ISOMETRIES AND PRIMORDIAL FLUCTUATIONS

BERNARDO FINELLI

Universiteit Utrecht

On how exact and broken spacetime symmetries probe inflation

Master's Thesis

Supervisor: Dr. Enrico Pajer

Institute for Theoretical Physics
Department of Physics and Astronomy
Faculty of Science
Utrecht University

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The scepter of power is fragile in a calloused hand.

— Cho-Arrim saying

ABSTRACT

Inflation has become a basic paradigm of modern physical cosmology. One of the most widely used description of such an accelerating expanding universe is that of the de Sitter space (dS). We explicitly derive the exact form of the dS isometries by considering an embedding in a higher dimensional Minkowski spacetime; these are spatial translations and rotations and spacetime dilations and boosts. For the boost in particular, we obtain its finite form, while in the literature we could find only its infinitesimal form. We then proceed to consider these same isometries in the context of quasi-de Sitter space via the slow roll formalism, where the dilations and boosts are broken. Furthermore, we discuss some insights these isometries provide, such as the dS/CFT duality and the consistency relation for inflaton correlators, which can be applied even outside of dS.

ACKNOWLEDGMENTS

I'd like to thank my supervisor Enrico Pajer whose advice was invaluable to this thesis. Our weekly meetings were always constructive and helped me keep the big picture of what we were doing in mind.

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Part I

INTRODUCTION

1 PREFACE

Exactly one hundred years ago, Albert Einstein published his renowned theory of general relativity. Spacetime, which for Newton was nothing more than a fixed background in which the laws of physics played out their roles, became itself an actor. The metric of spacetime—the object determining distances between any two points—was no longer a God-given entity assumed a priori, but a dynamical field whose evolution is described by the famous Einstein field equation. Just as the evolution of matter is tied to the spacetime it inhabits, so too is the evolution of spacetime intertwined with the matter it contains.

General relativity remains, to this day, our best theory of gravity, with what we experience as the gravitational force being nothing more than the interplay between matter and spacetime. But general relativity can do far more than simply explaining why apples fall or why the Earth goes around the Sun.

Cosmology takes the idea of a dynamical spacetime and runs it to its logical conclusion to study the beginnings, development and end of the entire universe. By assuming the universe to be filled homogeneously with ordinary matter and electromagnetic radiation, we can determine its entire evolution.

Problem is, matter and radiation cannot handle the task. This is because these common substances lead to our usual understanding of gravity being attractive. Any clump of matter or radiation spread across the universe will tend to collapse in on itself. So spacetime would either be contracting or in decelerated expansion. We currently measure our universe to be expanding, but if it is decelerating *and has always been so*, that introduces two major concerns in cosmology.

First, the *flatness problem*. Supernova surveys set the universe's spatial curvature to zero, within 0.4% [3]. This corresponds to the density of the universe being close to a certain critical value. If our universe has always been decelerating, then it was expanding much faster in the past, in particular in the primordial past. The initial density back then must have been phenomenally close to the critical one. There are no known physical processes that would enforce that, other than sheer coincidence.

The second major issue is called the *horizon problem*. We assumed the universe is homogeneous at large scales, because we observe that to be the case. For instance, the universe is permeatted by a form of electromagnetic radiation called the cosmic microwave background (CMB). Pick a random point in the sky and average the frequency of CMB photons coming from that point, then do the same for the point on the opposite side on the sky. You will observe the two frequencies differ by no more than 1 part in 10^5 [4]. However, if the universe was indeed expanding faster in the past, these two points do not know of each other's existence right now. That is, if the first point had emitted a signal at the beginning of the universe, it would still not have arrived at the other even today. The two points have no way of knowing each other's CMB frequency, unless the universe was conveniently created homogeneously everywhere or faster-than-light communication is possible.

The hypothesis known as *cosmological inflation* solves both these issues, as well as others. The crux of the idea is to flip the situation on its head: rather than a decelerating universe, an accelerating one. In particular, we suppose that in the primordial era, just right after the universe's beginning, the cosmos underwent this accelerated expansion then later transitioned to ordinary matter and radiation expansion. Inflation would then consist of an extremely quick and increasingly quicker expansion period in the past. The flatness problem is then solved, because this procedure flattens the universe whatever the initial curvature may be. The horizon problem is equally fixed as everything we can currently see in the sky—the so-called observable patch—all occupied a very small region of space which had plenty of time to homogenize. It was then violently inflated into a vast region of the universe.

