# A New Model for Quantum Space in Diagrammatic Loop Quantum Gravity 

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#### Abstract

In loop quantum gravity space is assumed to consist of finite-size building blocks. Spin networks are used to model this microscopic structure of space. The goal is to create spin networks that have the properties of classical space as it is described by general relativity. Currently this is not possible. This thesis introduces a model such that it is possible to construct spin networks that have the properties of a classical metric. It uses spin networks with 6 -valent nodes in a cubic structure and all links have spin 1. This choice is based on the interpretation of the volume operator of loop quantum gravity as the creation operator for volume. A new set of operators is defined that measure length, area and volume in a spin network. Spin networks are constructed for flat space, the Schwarzschild metric and for plane gravitational waves. The first part of this thesis describes a diagrammatic version of loop quantum gravity, where wave functions are represented by diagrams and operators have a graphical action on these diagrams. The interpretation of the volume operator as a creation operator is based on its graphical form.


## Contents

1 Diagrammatic Loop Quantum Gravity ..... 3
1.1 Spin Networks ..... 3
1.2 The Fundamentals of Loop Quantum Gravity ..... 5
1.3 Rules for Diagrammatic Calculations ..... 18
1.4 Examples of Spin Network Wave Functions ..... 25
1.5 The Node Inner Product ..... 28
1.6 The Graphical Action of the Flux Operator ..... 31
1.7 The Graphical Action of the Holonomy Operator ..... 38
1.8 The Area and Volume Operators ..... 42
2 A New Model for Quantum Space ..... 50
2.1 The 6 -Valent Node ..... 50
2.2 The Volume Operator creates Volume ..... 56
2.3 New Geometrical Operators ..... 58
2.4 From a Classical Metric to a Spin Network ..... 66
2.5 Flat Space ..... 68
2.6 The Quantum Schwarzschild Metric ..... 69
2.7 Gravitational Waves ..... 76
A SU(2) Representations ..... 79
B Properties of the $3 j$ and $6 j$-Symbols ..... 80
References ..... 82

## 1 Diagrammatic Loop Quantum Gravity

The goal of loop quantum gravity is to give a description of general relativity in the language of quantum mechanics. According to general relativity, a mass curves the space that surrounds it and this influences the movement of objects through this space. General relativity does not describe a force of gravity, but the geometry of spacetime. A theory of quantum gravity should define the structure of spacetime on a small scale.

Spin networks are used in loop quantum gravity to model the microscopic structure of space. They are discussed in section 1.1. Section 1.2 summarizes of the formalism of loop quantum gravity. Wave functions are defined that are based on spin networks. These wave functions can be represented by a diagram. Section 1.3 contains the rules to construct and manipulate such diagrams. Section 1.4 gives three examples of how a spin network defines a wave function and how this wave function can be represented by a diagram.

In the remainder of the first part the rules for diagrammatic manipulation are used in various calculations. The inner product between node states of a spin network is evaluated diagrammatically in section 1.5. The two basic operators of the theory have a graphical action on the wave function diagrams. These graphical actions are defined in sections 1.6 and 1.7 and it is shown that the commutation relations between these operators are implemented in the diagrammatic version of the theory. The eigenvalues of the volume operator are calculated diagrammatically in section 1.8 .

### 1.1 Spin Networks

In general relativity space and time are treated equally, but in quantum mechanics time is a passive parameter: for every value of the time $t$, probabilities for outcomes of measurements are predicted. To give time the same passive role in loop quantum gravity the 4 -dimensional spacetime is cut into 3 -dimensional spacelike slices that are labeled by the parameter time. The part of loop quantum gravity that studies the properties of these timeslices is called the kinematics, in contrast to the dynamics, which describes the time evolution. In this thesis only the kinematics of loop quantum gravity will be discussed.

In loop quantum gravity a spin network represents a piece of the 3dimensional space in a timeslice. A spin network is a collection of nodes that are connected to each other by links. The number of links that is connected in a node is called the valence of the node, which is at least three and there is no maximum. A link must start and end in a node, there are no open ends, and a link can run between any two nodes in a network. Every link carries a half-integer spin value: $0,1 / 2,1,3 / 2$, etc.

A spin network defines the microscopic structure of space as it is perceived by one observer. The nodes are interpreted as indivisible pieces of


Figure 1: On the left a 2-dimensional picture of a spin network that is connected to the continuous space that surrounds it by open links. On the right the same spin network where the open links are now collected by the special node $n_{0}$. The spin networks that are used in loop quantum gravity are 3 -dimensional.
volume that are the building blocks of space. There is no spatial resolution inside such a volume. The links of a spin network show how these pieces of volume are connected to each other and a link represents the contact surface between two pieces of volume. Spin networks were introduced into loop quantum gravity in Rovelli and Smolin.

It is assumed that a building block of space that is represented by one node in a spin network has a size close to the Planck length $\left(10^{-35} \mathrm{~m}\right)$. A closed spin network with a limited number of nodes is then a very small piece of space that is not connected to the rest of space. Alternatively, a spin network can be drawn with open links that represent the connection between the network and the continuous space that surrounds it. Since open links are not allowed in a spin network, one can use a special node $n_{0}$ that collects all the links to the external space. This node can not be interpreted as a piece of volume, but the links to this node do represent the boundary between the continuous space and the spin network. This is shown in figure 1 .

A spin network can be constructed as the dual of a triangulation of space. In a triangulation, space is divided into polyhedra, which are volumes that are bounded by at least four faces. Every polyhedron is represented by a node in the spin network and every face that connects two polyhedra is represented by a link between two nodes. It is not required that a spin network is constructed in this way. In general, a link can connect any two nodes in a network. A spin network that is the dual of a triangulation has no long distance connections, no microscopic "wormholes".

### 1.2 The Fundamentals of Loop Quantum Gravity

Loop quantum gravity is an attempt to construct a quantum theory that has general relativity as its classical limit. This section describes the structure of this quantum theory and its relation to general relativity. This section is based on Rovelli and Vidotto, Thiemann, Dah-Wei Chiou, Jinsong Yang and Yongge Ma, Rovelli], Wald] and [Misner, Thorne and Wheeler].

## The Quantization of a Free Particle

The quantization of general relativity in loop quantum gravity follows the same steps as the quantization of classical mechanics. Some elements of the quantization of the free particle in one dimension are repeated here to show the similarities.

In classical mechanics a point mass can be described by its place and its momentum. These canonically conjugate variables are the coordinates on the phase space for the point mass, which means that they completely describe the point mass at any point in time. In quantum mechanics the variables position and momentum are converted to operators on a state space. The formalism of quantum mechanics requires that the commutation relation between these operators equals i $\hbar$ times the Poisson bracket of their classical counterparts:

$$
\begin{equation*}
\{x, p\}=1 \quad \rightarrow \quad[\hat{x}, \hat{p}]=\mathrm{i} \hbar \tag{1}
\end{equation*}
$$

The fact that the operators $\hat{x}$ and $\hat{p}$ do not commute means that the position and momentum of a particle can not be known at the same time with infinite precision. A particle with a definite momentum $p$ is described by the momentum eigenket $|p\rangle$. At one moment in time the wave function for this state is

$$
\begin{equation*}
\psi_{p}(x)=\langle x \mid p\rangle \sim \exp (\mathrm{i} p x / \hbar) \tag{2}
\end{equation*}
$$

This is the wave function for a free particle with a definite momentum, but totally unknown position.

## The ADM Formulation of General Relativity

The same quantization path will now be followed for general relativity. The first step is the Hamiltonian formulation of classical general relativity, where a configuration variable and momentum variable are identified. This formulation requires that spacetime is treated as a stack of 3-dimensional spacelike slices. In this $(3+1)$-split of spacetime, the time coordinate $t$ is chosen such that the 3 -dimensional slices are slices of constant $t$. The description of gravity in the $(3+1)$-split is called the ADM formulation. It was introduced in Arnowitt, Deser and Misner. The 4 -dimensional metric $g_{\mu \nu}\left(x^{\mu}\right)$ is written
in terms of the 3 -dimensional metric of the slices $q_{a b}\left(x^{\mu}\right)$, a lapse function $N\left(x^{\mu}\right)$ and a shift function $N^{a}\left(x^{\mu}\right)$ :

$$
g_{\mu \nu}=\left(\begin{array}{cc}
N_{c} N^{c}-N^{2} & N_{a}  \tag{3}\\
N_{b} & q_{a b}
\end{array}\right)
$$

Latin indices are 3 -dimensional, $a=1,2,3$, and they are raised and lowered by the 3-metric: $N_{a}=q_{a b} N^{b}$. The meaning of the shift function is as follows: In each slice a 3 -dimensional coordinate system is defined. A 4 -vector that is normal to the slice $\Sigma_{t}$ connects a point in $\Sigma_{t}$ to a point in $\Sigma_{t+d t}$. These points do not necessarily have the same coordinate values. $N^{a} d t$ is the difference between these coordinate values. The shift function describes how the coordinate systems in different timeslices are shifted relative to each other. The lapse function relates the proper time along the normal vector to the coordinate time: $d \tau=N d t$.

The Einstein equation follows from the variation of the Einstein-Hilbert action. This action can be written in the 3 -metric and the lapse and shift functions as follows:

$$
\begin{equation*}
S=\frac{1}{2 \kappa} \int d^{4} x \sqrt{-g}{ }^{(4)} R=\frac{1}{2 \kappa} \int d t \int d^{3} x \sqrt{q} N\left({ }^{(3)} R+K_{a b} K^{a b}-K^{2}\right) \tag{4}
\end{equation*}
$$

Here is $\kappa=8 \pi G / c^{4}$ and $g$ and $q$ are the determinants of the 4 -metric and the 3 -metric. $K_{a b}\left(x^{\mu}\right)$ is the extrinsic curvature that describes how the 3 dimensional slices are curved in the 4 -dimensional spacetime and $K=K_{a}^{a}$. When a 4 -vector that is normal to a slice is parallel transported along an infinitesimal path inside the slice, then the resulting vector is not necessarily equal to the normal vector at the new location. The difference between the two vectors is proportional to the displacement, this difference is $K^{a}{ }_{b} d x^{b}$. The extrinsic curvature expressed in terms of the variables of the $(3+1)$-split is

$$
\begin{equation*}
K_{a b}=\frac{1}{2 N}\left(\dot{q}_{a b}-D_{a} N_{b}-D_{b} N_{a}\right) \tag{5}
\end{equation*}
$$

The dot indicates the derivative with respect to the time coordinate and $D_{a}$ is the covariant derivative defined by the 3 -metric.

The configuration variables in this formulation are the constituents of the 4 -metric in equation (3): the 3 -metric and the lapse and shift functions. The momentum that is conjugate to a configuration variable follows from the Lagrangian density. The action (4) defines the momentum that is conjugate to the 3 -metric:

$$
\begin{equation*}
\pi^{a b}=\frac{\partial \mathcal{L}}{\partial \dot{q}_{a b}}=\sqrt{q}\left(K^{a b}-K q^{a b}\right) \tag{6}
\end{equation*}
$$

The lapse and the shift functions do not have a conjugate momentum, since their time derivatives are not present in the Lagrangian density. The action can be expressed in the configuration variables and the momentum of
equation (6) as

$$
\begin{equation*}
S=\frac{1}{2 \kappa} \int d t \int d^{3} x\left(\pi^{a b} \dot{q}_{a b}-N C-2 N^{a} C_{a}\right) \tag{7}
\end{equation*}
$$

The variation of this action with respect to the lapse and the shift functions results in the requirement that the Hamiltonian constraint and the diffeomorphism constraint vanish:

$$
\begin{equation*}
C=-{ }^{(3)} R+\pi^{a b} \pi_{a b} / q-\pi^{2} / 2 q=0 \quad C_{a}=D_{a}\left(\pi^{a b} / \sqrt{q}\right)=0 \tag{8}
\end{equation*}
$$

For a fixed value of $t$, the variables $q_{a b}\left(x^{\mu}\right)$ and $\pi^{a b}\left(x^{\mu}\right)$ describe a slice of spacetime. The constraints are relations between $q_{a b}$ and $\pi^{a b}$ such that the way that a slice is embedded in spacetime as defined by the extrinsic curvature is not independent of the internal structure of a slice as defined by the 3 -metric. The Poisson brackets on the phase space for the $t$-slice are:

$$
\begin{align*}
& \left\{q_{a b}(t, \vec{x}), \pi^{c d}(t, \vec{y})\right\}=\kappa \delta_{(c}^{a} \delta_{d)}^{b} \delta^{3}(\vec{x}, \vec{y})  \tag{9}\\
& \left\{\pi^{a b}(t, \vec{x}), \pi^{c d}(t, \vec{y})\right\}=\left\{q_{a b}(t, \vec{x}), q_{c d}(t, \vec{y})\right\}=0
\end{align*}
$$

## The Ashtekar Variables

New variables will now be introduced in the 3-dimensional timeslices. A triad is a set of three vector fields $e_{i}^{a}(\vec{x})$ that are orthonormal, $q_{a b} e_{i}^{a} e_{j}^{b}=\delta_{i j}$, where the index $i=1,2,3$ labels the three vector fields. The triad vectors are the spatial coordinate axes of a local laboratory frame. The $e_{i}^{a}$ 's map spacetime indices to indices of the local frame, for example $e_{i}^{a} v_{a}=v_{i}$ The index $i$ is raised or lowered with the delta function $\delta^{i j}$ or $\delta_{i j}$. The 3 -metric can be expressed in the inverse of these fields, $e_{a}^{i}$ :

$$
\begin{equation*}
q_{a b}=e_{a}^{i} e_{b}^{j} \delta_{i j} \tag{10}
\end{equation*}
$$

A vector $v^{i}$ in the local frame also defines an $\mathrm{SU}(2)$ element. $v^{i}$ can be contracted with the generators of $\mathrm{SU}(2)$ to form an element of $\operatorname{su}(2), v^{i} \tau_{i} \in$ $\mathrm{su}(2)$, and this element of $\mathrm{su}(2)$ can then be exponentiated to an element of $\operatorname{SU}(2)$, see appendix A. A version of the extrinsic curvature with one local index can be defined as follows:

$$
\begin{equation*}
K_{a}^{i}=K_{a b} e_{j}^{b} \delta^{j i} \tag{11}
\end{equation*}
$$

There is a covariant derivative for an object with a local index:

$$
\begin{equation*}
D_{a} B_{b}^{i}=\partial_{a} B_{b}^{i}-\Gamma_{a b}^{c} B_{c}^{i}+\delta^{i j} \varepsilon_{j k l} \Gamma_{a}^{k} B_{b}^{l} \tag{12}
\end{equation*}
$$

The spin connection $\Gamma_{a}^{k}$ defines parallel transport for a vector with a local index. The Ashtekar variables that were introduced in Ashtekar are a
connection $A_{a}^{i}$, which is a linear combination of the spin connection and the extrinsic curvature, and the densitized triad field $E_{i}^{a}$ :

$$
\begin{equation*}
A_{a}^{i}=\Gamma_{a}^{i}+\beta K_{a}^{i} \quad E_{i}^{a}=\sqrt{q} e_{i}^{a} \tag{13}
\end{equation*}
$$

$\beta$ is called the Immirzi parameter. The Hamiltonian constraint $C$ and the diffeomorphism constraint $C_{a}$ of the ADM formulation, equation (8), can be written in these variables as

$$
\begin{align*}
& C=\frac{1}{\sqrt{|\operatorname{det} E|}}\left(E_{i}^{a} E_{j}^{b} \varepsilon^{i j k} \delta_{k l} F_{a b}^{l}-\frac{2\left(1+\beta^{2}\right)}{\beta^{2}} E_{i}^{[a} E_{j}^{b]}\left(A_{a}^{i}-\Gamma_{a}^{i}\right)\left(A_{b}^{j}-\Gamma_{b}^{j}\right)\right) \\
& C_{a}=F_{a b}^{i} E_{i}^{b} \tag{14}
\end{align*} F_{a b}^{i}=\partial_{a} A_{b}^{i}-\partial_{b} A_{a}^{i}+\varepsilon^{i}{ }_{j k} A_{a}^{j} A_{b}^{k}
$$

$F_{a b}^{i}$ is the curvature of the connection $A_{a}^{i}$. The spin connection $\Gamma_{a}^{i}$ can be expressed in $E_{i}^{a}$. The original idea was to choose $\beta=\mathrm{i}$, which removes the second term from the Hamiltonian constraint. Nowadays a real $\beta$ is used to keep the Ashtekar connection real, but this gives a more complicated Hamiltonian constraint.

A new phase space can be defined with the variables $A_{a}^{i}$ and $E_{i}^{a}$ and the constraints of equation (14). The Poisson brackets between these variables are then by definition:

$$
\begin{align*}
& \left\{A_{a}^{i}(\vec{x}), E_{j}^{b}(\vec{y})\right\}=\kappa \beta \delta_{b}^{a} \delta_{i}^{j} \delta^{3}(\vec{x}, \vec{y})  \tag{15}\\
& \left\{A_{a}^{i}(\vec{x}), A_{b}^{j}(\vec{y})\right\}=\left\{E_{i}^{a}(\vec{x}), E_{j}^{b}(\vec{y})\right\}=0
\end{align*}
$$

The variables of the ADM phase space are functions on this new phase space: $q_{a b}\left(E_{i}^{a}\right)$ and $\pi^{a b}\left(A_{a}^{i}, E_{i}^{a}\right)$. According to Thiemann] the Poisson brackets between these functions on the $(A, E)$-phase space are the same as the original ADM Poisson brackets of equation (9), if the additional Gauss constraint is satisfied:

$$
\begin{equation*}
G_{i}=\nabla_{a} E_{i}^{a}=\partial_{a} E_{i}^{a}+\varepsilon_{i j k} \delta^{k l} A_{a}^{j} E_{l}^{a}=0 \tag{16}
\end{equation*}
$$

The ADM phase space is now replaced by a phase space with variables $A_{a}^{i}$ and $E_{i}^{a}$. The constraints (14) and (16) reduce this phase space to the physical configurations.

## Holonomy and Flux Vector

The variables $A_{a}^{i}$ and $E_{i}^{a}$ are not the ones used in loop quantum gravity, because they are still functions on a continuous space. The variables are adjusted to be used with spin networks.

Instead of the connection $A_{a}^{i}$, the holonomy $U_{l}$ is used. A holonomy is a rotation matrix that rotates a vector in the same way as the parallel transport of this vector along a path does. Here the path is a link of the spin network and the parallel transport is defined by the connection $A_{a}^{i}$. In
every point in the timeslice, $A_{a}^{i}(\vec{x}) \tau_{i}$ is an element of $\operatorname{su}(2)$. The holonomy associated to the link $l$ is the path ordered exponent of the integral of $A_{a}^{i}(\vec{x}) \tau_{i}$ along the link:

$$
\begin{equation*}
U_{l}=\mathcal{P} \exp \left(\int_{l} A_{a}^{i} \tau_{i} d s^{a}\right) \tag{17}
\end{equation*}
$$

This $\mathrm{SU}(2)$ matrix defines the 3 d rotation of a spinor. The set of these holonomies $U_{l}$ is a discretization of the continuous field $A_{a}^{i}$.

The Ashtekar connection does not define parallel transport of a vector in space. This is defined by the Christoffel symbols $\Gamma_{a b}^{c}$ in equation (12). The spin connection $\Gamma_{a}^{i}$ in the same equation defines the parallel transport of a vector with a local index. The holonomy of equation (17) describes the effect of the connection $A_{a}^{i}=\Gamma_{a}^{i}+\beta K_{a}^{i}$, which is a combination of the spin connection and the extrinsic curvature of the timeslice. The physical interpretation of this connection and the holonomy $U_{l}$ is unclear.

The second variable that is defined on spin networks is based on the continuous variable $E_{i}^{a}$. As discussed in section 1.1, a spin network can be constructed as the dual of a triangulation of space. A node represents a polyhedron and a link represents a face that is shared by two polyhedra. In this way a surface is associated to each link. The area of this surface $S$ can be expressed in the variable $E_{i}^{a}$ :

$$
\begin{equation*}
\operatorname{Area}(S)=\int_{S} d^{2} u \sqrt{E_{i}^{a} n_{a} E_{j}^{b} n_{b} \delta^{i j}} \tag{18}
\end{equation*}
$$

The vector $n^{a}$ is normal to the surface. The 3 -metric is related to $E_{i}^{a}$ as $(\operatorname{det} q) q_{a b}=E_{a}^{i} E_{b}^{j} \delta_{i j}$ and the expression in the integral under the square root is equal to the determinant of the induced metric on the surface. The flux of the $E$-fields through the surface $S$ defines a vector $L^{i}$ with a local index:

$$
\begin{equation*}
L_{l}^{i}=\int_{S} \vec{E}^{i} \cdot d \vec{S} \tag{19}
\end{equation*}
$$

For a small surface, such that $E_{i}^{a}$ is constant, this vector is

$$
\begin{equation*}
L_{l}^{i}=\delta^{i j} E_{j}^{a} n_{a} d^{2} u \tag{20}
\end{equation*}
$$

From equation (18) it follows that the length of this flux vector, $\sqrt{ } L_{i} L^{i}$, is equal to the area of the surface:

$$
\begin{equation*}
\operatorname{Area}(S)^{2}=L_{l i} L_{l}^{i} \tag{21}
\end{equation*}
$$

The volume of a tetrahedron of the triangulation can also be expressed in the flux vectors. For each face of a Euclidean tetrahedron a vector $\vec{v}$ can be constructed that is normal to the face and which length is equal to the area of the face. The square of the volume of the tetrahedron is then equal to $2 / 9$ times $\overrightarrow{v_{1}} \cdot\left(\overrightarrow{v_{2}} \times \overrightarrow{v_{3}}\right)$. Up to a sign this holds for any three of the four
vectors associated with the tetrahedron. Here the vector $n^{a} d^{2} u$ is normal to a face and its length is equal to the coordinate area of the face. The square of the coordinate volume of a tetrahedron of the triangulation is then

$$
\begin{equation*}
\left(d^{3} x\right)^{2}=\frac{2}{9} \varepsilon^{a b c} n_{1 a} n_{2 b} n_{3 c} d^{2} u d^{2} v d^{2} w \tag{22}
\end{equation*}
$$

The volume of the tetrahedron can now be expressed in the flux vectors that correspond to three faces of the tetrahedron:

$$
\begin{align*}
\text { Volume }^{2} & =\frac{2}{9} \varepsilon_{i j k} L_{1}^{i} L_{2}^{j} L_{3}^{k}=\frac{2}{9} \varepsilon^{i j k} E_{i}^{a} E_{j}^{b} E_{k}^{c} n_{1 a} n_{2 b} n_{3 c} d^{2} u d^{2} v d^{2} w  \tag{23}\\
& =\frac{2}{9} \frac{1}{3!} \varepsilon^{i j k} \varepsilon_{a b c} E_{i}^{a} E_{j}^{b} E_{k}^{c} \varepsilon^{d e f} n_{1 d} n_{2 e} n_{3 f} d^{2} u d^{2} v d^{2} w \\
& =\frac{1}{3!} \varepsilon^{i j k} \varepsilon_{a b c} E_{i}^{a} E_{j}^{b} E_{k}^{c}\left(d^{3} x\right)^{2}=\operatorname{det}\left(E_{i}^{a}\right)\left(d^{3} x\right)^{2}=\operatorname{det}\left(q_{a b}\right)\left(d^{3} x\right)^{2}
\end{align*}
$$

In the step from the first to the second line the same sum of terms is written in a different way.

