How to audit income tax - two models combined

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Abstract

A basic model of linear income taxation is proposed for analyzing optimality in auditing income tax reports. A game-theoretical approach is used for deriving the optimal strategy for the tax administration. Taxpayers minimize expected costs including possible penalties for under-reporting while the tax administration chooses its audit policy to maximize expected revenue. A simple strategy of only two audit rates is shown to be optimal in most cases.

1 Introduction

Wherever there is a system of taxes, there is tax evasion. Governments usually set up tax administrations in order to monitor taxes and detect evasion. In most Western countries, civilians have the obligation to report their income to the tax administration. Based on these reported incomes, the civilian has to pay a certain amount of tax. However, it is possible that the civilian does not report his actual income, but declares a lower amount so that he needs to pay less tax over his earnings. In order to prevent this, the tax administration performs audits on the reported incomes to check if they are true. If under-reporting is detected, the taxpayer is obliged to pay the tax over the unreported income plus a fine. Unfortunately, these audits come at a certain price, so that it can be not optimal to audit every reported income. But how to decide which incomes to audit and which not? This is what we are going to investigate.

There is a considerable amount of literature focusing on how to optimize the tax system, looking at tax rates, equity principles or penalty rates when evasion is detected. Most of this literature is written by economists, psychologists or sociologists. Some mathematical articles have also been published. Kaplow [6] tries to find rules for optimal tax rates and enforcement expenditures. Scotchmer [4] investigates the case were audits do not with full certainty determine the true income and shows that some randomness in these audits is optimal. Lambert [3] has published a book in which he brings together many strands of the analysis of income distribution and redistribution. He among other things discusses the analysis of the income distribution and social welfare functions.

In this article we focus on the mathematical approach of the strategy for auditing reported incomes in order to optimize the income tax system. We combine two mathematical models to lay a foundation for further research on the matter. Cremer's paper (1990) [1] presents the "cutoff rule" and proves for which values it is optimal. He then considers the government as a third actor besides the taxpayer and the tax administration and extends his findings to the case where there are several audit groups of taxpayers. Vasin, in his "Mathematical Models for Organizing the Tax Service" (1999) [2], proves that any audit strategy can be replaced by a cutoff rule without any losses for the tax service. He then considers the case where tax inspectors can be bribed.

In this paper, we combine the mathematical foundations of these two models to set up a uniform basic model for the auditing of income tax. We do not embark into further application of the model, since the model must first be further improved in order to really stroke with reality. We start the paper with a brief explanation of the model and its structure in section 2. We also state the made assumptions. In section 3 we derive the optimal audit strategy combining the results from Cremer and Vasin. We show that in most cases a cutoff rule as an audit strategy is optimal. In such a cutoff rule, a reported income below a certain cutoff level t is audited with a constant probability p_1, and a reported income above t is audited with probability p_2. Section 3.1 focuses on the determination of the optimal values for p_1 and p_2. Section 3.2 aims to prove the optimality of this cutoff rule and in section 3.3, we show how the optimal cutoff level t can be derived. In section 4, we critically discuss the posed model and suggest on which regards it might be improved. In section 5, we aim to clarify the matter by presenting two examples.

Though we use the general environment that was provided by Vasin, we chose to use Cremer's notation in the model. In section 3.1, we slightly relaxed some assumptions of Cremer's model, but this did not have an essential influence in the proofs or the results derived from the model. The theorems and the arguments of the proofs in section 3.1 are thus due to Cremer. In section 3.2, we implement Vasin's results into our model. His proofs are however not completely followed in this paper, since they contained many inaccuracies, errors and loopholes. We have corrected these errors and filled the loopholes wherever it was possible. One problem however still remains unsolved, which makes the theorem hold only under certain circumstances. We thus pose the theorem as a conjecture. In section 3.2.2, we discuss how we can still derive the desired result in the case that the necessary conditions are not satisfied. The expressions in section 3.3 are also due to Vasin, although in his paper, they were not completely correct. We have corrected his errors, which were probably just typos, and have explained the theorems and their proofs.

2 The model

In this section we describe the model, introducing it by presenting a number of definitions. These provide a general structure of the model. We then present some assumptions which give the model more meaning. When the assumptions are changed, the outcome of this model is no longer guaranteed. We prepare for the next section, in which the optimal behaviour of the agents is derived.

2.1 Structure

In the model we consider two agents: the taxpayer and the tax administration.

We consider a homogeneous group of taxpayers. We can thus speak of the behaviour of "the taxpayer". The taxpayer has a pretax income $w \in [0, \infty)$, which is distributed according to a distribution function $F(w)$, with $f(w) = F'(w)$ being the probability density function. The taxpayer has the obligation to report his income to the tax administration. He can however choose not to report his true income w, but to conceal a part. This concealing is described by the taxpayer behaviour function $x(w)$, which denotes the reported income of a taxpayer with income w. The outcome of this taxpayer behaviour function is denoted by x. The tax administration knows $F(w)$. It does not know w. It only knows x, the reported income. The tax administration can decide to perform an audit, at cost c, after which the true income w will become known. It establishes an audit policy $p(x)$, where $p(x)$ is the probability that a taxpayer who reports an income of x will be audited.

Each taxpayer is subject to income tax according to some function $\theta(x)$ where x denotes reported income ($x \leq w$); if an audit on the reported income is performed by the tax administration and under-reporting is detected, he must pay a penalty $q(w, x)$, which depends on the (audited) true income w and the reported income x. We denote the expected tax-schedule for the taxpayer $T(w, x)$.

The taxpayer's objective is to maximize expected total income. He is supposed to be risk-neutral. For the taxpayer, tax and audit parameters as well as his pretax income are given. His choice is thus limited to part the of this income he wants to conceal. The taxpayer's behaviour will thus come down to minimizing his expected tax-schedule $T(x, w) = \theta(x) + p(x)q(w, x)$ with respect to x. This means that each taxpayer will conceal part of his income if he expects this to be profitable.

The tax administration's objective is maximizing total revenue. For any audit strategy $p(x)$, the net tax revenue per taxpayer is

$$R(p(x)) = \int [\theta(x) + p(x)(q(w, x) - c)]\ f(w)\ dw$$

The administration's goal is to find a strategy $p^*(x)$ maximizing this revenue.

2.2 Assumptions

The assumptions made in this section are essential for the final results of the model. Some of the assumptions are based on logical argumentation. There are however some assumptions that are general, but quite restrictive and therefore questionable. These are necessary to be able to calculate with the model.

We consider the case where we have a linear income tax: $\theta(x) = \theta x - \gamma$, for some $\theta \geq 0$ and $\gamma \geq 0$. The penalty is assumed proportional to income: $q(w, x) = (\pi + \theta)(w - x)$, $\pi \geq 0$. We note that the penalty includes tax on the under-reported income. The purpose of γ in the tax obligation function can be interpreted as a minimal amount of income over which no tax has to paid. We need to make note that it does not account for social support by the government for civilians with an income $w < \frac{\gamma}{\theta}$. We thus assume γ such that $\theta(x) \geq 0$. The most particular purpose of γ is that it is an instrument for the government (not the tax administration) to influence the social welfare of its people. By adjusting the value of γ, the government can directly increase the after-tax income of the taxpayer, thus influencing his social welfare function. In this model, we do not consider the role of the government in the game of reporting and auditing income tax. We can also derive all the results of this model by assuming $\gamma = 0$, but for the sake of completeness, we leave it as a constant.

We only consider audit policies by the tax administration where the audit probability is decreasing with increasing reported income. If we would have an increasing audit probability, it would become more and more profitable for an individual to report a lower income than his actual income w.

The assumptions in which we part from Cremer are those on the distribution of income $F(w)$ and the function for the penalty $q(w, x)$. Cremer assumed a maximum income w^+, but we show that we do not need this restriction to let the result hold. We thus suppose $w \in [0, \infty)$. He also restricted $F(w)$ in order to prevent a fat tail. He used this restriction in determining the optimal cutoff level. We do not make any such restriction until section 3.3, where we show how the optimal cutoff level can be derived. As for the penalty function $q(w, x)$, Cremer's definition did not explicitly state that the penalty included the unpaid tax over the concealed income. Where he used $\pi(w - x)$, we define $q(w, x) = (\pi + \theta)(w - x)$. The main difference between the assumptions in our model and those from Vasin is that of the use of γ. He does not define γ, thus proving the results for $\gamma = 0$.

