Brownian motion and Topological Field theories

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Author:

Peter Kristel

Advisors:

Dr. Emilio Cobanera
Dr. Bas Janssens
Dr. Johan van de Leur
Prof. Cristiane de Morais Smith

Utrecht University

Abstract

This thesis consists of two parts. In Part I we give an introduction to some of the mathematical aspects of gauge theory. In particular we will introduce the concept of a connection, which allows one to define a notion of curvature and a notion of parallel transport in an arbitrary vector bundle. In this context we will treat Yang-Mills theory and Chern-Simons theory, where the notion of a connection is a fundamental one. We will introduce the notion of a principal fiber bundle, which is the right tool to formalize the notion of local symmetries, which is central in gauge theory in physics. After this, we will consider 4-dimensional BF-theory. First from the point of view of canonical quantization and Feynman path-integral quantization, and then from a categorical perspective. From the categorical perspective we will find a simple criterion that characterizes 4-dimensional BF-theory up to isomorphism. In Part II we start with a description and an explanation of the integer quantum Hall effect. The explanation we give starts from a description of a single electron in a magnetic field. If the magnetic field is extremely strong, then (in a quantum mechanical description) the coordinates of the electron cease to commute. After this we introduce the independent oscillator model for Brownian motion and quantize it. It turns out that there is an interesting way to combine this quantum mechanical version of Brownian motion, with the theory of an electron in a strong magnetic field. We compute the mean squared displacement of the particle as a function of time for this system. Consequently, we analyse the result in two different regimes for the friction constant, associated with the Brownian motion. If the friction constant is very large, then the result is very similar to classical Brownian motion. If, however, the friction constant is small, then the movement of the particle is suppressed. In particular, the mean squared displacement grows linearly with time, and the coefficient is proportional to the friction constant. We conclude with a final consideration of the integer quantum Hall effect, and in particular we point out a connection between the integer quantum Hall effect and the Chern class of a vector bundle.

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Introduction

One might view this thesis as a study of the electron. The electron is, perhaps, along with the photon, the most well-known fundamental particle. Far too much is known about the electron for this thesis to be a comprehensive study of it. We will not even talk about the electron's spin, surely one of its most fundamental and interesting properties. We will consider the electron from two different points of view. On the one hand we will describe a tiny part of the mathematical framework that encompasses the most sophisticated point of view of the electron and of electromagnetism in general, this is done in part I. On the other hand we will analyse some very concrete physical systems, where the electron is the main ingredient, this is the content of part II. Chapters 7 through 9 can be read independently of part I.

Vector bundles are the natural mathematical setting for electromagnetism and, in some cases, quantum mechanics. In Chapter 1 we give a construction of a vector bundle from data that is naturally available in physics, namely a representation of a group and a collection of transition maps. We will actually see an example of such data and the corresponding construction in Chapter 10.

We continue with an introduction to the mathematical aspects of gauge theories in Chapter 2. Gauge theory is an immensely fruitful area of cooperation between mathematics and physics. After this introduction, two important examples of gauge theories are considered in Chapters 3 and 4. The first example is Yang-Mills theory, which encompasses the standard model of particle physics. Secondly we consider Chern-Simons theory, this is an example of a topological quantum field theory. It was shown by Witten, in Ref. [Witt89], that Chern-Simons theory can be used to obtain knot invariants. The Chern-Simons actions appears in the effective action used to describe the fractional quantum Hall effect, see Ref. [Zhang92]. Unfortunately, both of these interesting avenues of research fall beyond the scope of this thesis. We will be content to give the definition of the Chern-Simons action and prove some of its elementary properties.

In Chapter 5 we introduce the concept of a principal fiber bundle. These objects are closely related to the G-bundles introduced in Chapter 1. We will try to make this connection explicit, by translating some of the concepts introduced in the context of vector bundles to principal fiber bundles.

Finally, in Chapter 6 we will consider the so-called BF-theory in four dimensions. This theory is a gauge theory that is also a topological quantum field theory, just like Chern-Simons theory. In this chapter we follow the work [Baez95] very closely. Some slight familiarity with category theory is assumed in Chapter 6. Motivated by an analysis of this theory we construct a functor from the cobordism category to the category of vector spaces. This functor will have some special properties that make it a topological quantum field theory in the sense of Atiyah, see Ref. [Atiy88].

After this quite abstract story we will change gears a bit and consider some concrete physical systems in Part II.

In Chapter 7 we will describe and explain the integer quantum Hall effect (IQHE), originally discovered by K.v.Klitzing, G.Dorda & M.Pepper, see Ref. [KDP80]. The integer quantum Hall effect is, along with for example superconductivity, a macroscopic quantum phenomenon; a macroscopic quantity, in this instance the resistance, is quantized. The quantum Hall effect allows the international standard for resistance to be defined in terms of the electron charge and Planck's constant alone, and furthermore it allows the most accurate measurements of resistance today, see Refs. [RS10] and [SI06]. The theory we develop in Chapter 7 forms the basis of our analysis in Chapter 9. More specifically, in Section 7.4 we consider an electron in an extremely strong magnetic field, a system that may be described by so-called topological quantum mechanics. These theories were constructed by G.V.Dunne, R.Jackiw and C.A.Trugenberger in Ref. [DJT90], in analogy to Chern-Simons gauge theories. The theory of an electron in an extremely strong magnetic field has two striking properties. Firstly, in the absence of an external potential, the Hamiltonian vanishes identically, this is the reason for the name *topological* quantum mechanics. Secondly, the spatial coordinates of the particle cease to commute. We will have more to say about these facts in the text.

In Chapter 8 we give a brief recollection of the classical theory of Brownian motion and then go on to prescribe a way to quantize such a system. Because the energy of a particle undergoing Brownian motion is not conserved, some care must be taken in the quantization of such a system. A well-studied model that allows quantization of Brownian motion is the so-called independent oscillator (IO) model. We will introduce this model in Section 8.2 and study its quantization in Section 8.4. We will take a canonical approach to its quantization, advanced by Ford, Lewis and O'Connell in Ref. [FLO88]. The main result of this approach is a quantum mechanical version of the Langevin equation, called the operator Langevin equation, or the quantum Langevin equation by Ford, Lewis and O'Connell. An alternative approach that is well-studied is described by Caldeira and Leggett in Ref. [CL83].

The work done in Chapters 7 and 8 is synthesized in Chapter 9, where we consider Brownian motion in an extremely strong magnetic field. The main motivation in studying this problem is the question of dissipation in topological quantum mechanics. This question is especially interesting, because at first sight it appears to be nonsensical. How can a topological system dissipate energy? It had no energy to start with. The answer will turn out to be that in the independent oscillator model we have to give the topological system a certain interaction energy, which it can then exchange with the heat bath. The main result of Chapter 9 is the operator Langevin equation. We solve the operator Langevin equation in some especially simple situations, and compare the results to the classical theory of Brownian motion.

In Chapter 10 we give an alternative computation of the Hall conductance different from the one done in Section 7.3.3. The presentation given in Chapter 10 is based on Ref. [Kohm85]. The computation makes use of techniques from the theory of vector bundles treated in part I, in particular it relates the Hall current to the first Chern class of a particular vector bundle. As such, it lays bare the connection between the quantum Hall effect and topology.

Prerequisites and conventions

The prerequisites for Parts I and II are mostly disjoint. As one might guess by the titles, the prerequisites for Part I are mostly of a mathematical nature and the prerequisites for II are mostly of a physical nature.

For Part I we assume that the reader is familiar with the following concepts from differential geometry: differential forms, tangent bundles, vector bundles and fiber bundles. Furthermore, some familiarity with Lie groups is required. In Chapter 6 we will assume some familiarity with category theory, the relevant notions can be found in the rather short document [Baez04].

In Part II we assume that the reader is familiar with quantum mechanics, in particular, the quantum mechanical harmonic oscillator. Furthermore, the reader should be comfortable with the Lagrangians and Hamiltonians, and in extension of this, with the Heisenberg picture of quantum mechanics. In Chapter 8 we recall some basic facts of classical Brownian motion, our treatment is, however, far too short to serve as an introduction. We do not assume any familiarity with the quantum mechanical analog of Brownian motion.

The following conventions will be in effect throughout the text, unless explicitly stated otherwise. We abbreviate smooth manifold to manifold. We will use the Einstein summation convention for repeated indices, whenever the repeated index appears exactly once as a superscript and exactly once as a subscript. Any quantity expressed by a single boldface letter, for example $\mathbf{p}$ is a vector. (However, not all vectors will be represented in this way.) The square of a vector is short for the inner product of the vector with itself, i.e. if $\mathbf{p}$ is a vector, then $\mathbf{p}^2 = \mathbf{p} \cdot \mathbf{p}$. Finally, in Part II we will use Dirac's bra-ket notation.

Finally, k_B is Boltzmann's constant and $\hbar$ is the reduced Planck constant.

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Part I

Bundles and gauge fields

Chapter 1

Vector bundles

In this chapter we will describe a way to construct vector bundles that is often found in problems from physics. We will see an example of such a problem and the corresponding construction in Chapter 10. The presentation here will be given in coordinates, because this is the way that the data is usually given in physics.

1.1 G-Bundles

Let M be a manifold and $\{U_\alpha\}$ an open cover of M. Let G be a group and let (V, ρ) be a representation of G in the vector space V, that is $\rho : G \to \mathrm{GL}(V)$ is a group homomorphism. We denote the identity of the group G by e. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ be a collection of smooth maps satisfying $g_{\alpha\alpha} = e$ and $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = e$ for all α, β, γ. The relation $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = e$ is called the cocycle condition. Note that the cocycle condition together with the condition that $g_{\alpha\alpha} = e$ implies $g_{\alpha\beta}g_{\beta\alpha} = e$.

Definition 1.1.1. Given the data as specified above, the G-bundle E is defined as

$$E := \left(\bigcup_\alpha U_\alpha \times V \right) / \sim,$$

where the points $(x, v) \in U_\alpha \times V$ and $(x', v') \in U_\beta \times V$ are equivalent if and only if $x = x'$ and

$$v = \rho(g_{\alpha\beta}(x))v'.$$

The equation above is also abusively written as

$$v = g_{\alpha\beta}v'.$$

One may show that the relation $\sim$ is an equivalence relation by making use of the cocycle condition.

Let $M, \{U_\alpha\}, G, \rho$ and $g_{\alpha\beta}$ be as above.

Theorem 1.1.2. *The space E as defined above is actually a vector bundle.*

Proof. To prove the claim we need to show two things: the fibers E_x can be endowed with the structure of a vector space, and the space E is a fiber bundle.

First, let us show that the space E is a fiber bundle, i.e. is locally trivializable. Actually, if the pair (E, M) is to be a fiber bundle there should also be a smooth surjection $\pi : E \to M$. The map $\tilde{\pi} : \cup_\alpha U_\alpha \times V \to M, (x, v) \mapsto x$, factors through a map $\pi : E \to M$, since if $(x, v) \sim (x', v')$ then $\tilde{\pi}(x, v) = \tilde{\pi}(x', v')$. In other words, π is the map obtained from the universal property of the quotient with respect to the following diagram

$$\begin{array}{ccc} \cup_\alpha U_\alpha \times V & \xrightarrow{\tilde{\pi}} & M \\ {\scriptstyle q}\downarrow & \nearrow_{\exists ! \pi} & \\ E & & \end{array}$$

where we have denoted the quotient map by q.

Let $x \in M$, then x lies in some open U_α. The set $U_\alpha \times V$ is contained in the preimage of U_α under $\tilde{\pi}$, that is $U_\alpha \times V \subset \tilde{\pi}^{-1}(U_\alpha)$. The map q maps $U_\alpha \times V$ surjectively onto $\pi^{-1}(U_\alpha)$. When restricted to $U_\alpha \times V$, the map q is a diffeomorphism onto its image, that is

$$q\big|_{U_\alpha \times V} : U_\alpha \times V \xrightarrow{\simeq} q(U_\alpha \times V) \subseteq E.$$

The commutativity of the diagram above, and the fact that the maps $q, \tilde{\pi}$ and π are surjections implies that $q(U_\alpha \times V) = \pi^{-1}(U_\alpha)$. We conclude that q restricts to a diffeomorphism $U_\alpha \times V \simeq \pi^{-1}(U_\alpha)$.

Second, let us show that each fiber E_x can be endowed with the structure of a vector space.

We denote the image of $(x, v) \in U_\alpha \times V$ in E under the canonical projection map by $q(x, v) = [x, v]_\alpha$. It follows that the fiber E_x takes the form

$$E_x = \{[x, v]_\alpha \big| v \in V\}.$$

The fiber E_x can be endowed with the structure of a vector space as follows, let $\lambda \in \mathbb{R}$ and let $[x, v]_\alpha \in E_x$ and $[x, v']_\alpha \in E_x$ then we define

$$\lambda[x, v]_\alpha = [x, \lambda v]_\alpha,$$
$$[x, v]_\alpha + [x, v']_\alpha = [x, v + v']_\alpha.$$

Let us note that if $[x, v]_\alpha \in E_x$ and $[x, v']_\beta \in E_x$, then one might use

$$[x, v]_\alpha + [x, v']_\beta = [x, v]_\alpha + [x, g_{\alpha\beta} v']_\alpha.$$

This makes E_x into a vector space. The rules above are well defined since $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and since $g_{\alpha\beta}(v + v') = g_{\alpha\beta} v + g_{\alpha\beta} v'$. □

The group G is called the structure group of the vector bundle E.

1.2 The endomorphism bundle

If (E, π) is a bundle over a manifold M, we denote by $\Gamma(E)$ the space of (smooth) sections of E over M, i.e.

$$\Gamma(E) = \{s \in C^\infty(M, E) | \pi \circ s = \mathrm{Id}_M\}.$$

Definition 1.2.1 (Endomorphism bundle)**.** Let E be a vector bundle over M. The endomorphism bundle $\mathrm{End}(E)$ is the bundle $E \otimes E^*$. The fiber of $\mathrm{End}(E)$ is equal to the vector space $\mathrm{End}(E_x)$.

Suppose that $T \in \Gamma(\mathrm{End}(E))$, thus for each $x \in M$ we have

$$T(x) : E_x \to E_x, v \mapsto T(x)v,$$

this allows us to view T as a map

$$T : \Gamma(E) \to \Gamma(E),$$

by the equation

$$(Ts)(x) = T(x)s(x), \quad (s \in \Gamma(E), x \in M).$$

The map $T : \Gamma(E) \to \Gamma(E)$ is C^∞-linear, that is, for all $f \in C^\infty(M)$ and all $s \in \Gamma(E)$ we have $T(fs) = fT(s)$.

Theorem 1.2.2. *This map describes a bijection $\Gamma(End(E)) \simeq End(\Gamma(E))$, (where we see $\Gamma(E)$ as a $C^\infty(M)$-module).*

Proof. Let us denote the map described above by

$$\begin{aligned} \varphi :&\Gamma(\mathrm{End}(E)) \to \mathrm{End}(\Gamma(E)), \\ &(\varphi T)(s)(x) = T(x)s(x), \qquad (T \in \Gamma(\mathrm{End}(E)), s \in \Gamma(E), x \in M). \end{aligned}$$

Let us now describe a map $\psi : \mathrm{End}(\Gamma(E)) \to \Gamma(\mathrm{End}(E))$, which will turn out to be the inverse of φ. Let $\tau \in \mathrm{End}(\Gamma(E))$ be arbitrary. We will describe its image under ψ, that is $\psi\tau$, by giving its value at each $x \in M$, that is, we will give a linear map

$$\psi\tau(x) : E_x \to E_x.$$

Let $x \in M$ and $v \in E_x$ be arbitrary. Let $s \in \Gamma(E)$ be an arbitrary section with $s(x) = v$. We now define $\psi\tau$ by

$$(\psi\tau)(x) : E_x \to E_x, v \mapsto (\tau s)(x).$$

Using the fact that τ is $C^\infty(M)$-linear, one may show that this does not depend on the choice of section s as follows.

Suppose that we have two sections, $s, s' \in \Gamma(E)$, with the property that $s(x) = s'(x) = v$. We define the section $\Delta s := s - s'$. Then it suffices to show that $(\tau\Delta s)(x) = 0$. Let U be a trivializable open neighborhood of x. Then choose a basis of sections $\{e_i\}$ of E over U. We expand the section Δs on this basis, i.e. we write

$$\Delta s(m) = \sum_i \chi_i(m) e_i(m), \quad (m \in U),$$

where χ_i are smooth functions on U. Note that $\chi_i(x) = 0$ for all i, since $\Delta s(x) = 0$. Let us pick a smooth bump function f on M that satisfies $f(x) = 1$ and $\operatorname{supp}(f) = U$. Now we may compute

$$\begin{aligned} \Delta s &= (1-f)\Delta s + f\Delta s \\ &= (1-f)\Delta s + \sum_i f\chi_i e_i. \end{aligned}$$

We complete the proof by using the fact that τ is $C^\infty(M)$-linear,

$$\begin{aligned} (\tau\Delta s)(x) &= \tau\left((1-f)\Delta s + \sum_i f\chi_i s_i\right)(x) \\ &= (1-f)(x)(\tau\Delta s)(x) + \sum_i f(x)\chi_i(x)(\tau e_i)(x) \\ &= 0. \end{aligned}$$

We now show that ψ is left and right inverse of φ. Let $T \in \Gamma(\operatorname{End}(E))$, $x \in M$, and $v \in E_x$ and $s \in \Gamma(E)$ with $s(x) = v$, then we compute

$$(\psi\varphi T)(x)(v) = (\varphi T)(s)(x) = T(x)s(x) = T(x)v,$$

hence $\psi\varphi T = T$. Next, let $\tau \in \operatorname{End}(\Gamma(E))$, $s \in \Gamma(E)$ and $x \in M$, then we compute

$$(\varphi\psi\tau)(s)(x) = (\psi\tau)(x)(s(x)) = (\tau s)(x),$$

hence $\varphi\psi\tau = \tau$.

□

Given two vector bundles E, F over M we may define the homomorphism bundle $\operatorname{Hom}(E, F) = F \otimes E^*$. We now have the following theorem.

Theorem 1.2.3. *There is a bijection* $\Gamma(\mathit{Hom}(E,F)) \simeq \mathit{Hom}(\Gamma(E), \Gamma(F))$*, (where we view* $\Gamma(E)$ *and* $\Gamma(F)$ *as* $C^\infty(M)$*-modules.*

The proof of Theorem 1.2.2 can be adapted to this situation with minimal changes.

Chapter 2

Gauge theory

The subject of gauge theories is an immense one, there are many theories in physics which may be called gauge theories. A familiar example might be the classical theory of electromagnetism, but the standard model of physics is also a gauge theories. Even general relativity may, in some sense, be viewed as a gauge theory, though in this case there are many caveats. In this chapter we will give an introduction to some of the most basic notions that appear in gauge theory, our goal is not to be complete, but rather to give a leisurely introduction to the mathematical framework of gauge theories.

Let E be a vector bundle over M with fiber V. We denote the space of smooth vector fields on M by $\mathfrak{X}(M)$.

2.1 Connections

If M is any smooth manifold, then we may view the smooth functions $C^\infty(M)$ as sections of the trivial line bundle over M, i.e. $C^\infty(M) = \Gamma(M \times \mathbb{R})$. Given a vector field $v \in \mathfrak{X}(M)$, there is then a canonical way to take the derivative of any section $f \in \Gamma(M \times \mathbb{R})$ along v, yielding another section $\mathrm{d}f(v) \in \Gamma(M \times \mathbb{R})$. Hence, we may see the derivative as a map $\mathrm{d} : \mathfrak{X}(M) \to \mathrm{End}(\Gamma(M \times \mathbb{R}))$. In this section we will give the definition of a connection on a vector bundle, which may be thought of as a generalization of this construction.

Definition 2.1.1 (Connection on a vector bundle)**.** A connection D on M is a map $D : \mathfrak{X}(M) \to \mathrm{End}(\Gamma(E)), v \mapsto D_v$. Apart from the usual properties required for an endomorphism (of vector spaces) the map D satisfies the additional properties, for all $v, w \in \mathfrak{X}(M), s \in \Gamma(E)$ and $f \in C^\infty(M)$:

$$\begin{aligned} D_v(fs) &= v(f)s + fD_v s, \\ D_{v+w}s &= D_v s + D_w s, \\ D_{fv}s &= fD_v s. \end{aligned}$$

The map $D_v : \Gamma(E) \to \Gamma(E)$ is called the covariant derivative along v. One may check that, indeed, the usual exterior derivative d is a connection on the trivial line bundle $M \times \mathbb{R}$.

We investigate this definition in local coordinates. Let $U \subseteq M$ be an open neighborhood with coordinates x^μ and let ∂_μ be the corresponding basis of coordinate vector fields, and let e_i be a basis of sections of E over U. We write D_μ for D_{∂_μ}. We may expand the sections $D_\mu e_j$ over the basis e_i and write

$$D_\mu e_j = A^i_{\mu j} e_i,$$

where $A^i_{\mu j}$ are functions on U, they are the components of the so-called vector potential. For an arbitrary vector field $v = v^\mu \partial_\mu$ on U and an arbitrary section $s = s^i e_i$ of E over U we obtain the coordinate description of the covariant derivative of s in the direction of v:

$$\begin{aligned} D_v s &= D_{v^\mu \partial_\mu} s \\ &= v^\mu D_\mu s \\ &= v^\mu D_\mu (s^i e_i) \\ &= v^\mu ((\partial_\mu s^i) e_i + A^j_{\mu i} s^i e_j) \\ &= v^\mu (\partial_\mu s^i + A^i_{\mu j} s^j) e_i. \end{aligned}$$

We thus see that $A^i_{\mu j} v^\mu s^j e_i$ is a section of E over U. This allows one to view the vector potential A as an End(E)-valued 1-form. That is, as a section of the bundle

$$\mathrm{End}(E|_U) \otimes T^*U.$$

The justification for this viewpoint is that we may write

$$A = A^j_{\mu i} e_j \otimes e^i \otimes \mathrm{d}x^\mu.$$

Indeed, let $v \in \mathfrak{X}(U)$ and let s be a section of E over U, then we obtain

$$A(v)s = A^j_{\mu i} v^\mu s^i e_j.$$

Definition 2.1.2 (Standard flat connection)**.** Given a trivial open neighborhood U of M we define the standard flat connection D^0 on $E|_U$ by

$$D^0_v s = v(s^j) e_j.$$

A remark that will turn out to be important later is that the standard flat connection depends on the choice of trivialization.

If $E = M \times \mathbb{R}$, then the standard flat connection is actually just the exterior derivative. In some cases it will be possible to find a globally defined flat connection, in those cases we will write D^0 for this connection, if there is no globally defined flat connection we will use D^0 to denote any fixed connection. We will see more about what it means for a connection to be flat later.

End(E)-valued differential forms

Definition 2.1.3 (End(E)-valued differential forms)**.** An End(E)-valued differential form is a section of the bundle

$$\mathrm{End}(E) \otimes \Lambda T^*M.$$

Remark 2.1.4. It turns out that this object is very important to us, it is therefore convenient to introduce the following common shorthand notation

$$\Gamma(\Lambda T^*M \otimes \mathrm{End}(E)) = \Omega(M, \mathrm{End}(E)),$$
$$\Gamma(\Lambda^k T^*M \otimes \mathrm{End}(E)) = \Omega^k(M, \mathrm{End}(E)).$$

Elements of $\Omega^k(M, \mathrm{End}(E))$ are referred to as End(E)-valued k-forms. We should stress that these spaces are defined for any vector bundle E in the place of End(E).

The reader may notice that we have exchanged the order of the factors in the tensor product above, sometimes one order is more natural than the other, so we will do so freely.

Proposition 2.1.5. *If D' is a connection on E, then any other connection on E can be written as $D' + A$, where A is some End(E)-valued 1-form.*

Here we give the details of the proof sketched in Ref. [Baez94].

Proof. Suppose we are given two connections D' and D, we have to show that there exists an End(E)-valued 1-form A with the property that $D' + A = D$. Let us show that $A := D - D'$ does the job. Let $v \in \mathfrak{X}$, and $s, t \in \Gamma(E)$ be arbitrary, we claim that A_v is a $C^\infty(M)$-linear map $A_v : \Gamma(E) \to \Gamma(E)$, indeed,

$$A_v(fs) = fD_v(s) + v(f)s - fD'_v(s) - v(f)s = fD_v(s) - fD'_v(s) = fA_v(s),$$
$$A_v(s + t) = D_v(s) + D_v(t) - D'_v(s) - D'_v(t) = A_v(s) + A_v(t).$$

So we may identify $A_v \in \Gamma(\mathrm{End}(E))$. Furthermore if we fix $v, w \in \mathfrak{X}(M)$, $f \in C^\infty(M)$ and $s \in \Gamma(E)$ we also have that

$$A_{fv}(s) = fA_v(s), \quad \text{and}$$
$$A_{v+w}(s) = A_v(s) + A_w(s).$$

Which shows that if we consider A as a map

$$A : \mathfrak{X}(M) = \Gamma(TM) \to \Gamma(\mathrm{End}(E)),$$

it is $C^\infty(M)$-linear. So we may identify A as being $A \in \mathrm{Hom}(\Gamma(TM), \Gamma(\mathrm{End}(E)))$. An application of Theorem 1.2.3 now shows that $\mathrm{Hom}(\Gamma(TM), \Gamma(\mathrm{End}(E))) \simeq \Gamma(T^*M \otimes \mathrm{End}(E))$, which completes the proof. □

Since the difference of any two connections, D and D', gives an element $D - D' \in \Omega^1(M, \mathrm{End}(E))$ and furthermore since $D + A$ is a connection for any $A \in \Omega^1(M, \mathrm{End}(E))$, we say that the space of connections is an affine space for the space of End(E)-valued 1-forms on M.

2.2 Gauge transformations

In physics, gauge theories arose as a tool to formalize the notion of so-called local symmetries. At this point we will not be very concrete about this, but we will see examples later. The mathematical object that formalizes the notion of symmetry is a group, which we call the gauge group in our current context. The gauge group is the structure group G of our G-bundle. The mathematical object that formalizes the notion of *local* symmetry is called the group of gauge transformations, denoted by $\mathcal{G}$. We should warn the reader that the distinction between the gauge group and the group of gauge transformations is not always made very clear. In this thesis we will reserve the name 'gauge group' for the structure group G, this means that we will always, rather verbosely, refer to the group $\mathcal{G}$ as the 'group of gauge transformations'. We will give a definition of the group of gauge transformations similar to definition 1.1.1.

Definition 2.2.1 (Group of gauge transformations)**.** Let G be a Lie group and E a G-bundle over the manifold M with fiber V. Let $\{U_\alpha\}$ be an open cover of M that trivializes E. That is, the vector bundle E is obtained by the quotient

$$E := \left(\bigcup_\alpha U_\alpha \times V \right) / \sim .$$

See Definition 1.1.1 for the equivalence relation $\sim$. Then, the group of gauge transformations $\mathcal{G}$ consists of collections of maps

$$h_\alpha \in C^\infty(U_\alpha, G),$$

that satisfy the relation

$$h_\beta = g_{\beta\alpha} h_\alpha g_{\alpha\beta}, \tag{2.1}$$

where $g_{\alpha\beta}$ are the maps used to give the equivalence relation $\sim$ as in 1.1.1.

We should stress that an element of the group of gauge transformations $\mathcal{G}$ is a *collection* $\{h_\alpha\}$, that is, if we are given an element $\{h_\alpha\} \in \mathcal{G}$ we have a map $h_\alpha \in C^\infty(U_\alpha, G)$ for each member of the trivializing cover $\{U_\alpha\}$.

Eq. (2.1) tells us how gauge transformations transform under coordinate transformations.

The group of gauge transformations is actually a group. Indeed, if $\{h_\alpha\}, \{h'_\alpha\} \in \mathcal{G}$ then their product is simply the pointwise product, which exhibits the correct transformation behaviour

$$\begin{aligned} g_{\beta\alpha}(h \cdot h')_\alpha g_{\alpha\beta} &= g_{\beta\alpha} h_\alpha h'_\alpha g_{\alpha\beta} \\ &= (g_{\beta\alpha} h_\alpha g_{\alpha\beta})(g_{\beta\alpha} h'_\alpha g_{\alpha\beta}) \\ &= (h \cdot h')_\beta. \end{aligned}$$

The unit element of $\mathcal{G}$ is the collection of maps identically equal to the unit element of G.

The group of gauge transformations, $\mathcal{G}$, acts from the left on the space of sections of E as follows. Let $s \in \Gamma(E)$, then in a local trivialization $U_\alpha \times V$ of E we have

$$s_\alpha(x) = (x, v_\alpha(x)) \in U_\alpha \times V,$$

where $v_\alpha : U_\alpha \to V$. Let $h = \{h_\alpha\} \in \mathcal{G}$, then we define

$$(h \cdot s)_\alpha(x) = (x, h_\alpha(x) v_\alpha(x)) \in U_\alpha \times V.$$

This action is well-defined, as we will show here. For any section $s \in \Gamma(E)$ we must have $s_\alpha(x) \sim s_\beta(x)$ if $x \in U_\alpha \cap U_\beta$, that is,

$$v_\alpha(x) = g_{\alpha\beta} v_\beta(x).$$

So let us compute

$$\begin{aligned} h_\alpha(x) v_\alpha(x) &= h_\alpha(x) g_{\alpha\beta} v_\beta(x) \\ &= g_{\alpha\beta} h_\beta(x) g_{\beta\alpha} g_{\alpha\beta} v_\beta(x) \\ &= g_{\alpha\beta} h_\beta(x) v_\beta(x), \end{aligned}$$

which tells us that $(h \cdot s)_\alpha(x) \sim (h \cdot s)_\beta(x)$, as required.

One can apply gauge transformations to connections as follows. Let D be a connection on M and let $g \in \mathcal{G}$ be a gauge transformation. The gauge transform of D by g is then given by

$$D'_v(s) = g D_v(g^{-1} s).$$

Definition 2.2.2 (G-connection)**.** We say that D is a G-connection if for each U_α we have that in local coordinates the components $A_\mu \in \mathrm{End}(E)$ live in $\mathfrak{g}$, the Lie algebra of G.

Let us explain what it exactly means that A_μ lives in $\mathfrak{g}$. Just like before, if $s \in \Gamma(E)$, then in a local trivialization $U_\alpha \times V$ of E we have

$$s_\alpha(x) = (x, v_\alpha(x)) \in U_\alpha \times V,$$

now A_μ acts on s by

$$(A_\mu \cdot s)_\alpha(x) = (x, A_{\alpha,\mu}(x) v_\alpha(x)).$$

So $A_{\alpha,\mu}(x) \in \mathrm{End}(V)$. Recall that we have a representation $\rho : G \to \mathrm{GL}(V)$, which induces a representation $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$. If $A_{\alpha,\mu}$ lies in the image of ρ_*, i.e. $A_{\alpha,\mu} \in \rho_*(\mathfrak{g})$ then we say that $A_{\alpha,\mu}$ lives in $\mathfrak{g}$.

Proposition 2.2.3. *If D is any G-connection and $g \in \mathcal{G}$ an arbitrary gauge transformation, then the gauge transform D' of D is again a G-connection.*

Proof. First, let us check that D' is actually a connection. We fix $v \in \mathfrak{X}(M)$ and check that D'_v is actually an element of $\mathrm{End}(\Gamma(E))$. Let $r, s \in \Gamma(E)$ and let $f \in C^\infty(M)$. We compute

$$\begin{aligned} D'_v(r+s) &= gD_v(g^{-1}(r+s)) \\ &= gD_v(g^{-1}r + g^{-1}s) \\ &= gD_v(g^{-1}r) + gD_v(g^{-1}s) \\ &= D'_v(r) + D'_v(s), \quad \text{and,} \\ D'_v(fr) &= gD_v(g^{-1}(fr)) \\ &= gD_v(fg^{-1}r) \\ &= fgD_v(g^{-1}r) + gv(f)g^{-1}r, \\ &= fD'_v(r) + v(f)r, \end{aligned}$$

as desired. The other properties are proven similarly.

Next, let us show that D' is actually G-connection. We consider D on the neighborhood U_α, thus $D = D^0 + A$ and $D' = D^0 + A'$. We have seen that $(D_\mu s)^i = \partial_\mu s^i + A^i_{\mu j} s^j$. By definition of D' we obtain

$$\begin{aligned} (D'_\mu s)^i = g(D'_\mu g^{-1}s)^i &= g\partial_\mu(g^{-1}s^i) + gA^i_{\mu j}g^{-1}s^j \\ &= \partial_\mu s^i + g(\partial_\mu g^{-1})s^i + gA^i_{\mu j}g^{-1}s^j, \end{aligned}$$

thus we conclude that

$$A'_\mu = g(\partial_\mu g^{-1}) + gA_\mu g^{-1}.$$

Now let us consider a local chart U such that we may identify g and A_μ with maps

$$g : U \to \rho(G) \subseteq \mathrm{GL}(V), \quad A_\mu : U \to \rho_*(\mathfrak{g}) \subseteq \mathrm{End}(V).$$

It is then a standard fact of the theory of Lie groups that $(gA_\mu g^{-1})(x) \in \rho_*(\mathfrak{g})$. Let us show that if $x \in U$, then $(g(\partial_\mu g^{-1}))(x) \in \rho_*(\mathfrak{g})$. We consider the map

$$h : U \to G, y \mapsto g(x)g^{-1}(y),$$

it follows that $h(x) = e$, hence

$$(g(\partial_\mu g^{-1}))(x) = \partial_\mu h(x) \in \rho_*(\mathfrak{g}),$$

as required. □

2.3 Parallel transport and holonomy

In this section we assume that $\pi : E \to M$ is a vector bundle over a smooth manifold equipped with a connection D. Furthermore $\gamma : [0, T] \to M$ will be a smooth path in M, with endpoints $\gamma(0) = p$ and $\gamma(T) = q$.

2.3.1 Parallel transport

Let v be a vector in the fiber of E over $\gamma(t)$. We would like to find a path $u : [0, T] \to E$ that is the parallel translate of v along γ. We want to drag the vector v along the path γ, it turns out that we can do this using a connection. For this we will work in local trivialization $E|_U \simeq U \times V$, for the moment we suppose that γ maps into U. We suppose that u is a lift of γ, and define the map $v : [0, T] \to V$ such that we may write $u(t) = (\gamma(t), v(t))$. Now $u(t)$ is called the parallel translate of v along γ if $u(0) = (\gamma(0), v)$ and if the equation

$$D_{\gamma'(t)}u(t) := (\gamma(t), \frac{\mathrm{d}}{\mathrm{d}t}v(t)) + A(\gamma'(t))u(t) = (\gamma(t), 0),$$

holds for all $t \in [0, T]$. The idea is that we transport the vector along the path γ in such a way that the change in the direction of the vector, as measured by the connection D, is zero. We might slightly abuse notation and instead write

$$D_{\gamma'(t)}u(t) = \frac{\mathrm{d}}{\mathrm{d}t}u(t) + A(\gamma'(t))u(t).$$

If the image of γ is not trivializable, then it will at least lie in the finite union of trivializable sets, the above procedure can then be done in each of these patches separately and then glued together.

The solution to the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}u(t) + A(\gamma'(t))u(t) = 0$ always exists and can be found by the (always convergent) infinite sum

$$u(t) = \sum_{n=0}^{\infty}\left((-1)^n \int_{t\geqslant t_1\geqslant ...\geqslant t_n \geqslant 0} A(\gamma'(t_1))...A(\gamma'(t_n))\mathrm{d}t_n...\mathrm{d}t_1\right)u(0).$$

There is a concise way to denote this solution. Let $t_1, ..., t_n \in [0, T]$ be a collection of (not necessarily ordered) times. Let $\sigma : \{1, ..., n\} \to \{1, ..., n\}$ be the permutation that orders the times by increasing value, i.e. $t_{\sigma(1)} \geqslant ... \geqslant t_{\sigma(n)}$, then the path-ordered product is defined as

$$P\left[A(\gamma'(t_1)...A(\gamma'(t_n))\right] := A(\gamma'(t_{\sigma(1)}))...A(\gamma'(t_{\sigma(n)})).$$

In this way, we may write

$$\begin{aligned}&\sum_{n=0}^{\infty}\left((-1)^n \int_{t\geqslant t_1\geqslant ...\geqslant t_n \geqslant 0} A(\gamma'(t_1))...A(\gamma'(t_n))\mathrm{d}t_n...\mathrm{d}t_1\right)\\ =&\frac{1}{n!}\int_{t_i\in[0,t]} P\left[A(\gamma'(t_1)...A(\gamma'(t_n))\right] =: \frac{1}{n!}P\left(\int_0^t A(\gamma'(s))\mathrm{d}s\right)^n\end{aligned}$$

The final expression must be interpreted as short hand for the intermediate expression.

Finally, we define the path-ordered exponential by

$$Pe^{-\int_0^t A(\gamma'(s))\mathrm{d}s} := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} P\left(\int_0^t A(\gamma'(s))\mathrm{d}s\right)^n.$$

2.3.2 Holonomy

Definition 2.3.1 (Holonomy)**.** The holonomy of the connection D along the curve γ is the linear map

$$H(\gamma, D) : E_p \to E_q,$$

which sends the vector $v \in E_p$ to the result of parallel transport of v to E_q along γ with respect to the connection D.

If the path γ is only piecewise smooth, parallel transport may be defined in the obvious manner. If $\alpha : [0,T] \to M$ and $\beta : [0,S] \to M$ are paths in M with the property that the endpoint of α is the startpoint of β, i.e. $\alpha(T) = \beta(0)$, then we denote the product, i.e. the concatenation, of the paths α and β by $\beta\alpha$.