The above variables $U$ and $L^{i}$ are defined on the discrete structure of a spin network. Associated to every link is an $\mathrm{SU}(2)$ element $U$ and a vector $L^{i}$ that defines an $\operatorname{su}(2)$ element through $L^{i} \tau_{i} . U$ and $L^{i}$ are functions on the phase space with variables $A_{a}^{i}$ and $E_{i}^{a}$ and the Poisson brackets between them are

$$
\begin{align*}
& \left\{U_{l}, U_{l^{\prime}}\right\}=0 \\
& \left\{U_{l}, L_{l^{\prime}}^{i}\right\}=\kappa \delta_{l l^{\prime}} U_{l, 1} \tau^{i} U_{l, 2}  \tag{24}\\
& \left\{L_{l}^{i}, L_{l^{\prime}}^{j}\right\}=\kappa \delta_{l l^{\prime}} \varepsilon^{i j k} \delta_{k m} L_{l}^{m}
\end{align*}
$$

On the right hand side of the second bracket a generator $\tau^{i}$ of $\operatorname{SU}(2)$ is inserted in the holonomy $U_{l}=U_{l, 1} U_{l, 2}$. This generator is inserted at the point where the surface $S$ of equation (19) cuts the link $l$. To obtain a result that does not depend on an embedding in a continuous space, the generator can be inserted at an endpoint of the link, resulting in a right hand side that is proportional to $\tau^{i} U_{l}$ or $U_{l} \tau^{i}$. Here the choice is made to cut the holonomy into two equal parts: $U_{l, 1}=U_{l, 2}=U_{l}^{1 / 2}$. This gives a more appealing graphical action of the $L$-operator in section 1.6.

The second Poisson bracket in (24) can be calculated from the expressions of $U$ and $L^{i}$ in $A_{a}^{i}$ and $E_{i}^{a}$. One would think that the third Poisson bracket vanishes since the flux vector does not depend on the connection $A_{a}^{i}$, but in the literature this Poisson bracket is justified by the fact that the structure of the phase space of the discretized variables is $\left(U_{l}, L_{l}^{i} \tau_{i}\right) \in \mathrm{SU}(2) \times \mathrm{su}(2)$. This third Poisson bracket is identical to the Poisson bracket between two angular momenta.

## The Quantization of General Relativity

As discussed at the beginning of this section, the first step in constructing a quantum theory is to identify a canonical pair of variables. For loop quantum gravity the Ashtekar variables (13) form this canonical pair and the Poisson brackets between them in equation 15 are equivalent to the Poisson brackets (11) for the position and momentum of a particle. Subsequently functions of these variables were created that are defined on spin networks. These are the holonomy $U$ and the flux vector $L^{i}$. In the case of the quantization of the particle the position and momentum variables are converted to operators that obey the commutation relation in equation (1). Similarly, the operator versions of the holonomy and the flux vector should obey commutation relations that are derived from the Poisson brackets (24):

$$
\begin{align*}
{\left[\hat{U}_{l}, \hat{U}_{l^{\prime}}\right] } & =0 \\
{\left[\hat{U}_{l}, \hat{L}_{l^{\prime}}^{i}\right] } & =\mathrm{i} \hbar \kappa \delta_{l l^{\prime}} \hat{U}_{l}^{1 / 2} \tau^{i} \hat{U}_{l}^{1 / 2}  \tag{25}\\
{\left[\hat{L}_{l}^{i}, \hat{L}_{l^{\prime}}^{j}\right] } & =\mathrm{i} \hbar \kappa \delta_{l l^{\prime}} \varepsilon^{i j k} \delta_{k m} \hat{L}_{l}^{m}
\end{align*}
$$

Operators $U$ and $L^{i}$ that act on the same link do not commute. This means that the quantities that they measure can not be known at the same time with infinite precision. In the case of the quantization of the particle two classical observables become non-commuting operators, the position and the momentum. Here the flux operator is connected to the classical observables of area and volume, but the holonomy does not correspond to a clear physical observable.

The wave functions of loop quantum gravity are based on spin networks. As described in section 1.1 these spin networks consist of two elements: First there is the graph $\Gamma$, which defines the number of nodes and links, and which nodes are connected to each other by which links. The links are oriented. The graph is a combinatorial structure, not an embedded structure. Second, there is a spin $j$ assigned to every link.

Given the state $\left|\Gamma, j_{l}\right\rangle$ an attempt can be made to create a wave function for such a state in the following way: Assign a holonomy to each oriented link and multiply these holonomies with each other as prescribed by the graph, where the spin on a link determines the representation of the $\mathrm{SU}(2)$ element on that link. The indices of $\mathrm{SU}(2)$ matrices in spin $j_{l}$ representations are contracted as defined by the graph:

$$
\begin{equation*}
" \quad \psi_{\Gamma, j_{l}}\left(\left\{U_{l}\right\}\right)=\left\langle\left\{U_{l}\right\} \mid \Gamma, j_{l}\right\rangle=\otimes_{l, \Gamma} D_{j_{l}}\left(U_{l}\right) " \tag{26}
\end{equation*}
$$

The notation $D_{j}(U)$ means the matrix representation of the group element $U$ in the spin $j$ representation. This definition is between quotation marks, because the index contraction is not possible in this way.

## Intertwiners and Node Decomposition

The object that allows for a multiplication of multiple group elements in different representations and that is itself invariant under the group action is called an intertwiner:

$$
\begin{equation*}
i_{m_{1} m_{2} \cdots m_{N}}^{\left(j_{1} j_{2} \cdots j_{N}\right)} D_{j_{1}}\left(U_{1}\right)_{n_{1}}^{m_{1}} D_{j_{2}}\left(U_{2}\right)_{n_{2}}^{m_{2}} \cdots D_{j_{N}}\left(U_{N}\right)_{n_{N}}^{m_{N}} \tag{27}
\end{equation*}
$$

$\mathrm{SU}(2)$ matrices in the spin $j$ representation are square matrices with dimension $2 j+1 . \quad m$ and $n$ are here the matrix indices expressed as magnetic numbers: $m=j, j-1, \ldots,-j$. The $j$ 's in this expression are not indices and the summation convention does not apply to them.

The only object that connects $\mathrm{SU}(2)$ matrices like this in a way that is invariant under the action of $\mathrm{SU}(2)$ is the Wigner $3 j$-symbol:

$$
\left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3}  \tag{28}\\
m_{1} & m_{2} & m_{3}
\end{array}\right) D_{j_{1}}\left(U_{1}\right)^{m_{1}} D_{n_{1}}\left(D_{j_{2}}\right)^{m_{2}}{ }_{n_{2}} D_{j_{3}}\left(U_{3}\right)_{n_{3}}^{m_{3}}
$$

The $3 j$-symbol connects three $\mathrm{SU}(2)$ matrices. It is a real-valued function of the six numbers listed. Its values can for example be found with the function ThreeJSymbol[] in Wolfram Mathematica. The $3 j$-symbol can only be nonzero if these conditions hold: the sum of the spins is integer, no spin is larger than the sum of the other two and the sum of the magnetic numbers is zero:

$$
\begin{equation*}
j_{1}+j_{2}+j_{3}=\text { integer } \quad\left|j_{1}-j_{2}\right| \leq j_{3} \leq j_{1}+j_{2} \quad m_{1}+m_{2}+m_{3}=0 \tag{29}
\end{equation*}
$$

If the Wigner $3 j$-symbol is contracted on every index with the same $\mathrm{SU}(2)$ element, in different representations, then its value is unchanged. This is the $\mathrm{SU}(2)$ invariance of the $3 j$-symbol:

$$
\left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3}  \tag{30}\\
m_{1} & m_{2} & m_{3}
\end{array}\right) D_{j_{1}}(U)^{m_{1}}{ }_{m_{1}^{\prime}} D_{j_{2}}(U)^{m_{2}}{ }_{m_{2}^{\prime}} D_{j_{3}}(U)^{m_{3}}{ }_{m_{3}^{\prime}}=\left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3} \\
m_{1}^{\prime} & m_{2}^{\prime} & m_{3}^{\prime}
\end{array}\right)
$$

In the sketch of the wave function, equation (26), the intertwiner for a 3valent node of the spin network is the $3 j$-symbol. Since the $3 j$-symbol is the only invariant object, all intertwiners have to be constructed using only the $3 j$-symbol. Therefore for nodes with more than three links a decomposition into 3 -nodes is introduced. For example, a 4 -valent node can be decomposed into two 3 -valent nodes and an internal link between them:


In an embedding of the spin network the two nodes of the decomposition are in the same point and the internal link covers no distance. The spin $k$ on the internal link is restricted by the conditions of equation (29) on both nodes of the decomposition. The possible values for $k$ start at $k_{\min }$ and go in integer steps to $k_{\text {max }}$ :

$$
\begin{equation*}
k_{\min }=\max \left\{\left|j_{1}-j_{2}\right|,\left|j_{3}-j_{4}\right|\right\} \quad k_{\max }=\min \left\{j_{1}+j_{2}, j_{3}+j_{4}\right\} \tag{32}
\end{equation*}
$$

The holonomy on an internal link is the identity element, because in an embedding an internal link has zero length. The intertwiner for the 4 -node of equation (31) is then a product of two $3 j$-symbols which are contracted over one index:

$$
i_{m_{1} m_{2} m_{3} m_{4}}^{\left(j_{1} j_{j} j_{3} j_{4}\right)}=\sum_{m}\left(\begin{array}{ccc}
j_{1} & j_{2} & k  \tag{33}\\
m_{1} & m_{2} & m
\end{array}\right)\left(\begin{array}{ccc}
m & j_{3} & j_{4} \\
k & m_{3} & m_{4}
\end{array}\right)
$$

The second $3 j$-symbol contains a raised index $m$. Equation (57) is the definition of a $3 j$-symbol with a raised index. More properties of the Wigner $3 j$-symbol are discussed in section 1.3 .

## The Quantization of General Relativity - Continued

The spin network states have now acquired more structure. Next to the oriented graph $\Gamma$ and the spins $j$, every node has an internal structure. Such a node state $|n\rangle$ consists of a certain decomposition of the node into 3 -valent nodes and spins on the internal links. In total, a spin network state is:

$$
\begin{equation*}
\left.\left.|\mathrm{SNW}\rangle=\left|\Gamma, j_{l},\right| n\right\rangle\right\rangle \tag{34}
\end{equation*}
$$

A spin network state contains a spin for every link and a node state for every node. As discussed in section 1.8, a spin network state represents a piece of space with well defined geometric properties area and volume. Spin network states are analogous to the momentum eigenkets $|p\rangle$ for the free particle in equation (2).

The spin network wave function that is based on this spin network state is a function of holonomies, one for each link. It is defined as the multiplication of these holonomies with the intertwiners for the nodes, where the holonomies are in the representations defined by the spins on the links and the index contraction follows the graph of the spin network:

$$
\begin{equation*}
\psi_{\mathrm{SNW}}\left(\left\{U_{l}\right\}\right)=\left\langle\left\{U_{l}\right\} \mid \mathrm{SNW}\right\rangle=C_{\mathrm{SNW}}\left(\prod_{l} D_{j_{l}}\left(U_{l}\right)\right) \cdot \Gamma\left(\prod_{n} i_{n}\right) \tag{35}
\end{equation*}
$$

This is the wave function for a spin network in the well-defined spin network state $|\mathrm{SNW}\rangle$. In the classical discretization of the Ashtekar variables on a spin network a holonomy was assigned to each link. For the spin network
state $|\mathrm{SNW}\rangle$, the holonomies on the links are not well-defined. The wave function is a function of the holonomies. This wave function is analogous to the free particle wave function of equation (2), which is a function of $x$ and describes a particle with definite momentum but totally undefined position. $C_{\text {SNW }}$ is the normalization constant for the spin network wave function. Its value is given in equation (42). Examples of spin network wave functions are given in section 1.4.

For the free particle it is postulated that a complex-valued function of the position describes the state of the particle at every moment in time. The Schrödinger equation determines which function describes the particle in a given situation. Here it is postulated that functions of the form (35) describe the state of a piece of space, but there is no equation that connects a certain spin network state to a given situation.

## Inner Product

In quantum mechanics an inner product has to be defined on the state space. For the spin network states an inner product is defined that is based on the following integral over $\mathrm{SU}(2)$ :

$$
\begin{equation*}
\int_{\mathrm{SU}(2)} d \mu_{H}(U) D_{j^{\prime}}^{*}(U)^{m^{\prime}}{ }_{n^{\prime}} D_{j}(U)^{m}{ }_{n}=\frac{1}{2 j+1} \delta_{j, j^{\prime}} \delta^{m, m^{\prime}} \delta_{n, n^{\prime}} \tag{36}
\end{equation*}
$$

The components of an $\mathrm{SU}(2)$ matrix are orthogonal with respect to this integral defined by the Haar measure on $\mathrm{SU}(2), d \mu_{H}(U)$. The inner product between spin network states is defined as

$$
\begin{align*}
\left\langle\mathrm{SNW}^{\prime} \mid \mathrm{SNW}\right\rangle & =\delta_{\Gamma, \Gamma^{\prime}} \int_{\mathrm{SU}(2)^{L}} \prod_{l} d \mu_{H}\left(U_{l}\right)\left\langle\mathrm{SNW}^{\prime} \mid\left\{U_{l}\right\}\right\rangle\left\langle\left\{U_{l}\right\} \mid \mathrm{SNW}\right\rangle \\
& =\delta_{\Gamma, \Gamma^{\prime}} \int_{\mathrm{SU}(2)^{L}} \prod_{l} d \mu_{H}\left(U_{l}\right) \psi_{\mathrm{SNW}^{\prime}}^{*}\left(\left\{U_{l}\right\}\right) \psi_{\mathrm{SNW}}\left(\left\{U_{l}\right\}\right) \tag{37}
\end{align*}
$$

Due to the $\delta_{\Gamma, \Gamma^{\prime}}$ this inner product can only be nonzero when the graphs of the spin networks are the same. A node of $\Gamma$ can then be identified with a node of $\Gamma^{\prime}$. When the spin network wave function of equation (35) is entered in this integral, the terms in the integral that belong to one $N$-valent node are

$$
\begin{equation*}
i_{m_{1}^{\prime} \cdots m_{N}^{\prime}}^{\left(j_{1}^{\prime} \cdots j_{N}^{\prime}\right)} D_{j_{1}^{\prime}}^{*}\left(U_{1}\right)^{m_{1}^{\prime}} \cdots D_{j_{1}^{\prime}}^{*}\left(U_{N}\right)^{m_{N}^{\prime}}{ }_{n_{N}^{\prime}}^{\prime} i_{m_{1} \cdots m_{N}}^{\left(j_{1} \cdots j_{N}\right)} D_{j_{1}}\left(U_{1}\right)^{m_{1}} \cdots D_{j_{N}}\left(U_{N}\right)_{\substack{m_{N} \\ m_{N}}}^{m_{N}} \tag{38}
\end{equation*}
$$

The primed intertwiner has no star, because the intertwiners are constructed from Wigner $3 j$-symbols, which are real-valued. The integral is evaluated by using equation (36) and the terms of the last equation reduce to

$$
\begin{equation*}
\left(\prod_{l=1}^{N} \frac{1}{2 j_{l}+1} \delta_{j_{l}, j_{l}} \delta_{n_{l}, n_{l}^{\prime}}\right) i_{m_{1}^{\prime} \cdots m_{N}^{\prime}}^{\left(j_{1} \cdots j_{N}\right)} \delta^{m_{1}^{\prime}, m_{1}} \cdots \delta^{m_{N}^{\prime}, m_{N}} i_{m_{1} \cdots m_{N}}^{\left(j_{1} \cdots j_{N}\right)} \tag{39}
\end{equation*}
$$

The terms after the brackets are identified as an inner product between node states, which is denoted as $\left\langle n^{\prime} \mid n\right\rangle$. The inner product between spin network states is then

$$
\begin{equation*}
\left\langle\mathrm{SNW}^{\prime} \mid \mathrm{SNW}\right\rangle=C_{\mathrm{SNW}} C_{\mathrm{SNW}} \delta_{\Gamma, \Gamma^{\prime}} \prod_{l} \frac{1}{2 j_{l}+1} \delta_{j_{l}, j_{l}^{\prime}} \prod_{n}\left\langle n^{\prime} \mid n\right\rangle \tag{40}
\end{equation*}
$$

The products run over all the links and over all the nodes of the spin network. The inner product between node states will be evaluated diagrammatically in section 1.5. The result is equation (99):

$$
\begin{equation*}
\left\langle n^{\prime} \mid n\right\rangle=C_{n^{\prime}} C_{n} \prod_{\substack{\text { internal } \\ \text { links }}} \frac{1}{2 k+1} \delta_{k, k^{\prime}} \tag{41}
\end{equation*}
$$

The product runs here over all the internal links of the node and the $k$ 's are the spins on these internal links. The node states are orthogonal with respect to this inner product in the sense that the inner product is always zero when the node states are not identical. The spin network states are orthogonal with respect to the inner product of equation (37). The spin network states and the node states are orthonormal with the normalization constants

$$
\begin{equation*}
C_{\mathrm{SNW}}=\prod_{l} \sqrt{2 j_{l}+1} \quad C_{n}=\prod_{\substack{\text { internal } \\ \text { links }}} \sqrt{2 k+1} \tag{42}
\end{equation*}
$$

The inner product between normalized spin network states is

$$
\begin{equation*}
\left\langle\mathrm{SNW}^{\prime} \mid \mathrm{SNW}\right\rangle=\delta_{\Gamma, \Gamma^{\prime}} \prod_{l} \delta_{j_{l}, j_{l}^{\prime}} \prod_{n} \prod_{\substack{\text { interanal } \\ \text { links }}} \delta_{k, k^{\prime}} \tag{43}
\end{equation*}
$$

An inner product between spin network states is defined and the spin network states are orthonormal with respect to this inner product.

## Operators

The next step is the construction of a pair of operators, a holonomy operator and a flux operator, that act on spin network wave functions and that satisfy the commutation relations of equation (25). In the quantum mechanical description of the free particle there are the position operator and the momentum operator and one of them is a multiplicative operator and the other one is a derivative operator. The same is true here.

The holonomy operator acts on a function of the holonomy as a multiplicative operator:

$$
\begin{equation*}
\left(\hat{U}_{l}\right)^{m}{ }_{n} f\left(U_{l}\right)=D_{1}\left(U_{l}\right)^{m}{ }_{n} f\left(U_{l}\right) \tag{44}
\end{equation*}
$$

The operator multiplies the function by the holonomy in a fixed representation. Commonly this representation is chosen to be the fundamental spin$1 / 2$ representation, but in view of the model of the second part of this thesis the spin- 1 representation is chosen here.

The flux operator acts on the same function as a derivative operator:

$$
\begin{equation*}
\hat{L}_{l}^{i} f\left(U_{l}\right)=-\left.\mathrm{i} \frac{d}{d t} f\left(U_{l}^{1 / 2} \exp \left(t \tau^{i}\right) U_{l}^{1 / 2}\right)\right|_{t=0} \tag{45}
\end{equation*}
$$

The $\tau^{i}$ 's in this definition are the generators of $\mathrm{SU}(2)$. The commutation relations between these two operators will be analysed in sections 1.6 and 1.7.

## Constraints

Earlier in this section the classical phase space of the Ashtekar variables was discussed. On this phase space three constraints were defined, the Hamiltonian and diffeomorphism constraints of equation (14) and the Gauss constraint of equation 161. In general a constraint $C$ is a function of phase space variables $q$ and $p$ and only the configurations that satisfy the constraint equation $C(q, p)=0$ are physical. In the quantum version of the theory the phase space variables become operators that act on a wave function and the constraint also becomes an operator. Only the wave functions that satisfy $\hat{C} \psi=0$ describe physical states of the system.

First there is the Gauss constraint. It requires that the spin network wave functions are invariant under local $\operatorname{SU}(2)$ transformations. If a link $l$ is oriented from a node $n_{1}$ to a node $n_{2}$, then the holonomy of this link transforms under a local $\operatorname{SU}(2)$ action $\Lambda$ as

$$
\begin{equation*}
U_{l} \rightarrow U_{l}^{\prime}=\Lambda\left(n_{2}\right) U_{l} \Lambda\left(n_{1}\right)^{-1} \tag{46}
\end{equation*}
$$

Spin network wave functions were defined in equation (35). In these wave functions a holonomy is contracted with two Wigner $3 j$-symbols:

$$
\cdots\left(\begin{array}{ccc}
j_{4} & j_{5} & j_{1}  \tag{47}\\
m_{4} & m_{5} & m_{1}^{\prime}
\end{array}\right) D_{j_{1}}(U)^{m_{1}^{\prime}}{ }_{m_{1}}^{\prime}\left(\begin{array}{ccc}
m_{1} & j_{2} & j_{3} \\
j_{1} & m_{2} & m_{3}
\end{array}\right) \cdots
$$

Equation (30) showed that the $3 j$-symbol with three lower indices is invariant under an $\operatorname{SU}(2)$ transformation. A $3 j$-symbol with an upper index is invariant if this upper index is contracted with the inverse of the locally acting $\operatorname{SU}(2)$ element:

$$
D_{j_{1}}\left(\Lambda^{-1}\right)^{m_{1}^{\prime}}{ }_{m_{1}}^{\prime}\left(\begin{array}{ccc}
m_{1} & j_{2} & j_{3}  \tag{48}\\
j_{1} & m_{2} & m_{3}
\end{array}\right) D_{j_{2}}(\Lambda)_{m_{2}^{\prime}}^{m_{2}} D_{j_{3}}(\Lambda)_{m_{3}}^{m_{3}^{\prime}}=\left(\begin{array}{ccc}
m_{1}^{\prime} & j_{2} & j_{3} \\
j_{1} & m_{2}^{\prime} & m_{3}^{\prime}
\end{array}\right)
$$

This shows that a wave function that is constructed as in equation (47) is invariant when all the holonomies transform as in equation (46). The Gauss
constraint is implemented by representing a 3 -valent node by the $3 j$-symbol and by using a decomposition into 3 -valent nodes for nodes with more than three links.

The second constraint is the diffeomorphism constraint. When a spin network is embedded in a continuous space, a diffeomorphism deforms the spin network in a smooth way. The diffeomorphism constraint requires that the wave function is invariant under this deformation of the spin network. This is indeed the case, since the spin network state of equation (34) does not contain information about a specific embedding. The spin network state depends only on the combinatorial structure of the spin network and the spin labels on the links. The wave function of equation (35) is based on this spin network state and does also not depend on a specific embedding.

The spin network wave functions satisfy the Gauss constraint and the diffeomorphism constraint. The last constraint is the Hamiltonian constraint. According to Wald, the origin of this constraint is the freedom one has in choosing the slicing in the $(3+1)$-split of spacetime. The expression for the Hamiltonian constraint $C$ in equation (14) has to be converted to an operator expression that acts on the spin network wave functions and only the wave functions that are annihilated by this operator are physical:

$$
\begin{equation*}
\hat{C} \psi_{\mathrm{SNW}}\left(\left\{U_{l}\right\}\right)_{\text {physical }}=0 \tag{49}
\end{equation*}
$$

The spin network states that were described in this section form an extended state space that still has to be restricted by the Hamiltonian constraint operator. The inner product defined in equation (37) is an inner product on this extended state space. How the Hamiltonian operator acts on the quantum states and which states it selects is an open problem.

