On Inflection Points of Plane Curves

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Introduction

The field of algebraic geometry has a long history. Its origin lies in simple objects such as lines and circles hundreds (or even thousands) of years ago. Nowadays, algebraic geometry has developed to be an abstract part of modern mathematics. In particular *projective geometry* has allowed unification and generalization. Its sense of 'points at infinity', which are not defined in the more restricted, yet intuitive, affine geometry. Algebraic geometry is a significant part of mathematics with neighbouring fields, such as number theory, differential geometry and even algebraic topology.

Curves are a well-known subject in algebraic geometry. In particular this thesis will treat curves in the projective plane. Not only the real or complex projective plane, as generally studied in topology, but even the projective plane over a field of positive characteristic. This is an even less intuitive space, nonetheless it entails compelling consequences. In particular we will investigate some specific points called *inflection points* on our plane curves. These points will be closely examined in relation to the Hessian. The Hessian can be considered as a multivariate generalization of the second derivative. The Hessian is not exclusively considered in algebraic geometry, a basic calculus course is often the first place to get acquainted with this notion. However, we consider it in a formal way.

The subject of this bachelor thesis originates from my supervisor Prof. Dr. C. Faber. For that and his continuing stimulating supervision I am very grateful. He first introduced some notions of algebraic geometry and then suggested some possible directions for me. He recommended the highly informative book *Algebraic Curves* by W. Fulton [4]. This book is the main source for this thesis. It has given me an introduction to algebraic geometry and layout for the proof of one theorem which plays a significant role in this thesis. The idea for the generalization of this theorem came from my supervisor, yet the proof is my own. The first two interesting plane curves and the original idea for the *pseudo-Hessian* were suggested by my supervisor, the examination I did myself. Similarly, the idea to investigate Bézout's theorem was my supervisor's, yet I found the results given there.

This thesis is organized as follows. The first chapter will cover some introductory material on algebraic geometry, hence it contains no new material. In the second chapter the relation between inflection points and the Hessian will be thoroughly examined in characteristic zero. This includes the proof of a general theorem. To continue our investigation in positive characteristic, we will explore some remarkable plane curves in characteristic two and three in the third chapter. Here we will introduce the notion of the pseudo-Hessian. In the fourth chapter we will regard Bézout's Theorem for plane curves of degree four and the corresponding (pseudo-)Hessian. Not only will we present a general result in characteristic zero, we will give a few examples in positive characteristic to illustrate the possibilities.

Contents

Chapter 1

Introductory material

This thesis will use some notions and theorems commonly occurring in algebraic geometry. The first section will discuss some algebra needed. The second section will discuss some notions of a more geometric nature. If one is familiar with this material, this chapter may be skipped. Though, I will refer to notions and definitions from the second section in particular.

1.1 Algebra

We will assume same basic knowledge of rings. If one is not familiar with ring theory, one may turn to *Rings and Galois theory*[1]. For those familiar with the basics of ring theory, here is a short summary of some important notions to freshen up. This section will be based on material from the previously suggested book.

Definition 1.1.1. *A commutative ring with 1 is a* ***field*** *if all nonzero elements have a multiplicative inverse and* $1 \neq 0$.

This means in particular that $\mathbb{Z}$ is not a field, since all elements, except for ± 1 have no inverse. However, $\mathbb{Q}$ is a field, because it contains $\frac{q}{p}$ for all $\frac{p}{q}$, with $p, q \neq 0$. Many known rings are in fact fields, such as $\mathbb{C}$ or even $\mathbb{Z}/p\mathbb{Z}$, when p is a prime number. We will always denote a field with the letter K. Rings that are not fields will not be considered in this thesis. We continue with some more theory on fields.

Definition 1.1.2. *A set* $K' \subset K$*, with* K *a field, is called a* ***subfield*** *of* K*, if it is a field with the inherited operations of* K*.*

Similarly we can define the opposite of a subfield.

Definition 1.1.3. *If* L *is a field and* $K \subset L$ *is a subfield, then* L *is called a* ***field extension*** *of* K*.*

In this thesis we will examine different types of fields. An important notion used to distinguish fields is the characteristic. This concept will be used extensively, so a good understanding of this is necessary.

Definition 1.1.4. *Let R be a ring with 1. If there exists a positive integer n such that the multiplicative unit n times added to itself is the additive unit, then the* ***characteristic*** *of R is n. If no such integer exists, the characteristic of R is defined to be 0. Notation: $Char(R)$.*

Recall that the characteristic is either 0 or a prime number. The characteristic of a ring is in most cases easy to determine. For example, $\text{Char}(\mathbb{Q}) = 0$ and $\text{Char}(\mathbb{Z}/p\mathbb{Z}) = p$, for a prime number p.

However, there are some fields, that we are especially interested in, which are a bit more complicated. We will examine the differences that arise from fields with different characteristic, e.g. the difference between $\text{Char}(K) = 0$ and $\text{Char}(K) = p$. The most common fields with prime characteristics are $\mathbb{Z}/p\mathbb{Z}$, with p prime, these are examples of finite fields.

Definition 1.1.5. *A* ***finite field*** *is a field with finite cardinality. Notation $\mathbb{F}_q$, where q is the number of elements.*

In this thesis we will explore some properties of polynomials over different fields. In order to do so, we will need a formal definition of a polynomial.

Definition 1.1.6. *A* ***polynomial*** *in X with coefficients in a ring R is an expression of the form*

$$a_n X^n + \ldots + a_1 X + a_0,$$

with $n \in \mathbb{Z}_{\geq 0}$ and $a_i \in R$ for all $i \in \{0, \ldots, n\}$. When $a_n \neq 0$, the degree of the polynomial is n. The set of all polynomials in X with coefficients in R is denoted by $R[X]$.

We call a polynomial **monic**, if $a_n = 1$.
An important notion we will require is whether a field is closed or not.

Definition 1.1.7. *A field K is called* ***algebraically closed*** *if every non-constant polynomial in $K[X]$ has a zero in K.*

This definition actually implies that any non-constant polynomial in $K[X]$ can be written as a product of linear factors if K is algebraically closed (see Th. 8.13 from *A First Course in Abstract Algebra* [3]). Later on, we will only work with algebraically closed fields. Finite fields are going to be some of the main fields examined in this thesis. The following theorem summarizes some very useful properties of finite fields.

Theorem 1.1.1 (Finite fields, [1], 11.1.2)**.** *Let K be a finite field with q elements.*

1. *Then there is a prime p and a positive integer n, such that $q = p^n$. Also, $Char(K) = p$.*

2. *For every prime p and $n \geq 1$ there exists exactly one, up to isomorphism, field with p^n elements. There are no other finite fields.*

3. *The subfields of $\mathbb{F}_{p^n}$ are exactly $\mathbb{F}_{p^m}$, with $m \mid n$.*

For the proof one may turn to *Rings and Galoistheory* [1].

1.2 Geometry

Once we have a sufficient foundation in algebra, we can move on to the geometry. We will again summarize a few basic notions needed to form a sufficient amount of knowledge on the required geometry. This section is almost entirely based on material from the book *Algebraic Curves* by W. Fulton [4]. The examples are my own, but are inspired by those given by Fulton. Later in this thesis some other notions will be introduced. Those are not standard, which is why they are not introduced in this chapter. We start off with the *projective plane*.

Definition 1.2.1. *We define the* ***projective plane*** *over a field K to be*

$$\mathbb{P}^2_K = \{l \subset K^3 \mid l \text{ is a line through the origin}\}.$$

A line is a one dimensional object. In our case, one can think of our elements in $\mathbb{P}^2_K$ as a conjugacy class of elements in K^3. Let (a, b, c) be some point in K^3, not the origin. The line can be though of as

$$l = \{\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid \lambda \in K\}.$$

The most common example of a projective plane is the projective plane with $K = \mathbb{R}$, where all elements in $\mathbb{P}^2_{\mathbb{R}}$ are in fact the straight lines through the origin. This is one of the first spaces one encounters when studying topology.

Definition 1.2.2. *The elements in the projective plane are represented by* ***projective coordinates*** *or* ***homogeneous coordinates****. The point $(a : b : c)$, where $a, b, c \in K$, not all zero, represents the line in K^3 that goes through the origin and (a, b, c). Therefore, homogeneous coordinates have the property $(a : b : c) = (\lambda a : \lambda b : \lambda c)$ for all nonzero $\lambda \in K$. Mathemaically speaking:*

$$(a : b : c) = \{\lambda \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid \lambda \in K\}.$$

Sometimes we might want to change the coordinate system, to simplify some computations.

Definition 1.2.3. *A* ***projective change of coordinates*** *T is a linear transformation of coordinates in the projective plane. It is a function from $\mathbb{P}^2_K$ to $\mathbb{P}^2_K$.*

Later we introduce an *affine change of coordinates*, these two notions are very similar. We can then explain more precisely what such a transformation entails.
In the projective plane, we will study *plane curves*. To define these, we first need to introduce some basic concepts on polynomials.

Definition 1.2.4. *The* ***degree*** *of a polynomial F is the highest degree of its terms, where the degree of a term is the sum of the exponents of all variables in that term. Notation:* $\deg F$*.*

For example, take the polynomial $F = XYZ + 2X^2Y^3 + 6Z^4Y$. The first term has degree 3, the last two terms both have degree 5. That means $\deg F = 5$. Polynomials always consist of a finite number of terms and a finite degree per variable. This means the degree of a polynomial is always finite.

Definition 1.2.5. *A polynomial $F \in K[X_1, \ldots, X_n]$ is called* ***homogeneous****, if all nonzero terms have the same degree.*

To clarify this definition, let us look at two examples. The polynomial $F = X^2Y^3 + 4XYZ^3 + 7Z^5$ is homogeneous, because all terms have degree 5. However, the polynomial $G = 2XZ^3 + 3Z^2Y^2 + 4X^5$, is not homogeneous, because the first two terms have degree 4 and the last has degree 5.
We can now define our curves of interest.

Definition 1.2.6. *An* ***algebraic plane curve*** *is a set C contained in a plane X, defined by a non-constant homogeneous polynomial F, of the form*

$$C = \{P \in X \mid F(P) = 0\}.$$

When we want to say that some point P is a zero of F, we write $P \in C$. We will be particularly interested in curves in the projective plane. Those curves will always have the following structure.

$$C = \{P = (x : y : z) \in \mathbb{P}^2_K \mid F(P) = 0\},$$

where F is a polynomial with coefficients in K. The degree of F is the degree of the curve C. Also, often we will denote the curve defined by the polynomial F with C_F. All curves observed in this thesis are algebraic plane curves. We will often call them plane curves or just curves.

We will be specifically interested in *smooth* curves and also, some objects derived form the original curve, such as the *projective tangent line* and the *Hessian.* To define these notions, we first need to introduce the formal derivative.

Definition 1.2.7. *Let R be a commutative ring and $R[X_1, \ldots, X_k]$ its polynomial ring. Let $X_j \in \{X_1, \ldots, X_k\}$. The* ***partial derivative*** *with respect to X_j is a map*

$$\frac{\partial}{\partial X_j} : R[X_1, \ldots, X_k] \longrightarrow R[X_1, \ldots, X_k],$$

defined by

$$\sum_{i=1} \alpha_i X_1^{n_{1,i}} \cdot \ldots \cdot X_k^{n_{k,i}} \longmapsto \sum_{i=1} \alpha_i n_{j,i} X_1^{n_{1,i}} \cdot \ldots \cdot X_j^{n_{j,i}-1} \cdot \ldots \cdot X_k^{n_{k,i}}.$$

When taking the partial derivative with respect to X_j and then with respect to X_l, we mean the composition $\frac{\partial}{\partial X_l} \circ \frac{\partial}{\partial X_j}$. We denote this by $\frac{\partial^2}{\partial X_l \partial X_j}$. This is called a mixed partial derivative, if $j \neq l$. Note that the order does not matter, so $\frac{\partial^2}{\partial X_l \partial X_j} = \frac{\partial^2}{\partial X_j \partial X_l}$ for all j, l. We can also take the derivative m times with respect to one variable X_j, i.e. the composition of m times $\frac{\partial}{\partial X_j}$. We will denote this with $\frac{\partial^m}{\partial X_j^m}$. We can combine as many derivatives as we want, but in this thesis we will only consider the first and second derivatives. This definition is formal, we do not involve limits. However, our definition completely agrees with how we intuitively expect to find the derivative of a polynomial.

In general, we will be interested in *smooth* curves.

Definition 1.2.8. *Let C be an algebraic plane curve, defined by $F = 0$, where $F \in K[X, Y, Z]$, is called* ***smooth*** *if*

$$\frac{\partial F}{\partial X}(P) \neq 0 \text{ or } \frac{\partial F}{\partial Y}(P) \neq 0 \text{ or } \frac{\partial F}{\partial Z}(P) \neq 0,$$

for all $P \in C$.

On a smooth curve we can define the *projective tangent line*:

Definition 1.2.9. *Let C be an algebraic plane curve defined by $F = 0$, with $F \in K[X, Y, Z]$. The* ***projective tangent line*** *of C at P is the curve $L = 0$ with*

$$L = \frac{\partial F}{\partial X}(P)X + \frac{\partial F}{\partial Y}(P)Y + \frac{\partial F}{\partial Z}(P)Z,$$

if not all partial derivatives are zero. Otherwise it is not defined.

On a plane curve which is not smooth, we can encounter the situation that all partial derivatives are zero. The projective tangent line is not defined. However, We can define a different tangent object in this situation.

Definition 1.2.10. *Let C be a curve defined by $F = 0$, where F is a homogeneous polynomial of degree n. The affine polynomial f, obtained from F by setting $Z = 1$ has terms of lowest degree. The sum of all terms of lowest degree is a polynomial T_F.*
The ***tangent cone*** *of C at a point P is the curve defined by $T_F = 0$.*

Note that since F must be non-constant, this lowest degree is at least one. In particular if the lowest degree is one, the projective tangent line will be defined for all points.

Now that we have seen some objects related to the first derivatives, we can also look at an object related to the second derivatives.

Definition 1.2.11. *The* ***Hessian*** *of a function F is the determinant of the matrix consisting of all second partial derivatives. Notation: H_F.*

$$H_F = \begin{vmatrix} \frac{\partial^2 F}{\partial X^2} & \frac{\partial^2 F}{\partial Y \partial X} & \frac{\partial^2 F}{\partial Z \partial X} \\ \frac{\partial^2 F}{\partial X \partial Y} & \frac{\partial^2 F}{\partial Y^2} & \frac{\partial^2 F}{\partial Z \partial Y} \\ \frac{\partial^2 F}{\partial X \partial Z} & \frac{\partial^2 F}{\partial Y \partial Z} & \frac{\partial^2 F}{\partial Z^2} \end{vmatrix}.$$

Note that in this thesis we will only compute the Hessian for polynomials, in that case: $\frac{\partial^2 F}{\partial Y \partial X} = \frac{\partial^2 F}{\partial X \partial Y}$, hence the matrix is symmetric.