There is, of course, one catch: In an accelerated universe, gravity must somehow become repulsive, rather than attractive. This is not forbidden by general relativity, but it cannot be accomplished using simple matter or radiation. The secret identity of the so-called inflaton, the field responsible for such a feat, remains unknown. Candidates include the renowned Higgs boson [5] or supersymmetric particles [6]. Perhaps more importantly, inflation has to *end*. While we do observe our universe to be presently accelerating, that is due to an unrelated dark energy. After inflation, the universe transitioned to the ordinary, decelerating matter-and-radiation evolution.

Not knowing the inflaton's nature does not tie our hands. We simply need an effective theory for the inflaton which displays all its known characteristics and symmetries in the regime we are considering. Then it does not matter if this theory is not fundamental and breaks down during, say, the very early beginning of the universe, as we will not study that period.

More than that, we can observe the consequecences of inflation today. The inflaton, like any field, is a *quantum* field, meaning that, even in its lowest energy state, the field fluctuates. This is in opposition to *classical* fields, whose lowest energy state are still and quiet. These primordial fluctuations at the early beginning of the universe, which were microscopic in origin, were then forcefully inflated to cosmological sizes. This leads to the slight deviation from homogeneity we see in the CMB and the matter distribution of the universe. In other words, inflation seeds the inhomogeneities of the universe at large scales we observe today. Thus, current cosmological observations indirectly probe the inflaton.

The goal of this thesis is to study the role spacetime symmetries (isometries) had on these primordial fluctuations during inflation. The reason for the focus on isometries are three. First, observations of the CMB and other cosmological sources have certain properties which correspond to the symmetries of a particular type of inflating universe, the *de Sitter space*.

Second, symmetries lead to insight into the basic observable quantities in a quantum field theory, the correlators. They indicate the probability of any particle process to happen. Symmetries can be used to enforce certain forms on the correlators, even if we cannot calculate them. Furthermore, even when symmetries are broken, it is still possible to use this breaking to relate different correlators in certain limits. This is called a consistency relation.

Third, a focus on symmetries highlights an important duality between gravitational theories living in a given spacetime and gauge field theories on the boundary of said spacetime. In cosmology, the relevant spacetime is once again de Sitter space, whereas the boundary field theory is conformal, meaning the theory is the same even if you zoom in or out. This duality is called the dS/CFT correspondence [7]. Its more famous cousin is the AdS/CFT correspondence, where de Sitter space is replaced by another space, the anti-de Sitter space [8]. AdS/CFT has been formidably successful as a dictionary, allowing conformal field theories to be converted into gravitational string theories and vice-versa, translating correlators that would be intractable in one language into the other where they can be computed with minor effort. While the dS/CFT duality still has a long way to go, future research in this area seems promising.

This thesis starts with exposition of background material in differential geometry, cosmology, quantum field theory and inflation, which is needed for understanding the ideas proposed here. It is then divided into two main parts. First, we will discuss the symmetries of the de Sitter spacetime. In particular, de Sitter space has one symmetry which is not evident, the boost. We will derive a closed form for this boost, as we could not find its expression in the literature. The second part consists of moving to quasi-de Sitter. We explore the symmetries of de Sitter in this context where they are slightly broken. We then discuss how these results could be used for deriving consistency relations in further work.

2 BACKGROUND

2.1 DIFFERENTIAL GEOMETRY

2.1.1 *Manifolds, charts and distances*

The basic entity differential geometry deals with are *manifolds* $\mathcal{M}$ of dimension n, which are nothing more than sets of "points." These points are abstract objects; they could be actual points in spacetime, matrices, gauge group elements, etc. For us to actually be able to manipulate them, we need to describe them with numbers. Therefore, each manifold $\mathcal{M}$ must also be equipped with a *chart* X, a map:

There are also additional considerations to make sure the manifold is smooth and doesn't intersect itself but we will ignore these technicalities.

$$X : \mathcal{M} \rightarrow \mathbb{R}^n \tag{2.1}$$
$$P \rightarrow x \tag{2.2}$$

.