### 1.3 Rules for Diagrammatic Calculations

In the last section the spin network wave function was defined as a product of $\mathrm{SU}(2)$ matrices and Wigner $3 j$-symbols, see equation $(35)$. The way that these elements are multiplied is defined by the structure of a spin network. The $3 j$-symbol has a graphical representation as a node with three links, see equation 50 . The properties of the $3 j$-symbol can be translated into rules for the manipulation of diagrams. In this way expressions involving $3 j$-symbols can be simplified by applying graphical rules. Since the spin network wave function contains $3 j$-symbols, it can be represented a diagram, which is called here a wave function diagram:

$$
\text { 3d spin network } \quad \longrightarrow \begin{gathered}
\text { spin network } \\
\text { wave function }
\end{gathered} \quad \longrightarrow \quad \begin{gathered}
\text { 2d wave function } \\
\text { diagram }
\end{gathered}
$$

This section describes the wave function diagrams, which are also called angular momentum diagrams, and the rules for manipulating them. In the following sections these rules will be used to do calculations. The next section contains examples of wave functions and wave function diagrams. The diagrammatic method for Wigner $3 j$-symbols was developed by I. B. Levinson and was first published in English in Yutsis, Levinson and Vanagas. This section is based on Wormer and Paldus and Jinsong Yang and Yongge Ma. There are differences between the diagrammatic notations in these three references. Here every link has an arrow as in the first and second reference. The direction of the arrow is defined here as in the third reference.

The basics of wave function diagrams are:

- A wave function diagram is a graphical representation of a product of $\mathrm{SU}(2)$ matrices and Wigner $3 j$-symbols.
- A node is a Wigner $3 j$-symbol and a link is an $\mathrm{SU}(2)$ matrix. The graphical structure determines how indices are contracted.
- A plus sign at a node means that indices are read off counterclockwise and a minus sign means clockwise.
- An arrow on a link points from an upper index of a $3 j$-symbol to a lower index of a $3 j$-symbol. In short, an arrow points away from an upper index.
- If a link represents the identity matrix, then a single arrow is used. If it represents an other $\mathrm{SU}(2)$ matrix, then a double arrow is used. If this $\mathrm{SU}(2)$ matrix is replaced by its inverse, then a triple arrow is used.
- A wave function diagram is a 2-dimensional object. Its meaning does not change if it is turned in the plane.


A link carries a half-integer spin value $j=0,1 / 2,1,3 / 2$, etc. The spin value determines the representation of the $\mathrm{SU}(2)$ matrix. The three spins that are connected in a node must satisfy the Clebsch-Gordan conditions: Their sum is integer and no spin is larger than the sum of the other two:

$$
\begin{equation*}
j_{1}+j_{2}+j_{3}=\text { integer } \quad\left|j_{1}-j_{2}\right| \leq j_{3} \leq j_{1}+j_{2} \tag{51}
\end{equation*}
$$

Sign factors that depend on spins that connect in one node are often encountered, for example:

$$
\begin{equation*}
(-1)^{4 j}=1 \quad(-1)^{2 j_{1}+2 j_{2}+2 j_{3}}=1 \quad(-1)^{2 j_{1}+2 j_{2}}=(-1)^{2 j_{3}} \tag{52}
\end{equation*}
$$

The three magnetic numbers $m=-j,-j+1, \ldots, j$ in one $3 j$-symbol must satisfy $m_{1}+m_{2}+m_{3}=0$. An even permutation of the columns of the Wigner $3 j$-symbol leaves its value unchanged, while an odd permutation gives a sign:

$$
\left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3}  \tag{53}\\
m_{1} & m_{2} & m_{3}
\end{array}\right)=(-1)^{j_{1}+j_{2}+j_{3}}\left(\begin{array}{ccc}
j_{1} & j_{3} & j_{2} \\
m_{1} & m_{3} & m_{2}
\end{array}\right)
$$

Graphically this is:


First the diagrammatic rules are considered where the links represent the identity matrix, such that the $3 j$-symbols are connected directly to each other. A link that runs between two nodes represents a contraction over one index. The magnetic numbers are the indices of a $3 j$-symbol, so a closed link implies a sum over $m$ :


Here a $3 j$-symbol with an upper index is used and this is defined as follows:

$$
\begin{align*}
\left(\begin{array}{ccc}
j & j_{1} & j_{2} \\
m & m_{1} & m_{2}
\end{array}\right) & =(-1)^{j+m^{\prime}} \delta_{m, m^{\prime}}\left(\begin{array}{ccc}
-m^{\prime} & j_{1} & j_{2} \\
j & m_{1} & m_{2}
\end{array}\right)  \tag{56}\\
\left(\begin{array}{ccc}
m & j_{1} & j_{2} \\
j & m_{1} & m_{2}
\end{array}\right) & =(-1)^{j-m^{\prime}} \delta^{m, m^{\prime}}\left(\begin{array}{ccc}
j & j_{1} & j_{2} \\
-m^{\prime} & m_{1} & m_{2}
\end{array}\right) \tag{57}
\end{align*}
$$

Any one of the indices of the $3 j$-symbol can be raised or lowered in this way. Above equations are the same as contracting the $3 j$-symbol with

$$
\begin{equation*}
C_{m m^{\prime}}^{(j)}=(-1)^{j-m^{\prime}} \delta_{m,-m^{\prime}} \quad C_{(j)}^{m m^{\prime}}=(-1)^{j+m^{\prime}} \delta^{m,-m^{\prime}} \tag{58}
\end{equation*}
$$

The arrow on a link points from a lower index to an upper index. By raising and lowering the $m$ index in equation (55) the arrow on the link can be reversed at the cost of a sign $(-1)^{2 j}$ :


The closed diagrams that can be build with zero, two and four $3 j$-symbols are:




Diagram 62 is the definition of the Wigner $6 j$-symbol, which is a contraction of four $3 j$-symbols, see (B.8). Diagram (61) is the normalization of the $3 j$-symbol:

$$
\left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3}  \tag{63}\\
m_{1} & m_{2} & m_{3}
\end{array}\right)\left(\begin{array}{ccc}
m_{1} & m_{2} & m_{3} \\
j_{1} & j_{2} & j_{3}
\end{array}\right)=1
$$

Here the summation convention applies and the sum over the magnetic numbers is not written explicitly. The summation line combines two links into one:


It is the graphical version of this property of the $3 j$-symbol:

$$
\delta_{m_{1}}^{m_{1}^{\prime}} \delta_{m_{2}}^{m_{2}^{\prime}}=\sum_{l}(2 l+1)\left(\begin{array}{ccc}
j_{1} & j_{2} & l  \tag{65}\\
m_{1} & m_{2} & m
\end{array}\right)\left(\begin{array}{ccc}
m & m_{1}^{\prime} & m_{2}^{\prime} \\
l & j_{1} & j_{2}
\end{array}\right)
$$

The sum over the spin $l$ is limited by the Clebsch-Gordan conditions (51). This rule can be applied to any two links in a diagram. The open boxes in equation (64) represent parts of a diagram, possibly with open links. The links that connect to an open box can be open links themselves. The open boxes can also be connected to each other.

If a product of Wigner $3 j$-symbols forms a loop, then this loop can be removed from the diagram. These are the rules for the removal of loops with two, three and four nodes. The first two are preceded by their algebraic version:

$$
\left(\begin{array}{ccc}
j_{1} & j_{3} & j_{4}  \tag{66}\\
m_{1} & m_{3} & m_{4}
\end{array}\right)\left(\begin{array}{ccc}
m_{3} & m_{4} & m_{2} \\
j_{3} & j_{4} & j_{2}
\end{array}\right)=\frac{\delta_{j_{1} j_{2}}}{2 j_{1}+1} \delta_{m_{1}}^{m_{2}}
$$



$$
\left(\begin{array}{ccc}
j_{1} & m_{5} & j_{6} \\
m_{1} & j_{5} & m_{6}
\end{array}\right)\left(\begin{array}{ccc}
j_{4} & j_{2} & m_{6} \\
m_{4} & m_{2} & j_{6}
\end{array}\right)\left(\begin{array}{ccc}
m_{4} & j_{5} & j_{3} \\
j_{4} & m_{5} & m_{3}
\end{array}\right)
$$



$$
=\left\{\begin{array}{lll}
j_{1} & j_{2} & j_{3}  \tag{68}\\
j_{4} & j_{5} & j_{6}
\end{array}\right\}\left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3} \\
m_{1} & m_{2} & m_{3}
\end{array}\right)
$$




On the right hand side of the last equation terms are placed above one another. This means a product between these terms from the top to the bottom. To derive (71), first insert a summation line (64) in the horizontal links and then remove the two loops with three nodes. Loops with more nodes can be removed in the same way, by inserting summation lines to reduce the number of nodes in a loop.

The rules for cutting a diagram into two pieces are given below. If a diagram consists of multiple disconnected pieces, then this implies a product between the algebraic expressions for these pieces. A closed box represents a connected part of a diagram that has no open links and is not connected to the other box. These identities are only valid if the closed box consists only of $3 j$-symbols that are connected by identity matrices. The $3 j$-symbols can not be connected by other $\operatorname{SU}(2)$ elements inside the box. These rules are not valid if the resulting diagram is still connected.




The cut over two links is derived from the cut over three links by first inserting a loop in one of the links of $(72)$ with equation (67). To derive the cut over four links, first insert a summation line (64) in two of the links and then apply the cut over three links. Cuts over more links can be done by using more summation lines. What remains is the proof of identity (73):

If a closed box with three links consists of a connected diagram that is build from $3 j$-symbols that are connected by identity matrices, then this diagram is proportional to a single node, since all the loops inside such a diagram can be removed:


The two closed boxes on the left and the right are the same. The proportionality constant $C$ is equal to the given diagram on the right, because the reduction of that diagram follows the same steps as the reduction of the closed box on the left and results in the theta diagram (61), which is equal to one.

If one of the spins in a $3 j$-symbol is zero, then the $3 j$-symbol is proportional to the delta function in its other indices. Diagrammatically the link with spin zero can be removed from a node or between two nodes:

$$
\left(\begin{array}{ccc}
j_{1} & m_{2} & j=0  \tag{76}\\
m_{1} & j_{2} & m=0
\end{array}\right)=\frac{\delta_{j_{1} j_{2}}}{\sqrt{2 j_{1}+1}} \delta_{m_{1}}^{m_{2}}
$$




Up to this point the Wigner $3 j$-symbols were connected directly to each other, but a spin network wave function is a product of $3 j$-symbols and $\mathrm{SU}(2)$ matrices, see equation (35). The $\mathrm{SU}(2)$ matrices are the holonomies defined in equation 17 ). If there is an $\mathrm{SU}(2)$ matrix between two $3 j$-symbols, then this is indicated in a diagram by a double arrow on the link. This arrow
defines both the way how the indices are contracted and the direction of the transport defined by the holonomy. The rule for reversing the arrow on such a link is calculated by raising and lowering an index in the $3 j$-symbols with (56) and (57):

$$
\begin{gather*}
\left(\begin{array}{ccc}
j_{1} & j_{2} & j \\
m_{1} & m_{2} & m_{3}
\end{array}\right) D_{j}(U)^{m_{3}}{ }_{m_{4}}\left(\begin{array}{ccc}
m_{4} & j_{5} & j_{6} \\
j & m_{5} & m_{6}
\end{array}\right)=\left(\begin{array}{ccc}
j_{1} & j_{2} & m_{3}^{\prime} \\
m_{1} & m_{2} & j
\end{array}\right) \times \\
\times(-1)^{2 j} \delta_{m_{3}^{\prime}, m_{3}}(-1)^{-m_{3}+m_{4}} D_{j}(U)^{-m_{3}}{ }_{-m_{4}} \delta^{m_{4}, m_{4}^{\prime}}\left(\begin{array}{ccc}
j & j_{5} & j_{6} \\
m_{4}^{\prime} & m_{5} & m_{6}
\end{array}\right) \\
=(-1)^{2 j}\left(\begin{array}{ccc}
j_{1} & j_{2} & m_{3} \\
m_{1} & m_{2} & j
\end{array}\right) D_{j}\left(U^{-1}\right)^{m_{4}}{ }_{m_{3}}\left(\begin{array}{ccc}
j & j_{5} & j_{6} \\
m_{4} & m_{5} & m_{6}
\end{array}\right) \tag{79}
\end{gather*}
$$

Here it is used that a magnetic number $m$ that is summed over can be replace by $-m$. In the last step it is used that an $\operatorname{SU}(2)$ matrix in any representation satisfies

$$
\begin{equation*}
(-1)^{-m_{3}+m_{4}} D_{j}(U)^{-m_{3}}{ }_{-m_{4}}=\left(D_{j}\left(U^{-1}\right)^{T}\right)^{m_{3}}{ }_{m_{4}} \tag{80}
\end{equation*}
$$

The conclusion is that the cost of reversing an arrow on a link with an $\mathrm{SU}(2)$ element is $(-1)^{2 j}$ and the $\mathrm{SU}(2)$ element is replaced by its inverse, which is the correct element for transport in the new direction of the arrow. Reversing the arrow on a link that represents the identity element also gives the sign $(-1)^{2 j}$, see equation (59). A link with an $\operatorname{SU}(2)$ element has a double arrow and its inverse is indicated by a triple arrow. Graphically the arrow reversal is


In the following sections these rules for diagrammatic calculations will be applied, first to examples of the wave function.

### 1.4 Examples of Spin Network Wave Functions

A spin network state defines a wave function, which is a product of $\mathrm{SU}(2)$ matrices and Wigner $3 j$-symbols. The general definition of this spin network wave function was given in equation (35). In this section three examples are given of simple spin networks and their corresponding wave functions.

In a wave function the order of the spins in a $3 j$-symbol is not defined by the spin network state. Different orderings can lead to wave functions that differ by a minus sign. Here the following choice is made for the construction of a wave function from a spin network state:

Order of Spins in $3 j$-symbols: The lowest link number goes first. Normal links go before internal links.

## Example 1

The most simple spin network state that can be constructed, consists of two 3 -valent nodes that are connected to each other by three links. Each link carries a spin value:


The wave function for this spin network state is

$$
\begin{align*}
& \psi_{\mathrm{SNW}}\left(U_{1}, U_{2}, U_{3}\right)=C\left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3} \\
m_{1}^{\prime} & m_{2}^{\prime} & m_{3}^{\prime}
\end{array}\right)_{2} \times \\
& \quad \times D_{j_{1}}\left(U_{1}\right)^{m_{1}^{\prime}}{ }_{m_{1}} D_{j_{2}}\left(U_{2}\right)^{m_{2}^{\prime}}{ }_{m_{2}}^{\prime} D_{j_{3}}\left(U_{3}\right)^{m_{3}^{\prime}}{ }_{m_{3}}\left(\begin{array}{ccc}
m_{1} & m_{2} & m_{3} \\
j_{1} & j_{2} & j_{3}
\end{array}\right)_{1} \tag{83}
\end{align*}
$$

A subscript is added to the $3 j$-symbols to indicate which node they represent. The normalization constant $C$ was given in equation (42):

$$
\begin{equation*}
C=\sqrt{\left(2 j_{1}+1\right)\left(2 j_{2}+1\right)\left(2 j_{3}+1\right)} \tag{84}
\end{equation*}
$$

The wave function itself can again be represented by a diagram. Such diagrams were described in section 1.3 . A $3 j$-symbol is represented by a node and a $\mathrm{SU}(2)$ matrix is represented by a link with a double arrow. This diagram is 2-dimensional, in contrast to the spin network, which is 3-dimensional. The diagrammatic version of the wave function of equation (83) is


## Example 2

In this second example the spin network consists of two 4 -valent nodes that are connected by four links. The spin network state defines an internal structure for each node: a decomposition into 3 -valent nodes and for the internal links it defines orientations and the spin values $k_{1}$ and $k_{2}$ :


The wave function:

$$
\begin{align*}
& \psi_{\mathrm{SNW}}\left(U_{1}, U_{2}, U_{3}, U_{4}\right)= C\left(\begin{array}{ccc}
j_{1} & j_{2} & m^{\prime} \\
m_{1}^{\prime} & m_{2}^{\prime} & k_{2}
\end{array}\right)_{2 a}\left(\begin{array}{ccc}
j_{3} & j_{4} & k_{2} \\
m_{3}^{\prime} & m_{4}^{\prime} & m^{\prime}
\end{array}\right)_{2 b} \times \\
& \times D_{j_{1}}\left(U_{1}\right)^{m_{1}^{\prime}}{ }_{m_{1}} D_{j_{2}}\left(U_{2}\right)^{m_{2}^{\prime}}{ }_{m_{2}} D_{j_{3}}\left(U_{3}\right)^{m_{3}^{\prime}}{ }_{m_{3}} D_{j_{4}}\left(U_{4}\right)^{m_{4}^{\prime}}{ }_{m_{4}} \times \\
& \times\left(\begin{array}{ccc}
m_{1} & m_{2} & m \\
j_{1} & j_{2} & k_{1}
\end{array}\right)_{1 a}\left(\begin{array}{ccc}
m_{3} & m_{4} & k_{1} \\
j_{3} & j_{4} & m
\end{array}\right)_{1 b} \tag{87}
\end{align*}
$$

The normalization constant:

$$
\begin{equation*}
C=\sqrt{\left(2 j_{1}+1\right)\left(2 j_{2}+1\right)\left(2 j_{3}+1\right)\left(2 j_{4}+1\right)\left(2 k_{1}+1\right)\left(2 k_{2}+1\right)} \tag{88}
\end{equation*}
$$

The wave function diagram:

$$
\begin{equation*}
\psi_{\mathrm{SNW}}\left(U_{1}, U_{2}, U_{3}, U_{4}\right)=C \tag{89}
\end{equation*}
$$



## Example 3: Superposition Principle for Wave Functions

The spin network in this third example is the same as in the second example: two 4 -valent nodes that are connected by four links. Now the node $n_{1}$ does not have a definite spin value on its internal link, instead the node is in a superposition of node states with different internal spins:


The spin network state is


The wave function in diagrammatic form:

$$
\begin{equation*}
\psi_{\mathrm{SNW}}\left(U_{1}, U_{2}, U_{3}, U_{4}\right)=\sum_{k_{1}} a\left(k_{1}\right) C\left(k_{1}\right) \tag{92}
\end{equation*}
$$

The wave function is now a sum of terms and each term itself is also a wave function for this spin network. This is the superposition principle for wave functions. The sum of two wave functions that are both based on the same spin network is again a wave function for that spin network.

### 1.5 The Node Inner Product

In section 1.2 an inner product between spin network states was discussed and in this context an inner product between node states was defined below equation (39) as

$$
\begin{equation*}
\left\langle n^{\prime} \mid n\right\rangle=C^{\prime} C i_{m_{1}^{\prime} \cdots m_{N}^{\prime}}^{\prime\left(j_{1} \cdots j_{N}\right)} \delta^{m_{1}^{\prime}, m_{1}} \cdots \delta^{m_{N}^{\prime}, m_{N}} i_{m_{1} \cdots m_{N}}^{\left(j_{1} \cdots j_{N}\right)} \tag{93}
\end{equation*}
$$

The $i$ 's are the intertwiners for the nodes and the $C$ 's are the normalization constants for the node states. This is an inner product between two node states of the same node. The nodes $n$ and $n^{\prime}$ have the same number of external links and the same node decomposition, only the spin values on the internal links can be different. This node inner product can be evaluated diagrammatically, but not in the current form, because the delta's with two upper indices do not have a diagrammatic representation. The node inner product is first evaluated for the 4 -valent node and then the general case is discussed.

## 4-Valent Node

Section 1.4 contains examples of spin network wave functions. The wave function for a spin network with two 4 -valent nodes was given in equation 87). A 4-node is represented in a wave function as an intertwiner such as the one on the right in equation (94). Its diagrammatic version in a wave function diagram is on the left:


$$
i_{m_{1} m_{2} m_{3} m_{4}}^{\left(j_{1} j_{2} j_{3} j_{4}\right)}=\left(\begin{array}{ccc}
j_{1} & j_{2} & k  \tag{94}\\
m_{1} & m_{2} & m
\end{array}\right)\left(\begin{array}{ccc}
m & j_{3} & j_{4} \\
k & m_{3} & m_{4}
\end{array}\right)
$$

The link with spin $k$ is the internal link of the node. To create a diagrammatic version of the node inner product (93) for this node, the indices of the primed intertwiner need to be raised:

$$
\begin{align*}
& i_{m_{1}^{\prime} m_{2}^{\prime} m_{3}^{\prime} m_{4}^{\prime}}^{\left(j_{j} j_{3} j_{4}\right)}=\left(\begin{array}{ccc}
j_{1} & j_{2} & k^{\prime} \\
m_{1}^{\prime} & m_{2}^{\prime} & m^{\prime}
\end{array}\right)\left(\begin{array}{ccc}
m^{\prime} & j_{3} & j_{4} \\
k^{\prime} & m_{3}^{\prime} & m_{4}^{\prime}
\end{array}\right)= \\
& \delta_{m_{1}^{\prime}, m_{1}^{\prime \prime} \delta_{m_{2}^{\prime}, m_{2}^{\prime \prime}} \delta_{m^{\prime}, m^{\prime \prime}}\left(\begin{array}{ccc}
m_{1}^{\prime \prime} & m_{2}^{\prime \prime} & m^{\prime \prime} \\
j_{1} & j_{2} & k^{\prime}
\end{array}\right)(-1)^{k^{\prime}+n} \delta^{m^{\prime},-n}\left(\begin{array}{ccc}
k^{\prime} & j_{3} & j_{4} \\
n & m_{3}^{\prime} & m_{4}^{\prime}
\end{array}\right)}^{\quad=\Delta^{4}\left(\begin{array}{ccc}
m_{1}^{\prime \prime} & m_{2}^{\prime \prime} & m^{\prime \prime} \\
j_{1} & j_{2} & k^{\prime}
\end{array}\right)(-1)^{k^{\prime}+n^{\prime}} \delta_{m^{\prime \prime},-n^{\prime}}\left(\begin{array}{ccc}
n^{\prime} & m_{3}^{\prime \prime} & m_{4}^{\prime \prime} \\
k^{\prime} & j_{3} & j_{4}
\end{array}\right)} \\
& =\Delta^{4}\left(\begin{array}{cccc}
m_{1}^{\prime \prime} & m_{2}^{\prime \prime} & k^{\prime} \\
j_{1} & j_{2} & n^{\prime}
\end{array}\right)\left(\begin{array}{ccc}
n^{\prime} & m_{3}^{\prime \prime} & m_{4}^{\prime \prime} \\
k^{\prime} & j_{3} & j_{4}
\end{array}\right) \\
& \Delta^{4}=\delta_{m_{1}^{\prime}, m_{1}^{\prime \prime}}^{\delta_{2}^{\prime}, m_{2}^{\prime \prime} \delta_{m_{3}^{\prime}, m_{3}^{\prime \prime}} \delta_{m_{4}^{\prime}, m_{4}^{\prime \prime}}} \tag{95}
\end{align*}
$$

The $3 j$-symbols are turned upside down with equation (B.4). Additionally, equation (57) is used in the first step and equation (56) is used in the last step to lower the index of the internal link. The result is that the internal indices stay the same, while the external indices have been raised. When (94) and (95) are entered in (93), the node inner product is
$\left\langle n^{\prime} \mid n\right\rangle=C^{\prime} C\left(\begin{array}{ccc}m_{1} & m_{2} & k^{\prime} \\ j_{1} & j_{2} & m^{\prime}\end{array}\right)\left(\begin{array}{ccc}m^{\prime} & m_{3} & m_{4} \\ k^{\prime} & j_{3} & j_{4}\end{array}\right)\left(\begin{array}{ccc}j_{1} & j_{2} & k \\ m_{1} & m_{2} & m\end{array}\right)\left(\begin{array}{ccc}m & j_{3} & j_{4} \\ k & m_{3} & m_{4}\end{array}\right)$
This expression for the node inner product has a diagrammatic version and it can be evaluated with the techniques of section 1.3 .


