If Γ is a set of $\mathcal{L}$-sentences and ϕ is an $\mathcal{L}$-sentence, then $\Gamma \Vdash \phi$ means that for any interpretation of $\mathcal{L}$ in a sheaf topos, if all sentences in Γ are true, then ϕ is true. We have the following theorem:

Theorem 3.3.7 (**Soundness theorem**). *If Γ is a set of $\mathcal{L}$-sentences and ϕ is an $\mathcal{L}$-sentence, then $\Gamma \Vdash \phi$ if $\Gamma \vdash \phi$.*

From now on, we won't specify the language we are working with explicitly. If F is a sheaf and we are quantifying over variables of a sort that is interpreted as the sheaf F, we will often write $\exists x \in F$ or $\forall x \in F$, to avoid confusion.
In section 3.2.5, we defined for a sheaf F the power sheaf $\mathcal{P}(F)$ and for a pair of sheaves F, G the exponential sheaf G^F. They came with a relation $\in$ and an evaluation map $\text{ev}_{F,G}$. We see that $[\![a \in A]\!]$, for $a \in F(U)$ and $A \in \mathcal{P}(F)(U)$, is equal to the greatest $V \subseteq U$ such that $a \upharpoonright V \in A(V)$. For a term f of a sort interpreted as G^F and a term t of a sort interpreted as F, we get a term $\text{ev}_{F,G}(f, t)$ of a sort G that is interpreted as $[\![f]\!]([\![t]\!])$. By abuse of notation, we will write $f(t)$ for $\text{ev}_{F,G}(f, t)$. The power sheaf and exponential sheaf allow us to interpret higher-order logic in a sheaf topos. (For example, we can quantify over functions, subsets, functions of functions, etc.) We have the following theorem:

Theorem 3.3.8 (**Soundness for higher-order logic, [3, p.380]**). *The axioms and inference rules of intuitionistic higher-order logic are valid for standard interpretations in sheaves.*

This means that any statement that can be proven intuitionistically holds in a sheaf topos.
In the rest of this chapter, we will look at quotients and the natural numbers object. Quotients of sheaves are similar to quotients in ordinary set theory and we need them in the next chapter, when we want to construct sheaves modeling the real numbers. A natural numbers object is an object with some extra structure in a category, which has properties that are similar to the properties of $\mathbb{N}$ in the category of sets. We will show that such an object exists in a sheaf topos and that it satisfies the (second order) Peano axioms. This sheaf will also be our starting point when we construct a sheaf that models the real numbers.
Before we go on, we take a quick look at (easier) ways of how to prove certain statements. It is easy to see that, to prove $[\![\neg\phi]\!] = T$, we only need to show that $[\![\phi]\!] = \emptyset$. To prove that $[\![\phi \to \psi]\!] = T$, we need to show that $[\![\phi]\!] \subseteq [\![\psi]\!]$, and to show that $[\![\phi \leftrightarrow \psi]\!] = T$, we need to show that $[\![\phi]\!] = [\![\psi]\!]$. For universal quantifiers, there also exist simplifications: if we were to prove $[\![\forall x_1 \in F_1, \ldots, \forall x_n \in F_n\ \phi(x_1, \ldots, x_n)]\!] = U$ directly using definition 3.3.4 or 3.3.6, then it would become quite clumsy, since we need to check something for every sequence $U_n \subseteq U_{n-1} \subseteq \ldots \subseteq U_1 \subseteq U$ of opens and all $a_i \in F(U_i)$ for all $1 \leq i \leq n$. Luckily, it is easy to show that $[\![\forall x_1 \in F_1, \ldots, \forall x_n \in F_n\ \phi(x_1, \ldots, x_n)]\!] = U$ if and only if for all $V \subseteq U$ and all $a_i \in F_i(V)$ for every $1 \leq i \leq n$, we have $[\![\phi(a_1, \ldots, a_n)]\!] = V$. Similarly, if we have to show that $[\![\forall x \in F\ (\psi(x) \to \phi(x))]\!] = U$, then it is enough to show that, for every $V \subseteq U$ and every $a \in F(V)$ such that $[\![\psi(a)]\!] = V$, we have $[\![\phi(a)]\!] = V$.

3.3.3 Equivalence relations and quotients

Like in ordinary mathematics, we define an *equivalence relation* $\sim$ on F to be a 2-ary relation satisfying

1. $\forall x \in F\ (x \sim x)$
2. $\forall x, y \in F\ (x \sim y \to y \sim x)$
3. $\forall x, y, z \in F\ (x \sim y \wedge y \sim z \to x \sim z)$.

When working in a sheaf topos, we see that an equivalence relation on F corresponds to a subsheaf G of $F \times F$, such that for all $U \subseteq T$ and all sections $x, y, z \in G(U)$:

1. $(x, x) \in G(U)$
2. If $(x, y) \in G(U)$, then $(y, x) \in G(U)$
3. If $(x, y), (y, z) \in G(U)$, then $(x, z) \in G(U)$.

So the equivalence relation $\sim$ corresponds to a subsheaf G of $F \times F$, such that for every open $U \subseteq T$, the subset $G(U) \subseteq F(U) \times F(U)$ is an equivalence relation on $F(U)$ (in the traditional sense). To avoid confusion, we will denote this equivalence relation by $\approx$.

Definition 3.3.9. Let F be a sheaf and $\sim$ an equivalence relation on F. A *quotient* of F by $\sim$ is a sheaf H together with a morphism $p : F \to H$ such that

$$\forall x, y \in F\ (p(x) = p(y) \leftrightarrow x \sim y) \quad \text{and} \quad \forall x \in H\ \exists y \in F\ (p(y) = x)$$

hold.

We will first look at what this means for the sheaf H and the map p. The sentence $\forall x, y \in F\ (p(x) = p(y) \leftrightarrow x \sim y)$ holds if and only if $[\![p(x) = p(y)]\!] = [\![x \sim y]\!]$ for all sections $x, y \in F(U)$ for any open $U \subseteq T$. It is easily shown that this is equivalent to saying that the morphism p satisfies $p(x) = p(y)$ if and only if $x \approx y$ for all $x, y \in F(U)$, for every open $U \subseteq T$.
The second property of p means that for any open $U \subseteq T$ and any $x \in H(U)$, there exists a cover $\{U_i\}_{i \in I}$ of U such that for every $i \in I$, there is a $y \in F(U_i)$ satisfying $p(y) = x \upharpoonright U_i$. This does not imply that every $p_U : F(U) \to G(U)$ is surjective, but it does mean that every section $x \in G(U)$ is the amalgamation of a compatible family $(x_i)_{i \in I}$ such that $x_i = p(y_i)$ for some $y_i \in F(U)$. This leads us to the following lemma.

Lemma 3.3.10. *Let F be a sheaf and $\sim$ be an equivalence relation on F. Then any two quotients are isomorphic.*

Proof. Assume H_1 and H_2 together with the morphisms $p_1 : F \to H_1$ and $p_2 : F \to H_2$ are both quotients of F by $\sim$. We will show that there is an isomorphism σ making the diagram

$$\begin{array}{ccc} F & \xrightarrow{p_1} & H_1 \\ & {\scriptstyle p_2}\searrow & \updownarrow{\scriptstyle \sigma} \\ & & H_2 \end{array} \tag{5}$$

commute. Define subpresheaves H_1' and H_2' of H_1 and H_2 respectively by

$$H_1'(U) = p_1(F(U)) \quad \text{and } H_2'(U) = p_2(F(U)).$$

By the second property of quotients as defined in definition 3.3.9 and the definition of $\overline{H_1'}$ and $\overline{H_2'}$ as in lemma 3.2.9, we see that $\overline{H_1'} = H_1$ and $\overline{H_2'} = H_2$. By the first property of definition 3.3.9, we get an isomorphism $\phi : H_1' \to H_2'$, defined in the following way: if $x \in H_1'$, then $x = p_1(y)$ for some $y \in F$. Then let $\phi(x) = p_2(y)$. This is clearly well-defined, for if $p_1(y') = x$, then $y' \approx y$, hence $p_2(y') = p_2(y)$. To show that it is natural, let $V \subseteq U \subseteq T$ be opens and let $x \in H_1'(U)$. We need to show that $\phi(x) \restriction V = \phi(x \restriction V)$. Let $y \in F(U)$ be such that $p_1(y) = x$. Then $p_1(y \restriction V) = x \restriction V$ by naturality of p_1, hence $\phi(x \restriction V) = p_2(y \restriction V)$. We now see that

$$\phi(x) \restriction V = p_2(y) \restriction V = p_2(y \restriction V) = \phi(x \restriction V),$$

so $\phi : H_1' \to H_2'$ is indeed natural, hence a morphism. It is clear that ϕ is an isomorphism, since we can define a map $\psi : H_2' \to H_1'$ in exactly the same way, which is inverse to ϕ. We also see that $p_2 = \phi p_1$ and $p_1 = \psi p_2$, which follows directly from the definition of of ϕ and ψ.

The morphisms ϕ and ψ can be seen as morphisms with codomains H_2 and H_1, which are sheaves. By lemma 3.2.9, there are extensions $\overline{\phi} : H_1 \to H_2$ and $\overline{\psi} : H_2 \to H_1$. These morphisms are in fact isomorphisms. To see that $\overline{\psi\phi} : H_1 \to H_1$ is equal to id_{H_1}, we note that $\overline{\psi\phi}$ is an extension of $\psi\phi : H_1' \to H_1$. Since id_{H_1} is also an extension of $\psi\phi = \mathrm{id}_{H_1'}$, we see by lemma 3.2.9 that $\overline{\psi\phi} = \mathrm{id}_{H_1}$. We similarly see that $\overline{\phi\psi} = \mathrm{id}_{H_2}$, so H_1 and H_2 are isomorphic by morphisms making diagram (5) commute. ■

We of course want quotients to exist for any equivalence relation. The following lemma shows that we can construct a quotient as a subsheaf of $\mathcal{P}(F)$, similar to the way we construct the set of equivalence classes in set theory.

Lemma 3.3.11. *Let F be a sheaf and $\sim$ an equivalence relation on F. Then there exists a quotient of F by $\sim$.*

Proof. Define the subsheaf G of $\mathcal{P}(F)$ by

$$X \in G \leftrightarrow (\exists x \in F\ (x \in X) \wedge \forall x, y\ ((x \in X \to y \in X) \leftrightarrow x \sim y)),$$

so

$$G = [\![\exists x \in F\ (x \in X) \wedge \forall x, y\ ((x \in X \to y \in X) \leftrightarrow x \sim y)]\!].$$

This sheaf will turn out to be a quotient of $F/\sim$, with a map $p : F \to G$ defined later on. We will first look at what sections of the sheaf G look like. Let $X \in G(U)$. Then X is a subsheaf of $F \restriction U$. By definition, there is a cover $\{U_i\}_{i \in I}$ of U such that $X(U_i)$ is not empty for all $i \in I$. Furthermore, if $V \subseteq U$ is open and $X(V) \neq \emptyset$, then, for any $x \in X(V)$, we have that $y \in X(V)$ if and only if $x \approx y$, so for any open $V \subseteq U$, $X(V)$ is either empty or exactly one equivalence class of $F(V)$.

For a section $x \in F(U)$, we denote the equivalence class of x under $\approx$ by $[x]$. For any section $x \in F(U)$, we define the subsheaf X_x of $F \restriction U$ by

$$X_x(V) = [x \restriction V] \quad \text{for any open } V \subseteq U.$$

The proof that this is in fact a sheaf is left to the reader. Since it is clearly a section of G, we define $p : F \to G$ by $p(x) = X_x$. The proof that p is a morphism is also left to the reader.

We will now show that G together with p is a quotient of F by $\sim$. First, let $x, y \in F(U)$ be sections. Assume that $p(x) = p(y)$. Then $y \in X_x(U) = [x]$, so $x \approx y$. Now assume $x \approx y$. We need to show that $X_x = X_y$, so we need to show that $[x \upharpoonright V] = [y \upharpoonright V]$ for every $V \subseteq U$ open. Since these are equivalence classes, this amounts to showing that $x \upharpoonright V \approx y \upharpoonright V$. This follows directly from the fact that $\sim$ is a subsheaf of $F \times F$.
Now let $X \in G(U)$. By defininition, there is a cover $\{U_i\}_{i \in I}$ such that $X(U_i)$ is not empty for all i. If $y \in X(U_i)$, then it is easy to see that $p(y) = X \upharpoonright U_i$. Since $\{U_i\}_{i \in I}$ covers U, we see that $\forall X \in G\ \exists y \in F\ (p(y) = X)$ holds. ■

One can also prove that this set is a quotient using intuitionistic logic and the soundness theorem of the previous section. Since quotients always exist and are unique up to isomorphism, we will from now on, if we are given a sheaf F and an equivalence relation $\sim$ on F, simply speak of the quotient $F/\sim$.

3.3.4 Natural numbers object

A natural numbers object in a category $\mathcal{C}$ models, in a sense, the way the object $\mathbb{N}$ behaves in the category of sets. We will give a precise definition of what a natural numbers object is and show that there always exists one in a sheaf topos.

Definition 3.3.12 ([**7, p.69**]). Let $\mathcal{C}$ be a category with a terminal object $\mathbf{1}$. A *natural numbers object* (NNO) is a triple $(\mathbf{0}, N, S)$, where N is an object and $\mathbf{0} : \mathbf{1} \to N$ and $S : N \to N$ are arrows. These satisfy the property that for any such triple

$$\mathbf{1} \xrightarrow{\mathbf{x}} X \xrightarrow{f} X$$

there is a unique $\phi : N \to X$ making

$$\begin{array}{ccccc} \mathbf{1} & \xrightarrow{\mathbf{0}} & N & \xrightarrow{S} & N \\ & \searrow^{\mathbf{x}} & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ & & X & \xrightarrow{f} & X \end{array}$$

commute.

The arrow $\mathbf{0}$ can be seen as the 'zero element' of N and S can be seen as the successor function. In a sheaf topos, the triple $(\widetilde{\mathbf{0}}, \widetilde{\mathbb{N}}, \widetilde{S})$ is a natural numbers object, where $\widetilde{\mathbf{0}} : \mathbf{1} \to \widetilde{\mathbb{N}}$ picks the constant zero map in $\widetilde{\mathbb{N}}$ and $\widetilde{S}$ sends the map $n : U \to \mathbb{N}$ to $n + 1 : U \to \mathbb{N}$.