To do computations in the projective plane can be challenging. In the *affine* plane, these computations can be become easier.

Definition 1.2.12. *The* ***affine n-space*** *over a field K is the set of n-tuples of elements of K. Notation $A^n(K)$, also the K is omitted, if the choice of K is clear.*

The **affine plane** is the affine 2-space. It is indeed very similar to the projective plane. One of the coordinates in the projective plane can be set 1, to get the affine plane. The elements in the projective space where this coordinate is zero, can be considered as points at infinity in the affine plane.

Definition 1.2.13. *An* ***affine change of coordinates*** *on A^n is a bijective homogeneous polynomial map $T = (T_1, \ldots, T_n) : A^n \to A^n$, such that T_i is a polynomial of degree 1 for all i.*

Often we represent T by a $n \times n$ matrix, also denoted with T, where the entries of the matrix are exactly the coefficients of the polynomials. For example, let $K = \mathbb{R}$ and $n = 3$ and $T = (T_1, T_2, T_3)$, when $T_1 = X + 2Y - 3Z$, $T_2 = 4X + Y$ and $T_3 = 2Z$, we get matrix

$$T = \begin{pmatrix} 1 & 2 & -3 \\ 4 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

This is bijective, because $\det(T) \neq 0$. We can relate this notion to the *projective* change of coordinates. When we have an affine change of coordinates in A^{n+1}, this induces a projective change of coordinates in $\mathbb{P}^n$. We are only interested in the projective plane, i.e. $n = 2$.
The space and curves which we will study are now known. To study a curve is to study its points, in particular, some special points. Obviously, some introduction to the classification of points on curves needs to follow. In order to do so, we will first introduce some tools to distinguish points.

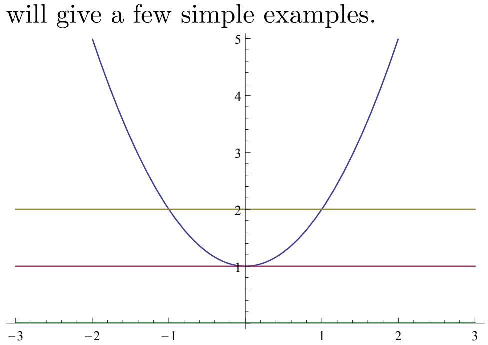
Definition 1.2.14. *Let C be an affine curve of degree n, defined by $F = 0$, where $F \in K[X, Y]$. Then there are $F_i \in K[X, Y]$ of degree i, such that*

$$F = \sum_{i=m}^{n} F_i.$$

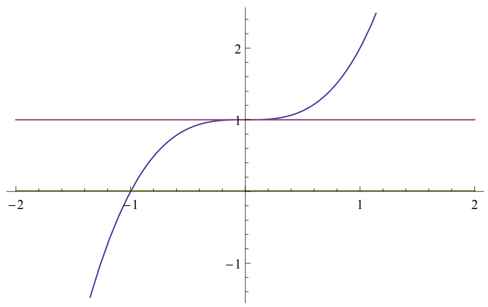
*The lowest degree m, such that $F_m \neq 0$, is the **multiplicity** of curve C at point P. Notation: $m_P(C)$.*

To explain this definition a little more, we will examine a few examples. First we examine curve C_F, defined by $F = 1 + 2X + 5XY + 3X^2 + XY^3$. We notice that the lowest degree occurring in F is 0, due to the constant term 1. A different curve C_G, defined by the polynomial $G = X^2Y^3 + 7X^4 + Y^3 + 12XY$, has no constant, nor linear term. The lowest term here is $12XY$, which has degree 2. We find $m_P(C_F) = 0$ and $m_p(C_G) = 2$.
Note that this definition only defines the multiplicity for $P = (0, 0)$. However, using a well chosen change of coordinates, i.e. one that translates P to $(0, 0)$, the multiplicity can be determined for any point. The multiplicity is independent of the chosen change of coordinates.
We can define the same notion for *projective* curves. Where F and F_i are now elements of $K[X, Y, Z]$ and $P = (0 : 0 : 1)$. The multiplicity of C_F at $(0 : 0 : 1)$ is defined as the multiplicity of C_f at $(0, 0)$. This f is the affine polynomial obtained from F by setting $Z = 1$.

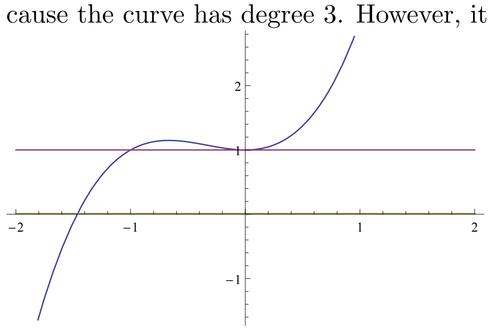
Once we have established that a point P is of interest, we want to investigate certain aspects. The intersection multiplicity of two curves at a point P is a number that attempts to reflect exactly what the name says. However, it is quite difficult to find a suitable definition that adheres to all rules we want it to. We not only want to include the 'obvious' intersection points, but also those at infinity. One can think of it as the order of contact between two curves. We will give a few simple examples.



We see the curves $Y = X^2 + 1$, $Y = 1$ and $Y = 2$. In $(0, 1)$ the tangent line to our curve is exactly $Y = 1$, we can see that the contact is tangent. We have contact of higher order, in this case it is two, since our curve is of degree two. However, the line $Y = 2$ intersects our curve in two points transversally. These intersections would both yield intersection multiplicity 1. We will continue with a different curve.



This is the curve defined by $Y = X^3 + 1$ and the tangent line $Y = 1$. Since the line is tangent at $(0, 1)$, we know that the intersection multiplicity is strictly larger than one. In this case we get intersection multiplicity 3, simply said because the curve has degree 3. However, it is a bit more complicated than this.



In this case we have $Y = X^3 + X^2 + 1$ and the line $Y = 1$. Only one term is added in comparison to the previous example. We still see that the line $Y = 1$ is tangent to our curve at $(0, 1)$. This implies the intersection multiplicity is larger than 1. Here it is 2, and the point $(-1, 1)$ has intersection multiplicity 1. One can consider the X^2 has a small change which has in some sense divided the intersection multiplicity over the two points, instead of just over the one we previously had.

Hopefully, the intuition for the concept of intersection multiplicity has been developed a little by these examples. We will now give the definition and give another example, where we will show how to calculate the intersection multiplicity of two curves.

Definition 1.2.15. *The* ***intersection multiplicity*** *of two plane curves F and G at the affine point P is defined by the following seven axioms. Notation: $I(P, F \cap G)$.*

1. *$I(P, F \cap G)$ is a non-negative integer if F and G intersect properly, i.e. F and G have no common factors. If they do not intersect properly, $I(P, F \cap G) = \infty$.*

2. *$I(P, F \cap G) = 0$ if and only if $P \notin F \cap G$.*

3. *If T is a affine change of coordinates on A^2 and $T(Q) = P$, then $I(Q, F(T) \cap G(T)) = I(P, F \cap G)$.*

4. $I(P, F \cap G) = I(P, G \cap F)$.

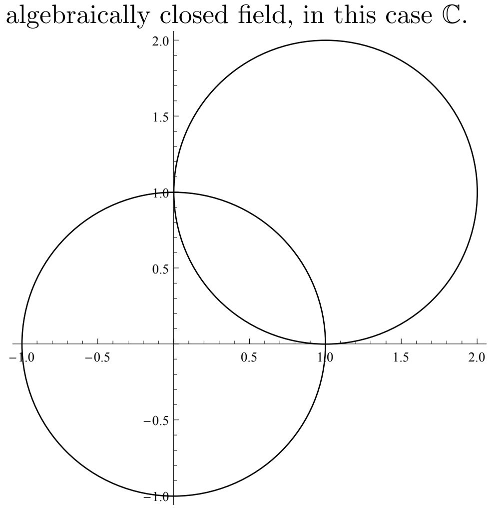
5. $I(P, F \cap G) \geq m_P(F)m_P(G)$. *Equality occurs if and only if the tangent cones of* F *and* G *have no common components.*

6. *If* $F = \prod_i F_i^{r_i}$ *and* $G = \prod_j G_j^{s_j}$, *with* F_i *and* G_j *polynomials, then* $I(P, F \cap G) = \sum_{i,j} r_i s_j I(P, F_i \cap G_j)$.

7. $I(P, F \cap G) = I(P, F \cap (G + HF))$, *for any* $H \in K[X, Y]$.

These axioms lead to a unique, yet unintuitive, definition, which we will not use.

Note that we can also talk about the intersection multiplicity of two projective curves. All axioms stay the same, except for nr. 3, where we should now talk about the projective instead of the affine change of coordinates, and nr. 7, where H is now an element of $K[X, Y, Z]$.
These axioms might seem quite abundant and not very intuitive. Therefore, we will present an example to clarify how to use the intersection multiplicity. We will consider two circles intersecting in two real points. We will always work in an algebraically closed field, in this case $\mathbb{C}$.



Let the first circle, denoted by C_1, be defined by $X^2 + Y^2 - 1 = 0$. The second circle C_2 by $(X - 1)^2 + (Y - 1)^2 - 1 = 0$. Our goal is to determine $I(P, C_1 \cap C_2)$ for P in the intersection. The expressions to define C_1 and C_2 are not homogeneous. We can make them homogeneous by adding a Z factor, when necessary. That way C_1 becomes $X^2 + Y^2 - Z^2 = 0$ and C_2 will be $(X - Z)^2 + (Y - Z)^2 - Z^2 = 0$. We can find four intersection points: $(0 : 1 : 1)$, $(1 : 0 : 1)$, $(1 : i : 0)$ and $(1 : -i : 0)$.
Let us calculate the intersection multiplicity for $P = (0 : 1 : 1)$. We need to go from this projective point to its affine companion. We may set $Z = 1$. We get c_1, defined by $X^2 + Y^2 - 1 = 0$ and c_2 by $(X - 1)^2 + (Y - 1)^2 - 1 = 0$. Where

our $P = (0:1:1)$ becomes its affine counterpart $p = (0,1)$. We now have that $I((0:1:1), C_1 \cap C_2) = I((0,1), c_1 \cap c_2)$. The calculation becomes much easier for $(0,0)$. That is why we will need an affine change of coordinates that substitutes Y by $Y'+1$. Note that property 3 allows us to do this. We get c_1' defined by $X^2 + (Y'+1)^2 - 1 = 0$ and c_2' by $(X-1)^2 + Y'^2 - 1 = 0$. We now have that $I((0,1), c_1 \cap c_2) = I((0,0), c_1' \cap c_2')$. This last expression is quite easy to determine using some other properties. We rewrite c_1' and c_2'.

$$\begin{aligned} c_1' &= X^2 + (Y'+1)^2 - 1 \\ &= X^2 + Y'^2 + 2Y'; \end{aligned}$$

$$\begin{aligned} c_2' &= (X-1)^2 + Y'^2 - 1 \\ &= X^2 + Y'^2 - 2X. \end{aligned}$$

Now we will compute the intersection multiplicity:

$$\begin{aligned} I((0,0), c_1' \cap c_2') =& I((0,0), X^2 + Y'^2 + 2Y' \cap X^2 + Y'^2 - 2X) \\ =& I((0,0), 2Y' - 2X \cap X^2 + Y'^2 - 2X) \\ =& I((0,0), 2 \cap X^2 + Y'^2 - 2X) \\ &+ I((0,0), Y' - X \cap X^2 + Y'^2 - 2X) \\ =& 0 + I((0,0), Y' - X \cap X^2 + XY' - 2X) \\ =& I((0,0), Y' - X \cap X^2 + X^2 - 2X) \\ =& I((0,0), Y' - X \cap 2) + I((0,0), Y' - X \cap X^2 - X) \\ =& 0 + I((0,0), Y' - X \cap X) + I((0,0), Y' - X \cap X - 1) \\ =& I((0,0), Y' \cap X) + 0 \\ =& m_{(0,0)}(Y') \cdot m_{(0,0)}(X) \\ =& 1 \cdot 1 = 1. \end{aligned}$$

First we have simply filled in c_1' and c_2', then we have used the properties of the intersection multiplicity. One may check that we used several properties in the following order: 7, 6, 2 and 7, 7, 6, 2 and 6, 2, 5 and we end with the use of the definition for $m_P(F)$. Which clearly must be 1, since Y' and X are both linear. We end up with an intersection multiplicity of 1. This make sense, intuitively, because we can see that the lines truly intersect, and have no tangent lines in common. The calculation above was done very precisely, in general we will do these type of calculations less explicitly and omit certain easy steps.

The intersection multiplicity gives a lot of information on the points on curves. We introduce some definitions to classify these different kinds of points. In particular, we will finally introduce *flexes*, the main type of points studied in this thesis.

Definition 1.2.16. *A point P is called* ***simple*** *on a curve F, if $\frac{\partial F}{\partial X}(P) \neq 0$ or $\frac{\partial F}{\partial Y}(P) \neq 0$. If a point is not simple, it is a multiple point.*

Definition 1.2.17. *A simple point P on a curve F is an* ***inflection point*** *if $I(P, F \cap L) \geq 3$, where L is the tangent line to F at P.*

We will often abbreviate inflection point to **flex**. When $I(P, F \cap L) = 3$, we call P an ordinary flex. When $I(P, F \cap L) = 4$, it is called a hyperflex. In general we call P a multiflex if $I(P, F \cap L) \geq 4$.

Chapter 2

Flexes and the Hessian in characteristic zero

There is a significant relation between the flexes of a curve F and its Hessian. This relation is represented in the following theorem.

Theorem 2.0.1 ([4], p. 116)**.** *Let C_F be a projective plane curve defined by $F = 0$ in $\mathbb{P}^2_K$, where K is an algebraically closed field with characteristic zero. Assume C_F contains no lines. Let H be the Hessian of F and C_H the curve defined by $H = 0$.*

1. *Then $P \in C_F \cap C_H$ if and only if P is either a flex or a multiple point of C_F.*
2. *$I(P, C_F \cap C_H) = 1$ if and only if P is an ordinary flex.*

In Fulton's book [4] this theorem is part of an exercise. The exercise gives a step-by-step plan to prove the theorem. We will prove something more general than this theorem. Both statements will actually be just a lemma and a short proof away, once we have proven this general theorem.