The bra node is the mirrored image of the ket node, where the node signs are changed, the arrows on the open links are reversed and the arrows on the internal links stay the same. To reduce this diagram the arrow on the $k^{\prime}$ link is reversed with equation (59) and the node signs of the two nodes at the top are changed with equation (54). The loop with two nodes at the top is then removed with equation (67) and the resulting diagram is the theta diagram (61), which is equal to one. The sign factor disappears because the sum of the spins $j_{1}, j_{2}$ and $k$ is integer, since they connect in the same node. The node states are orthonormal in this inner product with the normalization constant $C=\sqrt{2 k+1}$.

## General Case

Above, a diagrammatic version of the node inner product for the 4 -valent node was constructed. Now this will be extended to a node inner product for any node. The inner product of equation (93) is defined for intertwiners with only lower indices, so for nodes with only incoming links. An outgoing link appears in the node inner product as an upper index on the intertwiners and
these are upper indices in $3 j$-symbols that can be lowered with equation (57):

$$
\begin{align*}
i_{(j)}^{\prime m^{\prime}} \delta_{m^{\prime} m} i_{(j)}^{m} & =\binom{m^{\prime}}{j} \delta_{m^{\prime} m}\binom{m}{j}=(-1)^{2 j-2 n}\binom{j}{-n^{\prime}} \delta^{n^{\prime} n}\binom{j}{-n} \\
& =(-1)^{2 j+2 n} i_{n^{\prime}}^{(j)} \delta^{n^{\prime} n} i_{n}^{(j)} \tag{98}
\end{align*}
$$

Only the relevant indices and the relevant columns in $3 j$-symbols are shown. It is used that a magnetic number $m$ that is summed over can be replaced with $-m$. The final sign factor is +1 , since the sum of a spin and its magnetic number is always an integer. The result is that the upper indices on the intertwiners can be moved down for free and this makes equation (93) the most general form of the node inner product.

The next step is to raise the indices of the primed intertwiner to obtain an expression that has a diagrammatic version. This was done for the 4 -node in equation (95) and for any intertwiner the result is similar: The external indices are raised while the internal indices stay the same and no sign factor is picked up.

This means that the diagrammatic version of a general node inner product has the same features as the diagram of equation (97). It is constructed as follows:

- Draw the node as it appears in the wave function diagram, but with incoming external links and bring all these external links to the left side. This is the ket diagram.
- Mirror the ket diagram. Change the node signs and reverse the arrows on the open links, but not on the internal links. The internal links have a primed spin number. This is the bra diagram.
- Combine the open links of the bra and the ket diagram and multiply the diagram by normalization constants for the bra and the ket node.
The resulting diagram can again be reduced with the techniques of section 1.3 . Node decompositions have no loops in them, they are tree structures. It seems to be the case that the diagrammatic reduction of the node inner product for any node with a tree structure leads to a result that is similar to the result in equation (97):

$$
\begin{equation*}
\left\langle n^{\prime} \mid n\right\rangle=C^{\prime} C \prod_{\substack{\text { internal } \\ \text { links }}} \frac{\delta_{k, k^{\prime}}^{2 k+1}}{2 k+} \quad C=\prod_{\substack{\text { internal } \\ \text { links }}} \sqrt{2 k+1} \tag{99}
\end{equation*}
$$

For the given value of the normalization constant the node states are orthonormal in the sense that the inner product is one if all the internal spins of the two node states are equal and zero otherwise. In appendix A. 3 of Jinsong Yang and Yongge Ma this result is obtained graphically for a special class of node decompositions, where every 3 -node of the decomposition has at least one external link.

### 1.6 The Graphical Action of the Flux Operator

In section 1.2 general relativity was defined on the discretized structure of a 3-dimensional spin network. In this description the configuration variables are an $\mathrm{SU}(2)$ matrix and a flux vector, both defined for each link of the spin network. Loop quantum gravity is the quantum theory that is based on this description, where a wave function contains the information about the state of the timeslice and where the configuration variables are promoted to operators that act on this wave function.

In sections 1.3 and 1.4 it was described how a spin network wave function can be represented by a diagram and how such diagrams can be manipulated. Operators act on the wave function and they have a graphical action on the wave function diagrams. This results in a diagrammatic version of loop quantum gravity where calculations are reduced to manipulating diagrams.

The subject of this section is the graphical version of the flux operator and the next section treats the holonomy operator. The goal is to shown that the commutation relations of equation (25) between these two operators are implemented in the diagrammatic version of the theory.

## The Flux Operator

The action of the flux operator on a wave function was defined in equation (45). It acts on a holonomy as

$$
\begin{equation*}
\hat{L}_{i} U=-\left.\mathrm{i} \frac{d}{d t} U^{1 / 2} \exp \left(t \tau_{i}\right) U^{1 / 2}\right|_{t=0}=-\mathrm{i} U^{1 / 2} \tau_{i} U^{1 / 2} \tag{100}
\end{equation*}
$$

The holonomy is an $\mathrm{SU}(2)$ matrix in the spin $j$ representation. The $\tau_{i}$ is a generator of $\mathrm{SU}(2)$, also in the spin $j$ representation, see appendix A. This equation can be written in a semi-graphical way as


An $\mathrm{SU}(2)$ matrix is represented by a link with a double arrow. The flux operator obeys the correct commutation relation of equation 25 if the repeated action of this operator on the same link is as follows:


The first operator $L_{j}$ divides the holonomy in two equal parts and inserts a $\tau_{j}$ between these parts. The second operator $L_{i}$ also inserts a $\tau_{i}$ in the center of the holonomy, but upstream of the $\tau_{j}$ that is already present, with respect to the direction of the arrow on the link. With this definition the commutation relation between two flux operators is:


Here the commutation relation between the generators of $\mathrm{SU}(2)$ is used, $\left[\tau_{i}, \tau_{j}\right]=\varepsilon_{i j k} \tau^{k}$. Equation (104) is the same commutation relation as in equation (25) for a dimensionless $L$-operator.

## Graphical Action

In section 1.3 a diagrammatic representation of the spin network wave function was defined where Wigner $3 j$-symbols were represented as nodes and $\mathrm{SU}(2)$ matrices as links. The semi-graphical action of the flux operator in the form of equation (101) does not follow the rules of the wave function diagrams, since the right hand side is not completely diagrammatic. In appendix B of Jinsong Yang and Yongge Ma a graphical action of the flux operator on wave function diagrams is derived. This result is repeated here:

The generators $\tau_{i}$ of $\mathrm{SU}(2)$ are labeled by the index $i=1,2,3$. The spherical versions of these generators are labeled by the magnetic number $m_{i}=+1,0,-1$ and are defined as

$$
\begin{equation*}
\tau_{+1}=-\frac{1}{\sqrt{2}}\left(\tau_{1}+\mathrm{i} \tau_{2}\right) \quad \tau_{-1}=+\frac{1}{\sqrt{2}}\left(\tau_{1}-\mathrm{i} \tau_{2}\right) \quad \tau_{0}=\tau_{3} \tag{105}
\end{equation*}
$$

The $\tau_{m}$ 's are proportional to the Wigner $3 j$-symbol and thus have a graphical representation as in equation (50). The matrix elements and graphical versions of the $\tau_{m}$ 's are

$$
\begin{align*}
& \left.D_{j}\left(\tau_{m_{i}}\right)\right)^{m^{\prime}}=-\mathrm{i} \alpha(j)\left(\begin{array}{ccc}
m^{\prime} & 1 & j \\
j & m_{i} & m
\end{array}\right)  \tag{106}\\
& D_{j}\left(\tau^{m_{i}}\right)^{m^{\prime}}{ }_{m}=-\mathrm{i} \alpha(j)\left(\begin{array}{ccc}
m^{\prime} & m_{i} & j \\
j & 1 & m
\end{array}\right)  \tag{107}\\
& \alpha(j)=\sqrt{j(j+1)(2 j+1)} \tag{108}
\end{align*}
$$

Equation (106) can be derived from the definitions (105) together with the expressions for the matrix elements of the $\tau_{i}$ 's in A.3) and the expressions for the $3 j$-symbols that are involved, given in (B.5) to (B.7). Equation (107) defines the spherical $\tau$ 's with upper indices. The index $m_{i}$ of the $3 j$-symbol can be lowered with (57) to give

$$
\begin{equation*}
\tau^{m_{i}}=(-1)^{1-m_{i}^{\prime}} \delta^{m_{i}, m_{i}^{\prime}} \tau_{-m_{i}^{\prime}} \quad \tau^{+1}=\tau_{-1} \quad \tau^{-1}=\tau_{+1} \quad \tau^{0}=-\tau_{0} \tag{111}
\end{equation*}
$$

The following relations between the $\tau_{i}$ 's and $\tau_{m}$ 's can be checked by entering (105) and (111) in the right hand sides:

$$
\begin{align*}
\tau_{i} \tau^{i} & =-\tau_{m_{i}} \tau^{m_{i}}  \tag{112}\\
\varepsilon_{i j k} \tau^{i} \tau^{j} \tau^{k} & =\mathrm{i} \varepsilon_{m_{i} m_{j} m_{k}} \tau^{m_{i}} \tau^{m_{j}} \tau^{m_{k}}=\mathrm{i} \varepsilon^{m_{i} m_{j} m_{k}} \tau_{m_{i}} \tau_{m_{j}} \tau_{m_{k}}  \tag{113}\\
\varepsilon_{+10-1} & =+1 \tag{114}
\end{align*}
$$

In equations 102 to 104 there is a matrix product between the $\tau$ 's, but in equations 112 and 113 there is no matrix product. There is only an index contraction over the indices that are shown. This means that the full expression for equation 112 is

$$
\begin{equation*}
D_{j}\left(\tau_{i}\right)^{m}{ }_{n} D_{j^{\prime}}\left(\tau^{i}\right)^{m^{\prime}}{ }_{n^{\prime}}=-D_{j}\left(\tau_{m_{i}}\right)^{m}{ }_{n} D_{j^{\prime}}\left(\tau^{m_{i}}\right)^{m^{\prime}}{ }_{n^{\prime}} \tag{115}
\end{equation*}
$$

The $\varepsilon$-symbol with magnetic indices is also proportional to a $3 j$-symbol. The $3 j$-symbol has the same permutation properties, for both the sum of the three magnetic numbers must vanish and

$$
\left(\begin{array}{ccc}
1 & 1 & 1  \tag{116}\\
+1 & 0 & -1
\end{array}\right)=-\frac{1}{\sqrt{6}} \quad\left(\begin{array}{ccc}
1 & 1 & 1 \\
0 & 0 & 0
\end{array}\right)=0
$$

This means that the graphical version of the $\varepsilon$-symbol as defined in 114 is

$$
\varepsilon_{m_{i} m_{j} m_{k}}=-\sqrt{6}\left(\begin{array}{ccc}
1 & 1 & 1  \tag{117}\\
m_{i} & m_{j} & m_{k}
\end{array}\right)=-\sqrt{6} \underbrace{1}_{m_{j}} \underbrace{1}_{1} \underbrace{m_{i}}_{i}
$$

The goal was to construct a graphical version of the action of the flux operator. This is possible when the action of the flux operator can be expressed in the $\tau_{m}$ 's and $\varepsilon_{m_{i} m_{j} m_{k}}$. When the semi-graphical action of the flux operator of equation 101 is combined with equations 112 and 113 it turns out that there is a graphical version of the following combinations of $L$-operators:

$$
\begin{align*}
\hat{L}_{i} \hat{L}^{i} & \sim(-\mathrm{i})^{2} \tau_{i} \tau^{i}=\tau_{m_{i}} \tau^{m_{i}}  \tag{118}\\
\varepsilon_{i j k} \hat{L}^{i} \hat{L}^{j} \hat{L}^{k} & \sim(-\mathrm{i})^{3} \varepsilon_{i j k} \tau^{i} \tau^{j} \tau^{k}=-\varepsilon_{m_{i} m_{j} m_{k}} \tau^{m_{i}} \tau^{m_{j}} \tau^{m_{k}} \tag{119}
\end{align*}
$$

Again there is no matrix product between the $\tau$ 's. Consider now the inner product between two flux operators, $L \cdot L=L_{i} L^{i}$, where the two operators
act on different links. With the diagrammatic versions of the $\tau_{m}$ 's of 109 and (110), the graphical action of this inner product is


The labels on the $L$ 's indicate on which link they act. In the last step the $\tau_{m}$ 's create a node with an open link in the links 1 and 2 and the sum over the magnetic number is represented by connecting these open links, as was defined in equation (55). The link with spin 1 has a single arrow because an index is contracted directly between two $3 j$-symbols without an $\operatorname{SU}(2)$ matrix in between. The graphical version of equation (119) is shown in (127).

## The Square of the Flux Operator

The graphical method described above will now be applied to the square of the flux operator. The operator $L^{2}$ is $L_{i} L^{i}$, where both $L$ 's act on the same link. When 120 is combined with the ordering rule of equation (103) the graphical action of $L^{2}$ is

$$
\begin{align*}
& \left.=-(-1)^{2 j+1}(-1)^{2 j} \alpha(j)^{2}\right] \xrightarrow[+]{j} \\
& \left.=\frac{\alpha(j)^{2}}{2 j+1}\right] \quad \stackrel{j}{\longleftrightarrow}[=j(j+1)] \stackrel{j}{\longleftrightarrow} L \tag{121}
\end{align*}
$$

A node sign is changed with rule (54) and the arrow on a link is reversed with (59). The loop with two nodes is then removed with (67). This result is the expected result for the square of an angular momentum operator.

## Commutation Relation

As a second application of the graphical method, a diagrammatic calculation will be done that shows that the commutation relation between the flux operators is implemented on the spin network wave functions. Instead of
the commutation relation $(104)$, the following operator expression will be investigated:

$$
\begin{equation*}
\left[\hat{L}_{1 i}, \hat{L}_{1 j}\right] \hat{L}_{2}^{i} \hat{L}_{3}^{j}=\mathrm{i} \varepsilon_{i j k} \hat{L}_{1}^{k} \hat{L}_{2}^{i} \hat{L}_{3}^{j}=\mathrm{i} \varepsilon_{i j k} \hat{L}_{1}^{i} \hat{L}_{2}^{j} \hat{L}_{3}^{k} \tag{122}
\end{equation*}
$$

Both sides of equation (104) are multiplied by $\hat{L}_{2}^{i} \hat{L}_{3}^{j}$ to create combinations of flux operators that have a graphical action on the wave function diagrams. The numerical labels of the $L$-operators indicate on which link they act. The operators $L_{1}, L_{2}$ and $L_{3}$ act on any three links of a wave function diagram. Flux operators that act on different links commute. Using (120) and the ordering rule 103 , the first term of the commutator acts as


First a summation line (64) is entered in the vertical links. The sum is limited to $l=0,1,2$ by the Clebsch-Gordan conditions (51). A loop with three nodes is removed from the diagram with the graphical rule (69). Node signs are changed with (50) and (54) and arrows are reversed with (59) and (81). The constants that multiply the diagrams are placed on top of each other and this should be read as a product from the top to the bottom. The last diagram is called diagram B. The second term of the commutator in (122) is
$\left(\hat{L}_{1} \cdot \hat{L}_{3}\right)\left(\hat{L}_{1} \cdot \hat{L}_{2}\right)$


$$
=(-1)^{l+2} \times \text { diagram } \mathrm{A}
$$

The two spin 1 links of the central +-node are interchanged at the cost of a node sign change, using (50). Diagram C is then proportional to the second version of diagram A in equation (123). The complete commutator is now

$$
\begin{align*}
& \left(\hat{L}_{1} \cdot \hat{L}_{2}\right)\left(\hat{L}_{1} \cdot \hat{L}_{3}\right)-\left(\hat{L}_{1} \cdot \hat{L}_{3}\right)\left(\hat{L}_{1} \cdot \hat{L}_{2}\right) \\
& \times \sum_{l=0,1,2}(2 l+1)(-1)^{2 j_{1}}\left[(-1)^{3 l}-(-1)^{4 l}\right]\left\{\begin{array}{ccc}
1 & 1 & l \\
j_{1} & j_{1} & j_{1}
\end{array}\right\} \times \text { diagram B } \\
& =\alpha\left(j_{1}\right)^{2} \alpha\left(j_{2}\right) \alpha\left(j_{3}\right)(-1)^{2 j_{1}}(-6)\left\{\begin{array}{ccc}
1 & 1 & 1 \\
j_{1} & j_{1} & j_{1}
\end{array}\right\} \times \text { diagram D } \\
& =-\sqrt{6} \alpha\left(j_{1}\right) \alpha\left(j_{2}\right) \alpha\left(j_{3}\right) \times \operatorname{diagram} \mathrm{D} \tag{125}
\end{align*}
$$

The terms inside the square brackets reduce the sum to the $l=1$ term. Diagram D is diagram B with $l=1$ and it is given below in equation (127). In the last step the following expression for the $6 j$-symbol is used. It can be
found with the function SixJSymbol [] in Wolfram Mathematica.

$$
\left\{\begin{array}{ccc}
1 & 1 & 1  \tag{126}\\
j_{1} & j_{1} & j_{1}
\end{array}\right\}=(-1)^{-2 j_{1}} \frac{1}{\sqrt{6}} \frac{1}{\alpha\left(j_{1}\right)}
$$

For the graphical action of the right hand side of equation $(122),(119)$ is used in combination with the graphical version of $\tau^{m}$ in equation 110) and the $\varepsilon$-symbol in equation (117). The $\tau^{m}$ 's create a node with an open link in the links 1,2 and 3 and the $\varepsilon$-symbol is a new node that collects these open links. The result is


The left hand side of the commutation relation, equation (125), is equal to the right hand side (127). A diagrammatic calculation has confirmed that the commutation relation 122 between the flux operators holds for the graphical action of the flux operator on wave function diagrams.

### 1.7 The Graphical Action of the Holonomy Operator

In the last section the graphical action of the flux operator on wave function diagrams was discussed. The holonomy operator has also a graphical action, which will be used in this section to calculated the commutation relations that involve the holonomy operator: $[U, U],[U, L]$ and $\left[U, L^{2}\right]$.

The holonomy operator is a multiplicative operator on the spin network wave function. Its action was defined in equation (44):

$$
\begin{equation*}
\left(\hat{U}_{l}\right)^{m}{ }_{n} \psi_{\mathrm{SNW}}\left(\left\{U_{l}\right\}\right)=D_{1}\left(U_{l}\right)^{m}{ }_{n} \psi_{\mathrm{SNW}}\left(\left\{U_{l}\right\}\right) \tag{128}
\end{equation*}
$$

On the right hand side the wave function is multiplied by an $\mathrm{SU}(2)$ matrix in a fixed representation. This representation does not depend on the wave function. Here the spin 1 representation is chosen. Graphically the holonomy operator acts as follows on a wave function diagram, of which only the relevant part is shown:


The link with the double arrow and labeled by spin $j_{1}$ represents the $\mathrm{SU}(2)$ matrix $U_{1}$ in the spin $j_{1}$ representation that is contracted with two Wigner $3 j$-symbols. The holonomy operator copies this element $U_{1}$ in the spin 1 representation and the open ends of this new link carry the magnetic numbers $m$ and $n$.

The right hand side of equation $\sqrt{128}$ ) is again a function of $U_{l}$, so according to 44 the repeated action of $U$ on the same link is

$$
\begin{equation*}
\left(\hat{U}_{l}\right)^{m^{\prime}}{ }_{n^{\prime}}\left(\hat{U}_{l}\right)^{m}{ }_{n} \psi_{\operatorname{SNW}}\left(\left\{U_{l}\right\}\right)=D_{1}\left(U_{l}\right)^{m^{\prime}}{ }_{n^{\prime}} D_{1}\left(U_{l}\right)^{m}{ }_{n} \psi_{\mathrm{SNW}}\left(\left\{U_{l}\right\}\right) \tag{130}
\end{equation*}
$$

The action of the holonomy operator commutes with itself and this gives the required commutation relation of equation 25 :

$$
\begin{equation*}
\left[\left(\hat{U}_{l}\right)^{m}{ }_{n},\left(\hat{U}_{l^{\prime}}\right)^{m^{\prime}}{ }_{n^{\prime}}\right]=0 \tag{131}
\end{equation*}
$$

## Commutation Relation $[U, L]$

The commutator between the holonomy operator and the flux operator will now be calculated. First the action of the flux operator on an $\mathrm{SU}(2)$ matrix of the spin network wave function is repeated here from equations 45 and (100):

$$
\begin{align*}
\hat{L}^{i} f\left(U^{m}{ }_{n}\right) & =-\left.\mathrm{i} \frac{d}{d t} f\left(\left(U^{1 / 2}\right)^{m}{ }_{p} \exp \left(t \tau^{i}\right)^{p}{ }_{q}\left(U^{1 / 2}\right)^{q}{ }_{n}\right)\right|_{t=0}  \tag{132}\\
\hat{L}^{i} U^{m}{ }_{n} & =-\mathrm{i}\left(U^{1 / 2}\right)^{m}{ }_{p}\left(\tau^{i}\right)^{p}{ }_{q}\left(U^{1 / 2}\right)^{q}{ }_{n}
\end{align*}
$$

To obtain a shorter notation the matrix representation is not explicitly written. The complete notation is $U^{m}{ }_{n}=D_{j}(U)^{m}{ }_{n}$ and a matrix in the spin 1 representation will be denoted below as $\dot{U}^{m}{ }_{n}=D_{1}(U)^{m}{ }_{n}$. The indices $m, n, p, q, r, \ldots$ are magnetic numbers.