Lemma 3.3.13. *In a sheaf topos, the triple* $(\widetilde{\mathbf{0}}, \widetilde{\mathbb{N}}, \widetilde{S})$ *is a natural numbers object.*

Proof. Assume we are given a triple (x, X, f), where $x : \mathbf{1} \to X$ and $f : X \to X$. We need to define a map $\phi : \widetilde{\mathbb{N}} \to X$ such that $\phi(\text{const}(0)) = x$ and $f\phi = \phi\widetilde{S}$. This in particular means that, if we are given the constant function $\text{const}(n)$ for $n \in \mathbb{N}$, then $\phi(\text{const}(n+1)) = f\phi(\text{const}(n))$. We see inductively that this implies $\phi(\text{const}(n)) = f^n(x)$, where $f^n(x)$ means that f has been applied n times to x. Denote the subpresheaf of constant functions $U \to \mathbb{N}$ by N. It is then clear that $\overline{N} = \widetilde{\mathbb{N}}$ (see lemma 3.2.9), since any continuous $m : U \to \mathbb{N}$ can be written as the amalgamation

$$\bigvee_{n \in \mathbb{N}} \operatorname{const}(n)|_{U_n},$$

where $U_n = m^{-1}(n)$.
We can define a morphism $\psi : N \to X$ by $\psi(\operatorname{const}(n)) = f^n(x)$. By lemma 3.2.9, we see that there is a unique extension $\overline{\psi} : \widetilde{\mathbb{N}} \to X$, so a morphism $\phi : \widetilde{\mathbb{N}} \to X$ satisfying $\phi(\operatorname{const}(0)) = x$ and $f\phi = \phi\widetilde{S}$ has to be unique. It is left to the reader to prove that the morphism $\overline{\psi} : \widetilde{\mathbb{N}} \to X$ defined here indeed satisfies this. ■

We can show that this triple in fact satisfies the Peano axioms as given in [11, p. 694]:

1. $\forall x, y \in \mathbb{N}\ (x = y \leftrightarrow S(x) = S(y))$
2. $\forall x \in \mathbb{N}\ \neg(0 = S(x))$
3. $\forall z \in \mathcal{P}(\mathbb{N})\ (0 \in z \wedge (\forall x \in \mathbb{N}\ (x \in z \to S(x) \in z)) \to z = \mathbb{N})$

We can also show that $(\widetilde{\mathbf{0}}, \widetilde{\mathbb{N}}, \widetilde{S})$ is the only triple satisfying these axioms, in the sense that any other triple $(\mathbf{0}', N', S')$ satisfying these axioms is isomorphic to $(\widetilde{\mathbf{0}}, \widetilde{\mathbb{N}}, \widetilde{S})$. This can be proved in intuitionistic logic (see [12, p.165]), so by the soundness theorem this has to hold in any sheaf topos. We will nevertheless provide a direct proof below, since such a proof is also a good illustration of how to work with the interpretation of logic in a sheaf topos.

Lemma 3.3.14. *The triple $(\widetilde{\mathbf{0}}, \widetilde{\mathbb{N}}, \widetilde{S})$ satisfies the Peano axioms and any other triple $(\mathbf{0}', N', S')$ satisfying these axioms is isomorphic to $(\widetilde{\mathbf{0}}, \widetilde{\mathbb{N}}, \widetilde{S})$.*

Proof. To show that $(\widetilde{\mathbf{0}}, \widetilde{\mathbb{N}}, \widetilde{S})$ indeed satisfies the first axiom, let $n, m \in \widetilde{\mathbb{N}}(U)$. If $n = m$ on some open $V \subseteq U$, then $n(x) + 1 = m(x) + 1$ for all $x \in V$, so $S(n) = S(m)$ on V, hence $[\![n = m]\!] \subseteq [\![S(n) = S(m)]\!] = U$. The other direction is similar, so we see that $[\![n = m \leftrightarrow S(n) = S(m)]\!] = U$, hence

$$[\![\forall x, y \in \widetilde{\mathbb{N}}\ (x = y \leftrightarrow S(x) = S(y))]\!] = T.$$

For the second axiom, let $n \in \widetilde{\mathbb{N}}(U)$. Then $[\![\widetilde{\mathbf{0}} = S(n)]\!] = \emptyset$, since $S(n)(x) \geq 1$ for all $x \in U$, hence $S(n)$ is nowhere zero. Then $[\![\neg(\widetilde{\mathbf{0}} = S(n))]\!] = U$, hence

$$[\![\forall x \in \widetilde{\mathbb{N}}\ \neg(\widetilde{\mathbf{0}} = S(x))]\!] = T.$$

For the third axiom, let Z be a subsheaf of $\widetilde{\mathbb{N}} \upharpoonright U$. Assume that

$$[\![\widetilde{\mathbf{0}} \in Z \wedge (\forall x \in \widetilde{\mathbb{N}}\ (x \in Z \to S(x) \in Z))]\!] = U.$$

Then the constant function $\widetilde{\mathbf{0}}$ is in $Z(U)$, and for all $V \subseteq U$ and all $n \in Z(V)$, we have $S(n) \in Z(V)$. Note that this implies that $Z(V)$ contains all constant functions for all $V \subseteq U$. Since any $n \in \widetilde{\mathbb{N}}(V)$ can be written as an amalgamation of constant functions (see the proof of lemma 3.3.13), we see that $Z(V) = \widetilde{\mathbb{N}}(V)$ for all $V \subseteq U$ open, so we conclude that $Z = \widetilde{\mathbb{N}} \restriction U$. This proves that

$$[\![\forall z \in \mathcal{P}(\widetilde{\mathbb{N}})\ (\widetilde{\mathbf{0}} \in z \wedge (\forall x \in \widetilde{\mathbb{N}}\ (x \in z \to S(x) \in z)) \to z = \widetilde{\mathbb{N}})]\!] = T,$$

so the triple $(\widetilde{\mathbf{0}}, \widetilde{\mathbb{N}}, \widetilde{S})$ satisfies the Peano axioms.

Assume another triple $(\mathbf{0}', N', S')$ also satisfies these axioms. By lemma 3.3.13, there exists a morphism $\phi : \widetilde{\mathbb{N}} \to N'$ such that $\phi(\text{const}(0)) = \mathbf{0}'$ and $S'\phi = \phi\widetilde{S}$. To show that this map is injective, let $m, n \in \widetilde{\mathbb{N}}(U)$ and assume $m \neq n$. There exists an $x \in U$ such that both $m(x) \neq n(x)$, assume without loss of generality that $n(x) < m(x)$. Since $\mathbb{N}$ is discrete, we can take an open neighborhood $U_x \subseteq U$ of x on which m and n are constant. Then $n|_{U_x} = S^i(\widetilde{\mathbf{0}}|_{U_x})$ and $m|_{U_x} = S^j(\widetilde{\mathbf{0}}|_{U_x})$ for some $i < j \in \mathbb{N}_{>0}$. We then see that

$$\phi(n|_{U_x}) = S'^i(\mathbf{0}' \restriction U_x) \quad \text{and} \quad \phi(m|_{U_x}) = S'^j(\mathbf{0}' \restriction U_x).$$

If $\phi(n|_{U_x}) = \phi(m|_{U_x})$, then $S'^i(\mathbf{0}' \restriction U_x) = S'^j(\mathbf{0}' \restriction U_x)$. By the first axiom, we see that $\mathbf{0}' \restriction U_x = S'^{j-i}(\mathbf{0}' \restriction U_x)$, which by the second axiom is impossible if $i < j$. We therefore conclude that $\phi(n) \neq \phi(m)$.
Since all maps ϕ_U are injective, ϕ is an isomorphism between $\widetilde{\mathbb{N}}$ and a subsheaf of Z of N'. To show surjectivity, note that $\mathbf{0}' \in Z(T)$ and that Z is closed under S' (if $x \in Z(U)$, then $S(x) \in Z(U)$), hence by the third axiom, $Z = N'$, so ϕ is an isomorphism between $\widetilde{\mathbb{N}}$ and N'. ∎

We define $<$, $+$ and $\cdot$ pointwise on $\widetilde{\mathbb{N}}$. This means that we interpret the function symbols $+$ and $\cdot$ as morphisms $\widetilde{\mathbb{N}} \times \widetilde{\mathbb{N}} \to \widetilde{\mathbb{N}}$ which correspond to pointwise addition and pointwise multiplication. We define $<$ by setting, for $m, n \in \widetilde{\mathbb{U}}$,

$$[\![m < n]\!] = \{x \in U | m(x) < n(x)\}.$$

It is easy to check that $<$ is well defined and that these operations satisfy the usual properties of $+$, $\cdot$ and $<$ in Peano arithmatic, namely

1. $\forall x \in \widetilde{\mathbb{N}}\ (x + 0 = x)$ and $\forall x, y \in \widetilde{\mathbb{N}}\ (x + \widetilde{S}(y) = \widetilde{S}(x + y))$
2. $\forall x \in \widetilde{\mathbb{N}}\ (x \cdot 0 = 0)$ and $\forall x, y \in \widetilde{\mathbb{N}}\ (x \cdot \widetilde{S}(y) = x + x \cdot y)$
3. $\forall x, y \in \widetilde{\mathbb{N}}\ (x < y \leftrightarrow \exists z \in \widetilde{\mathbb{N}}\ (x + S(z) = y))$.

4 Some applications of sheaf toposes

In this section we will look at some applications of the theory developed in chapter 3. These applications will concern intuitionistic logic, and intuitionistic analysis in particular. In this chapter, we will sometimes consider sheaves over the Baire space. A small introduction on the Baire space can be found in [4, p.42]. For this thesis, it is not very important to know what the Baire space is. We only explicitly use two properties, namely that it has a countable basis of clopen sets and that it is homeomorphic to the space of irrational numbers [4, p.42].

Section 4.1 is mostly based on the article by Fourman and Scott [3] and the article by Fourman and Hyland [2]. Theorem 4.1.11 can be seen as a generalization of the proof Scott gives in [10], and the author has not been able to find an equivalent theorem in the literature. The author also has not been able to find a theorem similar to theorem 4.2.7 in the literature. Section 4.2 is mostly based on the article by Fourman and Hyland [2].

4.1 Constructing the continuum in a sheaf topos

In this section we will look at the construction of the real numbers in a sheaf topos. We will first construct an object that models the rational numbers from the natural numbers object. We will then, using Dedekind cuts, construct a sheaf $\mathcal{R}$ that models the real numbers.
In a topos there is a natural interpretation of higher order logic, since it has power objects and exponential objects. The total real functions can be interpreted as the sheaf $\mathcal{R}^{\mathcal{R}}$, for example. We will take a look at Brouwer's continuity theorem, which states that every total real function is continuous. At the end of this section we will compare our model for the continuum, the sheaf $\mathcal{R}$, with the model Dana Scott defined in [9]. This section is mostly based on §8 of the article by Fourman and Scott [3] and the article [2] by Fourman and Hyland.

4.1.1 Construction of the rationals

Assume a space T is given. We will consider sheaves over T in this part. As we saw in section 3.3.4, the natural numbers are modeled by $\widetilde{\mathbb{N}}$. The rationals will be constructed from $\widetilde{\mathbb{N}}$ in the same way as in classical mathematics. We will then show that this is in fact the sheaf $\widetilde{\mathbb{Q}}$, where $\mathbb{Q}$ is given the discrete topology. We will construct the rational numbers as a quotient of

$$X := \widetilde{\mathbb{N}} \times \widetilde{\mathbb{N}} \times (\widetilde{\mathbb{N}} \setminus \{0\}).$$

Here $\widetilde{\mathbb{N}} \setminus \{0\}$ is the subsheaf of $\widetilde{\mathbb{N}}$ defined by $[\![x \in \widetilde{\mathbb{N}} \wedge \neg(x = 0)]\!]$. It is easily shown that this is the sheaf $\widetilde{\mathbb{N}_{>0}}$, this is left to the reader. We define a relation on $X \times X$ in the following way: let (a, b, c) and (a', b', c') in X. We define

$$(a, b, c) \sim (a', b', c') \leftrightarrow ac' + b'c = bc' + a'c.$$

This is clearly an equivalence relation on X.