3 Optimal audit policy

In this section, we derive the optimal audit policy of the the tax administration, taking the behaviour of the taxpayer into account. We first consider a certain class of audit policies and then generalize the result to a much wider class of policies. The result may seem somewhat surprising and unfair. This is further discussed in section 4.

3.1 The cutoff rule

In this section, audit policy is restricted to a particular type: the probability of an audit is different if the reported income is below or above a certain cutoff level of income t. For a reported income below t, the probability of an audit is of a constant rate p_1. For a reported income above t this is of rate p_2. Hence, the audit probability $p(x)$ of an individual reporting x is given by

$$p(x) = \begin{cases} p_1 & \text{if} \quad x < t \\ p_2 & \text{if} \quad x \geq t \end{cases} \tag{1}$$

The main structure of this section and the proofs in it are due to Cremer.

3.1.1 The theorem

The objective of the tax administration is to maximize total tax revenue, net of audit costs. Tax parameters, audit costs and penalty rates are given; its instruments are t, p_1 and p_2.

The taxpayer's goal is to minimize the amount to be paid to the tax administration. This amount to be paid by an individual with income w, who reports x is denoted $T(x, w) = \theta x + p(x)(\pi + \theta)(w - x) - \gamma$. In the case of a truthful report, this becomes $T(w, w) = \theta w - \gamma$. Over-reporting is ruled out by the assumption that $x \leq w$. A positive fine is thus excluded. The objective function of the tax administration, that is, tax revenue net of audit costs, is given by

$$R(p_1, p_2, t) = \int_0^\infty [\, T(x(w), w) - p(x(w))\, c\,]\, f(w)\; dw \tag{2}$$

We denote p_1*, p_2*, $t*$ as the auditing parameters that maximize the tax net revenue. We start by determining the optimal audit probabilities p_1* and p_2*. Even though we use a simple model, this turns out to be quite a complex problem.

THEOREM 1.
The optimal audit probabilities of a cutoff policy are given by $p_1^* = \frac{\theta}{\pi+\theta}$ and $p_2^* = 0$.

We will explain Theorem 1 in this section by providing some properties and lemmas. The proofs for these can be found in section 3.1.2.

Property 1) The minimum audit probability for which each taxpayer declares his income truthfully is given by $\frac{\theta}{\pi+\theta}$ *, i.e.* $p(x) \geq \frac{\theta}{\pi+\theta} \Rightarrow \forall x \leq w,\, x(w) = x$

From now on, we define $\hat{p} \equiv \frac{\theta}{\pi+\theta}$ as the minimal deterrent probability. Logically, it is never profitable for the tax administration to audit a reported income with a probability higher than the minimal deterrent probability, since it does not change the taxpayer's report, but does increase expected audit costs. This leads to the next property.

Property 2) $\max(p_1^*, p_2^*) \leq \hat{p}$

We already shortly stated that only strategies of decreasing audit probability are to be considered. This means that high reports should not be audited with a higher probability than low reports. This statement is formalized in the following property.

Property 3) Without any loss of generality, we can restrict our attention to audit probabilities such that $p_2 \leq p_1$.

The proofs for these properties can mostly be derived from definitions and some simple argumentation. Note that we have already restricted our possibilities to the condition that $\hat{p} \geq p_1 \geq p_2$. We now define $\hat{t} \equiv t\left(\frac{\hat{p}-p_2}{p_1-p_2}\right)$ as the income at which the taxpayer is indifferent to reporting either 0 or t. It appears that the taxpayers are divided into two subgroups: those with income above $\hat{t}$ who report t and those with income less than $\hat{t}$ who report nothing. Taxpayers with income $\hat{t}$ are indifferent between reporting zero and reporting t and report t by convention. In the particular case where $\hat{t} \to \infty$, the first subgroup vanishes.

We are now left with two alternatives: either $p_1^* = \hat{p}$ and $p_2^* < \hat{p}$ or $p_2^* < p_1^* < \hat{p}$. We ignore the option where $p_1^* = p_2^* = \hat{p}$, since it is just a special case of the first with $t = \infty$.

With the results of Properties 1-3, we rewrite the expression for the revenue of the tax-administration of (2):

$$
\begin{aligned}
R(t,p_1,p_2) &= \int_0^\infty \left[T\left(x(w),w\right) - p\left(x(w)\right)c\right]\ dw \\
&= \int_0^{\hat{t}} [T(0,w) - p_1 c]\ f(w)\ dw + \int_{\hat{t}}^\infty [T(t,w) - p_2 c]\ f(w)\ dw \\
&= p_1\left[(\pi+\theta)\int_0^{\hat{t}} w f(w)\ dw - c\int_0^{\hat{t}} f(w)\ dw\right] + p_2(\pi+\theta)\int_{\hat{t}}^\infty (w-t) f(w)\ dw \\
&\quad - \int_{\hat{t}}^\infty (p_2 c - \theta t)\, f(w)\ dw - \int_0^\infty \gamma f(w)\ dw \\
&= p_1\left[(\pi+\theta)\int_0^{\hat{t}} dF(w) - F(\hat{t})c\right] + \left[p_2(\pi+\theta)\int_{\hat{t}}^\infty (w-t)\, dF(w) - (p_2 c - \theta t)(1-F(\hat{t}))\right] - \gamma
\end{aligned}
\tag{3}
$$

We need to make a comment on this expression. As noticed before, people with income below t report truthfully in the case where $p_1 = \hat{p}$, while they report no income with $p_1 < \hat{p}$. However, this discontinuity of $x(w)$ can be ignored in (2) since $p_1 = \hat{p}$ implies $T(0, w) = p_1(\pi + \theta)w = \theta w = T(w, w)$ for $w < t$. We can thus conclude that $R(t, p_1, p_2)$ is a continuous function on S, where S is the following set of possible optimal solutions (t^*, p_1^*, p_2^*):

$$S = \left\{(t, p_1, p_2) \in \mathbb{R}_+^3 \;\middle|\; 0 \leq p_2 \leq p_1 \leq \hat{p} \;\text{ and }\; t \in [0, \infty)\right\}$$

We are now ready to prove the theorem. We will show that $R(t, p_1, p_2)$ finds a maximum on S for $p_1 = \hat{p}$ and $p_2 = 0$. This maximum is also the global maximum of $R(t, p_1, p_2)$. The existence of a maximum on S follows from the continuity of R and the compactness of S. Properties 2 and 3 imply that this maximum on S is actually the global maximum. We will now first show that $p_2^* = 0$. With this result, we can conclude in a similar way that $p_1^* = \hat{p}$.

Let t and $\hat{t}$ be fixed such that $\hat{t} < \infty$ ($\hat{t} < \infty$ is assumed only for notational convenience. A slight modification argument is sufficient to show the same results for $t = \infty$). Using the definition of $\hat{t}$, we find:

$$p_2 = \frac{\hat{t}}{\hat{t} - t} p_1 - \hat{p} \frac{t}{\hat{t} - t} \tag{4}$$

and

$$\frac{dp_2}{dp_1} = \frac{\hat{t}}{\hat{t} - t} > 1 \tag{5}$$

We then differentiate R with respect to p_1, given t, $\hat{t}$, and (4).

$$\frac{dR}{dp_1} = (\pi + \theta) \int_0^{\hat{t}} dF(w) - F(\hat{t})c + \frac{\hat{t}}{\hat{t} - t} \left[(\pi + \theta) \int_{\hat{t}}^{\infty} (w - t) dF(w) - \big(1 - F(\hat{t})\big)\right] \tag{6}$$

We see that (6) is independent of p_1. R thus increases/decreases linearly with p_1. This implies that the maximum of R is achieved on the boundary of S. If (6) is positive, $p_1^* = \hat{p}$; if negative, $p_2^* = 0$. From (5), we have that on decreasing p_1, p_2 reaches zero before p_1. This brings us to two alternatives: (a) $p_1^* = \hat{p}$, $p_2^* < \hat{p}$ or (b) $p_1^* < \hat{p}$, $p_2^* = 0$. Furthermore, from the definition of $\hat{t}$, it follows that $p = \hat{p}$ implies $\hat{t} = t$. In case (a), R is a linear function of p_2, which is thus maximized for $p_2 = 0$ or $p_2 = \hat{p}$. As above, we ignore the option $p_2 = p_1 = \hat{p}$, since it is a special case of the the first option with $t = \infty$. We finds that in both case (a) and case (b), $p_2^* = 0$.