The action of $U$ after $L$ on the same link is

$$
\begin{align*}
\left(\hat{U}_{1}\right)^{m}{ }_{n} \hat{L}_{1}^{i}\left(U_{1}\right)^{p}{ }_{q} & =-\mathrm{i}\left(\hat{U}_{1}\right)^{m}{ }_{n}\left(U_{3}\right)^{p}{ }_{r}\left(\tau^{i}\right)^{r}{ }_{s}\left(U_{2}\right)^{s}{ }_{q} \\
& =-\mathrm{i}\left(\hat{U}_{3}\right)^{m}{ }_{u}\left(\hat{U}_{2}\right)^{u}{ }_{n}\left(U_{3}\right)^{p}{ }_{r}\left(\tau^{i}\right)^{r}{ }_{s}\left(U_{2}\right)^{s}{ }_{q} \\
& =-\mathrm{i}\left(\dot{U}_{1}\right)^{m}{ }_{n}\left(U_{3}\right)^{p}{ }_{r}\left(\tau^{i}\right)^{r}{ }_{s}\left(U_{2}\right)^{s}{ }_{q} \tag{133}
\end{align*}
$$

The $L$-operator cuts $U_{1}$ into two equal parts: $U_{1}=U_{3} U_{2}$ where $U_{2}=U_{3}=$ $U_{1}^{1 / 2}$. Next it is assumed that if the link on which $U$ acts is made up of multiple segments, $l_{1}=l_{2}+l_{3}$, that then the operator $U$ can be written as a multiplication of the operators for each segment:

$$
\begin{equation*}
\left(\hat{U}_{1}\right)^{m}{ }_{n}=\left(\hat{U}_{3}\right)^{m}{ }_{u}\left(\hat{U}_{2}\right)_{n}^{u} \tag{134}
\end{equation*}
$$

In the semi-graphical notation of equation 101 the action of $U$ after $L$ is


When evaluating the action of $L$ after $U$, it turns out that the product rule applies to the flux operator, since it is a derivative operator, see equation 132 :

$$
\begin{align*}
& \hat{L}^{i} \hat{U}^{m}{ }_{n} U^{p}{ }_{q}=\hat{L}^{i} \stackrel{\circ}{U}^{m}{ }_{n} U_{q}^{p}=  \tag{136}\\
& \quad-\left.\mathrm{i} \frac{d}{d t}\left(\left(\stackrel{\circ}{U}^{1 / 2}\right)^{m}{ }_{r} \exp \left(t \tau^{i}\right)^{r}{ }_{s}\left(\stackrel{\circ}{U}^{1 / 2}\right)^{s}{ }_{n}\left(U^{1 / 2}\right)^{p}{ }_{u} \exp \left(t \tau^{i}\right)^{u}{ }_{v}\left(U^{1 / 2}\right)^{v}{ }_{q}\right)\right|_{t=0} \\
& \quad=-\mathrm{i}\left(\stackrel{\circ}{U}^{1 / 2}\right)^{m}{ }_{r}\left(\stackrel{\circ}{\tau}^{i}\right)^{r}{ }_{s}\left(\stackrel{\circ}{U}^{1 / 2}\right)^{s}{ }_{n} U_{q}^{p}-\mathrm{i} \stackrel{\circ}{U}^{m}{ }_{n}\left(U^{1 / 2}\right)^{p}{ }_{u}\left(\tau^{i}\right)^{u}{ }_{v}\left(U^{1 / 2}\right)^{v}{ }_{q}
\end{align*}
$$

The second term is equal to the result of equation $\sqrt[133]{ }$ ). The commutator [ $U, L$ ] is then equal to minus the first term of 136 and in semi-graphical form this is:


The right hand side of this equation is also realized by the following operator expression:

$$
\begin{equation*}
\left[\left(\hat{U}_{1}\right)^{m}{ }_{n}, \hat{L}_{1}^{i}\right]=\mathrm{i}\left(\hat{U}_{3}\right)^{m}{ }_{r} D_{1}\left(\tau^{i}\right)^{r}{ }_{s}\left(\hat{U}_{2}\right)^{s}{ }_{n} \tag{138}
\end{equation*}
$$

The assumption of equation $(134)$ is used here to split the operator $U_{1}$ into two parts. This commutation relation between the holonomy operator and the flux operator is the one that was required in equation 25 .

## Commutation Relation $\left[U, L^{2}\right.$ ]

The last commutation relation that will be evaluated is the one between the holonomy operator and the square of the flux operator. The operator $L^{2}=L_{i} L^{i}$ of equation (121) commutes with the flux operator, but it does not commute with the holonomy operator. Using equation 138 the commutation relation is

$$
\begin{align*}
& {\left[\left(\hat{U}_{1}\right)^{m}{ }_{n}, \hat{L}_{1}^{2}\right]=\left[\left(\hat{U}_{1}\right)^{m}{ }_{n}, \hat{L}_{1 i}\right] \hat{L}_{1}^{i}+\hat{L}_{1 i}\left[\left(\hat{U}_{1}\right)^{m}{ }_{n}, \hat{L}_{1}^{i}\right]} \\
& \quad=\mathrm{i}\left(\hat{U}_{3}\right)^{m}{ }_{p} D_{1}\left(\tau_{i}\right)^{p}{ }_{q}\left(\hat{U}_{2}\right)^{q}{ }_{n} \hat{L}_{1}^{i}+\mathrm{i} \hat{L}_{1 i}\left(\hat{U}_{3}\right)^{m}{ }_{p} D_{1}\left(\tau^{i}\right)^{p}{ }_{q}\left(\hat{U}_{2}\right)^{q}{ }_{n} \tag{139}
\end{align*}
$$

When the first term on the right hand side acts on a link of a wave function diagram as before, the result is



There is a sum over the index $i$ on the left hand side. The graphical expression on the right hand side is obtained from 120 . When the last term of equation (139) acts on a wave function diagram, the product rule applies to the action of the $L$-operator and the result is, again with a sum over $i$ :


In the first step the ordering rule 103 is used to position the $\tau_{i}$ in the top link. The first term is equal to the action of $-L^{2}$ on the top link. In 121) it was calculated that the eigenvalue of $L^{2}$ is $j(j+1)$. The second term is equal to equation 140 , since the arrow on a link with integer spin can be reversed for free, see equation (59).

The action of the complete commutator on a link is then



It was checked numerically that these terms do not cancel each other. The right hand side can also be obtained by acting with the following operator expression:

$$
\begin{equation*}
\left[\left(\hat{U}_{1}\right)^{m}{ }_{n}, \hat{L}_{1}^{2}\right]=-2\left(\hat{U}_{1}\right)^{m}{ }_{n}+2 \mathrm{i}\left(\hat{U}_{3}\right)^{m}{ }_{p} D_{1}\left(\tau_{i}\right)^{p}{ }_{q}\left(\hat{U}_{2}\right)^{q}{ }_{n} \hat{L}_{1}^{i} \tag{143}
\end{equation*}
$$

The holonomy operator and the square of the flux operator do not commute, $\left[U, L^{2}\right] \neq 0$. The formalism of quantum mechanics states that the physical properties that correspond to these operators can not known together at the same time with infinite precision.

## Conclusion

In the last two sections it was shown that the classical Poisson brackets of equation (24) are implemented as the commutation relations between the operators $L$ and $U$ when they act on the spin network wave functions. The assumptions that were made to achieve this were the ordering rule for the $L$-operator of equation (103) and the ability of the $U$-operator to be split into multiple operators for segments of a link, as defined in equation (134).

### 1.8 The Area and Volume Operators

Spin networks were described in section 1.1 and it was stated there that the nodes of a spin network represent pieces of volume and that links represent the contact surfaces between these pieces of volume. In a quantum theory, operators that correspond to observables determine the physical interpretation of the theory, so the geometrical interpretation of spin networks is only justified if this is supported by geometrical operators. These geometrical operators exist and in this way loop quantum gravity succeeds in creating a quantum version of space, but the classical limit that connects this quantum space to a classical space has not been established. In this section the conventional area and volume operators of loop quantum gravity are described and the spectrum of the volume operator is calculated diagrammatically.

## Area operator

In section 1.2 classical general relativity was rewritten in the Ashtekar variables. From these Ashtekar variables new variables were constructed that were defined on spin networks. For every link there was an $\mathrm{SU}(2)$ holonomy $U_{l}$ and a flux vector $L_{l}^{i}$. A link of a spin network represents a surface that connects two pieces of volume and the length of the flux vector $L_{l}^{i}$ is equal to the area of this surface, see equation 21 .

In the quantum theory, a link of a spin network carries a spin number $j$ and the link is an eigenstate of the operator $L_{l}^{2}$ with eigenvalue $j(j+1)$, see equation 121. These eigenvalues are interpreted as the square of the area of the surface that is represented by the link. The area operator that acts on a link $l$ and its eigenvalues are

$$
\begin{gather*}
\text { Area }(l)^{2}=\left(l_{0}\right)^{4} \hat{L}_{l}^{2} \quad \operatorname{Area}(j)=\left(l_{0}\right)^{2} \sqrt{j(j+1)}  \tag{144}\\
l_{0}=\sqrt{8 \pi \gamma} l_{P}=\sqrt{\frac{8 \pi G \hbar \gamma}{c^{3}}}
\end{gather*}
$$

Here is $\gamma$ the dimensionless Immirzi parameter that sets the length scale $l_{0}$ relative to the Planck length $l_{P}$.

The spectrum of the area operator is discrete. This means that indivisible pieces of area can only have one of these discrete area values. Any surface is composed of multiple of these pieces, such that the area of any surface can only take discrete values.

## Volume operator

A tetrahedron is a volume that is bounded by four triangular faces. In a spin network it is represented by a node with four links, see figure 2. Such a 4 -valent node is the most simple building block of volume in a spin network.


Figure 2: On the left a tetrahedron with arrows that are normal to the faces. On the right the 4 -valent node of a spin network that is dual to this tetrahedron.

The volume of a tetrahedron was expressed in the flux vectors in equation (23). Following sections 1.3 and 7.5 .3 of Rovelli and Vidotto the same expression can be reached in the following way: Consider a tetrahedron in Euclidean space. Construct for every face of the tetrahedron a vector $\vec{L}$ that is normal to the face and which length is equal to the area of the face. Express the volume of the tetrahedron in these vectors $\vec{L}$. The operator expression for the volume of a 4 -valent node is then obtained by replacing the vectors $\vec{L}$ by the operators $\left(l_{0}\right)^{2} \hat{L}^{i}$.

The volume of a tetrahedron in terms of the vectors $\vec{L}$ is

$$
\begin{equation*}
\text { Volume }^{2}=\frac{2}{9}\left(\vec{L}_{1} \times \vec{L}_{2}\right) \cdot \vec{L}_{3}=\frac{2}{9} \varepsilon_{i j k} L_{1}^{i} L_{2}^{j} L_{3}^{k} \tag{145}
\end{equation*}
$$

The labels 1,2 and 3 indicate to which face the vectors belong. This equation is valid for any combination of three $L$-vectors, since $\vec{L}_{1}+\vec{L}_{2}+\vec{L}_{3}+\vec{L}_{4}=0$. The square of the volume can be positive or negative depending on the combination of $L$-vectors.

The vectors are now replaced by operators such that the volume operator acts on three of the four links that connect in a 4 -valent node. An operator $Q$ with eigenvalues $q$ is defined for convenience. The volume operator and the volume eigenvalues of a node $n$ are

$$
\begin{equation*}
\hat{Q}_{n}=\varepsilon_{i j k} \hat{L}_{1}^{i} \hat{L}_{2}^{j} \hat{L}_{3}^{k} \tag{146}
\end{equation*}
$$

$$
\operatorname{Volûme}(n)^{2}=\frac{2}{9}\left(l_{0}\right)^{6} \hat{Q}_{n} \quad \text { Volume }(n)=\frac{\sqrt{2}}{3}\left(l_{0}\right)^{3} \sqrt{\left|q_{n}\right|}
$$

## Volume Calculation

In section 7.5.3 of Rovelli and Vidotto, the spectrum of the volume operator is calculated by placing the volume operator of equation (146) inside a
node inner product: $\left\langle n^{\prime}\right| \hat{V}|n\rangle$. This is done because volume is a property of a node.

Here the volume spectrum will be calculated in a different way. The volume operator consists of $L$-operators and here these $L$-operators are defined to act on spin network wave functions, not on node states. The $L$-operator has a graphical action on wave function diagrams, which was described in section 1.6, and this defines also a graphical action for the volume operator on wave function diagrams. This graphical action is used here together with the techniques for diagrammatic manipulation of section 1.3 to calculate the spectrum of the volume operator.

However, there is a problem. The diagrammatic manipulation of section 1.3 is only possible when the links represent the identity matrix and not when the links represent a general $\operatorname{SU}(2)$ matrix. This can be solved by letting the $L$-operator act at the end of a link, near the node, but here the choice was made to let the $L$-operator act in the middle of a link, see equation (45). The calculation can still be done if the volume operator acts on a spin network wave function where all the holonomies are the identity element:

$$
\begin{equation*}
\hat{Q}_{n} \psi_{\mathrm{SNW}}\left(\left\{U_{l}=\mathbb{1}\right\}\right) \tag{147}
\end{equation*}
$$

This is not the most general spin network wave function and in this sense the calculation is incorrect, but this approach does lead to the usual spectrum for the volume operator.

The graphical action of the $Q$-operator was given in equation (127). It is a 3 -valent node that grasps three links. Here it acts on a 4 -valent node with an internal link with spin $k$. Diagrammatic manipulation is then used to reduce the diagram back to the form of the initial 4 -node. The diagram is multiplied by the normalization constant for the node, which is $\sqrt{2 k+1}$, see equation (97). The diagrammatic evaluation of the action of the $Q$-operator is shown on the next page, where only one node of the wave function diagram is shown.

The links with spin $j_{1}$ to $j_{4}$ have a single arrow to indicate that these links represent the identity matrix, $U_{1}=U_{2}=U_{3}=U_{4}=\mathbb{1}$. To reduce the diagram, two times a loop with four nodes is removed with (71). On the third line the calculation is continued with only the diagram. During the calculation, node signs are changed at the cost of $(-1)^{j_{1}+j_{2}+j_{3}}$ and arrows on the links are reversed at the cost of $(-1)^{2 j}$, see equations (54) and (59).

The limits on the sums are set by the Clebsch-Gordan conditions (51). The spins $l$ and $h$ take integer steps between these values:

$$
\begin{array}{ll}
l_{\text {min }}=\max \left\{\left|j_{1}-1\right|,\left|j_{2}-k\right|\right\} & h_{\min }=\max \left\{\left|j_{1}-j_{2}\right|,\left|j_{3}-j_{4}\right|\right\}  \tag{148}\\
l_{\max }=\min \left\{j_{1}+1, j_{2}+k\right\} & h_{\max }=\min \left\{j_{1}+j_{2}, j_{3}+j_{4}\right\}
\end{array}
$$

The limits on the final internal spin $h$ are the same as on the initial internal spin $k$.




The complete action of the $Q$-operator is:


$$
\begin{aligned}
& c_{h}(k)=-\mathrm{i} \sqrt{6} \alpha\left(j_{1}\right) \alpha\left(j_{2}\right) \alpha\left(j_{3}\right)(-1)^{2 j_{1}+j_{3}+j_{4}+k} \sqrt{2 k+1} \sqrt{2 h+1} \times \\
& \times\left\{\begin{array}{lll}
1 & k & h \\
j_{4} & j_{3} & j_{3}
\end{array}\right\} \sum_{l}(2 l+1)\left\{\begin{array}{lll}
1 & 1 & 1 \\
l & j_{1} & j_{1}
\end{array}\right\}\left\{\begin{array}{lll}
1 & j_{1} & l \\
k & j_{2} & j_{2}
\end{array}\right\}\left\{\begin{array}{ccc}
1 & k & h \\
j_{2} & j_{1} & l
\end{array}\right\}
\end{aligned}
$$

The spins in the $6 j$-symbols are rearranged using equation (B.10). The action of the volume operator is not diagonal in the sense that the resulting node does not have the same spin on the internal link. The result is a sum over nodes with different internal spins. This results in a matrix for the volume operator. The eigenvectors of this matrix are the superpositions of node states on which the volume operator is diagonal. An eigenvalue gives the physical volume of such a volume eigenstate.

The matrix notation associates the unit vector $\vec{e}_{1}$ with $k_{\text {min }}$ and column $\left(k-k_{\min }+1\right)$ of the $Q$-matrix is the standing vector $c_{h}(k)$. When the spins $j$ are specified, the $6 j$-symbols in the coefficients $c_{h}(k)$ can be calculated. For the following matrices the Java source code that is available with the article Mathar was used to calculate the values of the $6 j$-symbols.

For the monochromatic node, where all spins $j$ are equal, the $Q$-matrices for the lowest three spin values are

$$
\begin{gather*}
Q_{j=1 / 2}=\frac{\sqrt{3}}{4}\left(\begin{array}{cc}
0 & -\mathrm{i} \\
\mathrm{i} & 0
\end{array}\right) \quad Q_{j=1}=\frac{1}{\sqrt{3}}\left(\begin{array}{ccc}
0 & -2 \mathrm{i} & 0 \\
2 \mathrm{i} & 0 & -\sqrt{5} \mathrm{i} \\
0 & \sqrt{5} \mathrm{i} & 0
\end{array}\right) \\
Q_{j=3 / 2}=\left(\begin{array}{cccc}
0 & -2.165 \mathrm{i} & 0 & 0 \\
2.165 \mathrm{i} & 0 & -3.098 \mathrm{i} & 0 \\
0 & 3.098 \mathrm{i} & 0 & -2.662 \mathrm{i} \\
0 & 0 & 2.662 \mathrm{i} & 0
\end{array}\right) \tag{151}
\end{gather*}
$$

The values in the last matrix are rounded to three decimal places. These matrices are the same as the ones found in section 7.5 .3 of Rovelli and Vidotto]. The node state of the 4 -valent node is denoted as $|k\rangle$ where $k$ is the spin on the internal link. The eigenvalues and the eigenstates of the $Q$-operator for the lowest three spin values are:

$$
\begin{array}{rll}
j=1 / 2: & q_{ \pm}= \pm \sqrt{3} / 4 & \left|q_{ \pm}\right\rangle=\mathrm{i} / \sqrt{2}|0\rangle \mp 1 / \sqrt{2}|1\rangle  \tag{152}\\
j=1: & q_{0}=0 & \left|q_{0}\right\rangle=\sqrt{5} / 3|0\rangle+2 / 3|2\rangle \\
& q_{ \pm}= \pm \sqrt{3} & \left|q_{ \pm}\right\rangle= \pm \sqrt{2} / 3|0\rangle+\mathrm{i} / \sqrt{2}|1\rangle \mp \sqrt{5} / 3 \sqrt{2}|2\rangle \\
j=3 / 2: & & \\
& \\
q_{1, \pm}= \pm 1.299 & \left|q_{1, \pm}\right\rangle= & 0.645 \mathrm{i}|0\rangle \mp 0.387|1\rangle+0.289 \mathrm{i}|2\rangle \mp 0.592|3\rangle \\
q_{2, \pm}= \pm 4.437 & \left|q_{2, \pm}\right\rangle=-0.289 \mathrm{i}|0\rangle \pm 0.592|1\rangle+0.645 \mathrm{i}|2\rangle \mp 0.387|3\rangle
\end{array}
$$

## Volume Eigenvalues

The volume eigenvalues are related to the eigenvalues of the $Q$-operator as described by equation (146). Monochromatic 4-nodes with integer spin have a zero eigenvalue. The nonzero volume eigenvalues are double degenerate, the zero eigenvalues are not degenerate. The volume eigenstates are the same as the eigenstates of $\hat{Q}$. A 4 -valent node in a volume eigenstate is a linear combination of 4 -nodes with different internal spins $k$.


Figure 3: Volume eigenvalues of the monochromatic 4-valent node (crosses) and the volume of a regular Euclidean tetrahedron (line).

Figure 3 is a plot of the volume eigenvalues for the monochromatic 4valent node, up to spin $j=10$. Similar plots can be found in chapter 4 of Haggard. Section 7.3 of Brunnemann and Rideout contains a formula for the characteristic polynomial of the $Q$-matrix for any spin $j$. This formula was used to calculate the volume eigenvalues for this plot.

The monochromatic 4-node represents an equal-sided tetrahedron. The tetrahedron that has six edges of equal length is equal-sided and is called the regular tetrahedron. The relation between the face area and the volume for a regular Euclidean tetrahedron with edge length $a$ is:

$$
\begin{gather*}
\text { Face Area }=\frac{\sqrt{3}}{4} a^{2} \quad \text { Volume }=\frac{a^{3}}{6 \sqrt{2}} \\
\text { Volume }=\frac{2^{3 / 2}}{3^{7 / 4}}(\text { Face Area })^{3 / 2} \tag{153}
\end{gather*}
$$

When the area eigenvalues of equation (144) are entered here for the face area, the result is an expression for the volume of a regular Euclidean tetrahedron in terms of $j$ :

$$
\begin{equation*}
\text { Area } /\left(l_{0}\right)^{2}=\sqrt{j(j+1)} \quad \text { Volume } /\left(l_{0}\right)^{3}=\frac{2^{3 / 2}}{3^{7 / 4}}(j(j+1))^{3 / 4} \tag{154}
\end{equation*}
$$

This relation is plotted as the line in figure 3. All the quantum volume eigenvalues are below this line.

## Interpretation

How should this volume spectrum be interpreted? A triangle is uniquely defined by the lengths of its three edges. A Euclidean tetrahedron is not uniquely defined by the areas of its four faces. A tetrahedron has six degrees of freedom, for example the lengths of the six edges. This means that the volume is still a free parameter when the areas of the faces are fixed. In the case that is treated here, where all four faces have equal area, the volume is maximal for the regular tetrahedron, but it can be arbitrarily small. Consider for example the following tetrahedron that is defined by the coordinates of its four corners and that is pictured here as seen from the positive $z$-direction. This tetrahedron has four faces of equal area and a small volume:


$$
\begin{array}{ll}
\vec{v}_{1}=(0 ; 0 ; 0) & \vec{v}_{2}=(1 ; 0 ; 0) \\
\vec{v}_{3}=(0.001 ; 1 ; 0) & \vec{v}_{4}=(0.999 ; 0.998 ; 0.06) \\
\text { Area }=0.5 & \text { Volume }=0.01
\end{array}
$$

This means that every point in the area below the line and including the line in figure 3 corresponds to a valid classical tetrahedron. Every point below the line corresponds in fact to a 1-dimensional family of tetrahedra. When the face areas and volume are specified, there is still one degree of freedom left, for example the length of the edge from $\vec{v}_{1}$ to $\vec{v}_{2}$.

This is different in the quantum theory. The area operator limits the face areas of a quantum tetrahedron to discrete values, because the spin values are discrete. The volume operator limits the allowed tetrahedra to the ones that have one of the discrete volumes that are plotted as the crosses in figure 3. All these quantum tetrahedra are Euclidean tetrahedra. A spin network that consists of only 4 -valent nodes resembles then the piecewise flat geometry of a 3-dimensional Regge space, except for the following:

The state of a node is uniquely determined by the observables area and volume. This state is labeled by the spins on the links that connect in the node and by the internal structure and internal spin of the node. Therefore, the area operator and the volume operator form a complete set of observables for a node of a spin network. This means that you can know at the same time the face areas and the volume of a quantum tetrahedron precisely, but the last degree of freedom is then quantum spread. A classical angular momentum vector $\vec{L}$ has three degrees of freedom. Quantum-mechanically only two degrees of freedom can be known at the same time: $L^{2}$ and $L_{z}$. Here the classical tetrahedron has six degrees of freedom and for the quantum tetrahedron only five of these can be known at the same time.

To sum up, the area and volume operators of loop quantum gravity define quantum building blocks of 3-dimensional space that are Euclidean tetrahedra with discretized values for face area and volume that are quantum spread in at least one degree of freedom.