Theorem 2.0.2. *Let C_F be a projective plane curve of degree n, defined by $F = 0$, with F a homogeneous polynomial. Assume C_F contains no lines. Let H be the Hessian of F and C_H the curve defined by $H = 0$. Let P be a simple point on C_F and C_L is the projective tangent line, defined by $L = 0$, then*

$$I(P, C_F \cap C_H) = I(P, C_F \cap C_L) - 2,$$

where C_L represents the projective tangent line of C_F.

The steps given by Fulton will be taken. However, some steps can be done more generally than Fulton suggests. We will then take a different route. Once we have proved this general statement, we will prove the theorem in Fulton's. As mentioned, the proof consists of several steps. We will dedicate each of the following sections to one of the steps mentioned in the book. To make it easier to follow, we will try to explain the necessity of each step. However, sometimes several steps are needed, so not all steps can be directly used after being proven.

2.1 Change of coordinates

The theorem holds for all P. However, many things are easier for $P = (0:0:1)$. For example the intersection multiplicity (see Def. 1.2.15) as one can see in the example given there. We want to use any affine or projective change of coordinates, without it complicating the computations too much. Because we want to translate any P to the projective $(0:0:1)$ or the affine $(0,0)$. We know that this works for F, but we want to manipulate the Hessian too. We have a lemma that expresses a very nice property, which will allow us to work with any projective or affine change of coordinates freely.

Lemma 2.1.1. *Let T be a projective change of coordinates in $\mathbb{P}^2_K$, where K is an algebraically closed field with characteristic 0. Then the Hessian of $F(T)$, $H_{F(T)}$, equals* $\det(T)^2 H_F(T)$.

Proof. Since T is a linear transformation, the chain rule can be applied. We write $T = (T_1, T_2, T_3)$, where $T_i(Y_1, Y_2, Y_3)$ and $F(X_1, X_2, X_3)$. Recall that $F(T) = F(T_1, T_2, T_3)$.
The Hessian is the determinant of a matrix consisting of all second partial derivatives, this matrix will be denoted with a tilde. So, $H_F = \det(\tilde{H}_F)$ and similarly $H_{F(T)} = \det(\tilde{H}_{F(T)})$. First we compute $\tilde{H}_{F(T)}$, starting with the first and second partial derivatives of $F(T)$. These are quite complicated, so we will determine them step by step. We determine the first partial derivative of $F(T)$ with respect to Y_j with the chain rule:

$$\begin{aligned} \frac{\partial F(T)}{\partial Y_j} &= \frac{\partial T_1}{\partial Y_j}\frac{\partial F}{\partial X_1}(T) + \frac{\partial T_2}{\partial Y_j}\frac{\partial F}{\partial X_2}(T) + \frac{\partial T_3}{\partial Y_j}\frac{\partial F}{\partial X_3}(T) \\ &= \sum_{n=1}^{3} \frac{\partial T_n}{\partial Y_j}\frac{\partial F}{\partial X_n}(T). \end{aligned}$$

The second partial derivatives can be determined. We start by determining the partial derivative of $\frac{\partial F}{\partial X_n}(T)$ with respect to Y_i. As before, we use the chain rule:

$$\frac{\partial}{\partial Y_i}\left(\frac{\partial F}{\partial X_n}(T)\right) = \sum_{k=1}^{3} \frac{\partial T_k}{\partial Y_i}\frac{\partial^2 F}{\partial X_n \partial X_k}(T).$$

The computation of the second derivatives of $F(T)$ consists of a few steps. We use some basic properties of derivatives, such as the derivative of a sum is the sum of the derivatives and the common product rule. Another nice fact is that the second derivatives of T_i are always zero, because T_i is a linear polynomial. These rules together with the two expressions above give us the following computation.

$$\begin{aligned}
\frac{\partial^2 F(T)}{\partial Y_i \partial Y_j} &= \frac{\partial}{\partial Y_i}\left(\frac{\partial F(T)}{\partial Y_j}\right) \\
&= \sum_{n=1}^{3} \frac{\partial}{\partial Y_i}\left(\frac{\partial T_n}{\partial Y_j}\frac{\partial F}{\partial X_n}(T)\right) \\
&= \sum_{n=1}^{3} \left(\frac{\partial}{\partial Y_i}\left(\frac{\partial T_n}{\partial Y_j}\right)\frac{\partial F}{\partial X_n}(T) + \frac{\partial T_n}{\partial Y_j}\frac{\partial}{\partial Y_i}\left(\frac{\partial F}{\partial X_n}(T)\right)\right) \\
&= \sum_{n=1}^{3} \left(\frac{\partial^2 T_n}{\partial Y_j \partial Y_i}\frac{\partial F}{\partial X_n}(T) + \frac{\partial T_n}{\partial Y_j}\frac{\partial}{\partial Y_i}\left(\frac{\partial F}{\partial X_n}(T)\right)\right) \\
&= \sum_{n=1}^{3} \left(0 + \frac{\partial T_n}{\partial Y_j}\left(\sum_{k=1}^{3}\frac{\partial T_k}{\partial Y_i}\frac{\partial^2 F}{\partial X_n \partial X_k}(T)\right)\right) \\
&= \sum_{n=1}^{3} \left(\frac{\partial T_n}{\partial Y_j}\left(\sum_{k=1}^{3}\frac{\partial T_k}{\partial Y_i}\frac{\partial^2 F}{\partial X_n \partial X_k}(T)\right)\right).
\end{aligned}$$

Note that in this notation the variables Y_m are omitted, to avoid larger expressions. We have now found the (i, j)-th entry of $\tilde{H}_{F(T)}$. We wish to recognize some matrix multiplication with $\tilde{H}_F(T)$ and T. Let us first determine T, where we use that $t_{i,j} = \frac{\partial T_i}{\partial Y_j}$ (Def. 1.2.13).

$$T = \begin{pmatrix} \frac{\partial T_1}{\partial Y_1} & \frac{\partial T_1}{\partial Y_2} & \frac{\partial T_1}{\partial Y_3} \\ \frac{\partial T_2}{\partial Y_1} & \frac{\partial T_2}{\partial Y_2} & \frac{\partial T_2}{\partial Y_3} \\ \frac{\partial T_3}{\partial Y_1} & \frac{\partial T_3}{\partial Y_2} & \frac{\partial T_3}{\partial Y_3} \end{pmatrix}.$$

We also need $\tilde{H}_F(T)$. That is the matrix of all second derivatives of F with the change of coordinates T implemented at the end.

$$\tilde{H}_F(T) = \begin{pmatrix} \frac{\partial^2 F}{\partial X_1^2}(T) & \frac{\partial^2 F}{\partial X_1 \partial X_2}(T) & \frac{\partial^2 F}{\partial X_1 \partial X_3}(T) \\ \frac{\partial^2 F}{\partial X_1 \partial X_2}(T) & \frac{\partial^2 F}{\partial X_2^2}(T) & \frac{\partial^2 F}{\partial X_2 \partial X_3}(T) \\ \frac{\partial^2 F}{\partial X_1 \partial X_3}(T) & \frac{\partial^2 F}{\partial X_2 \partial X_3}(T) & \frac{\partial^2 F}{\partial X_3^2}(T) \end{pmatrix}.$$

We can now calculate $T^t \tilde{H}_F(T) T$. Doing the complete computation gives a very large complicated matrix. We rather just compute one entry of the product. Let us compute the (i, j)-th entry. This results from the i-th row of T^t, $\tilde{H}_F(T)$ and the j-th column of T.

$$\begin{pmatrix} \frac{\partial T_1}{\partial Y_i} & \frac{\partial T_2}{\partial Y_i} & \frac{\partial T_3}{\partial Y_i} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 F}{\partial X_1^2}(T) & \frac{\partial^2 F}{\partial X_1 \partial X_2}(T) & \frac{\partial^2 F}{\partial X_1 \partial X_3}(T) \\ \frac{\partial^2 F}{\partial X_1 \partial X_2}(T) & \frac{\partial^2 F}{\partial X_2^2}(T) & \frac{\partial^2 F}{\partial X_2 \partial X_3}(T) \\ \frac{\partial^2 F}{\partial X_1 \partial X_3}(T) & \frac{\partial^2 F}{\partial X_2 \partial X_3}(T) & \frac{\partial^2 F}{\partial X_3^2}(T) \end{pmatrix} \begin{pmatrix} \frac{\partial T_1}{\partial Y_j} \\ \frac{\partial T_2}{\partial Y_j} \\ \frac{\partial T_3}{\partial Y_j} \end{pmatrix}.$$

Let us first calculate the i-th row multiplied with $\tilde{H}_F(T)$.

$$\begin{pmatrix} \frac{\partial T_1}{\partial Y_i} & \frac{\partial T_2}{\partial Y_i} & \frac{\partial T_3}{\partial Y_i} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 F}{\partial X_1^2}(T) & \frac{\partial^2 F}{\partial X_1 \partial X_2}(T) & \frac{\partial^2 F}{\partial X_1 \partial X_3}(T) \\ \frac{\partial^2 F}{\partial X_1 \partial X_2}(T) & \frac{\partial^2 F}{\partial X_2^2}(T) & \frac{\partial^2 F}{\partial X_2 \partial X_3}(T) \\ \frac{\partial^2 F}{\partial X_1 \partial X_3}(T) & \frac{\partial^2 F}{\partial X_2 \partial X_3}(T) & \frac{\partial^2 F}{\partial X_3^2}(T) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=1}^{3} \frac{\partial T_k}{\partial Y_j} \frac{\partial^2 F}{\partial X_1 \partial X_k}(T) & \sum_{k=1}^{3} \frac{\partial T_k}{\partial Y_j} \frac{\partial^2 F}{\partial X_2 \partial X_k}(T) & \sum_{k=1}^{3} \frac{\partial T_k}{\partial Y_j} \frac{\partial^2 F}{\partial X_3 \partial X_k}(T) \end{pmatrix}.$$

Then we multiply this result with the j-th column of T to finally get the (i, j)-th entry of $T^t \tilde{H}_F(T)T$.

$$\begin{pmatrix} \sum_{k=1}^{3} \frac{\partial T_k}{\partial Y_i} \frac{\partial^2 F}{\partial X_1 \partial X_k}(T) & \sum_{k=1}^{3} \frac{\partial T_k}{\partial Y_i} \frac{\partial^2 F}{\partial X_2 \partial X_k}(T) & \sum_{k=1}^{3} \frac{\partial T_k}{\partial Y_i} \frac{\partial^2 F}{\partial X_3 \partial X_k}(T) \end{pmatrix} \begin{pmatrix} \frac{\partial T_1}{\partial Y_j} \\ \frac{\partial T_2}{\partial Y_j} \\ \frac{\partial T_3}{\partial Y_j} \end{pmatrix}$$

$$= \sum_{n=1}^{3} \left(\frac{\partial T_n}{\partial Y_j} \left(\sum_{k=1}^{3} \frac{\partial T_k}{\partial Y_i} \frac{\partial^2 F}{\partial X_n \partial X_k}(T) \right) \right).$$

This leads to the conclusion that

$$\tilde{H}_{F(T)} = T^t \tilde{H}_F(T)T.$$

We may now take the determinant and finally obtain the identity presented in the lemma:

$$\begin{aligned} H_{F(T)} &= \det(\tilde{H}_{F(T)}) \\ &= \det(T^t \tilde{H}_F(T)T) \\ &= \det(T^t) \det(\tilde{H}_F(T)) \det(T) \\ &= \det(T) H_F(T) \det(T) \\ &= \det(T)^2 H_F(T). \end{aligned}$$

□

T is an invertible matrix, so $\det(T) \neq 0$. In the projective plane, we may freely multiply a representative by any scalar. In this case we are talking about the factor $\det{(T)}^2$, which leads to $H_{F(T)} = H_F(T)$. This equality is valid for any projective change of coordinates T, in particular the transformation $\tilde{T}$, such that our point of interest P is translated to $P_{\tilde{T}} = (0 : 0 : 1)$. As we will encounter later, this is a very useful translation for computations. We can even ensure the linear part only consists of a term in Y with the linear transformation which replaces the linear term with Y. This $F(\tilde{T})$ we obtain meets the requirements of our lemma. This last change leads us to the affine functions f and h, such that $f(X,Y) = \tilde{F}(X : Y : 1)$ and $h(X,Y) = H(\tilde{T})(X : Y : 1)$. Note that the linear term of f only consists of a term in Y and the constant term is zero. This allows us to use f and h instead of F and H. From now on, anything we prove for f, h and $P = (0,0)$ is also true for F, H and any P.