We note that $p_2 = 0$, together with the definition of $\hat{t}$ implies that $\hat{t} = \frac{t\theta}{(\pi+\theta)p_1}$. We can now write our revenue function (3) more simply as

$$R(t, p_1, 0) = p_1(\pi + \theta) \int_0^{\frac{t\theta}{(\pi+\theta)p_1}} w dF(w) - F\left(\frac{t\theta}{(\pi + \theta)p_1}\right) p_1 c + \left[1 - F\left(\frac{t\theta}{(\pi + \theta)p_1}\right)\right] \theta t - \gamma \tag{7}$$

We proceed to show that $p_1^* = \hat{p}$ in a very similar way as we did for p_2^*. We fix t and $\hat{t}$ such that $\hat{t} < \infty$ and then differentiate (7) with respect to p_1. We find again that this derivative is independent of p_1 so that the optimal value of p_1 must lie on the boundary of S. We find that $p_1^* = \hat{p}$. Finally we conclude that Theorem 1 is true.

■

With this policy, all individuals that report an income below the cutoff t are audited with probability $\hat{p}$, which is the minimal deterrent policy. Individuals that report an income above the cutoff will never be audited. This results in a division of two groups of taxpayers. Individuals with an income below the cutoff t will report their true income. Individuals with an income above this cutoff will report an income of t.

3.1.2 Proofs

In this section, we provide the proofs of the properties stated in the section 3.1.1 and some statements that have been made casually. We start by stating the property once more, followed by a mathematical proof. Some properties in this section you will not recognize from section 3.1.1, since they were never explicitly stated. They are however necessary for the complete proof of Theorem 1. We will then specify what is the result of this property.

Property 1) $\hat{p} \equiv \frac{\theta}{\pi+\theta}$ *is the minimum audit probability for which each taxpayer declares his income truthfully, i.e.* $p(x) \geq \hat{p} \;\Rightarrow\; \forall x \leq w,\, x(w) = x$

Proof: It is sufficient to compare the tax burden in the case of truthful reporting and the case of under-reporting. If $p(x) \geq \hat{p}$ and the taxpayer under-reports his income, i.e. if $x < w$, then he has to pay more to the tax-administration then if he had reported his true income w.

$$T(x, w) = \theta x + p(x)(\pi + \theta)(w - x) - \gamma \geq \theta x + \theta(w - x) - \gamma = \theta w - \gamma = T(w, w)$$

■

Property 2) $\max(p_1^*, p_2^*) \leq \hat{p}$

Proof: We prove this property by contradiction. Assume that Property 2 does not hold. The following cases are possible:

$$(i)\; p_1^* \geq \hat{p},\; p_2^* > \hat{p}$$
$$(ii)\; p_1^* < \hat{p},\; p_2^* > \hat{p}$$
$$(iii)\; p_1^* > \hat{p},\; p_2^* \leq \hat{p}$$

In any of these cases we can define $p_i' = min(p_i^*, \hat{p}),\; i = 1, 2$. We then see that if $p_i' = p_i^*$, the audit strategy does not change, so the taxpayer behaviour also remains unchanged. If $p_i' = \hat{p}$, the original audit probability was higher than the minimal deterrent probability. By Property 1, the original taxpayer behaviour is given by the taxpayer declaring his true income w. With our new audit probability $p_i' = \hat{p}$, the taxpayer still declares $x = w$. The taxpayer behaviour thus does not change. Expected audit costs will however decrease. We show this by showing that the integrand of the expression for the expected revenue of (2) increases when p_i^* is replaced by p_i' for $i = 1, 2$. The following expression holds whenever $p_i' = \hat{p}$. Note that then $p_i' \leq p_i^*$.

$$[\theta x + p_i^*((\pi + \theta)(w - x) - c) - \gamma]\, f(w) = [\theta w - p_i^* c - \gamma]\, f(w) \leq [\theta w - p_i' c - \gamma]\, f(w)$$

In the case of $p_i' = p_i^*$, the revenue does not change. We conclude that if we replace p_1^*, p_2^* by p_1', p_2', the taxpayer's behaviour will never change. However, the expected audit costs will decrease, which increases the expected total revenue for the tax administration. This contradicts the definition that p_1^* and p_2^* are the optimal audit policy. We conclude that indeed $\max(p_1^*, p_2^*) \leq \hat{p}$

■

Property 3) Without any loss of generality, we can restrict our attention to audit probabilities such that $p_2 \leq p_1$.

Proof: If $p_2 > p_1$, it means that for all taxpayers with an income higher than the cutoff income t, the probability of being audited is lower when reporting an income smaller than t. It even implies that for any policy (p_1, p_2, t) with $p_2 < p_2$ and taking Property 2 into account, the best reply for all taxpayers is to report no income, i.e. $\forall w \in [0, \infty), x(w) = 0$. The policy can thus be replaced by (p_1, p_2') with $p_2' \leq p_1$ which leaves the taxpayers' report unchanged and does not increase audit costs.

■

Property 4 and 5 are both necessary for the derivation of the reply of the taxpayer.

Property 4) If $p(x) = p$ *for any* $x \in [a, b]$, *then* $\forall x \in (a, b)$:

$$\frac{\partial}{\partial x} T(x, w) \geq 0 \Leftrightarrow p \leq \hat{p} \tag{8}$$

$$\frac{\partial}{\partial x} T(x, w) < 0 \Leftrightarrow p > \hat{p} \tag{9}$$

Proof: We first prove (8). The proof of the equivalence of (9) is similar. Firstly we note that $\forall x \in (a, b)$, $T(x, w) = \theta x + p(\pi + \theta)(w - x)$, thus $\forall x \in (a, b)$, $\frac{\partial}{\partial x} T(x, w) = \theta - p(\pi + \theta)$. Suppose $p \geq \hat{p}$, then $\frac{\partial}{\partial x} T(x, w) = \theta - p(\pi + \theta) \leq \theta - \hat{p}(\pi + \theta) = 0$. Now suppose $\frac{\partial}{\partial x} T(x, w) \leq 0$, then $\theta - p(\pi + \theta) \leq 0$, so we find that $p \geq \hat{p}$.

■

Property 5)

$$(i) \;\; T(0, w) \gtreqless T(t, w) \Leftrightarrow w \gtreqless \hat{t} \equiv t \left(\frac{\hat{p} - p_2}{p_1 - p_2} \right) \tag{10}$$

$$(ii) \;\; \frac{\partial [T(0, w) - T(t, w)]}{\partial w} > 0 \tag{11}$$

Proof: Both results are directly obtained from the definition of $T(x, w)$:

$$(i): \; T(0, w) = p_1(\pi + \theta)w - \gamma, \;\; T(t, w) = \theta t + p_2(\pi + \theta)(w - t) - \gamma$$

and so,

$$\begin{aligned} &T(0, w) \geq T(t, w) \Rightarrow p_1(\pi + \theta)w \geq \theta t + p_2(\pi + \theta)(w - t) \Rightarrow \\ &w(p_1(\pi + \theta) - p_2(\pi + \theta)) \geq \theta t - t p_2(\pi + \theta) \Rightarrow \\ &w \geq t \left(\frac{\theta - p_2(\pi + \theta)}{p_1(\pi + \theta) - p_2(\pi + \theta)} \right) = t \left(\frac{\hat{p} - p_2}{p_1 - p_2} \right) \end{aligned}$$

$$\begin{aligned} (ii): \; &\frac{\partial}{\partial w} \left(T(0, w) - T(t, w) \right) = \frac{\partial}{\partial w} \left(p_1(\pi + \theta)w + \theta t - p_2(\pi + \theta)(w - t) \right) \\ &= p_1(\pi + \theta) - p_2(\pi + \theta) > 0 \end{aligned}$$

■

We note that (ii) implies that the expected tax payments increase with effective income, hence the gain from reporting zero relative to reporting the cutoff income t is decreasing with effective income. We can thus conclude the division of the subgroup of individuals with an income below $\hat{t}$ reporting nothing and the subgroup of individuals with an income above $\hat{t}$ reporting t.

The rest of the proof of Theorem 1 can be found in section 3.1.1.