There is no canonical way to choose a parametrization for the product of two paths, so let us show that the holonomy does not depend on the parametrization.

Proposition 2.3.2. *The holonomy along a path γ does not depend on the parametrization chosen.*

Proof. Let $\alpha : [0,T] \to M$ be a smooth path and let $f : [0,S] \to [0,T]$ be a smooth function with $f(0) = 0$ and $f(S) = T$. We set $\beta = \alpha \circ f$. The claim is equivalent to the statement that $H(\alpha, D) = H(\beta, D)$ for any connection D. Now let $u_\alpha : [0,T] \to E$ obey the formula

$$D_{\alpha'(t)} u_\alpha(t) = 0.$$

We define $u_\beta : [0,S] \to E$ by $u_\beta := u_\alpha \circ f$. If the path u_β satisfies the differential equation $D_{\beta'(t)} u_\beta(t) = 0$ we are done. So let us compute this

$$\begin{aligned} D_{\beta'(t)} u_\beta(t) &= \frac{\mathrm{d}}{\mathrm{d}t} u_\alpha(f(t)) + A((\alpha \circ f)'(t)) u_\alpha(f(t)) \\ &= u_\alpha'(f(t)) f'(t) + A(\alpha'(f(t)) f'(t)) u_\alpha(f(t)) \\ &= f'(t) \left[u_\alpha'(f(t)) + A(\alpha'(f(t))) u_\alpha(f(t)) \right] \\ &= f'(t) \left[D_{\alpha'(f(t))} u_\alpha(f(t)) \right] = 0, \end{aligned}$$

as desired. □

Suppose that $u(t) \in E_{\gamma(t)}$ satisfies the parallel transport equation

$$D_{\gamma'(t)}u(t) = 0.$$

Let us work in a local trivialization and write

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = -A(\gamma'(t))u.$$

Which we will write as

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = -\gamma'^{\mu}(t)A_{\mu}u(t),$$

where $A_\mu = A_\mu(\gamma(t))$. If we now set $w(t) = g(\gamma(t))u(t)$, where g is a gauge transformation it follows that w satisfies the equation

$$D'_{\gamma'(t)}w(t) = 0,$$

where $D'_{\gamma'(t)}w(t)$ is the connection obtained by transforming the vector potential A as follows:

$$A'_\mu = gA_\mu g^{-1} + g\partial_\mu g^{-1}.$$

It follows that

$$H(\gamma, D') = g(\gamma(T))H(\gamma, D)g(\gamma(0))^{-1}.$$

Suppose that $\gamma : [0,T] \to M$ is a smooth path in M with $\gamma(0) = \gamma(T) = p$. In this case the map $H(\gamma, D) : E_{\gamma(0)} \to E_{\gamma(T)}$ is a map from E_p to itself, i.e. $H(\gamma, D) \in \mathrm{End}(E_p)$. It follows by cyclicity of the trace that the trace of such a map is gauge invariant:

$$\mathrm{tr}\left[H(\gamma, D')\right] = \mathrm{tr}\left[g(p)H(\gamma, D)g(p)^{-1}\right] = \mathrm{tr}\left[H(\gamma, D)\right].$$

For this reason the trace of the holonomy around a loop might be physically interesting object, and it actually is. It even has a special name: a Wilson loop.

2.4 Curvature

Suppose that $\pi : E \to M$ is a vector bundle equipped with a connection D.

Definition 2.4.1 (Curvature)**.** Given two vector fields, $v, w \in \mathfrak{X}(M)$, we define the curvature $F(v, w) \in \mathrm{End}(\Gamma(E))$ by

$$F(v, w)s = D_v D_w s - D_w D_v s - D_{[v,w]}s, \quad (s \in \Gamma(E)).$$

That the object $F(v, w)$ defined in this way actually lives in $\mathrm{End}(\Gamma(E))$ is shown in [Baez94], in Part II, Chapter 3.

A connection such that $F(v, w)s = 0$ for all $v, w \in \mathfrak{X}(M)$ and all $s \in \Gamma(E)$ is called a flat connection. By the isomorphism $\Gamma(\mathrm{End}(E)) \simeq \mathrm{End}(\Gamma(E))$, described in Theorem 1.2.3, it follows that $F(v, w)$ corresponds to a section of $\mathrm{End}(E)$, hence we may view F as an $\mathrm{End}(E)$-valued two-form, $F \in \Omega^2(M, \mathrm{End}(E))$. The curvature measures the holonomy of a connection around an infinitesimal loop, we will make this statement more precise. Let us work in local coordinates on a neighborhood U, with a given point $x \in M$ as the origin. If v, w are vector fields on M then $F(v, w)$ is a section of $\mathrm{End}(E)$. On U we can pick the special vector fields ∂_μ and ∂_ν, using these we introduce the notation

$$F_{\mu\nu} = F(\partial_\mu, \partial_\nu),$$

a calculation, (given in [Baez94], Part II, Chapter 3), then shows

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

Let $v \in E_p$ and let us perform the parallel transport of v around a small square with sides of length ε in the $x^\mu - x^\nu$-plane. Let us call the result v'.

Proposition 2.4.2. *Up to terms of order* ε^2

$$v - v' = \varepsilon^2 F_{\mu\nu}.$$

Or, in other words, if $\gamma : [0, 4] \to M$ *denotes the path around the square:*

$$H(\gamma, D) = 1 - \varepsilon^2 F_{\mu\nu}.$$

Proof. We use the path ordered exponential formula, or rather the power series that defines it

$$v' = u(4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} P \left(\int_0^4 A(\gamma'(s)) \mathrm{d}s \right)^n v.$$

Suppressing all the coordinates but x^μ and x^ν we can write down explicit expressions for $\gamma(s)$:

$$\begin{aligned} \gamma(s) &= (s\varepsilon, 0), \quad 0 \leqslant s \leqslant 1, \\ \gamma(s) &= (\varepsilon, (s-1)\varepsilon), \quad 1 < s \leqslant 2, \\ \gamma(s) &= ((3-s)\varepsilon, \varepsilon), \quad 2 < s \leqslant 3, \\ \gamma(s) &= (0, (4-s)\varepsilon), \quad 3 < s \leqslant 4. \end{aligned}$$

We compute $\gamma'(s)$:

$$\begin{aligned} \gamma'(s) &= \varepsilon\partial_\mu, \quad 0 \leqslant s \leqslant 1, \\ \gamma'(s) &= \varepsilon\partial_\nu, \quad 1 < s \leqslant 2, \\ \gamma'(s) &= -\varepsilon\partial_\mu, \quad 2 < s \leqslant 3, \\ \gamma'(s) &= -\varepsilon\partial_\nu, \quad 3 < s \leqslant 4. \end{aligned}$$

So we have that $A(\gamma'(s)) \propto \varepsilon$ for all $s \in [0,4]$, so if we are interested in v' up to order ε^2 we may write

$$v' = \left[1 - \int_0^4 A(\gamma'(s))\mathrm{d}s + \frac{1}{2}P\left(\int_0^4 A(\gamma'(s))\mathrm{d}s\right)^2\right]v.$$

In what follows it will be important to remember that $A(\gamma'(s))$ actually depends on $\gamma(s)$ as well, so we will add this as an argument and write $A(\gamma'(s), \gamma(s))$. So let us compute the second term on the right hand side of this equation

$$\begin{aligned} -\int_0^4 A(\gamma'(s))\mathrm{d}s = &-\varepsilon\int_0^1 A(\partial_\mu, \gamma(s))\mathrm{d}s - \varepsilon\int_1^2 A(\partial_\nu, \gamma(s))\mathrm{d}s \\ &+\varepsilon\int_2^3 A(\partial_\mu, \gamma(s))\mathrm{d}s + \varepsilon\int_3^4 A(\partial_\nu, \gamma(s))\mathrm{d}s. \end{aligned}$$

We introduce the notation $A_\mu(x,y) := A(\partial_\nu, (x,y))$. We obtain

$$\begin{aligned} -\int_0^4 A(\gamma'(s))\mathrm{d}s = &-\varepsilon\int_0^1 A_\mu(s\varepsilon, 0)\mathrm{d}s - \varepsilon\int_1^2 A_\nu(\varepsilon, (s-1)\varepsilon)\mathrm{d}s \\ &+\varepsilon\int_2^3 A_\mu((3-s)\varepsilon, \varepsilon)\mathrm{d}s + \varepsilon\int_3^4 A_\nu(0, (4-s)\varepsilon)\mathrm{d}s. \end{aligned}$$

At this point one might Taylor expand the integrands around $\varepsilon = 0$ and obtain

$$-\int_0^4 A(\gamma'(s))\mathrm{d}s = -\varepsilon^2(\partial_\mu A_\nu - \partial_\nu A_\mu),$$

where $A_\mu = A_\mu(0,0)$. Next we compute the term

$$\frac{1}{2}P\left(\int_0^4 A(\gamma'(s))\mathrm{d}s\right)^2 = \int_0^4 \mathrm{d}s\int_0^s \mathrm{d}s'\left[A(\gamma'(s))A(\gamma'(s'))\right].$$

We compute the inner integral

$$\int_0^s A(\gamma'(s'))\mathrm{d}s' = \varepsilon\begin{cases} sA_\mu & 0 \leqslant s \leqslant 1, \\ A_\mu + (s-1)A_\nu & 1 < s \leqslant 2, \\ (3-s)A_\mu + A_\nu & 2 < s \leqslant 3, \\ (4-s)A_\nu & 3 < s \leqslant 4. \end{cases}$$

So we obtain

$$\begin{aligned}\int_0^4 \mathrm{d}sA(\gamma'(s))\int_0^s \mathrm{d}s'A(\gamma'(s')) &= \varepsilon^2\int_0^1 sA_\mu^2\mathrm{d}s\\ &+\varepsilon^2\int_1^2 A_\nu\left(A_\mu+(s-1)A_\nu\right)\mathrm{d}s\\ &-\varepsilon^2\int_2^3 A_\mu\left((3-s)A_\mu+A_\nu\right)\mathrm{d}s\\ &-\varepsilon^2\int_3^4(4-s)A_\nu^2\mathrm{d}s\\ &=\varepsilon^2[A_\nu,A_\mu].\end{aligned}$$

Thus, we conclude that

$$v' = \left[1-\varepsilon^2(\partial_\mu A_\nu-\partial_\nu A_\mu)-\varepsilon^2[A_\mu,A_\nu]\right]v = \left[1-F_{\mu\nu}\right]v,$$

as desired. □

If we recall that we defined

$$F_{\mu\nu} := F(\partial_\mu,\partial_\nu),$$

it follows that the curvature 2-form, F, is given by

$$F = \frac{1}{2}F_{\mu\nu}dx^\mu\wedge dx^\nu.$$

We mention this notation because it is standard in the physics literature.

Let us consider how the curvature transforms under a gauge transformation.

Proposition 2.4.3. *Let $g\in\mathcal{G}$ be a gauge transformation. Let D be a connection. The gauge transform of D is denoted by $D' = gDg^{-1}$. Let F denote the curvature of D, and let F' denote the curvature of D'. Then the following relation holds*

$$F' = gFg^{-1}. \tag{2.2}$$

The proof can be found in [Baez94], we repeat it here.

Proof. Let $u,v\in\mathfrak{X}(M)$ and let $s\in\Gamma(E)$, then claim follows from the computation

$$\begin{aligned}F'(u,v) &= D'_uD'_vs - D'_vD'_us - D'_{[u,v]}s\\ &= gD_uD_vg^{-1}s - gD_vD_ug^{-1}s - gD_{[u,v]}g^{-1}s\\ &= gF(u,v)g^{-1}s.\end{aligned}$$

□

2.5 The exterior covariant derivative

In this section we will, given a connection D, construct a map

$$\mathrm{d}_D : \Gamma(\Lambda T^*M \otimes E) \to \Gamma(\Lambda T^*M \otimes E),$$

called the exterior covariant derivative. We first prove the following lemma, that simplifies much of the notation.

Lemma 2.5.1. *Any E-valued differential form can be written as a sum of those of the form $s \otimes \omega$, where s is a section of E and ω is an ordinary differential form on M.*

This is a direct consequence of the more general claim:

Lemma 2.5.2. *Suppose that E and E' are vector bundles over M. Any section of $E \otimes E'$ can be written as a locally finite sum of sections of the form $s \otimes s'$, where $s \in \Gamma(E)$ and $s' \in \Gamma(E')$.*

Proof. Let $r \in \Gamma(E \otimes E')$ and $x \in M$ be arbitrary but fixed. Let $U \subseteq M$ be a neighborhood of x such that $E \otimes E'$ is trivializable, i.e. $\pi^{-1}(U) \simeq (E_x \otimes E'_x) \times U$. Let us choose a basis for sections of E over U, denoted by $\{e_i\}$ and a basis for sections of E' over U, denoted by $\{e'_i\}$. Then the set $\{e_i \otimes e_j\}$ is a basis for sections of $E \otimes E'$ over U, which is the statement that is required.

□

We started our discussion of connections with the idea of generalizing the exterior derivative d, which could be viewed as a map $\mathrm{d} : \mathfrak{X}(M) \to \mathrm{End}(\Gamma(M \times \mathbb{R}))$ to non-trivial vector bundles instead of $M \times \mathbb{R}$. Of course, one may also view the exterior derivative as a map $\mathrm{d} : C^\infty(M) \to \Gamma(\Lambda^1 T^*M)$, or, equivalently, as a map $\mathrm{d} : \Gamma(M \times \mathbb{R}) \to \Gamma(\Lambda^1 T^*M)$. In the sequel, we will generalize the exterior derivative from this point of view.

Definition 2.5.3. Given a section s of E we define the E-valued 1-form $d_D s$ as follows

$$d_D s(v) = D_v s, \quad (v \in \mathfrak{X}(M)).$$

The map

$$d_D : \Gamma(E) \to \Gamma(\Lambda^1 T^*M \otimes E)$$

is called the exterior covariant derivative (with respect to the connection D).

We will shortly see how to extend the exterior covariant derivative to a map

$$d_D : \Gamma(\Lambda T^*M \otimes E) \to \Gamma(\Lambda T^*M \otimes E).$$

As preparation we have the following proposition:

Proposition 2.5.4. *In local coordinates on some open set $U \subseteq M$ definition 2.5.3 is equivalent to setting*

$$d_D s = D_\mu s \otimes dx^\mu.$$

Proof. Let $v \in \mathfrak{X}(U)$, then $v = v^\mu \partial_\mu$, thus

$$[D_\mu s \otimes dx^\mu]\, v = v^\mu D_\mu s = D_v s = d_D s,$$

as required. □

Definition 2.5.5. To define d_D on arbitrary E-valued differential forms it suffices, by Lemma 2.5.2, to define it on those of the form $s \otimes \omega$, where $s \in \Gamma(E)$ and $\omega \in \Gamma(\Lambda T^* M)$. The definition is this:

$$d_D(s \otimes \omega) = d_D s \wedge \omega + s \otimes d\omega.$$

The square of the exterior covariant derivative, d_D^2 is proportional to the curvature of D, or, more precisely,

$$d_D^2 \eta = F \wedge \eta,$$

for any E-valued form η, a proof may be found in Ref. [Baez94].

2.6 The Bianchi identity

Let D be a connection on a vector bundle E over M. We define the dual connection D^* on E^* by

$$(D_v^* \lambda)(s) = v(\lambda(s)) - \lambda D_v(s), \quad (v \in \mathfrak{X}(M), s \in \Gamma(E), \lambda \in \Gamma(E^*)).$$

Using this we can define a new connection on $E \otimes E^*$, denoted $D \otimes D^*$, or more abusively just by D, it acts as

$$(D \otimes D^*)_v (s \otimes s^*) = (D_v s) \otimes s^* + s \otimes (D_v^* s^*),$$

for $v \in \mathfrak{X}(M), s \in \Gamma(E), s^* \in \Gamma(E^*)$.

Proposition 2.6.1 (Chain rule for covariant derivatives)**.** *Denote the connection on $E \otimes E^* = End(E)$ by D. Then for all $T \in \Gamma(End(E))$, all $v \in \mathfrak{X}$, and all $s \in \Gamma(E)$ we have*

$$(D_v T)(s) = D_v(Ts) - T(D_v s).$$

Proof. We write $T = t \otimes t^*$ and compute

$$\begin{aligned} D_v(Ts) - T(D_v s) &= D_v(t^*(s)t) - t^*(D_v s)t \\ &= v(t^*(s))t + t^*(s) D_v(t) - t^*(D_v s)t \\ &= (v(t^*(s)) - t^*(D_v s))\, t + t^*(s) D_v(t) \\ &= t^*(s) D_v(t) + (D_v^* t^*)(s) t \\ &= (D_v T)(s). \end{aligned}$$

□

Theorem 2.6.2 (Bianchi identity)**.** *Let D be a connection on a vector bundle E over a manifold M and let F be the curvature of D. Then the following equation holds*

$$d_D F = 0.$$

This expression is called the Bianchi identity.

The proof of this theorem will be the content of the rest of this section. In the rest of this section we will suppose that we have fixed some vector bundle E on a manifold M equipped with a connection D. As a preparation we first show some other identities.

Lemma 2.6.3. *There is a unique way to define the wedge product of two End(E)-valued forms such that the wedge of the End(E)-valued forms $S \otimes \omega$ and $T \otimes \mu$ is given by*

$$(S \otimes \omega) \wedge (T \otimes \mu) = ST \otimes (\omega \wedge \mu),$$

and such that the wedge product depends $C^\infty(M)$-linearly on each factor.

Proof. If we extend the above relation bilinearly we have defined the wedge product uniquely on all End(E)-valued forms. What remains is to show that this actually gives a wedge product, i.e. that this product is associative, and that it depends $C^\infty(M)$-linearly on each factor. So let us show this, let $R \otimes \nu$ be another End(E)-valued form

$$\begin{aligned}(S \otimes \omega) \wedge ((T \otimes \mu) \wedge (R \otimes \nu)) &= S \otimes \omega \wedge [TR \otimes (\mu \wedge \nu)] \\ &= STR \otimes (\omega \wedge \mu \wedge \nu) \\ &= ((S \otimes \omega) \wedge (T \otimes \mu)) \wedge (R \otimes \nu).\end{aligned}$$

The fact that this wedge product depends $C^\infty(M)$-linearly on each factor is obvious. □

Lemma 2.6.4. *Let ω be an End(E)-valued p-form and μ an End(E)-valued form, the following relation holds*

$$d_D(\omega \wedge \mu) = d_D\omega \wedge \mu + (-1)^p \omega \wedge d_D\mu.$$

We say that the exterior covariant derivative is a graded derivation of $\Omega(M, End(E))$.

Proof. We set $\omega = T \otimes \rho$ and $\mu = S \otimes \sigma$, where T, S are sections of End(E) and ρ is a differential p-form and σ an arbitrary differential form. Let us compute

$$\begin{aligned}
\mathrm{d}_D(\omega\wedge\mu) &= \mathrm{d}_D((T\otimes\rho)\wedge(S\otimes\sigma)) \\
&= \mathrm{d}_D(TS\otimes(\rho\wedge\sigma)) \\
&= \mathrm{d}_D(TS)\wedge\rho\wedge\sigma + TS\otimes \mathrm{d}(\rho\wedge\sigma) \\
&\overset{!}{=} (\mathrm{d}_D(T)S + T\mathrm{d}_D(S))\wedge\rho\wedge\sigma + TS\otimes((\mathrm{d}\rho)\wedge\sigma + (-1)^p\rho\wedge \mathrm{d}\sigma) \\
&= (\mathrm{d}_D(T)S\wedge\rho + TS\otimes \mathrm{d}\rho)\wedge\sigma + T(\mathrm{d}_D(S)\wedge\rho\wedge\sigma + (-1)^p S\otimes\rho\wedge \mathrm{d}\sigma) \\
&= (\mathrm{d}_D(T)\wedge\rho + T\otimes \mathrm{d}\rho)\wedge(S\otimes\sigma) + (-1)^p(T\otimes\rho)\wedge(\mathrm{d}_D(S)\wedge\sigma + S\otimes \mathrm{d}\sigma) \\
&= (\mathrm{d}_D(T\otimes\rho))\wedge(S\otimes\sigma) + (-1)^p(T\otimes\rho)\wedge(\mathrm{d}_D(S\otimes\sigma)) \\
&= \mathrm{d}_D\omega\wedge\mu + (-1)^p\omega\wedge \mathrm{d}_D\mu
\end{aligned}$$

This shows the claim. In the step with the exclamation mark we have used the rule

$$\mathrm{d}_D(TS) = \mathrm{d}_D(T)S + T\mathrm{d}_D(S).$$

Which follows from Proposition 2.6.1, by a straightforward computation. □

Proposition 2.6.5. *Let D and D' be two connections on E and let $A = D - D' \in \Omega^1(M, End(E))$ be their difference. Let ω be any E-valued form and let η be any $End(E)$-valued form, the following relations hold*

$$\begin{aligned}
d_D\omega &= d_{D'}\omega + A\wedge\omega, \\
d_D\eta &= d_{D'}\eta + [A,\eta].
\end{aligned}$$

These relations are proved in Ref. [Baez94], Part II, Chapter 3, here we will give a proof of the second one.

Proof. Let η be any End(E)-valued form. Then we can write $\eta = \eta_I \otimes \mathrm{d}x^I$, and compute

$$d_D(\eta) = D_\mu(\eta_I)\wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^I.$$

Now we use Proposition 2.6.1, which implies

$$(D_\mu\eta_I)(s) = D_\mu(\eta_I s) - \eta_I(D_\mu s),$$

or in other words

$$D_\mu(\eta_I) = [D_\mu, \eta_I],$$

where the multiplication implied by the commutator brackets on the right hand side is composition of maps. We go on to compute

$$\begin{aligned}
d_D\eta &= [D_\mu, \eta_I]\otimes \mathrm{d}x^\mu\wedge \mathrm{d}x^I \\
&= [D'_\mu + A_\mu, \eta_I]\otimes \mathrm{d}x^\mu\wedge \mathrm{d}x^I \\
&= \mathrm{d}_{D'}\eta + A\wedge\eta - (-1)^p\eta\wedge A \\
&= \mathrm{d}_{D'}\eta + [A,\eta].
\end{aligned}$$

□

Note that the Bianchi identity is a local statement, so to prove it we may work in a local trivialization of E and thus write $D = D^0 + A$, where D^0 is the standard flat connection. In the rest of this section we will work in such a local trivialization and write $\mathrm{d} = \mathrm{d}_{D^0}$.

To complete the proof of the Bianchi identity we use the following expression for the curvature

$$F = \mathrm{d}A + A \wedge A,$$

which leads to

$$\begin{aligned}
\mathrm{d}_D F &= \mathrm{d}F + [A, F] \\
&= \mathrm{d}(\mathrm{d}A + A \wedge A) + [A, (\mathrm{d}A + A \wedge A)] \\
&= \mathrm{d}A \wedge A - A \wedge \mathrm{d}A + [A, \mathrm{d}A] + [A, A \wedge A] \\
&= [\mathrm{d}A, A] + [A, \mathrm{d}A] \\
&= 0,
\end{aligned}$$

which is the Bianchi identity.

Alternative proofs of the Bianchi identity can be found in Ref. [Baez94], Part II, Chapter 3.

Chapter 3

Yang-Mills theory

In this chapter, we consider a specific gauge theory, namely Yang-Mills theory. It is the result of the efforts of Chen Ning Yang and Robert Mills to generalise the gauge theory for Abelian groups, for example Electrodynamics, to non-Abelian groups, [YM54].

The importance of Yang-Mills theory in physics can hardly be overstated, the standard model of particle physics is a Yang-Mills theory with as gauge group a certain quotient of $U(1) \times SU(2) \times SU(3)$, by a finite normal subgroup, see Ref. [Baez05].

3.1 Maxwell's equations

Maxwell's equations read

$$\mathrm{d}F = 0 \qquad \star\,\mathrm{d}\star F = J.$$

Yang-Mills theory should be a generalization of Maxwell theory, which lives on the trivial $U(1)$-bundle over some semi-Riemannian manifold M, to any G-bundle over some semi-Riemannian manifold M. We have already done all the work required to generalize the first of Maxwell's equations, it becomes

$$\mathrm{d}_D F = 0.$$

To generalize the second of Maxwell's equations we need to generalize the Hodge star operator to $\mathrm{End}(E)$-valued differential forms.

Definition 3.1.1 (Hodge star operator)**.** Let T be any section of $\mathrm{End}(E)$ and let ω be any differential form, the Hodge star operator is then given by

$$\star(T \otimes \omega) = T \otimes \star\omega,$$

where the $\star$ on the right hand side is the usual Hodge star operator on differential forms.

The second of Maxwell's equations now generalizes to

$$\star \mathrm{d}_D \star F = J,$$

where J is an $\mathrm{End}(E)$-valued 1-form, called the current.

3.1.1 Gauge invariance

Here we will show the gauge invariance of the Yang-Mills equations. Let $\pi : E \to M$ be a vector bundle with connection D. Let $g \in \mathcal{G}$ be a gauge transformation and let D' be the gauge transform of D by g, i.e. $D'_v s = gD_v(g^{-1}s)$ for any section s and any vector field v. Let us compute the exterior derivative corresponding to the transformed connection, so let $\omega = \omega_I \otimes \mathrm{d}x^I$ be an E-valued differential form,

$$\begin{aligned}
\mathrm{d}_{D'}\omega &= D'_\mu(\omega_I) \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^I \\
&= gD_\mu(g^{-1}\omega_I) \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^I \\
&= g\mathrm{d}_D(g^{-1}\omega).
\end{aligned}$$

Next we claim that for any section T of $\mathrm{End}(E)$ we have

$$D'_v T = \mathrm{Ad}(g) D_v(\mathrm{Ad}(g^{-1})T),$$

where $\mathrm{Ad}(g)T = g \circ T \circ g^{-1}$. Let s be a section of E. We start from the right hand side

$$\begin{aligned}
\mathrm{Ad}(g)D_v(\mathrm{Ad}(g^{-1})T)(s) &= gD_v(g^{-1}Tg)g^{-1}s \\
&= g\big[D_v(g^{-1}T)s + g^{-1}TD_v(g)g^{-1}s\big] \\
&= gD_v(g^{-1}T)s + T\big[D_v(s) - gD_v(g^{-1}s)\big] \\
&= gD_v(g^{-1}Ts) - TgD_v(g^{-1}s) \\
&= D'_v(Ts) - TD'_v(s) \\
&= (D'_v T)(s).
\end{aligned}$$

This can also be written as

$$\mathrm{d}_{D'}(\mathrm{Ad}(g)\eta) = \mathrm{Ad}(g)\mathrm{d}_D(\eta).$$

Let $\eta = \eta_I \otimes \mathrm{d}x^I$ be an $\mathrm{End}(E)$-valued form, we compute,

$$\begin{aligned}
\mathrm{d}_{D'}\eta &= D'_\mu \eta_I \otimes \mathrm{d}x^\mu \wedge x^I \\
&= \mathrm{Ad}(g)D_v(\mathrm{Ad}(g^{-1})\eta_I) \otimes \mathrm{d}x^\mu \wedge \mathrm{d}x^I \\
&= \mathrm{Ad}(g)\mathrm{d}_D(\mathrm{Ad}(g^{-1})\eta).
\end{aligned}$$

We are now in a position to prove the gauge invariance of the Yang-Mills equations. Suppose that

$$\star \mathrm{d}_D \star F = J,$$

then it follows that

$$\begin{aligned} J' &= gJg^{-1} \\ &= g \star \mathrm{d}_D \star Fg^{-1} \\ &= \star \mathrm{Ad}(g)[\mathrm{d}_D \star F] \\ &= \star [\mathrm{d}_{D'} \star \mathrm{Ad}(g)F] \\ &= \star \mathrm{d}_{D'} \star F', \end{aligned}$$

in the last step we have used Eq. (2.2). We conclude that the second Yang-Mills equation is gauge invariant.

3.2 Yang-Mills Lagrangian

In physics, one usually does not just postulate equations of motion, but rather, one tries to find them from some Lagrangian, using the so-called action principle. To define the Yang-Mills Lagrangian we first define a map $\mathrm{tr} : \Gamma(\mathrm{End}(E)) \to C^\infty(M)$ as follows. Let $x \in M$ and let T be a section of the endomorphism bundle, then

$$\mathrm{tr}(T)(x) = \mathrm{tr}(T(x)),$$

where the tr on the right hand side is the usual trace operator defined for an endomorphism of any vector space. We extend this to a trace for $\mathrm{End}(E)$-valued forms. If T is a section of $\mathrm{End}(E)$ and ω is a differential form we define

$$\mathrm{tr}(T \otimes \omega) = \mathrm{tr}(T)\omega.$$

Now we can give the Yang-Mills Lagrangian: if D is a connection on E, this is the 4-form given by

$$\mathcal{L}_{YM} = \frac{1}{2}\mathrm{tr}(F \wedge \star F).$$

Let us show how one finds the Yang-Mills equations from the Yang-Mills Lagrangian and the action principle. Let us fix a connection D^0 (not necessarily flat), such that any other connection D can be written as

$$D = D^0 + A,$$

with A an $\mathrm{End}(E)$-valued 1-form. The Yang-Mills action is

$$S_{YM}(A) = \frac{1}{2}\int_M \mathrm{tr}(F \wedge \star F).$$

The action principle now tells us that we can find the equations of motion for A by finding the extremum of the action, i.e. from the equation

$$\delta S_{YM} = 0.$$

Let us determine what this actually means. Let δA be an arbitrary End(E)-valued 1-form. We define

$$A_s := A + s\delta A,$$

and then

$$\delta G = \frac{\mathrm{d}}{\mathrm{d}s} G(A_s)\big|_{s=0},$$

for any function G. Clearly δG may depend on δA, but when we write $\delta G = 0$ we mean that this is true for all choices of δA. First let us compute the variation of F. To this end we first note that we have

$$F = F_0 + \mathrm{d}A + A \wedge A,$$

where F_0 is the curvature of D^0. So we have

$$\begin{aligned}
\delta F &= \frac{\mathrm{d}}{\mathrm{d}s}(F_0 + \mathrm{d}A_s + A_s \wedge A_s)\big|_{s=0} \\
&= \mathrm{d}\left(\frac{\mathrm{d}}{\mathrm{d}s}A_s\right) + \left(\frac{\mathrm{d}}{\mathrm{d}s}A_s\right) \wedge A + A \wedge \left(\frac{\mathrm{d}}{\mathrm{d}s}A_s\right)\big|_{s=0} \\
&= \mathrm{d}\delta A + \delta A \wedge A + A \wedge \delta A \\
&= \mathrm{d}\delta A + [A, \delta A] \\
&= \mathrm{d}_D \delta A.
\end{aligned}$$

To compute the variation of the Yang-Mills action we will need some additional information about the trace. Suppose that $\eta = \eta_I \otimes \mathrm{d}x^I$ is an End(E)-valued p-form, and $\mu = \mu_I \otimes \mathrm{d}x^I$ is an End(E)-valued q-form, we compute

$$\begin{aligned}
\mathrm{tr}(\eta \wedge \mu) &= \mathrm{tr}(\eta_I \mu_J)\mathrm{d}x^I \wedge \mathrm{d}x^J \\
&= (-1)^{pq}\mathrm{tr}(\mu_J \eta_I)\mathrm{d}x^J \wedge \mathrm{d}x^I \\
&= (-1)^{pq}\mathrm{tr}(\mu \wedge \eta).
\end{aligned}$$

As a direct consequence we have

$$\mathrm{tr}([\omega, \mu]) = 0.$$

Next, if η is an End(E) valued form we compute

$$\begin{aligned}
\mathrm{tr}(\mathrm{d}_D \eta) &= \mathrm{tr}(\mathrm{d}\eta + [A, \eta]) \\
&= \mathrm{tr}(\mathrm{d}\eta) \\
&= \mathrm{d}\,\mathrm{tr}(\eta).
\end{aligned}$$

Now let us derive the Yang-Mills equations, using the action principle. We compute the variation of the Yang-Mills action

$$\begin{aligned}\delta S_{YM} &= \frac{1}{2}\delta \int_M \mathrm{tr}(F \wedge \star F) \\ &= \frac{1}{2}\int_M \mathrm{tr}(\delta F \wedge \star F + F \wedge \star \delta F) \\ &= \int_M \mathrm{tr}(\delta F \wedge \star F) \\ &= \int_M \mathrm{tr}(\mathrm{d}_D \delta A \wedge \star F).\end{aligned}$$

If M has no boundary, then Stokes' theorem tells us

$$\delta S_{YM} = \int_M \mathrm{tr}(\delta A \wedge \mathrm{d}_D \star F),$$

if this is to vanish for arbitrary δA we conclude

$$\mathrm{d}_D \star F = 0.$$

Which are the Yang-Mills equations in vacuum.

In section 3.1.1 we showed that the Yang-Mills equations are gauge-invariant. Here we will show that the Yang-Mills action is gauge invariant, which has as a direct consequence that the Yang-Mills equations are gauge-invariant as well. Let $g \in \mathcal{G}$ be a gauge transformation. Let A be an $\mathrm{End}(E)$-valued 1-form and let $A' = g(\mathrm{d}g) + gAg^{-1}$ be its gauge-transform. Then we may compute, using Eq. (2.2)

$$\begin{aligned}S_{YM}(A') &= \frac{1}{2}\int_M \mathrm{tr}(F' \wedge \star F') \\ &= \frac{1}{2}\int_M \mathrm{tr}(gFg^{-1} \wedge \star gFg^{-1}) \\ &= \frac{1}{2}\int_M \mathrm{tr}(F \wedge \star F) \\ &= S_{YM}(A).\end{aligned}$$

Chapter 4

Chern-Simons theory

In this chapter we describe a second gauge theory, called Chern-Simons theory, that has interesting uses in topology and in physics, as mentioned in the introduction. Chern-Simons theory will be one of the main ingredients in the analysis of 4-dimensional BF theory, which is the subject of Chapter 6.

4.1 The Chern form

Let $\pi : E \to M$ be a trivial vector bundle over M, and let D be a connection on E. We write

$$D = D^0 + A,$$

where D^0 is the standard flat connection, and A is an $\mathrm{End}(E)$-valued 1-form. Recall that

$$d_D\omega = \mathrm{d}\omega + A \wedge \omega$$

for any E-valued form ω.

Proposition 4.1.1. *In the present context, the k-th Chern form, $tr(F^k)$, is exact for any k.*

Proof. Let $A_s = sA$ and let

$$F_s = s\mathrm{d}A + s^2 A \wedge A$$

be the curvature of A_s. First note that

$$\frac{\mathrm{d}F_s}{\mathrm{d}s} = \mathrm{d}A + 2sA \wedge A = \mathrm{d}A + [A, A_s] = d_{D_s}A,$$

so it follows by the Bianchi identity that

$$\mathrm{tr}(\frac{\mathrm{d}F_s}{\mathrm{d}s} \wedge F_s^{k-1}) = \mathrm{tr}(\mathrm{d}_{D_s}A \wedge F_s^{k-1}) = \mathrm{tr}(\mathrm{d}_{D_s}[A \wedge F_s^{k-1}]) = \mathrm{dtr}(A \wedge F_s^{k-1}).$$

We now compute

$$\begin{aligned}\operatorname{tr}(F^k) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\operatorname{tr}(F_s^k)\mathrm{d}s \\ &= k\int_0^1 \operatorname{tr}(\frac{\mathrm{d}F_s}{\mathrm{d}s} \wedge F_s^{k-1})\mathrm{d}s \\ &= k\mathrm{d}\int_0^1 \operatorname{tr}(A \wedge F_s^{k-1})\mathrm{d}s.\end{aligned}$$

□

Let E be a $U(1)$-bundle over an arbitrary smooth orientable manifold M with standard fiber given by the fundamental representation of $U(1)$. Suppose that the dimension of M is at least 2.

Proposition 4.1.2. *The first Chern class of E, that is,*

$$\frac{i}{2\pi}F,$$

is integral.

Proof. The proposition says that for any compact two-dimensional oriented submanifold without boundary $\Sigma \subseteq M$ we have that

$$\frac{i}{2\pi}\int_\Sigma F$$

is an integer. So let $\Sigma \subseteq M$ be an arbitrary submanifold without boundary. Choose an arbitrary point $\xi \in \Sigma$ and let $U \subseteq \Sigma$ be a coordinate patch around ξ. Let $\gamma : S^1 \to U$ be a loop around ξ. The loop γ cuts the surface Σ into two pieces Σ^+ and Σ^- such that $\Sigma^+ \cap \Sigma^- = \gamma$ (we use γ interchangeably for both the map $\gamma : S^1 \to \Sigma$ and for the submanifold defined by the map), and such that the boundary of Σ^+ is $\partial\Sigma^+ = \gamma$ and the boundary of Σ^- is $\partial\Sigma^- = \gamma^{-1}$, where γ^{-1} denotes the loop γ, but with opposite orientation.