Definition 4.1.1. We define a *model for the rational numbers Q* in a sheaf topos to be the quotient

$$(\widetilde{\mathbb{N}} \times \widetilde{\mathbb{N}} \times (\widetilde{\mathbb{N}} \setminus \{0\}))/\sim,$$

where $\sim$ is defined by

$$(a,b,c) \sim (a',b',c') \leftrightarrow ac' + b'c = bc' + a'c.$$

For the rest of this section, we write $X = \widetilde{\mathbb{N}} \times \widetilde{\mathbb{N}} \times (\widetilde{\mathbb{N}} \setminus \{0\})$. To show that $\widetilde{\mathbb{Q}}$ is a model for the rational numbers, we use the following lemma:

Lemma 4.1.2. *There is a 1-1 correspondence between the equivalence classes in $X(U)$ and the continuous functions $U \to \mathbb{Q}$ for any open $U \subseteq T$, where $\mathbb{Q}$ has the discrete topology. This correspondence is defined by $(a,b,c) \mapsto (a-b)/c$ for $(a,b,c) \in X$.*

Proof. Let $U \subseteq T$ be open. If $[(a,b,c)] \subseteq X(U)$ is an equivalence class, define the continuous map $q : U \to \mathbb{Q}$ by $q(x) = (a(x)-b(x))/c(x)$ for all $x \in U$. Note that c is nowhere zero, so this q is everywhere defined. It is easily show that this is well-defined on equivalence classes. Assume $(a,b,c) \sim (a',b',c')$ in $X(U)$. Then

$$a(x)c'(x) + b'(x)c(x) = b(x)c'(x) + a'(x)c(x)$$

for all $x \in U$, hence

$$(a(x) - b(x))/c(x) = (a'(x) - b'(x))/c'(x)$$

for all $x \in U$.
We will now show the converse. Given a number $x \in \mathbb{Q}$, there is one unique way of writing it as $p_1(x)/p_2(x)$ where $\gcd(p_1(x), p_2(x)) = 1$ and $p_2(x) > 0$. This gives us two continuous maps p_1 and p_2 (they are continuous since $\mathbb{Q}$ has the discrete topology). Given a number $x \in \mathbb{Z}$, we can associate a $(n_1(x), n_2(x)) \in \mathbb{N}^2$ to it in the following way. If $x \geq 0$, set $n_1(x) = x$ and $n_2(x) = 0$. If $x < 0$, set $n_1(x) = 0$ and $n_2(x) = -x$. This gives us two continuous maps n_1 and n_2. Assume a continuous $q : U \to \mathbb{Q}$ is given. Consider the following maps:

$$\begin{aligned} q_1(x) &= n_1(p_1(q(x))) \\ q_2(x) &= n_2(p_1(q(x))) \\ q_3(x) &= p_2(q(x)). \end{aligned}$$

This is a triple (q_1, q_2, q_3) of continuous maps $U \to \mathbb{N}$, where $q_3 > 0$, hence an element of $X(U)$. We associate to q the equivalence class $[(q_1, q_2, q_3)]$. It is easily shown that this operation is inverse to the one defined above, hence there is a 1-1 correspondence between continuous maps $U \to \mathbb{Q}$ and equivalence classes in $X(U)$. ∎

Lemma 4.1.3. *The sheaf $\widetilde{\mathbb{Q}}$, where $\mathbb{Q}$ has the discrete topology, is a model for the rational numbers.*

Proof. Define a morphism $p : X \to \widetilde{\mathbb{Q}}$ by $p(a,b,c) = (a-b)/c$. This is clearly a morphism. By lemma 4.1.2, we see that the maps $p_U : X(U) \to \widetilde{\mathbb{Q}}(U)$ are surjective, hence the second property of quotients as defined in definition 3.3.9 is satisfied. The first property is also satisfied, for by lemma 4.1.2, we see that for all $x, y \in X(U)$, we have $p(x) = p(y)$ if and only if $x \approx y$. So $\widetilde{\mathbb{Q}}$ with the map $p : X \to \widetilde{\mathbb{Q}}$ is a quotient of X by $\sim$. ∎

Like in ordinary mathematics, we can define $+$, $\cdot$ and $<$ on $\widetilde{\mathbb{Q}}$ in terms of the definition of $+$, $\cdot$ and $<$ on $\widetilde{\mathbb{N}}$. Note that every $q \in \widetilde{\mathbb{Q}}$ can be written as $p(a, b, c)$ for some $(a, b, c) \in X$. We now define

1. $p(a_1, b_1, c_1) + p(a_2, b_2, c_2) = p(a_1c_2 + a_2c_1, b_1c_2 + b_2c_1, c_1c_2)$
2. $p(a_1, b_1, c_1) \cdot p(a_2, b_2, c_2) = p(a_1a_2 + b_1b_2, a_1b_2 + a_2b_1, c_1c_2)$
3. $p(a_1, b_1, c_1) < p(a_2, b_2, c_2) \leftrightarrow a_1c_2 + b_2c_1 < a_2c_1 + b_1c_2$.

It is easy to check that this corresponds to pointwise addition and multiplication and that for $p, q \in \widetilde{\mathbb{Q}}(U)$,

$$[\![p < q]\!] = \{x \in U | p(x) < q(x)\}.$$

4.1.2 Constructing the Dedekind reals from the rationals

In classical mathematics, the real numbers are often constructed from the rationals using Dedekind cuts. The set of Dedekind cuts is then constructed as a subset of $\mathcal{P}(\mathbb{Q}) \times \mathcal{P}(\mathbb{Q})$ having certain properties. Another popular way to construct the real numbers uses Cauchy sequences. We will use the first approach here, but we will come back to the second approach in section 4.2. We will construct a subsheaf $\mathcal{R}$ of $\mathcal{P}(\widetilde{\mathbb{Q}}) \times \mathcal{P}(\widetilde{\mathbb{Q}})$ similar to the classical approach. It is then shown that this subsheaf is isomorphic to $\widetilde{\mathbb{R}}$, where $\mathbb{R}$ has its usual topology (open intervals as a basis). Comparing this sheaf, interpreting logic as in section 3.3, to Dana Scott's model of the intuitionistic continuum [9], we see that these models, although similar, are not equivalent for all topological spaces T. This is because the interpretation of the quantifiers differs. In Scott's model, quantifiers only range over functions $f : T \to \mathbb{R}$, so global sections of $\widetilde{\mathbb{R}}$. In the interpretation defined in section 3.3, however, they also quantify over other sections of $\widetilde{\mathbb{R}}$.

In classical mathematics, a Dedekind cut is a pair (A, B) of disjoint subsets of $\mathbb{Q}$. Here A is a downward closed subset with no greatest element and B is an upward closed subset subset with no smallest element. Furthermore, for any two rational numbers p, q such that $p < q$, we have $p \in A$ or $q \in B$. This leads us to the following definition of a Dedekind cut, as given in §8 of [3].

Definition 4.1.4. A dedekind cut (A, B) is defined by the following axioms

1. $\exists p, q \in Q\ (p \in A \wedge q \in B)$
2. $\forall p \neg(p \in A \wedge p \in B)$
3. $\forall p(p \in A \leftrightarrow \exists q \in A\ q > p)$
4. $\forall p(p \in B \leftrightarrow \exists q \in A\ q < p)$
5. $\forall p, q(p < q \to p \in A \vee q \in B)$

We will write this as $\mathrm{Cut}(A, B)$. We define the *sheaf of (Dedekind) real numbers* $\mathcal{R}$ in a sheaf topos to be the subsheaf $[\![\mathrm{Cut}(A, B)]\!]$ of $\mathcal{P}(\widetilde{\mathbb{Q}}) \times \mathcal{P}(\widetilde{\mathbb{Q}})$.

In the remaining part of this section, we will look at the structure of $\mathcal{R}$ in detail, proving that it is isomorphic to $\widetilde{\mathbb{R}}$.

Let $U \subseteq T$ be open and let $(A, B) \in \mathcal{R}(U)$. Then A and B are subsheaves of $\widetilde{\mathbb{Q}} \upharpoonright U$. If $q : U \to \mathbb{Q}$ is a constant function, then by definition $[\![q \in A]\!]$ is the largest open $V \subseteq T$ such that $q|_V \in \mathcal{R}(V)$. This allows us, for $x \in U$, to define the set

$$A_x = \{q \in \mathbb{Q} \mid x \in [\![q \in A]\!]\}$$

and similarly

$$B_x = \{q \in \mathbb{Q} \mid x \in [\![q \in B]\!]\}.$$

We will show that (A_x, B_x) is actually a Dedekind cut in the traditional sense. This way, we obtain a function $f : U \to \mathbb{R}$ defined by $f(x) = (A_x, B_x)$. We will then show that this function is continuous, and that any continuous $f : U \to \mathbb{R}$ defines a Dedekind cut $(A, B) \in \mathcal{R}(U)$, obtaining a 1-1 correspondence between elements of $\mathcal{R}(U)$ and continuous maps $f : U \to \mathbb{R}$. This will turn out to be an isomorphism between $\mathcal{R}$ and $\widetilde{\mathbb{R}}$.

Lemma 4.1.5. *For any $(A, B) \in \mathcal{R}(U)$ and $x \in U$, the pair of sets (A_x, B_x) as defined above is a Dedekind cut. Hence, from any $(A, B) \in \mathcal{R}(U)$, we obtain a map $f : U \to \mathbb{R}$.*

Proof. Let $U \subseteq T$ be open and let $(A, B) \in \mathcal{R}(U)$. We will start by showing that the set A_x, for $x \in U$, is a lower Dedekind cut. The proof that B_x is an upper Dedekind cut is similar and will be left to the reader.
Let A_x be as defined above. By the first axiom, there is an open $V \subseteq U$ containing x and a continuous $p : V \to \mathbb{Q}$ such that $p \in A(V)$. Then $V' := p^{-1}(p(x)) \subseteq T$ is open (since $\mathbb{Q}$ has the discrete topology). Since $x \in V'$ and $p|_{V'}$ is constant, we conclude that $p(x) \in A_x$, hence A_x is not empty.
We now need to show that A_x is downward closed and has no greatest element, which is equivalent to proving that, for all $p \in \mathbb{Q}$, there is a $q \in A_x$ such that $q > p$ if and only if $p \in A_x$.
First, assume $p \in A_x$. Then for some $V \subseteq U$, the constant map $p : V \to \mathbb{Q}$ is in $A(V)$. Then, by axiom 3, there is a $V' \subseteq V$ and a continuous $q : V' \to \mathbb{Q}$, such that $x \in V'$ and $q > p$, in particular $q(x) > p$. By the same argument as the one above showing that $A_x \neq \emptyset$, we conclude $q(x) \in A_x$.
Now assume there is some $q \in A_x$ such that $p < q$. Then in some neighborhood $V \subseteq U$ of x, the constant map q is in $A(V)$. By axiom 3, this implies the constant map p is in $A(V)$, hence $p \in A_x$.
This proves that A_x is a lower Dedekind cut and we see similarly that B_x is an upper Dedekind cut. We will now show that they are disjoint. Assume $q \in A_x$ and $q \in B_x$. Then $x \in [\![q \in A]\!] \cap [\![q \in B]\!] = [\![q \in A \wedge q \in B]\!]$. But by axiom 2, $[\![q \in A \wedge q \in B]\!] = \emptyset$, so this is not possible. Hence A_x and B_x must be disjoint.
The last thing we have to prove is that, for $p < q \in \mathbb{Q}$, we have $p \in A_x$ or $q \in B_x$. From axiom 5, we see that for the constant functions $p, q : U \to \mathbb{Q}$, we have $[\![p < q \to p \in A \vee q \in B]\!] = U$, hence

$$[\![p < q]\!] \subseteq [\![p \in A]\!] \cup [\![q \in B]\!].$$

Since $[\![p < q]\!] = U$, we see that either $x \in [\![p \in A]\!]$ or $x \in [\![q \in B]\!]$, or, equivalently, $p \in A_x$ or $q \in B_x$.

We see that for any $x \in U$, the pair (A_x, B_x) is a Dedekind cut, hence we obtain a function $f : U \to \mathbb{R}$. ■

Lemma 4.1.6. *The map $f : U \to \mathbb{R}$ obtained in lemma 4.1.5 is continuous, where $\mathbb{R}$ has the standard topology of the reals.*

Proof. Let $U \subseteq T$ be open and let $(A, B) \in \mathcal{R}(U)$. Assume $f : U \to \mathbb{R}$ is constructed as in lemma 4.1.5. Let $(a, b) \subseteq \mathbb{R}$ be an open interval and assume $x \in f^{-1}((a, b))$. Then $a < f(x) < b$, so there are rational numbers p, q such that $a < p < f(x) < q < b$. Since $p \in A_x$ and $q \in B_x$, there is an open neighborhood $V \subseteq U$ of x such that the constant functions $p, q : V \to \mathbb{Q}$ are elements of A and B respectively. Hence, for any $y \in V$, we have $p \in A_y$ and $q \in B_y$, so $a < p < f(y) < q < b$. Therefore $V \subseteq f^{-1}((a, b))$. Since this holds for any $x \in f^{-1}((a, b))$, we see that $f^{-1}((a, b))$ is open, hence f is continuous. ■

We define, for any open $U \subseteq T$ and any continuous $f : U \to \mathbb{R}$, a pair of subsheaves A, B of $\widetilde{\mathbb{Q}} \restriction U$ in the following way:

$$A(V) = \{q \in \widetilde{\mathbb{Q}}(V) \mid q < f|_V\}$$
$$B(V) = \{q \in \widetilde{\mathbb{Q}}(V) \mid q > f|_V\}.$$

Lemma 4.1.7. *For any continuous map $f : U \to \mathbb{R}$, the pair (A, B) defined above is an element of $\mathcal{R}(U)$. Combined with the correspondence of lemma 4.1.5, we obtain a 1-1 correspondence. This correspondence is an isomorphism between $\mathcal{R}$ and $\widetilde{\mathbb{R}}$.*

Proof. We will first show that $\mathrm{Cut}(A, B)$ holds, i.e. $[\![\mathrm{Cut}(A, B)]\!] = U$.
For axiom 1, let $x \in U$. Let p, q be rational numbers such that $p < f(x) < q$. Say $V := f^{-1}((p, q))$. Then $x \in V$ and the constant maps $p, q : V \to \mathbb{Q}$ satisfy $p < f|_V < q$, hence $p \in A(V)$ and $q \in B(V)$. Since this holds for any $x \in U$, we see that axiom 1 is satisfied.
For axiom 2, let $p \in \widetilde{\mathbb{Q}}(V)$, where $V \subseteq U$. We see that, by definition,

$$[\![p \in A]\!] = \{x \in V \mid p(x) < f(x)\} \text{ and } [\![p \in B]\!] = \{x \in V \mid p(x) > f(x)\}.$$

These sets are clearly disjoint, so $[\![p \in A \wedge p \in B]\!] = \emptyset$, hence $[\![\neg(p \in A \wedge p \in B]\!])]\!] = V$. Since this holds for any open $V \subseteq U$ and any $p \in \widetilde{\mathbb{Q}}(V)$, we conclude that axiom 2 holds.
Now let $p \in \widetilde{\mathbb{Q}}(V)$, where $V \subseteq U$ is open. We need to show that $[\![p \in A]\!] = [\![\exists q \in A\ p < q]\!]$. Assume $x \in [\![p \in A]\!]$. Then for some open neighborhood $V' \subseteq V$ of x, we have $p \in A(V')$. Since $p^{-1}(p(x))$ is open, we can assume without loss of generality that p is constant on V'. Furthermore, $f^{-1}((1/2(p(x) + f(x)), \infty))$ is open and contains x, so we can furthermore assume that $f > 1/2(f(x) + p(x))$ on V'. Now let q be a rational number such that $p(x) < q < 1/2(f(x) + p(x))$. Let $q : V' \to \mathbb{Q}$ be the constant function. Then $q \in A(V')$ by definition and $p < q$, so $x \in [\![\exists q \in A\ p < q]\!]$.
For the converse, assume $x \in [\![\exists q \in A\ p < q]\!]$. Then for some neighborhood $V' \subseteq V$ of x, there exists a $q \in A(V')$, such that $p|_{V'} < q$. Since $q \in A(V')$, we see that $p|_{V'} < q < f|_{V'}$, hence $p|_{V'} \in A(V')$. Therefore $x \in [\![p \in A]\!]$. We therefore conclude that axiom 3 holds.
The proof for axiom 4 is symmetrical to the proof of axiom 3.
For the last axiom, let $V \subseteq U$ open and let $p, q \in \widetilde{\mathbb{Q}}(V)$. Assume $x \in [\![p < q]\!]$, so

$p(x) < q(x)$. Then $p(x) < f(x)$ or $q(x) > f(x)$. In the first case, since f is continuous, the set $f^{-1}((p(x), \infty))$ is open, contains x and $p(x) < f$ on this set. Since p is constant on a neighborhood of x, there is a neighborhood W of x such that $p < f$ on W, hence $p|_W \in A(W)$, hence $x \in [\![p \in A]\!]$. We see similarly that, if $q(x) > f(x)$, then $x \in [\![q \in B]\!]$. Hence $x \in [\![p \in A \vee q \in B]\!]$. We see that $[\![p < q]\!] \subseteq [\![p \in A \vee q \in B]\!]$ for all $V \subseteq U$ open and $p, q \in \widetilde{\mathbb{Q}}(V)$, hence axiom 5 holds.