3.2 General audit probabilities

We now consider a wider class of audit policies. In this section we will discuss the conjecture that any decreasing audit probability can be replaced by the audit policy with a cutoff rule from section 3.1 without a decrease in the net revenue of the tax administration. Vasin provided a proof for this conjecture, but in that proof he implicitly made an assumption that does not always hold. Without this assumption, the proof is no longer completely correct. In section 3.2.1, we assume the sufficient conditions and present Theorem 2. The main construction of the proof for Theorem 2 is due to Vasin, though in his paper he simply skipped all the lemmas and propositions we present in this section. Theorem 2 states some conditions, any mention of these conditions is left out by Vasin. If these conditions were to be left out and the result could still be proved, then the proof of the conjecture would follow. Vasin's proof however does not suffice to do so. We try to make as clear as possible when complications may be encountered and discuss what happens then in section 3.2.2. Here we show what will happen if the conditions are not satisfied. We show what the taxpayer behaviour will look like and we illustrate how to still find a dominant cutoff strategy with a general example. In section 3.2.3, complete proofs of all theorems, lemmas and propositions is provided.

3.2.1 Vasin's theorem

We consider any decreasing audit probability $p(x)$. Our goal is to show when the policy of using this probability is matched or improved by the policy of the cutoff rule. In order to do so, we need to introduce some definitions and notations that help us describing the proper conditions for this to hold. For any $t_1 < t_2 < \cdots < t_n$, and $\hat{p} \geq p_1 > p_2 > \cdots > p_n \geq 0$, consider an n-level strategy $p(x)$ such that:

$$p(x) = \begin{cases} \hat{p} & \text{if } x < t_1 \\ p_k & \text{if } x \in [t_k, t_{k+1}) \\ p_n & \text{if } x \geq t_n \end{cases} \qquad (k = 1, \dots, n-1) \tag{12}$$

We define $S(t_1, \dots, t_n, p_1, \dots, p_n)$ as such an n-level strategy. We now introduce

$$\hat{t}_k \equiv \frac{\theta}{\theta + \pi} \left(\frac{t_{k+1} - t_k}{p_k - p_{k+1}} \right) + \frac{p_k t_k - p_{k+1} t_{k+1}}{p_k - p_{k+1}} \text{ for } k \in \{1, \dots, n-1\} \tag{13}$$

We choose this value of $\hat{t}_k$ so that an individual with income $w = \hat{t}_k$ is indifferent to declaring $x = t_k$ or $x = t_{k+1}$, i.e.

$$\theta t_k + p_k(\theta + \pi)(\hat{t}_k - t_k) = \theta t_{k+1} + p_{k+1}(\theta + \pi)(\hat{t}_k - t_{k+1})$$

We define $\hat{t}_n \equiv \infty$.

In many cases where the values of the p_i and t_i are somewhat evenly spread, $\hat{t}_i$ will be strictly increasing so that $\hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_n$. However, it is possible that this will not hold. In his paper, Vasin proves that whenever $\hat{t}_i$ strictly increases with i, any n-level strategy can be matched or improved by the cutoff rule. He then concludes that any strategy of decreasing audit probabilities can also be replaced by the cutoff rule, by the argument that we can approach any such strategies in $\mathbb{R}$ by an n-level strategy for n sufficiently large. We will first prove that this indeed holds in the case that the $\hat{t}_i$ are strictly increasing and then show what will happen when $\hat{t}_i$ is not strictly increasing with i in section 3.2.2.

THEOREM 2.
Any admissible n-level audit strategy $S(t_1,\ldots,t_n,p_1,\ldots,p_n)$ for which $\hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_n$, can be at least matched by the strategy

$$p(x) = \begin{cases} \hat{p} & \text{if } x < t \\ 0 & \text{if } x \geq t \end{cases} \tag{14}$$

for some $t \in \mathbb{R}$.

Let $t_1 < \cdots < t_n$ and $p_1 > \cdots > p_n \geq 0$, such that $\hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_{n-1} < \hat{t}_n \equiv \infty$. We will prove Theorem 2 by induction on n.

Proof.
We first show what the taxpayer's behaviour will now look like. For an individual earning w, the possibilities of declaring income can be divided into intervals $[0,t_1),[t_1,t_2),[t_2,t_3),\ldots,[t_{n-1},t_n)$. A declaration of $x \in [0,t_1)$ will be audited with the minimal deterrent probability $\hat{p}$, so that for $w \in [0,t_1)$, $x(w) = w$. However, declarations of x in any other interval will be audited with a probability less than $\hat{p}$ so that the expected tax schedule decreases with x. Within an interval, the audit probability is constant so that the taxpayer is always better off declaring the infimum of the interval than declaring any other value within the interval. It thus only makes sense for the taxpayer to declare either his true income or to declare t_k for some $k \in \{1,\ldots,n-1\}$. We can therefore proceed considering only these values for $x(w)$. We now define $T_k(w)$ as the expected tax schedule for a taxpayer with income w when he declares t_k. Define $T_0(w)$ as the expected tax schedule when declaring truthfully. We now state a proposition which will lead to the determination of the taxpayer's behaviour function.

Proposition 1
If the true income w is above the level of income with which the taxpayer is indifferent to reporting either t_k or t_{k+1}, then it is more profitable for the taxpayer to report t_{k+1} than to report t_k. It is the other way around when w is below this point of indifference, i.e.

$$w \gtrless \hat{t}_k \Rightarrow T_k(w) \gtrless T_{k+1}(w)$$

We can now see that the taxpayer's behaviour is described as follows:

$$x(w) = \begin{cases} w & \text{if } w \in [0,t_1) \\ t_1 & \text{if } w \in [t_1,\hat{t}_1) \\ t_k & \text{if } w \in [\hat{t}_{k-1},\hat{t}_k), \qquad (k = 2,\ldots,n) \end{cases} \tag{15}$$

We assume that if several declared income values correspond to the same expected income after taxes and penalties, the taxpayer declares a value closest to the actual income. If $\hat{t}_k \leq t_{k-1}$ for some k, then the taxpayer never declares t_k. There exists an audit strategy of level $l < n$, that produces the same taxpayer behaviour and the same revenue. We will then consider an audit strategy that differs from the initial strategy only in that $p(x) = p_{k-1}$ for $t_k \leq x < t_{k+1}$.

Lemma 1
The 2-level strategy $S(t_1,t_2,p_1,p_2)$ is dominated by one of the following strategies:
(i) The strategy of the random audit rule where any reported income x is audited with probability $\hat{p}$
(ii) The 1-level strategy $S(t_1,0)$
(iii) The 1-level strategy $S(\hat{t}_1,0)$

The proof of this Lemma is provided in section 3.2.3.

We finish the proof by assuming the result for any strategy of level $n-1$ and showing that the result then also holds for a strategy of level n with a similar method as we used in the proof for the 2-level strategy.

3.2.2 Complications of the conjecture

Now, what will happen if $\hat{t}_i$ is not strictly increasing with i? We will show how the taxpayer behaviour functions alters in this case. We first suppose there is at least one disruption of these increasing $\hat{t}_i$'s and derive the altered taxpayer behaviour function. It should then be clear how the behaviour function changes in the case of more disruptions. We then show in which cases we can successfully apply the proof of Theorem 2 on a 3-level strategy where $\hat{t}_i$ is not strictly increasing with i.

We again consider the audit probability

$$p(x) = \begin{cases} \hat{p} & \text{if } x < t_1 \\ p_k & \text{if } x \in [t_k, t_{k+1}) \qquad\qquad (k = 1, \dots, n-1) \\ p_n & \text{if } x \geq t_n \end{cases}$$

where $\hat{p} > p_1 > \cdots > p_n > 0$ and $t_1 < \cdots < t_n$. We use the same definition of $\hat{t}_k$ as in (13) for $k \in \{1, \dots, n-1\}$ and define $\hat{t} \equiv \infty$.

Now, suppose that $\exists k \in \{2, \dots, n-1\}$ such that $\hat{t}_k \leq \hat{t}_{k-1}$. Let l be the smallest natural number in $\{k+1, \dots, n\}$ for which $\hat{t}_{k-1} < \hat{t}_l$. We will show that with this audit probability policy, the taxpayer will never declare income t_k:

if $w < \hat{t}_{k-1}$, then $T_{k-1}(w) < T_k(w) \Rightarrow t_k$ will not be declared

if $w \geq \hat{t}_{k-1}$, the also $w > \hat{t}_k$ so that $T_{k-1}(w) \geq T_k(w) > T_l(w) \Rightarrow t_k$ will not be declared

We conclude that t_k will never be declared. We thus find the following taxpayer behaviour function:

$$x(w) = \begin{cases} w & \text{if } w \in [0, t_1) \\ t_1 & \text{if } w \in [t_1, \hat{t}_1) \\ t_i & \text{if } w \in [\hat{t}_{i-1}, \hat{t}_i) \qquad\qquad (i = 2, \dots, k-1) \\ t_l & \text{if } w \in [\hat{t}_{k-1}, \hat{t}_l) \\ t_j & \text{if } w \in [\hat{t}_j, \hat{t}_{j+1}) \qquad\qquad (j = l, \dots, n-1) \end{cases}$$

We see that the behaviour function simply skips the cutoff levels where the disruption of the strict increasing of $\hat{t}_i$ is caused. Now, how does this affect the proof we have seen above? Unfortunately we cannot prove that we can always replace an n-level strategy with a 1-level strategy. We can though show under which circumstances we can replace a 3-level strategy with a 1-level strategy. A general example can be found in section 3.2.3. This should provide an idea of when we can successfully apply Theorem 2. In section 5 we present another, more specific example to clarify this even more.