In this formalism the building blocks of quantum space do not have a well defined shape. As a result it is difficult to construct meaningful spin networks. The goal is to construct spin networks that have the geometrical properties of a spatial slice of a classical spacetime, but there are no results in this direction. It is not clear how spin networks should be used.

The second part of this thesis describes a model for quantum space that is based on a different interpretation of the volume operator and in this model it is possible to make the connection to classical metrics.


Figure 4: On the left a 6 -valent node of a spin network and the piece of volume that it represents. On the right a 3 -dimensional sketch of the node decomposition, where the dashed lines are internal links.

## 2 A New Model for Quantum Space

In this part a new model for quantum space in loop quantum gravity is introduced. In this model it is possible to put the information of a metric into a spin network. The spin network defines then the microscopic structure of this metric. It uses spin networks with 6 -valent nodes in a cubic structure and all the links have spin 1 . First the 6 -valent node is studied in section 2.1. In section 2.2 the conventional volume operator is re-interpreted as the creation operator for volume. New operators that measure the geometrical quantities length, area and volume in a spin network are defined in section 2.3. In this new formalism, spin network states can be constructed that have the geometrical properties of the spatial part of a classical metric. How this can be done is discussed in section 2.4. This method is used to construct spin networks for flat space in section 2.5, for the Schwarzschild metric in section 2.6 and for plane gravitational waves in section 2.7 .

### 2.1 The 6-Valent Node

In loop quantum gravity a spin network represents a piece of a 3 -dimensional spacelike slice of spacetime. Nodes represent pieces of volume and links indicate how these volumes are connected to each other and they represent the contact surface between the pieces of volume. This was discussed in section 1.1 .

The volume eigenvalues of a 4 -valent node, the quantum version of a tetrahedron, were calculated in section 1.8, A disadvantage of using the tetrahedron as the fundamental building block is the fact that the regular tetrahedron can not fill 3-dimensional Euclidean space. The only regular
polyhedron that can fill $\mathbb{R}^{3}$ is the regular hexahedron, the cube. For this reason the 6 -valent node is an interesting object to study.

The 6 -node is the dual of a hexahedron, which is a volume that is bounded by six faces. The general hexahedron is a complicated object. For example, a hexahedron can be constructed as follows: Start with a tetrahedron and cut off one corner. This is a pentahedron. Then cut off one of the corners of the pentahedron. The resulting object is a hexahedron. In general the six faces are a mix of triangles, squares and pentagons. Because of its complexity, the physical relevance of such a general hexahedron is unclear.

More interesting than the general hexahedron is the parallelepiped, which has six faces that are parallelograms. One can fill 3-dimensional space by repeating the same parallelepiped in all directions. It is also always clear which two faces are opposite faces, because they are parallel to each other, and these opposite faces have the same area.

It was discussed in section 1.2 that a node of a spin network state carries an internal decomposition into 3 -valent nodes. Multiple decompositions are possible for a 6 -valent node, but the properties of the parallelepiped lead to a natural choice: Connect two links that represent opposite faces together in a 3 -valent node. For the three pairs of opposite faces this gives three internal links that can be joined in an internal node. This is shown in figure 4 for a special parallelepiped, the cube.

This is the natural choice for the decomposition, since no additional information is needed to decide which two links are connected to each other in the decomposition.

## Spin Network Wave Function

A spin network state was defined in equation (34) as an unembedded spin network where the links are oriented and carry a half-integer spin value, and where every node has an internal decomposition into 3 -valent nodes. A 6 -valent node with the decomposition of figure 4 looks as follows in a spin network state:


The $j$ 's are spins on normal links and the $k$ 's are spins on internal links. The $n$ 's label the nodes. The spin network wave function was defined in equation (35). It assigns an $\mathrm{SU}(2)$ element to each link of a spin network.

Every 3 -valent node is represented in the wave function by a Wigner $3 j$ symbol and a link with spin $j$ is represented by an $\mathrm{SU}(2)$ matrix in the spin $j$ representation. An internal link is represented by the identity matrix. The arrows of the orientation of the links determine the index contraction: An arrow points away from an upper index. The order of the spins in a $3 j$-symbol was defined in section 1.4. The lowest link number goes first and normal links go before internal links. The 6 -node of equation (155) is represented in a spin network wave function as

$$
\begin{align*}
& \psi_{\mathrm{SNW}}\left(\left\{U_{l}\right\}\right) \sim\left(\begin{array}{ccc}
j_{x 1} & m_{x 2} & m_{x} \\
m_{x 1} & j_{x 2} & k_{x}
\end{array}\right)_{x}\left(\begin{array}{ccc}
j_{y 1} & m_{y 2} & m_{y} \\
m_{y 1} & j_{y 2} & k_{y}
\end{array}\right)_{y} \times \\
& \times\left(\begin{array}{ccc}
j_{z 1} & m_{z 2} & m_{z} \\
m_{z 1} & j_{z 2} & k_{z}
\end{array}\right)_{z}\left(\begin{array}{ccc}
k_{x} & k_{y} & k_{z} \\
m_{x} & m_{y} & m_{z}
\end{array}\right)_{i} \tag{156}
\end{align*}
$$

This expression does not include the links that are connected to the node. The internal links are represented by the summations over the indices $m_{x}$, $m_{y}$ and $m_{z}$.

In section 1.3 it was described how a spin network wave function can be represented by a 2 -dimensional diagram. Section 1.4 contains examples of these wave function diagrams. A $3 j$-symbol is represented by a node with three links. A node sign indicates how the indices are read off from the node, where a + -sign means counterclockwise. The arrows point again away from an upper index and if a link has a single arrow, then it represents the identity matrix. If it has a double arrow, it represents an $\operatorname{SU}(2)$ matrix. The diagrammatic version of the wave function of equation 156 is


The spin of a link is a measure for the area of the surface that it represents, see equation (144). For a 6 -node that represents a parallelepiped, the two non-internal links that are joined in a 3 -node of the decomposition need to have the same spin, since opposite faces of a parallelepiped have the same area. For example, $j_{x 1}=j_{x 2}=j_{x}$. The spin values on the internal links are limited by the Clebsch-Gordan conditions of equation (51): The sum of three connected spins must be integer and no spin can be larger than the sum of the other two. The possible spins on the internal links are

$$
\begin{equation*}
k_{a}=0,1, \ldots, 2 j_{a} \tag{158}
\end{equation*}
$$

The three internal links are connected in a node, so $k_{x}, k_{y}$ and $k_{z}$ also have to satisfy the condition

$$
\begin{equation*}
\left|k_{x}-k_{y}\right| \leq k_{z} \leq k_{x}+k_{y} \tag{159}
\end{equation*}
$$

The representation of a 6 -valent node in a wave function diagram as in equation (157) will be important in section 2.2 .

## Volume

The volume eigenvalues of a 4 -valent node were calculated in section 1.8 , First the classical volume of a tetrahedron was expressed in the vectors $L$ that are normal to the faces of the tetrahedron and have a length that is equal to the area of the face. Then these vectors were replaced by the $L$-operators that were discussed in sections 1.2 and 1.6, to arrive at the expression for the volume operator.

The volume of a Euclidean parallelepiped is related to the volume of a tetrahedron. Three edges that connect in one point define both a parallelepiped and a tetrahedron. The faces of the parallelepiped have twice the area of the faces of the tetrahedron and the parallelepiped has six times the volume of the tetrahedron. The expression for the volume of a tetrahedron in equation (145) leads then to an expression for the volume of a parallelepiped:

$$
\begin{equation*}
\text { Volume }{ }^{2}=6^{2} \frac{2}{9}\left(\frac{1}{2}\right)^{3}\left(\vec{L}_{1} \times \vec{L}_{2}\right) \cdot \vec{L}_{3}=\left(\vec{L}_{1} \times \vec{L}_{2}\right) \cdot \vec{L}_{3} \tag{160}
\end{equation*}
$$

The vectors $\vec{L}$ are here the vectors that belong to three faces of the parallelepiped that connect in one corner. They are normal to these faces and their length is equal to the area of the face. When the vectors $\vec{L}_{l}$ are replaced by the flux operators $\hat{L}_{l}^{i}$ the volume operator for a parallelepiped is

$$
\begin{equation*}
\text { Volûme }^{2}=\left(l_{0}\right)^{6} \hat{Q}=\left(l_{0}\right)^{6} \varepsilon_{i j k} \hat{L}_{1}^{i} \hat{L}_{2}^{j} \hat{L}_{3}^{k} \tag{161}
\end{equation*}
$$

The calculation of the volume eigenvalues of section 1.8 can be repeated for the parallelepiped by acting with the diagrammatic version of this volume operator on the wave function diagram of equation (157). The result for the 4 -valent node was given in equation (150) and similarly the result for the parallelepiped is:



Figure 5: Volume eigenvalues of the monochromatic 6-valent node (crosses) and the Euclidean volume of a cube (line).

The normalization constant for this node is $C_{j k l}=\sqrt{2 j+1} \sqrt{2 k+1} \sqrt{2 l+1}$, see section 1.5. The coefficients are

$$
\begin{align*}
& a_{l_{x} l_{y} l_{z}}\left(k_{x}, k_{y}, k_{z}\right)=\mathrm{i} \sqrt{6} \alpha\left(j_{x}\right) \alpha\left(j_{y}\right) \alpha\left(j_{z}\right)(-1)^{2 j_{x}+2 j_{y}+2 j_{z}}(-1)^{l_{x}+l_{y}+l_{z}} \times \\
& \times \sqrt{2 k_{x}+1} \sqrt{2 k_{y}+1} \sqrt{2 k_{z}+1} \sqrt{2 l_{x}+1} \sqrt{2 l_{y}+1} \sqrt{2 l_{z}+1} \times \\
& \times\left\{\begin{array}{ccc}
1 & k_{x} & l_{x} \\
j_{x} & j_{x} & j_{x}
\end{array}\right\}\left\{\begin{array}{ccc}
1 & k_{y} & l_{y} \\
j_{y} & j_{y} & j_{y}
\end{array}\right\}\left\{\begin{array}{ccc}
1 & k_{z} & l_{z} \\
j_{z} & j_{z} & j_{z}
\end{array}\right\} \times \sum_{h}(2 h+1)\left\{\begin{array}{ccc}
1 & k_{x} & l_{x} \\
l_{y} & l_{z} & h
\end{array}\right\}\left\{\begin{array}{ccc}
1 & k_{y} & l_{y} \\
k_{x} & h & k_{z}
\end{array}\right\}\left\{\begin{array}{ccc}
1 & k_{z} & l_{z} \\
h & 1 & 1
\end{array}\right\}
\end{align*}
$$

The sum over $h$ goes in integer steps from $h_{\min }$ to $h_{\text {max }}$ :

$$
\begin{equation*}
h_{\min }=\max \left\{\left|l_{z}-1\right|,\left|k_{x}-l_{y}\right|\right\} \quad h_{\max }=\min \left\{l_{z}+1, k_{x}+l_{y}\right\} \tag{164}
\end{equation*}
$$

The monochromatic 6 -valent node connects six links that carry the same spin value, $j_{x}=j_{y}=j_{z}=j$. In that case is the matrix version of the operator $Q$ a Hermitian matrix with real eigenvalues $q$. The volume eigenvalues of the monochromatic 6 -valent node are then

$$
\begin{equation*}
\text { Volume }=\left(l_{0}\right)^{3} \sqrt{|q|} \tag{165}
\end{equation*}
$$

The Euclidean cube has a certain relation between the area of its faces and its volume. With the area spectrum of equation (144) the relation between the spin $j$ and the volume of a Euclidean cube is

$$
\begin{equation*}
\text { Area }=\left(l_{0}\right)^{2} \sqrt{j(j+1)} \quad \text { Volume }=\text { Face Area }{ }^{3 / 2}=\left(l_{0}\right)^{3}(j(j+1))^{3 / 4} \tag{166}
\end{equation*}
$$

The volume eigenvalues of the monochromatic 6 -valent node up to spin $j=3$ are plotted in figure 5 together with this relation.

The interpretation of this volume spectrum is similar to the interpretation of the volume spectrum of the 4 -valent node. There is no equal-sided parallelepiped with more volume than the cube, given a certain face area, and the volume can be as small as zero. Every equal-sided Euclidean parallelepiped corresponds to a point on or below the line in figure 5. The face areas are discretized by the area operator and the volume operator again selects the Euclidean parallelepipeds that correspond to the crosses as the volume eigenstates of the monochromatic 6 -node in the quantum theory. The parallelepiped and the tetrahedron both have six degrees of freedom, since they are both defined by three edges that connect in one corner. Again the area and volume operators do not describe all these degrees of freedom. When these 6 -valent nodes are used as the quantum building blocks of space is it again unclear how one can make the connection between spin networks and classical space in an appropriate classical limit.

### 2.2 The Volume Operator creates Volume

It is difficult to construct physically meaningful spin network states based on the building blocks defined by the volume spectra of section 1.8 and the last section. Therefore, a different interpretation of the volume operator could be justified.

The diagrammatic approach to loop quantum gravity makes the following observation possible: The volume operator for the parallelepiped was defined in equation 161). It is proportional to $\varepsilon_{i j k} \hat{L}_{1}^{i} \hat{L}_{2}^{j} \hat{L}_{3}^{k}$. The graphical action of this operator on a wave function diagram was derived in section 1.6. This graphical action is a 3 -valent node that grabs three existing links. Equation (127) is repeated here:


On the other hand, a 6 -valent node with the decomposition of the figure on page 50 is represented in a wave function diagram as in equation (157):


When these two diagrams are compared it turns out that the diagrammatic version of the volume operator has the same structure as a 6 -valent node. Three links connect in a 3 -valent node and the other ends of these links connect to three different links. The conclusion is that the volume operator adds a 6 -valent node to a spin network. Since a node of a spin network represents a piece of volume, the volume operator adds volume to a spin network. In this way the volume operator does not measure the volume, it can be interpreted as the creation operator for volume.

The volume operator adds a node to a spin network as follows: The volume operator in equation 167 grabs three links. The spin 1 links in that diagram are interpreted as internal links. In an embedding of a spin network, the internal structure of a node is not visible. This means that the links on which the volume operator acts are pulled together to form a new node. For example, the volume operator that acts on the links $l_{1}, l_{3}$ and $l_{5}$ of the following spin network creates the new 6 -valent node at the center:


The expression for the volume operator for the 6 -valent node was derived from the classical expression for the volume of a Euclidean parallelepiped, see equation (160). Therefore, the next assumption is that the node that is created by the volume operator represents a piece of Euclidean space. It follows from the diagram in equation (167) that the 6 -valent node with spin 1 on the three internal links is a piece of flat space.

This interpretation of the volume operator is only possible when the $L$-operator acts inside a link and not at an endpoint of a link. This is the reason why this choice was made in equation (24).

To summarize, this new interpretation of the volume operator is:

- The volume operator does not measure the volume, it is the creation operator for volume, since it adds a 6 -valent node to a spin network.
- The 6 -valent node with the decomposition of the figure on page 50 with spin 1 on the three internal links represents a piece of flat space.

Two remarks have to made. The graphical version of the volume operator is interpreted as the creation operator for volume, but this graphical operator is not derived from the expression for the volume, but from the expression for the square of the volume. Also the numerical constant in front of the last diagram in equation (167) is not used in this interpretation.

In the next section a new set of operators that measure area, volume and length in a spin network will be defined in such a way that the 6 -valent node with spin 1 on the internal links has the geometrical properties of a piece of flat space.


Figure 6: A 2-dimensional picture of a spin network with a cubic structure. The real spin network is 3-dimensional. All the links have spin 1. The dashed lines are the boundaries of the cubic regions represented by the nodes.

### 2.3 New Geometrical Operators

In the last section the standard volume operator of loop quantum gravity was interpreted as the creation operator for volume. It creates a 6 -valent node that has spin 1 on the internal links and this node is interpreted as a piece of flat space. Because of this alternative interpretation there is no longer an operator that measures volume. In this section a new set of operators will be defined that measure the geometrical quantities length, area and volume in a spin network.

The goal of the following construction is to create spin networks that have the geometrical properties of classical space. General relativity differentiates between coordinate distances and physical distances. The 6 -valent nodes are placed in a cubic structure as in figure 6. The distances in this figure are coordinate distances, where the coordinates are chosen such that coordinate distances are equal to physical distances when the masses go to zero and space is flat. The spins on the internal links of a node define the node state and these node states determine the physical distances in the spin network. Operators extract this geometrical information from the spin network. But regardless of the node state, a node always represents a cube in coordinate distances as in figure 6.

The 6 -valent node with spin 1 on the internal links that is created by the volume operator is interpreted as a piece of flat space and this node state is assumed to be the central state of the spectrum. This is realized if all the links carry a spin 1 . That is, all the normal links that are visible in figure 6, not the internal links of the node decompositions. The spins on the internal links are then limited to $k=0,1,2$, see equation 158 , and the 6 -node with spin 1 on the internal links is at the center of the spectrum.


Figure 7: On the left a 6 -valent node and the definitions of length, area and volume. On the right a 3 -dimensional sketch of the node decomposition, where the dashed lines are internal links.

The node with spin 1 on the internal links must be physically a Euclidean cube, while the other possible node states allow the nodes to have physical volumes that differ from the Euclidean volume, while these nodes still represent a cube in coordinate distances. A spin network can represent flat space in this way and it can also deviate from it as is allowed in general relativity.

## Length, Area and Volume

A set of geometrical operators will now be constructed that implement the above ideas. On the right in figure 7 the decomposition of the 6 -valent node is shown. The three internal links are each associated with one of the three orthogonal directions defined by the cube. For example, the internal link that is labeled by $y$ is connected to the two links that run in the $y$-direction. The following model is based on this assumption:

The spin on the internal link that is labeled by $y$ is related to the physical length in the $y$-direction inside the cube between the two opposing faces.

The definition of the length in the $y$-direction is shown on the left in the figure 7. Suppose that there is a length operator that measures this distance. It acts on an internal link and its eigenvalue is a function of the spin on that link. This length operator defines an area operator that measures the area in the $x-y$ plane inside the node. Let the length operator act on the internal links labeled by $x$ and $y$ and take the product:

$$
\begin{equation*}
\hat{\operatorname{Area}}(x, y)=\prod_{a=x, y} \hat{\operatorname{Length}}(a) \tag{169}
\end{equation*}
$$

This area operator is also defined for the combinations $(x, z)$ and $(y, z)$. The area that is measured by the operator $\operatorname{Area}(x, z)$ is shown in figure 7 .

Let the length operator act on all three internal links. The product is the physical volume represented by the node:

$$
\begin{equation*}
\text { Volûme }=\prod_{a=x, y, z} \hat{\text { Length }(a)} \tag{170}
\end{equation*}
$$

This volume is shown as the cube in figure 7 .

## Length Operator

An area and volume operator are now defined in terms of a length operator. The next step is to construct this length operator. It should act on a link and its eigenvalues should be a function of the spin on that link. The most simple form for this operator is a linear combination of the square of the flux operator and the identity operator:

$$
\begin{equation*}
\text { Length }=l_{P} a\left(\hat{L}^{2}+b \hat{\mathbb{1}}\right) \quad l_{P}=\sqrt{\frac{\hbar G}{c^{3}}} \approx 10^{-35} m \tag{171}
\end{equation*}
$$

The Planck length $l_{P}$ is included, since the granulation of space is expected to happen around this length scale. $a$ and $b$ are free parameters. $a$ sets the scale relative to the Planck length.

A link that carries a spin $k$ is an eigenstate of the operator $\hat{L}^{2}$ with eigenvalue $k(k+1)$, see equation (121). This link is then also an eigenstate of this length operator and the length eigenvalue is:

$$
\begin{equation*}
\operatorname{Length}(k)=l_{P} a(k(k+1)+b) \tag{172}
\end{equation*}
$$

The parameter $b$ can be fixed by making an assumption about the spectrum of the length operator. Internal links can have spin 0,1 or 2 and spin 1 is associated with flat space. At the beginning of this section it was assumed that the Euclidean length element is the center of the spectrum. Now it is assumed that the length eigenvalues for the spins 0 and 2 are inversely proportional to each other. The reason for this choice will be given on the next page. Inverse proportionality means that if the maximum length is twice the Euclidean length, that then the minimum length is half the Euclidean length:

$$
\begin{equation*}
\frac{\operatorname{Length}(k=2)}{\operatorname{Length}(k=1)}=\frac{\operatorname{Length}(k=1)}{\operatorname{Length}(k=0)} \tag{173}
\end{equation*}
$$

Equation (172) gives the following length eigenvalues:

$$
\begin{align*}
& \operatorname{Length}(k=0)=l_{P} a b \\
& \operatorname{Length}(k=1)=l_{P} a(2+b)  \tag{174}\\
& \operatorname{Length}(k=2)=l_{P} a(6+b)
\end{align*}
$$

With these length eigenvalues, the requirement of equation (173) leaves only one possible value for $b$ :

$$
\begin{equation*}
\frac{6+b}{2+b}=\frac{2+b}{b} \quad \Rightarrow \quad b=2 \tag{175}
\end{equation*}
$$

An equidistant length spectrum is not possible with the linear combination of equation (171). If the length scale of the theory is defined as $l_{L Q G}=4 l_{P} a$, the operators for length, area and volume are

$$
\begin{gather*}
\operatorname{Length}(n, a)=l_{L Q G}\left(\hat{L}_{a}^{2}+2 \hat{\mathbb{1}}\right) / 4  \tag{176}\\
\operatorname{Area}(n, a, b)=\prod_{2 \text { internal links }} \operatorname{Length}(n, c) \quad \hat{\operatorname{Lolume}}(n)=\prod_{3 \text { internal links }} \operatorname{Length}(n, a)
\end{gather*}
$$

These operators act on a node $n$ and on internal links of that node that are labeled by $a=x, y, z$.

## Inverse Proportionality

The minimum and maximum length eigenvalues were required to be the inverse of each other in equation 173 . This choice is based on the fact that one often encounters inverses when working with general relativity. For example in the following situation: An observer $A$ at infinity in the Schwarzschild space uses coordinates such that coordinate distances are equal to physical distances in his vicinity. At different locations this is no longer the case. In particular the relation between the coordinate distance and the physical distance on the radial line through the observer's location is

$$
\begin{equation*}
d s^{2}=g_{r r}(r) d r^{2}=\frac{d r^{2}}{1-r_{s} / r} \tag{177}
\end{equation*}
$$

A second observer $B$ who is located along the same radial line and is stationary at the location $r=r_{0}$, uses a different coordinate $z$ along this radial line. The radial length element in this coordinate is

$$
\begin{equation*}
d s^{2}=g_{r r}(r) d r^{2}=g_{r r}(r(z))\left(\frac{\partial r}{\partial z}\right)^{2} d z^{2}=g_{z z}(z) d z^{2} \tag{178}
\end{equation*}
$$

Observer $B$ chooses the coordinate $z$ such that he is at the location $z=0$ and such that coordinate distances are equal to physical distances in his vicinity. The linear coordinate transformation that achieves this is

$$
\begin{equation*}
z(r)=\frac{r-r_{0}}{\sqrt{1-r_{s} / r_{0}}} \quad r(z)=z \sqrt{1-r_{s} / r_{0}}+r_{0} \tag{179}
\end{equation*}
$$

The metric component that observer $B$ uses to describe radial distances is

$$
\begin{equation*}
g_{z z}(z)=g_{r r}(r(z))\left(\frac{\partial r}{\partial z}\right)^{2}=\frac{r_{0}+z \sqrt{1-r_{s} / r_{0}}}{r_{0}+z / \sqrt{1-r_{s} / r_{0}}} \tag{180}
\end{equation*}
$$



$$
\begin{array}{rcc}
r= & r_{0} & \infty \\
g_{r r} & = & 1 /\left(1-r_{s} / r_{0}\right) \\
g_{z z} & = & 1
\end{array}
$$

Figure 8: Two observers $A$ and $B$ on the same radial line in Schwarzschild space and the values of the metric components that they use.