The chart is simply a function that given a point P as input returns a tuple $X(P) = x = (x^1, ..., x^n)$ of numerical values. In other words, it is a *coordinate system* and each x^μ is one of the coordinates. A simple example is the Mercator chart, which maps each point on the surface of the Earth (except the poles) to two coordinates on a rectangular plane.

Manifolds may also be endowed with a metric map G which, given two points A and B on the manifold, returns the distance squared $||AB||^2$ between them:

$$G : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R} \tag{2.3}$$
$$(A, B) \rightarrow ||AB||^2 \tag{2.4}$$

Once again, without actual numerical values, we cannot employ the metric to perform any sort of calculation, as G, A and B are abstract entities if left as they are. By inverting the chart we can obtain a point given its numerical coordinates x:

$$X^{-1} : \mathbb{R}^n \rightarrow \mathcal{M} \tag{2.5}$$
$$x \rightarrow P \tag{2.6}$$

Charts in reality may not cover the whole manifold when inverting, which is why we require an atlas, *a collection of charts that together cover it in its entirety. For simplicity, we will work only with global charts that require no atlas.*

So if A and B have coordinates x and y respectively in a certain chart X, we have:

$$||AB||^2 = G\left(X^{-1}(x)\,,\, X^{-1}(y) \right) = S(x,y) \tag{2.7}$$

Suppose now we move the tuple y very close to x, i.e., $y = x + dx$. We can then expand $S(x,y)$ to second order:

$$S(x, x+dx) = S(x,x) + dx^\mu \frac{\partial S(x,y)}{\partial y^\mu}\Big|_{y=x} + dx^\mu dx^\nu \frac{\partial^2 S(x,y)}{\partial y^\nu \partial y^\mu}\Big|_{y=x} \tag{2.8}$$

Notice that x is a tuple of n entries, so whenever we take a derivative with respect to x, we are really taking a derivative with respect to each entry. We remind the reader that repeated indices imply summation, i.e., $a^\mu b_\mu = \sum a^\mu b_\mu$.

We will always impose that the first derivative $\frac{\partial S(x,y)}{\partial y^\mu}\Big|_{y=x}$ should vanish. This is obvious from a spatial analogy: The distance between two points is minimal when the two points coincide. Space*time* distances, however, can be negative. Typically in physics, the Hessian $\frac{\partial^2 S(x,y)}{\partial y^\nu \partial y^\mu}\Big|_{y=x} = g_{\mu\nu}(x)$ has determinant -1, so together with the condition of vanishing first derivative this implies that, when two spacetime points coincide, the distance between them is an inflection, not minimum, point. In any case, we can call $g_{\mu\nu}(x)$ the coordinates of the metric in the coordinate chart we are using, and the infinitesimal squared distance interval $S(x, x+dx) - S(x,x) = ds^2(x)$. Thus we get one of the most important equations in differential geometry:

$$ds^2(x) = g_{\mu\nu}(x) dx^\mu dx^\nu \tag{2.9}$$

2.1.2 *Coordinate changes*

We know that a given manifold $\mathcal{M}$ can be described in different coordinates frames. For instance, flat space can be described with rectangular or polar coordinates. This should correspond to the existence of multiple different charts for $\mathcal{M}$ which is hardly a surprising fact.

If in addition to the chart X we add a new one to our manifold, say U:

$$U : \mathcal{M} \rightarrow \mathbb{R}^n \tag{2.10}$$
$$P \rightarrow x \tag{2.11}$$

Then we can construct a transition map, namely the composition $f = U \circ X^{-1}$:

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n \tag{2.12}$$
$$x \rightarrow y \tag{2.13}$$