3.2.3 Proofs

In this section we prove the proposition and the lemma from section 3.2.1 in order to complete the proof of Theorem 2. We then discuss an example of a 3-level strategy and show how we can find a 1-level strategy that dominates it. We first state the statements to prove once more and then provide the mathematical proofs.

Proposition 1
If the true income w is above the level of income with which the taxpayer is indifferent to reporting either t_k or t_{k+1}, then it is more profitable for the taxpayer to report t_k than to report t_{k+1}. It is the other way around when w is below this point of indifference, i.e.

$$w \gtrless \hat{t}_k \Rightarrow T_k(w) \gtrless T_{k+1}(w) \tag{16}$$

Proof.
Suppose $w = \hat{t}_k + \epsilon$, $\epsilon > 0$. Then:

$$\begin{aligned} T_k(w) &= \theta t_k + p_k(\theta+\pi)(\hat{t}_k + \epsilon - t_k) - \gamma \\ &= \theta t_k + p_k(\theta+\pi)(\hat{t}_k - t_k) + p_k(\theta+\pi)\epsilon - \gamma \\ &= T_k(\hat{t}_k) + p_k(\theta+\pi)\epsilon - \gamma \\ &= T_{k+1}(\hat{t}_k) + p_k(\theta+\pi)\epsilon - \gamma \\ &> T_{k+1}(\hat{t}_k) + p_{k+1}(\theta+\pi)\epsilon - \gamma = T_{k+1}(w) \end{aligned}$$

We thus see that for any $w > \hat{t}_k$, $T_k(w) > T_{k+1}(w)$. Now, if $w = \hat{t}_k - \epsilon$, with $\epsilon > 0$, we find that $T_k(w) < T_{k+1}(w)$ in a similar way.

■

Lemma 1
The 2-level strategy $S(t_1, t_2, p_1, p_2)$ is dominated by one of the following strategies:
(i) The strategy of the random audit rule where any reported income x is audited with probability $\hat{p}$
(ii) The 1-level strategy $S(t_1, 0)$
(iii) The 1-level strategy $S(\hat{t}_1, 0)$

Proof.
Consider the 2-level strategy $S(t_1, t_2, p_1, p_2)$. Then $\hat{t}_1$ is obtained from (13). Denote by $dp = (d_1, d_2)$ a change such that $\hat{t}_1$ remains unchanged for any admissible strategy $S(t_1, t_2, p_1 + zd_1, p_2 + zd_2)$. We now have the following equations:

$$\hat{t}_1 = \hat{p}\frac{t_2 - t_1}{p_1 - p_2} + \frac{p_1 t_1 - p_2 t_2}{p_1 - p_2} \tag{17}$$

$$\hat{t}_1 = \hat{p}\frac{t_2 - t_1}{p_1 + zd_1 - p_2 - zd_2} + \frac{(p_1 + zd_1)t_1 - (p_2 + zd_2)t_2}{p_1 + zd_1 - p_2 - zd_2} \tag{18}$$

From (17) we derive that $p_1 = p_2 \frac{\hat{t}_1 - t_2}{\hat{t}_1 - t_1} + \hat{p}\frac{t_2 - t_1}{\hat{t}_1 - t_1}$. With this result, we find from (18) that $d_1 = d_2 \frac{\hat{t}_1 - t_2}{\hat{t}_1 - t_1}$.

Let $p_z = p + zdp$ with $d_2 = 1$. We consider the audit strategy $S(t_1, t_2, p_1 + zd_1, p_2 + z)$. Note that we now have the following audit rule and behaviour function:

$$p_z(x) = \begin{cases} \hat{p} & \text{if } x < t_1 \\ p_1 + z\frac{\hat{t}_1 - t_2}{\hat{t}_1 - t_1} & \text{if } x \in [t_1, t_2) \\ p_2 + z & \text{if } x \geq t_2 \end{cases}$$

$$x(w) = \begin{cases} w & \text{if } w \in [0, t_1) \\ t_1 & \text{if } w \in [t_1, \hat{t}_1) \\ t_2 & \text{if } w \geq \hat{t}_1 \end{cases}$$

The net revenue per taxpayer of the tax service becomes

$$
\begin{aligned}
R(p_z(\cdot)) &= \int \left[\theta x(w) + p_z(x)\big((\pi+\theta)(w-x(w))-c\big)-\gamma\right] f(w)\,dw \\
&= \int_0^{t_1} [\theta w - \hat{p}c]\, f(w)\,dw + \int_{t_1}^{\hat{t}_1} \left[\theta t_1 + (p_1 + z d_1)\big((\pi+\theta)(w-t_1)-c\big)\right] f(w)\,dw \\
&\quad + \int_{\hat{t}_1}^{\infty} \left[\theta t_2 + (p_2 + z)\big((\pi+\theta)(w-t_2)-c\big)\right] f(w)\,dw - \gamma \qquad (19)
\end{aligned}
$$

The greatest possible z corresponds to $p_{z_{max},1} = p_{z_{max},2} = \hat{p}$, which represents the random audit rule. With the least possible z, we have $p_{z_{min},2} = 0$, $p_{z_{min},1} = \hat{p}\frac{t_2 - t_1}{\hat{t}_1 - t_1}$. We are going to study the derivative of (19), $\frac{d}{dz}R(p_z(\cdot))$.

$$
\begin{aligned}
\frac{d}{dz}R(p_z(\cdot)) &= \int_{t_1}^{\hat{t}_1} d_1\big((\pi+\theta)(w-t_1)-c\big) f(w)\,dw + \int_{\hat{t}_1}^{\infty} \big((\pi+\theta)(w-t_2)-c\big) f(w)\,dw \\
&= \frac{\hat{t}_1 - t_2}{\hat{t}_1 - t_1} \int_{t_1}^{\hat{t}_1} \big((\pi+\theta)(w-t_1)-c\big) f(w)\,dw + \int_{\hat{t}_1}^{\infty} \big((\pi+\theta)(w-t_2)-c\big) f(w)\,dw \qquad (20)
\end{aligned}
$$

This partial derivative is independent of z, the net revenue is thus a linear function of z. If $\frac{d}{dz}R(p_z(\cdot)) > 0$, then the revenue from $S(t_1, t_2, p_1, p_2)$ is less than the revenue from $S(t_1, t_2, \hat{p}, \hat{p})$. Hence, the tax service is better off when using the random audit rule. If $\frac{d}{dz}R(p_z(\cdot)) \leq 0$, the strategy is dominated by $S(t_1, t_2, \hat{p}\frac{t_2 - t_1}{\hat{t}_1 - t_1}, 0)$. Define $\lambda = \frac{t_2 - t_1}{\hat{t}_1 - t_1}$. We find that the revenue of this strategy yields:

$$
\begin{aligned}
R\big(S(t_1, t_2, \lambda\hat{p}, 0)\big) &= \int_0^{t_1} [\theta w - \hat{p}c] f(w)\,dw + \int_{t_1}^{\hat{t}_1} [\theta t_1 + \lambda\theta(w - t_1) - \hat{p}c\lambda] f(w)\,dw \\
&\quad + \int_{\hat{t}_1}^{\infty} \theta t_2\, f(w)\,dw - \gamma \qquad (21)
\end{aligned}
$$