We will now show that this is a 1-1 correspondence, combined with the correspondence of lemma 4.1.5. First let $f : U \to \mathbb{R}$ be continuous. We obtain $(A, B) \in \mathcal{R}(U)$ and we need to show that (A_x, B_x), for $x \in U$, is equal to $f(x)$. For this we need to prove that A_x is the set of rational numbers q such that $q < f(x)$. Let $q < f(x)$. Then, as shown above, there is a neighborhood V of x such that $q < f$ on V. Therefore the constant function q is in $A(V)$, so $x \in [\![q \in A]\!]$, hence $q \in A_x$. Now assume $q \in A_x$. Then $q \in A(V)$ for some neighborhood V of x, so in particular $q < f(x)$. We similarly see that B_x is precisely those rationals which are greater than $f(x)$, so the map $x \mapsto (A_x, B_x)$ is in fact equal to f.
Now assume $(A, B) \in \mathcal{R}(U)$ is given. We then obtain a function $f : x \mapsto (A_x, B_x)$, which in turn gives us $(A', B') \in \mathcal{R}(U)$. We will show that $A' = A$, the proof for $B' = B$ is similar. Let $V \subseteq U$ be open and let $q \in A(V)$. Then $q(x) \in A_x$ for all $x \in V$, since q is locally constant, so $q < f|_V$, so $q \in A'(V)$ by definition. Now assume $q \in A'(V)$. Then $q < f|_V$, so $q(x) \in A_x$ for all $x \in V$. Hence every $x \in V$ has a neighborhood W on which the constant function equal to $q(x)$ is in $A(W)$. Furthermore, every x has a neighborhood on which q is constant, hence every $x \in V$ has a neighborhood $V_x \subseteq V$ on which q is constant and the constant function $q|_{V_x}$ is in $A(V_x)$. Since $\{V_x\}_{x \in V}$ is a cover of V and q is the amalgamation of all these constant functions, we see that $q \in A(V)$. We conclude that $A(V) = A'(V)$.
For every open $U \subseteq T$, we have a 1-1 correspondence $\Phi_U : \mathcal{R}(U) \to \widetilde{\mathbb{R}}$. We want to show that this correspondence is natural, in the sense that, for $V \subseteq U$ open,

$$\Phi_U((A, B)) \restriction V = \Phi_V((A, B) \restriction V) = \Phi_V((A \restriction V, B \restriction V).$$

Since $(A \restriction V)(W) = A(W)$ and $(B \restriction V)(W) = B(W)$ for any open $W \subseteq V$, we see that the obtained functions $\Phi_U((A, B))$ and $\Phi_V((A, B) \restriction V)$ must be equal on V, hence the above equality must hold. This shows that the 1-1 correspondence Φ is natural, so it is a sheaf morphism. We conclude that $\mathcal{R}$ and $\widetilde{\mathbb{R}}$ are isomorphic. ■

We have now proved the following theorem:

Theorem 4.1.8 (**[3, theorem 8.8]**). *For any topological space T, the sheaf $\mathcal{R}$ is isomorphic to $\widetilde{\mathbb{R}}$, where $\mathbb{R}$ has the standard topology of the reals.*

Under this isomorphism, addition, multiplication and the ordering of $\mathcal{R}$ correspond to pointwise addition, multiplication and ordering in $\widetilde{\mathbb{R}}$. From $(A, B) \in \mathcal{R}(U)$, we obtain a map $f : U \to \mathbb{R}$, which is in fact just a collection of Dedekind cuts $\{(A_x, B_x)\}_{x \in U}$. The definition of addition, multiplication and the order are the same for (A, B) as for the cuts (A_x, B_x), which corresponds to the pointwise definition on $\widetilde{\mathbb{R}}(U)$.

4.1.3 Brouwer's continuity theorem

In 1924, L.E.J. Brouwer published the paper *Beweis, dass jede volle Funktion gleichmässig stetig ist* (Proof, that every total function is uniformly continuous) [1], in which he proved that, in his (intuitionistic) interpretation of the continuum, every function $f : \mathbb{R} \to \mathbb{R}$ is continuous. This statement is called *Brouwer's continuity theorem* or *Brouwer's continuity principle*. We will show that, if our base space T has certain properties, this holds in the sheaf topos $\mathrm{Shv}(T)$. By this, we mean that

$$\begin{aligned} &[\![\forall f \in \mathcal{R}^{\mathcal{R}}\ \forall a \in \mathcal{R}\ \forall \epsilon \in \mathcal{R}\ (\epsilon > 0 \to \exists \delta \in \mathcal{R}\ (\delta > 0 \wedge \forall b \in \mathcal{R} \\ &(a - \delta < b < a + \delta \to f(a) - \epsilon < f(b) < f(a) + \epsilon)))]\!] = T \end{aligned}$$

in the topos $\mathrm{Shv}(T)$.
In part A.1 of the appendix, we define the sheaves $\mathcal{F}(\mathbb{R}, \mathbb{R})$ and $\mathcal{C}(\mathbb{R}, \mathbb{R})$ by

$$\mathcal{F}(\mathbb{R}, \mathbb{R})(U) = \{\text{all functions } U \times \mathbb{R} \to \mathbb{R}\}$$

and

$$\mathcal{C}(\mathbb{R}, \mathbb{R})(U) = \{\text{all continuous maps } U \times \mathbb{R} \to \mathbb{R}\}.$$

In part A.2 and A.1 of the appendix, we show that $\mathcal{R}^{\mathcal{R}}$ is sometimes isomorphic to a subsheaf of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ containing $\mathcal{C}(\mathbb{R}, \mathbb{R})$. In particular, if T is first-countable, normal, Hausdorff and has no isolated points, then by lemma A.2.3, we see that $\mathcal{R}^{\mathcal{R}}$ is isomorphic to $\mathcal{C}(\mathbb{R}, \mathbb{R})$. We will now prove the following theorem:

Theorem 4.1.9. *Let $Shv(T)$ be a sheaf topos such that $\mathcal{R}^{\mathcal{R}}$ is isomorphic to $\mathcal{C}(\mathbb{R}, \mathbb{R})$, where the evaluation map $ev_{\mathcal{R},\mathcal{R}} : \mathcal{C}(\mathbb{R}, \mathbb{R}) \times \mathcal{R} \to \mathcal{R}$ is given by*

$$ev_{\mathcal{R},\mathcal{R}}(f, g)(x) = f(x, g(x))$$

for all $x \in T$. Then Brouwer's continuity theorem holds in $Shv(T)$.

The following proof a modified version of the on the one given in [6, p. 40].

Proof. Let $U \subseteq T$ open and let $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})(U)$ and $a, \epsilon \in \mathcal{R}(U)$ such that $\epsilon(x) > 0$ for all $x \in U$. We have three continuous maps $f : U \times \mathbb{R} \to \mathbb{R}$, $a : U \to \mathbb{R}$ and $\epsilon : U \to \mathbb{R}_{>0}$, hence by lemma 4.1.10, for every $x \in U$ there exists a neighborhood $U_x \subseteq U$ and a $\delta_x > 0$ such that for every $\xi \in U_x$ and every $t \in (a(x) - \delta_x, a(x) + \delta_x)$ the following inequalities hold

$$|a(\xi) - a(x)| < 1/2\delta_x \tag{6}$$

$$|f(\xi, t) - f(\xi, a(x))| < 1/2\epsilon(\xi). \tag{7}$$

We will now show that

$$U_x \subseteq [\![\forall b \in \mathcal{R}\ (a - 1/2\delta_x < b < a + 1/2\delta_x \to f(a) - \epsilon < f(b) < f(a) + \epsilon)]\!].$$

Note that this proves that Brouwer's continuity theorem holds in $\mathrm{Shv}(T)$, since such a neighborhood U_x and such a δ_x exist for every $x \in U$.
To prove this, let $V \subseteq U_x$ and $b \in \mathcal{R}(V)$ such that

$$V = [\![a - 1/2\delta_x < b < a + 1/2\delta_x]\!],$$

so $|b(\zeta) - a(\zeta)| < 1/2\delta_x$ for all $\zeta \in V \subseteq U_x$. Combining this with inequality (6), which says $|a(\zeta) - a(x)| < 1/2\delta_x$, we see that $|b(\zeta) - a(x)| < \delta_x$, hence $a(\zeta), b(\zeta) \in (a(x) - \delta_x, a(x) + \delta_x)$. Using inequality (7), we see that

$$|f(\zeta, a(\zeta)) - f(\zeta, a(x))| < 1/2\epsilon(\zeta) \quad \text{and} \quad |f(\zeta, b(\zeta)) - f(\zeta, a(x))| < 1/2\epsilon(\zeta),$$

so $|f(\zeta, a(\zeta)) - f(\zeta, b(\zeta))| < \epsilon(\zeta)$. Since this holds for all $\zeta \in V$, we see that $V = [\![f(a) - \epsilon < f(b) < f(a) + \epsilon]\!]$, hence

$$V = [\![(a - 1/2\delta_x < b < a + 1/2\delta_x \to f(a) - \epsilon < f(b) < f(a) + \epsilon]\!].$$

Since V and b were arbitrary, we see that

$$U_x \subseteq [\![\forall b \in \mathcal{R}\ (a - 1/2\delta_x < b < a + 1/2\delta_x \to f(a) - \epsilon < f(b) < f(a) + \epsilon)]\!],$$

so we have proven that Brouwer's continuity theorem holds in Shv(T). ■

Lemma 4.1.10. *Let U be a topological space and let $f : U \times \mathbb{R} \to \mathbb{R}$, $a : U \to \mathbb{R}$ and $\epsilon : U \to \mathbb{R}_{>0}$ be continuous. Then for every $x \in U$ there exists a neighborhood U_x of x and a $\delta_x > 0$, such that for every $\xi \in U_x$ and every $t \in (a(x) - \delta_x, a(x) + \delta_x)$, the following inequalities hold:*

$$|a(\xi) - a(x)| < 1/2\delta_x$$
$$|f(\xi, t) - f(\xi, a(x))| < 1/2\epsilon(\xi).$$

Proof. Let $x \in U$ and set $x' = \epsilon(x)/2 > 0$. Then $\epsilon > x'$ on a neighborhood V_x of x by continuity of ϵ. By continuity of f, there exist a $\delta_x > 0$ and a neighborhood V_x' of x such that $|f(\xi, t) - f(\xi, a(x))| < x'/2$ for all $t \in (a(x) - \delta_x, a(x) + \delta_x)$ and all $\xi \in V_x'$. Now let W_x be a neighborhood of x such that $|a(\xi) - a(x)| < 1/2\delta_x$ for $\xi \in W_x$, which exists by continuity of a. If we define $U_x = V_x \cap V_x' \cap W_x$, then the inequalies

$$|a(\xi) - a(x)| < 1/2\delta_x$$
$$|f(\xi, t) - f(\xi, a(x))| < 1/2\epsilon(\xi)$$

hold for all $\xi \in U_x$ and $t \in (a(x) - \delta_x, a(x) + \delta_x)$. ■

Combining theorem 4.1.9 with lemma A.2.3 from the appendix, we get the following result:

Theorem 4.1.11. *Let T be a first-countable, normal Hausdorff space with no isolated points. Then Brouwers continuity theorem holds in Shv(T).*

4.1.4 Comparing our model to Dana Scott's model

In [9], Dana Scott introduces a model for the (single-sorted) language $\mathcal{L} = \{0, 1, +, \cdot, <\}$ that is similar to the sheaf $\mathcal{R}$. We will denote Scott's model by $\mathcal{R}_S$. It consists of the continuous functions $T \to \mathbb{R}$ for a topological space T, and the symbols $0, 1, +, \cdot$ and $<$ are interpreted pointwise. So this model can be seen as $\mathcal{R}(T)$, where $\mathcal{R}$ is the sheaf of Dedekind reals. Dana Scott interprets terms in the same way as we do, though he interprets formulas slightly different. It is natural to ask whether these models are really different, so if they really satisfy different $\mathcal{L}$-sentences. We will try to answer this question in this section. Dana Scott's interpretation will be denoted by $[\![\cdot]\!]_S$, to avoid confusion.

Definition 4.1.12. For a term t be a term with free variables $(x_1, \ldots, x_n)$, we define its interpretation $[\![t]\!]_S : \mathcal{R}_S^n \to \mathcal{R}_S$ by

1. If $t = x$ for a variable, then $[\![t]\!]_S = \mathrm{id}_{\mathcal{R}_S}$.
2. If $t = 0$ or $t = 1$, then $[\![t]\!]_S$ is the constant 0 function or the constant 1 function respectively.
3. If $t = f(t_1, \ldots, t_m)$ (so $t = t_1 + t_2$ or $t = t_1 \cdot t_2$), then $[\![t]\!]_S$ is defined by

$$[\![t]\!]_S(a_1, \ldots, a_n) = [\![f]\!]_S([\![t_1]\!]_S(a_1, \ldots, a_n), \ldots, [\![t_m]\!]_S(a_1, \ldots, a_n))$$

for $a_1, \ldots, a_n \in \mathcal{R}_S$.