Now, let D be the connection with curvature F. Let $(U_i)_{i\in I}$ be a finite collection of coordinate charts covering Σ^+ and trivializing E over Σ. Let $(\gamma_i)_{i\in I}$ be a collection of loops such that

$$\prod_{i\in I}\gamma_i = \gamma,$$

and such that for each $i \in I$ the loop γ_i lands in U_i. It follows that

$$\prod_{i\in I} H(\gamma_i, D) = H(\gamma, D).$$

Then, locally, that is for each U_i, we can write $D = D^0 + A$, where D^0 is the standard flat connection on U_i and A is an $\mathrm{End}(E)$-valued 1-form, with

$F = \mathrm{d}A$. We may now compute the holonomy of the connection D around the loop γ_i as follows

$$H(\gamma_i, D) = e^{-\int_{\gamma_i} A}.$$

Now let $\Sigma_i \subseteq \Sigma$ be the surface bounded by the loop γ_i, i.e. such that $\partial\Sigma_i = \gamma_i$. By Stokes' theorem it follows that

$$H(\gamma_i, D) = e^{-\int_{\Sigma_i} \mathrm{d}A} = e^{-\int_{\Sigma_i} F}.$$

Thus we see that

$$H(\gamma, D) = \prod_{i\in I} H(\gamma_i, D) = e^{-\sum_{i\in I}\int_{\Sigma_i} F} = e^{-\int_{\Sigma^+} F}.$$

The same procedure shows that

$$H(\gamma^{-1}, D) = e^{-\int_{\Sigma^-} F}.$$

So we conclude that

$$1 = H(\gamma, D)H(\gamma^{-1}, D) = e^{-\int_{\Sigma^+} F - \int_{\Sigma^-} F} = e^{-\int_{\Sigma} F},$$

thus $-\int_\Sigma F = 2\pi i k$, with $k \in \mathbb{Z}$, as required. □

4.2 Chern-Simons action

Definition 4.2.1 (Chern-Simons action)**.** Let E be a trivial G-bundle over a three-dimensional manifold Σ without boundary. The Chern-Simons action is given by

$$S_{CS}(A) = \int_\Sigma \mathrm{tr}(A \wedge \mathrm{d}_{D^0} A + \frac{2}{3} A \wedge A \wedge A),$$

where A is the vector potential corresponding to a G-connection $D = D^0 + A$, and D^0 is the canonical flat connection on the trivial bundle E.

Remark 4.2.2. At this point it is important to stress that we distinguish between trivial fiber bundles and trivializable fiber bundles. A trivial fiber bundle is a fiber bundle $E \to M$ with fiber F, equipped with a trivialization $\phi : E \xrightarrow{\simeq} M \times F$. A trivializable fiber bundle is a fiber bundle $E \to M$ with fiber F, for which there *exists* a trivialization $\phi : E \xrightarrow{\simeq} M \times F$. There might however be another trivialization $\phi' : E \xrightarrow{\simeq} M \times F$, which is just as good. This distinction is important for the Chern-Simons action, since it depends on the choice of trivialization because the choice of global flat connection D^0 depends on the choice of trivialization.

In the rest of this chapter we will write $\mathrm{d} = \mathrm{d}_{D^0}$ for the exterior covariant derivative with respect to the flat connection.

Proposition 4.2.3. *The variation of the Chern-Simons action is given by*

$$\begin{aligned}\delta S_{CS} &= 2\int_\Sigma tr((dA + A\wedge A)\wedge \delta A)\\ &= 2\int_\Sigma tr(F\wedge\delta A)\,.\end{aligned}$$

Hence, the connections that extremize this action are the flat connections.

Proof. The result follows from the computation

$$\begin{aligned}\delta S_{CS} &= \int_\Sigma \mathrm{tr}\left(\delta(A\wedge \mathrm{d}A) + \frac{2}{3}\delta(A\wedge A\wedge A)\right)\\ &= \int_\Sigma \mathrm{tr}\left(\delta A\wedge \mathrm{d}A + A\wedge \mathrm{d}\delta A + 2A\wedge A\wedge\delta A\right)\\ &= 2\int_\Sigma \mathrm{tr}\left((\mathrm{d}A + A\wedge A)\wedge\delta A\right).\end{aligned}$$

□

Let $E\to M$ be a trivial vector bundle over M. Suppose that the boundary of M is a three-dimensional manifold Σ. From the proof of Proposition 4.1.1 it follows that the second Chern-Form is given by

$$\mathrm{tr}(F\wedge F) = \mathrm{d}\,\mathrm{tr}\left(A\wedge\mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right),$$

hence it follows that

$$\int_M \mathrm{tr}(F\wedge F) = \int_\Sigma \mathrm{tr}\left(A\wedge\mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right) = S_{CS}(A).$$

Proposition 4.2.4. *The Chern-Simons action is invariant under so-called small gauge transformations, that is, gauge transformations that are connected to the identity.*

Proof. Let g be a gauge transformation connected to the identity and let g_s be the corresponding one-parameter family of gauge transformations, that is g_s is defined for $s\in[0,1]$ and $g_0 = e$ and $g_1 = g$. Let A_s be the corresponding gauge-transformed vector potential

$$A_s = g_s A g_s^{-1} + g_s\mathrm{d}(g_s^{-1}).$$

We will show that

$$\frac{\mathrm{d}}{\mathrm{d}s}S_{CS}(A_s) = 0,$$

it is sufficient to show this at $s = 0$ since

$$\frac{\mathrm{d}}{\mathrm{d}s}S_{CS}(A_s)\big|_{s=t} = \frac{\mathrm{d}}{\mathrm{d}s}S_{CS}((A_t)_s)\big|_{s=0}.$$

We introduce the notation

$$T = \frac{\mathrm{d}}{\mathrm{d}s} g_s \big|_{s=0},$$

from which follows

$$\frac{\mathrm{d}}{\mathrm{d}s} g_s^{-1} = -T.$$

We compute

$$\frac{\mathrm{d}}{\mathrm{d}s} A_s \big|_{s=0} = [T, A] - \mathrm{d}T.$$

Now we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} S_{CS}(A_s)\big|_{s=0} =& \frac{\mathrm{d}}{\mathrm{d}s} \int_\Sigma \mathrm{tr}(A_s \wedge \mathrm{d}A_s + \frac{2}{3} A_s \wedge A_s \wedge A_s)\big|_{s=0} \\
=& \int_\Sigma \mathrm{tr}(\frac{\mathrm{d}A_s}{\mathrm{d}s} \wedge \mathrm{d}A_s + A_s \wedge \mathrm{d}\frac{\mathrm{d}A_s}{\mathrm{d}s} + 2A_s \wedge A_s \wedge \frac{\mathrm{d}A_s}{\mathrm{d}s})\big|_{s=0} \\
=& \int_\Sigma \mathrm{tr}\big([T, A] \wedge \mathrm{d}A - \mathrm{d}T \wedge \mathrm{d}A + A \wedge \mathrm{d}[T, A] \\
& + 2A \wedge A \wedge ([T, A] - \mathrm{d}T)\big).
\end{aligned}$$

First note that

$$\int_\Sigma \mathrm{tr}(-\mathrm{d}T \wedge \mathrm{d}A) = 0,$$

by Stokes' theorem. We consider the term

$$\int_\Sigma \mathrm{tr}(A \wedge \mathrm{d}[T, A]) = \int_\Sigma \mathrm{tr}(\mathrm{d}A \wedge AT - \mathrm{d}A \wedge TA + A \wedge A \wedge \mathrm{d}T + A \wedge \mathrm{d}T \wedge A).$$

Now note that

$$0 = \int_\Sigma \mathrm{tr}(\mathrm{d}(A \wedge AT)) = \int_\Sigma \mathrm{tr}(\mathrm{d}A \wedge AT - \mathrm{d}A \wedge TA + A \wedge A \wedge \mathrm{d}T).$$

This implies that

$$\int_\Sigma \mathrm{tr}(A \wedge \mathrm{d}T \wedge A) = \int_\Sigma \mathrm{tr}([T, A] \wedge \mathrm{d}A).$$

Putting all this together we see that

$$\frac{\mathrm{d}}{\mathrm{d}s} S_{CS}(A_s)\big|_{s=0} = 2 \int_\Sigma \mathrm{tr}([T, A] \wedge \mathrm{d}A + A \wedge A \wedge ([T, A] - \mathrm{d}T)).$$

The graded cyclic property of the trace implies that

$$\mathrm{tr}(A \wedge A \wedge [T, A]) = 0,$$

so we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} S_{CS}(A_s)\big|_{s=0} &= 2 \int_\Sigma \mathrm{tr}(TA \wedge \mathrm{d}A - AT \wedge \mathrm{d}A - A \wedge A \wedge \mathrm{d}T) \\
&= 2 \int_\Sigma \mathrm{tr}(\mathrm{d}(AT \wedge A)) \\
&= 0.
\end{aligned}$$

□

Chapter 5

Principal fiber bundles

5.1 Principal G-bundles

In the above we have considered vector bundles and in particular G-bundles. In this section we consider the related notion of principal G-bundles.

Definition 5.1.1 (Principal G-bundle)**.** Let G be a Lie group. A principal G-bundle P over a manifold M consists of the following data:

- A manifold P with a right action of G on P

$$P \times G \to P, (p, g) \mapsto pg.$$

- A surjective map $\pi : P \to M$, which is G-invariant, that is $\pi(pg) = \pi(g)$ for all $p \in P$ and all $g \in G$.

The data must satisfy the local triviality condition: for each $x \in M$, there exists an open neighborhood U of x and a diffeomorphism

$$\psi : \pi^{-1}(U) \to U \times G$$

which maps each fiber $\pi^{-1}(u)$ to the fiber $\{u\} \times G$ and which is G-equivariant, that is

$$\psi(pg) = \psi(p)g,$$

for each $p \in \pi^{-1}(U)$ and each $g \in G$.

The local triviality condition, together with the G-equivariance of the induced diffeomorphism implies that each fiber $P_x = \pi^{-1}(x)$ is, as a right G-space, isomorphic to the group G. Given a fixed $p_0 \in P_x$ the map

$$\varphi_{p_0} : G \to P_x, g \mapsto p_0 g,$$

is a diffeomorphism, that intertwines the right G-actions $G \circlearrowleft G$ and $P_x \circlearrowleft G$. Its inverse is given by

$$l_{\psi(p_0)^{-1}} \circ \psi,$$

where $l_{(u,g)}$ is left multiplication by g. Let us verify this, so let $g \in G$ be arbitrary, then

$$l_{\psi(p_0)^{-1}} \circ \psi \circ \varphi_{p_0}(g) = \psi(p_0)^{-1}(\psi(p_0 g)) = \psi(p_0)^{-1}\psi(p_0)g = g.$$

Conversely, if $p \in P_x$ is arbitrary we see that

$$(\varphi_{p_0} \circ l_{\psi(p_0)^{-1}} \circ \psi)(p) = p_0\psi(p_0)^{-1}\psi(p) = (\psi^{-1} \circ \psi)\left(p_0\psi(p_0)^{-1}\psi(p)\right) = p,$$

as required. In some sources the condition that the diffeomorphism ψ is G-equivariant is replaced by the condition that the right action of G on P acts freely and transitively on the fibers, in the above we have shown that if P is a principal G-bundle as in the definition given here, it most definitely is in the definition given in other sources. (And in fact, as one might hope, the different definitions are equivalent.)

Example 5.1.2 (Frame Bundle)**.** Let E be a vector bundle over a manifold M. Suppose that the fiber of E is a vector space V. For each $x \in M$ we may choose an ordered basis for E_x, such a basis is called a frame. For a vector space the frame bundle $\mathrm{Fr}(V)$ is the space of frames of V. The frame bundle of the vector bundle E is the space

$$\mathrm{Fr}(E) := \{(x, u) : x \in M, u \text{ a frame of } E_x\}.$$

Local triviality of the frame bundle follows from local triviality of the vector bundle E, i.e. the frame bundle is locally diffeomorphic to $U \times \mathrm{Fr}(V)$, (where $U \subseteq M$). The frame bundle comes equipped with a natural right action of $\mathrm{GL}(V)$. Locally described by

$$\mathrm{Fr}(E_x) \times G \to \mathrm{Fr}(E_x), ((x, u), g) \mapsto (x, ug).$$

If $u, u' \in \mathrm{Fr}(V)$ there exists a unique element $g \in \mathrm{GL}(V)$ such that $u' = ug$. In this way we obtain a (non-canonical) isomorphism between $\mathrm{Fr}(V)$ and $\mathrm{GL}(V)$, in this way $\mathrm{Fr}(E)$ becomes a principal $\mathrm{GL}(V)$-bundle.

Definition 5.1.3 (Vertical vectors)**.** We say that a vector $v_p \in T_pP$ is vertical if it is tangent to the fiber of π, that is, if $(\mathrm{d}\pi)(v_p) = 0$.

Definition 5.1.4 (Infinitesimal action)**.** If $X \in \mathfrak{g}$, then

$$p \mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} p\exp(tX)$$

defines a vector field on P, we denote this vector field by v_X. The map

$$\mathfrak{a} : \mathfrak{g} \to \mathfrak{X}(P), X \mapsto v_X$$

is called the infinitesimal action of $\mathfrak{g}$ on P.

Note that, conveniently, if $X \in \mathfrak{g}$ then $\phi_t : p \mapsto p\exp(tX)$ is the flow of $\mathfrak{a}(X) = v_X$.

Let $p \in P$ and $X \in \mathfrak{g}$ be arbitrary, we denote $\varphi_p : \mathfrak{g} \to P, X \mapsto p\exp X$, this means that we have

$$(v_X)_p = \mathrm{d}(\varphi_p)X.$$

Now also note that $\pi \circ \varphi_p : \mathfrak{g} \to M, X \mapsto p$ is the constant map. Thus we see

$$\mathrm{d}\pi(v_X)_p = d(\pi \circ \varphi_p)X = 0,$$

since this holds for any $p \in P$, we have that v_X is a vertical vector field, for any $X \in \mathfrak{g}$.

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of the Lie group G in the vector space V.

Definition 5.1.5 (Basic forms)**.** A differential form $A \in \Omega^k(P, V)$ is horizontal if $i_v(A) = 0$ for all vertical vector fields $v \in \mathfrak{X}(P)$. If a differential form $A \in \Omega^k(P, V)$ is both horizontal and G-equivariant, ($R_g^*(A) = \rho(g)^{-1}(A)$), it is called *basic*. The space of basic differential forms is denoted $\Omega^k(P, V)_{\mathrm{bas}}$. The space of equivariant differential forms is denoted by $\Omega^k(P, V)^G$.

5.2 The associated vector bundle

There is a way to construct a vector bundle, called the associated vector bundle, from a principal G-bundle. Let M be a manifold, let G be a Lie group and let $\pi : P \to M$ be a principal G-bundle over M. Let (ρ, V) be a representation of G in V, i.e. V is a vector space (over an arbitrary field $\mathbb{K}$) and ρ is a group homomorphism $\rho : G \to GL(V)$. We define a right action of G on the product space $P \times V$ by

$$(P \times V) \times G \to P \times V, ((p, v), g) \mapsto (pg, \rho(g)^{-1}v).$$

Definition 5.2.1 (Associated vector bundle)**.** Given the data as above, we define the associated vector bundle as the space

$$E(P, V) := (P \times V)/G$$

Explicitly, the following elements of $E(P, V)$ are identified

$$[pg, v] = [p, \rho(g)v].$$

Proposition 5.2.2. *The associated vector bundle construction does indeed produce a vector bundle.*

Proof. The projection map $\tilde{\pi} : E(P, V) \to M$ is given by

$$\tilde{\pi}[p, v] = \pi(p),$$

for all $[p, v] \in E(P, V)$, this map is well defined as a consequence of the G-invariance of $\pi : P \to M$. Let $x \in M$ be arbitrary, then the fiber

$$E(P, V)_x = \{[p, v] : p \in P_x, v \in V\},$$

has a natural structure of vector space, to wit

$$[p, v] + [p, w] = [p, v + w], \quad \lambda[p, v] = [p, \lambda v],$$

for any $[p, v], [p, w] \in E(P, V)_x$ and $\lambda \in \mathbb{K}$. Now suppose that $[p, v], [q, w] \in E(P, V)_x$, then there exists an element $g \in G$ such that $p = qg$, (since the group G acts transitively on the fibers of P), and we see that $[q, w] = [p, g^{-1}w]$, so

$$[p, v] + [q, w] = [p, v + g^{-1}w].$$

Finally, let us consider the smooth structure on $E(P, V)$. Let $x \in M$ be arbitrary, then the local trivialization of P described as

$$\psi : \pi^{-1}(U) \to U \times G,$$

leads to a local trivialization of $E(P, V)$:

$$\tilde{\psi} : \tilde{\pi}^{-1}(U) \to U \times V, p \mapsto (\pi(p), \psi_2(p)v),$$

where $\psi_2(p) \in G$ is the second component of $\psi(p) \in U \times G$. This gives us the smooth structure and at the same time shows that the local triviality condition holds. $\square$

Let M be a smooth manifold and let $n \in \mathbb{N}_{\geqslant 1}$. Let $\mathcal{C}$ be the category of Vector bundles of rank n over M. Here the vector bundle morphisms are taken to be fiberwise invertible linear smooth maps covering the identity map of M. In other words if E and E' are objects in $\mathcal{C}$ and $f \in \mathrm{Hom}(E, E')$ then the following triangle commutes

$$\begin{array}{ccc} E & \xrightarrow{\quad f \quad} & E' \\ & {\scriptstyle \pi_E} \searrow \quad \swarrow {\scriptstyle \pi_{E'}} & \\ & M & \end{array}$$

And for each $x \in M$ the map f restricts to an invertible linear map $f : E_x \to E'_x$. Let $\mathcal{D}$ be the category of principal $\mathrm{GL}(n, \mathbb{R})$-bundles over M. Here the morphisms are taken to be invertible right $\mathrm{GL}(n, \mathbb{R})$-equivariant smooth maps that cover the identity of M. In other words, if P and P' are objects in $\mathcal{D}$ and $f \in \mathrm{Hom}(P, P')$ then the following triangle commutes

$$\begin{array}{ccc} P & \xrightarrow{\quad f \quad} & P' \\ & {\scriptstyle \pi_P} \searrow \quad \swarrow {\scriptstyle \pi_{P'}} & \\ & M & \end{array}$$

and for all $g \in \mathrm{GL}(n, \mathbb{R})$ and $p \in P$ we have

$$f(p \cdot g) = f(p) \cdot g.$$

Note that in this way both $\mathcal{C}$ and $\mathcal{D}$ are groupoids. In the following we will argue that these categories are essentially equivalent. To begin, we need functors $\mathrm{Fr} : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$.

Definition 5.2.3. We define the functor $\mathrm{Fr} : \mathcal{C} \to \mathcal{D}$ as follows.

Objects — Given an object E of $\mathcal{C}$, we define $\mathrm{Fr}(E)$ to be the frame bundle of E. The frame bundle may be defined as

$$\mathrm{Fr}(E) = \{\phi : \mathbb{R}^n \xrightarrow{\sim} E_x | x \in M, \phi \text{ a linear bijection}\}.$$

If $\phi \in \mathrm{Fr}(E)$ and $g \in \mathrm{GL}(n, \mathbb{R})$, then the right $\mathrm{GL}(n, \mathbb{R})$-action is given by $(\phi, g) \mapsto \phi \circ g$, i.e. it is given by precomposition with the linear map $g : \mathbb{R}^n \to \mathbb{R}^n$.

Morphisms — Given a morphism $f : E \to E'$, we define a map $\mathrm{Fr}(f) : \mathrm{Fr}(E) \to \mathrm{Fr}(E')$ as follows. Given a frame $\phi : \mathbb{R}^n \to E_x$ we obtain a frame $\mathbb{R}^n \to E'_x$ simply by composition with the linear map $f_x : E_x \to E'_x$ as in the diagram

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\phi} & E_x \\ & \underset{\mathrm{Fr}(f)\phi}{\searrow} & \downarrow f_x \\ & & E'_x \end{array}$$

Since composition of maps is associative, this map is right $\mathrm{GL}(n, \mathbb{R})$-equivariant, i.e. if $g \in \mathrm{GL}(n, \mathbb{R})$, then $(\mathrm{Fr}(f)_x \phi) \circ g = \mathrm{Fr}(f)_x(\phi \circ g)$.

We see that the inverse of $\mathrm{Fr}(f)$ is given by $\mathrm{Fr}(f^{-1})$. Furthermore we have that if $U \subseteq M$ with $E|_U \simeq U \times \mathbb{R}^n$ and $E'|_U \simeq \mathbb{R}^n$, then $f : E \to E'$ induces a smooth map $\hat{f} : U \to \mathrm{GL}(n, \mathbb{R})$. One may show that with respect to the induced trivializations of the frame bundles we obtain

$$\begin{aligned} \mathrm{Fr}(f) : U \times \mathrm{GL}(n, \mathbb{R}) &\to U \times \mathrm{GL}(n, \mathbb{R}), \\ (x, g) &\mapsto (x, \hat{f}(x) g), \end{aligned}$$

hence showing that $\mathrm{Fr}(f)$ is smooth. One may furthermore verify that Fr preserves compositions, and is thus a functor.

Definition 5.2.4. We define the functor $G : \mathcal{D} \to \mathcal{C}$ as follows.

Objects — Given an object P of $\mathcal{D}$, we define $G(P)$ to be the vector bundle associated with the natural representation of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$, that is

$$G(P) := (P \times \mathbb{R}^n)/\mathrm{GL}(n, \mathbb{R}).$$

Morphisms — Given a morphism $f : P \to P'$, we consider the map

$$\begin{aligned} P \times \mathbb{R}^n &\to (P' \times \mathbb{R}^n)/\mathrm{GL}(n, \mathbb{R}) = G(P'), \\ (p, v) &\mapsto [f(p), v]. \end{aligned}$$

This is a smooth map that descends to the quotient (by right $\mathrm{GL}(n, \mathbb{R})$-equivariance of f), as in the following diagram

$$\begin{array}{ccc} P \times \mathbb{R}^n & \longrightarrow & G(P') \\ \big\downarrow & \nearrow_{G(f)} & \\ G(P) & & \end{array}$$

This defines the, automatically smooth, map $G(f) : G(P) \to G(P')$.

The map $G(f)$ is fiberwise linear, because it does nothing to the second component. The inverse of the map $G(f)$ is $G(f^{-1})$. Finally, one may show that G preserves compositions.

Proposition 5.2.5. *The category $\mathcal{C}$ of Vector bundles of rank n over M is essentially equivalent to the category $\mathcal{D}$ of principal $GL(n, \mathbb{R})$-bundles over M. In particular, there are natural transformations*

$$\alpha : G \circ Fr \Rightarrow Id_{\mathcal{C}} \quad \text{and} \quad \beta : Fr \circ G \Rightarrow Id_{\mathcal{D}}.$$

Sketch of proof. We will only give the components of the natural transformations α and β and leave the verifications to the reader. Let E be an object of $\mathcal{C}$. The component of α at E should be a morphism $\alpha_E : G \circ \mathrm{Fr}(E) \to E$. Let $[f_x, v] \in G \circ \mathrm{Fr}(E) = (\mathrm{Fr}(E) \times \mathbb{R}^n)/\mathrm{GL}(n, \mathbb{R})$, then we define

$$\alpha_E([f_x, v]) = f_x(v).$$

One may verify that this is well-defined, actually yields a morphism $\alpha_E : G \circ \mathrm{Fr}(E) \to E$, and that the naturality square commutes.

Next, let P be an object of $\mathcal{D}$. The component of β at P is a morphism $\beta_P : \mathrm{Fr} \circ G(P) \to P$, it is actually simpler to write down the inverse $\beta_P^{-1} : P \to \mathrm{Fr} \circ G(P)$. We now define the component of β^{-1} at P,

$$\begin{aligned} \beta_P^{-1} : P &\to \mathrm{Fr}(P \times \mathbb{R}^n/\mathrm{GL}(n, \mathbb{R})) \\ p &\mapsto (v \mapsto [p, v]). \end{aligned}$$

Again, one may verify that, indeed β_P is a morphism $P \to \mathrm{Fr} \circ G(P)$ and that the naturality square commutes. □

The following isomorphism is sometimes implicitly used when talking about associated vector bundles.

Proposition 5.2.6. *If $E = E(P, V)$ is the associated vector bundle, then one has a linear isomorphism*

$$\pi^\bullet : \Omega^k(M, E) \xrightarrow{\sim} \Omega^k(P, V)_{bas}.$$

Proof. We will give an explicit construction of the map $\pi^\bullet$. To do so we first consider the pullback bundle

$$\pi^*(E) := \{(p, x) \in P \times E | \pi(p) = \tilde{\pi}(x)\} \subseteq P \times E,$$

sometimes also denoted as $\pi^*(E) = P \times_\pi E$. Notice that there is a canonical isomorphism

$$i : P \times V \to \pi^*(E), (p, v) \mapsto (p, [p, v]).$$

For fixed $p \in P$, the map i induces an invertible linear map

$$i_p : V \to E_{\pi(p)}, v \mapsto [p, v].$$

Next we note that the map $\pi : P \to M$ induces a map

$$\pi^* : \Omega^k(M, E) \to \Omega^k(P, \pi^* E),$$

given by

$$\pi^*(\omega)(v^1, ..., v^k) = \omega((\mathrm{d}\pi)_p(v^1), ..., (\mathrm{d}\pi)_p(v^k)) \in E_{\pi(p)} = (\pi^* E)_p,$$

for $v^1, ..., v^k \in T_p P$. We claim now that the map $\pi^\bullet$ is given by

$$\pi^\bullet := (\mathrm{Id} \otimes i^{-1}) \circ \pi^* : \Omega^k(M, E) \xrightarrow{\sim} \Omega^k(P, V).$$

Let us be entirely explicit about what we mean by $\mathrm{Id} \otimes i^{-1}$. Let $T \in \Omega^k(P, \pi^* E) = \Gamma(\Lambda^k T^* P \otimes \pi^* E)$, $p \in P$, and $v^1, ..., v^k \in T_p P$, then the following equation holds

$$((\mathrm{Id} \otimes i^{-1})T)_p(v^1, ..., v^k) = i_p^{-1}(T_p(v^1, ..., v^k)).$$

There are now a number of checks to do to make sure that this map is an isomorphism.

- First we check that $\pi^\bullet$ actually maps into $\Omega^k(P, V)_{\mathrm{bas}}$. First fix $p \in P$, then let $v \in T_p P$ be vertical and let $v^2, ..., v^k \in T_p P$ be arbitrary, we now compute for an arbitrary $\omega \in \Omega^k(M, E)$

  $$\iota_v \pi^\bullet(\omega)_p(v^2, ..., v^k) = i^{-1}\omega((\mathrm{d}\pi)_p(v), (\mathrm{d}\pi)_p(v^2), ..., (\mathrm{d}\pi)_p(v^k)) = 0,$$

  since $(\mathrm{d}\pi)_p(v) = 0$. Next let us show that the resulting V-valued form is G-equivariant. So let $g \in G$ be arbitrary, and let $v^1, ..., v^k \in T_p P$

be arbitrary. First we show that the map $i : P \times V \to \pi^*(E)$ has the following property

$$i_{pg}^{-1} = \rho(g)^{-1} i_p^{-1},$$

since

$$\rho(g)^{-1} i_p^{-1}[p, v] = \rho(g)^{-1} v = i_{pg}^{-1} i_{pg} \rho(g)^{-1} v = i_{pg}^{-1}[pg, \rho(g)^{-1} v] = i_{pg}^{-1}[p, v].$$

Now we show the G-equivariance,

$$\begin{aligned} R_g^*(i^{-1}\pi^*\omega)(v^1, ..., v^k) &= i_{pg}^{-1}(\pi^*\omega)_{pg}((\mathrm{d}R_g)_p v^1, ..., (\mathrm{d}R_g)_p v^k) \\ &= \rho(g)^{-1} i_p^{-1} \omega_{\pi(pg)}(\mathrm{d}(\pi \circ R_g)_p v^1, ..., \mathrm{d}(\pi \circ R_g)_p v^k) \\ &= \rho(g)^{-1}(i^{-1}\pi^*\omega)(v^1, ..., v^k). \end{aligned}$$

- Let us show that π^* is injective (since i is an isomorphism it follows that $\pi^\bullet$ is injective). Since π^* is linear it suffices to show that if $\omega \neq 0$ then $\pi^*\omega \neq 0$. So let $\omega \neq 0$, this means that there exists a $p \in P$ such that we can pick $x^1, ..., x^k \in T_{\pi(p)}M$ such that $\omega(x^1, ..., x^k) \neq 0$. Since $\mathrm{d}\pi$ is surjective there exist $v^1, ..., v^k \in T_pP$ such that $(\mathrm{d}\pi)_p v^i = x^i$ for $0 < i \leqslant k$. It follows that

$$\pi^*\omega(v^1, ..., v^k) = \omega(x^1, ..., x^k) \neq 0.$$

- Our next order of business is to show that $\pi^\bullet$ is surjective. Since i is an isomorphism this can be expressed as follows: we claim that for each $\eta \in \Omega^k(P, V)_{\mathrm{bas}}$ there exists an $\omega \in \Omega^k(M, E)$ such that

$$\omega_{\pi(p)}((\mathrm{d}\pi)_p(v^1), ..., (\mathrm{d}\pi)_p(v^k)) = i_p(\eta_p(v^1, ..., v^k)), \tag{5.1}$$

for each $p \in P$ and all $v^i \in T_pP$. So let $\eta \in \Omega^k(P, V)_{\mathrm{bas}}$ be arbitrary. Then we define $\omega \in \Omega^k(M, E)$ by

$$\omega_x(x^1, ..., x^k) = i_p(\eta_p(v^1, ..., v^k)),$$

where p is chosen such that $\pi(p) = x$ and where each $v^i \in T_pP$ is an arbitrary choice of a vector such that $(\mathrm{d}\pi)_p(v^i) = x^i$. It is clear that with this choice of ω Eq. (5.1) is satisfied.

Let us now show that this definition does not depend on the choices for p and $v^1, ..., v^k$.

First suppose that p is fixed, but we have two different choices, say $v^i \in T_pP$, and $w^i \in T_pP$ such that $(\mathrm{d}\pi)_p(v^i) = x^i = (\mathrm{d}\pi)_p(w^i)$ for each i. Then it follows that $w^i = v^i + a^i$ with $(\mathrm{d}\pi)_p(a^i) = 0$, but η was assumed to be horizontal, thus $\iota_{a^i}\eta = 0$ for each i and thus

$$\eta_p(v^1, ..., v^k) = \eta_p(v^1 + a^1, ..., v^k) = \eta_p(w^1, ..., w^k).$$

Now suppose that we have two points $p, p' \in P$ with $\pi(p) = \pi(p')$, then there exists an element $g \in G$, such that $p' = pg$. Suppose furthermore that we have chosen $v^i \in T_{p'}P$ such that $(\mathrm{d}\pi)_{p'}(v^i) = x^i$, then we compute

$$\begin{aligned} i_{p'}\eta_{p'}(v^1, ..., v^k) &= i_{pg}\eta_{pg}(v^1, ..., v^k) \\ &= i_p\rho(g)\eta_{pg}(v^1, ..., v^k) \\ &= i_p R^*_{g^{-1}}\eta_{pg}(v^1, ..., v^k) \\ &= i_p\eta_p((\mathrm{d}R_g^{-1})_{p'}v^1, ..., (\mathrm{d}R_g^{-1})_{p'}v^k). \end{aligned}$$

Now note that

$$(\mathrm{d}\pi)_p(\mathrm{d}R_g^{-1})_{p'}v^i = \mathrm{d}(\pi \circ R_g^{-1})_{p'}v^i = (\mathrm{d}\pi)_{p'}v^i = w^i,$$

so if $w^i \in T_pP$ with $(\mathrm{d}\pi)_p w^i = x^i$ then

$$i_{p'}\eta_{p'}(v^1, ..., v^k) = i_p\eta_p(w^1, ..., w^k),$$

this concludes the proof.

□

5.3 The adjoint bundle

The adjoint bundle is an example of an associated vector bundle. Let G be a Lie group and let $\mathfrak{g}$ denote its Lie algebra. Let P be a principal G-bundle over a manifold M. If we now construct the associated vector bundle for the representation $(\mathrm{Ad}, \mathfrak{g})$ we obtain the adjoint bundle. That is,

$$\mathrm{ad}P = (P \times \mathfrak{g})/G,$$

so two elements $(p, X), (q, Y) \in P \times \mathfrak{g}$ are equivalent if there is an element $g \in G$ such that

$$(q, Y) = (pg, \mathrm{Ad}(g)X).$$

Definition 5.3.1 (Wedge product)**.** Let $B = B_I \otimes \mathrm{d}x^I \in \Gamma(\mathrm{ad}P \otimes \Lambda^k T^*M)$ and $F = F_I \otimes \mathrm{d}y^I \in \Gamma(\mathrm{ad}P \otimes \Lambda^l T^*M)$. Then we define the wedge product $B \wedge F \in \Gamma(\mathrm{ad}P \otimes \mathrm{ad}P \otimes \Lambda^{k+l}T^*M)$ to be

$$B \wedge F = B_I \otimes F_J \otimes \mathrm{d}x^I \wedge \mathrm{d}y^J.$$

Suppose now that the Lie algebra $\mathfrak{g}$ is equipped with an invariant non-degenerate symmetric bilinear form $\langle\, , \rangle$. In this case we can define the following map.

Definition 5.3.2 (Trace)**.** The trace is the map $\mathrm{tr} : \Gamma(\mathrm{ad}P \otimes \mathrm{ad}P \otimes \Lambda^k T^*M) \to \Gamma(\Lambda^k T^*M)$ defined by

$$\mathrm{tr}(B_I \otimes F_J \otimes \mathrm{d}x^{I,J})_x = \langle B_{I,x}, F_{J,x}\rangle \mathrm{d}x^{I,J}_x,$$

for each $x \in M$.

5.4 Connections on principal bundles

Let G be a Lie group and let $\pi : P \to M$ be a principal G-bundle over a manifold M.

Definition 5.4.1 (Horizontal subspace)**.** A horizontal subspace of P at $p \in P$ is a subspace

$$\mathcal{H}_p \subseteq T_p P$$

with the property that the map

$$(\mathrm{d}\pi)_p\big|_{\mathcal{H}_p} : \mathcal{H}_p \to T_{\pi(p)} M$$

is an isomorphism. Note that given a horizontal subspace at p we can translate it to a horizontal subspace at gp as follows

$$R_g(\mathcal{H}_p) := \{(\mathrm{d}R_g)(X_p) : X_p \in \mathcal{H}_p\}.$$

Definition 5.4.2 (Connections as horizontal distributions)**.** A connection on P is a vector sub-bundle $\mathcal{H} \subseteq TP$, (also called a distribution), with the property that each fiber $\mathcal{H}_p$ is a horizontal subspace of P and

$$\mathcal{H}_{pg} = R_g(\mathcal{H}_p),$$

for all $p \in P$ and $g \in G$.

The following is an equivalent definition of a connection on the principal G-bundle P as a 1-form.

Definition 5.4.3 (Connections as 1-forms)**.** A connection on P is a $\mathfrak{g}$-valued 1-form

$$\alpha \in \Omega^1(P, \mathfrak{g}),$$

which is G-equivariant, i.e. it satisfies

$$R_g^*(\alpha) = \mathrm{Ad}_{g^{-1}}(\alpha),$$

for any $g \in G$. Here R_g^* is the pullback of the map $R_g : P \to P, p \mapsto pg$. Furthermore α should satisfy

$$\alpha_p \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} p \exp(tX) \right) = X,$$

for any $p \in P$ and any $X \in \mathfrak{g}$.

Thus if α is a connection 1-form on P it follows from the definition that

$$\alpha(v_X) = X,$$

(for the definition of v_X recall definition 5.1.4). We furthermore note that any vertical vector $v \in T_p P$ is of the form $(v_X)_p$ for some $X \in \mathfrak{g}$.

Proposition 5.4.4. *There is a bijection between connection* 1*-forms* $\alpha \in \Omega^1(P, \mathfrak{g})^G$ *and connections* $\mathcal{H}$ *on* P*. The bijection respects the following equality*

$$\mathcal{H}_p = \{X_p \in T_pP : A_p(X_p) = 0\}.$$

The proof to this proposition can be found in Ref. [G-struc].

An important use of a connection on a vector bundle was to give a notion of parallel transport, so one might expect that given a connection $\mathcal{H} \subseteq TP$ on the principal G-bundle P we might use $\mathcal{H}$ to give a notion of parallel transport. This is indeed possible:

Definition 5.4.5 (Horizontal curves)**.** Given a connection $\mathcal{H} \subseteq TP$ and a curve $u : I \to P$, we say that u is horizontal if

$$\dot{u}(t) \in \mathcal{H}_{\gamma(t)}$$

for all $t \in I$. Here $\gamma(t) = \pi(u(t))$.

From Ref. [G-struc] we have the following theorem:

Theorem 5.4.6 (Existence of horizontal curves)**.** *Let* $\gamma : I \to M$ *be a curve in* M*, and* $t_0 \in I$*. Then for any* $u_0 \in P_{\gamma(t_0)}$ *there exists a unique horizontal curve* $u : I \to P$ *above* γ*, (i.e.* $\pi(u(t)) = \gamma(t)$*) with the property that* $u(t_0) = u_0$*.*

The Lie bracket $[\bullet, \bullet]$ of the Lie algebra $\mathfrak{g}$ extends to a bracket defined as follows.

Definition 5.4.7 (Bracket)**.** We define the bracket

$$[\bullet, \bullet] : \Omega^1(P, \mathfrak{g}) \times \Omega^1(P, \mathfrak{g}) \to \Omega^2(P, \mathfrak{g}),$$

by the formula

$$[\alpha, \beta](v, w) := [\alpha(v), \beta(w)] - [\alpha(w), \beta(v)], \quad (v, w \in \mathfrak{X}(P)).$$

Definition 5.4.8 (Curvature)**.** The curvature of a connection 1-form α is defined as

$$F = \mathrm{d}\alpha + \frac{1}{2}[\alpha, \alpha] \in \Omega^2(P, \mathfrak{g}).$$

The curvature has some special properties that we would like to discuss, recall the terminology introduced in definition 5.1.5.