2.2 Euler's identity

In the proof we define a new function g. The next section is dedicated to this g. We want to prove that this g has a certain property. To prove that in the next section, we need Euler's identity. We will first prove this identity before we define this g. We will express Euler's identity in a lemma:

Lemma 2.2.1. *Let F be a non-constant homogeneous polynomial of degree n with coefficients in a field K and k variables $X_1, \ldots, X_k$ with degree n, then*

$$nF = \sum_{j=1}^{k} X_j \frac{\partial F}{\partial X_j}. \tag{2.1}$$

Proof. Since F is a polynomial of degree n, the polynomial $\frac{\partial F}{\partial X_j}$ is homogeneous polynomial of degree $n-1$ or identically zero. The latter happens if and only if F contains no X_j in any of its terms. Let us write F as the sum of its terms:

$$F = \sum_i \alpha_i X_1^{n_{1,i}} \cdots X_k^{n_{k,i}}$$

where α_i is a nonzero element of K and $n_{m,i}$ a non negative integer. We can express the homogeneity of F more algebraically:

$$n = \sum_{m=1}^{k} n_{m,i} \text{ , for all } i$$

As mentioned before, we know $\frac{\partial F}{\partial X_j}$ is identically zero if and only if F contains no X_j. We can rephrase this to $n_{j,i} = 0$ for all i. On the other hand, if $n_{j,i} \neq 0$ for some i, we obtain a homogeneous polynomial of degree $n-1$. Let us take the partial derivative of F with respect to X_j.

$$\frac{\partial F}{\partial X_j} = \sum_i \alpha_i X_1^{n_{1,i}} \cdots n_{j,i} X_j^{n_{j,i}-1} \cdots X_k^{n_{k,i}}.$$

To find our identity, we need to multiply by X_j

$$\frac{\partial F}{\partial X_j} X_j = \sum_i \alpha_i n_{j,i} X_1^{n_{1,i}} \cdots X_j^{n_{j,i}} \cdots X_k^{n_{k,i}}.$$

Now we sum over all variables X_j and after some manipulations we obtain our desired identity:

$$\begin{aligned} \sum_{j=1}^{k} \frac{\partial F}{\partial X_j} X_j &= \sum_i \left(\sum_{j=1}^{k} \alpha_i n_{j,i} X_1^{n_{1,i}} \cdots X_k^{n_{k,i}} \right) \\ &= \sum_i \left(\alpha_i X_1^{n_{1,i}} \cdots X_k^{n_{k,i}} \sum_{j=1}^{k} n_{j,i} \right) \\ &= \sum_i \alpha_i X_1^{n_{1,i}} \cdots X_k^{n_{k,i}} \cdot n \\ &= n \sum_i \alpha_i X_1^{n_{1,i}} \cdots X_k^{n_{k,i}} \\ &= nF. \end{aligned}$$

□

For the proof of the general theorem (2.0.2) we will use Euler's identity, where $k = 3$ and $X_1 = X$, $X_2 = Y$ and $X_3 = Z$, leading to the following equality:

$$X_1 \frac{\partial F}{\partial X_1} + X_2 \frac{\partial F}{\partial X_2} + X_3 \frac{\partial F}{\partial X_3} = nF. \tag{2.2}$$

Furthermore, we also want a similar expression for the first partial derivatives. To do that these first partial derivatives must have degree greater or equal to 1, in particular these partial derivatives cannot be identically zero. Note that the degree of these first derivatives is exactly $n - 1$, when the degree of F is n. We obtain the following identity for all $j = 1, 2, 3$

$$X_1 \frac{\partial^2 F}{\partial X_j \partial X_1} + X_2 \frac{\partial^2 F}{\partial X_j \partial X_2} + X_3 \frac{\partial^2 F}{\partial X_j \partial X_3} = (n-1) \frac{\partial F}{\partial X_j}. \tag{2.3}$$

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2.3 Auxiliary function g

As mentioned in the previous section, we define a new curve g. We need to do this because the Hessian is a function, which does not allow certain computations to be as easy as we want them to be. This g however, looks very different, but in fact captures the same information has the Hessian. Let us define this mysterious g:

$$g := \left(\frac{\partial f}{\partial Y}\right)^2 \frac{\partial^2 f}{\partial X^2} + \left(\frac{\partial f}{\partial X}\right)^2 \frac{\partial^2 f}{\partial Y^2} - 2\frac{\partial f}{\partial X}\frac{\partial f}{\partial Y}\frac{\partial^2 f}{\partial X \partial Y}. \tag{2.4}$$

We can see that this is a relatively small expression compared to what the Hessian looks like. One can imagine that this function is easier to use in computations. It turns out that the intersection multiplicity for g and h with f is the same. This is expressed in the following lemma:

Lemma 2.3.1. *For functions f, h and g the following equality holds for all P:*

$$I(P, f \cap h) = I(P, f \cap g).$$

Proof. To form g, we will look at h as the Hessian of f in matrix form, before reduction. In the determinant we may freely add rows and columns. Also, because we are interested in $I(P, F \cap H)$, with $P = (0,0,1)$, we may use $Z \neq 0$.

We start with h:

$$\begin{vmatrix} \frac{\partial^2 f}{\partial X^2} & \frac{\partial^2 f}{\partial Y \partial X} & \frac{\partial^2 f}{\partial Z \partial X} \\ \frac{\partial^2 f}{\partial X \partial Y} & \frac{\partial^2 f}{\partial Y^2} & \frac{\partial^2 f}{\partial Z \partial Y} \\ \frac{\partial^2 f}{\partial X \partial Z} & \frac{\partial^2 f}{\partial Y \partial Z} & \frac{\partial^2 f}{\partial Z^2} \end{vmatrix}.$$

We multiply the last row with Z. Now we are computing hZ. Then add the first row times X and the second row times Y to the third. This gives us the following matrix:

$$\begin{vmatrix} \frac{\partial^2 f}{\partial X^2} & \frac{\partial^2 f}{\partial Y \partial X} & \frac{\partial^2 f}{\partial Z \partial X} \\ \frac{\partial^2 f}{\partial X \partial Y} & \frac{\partial^2 f}{\partial Y^2} & \frac{\partial^2 f}{\partial Z \partial Y} \\ X\frac{\partial^2 f}{\partial X^2} + Y\frac{\partial^2 f}{\partial X \partial Y} + Z\frac{\partial^2 f}{\partial X \partial Z} & X\frac{\partial^2 f}{\partial Y \partial X} + Y\frac{\partial^2 f}{\partial Y^2} + Z\frac{\partial^2 f}{\partial Y \partial Z} & X\frac{\partial^2 f}{\partial Z \partial X} + Y\frac{\partial^2 f}{\partial Z \partial Y} + Z\frac{\partial^2 f}{\partial Z^2} \end{vmatrix}.$$

We use Euler's identity (see equation 2.3) to reduce this to a matrix with simpler entries:

$$\begin{vmatrix} \frac{\partial^2 f}{\partial X^2} & \frac{\partial^2 f}{\partial Y \partial X} & \frac{\partial^2 f}{\partial Z \partial X} \\ \frac{\partial^2 f}{\partial X \partial Y} & \frac{\partial^2 f}{\partial Y^2} & \frac{\partial^2 f}{\partial Z \partial Y} \\ (n-1)\frac{\partial f}{\partial X} & (n-1)\frac{\partial f}{\partial Y} & (n-1)\frac{\partial f}{\partial Z} \end{vmatrix}.$$

Then we do the same thing to the columns. First multiplying the last column with Z. The result is hZ^2. Again we add the first column times X and the second row times Y to the third.

$$\begin{vmatrix} \frac{\partial^2 f}{\partial X^2} & \frac{\partial^2 f}{\partial Y \partial X} & X\frac{\partial^2 f}{\partial X^2} + Y\frac{\partial^2 f}{\partial X \partial Y} + Z\frac{\partial^2 f}{\partial X \partial Z} \\ \frac{\partial^2 f}{\partial X \partial Y} & \frac{\partial^2 f}{\partial Y^2} & X\frac{\partial^2 f}{\partial Y \partial X} + Y\frac{\partial^2 f}{\partial Y^2} + Z\frac{\partial^2 f}{\partial Y \partial Z} \\ \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} & (n-1)(X\frac{\partial f}{\partial X} + Y\frac{\partial f}{\partial Y} + Z\frac{\partial f}{\partial Z}) \end{vmatrix}.$$

We can use Euler's identity again on all right components (see Eq. 2.2 and 2.3).

$$\begin{vmatrix} \frac{\partial^2 f}{\partial X^2} & \frac{\partial^2 f}{\partial Y \partial X} & (n-1)\frac{\partial f}{\partial X} \\ \frac{\partial^2 f}{\partial X \partial Y} & \frac{\partial^2 f}{\partial Y^2} & (n-1)\frac{\partial f}{\partial Y} \\ \frac{\partial f}{\partial X} & \frac{\partial f}{\partial Y} & (n-1)nf \end{vmatrix}.$$

Now we can calculate hZ^2 using the determinant formula for 3×3 matrices:

$$\begin{aligned} hZ^2 =&(n-1)\frac{\partial f}{\partial X}\left(\frac{\partial^2 f}{\partial X \partial Y}\frac{\partial f}{\partial Y} - \frac{\partial f}{\partial X}\frac{\partial^2 f}{\partial Y^2}\right) \\ &+ (n-1)\frac{\partial f}{\partial Y}\left(\frac{\partial^2 f}{\partial X \partial Y}\frac{\partial f}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial^2 f}{\partial X^2}\right) \\ &+ (n-1)nf\left(\frac{\partial^2 f}{\partial X^2}\frac{\partial^2 f}{\partial Y^2} - \left(\frac{\partial^2 f}{\partial X \partial Y}\right)^2\right). \end{aligned}$$

Rearranging the terms leads to a simple expression for hZ^2:

$$hZ^2 = (1-n)g + (n-1)nf\left(\frac{\partial^2 f}{\partial X^2}\frac{\partial^2 f}{\partial Y^2} - \left(\frac{\partial^2 f}{\partial X \partial Y}\right)^2\right). \tag{2.5}$$

We can see that $hZ^2 = (1-n)g + f\phi$, with ϕ some polynomial, if $n > 1$. Since F contains no lines, in particular F cannot be a line, therefore we know $n > 1$. Let us consider $I(P, C_f \cap C_{hZ^2})$. We can calculate this in two ways. The first method:

$$\begin{aligned} I(P, C_f \cap C_{hZ^2}) &= I(P, C_f \cap C_h) + I(P, C_f \cap Z^2) \\ &= I(P, C_f \cap C_h) + 2 \cdot I(P, C_f \cap Z) \\ &= I(P, C_f \cap C_h). \end{aligned}$$

Note that $Z \neq 0$ implies $I(P, C_f \cap Z) = 0$. Now the second method:

$$\begin{aligned} I(P, C_f \cap C_{hZ^2}) &= I(P, C_f \cap C_{(1-n)g+f\phi}) \\ &= I(P, C_f \cap C_{(1-n)g}) \\ &= I(P, C_f \cap C_g). \end{aligned}$$

□

The intersection multiplicity $I(P, F \cap H)$ is invariant under affine change of coordinates. Thus $I(P, F \cap H) = I(P, f \cap h)$. We shall use the following identity many times:

$$I(P, C_F \cap C_H) = I(P, C_f \cap C_g). \tag{2.6}$$

2.4 Proof of general theorem

We have done enough to finally prove the general theorem (2.0.2), which said:

Let C_F be a projective plane curve of degree n, defined by $F = 0$, with F a homogeneous polynomial. Assume C_F contains no lines. Let H be the Hessian of F and C_H the curve defined by $H = 0$. Let P be a simple point on C_F, then

$$I(P, C_F \cap C_H) = I(P, C_F \cap C_L) - 2.$$

Proof. We use a change of coordinates such that P is moved to $(0 : 0 : 1)$. Then we let $Z = 1$ to obtain the affine curve. Recall that the linear term must be nonzero, because P is simple. We may use a change of coordinates such that the linear term of our affine curve is Y. We now have an affine function f, defining a affine curve C_f. We have proven $I(P, C_H \cap C_F) = I(P, C_f \cap C_g)$. Note that the C_L is translated to only Y, because this is the linear term. We denote this affine curve with C_l, where $l = Y$. First we find a nice expression for $I(P, C_f \cap C_l)$, where $P = (0, 0)$. Remember that C_F contains no lines, and therefore F and f cannot be divisible by Y. This means f has at least one pure X term. Let N be the degree of this term. We know $N > 1$, since the linear term only consists of Y. Let $f = f_1 + f_2$, where f_1 only contains terms not divisible by Y and f_2 contains all other terms, which are of course all divisible by Y. Using property 7 of the intersection multiplicity, we obtain that

$$I(P, C_f \cap C_l) = I(P, (C_{f_1} + C_{f_2}) \cap C_l) = I(P, C_{f_1} \cap Y).$$

Since f_1 only consists of terms in X, we know that $f_1 = a_1X + \ldots + a_nX^n$. Remember that $N = \min\{k \mid a_k \neq 0\}$. This allows us to write $f_1 = a_NX^N + \ldots + a_nX^n$, where per definition $a_N \neq 0$. We will show that $I(P, C_{f_1} \cap C_l) = N$.

$$\begin{aligned}
I(P, C_{f_1} \cap C_l) &= I(P, a_NX^N + \ldots + a_nX^n \cap Y) \\
&= I(P, X^N(a_N + \ldots + a_nX^{n-N}) \cap Y) \\
&= I(P, X^N \cap Y) + I(P, a_N + \ldots + a_nX^{n-N} \cap Y) \\
&= I(P, X^N \cap Y) + 0 \\
&= NI(P, X \cap Y) \\
&= N.
\end{aligned}$$

In the calculation above we used properties 6 (twice) and 2 of the intersection multiplicity (see Def. 1.2.15). Now we found that

$$I(P, C_f \cap C_l) = N. \tag{2.7}$$

We now wish to show that $I(P, C_f \cap C_g) = N - 2$. Let us observe where the term a_NX^N of f 'appears' in g.

$$g := \left(\frac{\partial f}{\partial Y}\right)^2 \frac{\partial^2 f}{\partial X^2} + \left(\frac{\partial f}{\partial X}\right)^2 \frac{\partial^2 f}{\partial Y^2} - 2\frac{\partial f}{\partial X}\frac{\partial f}{\partial Y}\frac{\partial^2 f}{\partial X \partial Y}.$$

Since $\frac{\partial f}{\partial Y}$ yields the constant term 1 and $\frac{\partial^2 f}{\partial X^2}$ gives us $a_N N(N-1)X^{N-2}$ as lowest term not divisible by Y. All other terms in g are either divisible by Y or are of higher degree than $N-2$.
We shall (recursively) define functions g_i, t_i and ϕ_i. We start with $g_0 := g$, and $g_{i+1} = g_i - f\phi_i$. This ϕ_i is defined through g_i and t_i.

$$\phi_i := \begin{cases} t_i/Y & \text{if } Y \mid t_i \\ 0 & \text{otherwise.} \end{cases}$$

This t_i is the sum of all terms of lowest degree of g_i. The curve C_{t_i} defined by $t_i = 0$ is the tangent cone (see Def. 1.2.10). We are interested in the curves C_{g_i}, which are defined through $g_i = 0$, the curves C_{t_i}, defined by $t_i = 0$, and lastly the curves C_{ϕ_i}, defined by $\phi_i = 0$.
The choice of such functions might be unclear, let us clarify. We want to obtain a curve C_{g_ι} such that its tangent cone is not divisible by Y. However, we want to achieve this solely by subtracting multiples of f, to preserve the intersection multiplicity.
We shall now prove the general theorem using these functions. We know the lowest term in f is Y, all other terms are of degree 2 or higher. We write $f = Y + \chi$, where $\deg(\chi) \geq 2$. As mentioned before we have chosen g_i such that the intersection multiplicity is not influenced. Let us say this in a more mathematical way for all $i \geq 0$:

$$\begin{aligned} I(P, C_f \cap C_{g_{i+1}}) &= I(P, C_f \cap C_{g_i} - C_f C_\phi) \\ &= I(P, C_f \cap C_{g_i}). \end{aligned}$$