Note that if t has no free variables, then t is an element of $\mathcal{R}_S$, so a continuous map $T \to \mathbb{R}$.

Definition 4.1.13. For a formula ϕ with free variables $x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in \mathcal{R}_S$, (write $\mathbf{a} = (a_1, \ldots, a_n)$), we define the open subset $[\![\phi]\!]_S \subseteq T$ inductively by

1. $[\![\bot]\!]_S = \emptyset$.
2. $[\![(t = s)(\mathbf{a})]\!]_S = \mathrm{int}(\{x \in T \mid [\![t]\!]_S(\mathbf{a})(x) = [\![s]\!]_S(\mathbf{a})(x)\})$.
3. $[\![(t < s)(\mathbf{a})]\!]_S = \{x \in T \mid [\![t]\!]_S(\mathbf{a})(x) < [\![s]\!]_S(\mathbf{a})(x)\}$
4. $[\![(\phi \wedge \psi)(\mathbf{a})]\!]_S = [\![\phi(\mathbf{a})]\!]_S \cap [\![\psi(\mathbf{a})]\!]_S$.
5. $[\![(\phi \vee \psi)(\mathbf{a})]\!]_S = [\![\phi(\mathbf{a})]\!]_S \cup [\![\psi(\mathbf{a})]\!]_S$.
6. $[\![(\phi \to \psi)(\mathbf{a})]\!]_S = \mathrm{int}((T \setminus [\![\phi(\mathbf{a})]\!]_S) \cup [\![\psi(\mathbf{a})]\!]_S)$.
7. $[\![\neg\phi(\mathbf{a})]\!]_S = \mathrm{int}(T \setminus [\![\phi(\mathbf{a})]\!]_S)$.
8. $[\![\exists x^S \phi(x^S, \mathbf{a})]\!]_S = \bigcup\{[\![\phi(a, \mathbf{a})]\!]_S \mid a \in \mathcal{R}_S\}$.
9. $[\![\forall x^S \phi(x^S, \mathbf{a})]\!]_S = \mathrm{int}\left(\bigcap\{[\![\phi(a, \mathbf{a})]\!]_S \mid a \in \mathcal{R}_S\}\right)$.

We see that this interpretation is the same as ours, except for the interpretation of quantifiers. To see that this really results in a different model, define the space $T = \mathbb{R} \cup \{*\}$, where $U \subseteq T$ is open if and only if $U \subseteq \mathbb{R}$ is open in the usual topology of $\mathbb{R}$, or $U = T$. This means that, if $* \in U$, then $U = T$.

Lemma 4.1.14. *If $T = \mathbb{R} \cup \{*\}$ is defined as above, then there exist $\mathcal{L}$-sentences which are valid in $\mathcal{R}_S$ but not in $\mathcal{R}$.*

Proof. Note that any continuous map $f : T \to \mathbb{R}$ is constant, so $\mathcal{R}_C$ can be seen as the classical reals. In particular, $\forall x\ (x < 0 \vee x = 0 \vee x > 0)$ is valid in $\mathcal{R}_C$. This statement is not valid in $\mathcal{R}$. Note that $\mathbb{R} \subseteq T$ is open and $f = \mathrm{id}_{\mathbb{R}} \in \mathcal{R}(\mathbb{R})$. We also see that

$$[\![f < 0 \vee f = 0 \vee f > 0]\!] \neq \mathbb{R},$$

which implies

$$[\![\forall x\ (x < 0 \vee x = 0 \vee x > 0)]\!] \neq T.$$

■

In the case that T is normal and Hausdorff however, we can locally extend maps $f : V \to \mathbb{R}$, by which we mean that for any $x \in V$, there exists a neighborhood V_x of x and a map $\overline{f}_x : T \to \mathbb{R}$ such that $\overline{f}_x$ and f agree on V_x. This is proved in the proof of lemma A.2.1. This allows us to prove the following theorem:

Theorem 4.1.15. *If T is a normal Hausdorff space, then for any $\mathcal{L}$-formula ϕ and any $a_1, \ldots, a_n \in \mathcal{R}(T) = \mathcal{R}_S$, we have*

$$[\![\phi(a_1, \ldots, a_n)]\!] = [\![\phi(a_1, \ldots, a_n)]\!]_S$$

Proof. We will prove this by induction on $\mathcal{L}$-formulas. Note that we don't have to consider the steps $\perp$, $=$, $<$, $\wedge$, $\vee$, $\to$ and $\neg$, since these are defined the same way for $[\![\cdot]\!]_S$ as for $[\![\cdot]\!]$. So all that is left are the quantifiers. Assume ϕ is a formula such that the hypothesis holds. Assume $x \in [\![\exists a \phi(a)]\!]_S$. Then there is a continuous $f : T \to \mathbb{R}$ such that $x \in [\![\phi(f)]\!]_S$. This is equal to $[\![\phi(f)]\!]$ by the hypothesis, hence $x \in [\![\exists a \phi(a)]\!]$.
Now assume $x \in [\![\exists a \phi(a)]\!]$. Then for some neighborhood U of x, there exists a map $f : U \to \mathbb{R}$ such that $x \in [\![\phi(f)]\!]$. Extend this map on a neighborhood V_x of x to a map $\overline{f}_x : T \to \mathbb{R}$. Then $x \in [\![\phi(f|_{V_x})]\!] \subseteq [\![\phi(\overline{f}_x)]\!] = [\![\phi(\overline{f}_x)]\!]_S$, so $x \in [\![\exists a \phi(a)]\!]_S$.

The proof for $\forall$ is similar. Note that if $V \subseteq [\![\forall a \phi(a)]\!]$, then for every $f : V \to \mathbb{R}$, we have $[\![\phi(f)]\!] = V$. In particular, for every $f : T \to \mathbb{R}$, we have $V \subseteq [\![\phi(f)]\!] = [\![\phi(f)]\!]_S$, since $V = [\![\phi(f|_V)]\!]$. We therefore see that $V \subseteq [\![\forall a \phi(a)]\!]_S$, hence $[\![\forall a \phi(a)]\!] \subseteq [\![\forall a \phi(a)]\!]_S$.
For the converse, note that proving $[\![\forall a \phi(a)]\!]_S \subseteq [\![\forall a \phi(a)]\!]$ amounts to proving that for any $V \subseteq [\![\forall a \phi(a)]\!]_S$ and any $f : V \to \mathbb{R}$, we have $[\![\phi(f)]\!] = V$. Let V and f be like this. Then for any $x \in V$ there exist a neighborhood V_x and a $\overline{f}_x : T \to \mathbb{R}$ agreeing with f on V_x. Note that $f = \bigvee_{x \in V} \overline{f}_x|_{V_x}$. We therefore see that

$$[\![\phi(f)]\!] = \left[\!\!\left[\phi\left(\bigvee_{x \in V} \overline{f}_x|_{V_x}\right)\right]\!\!\right] = \bigcup_{x \in V} [\![\phi(\overline{f}_x|_{V_x})]\!].$$

We see that $[\![\phi(\overline{f}_x|_{V_x})]\!] = V_x$, since

$$V_x \subseteq V \subseteq [\![\forall a \phi(a)]\!]_S \subseteq [\![\phi(\overline{f}_x)]\!]_S = [\![\phi(\overline{f}_x)]\!],$$

so $[\![\phi(f)]\!] = \bigcup [\![\phi(\overline{f}_x|_{V_x})]\!] = \bigcup V_x = V$. ■

4.2 Constructing the reals using Cauchy sequences

As promised in section 4.1.2, we will now look at another construction of the real numbers that is often seen in mathematics, using Cauchy sequences. In classical mathematics, this method provides us with a model for the real numbers which is isomorphic to the Dedekind reals. In intuitionistic mathematics, these models need not be isomorphic. In [12, p.274], Troelstra and Van Dalen prove that the Cauchy reals are isomorphic to the Dedekind reals if the axiom of countable choice (ACN) is assumed.

In this section will first show that the axiom of countable choice holds in Shv(T) if T is the Baire space, which implies that the Cauchy reals and the Dedekind reals are isomorphic in this topos. We will then look at the construction of the Cauchy reals in a topos. We denote the sheaf of Dedekind reals by $\mathcal{R}_D$ and the sheaf of Cauchy reals by $\mathcal{R}_C$, to avoid confusion. We will prove that there is a topos in which $\mathcal{R}_D \not\simeq \mathcal{R}_C$, showing that the Dedekind and Cauchy reals are indeed not intuitionistically equivalent. We will even show that in this topos, Brouwer's continuity theorem does not hold for $\mathcal{R}_C$, yet it does for $\mathcal{R}_D$. The author has not been able to find this particular statement in the literature. We will then construct a topos in which $\mathcal{R}_D \simeq \mathcal{R}_C$, yet ACN does not hold, showing that these statements are not equivalent. This section is based on page 289 and 290 of [2] by Fourman and Hyland.

4.2.1 The axiom of countable choice in the topos of sheaves over the Baire space

The *axiom of countable choice* states that, if we are given sequence of nonempty subsets $(A_n)_{n\in\mathbb{N}}$ of a set A, then there exists a sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in A_n$ for all $n \in \mathbb{N}$. If we write $\phi(n, a)$ for $a \in A_n$, then it is easy to see that the axiom of countable choice can be formulated as

$$\forall n \in \mathbb{N}\, \exists a \in A\ \phi(n, a) \to \exists f \in A^{\mathbb{N}}\, \forall n \in \mathbb{N}\ \phi(n, f(n)).$$

We say that the axiom of countable choice holds in Shv(T) if for every sheaf A and every formula $\phi(x, y)$ where x is of sort $\widetilde{\mathbb{N}}$ and y of sort A, the following holds:

$$\forall n \in \widetilde{\mathbb{N}}\, \exists a \in A\ \phi(n, a) \to \exists f \in A^{\widetilde{\mathbb{N}}}\, \forall n \in \widetilde{\mathbb{N}}\ \phi(n, f(n)).$$

By the soundness theorem from section 3.3 and the (constructive) proof that the Cauchy reals and Dedekind reals are isomorphic by Troelstra and Van Dalen [12, p.274], we see that in such a topos, $\mathcal{R}_C$ and $\mathcal{R}_D$ are isomorphic. We will now show that ACN holds in Shv(T) if T is the Baire space, hence $\mathcal{R}_C$ and $\mathcal{R}_D$ are isomorphic in this topos. In the later sections, we will look at a topos where $\mathcal{R}_C$ and $\mathcal{R}_D$ are not isomorphic and a topos where $\mathcal{R}_C$ and $\mathcal{R}_D$ are isomorphic, yet ACN does not hold.
From now on, let T be the Baire space. To show that ACN holds in Shv(T), we need the following lemma:

Lemma 4.2.1. *Let T be the Baire space and let $\{U_i\}_{i\in I}$ be a collection of opens. Then there exists a countable refinement $\{W_n\}_{n\in\mathbb{N}}$ such that the sets W_n are clopen and pairwise disjoint.*

Proof. The basis of the topology of the Baire space, as defined on page 42 of [4], consists of clopen sets and is countable (since the collection of all finite sequences of natural numbers is countable). Since T has a countable basis of clopen sets, every collection of opens $\{U_i\}_{i\in I}$ has a countable refinement of clopen sets $\{V_n\}_{n\in\mathbb{N}}$. Now define $W_0 = V_0$ and $W_n = V_n \setminus W_{n-1}$. Then the sets W_n are clearly pairwise disjoint and $\cup W_n = \cup V_n$. We see inductively that all W_n are clopen, since W_0 is clopen, and if W_{n-1} is clopen, then $W_n = V_n \setminus W_{n-1} = V_n \cap W_{n-1}^c$ is clopen. We conclude that $\{W_n\}_{n\in\mathbb{N}}$ is a countable refinement of $\{U_i\}_{i\in I}$ such that the sets W_n are clopen and pairwise disjoint. ■

We will now prove that ACN holds in Shv(T). We need to show that

$$[\![\forall n \in \widetilde{\mathbb{N}}\ \exists a \in A\ \phi(n,a)]\!] \subseteq [\![\exists f \in A^{\widetilde{\mathbb{N}}}\ \forall n \in \widetilde{\mathbb{N}}\ \phi(n, f(n))]\!].$$

Before we prove this, note that every function $f : \mathbb{N} \to A(U)$, for $U \subseteq T$ open, extends to a morphism $\overline{f} : \widetilde{\mathbb{N}} \restriction U \to A$. If $x \in \widetilde{\mathbb{N}}(V)$, then $\{x^{-1}(n)\}_{n\in\mathbb{N}}$ is a cover of V such that x is constant on each open of this cover. Then define $\overline{f}(x) = \bigvee (f(n) \restriction x^{-1}(n))$. The details are left to the reader, but note that this can be seen as a special case of lemma 3.2.9.