Proposition 5.4.9. *The curvature* F *of a connection* 1*-form* α *is a basic form.*

Proof. Let $g \in G$ be arbitrary, we compute

$$\begin{aligned} R_g^*(F) &= R_g^*\left(\mathrm{d}\alpha + [\alpha,\alpha]\right), \\ &= \mathrm{d}R_g^*\alpha + [R_g^*\alpha, R_g^*\alpha] \\ &= \mathrm{d}\mathrm{Ad}_g^{-1}\alpha + [\mathrm{Ad}_g^{-1}\alpha, \mathrm{Ad}_g^{-1}\alpha] \\ &= \mathrm{Ad}_g^{-1}\left(\mathrm{d}\alpha + [\alpha,\alpha]\right) \\ &= \mathrm{Ad}_g^{-1}F, \end{aligned}$$

which establishes the G-invariance. Let $v \in \mathfrak{X}(P)$ be a vertical vector field. Then there exists an $X \in \mathfrak{g}$ such that $v = v_X$. Let us compute

$$\begin{aligned} \iota_{v_X} F &= \iota_{v_X}\mathrm{d}\alpha + \frac{1}{2} i_{v_X}[\alpha,\alpha] \\ &= \mathscr{L}_{v_X}\alpha - \mathrm{d}\iota_{v_X}\alpha + [\alpha(v_X),\alpha] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\phi_t^*\alpha) + [X,\alpha], \end{aligned}$$

we recall that the flow corresponding to the vector field v_X is $\phi_t : p \mapsto p\exp(tX)$, hence $\phi_t = R_{\exp(tX)}$ and we see

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(R^*_{\exp(tX)}\alpha) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\mathrm{Ad}(\exp(-tX))\alpha\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\left(\exp(-t\mathrm{ad}X)\alpha\right) \\ &= -[X,\alpha], \end{aligned}$$

so we conclude that $\iota_{v_X}F = 0$. □

Remark 5.4.10. Together with Proposition 5.2.6, the proposition above says that we may identify the curvature $F \in \Omega^2(P,\mathfrak{g})$ with an ad$P$-valued two-form $F \in \Omega^2(M, \mathrm{ad}P)_{\mathrm{bas}}$. In a similar vein, if the principal bundle P is trivial we may fix the canonical flat connection α_0, and any other connection α on P may be written as $\alpha = \alpha_0 + A$, where $A \in \Omega^1(P,\mathfrak{g})_{\mathrm{bas}}$ is a basic form. Again using Proposition 5.2.6 we may thus identify A with an ad$P$-valued one-form $A \in \Omega^1(M, \mathrm{ad}P)$.

Note that the space of connection 1-forms is an affine space for $\Omega^1(M, \mathrm{ad}P)$.

Chapter 6

4-dimensional BF theory

In this chapter we consider another example of a topological field theory, namely BF theory. The goal of this chapter is to construct a topological quantum field theory in the sense defined by Atiyah in Ref. [Atiy88]. We will start by considering the canonical and path integral quantization of the BF Lagrangian. Using the results we obtain here, we define a functor from the cobordism category to the category of vector spaces. This functor turns out to have some nice properties, which turn it into a topological quantum field theory. The specific theory we treat here was studied in Ref. [Baez95], we essentially work out the details.

6.1 The 4-dimensional BF Lagrangian

In this section we give the definition of the 4-dimensional BF Lagrangian and study its canonical and path integral quantization.

Let G be a Lie group with an invariant nondegenerate symmetric bilinear form on its Lie algebra $\mathfrak{g}$. Let M be an oriented 4-manifold equipped with a principal G-bundle, P, over it. Let α be a connection 1-form on P and B an $\mathrm{ad}P$-valued 2-form on M. Let F be the $\mathrm{ad}P$-valued curvature 2-form, (recall Proposition 5.2.6). Then, the 4-dimensional BF action with cosmological constant Λ is given by

$$S_{BF}(B,F) = \int_M \mathrm{tr}(B \wedge F + \frac{\Lambda}{12} B \wedge B).$$

The name 'BF-theory' refers to the fact that the fields involved are usually called B and F. Varying with respect to α gives

$$\begin{aligned}\delta_\alpha S(B,F) &= \int_M \mathrm{tr}(B \wedge \delta_\alpha F) \\ &= \int_M \mathrm{tr}(B \wedge \mathrm{d}_\alpha \delta\alpha),\end{aligned}$$

in the case that M is without boundary, Stokes' theorem tells us that the equation of motion becomes $\mathrm{d}_\alpha B = 0$.

The BF Lagrangian is given by

$$\mathcal{L}_{BF} = \mathrm{tr}(B \wedge F + \frac{\Lambda}{12} B \wedge B),$$

or, in local coordinates,

$$\begin{aligned}\mathcal{L}_{BF} =& B^a_{ij} F^b_{kl} \langle X_a, X_b \rangle \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^l \\ &+ \frac{\Lambda}{12} B^a_{ij} B^b_{kl} \langle X_a, X_b \rangle \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^l.\end{aligned}$$

Here the X_a's form a basis for the Lie algebra $\mathfrak{g}$. Furthermore F is given by the familiar formula $F = \mathrm{d}\alpha + \frac{1}{2}[\alpha, \alpha]$, or in local coordinates

$$F = -\frac{\partial A^a_i}{\partial x^j} X_a \otimes \mathrm{d}x^i \wedge \mathrm{d}x^j + A^a_i A^b_j [X_a, X_b] \otimes \mathrm{d}x^i \wedge \mathrm{d}x^j,$$

recall that, locally, there is always a flat connection α_0, and we write $\alpha - \alpha_0 = A \in \Omega^1(M, \mathrm{ad}P)$. If f^a_{bc} are the structure constants of the Lie algebra $\mathfrak{g}$, i.e. $[X_b, X_c] = f^a_{bc} X_a$, we can read off the components of F:

$$F^a_{kl} = \frac{1}{2}\left(\frac{\partial A^a_l}{\partial x^k} - \frac{\partial A^a_k}{\partial x^l}\right) + f^a_{bc} A^b_k A^c_l.$$

We would like to quantize this theory using the canonical formalism. To that end we assume that the manifold M is of the form $\Sigma \times \mathbb{R}$, where Σ is a closed three-dimensional manifold that represents space and $\mathbb{R}$ represents time. The space of physical states now becomes the space of connections. The momentum conjugate to A is defined by

$$(\Pi_A)^m_c = \frac{\partial \mathcal{L}_{BF}}{\partial(\partial_0 A^c_m)}.$$

For this computation, it is convenient to write

$$\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}x^l = \varepsilon^{ijkl} \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3,$$

this allows us to work with the Lagrangian density, instead of the Lagrangian differential form. By this we mean that we define the Lagrangian density

$$\tilde{\mathcal{L}}_{BF} = \left(B^a_{ij} F^b_{kl} \langle X_a, X_b \rangle + \frac{\Lambda}{12} B^a_{ij} B^b_{kl} \langle X_a, X_b \rangle\right) \varepsilon^{ijkl}.$$

Using this we compute

$$\begin{aligned}\frac{\partial \tilde{\mathcal{L}}_{BF}}{\partial(\partial_0 A^c_m)} &= B^a_{ij} \frac{\partial F^b_{kl}}{\partial(\partial_0 A^c_m)} \langle X_a, X_b \rangle \varepsilon^{ijkl} \\ &= -B^a_{ij} \delta^0_l \delta^m_k \delta^b_c \langle X_a, X_b \rangle \varepsilon^{ijkl} \\ &= -B^a_{ij} \langle X_a, X_c \rangle \varepsilon^{ijm0}.\end{aligned}$$

If we restrict i, j, k to spacelike indices, i.e. $i, j, k = 1, 2, 3$, then we obtain

$$\frac{\partial \tilde{\mathcal{L}}_{BF}}{\partial(\partial_0 A^c_m)} = B^a_{ij} \langle X_a, X_c \rangle \varepsilon^{ijm}.$$

It follows that we have the following relation

$$\delta^m_i \delta^a_c \delta(x - y) = \{A^a_i(x), (\Pi_A)^m_c(y)\} = \{A^a_i(x), B^b_{kl}(y)\} \langle X_b, X_c \rangle \varepsilon^{lkm}.$$

We may view $\langle X_a, X_b \rangle$ as a metric, and use it to raise and lower Lie algebra indices, in other words, we write

$$B_{a,ij} = B^b_{ij} \langle X_a, X_b \rangle.$$

We thus see

$$\delta^m_i \delta^a_c \delta(x - y) = \{A^a_i(x), B_{c,kl}(y)\} \varepsilon^{lkm}.$$

The canonical momentum conjugate to A is B. The equations of motion for B that follow from $\tilde{\mathcal{L}}_{BF}$ are

$$F^a_{ij} + \frac{\Lambda}{6} B^a_{ij} = 0.$$

Since this equation of motion does not contain any time derivatives it is a *constraint*. Upon canonical quantization the states become functions on the space of connections on Σ, denoted by $\mathcal{A}_\Sigma$. We promote B^a_{ij} to the operator

$$B^a_{ij} = -i \varepsilon_{ijk} \frac{\delta}{\delta A_{ka}}.$$

The constraint thus becomes

$$\left(F^a_{ij} - i \frac{\Lambda}{6} \varepsilon_{ijk} \frac{\delta}{\delta A_{ka}} \right) \psi = 0. \tag{6.1}$$

If $P\big|_\Sigma$ is trivializable the solution is given by

$$\psi(A) = e^{\frac{-3i}{\Lambda} S_{CS}(A)},$$

where S_{CS} is the Chern-Simons action given by

$$S_{CS}(A) = \int_\Sigma \mathrm{tr}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A).$$

Here we recall Remark 5.4.10, to make sense of this expression. Note that two different choices of trivialization for $P\big|_\Sigma$ will give the same Chern-Simons action, up to a constant, this means that $\psi(A)$ is determined up to a factor. This factor is, however, not interesting, since the constraint, Eq. (6.1), only fixes ψ up to a factor. Verifying that this expression for ψ

actually solves Eq. (6.1) is straightforward if one uses the result of Proposition 4.2.3, which reads

$$\frac{\delta S_{CS}}{\delta A_{ka}(x)} = \varepsilon^{ijk} F^a_{ij}(x).$$

Next we consider the path-integral quantization of this BF-theory. If M has boundary $\partial M = \Sigma$, we expect to obtain a vector ψ in the space of states on Σ as follows:

$$\psi(A_\Sigma) = \int_{A|_\Sigma = A_\Sigma} \mathcal{D}A \int \mathcal{D}B e^{i \int_M \mathrm{tr}(B \wedge F + \frac{\Lambda}{12} B \wedge B)}.$$

If we complete the square using the substitution $B \to B - 6F/\Lambda$ we can perform the, then quadratic, integral over B and obtain

$$\psi(A_\Sigma) \propto \int_{A|_\Sigma = A_\Sigma} \mathcal{D}A e^{-\frac{3i}{\Lambda} \int_M \mathrm{tr}(F \wedge F)}.$$

We now derive a result that will be convenient in the sequel.

Proposition 6.1.1. *Let $\alpha, \alpha' \in \Omega^1(P, \mathfrak{g})$ be arbitrary connections on a principal fiber bundle $P \to M$. Denote the boundary of M by Σ. Suppose that $P\big|_\Sigma$ is trivializable and fix a trivialization of $P\big|_\Sigma$. Let F be the curvature of α and F' the curvature of α'. Define $A = \alpha - \alpha' \in \Omega^1(M, adP)$. Then the identity*

$$\int_M tr(F \wedge F) = S_{CS}(A|_\Sigma) + \int_M tr(F' \wedge F') + 2 \int_\Sigma tr(A \wedge F'), \qquad (6.2)$$

holds, where $A|_\Sigma$ is the restriction of A to Σ.

Proof. We define a family of $\mathfrak{g}$-valued 1-forms on P as follows

$$\alpha_s := \alpha' + sA, \quad (s \in I),$$

so in particular $\alpha_1 = \alpha$ and $\alpha_0 = \alpha'$. The curvature of α_s is

$$\begin{aligned} F_s &= \mathrm{d}\alpha_s + \frac{1}{2}[\alpha_s, \alpha_s] \\ &= F_0 + s\mathrm{d}A + \frac{s^2}{2}[A, A] + s[\alpha_0, A]. \end{aligned}$$

Let us write d_{D_s} for the exterior covariant derivative induced by the connection α_s. We now compute

$$\begin{aligned}
\operatorname{tr}(F_1 \wedge F_1) - \operatorname{tr}(F_0 \wedge F_0) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \operatorname{tr}(F_s \wedge F_s) \\
&= 2\int_0^1 \operatorname{tr}(\frac{\mathrm{d}F_s}{\mathrm{d}s} \wedge F_s) \\
&= 2\int_0^1 \operatorname{tr}\left((\mathrm{d}A + s[A,A] + [\alpha_0, A]) \wedge F_s\right) \\
&= 2\int_0^1 \operatorname{tr}(\mathrm{d}_{D_s} A \wedge F_s) \\
&= 2\mathrm{d}\int_0^1 \operatorname{tr}(A \wedge F_s) \\
&= \mathrm{d}\operatorname{tr}(A \wedge \mathrm{d}A + \frac{1}{3} A \wedge [A,A] + 2A \wedge F_0 + A \wedge [\alpha_0, A]) \\
&= \mathrm{d}\operatorname{tr}(A \wedge \mathrm{d}_{D_0} A + \frac{2}{3} A \wedge A \wedge A + 2A \wedge F_0).
\end{aligned}$$

Now, if we integrate both sides over M we obtain

$$\begin{aligned}
\int_M \operatorname{tr}(F_1 \wedge F_1) &= \int_M \operatorname{tr}(F_0 \wedge F_0) + \int_M \mathrm{d}\operatorname{tr}(A \wedge \mathrm{d}_{D_0} A + \frac{2}{3} A \wedge A \wedge A + 2A \wedge F_0) \\
&= \int_M \operatorname{tr}(F_0 \wedge F_0) + \int_\Sigma \operatorname{tr}(A \wedge \mathrm{d}_{D_0} A + \frac{2}{3} A \wedge A \wedge A + 2A \wedge F_0) \\
&= \int_M \operatorname{tr}(F_0 \wedge F_0) + 2\int_\Sigma \operatorname{tr}(A \wedge F_0) + S_{CS}(A_\Sigma)
\end{aligned}$$

If we recall that $F_0 = F'$ and $F_1 = F$ we obtain the desired result. □

The applications of this result in the sequel will be in the form of the following corollaries.

Corollary 6.1.2. *Let $P \to M$ be a principal fiber bundle. Denote the boundary of M by Σ. Suppose that $P|_\Sigma$ is trivializable and fix a trivialization of $P|_\Sigma$. Let $\alpha \in \Omega^1(P, \mathfrak{g})$ be an arbitrary connection on P, and let $\alpha' \in \Omega^1(P, \mathfrak{g})$ be an arbitrary extension of a flat connection on $P|_\Sigma$. Define $A = \alpha - \alpha' \in \Omega^1(M, adP)$. Then the identity*

$$\int_M tr(F \wedge F) = S_{CS}(A|_\Sigma) + \int_M tr(F' \wedge F')$$

holds, where $A|_\Sigma$ is the restriction of A to Σ.

Note that the left-hand side of Eq. (6.2) does not depend on the choice of trivialization, so neither does the right hand side. The pieces $S_{CS}(A_\Sigma)$ and $2\int_\Sigma \operatorname{tr}(A \wedge F')$, thus depend on the choice of trivialization of P_Σ in such a way that their sum does not. Similarly, the left-hand side of Eq. (6.2) does not depend on α' so neither does the right-hand side.

6.2 The BF functor

Motivated by the results of the previous section we translate the description of 4-dimensional BF theory in terms of a Lagrangian into a description in the style of Atiyah's TQFT, see Ref. [Atiy88], thus in terms of a functor Z_{BF}. For a summary of the definitions from category theory used here we refer the reader to Ref. [Baez04]. First let us describe the domain of the functor.

Definition 6.2.1. Let $\mathbf{C}$ be the category determined by the following data:

- An object Σ of $\mathbf{C}$ is a compact oriented 3-manifold, equipped with a trivializable principal G-bundle $P_\Sigma \to \Sigma$.
- Given two objects Σ and Σ', a morphism $M : \Sigma \to \Sigma'$ is an equivalence class of compact oriented 4-manifolds M with boundary, equipped with principal G-bundle $P_M \to M$ and bundle isomorphism $\tilde{f}_M : P_{\bar{\Sigma}} \cup P_{\Sigma'} \to P_M\big|_{\partial M}$ lifting an orientation-preserving diffeomorphism $f_M : \bar{\Sigma} \cup \Sigma' \to \partial M$.

The equivalence relation is that $M \sim M'$ if there is a bundle isomorphism $\alpha : P_M \to P_{M'}$ such that $\tilde{f}_{M'} = \alpha \circ \tilde{f}_M$. Furthermore, if Σ is any oriented 3-manifold, then $\bar{\Sigma}$ is the same manifold, with its orientation reversed.

Remark 6.2.2. If we endow $\mathbf{C}$ with the disjoint union $\cup : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, then the category $\mathbf{C}$ becomes a symmetric monoidal category. The unit object is the empty set $\emptyset$. If Σ is an object of $\mathbf{C}$, then the dual object is $\bar{\Sigma}$, (here the principal G-bundle P_Σ is untouched). In this way $\mathbf{C}$ becomes a rigid symmetric monoidal category. The associator and braiding are the obvious maps.

The codomain of the functor Z_{BF} will be the symmetric monoidal category **Vect** of vector spaces and linear maps, equipped with the usual tensor product $\otimes$.

If Σ is an object in $\mathbf{C}$, let $\mathcal{A}_\Sigma$ denote the space of connections on P_Σ. We recall that by Proposition 5.2.6 and the fact that P_Σ is trivializable there is a non-canonical isomorphism $\mathcal{A}_\Sigma \simeq \Omega^1(\Sigma, \mathrm{ad}P_\Sigma)$.

Definition 6.2.3 (The BF functor)**.** Here, we will give the definition of the functor $Z_{BF} : \mathbf{C} \to \mathbf{Vect}$ and show that it is well-defined.

Objects — Let Σ be an object in $\mathbf{C}$. We define $Z_{BF}(\Sigma)$ to be the space of functions on $\mathcal{A}_\Sigma$ that are multiples of $\exp(-3iS_{CS}(A_\Sigma)/\Lambda)$. That is

$$Z_{BF}(\Sigma) = \{\phi : \mathcal{A}_\Sigma \to \mathbb{C} | \phi(A_\Sigma) = \lambda e^{-\frac{3i}{\Lambda} S_{CS}(A_\Sigma)}, A_\Sigma \in \mathcal{A}_\Sigma, \lambda \in \mathbb{C}\}.$$

Here, we use the non-canonical isomorphism $\Omega^1(P_\Sigma, \mathfrak{g})^G \simeq \Omega^1(\Sigma, \mathrm{ad}P_\Sigma)$. Because the Chern-Simons actions corresponding to different trivializations only differ by a constant, this space is independent of the choice of trivialization of P_Σ. Separately we define $Z_{BF}(\emptyset) = \mathbb{C}$.

Morphisms — If $M : \emptyset \to \Sigma$ is a morphism in $\mathbf{C}$, then we define

$$Z_{BF}(M) : \mathbb{C} \to Z_{BF}(\Sigma), \lambda \mapsto \lambda\psi,$$

where ψ is the function

$$\begin{aligned} \psi : \mathcal{A}_\Sigma &\to \mathbb{C}, \\ \alpha_\Sigma &\mapsto e^{-\frac{3i}{\Lambda}\int_M \mathrm{tr}(F\wedge F)}, \end{aligned}$$

where F is the curvature of any connection α extending α_Σ to all of P_M. Note that to define the map $\psi : \mathcal{A}_\Sigma \to \mathbb{C}$ we made no mention of a trivialization of P_Σ, so even though we will use a trivialization of P_Σ to prove that $\psi \in Z_{BF}(\Sigma)$, ψ does not depend on the trivialization of P_Σ.

Let us show that this does not depend on the choice of extension of α_Σ and that $\psi \in Z_{BF}(\Sigma)$. We fix a trivialization of P_Σ, this induces a flat connection on P_Σ. Let us extend this flat connection to all of P_M, (the extension need not be flat), we denote this extension by α'. We denote the curvature of α' by F'. Let us write $A = \alpha - \alpha'$. From Corollary 6.1.2 it follows that

$$\int_M \mathrm{tr}(F\wedge F) = S_{CS}(A|_\Sigma) + \int_M \mathrm{tr}(F'\wedge F').$$

The right-hand side does not depend on the extension α of α_Σ, thus neither does the left-hand side. To use Corollary 6.1.2 we have to choose a trivialization of $P_M|_\Sigma$. The manifold M comes equipped with a bundle isomorphism $\tilde{f}_M : P_\Sigma \to P_M|_\Sigma$. Via this bundle isomorphism the trivialization of P_Σ, that we have already chosen, induces a trivialization of $P_M|_\Sigma$. Using this trivialization it follows that

$$S_{CS}(A_\Sigma) = S_{CS}(A|_\Sigma). \tag{6.3}$$

It follows that indeed $\psi \in Z_{BF}(\Sigma)$.

Remark 6.2.4. Eq. (6.3) is not entirely trivial. By $S_{CS}(A_\Sigma)$ we mean the Chern-Simons action with respect to the fixed trivialization of P_Σ and by $S_{CS}(A|_\Sigma)$ we mean the Chern-Simons action with respect to the trivialization induced on $P_M|_\Sigma$ by the map $\tilde{f}_M : P_\Sigma \to P_M|_\Sigma$.

Now suppose that $M : \Sigma \to \Sigma'$, thus $\partial M = \bar{\Sigma} \cup \Sigma'$. In this case we define

$$Z_{BF}(M) : Z_{BF}(\Sigma) \to Z_{BF}(\Sigma'), \phi \mapsto \psi\phi, \tag{6.4}$$

where ψ is the function

$$\begin{aligned} \psi : \mathcal{A}_{\bar{\Sigma}\cup\Sigma'} &\to \mathbb{C}, \\ \alpha_{\bar{\Sigma}\cup\Sigma'} &\mapsto e^{-\frac{3i}{\Lambda}\int_M \mathrm{tr}(F\wedge F)} \end{aligned}$$

where F is the curvature of any connection α extending $\alpha_{\bar{\Sigma}\cup\Sigma'}$ to all of P_M. We have already shown that this in fact does not depend on the choice of extension α. This map is clearly linear. It remains to show that it does indeed map $Z_{BF}(\Sigma)$ into $Z_{BF}(\Sigma')$. Let $\phi \in Z_{BF}(\Sigma)$ and let $\alpha_{\Sigma'} \in \mathcal{A}_{\Sigma'}$ be arbitrary. Fix a trivialization of P_Σ and $P_{\Sigma'}$, hence of $P_{\Sigma\cup\Sigma'}$. Fix the standard flat connection on $P_{\Sigma\cup\Sigma'}$ and denote an arbitrary extension of this flat connection to all of P_M by α'. Denote the curvature of α by F and of α' by F'. We define $A = \alpha - \alpha'$. Using Corollary 6.1.2 we obtain

$$\begin{aligned}\int_M \operatorname{tr}(F \wedge F) &= S_{CS}(A|_{\bar{\Sigma}\cup\Sigma'}) + \int_M \operatorname{tr}(F' \wedge F') \\ &= -S_{CS}(A_\Sigma) + S_{CS}(A_{\Sigma'}) + \int_M \operatorname{tr}(F' \wedge F'),\end{aligned} \tag{6.5}$$

note that, in writing the second line, we have used the fact that the trivializations of $P_{\bar{\Sigma}\cup\Sigma'}$ and $P_M|_{\bar{\Sigma}\cup\Sigma}$ agree. The minus sign in the identity $S_{CS}(A_{\bar{\Sigma}}) = -A_{CS}(A_\Sigma)$ is the reason that this definition works out. It is a result of the fact that when we include the in-boundary Σ into M we flip its orientation. Because $\phi \in Z_{BF}(\Sigma)$, there exists some $\lambda \in \mathbb{C}$ such that

$$\phi(\alpha_\Sigma) = \lambda e^{-\frac{3i}{\Lambda} S_{CS}(A_\Sigma)}.$$

We now complete the proof, using Eq. (6.5),

$$\psi(\alpha_{\Sigma\cup\Sigma'})\phi(\alpha_\Sigma) = \lambda e^{-\frac{3i}{\Lambda}\left(S_{CS}(A_{\Sigma'}) - \int_M (F' \wedge F')\right)},$$

it follows that $\psi(\alpha_{\Sigma\cup\Sigma'})\phi(\alpha_\Sigma)$ does not depend on α_Σ and indeed $\psi\phi \in Z_{BF}(\Sigma')$.

We would like to show that Z_{BF} indeed preserves the identity morphisms. To do so, we will first give a description of the identity morphisms in **C**.

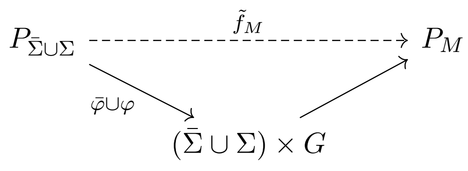
Let us fix an object Σ of **C**. We will describe the identity morphism $M : \Sigma \to \Sigma$. As a manifold it is given by $M = \Sigma \times I$. There is an obvious inclusion

$$f_M : \bar{\Sigma} \cup \Sigma \to \Sigma \times I.$$

The principal fiber bundle over M is given by

$$P_M = \Sigma \times I \times G.$$

Now let us fix a trivialization $\varphi : P_\Sigma \to \Sigma \times G$, which induces a trivialization $\bar{\varphi} : P_{\bar{\Sigma}} \to \bar{\Sigma} \times G$. The lift $\tilde{f}_M$ of f_M is obtained by declaring the following diagram to commute



One may verify that indeed $\tilde{f}_M$ lifts f_M. If one picks another trivialization $\varphi' : P_\Sigma \to \Sigma \times G$ one obtains another map $\tilde{f}'_M : P_{\bar{\Sigma}\cup\Sigma} \to P_M$. We investigate the dependence of the resulting morphism $M : \Sigma \to \Sigma$, (remember that this was an equivalence class). To that end we define the map χ to be that map that makes the following diagram commute

$$\begin{array}{ccccc} & & (\bar{\Sigma}\cup\Sigma)\times G & \longrightarrow & P_M|_{\partial M} \\ & \overset{\bar{\varphi}'\cup\varphi'}{\nearrow} & & \overset{\tilde{f}'_M}{\nearrow} & \\ P_{\bar{\Sigma}\cup\Sigma} & & & & \downarrow \chi \\ & \underset{\bar{\varphi}\cup\varphi}{\searrow} & & \underset{\tilde{f}_M}{\searrow} & \\ & & (\bar{\Sigma}\cup\Sigma)\times G & \longrightarrow & P_M|_{\partial M}. \end{array} \tag{6.6}$$

We might expect the pairs $(M, \tilde{f}_M) : \Sigma \to \Sigma$ and $(M, \tilde{f}'_M) : \Sigma \to \Sigma$ to represent the same morphism. Using the identities $P_M|_{\partial M} = \Sigma \times \{0,1\} \times G$ and $P_M = \Sigma \times I \times G$, we see that this is the case if and only if there exists a bundle isomorphism $\xi : P_M \to P_M$ such that the following diagram commutes

$$\begin{array}{ccc} \Sigma \times \{0,1\} \times G & \hookrightarrow & \Sigma \times I \times G \\ \downarrow \chi & & \downarrow \xi \\ \Sigma \times \{0,1\} \times G & \hookrightarrow & \Sigma \times I \times G \end{array}$$

This essentially means that ξ is a homotopy between

$$\chi(-,0,-) : \Sigma \times G \to \Sigma \times G, (s,g) \mapsto \chi(s,0,g), \qquad \text{and}$$
$$\chi(-,1,-) : \Sigma \times G \to \Sigma \times G, (s,g) \mapsto \chi(s,1,g).$$

Using the diagram 6.6 and the fact that $\bar{\varphi}' = \varphi'$ and $\bar{\varphi} = \varphi$, we see that $\chi(-,0,-) = \chi(-,1,-)$, hence the map ξ exists.

Proposition 6.2.5. *If Σ is any object in* $\mathbf{C}$ *and $M : \Sigma \to \Sigma$ is the corresponding identity morphism, then $Z_{BF}(M) = Id_{Z_{BF}\Sigma}$.*

Proof. For notational clarity we distinguish the source and target of M by a prime, i.e. $M : \Sigma \to \Sigma'$, however the reader is urged to keep in mind that Σ and Σ' are one and the same, viewed as objects of $\mathbf{C}$. Let $\phi \in Z_{BF}(\Sigma)$ and $\alpha_{\Sigma'} \in \mathcal{A}_{\Sigma'}$ be arbitrary. We follow the procedure described above leading up to Eq. (6.5), however instead of picking an arbitrary extension α' of the flat connection on $P_{\bar{\Sigma}\cup\Sigma'}$ we take α' to be the flat connection on P_M. It follows by Eq. (6.5) that

$$\int_M \mathrm{tr}(F \wedge F) = -S_{CS}(A_\Sigma) + S_{CS}(A_{\Sigma'}),$$

where, we have used Eq. (6.3). We compute

$$\psi(\alpha_{\bar{\Sigma}\cup\Sigma'})\phi(\alpha_\Sigma) = \lambda e^{-\frac{3i}{\Lambda}S_{CS}(A_{\Sigma'})} = \phi(\alpha_{\Sigma'}).$$

Hence, we conclude that indeed $Z_{BF}(M) = \mathrm{Id}_{Z_{BF}(\Sigma)}$. □

Remark 6.2.6. In the proof above we have used that the trivializations of $P_{\bar{\Sigma}\cup\Sigma}$ and of $P_M|_{\partial M}$ are compatible. In fact, this is what allowed us to use Eq. (6.3).

Let us fix a quintuple

$$\Sigma_0 \xrightarrow{M_1} \Sigma_1 \xrightarrow{M_2} \Sigma_2$$

in **C**. Let us denote $M_2 \circ M_1 = M_2 \cup_{\Sigma_1} M_1 : \Sigma_0 \to \Sigma_2$ for the composite of M_1 and M_2 in **C**. Then, we have, for any curvature two-form F,

$$\int_{M_2\cup_{\Sigma_1}M_1} \mathrm{tr}(F\wedge F) = \int_{M_2} \mathrm{tr}(F\wedge F) + \int_{M_1} \mathrm{tr}(F\wedge F).$$

Using this equation one might check that $Z_{BF}(M_2 \circ M_1) = Z_{BF}(M_2) \circ Z_{BF}(M_1)$.

In summary, we have shown that Z_{BF} is well-defined and preserves both the identity and composition. This completes the description of the functor $Z_{BF} : \mathbf{C} \to \mathbf{Vect}$.

Theorem 6.2.7. *The functor $Z_{BF} : \mathbf{C} \to \mathbf{Vect}$ is symmetric monoidal.*

Proof. If $\Sigma, \Sigma' \in \mathbf{C}$ then there is a natural isomorphism

$$\Phi_{\Sigma,\Sigma'} : Z_{BF}(\Sigma) \otimes Z_{BF}(\Sigma') \to Z_{BF}(\Sigma \cup \Sigma'),$$

namely the one that says

$$\Phi_{\Sigma,\Sigma'} : \lambda(\phi_\Sigma \otimes \phi_{\Sigma'}) \mapsto \lambda(\phi_\Sigma\phi_{\Sigma'}),$$

this map does as advertised, since

$$S_{CS}(A_\Sigma) + S_{CS}(A_{\Sigma'}) = S_{CS}(A_\Sigma \cup A_{\Sigma'}),$$

so that $\phi_\Sigma\phi'_\Sigma \in \mathcal{A}_{\Sigma\cup\Sigma'}$. □

According to Ref. [CY99] any symmetric monoidal functor preserves duals up to isomorphism.

6.3 The case $G = \mathbf{GL}(4, \mathbb{R})$

An interesting feature of 3-dimensional compact manifolds is that they have trivializable tangent bundles, see for example exercise 12-B in Ref. [MilSt]. We will take advantage of this fact to obtain a functor $Z : 4\mathbf{Cob} \to \mathbf{Vect}$. Here $4\mathbf{Cob}$ is the 4-dimensional cobordism category, it is very much like $\mathbf{C}$, where we simply forget about the bundles. More specifically, the definition is,

Definition 6.3.1 (4**Cob**)**.** The category 4**Cob** is the category determined by the following data:

- An object Σ of 4**Cob** is a compact oriented 3-manifold.
- Given two objects Σ and Σ', a morphism $M : \Sigma \to \Sigma'$ is an equivalence class of compact oriented 4-manifolds M with boundary, equipped with an orientation-preserving diffeomorphism $f_M : \bar{\Sigma} \cup \Sigma' \to \partial M$.

Just like $\mathbf{C}$, this is a rigid symmetric monoidal category. We would like to design a functor $F : 4\mathbf{Cob} \to \mathbf{C}$, so that we could obtain a functor $Z = Z_{BF} \circ F : 4\mathbf{Cob} \to \mathbf{Vect}$. We will obtain a map $F : 4\mathbf{Cob} \to \mathbf{C}$ that is not quite a functor, but it will turn out that $Z = Z_{BF} \circ F$ will be a functor.

So let us give a description of the map F. Let Σ be an object in 4**Cob**, that is, Σ is a compact oriented three-dimensional manifold. Let $T\Sigma$ be its tangent bundle and let $L\Sigma$ be the trivial line bundle over Σ. Since $T\Sigma$ and $L\Sigma$ are trivializable, so is $T\Sigma \oplus L\Sigma$. Thus, the frame bundle $\mathrm{Fr}(T\Sigma \oplus L\Sigma)$ is trivializable as well, recall example 5.1.2. The fiber of $T\Sigma \oplus L\Sigma$ is $\mathbb{R}^4$, thus $P_\Sigma := \mathrm{Fr}(T\Sigma \oplus L\Sigma)$ is a trivializable principal $\mathrm{GL}(4, \mathbb{R})$-bundle, we define the image of Σ under F to be this object in $\mathbf{C}$. Now let M be a morphism in 4**Cob** going from Σ to Σ'. That is, M is a 4-dimensional compact oriented manifold, equipped with an orientation-preserving diffeomorphism $f : \bar{\Sigma} \cup \Sigma' \to \partial M$. Let P_M be the frame bundle of M. Now we note that morphisms in $\mathbf{C}$ are equipped with a bit of extra structure, namely a lift $\tilde{f} : P_{\bar{\Sigma}} \cup P_{\Sigma'} \to P_M|_{\partial M}$, of the diffeomorphism $f : \bar{\Sigma} \cup \Sigma' \to \partial M$.

First, let us work on the level of the vector bundle $T\bar{\Sigma} \oplus L\bar{\Sigma} \cup T\Sigma' \oplus L\Sigma'$. There is an obvious lift of $f : \bar{\Sigma} \cup \Sigma' \to \partial M$ for the first component, namely $\mathrm{d}f : T\bar{\Sigma} \cup T\Sigma' \to T\partial M \subset TM|_{(\bar{\Sigma} \cup \Sigma')}$. What remains is to construct a map $\mu : L\bar{\Sigma} \cup L\Sigma' \to TM|_{(\bar{\Sigma} \cup \Sigma')}$. The construction of this map amounts to the choice of two sections, $v : f(\bar{\Sigma}) \to TM|_{\bar{\Sigma}}$ and $w : f(\Sigma') \to TM|_{\Sigma'}$, we will see in a moment that v and w need to have some special properties. Given such a choice, the map μ restricted to $L\bar{\Sigma}$ will become

$$\mu|_{L\bar{\Sigma}} : L\bar{\Sigma} = \bar{\Sigma} \times \mathbb{R} \to TM|_{\bar{\Sigma}}, (x, t) \mapsto tv(f(x)).$$

The map $\mathrm{d}f$ maps $T\bar{\Sigma}$ into the subbundle of vectors tangent to $\bar{\Sigma}$, thus if v is nowhere tangent to $\bar{\Sigma}$, (thus in particular nowhere vanishing), the linear map $(\mathrm{d}f \oplus \mu)|_{T\bar{\Sigma}\oplus L\bar{\Sigma}}$ is an isomorphism. The same holds for w and $(\mathrm{d}f \oplus \mu)|_{T\Sigma'\oplus L\Sigma'}$. Now if v is inwards-pointing and w is outwards-pointing the map

$$\mathrm{d}f \oplus \mu : T\bar{\Sigma} \oplus L\bar{\Sigma} \cup T\Sigma' \oplus L\Sigma' \to TM|_{\Sigma\cup\Sigma'} \tag{6.7}$$

is an orientation-preserving isomorphism, lifting $f : \bar{\Sigma} \cup \Sigma' \to \partial M$.

It remains to show that the isomorphism $\mathrm{d}f \oplus \mu$ gives us an isomorphism $\tilde{f} : P_{\bar{\Sigma}} \cup P_{\Sigma} \to P_M|_{\partial M}$, but this follows directly from prosition 5.2.5.

This completes the description of the map $F : 4\mathbf{Cob} \to \mathbf{C}$.

The procedure above does give us a functor $F' : 4\mathbf{Cob}' \to \mathbf{C}$, where $4\mathbf{Cob}'$ is the category with the same objects and morphisms as $4\mathbf{Cob}$, except that a morphism M in $4\mathbf{Cob}'$ is equipped with a section v of TM over ∂M, that is nowhere tangent to ∂M and inwards pointing on the in-boundary and outwards pointing on the out-boundary.