We now note that this revenue is the same as the convex combination of the revenue of two 1-level cutoff rules:

$$
\begin{aligned}
&\lambda R\big(S(\hat{t}_1, 0)\big) + (1-\lambda) R\big(S(t_1, 0)\big) = \lambda \left[\int_0^{\hat{t}_1} [\theta w - \hat{p}c - \gamma] f(w)\,dw + \int_{\hat{t}_1}^{\infty} [\theta \hat{t}_1 - \gamma] f(w)\,dw\right] \\
&\quad + (1-\lambda)\left[\int_0^{t_1} [\theta w - \hat{p}c - \gamma] f(w)\,dw + \int_{t_1}^{\infty} [\theta t_1 - \gamma] f(w)\,dw\right] \\
&= \left(\lambda \int_0^{t_1} [\theta w - \hat{p}c] f(w)\,dw + (1-\lambda)\int_0^{t_1} [\theta w - \hat{p}c] f(w)\,dw\right) + \left(\lambda \int_{t_1}^{\hat{t}_1} [\theta w - \hat{p}c] f(w)\,dw \right. \\
&\quad \left. + (1-\lambda)\int_{t_1}^{\hat{t}_1} [\theta t_1] f(w)\,dw\right) + \left(\lambda \int_{\hat{t}_1}^{\infty} [\theta \hat{t}_1] f(w)\,dw + (1-\lambda)\int_{\hat{t}_1}^{\infty} [\theta t_1] f(w)\,dw\right) - \gamma \\
&= \int_0^{t_1} [\theta w - \hat{p}c] f(w)\,dw + \int_{t_1}^{\hat{t}_1} [\theta t_1 + \lambda\theta(w - t_1) - \hat{p}c\lambda] f(w)\,dw \\
&\quad + \int_{\hat{t}_1}^{\infty} [\theta\lambda(\hat{t}_1 - t_1) + \theta t_1] f(w)\,dw - \gamma \\
&= R\big(S(t_1, t_2, \lambda\hat{p}, 0)\big)
\end{aligned}
$$

So at least one of these 1-level cutoff rules is not worse than our 2-level strategy. In conclusion, any 2-level strategy can be replaced by a 1-level strategy.

■

Now assume that any $(n-1)$-level strategy can be replaced by a 1-level strategy for some $n \geq 2$. We consider any non-increasing n-level strategy $S(t_1, \dots, t_n, p_1, \dots, p_n)$. If $p_n > 0$, denote $(p_1, \dots, p_n)$ by $\bar{p}$. Denote by $d\bar{p} = (d_1, \dots, d_n)$ a change such that $\hat{t}_1, \dots, \hat{t}_n$ are the same for every strategy $S(t_1, \dots, t_n, \bar{p}(z))$, where $\bar{p}(z) = \bar{p} + zd\bar{p}$.

It suffices to set $d_n = 1$, $d_{k-1} = d_k \frac{\hat{t}_{k-1} - t_k}{\hat{t}_{k-1} - t_{k-1}}$, $k = n, \dots, 2$. The supremum z_{max} of all possible z is obtained from $p_{z_{max},k} = \hat{p}$ for every $k = 1, \dots, n$, and the infimum is obtained from $p_{z_{min},n} = 0$. The rest of the argument is the same as for $n = 2$.

We now discuss the example of some 3-level strategy and show how we can find a 1-level strategy that dominates it. Suppose some audit strategy $S(t_1, t_2, t_3, p_1, p_2, p_3)$, such that $\hat{t}_2 \leq \hat{t}_1$. The taxpayer behaviour function thus becomes:

$$x(w) = \begin{cases} w & \text{if } w \in [0, t_1) \\ t_1 & \text{if } w \in [t_1, \hat{t}_1) \\ t_3 & \text{if } w \geq \hat{t}_1 \end{cases} \tag{22}$$

As in the proof above, we again define a change $dp = (d_1, d_2, d_3)$ in the audit probability such that the values of $\hat{t}_1$ and $\hat{t}_2$ do not change. We consider an admissible strategy $S(t_1, t_2, t_3, p_1 + zd_1, p_2 + zd_2, p_3 + zd_3)$. We now let $d_3 = 1$ and find that $d_i = d_{i+1} \frac{\hat{t}_i - t_{i+1}}{\hat{t}_i - t_i}$ for $i = 1, 2$. The net revenue of this strategy is then linear in z, so we need to look at its derivative to find a strategy that yields a higher net revenue. If this derivative is positive, our strategy is dominated by $S(t_1, t_2, t_3, \hat{p}, \hat{p}, \hat{p})$, which is the random audit rule. If this derivative is negative, our strategy is dominated by $S\left(t_1, t_2, t_3, \hat{p}\left(\frac{t_3 - t_2}{\hat{t}_2 - t_2} + \frac{t_2 - t_1}{\hat{t}_1 - t_1}\right), \hat{p}\left(\frac{t_3 - t_2}{\hat{t}_2 - t_2}\right), 0\right)$. Unfortunately we cannot prove that this strategy is always dominated or matched by a 1-level strategy, nor can we prove that it is never dominated. It is hard to prove such dominance, since there are simply too many variables on which the expected revenue of this strategy depends and we cannot rewrite it in a smart way as we did in the proof of Theorem 2. What we can do, is try to find a 2-level strategy that dominates our 3-level strategy. We can try this in several ways. One method is to construct a 2-level strategy that yields the same taxpayer behaviour as our 3-level strategy. We show how this can be done below. In Example 2 of section 5, we show another method that can be attempted.

In order to find a strategy $S(t'_1, t'_2, p'_1, p'_2)$ that yields the same taxpayer behaviour function, we let $t'_1 = t_1$, $t'_2 = t_3$. We now need to choose p'_1 and p'_2 such that the income with which the taxpayer is indifferent to declaring t_1 or t_3 is equal to $\hat{t}_1$. After some calculations, we find that this is so if

$$p'_1 = \hat{p}\left(\frac{t_3 - t_1}{\hat{t}_1 - t_1}\right) + p'_2\left(\frac{\hat{t}_1 - t_3}{\hat{t}_1 - t_1}\right) \tag{23}$$

We note that we still are bound to the condition that $\hat{p} > p'_1 \geq p'_2 > 0$. Now, for any 2-level strategy for which (23) holds, the taxpayer behaviour function is the same as (22). Now, all that is left to do, is to pick the right values of p'_1, p'_2, such that the revenue of $S(t_1, t_3, p'_1, p'_3)$ is higher than the revenue of $S(t_1, t_2, t_3, p_1, p_2, p_3)$, i.e.

$$
\begin{aligned}
&\int_0^{t_1} [\theta w - \hat{p}c - \gamma] f(w)\, dw + \int_{t_1}^{\hat{t}_1} [\theta t_1 + p_1'((\theta+\pi)(w-t_1) - c) - \gamma] f(w)\, dw \\
&\qquad + \int_{\hat{t}_1}^{\infty} \theta t_3 + p_2'((\theta+\pi)(w-t_3) - c - \gamma)\, f(w)\, dw \\
\geq &\int_0^{t_1} [\theta w - \hat{p}c - \gamma] f(w)\, dw + \int_{t_1}^{\hat{t}_1} [\theta t_1 + p_1((\theta+\pi)(w-t_1) - c) - \gamma] f(w)\, dw \\
&\qquad + \int_{\hat{t}_1}^{\infty} [\theta t_3 + p_3((\theta+\pi)(w-t_3) - c) - \gamma]\, f(w)\, dw \\
\Leftrightarrow &\int_{t_1}^{\hat{t}_1} [\theta t_1 + p_1'((\theta+\pi)(w-t_1) - c)] f(w)\, dw + \int_{\hat{t}_1}^{\infty} [\theta t_3 + p_2'((\theta+\pi)(w-t_3) - c)]\, f(w)\, dw \\
\geq &\int_{t_1}^{\hat{t}_1} [\theta t_1 + p_1((\theta+\pi)(w-t_1) - c)] f(w)\, dw + \int_{\hat{t}_1}^{\infty} [\theta t_3 + p_3((\theta+\pi)(w-t_3) - c)]\, f(w)\, dw
\end{aligned} \tag{24}
$$

The values of p_1', p_2' that satisfy (24) completely depend on the tax and penalty ratios as well as the distribution of income. In general we can find some sufficient 2-level strategy. Then we can apply Theorem 2 so that we can find a 1-level strategy dominating our 3-level strategy.