Theorem 4.2.2. *Let T be the Baire space. Then ACN holds in Shv(T).*

Proof. Let $U = [\![\forall n \in \widetilde{\mathbb{N}}\ \exists a \in A\ \phi(n,a)]\!]$ and let $m \in \mathbb{N}$. Denote the constant function $U \to \mathbb{N}$ equal to m, simply by m. Then there is a cover $\{U_i\}_{i\in I}$ of U, such that for every $i \in I$, there exists an $a \in A(U_i)$ such that $[\![\phi(m \restriction U_i, a)]\!] = U_i$. By lemma 4.2.1, there exists a countable refinement $\{W_n\}_{n\in\mathbb{N}}$ of pairwise disjoint clopen sets. Then for every $n \in \mathbb{N}$, there exists an $a_n \in A(W_n)$ such that $[\![\phi(m \restriction W_n, a_n)]\!] = W_n$. Pick such an a_n for every $n \in \mathbb{N}$. Then $(a_n)_{n\in\mathbb{N}}$ is a compatible family with respect to $\{W_n\}_{n\in\mathbb{N}}$, since they are pairwise disjoint, so define $a' = \bigvee a_n$. We see that $[\![\phi(m,a')]\!] = \cup[\![\phi(m \restriction W_n, a_n)]\!] = \cup W_n = U$. Hence for every constant $m \in \mathbb{N}(U)$, there exists an $a'_m \in A(U)$ for which $[\![\phi(m, a'_m)]\!] = U$. Pick such a section a'_m for every $m \in \mathbb{N}$ and define $f : \mathbb{N} \to A(U)$ by setting $f(m) = a'_m$. Then $[\![\phi(m, f(m))]\!] = U$ for every $m \in \mathbb{N}$. This function extends to a morphism $\overline{f} : \widetilde{\mathbb{N}} \restriction U \to A$.
It is easy to see that $[\![\forall n \in \widetilde{\mathbb{N}}\ \phi(n, \overline{f}(n))]\!] = U$. Let $x \in \widetilde{\mathbb{N}}(V)$ for some open $V \subseteq U$ and write V_n for $x^{-1}(n)$. Then x is constant and equal to n on V_n, hence $[\![\phi(x \restriction V_n, f(n) \restriction V_n)]\!] = V_n$ for every $n \in \mathbb{N}$. We now see that

$$[\![\phi(x, \overline{f}(x))]\!] = [\![\phi(x, \bigvee (f(n) \restriction V_n))]\!] = \bigcup [\![\phi(x \restriction V_n, f(n) \restriction V_n)]\!] = \bigcup V_n = V$$

by definition of $\overline{f}$. Since this holds for all open $V \subseteq U$ and all $x \in \widetilde{\mathbb{N}}(V)$, we see that $[\![\forall n \in \widetilde{\mathbb{N}}\ \phi(n, f(n))]\!] = U$. Therefore

$$[\![\forall n \in \widetilde{\mathbb{N}}\ \exists a \in A\ \phi(n,a)]\!] = U \subseteq [\![\exists f \in A^{\widetilde{\mathbb{N}}}\ \forall n \in \widetilde{\mathbb{N}}\ \phi(n, f(n))]\!],$$

so we conclude that ACN holds in Shv(T). ■

4.2.2 A topos where $\mathcal{R}_{\textbf{Dedekind}}$ and $\mathcal{R}_{\textbf{Cauchy}}$ are not isomorphic

We will construct the Cauchy reals as a quotient of a subsheaf of $\widetilde{\mathbb{Q}}^{\widetilde{\mathbb{N}}}$. A *fundamental sequence* or a rational Cauchy sequence is a sequence $(a_n)_{n\in\mathbb{N}}$ of rational numbers such that for all rational numbers $q > 0$, there exists an $N \in \mathbb{N}$, such that for all $m, n > N$, we have $|a_m - a_n| < q$. This leads us to the following definition:

Definition 4.2.3. We define the *sheaf of fundamental sequences* Q_{C} to be the subsheaf of $\widetilde{\mathbb{Q}}^{\widetilde{\mathbb{N}}}$ defined by

$$a \in Q_{\mathrm{C}} \leftrightarrow \forall q \in \widetilde{\mathbb{Q}}\ (q > 0 \to \exists N \in \widetilde{\mathbb{N}}\ \forall m, n\ (m, n > N \to -q < a_m - a_n < q)).$$

We now define the Cauchy reals as a quotient of Q_{C}.

Definition 4.2.4. Define an equivalence relation $\sim$ on Q_{C} by

$$a \sim b \leftrightarrow \forall q \in \widetilde{\mathbb{Q}}\ (q > 0 \to \exists N \in \widetilde{\mathbb{N}}\ \forall n (n > N \to -q < a_n - b_n < q)).$$

We define *a model $\mathcal{R}_C$ of the Cauchy reals* to be the quotient $Q_{\mathrm{C}}/\sim$.

Note that $\sim$ defines an equivalence relation on the set $Q_C(U)$ for every open $U \subset T$. We will denote this equivalence relation by $\approx$, to avoid confusing it with the subsheaf $\sim$ of $Q_C \times Q_C$.

Let $p : Q_C \to \mathcal{R}_C$ be a model for the Cauchy reals. In this sheaf, we define the interpretations of 0 and 1 to be $p(0')$ and $p(1')$ respectively, where $0'$ and $1'$ are the constant morphisms $\widetilde{\mathbb{N}} \to \widetilde{\mathbb{Q}}$ equal to the 0 and 1. We define $+$ and $\cdot$ to be 'pointwise' addition and multiplication, so for $s_1, s_2 \in Q_C(U)$, we define $(s_1 + s_2) : \widetilde{\mathbb{N}} \upharpoonright U \to \widetilde{\mathbb{Q}}$ to be the morphism $(s_1 + s_2)(x)(n) = s_1(x)(n) + s_2(x)(n)$ and we set $p(s_1) + p(s_2) = p(s_1 + s_2)$. This is done similarly for multiplication. Since every $a \in \mathcal{R}_C$ is an amalgamation of sections of the form $p(s)$, this extends to a definition on the whole of $\mathcal{R}_C$. The proof that this is well-defined is left to the reader. Lastly, we define $<$ by

$$p(a) < p(b) \leftrightarrow \exists N \in \widetilde{\mathbb{N}}\ (\forall n \in \widetilde{\mathbb{N}}\ (n > N \to a_n < b_n)).$$

The proof that this is well-defined is also left to the reader.
Now assume that T is locally connected. Note that by lemma A.3.1, the exponent $\widetilde{\mathbb{Q}}^{\widetilde{\mathbb{N}}}$ is given by $\widetilde{\mathbb{Q}^{\mathbb{N}}_{\mathrm{dis}}}$, so Q_C can be seen as a subsheaf of $\widetilde{\mathbb{Q}^{\mathbb{N}}_{\mathrm{dis}}}$. This allows us to prove to following:

Lemma 4.2.5. *If T is locally connected, then $\widetilde{\mathbb{R}_{dis}}$ is a model for the Cauchy reals, where $\mathbb{R}_{dis}$ is the set of the real numbers with the discrete topology.*

Proof. By writing out the definition of the sheaf of fundamental sequences Q_C as a subsheaf of $\widetilde{\mathbb{Q}^{\mathbb{N}}_{\mathrm{dis}}}$, we see that $Q_C(U)$ consists of all locally constant functions $s : U \to \mathbb{Q}^{\mathbb{N}}$ such that $s(x)$ is a fundamental sequence for every $x \in U$, in the classical sense. It is well know that these sequences converge to a unique real number, write $s_x = \lim_{n\to\infty} s(x)(n)$. Define the map $p : Q_C \to \widetilde{\mathbb{R}_{\mathrm{dis}}}$ by letting $p(s)$ be the function $x \mapsto s_x$. Since s is locally constant, we see that $p(s)$ is also locally constant, hence $p(s)$ is indeed a section of $\widetilde{\mathbb{R}_{\mathrm{dis}}}$ for all sections s of Q_C. We will now show that

$$\forall s, s' \in Q_C (p(s) = p(s') \leftrightarrow s \sim s') \quad \text{and} \quad \forall r \in \widetilde{\mathbb{R}_{\mathrm{dis}}}\ \exists s \in Q_C\ (p(s) = r)$$

hold, proving that $\widetilde{\mathbb{R}_{\mathrm{dis}}}$ is a model for the Cauchy reals. To see that the first holds, let $s, s' \in Q_C(U)$ for some open $U \subseteq T$. We need to show that $s \approx s'$ on an open V if and only if $p(s) = p(s')$ on this open V. So let $V \subseteq U$ be open such that $s|_V \approx s'|_V$. Then, by definition 4.2.4, for every rational $q > 0$ and every $x \in V$, there exists a $N \in \mathbb{N}$ such that

for all $n > N$, we have $|s(x)(n) - s'(x)(n)| < q$. This implies $\lim_{n\to\infty} s(x) = \lim_{n\to\infty} s'(x)$, hence $p(s)(x) = p(s')(x)$ for every $x \in V$.
For the other direction, assume $p(s)(x) = p(s')(x)$ for every $x \in V$. Let $q \in \widetilde{\mathbb{Q}}(W)$ for some $W \subseteq V$ such that $q > 0$ on W. Now let $x \in W$. Then s, s' and q are constant on some neighborhood W_x of x. Since they converge to the same limit, there exists an N such that for every $n > N$ and $y \in W_x$, we have $|s(n)(y) - s(n)(y)| < q(y)$. This implies $s|_V \approx s'|_V$ by definition 4.2.4, so we conclude that the first statement holds.
To prove the second statement, we need to show that every $r \in \widetilde{\mathbb{R}_{\text{dis}}}(U)$ is an amalgamation of elements of the form $p(s)$ for sections s of Q_C. Let $U \subseteq T$ be open and let $r \in \widetilde{\mathbb{R}_{\text{dis}}}(U)$. We find a cover $\{U_i\}_{i\in\mathbb{R}}$ defined by $U_i = r^{-1}(i)$, since r is continuous with respect to the discrete topology on $\mathbb{R}$. For every $i \in \mathbb{R}$, we see that $r|_{U_i}$ is constant. Pick a fundamental sequence a_n converging to this constant and define $s \in Q_C(U_i)$ by $s(x)(n) = a_n$ for every $x \in U_i$. Then $p(s) = r|_{U_i}$, so $r|_{U_i}$ is of the form $p(s)$ for a section s of Q_C for every $i \in \mathbb{R}$. Since $r = \bigvee r|_{U_i}$, we conclude that r is an amalgamation of elements of the form $p(s)$. We have now proven the second statement, hence $\widetilde{\mathcal{R}_{\text{dis}}}$ is a model for the Cauchy reals. ■

From now on, if our base space T is locally connected, we write $\mathcal{R}_C$ for $\widetilde{\mathbb{R}_{\text{dis}}}$. In $\mathcal{R}_C$, it is easy to see that 0 and 1 are represented by the constant functions equal to 0 and 1 and that $+$, $\cdot$ and $<$ as defined above, correspond to their pointwise definition. We will show this for $+$ and $<$, the others are left to the reader. Let $r, r' \in \mathcal{R}_C(U)$, let $x \in U$ and pick a neighborhood U_x on which r and r' are constant. Let a_n and a'_n be fundamental sequences converging to these constants. Then define $s, s' \in Q_C(U_x)$ by $s(y)(n) = a_n$ and $s'(y)(n) = a'_n$ for all $y \in U_x$. Then $p(s) = r|_{U_x}$ and $p(s') = r|_{U_x}$. We see that $p(s) + p(s') = p(s + s')$, so

$$(p(s) + p(s'))(x) = p(s + s')(x) = \lim_{n\to\infty} (a_n + a'_n) = r(x) + r'(x)$$

which is indeed pointwise addition. For $<$, note that $r|_{U_x} < r'|_{U_x}$ if and only if there exists an $N \in \mathbb{N}$ such that for all $n > N$, $a_n < a'_n$, which is obvious from the fact that $\lim_{n\to\infty} a_n = r(x)$, $\lim_{n\to\infty} a'_n = r'(x)$ and that $r|_{U_x}$ and $r'|_{U_x}$ are constant.

When we say that $\mathcal{R}_D$ and $\mathcal{R}_C$ are isomorphic, we mean that there is an isomorphism preserving the structure, so preserving 0, 1, $<$, $\cdot$ and $+$. We will now show that, if we take $\mathbb{R}$ as our base space, that $\mathcal{R}_\text{D} \not\cong \mathcal{R}_\text{C}$.

Theorem 4.2.6. *In the sheaf topos Shv($\mathbb{R}$), where $\mathbb{R}$ has the usual topology, $\mathcal{R}_D$ and $\mathcal{R}_C$ are not isomorphic.*

Proof. For any $x, y \in \mathcal{R}_\text{C}(T)$, we have $[\![x < y]\!] = T$, $[\![x = y]\!] = T$ or $[\![x > y]\!] = T$, since all elements of $\mathcal{R}_\text{C}(T)$ are constant functions. This does not hold in $\mathcal{R}_\text{D}(T)$, since $[\![0 < \text{id}_\mathbb{R}]\!] = (0, \infty)$. Hence there cannot be an isomorphism between $\mathcal{R}_\text{D}$ and $\mathcal{R}_\text{C}$ preserving their structure. ■

The above proof in fact works for any base space T which is locally connected and for which there is a non-constant continuous map $f : U \to \mathbb{R}$ for some connected open $U \subseteq T$.

An interesting thing to note, is that in Shv($\mathbb{R}$), Brouwer's continuity theorem is not true for $\mathcal{R}_C$, yet it is true for $\mathcal{R}_D$.

Theorem 4.2.7. *In Shv($\mathbb{R}$), Brouwer's continuity theorem is not true for $\mathcal{R}_C$, yet it is true for $\mathcal{R}_D$.*

Proof. The fact that Brouwer's continuity theorem is true for $\mathcal{R}_D$ is a consequence of theorem 4.1.11. To show that Brouwer's continuity theorem does not hold for $\mathcal{R}_C$, we will simply pick a counterexample $f \in \mathcal{R}_C^{\mathcal{R}_C}(\mathbb{R})$. Note that $\mathcal{R}_C^{\mathcal{R}_C}$ can be seen as $\widetilde{\mathbb{R}_{\text{dis}}^{\mathbb{R}}}$ by lemma A.3.1. Let $f \in \widetilde{\mathbb{R}_{\text{dis}}^{\mathbb{R}}}(U)$ be the function $f : \mathbb{R} \to \mathbb{R}^{\mathbb{R}}$ defined by

$$f(x)(y) = \begin{cases} 0 & \text{if} \quad y \leq 0 \\ 1 & \text{if} \quad y > 0. \end{cases}$$

If a is the constant zero function and ϵ is the constant function $1/2$, then it is easy to check that

$$[\![\exists \delta \in \mathcal{R}_C\ (\delta > 0 \wedge \forall b \in \mathcal{R}_C\ (a - \delta < b < a + \delta \to f(a) - \epsilon < f(b) < f(a) + \epsilon))]\!] = \emptyset$$

hence

$$[\![\forall f \in \mathcal{R}_C^{\mathcal{R}_C}\ \forall a \in \mathcal{R}_C\ \forall \epsilon \in \mathcal{R}_C\ (\epsilon > 0 \to \exists \delta \in \mathcal{R}_C\ (\delta > 0 \wedge \forall b \in \mathcal{R}_C$$
$$(a - \delta < b < a + \delta \to f(a) - \epsilon < f(b) < f(a) + \epsilon)))]\!] = \emptyset,$$

so Brouwer's continuity theorem is definitely not true for $\mathcal{R}_C$. ■

4.2.3 A proof that $\mathcal{R}_{\textbf{Dedekind}} \simeq \mathcal{R}_{\textbf{Cauchy}}$ does not imply ACN

In [2, p.290], Fourman and Hyland construct a topos in which ACN does not hold, yet $\mathcal{R}_{\text{Dedekind}}$ and $\mathcal{R}_{\text{Cauchy}}$ are isomorphic. Let T be the space $\mathbb{R} \cup \{*\}$, where $U \subseteq T$ is open if and only if $U = \emptyset$ or $U \cap \mathbb{R}$ is open in $\mathbb{R}$ and $* \in U$.