We can use property 7 of the intersection multiplicity, because ϕ was either zero or a polynomial. We claim there is a smallest $\iota \in \mathbb{Z}_{\geq 0}$, such that $g_{\iota+1} = g_\iota$.
Let us prove this claim by contradiction. Assume there is no such ι. That means $g_{i+1} \neq g_i$, for all $i \in \mathbb{Z}_{\geq 0}$. That can only happen if $\phi \neq 0$, which can only happen if $Y \mid t_i$ for all $i \in \mathbb{Z}_{\geq 0}$. The tangent cone can only be divisible if no pure X term is part of the tangent cone. We shall prove that $\deg t_{i+1} > \deg t_i$ if $g_{i+1} \neq g_i$. By assumption $g_{i+1} \neq g_i$. Thus $Y \mid t_i$. We obtain

$$\begin{aligned} g_{i+1} &= g_i - f\phi_i \\ &= g_i - (Y + \chi)\phi_i \\ &= g_i - Y\phi_i - \chi\phi_i \\ &= g_i - t_i - \chi\phi_i. \end{aligned}$$

We know $g_i - t_i$ only consists of terms with degree strictly larger than the degree of t_i, per definition of t_i. Since χ only consists of terms with degree 2 or higher, the product $\chi\phi_i$ can only contain terms of degree strictly larger than $\phi_i + 1$. Recall that t_{i+1} is the sum of all terms with lowest degree in g_{i+1}, i.e.

$$\begin{aligned}\deg(t_{i+1}) &\geq \min\{\deg(g_i - t_i), \deg(\chi\phi_i)\}\\ &= \min\{\deg(g_i - t_i), \deg(\chi) + \deg(\phi_i)\}\\ &= \min\{\deg(g_i - t_i), \deg(\chi) + \deg(t_i) - 1\}\\ &\geq \min\{\deg(g_i - t_i), 2 + \deg(t_i) - 1)\}\\ &= \min\{\deg(g_i - t_i), 1 + \deg(t_i)\}\\ &> \deg(t_i).\end{aligned}$$

We get an strictly increasing sequence of degrees here. This, however, is impossible. Remember the $a_N N(N-1)X^{N-2}$ term in $g = g_0$. When f is multiplied with ϕ_0, it is impossible to eliminate this term, since the lowest purely X term in f is $a_N X^N$. We therefore know this X^{N-2} term will survive all manipulations with f. More precisely, we know we will have a term of degree $N-2$ in g_i for all i. This makes it impossible for the degree of the lowest terms to exceed $N-2$. However, we had just proven the degree is strictly increasing for all i. This is the contradiction we were looking for.
This means we now know that there must be a $\iota \in \mathbb{Z}_{\geq 0}$, such that $g_{\iota+1} = g_\iota$. We let ι be the smallest integer with this property.The fact that $g_{\iota+1} = g_\iota$ implies $\phi_\iota = 0$, since f is nonzero. We know $\phi_\iota = 0$ if and only if $Y \nmid t_\iota$. Which means there must a term purely consisting of X in t_ι. We know the lowest term purely consisting of X is $a_N N(N-1)X^{N-2}$. Therefore this specific term must be part of t_ι. Recall that the tangent cone of ϕ_ι is defined through $t_\iota = 0$. We know there is a purely X term in the tangent cone of ϕ_ι, which means it is not divisible by Y. This is very useful, because this means the tangent cones share no common lines. We can use property 5 of the intersection multiplicity with equality.

$$\begin{aligned}I(P, C_f \cap C_{g_\iota}) &= m_P(C_f) \cdot m_P(C_{g_\iota})\\ &= 1 \cdot m_P(C_{g_\iota})\\ &= \deg(t_\iota)\\ &= N - 2.\end{aligned}$$

We are almost there, because we only need the fact that $I(P, C_f \cap C_{g_\iota}) = I(P, C_f \cap C_g)$, which is a result of applying $I(P, C_f \cap C_{g_{i+1}}) = I(P, C_f \cap C_{g_i})$ a finite number of times. We conclude:

$$\begin{aligned}I(P, C_f \cap C_g) &= I(P, C_f \cap C_{g_\iota})\\ &= N - 2.\end{aligned}$$

□

We have proven our general theorem. Its strength might not be as evident now, but in the next section we will prove the theorem from Fulton's [4] quite easily. We only need one extra lemma regarding multiple points.

2.5 Multiple points

The theorem in Fulton's book makes a distinction between (ordinary) flexes and multiple points. This can be done using the following lemma:

Lemma 2.5.1. *If P is a multiple point on C_F, then $I(P, C_F \cap C_H) \geq 2$.*

Proof. If P is multiple point on C_F, it is not simple on C_F. Then it cannot be simple on C_f. That means $\frac{\partial f}{\partial X}(P) = \frac{\partial f}{\partial Y}(P) = 0$. This leads to the easy computation of $g(P)$:

$$\begin{aligned} g(P) &= \left(\frac{\partial f}{\partial Y}(P)\right)^2 \frac{\partial^2 f}{\partial X^2}(P) + \left(\frac{\partial f}{\partial X}(P)\right)^2 \frac{\partial^2 f}{\partial Y^2}(P) - 2\frac{\partial f}{\partial X}(P)\frac{\partial f}{\partial Y}(P)\frac{\partial^2 f}{\partial X \partial Y}(P) \\ &= 0^2 \frac{\partial^2 f}{\partial X^2}(P) + 0^2 \frac{\partial^2 f}{\partial Y^2}(P) - 2 \cdot 0^2 \frac{\partial^2 f}{\partial X \partial Y}(P) \\ &= 0. \end{aligned}$$

This means $P \in C_g$ and now we know that $m_P(C_g) \geq 1$. Also, $m_P(C_f) \geq 2$, because $P \in C_f$ and $m_P(C_f) = 1$ if and only if P is simple on C_f, which it is not, since P is not simple on F. The last step uses a property of the intersection number. $I(P, C_f \cap C_g) \geq m_P(C_f) m_P(C_g) \geq 2$. Which means $I(P, C_F \cap C_H) \geq 2$. □

2.6 Proof of the first statement of Fulton's theorem

We have now gathered enough to prove the two statements in Fulton's book [4]. The first statement in the theorem stated the following:

> $P \in C_H \cap C_F$ *if and only if* P *is either a flex or a multiple point of* C_F

Proof. First, we will prove the ($\Rightarrow$) implication. We assume $P \in C_H \cap C_F$ and P is not a multiple point of C_F, we want to prove that P must be a flex. We assume that P is not a multiple point, then it must be simple. This allows us to use our general theorem. Since $P \in C_H \cap C_F$, we know that $I(P, C_H \cap C_F) \geq 1$, by definition. We apply the general theorem and get that $I(P, C_f \cap Y) = I(P, C_F \cap Y) \geq 1 + 2 = 3$. That is precisely the definition of a flex.

Secondly, the ($\Leftarrow$) implication. We will prove this first for P is a flex and then for P is a multiple point.
If P is a flex, then $I(P, C_F \cap Y) \geq 3$. Because P is a flex, it must be simple. We use our general theorem to say that $I(P, C_F \cap C_H) \geq 1$. That is precisely saying that $P \in C_H \cap C_F$.
If P is multiple point on C_F, we use the lemma on multiple points, that said $I(P, C_F \cap C_H) \geq 2 \geq 1$. So $P \in C_F \cap C_H$. □

2.7 Proof of the second statement of Fulton's theorem

The second statement of Fulton's book, said the following:

$I(P, C_H \cap C_F) = 1$ *if and only if* P *is an ordinary flex*

Proof. Let us make a distinction between whether P is a simple or a multiple point.
If P is simple, we may use our general theorem, which says $I(P, C_F \cap C_H) = I(P, C_F \cap C_L) - 2$. We see that $I(P, C_H \cap C_F) = 1$ if and only if $I(P, C_F \cap Y) = 3$, which means P is an ordinary flex.
If P is not simple, thus multiple, we use the lemma on multiple points. That said $I(P, C_F \cap C_H) \geq 2$. So, it can never be 1 for a multiple point.
We see that indeed ordinary flexes are the only points that have an intersection multiplicity of 1 with the Hessian. □

Chapter 3

Interesting curves in nonzero characteristic

The general theorem 2.0.2 was only stated for $\mathbb{P}^2_K$, where K was a field of characteristic zero. However, we are also interested in fields with positive characteristic. In this chapter we examine a few interesting curves in $\mathbb{P}^2_K$, with K a field, which has positive characteristic. We are particularly interested in smooth curves, because that way we know all points are simple, and thus are eligible to be a flex.

3.1 An interesting curve in characteristic 3

We will consider the polynomial $F = X^3Y + Y^3Z + Z^3X$ and its curve $C_F = \{P = (a : b : c) \in \mathbb{P}^2_K \mid F(P) = 0\}$, where K is an algebraically closed field of characteristic 3.

First we check whether this curve is smooth or not. Let us determine the first derivatives

$$\frac{\partial F}{\partial X} = 3X^2Y + Z^3 = Z^3;$$
$$\frac{\partial F}{\partial Y} = 3Y^2Z + X^3 = X^3;$$
$$\frac{\partial F}{\partial Z} = 3Z^2X + Y^3 = Y^3.$$

Only when all partial derivatives equal zero in a point $P = (a : b : c)$, the curve is not smooth at P. This happens, when $a^3 = b^3 = c^3 = 0$, which implies $a = b = c = 0$. However, $P = (0 : 0 : 0) \notin \mathbb{P}^2_K$. We conclude that C_F is a smooth plane curve.

3.1.1 Multiplicity

To find the multiplicity of a flex, we need the projective tangent line (Def. 1.2.9):

$$\frac{\partial F}{\partial X}(P)X + \frac{\partial F}{\partial Y}(P)Y + \frac{\partial F}{\partial Z}(P)Z = 0.$$

For a point $P = (a : b : c) \in C_F$, the projective tangent line C_L is defined by

$$c^3X + a^3Y + b^3Z = 0.$$

For any point P in C_F we can calculate the intersection multiplicity of C_F and the projective tangent line C_L at P. Let $P = (a : b : c) \in C_F$. Then a, b and c are not all equal to zero, therefore we may assume $c \neq 0$, due to the symmetry of F. This implies c^{-1} exists and multiplying the coordinates of P by c^{-1} does not change P, thus we may assume $c = 1$, hence $P = (a : b : 1)$. The tangent line C_L at P is

$$X + a^3Y + b^3Z = 0.$$

The fact that $P \in C_F$ implies

$$a^3b + b^3 + a = 0. \tag{3.1}$$

Substituting $X = -a^3Y - b^3Z$ in $F = 0$, we obtain a homogeneous polynomial G of degree 4 in Y and Z. We will use $(\alpha + \beta)^3 = \alpha^3 + \beta^3$ (characteristic is 3) and $a^3b = -b^3 - a$ (see Eq. 3.1) to factorize G:

$$\begin{aligned}
G &= (-a^3Y - b^3Z)^3Y + Y^3Z + Z^3(-a^3Y - b^3Z) \\
&= ((-a^3Y)^3 + (-b^3Z)^3)Y + Y^3Z + Z^3(-a^3Y - b^3Z) \\
&= (-a^3Y)^3Y + (-b^3Z)^3Y + Y^3Z - a^3YZ^3 - b^3Z^4 \\
&= -a^9Y^4 - b^9YZ^3 + Y^3Z - a^3YZ^3 - b^3Z^4 \\
&= -a^9Y^4 + (-a^3 - b^9)YZ^3 + Y^3Z - b^3Z^4 \\
&= -a^9Y^4 + (-a - b^3)^3YZ^3 + Y^3Z - b^3Z^4 \\
&= -a^9Y^4 + (a^3b)^3YZ^3 + Y^3Z - b^3Z^4 \\
&= (-Y^3)(a^9Y) + (a^9Y)(bZ)^3 + (-Y^3)(-Z) + (bZ)^3(-Z) \\
&= (-Y^3 + (bZ)^3)(a^9Y - Z) \\
&= (-Y + bZ)^3(a^9Y - Z).
\end{aligned}$$

We are looking for the multiplicity at $P = (a : b : 1)$, the first factors $-Y + bZ$ clearly yield a multiplicity of 3 for all P. The $a^9Y - Z$ factor can increase the multiplicity to 4. This happens if and only if $a^9b = 1$. This can only happen if $a \neq 0$, hence we may substitute $b = a^{-9}$:

$$\begin{aligned}
a^3b + b^3 + a &= 0 \\
a^3a^{-9} + (a^{-9})^3 + a &= 0 \\
a^{-6} + a^{-27} + a &= 0 \\
a^{21} + 1 + a^{28} &= 0
\end{aligned}$$

This is a function in a, so we can determine the derivative $21a^{20}+28a^{27}=a^{27}$. The derivative is zero if and only if $a=0$. However, $a\neq 0$, which means the derivative is never vanishing. This implies all zeros are distinct. There are exactly 28 different hyperflexes. All other points in C_F are ordinary flexes. This is quite extraordinary. If we were to consider a field with characteristic zero, it is impossible for all points on a smooth curve to be inflection points. There is a theorem that implies this fact.

Theorem 3.1.1 (Sard's Theorem [5], 6.10)**.** *Suppose M and N are smooth manifolds with or without boundary and $F: M\to N$ is a smooth map. Then the set of critical values of F has measure zero in N.*

Remark. *One might not immediately see the consequences of this theorem. But it basically says that any smooth function between smooth manifolds, for example $\mathbb{R}$, $\mathbb{C}$, can only have a 'small' set of critical points, i.e. where all partial derivatives are zero. Small is not the correct formal terminology, saying that the Lebesgue measure of a set is zero, however, is the formal way of saying that. For those unfamiliar with measure theory, we present two examples in $\mathbb{R}$ with Euclidean geometry to get a feeling for what it means for the Lebesgue measure to be zero: any countable set has measure zero and any interval containing more than one element has measure greater than zero. One can think of this theorem as basically saying that only a few points, not an interval, can be critical on a smooth manifold. In our case we are considering a smooth curve. We see that much more is possible, when the characteristic is positive, compared to sets such as $\mathbb{R}$ and $\mathbb{C}$ which have characteristic zero.*
If one is interested in the particulars of measure theory and the Lebesgue measure, one may turn to Measures, Integrals and Martingales by R.L. Schilling [6].