3.3 Optimal cutoff level t

From section 3.1.1 and 3.2.1, we can conclude that in most cases, we find an optimal audit probability of the class of cutoff rules. The question now rises how to determine the optimal cutoff level, i.e: With an audit policy of the form

$$
p(x) = \begin{cases} \hat{p} & \text{if } x < t \\ 0 & \text{if } x \geq t \end{cases}
$$

What is the optimal t? We note that the $t = \infty$ is allowed. In this section we will derive conditions on the hand of which we can determine the optimal cutoff t. We do not derive an explicit expression for t. The theorem and the propositions in this section as well as the general outline of their proofs are due to Vasin. Since we will only discuss cutoff rules as audit strategies, it is sufficient to only denote the cutoff level t in order to specify which strategy is meant. We from hereon denote the revenue per taxpayer from a cutoff rule as an audit strategy with cutoff level t shortly by $R(t)$.

THEOREM 3.
In the class of cutoff rules, if the inequality

$$
\int_{w \geq t} \left[\theta(w-t) - \frac{c\theta}{\pi+\theta} \right] f(w)\, dw \geq 0 \tag{25}
$$

is satisfied for all $t \in \mathbb{R}$, then the optimal cutoff is found at $t = \infty$.

Proof.
We will prove this theorem by comparing the revenue from a cutoff at ∞ with the revenue from a cutoff at $t < \infty$. Suppose $t < \infty$. We thus compare $R(\infty)$ and $R(t)$. Note that in the first case, all

taxpayers will always declare their true income, where in the second case, whis will only happen for an income $w < t$. We find that

$$
\begin{aligned}
R(\infty) - R(t) &= \int_0^\infty [\theta w - \hat{p}c - \gamma] f(w)\, dw - \left(\int_0^t [\theta w - \hat{p}c - \gamma] f(w)\, dw + \int_t^\infty [\theta t - \gamma] f(w)\, dw \right) \\
&= \int_{w \geq t} [\theta(w - t) - \frac{c\theta}{\pi + \theta}] f(w)\, dw \geq 0
\end{aligned}
$$

We find that for every $t < \infty$, $R(\infty) > R(t)$, so the revenue for the audit rule where $t = \infty$ is greater than the revenue if $t < \infty$. We conclude that the optimal cutoff level is found at $t = \infty$.

■

Let us consider relationship (25) in more detail in order to find out when it does and does not hold. We find that it reduces to an equality of $f(w) = ke^{-w\frac{\theta+\pi}{c}}$ and holds as an inequality if $|\frac{f'(w)}{f(w)}| < \frac{\pi+\theta}{c}$ for $w > t$, i.e., if the income distribution has a fat tail.

We have derived the optimal cutoff in the case that (25) holds. We now proceed assuming that inequality (25) does not hold for some t. The optimization problem for t now in general has multiple solutions. However, for a wide class of distributions, the optimal value of t is unique. We show this by the following proposition:

Proposition 2
Assume that the probability density function of income distribution $f(w)$ has a single maximum and $|\frac{f'(w)}{f(w)}|$ increases in w for $f'(w) < 0$. Then $R(t)$ has at most one local maximum.

Before we start on the proof of this proposition, we first explain the assumptions that are made. A distribution density function that has a single maximum generally has a peak at some income level M, often called the modal income, and from thereon decreases towards 0. The part of the domain where $f'(w) < 0$ is thus the part where $w > M$. The assumption that $|\frac{f'(w)}{f(w)}|$ increases in w for $w > M$ prevents the distribution from having a fat tail.

Proof. We start by calculating $\frac{d^2}{dt^2}R(t)$:

$$
\begin{aligned}
\frac{d^2}{dt^2}R(t) &= \frac{d^2}{dt^2}\left(\int_0^t [\theta w - \hat{p}c] f(w)\, dw + \int_t^\infty \theta t f(w)\, dw - \gamma \right) \\
&= \frac{d^2}{dt^2}\left(\int_0^\infty [\theta w - \frac{\theta c}{\theta + \pi}] f(w) dw + \theta t (F(\infty) - F(t)) - \gamma \right) \\
&= \frac{d}{dt}\left(\theta t f(t) - \frac{\theta c}{\theta + \pi} f(t) + \theta(1 - F(t)) - \theta t f(t) \right) \\
&= -\frac{\theta c}{\theta + \pi} f'(t) - \theta f(t) = -\theta f(t) \left(1 + \frac{c f'(t)}{(\theta + \pi) f(t)} \right)
\end{aligned}
$$

Define t' as the supremum of the cutoff levels t for which $\frac{f'(t)}{f(t)} = -\frac{\theta+\pi}{c}$. We see that on the interval where $\frac{f'(t)}{f(t)} \geq -\frac{\theta+\pi}{c}$, the second derivative of the revenue function $R(t)$ is non-positive, so we have a concave revenue $R(t)$. The interval where $\frac{f'(t)}{f(t)} < -\frac{\theta+\pi}{c}$ is found when $t > t'$. Here, the revenue function is convex. On the first interval, there is at most one local maximum. In the second interval, the only possible optimum is $t = \infty$. Comparing these two variants. we find the optimal cutoff level t.

■

4 Comments

In this section we discuss the results found in the previous sections. We also discuss the the model itself. Is it a realistic model? Do the assumptions stroke with reality or are they too restrictive? We suggest what could be improved or what could be surveyed more carefully in future research.

Some of the results that are derived in this model may seem somewhat surprising. The optimal audit policy as a cutoff rule indeed implies that the tax administration only audits 'poor' individuals, though a higher return could be expected from auditing 'rich' individuals. The explanation for this paradox is that the tax administration does not know which individuals are rich and which are poor. It thus develops a strategy that holds for all taxpayers. If it was to reduce the audits on people who report a lower income, it might be advantageous for individuals with a high income to pretend they are poor. In this model, audit decisions can only be based on reported income. It may very well be that in reality, the tax administration can perform some screening at low cost to separate several audit groups of taxpayers which are more likely to have a high/low income. The administration can then develop a different audit policy for the individuals in each separate group. We however assumed a *homogeneous group of taxpayers*.

We also have to make a comment on the fact that we assumed the same audit cost c for every taxpayer, though it may very well be possible that audits on some individuals will be more costly than the audit on others. We can for instance imagine that individuals who are on payroll at a company are easier to audit than freelancers who work independently. It would be interesting to investigate how the model can be applied on several audit groups with different parameters.

As we mentioned before, some of the assumptions that we make are somewhat restrictive. Relaxing these assumptions will however undoubtedly complicate the treatment of the problem at hand. It is necessary that we critically look upon the assumptions to check in which regards the model may be improved. One of the most influential assumptions that is made is the *risk neutrality of the taxpayers*. In reality, many individuals will report their true income by moral incentives also, instead of only taking monetary incentives into consideration. One must also note that the penalty on the detection of tax evasion will in many cases not only be monetary. Other losses, such as loss of time, aggravation and damage to reputation will usually be involved in affecting the taxpayer's behaviour. Vasin includes these additional losses in his model by assuming they are equivalent to a certain additional amount of money. These implicit additional penalties are however usually a priori unknown and very subjective to the individual concerned.

Another difference between the model and reality that is easily noticed, is the assumption of *linearity in the tax obligation* $\theta(x)$. Many Western countries use a stepwise increasing tax obligation. Individuals with high income thus need to pay a relatively greater part of their income to the tax administration than individuals with a lower income do. Replacing our linear tax rate with such a stepwise increasing tax function would make calculation with the model much more difficult, but it would undoubtedly have a large impact on its results.

In our model, we only consider the taxpayer and the tax administration as actors in the income tax game. The government, who decides the tax obligation, penalty rates and the constant γ, is not taken into account. It however plays a large role in the process of redistribution of income through income tax. When implementing the role of the government in a model, one will find that formulating its objective can be complicated. One may try to measure it by some sort of social welfare function which is bounded to a minimal revenue. A restriction of the penalty rate is then also in order, since it may be unethical to fine individuals with an disproportional high amount of money, though it does effectively increase the revenue. When comparing our model with reality, we find that a penalty on tax evasion that is proportioned to the concealed income $w-x$ is quite realistic. One can however wonder whether this penalty proportionate to $w-x$ is desirable for the government. Are there more efficient penalty functions that are morally justified? One could also try to analyze in detail the intricate relationship between government and tax administration.

Future research on the subject could focus on improving some of the matters mentioned above. Especially adjustments to the tax function $\theta(x)$ and to the risk neutrality of the taxpayer seem promising. One could also focus on completing the proof of the optimality of the cutoff rule as an audit probability policy. When the model becomes more complicated, performing computer simulation might help finding optimality.