Let us define $G : 4\mathbf{Cob} \to 4\mathbf{Cob}'$ to be the map that assigns to each object in $4\mathbf{Cob}$ the same object in $4\mathbf{Cob}'$ and to each morphism of $4\mathbf{Cob}$ the same morphism in $4\mathbf{Cob}'$ with arbitrarily chosen section v of TM over ∂M with the necessary properties.

We define $Z : 4\mathbf{Cob} \to \mathbf{Vect}$ to be given by $Z = Z_{BF} \circ F' \circ G$. The map Z is the map that we are actually interested in, and not its constituents F' and G. Hence we will prove that Z is actually a functor directly.

Theorem 6.3.2. *The map $Z : 4\mathbf{Cob} \to \mathbf{Vect}$ is a symmetric monoidal functor.*

Proof. It is clear that Z preserves composition. Let us show that Z preserves the identity morphism. Let Σ be an arbitrary object in $4\mathbf{Cob}$. The identity morphism is represented by the manifold $\Sigma \times I$. We recall that any compact oriented three-manifold has trivializable tangent bundle. Let us fix a trivialization of $T\Sigma$, which induces a trivialization on P_Σ and on $P_{\bar{\Sigma}}$. Let us recall the definition 6.2.3, we see that the map $Z(M) : Z(\Sigma) \to Z(\Sigma)$ is fully determined by a function $\psi \in \mathcal{A}(P_{\bar{\Sigma}} \cup P_\Sigma)$. By Eq. (6.4) it suffices to show that for all connections $\alpha_{\bar{\Sigma}\cup\Sigma}$ on $P_{\bar{\Sigma}} \cup P_\Sigma$ it is true that

$$\psi(\alpha_{\bar{\Sigma}\cup\Sigma}) = e^{-\frac{3i}{\Lambda} S_{CS}(A_{\bar{\Sigma}\cup\Sigma})}.$$

According to the prescription of G we have equipped $\partial M = \{0,1\} \times \Sigma$ with an arbitrary section v of $TM|_{\partial M}$, which is nowhere tangent to the boundary of M and inwards pointing on $\{0\} \times \Sigma$ and outwards pointing on $\{1\} \times \Sigma$. According to the prescription of F' we have used the vector field v to obtain an isomorphism

$$\tilde{f}_M : P_{\bar{\Sigma}} \cup P_\Sigma \to P_M|_{\partial M}.$$

According to the prescription of Z_{BF} the function $\psi : \mathcal{A}(P_{\bar{\Sigma}} \cup P_\Sigma) \to \mathbb{C}$ is given by

$$\psi(\alpha_{\bar{\Sigma}\cup\Sigma}) = e^{-\frac{3i}{\Lambda} \int_M \mathrm{tr}(F\wedge F)},$$

where F is the curvature of any connection α extending $(\tilde{f}_M^*)^{-1}(\alpha_{\bar{\Sigma}\cup\Sigma})$ to all of P_M. Because we have already fixed a trivialization of $T\Sigma$, there is an obvious trivialization of $TM = T\Sigma \times TI$, and hence of P_M. Note that this trivialization induces a trivialization of $P_M|_{\partial M}$ that need not agree with the trivialization of $P_M|_{\partial M}$ induced by $\tilde{f}_M$. Taking α' in Eq. (6.5) to be the standard flat connection on P_M we obtain

$$\psi(\alpha_{\bar{\Sigma}\cup\Sigma}) = e^{-\frac{3i}{\Lambda}S_{CS}(A|_{\bar{\Sigma}\cup\Sigma})}.$$

The desired result follows if we can show that

$$S_{CS}(A_{\bar{\Sigma}\cup\Sigma}) = S_{CS}(A|_{\bar{\Sigma}\cup\Sigma}),$$

here $A_{\bar{\Sigma}\cup\Sigma}$ and $A|_{\bar{\Sigma}\cup\Sigma}$ need not be equal, because they were obtained from $\alpha_{\bar{\Sigma}\cup\Sigma}$ via different trivializations of $P_M|_{\partial M}$. Let us compare this with Eq. (6.3), there the result followed because the trivialization of $P_M|_\Sigma$ and P_Σ were constructed via the map $\tilde{f}_M$. In our current setup, the map $\tilde{f}_M$ does not carry the trivialization of $P_{\bar{\Sigma}\cup\Sigma}$ to the trivialization of $P_M|_{\partial M}$. We will show however that $\tilde{f}_M$ does carry the trivialization of $P_{\bar{\Sigma}\cup\Sigma}$ to the trivialization of $P_M|_{\partial M}$, *up to a small gauge transformation.* Then, since the Chern-Simons action is invariant under small gauge transformations, see Proposition 4.2.4, we are done. For notational convenience, let us write $S = \bar{\Sigma} \cup \Sigma$. Let us work on the level of the vector bundles $TM|_{\partial M}$ and $TS \oplus LS$, and simply identify the map $\tilde{f}_M : P_S \to P_M|_{\partial M}$ with the map $TS \oplus LS \to TM|_{\partial M}$ that produced it. We consider the following diagram

$$\begin{array}{ccc} TS \oplus LS & \longrightarrow & \Sigma \times \mathbb{R}^4 \\ \big\downarrow \tilde{f}_M & & \big\downarrow h \\ TM|_{\partial M} & \longrightarrow & \Sigma \times \mathbb{R}^4 \end{array}$$

The map $h : \Sigma \times \mathbb{R}^4 \to \Sigma \times \mathbb{R}^4$ is defined to be the map that makes the square commute. The map h will be a bundle isomorphism, we may identify it as a map $h : \Sigma \to \mathrm{GL}(\mathbb{R}, 4)$. Using Eq. (6.7) and the fact that $f : S \to \partial M$ was the obvious inclusion, one may verify that the map h takes the form

$$h(x) = \begin{pmatrix} 1 & 0 & 0 & v_1(f(x)) \\ 0 & 1 & 0 & v_2(f(x)) \\ 0 & 0 & 1 & v_3(f(x)) \\ 0 & 0 & 0 & v_4(f(x)) \end{pmatrix},$$

for all $x \in S$. Here v is (the image in the trivialization of) the supplied section of TM over ∂M. We now consider the one-parameter family of linear maps $h_t : \Sigma \times \mathbb{R}^4 \to \Sigma \times \mathbb{R}^4$, for $t \in I$, given by

$$h(x) = \begin{pmatrix} 1 & 0 & 0 & tv_1(f(x)) \\ 0 & 1 & 0 & tv_2(f(x)) \\ 0 & 0 & 1 & tv_3(f(x)) \\ 0 & 0 & 0 & (1-t) + tv_4(f(x)) \end{pmatrix}.$$

It is clear that $h_0 = \mathrm{Id}$ and $h_1 = h$. The determinant of h_t is given by $(1-t) + tv_4(f(x))$. The assumption that v is inwards pointing on $\bar{\Sigma}$ and outwards pointing on Σ is equivalent to the assumption that $v_4(f(x)) > 0$ for all $x \in S$. Hence h_t is a homotopy from Id to h, thus h is a small gauge transformation. □

6.4 Uniqueness of Z

In the article that we are following, [Baez95], another topological quantum field theory is discussed, namely Crane-Yetter-Broda theory [CY93]. Unfortunately, a description of this theory is beyond the scope of this thesis. The article then goes on to prove the equivalence of the Chern-Simons theory, Z, that we discussed, and Crane-Yetter-Broda theory. We can, however give a simple, but highly non-exhaustive, criterion that may be used to prove that two topological quantum field theories are the same. This criterion is used to prove the equivalence. Before we do so we will have something more to say about the theory described by Z.

One of the main interests of topological quantum field theories, from a mathematical point of view, is that they produce invariants of manifolds as follows. The functor $Z : 4\mathbf{Cob} \to \mathbf{Vect}$ takes morphisms in $4\mathbf{Cob}$, that is 4-manifolds with boundary, to morphisms in $\mathbf{Vect}$. Actually, a morphism in $4\mathbf{Cob}$ is an equivalence class of a 4-manifold. This means that if M and M' are diffeomorphic 4-manifolds, then Z does not distinguish between them. In particular if $M : \emptyset \to \emptyset$, then $Z(M) : \mathbb{C} \to \mathbb{C}$, hence we may identify $Z(M)$ with a complex number. This number depends only on the diffeomorphism class of M. In our current setting we have

$$Z(M) = e^{-\frac{3i}{\Lambda}\int_M \mathrm{tr}(F\wedge F)},$$

where F is the curvature of a connection on the tangent bundle of M. According to the Hirzebruch signature theorem, see [HBJ92] or [Hirz78], we have

$$\int_M \mathrm{tr}(F \wedge F) = 12\pi^2\sigma(M),$$

where $\sigma(M)$ is the signature of M. We conclude that

$$Z(M) = e^{-\frac{36i}{\Lambda}\pi^2\sigma(M)}.$$

We have the following convenient lemma that is true for any topological quantum field theory, where the objects of $\mathbf{Vect}$ are finite-dimensional vector spaces.

Lemma 6.4.1. *Let $Z : 4\mathbf{Cob} \to \mathbf{Vect}$ be any topological quantum field theory. Suppose that Σ is an object of $4\mathbf{Cob}$ and $M : \Sigma \to \Sigma$ a morphism. We can construct a morphism $M^\circ : \emptyset \to \emptyset$ by glueing M to itself along* *its boundary, that is, we identify $\bar{\Sigma} \subset \partial M$ with $\Sigma \subset \partial M$. The following equation now holds:*

$$Z(M^\circ) = tr(Z(M)).$$

Proof. On the one hand we may identify $Z(M) \in \mathrm{End}(Z(\Sigma))$ with $\widetilde{Z(M)} \in Z(\Sigma) \otimes Z^*(\Sigma)$. There exist vectors $\psi^i \in Z(\Sigma)$ and $\alpha_i \in Z^*(\Sigma)$ such that

$$\widetilde{Z(M)} = \psi^i \otimes \alpha_i,$$

in terms of which the trace of $Z(M)$ now reads

$$\mathrm{tr}(Z(M)) = \alpha_i(\psi^i) = \mathrm{tr}(\widetilde{Z(M)}).$$

Similarly there is a canonical isomorphism

$$\begin{gathered} \mathrm{End}(Z(\Sigma)) \simeq \mathrm{Hom}(Z^*(\Sigma) \otimes Z(\Sigma), \mathbb{C}), \\ (T : Z(\Sigma) \to Z(\Sigma)) \mapsto (\alpha \otimes v \mapsto \alpha(T(v))). \end{gathered}$$

Given $T \in \mathrm{End}(Z(\Sigma))$ we denote its image in $\mathrm{Hom}(Z^*(\Sigma) \otimes Z(\Sigma), \mathbb{C})$ by $\utilde{T}$. We see that

$$\utilde{\mathrm{Id}_{Z(\Sigma)}} : \alpha \otimes v \mapsto \alpha(v),$$

hence $\utilde{\mathrm{Id}_{Z(\Sigma)}} = \mathrm{tr}$. On the other hand, given $M : \Sigma \to \Sigma$ we may form $\tilde{M} : \emptyset \to \Sigma \cup \bar{\Sigma}$, and similarly we may form $\utilde{\mathrm{Id}}_\Sigma : \Sigma \cup \bar{\Sigma} \to \emptyset$, from the identity on Σ. It is clear that $M^\circ = \utilde{\mathrm{Id}}_\Sigma \circ \tilde{M}$. Using the fact that Z is functor that preserves duals we now obtain $Z(\utilde{M}) = \utilde{Z(M)}$ and $Z(\utilde{\mathrm{Id}}_\Sigma) = \utilde{Z(\mathrm{Id}_\Sigma)}$, hence

$$Z(M^\circ) = Z(\utilde{\mathrm{Id}}_\Sigma) \circ Z(\tilde{M}) = \utilde{Z(\mathrm{Id}_\Sigma)} \circ \widetilde{Z(M)} = \utilde{\mathrm{Id}_{Z(\Sigma)}} \circ \widetilde{Z(M)} = \mathrm{tr}(Z(M)). \quad \square$$

The finite-dimensionality of the vector spaces involved is required to make sense of the notion of trace.

As a particular application of Lemma 6.4.1 we consider the identity morphism $\mathrm{Id}_\Sigma = \Sigma \times I$. We have $\mathrm{Id}_\Sigma^\circ = \Sigma \times S^1$. It follows that

$$Z(\Sigma \times S^1) = \mathrm{tr}(\mathrm{Id}_{Z(\Sigma)}) = \dim(Z(\Sigma)). \tag{6.8}$$

We are now ready to state and prove the criterion mentioned above. Note that we are *not* talking about extended topological quantum field theories.

Theorem 6.4.2. *If $Z, Z' : 4\mathbf{Cob} \to \mathbf{Vect}$ are topological quantum fields theories which obey, for all compact oriented 4-manifolds M,*

$$Z(M) = Z'(M) = y^{\sigma(M)},$$

with $y \neq 0$, then there is a monoidal natural isomorphism $F : Z \to Z'$. It is also said that Z and Z' are equivalent as topological quantum field theories.

Proof. Suppose that $Z, Z' : 4\mathbf{Cob} \to \mathbf{Vect}$ are functors as in the assumption of the theorem. The claim is now that for each object Σ of $4\mathbf{Cob}$ there exists an isomorphism F_Σ, such that for all triples $M : \Sigma_1 \to \Sigma_2$, the following diagram commutes

$$\begin{array}{ccc} Z(\Sigma_1) & \xrightarrow{F_{\Sigma_1}} & Z'(\Sigma_1) \\ {\scriptstyle Z(M)}\downarrow & & \downarrow{\scriptstyle Z'(M)} \\ Z(\Sigma_2) & \xrightarrow{F_{\Sigma_2}} & Z'(\Sigma_2) \end{array} \tag{6.9}$$

this is the naturality square of F. Furthermore the isomorphisms F_Σ should be compatible with the monoidal structure, we will elaborate on this compatibility later in the proof.

Before we construct F, we argue that $Z(\Sigma)$ and $Z'(\Sigma)$ are one-dimensional for any Σ. This follows from the fact that $\sigma(S^1 \times \Sigma) = 0$ and from equation 6.8 as follows

$$\dim(Z(\Sigma)) = Z(\Sigma \times S^1) = y^{\sigma(M)} = 1.$$

Because the oriented cobordism group in three dimensions is trivial there is, for each Σ, a morphism $M : \emptyset \to \Sigma$. Moreover, by the computation

$$Z(\bar{M})Z(M)1 = Z(\bar{M} \circ M)1 = y^{\sigma(\bar{M} \circ M)} \neq 0,$$

the vector $Z(M)1 \in Z(\Sigma)$ is non-zero. We now define the map $F_\Sigma : Z(\Sigma) \to Z'(\Sigma)$ by the equation

$$F_\Sigma(Z(M)1) = Z'(M)1, \tag{6.10}$$

for any $M : \emptyset \to \Sigma$. There are now a number of checks we need to do.

- First note that, since $Z(\Sigma)$ and $Z'(\Sigma)$ are one-dimensional, the equation (6.10) completely determines the map F_Σ.
- Let us check that in fact if Eq. (6.10) holds for one choice of $M : \emptyset \to \Sigma$ it holds for all such M. That is, the definition of F does not depend on the choice of M. Let $M_1, M_2 : \emptyset \to \Sigma$ be arbitrary but fixed. By the one-dimensionality of $Z(\Sigma)$ and $Z'(\Sigma)$ there exist constants $\alpha \in \mathbb{C}$ and $\alpha' \in \mathbb{C}$ such that

  $$Z(M_1) = \alpha Z(M_2), \qquad Z'(M_1) = \alpha' Z(M_2).$$

  Since $\bar{M}_1 \circ M_1$ and $\bar{M}_1 \circ M_2$ are closed 4-manifolds we have

  $$Z(\bar{M}_1 \circ M_1) = \alpha Z(\bar{M}_1 \circ M_2), \quad \text{and} \quad Z'(\bar{M}_1 \circ M_1) = \alpha' Z'(\bar{M}_1 \circ M_2),$$

  hence $\alpha = \alpha'$.

- We consider the naturality of F. Fix an arbitrary triple $M : \Sigma_1 \to \Sigma_2$ in 4**Cob**. Let us show that the naturality square (6.9) indeed commutes

$$\begin{aligned} Z'(M)F_{\Sigma_1}Z(N)1 &= Z'(M)Z'(N)1 \\ &= Z'(M \circ N)1 \\ &= F_{\Sigma_2}Z(M \circ N)1 \\ &= F_{\Sigma_2}Z(M)Z(N)1. \end{aligned}$$

- Finally, let us show that F is compatible with the monoidal structure. Fix the pairs $M_1 : \emptyset \to \Sigma_1$ and $M_2 : \emptyset \to \Sigma_2$. Then it follows

$$\begin{aligned} F_{\Sigma_1 \cup \Sigma_2}Z(M_1 \cup M_2)1 &= Z'(\Sigma_1 \cup \Sigma_2)1 \\ &= Z'(\Sigma_1) \otimes Z'(\Sigma_2)1 \\ &= F_{\Sigma_1}Z(\Sigma_1)1 \otimes F_{\Sigma_2}Z(\Sigma_2)1 \\ &= (F_{\Sigma_1} \otimes F_{\Sigma_2})Z(M_1 \cup M_2)1. \end{aligned}$$

  Fix $M : \emptyset \to \emptyset$. It follows

$$F_\emptyset Z(M)1 = Z'(M)1 = Z(M).$$

  These results show that F is a *monoidal* natural transformation. □

This result is in a certain sense a uniqueness result. Any four-dimensional topological quantum field theory $G : 4\mathbf{Cob} \to \mathbf{Vect}$ that has

$$G(M) = e^{-\frac{36i}{\Lambda}\pi^2\sigma(M)}$$

for all closed four-manifolds M, is equivalent to four-dimensional BF-theory with cosmological constant Λ, also called Z. We might interpret this as saying that Z is the *only* theory with

$$Z(M) = e^{-\frac{36i}{\Lambda}\pi^2\sigma(M)}$$

for all closed four-manifolds M.

Let us conclude this chapter with the remark that the Crane-Yetter-Broda topological quantum field theory does indeed assign the number $y^{\sigma(M)}$ to each compact oriented 4-manifold M, and hence, is equivalent to 4-dimensional BF theory. For a discussion on this point we refer to reader to [Baez95].

Part II

Magnetic fields and Brownian motion

Chapter 7

Integer quantum Hall effect

In 1985, Klaus von Klitzing was awarded the Nobel prize in physics for his discovery of the integer quantum Hall effect, [KDP80]. An introduction to the theory behind the integer quantum Hall effect will be the subject of this chapter. It is, of course, a very interesting subject in its own right, and moreover, it will be one of the two main ingredients in the theory we wish to develop. There is much that can be said about the IQHE and its variants, like the fractional quantum Hall effect (FQHE), but we will restrict ourselves to just one aspect of the theory, namely the so-called Landau quantization. The interested reader might find a more complete introduction to the subject in Ref. [Goer09], which is also the main reference for this chapter.

7.1 Landau quantization

We start by setting up the problem. Consider a particle of mass m and charge $-e$ constrained to move in the plane. Its position is described by the vector $\mathbf{r}$. A homogeneous magnetic field, of magnitude $B > 0$, is applied perpendicularly to the plane. Note that the force exerted on the particle will always lie in the plane. The magnetic field may be described by a vector potential $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$, such that $\nabla \times \mathbf{A} = \mathbf{B} = B\hat{\mathbf{z}}$. Any function $\mathbf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ with the property that $\nabla \times \mathbf{A} = \mathbf{B}$ is called a vector potential for $\mathbf{B}$. There are many such vector potentials, the selection of any particular one of them is called a choice of gauge. We will make use of two particular choices, namely the symmetric gauge

$$\mathbf{A}(x, y, z) = \frac{B}{2}(-y, x, 0),$$

and the Landau gauge

$$\mathbf{A}(x, y, z) = B(-y, 0, 0).$$

If $\xi : \mathbb{R}^3 \to \mathbb{R}$ is any function, and $\mathbf{A}$ is a vector potential for $\mathbf{B}$, then so is $\mathbf{A}+\nabla\xi$. The assignment $\mathbf{A} \to \mathbf{A} + \nabla\xi$ is called a gauge transformation. In fact, any two vector potentials for $\mathbf{B}$ can be related by a gauge transformation. Using any choice of gauge for $\mathbf{A}$, the dynamics of the particle are governed by the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - e\mathbf{A}\cdot\dot{\mathbf{r}}. \tag{7.1}$$

One might protest that the Lagrangian is not gauge invariant, however, using the chain rule for derivatives, one can show that under a gauge transformation the Lagrangian only changes by a total derivative. The Lagrangian description is not important for us at the moment, but it will turn out to be convenient later. Let us thus pass to the Hamiltonian description. We first compute the canonical momentum

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} - e\mathbf{A}. \tag{7.2}$$

Note that the canonical momentum is not the usual momentum $m\dot{\mathbf{r}}$. In the above computation, we have assumed that the vector potential does not contain any time derivatives of $\mathbf{r}$, an assumption that was already implicit in writing $A : \mathbb{R}^3 \to \mathbb{R}^3$. The Hamiltonian becomes

$$H = \left(\frac{\partial L}{\partial \dot{\mathbf{r}}}\dot{\mathbf{r}} - L\right)\Bigg|_{\dot{\mathbf{r}}=(\mathbf{p}+e\mathbf{A})/m} = \frac{1}{2}\frac{(\mathbf{p}+e\mathbf{A})^2}{m}. \tag{7.3}$$

A striking feature of this system is that the canonical momentum is not gauge invariant. One might conclude that something has gone awfully wrong; however, the Hamiltonian, miraculously, is gauge invariant! So all hope is not lost, and we will, of course, see that in the end everything works out, as long as one promises never to try to *measure* the canonical momentum. An interesting remark is that the Hamiltonian in Eq. (7.3) is exactly the Hamiltonian that one obtains by applying minimal coupling, $\mathbf{p} \to \mathbf{p} + e\mathbf{A}$, to the free particle Hamiltonian, that is

$$\frac{1}{2}\frac{\mathbf{p}^2}{m} \to \frac{1}{2}\frac{(\mathbf{p}+e\mathbf{A})^2}{m}.$$

7.1.1 Classical solution

We are interested in the quantum mechanical solution to the problem described by Eq. (7.3). However, there are some valuable lessons to be learned from the classical picture. We may derive the equations of motion from either the Lagrangian (7.1) or the Hamiltonian (7.3). Either way, the equation of motion is given by the Lorentz force law

$$m\ddot{\mathbf{r}} = -e(\dot{\mathbf{r}} \times \mathbf{B}).$$

We introduce the cyclotron frequency

$$\omega_c := \frac{eB}{m},$$

and write out the equation of motion in components

$$\ddot{\mathbf{r}}_x = -\omega_c \dot{\mathbf{r}}_y,$$
$$\ddot{\mathbf{r}}_y = \omega_c \dot{\mathbf{r}}_x.$$

One may verify that the equations

$$\begin{aligned} \mathbf{r}_x(t) &= C_x - R\sin(\omega_c t + \phi), \\ \mathbf{r}_y(t) &= C_y + R\cos(\omega_c t + \phi), \end{aligned} \tag{7.4}$$

solve the equation of motion, for any $C_x, C_y, R, \phi \in \mathbb{R}$. We see that the particle describes a circular motion of radius R around the point (C_x, C_y). We can express C_x and C_y in terms of $\dot{\mathbf{r}}$ and $\mathbf{r}$ as follows

$$\begin{aligned} C_x &= \mathbf{r}_x(t) - \frac{1}{\omega_c}\dot{\mathbf{r}}_y(t), \\ C_y &= \mathbf{r}_y(t) + \frac{1}{\omega_c}\dot{\mathbf{r}}_x(t). \end{aligned} \tag{7.5}$$

The numbers C_x and C_y are the so-called guiding center coordinates, which will turn out to be quite relevant for the quantum mechanical problem. Their importance might be expected because C_x and C_y are constants of motion, and thus, commute with the Hamiltonian.

7.1.2 Canonical quantization

We now perform canonical quantization by promoting $\mathbf{r}$ and $\mathbf{p}$ to operators, obeying the canonical commutation relations

$$[\mathbf{r}_i, \mathbf{p}_j] = i\hbar\delta_{ij},$$

with the additional condition that all commutators not determined by this equation vanish. We define the gauge-invariant momentum

$$\Pi = \mathbf{p} + e\mathbf{A}.$$

Let us compute the commutator between the different components of Π,

$$\begin{aligned} [\Pi_x, \Pi_y] &= [\mathbf{p}_x + e\mathbf{A}_x, \mathbf{p}_y + e\mathbf{A}_y] \\ &= -ie\hbar\left(\nabla \times A\right)_z \\ &= -ie\hbar B. \end{aligned}$$

It is conventional to introduce the so-called magnetic length

$$l_B = \sqrt{\frac{\hbar}{eB}}.$$

In terms of the magnetic length, the commutator above reads

$$[\Pi_x, \Pi_y] = -i\frac{\hbar^2}{l_B^2}. \tag{7.6}$$

The Hamiltonian (7.3) can be written as

$$H = \frac{\Pi_x^2}{2m} + \frac{\Pi_y^2}{2m}. \tag{7.7}$$

In this way, we see that the system is essentially a harmonic oscillator! We define the analogs of the creation and annihilation operators

$$a = \frac{l_B}{\hbar\sqrt{2}}\left(\Pi_x - i\Pi_y\right), \qquad a^\dagger = \frac{l_B}{\hbar\sqrt{2}}\left(\Pi_x + i\Pi_y\right). \tag{7.8}$$

Up to this point, there were no restrictions on the number e; however, at this point we have assumed that e is positive, if it is not, the expression for $a^\dagger$ would be inconsistent with the expression for a. It is not difficult to modify the theory developed here to the case that e is negative. Nothing essential changes, so for the sake of clarity, let us fix e to be positive from now on. One can show, using Eq. (7.6), that a and $a^\dagger$ satisfy the following relation

$$\left[a, a^\dagger\right] = 1. \tag{7.9}$$

The Hamiltonian (7.3) may be expressed as

$$H = \frac{\hbar^2}{ml_B^2}\left(a^\dagger a + \frac{1}{2}\right). \tag{7.10}$$

We can now provide a set of orthogonal vectors in the Hilbert space of the problem, that we will later extend to a basis of the Hilbert space. We will refer to this set as a partial basis. The partial basis states are indexed by a natural number $n \in \mathbb{N}$ and obey the relations

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \text{and} \quad a|n\rangle = \sqrt{n}|n-1\rangle,$$

and the annihilation operator annihilates the vacuum,

$$a|0\rangle = 0.$$

This basis is a partial basis because we started with a particle in two spatial dimensions, and this basis describes a one dimensional harmonic oscillator. The spectrum of the system described by Eqs. (7.9) and (7.10) is now given by

$$E_n = \frac{\hbar^2}{ml_B^2}\left(n + \frac{1}{2}\right) = \hbar\omega_c\left(n + \frac{1}{2}\right),$$

where $\omega_c = eB/m = \hbar/(ml_B^2)$ is the cyclotron frequency . The space of states corresponding to a fixed $n \in \mathbb{N}$ is called a Landau level.

In the following sections, we will find two distinct ways to characterise the remaining degree of freedom, or equivalently to complete the partial basis constructed above.

7.1.3 Guiding center coordinates

The partial basis defined above diagonalizes the Hamiltonian. Hence, if we want to complete it, we might want to find operators that commute with the Hamiltonian and try to diagonalize those. Now, we know that an operator commutes with the Hamiltonian if and only if it is a constant of motion. The astute reader might notice that this is a remark that we actually have made already! Recall section 7.1.1, where we found the guiding center coordinates, C_x and C_y, which were constants of motion. We are, however, not yet in the position to make use of C_x and C_y, since Eq. (7.5) gives us an expression in terms of $\dot{\mathbf{r}}$, which is surely not admissible in a Hamiltonian setup. We recall Eq. (7.2) and write

$$m\dot{\mathbf{r}} = \mathbf{p} + e\mathbf{A}.$$

This leads to the expressions

$$\begin{aligned} C_x &= \mathbf{r}_x - \frac{\mathbf{p}_y}{m\omega_c} - \frac{e\mathbf{A}_y}{m\omega_c} = \mathbf{r}_x - \frac{\mathbf{p}_y}{eB} - \frac{\mathbf{A}_y}{B} = \mathbf{r}_x - \frac{\Pi_y}{eB}, \\ C_y &= \mathbf{r}_y + \frac{\mathbf{p}_x}{m\omega_c} + \frac{e\mathbf{A}_x}{m\omega_c} = \mathbf{r}_y + \frac{\mathbf{p}_x}{eB} + \frac{\mathbf{A}_x}{B} = \mathbf{r}_y + \frac{\Pi_x}{eB}. \end{aligned} \tag{7.11}$$

As expected, one may verify by direct computation

$$\begin{aligned} [C_x, \Pi_x] &= 0, \qquad [C_x, \Pi_y] = 0, \\ [C_y, \Pi_x] &= 0, \qquad [C_y, \Pi_y] = 0. \end{aligned}$$

Thus, the guiding center operators, C_x and C_y, commute with the Hamiltonian. This implies that we may, in principle, simultaneously diagonalize either C_x or C_y and the Hamiltonian. However, we may not diagonalize both C_x and C_y

$$\begin{aligned} [C_x, C_y] &= \left[\mathbf{r}_x - \frac{\mathbf{p}_y}{eB} - \frac{\mathbf{A}_y}{B}, \mathbf{r}_y + \frac{\mathbf{p}_x}{eB} + \frac{\mathbf{A}_x}{B}\right] \\ &= \frac{i\hbar}{eB} + \frac{i\hbar}{eB} + \frac{i\hbar}{eB^2}\frac{\partial A_x}{\partial y} - \frac{i\hbar}{eB^2}\frac{\partial A_y}{\partial x} \\ &= \frac{i\hbar}{eB} \\ &= il_B^2. \end{aligned}$$

Instead of trying to diagonalize either C_x or C_y – diagonalizing C_x seems somewhat unfair to C_y and vice versa – we notice the similarity of the expression $[C_x, C_y]$ with Eq. (7.6). Analogously to the harmonic oscillator, we define creation and annihilation operators

$$b = \frac{1}{l_B\sqrt{2}}\left(C_x + iC_y\right), \qquad b^\dagger = \frac{1}{l_B\sqrt{2}}\left(C_x - iC_y\right). \tag{7.12}$$

One may then verify that

$$\left[b, b^\dagger\right] = 1.$$

Therefore we may diagonalize $b^\dagger b$, that is, we introduce the set of states indexed by $m \in \mathbb{N}$ that obey

$$b^\dagger b|m\rangle = m|m\rangle.$$

We now have a complete orthonormal basis for the Hilbert space, indexed by two integers, $n, m \in \mathbb{N}$, constructed as follows. We define the vacuum state as the unique vector obeying

$$a|0,0\rangle = b|0,0\rangle = 0. \tag{7.13}$$

Then, all other states are constructed as

$$|n,m\rangle = \frac{\left(a^\dagger\right)^n \left(b^\dagger\right)^m}{\sqrt{(n!)(m!)}}|0,0\rangle. \tag{7.14}$$

Using the commutation relations $[a, a^\dagger] = 1$ and $[b, b^\dagger] = 1$, one may verify the following useful identities

$$\begin{aligned} a^\dagger a|n,m\rangle &= n|n,m\rangle, & b^\dagger b|n,m\rangle &= m|n,m\rangle, \\ a|n,m\rangle &= \sqrt{n}|n-1,m\rangle, & a^\dagger|n,m\rangle &= \sqrt{n+1}|n+1,m\rangle, \\ b|n,m\rangle &= \sqrt{m}|n,m-1\rangle, & b^\dagger|n,m\rangle &= \sqrt{m+1}|n,m+1\rangle. \end{aligned}$$

Note that for any $n, m, m' \in \mathbb{N}$, the two states $|n,m\rangle$ and $|n,m'\rangle$ have the same energy, so the Landau levels are (infinitely) degenerate.

7.1.4 Wave functions

One might be interested in the wave functions corresponding to the system defined by Eqs. (7.13) and (7.14). Formally, we know that the wave functions should be elements of $L^2(\mathbb{R}^2)$, that is, complex-valued square integrable functions of two real variables. We will denote the wave function corresponding to the state $|n,m\rangle$ by $\varphi_{(n,m)}(x,y)$. According to the usual rules of canonical quantization, we may represent the position and momentum operators as

$$\begin{aligned} \mathbf{r}_x &\to x, & \mathbf{r}_y &\to y, \\ \mathbf{p}_x &\to -i\hbar\partial_x, & \mathbf{p}_y &\to -i\hbar\partial_y. \end{aligned} \tag{7.15}$$

Note that the expressions for p_x and p_y do not depend on the gauge potential, $\mathbf{A}$. However, we know that upon a gauge transformation

$$\mathbf{A} \to \mathbf{A} + \nabla\xi, \tag{7.16}$$

the canonical momentum should transform as

$$\mathbf{p} \to \mathbf{p} - e\nabla\xi.$$

Instead of trying to modify the representation of the operators we ask that the wave functions $\varphi \in L^2(\mathbb{R}^2)$ obey a certain transformation behaviour, namely, under the transformation (7.16) the wave function should transform as

$$\varphi \to e^{-e\frac{i}{\hbar}\xi}\varphi. \tag{7.17}$$

Using Eqs. (7.8) and (7.12) for a and b, one may write

$$\begin{aligned} a' &:= \frac{\hbar\sqrt{2}}{l_B}a = \Pi_x - i\Pi_y, \\ b' &:= eBl_B\sqrt{2}b = eB(x+iy) - \Pi_y + i\Pi_x. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{ia'-b'}{2} &= \Pi_y - \frac{eB}{2}(x+iy), \\ \frac{a'-ib'}{2} &= \Pi_x + \frac{eB}{2}(y-ix). \end{aligned}$$

From Eq. (7.13), it now follows that the ground state satisfies the equations

$$\begin{aligned} \left(\Pi_y - \frac{eB}{2}(x+iy)\right)\varphi_{(0,0)} &= 0, \\ \left(\Pi_x + \frac{eB}{2}(y-ix)\right)\varphi_{(0,0)} &= 0. \end{aligned}$$

The canonical quantization rules in Eq. (7.15) tell us that

$$\left(-i\hbar\partial_y + eA_y(x,y) - \frac{eB}{2}(x+iy)\right)\varphi_{(0,0)}(x,y) = 0, \tag{7.18}$$

$$\left(-i\hbar\partial_x + eA_x(x,y) + \frac{eB}{2}(y-ix)\right)\varphi_{(0,0)}(x,y) = 0. \tag{7.19}$$

Eq. (7.18) is solved by

$$\begin{aligned} \varphi_{(0,0)}(x,y) &= f(x)\exp\left\{\frac{1}{i\hbar}\int_0^y \left[eA_y(x,k) - \frac{eB}{2}(x+ik)\right]\mathrm{d}k\right\} \\ &= f(x)\exp\left\{\frac{1}{i\hbar}\left(\int_0^y eA_y(x,k)\mathrm{d}k - \frac{eB}{4}iy^2 - \frac{eB}{2}xy\right)\right\}, \end{aligned} \tag{7.20}$$

where f is an arbitrary function of x, (but independent of y). We substitute this result into Eq. (7.19), divide both sides by $\varphi_{(0,0)}(x,y)$, and obtain

$$i\hbar\frac{\partial_x f(x)}{f(x)} = -\int_0^y e\frac{\partial A_y(x,k)}{\partial x}\mathrm{d}k + eA_x(x,y) + eBy - \frac{eB}{2}ix. \tag{7.21}$$

Because the left-hand side does not depend on y, the right-hand side may not depend on y either. Taking the derivative with respect to y of the right hand side and using the fact that

$$\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B,$$

one sees that the right hand side is indeed independent of y. The differential equation Eq. (7.21) is solved by

$$\begin{aligned} f(x) &= \exp\left\{ \frac{1}{i\hbar} \int_0^x \left[-\int_0^y e\frac{\partial A_y(l,k)}{\partial l} \mathrm{d}k + eA_x(l,y) + eBy - \frac{eB}{2} il \right] \mathrm{d}l \right\} \\ &= \exp\left\{ \frac{1}{i\hbar} \left(-\int_0^y eA_y(x,k)\mathrm{d}k + \int_0^x eA_x(l,y)\mathrm{d}l + eBxy - \frac{eB}{4} ix^2 \right) \right\}. \end{aligned}$$

Finally, we substitute this back into Eq. (7.20), and obtain

$$\varphi_{(0,0)}(x,y) = \exp\left\{ -\frac{eB}{4\hbar}(x^2+y^2) + \frac{eB}{2i\hbar}xy + \int_0^x \frac{eA_x(l,y)}{i\hbar}\mathrm{d}l \right\}.$$

All other states can now be found using Eq. (7.14). Some remarks are now in order. First, the wave functions are not gauge invariant. In fact, under a gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla\xi$, the ground state wave function transforms as

$$\varphi_{(0,0)}(x,y) \to e^{-e\frac{i}{\hbar}\xi(x,y)} \varphi_{(0,0)}(x,y).$$

We see that this solution indeed exhibits the required transformation behaviour, as stated in Eq. (7.17). Next, notice that there is a choice of gauge for which the ground state wave function becomes particularly simple, namely the symmetric gauge, $\mathbf{A}(x,y,z) = B(-y,x,0)/2$. In this case we obtain

$$\begin{aligned} \varphi_{(0,0)}(x,y) &= \exp\left(-\frac{eB}{4\hbar}(x^2+y^2) \right) \\ &= \exp\left(-\frac{1}{4l_B^2}(x^2+y^2) \right). \end{aligned}$$

Using Eq. (7.14) we now compute

$$\varphi_{(0,m)}(x,y) = \left(\frac{x+iy}{l_B} \right)^m \exp\left(-\frac{1}{4l_B^2}(x^2+y^2) \right). \tag{7.22}$$

In Chapter 10 we shall work in the symmetric gauge. Finally, we should say that we have not bothered to normalize our states.