3.2 An interesting curve in characteristic 2

We will now examine the properties of the polynomial $F=X^4+XY^3+YZ^3$ and its curve C_F defined by $F=0$. We will consider it in $\mathbb{P}^2_K$, where K is an algebraically closed field of characteristic 2. We will again see some remarkable results, yet for different reasons than the curve in the previous section. Nonetheless, we want our curve to be smooth. This holds if and only if

$$\frac{\partial F}{\partial X}(P)\neq 0 \text{ or } \frac{\partial F}{\partial Y}(P)\neq 0 \text{ or } \frac{\partial F}{\partial Z}(P)\neq 0.$$

We start by determining these derivatives:

$$\begin{aligned}\frac{\partial F}{\partial X}&=4X^3+Y^3=Y^3;\\ \frac{\partial F}{\partial Y}&=3XY^2+Z^3=XY^2+Z^3;\\ \frac{\partial F}{\partial Z}&=3YZ^2=YZ^2.\end{aligned}$$

If all first derivatives are zero at $P = (a : b : c)$, then the derivative with respect to X gives us $c^3 = 0 \Rightarrow c = 0$. The derivative with respect to Y then yields $ab^2 + c^3 = 0 \Rightarrow c^3 = 0 \Rightarrow c = 0$. However, to be an element of C_F, $a^4 + ab^3 + bc^3 = 0 \Rightarrow a^4 = 0 \Rightarrow a = 0$. Remember that $(0 : 0 : 0) \notin C_F$, so we see that for all elements in C_F not all first derivatives are zero. Therefore, C_F is a smooth curve.

3.2.1 Hessian and pseudo-Hessian

Let us determine the Hessian H of F. First all second derivatives

$$\begin{aligned}
&\frac{\partial^2 F}{\partial X^2} = 0; \\
&\frac{\partial^2 F}{\partial Y^2} = 2XY = 0; \\
&\frac{\partial^2 F}{\partial Z^2} = 2YZ = 0; \\
&\frac{\partial^2 F}{\partial X \partial Y} = 3Y^2 = Y^2; \\
&\frac{\partial^2 F}{\partial X \partial Z} = 0; \\
&\frac{\partial^2 F}{\partial Y \partial Z} = Z^2; \\
&.
\end{aligned}$$

The Hessian can be computed:

$$H = \begin{vmatrix} 0 & Y^2 & 0 \\ Y^2 & 0 & Z^2 \\ 0 & Z^2 & 0 \end{vmatrix} = 0.$$

We can see that the Hessian is identically zero. This is in fact not very special. All degree 4 polynomials yield an identically zero Hessian in characteristic 2. We express this in a lemma:

Lemma 3.2.1. *Let F be a homogeneous polynomial in three variables X_1, X_2 and X_3 and coefficients in K, where K is an algebraically closed field of characteristic 2, then the Hessian H_F of F is identically zero.*

Proof. Let us first write F as a sum of its terms:

$$F = \sum_{i=1} \alpha_i X_1^{n_{1,i}} X_2^{n_{2,i}} X_3^{n_{3,i}}.$$

We want to determine $\frac{\partial^2 F}{\partial X_j^2}$. For all j any term will get a factor $n_{j,i} \cdot (n_{j,i} - 1)$, when taking the derivative with respect to the same variable twice. We know either $n_{j,i}$ or $n_{j,i} - 1$ is even, resulting in an even product. This means we get an even scalar in front of every term. In characteristic 2, that means all terms are zero. We see that $\frac{\partial^2 F}{\partial X_j^2} = 0$ for all j. This simplifies the Hessian nearly enough. Let us determine H_F.

$$H_F = \begin{vmatrix} \frac{\partial^2 F}{\partial X_1^2} & \frac{\partial^2 F}{\partial X_2 \partial X_1} & \frac{\partial^2 F}{\partial X_3 \partial X_1} \\ \frac{\partial^2 F}{\partial X_1 \partial X_2} & \frac{\partial^2 F}{\partial X_2^2} & \frac{\partial^2 F}{\partial X_3 \partial X_2} \\ \frac{\partial^2 F}{\partial X_1 \partial X_3} & \frac{\partial^2 F}{\partial X_2 \partial X_3} & \frac{\partial^2 F}{\partial X_3^2} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial^2 F}{\partial X_2 \partial X_1} & \frac{\partial^2 F}{\partial X_3 \partial X_1} \\ \frac{\partial^2 F}{\partial X_1 \partial X_2} & 0 & \frac{\partial^2 F}{\partial X_3 \partial X_2} \\ \frac{\partial^2 F}{\partial X_1 \partial X_3} & \frac{\partial^2 F}{\partial X_2 \partial X_3} & 0 \end{vmatrix}$$

The determinant can be computed now. A lot of terms of the Hessian will be zero, due to the zero valued entries.

$$\begin{aligned} H_F &= -\frac{\partial^2 F}{\partial X_1 \partial X_2}\left(0 - \frac{\partial^2 F}{\partial X_2 \partial X_3}\frac{\partial^2 F}{\partial X_3 \partial X_1}\right) + \frac{\partial^2 F}{\partial X_1 \partial X_3}\left(\frac{\partial^2 F}{\partial X_2 \partial X_1}\frac{\partial^2 F}{\partial X_3 \partial X_2} - 0\right) \\ &= \frac{\partial^2 F}{\partial X_1 \partial X_2}\frac{\partial^2 F}{\partial X_2 \partial X_3}\frac{\partial^2 F}{\partial X_3 \partial X_1} + \frac{\partial^2 F}{\partial X_1 \partial X_3}\frac{\partial^2 F}{\partial X_2 \partial X_1}\frac{\partial^2 F}{\partial X_3 \partial X_2} \\ &= 2 \cdot \frac{\partial^2 F}{\partial X_1 \partial X_2}\frac{\partial^2 F}{\partial X_2 \partial X_3}\frac{\partial^2 F}{\partial X_3 \partial X_1} \\ &= 0 \end{aligned}$$

□

In Chapter 2 we have seen some very nice properties regarding flexes and the Hessian. This lemma basically says that we cannot extract any information from the Hessian.
However, we can define a new notion, similar to the Hessian, which is not identically zero for all polynomials. To make that happen we have to eliminate some of these factors that are divisible by 2. To define this *pseudo-Hessian*, we first need to define an alternative to the regular second derivative.

Definition 3.2.1. *The **pseudo-second derivative** with respect to X_j is a map*

$$\frac{\tilde{\partial}^2}{\partial X_j^2} : K[X_1, \dots X_k] \longrightarrow K[X_1, \dots X_k],$$

with K an algebraically closed field, which sends

$$\sum_{i=1} \alpha_i X_1^{n_{1,i}} \cdot \ldots \cdot X_k^{n_{k,i}} \longmapsto \begin{cases} \sum_{i=1} \alpha_i \binom{n_{j,i}}{2} X_1^{n_{1,i}} \cdot \ldots \cdot X_j^{n_{j,i}-2} \cdot \ldots \cdot X_k^{n_{k,i}} & \text{if } n_{j,i} \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

We will use this definition for a field K with characteristic 2. Since $n_{j,i}$ is an integer, we know $\binom{n_{j,i}}{2}$ is an integer and thus $\alpha_i \binom{n_{j,i}}{2}$ is an element of K. This map is well defined, now we can move on to the definition of the pseudo-Hessian.

Definition 3.2.2. *The **pseudo-Hessian** of a polynomial $F \in K[X_1, X_2, X_3]$, with K an algebraically closed field. is a map $\tilde{H}_F : \mathbb{P}^2_K \to K$, defined by*

$$\begin{aligned}\tilde{H}_F := &\frac{\partial^2 F}{\partial X_1 \partial X_2}\frac{\partial^2 F}{\partial X_2 \partial X_3}\frac{\partial^2 F}{\partial X_3 \partial X_1} + \frac{\tilde{\partial}^2 F}{\partial X_1^2}\left(\frac{\partial^2 F}{\partial X_2 \partial X_3}\right)^2 \\ &+ \frac{\tilde{\partial}^2 F}{\partial X_2^2}\left(\frac{\partial^2 F}{\partial X_1 \partial X_3}\right)^2 + \frac{\tilde{\partial}^2 F}{\partial X_3^2}\left(\frac{\partial^2 F}{\partial X_1 \partial X_2}\right)^2.\end{aligned}$$

We will often we interested in the curve $\tilde{H} = 0$, we will denote this by $C_{\tilde{H}}$.

Remark. *Of course, we have defined this pseudo-Hessian for a field K with characteristic 2. The choice of such a pseudo-Hessian might be unclear, let us try to clarify its definition. Recall the proof of Lemma 3.2.1, in which we saw that all pure second derivatives are identically zero. The $\binom{n_{j,i}}{2}$ factor in the pseudo-second derivative prevents this from happening. Let us write $\binom{n_{j,i}}{2} = \frac{n_{j,i}(n_{j,i}-1)}{2}$. In a way, we simple 'divided' the regular second derivative by 2, but we did it in such a way that it is well defined. One could wonder why the pseudo-Hessian has less terms than the regular Hessian. This sense of dividing by 2 is what we want for the entire Hessian, but not for every factor in every term of the Hessian. Let us clarify, we only want to eliminate one factor 2 from each term. The term containing all pure second derivatives is then still divisible by 2, i.e. identically zero. Also there is one entirely mixed term, containing no pure second derivatives, which occurs twice. These terms will be the same and therefore, in a field of characteristic 2, the sum of these will always be zero. However, when we divide by two, we only have one mixed term left. If one combines all these facts, the definition of the pseudo-Hessian can be thought of as the regular Hessian divided by 2, before reduction modulo 2.*

We can extract no information from the regular Hessian of a function, nevertheless we claim that this pseudo-Hessian does contain relevant information about a curve and its flexes. We will continue with C_F as previously defined by $X^4 + XY^3 + YZ^3 = 0$. Let us determine the pseudo-Hessian $\tilde{H}_F$. First we determine the pseudo-second derivatives:

$$\begin{aligned}\frac{\tilde{\partial}^2 F}{\partial X^2} &= \binom{4}{2} X^2 = 6X^2 = 0; \\ \frac{\tilde{\partial}^2 F}{\partial Y^2} &= \binom{3}{2} XY = 3XY = XY; \\ \frac{\tilde{\partial}^2 F}{\partial Z^2} &= \binom{3}{2} YZ = 3YZ = YZ.\end{aligned}$$

Now the (regular) mixed second derivatives are given by:

$$\frac{\partial^2 F}{\partial X \partial Y} = 3Y^2 = Y^2;$$
$$\frac{\partial^2 F}{\partial X \partial Z} = 0;$$
$$\frac{\partial^2 F}{\partial Y \partial Z} = 3Z^2 = Z^2.$$

Then we get $\tilde{H}_F$ using the definition of the pseudo-Hessian (Def. 3.2.2):

$$\tilde{H}_F = Y^2 \cdot Z^2 \cdot 0 + 0 \cdot (Z^2)^2 + XY \cdot 0^2 + YZ \cdot (Y^2)^2 = Y^5 Z.$$

The pseudo-Hessian is not identically zero, that allows for a lot more than the regular Hessian. In the next section we will show that in fact the pseudo-Hessian contains similar information in characteristic 2 as the regular Hessian does when the characteristic is zero.

3.2.2 Flexes and the pseudo-Hessian

Our general theorem was only valid for characteristic zero. We want to attribute similar qualities to our pseudo-Hessian, when the characteristic is 2. Currently we have a curve C_F defined by $F = 0$ and the pseudo-Hessian $\tilde{H}_F$. We will define the curve $C_{\tilde{H}}$ by $\tilde{H}_F = 0$. We will show that $P \in C_F \cap C_{\tilde{H}}$ if and only if P is a flex.

First we will prove ($\Rightarrow$) Let us first examine the elements in $C_F \cap C_{\tilde{H}}$. Let $P = (a : b : c)$, where a, b and c are not all zero. We know that $P \in C_{\tilde{H}}$ if and only if $b = 0$ or $c = 0$. If we want $P \in C_F$ and $b = 0$, then we get $a^4 = 0$, which means $a = 0$. Since not all coordinates can be zero $c \neq 0$. $P = (0 : 0 : c) = (0 : 0 : 1)$. If we have $P \in C_F \cap C_{\tilde{H}}$ and $b \neq 0$, then $c = 0$. We get $a^4 + ab^3 = 0$. So all $P = (a : b : 0) = (\alpha : 1 : 0) \in C_F \cap C_{\tilde{H}}$ with $\alpha^4 + \alpha = 0$. We find

$$C_F \cap C_{\tilde{H}} = \{(0 : 0 : 1)\} \cup \{(a : 1 : 0) \mid a^4 + a = 0\}. \tag{3.2}$$

We can consider $a^4 + a = 0$ as a function with variable a, the derivative with respect to a is the constant 1, which is obviously never zero. All zeros of this function are distinct, so we have exactly 4 different solutions for $a^4 + a = 0$. In total there are 5 points in the intersection. To find that these points are indeed flexes, we will calculate $I(P, C_F \cap C_{\tilde{H}})$, where C_L is the projective tangent line to C_F at P. We start with $(0 : 0 : 1)$. First we find the projective tangent line (Def. 1.2.9) defined by $L = 0$, where

$$L = 0^3 X + (0^3 + 1^3) Y + 1^2 Z = Y.$$

So C_L is defined by $Y = 0$. We will work with $f(X, Y) = F(X : Y : 1)$ and $l(X, Y) = L(X : Y : 1)$. The curves defined by f and l will be denoted by small letters: C_f and C_l, respectively.

$$f = X^4 + XY^3 + Y;$$
$$l = Y.$$

Notice the use of properties 3,6 and 7 of the intersection multiplicity in the computation below.

$$\begin{aligned} I((0:0:1), C_F \cap C_L) &= I((0,0), C_f \cap C_l) \\ &= I((0,0), X^4 + XY^3 + Y \cap Y) \\ &= I((0,0), X^4 \cap Y) \\ &= 4 \cdot I((0,0), X \cap Y) \\ &= 4. \end{aligned}$$

We see that $(0:0:1)$ is in fact a hyperflex.
We will now examine the other points in $C_F \cap C_{\tilde{H}}$. We will write $P = (a:1:0)$, with $a^4 + a = 0$. First the projective tangent line to P is given by

$$L = 1^3X + (a1^2 + 0^3)Y + 1 \cdot 0^2 = X + aY.$$

We will use a change of coordinates T, which replaces X by $X' + aY$. The original $P = (a:1:0)$ will become $P' = (0:1:0)$. We do this such that the affine point becomes $(0,0)$, which makes computations a lot easier. We need to rewrite F and L.

$$\begin{aligned} F' &= (X' + aY)^4 + (X' + aY)Y^3 + YZ^3 \\ &= X'^4 + a^4Y^4 + X'Y^3 + aY^4 + YZ^3; \\ L' &= X' + aY + aY = X'. \end{aligned}$$

We will use the affine functions $f'(X', Z) = F'(X' : 1 : Z)$ and $l'(X', Z) = L'(X' : 1 : Z)$

$$f' = X'^4 + a^4 + X' + a + Z^3 = X'^4 + X' + Z^3;$$
$$l' = X'.$$

Above we used that $a^4 + a = 0$ to find the first expression. These two functions give us two affine curves $C_{f'}$ and $C_{l'}$, defined by $f' = 0$ and $l' = 0$, respectively. Now the intersection multiplicity. We will use properties 3, 6 and 7 (see Def. 1.2.15).

$$\begin{aligned} I((a:1:0), C_F \cap C_L) &= I((0,0), C_{f'} \cap C_{l'}) \\ &= I((0,0), X'^4 + X' + Z^3 \cap X') \\ &= I((0,0), Z^3 \cap X') \\ &= 3 \cdot I((0,0), Z \cap X') \\ &= 3. \end{aligned}$$

We see that in fact all these points are ordinary flexes. This means that all P in $C_F \cap C_{\tilde{H}}$ are flexes.