5 Examples

Some of the theorems above may be quite difficult to read due to the many notations that are used. In this section, we aim to clarify some notational difficulties by illustrating the proofs and statements of section 3.2 with examples. In Example 1, we show the result of Theorem 2 for a 2-level strategy. The main purpose of this example is to help the reader understand the proof by executing it. In Example 2, we work out a 3-level strategy under conditions for which Theorem 2 does not hold. We aim to give an idea of how we can still try to derive the optimality of the cutoff rule.

Example 1.
Let the tax rate $\theta = 0.2$ and the penalty rate $\theta = 0.3$. We then find the minimal deterrent probability $\hat{p} = \frac{\theta}{\theta+\pi} = 0.4$. We define the audit strategy

$$p(x) = \begin{cases} \hat{p} & \text{if } x < 10 \\ 0.3 & \text{if } 10 \leq x < 20 \\ 0.2 & \text{if } x \geq 20 \end{cases} \tag{26}$$

This corresponds to $t_1 = 10$, $t_2 = 20$, $p_1 = 0.3$, $p_2 = 0.2$.

We then derive $\hat{t}_1 = \hat{p}\frac{t_2 - t_1}{p_2 - p_1} + \frac{p_1 t_1 - p_2 t_2}{p_2 - p_1} = 0.4\frac{10}{0.1} + \frac{-1}{0.1} = 30$. An individual with an income $w = 30$ is thus indifferent to declaring $x = 10$ or $x = 20$. By convention he declares $x = 20$, since it is closer to his actual income. This gives the following taxpayer behaviour function:

$$x(w) = \begin{cases} w & \text{if } w < 10 \\ 10 & \text{if } 10 \leq w < 30 \\ 20 & \text{if } w \geq 30 \end{cases} \tag{27}$$

We will now consider a change $dp = (p_1 + d_1, p_2 + d_2)$ in the probability scheme for which $\hat{t}_1 = 30$ does not change. We consider the strategy $S(10, 20, 0.3 + zd_1, 0.2 + zd_2)$ for some z such that the strategy is admissible. We then suppose $d_2 = 1$ and find:

$$d_1 = d_2 \frac{\hat{t}_1 - t_2}{\hat{t}_1 - t_1} = \frac{20}{10} = \frac{1}{2} \tag{28}$$

The audit probability has now changed to

$$q(x) = \begin{cases} \hat{p} & \text{if } x < 10 \\ 0.3 + \frac{1}{2}z & \text{if } 10 \leq x < 20 \\ 0.2 + z & \text{if } x \geq 20 \end{cases} \tag{29}$$

This strategy leads to the same taxpayer behaviour function as the previous. We still have $x(w)$ as in (27). Note that our changed strategy is still bound to the restriction of $\hat{p} \geq p_1 \geq p_2 \geq 0$, which gives $-0.2 \leq z \leq 0.2$.

We now consider the revenue per taxpayer, gained from this strategy and find that its derivative is independent of z, which makes the revenue linear in z. It is thus dominated by the strategy where $z = 0.2$ or by the strategy where $z = -0.2$. In the first case, we find that $R(S(10, 20, 0.4, 0.4))$ is the random audit rule, which is 1-level cutoff probability with $t = \infty$. In the latter case, we find:

$$\begin{aligned}R(S(10,20,0.2,0)) &= \int_0^{10} [0.2w - 0.4c - \gamma]f(w)\,dw + \int_{10}^{30} [2 + 0.1(w-10) - 0.2c - \gamma]f(w)\,dw \\ &\quad + \int_{30}^{\infty} [4-\gamma]f(w)\,dw \\ &= \int_0^{10} [0.2w - 0.4c]f(w)\,dw + \int_{10}^{30} [1 + 0.1w - 0.2c]f(w)\,dw \\ &\quad + \int_{30}^{\infty} 4f(w)\,dw - \gamma\end{aligned}$$

We note that $\frac{1}{2}R(S(10,0)) + \frac{1}{2}R(S(30,0))$ yields the same revenue:

$$\begin{aligned}&\frac{1}{2}R(S(10,0)) + \frac{1}{2}R(S(30,0)) = \frac{1}{2}\int_0^{10} [0.2w - 0.4c - \gamma]f(w)\,dw + \frac{1}{2}\int_{10}^{\infty} [2-\gamma]f(w)\,dw \\ &+ \frac{1}{2}\int_0^{30} [0.2w - 0.4c - \gamma]f(w)\,dw + \frac{1}{2}\int_{30}^{\infty} [6-\gamma]f(w)\,dw \\ &= \int_0^{10} [0.2w - 0.4c]f(w)\,dw + \int_{10}^{30} [1 + 0.1w - 0.2c]f(w)dw + \int_{30}^{\infty} 4f(w)\,dw - \gamma\end{aligned}$$

So we know that one of these 1-level strategies yields a revenue that is greater than or equal to the revenue of our 2-level strategy. This concludes the example.

Example 2.
We again let the tax rate be $\theta = 0.2$ and the penalty rate $\pi = 0.3$, which gives the minimal deterrent probability $\hat{p} = 0.4$. We now let $t_1 = 10$, $t_2 = 20$, $t_3 = 30$ and $p_1 = 0.3$, $p_2 = 0.25$, $p_3 = 0.05$.

We now find the following values of indifference: $\hat{t}_1 = 40$, $\hat{t}_2 = 37.5$. We note that for these values, $\hat{t}_1 > \hat{t}_2$, so that Theorem 2 cannot be applied. We find the following taxpayer behaviour:

$$x(w) = \begin{cases} w & \text{if } w < 10 \\ 10 & \text{if } 10 \leq w < 40 \\ 30 & \text{if } w \geq 30 \end{cases} \tag{30}$$

We note that $t_2 = 20$ is never declared. One may wonder why we cannot just treat this case as a 2-level strategy on which we can apply Theorem 2. This will not hold, since although the taxpayer behaviour function acts as if we are dealing with a 2-level strategy, we need to realise that (30) is still the result of a 3-level strategy. Although the taxpayer will never declare $t_2 = 20$ and as a result the tax administration will never audit with probability $p_2 = 0.25$, the fact that it *would* do so if the taxpayer were to declare $x \in [20, 30)$ still affects the taxpayer behaviour. More concretely, if were to consider $S(10, 30, 0.3, 0.05)$, the value of the point of indifference would change to $\hat{t}_1 = 38$, which would then result in a different taxpayer behaviour function than we found in (30). If we can however prove that this 2-level strategy dominates our original 3-level strategy, we can still conclude the optimality of the cutoff rule. We compare the strategies by comparing their revenues. We find:

$$\begin{aligned}&R(S(10,20,30,0.3,0.25,0.05)) = \\ &\int_0^{10} [0.2w - 0.4c]f(w)dw + \int_{10}^{40} [0.2 \cdot 10 + 0.3 \cdot 0.5(w-10) - 0.3c]f(w)dw \\ &+ \int_{40}^{\infty} [0.2 \cdot 30 + 0.05 \cdot 0.5(w-30) - 0.05c]f(w)dw - \gamma\end{aligned} \tag{31}$$

and

$$R(S(10,30,0.3,0.05)) =$$
$$\int_0^{10}[0.2w - 0.4c]f(w)dw + \int_{10}^{38}[0.2\cdot 10 + 0.3\cdot 0.5(w-10) - 0.3c]f(w)dw$$
$$+\int_{38}^{\infty}[0.2\cdot 30 + 0.05\cdot 0.5(w-30) - 0.05c]f(w)dw \qquad (32)$$

so that

$$R(S(10,30,0.3,0.05)) - R(S(10,20,30,0.3,0.25,0.05)) = \int_{38}^{40}[4.75 + 0.25c - 0.125w]f(w)dw \qquad (33)$$

We see that expression (33) is dependent of the audit cost and the probability density function $f(w)$. We note however that in many cases we will find that our our original strategy $R(S(10,20,30,0.3,0.25,0.05))$ is in fact dominated by the 2-level strategy $R(S(10,30,0.3,0.05))$, on which we can successfully apply Theorem 2 to conclude that the original strategy is dominated by the cutoff rule.

If this does not work, we can also try to find another 2-level audit strategy $S(10,30,q_1,q_2)$ that yields the same point of indifference $\hat{t}_1 = 40$ but produces a larger net revenue. We show this approach for any 3-level strategy in section 3.2.3.

We can conclude that for many admissible strategies, we can find a way to use Theorem 2 in order to conclude that the cutoff rule is optimal. We can unfortunately not yet prove that this is always the case.

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