7.1.5 Landau gauge

Up to now we have avoided choosing a gauge. In this section, we will choose one. Specifically, we will work in the Landau gauge, $\mathbf{A}(x,y,z) = B(-y,0,0)$. In this gauge, the gauge-invariant momentum becomes

$$\Pi_x = \mathbf{p}_x - eBy,$$
$$\Pi_y = \mathbf{p}_y.$$

We see that the x-component of the canonical momentum, $\mathbf{p}_x$, commutes with both Π_x and Π_y, and thus with the Hamiltonian, Eq. (7.7). Hence, we may simultaneously diagonalize H and $\mathbf{p}_x$. This choice of gauge is thus particularly natural if we have a potential that breaks translational invariance in the y-direction, but not in the x-direction. The functions, parametrized by $k \in \mathbb{R}$, e^{ikx} diagonalize the action of p_x, i.e. $p_x e^{ikx} = \hbar k e^{ikx}$. Motivated by this fact we take the following ansatz for the eigenfunctions of the Hamiltonian

$$\psi_{n,k}(x,y) = e^{ikx}\chi_{n,k}(y). \tag{7.23}$$

Note that these functions are, in principle, not normalizable. Let us now consider the time-independent Schrödinger equation

$$\frac{\Pi_x^2 + \Pi_y^2}{2m}\psi_{n,k}(x,y) = E_{n,k}\psi_{n,k}(x,y).$$

Which, using the ansatz (7.23), becomes

$$\left[\frac{p_y^2}{2m} + m\frac{\omega_c^2}{2}\left(y - l_B^2 k\right)^2\right]\chi_{n,k}(y) = E_{n,k}\chi_{n,k}(y). \tag{7.24}$$

This is effectively a one-dimensional harmonic oscillator, with angular frequency ω_c, centered at $l_B^2 k$. It follows that $E_{n,k} = \hbar\omega_c\,(n + 1/2)$, as in fact we already knew. Let us define $y_0 := l_B^2 k$. The wave functions thus become

$$\chi_{n,k}(y) = \frac{1}{\sqrt{2^n n!}}\left(\frac{1}{2}\right)^{1/4}\frac{1}{\sqrt{l_B}}e^{-\frac{(y-y_0)^2}{2l_B^2}}\,\mathrm{Her}_n\left(\frac{y-y_0}{l_B}\right),$$

where Her_n are the Hermite polynomials

$$\mathrm{Her}_n(y) = (-1)^n e^{y^2}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\left(e^{-y^2}\right).$$

7.2 Integer quantum Hall effect: phenomenology

In this section we describe the phenomenology of the integer Quantum Hall effect.

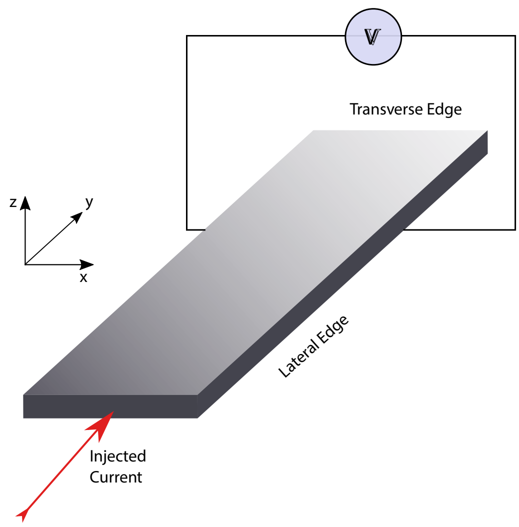


Figure 7.1: Schematic overview of the quantum Hall setup.

We consider a rectangular conductor of negligible thickness. The conductor is imperfect in the sense that there are some impurities in the material, this will turn out to be essential for the effect we are about to describe, as will be explained in section 7.3.4. The length of the conductor is supposed to be much greater than its width. A current is applied along the length of the conductor and a homogeneous magnetic field is applied perpendicularly to the surface of the conductor. One measures the voltage drop across the width of the conductor as in Fig. 7.1.

The resistance corresponding to this voltage drop is called the Hall resistance. We now consider the Hall resistance as a function of the magnetic field. It turns out that the Hall resistance is piece-wise constant as in Fig. 7.2. This effect is called the *integer* quantum Hall effect because the resistance is of the form $2\pi\hbar/(Ne^2) =: R_K/N$, where $N \in \mathbb{N}$. The constant R_K is called the von Klitzing constant, in honor of von Klitzing's discovery of the integer quantum Hall effect.

This effect is rather striking, because it is a macroscopic phenomenon that can not be explained classically. In fact, the straight blue line in Fig. 7.2 shows the classical prediction of the resistance as a function of magnetic field. The classical and the quantum mechanical result agree if the magnetic field takes the form

$$\frac{B}{B_0} = \frac{1}{N}, \quad (N \in \mathbb{Z})$$

where B_0 is some material dependent quantity. This feature will be explained in the sequel.

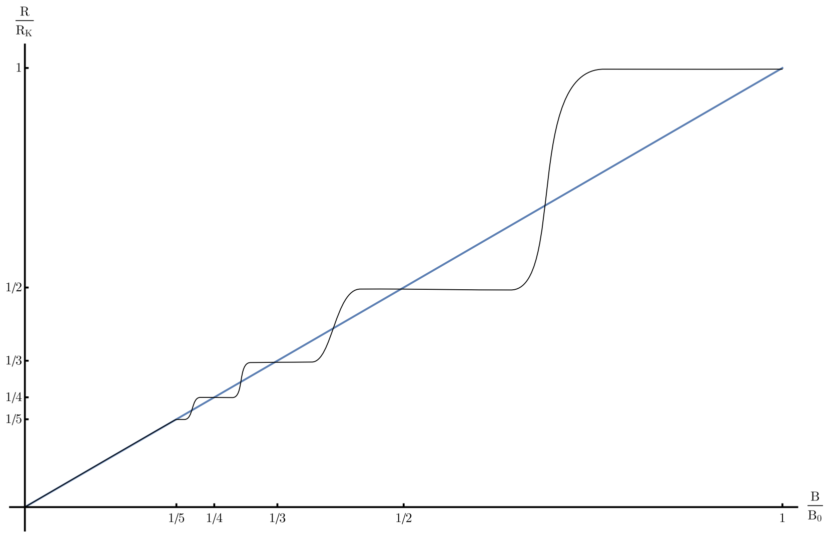


Figure 7.2: The Hall resistance as a function of the magnetic field.

7.3 Integer quantum Hall effect: theory

To accommodate for the quantum Hall effect we need to modify the setup we described in the previous sections in three ways. First, we need to add some confining potential, say in the y-direction, such that we indeed model a long and thin conductor. Second, we should include a disorder potential that describes the impurities of the conductor. Finally, we need to include the effect of the injected current. We include the first two effects by adding the following term to the Hamiltonian, Eq. (7.24),

$$V(x,y) = V_{\text{conf}}(y) + V_{\text{imp}}(x,y),$$

where $V_{\text{conf}}(y)$ is the term that describes the confining potential and $V_{\text{imp}}(x,y)$ the term that describes the impurities, or disorder.

7.3.1 Semi-classical treatment of disorder

In this section we give a semi-classical picture of the effect of disorder on the electrons in the conductor. We assume that the disorder potential V_{imp} varies smoothly and does not contribute to mixing of Landau levels. In this case we may make the approximation

$$V_{\text{imp}}(x,y) \approx V_{\text{imp}}(C_x, C_y), \tag{7.25}$$

see section 7.4.2 for more about this approximation. More concretely, we make the assumption that

$$|\nabla V| \ll \frac{\hbar\omega_c}{l_B}. \tag{7.26}$$

We now consider the Heisenberg equations of motion for the guiding center coordinates in this approximation.

$$\dot{C}_x = \frac{i}{\hbar}[H, C_x] = \frac{i}{\hbar}[V_{\text{imp}}(C_x, C_y), C_x] = \frac{l_B^2}{\hbar}\frac{\partial V_{\text{imp}}(C_x, C_y)}{\partial C_y},$$
$$\dot{C}_y = \frac{i}{\hbar}[H, C_y] = \frac{i}{\hbar}[V_{\text{imp}}(C_x, C_y), C_y] = -\frac{l_B^2}{\hbar}\frac{\partial V_{\text{imp}}(C_x, C_y)}{\partial C_x}.$$

We compute

$$(\langle \dot{C}_x \rangle, \langle \dot{C}_y \rangle) \cdot \langle \nabla V \rangle = 0,$$

or in other words, the expectation value of the velocity of the guiding center coordinates is perpendicular to the expectation value of the gradient of the potential. This means that in the semi-classical picture, the guiding center coordinates of an electron follow the equipotential lines. It follows that an electron can only make it from one edge of the sample to the other if it is on an open equipotential line. From this point on we make the assumption that in the bulk of the conductor all equipotential lines are closed, the only equipotential lines that extend entirely through the conductor are located at the lateral edges. This assumption is generically fulfilled.

7.3.2 Confining potential

In this section we consider the effect of a confining potential on the system. The confining potential models the edge of the conductor, it should be a smooth version of the infinite square well potential. If we forget about the effect of disorder for a moment and again decompose the wave functions as in Eq. (7.23), that is, we write

$$\psi_{n,k}(x, y) = e^{ikx}\chi_{n,k}(y).$$

Then, the effective time-independent Schrödinger equation, (see Eq. (7.24)), for $\chi_{n,k}(y)$ becomes

$$H_k\chi_{n,k}(y) = \left[\frac{p_y^2}{2m} + \frac{1}{2}m\omega_c^2(y - y_0)^2 + V_{\text{conf}}(y)\right]\chi_{n,k}(y) = E_{n,k}\chi_{n,k}(y), \tag{7.27}$$

recall that $y_0(k) = l_B^2 k$. Let us expand the confining potential up to first order,

$$V_{\text{conf}}(y) = V_{\text{conf}}(y_0) + \left.\frac{\partial V_{\text{conf}}}{\partial y}\right|_{y=y_0}(y - y_0) + \mathcal{O}\left(\frac{\partial^2 V_{\text{conf}}}{\partial y^2}\right)(y - y_0)^2.$$

We recall our assumption, Eq. (7.26), that the potential varies slowly, this means that

$$\frac{\partial^2 V_{\text{conf}}}{\partial y^2}, \quad \text{and} \quad \left(\frac{\partial V_{\text{conf}}}{\partial y}\right)^2,$$

are sufficiently small, thus we can write

$$V_{\mathrm{conf}}(y) = V_{\mathrm{conf}}(y_0) + \left.\frac{\partial V_{\mathrm{conf}}}{\partial y}\right|_{y=y_0} (y - y_0).$$

We now define

$$y_0' := y_0 - \frac{1}{m\omega_c^2} \left.\frac{\partial V_{\mathrm{conf}}}{\partial y}\right|_{y=y_0},$$

then, one may verify that

$$\frac{1}{2}m\omega_c^2(y - y_0)^2 + V_{\mathrm{conf}}(y) = \frac{1}{2}m\omega_c^2(y - y_0')^2 + V_{\mathrm{conf}}(y_0).$$

We thus see that the time-independent Schrödinger equation, (see Eq. (7.27)), becomes

$$\left[\frac{p_y^2}{2m} + \frac{1}{2}m\omega_c^2(y - y_0')^2 + V_{\mathrm{conf}}(y_0)\right] \chi_{n,k}(y) = E_{n,k}\chi_{n,k}(y).$$

We conclude that the solution is that $\chi_{n,k}(y)$ describes a harmonic oscillator centered at $y_0'(k)$ with energy

$$E_{n,k}' = \hbar\omega_c \left(n + \frac{1}{2}\right) + V(y_0).$$

This result can be understood intuitively. The magnetic field forces the particles to localize on a length scale l_B, we have assumed that the potential is approximately constant on this length scale. Thus, the result is that the wave functions are not affected by the confining potential, and the energy is shifted by the, approximately constant, confining potential $V(y_0)$. The upshot of this analysis is that the confining potential is fully compatible with the semi-classical treatment of disorder from section 7.3.1.

7.3.3 Conductance of a filled Landau level

In this section we assume that the first n Landau levels are completely filled. We will compute the current induced by a potential difference between the opposing lateral edges of the sample, see Fig. 7.1. The current of the n-th Landau level, (as in Fig. 7.1) is given by

$$I_n^x = -\frac{e}{L}\sum_k \langle n, k|\dot{x}|n, k\rangle,$$

where $|n, k\rangle$ is the state $\langle x, y|n, k\rangle = \psi_{n,k}(x, y) = e^{ikx}\chi_{n,k}(y)$, and L is the length of the sample. Let us recall that $y_0 = kl_B^2$, thus the range of the sum over k is dictated by the width of the sample, let us suppose that it runs from k_{min} to k_{max}. The values that k is allowed to take are dictated by the boundary conditions $\psi_{n,k}(0,y)=0$ and $\psi_{n,k}(L,y)=0$, we will not need the explicit values that k is allowed to take, only the separation between two adjacent ones, which is $\Delta k = 2\pi/L$. Now, we use the Heisenberg equations of motion and compute

$$i\hbar\dot{x} = [x,H] = i\hbar\frac{\partial H}{\partial p_x} = i\frac{\partial H}{\partial k},$$

where in the last step we have used that $p_x|n,k\rangle = \hbar k|n,k\rangle$. It then follows that

$$\langle n,k|\dot{x}|n,k\rangle = \frac{1}{\hbar}\frac{\partial E_{n,k}}{\partial k}.$$

We approximate the derivative

$$\frac{\partial E_{n,k}}{\partial k} \approx \frac{E_{n,k+\Delta k}-E_{n,k}}{\Delta k} = \frac{L}{2\pi}(E_{n,k+\Delta k}-E_{n,k}),$$

which leads to

$$I_n^x = -\frac{e}{\hbar L}\sum_k \frac{\partial E_{n,k}}{\partial k} = -\frac{e}{2\pi\hbar}\sum_k (E_{n,k+\Delta k}-E_{n,k}) = -\frac{e}{2\pi\hbar}(E_{n,k_{\max}}-E_{n,k_{\min}}).$$

This is essentially the discrete version of Stokes' theorem. The difference between the energies is nothing but the Hall voltage

$$E_{n,k_{\max}} - E_{n,k_{\min}} = -eV.$$

So we conclude that the Hall conductance of a single Landau level is

$$\frac{I_n^x}{V} = \frac{e^2}{2\pi\hbar}. \tag{7.28}$$

Comparison of this result to the discussion in section 7.3.1 leads to the conclusion that each Landau level corresponds to exactly one of the open equipotential lines, that are located at the lateral edges. Based on reflection symmetry of the confining potential along the x-axis, one might expect there to be an even number of open equipotential lines, this is however not the case since the magnetic field breaks this symmetry; the electrons are chiral.

7.3.4 Partially filled Landau levels

In this section we will discuss the emergence of the plateaus in the Hall resistance, as depicted in Fig. 7.2. We are thus looking for the resistance, or equivalently the conductance, as a function of the magnetic field. For this we require the so-called filling factor, the integer part of which is the number of completely filled Landau levels, and the fractional part of which

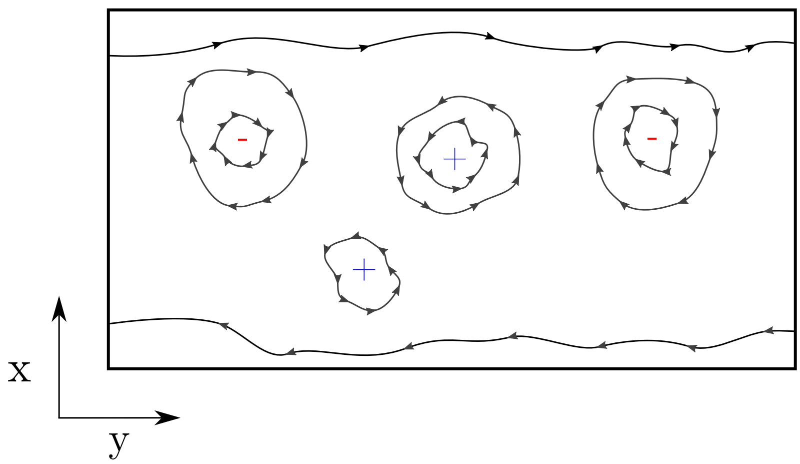


Figure 7.3: Top down view of a part of the conductor. The lines with arrows represent equipotential lines of a typical effective potential.

is the filling fraction of the hightest unfilled Landau level. According to Ref. [Goer09], it is given by

$$\nu = \frac{2\pi\hbar n_{\mathrm{el}}}{eB}, \tag{7.29}$$

where n_{el} is the number of electrons, which we assume to be fixed. We see that if we increase the strength of the magnetic field the number of completely filled Landau levels decreases, thus the conductance should decrease with it. Let us now qualitatively discuss the emergence of plateaus in the Hall conductance. In the previous sections, we have argued that the conductance is due to the existence of open equipotential lines. We have furthermore argued that each Landau level has exactly one open equipotential line available for electron transport. Now suppose that the n-th Landau level is partially filled. The $n-1$ levels beneath it each contribute $e^2/(2\pi\hbar)$ to the conductance. The electrons in the n-th level only have one open equipotential line available, which they may or may not use. If they do, then the total conductance is $ne^2/(2\pi\hbar)$, if they do not, then the total conductance is $(n-1)e^2/(2\pi\hbar)$. This essentially explains the integer quantum Hall effect. Let us give a more intuitive picture of what is happening. We will consider the effective potential seen by an electron in the n-th Landau level, this potential will include the effects of all electrons in lower Landau levels on our special electron, all effects of the material properties and the confining potential. We will assume that this effective potential fulfills the generic condition as described at the end of section 7.3.1. In Fig. 7.3 we have a drawn piece of the conductor. The lines with arrows represent the equipotential lines of the effective potential as seen by an electron in the

n-th Landau level. The arrows indicate the direction of the electron orbits as follows from the analysis in section 7.3.1. A plus sign indicates a local maximum of the effective potential and a minus indicates a local minimum. Let us suppose that we apply an electric field in the y-direction, then we essentially shut off one of the open equipotential lines. Now, imagine that we start in the situation that the n-th Landau level is empty. By turning down the magnetic field we start filling the n-th Landau level. The first electron will look for the lowest equipotential line, which in our picture 7.3 is closed. As we turn down the magnetic field the electrons will keep filling the closed equipotential lines centered around local minima, (in our picture the clockwise oriented ones), until the lowest unoccupied equipotential line is an open one, at which point the conductance jumps. If we keep decreasing the magnetic field the electrons will start filling the closed equipotential lines around local maxima, until the n-th Landau level is filled. At this point the same story holds for the $(n+1)$-th Landau level. We conclude that this semi-classical picture essentially explains the integer quantum Hall effect. We may also conclude that the point at which the conductance jumps, depends crucially on the microscopic properties of the material, but the *existence* of the plateaus is independent of these properties.

7.4 Strong magnetic field

7.4.1 Lagrangian approach

In this section we consider the strong magnetic field limit of the theory described by the Lagrangian (7.1). We will show that this limit is a "topological" limit, in the sense that after taking this limit, the Hamiltonian vanishes. We will work in the Landau gauge, $\mathbf{A}(x, y, z) = (-yB, 0, 0)$. In this case the Lagrangian (7.1) reads

$$L = \frac{1}{2}\dot{\mathbf{r}}^2 + eB\dot{\mathbf{r}}_x\mathbf{r}_y - V(\mathbf{r}_x, \mathbf{r}_y),$$

here, we have included some arbitrary external potential $V(\mathbf{r}_x, \mathbf{r}_y)$. We claim that in the strong magnetic field limit we may discard the kinetic term $m\dot{\mathbf{r}}^2/2$,

$$L = -e\mathbf{A} \cdot \dot{\mathbf{r}} - V(\mathbf{r}). \tag{7.30}$$

This approximation will be valid if

$$\frac{1}{2}m\dot{\mathbf{r}}^2 \ll eB\dot{\mathbf{r}}_x\mathbf{r}_y. \tag{7.31}$$

Let us write $v_0 = \alpha c$, for the typical velocity of an electron, where α is a dimensionless constant. Let us thus write $\dot{\mathbf{r}}_x = \dot{\mathbf{r}}_y = \alpha c$, in which case we obtain

$$\alpha mc \ll eB\mathbf{r}_y. \tag{7.32}$$

If we now set $\mathbf{r}_y = l_B = \sqrt{\hbar/(eB)}$, we get the following criterion

$$\alpha mc \ll \frac{\hbar}{l_B},$$

or, equivalently,

$$l_{\mathrm{C}} \gg \alpha l_B, \tag{7.33}$$

where $l_{\mathrm{C}} = \hbar/(mc)$ is the characteristic length. We conclude that if the characteristic length is far larger than the magnetic length, l_B, times α we may work with the approximate Lagrangian (7.30). For electrons in the lowest conduction band of gallium arsenide at room temperature we have, from Ref. [Blak82],

$$\alpha \approx 0.001 \qquad m \approx 0.06 m_{\mathrm{el}},$$

where m_{el} is the electron mass, (m is the effective mass of electrons in gallium arsenide). Using this data we conclude that the equality $l_C = \alpha l_B$ is satisfied at $B \approx 20$ Tesla.

We compute the canonical momentum corresponding to $\mathbf{r}_x$,

$$\mathbf{p}_x = \frac{\partial L}{\partial \dot{\mathbf{r}}_x} = eB\mathbf{r}_y.$$

We see that the momentum canonically conjugate to the observable $\mathbf{r}_x$ is proportional to $\mathbf{r}_y$. Hence, $\mathbf{r}_x$ and $\mathbf{r}_y$ are the only dynamical variables. Before discarding the kinetic term $m\dot{\mathbf{r}}^2/2$ there were four dynamical variables, $\mathbf{r}_x$, $\mathbf{r}_y$, $\dot{\mathbf{r}}_x$ and $\dot{\mathbf{r}}_y$. Thus, if we discard the kinetic term $m\dot{\mathbf{r}}^2/2$, then the system undergoes dimensional reduction. The approximation we made is thus far from innocent. The Hamiltonian corresponding to the approximate Lagrangian (7.30) reads

$$H = \mathbf{p} \cdot \dot{\mathbf{r}} - L = V(\mathbf{r}).$$

Hence, we see that in the absence of an external potential V the Hamiltonian vanishes identically in this approximation. Upon quantization, $\mathbf{p}_x = eB\mathbf{r}_y$ and $\mathbf{r}_x$ turn into operators obeying the canonical commutation relations

$$[\mathbf{r}_x, \mathbf{r}_y] = i\frac{\hbar}{eB} = il_B^2. \tag{7.34}$$

An important consequence of this commutation relation is that, by the Heisenberg uncertainty principle, we can no longer localize the particle with arbitrary precision. Denoting the variance of $\mathbf{r}_x$ by σ_x and the variance of $\mathbf{r}_y$ by σ_y the uncertainty relation reads

$$\sigma_x \sigma_y \geqslant \frac{l_B^2}{2}.$$

In view of the demand $l_{\mathrm{C}} \gg \alpha l_B$ this is likely a mild requirement, unless the electrons are especially slow. The commutation relation (7.34) is identical to the commutation relation $[C_x, C_y] = il_B^2$. In fact, Eq. (7.5) implies that in the large magnetic field limit the guiding center coordinates coincide with the coordinates of the particle. At this point, a word of warning is in order. Based on Eq. (7.4) one might be inclined to say that $\dot{\mathbf{r}}$ is proportional to ω_c, thus to B, such that in the large magnetic field limit the coordinates C_x and $\mathbf{r}_x$ will not coincide. This reasoning is erroneous, since Eq. (7.4) is a solution to an initial value problem, i.e. one should specify $\dot{\mathbf{r}}$ at some time t_0 and thus $\dot{\mathbf{r}}$ may not depend on B. Furthermore, our description of the problem is non-relativistic, and will thus break down if the velocities become too large.

7.4.2 Projecting onto a Landau level

There is another way to understand the strong magnetic field limit. We recall that the energy associated to the n-th Landau level is $E_n = eB(n + 1/2)/m$, in particular, it is linear in B, so if B is very large we are essentially separating the Landau levels by a large amount. Based on this argument we conclude that taking the strong magnetic field limit is equivalent to projecting onto a Landau level. We recall the basis for the Hilbert space as constructed in Eq. (7.14). Using this notation we see that the operator that projects a wave function onto the n-th Landau level is given by

$$P_n := \sum_{m=0}^{\infty} |n, m\rangle\langle n, m|.$$

One may verify that

$$P_n a P_n = 0, \qquad \text{and} \qquad [P_n, b] = 0. \tag{7.35}$$

Combining Eqs. (7.8) and (7.11) we see that

$$\begin{aligned} \mathbf{r}_x &= C_x + i\frac{l_B}{\sqrt{2}}(a - a^\dagger), \\ \mathbf{r}_y &= C_y - \frac{l_B}{\sqrt{2}}(a + a^\dagger). \end{aligned}$$

Using Eqs. (7.12) and (7.35) one sees that

$$\begin{aligned} P_n \mathbf{r}_x P_n &= C_x P_n = P_n C_x, \\ P_n \mathbf{r}_y P_n &= C_y P_n = P_n C_y. \end{aligned}$$

Using the fact that $[C_x, C_y] = il_B^2$ it follows that

$$[P_n \mathbf{r}_x P_n, P_n \mathbf{r}_y P_n] = il_B^2 P_n.$$

And finally, we may also project the Hamiltonian onto the n-th Landau level

$$P_n H(X, Y) P_n = \hbar\omega_c \left(n + \frac{1}{2}\right) P_n + P_n V(\mathbf{r}_x, \mathbf{r}_y) P_n. \tag{7.36}$$

Here, we see that the approximation (7.25) is equivalent to the approximation that

$$P_n V_{\mathrm{imp}}(\mathbf{r}_x, \mathbf{r}_y) P_n \approx V_{\mathrm{imp}}(P_n \mathbf{r}_x P_n, P_n \mathbf{r}_y P_n) = V_{\mathrm{imp}}(C_x P_n, C_y P_n).$$

Hence, our intuitive argument that the strong magnetic field limit is equivalent to the projection onto a Landau level seems to be correct. Further discussion of this "topological" limit may be found in Ref. [DJ93].

Chapter 8

Quantum Brownian motion

In this Chapter, we study the quantization of the Brownian motion of a single particle in two dimensions. Since the Brownian motion of a particle is dissipative, the system does not admit a Lagrangian or a Hamiltonian description. This fact complicates the quantization of the model, since the methods of quantization that are well understood rely on the existence of a Lagrangian or a Hamiltonian, (i.e. the Feynman path integral or canonical quantization).

We will first give a brief description of the classical theory of Brownian motion. Then, we will describe the independent oscillator model, which is a Hamiltonian model used to quantize Brownian motion. Finally, we will discuss the canonical quantization of the independent oscillator model. This model was extensively discussed by Ford, Lewis and O'Connell (FLO), in a series of papers, of which [FLO88] is the most important one for us.

8.1 Classical Brownian motion

Brownian motion describes the effective random movement of a large (macroscopic) particle in a bath of microscopic particles. It is thus an effective (or statistical) theory, where the macroscopic degree of freedom is the position of the large particle. One might thus see the position of the large particle as analogous to, for example, the temperature or the pressure of a gas in thermodynamics. The time evolution of a particle undergoing Brownian motion is described by the so-called Langevin equation. It is this stochastic differential equation that will eventually guide us through the quantization of Brownian motion. If m is the mass of the particle and $\mathbf{r}(t)$ denotes the probability distribution of the position of the particle at time t, then the Langevin equation is

$$m\ddot{\mathbf{r}} = -\eta\dot{\mathbf{r}} + \mathbf{f}(t). \tag{8.1}$$

Here, $\eta > 0$ is the damping coefficient and $\mathbf{f}(t)$ is the random force exerted on the particle by the bath. We assume that the force exerted on the large particle by the bath consists of a frictional force that is linear to the velocity of the large particle, hence the term $-\eta\dot{\mathbf{r}}$, plus a random force that depends only on the time, hence the term $\mathbf{f}(t)$.

The force $\mathbf{f}(t)$ has a Gaussian probability distribution and obeys

$$\langle \mathbf{f}_\alpha(t)\mathbf{f}_\beta(t')\rangle = 2\eta k_B T \delta_{\alpha\beta}\delta(t-t'), \tag{8.2}$$

where k_B is Boltzmann's constant, T is the temperature, $\delta_{\alpha\beta}$ is Kronecker's delta symbol and $\delta(t-t')$ is the Dirac delta distribution.

Let us give a brief derivation of the fact that $\langle \mathbf{r}^2(t)\rangle \propto t$ for large times t. The general solution to Eq. (8.1) is

$$\mathbf{r}(t) = \mathbf{r}(0) + \frac{m}{\eta}\dot{\mathbf{r}}(0)(1-e^{-\frac{\eta}{m}t}) + \frac{1}{\eta}\int_0^t \left(1-e^{-\frac{\eta}{m}(t-t')}\right)\mathbf{f}(t')\mathrm{d}t'. \tag{8.3}$$

Let us assume that $\mathbf{r}(0) = 0$ and $\dot{\mathbf{r}}(0) = 0$, this assumption is not necessary, but it simplifies the computations. In this case we obtain

$$\begin{aligned}
\langle \mathbf{r}^2(t)\rangle &= \frac{1}{\eta^2}\int_0^t\int_0^t \left(1-e^{-\frac{\eta}{m}(t-t')}\right)\left(1-e^{-\frac{\eta}{m}(t-t'')}\right)\langle \mathbf{f}(t')\cdot\mathbf{f}(t'')\rangle \mathrm{d}t'\mathrm{d}t'' \\
&= \frac{2nk_BT}{\eta}\int_0^t \left(1-e^{-\frac{\eta}{m}(t-t')}\right)^2 \mathrm{d}t' \\
&= \frac{2nk_BT}{\eta}\left(t - m\frac{2-2e^{-\frac{\eta}{m}t}}{\eta} + m\frac{1-e^{-2\frac{\eta}{m}t}}{\eta}\right),
\end{aligned}$$

where n is the number of dimensions in which the particle is allowed to move. We see that for large times t we have

$$\langle \mathbf{r}^2(t)\rangle \approx \frac{2nk_BT}{\eta}t. \tag{8.4}$$

8.2 The independent oscillator model

The idea of the independent oscillator (IO) model is to give a Hamiltonian description of the entire system, that is, including both microscopic and macroscopic degrees of freedom. We can then quantize this total description using any of the usual methods, that is, by Feynman path integrals or canonical quantization. The quantization of this system using Feynman path integrals and influence functionals was extensively studied by Caldeira and Leggett in Ref. [CL83]. The method of canonical quantization was studied by Ullersma in Ref. [Ull66] and by Ford, Lewis and O'Connell in Ref. [FLO88]. After quantization, we assume the microscopic part of the system to be in some known thermodynamical state, and consider only the way in which it affects our macroscopic degree of freedom. We should stress that the microscopic part of the description is not meant to actually reflect the microscopic degrees of freedom of the system. It should only be such that the effective theory for the macroscopic degree of freedom is exactly the theory we want to quantize, in our case, the Langevin equation.

We imagine that our Brownian particle is coupled to an infinite number of particles by ideal springs. The Lagrangian corresponding to this model is

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \sum_i \frac{1}{2}m_i\dot{\mathbf{r}}_i^2 - \sum_i \frac{1}{2}m_i\omega_i^2\left(\mathbf{r}_i - \mathbf{R}\right)^2. \tag{8.5}$$

Here $\mathbf{R}$ is the position of the Brownian particle, $\mathbf{r}_i$ is the position of the i-th particle in the bath, M is the mass of the Brownian particle, m_i is the mass of the i-th particle and ω_i is the frequency corresponding to the spring coupling the i-th particle to the Brownian particle.

Before we quantize this system, we should verify that it does indeed describe Brownian motion at the classical level. The argument presented here is adapted from Ref. [FLO88]. The Euler-Lagrange equations corresponding to the Lagrangian (8.4) are

$$M\ddot{\mathbf{R}} = \sum_i m_i\omega_i^2(\mathbf{r}_i - \mathbf{R}), \tag{8.6}$$

$$\ddot{\mathbf{r}}_i = \omega_i^2(\mathbf{R} - \mathbf{r}_i). \tag{8.7}$$

The solution of Eq. (8.6) reads

$$\mathbf{r}_i(t) = \mathbf{r}_i^h(t) + \mathbf{R}(t) - \int_0^t \cos[\omega_i(t-s)]\dot{\mathbf{R}}(s)\mathrm{d}s, \tag{8.8}$$

where $\mathbf{r}_i^h(t)$ is the solution to the homogeneous equations of motion, i.e. it satisfies

$$\ddot{\mathbf{r}}_i^h(t) + \omega_i^2\mathbf{r}_i^h(t) = 0.$$

An explicit expression for the homogeneous solution is

$$\mathbf{r}_i^h(t) = \mathbf{r}_i(0)\cos(\omega_i t) + \dot{\mathbf{r}}_i(0)\frac{\sin(\omega_i t)}{\omega_i}. \tag{8.9}$$

Combining Eqs. (8.5) and (8.7) leads to

$$M\ddot{\mathbf{R}}(t) = \sum_i m_i\omega_i^2\left\{\mathbf{r}_i^h(t) - \int_0^t \cos[\omega_i(t-s)]\dot{\mathbf{R}}(s)\mathrm{d}s\right\}. \tag{8.10}$$

Now we define

$$\mathbf{f}(t) := \sum_i m_i\omega_i^2\mathbf{r}_i^h(t), \tag{8.11}$$

$$\mu(t-s) := \sum_i m_i\omega_i^2\cos[\omega_i(t-s)]. \tag{8.12}$$

Thus, Eq. (8.9) can be written as

$$M\ddot{\mathbf{R}}(t) + \int_0^t \mu(t-s)\dot{\mathbf{R}}(s)\mathrm{d}s = \mathbf{f}(t). \tag{8.13}$$

Indeed, the term $\mathbf{f}(t)$ is the random force. The function $\mu(t-s)$ is called the memory kernel, it describes the way the system depends on its history. We see that if we have

$$\int_0^t \mu(t-s)\dot{\mathbf{R}}(s)\mathrm{d}s = \eta\dot{\mathbf{R}}(t), \tag{8.14}$$

we have successfully reproduced the Langevin equation, Eq. (8.1), at least in form. We claim that setting

$$m_k = \frac{2\eta}{\pi k}, \quad \text{and} \tag{8.15}$$

$$\omega_k = k, \tag{8.16}$$

indeed gives us (8.13). To show this, let us take the continuum limit of Eq.(8.11), in this case it reads

$$\mu(t-s) = \frac{2\eta}{\pi}\sum_k k\cos[k(t-s)] \longrightarrow \frac{2\eta}{\pi}\int_0^\infty \cos[k(t-s)]\mathrm{d}k = 2\eta\delta(t-s).$$

Substitution of this result into Eq. (8.9) yields

$$M\ddot{\mathbf{R}}(t) + 2\theta(0)\dot{\mathbf{R}}(t) = \mathbf{f}(t),$$

where $\theta(t)$ is the Heaviside theta function defined by

$$\theta(t) := \int_{-1}^t \delta(s)\mathrm{d}s.$$

Now a subtle point arises. There are different possible conventions for the value $\theta(0)$, to decide which one we should use, we should consider the limiting procedure that we used to obtain the Dirac delta distribution. We thus compute

$$\begin{aligned}
\theta(0) &= \int_{-1}^0 \delta(s)\mathrm{d}s \\
&= \frac{1}{\pi}\int_{-1}^0 \mathrm{d}s \int_0^\infty \mathrm{d}k\cos[ks] \\
&= \frac{1}{\pi}\int_0^\infty \mathrm{d}k \int_{-1}^0 \cos[ks] \\
&= \frac{1}{\pi}\int_0^\infty \mathrm{d}k\frac{\sin[k]}{k} \\
&= \frac{1}{2}.
\end{aligned}$$

Hence, the distribution described by Eqs. (8.14) and (8.15) does indeed reproduce the Langevin equation. Now let us show that, under the right assumptions, we also get that $\mathbf{f}$ obeys Eq. (8.2). For notational convenience, we will assume that there is just one spatial dimension, generalization to more spatial dimensions is straightforward. First, let us suppose that we keep the Brownian particle fixed at the origin up to some time $t_0 > 0$. This is a technical assumption, that ensures that we have $\mathbf{r}_i(0) = \mathbf{r}_i^h(0)$ and $\dot{\mathbf{r}}_i(0) = \dot{\mathbf{r}}_i^h(0)$. Furthermore, we assume that at $t = 0$ the harmonic oscillators $\mathbf{r}_k$ are pairwise uncorrelated, and are canonically distributed, at temperature T, with energy

$$\begin{aligned} E_k &= \frac{1}{2}m_k\dot{\mathbf{r}}_k^2 + \frac{1}{2}m_k\omega_k^2\mathbf{r}_k^2 \\ &= \frac{\eta}{\pi k}\dot{\mathbf{r}}_k^2 + \frac{\eta}{\pi}k\mathbf{r}_k^2. \end{aligned}$$

The equipartition theorem then tells us that

$$\langle \mathbf{r}_k(t)\mathbf{r}_{k'}(t)\rangle = \delta_{kk'}\frac{\pi k_B T}{2\eta k}, \qquad (t < t_0), \tag{8.17}$$

$$\langle \dot{\mathbf{r}}_k(t)\dot{\mathbf{r}}_{k'}(t)\rangle = \delta_{kk'}\frac{\pi k k_B T}{2\eta}, \qquad (t < t_0). \tag{8.18}$$

Taking the derivative with respect to time of Eq. (8.16) yields

$$\langle \mathbf{r}_k(t)\dot{\mathbf{r}}_{k'}(t)\rangle = 0, \qquad (t < t_0).$$

We now set out to compute $\langle \mathbf{f}(t)\mathbf{f}(t')\rangle$. First, we take the continuum limit in Eq. (8.10)

$$\mathbf{f}(t) \longrightarrow \frac{2\eta}{\pi}\int_0^\infty \mathbf{r}_k^h(t)\mathrm{d}k.$$

Now, we use Eq. (8.8) and write

$$\begin{aligned} \langle \mathbf{f}(t)\mathbf{f}(t')\rangle =& \frac{4\eta^2}{\pi^2}\int_0^\infty\int_0^\infty \langle \mathbf{r}_k^h(t)\mathbf{r}_{k'}^h(t')\rangle \mathrm{d}k\mathrm{d}k' \\ =& \frac{4\eta^2}{\pi^2}\int_0^\infty\int_0^\infty \Big[\langle \mathbf{r}_k(0)\mathbf{r}_{k'}(0)\rangle \cos(kt)\cos(k't') \\ &+ \langle \dot{\mathbf{r}}_k(0)\dot{\mathbf{r}}_{k'}(0)\rangle \frac{\sin(kt)\sin(k't')}{k^2}\Big]\mathrm{d}k\mathrm{d}k' \\ =& k_B T\frac{2\eta}{\pi}\int_0^\infty \Big[\cos(kt)\cos(kt') + \sin(kt)\sin(kt')\Big]\mathrm{d}k \\ =& k_B T\frac{2\eta}{\pi}\int_0^\infty \cos[k(t-t')]\mathrm{d}k \\ =& 2k_B T\eta\delta(t-t'). \end{aligned}$$

Here we have used that in the continuum limit $\delta_{kk'} \to k\delta(k-k')$.