This metric component has the property that $g_{z z}(z=0)=1$ as required and in the limit where $z$ goes to infinity it is

$$
\begin{equation*}
\lim _{z \rightarrow \infty} g_{z z}(z)=1-r_{s} / r_{0} \tag{181}
\end{equation*}
$$

When this limit is compared to $g_{r r}\left(r_{0}\right)$ in equation 177 the conclusion is that the deformation of the length element that the two observers see at each other's location is the inverse of each other. This is illustrated in figure 8 . It seems therefore appropriate to use a length spectrum where the eigenvalues are inversely proportional to each other.

## Spectra

Operators that measure the physical length, area and volume in a spin network are now defined. These operators commute with each other and the node states are simultaneous eigenstates of length, area and volume. The eigenvalues are given on page 63 .

Some remarks about these spectra. The length operator measures the physical distance inside a coordinate cube between two opposing faces. The spins on the three internal links contain this length information for the three pairs of opposing faces. The length operator acts on a node $n$ and on an internal link of that node with spin $k_{a}$. It is denoted as $\operatorname{Length}(n, a)$. The spin on an internal link can be 0,1 or 2 and the length eigenvalues are respectively $1 / 2,1$ and 2 times the loop quantum gravity length scale. A higher spin value gives more length in that direction.

A special node decomposition was defined for the nodes of the cubic spin network in section 2.1. Once the decomposition of a node is given, the node states are labeled by the three spins on the internal links. These three internal spins have to satisfy the Clebsch-Gordan conditions, since the internal links connect in an internal 3-node. One internal spin can not be larger than the sum of the other two, see equation 159 . This leads to the fifteen volume eigenstates for a node that are listed in the table. The

Eigenvalues for Length, Area and Volume
$\operatorname{Length}(n, a)=l_{L Q G}\left(k_{a}\left(k_{a}+1\right)+2\right) / 4$
$\operatorname{Area}(n, a, b)=\prod \operatorname{Length}(n, c) \quad \operatorname{Volume}(n)=\prod \operatorname{Length}(n, a)$
$\underset{c=a, b}{2 \text { internal links }} 3$ internal links
Length

| Length |  |
| :---: | :---: |
| Spin $k$ | Length $/ l_{L Q G}$ |
| 0 | $1 / 2$ |
| 1 | 1 |
| 2 | 2 |


| Area |  |
| :---: | :---: |
| Spins $\left(k_{a}, k_{b}\right)$ | Area $/\left(l_{L Q G}\right)^{2}$ |
| $(0,0)$ | $1 / 4$ |
| $(0,1)$ | $1 / 2$ |
| $(1,0)$ | $1 / 2$ |
| $(1,1)$ | 1 |
| $(0,2)$ | 1 |
| $(2,0)$ | 1 |
| $(1,2)$ | 2 |
| $(2,1)$ | 2 |
| $(2,2)$ | 4 |

Volume
Number of states $=15$

| State $\left\|k_{x}, k_{y}, k_{z}\right\rangle$ | Volume $/\left(l_{L Q G}\right)^{3}$ |
| :---: | :---: |
| $\|0,0,0\rangle$ | $1 / 8$ |
| $\|0,1,1\rangle$ | $1 / 2$ |
| $\|1,0,1\rangle$ | $1 / 2$ |
| $\|1,1,0\rangle$ | $1 / 2$ |
| $\|1,1,1\rangle$ | 1 |
| $\|0,2,2\rangle$ | 2 |
| $\|2,0,2\rangle$ | 2 |
| $\|2,2,0\rangle$ | 2 |
| $\|2,1,1\rangle$ | 2 |
| $\|1,2,1\rangle$ | 2 |
| $\|1,1,2\rangle$ | 2 |
| $\|1,2,2\rangle$ | 4 |
| $\|2,1,2\rangle$ | 4 |
| $\|2,2,1\rangle$ | 4 |
| $\|2,2,2\rangle$ | 8 |

physical volume that is measured by the volume operator is equal to the product of the three length eigenvalues of the node.

An area eigenvalue depends on the spins on two internal links. The physical area is the product of the length eigenvalues of these two links.

In the previous section it was argued that the 6 -valent node with spins 1 on the three internal links should be a piece of flat space. This is indeed implemented by these geometrical operators. A node in the node eigenstate $|1,1,1\rangle$ has a volume of $\left(l_{L Q G}\right)^{3}$, the distance between the three pairs of opposing faces is $l_{L Q G}$ and the cross-sections have an area of $\left(l_{L Q G}\right)^{2}$. This describes a Euclidean cube with edge length $l_{L Q G}$. When an internal spin differs from 1 , the length, area and volume eigenvalues change in sync. The node states other than $|1,1,1\rangle$ do not give the node a different shape in a coordinate picture. These nodes represent pieces of space that are not Euclidean.

## Discussion

This model is based on the interpretation of the conventional volume operator as the creation operator for volume. In this section the following elements were added: The 6 -valent nodes are placed in a cubic structure, the the non-internal links have spin 1 , there is a connected set of operators for length, area and volume, a length operator was constructed and the free parameter in this operator was fixed. Only the length scale $l_{L Q G}$ remains undetermined.

In conventional loop quantum gravity, nodes represent pieces of volume and links represent the contact surfaces between these pieces of volume. The conventional area operator measures the area of these contact surfaces. In this model, areas are not defined for the contact surfaces, but inside the volume that is represented by a node. This construction seems justified for the following reason: A surface is a collection of points in space. When space is granulated, the smallest element in space is no longer a point, but a small volume. A surface is then a collection of these building blocks and it is no longer infinitely thin, but it has a finite thickness. The area can then be defined inside a building block. In the same way, the thinnest line in a granulated space still has a finite cross-section. In this model a link represents only the way that building blocks are connected, not a contact surface between them.

In section 1.2 , around equation (49), the Hamiltonian constraint operator of loop quantum gravity was discussed. This constraint describes an unknown reduction of the state space. The model that is described in this section is defined on a reduced state space. The original state space consists of all spin networks with nodes of arbitrary valence, with links that can run between any two nodes in the network and with any spin value on the links. In the model of this section all the nodes are 6 -valent nodes that have only
connections with their direct neighbours in the cubic structure and all the links have spin 1. It is unclear whether the Hamiltonian constraint describes this reduction, but the node states of this section have interesting properties:

The gravitational field has two degrees of freedom in each point of space. This is discussed for example at the end of chapter 10 in Wald. The node states on page 63 have also two degrees of freedom. First there are seven distinct node states and second, these node states have an orientation. They can be oriented in three possible ways. For example, the node state $|2,1,1\rangle$ can also be oriented as $|1,2,1\rangle$ or $|1,1,2\rangle$.

A plane gravitational wave is invariant under a rotation of $\pi$ around its direction of propagation, see for example section 35.6 of [Misner, Thorne and Wheeler]. This property is related to the spin 2 of the graviton. The node states have this same property. An internal spin is connected to an axis, not to a vector. The node state $\left|k_{x}, k_{y}, k_{z}\right\rangle$ is unchanged if it is rotated over $\pi$ along the $x, y$ or $z$-axis.

### 2.4 From a Classical Metric to a Spin Network

In the last section an operator was defined that measures physical lengths in a spin network. In general relativity it is the metric tensor that contains information about physical distances in space. A connection can now be made between classical metrics and spin networks. A spin network is a model for the microscopic structure of a 3-dimensional timeslice of spacetime. The goal is to construct spin network states that have the geometrical properties of the spatial part of a classical metric.

## The Metric Operator

The components of the metric are functions of the coordinates. These functions have two influences: the coordinate system and the curvature of space. For distances in the timeslice, these two contributions can be split when the spatial part of the metric is diagonal:

$$
\begin{equation*}
d s^{2}=\sum_{a=1}^{3} g_{a a}(\vec{x})\left(d x^{a}\right)^{2}=\sum_{a} q_{a}(\vec{x}) f_{a}(\vec{x})\left(d x^{a}\right)^{2} \tag{182}
\end{equation*}
$$

The functions $f_{a}(\vec{x})$ contain the information about the coordinate system. They are the metric components of flat space in these coordinates. The functions $q_{a}(\vec{x})$ describe the deviation from flat space. For example, for the Schwarzschild metric these functions are

$$
\begin{array}{ll}
f_{r}(\vec{r})=1 \quad f_{\theta}(\vec{r})=r^{2} \quad f_{\phi}(\vec{r})=r^{2} \sin ^{2} \theta  \tag{183}\\
q_{r}(\vec{r})=1 /\left(1-r_{s} / r\right) & q_{\theta}(\vec{r})=q_{\phi}(\vec{r})=1
\end{array}
$$

A length operator was defined in the last section that measures the length in a node in one of three directions. The eigenvalue of this operator depends on the spin of one of the internal links of the node. A node in the node state $|1,1,1\rangle$ is a piece of flat space and a deviation of a spin from 1 gives the node more or less length in that direction. The operator version of the function $q_{a}(\vec{x})$ is the operator that measures the square of the length divided by the square of the Euclidean length. It acts on a node $n$ :

$$
\begin{equation*}
\hat{q}_{a}(n)=\frac{\text { Length }(n, a)^{2}}{\operatorname{Length}(k=1)^{2}}=\frac{\text { Length }(n, a)^{2}}{\left(l_{L Q G}\right)^{2}} \tag{184}
\end{equation*}
$$

The length eigenvalues of a node are defined for three orthogonal directions. If the coordinate system that is used is orthogonal and the six links that connect to a node $n$ at the coordinate location $\vec{x}$ are aligned with the coordinate directions, then the metric operator for this node is

$$
\begin{equation*}
\hat{g}_{a a}(n(\vec{x}))=f_{a}(\vec{x}) \hat{q}_{a}(n) \tag{185}
\end{equation*}
$$

When this operator acts on a node eigenstate of page 63, the eigenvalues are the three components of the metric that are defined for the node.

## From a Classical Metric to a Spin Network State

In general relativity a mass-energy distribution determines the geometry of spacetime. Similarly, a theory of quantum gravity should relate a mass distribution to the quantum structure of space. This is currently not possible, but if a spin network is constructed that has the geometrical properties of a classical metric, then this spin network defines the microscopic structure of the classical space.

In this model the connection between a classical metric and a spin network state can only be made when the coordinate system that is being used satisfies two requirements in the region of interest:

- The coordinate system must be orthogonal.
- The metric must be diagonal in these coordinates.

Since the metric is a symmetric tensor, the orthogonal coordinate system can always be rotated at every point such that the metric is diagonal everywhere.

Given such a coordinate system, replace in a region of space the continuous space by a cubic spin network, such that the orthogonal directions of the links in the spin network are aligned with the coordinate directions. The geometrical properties of a spin network are not encoded in the structure of the spin network, since this structure is always cubical. The geometrical properties are encoded in the node states of the spin network.

A spin network state has the geometrical properties of the spatial part of a classical metric if, for all three orthogonal directions, the expectation value of the length operator is equal to the square root of the classical curvature function $q_{a}(\vec{x})$ :

$$
\begin{equation*}
\frac{\langle n| \operatorname{Length}(n, a)|n\rangle}{l_{L Q G}}=\sqrt{q_{a}(\vec{x})} \tag{186}
\end{equation*}
$$

This equation connects the node states of a spin network to a classical metric. It can be used to construct a spin network state that has the geometrical properties that are described by the metric. The expectation value is taken with respect to the node state $|n\rangle$, which is a superposition of the node eigenstates that are listed on page 63 .

$$
\begin{equation*}
|n\rangle=\sum c_{n}\left(k_{1}, k_{2}, k_{3}\right)\left|k_{1}, k_{2}, k_{3}\right\rangle \tag{187}
\end{equation*}
$$

The eigenvalue of the length operator on an internal link with spin $k_{a}$ were also given on page 63 for the three possible spin values:

$$
\begin{array}{c|ccc}
k_{a} & 0 & 1 & 2 \\
\hline \text { Length }(n, a) / l_{L Q G} & 1 / 2 & 1 & 2
\end{array}
$$



$$
\square=|1,1,1\rangle
$$

Figure 9: A 2-dimensional picture of a spin network in the $e=0$ flat state. One small square is a building block of space that is represented by one node in the cubic spin network. All the nodes are in the state $|x, y, z\rangle=|1,1,1\rangle$.

### 2.5 Flat Space

The formalism of the last section can be applied to flat space. Cartesian coordinates are orthogonal coordinates in a spatial slice of constant time and the metric of flat space is diagonal in these coordinates, $g_{a a}(\vec{x})=f_{a}(\vec{x})=$ $q_{a}(\vec{x})=1$ for $a=x, y, z$. All nodes are assumed to be in the same superposition of node eigenstates. Equation (186) is satisfied if this superposition leads to the following probability distribution for the internal spins:

$$
k_{a}= \begin{cases}0 & 2 e_{a}  \tag{188}\\ 1 \\ 2 & \text { with probability } \\ 1-3 e_{a} \\ e_{a}\end{cases}
$$

The parameters $e_{x}, e_{y}$ and $e_{z}$ have values in the interval $[0,1 / 3]$. The probability for spin 0 is twice the probability for spin 2 because the length eigenvalues for these spins are $1 / 2$ and 2 times $l_{L Q G}$. The expectation value for the length operator is

$$
\begin{equation*}
\langle n| \operatorname{Length}(n, a)|n\rangle / l_{L Q G}=2 e_{a} \cdot 1 / 2+\left(1-3 e_{a}\right) \cdot 1+e_{a} \cdot 2=1 \tag{189}
\end{equation*}
$$

Uniformity in the directions $x, y$ and $z$ requires that $e_{x}=e_{y}=e_{z}=e$. There are multiple possibilities for the flat state. If $e=0$ then all the internal links have spin 1. This means that all the nodes are in the state $|x, y, z\rangle=|1,1,1\rangle$ and the spin network contains only flat nodes, see figure 9 . This $e=0$ flat state will be used in the following sections.

The other possibility is a flat state with $e \neq 0$. The nodes are then in a superposition of node eigenstates. After the collapse of these superpositions, the spin network contains a mix of nodes in different node eigenstates, in such a way that the contributions to a length measurement of internal spins that differ from 1 cancel each other out for large distances.

### 2.6 The Quantum Schwarzschild Metric

Section 2.4 introduced a method to construct spin networks that have the geometrical properties of a classical metric. In this section this method will be applied to the Schwarzschild metric. The usual coordinates for the Schwarzschild metric are orthogonal except for the $(\theta=0)$-axis. The metric is diagonal in these coordinates:

$$
\begin{equation*}
d s^{2}=-\left(1-r_{s} / r\right) c^{2} d t^{2}+\frac{d r^{2}}{1-r_{s} / r}+r^{2}\left(d \theta^{2}+\sin ^{2} \theta d \phi^{2}\right) \tag{190}
\end{equation*}
$$

The curvature functions $q_{a}(\vec{x})$ for the Schwarzschild metric were given in equation 183):

$$
\begin{equation*}
q_{r}(\vec{r})=1 /\left(1-r_{s} / r\right) \quad q_{\theta}(\vec{r})=q_{\phi}(\vec{r})=1 \tag{191}
\end{equation*}
$$

The node states are determined from equation (186):

$$
\begin{gather*}
\frac{\langle n| \operatorname{Length}(n, r)|n\rangle}{l_{L Q G}}=\sqrt{q_{r}(\vec{r})}=\frac{1}{\sqrt{1-r_{s} / r}}  \tag{192}\\
\langle n| \operatorname{Length}(n, \theta)|n\rangle / l_{L Q G}=\langle n| \operatorname{Length}(n, \phi)|n\rangle / l_{L Q G}=1 \tag{193}
\end{gather*}
$$

When the $e=0$ flat state of section 2.5 is used, equation (193) states that $k_{\theta}=k_{\phi}=1$. The superposition of node eigenstates $|r, \theta, \phi\rangle$ that can solve equation 192 is

$$
\begin{equation*}
|n\rangle=a(\vec{r})|1,1,1\rangle+b(\vec{r})|2,1,1\rangle \tag{194}
\end{equation*}
$$

There are two equations for the coefficients $a$ and $b$. First, the node eigenstates are orthonormal, see section 1.5, and the node state $|n\rangle$ is normalized: $|a|^{2}+|b|^{2}=1$. Second, $|a|^{2}$ and $|b|^{2}$ are the probabilities that a node collapses to the state $|1,1,1\rangle$ or $|2,1,1\rangle$ :

$$
\begin{equation*}
\frac{\langle n| \operatorname{Length}(n, r)|n\rangle}{l_{L Q G}}=|a|^{2} \cdot 1+|b|^{2} \cdot 2=\frac{1}{\sqrt{1-r_{s} / r}} \tag{195}
\end{equation*}
$$

The solution for $a$ and $b$ is

$$
\begin{equation*}
|a(\vec{r})|^{2}=2-\frac{1}{\sqrt{1-r_{s} / r}} \quad|b(\vec{r})|^{2}=-1+\frac{1}{\sqrt{1-r_{s} / r}} \tag{196}
\end{equation*}
$$

The probabilities $|a|^{2}$ and $|b|^{2}$ for finding a node in a certain node eigenstate are plotted in figure 10 . For $r \rightarrow \infty,|a|^{2}=1$ and $|b|^{2}=0$ and this is the $e=0$ flat state. For $r=4 r_{s} / 3,|a|^{2}=0$ and $|b|^{2}=1$ and this this is also the case for $r<4 r_{s} / 3$, since probabilities are limited to the interval $[0,1]$.

Figure 11 shows spin networks for different values of the radial coordinate. A piece of continuous space is replaced by a cubic spin network that is aligned with the coordinate directions. One small square in the figure is


Figure 10: The probabilities for finding a node in the state $|r, \theta, \phi\rangle=|1,1,1\rangle$ (line) or the state $|2,1,1\rangle$ (dashed line) in a spin network that represents a piece of the Schwarzschild metric, as a function of the radial coordinate $r$. Plotted are the coefficients $|a|^{2}$ (line) and $|b|^{2}$ (dashed line) of equation 196 ). The dotted line is $\sqrt{-g_{t t}}$.


Figure 11: 2-dimensional pictures of spin networks for the Schwarzschild metric for different values of $r$. One small square is a building block of space that is represented by one node in the cubic spin network. A white square is a node in the state $|r, \theta, \phi\rangle=|1,1,1\rangle$ which is a piece of flat space and a gray square is a node in the state $|2,1,1\rangle$ which has more length in the radial direction.
a building block of space that is represented by one node. These building blocks are cubes in coordinate distances, while their physical geometrical properties are determined by the node states. The probabilities $|a|^{2}$ and $|b|^{2}$ are treated as constants for each spin network, since the size of the spin networks is much smaller than the distances over which the metric changes. Initially, all the nodes of a spin network are in the same superposition that is described by equations (194) and (196). When these superpositions collapse to node eigenstates following the probabilities $|a|^{2}$ and $|b|^{2}$, the result is a random pattern of nodes in the states $|r, \theta, \phi\rangle=|1,1,1\rangle$ and $|2,1,1\rangle$. This is shown in the figure. The spin network states in the figure are chosen such that the number of $|2,1,1\rangle$-nodes is close to the expectation value of $64|b|^{2}$.

## Discussion

In figure 11 coordinate distances are shown. The physical distances are determined by the node states. The spin network for $r \rightarrow \infty$ is the same as the spin network for the $e=0$ flat state. All the nodes are in the $|1,1,1\rangle$-state. The other spin networks in the figure also contain nodes in the $|2,1,1\rangle$-state. These nodes have twice as much physical length in the radial direction as the $|1,1,1\rangle$-nodes, while the physical length in the $\theta$ and $\phi$ direction is the same. Every column in the figure represents the same amount of coordinate length. When there are more $|2,1,1\rangle$-nodes there is more physical length in the same piece of radial coordinate length. This is what the Schwarzschild metric describes.

The smaller the value of $r$, the more $|2,1,1\rangle$-nodes there are in the spin network. For $r \leq 4 r_{s} / 3$ the spin network is saturated with $|2,1,1\rangle$-nodes. The microscopic structure defined by this model limits the deformation of the length element. The length element is constant inside $r=4 r_{s} / 3$. This means that there is a deviation from general relativity in the macroscopic region $r<4 r_{s} / 3$.

This picture of space is similar to the picture in the perturbative approach to quantum gravity, where the metric is split into a flat part and a small deviation:

$$
\begin{equation*}
g_{\mu \nu}=\eta_{\mu \nu}+h_{\mu \nu} \tag{197}
\end{equation*}
$$

The cubic spin network provides a flat background. With $|1,1,1\rangle$-nodes this is flat space. When a node excitation, an excitation of the gravitational field, is present this changes the geometrical properties. A difference with the perturbative approach is that the deviations from flat space are not small. A node in the $|2,1,1\rangle$-state is not close to flat space. Every node in these spin networks represents a piece of space that is either flat or maximally curved. Another difference is that in the perturbative approach the excitations of the gravitational field have a well defined momentum and here the excitations have a well defined position.

The distance between two nodes in the spin networks of figure 11 is only well defined when the two nodes are in the same row or column. For two nodes that are not in the same row or column the Pythagorean theorem can be applied to a path between the two nodes along a row and a column. The resulting physical distance is different for different paths because of the random pattern of node states. For larger distances the fluctuations average out and the Pythagorean theorem holds approximately.

## Weak Field Limit

It is interesting to see that in the weak field limit $r \gg r_{s}$, the probability to find a $|2,1,1\rangle$-node is proportional to the gravitational potential:

$$
\begin{equation*}
|b|^{2}=-1+\frac{1}{\sqrt{1-r_{s} / r}} \approx-1+1+\frac{r_{s}}{2 r}=\frac{r_{s}}{2 r}=\frac{G M}{c^{2} r}=-\Phi(\vec{r}) / c^{2} \tag{198}
\end{equation*}
$$

Here it is used that the Schwarzschild radius is $r_{s}=2 G M / c^{2}$. In the same limit the probability to find a $|1,1,1\rangle$-node is related to the time component of the classical metric:

$$
\begin{equation*}
|a|^{2}=2-\frac{1}{\sqrt{1-r_{s} / r}} \approx 2-1-\frac{r_{s}}{2 r}=1-\frac{r_{s}}{2 r} \approx \sqrt{1-\frac{r_{s}}{r}}=\sqrt{-g_{t t}(\vec{r})} \tag{199}
\end{equation*}
$$

$\sqrt{-g_{t t}}$ is plotted as the dotted line in figure 10 . This relation can be interpreted as follows: The lapse function of equation (3) is an average over many nodes, like the smooth classical metric, and its value is equal to the fraction of $|1,1,1\rangle$-nodes. Alternatively, if every node has its own eigentime like a stationary observer, then this eigentime advances only when the node is in the flat state $|1,1,1\rangle$ :

$$
\begin{equation*}
d \tau=\sqrt{-g_{t t}(\vec{r})} d t \approx|a(\vec{r})|^{2} d t \tag{200}
\end{equation*}
$$

The expressions for the weak field limits of $|a|^{2}$ and $|b|^{2}$ depend in fact on the spectrum of the length operator. If $\alpha$ is the maximum length eigenvalue relative to the flat length eigenvalue, then $|a|^{2}$ and $|b|^{2}$ are

$$
\begin{gather*}
|a|^{2} \approx 1-\frac{1}{2(\alpha-1)} \frac{r_{s}}{r} \quad|b|^{2} \approx \frac{1}{2(\alpha-1)} \frac{r_{s}}{r}  \tag{201}\\
\alpha=\frac{\operatorname{Length}(k=2)}{\text { Length }(k=1)}
\end{gather*}
$$

The weak field limits of (198) and (199) are only obtained when $\alpha=2$. The spectrum of the length operator was obtained in section 2.3 from the most simple expression for the length operator, equation (171), and the requirement of inverse proportionality. The value $\alpha=2$ was not put in by hand.