Theorem 4.2.8 (**[2, p.290]**)**.** *In the topos Shv(T), ACN does not hold, yet $\mathcal{R}_D$ and $\mathcal{R}_C$ are isomorphic. In particular, $\mathcal{R}_{Dedekind} \simeq \mathcal{R}_{Cauchy}$ does not imply ACN.*

Proof. Let $U \subseteq T$ be open and nonempty. Then $* \in U$. Let $f \in \mathcal{R}_{\text{D}}(U)$. Note that $O := \mathbb{R} \setminus \{f(*)\}$ is open in $\mathbb{R}$, hence $f^{-1}(O)$ is open. Since $* \notin f^{-1}(O)$, this implies that $f^{-1}(O) = \emptyset$, hence $f^{-1}(f(*)) = U$. We conclude that f is constant. Similarly, we see that any $f \in \mathcal{R}_{\text{C}}(U)$ is constant, hence $\mathcal{R}_{\text{C}}(U) = \mathcal{R}_{\text{D}}(U)$ for all open $U \subseteq T$, hence $\mathcal{R}_{\text{D}}$ and $\mathcal{R}_{\text{C}}$ are isomorphic.
To show that ACN does not hold, we construct a sequence of subsheaves of $\widetilde{\mathbb{Q}}$. We see that $\widetilde{\mathbb{Q}}(U)$ consists of constant functions $U \to \mathbb{Q}$, so we might as well view $\widetilde{\mathbb{Q}}(U)$ as the set of rational numbers when $U \neq \emptyset$ (note that $\widetilde{\mathbb{Q}}(\emptyset) = \{*\}$). We will also view $\widetilde{\mathbb{N}}(U)$ as the set of natural numbers if $U \neq \emptyset$. We define, for $n \in \mathbb{N}$, the subsheaf A_n of $\widetilde{\mathbb{Q}}$ by

$$A_n(U) = \{q \in \mathbb{Q} \mid U \subseteq (q - 1/n, q + 1/n) \cup \{*\}\}$$

for $U \neq \emptyset$. The proof that this is a sheaf is left to the reader. If we define $\phi(n, q) \leftrightarrow q \in A_n$ (where n is of sort $\widetilde{\mathbb{N}}$ and q of sort $\widetilde{\mathbb{Q}}$), then it is clear that

$$[\![\forall n \in \widetilde{\mathbb{N}}\ \exists q \in \widetilde{\mathbb{Q}}\ \phi(n, y)]\!] = T.$$

If ACN holds, then

$$[\![\exists f \in \widetilde{\mathbb{Q}}^{\widetilde{\mathbb{N}}}\ \forall n \in \widetilde{\mathbb{N}}\ \phi(n, f(n))]\!] = T,$$

so any $x \in T$ has a neighborhood U such that there is a morphism $f \in \widetilde{\mathbb{Q}}^{\widetilde{\mathbb{N}}}(U)$ for which $f(n) \in A_n(U)$ for every $n \in \mathbb{N}$. We will show that this is not the case if $x \in \mathbb{R} \subseteq T$, proving that ACN does not hold in Shv(T). Let $x \in \mathbb{R}$ and let $U \subseteq T$ be an open neighborhood of x. Then there is an $n \in \mathbb{N}$ such $(x - 2/n, x + 2/n) \subseteq U$. Then for any rational number q, it is clear that $U \not\subseteq (q - 1/n, q + 1/n) \cup \{*\}$, since this would imply $(x - 2/n, x + 2/n) \subseteq (q - 1/n, q + 1/n)$, implying $4/n < 2/n$. Hence $A_n(U) = \emptyset$. Note that a morphism $f : \widetilde{\mathbb{N}} \restriction U \to \widetilde{\mathbb{Q}}$ is simply a map $f : \mathbb{N} \to \mathbb{Q}$. Since $A_n(U) = \emptyset$, there cannot exists such a map with $f(n) \in A_n(U)$ for all $n \in \mathbb{N}$. Since this holds for any neighborhood U, we conclude that ACN cannot hold in Shv(T). ■

A Appendix: the sheaf $\widetilde{B}^{\widetilde{A}}$ for topological spaces A, B

When studying the interpretation of higher-order logic in $\mathrm{Shv}(T)$, the category of sheaves over a given topological space T, the exponential sheaf G^F plays an important role. For two sheaves F and G, it is defined by

$$G^F(U) = \{\text{all morphisms } \phi : F \restriction U \to G\},$$

where the restriction to $V \subseteq U$ of a morphism $\phi \in G^F(U)$ is simply defined by

$$(\phi \restriction V)(x) = \phi(x),$$

for all $x \in F(W)$ for every $W \subseteq V$ open. The evaluation morphism $\mathrm{ev}_{F,G} : G^F \times F \to G$ is given by $\mathrm{ev}_{F,G}(\phi, x) = \phi(x)$.
This description often proves to be quite hard to work with, so it is natural to ask whether an easier description exists. In this thesis, we often work with sheaves of the form $\widetilde{A}$ for a topological space A. Our aim is to find an easier description for the sheaf $\widetilde{B}^{\widetilde{A}}$, where A and B are topological spaces. Although this turns out to be quite hard to do in general, some partial results are presented in this appendix.
A section $\phi \in \widetilde{B}^{\widetilde{A}}(U)$ is a collection of maps $\phi_V : \widetilde{A}(V) \to \widetilde{B}(V)$ for all $V \subseteq U$ open which is natural, so $\phi_V(f)|_W = \phi_W(f|_W)$ for all opens $W \subseteq V \subseteq U$. This implies that for any two $f, g \in \widetilde{A}(V)$, if $f|_W = g|_W$, then $\phi(f)|_W = \phi(g)|_W$, so

$$[\![f = g]\!] \subseteq [\![\phi(f) = \phi(g)]\!]. \tag{8}$$

There are two interesting questions that arise, which might allow for an easier description for the sheaf $\widetilde{B}^{\widetilde{A}}$. The first question is whether (8) can be replaced by the statement that $f(x) = g(x)$ implies $\phi(f)(x) = \phi(g)(x)$. The second question is whether the maps ϕ_V for $V \subseteq U$ are fully determined by ϕ_U.
We can prove the following lemma, concerning the second question.

Lemma A.0.9. *If for every open $V \subseteq T$, every continuous map $f : V \to A$ can be locally extended to a map $\overline{f} : T \to A$, then for every $U \subseteq T$, a morphism $\phi : \widetilde{A} \restriction U \to \widetilde{B}$ is fully determined by ϕ_U. Furthermore, any mapping $\phi_U : \widetilde{A}(U) \to \widetilde{B}(U)$ such that (8) holds, uniquely determines a morphism $\phi : \widetilde{A} \restriction U \to \widetilde{B}$.*

By locally extended, we mean that every $x \in V$ has an open neighborhood V_x such that $f|_{V_x}$ can be extended to a map $\overline{f} : T \to A$.

Proof. First, let $\phi : \widetilde{A} \restriction U \to \widetilde{B}$, let $f \in \widetilde{A}(V)$ for some $V \subseteq U$ and let $x \in V$. Then extend $f|_{V_x}$ to a map $\overline{f} : T \to A$. We see that

$$\phi_V(f)(x) = \phi_{V_x}(f|_{V_x})(x) = \phi_U(\overline{f}|_U)(x),$$

so $\phi_V(f)(x)$ is determined by ϕ_U. Since this holds for all $x \in V$ and all $f \in \widetilde{A}(V)$, we see that ϕ_V is fully determined by ϕ_U for all $V \subseteq U$.
Now assume we are given a mapping $\phi_U : \widetilde{A}(U) \to \widetilde{B}(U)$ such that (8) holds. We define $\phi : \widetilde{A} \restriction U \to \widetilde{B}$ by

$$\phi_V(f)(x) = \phi_U(\overline{f})(x)$$

for every $x \in V$ and every $f \in \widetilde{A}(V)$, where $\overline{f}$ is any function that extends $f|_{V_x}$ for some neighborhood V_x of x. The fact that this is well-defined is a direct consequence of (8). The fact that ϕ is natural also follows directly from the definition. ■

This lemma shows that, if functions can locally extended, then we can view $\widetilde{B}^{\widetilde{A}}$ as the sheaf whose sections over U are the mappings $\phi_U : \widetilde{A} \restriction U \to \widetilde{B}$ satisfying (8).

In the next sections, we will prove some specific results for the sheaves $\mathcal{R}^{\mathcal{R}}$, where $\mathcal{R} = \widetilde{\mathbb{R}}$, and $\widetilde{B}^{\widetilde{A}}$, where A and B are discrete spaces. The author found it really hard to answer the question of what $\widetilde{B}^{\widetilde{A}}$ looks like for general topological spaces A and B, though this might be an interesting starting point for further research.

A.1 The sheaf of functions $T \times A \to B$.

Assume we are considering sheaves over the topological space T and let A and B be two given topological spaces. We define the sheaf $\mathcal{F}(A, B)$ of functions $T \times A \to B$ by

$$\mathcal{F}(A, B)(U) = \{\text{all functions } U \times A \to B\},$$

where the restriction maps $(\cdot) \restriction V$ are ordinary function restrictions. We define the subsheaf $\mathcal{C}(A, B)$ of $\mathcal{F}(A, B)$ by

$$\mathcal{C}(A, B)(U) = \{\text{all continuous maps } U \times A \to B\}.$$

We have the following lemma:

Lemma A.1.1. *If for any open U and any $\phi \in \widetilde{B}^{\widetilde{A}}(U)$,*

$$f(x) = f'(x) \quad \textit{implies} \quad \phi(f)(x) = \phi(f')(x)$$

for all sections f, f' of $\widetilde{A}$ and all x, then $\widetilde{B}^{\widetilde{A}}$ can be seen as a subsheaf F of $\mathcal{F}(A, B)$. The evaluation map $ev_{\widetilde{A},\widetilde{B}} : F \times \widetilde{A} \to \widetilde{B}$ is then given by

$$ev_{\widetilde{A},\widetilde{B}}(g, f)(x) = g(x, f(x))$$

for $f \in \widetilde{A}(U)$, $g \in F(U)$ and $x \in T$. Furthermore, the subsheaf $\mathcal{C}(A, B)$ of $\mathcal{F}(A, B)$ is contained in F.

Proof. Assume the hypothesis of this lemma and denote for a section $f \in \widetilde{A}(V)$ and an $x \in V$ the constant map $U \to A$ equal to $f(x)$ by $\text{const}(f(x))$. Then $\phi(f)(x) = \phi(\text{const}(f(x)))(x)$. This holds for all sections f and all $x \in V$, so ϕ is fully determined by what it does to constant functions. We can associate the function $g_\phi : U \times A \to B$ defined by $g_\phi(x, a) = \phi(\text{const}(a))(x)$. Since ϕ is fully determined by what it does to constant functions, this correspondence is injective, hence $\widetilde{B}^{\widetilde{A}}$ can be seen as a subsheaf of $\mathcal{F}(A, B)$. We see that

$$g_\phi(x, f(x)) = \phi(\text{const}(f(x))(x) = \phi(f)(x) = \text{ev}_{\widetilde{A},\widetilde{B}}(\phi, f)(x),$$

so the evaluation map is indeed given by

$$\text{ev}_{\widetilde{A},\widetilde{B}}(g, f)(x) = g(x, f(x)).$$

Denote the subsheaf of $\mathcal{F}(A,B)$ to which $\widetilde{B}^{\widetilde{A}}$ is isomorphic by F. To see that $\mathcal{C}(A,B)$ is a subsheaf of F, let $h \in \mathcal{C}(A,B)(U)$. Then h induces a morphism $\widetilde{h} : \widetilde{A} \restriction U \to \widetilde{B}$ defined by

$$\widetilde{h}(f)(x) = h(x, f(x))$$

for $f \in \widetilde{A}(V)$ and $x \in V \subseteq U$. The continuity of $\widetilde{h}(f)$ follows from the continuity of h. If we define $g_{\widetilde{h}} : U \times A \to B$ as above, it is clear that $h = g_{\widetilde{h}}$, hence $h = g_{\widetilde{h}} \in F$. ■

We see in particular that g_ϕ as defined above only depends on ϕ_U and not on ϕ_V for any $V \subseteq U$, so any $\phi : \widetilde{A} \restriction U \to \widetilde{B}$ is fully determined by ϕ_U.

A.2 The sheaf $\mathcal{R}^{\mathcal{R}}$

In this section, we denote the sheaf $\widetilde{\mathbb{R}}$ by $\mathcal{R}$. We will use the Tietze extension theorem from topology multiple times:

Theorem (**Tietze extension theorem**). *Let T be a normal space, let $X \subseteq T$ be closed and let $f : X \to \mathbb{R}$ be continuous with respect to the subspace topology. Then there exists a continuous extension $\overline{f} : T \to \mathbb{R}$ of f.*

When considering the sheaf $\mathcal{R}^{\mathcal{R}}$, we can show the following:

Lemma A.2.1. *If T is a normal Hausdorff space, then we can view sections $\phi \in \mathcal{R}^{\mathcal{R}}(U)$ as mappings $\phi : \mathcal{R}(U) \to \mathcal{R}(U)$ satisfying (8).*

Proof. We will show that any map $f : V \to \mathbb{R}$, for $V \subseteq T$ open, can be locally extended. The result is then a direct consequence of lemma A.0.9. To see that this is possible, let $f : V \to \mathbb{R}$ be given and let $x \in V$. Note that $\{x\}$ and V^c are closed, so there exist opens V_x and W such that $x \in V_x$, $V^c \subseteq W$ and $V_x \cap W = \emptyset$, by normality of T. Then $V_x \subseteq W^c \subseteq V$, where W^c is closed. By the Tietze extension theorem, there exists a function $\overline{f} : T \to \mathbb{R}$ such that $\overline{f}|_{W^c} = f|_{W^c}$. Since $V_x \subseteq W^c$, the function $\overline{f}$ extends f in a neighborhood of x. This can be done for any $x \in V$, so we conclude that f can be locally extended. ■

We can furthermore prove

Lemma A.2.2. *Let T be a first-countable, normal space and let $\phi \in \mathcal{R}(U)$ for some open $U \subseteq T$. Then for any sections $f, g \in \mathcal{R}(V)$, where $V \subseteq U$, and any $x \in V$, if $f(x) = g(x)$, then $\phi(f)(x) = \phi(g)(x)$.*

The following proof is based on the proof given in [10, p.248-249] by Dana Scott.