It is important to note that this derivation should not be seen as a justification of the Langevin equation, but rather, as a verification that the model described by the Lagrangian (8.4) does indeed describe Brownian motion.

8.3 Quantum harmonic oscillators in thermal equilibrium

The goal of this section is to derive analogs of Eqs. (8.16) and (8.17) for the quantum mechanical case. The results are Eqs. (8.20), (8.21) and (8.22) at the end of this section. The derivation presented here is adapted from Ref. [FKM65]. We consider a system of uncoupled harmonic oscillators canonically distributed, at temperature T, with respect to the Hamiltonian H_B given by

$$H_B = \sum_j \left[\frac{p_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 r_j^2 \right].$$

The operators r_j and p_j satisfy the usual commutation relations

$$[r_i, p_j] = i\hbar\delta_{ij}, \quad [r_i, r_j] = [p_i, p_j] = 0.$$

If O is any operator then its expectation value may be computed by

$$\langle O \rangle = \frac{\mathrm{Tr}\{O \exp(-H_B/(kT))\}}{\mathrm{Tr}\{\exp(-H_B/(kT))\}}.$$

We introduce the usual ladder operators

$$a_j = \sqrt{\frac{m_j \omega_j}{2\hbar}} \left(r_j + \frac{i}{m_j \omega_j} p_j \right).$$

The commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0,$$

follow directly from the commutation relations between q and p. We express the usual positions and momentum operators in terms of the ladder operators

$$r_j = \sqrt{\frac{\hbar}{2m_j\omega_j}} (a_j + a_j^\dagger), \tag{8.19}$$

$$p_j = i\sqrt{\frac{m_j\omega_j\hbar}{2}} (a_j^\dagger - a_j). \tag{8.20}$$

Thus, we obtain

$$H_B = \sum_j \hbar\omega_j (a_j^\dagger a_j + \frac{1}{2}).$$

We now use the fact that the eigenvalues of the j-th single particle Hamiltonian $H_{Bj} = \hbar\omega_j(a_j^\dagger a_j + 1/2)$ are given by $E_{Bjn} = \hbar\omega_j(n + 1/2)$. Which leads to

$$\begin{aligned}
\langle a_i^\dagger a_j \rangle &= \delta_{ij} \frac{\mathrm{Tr}\{a_i^\dagger a_j \exp(-H_B/(kT))\}}{\mathrm{Tr}\{\exp(-H_B/(kT))\}} \\
&= \delta_{ij} \frac{\sum_{n=0}^{\infty} n \exp\left[-\frac{\hbar\omega_j}{kT}\left(n + \frac{1}{2}\right)\right]}{\sum_{n=0}^{\infty} \exp\left[-\frac{\hbar\omega_j}{kT}\left(n + \frac{1}{2}\right)\right]} \\
&= \delta_{ij}\left[\exp\left(\frac{\hbar\omega_j}{kT}\right) - 1\right]^{-1}.
\end{aligned}$$

In other words

$$\begin{aligned}
\langle a_i^\dagger a_j \rangle &= \frac{1}{2}\delta_{ij}[\coth\left(\frac{\hbar\omega_j}{2kT}\right) - 1], \\
\langle a_i a_j^\dagger \rangle &= \frac{1}{2}\delta_{ij}[\coth\left(\frac{\hbar\omega_j}{2kT}\right) + 1],
\end{aligned}$$

here we have used the commutation relations for the ladder operators. Using Eqs. (8.18) and (8.19) we obtain the following expressions

$$\langle r_i r_j \rangle = \delta_{ij} \frac{\hbar}{2\omega_j m_j} \coth\left(\frac{\hbar\omega_j}{2kT}\right), \tag{8.21}$$

$$\langle p_i p_j \rangle = \delta_{ij} \frac{1}{2}\hbar\omega_j m_j \coth\left(\frac{\hbar\omega_j}{2kT}\right), \tag{8.22}$$

$$\langle r_i p_j \rangle = \frac{1}{2} i\hbar\delta_{ij}. \tag{8.23}$$

Since harmonic oscillators in multiple dimensions can be seen as uncoupled one dimensional harmonic oscillators, it is straightforward to generalize Eqs. (8.20), (8.21) and (8.22) to multiple dimensions. Let us also note that the expectation values of the position and momentum are equal to zero, i.e. $\langle r_i \rangle = 0$ and $\langle p_i \rangle = 0$.

8.4 Quantum Brownian motion

In this section we take the independent oscillator model, i.e. the Lagrangian (8.4), and perform canonical quantization. The derivation presented here follows very closely Ref. [FLO88]. The Hamiltonian corresponding to the Lagrangian (8.4) is

$$H = \frac{1}{2M}\mathbf{P}^2 + \sum_i \frac{1}{2m_i}\mathbf{p}_i^2 + \sum_i \frac{1}{2} m_i \omega_i^2 (\mathbf{r}_i - \mathbf{R})^2, \tag{8.24}$$

where $\mathbf{P}$ and $\mathbf{p}_i$ are the momenta of the Brownian particle and of the bath respectively. Canonical quantization now tells us that we have the commutation relations

$$[\mathbf{R}, \mathbf{P}] = i\hbar, \quad \text{and} \quad [\mathbf{r}_i, \mathbf{p}_j] = \delta_{ij} i\hbar, \tag{8.25}$$

and all other commutators vanish. We shall work in the Heisenberg picture, i.e. states are independent of time and operators obey the Heisenberg equation

$$\frac{\mathrm{d}}{\mathrm{d}t} A(t) = \frac{i}{\hbar}[H, A(t)].$$

The Heisenberg equations corresponding to the model defined by Eqs. (8.23) and (8.24), thus become

$$\ddot{\mathbf{r}}_j = -\omega_j^2(\mathbf{r}_j - \mathbf{R}), \tag{8.26}$$

$$M\ddot{\mathbf{R}} = \sum_i m_i \omega_i^2 (\mathbf{r}_i - \mathbf{R}). \tag{8.27}$$

Note that these equations are identical to the equations we obtained in the classical case, Eqs. (8.5) and (8.6). The differential equation (8.25) is solved, for $t > 0$, by

$$\mathbf{r}_j(t) = \mathbf{r}_j^h(t) + \omega_j \int_0^t \sin\left[\omega_j(t-s)\right] \mathbf{R}(s) \mathrm{d}s, \tag{8.28}$$

where $\mathbf{r}_j^h(t)$ is the solution to the homogeneous equation $\ddot{\mathbf{r}}(t) + \omega_j^2 \mathbf{r}(t) = 0$, given by

$$\mathbf{r}_j^h(t) = \mathbf{r}_j \cos(\omega_j t) + \mathbf{p}_j \frac{\sin(\omega_j t)}{m_j \omega_j}. \tag{8.29}$$

Here, $\mathbf{r}_j$ and $\mathbf{p}_j$ are the inital conditions of the Heisenberg equation of motion, i.e. they are operators obeying Eq. (8.24), that represent the initial state of the j-th particle. By partial integration of Eq. (8.27) one finds

$$\mathbf{r}_j(t) = \mathbf{r}_j^h(t) + \mathbf{R}(t) - \cos(\omega_j t)\mathbf{R}(0) - \int_0^t \cos[\omega_j(t-s)]\dot{\mathbf{R}}(s)\mathrm{d}s, \tag{8.30}$$

thus, the solution (8.27) is equivalent to the solution (8.8). Substitution of Eq. (8.29) in Eq. (8.26) yields

$$M\ddot{\mathbf{R}}(t) = \sum_i m_i \omega_i^2 \left[\mathbf{r}_i^h(t) - \cos(\omega_i t)\mathbf{R}(0) - \int_0^t \cos[\omega_i(t-s)]\dot{\mathbf{R}}(s)\mathrm{d}s\right]. \tag{8.31}$$

We now reintroduce the operator-valued random force

$$\mathbf{f}(t) := \sum_i m_i \omega_i^2 \mathbf{r}_i^h(t), \tag{8.32}$$

and the memory kernel

$$\mu(t) := \sum_i m_i \omega_i^2 \cos\left[\omega_i(t-s)\right]. \tag{8.33}$$

Thus, we may write Eq. (8.30) as

$$M\ddot{\mathbf{R}}(t) + \mu(t)\mathbf{R}(0) + \int_0^t \mu(t-s)\dot{\mathbf{R}}(s)\mathrm{d}s = \mathbf{f}(t), \tag{8.34}$$

in analogy with Eq. (8.12) we call this equation the quantum- or operator Langevin equation. Now let us take $\mu(t) = 2\eta\delta(t)$, see Eqs. (8.14) and (8.15) for a distribution of masses and frequencies that does this. Then, we obtain

$$M\ddot{\mathbf{R}}(t) + 2\eta\delta(t)\mathbf{R}(0) + \eta\dot{\mathbf{R}}(t) = \mathbf{f}(t). \tag{8.35}$$

Let us discard the term $2\eta\delta(t)\mathbf{R}(0)$, in doing so our solution will only be valid for $t > 0$. For notational convenience we shift our time parameter $t \mapsto t + t_0$ for some time $t_0 > 0$, then we obtain the differential equation

$$M\ddot{\mathbf{R}}(t) + \eta\dot{\mathbf{R}}(t) = \mathbf{f}(t), \qquad t > 0. \tag{8.36}$$

This is exactly the classical Langevin equation, Eq. (8.1), with solution Eq. (8.3), which reads

$$\mathbf{R}(t) = \mathbf{R}(0) + \frac{1}{\eta}\mathbf{P}(0)(1 - e^{-\frac{\eta}{M}t}) + \frac{1}{\eta}\int_0^t \left(1 - e^{-\frac{\eta}{M}(t-t')}\right)\mathbf{f}(t')\mathrm{d}t', \tag{8.37}$$

where we have used the relation

$$M\dot{\mathbf{R}} = \frac{i}{\hbar}[H, \mathbf{R}] = \mathbf{P}. \tag{8.38}$$

There are now two features that are different from the situation in section 8.1. The first, and most important, is that the equation

$$\langle \mathbf{f}_\alpha(t)\mathbf{f}_\beta(t')\rangle = 2\eta k_B T \delta_{\alpha\beta}\delta(t-t'),$$

no longer holds. We will find the correct expression for $\langle \mathbf{f}_\alpha(t)\mathbf{f}_\beta(t')\rangle$ in the sequel. Secondly, we may no longer assume that $\mathbf{R}(0) = 0$ and $\mathbf{P}(0) = 0$, since these initial conditions are inconsistent with the canonical commutation relation. We may not even assume that $\langle \mathbf{R}^2(0)\rangle = 0$ and $\langle \mathbf{P}^2(0)\rangle = 0$, since this would violate the Heisenberg uncertainty principle.

We will denote the anti-commutator using braces, i.e. $\{f(t), f(t')\} = f(t)f(t') + f(t')f(t)$.

For the rest of this section we will suppose that the bath oscillator are canonically distributed with respect to the Hamiltonian H_B, that is, the arguments from section 8.3 hold. Now, let us compute the symmetrized force correlator,

$$\frac{1}{2}\langle\{f(t), f(t')\}\rangle = \frac{1}{2}\sum_{k,l} m_k\omega_k^2 m_l\omega_l^2\langle\{r_k^h, r_l^h\}\rangle.$$

First, we compute $\langle\{r_k^h, r_l^h\}\rangle$, using Eqs. (8.28), (8.20), (8.21) and (8.22),

$$\begin{aligned}
\frac{1}{2}\langle\{r_k^h, r_l^h\}\rangle &= \frac{1}{2}\langle\{r_k\cos(\omega_k t) + p_k\frac{\sin(\omega_k t)}{m_k\omega_k}, r_l\cos(\omega_l t') + p_l\frac{\sin(\omega_l t')}{m_l\omega_l}\}\rangle \\
&= \frac{\delta_{kl}\hbar}{2\omega_k m_k}\left(\cos(\omega_k t)\cos(\omega_k t') + \sin(\omega_k t)\sin(\omega_k t')\right)\coth\left(\frac{\hbar\omega_k}{2k_BT}\right) \\
&= \frac{\delta_{kl}\hbar}{2\omega_k m_k}\cos[\omega_k(t-t')]\coth\left(\frac{\hbar\omega_k}{2k_BT}\right),
\end{aligned}$$

here, we have used that $\langle\{r_k, p_l\}\rangle = 0$. It follows that

$$\frac{1}{2}\langle\{f(t), f(t')\}\rangle = \frac{\hbar}{2}\sum_k m_k\omega_k^3\cos[\omega_k(t-t')]\coth\left(\frac{\hbar\omega_k}{2k_BT}\right).$$

The generalization to multiple dimensions reads

$$\frac{1}{2}\langle\{\mathbf{f}_\alpha(t), \mathbf{f}_\beta(t')\}\rangle = \delta_{\alpha\beta}\frac{\hbar}{2}\sum_k m_k\omega_k^3\cos[\omega_k(t-t')]\coth\left(\frac{\hbar\omega_k}{2k_BT}\right). \tag{8.39}$$

Suppose that the distribution of masses and frequencies of the bath is given by Eqs. (8.14) and (8.15), i.e. $m_k = 2\eta/(\pi k)$, and $\omega_k = k$, in this case we obtain

$$\begin{aligned}
\frac{1}{2}\langle\{\mathbf{f}_\alpha(t), \mathbf{f}_\beta(t')\}\rangle &= \delta_{\alpha\beta}\frac{\hbar\eta}{\pi}\sum_k k^2\cos[k(t-t')]\coth\left(\frac{\hbar k}{2k_BT}\right) \\
&= \delta_{\alpha\beta}\frac{\hbar\eta}{\pi}\int_0^\infty k\cos[k(t-t')]\coth\left(\frac{\hbar k}{2k_BT}\right)\mathrm{d}k \\
&= \eta k_BT\frac{\mathrm{d}}{\mathrm{d}t}\coth\left(\frac{\pi k_BT(t-t')}{\hbar}\right).
\end{aligned} \tag{8.40}$$

Strictly speaking, the integral

$$\int_0^\infty k\cos[k(t-t')]\coth\left(\frac{\hbar k}{2k_BT}\right)\mathrm{d}k,$$

diverges, one can, however, make sense of this expression by declaring it to be the Fourier cosine transform of

$$k\coth\left(\frac{\hbar k}{2k_BT}\right).$$

In the remainder of this section we will show that under the assumption that the distribution of masses and frequencies of the bath is given by Eqs. (8.14) and (8.15), we obtain

$$\langle \mathbf{R}^2(t)\rangle = \frac{nk_BT}{\eta}t,$$

for large times t. Using Eq. (8.36) one sees that

$$\langle \mathbf{R}^2(t)\rangle = \left\langle \left(\mathbf{R}(0) + \frac{1}{\eta}\mathbf{P}(0)(1-e^{-\frac{\eta}{M}t}) + \frac{1}{\eta}\int_0^t \left(1-e^{-\frac{\eta}{M}(t-t')}\right)\mathbf{f}(t')\mathrm{d}t'\right)^2\right\rangle. \tag{8.41}$$

We shall treat, one by one, the terms that one obtains when expanding the square in this expression. Let us suppose that

$$\langle \mathbf{R}(0)\rangle = 0, \quad \text{and} \quad \langle \mathbf{P}(0)\rangle = 0.$$

Furthermore, we define

$$\sigma^2_{\mathbf{R}} := \langle \mathbf{R}^2(0)\rangle, \quad \text{and} \quad \sigma^2_{\mathbf{P}} := \langle \mathbf{P}^2(0)\rangle. \tag{8.42}$$

We assume that at $t = 0$ the operators $\mathbf{R}(t)$ and $\mathbf{P}(t)$ do not act on the part of the Hilbert space belonging to the bath particles, this implies that

$$\langle \mathbf{R}(0)\cdot\mathbf{f}(t)\rangle = 0, \quad \text{and} \quad \langle \mathbf{P}(0)\cdot\mathbf{f}(t)\rangle = 0, \tag{8.43}$$

for all times $t \geqslant 0$. We now compute

$$\begin{aligned}
&\frac{1}{\eta^2}\left\langle \left(\int_0^t \left(1-e^{-\frac{\eta}{M}(t-t')}\right)\mathbf{f}(t')\mathrm{d}t'\right)^2\right\rangle \\
=&\frac{1}{2\eta^2}\int_0^t\int_0^t \left(1-e^{-\frac{\eta}{M}(t-s)}\right)\left(1-e^{-\frac{\eta}{M}(t-s')}\right)\sum_{\alpha=1}^{n}\langle\{\mathbf{f}_\alpha(s),\mathbf{f}_\alpha(s')\}\rangle\mathrm{d}s\mathrm{d}s' \\
=&\frac{nk_BT}{\eta}\int_0^t\int_0^t \left(1-e^{-\frac{\eta}{M}(t-s)}\right)\left(1-e^{-\frac{\eta}{M}(t-s')}\right)\frac{\mathrm{d}}{\mathrm{d}s}\coth\left(\frac{\pi k_BT(s-s')}{\hbar}\right)\mathrm{d}s\mathrm{d}s'.
\end{aligned}$$

This expression is difficult to handle analytically, however, we are interested in the behaviour of this expression only for large times $t \gg M/\eta$, so we approximate

$$\left(1-e^{-\frac{\eta}{M}(t-s)}\right) \approx 1.$$

In this case we obtain

$$\frac{1}{\eta^2}\left\langle \left(\int_0^t \left(1-e^{-\frac{\eta}{M}(t-t')}\right)\mathbf{f}(t')\mathrm{d}t'\right)^2\right\rangle = \frac{nk_BT}{\eta}\int_0^t \coth\left(\frac{\pi k_BT(t-s')}{\hbar}\right)\mathrm{d}s'.$$

This expression diverges, to handle this, we include a regulator $\varepsilon > 0$, and instead compute

$$\begin{aligned}&\frac{nk_BT}{\eta}\int_0^{t-\varepsilon}\coth\left(\frac{\pi k_BT(t-s')}{\hbar}\right)\mathrm{d}s'\\ =&\frac{n\hbar}{\pi\eta}\left[\log\sinh\left(\frac{\pi k_BTt}{\hbar}\right)+\log\operatorname{csch}\left(\frac{\pi k_BT\varepsilon}{\hbar}\right)\right].\end{aligned}$$

For large times t we may use the approximation

$$\begin{aligned}\log\sinh\left(\frac{\pi k_BTt}{\hbar}\right)&=\log\left[\frac{1}{2}\exp\left(\frac{\pi k_BTt}{\hbar}\right)-\frac{1}{2}\exp\left(-\frac{\pi k_BTt}{\hbar}\right)\right]\\ &\approx\log\left[\frac{1}{2}\exp\left(\frac{\pi k_BTt}{\hbar}\right)\right]\\ &=\frac{\pi k_BT}{\hbar}t-\log[2].\end{aligned}$$

We thus conclude that

$$\frac{1}{\eta^2}\left\langle\left(\int_0^t\left(1-e^{-\frac{\eta}{M}(t-t')}\right)\mathbf{f}(t')\mathrm{d}t'\right)^2\right\rangle=\frac{nk_BT}{\eta}t+\lim_{\varepsilon\downarrow 0}C(\varepsilon),\qquad(8.44)$$

where $C(\varepsilon)$ is given by

$$C(\varepsilon)=\frac{n\hbar}{\pi\eta}\log\left[\frac{1}{2}\operatorname{csch}\left(\frac{\pi k_BT\varepsilon}{\hbar}\right)\right].$$

Note that $\lim_{\varepsilon\downarrow 0}C(\varepsilon)$ diverges. It is, however, a constant with respect to t, so we will discard it in what follows. Using Eqs. (8.40), (8.41), (8.42) and (8.43) we conclude that, for $t \gg M/\eta$, we have

$$\begin{aligned}\langle\mathbf{R}^2(t)\rangle&\approx\sigma_{\mathbf{R}}^2+\frac{\sigma_{\mathbf{P}}^2}{\eta^2}+\frac{1}{\eta}\left(\mathbf{R}(0)\cdot\mathbf{P}(0)+\mathbf{P}(0)\cdot\mathbf{R}(0)\right)+\frac{nk_BT}{\eta}t\\ &\approx\frac{nk_BT}{\eta}t,\end{aligned}$$

which is the same as the classical result (**??**), up to a factor of two.

Chapter 9

Quantum Brownian motion in a Landau level

In this Chapter we combine the lessons learned from Chapters 7 and 8, specifically, we will consider the independent oscillator model from section 8.2 in the strong magnetic field limit as described in section 7.4.

9.1 Obtaining the operator Langevin equations

We start from the independent oscillator Lagrangian, Eq. (8.4), which we repeat here

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 + \sum_i \frac{1}{2}m_i\dot{\mathbf{r}}_i^2 - \sum_i \frac{1}{2}m_i\omega_i^2 \left(\mathbf{r}_i - \mathbf{R}\right)^2 ,$$

where the notation is as in section 8.2. We assume that the Brownian particle has charge $-e < 0$ and the bath particles carry no charge. We furthermore assume that the Brownian particle and the bath oscillators are constrained to move in the xy-plane and we apply a homogeneous magnetic field in the z-direction. We obtain the Lagrangian

$$L = \frac{1}{2}M\dot{\mathbf{R}}^2 - e\mathbf{A}\cdot\dot{\mathbf{R}} - V(\mathbf{R}) + \sum_i \frac{1}{2}m_i\dot{\mathbf{r}}_i^2 - \sum_i \frac{1}{2}m_i\omega_i^2 \left(\mathbf{r}_i - \mathbf{R}\right)^2 , \quad (9.1)$$

where we have also included an external potential $V(\mathbf{R})$ that works on the Brownian particle. We now take the strong magnetic field limit and perform canonical quantization as described in section 7.4. Let us summarize the assumptions and results of this procedure. We assume that the Compton length corresponding to the Brownian particle is far greater than the magnetic length corresponding to the Brownian particle. The, now quantum-mechanical, system is described by the Hamiltonian

$$H = V(\mathbf{R}) + \sum_i \frac{\mathbf{p}_i^2}{2m_i} + \sum_i \frac{1}{2}m_i\omega_i^2 \left(\mathbf{r}_i - \mathbf{R}\right)^2 , \quad (9.2)$$

where $\mathbf{p}_i$ is the canonical momentum vector corresponding to the i-th particle, with components denoted by $\mathbf{p}_i = (\mathbf{p}_i^x, \mathbf{p}_i^y)$. Furthermore, the following commutation relations hold,

$$[\mathbf{R}_x, \mathbf{R}_y] = il_B^2, \qquad [\mathbf{r}_i^\alpha, \mathbf{p}_j^\beta] = \delta_{ij}\delta^{\alpha\beta} i\hbar, \tag{9.3}$$

and all other commutators vanish. Because we are working in two spatial dimensions, it is at this point convenient to change our notation somewhat. We shall henceforth write

$$\begin{aligned} X &:= \mathbf{R}_x, & Y &:= \mathbf{R}_y, \\ x_i &:= \mathbf{r}_i^x, & y_i &:= \mathbf{r}_i^y, \\ p_i^x &:= \mathbf{p}_i^x, & p_i^y &:= \mathbf{p}_i^y. \end{aligned}$$

Eqs. (9.2) and (9.3) thus read

$$H = V(X,Y) + \sum_i \left[\frac{(p_i^x)^2}{2m_i} + \frac{1}{2}m\omega_i^2 (x_i - X)^2\right] + \sum_i \left[\frac{(p_i^y)^2}{2m_i} + \frac{1}{2}m\omega_i^2 (y_i - Y)^2\right], \tag{9.4}$$

and

$$[X, Y] = il_B^2, \qquad [x_i, p_j^x] = \delta_{ij} i\hbar, \qquad [y_i, p_j^y] = \delta_{ij} i\hbar.$$

We will work in the Heisenberg picture, like in section 8.4, the operators will depend on time and the states will be time-independent. The equations of motion for the bath read

$$\begin{aligned} \ddot{x}_i + \omega_i^2 x_i &= \omega_i^2 X, \\ \ddot{y}_i + \omega_i^2 y_i &= \omega_i^2 Y. \end{aligned} \tag{9.5}$$

These equations are solved by

$$\begin{aligned} x_i &= x_i^h(t) + X(t) - \cos(\omega_i t)X(0) - \int_0^t \cos[\omega_i (t-s)]\dot{X}(s)\mathrm{d}s, \\ y_i &= y_i^h(t) + Y(t) - \cos(\omega_i t)Y(0) - \int_0^t \cos[\omega_i (t-s)]\dot{Y}(s)\mathrm{d}s, \end{aligned} \tag{9.6}$$

where

$$x_i^h(t) = x_i \cos(\omega_i t) + p_i^x \frac{\sin(\omega_i t)}{m_i \omega_i}, \tag{9.7}$$

and similarly for $y_i^h(t)$. See section 8.4, in particular the argument leading to Eq. (8.29) for more details. The equations of motion for the Brownian particle read

$$\begin{aligned} \dot{X} &= \frac{l_B^2}{\hbar}\frac{\partial V}{\partial Y} + \frac{l_B^2}{\hbar}\sum_i m_i \omega_i^2 (y_i - Y), \\ \dot{Y} &= -\frac{l_B^2}{\hbar}\frac{\partial V}{\partial X} - \frac{l_B^2}{\hbar}\sum_i m_i \omega_i^2 (x_i - X). \end{aligned} \tag{9.8}$$

Before we continue with the derivation of the operator Langevin equations, we introduce some convenient, and physically meaningful, notation. First,

$$U_\alpha(t) := \frac{l_B^2}{\hbar}\sum_j m_j\omega_j^2\left[\mathbf{r}_j^\alpha\cos(\omega_j t) + \mathbf{p}_j^\alpha\frac{\sin(\omega_j t)}{m_j\omega_j}\right] \tag{9.9}$$

defines a random velocity field. The statistics of this random field is determined solely by the state of the oscillator bath. Second, we define the memory kernel

$$\mu(t) := \frac{l_B^2}{\hbar}\sum_i m_i\omega_i^2\cos[\omega_i t]. \tag{9.10}$$

Substitution of Eq. (9.6) into Eq. (9.8) yields

$$\begin{aligned}\dot{X}(t) - \frac{l_B^2}{\hbar}\frac{\partial V}{\partial Y}(t) &= U_y(t) - \mu(t)Y(0) - \int_0^t \mu(t-s)\dot{Y}(s)\mathrm{d}s,\\ \dot{Y}(t) + \frac{l_B^2}{\hbar}\frac{\partial V}{\partial X}(t) &= -U_x(t) + \mu(t)X(0) + \int_0^t \mu(t-s)\dot{X}(s)\mathrm{d}s.\end{aligned} \tag{9.11}$$

These equations are the operator Langevin equations corresponding to a charged particle in a very strong magnetic field.

9.2 Solving the operator Langevin equations

9.2.1 Thermal statistics of the velocity field

In the previous Chapter we have seen that the inequal time symmetric correlator of the random force played an essential role in determining quantities of physical interest. Thus, we devote this section to the inequal time symmetric correlator of the random velocity U, which now plays the role of random force. We assume that the bath oscillators are canonically distributed, at temperature T, with respect to the Hamiltonian

$$H_B = \sum_j\left[\frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2 r_j^2\right], \tag{9.12}$$

that is, the assumptions made in section 8.3 hold. Under these assumptions, one may show, using Eqs. (8.20), (8.21) and (8.22) that the following equation holds

$$\frac{1}{2}\langle U_\alpha(t)U_\beta(t')+U_\beta(t')U_\alpha(t)\rangle = \delta_{\alpha\beta}\frac{l_B^4}{2\hbar}\sum_i m_i\omega_i^3\cos[\omega_i(t-t')]\coth\left(\frac{\hbar\omega_i}{2k_BT}\right). \tag{9.13}$$

9.2.2 Memory-free system in the topological limit

In this section, we will consider the case that the system has no memory, that is, it is Markovian, that is, $\mu(t) = 2\gamma\delta(t)$, here γ is some dimensionless constant. We will furthermore set $V = 0$. We will show that in the limit of large time t we have

$$\langle X^2(t)\rangle = \frac{\gamma l_B^2 k_B T}{\hbar(1+\gamma^2)} t. \tag{9.14}$$

Setting $V = 0$ in Eq. (9.11) yields

$$\dot{X}(t) = U_y(t) - \mu(t)Y(0) - \int_0^t \mu(t-s)\dot{Y}(s)\mathrm{d}s, \tag{9.15}$$

$$\dot{Y}(t) = -U_x(t) + \mu(t)X(0) + \int_0^t \mu(t-s)\dot{X}(s)\mathrm{d}s. \tag{9.16}$$

Next, we assume that

$$\mu(t-s) = \frac{l_B^2}{\hbar}\sum_k m_k\omega_k^2 \cos[\omega_k(t-s)] = 2\gamma\delta(t-s). \tag{9.17}$$

This is essentially the assumption that the system has no memory, or that it is Markovian. A particular distribution of frequencies and masses that has this property is

$$\omega_k = k, \quad m_k = \frac{2}{k}\frac{\gamma\hbar}{\pi l_B^2}.$$

Comparison with Eqs. (8.14) and (8.15) gives us that γ is related to the usual friction constant η by

$$\gamma = \eta\frac{l_B^2}{\hbar}, \tag{9.18}$$

this is just a consequence of the fact that we have absorbed the factor $l_B^2/\hbar$ into our definition of $\mu(t)$ for notational convenience. We now obtain

$$\dot{X}(t) + \gamma\dot{Y}(t) = U_y(t) - \delta(t)\gamma 2Y(0), \tag{9.19}$$

$$\dot{Y}(t) - \gamma\dot{X}(t) = -U_x(t) + \delta(t)\gamma 2X(0). \tag{9.20}$$

To solve these equations, we first cast them in matrix form and solve for $\dot{X}$ and $\dot{Y}$. This yields

$$\begin{pmatrix} 1 & \gamma \\ -\gamma & 1 \end{pmatrix}\begin{pmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{pmatrix} = \begin{pmatrix} U_y(t) - \delta(t)\gamma 2Y(0) \\ -U_x(t) + \delta(t)\gamma 2X(0) \end{pmatrix}, \tag{9.21}$$

We obtain the solution of Eq. (9.21):

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \frac{\gamma(U_x(t)-\delta(t)\gamma 2X(0))+U_y(t)-\delta(t)\gamma 2Y(0)}{\gamma^2+1} \\ \frac{-U_x(t)+\delta(t)\gamma 2X(0)+\gamma(U_y(t)-\delta(t)\gamma 2Y(0))}{\gamma^2+1} \end{pmatrix}. \tag{9.22}$$

To find X and Y as a function of t, we just need to integrate this result and add constants, O_X and O_Y,

$$X(t) = \frac{1}{\gamma^2+1}\int_0^t [U_y(s)+\gamma U_x(s)]\,\mathrm{d}s - \gamma 2\frac{Y(0)+\gamma X(0)}{\gamma^2+1} + O_X \tag{9.23}$$

$$Y(t) = \frac{1}{\gamma^2+1}\int_0^t [-U_x(s)+\gamma U_y(s)]\,\mathrm{d}s + \gamma 2\frac{X(0)-\gamma Y(0)}{\gamma^2+1} + O_Y. \tag{9.24}$$

Requiring consistency of Eqs. (9.23) and (9.24) at $t = 0$ yields

$$X(t) = \frac{1}{\gamma^2+1}\int_0^t [U_y(s)+\gamma U_x(s)]\,\mathrm{d}s + X(0) \tag{9.25}$$

$$Y(t) = \frac{1}{\gamma^2+1}\int_0^t [-U_x(s)+\gamma U_y(s)]\,\mathrm{d}s + Y(0). \tag{9.26}$$

Let us recall that the operators $U_x(t)$ and $U_y(t)$ only work on the part of the Hilbert space belonging to the bath. This is the reason we did not absorb the terms $\cos(\omega_j^x t)X(0)$ and $\cos(\omega_j^y t)Y(0)$ into the homogeneous part of the solutions, since we would have had to redefine U_x and U_y to absorb these terms as well, and this would have invalidated the statement above. Next, we assume that the initial state is factorizable and that the operators $X(0)$ and $Y(0)$ only work on the part of the Hilbert space belonging to the Brownian particle. This implies that $\langle X(0)U_\alpha(t)\rangle = \langle X(0)\rangle\langle U_\alpha(t)\rangle$ and $\langle Y(0)U_\alpha(t)\rangle = \langle Y(0)\rangle\langle U_\alpha(t)\rangle$, for all t. Let us set $\langle X(0)\rangle = \langle Y(0)\rangle = 0$ and assume that the bath is canonically distributed with respect to the free Hamiltonian H_B, as in section 9.2.1. It then follows that $\langle U_x(t)\rangle = \langle U_y(t)\rangle = 0$, and thus that $\langle U_x(t)U_y(t')\rangle = 0$.