## Variation in Length Measurements

The node states are constructed such that the Schwarzschild metric is reproduced for distances that are much larger than the length scale $l_{L Q G}$. This means that in this model the classical limit is the limit where distances get large and consist of many nodes. The smooth classical metric is retrieved as the average over many nodes. The probabilistic nature of the collapse of the node superpositions causes deviations from the Schwarzschild metric on smaller length scales. When the physical length of a radial coordinate distance is measured in the spin networks of figure 11, different columns give different results because of the random pattern. When the physical length of a certain radial coordinate distance is measured, the probability function for the possible outcomes is the binomial distribution, since there are two different node states.

If the radial coordinate distance $l$ consists of $N$ nodes and $A$ is the number of nodes that are found in the state $|1,1,1\rangle$, then the standard deviation in the binomial distribution for $A$ is

$$
\begin{equation*}
\Delta A=\sqrt{N|a|^{2}|b|^{2}} \quad N=l / l_{L Q G} \tag{202}
\end{equation*}
$$

The coordinate distance $l$ is assumed to be much smaller than the scale on which the probabilities $|a|^{2}$ and $|b|^{2}$ change, such that they can be treated as constants. The physical length of this coordinate distance as a function of the number of flat nodes is

$$
\begin{equation*}
\operatorname{Length}(A)=(2 N-A) l_{L Q G} \tag{203}
\end{equation*}
$$

The standard deviation in the distribution for this length is

$$
\begin{equation*}
\Delta \text { Length }=\left|\frac{d \operatorname{Length}(A)}{d A}\right| \Delta A=l_{L Q G} \Delta A \tag{204}
\end{equation*}
$$

The expectation value for the physical length of $l$ is

$$
\begin{equation*}
\langle\operatorname{Length}(l)\rangle=l \sqrt{q_{r}(\vec{r})}=l / \sqrt{1-r_{s} / r} \tag{205}
\end{equation*}
$$

Finally the relative standard deviation for the physical length is

$$
\begin{equation*}
\frac{\Delta \text { Length }}{\langle\operatorname{Length}(l)\rangle}=\sqrt{\frac{l_{L Q G}}{l}} f(r) \quad f(r)=\left(2 \frac{r_{s}}{r}-3+3 \sqrt{1-\frac{r_{s}}{r}}\right)^{1 / 2} \tag{206}
\end{equation*}
$$

When the measured distance gets larger the relative standard deviation drops as $l^{-1 / 2}$. It is interesting to see that the relative standard deviation depends on the length scale $l_{L Q G}$. This makes the length scale of the microscopic structure of space potentially measurable.


Figure 12: The embedding of the radial component of the quantum-adjusted Schwarzschild metric (line) and the classical Schwarzschild metric (dashed line).

## Quantum Schwarzschild Metric

In this model the deformation of the length element away from the flat length element is limited. This can be seen directly from the spectrum of the length operator on page 63. The maximum length in one direction in a node is two times the Euclidean length. In the classical metric this maximum change to the radial length element is reached at

$$
\begin{equation*}
\sqrt{g_{r r}(\vec{r})}=\frac{1}{\sqrt{1-r_{s} / r}}=2 \quad \Rightarrow \quad r=\frac{4}{3} r_{s} \tag{207}
\end{equation*}
$$

The length element is constant for $r$ smaller than $4 / 3$ times the Schwarzschild radius. The radial length element of the quantum-adjusted Schwarzschild metric is then

$$
d s^{2}=\left\{\begin{array}{cc}
\frac{d r^{2}}{1-r_{s} / r} & r>4 r_{s} / 3  \tag{208}\\
4 d r^{2} & r \leq 4 r_{s} / 3
\end{array}\right.
$$

This metric can be visualized by drawing an embedding in the usual way, where the coordinate distance is along the horizontal axis and the physical distance is along a curve, see for example section 23.8 of [Misner, Thorne and Wheeler]. The distance along the curve $w(r)$ is the physical distance if

$$
\begin{equation*}
d s^{2}=\left[1^{2}+\left(\frac{d w(r)}{d r}\right)^{2}\right] d r^{2} \tag{209}
\end{equation*}
$$

The function $w(r)$ that reproduces the metric of equation (208) is

$$
w(r)=\left\{\begin{array}{cc}
2 r_{s} \sqrt{\frac{r}{r_{s}}-1} & r>4 r_{s} / 3  \tag{210}\\
\left(3 r-2 r_{s}\right) / \sqrt{3} & r \leq 4 r_{s} / 3
\end{array}\right.
$$

The embedding of the radial length element is shown in figure 12, for the classical and the quantum-adjusted metric.

The radial length element of equation (208) is finite everywhere. It does not diverge at the Schwarzschild radius or at $r=0$. This leads to an embedding in figure 12 that is continuous everywhere. This does not mean that the black hole does not exist. The time component of the metric still goes to zero.

The time component of the classical metric $g_{t t}(\vec{r})$ becomes zero at the Schwarzschild radius, while the radial component of the quantum metric changes its behavior at $r=4 r_{s} / 3$. Equation (199) showed that $|a|^{2}$ and $\sqrt{-g_{t t}}$ approach each other for large $r$. Both the radial and the time component of the metric change their behavior at $r=4 r_{s} / 3$ when $g_{t t}(\vec{r})$ is replaced by $-\left(|a|^{2}\right)^{2}$. With this adjustment to the time component, the quantum-adjusted Schwarzschild metric is

$$
d s^{2}=\left\{\begin{array}{cc}
-\left(2-\frac{1}{\sqrt{1-r_{s} / r}}\right)^{2} d t^{2}+\frac{\mathrm{d} r^{2}}{1-r_{s} / r}+r^{2} d \Omega^{2} & r>4 r_{s} / 3 \\
0 d t^{2}+4 d r^{2}+r^{2} d \Omega^{2} & r \leq 4 r_{s} / 3
\end{array}\right.
$$

Outside $r=4 r_{s} / 3$ this is approximately equal to the Schwarzschild metric and inside $r=4 r_{s} / 3$ the time component and the radial component of this metric are constant.

### 2.7 Gravitational Waves

In this section the method of section 2.4 will be applied to gravitational waves. The spacetime metric of the plane gravitational wave solution of linearized general relativity is, see for example chapter 35 of [Misner, Thorne and Wheeler]:

$$
g_{\mu \nu}\left(x^{\mu}\right)=\eta_{\mu \nu}+\left(\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{212}\\
0 & 0 & 0 & 0 \\
0 & 0 & A_{+} & A_{\times} \\
0 & 0 & A_{\times} & -A_{+}
\end{array}\right) \operatorname{Re}\left[e^{-\mathrm{i} \omega(c t-x)}\right]
$$

Minkowski coordinates are used and the wave propagates in the positive $x$-direction. $A_{+}$and $A_{\times}$are the amplitudes for the plus polarization and the cross polarization, $A_{+}, A_{\times} \ll 1$. The metric of a cross-polarized wave is not diagonal, but it can be made diagonal by rotating the spatial part of the coordinate system counter-clockwise around the $x$-axis over an angle of $\pi / 4$ :

$$
\begin{gather*}
R=R(\theta=\pi / 4)=\left(\begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array}\right)=\frac{1}{\sqrt{2}}\left(\begin{array}{cc}
1 & -1 \\
1 & 1
\end{array}\right)  \tag{213}\\
R^{-1}\left(\begin{array}{cc}
1 & A_{\times} \\
A_{\times} & 1
\end{array}\right) R=\left(\begin{array}{cc}
1+A_{\times} & 0 \\
0 & 1-A_{\times}
\end{array}\right)
\end{gather*}
$$

If both amplitudes $A_{+}$and $A_{\times}$are nonzero, then the coordinate system can still be rotated such that the metric is diagonal. In that case the angle depends on the ratio of the two amplitudes. The metric of a circularly polarized wave can be made diagonal by rotating the coordinate system differently for every value of $x$.

The metric of a plus-polarized wave, $A_{\times}=0$, is diagonal in equation (212). In the $(t=0)$-slice, the spatial part of this metric can be written in the product form of equation 182 :

$$
\begin{gather*}
g_{a a}(\vec{x})=f_{a}(\vec{x}) q_{a}(\vec{x}) \quad a=x, y, z  \tag{214}\\
f_{a}(\vec{x})=(1,1,1) \quad q_{a}(\vec{x})=\left(1,1+A_{+} \cos \omega x, 1-A_{+} \cos \omega x\right)
\end{gather*}
$$

There is no summation over the index $a$ in the first equation. The node state is determined from equation $\sqrt{186}$ ):

$$
\begin{equation*}
\frac{\langle n| \operatorname{Length}(n, a)|n\rangle}{l_{L Q G}}=\sqrt{q_{a}(\vec{x})} \approx\left(1,1+\frac{1}{2} A_{+} \cos \omega x, 1-\frac{1}{2} A_{+} \cos \omega x\right) \tag{215}
\end{equation*}
$$

The approximation in the last step is valid since $A_{+} \ll 1$. The superposition of node eigenstates $|x, y, z\rangle$ that can solve this equation is

$$
\begin{equation*}
|n\rangle=a|1,1,1\rangle+b|1,0,1\rangle+c|1,2,1\rangle+d|1,1,0\rangle+e|1,1,2\rangle \tag{216}
\end{equation*}
$$

The coefficients $a$ to $e$ are functions of $x$. The node eigenstates are orthonormal. The equations that follow from equation (215) are then

$$
\begin{align*}
|a|^{2}+|b|^{2}+|c|^{2}+|d|^{2}+|e|^{2} & =1 \\
\left(|a|^{2}+|d|^{2}+|e|^{2}\right) \cdot 1+|b|^{2} \cdot \frac{1}{2}+|c|^{2} \cdot 2 & =1+\frac{1}{2} A_{+} \cos \omega x  \tag{217}\\
\left(|a|^{2}+|b|^{2}+|c|^{2}\right) \cdot 1+|d|^{2} \cdot \frac{1}{2}+|e|^{2} \cdot 2 & =1-\frac{1}{2} A_{+} \cos \omega x
\end{align*}
$$

A solution is

$$
\begin{array}{rll}
|x, y, z\rangle= & |1,1,1\rangle: & |a|^{2}=1-\frac{3}{2} A_{+}  \tag{218}\\
& |1,0,1\rangle: & |b|^{2}=\frac{1}{2} A_{+}(1-\cos \omega x)=A_{+} \sin ^{2}\left(\frac{\omega x}{2}\right) \\
& |1,2,1\rangle: & |c|^{2}=\frac{1}{4} A_{+}(1+\cos \omega x)=\frac{1}{2} A_{+} \cos ^{2}\left(\frac{\omega x}{2}\right) \\
& |1,1,0\rangle: & |d|^{2}=\frac{1}{2} A_{+}(1+\cos \omega x)=A_{+} \cos ^{2}\left(\frac{\omega x}{2}\right) \\
& |1,1,2\rangle: & |e|^{2}=\frac{1}{4} A_{+}(1-\cos \omega x)=\frac{1}{2} A_{+} \sin ^{2}\left(\frac{\omega x}{2}\right)
\end{array}
$$

After the superpositions of equation (216) have collapsed to the node eigenstates, the spin networks are in states as the ones plotted in figure 13. In this figure the amplitude $A_{+}=4 / 36$ is used, such that the expectation value for the number of non-flat nodes is 6 for every 36 nodes. For $\omega x=0$ and $\omega x=\pi$ the physical length exceeds the coordinate length in one direction and there is less physical length than coordinate length in the other direction. For $\omega x=\pi / 2$ and $\omega x=3 \pi / 2$ the effects of the nodes that are not in the flat state cancel each other out for large distances, such that space appears flat. Two nodes in the state $|1,0,1\rangle$ cancel one node in the state $|1,2,1\rangle$, since the length eigenvalues of these nodes in the $y$-direction are $1 / 2$ and 2 times $l_{L Q G}$. Since $|b|^{2}+|d|^{2}=A_{+}$, the sum of the probabilities to find the node eigenstates $|1,0,1\rangle$ and $|1,1,0\rangle$ is the same for all $x$. The same is true for $|c|^{2}$ and $|e|^{2}$. When a node excitation is present inside a ring of test masses, this changes the physical distance between the test masses.


Figure 13: 2-dimensional pictures of spin networks for a plane gravitational wave, for different values of the $x$-coordinate. One small square is a building block of space that is represented by one node in the cubic spin network. The different types of squares indicate different node states. On the top the effect of the gravitational wave on a large ring of test masses in the $y-z$ plane at the various values of $x$. Physical distances are plotted for these rings.

## A $\mathrm{SU}(2)$ Representations

$\mathrm{An} \mathrm{SU}(2)$ matrix in the spin $j$ representation is a square matrix of dimension $2 j+1$. There are two possible indices: $i$ is ascending, $m_{i}$ is descending:

$$
\begin{gather*}
\left(\begin{array}{ccc}
i=1 & \cdots & i=2 j+1 \\
& & \\
& i=1+j-m_{i} \quad & \left(\begin{array}{lll}
m_{i}=j & \cdots & m_{i}=-j \\
& &
\end{array}\right)
\end{array} . \begin{array}{l} 
\\
\\
\end{array}\right) \tag{A.1}
\end{gather*}
$$

$\mathrm{SU}(2)$ has three generators $\tau_{1}, \tau_{2}, \tau_{3}$ and their matrix elements in the spin $j$ representation are:

$$
\begin{align*}
&\left.D_{j}\left(\tau_{1}\right)\right)_{m}^{m^{\prime}}=-\frac{\mathrm{i}}{2} \sqrt{j(j+1)-m^{\prime}\left(m^{\prime}+1\right)} \delta_{m-1}^{m^{\prime}} \\
&-\frac{\mathrm{i}}{2} \sqrt{j(j+1)-m^{\prime}\left(m^{\prime}-1\right)} \delta_{m+1}^{m^{\prime}} \\
& \begin{aligned}
D_{j}\left(\tau_{2}\right)^{m^{\prime}}{ }_{m}= & \frac{1}{2} \sqrt{j(j+1)-m^{\prime}\left(m^{\prime}+1\right)} \delta_{m-1}^{m^{\prime}} \\
& -\frac{1}{2} \sqrt{j(j+1)-m^{\prime}\left(m^{\prime}-1\right)} \delta_{m+1}^{m^{\prime}} \\
\left.D_{j}\left(\tau_{3}\right)\right)_{m}^{m^{\prime}}= & -\mathrm{i} m^{\prime} \delta_{m}^{m^{\prime}}
\end{aligned} \tag{A.3}
\end{align*}
$$

The $\tau$ matrices for $j=1 / 2$ and $j=1$ are:

$$
\begin{gather*}
\tau_{1}=\frac{1}{2}\left(\begin{array}{cc}
0 & -\mathrm{i} \\
-\mathrm{i} & 0
\end{array}\right) \quad \tau_{2}=\frac{1}{2}\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right) \quad \tau_{3}=\frac{1}{2}\left(\begin{array}{cc}
-\mathrm{i} & 0 \\
0 & \mathrm{i}
\end{array}\right) \quad \text { (A.4) }  \tag{A.4}\\
\tau_{1}=\frac{1}{\sqrt{2}}\left(\begin{array}{ccc}
0 & -\mathrm{i} & 0 \\
-\mathrm{i} & 0 & -\mathrm{i} \\
0 & -\mathrm{i} & 0
\end{array}\right) \quad \tau_{2}=\frac{1}{\sqrt{2}}\left(\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 0
\end{array}\right) \quad \tau_{3}=\frac{1}{\sqrt{2}}\left(\begin{array}{ccc}
-\mathrm{i} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \mathrm{i}
\end{array}\right) \tag{A.5}
\end{gather*}
$$

Properties of the $\tau$ matrices:
Only for $j=1 / 2$, with $\sigma_{i}$ the Pauli matrices:

$$
\begin{equation*}
\tau_{i}=-\frac{\mathrm{i}}{2} \sigma_{i} \quad \tau_{i}^{2}=-\frac{1}{4} \mathbb{1} \tag{A.6}
\end{equation*}
$$

For all $j$ :

$$
\begin{gather*}
\operatorname{Tr}\left(\tau_{i}\right)=0 \quad \tau_{i}^{\dagger}=\left(\bar{\tau}_{i}\right)^{T}=-\tau_{i} \quad \text { (anti-hermitian) }  \tag{A.7}\\
{\left[\tau_{i}, \tau_{j}\right]=\varepsilon_{i j k} \tau^{k} \quad \tau^{i}=\delta^{i j} \tau_{j}} \tag{A.8}
\end{gather*}
$$

The matrix $U$ is special unitary with a real unit vector $\hat{n}$ and an angle $\phi$ :

$$
\begin{equation*}
U=\exp (\vec{\tau} \cdot \hat{n} \phi) \in \operatorname{SU}(2) \quad U^{\dagger}=U^{-1} \quad \operatorname{det}(U)=1 \tag{A.9}
\end{equation*}
$$

Under a change of the angle $\phi, U$ transforms as

$$
\begin{align*}
& \phi \rightarrow \phi+2 \pi: U \rightarrow-U  \tag{A.10}\\
& \phi \rightarrow \phi+4 \pi: U \rightarrow U \tag{A.11}
\end{align*}
$$

## B Properties of the $3 j$ and $6 j$-Symbols

## $3 j$-symbol

The Wigner $3 j$-symbols are related to the Clebsch-Gordan coefficients:

$$
\left\langle j_{1} j_{2} ; m_{1} m_{2} \mid j_{1} j_{2} ; j m\right\rangle=(-1)^{j_{1}-j_{2}+m} \sqrt{2 j+1}\left(\begin{array}{ccc}
j_{1} & j_{2} & j  \tag{B.1}\\
m_{1} & m_{2} & -m
\end{array}\right)
$$

An even permutation of the columns does not change the value of a $3 j$ symbol. An odd permutation of the columns gives a sign factor:

$$
\left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3}  \tag{B.2}\\
m_{1} & m_{2} & m_{3}
\end{array}\right)=(-1)^{j_{1}+j_{2}+j_{3}}\left(\begin{array}{ccc}
j_{2} & j_{1} & j_{3} \\
m_{2} & m_{1} & m_{3}
\end{array}\right)
$$

The signs of all three magnetic numbers can be changed at the cost of the same sign factor:

$$
\left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3}  \tag{B.3}\\
m_{1} & m_{2} & m_{3}
\end{array}\right)=(-1)^{j_{1}+j_{2}+j_{3}}\left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3} \\
-m_{1} & -m_{2} & -m_{3}
\end{array}\right)
$$

By combining (B.3) with the raising of one index (56), $m_{1}+m_{2}+m_{3}=0$ and $j_{1}+j_{2}+j_{3}=$ integer, all three indices of a $3 j$-symbol can be raised as follows:

$$
\left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3}  \tag{B.4}\\
m_{1} & m_{2} & m_{3}
\end{array}\right)=\delta_{m_{1}, m_{1}^{\prime}} \delta_{m_{2}, m_{2}^{\prime}} \delta_{m_{3}, m_{3}^{\prime}}\left(\begin{array}{ccc}
m_{1}^{\prime} & m_{2}^{\prime} & m_{3}^{\prime} \\
j_{1} & j_{2} & j_{3}
\end{array}\right)
$$

The following $3 j$-symbols are needed in section 1.6. The source of these expressions is Wolfram Mathematica:

$$
\begin{align*}
& \left(\begin{array}{ccc}
j & 1 & j \\
-m^{\prime} & 0 & m
\end{array}\right)=(-1)^{-j+m^{\prime}} \frac{m^{\prime}}{\sqrt{j(j+1)(2 j+1)}} \delta_{m, m^{\prime}}  \tag{B.5}\\
& \left(\begin{array}{ccc}
j & 1 & j \\
-m^{\prime} & 1 & m
\end{array}\right)=-(-1)^{-j+m^{\prime}} \frac{1}{\sqrt{2}} \sqrt{\frac{j(j+1)-m^{\prime}\left(m^{\prime}-1\right)}{j(j+1)(2 j+1)}} \delta_{m, m^{\prime}-1}  \tag{B.6}\\
& \left(\begin{array}{ccc}
j & 1 & j \\
-m^{\prime} & -1 & m
\end{array}\right)=(-1)^{-j+m^{\prime}} \frac{1}{\sqrt{2}} \sqrt{\frac{j(j+1)-m^{\prime}\left(m^{\prime}+1\right)}{j(j+1)(2 j+1)}} \delta_{m, m^{\prime}+1} \tag{B.7}
\end{align*}
$$

## $6 j$-symbol

The $6 j$-symbol is defined as a contraction of four $3 j$-symbols:

$$
\begin{align*}
\left\{\begin{array}{lll}
j_{1} & j_{2} & j_{3} \\
j_{4} & j_{5} & j_{6}
\end{array}\right\}= & \left(\begin{array}{ccc}
j_{1} & j_{2} & j_{3} \\
m_{1} & m_{2} & m_{3}
\end{array}\right)\left(\begin{array}{ccc}
m_{1} & m_{5} & j_{6} \\
j_{1} & j_{5} & m_{6}
\end{array}\right) \times \\
& \times\left(\begin{array}{ccc}
m_{2} & m_{6} & j_{4} \\
j_{2} & j_{6} & m_{4}
\end{array}\right)\left(\begin{array}{ccc}
m_{3} & m_{4} & j_{5} \\
j_{3} & j_{4} & m_{5}
\end{array}\right) \tag{B.8}
\end{align*}
$$

The following triplets of spins have to satisfy the Clebsch-Gordan conditions (51):

$$
\begin{equation*}
\left(j_{1}, j_{2}, j_{3}\right) \quad\left(j_{1}, j_{5}, j_{6}\right) \quad\left(j_{2}, j_{4}, j_{6}\right) \quad\left(j_{3}, j_{4}, j_{5}\right) \tag{B.9}
\end{equation*}
$$

The $6 j$-symbol is invariant under any permutation of the columns and it is invariant if any two columns are turned upside down:

$$
\left\{\begin{array}{lll}
j_{1} & j_{2} & j_{3}  \tag{B.10}\\
j_{4} & j_{5} & j_{6}
\end{array}\right\}=\left\{\begin{array}{lll}
j_{1} & j_{5} & j_{6} \\
j_{4} & j_{2} & j_{3}
\end{array}\right\}
$$

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