We will now prove the ($\Leftarrow$) part. If P is a flex on C_F, then $P \in C_F \cap C_{\tilde{H}}$. We will prove the contraposition of this statement.
If $P \notin C_F$, then it cannot be a flex. Therefore, we will examine $P \in C_F$, with $P \notin C_{\tilde{H}}$. Take a point $P = (a : b : c) \in C_F$. If $P \notin C_{\tilde{H}}$, then $b \neq 0$ and $c \neq 0$. Which implies $a \neq 0$, otherwise $P \notin C_F$. We may assume one of the coordinates is 1. Choose $a = 1$. We now have a point $P = (1 : b : c)$, with $b \neq 0$ and $c \neq 0$. We want $P \in C_F$. We get that

$$1 + b^3 + bc^3 = 0.$$

Next, we determine the projective tangent line C_L defined at P by $L = 0$, where

$$L = b^3 X + (b^2 + c^3)Y + bc^3 Z.$$

We need a change of coordinates again. We will replace Y by $Y + bX$ and Z by $Z + cX$, we will leave X the same. This translates $P = (1 : b : c)$ to $P' = (1 : 0 : 0)$. Let us implement these changes in F and L.

$$\begin{aligned} F' &= X^4 + X(Y + bX)^3 + (Y + bX)(Z + cX)^3; \\ L' &= b^3 X + (b^2 + c^3)(Y + bX) + bc^2(Z + cX). \end{aligned}$$

These can be rewritten, but this is easier if we first go to affine coordinates, where we let $X = 1$. The affine functions become $f'(Y, Z) = F'(1 : Y : Z)$ and $l'(Y, Z) = L'(1 : Y : Z)$, with

$$\begin{aligned} f' =& 1 + (Y + b)^3 + (Y + b)(Z + c)^3 \\ =& 1 + (Y^2 + b^2)(Y + b) + (YZ + cY + bZ + bc)(Z^2 + c^2) \\ =& Y^3 + bY^2 + Y(Z^3 + cZ^2 + c^2 Z + b^2 + c^3) \\ &+ bZ^3 + bcZ^2 + bc^2 Z + bc^3 + b^3 + 1 \\ =& Y(Y^2 + bY + Z^3 + cZ^2 + c^2 Z + b^2 + c^3) \\ &+ Z(bZ^2 + bcZ + bc^2) \\ =& Y\phi_1 + Z\phi_2. \end{aligned}$$

These ϕ_1 and ϕ_2 are simply the factors omitted from the previous line. This is quite a large expression, but in the intersection multiplicity we can eliminate this $Y\phi_1$.

$$\begin{aligned} l' &= b^3 + (b^2 + c^3)(Y + b) + bc^2(Z + c) \\ &= b^3 + (b^2 + c^3)Y + b^3 + bc^3 + bc^2 Z + bc^3 \\ &= (b^2 + c^3)Y + bc^2 Z \\ &= (b^3 + bc^3)Y + b^2 c^2 Z \\ &= Y + b^2 c^2 Z. \end{aligned}$$

Since $b \neq 0$, multiplication with b is allowed, which has led us to the possibility to use $1+b^3+bc^3=0$ in the calculation above. Now we have found l' and f', we can define curves $C_{f'}$ and $C_{l'}$ by $f'=0$ and $l'=0$, respectively. We compute the intersection multiplicity.

$$\begin{aligned}
I((1:b:c),C_F \cap C_L) =& I((0,0),C_{f'} \cap C_{l'}) \\
=& I((0,0),Y\phi_1 + Z\phi_2 \cap Y + b^2c^2Z) \\
=& I((0,0),(b^2c^2\phi_1 + \phi_2)Z \cap Y + b^2c^2Z) \\
=& I((0,0),Z \cap Y + b^2c^2Z) \\
& + I((0,0),b^2c^2\phi_1 + \phi_2 \cap Y + b^2c^2Z) \\
=& I((0,0),Z \cap Y) \\
& + I((0,0),b^2c^2\phi_1 + \phi_2 \cap Y + b^2c^2Z) \\
=& 1 + I((0,0),b^2c^2\phi_1 + \phi_2 \cap Y + b^2c^2Z).
\end{aligned}$$

To continue, we must find out if there is a nonzero constant term in $b^2c^2\phi_1+\phi_2$. We find that the constant term is defined by

$$b^2c^2(b^2+c^3)+bc^2 = bc^2(b^3+bc^3+1) = 0$$

We observe that the constant term is identically zero. Hence, we must continue. However, to do this we rewrite $b^2c^2\phi_1+\phi_2$ to make it easier to see what we can eliminate.

$$\begin{aligned}
b^2c^2\phi_1 + \phi_2 =& bZ^2 + bcZ + bc^2 \\
& + b^2c^2(Y^2 + bY + Z^3 + cZ^2 + c^2Z + b^2 + c^3) \\
=& Y(b^2c^2Y + b^3c^2) \\
& + Z(b^2c^2Z^2 + b^2c^3Z + b^2c^4 + bZ + bc) \\
=& Y\psi_1 + Z\psi_2.
\end{aligned}$$

As we did before we can now eliminate the $Y\psi_1$ term in the intersection multiplicity.

$$\begin{aligned}
I((0,0),b^2c^2\psi_1 + \psi_2 \cap Y + b^2c^2Z) =& I((0,0),Y\psi_1 + Z\psi_2 \cap Y + b^2c^2Z) \\
=& I((0,0),b^2c^2Z\psi_1 + Z\psi_2 \cap Y + b^2c^2Z) \\
=& I((0,0),Z(b^2c^2\psi_1 + \psi_2) \cap Y + b^2c^2Z) \\
=& I((0,0),Z \cap Y + b^2c^2Z) + I((0,0),b^2c^2\psi_1 \\
& + \psi_2 \cap Y + b^2c^2Z) \\
=& I((0,0),Z \cap Y) \\
& + I((0,0),b^2c^2\psi_1 + \psi_2 \cap Y + b^2c^2Z) \\
=& 1 + I((0,0),b^2c^2\psi_1 + \psi_2 \cap Y + b^2c^2Z).
\end{aligned}$$

We will now again check whether the constant term is zero. We find the following

$$\begin{aligned}
b^2c^4 + bc + b^5c^4 &= bc(bc^3 + 1 + b^4c^3) \\
&= bc(b^3 + b^4c^3) \\
&= b^4c(1 + bc^3) \\
&= b^4cb^3 \\
&= b^7c.
\end{aligned}$$

We know that $b^7c = 0$ if and only if $b = 0$ or $c = 0$. Which by assumption that $P \notin C_h$ cannot happen. Therefore there is a nonzero constant term, which means $(0,0)$ is not an element of the intersection of the curves. The intersection multiplicity must be zero.

$$\begin{aligned}
I((1:b:c), C_F \cap C_L) &= I((0,0), C_{f'} \cap C_{l'}) \\
&= 1 + I((0,0), b^2c^2\phi_1 + \phi_2 \cap Y + b^2c^2Z) \\
&= 1 + 1 + I((0,0), b^2c^2\psi_1 + \psi_2 \cap Y + b^2c^2Z) \\
&= 2 + 0 = 2.
\end{aligned}$$

Earlier we have seen that the intersection multiplicity of a flex is three or higher. We see that this is not the case, therefore $(1:b:c)$ is not a flex for all $b \neq 0$ and $c \neq 0$. We know that all points in C_F, which are not in C_H are of this form. This concludes the proof. We see that indeed P is a flex on C_F if and only if $P \in C_F \cap C_H$.

3.2.3 Multiplicity of the flexes

The general theorem has only been proven for fields K, whose characteristic is zero. However, we can investigate what happens in positive characteristic. In the case where the characteristic is 2 the Hessian is identically zero for all curves of degree 4. Therefore, there is no point in checking anything for the Hessian. However, we can determine the intersection multiplicity with the pseudo-Hessian. We will find some results that one might not expect.

Let us first examine $(0:0:1)$. We wish to find the intersection multiplicity of C_F and C_H at $P = (0:0:1)$. As before we will use $f = X^4 + XY^3 + Y$ and with the same affine change of coordinates, i.e. $Z = 1$, we get $\tilde{h}_F = Y^5$. Again, we define the curve $C_{\tilde{h}}$ by $\tilde{h}_F = 0$.

$$\begin{aligned}
I((0:0:1), C_F \cap C_{\tilde{H}}) &= I((0,0), C_f \cap C_{\tilde{h}}) \\
&= I((0,0), X^4 + XY^3 + Y \cap Y^5) \\
&= 5 \cdot I((0,0), X^4 + XY^3 + Y \cap Y) \\
&= 5 \cdot I((0,0), X^4 \cap Y) \\
&= 20 \cdot ((0,0), X \cap Y) \\
&= 20.
\end{aligned}$$

We see an extraordinary intersection multiplicity of 20.
Now consider the other flexes of C_F: $(a:1:0)$, with $a^4 + a = 0$. We will use the same projective change of coordinates as we did to compute the intersection multiplicity with the projective tangent line of these points, i.e. X to $X' + aY$. We obtain the affine functions again by $Y = 1$. We get $f' = X'^4 + X' + Z^3$ and $\tilde{h}' = Z$.

$$\begin{aligned}
I((a:0:1), C \cap C_{\tilde{H}}) &= I((0,0), X'^4 + X' + Z^3 \cap Z) \\
&= I((0,0), X'(X'^3 + 1) \cap Z) \\
&= I((0,0), X' \cap Z) + I((0,0), (X'^3 + 1) \cap Z) \\
&= I((0,0), X' \cap Z) + 0 \\
&= 1.
\end{aligned}$$

We see that these curves, i.e. C_F and $C_{\tilde{H}}$, intersect in five points, four of which have intersection multiplicity 1 and the fifth has intersection multiplicity 20.

Chapter 4

Bézout's Theorem

In the previous chapters we have studied curves, the (pseudo-)Hessian and flexes. We have seen some interesting relations between these and the intersection multiplicity. In this chapter we will investigate the different ways to adhere to Bézout's theorem. Let us first give the theorem:

Theorem 4.0.2 (Bézout's theorem [4], p. 112)**.** *Let C_F and C_G be projective plane curves of degree n and m, respectively. Assume C_F and C_G have no common component, then*

$$\sum_P I(P, C_F \cap C_G) = mn.$$

This theorem says that two curves of certain degrees always intersect in the same number of points, multiplicity correctly counted. Where correctly counted means the very cleverly constructed intersection multiplicity we have seen before (see Def. 1.2.15). This may evoke some curiosity to the distribution of this intersection multiplicity among the intersection points. We will investigate this distribution in a specific situation. We will consider F and its Hessian or pseudo-Hessian as the defining polynomials of interest and we are particularly interested in flexes. To see some results we will look at curves C_F of degree 4. We will consider such C_F first in characteristic zero, and then in characteristic 2. We will discover that there are very few options for the Hessian if the characteristic is zero and much more for the pseudo-Hessian in characteristic 2. For the proof I recommend the one from *Algebraic Curves* ([4] p. 112). Especially since this book is also the source for our definition of the intersection multiplicity.

4.1 Bézout's theorem in characteristic zero

We will use the general theorem (see Th. 2.0.2), which says

$$I(P, C_F \cap C_H) = I(P, C_F \cap C_L) - 2.$$

Recall that $I(P, C_F \cap C_L) = I(P, C_f \cap Y)$ for the specified f in the proof. However, we also need a result from the proof (see Eq. 2.7), that said

$$I(P, C_f \cap Y) = N = \min\{k \mid k \geq 2, a_k \neq 0\}.$$

Our F has degree 4 by assumption, therefore f has degree four or lower, this means that $\min\{k \mid a_k \neq 0\}$ cannot exceed 4. However, we are only interested in intersection points. So, $I(P, C_F \cap C_H) \geq 1$. This leaves two options for $I(P, C_F \cap C_H)$: it must be 1 or 2.

4.2 Bézout's theorem in characteristic p

We will now consider some specific curves in positive characteristic, to show that the intersection multiplicities of C_F and the pseudo-Hessian do not obey such strict rules. We have seen that the curve $X^4 + XY^3 + YZ^3 = 0$ has 5 different intersection points with the pseudo-Hessian. The intersection multiplicity of one of those, is what made this curve so interesting. Four points had intersection multiplicity 1 and one point had intersection multiplicity 20. This last intersection multiplicity is quite extreme. What are the other possibilities? Can we actually find all values in between? Or even higher multiplicities? These are questions we will investigate in this section.

4.2.1 Intersection multiplicity: 8

To see if there are more intersection multiplicities possible, we can take a look at polynomial $F = X^4 + XY^3 + YZ^3 + X^2Y^2$ in $\mathbb{P}^2_K$ with K an algebraically closed field with characteristic 2. Let us investigate curve C_F defined by $F = 0$. We will see that this curve has an element with intersection multiplicity 8 with the pseudo-Hessian. We will start by checking whether this curve is smooth or not. Let us determine the first derivatives.

$$\begin{aligned}
\frac{\partial F}{\partial X} &= 4X^3 + Y^3 + 2XY^2 = Y^3; \\
\frac{\partial F}{\partial Y} &= 3XY^2 + Z^3 + 2X^2Y = XY^2 + Z^3; \\
\frac{\partial F}{\partial Z} &= 3YZ^2 = YZ^2.
\end{aligned}$$

We consider $P = (a : b : c)$. If all derivative are zero, then $b = 0$, because of the derivative with respect to X. This implies $c = 0$, because of the derivative with respect to Y. Finally, these two things imply that $a = 0$, because P needs to be an element of C_F. We obtain $(0 : 0 : 0)$, which does not exist in $\mathbb{P}^2_K$. Thus, C_F is a smooth curve.