We now compute the symmetrized expectation value

$$\begin{aligned}&\frac{\langle X(t)X(t')+X(t')X(t)\rangle}{2} \\ &= \frac{1}{2(\gamma^2+1)^2}\int_0^t \mathrm{d}s\int_0^{t'}\mathrm{d}s'\Big\langle U_y(s)U_y(s')+U_y(s')U_y(s)+\gamma^2 U_x(s)U_x(s')+\gamma^2 U_x(s')U_x(s)\Big\rangle \\ &\quad + \langle X(0)^2\rangle.\end{aligned} \tag{9.27}$$

We recall Eq. (9.13), which, with our current choice of distribution for the oscillator masses and frequencies, reads

$$\begin{aligned}\frac{1}{2}\langle U_\alpha(t)U_\beta(t')+U_\beta(t')U_\alpha(t)\rangle &= \delta_{\alpha\beta}\gamma\frac{l_B^2}{\pi}\sum_k k^2\cos[k(t-t')]\coth\left(\frac{\hbar k}{2k_BT}\right) \\ &= \delta_{\alpha\beta}\gamma\frac{l_B^2}{\pi}\int_0^\infty k\cos[k(t-t')]\coth\left(\frac{\hbar k}{2k_BT}\right)\mathrm{d}\omega \\ &= \delta_{\alpha\beta}\gamma k_BT\frac{l_B^2}{\hbar}\frac{\mathrm{d}}{\mathrm{d}t}\coth\left(\frac{\pi k_BT(t-t')}{\hbar}\right).\end{aligned} \tag{9.28}$$

Strictly speaking, the integral

$$\int_0^\infty k\cos[k(t-t')]\coth\left(\frac{\hbar k}{2k_BT}\right)\mathrm{d}\omega \tag{9.29}$$

diverges. However, one can make sense of this expression nonetheless, by declaring it to be the Fourier cosine transform of the function $k\coth(\hbar k/(2k_BT))$, in which case one obtains the result above. Hence Eq. (9.27) becomes

$$\frac{1}{2}\langle X(t)X(t')+X(t')X(t)\rangle = \frac{\gamma k_BTl_B^2}{\hbar(1+\gamma^2)}\int_0^{t'}\mathrm{d}s'\coth\left(\frac{\pi k_BT(t-s')}{\hbar}\right)+\langle X(0)^2\rangle. \tag{9.30}$$

We perform the integral in Eq. (9.30) for $0<t'<t$, and obtain

$$\begin{aligned}&\frac{1}{2}\langle X(t)X(t')+X(t')X(t)\rangle\\ =&\frac{\gamma l_B^2}{\pi(1+\gamma^2)}\log\left[\operatorname{csch}\left(\frac{\pi(t-t')Tk_B}{\hbar}\right)\sinh\left(\frac{\pi tTk_B}{\hbar}\right)\right]+\langle X(0)^2\rangle.\end{aligned}$$

We would like to compute $\langle X^2(t)\rangle$ to compare this result to the well-known result $\langle x^2(t)\rangle\propto t$ for Brownian motion. However, our current expression is not valid for $t=t'$. Hence, we set $t'=t-\varepsilon$, where ε is some small positive time. In this case, we obtain

$$\begin{aligned}&\frac{1}{2}\langle X(t)X(t-\varepsilon)+X(t-\varepsilon)X(t)\rangle\\ &=C(\varepsilon)+\frac{\gamma l_B^2}{\pi(1+\gamma^2)}\log\left[\sinh\left(\frac{\pi tTk_B}{\hbar}\right)\right]+\langle X(0)^2\rangle,\end{aligned} \tag{9.31}$$

where $C(\varepsilon)$ does not depend on t and is given by

$$C(\varepsilon)=\frac{\gamma l_B^2}{\pi(1+\gamma^2)}\log\left[\operatorname{csch}\left(\frac{\pi\varepsilon Tk_B}{\hbar}\right)\right]. \tag{9.32}$$

For large times t, we obtain

$$\begin{aligned}\log\left[\sinh\left(\frac{\pi tTk_B}{\hbar}\right)\right]&=\log\left[\frac{1}{2}\exp\left(\frac{\pi tTk_B}{\hbar}\right)+\frac{1}{2}\exp\left(-\frac{\pi tTk_B}{\hbar}\right)\right]\\ &\approx\log\left[\frac{1}{2}\exp\left(\frac{\pi tTk_B}{\hbar}\right)\right]\\ &=\frac{\pi Tk_B}{\hbar}t-\log[2],\end{aligned}$$

which leads to

$$\frac{1}{2}\langle X(t)X(t-\varepsilon)+X(t-\varepsilon)X(t)\rangle=\frac{\gamma l_B^2Tk_B}{\hbar(1+\gamma^2)}t+\langle X(0)^2\rangle+C^\vee(\varepsilon), \tag{9.33}$$

where

$$C^{\vee}(\varepsilon) = \frac{\gamma}{\pi(1+\gamma^2)} \log\left[\frac{1}{2}\text{csch}\left(\frac{\pi\varepsilon T k_B}{\hbar}\right)\right]. \tag{9.34}$$

Discarding the infinite term $\lim_{\varepsilon\downarrow 0} C^{\vee}(\varepsilon)$, we see that for large times t we have

$$\langle X^2(t)\rangle = \frac{\gamma l_B^2 T k_B}{\hbar(1+\gamma^2)} t.$$

Using Eq. (9.18), relating γ and η we obtain

$$\langle X^2(t)\rangle = \frac{k_B T}{\eta + \frac{\hbar^2}{l_B^4}\frac{1}{\eta}} t.$$

In the case that the friction constant η is large, that is, $\eta \gg \hbar/l_B^2$, we obtain

$$\langle X^2(t)\rangle \approx \frac{k_B T}{\eta} t, \tag{9.35}$$

This result is the same as the classical result without a magnetic field Eq. (**??**), which is the same as the quantum mechanical result without a magnetic field.

If the friction constant is small, that is, $0 < \eta \ll \hbar/l_B^2$, we see that

$$\langle X^2(t)\rangle \approx \frac{l_B^4 k_B T \eta}{\hbar^2} t = \frac{k_B T \eta}{(eB)^2} t, \tag{9.36}$$

which is in stark contrast with the classical result. We see that if η is small, the magnetic field suppresses fluctuations.

Chapter 10

Hall conductance as topological invariant

In this Chapter we give an alternative derivation of the expression for the Hall conductance of a single Landau level, Eq (7.28). Our main reference for this Chapter is Ref. [Kohm85].

10.1 Generalized Bloch waves

We consider a setup as described in section 7.1, but we include an external potential, $V(x,y)$, such that the time-independent Schrödinger equation becomes

$$H\psi(x,y) = \left[\frac{1}{2m}\left(\mathbf{p}+e\mathbf{A}\right)^2 + V(x,y)\right]\psi(x,y) = E\psi(x,y). \tag{10.1}$$

The potential is assumed to be (a,b)-periodic, that is,

$$V(x,y) = V(x+a,y) = V(x,y+b).$$

The system is invariant under a translation by a along the x-direction or by b along the y-direction, the Hamiltonian, however, is not. If the Hamiltonian was invariant under these translations we could apply Bloch's theorem. We will state and prove Bloch's theorem in this section, however, let us first give some useful definitions. We will specialize our definitions to the case of a system in two dimensions that is translation invariant under translations by $\mathbf{a} = a\hat{\mathbf{x}}$ and $\mathbf{b} = b\hat{\mathbf{y}}$. The Bravais lattice is the lattice, in $\mathbb{R}^2$, of vectors of the form $\mathbf{R} = n_x\mathbf{a} + n_y\mathbf{b}$ for $n_x, n_y \in \mathbb{Z}$.

Definition 10.1.1 (Translation Operator)**.** Let $\mathbf{R}$ be a Bravais lattice vector, then the *translation operator along* $\mathbf{R}$ is the operator $T_{\mathbf{R}}$, that, given a smooth function $f(\mathbf{r})$ shifts it by $\mathbf{R}$. That is,

$$T_{\mathbf{R}}f(\mathbf{r}) = f(\mathbf{r}+\mathbf{R}). \tag{10.2}$$

Each translation operator has an explicit form

$$T_{\mathbf{R}} = \exp\left[\frac{i}{\hbar}\mathbf{R}\cdot\mathbf{p}\right], \tag{10.3}$$

where $\mathbf{p}$ is the canonical momentum. One may verify that this explicit form indeed works on smooth functions, by a Taylor expansion and an application of the binomial theorem.

In the sequel all translation operators will be assumed to be translation operators along some Bravais lattice vector.

Theorem 10.1.2 (Bloch's theorem)**.** *Let H be a Hamiltonian invariant under all translations by Bravais lattice vectors, that is, H commutes with $T_{\mathbf{R}}$ for all Bravais lattice vectors $\mathbf{R}$. Suppose furthermore that there is a basis for the Hilbert space of the problem that diagonalizes the Hamiltonian. Then, there is a complete basis, $\psi_{n,}(\mathbf{r})$, for the Hilbert space of the problem with the following properties.*

- *Each basis vector is an eigenvector of the Hamiltonian,*

$$H\psi_n = E_n\psi_n.$$

- *Each basis vector $\psi_n(\mathbf{r})$ can be decomposed as*

$$\psi_n(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}u_{n,\mathbf{k}}(\mathbf{r}),$$

where $u_{\mathbf{k}}(\mathbf{r})$ is (a,b)-periodic and $\mathbf{k} \in \mathbb{R}^2$.

The theorem above is really a statement about eigenvectors of translation operators, formalized in the following lemma.

Lemma 10.1.3 (Diagonalizing translation operators)**.** *If $\psi(\mathbf{r})$ is a simultaneous eigenvector of the translation operator $T_{\mathbf{R}}$ for all Bravais lattice vectors $\mathbf{R}$, then $\psi(\mathbf{r})$ may be written as*

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}u_{\mathbf{k}}(\mathbf{r}),$$

where $u_{\mathbf{k}}(\mathbf{r})$ is (a,b)-periodic and $\mathbf{k} \in \mathbb{R}^2$.

Proof. From the explicit form of the translation operators, Eq. (10.3), we see that each translation operator is unitary, thus any eigenvalue of a translation operator is a complex number of modulus one. Now, suppose that $\psi(\mathbf{r})$ is a simultaneous eigenvector of all translation operators, thus, in particular $\psi(\mathbf{r})$ is an eigenvector of the translations along the basis vector $\mathbf{a}$ and $\mathbf{b}$, thus

$$\begin{aligned}\psi(\mathbf{r}+\mathbf{a}) &= T_{\mathbf{a}}\psi(\mathbf{r}) = e^{i\mathbf{k}_x}\psi(\mathbf{r}),\\ \psi(\mathbf{r}+\mathbf{b}) &= T_{\mathbf{b}}\psi(\mathbf{r}) = e^{i\mathbf{k}_y}\psi(\mathbf{r}),\end{aligned} \tag{10.4}$$

for some real numbers $\mathbf{k}_x$ and $\mathbf{k}_y$. We define

$$u_{\mathbf{k}}(\mathbf{r}) := e^{-i\mathbf{k}\cdot\mathbf{r}}\psi(\mathbf{r}) = e^{-i\mathbf{k}_x x - i\mathbf{k}_y y}\psi(\mathbf{r})$$

It follows that, for any Bravais lattice vector $\mathbf{R}$, we have

$$\begin{aligned} u_{\mathbf{k}}(\mathbf{r}+\mathbf{R}) &= T_{\mathbf{R}} u_{\mathbf{k}}(\mathbf{r}) \\ &= T_{\mathbf{R}}(e^{-i\mathbf{k}\cdot\mathbf{r}}\psi(\mathbf{r})) \\ &= e^{-i\mathbf{k}\cdot\mathbf{r}-i\mathbf{k}\cdot\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}}\psi(\mathbf{r}) \\ &= u_{\mathbf{k}}(\mathbf{r}). \end{aligned}$$

□

Bloch's theorem is now a straightforward consequence.

Proof of Bloch's theorem. If the Hamiltonian commutes with all translation operators, then there exists a basis, ψ_n, for the Hilbert space of the problem that diagonalizes both the Hamiltonian and all translation operators. Now, apply Lemma 10.1.3 to this basis. □

Bloch's theorem is a powerful tool in the study of systems with translational symmetry. However, the vector potential $\mathbf{A}$ does not have translational symmetry, even though the magnetic field it produces is homogeneous. Hence, we are not quite in the position to use Bloch's theorem, since the Hamiltonian in Eq. (10.1), given by

$$H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2 + V(x, y),$$

does not commute with the translation operators. Let us instead consider the so-called magnetic translation operators. The magnetic translation operators are most generally defined in terms of the guiding center coordinates, so let us recall their definition from 7.1.3 here,

$$C_x = \mathbf{r}_x - \frac{\mathbf{p}_y}{eB} - \frac{\mathbf{A}_y}{B}, \tag{10.5}$$

$$C_y = \mathbf{r}_y + \frac{\mathbf{p}_x}{eB} + \frac{\mathbf{A}_x}{B}. \tag{10.6}$$

We collect these operators in the vector $\mathbf{C} := (C_x, C_y)$. Let us furthermore recall the definition of the magnetic length from section 7.1.2

$$l_B^2 = \frac{\hbar}{eB}.$$

Definition 10.1.4 (Magnetic Translation Operators)**.** Let $\mathbf{R}$ be a Bravais lattice vector, then the *magnetic translation operator along* $\mathbf{R}$, denoted $\hat{T}_{\mathbf{R}}$, is defined by the formula

$$\hat{T}_{\mathbf{R}} = \exp\left[\frac{i}{\hbar} e\mathbf{R} \cdot (\mathbf{B} \times \mathbf{C})\right].$$

One may show that in our case, that is, when $\mathbf{B} = (0, 0, B)$, it follows straightforwardly from the definition that

$$\hat{T}_{\mathbf{R}} = \exp\left[\frac{i}{l_B^2}\left(\mathbf{R}_x C_y - \mathbf{R}_y C_x\right)\right]. \tag{10.7}$$

It is clear that the magnetic translation operators commute with the kinetic part of the Hamiltonian, since the operators C_x and C_y do, and in fact they also commute with the periodic potential, as we will show below. It is most easily seen in the symmetric gauge,

$$\mathbf{A} = (\mathbf{B} \times \mathbf{r})/2 = B(-y, x, 0)/2, \tag{10.8}$$

which we will use in the rest of this section. This choice makes the magnetic translation operators especially simple. We can rewrite the magnetic translation operators using the expressions for the guiding center operators, Eq. (10.5), the expression for the magnetic translation operator, Eq. (10.7), and the explicit epxressions for the symmetric gauge, Eq. (10.8),

$$\hat{T}_{\mathbf{R}} = \exp\left[\frac{i}{\hbar}\left(\mathbf{R}_x \mathbf{p}_x + \mathbf{R}_y \mathbf{p}_y\right) + \frac{1}{2}\frac{i}{l_B^2}\left(\mathbf{R}_x \mathbf{r}_y - \mathbf{R}_y \mathbf{r}_x\right)\right].$$

It follows from the identity

$$[\mathbf{R}_x \mathbf{p}_x + \mathbf{R}_y \mathbf{p}_y, \mathbf{R}_y \mathbf{r}_x - \mathbf{R}_x \mathbf{r}_y] = 0$$

that

$$\begin{aligned}
\hat{T}_{\mathbf{R}} &= \exp\left[\frac{i}{\hbar}\mathbf{R} \cdot \mathbf{p}\right] \exp\left[\frac{1}{2}\frac{i}{l_B^2}\left(\mathbf{R}_x \mathbf{r}_y - \mathbf{R}_y \mathbf{r}_x\right)\right] \\
&= T_{\mathbf{R}} \exp\left[\frac{1}{2}\frac{i}{l_B^2}\left(\mathbf{R}_x \mathbf{r}_y - \mathbf{R}_y \mathbf{r}_x\right)\right] \\
&= \exp\left[\frac{1}{2}\frac{i}{l_B^2}\left(\mathbf{R}_x \mathbf{r}_y - \mathbf{R}_y \mathbf{r}_x\right)\right] T_{\mathbf{R}}.
\end{aligned}$$

We have assumed that $T_{\mathbf{R}}$ commutes with V, i.e. V is periodic, and it is clear that

$$\exp\left[\frac{1}{2}\frac{i}{l_B^2}\left(\mathbf{R}_x \mathbf{r}_y - \mathbf{R}_y \mathbf{r}_x\right)\right]$$

commutes with V, hence $\hat{T}_{\mathbf{R}}$ commutes with the Hamiltonian, Eq. (10.1). Let us give explicit expressions for the magnetic translation operators along the Bravais lattice basis vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\begin{aligned}
\hat{T}_{\mathbf{a}} &= T_{\mathbf{a}} \exp\left[\frac{i}{2l_B^2} a \mathbf{r}_y\right], \\
\hat{T}_{\mathbf{b}} &= T_{\mathbf{b}} \exp\left[-\frac{i}{2l_B^2} b \mathbf{r}_x\right].
\end{aligned} \tag{10.9}$$

We would like find a basis for the Hilbert space of our problem that simultaneously diagonalizes the Hamiltonian and all magnetic translation operators. If such a basis is to exist, the magnetic translation operators must commute amongst themselves. Hence, we use the defining property of the translation operator, Eq. (10.2), and the expression for the magnetic translation operators, Eq. (10.9), to compute

$$\begin{aligned}\hat{T}_{\mathbf{a}}\hat{T}_{\mathbf{b}} &= T_{\mathbf{a}} \exp\left[\frac{i}{2l_B^2} a\mathbf{r}_y\right] T_{\mathbf{b}} \exp\left[-\frac{i}{2l_B^2} b\mathbf{r}_x\right] \\ &= T_{\mathbf{b}} \exp\left[-\frac{i}{2l_B^2} b(\mathbf{r}_x + a)\right] T_{\mathbf{a}} \exp\left[\frac{i}{2l_B^2} a(\mathbf{r}_y - b)\right] \\ &= \exp\left[-\frac{i}{l_B^2} ab\right] \hat{T}_{\mathbf{b}}\hat{T}_{\mathbf{a}}.\end{aligned} \tag{10.10}$$

We conclude that if ab/l_B^2 is an integer multiple of 2π, then the magnetic translation operators will commute. From now on, we will work under the assumption that this is the case. In fact, in the original paper by Kohmoto [Kohm85], the much weaker assumption that ab/l_B^2 is a rational multiple of 2π was used. Not much changes, we just take the more restrictive assumption for notational simplicity. One can also deal with the case that ab/l_B^2 is arbitrary, but this requires significantly more sophisticated tools, most notably noncommutative geometry as developed by Alain Connes, see for example Ref. [BES94].

Remark 10.1.5 (Spectra of the magnetic translation operators)**.** The expression (10.10) implies that the two one-parameter families $\hat{T}_{t\mathbf{a}}$ and $\hat{T}_{t\mathbf{b}}$, $(t \in \mathbb{R})$ fulfill the requirements of von Neumann's uniqueness theorem, (see for example theorem 2.1 in Ref. [Maas04]), hence their spectra are equal to the unit circle, $S^1 \subsetneq \mathbb{C}$.

Theorem 10.1.6 (Generalized Bloch waves)**.** *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be basis vectors for a Bravais lattice, such that the magnetic translation operators* $\hat{T}_{\mathbf{a}}$ *and* $\hat{T}_{\mathbf{b}}$ *commute. Let* H *be a Hamiltonian that commutes with these magnetic translation operators. Suppose furthermore that there exists a basis for the Hilbert space of the problem that diagonalizes* H*. Then, there exists a complete basis,* $\psi_n(\mathbf{r})$*, for the Hilbert space of the problem, with the following properties.*

- *Each basis vector is an eigenvector of the Hamiltonian,*

$$H\psi_n = E_n\psi_n.$$

- *For each basis vector,* $\psi_n(\mathbf{r})$*, there exist a vector* $\mathbf{k} \in \mathbb{R}^2$*, and a function*

$u_{n,\mathbf{k}} : \mathbb{R}^2 \to \mathbb{C}$, *with the property that*

$$u_{n,\mathbf{k}}(x+a,y) = e^{-\frac{i}{2l_B^2}ay} u_{n,\mathbf{k}}(x,y) \tag{10.11}$$

$$u_{n,\mathbf{k}}(x,y+b) = e^{\frac{i}{2l_B^2}bx} u_{n,\mathbf{k}}(x,y) \tag{10.12}$$

such that

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}}(\mathbf{r}).$$

Analogously to the situation in the regular Bloch's theorem, the statement above is really a statement about the eigenvectors of magnetic translation operators.

Lemma 10.1.7 (Diagonalizing magnetic translation operators)**.** *If $\psi(\mathbf{r})$ is a simultaneous eigenvector of the translation operator $T_{\mathbf{R}}$ for all Bravais lattice vectors $\mathbf{R}$, then there exist a vector $\mathbf{k} \in \mathbb{R}^2$, and a function $u_{\mathbf{k}} : \mathbb{R}^2 \to \mathbb{C}$, with the property that*

$$u_{\mathbf{k}}(x+a,y) = e^{-\frac{i}{2l_B^2}ay} u_{\mathbf{k}}(x,y),$$
$$u_{\mathbf{k}}(x,y+b) = e^{\frac{i}{2l_B^2}bx} u_{\mathbf{k}}(x,y)$$

such that

$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}}(\mathbf{r}).$$

Proof. Assume that the vector ψ is a simultaneous eigenvector for $\hat{T}_{\mathbf{a}}$ and for $\hat{T}_{\mathbf{b}}$ with eigenvectors $e^{ik_x a}$ and $e^{ik_y b}$, respectively. That is,

$$\hat{T}_{\mathbf{a}}\psi(x,y) = e^{ik_x a}\psi(x,y),$$
$$\hat{T}_{\mathbf{b}}\psi(x,y) = e^{ik_y b}\psi(x,y).$$

Because the operators $\hat{T}_{\mathbf{a}}$ and $\hat{T}_{\mathbf{b}}$ are unitary, the numbers k_x and k_y are real. We assemble the numbers k_x and k_y into the vector $\mathbf{k} = (k_x, k_y)$ and define

$$u_{\mathbf{k}}(x,y) = e^{-i\mathbf{k}\cdot\mathbf{r}}\psi(x,y).$$

Using Eq. (10.9) we now see that

$$\begin{aligned}
u_{\mathbf{k}}(x+a,y) &= T_{\mathbf{a}} u_{\mathbf{k}}(x,y) \\
&= T_{\mathbf{a}} e^{-i\mathbf{k}\cdot\mathbf{r}}\psi(x,y) \\
&= e^{-i\mathbf{k}\cdot\mathbf{r} - ik_x a} T_{\mathbf{a}}\psi(x,y) \\
&= e^{-i\mathbf{k}\cdot\mathbf{r} - ik_x a} e^{-\frac{i}{2l_B^2}\mathbf{a}\mathbf{r}_y} \hat{T}_{\mathbf{a}}\psi(x,y) \\
&= e^{-\frac{i}{2l_B^2}ay} e^{-i\mathbf{k}\cdot\mathbf{r}}\psi(x,y) \\
&= e^{-\frac{i}{2l_B^2}ay} u_{\mathbf{k}}(x,y),
\end{aligned}$$

a similar computation for $u_{\mathbf{k}}(x,y+b)$ completes the proof. □

Analogously to Bloch's theorem, the generalized version is a direct result of the lemma above.

Proof of Generalized Bloch waves. If the Hamiltonian commutes with all magnetic translation operators, then there exists a basis, ψ_n, for the Hilbert space of the problem that diagonalizes both the Hamiltonian and all translation operators. Now apply Lemma 10.1.7. □

Note that remark 10.1.5 implies that for each $\mathbf{k} \in \mathbb{R}^2$ there exists a function $u_{\mathbf{k}}$ which is a Bloch wave in the sense of theorem 10.1.6.

10.2 Hall conductance

We continue with our study of the Hamiltonian in Eq. (10.1). We recall that we have shown that it commutes with the magnetic translation operators, $\hat{T}_{\mathbf{a}}$ and $\hat{T}_{\mathbf{b}}$ as given in Eq. (10.9), and that these operators commute amongst each other. We are thus in the position to apply the generalized Bloch theorem, 10.1.6. We obtain a complete basis, ψ_n, for the Hilbert space of our problem, $L^2(\mathbb{R}^2)$, where each ψ_n is an eigenvector of H with eigenvalue E_n. Furthermore, for each n there exist a vector $\mathbf{k} \in \mathbb{R}^2$ and a vector $u_{n,\mathbf{k}} \in L^2(\mathbb{R}^2)$, with the property that

$$\begin{aligned} u_{n,\mathbf{k}}(x+a,y) &= e^{-\frac{i}{2l_B^2}ay} u_{n,\mathbf{k}}(x,y), \\ u_{n,\mathbf{k}}(x,y+b) &= e^{\frac{i}{2l_B^2}bx} u_{n,\mathbf{k}}(x,y), \end{aligned}$$

such that

$$\psi_n(\mathbf{r})(x,y) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{n,\mathbf{k}}(\mathbf{r}).$$

Using Eq. (10.1), one may show that

$$\left[\frac{1}{2m}\left(-i\hbar\nabla + \hbar\mathbf{k} + e\mathbf{A}\right)^2 + V(x,y)\right] u_{n,\mathbf{k}}(x,y) = E_n u_{n,\mathbf{k}}(x,y),$$

where ∇ is the gradient.

Let us now define the magnetic Brillouin zone

$$I^2 = \{\mathbf{k} \in \mathbb{R}^2 : 0 \leqslant k_x \leqslant 2\pi/a,\ 0 \leqslant k_y \leqslant 2\pi/b\}.$$

The vectors $\mathbf{c} := (2\pi/a, 0)$ and $\mathbf{d} := (0, 2\pi/b)$ are the basis vectors for the reciprocal lattice for the Bravais lattice, that is, they obey

$$\begin{aligned} \mathbf{a}\cdot\mathbf{c} &= 2\pi, & \mathbf{a}\cdot\mathbf{d} &= 0, \\ \mathbf{b}\cdot\mathbf{c} &= 0, & \mathbf{b}\cdot\mathbf{d} &= 2\pi. \end{aligned}$$

One may show, using linear response theory, that the Hall conductance of a completely filled Landau level, $\sigma_{n,xy}$, is given by

$$\sigma_{n,xy} = -\frac{e^2}{\hbar}\frac{i}{(2\pi)^2}\int_{I^2} \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{r} \left(\frac{\partial \bar{u}_{n,\mathbf{k}}}{\partial k_y}\frac{\partial u_{n,\mathbf{k}}}{\partial k_x} - \frac{\partial \bar{u}_{n,\mathbf{k}}}{\partial k_x}\frac{\partial u_{n,\mathbf{k}}}{\partial k_y} \right), \tag{10.13}$$

for a derivation of this result we refer to Ref. [Kohm85]. In what follows, the symbol $\nabla_{\mathbf{k}}$ denotes the gradient with respect to $\mathbf{k}$, that is $\nabla_{\mathbf{k}} = (\partial_{k_x}, \partial_{k_y})$. We now define the vector field, (not to be confused with the vector potential $\mathbf{A}$),

$$\hat{\mathbf{A}}(\mathbf{k}) = \int \mathrm{d}\mathbf{r}\, \bar{u}_{\mathbf{k}} \nabla_{\mathbf{k}} u_{\mathbf{k}} = \langle u_{\mathbf{k}} | \nabla_{\mathbf{k}} | u_{\mathbf{k}} \rangle, \tag{10.14}$$

in terms of which, the Hall conductance can be written as

$$\sigma_{n,xy} = -\frac{e^2}{\hbar}\frac{1}{(2\pi)^2}\int_{I^2} \mathrm{d}\mathbf{k} \left[\nabla_{\mathbf{k}} \times \hat{\mathbf{A}}(\mathbf{k}) \right]_z, \tag{10.15}$$

where the subscript z tells us that we take the z-component. At this point we would like to identify opposite edges of the square I^2 to obtain a torus $\mathbb{T}^2$. However, the function $\hat{\mathbf{A}} : I^2 \to \mathbb{C}$ need not descend to the torus, that is, we might have, for example

$$\hat{\mathbf{A}}(0, k_y) \neq \hat{\mathbf{A}}(2\pi/a, k_y),$$

for some $0 \leqslant k_y \leqslant 2\pi/b$. It turns out that we may view $\hat{\mathbf{A}}(\mathbf{k})$ as the components of a differential one-form on a certain complex line bundle over the torus. In the sequel we will explicitly construct this complex line bundle over the torus. First we require the following result on the Bloch wave functions $u_{n,\mathbf{k}}$.

Lemma 10.2.1. *For each n there exist functions $\theta_{n,x}, \theta_{n,y} : \mathbb{R}^2 \to \mathbb{R}$, such that for all $k \in \mathbb{R}^2$ the following relations hold*

$$u_{n,\mathbf{k}+\mathbf{c}}(\mathbf{r}) = e^{i\theta_{n,x}(\mathbf{k})} u_{n,\mathbf{k}}(\mathbf{r}),$$
$$u_{n,\mathbf{k}+\mathbf{d}}(\mathbf{r}) = e^{i\theta_{n,y}(\mathbf{k})} u_{n,\mathbf{k}}(\mathbf{r}).$$

An explanation can be found in Ref. [DZ85].

This result allows us to construct a complex line bundle over the torus in such a way that we may view the maps

$$\mathbb{R}^2 \to \mathbb{C}, \mathbf{k} \mapsto u_{n,\mathbf{k}}(\mathbf{r}),$$

as sections of this complex line bundle. Let us first construct an open cover of $I^2 \subsetneq \mathbb{R}^2$. Consider the the following subdivision of I^2 into four rectangles

$$V_1 := \{\mathbf{k} \in I^2 | k_x \leqslant \pi/a, k_y \leqslant \pi/b\},$$
$$V_2 := \{\mathbf{k} \in I^2 | k_x \leqslant \pi/a, \pi/b \leqslant k_y\},$$
$$V_3 := \{\mathbf{k} \in I^2 | \pi/a \leqslant k_x, k_y \leqslant \pi/b\},$$
$$V_4 := \{\mathbf{k} \in I^2 | \pi/a \leqslant k_x, \pi/b \leqslant k_y\}.$$

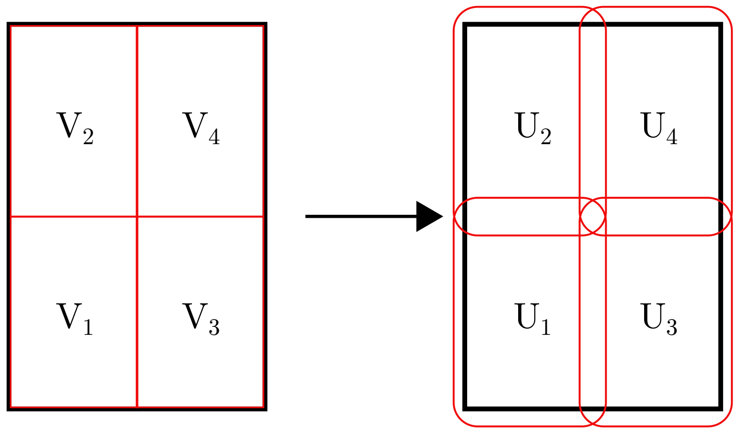


Figure 10.1: We divide the rectangle into four sets and then slightly grow each of these sets to obtain an open cover of the rectangle that descends to the torus.

For each $\alpha = 1, 2, 3, 4$ let U_α be an open set containing V_α. See Fig. 10.1 for a picture of the sets V_α and U_α.

The collection U_α forms an atlas for the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. That is, for each U_α the map projection map $q : \mathbb{R}^2 \to \mathbb{T}^2$ restricts to a map that is a diffeomorphism onto its image $U'_\alpha := q(U_\alpha)$,

$$q\big|_{U_\alpha} : U_\alpha \xrightarrow{\cong} U'_\alpha \subsetneq \mathbb{T}^2.$$

Let us assume that all zeroes of $u_{n,\mathbf{k}}(\mathbf{r})$ are located outside of the overlaps $q^{-1}(U'_\alpha \cap U'_\beta)$ for all $\alpha, \beta = 1, 2, 3, 4$. If $\alpha \neq \beta$, then for each point $\mathbf{k} \in U'_\alpha \cap U'_\beta$ there are two distinct points[1], $\mathbf{k}^\alpha \in U_\alpha$ and $\mathbf{k}^\beta \in U_\beta$, such that $q(\mathbf{k}^\alpha) = q(\mathbf{k}^\beta) = x$. For each $\alpha = 1, 2, 3, 4$ we now define the map $\phi_\alpha : U'_\alpha \to \mathbb{R}$ by

$$e^{i\phi_\alpha(\mathbf{k})} = \frac{u_{\mathbf{k}^\alpha}(\mathbf{r})}{|u_{\mathbf{k}^\alpha}(\mathbf{r})|},$$

this map might not be defined on all of U'_α, but by the assumption on the zeroes of $u_{n,\mathbf{k}}(\mathbf{r})$ it is certainly defined on the overlaps, and this will turn out to be sufficient. For each pair $\alpha, \beta = 1, 2, 3, 4$ we now define the map

$$\begin{aligned} g_{\alpha\beta} : U'_\alpha \cap U'_\beta &\to U(1), \\ \mathbf{k} &\mapsto e^{i(\phi_\alpha(\mathbf{k}) - \phi_\beta(\mathbf{k}))}. \end{aligned} \tag{10.16}$$

[1] We write $\mathbf{k}^\alpha$ instead of $\mathbf{k}_\alpha$ to avoid confusion with the *components* of $\mathbf{k}$ denoted by k_x and k_y, we will use this notation in the sequel. Also note that we break with our convention, in the rest of this section, no summation is implied over repeated indices.

In the sequel we will write $\phi_{\alpha\beta} = \phi_\alpha - \phi_\beta$. For all $\alpha, \beta, \gamma = 1, 2, 3, 4$ the maps $g_{\alpha\beta}$ obey the conditions $g_{\alpha\alpha} = 1$ and $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$. This allows us to construct a complex line bundle over the torus $\mathbb{T}^2$ as in definition 1.1.1, by taking the vector space $V = \mathbb{C}$ and the natural representation by left multiplication $\rho : U(1) \to GL(\mathbb{C})$. We denote this complex line bundle by $\pi : E \to \mathbb{T}^2$. By lemma 10.2.1 it follows that for all pairs $\alpha, \beta = 1, 2, 3, 4$, for all $\mathbf{k} \in U'_\alpha \cap U'_\beta$ we have

$$u_{n,\mathbf{k}^\alpha}(\mathbf{r}) = g_{\alpha\beta}(\mathbf{k}) u_{n,\mathbf{k}^\beta}(\mathbf{r}),$$

hence we may identify $u_{n,\mathbf{k}}(\mathbf{r})$ as a section of E. We now recall the definition (10.14) of the vector field $\hat{\mathbf{A}}(\mathbf{k})$

$$\hat{\mathbf{A}}(\mathbf{k}) = \langle u_\mathbf{k} | \nabla_\mathbf{k} | u_\mathbf{k} \rangle, \qquad (\mathbf{k} \in \mathbb{R}^2),$$

in terms of which we define the one-form, (not to be confused with the vector potential $\mathbf{A}$),

$$A(\mathbf{k}) = \hat{\mathbf{A}}(\mathbf{k}) \cdot \mathrm{d}\mathbf{k}, \qquad (\mathbf{k} \in \mathbb{R}^2).$$

We claim that the one-form A may be viewed as a connection one-form on E. It suffices to show that the one-form A exhibits the correct 'transformation behaviour'. Let $\alpha, \beta = 1, 2, 3, 4$ be arbitrary, let $k \in U'_\alpha \cap U'_\beta$, then we use the fact that $\langle u_{\mathbf{k}^\beta} | u_{\mathbf{k}^\beta} \rangle = 1$ and the definition of $g_{\alpha\beta}$, (Eq. (10.16)) to compute

$$\begin{aligned} A(\mathbf{k}^\alpha) &= \langle u_{\mathbf{k}^\alpha} | \nabla_\mathbf{k} | u_{\mathbf{k}^\alpha} \rangle \cdot \mathrm{d}\mathbf{k} \\ &= \langle u_{\mathbf{k}^\beta} | g_{\beta\alpha}(\mathbf{k}) \nabla_\mathbf{k} \left[g_{\alpha\beta}(\mathbf{k}) | u_{\mathbf{k}^\beta} \rangle \right] \cdot \mathrm{d}\mathbf{k} \\ &= A(\mathbf{k}^\beta) + \langle u_{\mathbf{k}^\beta} | g_{\beta\alpha}(\mathbf{k}) (\nabla_\mathbf{k} g_{\alpha\beta}(\mathbf{k})) | u_{\mathbf{k}^\beta} \rangle \rangle \cdot \mathrm{d}\mathbf{k} \\ &= A(\mathbf{k}^\beta) + i \, \mathrm{d}\phi_{\alpha\beta}(\mathbf{k}). \end{aligned}$$

On each chart U'_α we define a connection $\mathrm{d} - A(\mathbf{k}^\alpha)$. One may check that indeed for all $\alpha, \beta = 1, 2$ and all $\mathbf{k} \in U'_\alpha \cap U'_\beta$ we have

$$(\mathrm{d} - A(\mathbf{k}^\alpha)) u_{\mathbf{k}^\alpha} = g_{\alpha\beta}(\mathbf{k}) (\mathrm{d} - A(\mathbf{k}^\beta)) u_{\mathbf{k}^\beta}$$

The curvature F of the connection D is given by

$$F = \mathrm{d}A = \frac{\partial}{\partial k_\mu} \hat{\mathbf{A}}^\nu(\mathbf{k}) \, \mathrm{d}k_\mu \wedge \mathrm{d}k_\nu,$$

where μ and ν are summed over $\{x, y\}$. One may show that the curvature does descend to the torus, hence

$$\int_{I^2} F = \int_{\mathbb{T}^2} F.$$

We now compute

$$\begin{aligned}\int_{I^2} F &= \int_{I^2} \frac{\partial}{\partial k_\mu} \hat{\mathbf{A}}^\nu(\mathbf{k})\, \mathrm{d}k_\mu \wedge \mathrm{d}k_\nu \\ &= \int_{I^2} \mathrm{d}\mathbf{k} \left[\nabla_{\mathbf{k}} \times \hat{\mathbf{A}}(\mathbf{k})\right]_z ,\end{aligned}$$

hence using Eq. (10.15), it follows that

$$\sigma_{n,xy} = -e^2\hbar \frac{1}{(2\pi)^2} \int_{I^2} \mathrm{d}\mathbf{k} \left[\nabla_{\mathbf{k}} \times \hat{\mathbf{A}}(\mathbf{k})\right]_z = -e^2\hbar \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} F.$$

The number

$$\frac{1}{2\pi} \int_{\mathbb{T}^2} F$$

is nothing but the first Chern class, hence an integer by proposition 4.1.2. We conclude that

$$\sigma_{n,xy} = -\frac{e^2}{2\pi\hbar} N,$$

where N is an integer.

Chapter 11

Outlook

In Chapter 9 we considered a composite system, consisting of a topological part coupled to a harmonic oscillator part. The topological part being the charged particle in a strong magnetic field. If one considers this quantum mechanical theory to be a $(0+1)$-dimensional quantum field theory, then an interesting problem would be to consider an $(n+1)$-dimensional analog. In particular, one could take an n-dimensional topological quantum field theory and couple this to a quantum field theory. There are many choices available for both the topological quantum field theory and the non-topological quantum field theory. A natural choice for the topological quantum field theory would be Chern-Simons theory, because it is understood very well, and because it appears to be a natural analog of a particle in a strong magnetic field, see Ref [DJT90]. A natural way to couple this theory to a non-topological quantum field theory might be to follow a procedure similar to Stueckelberg coupling, see for example Ref. [KN05]. Another interesting approach might be to couple a topological quantum field theory to a conformal quantum field theory, because just like topological quantum field theories, conformal quantum field theories can be used to construct functors from something like the cobordism category to something like the category of vector spaces, see Ref. [Segal88]. It would be interesting to see of anything of this categorical description remains.

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