Proof. We will show that for any $f \in \mathcal{R}(V)$ and any $x \in V$, we have $\phi(f)(x) = \phi(\text{const}(f(x)))(x)$. Note that this implies what we want to prove. Let $f \in \mathcal{R}(V)$ and let $x \in V$. If f is constant on an open neighborhood V_x of x, then $\phi(f)(x) = \phi(\text{const}(f(x)))(x)$ is obvious, since

$$\phi(f)(x) = \phi(f|_{V_x})(x) = \phi(\text{const}(f(x)))(x).$$

Assume f is not constant on a neighborhood V_x of x. Since T is first-countable, there is a countable decreasing basis of neighborhoods $B_1 \supset B_2 \supset B_3 \supset \ldots$ for x. Note that for every $n \in \mathbb{N}$, there is a $y_n \in B_n$ such that $f(y_n) \neq f(x)$, otherwise f would be constant on B_n. If we pick such an y_n for ever n, we obtain a sequence $(y_n)_n$ converging to x. By continuity of f, we see that $(f(y_n))_n$ converges to $f(x)$, yet $f(y_n)$ is never equal to x. We can therefore pick a subsequence $(z_n)_n$ of $(y_n)_n$ such that $|f(x) - f(z_n)| < |f(x) - f(z_{n-1})|$ for every $n \in \mathbb{N}$, hence all $f(z_n)$ are distinct. Now pick a neighborhood $Z_n \subseteq \mathbb{R}$ of $f(z_n)$ for every n, such that $\mathrm{Cl}(Z_n) \cap \mathrm{Cl}(Z_m) = \emptyset$ for all $m, n \in \mathbb{N}$. Now define

$$W_1 = \bigcup_{n \text{ even}} Z_n \quad \text{and} \quad W_2 = \bigcup_{n \text{ odd}} Z_n.$$

It is easily checked that $\mathrm{Cl}(W_1) = (\cup_{n \text{ even}} \mathrm{Cl}(Z_n)) \cup \{f(x)\}$ and $\mathrm{Cl}(W_2) = (\cup_{n \text{ odd}} \mathrm{Cl}(Z_n)) \cup \{f(x)\}$, hence $\mathrm{Cl}(W_1) \cap \mathrm{Cl}(W_2) = \{f(x)\}$. We define $V_1 = f^{-1}(W_1)$ and $V_2 = f^{-1}(W_2)$. Note that these are disjoint and that $\mathrm{Cl}(V_1) \subseteq f^{-1}(\mathrm{Cl}(W_1))$ and $\mathrm{Cl}(V_2) \subseteq f^{-1}(\mathrm{Cl}(W_2))$, so $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) \subseteq f^{-1}(\{f(x)\})$. In particular, $x \in \mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2)$ and f and $\mathrm{const}(f(x))$ agree on $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2)$. Therefore the functions $f|_{\mathrm{Cl}(V_1)} : \mathrm{Cl}(V_1) \to \mathbb{R}$ and $\mathrm{const}(f(x))|_{\mathrm{Cl}(V_2)} : \mathrm{Cl}(V_2) \to \mathbb{R}$ can be glued together to form the continuous function

$$g : \mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) \to \mathbb{R}; \quad g(y) = \begin{cases} f(y) & \text{if } \; y \in \mathrm{Cl}(V_1), \\ f(x) & \text{if } \; y \in \mathrm{Cl}(V_2). \end{cases}$$

By the Tietze extension theorem, there exists a map $\overline{g} : T \to \mathbb{R}$ agreeing with g on $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2)$. Note that $\phi(f)$ agrees with $\phi(g)$ on V_1, hence also on $\mathrm{Cl}(V_1)$ and similarly $\phi(\mathrm{const}(f(x)))$ agrees with $\phi(g)$ on V_2, hence also on $\mathrm{Cl}(V_2)$. We now see that

$$\phi(f)(x) = \phi(g)(x) = \phi(\mathrm{const}(f(x)))(x).$$

∎

Combining the above lemma with lemma A.1.1, we see that we can consider sections of $\mathcal{R}^{\mathcal{R}}$ as maps $U \times \mathbb{R} \to \mathbb{R}$. In [10], Dana Scott proves that, if T is the Baire space, then these maps are continuous, proving that $\mathcal{R}^{\mathcal{R}}$ is isomorphic to $\mathcal{C}(\mathbb{R}, \mathbb{R})$. We will prove a generalized version of this:

Lemma A.2.3. *Let T be a first-countable, normal Hausdorff space with no isolated points. Then $\mathcal{R}^{\mathcal{R}}$ is isomorphic to $\mathcal{C}(\mathbb{R}, \mathbb{R})$, where the evaluation map $ev_{\mathcal{R},\mathcal{R}} : \mathcal{C}(\mathbb{R}, \mathbb{R}) \times \mathcal{R} \to \mathcal{R}$ is given by*

$$ev_{\mathcal{R},\mathcal{R}}(f, g)(x) = f(x, g(x))$$

for all $x \in T$.

The following proof is based on the proof given in [10, p.249-250] by Dana Scott.

Proof. By lemma A.1.1 and lemma A.2.2, we see that $\mathcal{R}^{\mathcal{R}}$ can be seen as a subsheaf F of $\mathcal{F}(\mathbb{R}, \mathbb{R})$ containing $\mathcal{C}(\mathbb{R}, \mathbb{R})$. To prove our claim, we need to show for every $U \subseteq T$ that any section of $F(U)$ is a continuous function $U \times \mathbb{R} \to \mathbb{R}$. Let $\Phi \in F(U)$ and let $\phi \in \mathcal{R} \restriction U \to \mathcal{R}$ be the sheaf morphism such that

$$\phi(f)(x) = \Phi(x, f(x))$$

for all $f \in \mathcal{R}(U)$ and all $x \in U$. Assume Φ is not continuous, say it is not continuous at $(x, a) \in U \times \mathbb{R}$. Since T is first-countable, there is a decreasing sequence of neighborhoods

$(U_n)_{n\in\mathbb{N}}$ of x. We can assume that $U_n \subseteq U$, otherwise we replace U_n by $U \cap U_n$. We easily see that $\cap_n U_n = \{x\}$: if $y \in U$, then x, y can be seperated by opens U_x and U_y. Since $U_n \subseteq U_x$ for some $n \in \mathbb{N}$, we see that $y \notin \cap_n U_n$. Since Φ is not continuous in (x, a), there is an $\epsilon > 0$ such that every neighborhood of (x, a) contains a (x', a') such that $|\Phi(x', a') - \Phi(x, a)| > \epsilon$. In particular, we can pick such a $(x_n, a_n) \in U_n \times (a - 1/n, a + 1/n)$ for every $n \in \mathbb{N}$, obtaining a sequence $(x_n, a_n)_n$ such that $(x_n)_n$ converges to x, $(a_n)_n$ converges to a and $|\Phi(x_n, a_n) - \Phi(x, a)| > \epsilon$ for all n. Note that the map $y \mapsto \Phi(y, a_n)$ is continuous for every fixed a_n, since it is equal to $\phi(\mathrm{const}(a_n))$. We can therefore assume that $x_n \neq x$ for every $n \in \mathbb{N}$, since $x_n \mapsto |\Phi(x_n, a_n) - \Phi(x, a)|$ is a continuous function, hence $|\Phi(x_n, a_n) - \Phi(x, a)| > \epsilon$ holds in a neighborhood of x_n. Since T has no isolated points, this neighborhood can never solely contain x. We can also assume all x_n to be distinct: for every x_n, there is a $m \in \mathbb{N}$ such that $x_n \notin U_m$. Then $x_m \neq x_n$. Since we can do this for ever $n \in \mathbb{N}$, we can pick a subsequence $(x'_n)_n$ such that all x'_n are distinct. So assume that all x_n are distinct. Let $X = \{x_n\}_{n\in\mathbb{N}} \cup \{x\} \subseteq U$. Then X is a closed set. We now define a function $\xi : X \to \mathbb{R}$ by $\xi(x_n) = a_n$ and $\xi(x) = a$. This function is easily seen to be continuous, so by the Tietze extension theorem, it can be extended to a continuous map $\overline{\xi} : T \to \mathbb{R}$. We then see that

$$\lim_{n\to\infty} \phi(\overline{\xi}|_U)(x_n) = \phi(\overline{\xi}|_U)(x)$$

by continuity of $\phi(\overline{\xi}|_U)$, yet

$$|\phi(\overline{\xi}|_U)(x_n) - \phi(\overline{\xi}|_U)(x)| = |\Phi(x_n, \xi(x_n)) - \Phi(x, \xi(x))| = |\Phi(x_n, a_n) - \Phi(x, a)| > \epsilon$$

for all $n \in \mathbb{N}$. This is a contradiction, so $\Phi : U \times \mathbb{R} \to \mathbb{R}$ is continuous. ■

A.3 The case where A and B are discrete spaces

If A and B are discrete spaces and T is locally connected, we can prove the following:

Lemma A.3.1 (**[2, lemma 2.3]**). *Assume T is locally connected and assume that A and B are discrete. Then $\widetilde{B}^{\widetilde{A}}$ is isomorphic to $\widetilde{B^A_{dis}}$, where B^A_{dis} is the set of all functions $A \to B$ with the discrete topology. The evaluation morphism $ev_{\widetilde{A},\widetilde{B}} : \widetilde{B^A_{dis}} \times \widetilde{A} \to \widetilde{B}$ is given by*

$$ev_{\widetilde{A},\widetilde{B}}(f, a)(x) = f(x)(a(x)).$$

Proof. Note that a map $f : U \to A$ or $f : U \to B$, for $U \subseteq T$, is continuous if and only if it is locally constant, since U is locally connected and A and B are discrete. Since every continuous map $f : U \to A$ is locally constant, it is clear that the hypothesis of lemma A.1.1 is satisfied (for any two locally constant maps $f, g : U \to A$ agree on a neighborhood of $x \in U$ if they agree in x). We can therefore see any $\phi \in \widetilde{B}^{\widetilde{A}}(U)$ as a map $\phi' : U \times A \to B$, which defines a map $f_\phi : U \to B^A_{\mathrm{dis}}$ by $f_\phi(x)(a) = \phi'(x, a)$ for $x \in U$ and $a \in A$. To show that f_ϕ is continuous, we need to show that it is locally constant. Let $x \in U$ and let $V \subseteq U$ be a connected neighborhood of x. Then

$$f_\phi(y)(a) = \phi(\mathrm{const}(a))(y) = \phi(\mathrm{const}(a))(x) = f_\phi(x)(a)$$

for all $a \in A$ and all $y \in V$, where we use that $\phi(\mathrm{const}(a))$ is constant on V.
We now show that any continuous (i.e. locally constant) map $f : U \to B^A_{\mathrm{dis}}$ induces a morphism $\phi_f : \widetilde{A} \restriction U \to \widetilde{B}$. We define, for $g \in \widetilde{A}(U')$ and $U' \subseteq U$,

$$\phi_f(g)(x) = f(x)(g(x)).$$

We need to check that $\phi_f(g)$ is locally constant. Let $x \in U'$ and let $V \subseteq U'$ be a connected neighborhood. Then f and g are constant on V, so

$$\phi_f(g)(y) = f(y)(g(y)) = f(x)(g(x)) = \phi_f(g)(x)$$

for any $y \in V$. We conclude that there is a 1-1 correspondence between $\widetilde{B}^{\widetilde{A}}$ and $\widetilde{B^A_{\mathrm{dis}}}$. The prove that this correspondence is an isomorphism (so the prove that it is natural) is left to the reader. We see that under this correspondence

$$\mathrm{ev}_{\widetilde{A},\widetilde{B}}(\phi, a)(x) = \phi(a)(x) = f_\phi(x)(a(x)).$$

■

Index

References

[1] L.E.J. Brouwer. "Beweis, dass jede volle Funktion gleichmässig stetig ist". In: *Proc. Amsterdam* 27 (1924), pp. 189–194.

[2] M.P. Fourman and J.M.E. Hyland. "Sheaf models for analysis". In: *Applications of sheaves*. Springer, 1979, pp. 280–301.

[3] M.P. Fourman and D.S. Scott. "Sheaves and logic". In: *Applications of sheaves*. Springer, 1979, pp. 302–401.

[4] T. Jech. *Set theory. The Third Millennium Edition, revised and expanded*. Springer-Verlag, 2003.

[5] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic: A first introduction to topos theory*. New York: Springer-Verlag, 1992.

[6] I. Moerdijk and J. van Oosten. *Topos Theory*. Lecture notes. Utrecht University, 2007. URL: `http://www.staff.science.uu.nl/~ooste110/syllabi/toposmoeder.pdf`.

[7] J. van Oosten. *Basic Category Theory*. Lecture notes. Revised edition, July 2002. Utrecht University, 2000. URL: `http://www.staff.science.uu.nl/~ooste110/syllabi/catsmoeder.pdf`.

[8] J. van Oosten. *Intuitionism.* Lecture notes. Revised edition, april 1996. 1995. URL: http://www.staff.science.uu.nl/~ooste110/syllabi/intuitionism.pdf.

[9] D.S. Scott. “Extending the topological interpretation to intuitionistic analysis”. In: *Compositio Mathematica* 20 (1968), pp. 194–210.

[10] D.S. Scott. “Extending the topological interpretation to intuitionistic analysis, II”. In: *Studies in Logic and the Foundations of Mathematics* 60 (1970), pp. 235–255.

[11] D.S. Scott. “Identity and existence in intuitionistic logic”. In: *Applications of sheaves.* Springer, 1979, pp. 660–696.

[12] A.S. Troelstra and D. van Dalen. *Constructivism in Mathematics.* Vol. 1. Utrecht: Elsevier, 1988.