Now we know that our curve is indeed smooth, we can check our other claim regarding the pseudo-Hessian. Let us determine the pseudo-Hessian. First, we must compute the pseudo-second derivatives:

$$\begin{aligned}
\frac{\tilde{\partial}^2 F}{\partial X^2} &= \binom{4}{2} X^2 + \binom{2}{2} Y^2 = 6X^2 + Y^2 = Y^2; \\
\frac{\tilde{\partial}^2 F}{\partial Y^2} &= \binom{3}{2} XY + \binom{2}{2} X^2 = 3XY + X^2 = XY + X^2; \\
\frac{\tilde{\partial}^2 F}{\partial Z^2} &= \binom{3}{2} YZ = 3YZ = YZ.
\end{aligned}$$

Secondly, we will determine the (regular) mixed partial derivatives:

$$\begin{aligned}
\frac{\partial^2 F}{\partial X \partial Y} &= 3Y^2 + 4XY = Y^2; \\
\frac{\partial^2 F}{\partial X \partial Z} &= 0; \\
\frac{\partial^2 F}{\partial Y \partial Z} &= 3Z^2 = Z^2.
\end{aligned}$$

We have gathered all ingredients to compute the pseudo-Hessian (see Def. 3.2.2):

$$\begin{aligned}
\tilde{H}_F &= Y^2 \cdot Z^2 \cdot 0 + Y^2 \cdot (Z^2)^2 + (XY + X^2) \cdot 0^2 + YZ(Y^2)^2 \\
&= Y^2 Z^4 + Y^5 Z \\
&= Y^2 Z(Z^3 + Y^3).
\end{aligned}$$

The pseudo-Hessian is $\tilde{H} = Y^2 Z(Z^3 + Y^3)$, which gives us the curve $C_{\tilde{H}}$, defined by $\tilde{H} = 0$. Now we determine which points lie on $C_{\tilde{H}} \cap C$. We write $P = (a : b : c)$. We first determine $C \cap C_{\tilde{H}}$. If $P \in C_{\tilde{H}}$, then $b^2 c(c^3 + b^3) = 0$. This leaves a few options: $b = 0$, $c = 0$ or $c^3 + b^3 = 0$.
First, if $b = 0 \Rightarrow a = 0 \Rightarrow c = 1$. We get $P = (0 : 0 : 1)$.
Secondly, if $c = 0 \Rightarrow a^4 + ab^3 + a^2 b^2 = 0$, so $a = 0$ is a possibility, which forces $b \neq 0$. We find $P = (0 : 1 : 0)$. If $a \neq 0$, we know $b \neq 0$ with $a^3 + b^3 + ab^2 = 0$. Since $a \neq 0$, we let it be 1. We get $1 + b^3 + b^2 = 0$. Let us determine how many options this leaves for b. The derivative with respect to b is $3b^2 + 2b = b^2$, which is zero only at $b = 0$, which was not an option. Therefore we know we have three different choices for b. So far we have five distinct elements:

$$\{(0 : 0 : 1), (0 : 1 : 0)\} \cup \{(1 : b : 0) \mid b^3 + b^2 + 1 = 0\}.$$

We can now move on to the points, where $c \neq 0$ and $b \neq 0$, but where $c^3 + b^3 = 0$. We let $b = 1$, because it is nonzero. We obtain $c^3 + 1 = 0$. Which gives us exactly 3 different options for c. We must now check which points lie on C_F. We fill in $P = (a : 1 : c)$, with $c^3 = 1$, we obtain $a^4 + a + c^3 + a^2 = 0$. This we can simplify, using $c^3 = 1$, to $a^4 + a + 1 + a^2 = 0$. Again we need to know how many distinct options this gives us. The derivative with respect to a is $4a^3+1+2a=1$, so it is never zero. We find exactly four different options for a, independent of our choice of c. We find twelve points $(a:1:c)$, with $c^3+1=0$ and $a^4+a^2+a+1=0$. Now we have determined all points in $C \cap C_{\tilde{H}}$:

$$\begin{aligned}&\{(0:0:1),(0:1:0)\} \cup \{(1:b:0) \mid b^3+b^2+1=0\}\\ &\cup \{(a:1:c) \mid c^3+1=0, a^4+a^2+a+1=0\}.\end{aligned}$$

This set contains exactly 17 distinct elements. Bézout's theorem (4.0.2) tells us we expect a total intersection multiplicity of 24. It turns out this total is reached thanks to $(0:0:1)$, which has a intersection multiplicity of 8. This we will show now.

We are interested in $I((0:0:1), C_F \cap C_{\tilde{H}}$. We will set $Z=1$ and obtain c and $c_{\tilde{H}}$: $X^4+XY^3+Y+X^2Y^2$ and $Y^2(Y^3+1)=0$, respectively.

$$\begin{aligned}I((0,0), C_f \cap C_{\tilde{h}}) =& I((0,0), C_f \cap Y^2(Y^3+1)\\ =& 2 \cdot I((0,0), C_f \cap Y) + I((0,0), C_f \cap Y^3+1)\\ =& 2 \cdot I((0,0), C_f \cap Y)\\ =& 2 \cdot I((0,0), X^4+XY^3+Y+X^2Y^2 \cap Y)\\ =& 2 \cdot I((0,0), X^4 \cap Y)\\ =& 8 \cdot I((0,0), X \cap Y)\\ =& 8.\end{aligned}$$

As claimed before, indeed the intersection multiplicity is 8. We know all other points must have intersection multiplicity 1, for Bézout's theorem to hold.

4.2.2 Intersection multiplicities: 4 and 17

Someone might expect certain numbers to come up like 4, 8 or even 16, but 17 is quite different. Let us examine the curve C_F defined by $F=X^4+XY^3+YZ^3+X^2YZ$. This curve has a point of exceptional multiplicity, different from what we found before. Since an intersection multiplicity of 4 has not been encountered yet, we will also show the corresponding calculation. Again, we start off by checking whether this curve is smooth.

$$\begin{aligned}\frac{\partial F}{\partial X} &= 4X^3+Y^3+2XYZ = Y^3;\\ \frac{\partial F}{\partial Y} &= 3XY^2+Z^3+X^2Z = XY^2+Z^3+X^2Z;\\ \frac{\partial F}{\partial Z} &= 3YZ^2+X^2Y = YZ^2+X^2Y.\end{aligned}$$

We consider $P=(a:b:c)$. We know that $b=0$, because of the derivative with respect to X. This implies that $a=0$, because we want P to be an element of C_F. Lastly, these things imply that $c = 0$, because of the derivative with respect to Y. We obtain $P = (0 : 0 : 0)$, which is not an element of $\mathbb{P}^2_K$. We see that C_F is a smooth curve.

Then we continue with determining the pseudo-Hessian. First, we compute the pseudo-second derivatives:

$$
\begin{aligned}
\frac{\tilde{\partial}^2 F}{\partial X^2} &= \binom{4}{2} X^2 + \binom{2}{2} YZ = YZ; \\
\frac{\tilde{\partial}^2 F}{\partial Y^2} &= \binom{3}{2} XY = XY; \\
\frac{\tilde{\partial}^2 F}{\partial Z^2} &= \binom{3}{2} YZ = YZ.
\end{aligned}
$$

Secondly, we determine the (regular) mixed partial derivatives:

$$
\begin{aligned}
\frac{\partial^2 F}{\partial X \partial Y} &= 3Y^2 = Y^2; \\
\frac{\partial^2 F}{\partial X \partial Z} &= 0; \\
\frac{\partial^2 F}{\partial Y \partial Z} &= 3Z^2 + X^2 = X^2 + Z^2.
\end{aligned}
$$

We have gathered all ingredients to determine the pseudo-Hessian:

$$
\begin{aligned}
\tilde{H}_F &= Y^2 \cdot (X^2 + Z^2) \cdot 0 + YZ \cdot (X^2 + Z^2)^2 + (XY) \cdot 0^2 + YZ(Y^2)^2 \\
&= YZ(X^4 + Z^4) + YZ \cdot Y^4 \\
&= YZ(X^4 + Y^4 + Z^4) \\
&= YZ(X + Y + Z)^4.
\end{aligned}
$$

We can define the curve $C_{\tilde{H}}$ by $\tilde{H} = 0$. We are interested in the intersection of C_F and $C_{\tilde{H}}$. We write $P = (a : b : c)$. If $P \in C_{\tilde{H}}$, then $bc(a + b + c)^4 = 0$. This equation gives us three options: $b = 0$, $c = 0$ or $a + b + c = 0$.
First, if $b = 0$, then for P to be an element of C_F, we know $a = 0$, which leaves $c = 1$, because not all coordinates can be zero. We obtain exactly one point $P = (0 : 0 : 1)$.
Secondly, if $c = 0$, we know $a^4 + ab^3 = 0$. The case where $b = 0$ contradicts with $c = 0$, therefore we may assume $b = 1$. We obtain $a^4 + a = 0$. This gives us exactly four points $P = (a : 1 : 0)$, with $a^4 + a = 0$.
Lastly, we assume both $b \neq 0$ and $c \neq 0$. Let us set $b = 1$. Then $a + 1 + c = 0$ and $a^4 + a + c^3 + a^2c = 0$. The second equation tells us $a \neq 0$, because $c^3 \neq 0$. However when we substitute $c = a + 1$ in the second equation, the only solution

is $a = 0$. Therefore, we obtain no more points here. We have determined the intersection of our curves.

$$C \cap C_{\tilde{H}} = \{(0:0:1)\} \cup \{(a:1:0) \mid a^4 + a = 0\}$$

We will show that $(0 : 0 : 1)$ has intersection multiplicity 4 on C_F and $C_{\tilde{H}}$. The most interesting point on this curve is in fact $(1 : 1 : 0)$, which has the remarkable intersection multiplicity 17. All other three points have intersection multiplicity 1. We will only show the calculation for the special ones.

First we consider $(0 : 0 : 1)$. We simply set $Z = 1$ and determine the affine curves C_f and $C_{\tilde{h}}$. We determine the affine version of F, we get $f = X^4 + XY^3 + Y + X^2Y$. Now C_f is defined by $f = 0$. Now the affine pseudo-Hessian: $\tilde{h} = Y(X + Y + 1)^4$. We define $C_{\tilde{h}}$ by $\tilde{h} = 0$. We will consider the affine point $(0, 0)$. Let us compute.

$$\begin{aligned}
I((0,0), C_f \cap C_{\tilde{h}}) &= I((0,0), C_f \cap Y(X+Y+1)^4) \\
&= I((0,0), C_f \cap Y) + 4 \cdot I((0,0), C_f \cap X + Y + 1) \\
&= I((0,0), C_f \cap Y) + 0 \\
&= I((0,0), X^4 + XY^3 + Y + X^2Y \cap Y) \\
&= I((0,0), X^4 \cap Y) \\
&= 4 \cdot I((0,0), X \cap Y) \\
&= 4.
\end{aligned}$$

We had not seen an intersection multiplicity of 4 before. This, however, is not as extraordinary as an intersection of 17. We will now commence the computation to show this claim.

The point $(1 : 1 : 0)$ needs some transformation. We let X get replaced by $X + Y$ and leave all others exactly the same. We then set $Y = 1$ to obtain the affine curves. Let us compute this for F, we obtain

$$\begin{aligned}
f' &= (X+1)^4 + (X+1) + Z^3 + (X+1)^2 Z \\
&= X^4 + X + X^2Z + Z^3 + Z.
\end{aligned}$$

Continuing with the pseudo-Hessian, we get

$$\begin{aligned}
\tilde{h}' &= Z(X + 1 + 1 + Z)^4 \\
&= Z(X+Z)^4,
\end{aligned}$$

with affine curves $C_{f'}$ and $C_{\tilde{h}'}$ defined by $f' = 0$ and $\tilde{h}' = 0$, respectively. Our affine point of interest again becomes $(0, 0)$. Computing the intersection, give us the following calculation.

$$\begin{aligned}
I((0,0),C_{f'}\cap C_{\tilde{h}'}) =& I((0,0),C_{f'}\cap Z(X+Z)^4)\\
=& I((0,0),C_{f'}\cap Z)+4\cdot I((0,0),C_{f'}\cap X+Z)\\
=& I((0,0),X^4+X+X^2Z+Z^3+Z\cap Z)\\
&+4\cdot I((0,0),C_{f'}\cap X+Z)\\
=& I((0,0),X^4+X\cap Z)\\
&+4\cdot I((0,0),C_{f'}\cap X+Z)\\
=& I((0,0),X\cap Z)+I((0,0),X^3+1\cap Z)\\
&+4\cdot I((0,0),C_{f'}\cap X+Z)\\
=& 1+4\cdot I((0,0),C_{f'}\cap X+Z).
\end{aligned}$$

We still need to compute $I((0,0),C_{f'}\cap X+Z)$. We will do this computation separately to avoid one big computation.

$$\begin{aligned}
I((0,0),C_{f'}\cap X+Z =& I((0,0),X^4+X+X^2Z+Z^3+Z\cap X+Z)\\
=& I((0,0),X^4+X^2Z+Z^3\cap X+Z)\\
=& I((0,0),X^3Z+X^2Z+Z^3\cap X+Z)\\
=& I((0,0),Z\cap X+Z)\\
&+I((0,0),X^3+X^2+Z^2\cap X+Z)\\
=& I((0,0),Z\cap X)\\
&+I((0,0),X^3\cap X+Z)\\
=& 1+3\cdot I((0,0),X\cap X+Z)\\
=& 1+3\cdot I((0,0),X\cap Z)\\
=& 4.
\end{aligned}$$

We can see that our claim is true:

$$\begin{aligned}
I((0,0),C_{f'}\cap C_{\tilde{H}'}) &= 1+4\cdot I((0,0),C_{f'}\cap X+Z)\\
&= 1+4\cdot 4 = 17.
\end{aligned}$$

The examples treated in this chapter give the suggestion that all outcomes are possible. There are no articles proving or disproving this claim, therefore the claim remains a conjecture.

Conclusion

We have seen some general results for plane curves in the projective plane over a field of characteristic zero. More general than given by Fulton. However, we have seen that the situation for positive characteristic is more complicated. Nonetheless, the examples in positive characteristic have shown that it yields a mathematical world worth exploring. We have introduced a new notion called the pseudo-Hessian, which seems to have some nice properties. Its definition is cleverly constructed to make sure it can do what we want it.

Unfortunately, general results on this pseudo-Hessian were not obtained. However, I hope the less general results support the relevance of the pseudo-Hessian. There are still some unanswered questions in the examination of the pseudo-Hessian. Is a point an inflection point on a curve if and only it is an element of the curve defined by the pseudo-Hessian? Is the intersection multiplicity between the curve and the pseudo-Hessian restricted to certain values as is the case in characteristic zero? If it is restricted, why and to which values? If these questions are answered, one might even consider fields of positive characteristic, other than two. Perhaps a pseudo-Hessian can be defined for all fields of positive characteristic. All very interesting questions for the future.

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