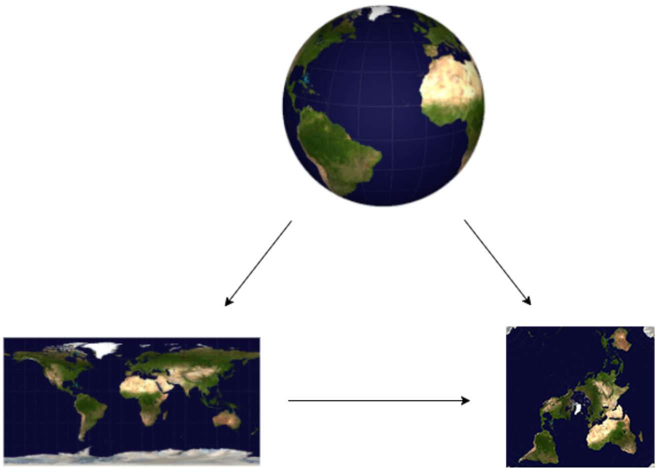

Figure 2.1: An example of a coordinate change. Two charts of the globe are the equirectangular (left) and the Peirce quincuncial (right). It is possible to map one to the other (except the poles) via a convenient function. Derivative work of [1, 2].

The function f is a coordinate change. It uses the inverse chart X^{-1} to map the coordinates of a point back onto the manifold, then applies U to obtain the coordinates of the very same point in a new system of coordinates.

The function $g_{\mu\nu}(x)$ was for the coordinate chart X. How will it change in the chart U? We are not remapping or transforming the manifold so the actual metric map G does not change at all. Thus distances cannot change under a coordinate change. Furthermore, the chain rule dictates that:

$$du^\mu = \frac{\partial u^\mu}{\partial x^\rho}(x)\, dx^\rho = J^\mu_\rho(x)\, dx^\rho \tag{2.14}$$

$$dx^\mu = \frac{\partial x^\mu}{\partial u^\rho}(u)\, du^\rho = [J^{-1}(u)]^\mu_\rho\, du^\rho \tag{2.15}$$

where J is the Jacobian of the coordinate change and can be calculated from $u = f(x)$. Then the invariance of ds^2 implies:

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = g_{\rho\sigma}(x)\, [J^{-1}(u)]^\rho_\mu [J^{-1}(u)]^\sigma_\nu\, du^\mu du^\nu \tag{2.16}$$

$$ds^2 = \tilde{g}_{\mu\nu}(u) du^\mu du^\nu \tag{2.17}$$

where $\tilde{g}$ is the new metric in the new coordinate chart. Then we can see that:

$$\tilde{g}_{\mu\nu}(u) = [J^{-1}(x)]^\rho_\mu\, [J^{-1}(x)]^\sigma_\nu\, g_{\rho\sigma}(x) \tag{2.18}$$

A change of coordinates is also called a passive transformation, because we do not actually change the manifold, simply our point of view.

2.1.3 *Isometric transformations*

We may also imagine transformations that act directly on the manifold to remap it:

$$\begin{aligned} T : \mathcal{M} &\rightarrow \mathcal{M} && (2.19)\\ P &\rightarrow Q && (2.20) \end{aligned}$$

If the manifold transforms into something else, so will its metric G. The new one will be given by $G \circ T$. But suppose we have

$$G \circ T = G \tag{2.21}$$

Then the transformation T actually leaves the metric unchanged; i.e., it preserves distances between points. It is called an *isometry*. This is all very well and good, but G and T are abstract maps. How does this all work in terms of coordinates?

Just like before, suppose we have a chart X. But there won't be another chart U, because we do not wish to change coordinates. Instead, we use X^{-1} to obtain a point given its coordinates, transform the point according to T then use the very same chart X to return to our coordinate system. This will thus give rise to a function $h = X \circ T \circ X^{-1}$:

$$\begin{aligned} h : \mathbb{R}^n &\rightarrow \mathbb{R}^n && (2.22)\\ x &\rightarrow y && (2.23) \end{aligned}$$

which represents how the coordinates of the point will change under the transformation. This is *not* a change of coordinates—we are still in the same chart X but in a new, altered manifold. Because of this, we call it an active transformation.

Using Equation 2.9, we can represent distances concretely using coordinates. Thus to check if h (and by extension, T) is an isometry, we must transform $x \rightarrow h(x)$. This is an active transformation in the same coordinate system, so the metric transforms actively as $g_{\mu\nu}(x) \rightarrow g_{\mu\nu}(h(x))$, not passively as described by Equation 2.18.

$$\begin{aligned} x &\rightarrow h(x) && (2.24)\\ dx^\mu &\rightarrow J^\mu_\rho(x)dx^\rho && (2.25)\\ ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu &\rightarrow g_{\mu\nu}(h(x))J^\mu_\rho(x)J^\nu_\sigma(x)dx^\rho dx^\sigma && (2.26)\\ &\rightarrow (ds^2)' && (2.27) \end{aligned}$$

An isometry is then $ds^2 = (ds^2)'$, which gives:

$$\begin{aligned} g_{\mu\nu}(x) &= J^\rho_\mu(x)J^\sigma_\nu(x)\, g_{\rho\sigma}(h(x)) && (2.28)\\ [J^{-1}(x)]^\rho_\mu\, [J^{-1}(x)]^\sigma_\nu\, g_{\mu\nu}(x) &= g_{\rho\sigma}(h(x)) && (2.29) \end{aligned}$$

But wait. The LHS is exactly what we would get after a change of coordinates $\tilde{x} = h(x)$ as dictated by Equation 2.18. Therefore, an isometry of the metric is equivalent to a change of coordinates that satisfies:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(\tilde{x}) \tag{2.30}$$

This means that under this coordinate change, the form of the metric is unchanged; we transform it to this new chart simply by replacing x with $\tilde{x}$.

To recap, any map h of the coordinates can be used passively to produce a new coordinate chart or actively to produce a new manifold. If, when used passively in a coordinate change, the form of the metric does not change, then the corresponding active transformation is an isometry.

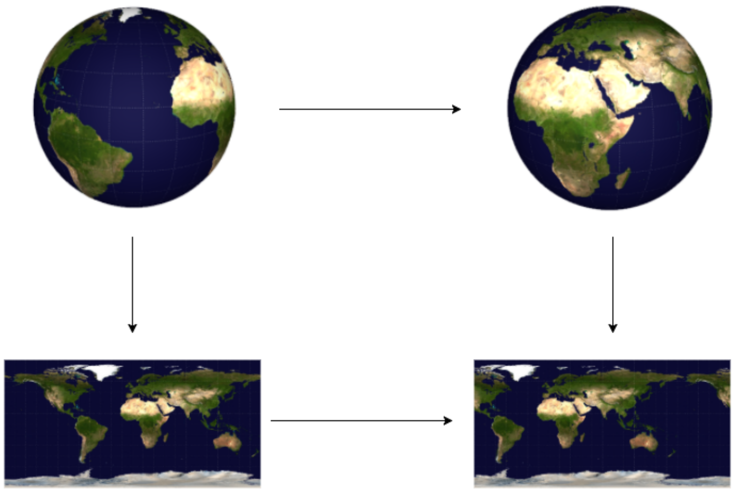

Figure 2.2: An example of an isometry. Charting the globe via the equirectangular projection, shifting the coordinates horizontally then projecting back onto the globe with the inverse chart is equivalent to simply rotating the planet, which leaves distances invariant. Derivative work of [1, 2].

As an example, consider the following 2D metric in a certain coordinate chart:

$$ds^2 = \frac{d\tau^2 - d\chi^2}{\tau^2} \tag{2.31}$$

or, in other words, $g_{\mu\nu}(\tau, \chi) = \frac{1}{\tau^2}\eta_{\mu\nu}$. Then under a space translation $\chi \to \chi + a$ we have $d\chi \to d\chi$ and the distance ds^2 is invariant. More specifically, $g_{\mu\nu}(\tau, \chi) \to g_{\mu\nu}(\tau, \chi)$ and the Jacobian for translations is just the identity so, by Equation 2.28, it is an isometry. In fact, whenever the metric does not depend on a variable, a translation of said variable will be an isometry. We say this isometry is manifest.

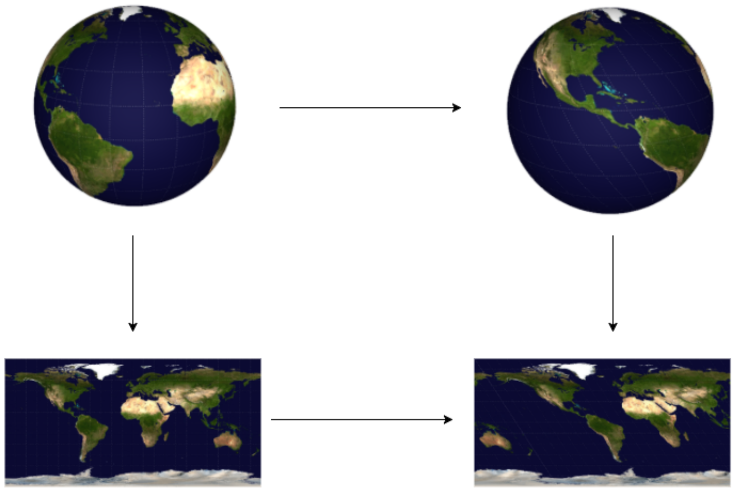

Figure 2.3: An example of a non-isometric transformation. As before, we chart the globe with the equirectangular projection, but now we shear it. Inverting back onto the globe results in a planet with warped distances. Derivative work of [1].

There is another isometry in Equation 2.31, namely the dilation $(\tau, \chi) \to (\Lambda\tau, \Lambda\chi)$. This one is not manifest, but it is still fairly straightforward to see it is indeed an isometry. However, there is yet a third isometry lurking there. This one is hidden and not obvious at all. We will return to in Section 3.2.3.

2.1.4 *Conformal transformations*

There is one final class of spacetime transformations that beckons our attention. If isometries are maps that preserve distances, conformal transformations are those which preserve angles. As before, consider a manifold map T with the coordinate representation h. Under the change of coordinates $\tilde{x} = h(x)$, we relax Equation 2.30; a simple rescaling is sufficient rather than exact equality:

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \Omega(\tilde{x}) g_{\mu\nu}(\tilde{x}) \tag{2.32}$$

If the above holds, then h is a conformal change of coordinates and the corresponding active transformation T is a conformal transformation. It should go without saying that all isometric transformations are also conformal.

We will be specially interested in the conformal transformations of flat Euclidean space. It should come as no surprise that rotations, translations and the dilation (rescaling of space) preserve shapes and are thus conformal. Euclidean space also has one additional conformal transformation, called the special conformal transformation (SCT). Geometrically, the SCT is an inversion $\vec{x} \to \frac{1}{\vec{x}} = \frac{\vec{x}}{|\vec{x}|^2}$, then a

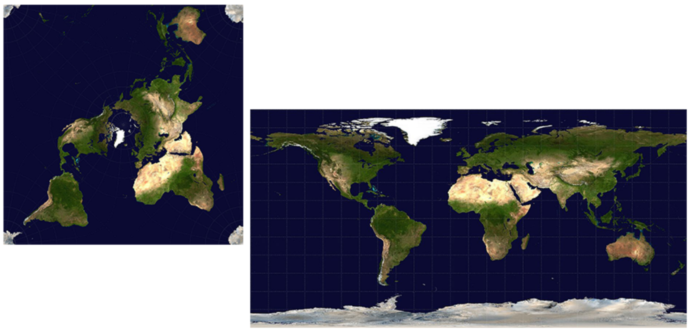

Figure 2.4: The Peirce quincuncial chart (left) is a conformal coordinate system of the globe—it locally preserves shapes. The equirectangular chart (right) is not, with the distortion being evident close to the poles. Derivative work of [1, 2].

translation by some vector $\vec{b}$, followed by a final inversion. In short, the SCT is:

$$\vec{x} \to \frac{1}{\vec{x}} \to \frac{1}{\vec{x}} + \vec{b} \to \frac{1}{\frac{1}{\vec{x}} + \vec{b}} \tag{2.33}$$

With proper simplification this reduces to:

$$\vec{x} \to \frac{\vec{x} + \vec{b}x^2}{1 + 2\vec{b} \cdot \vec{x} + b^2 x^2} \tag{2.34}$$

2.2 COSMOLOGY

2.2.1 *Friedmann equation*

We set the speed of light $c = 1$ from now on.

A formal derivation of the Friedmann equation requires working with the machinery of general relativity. Fortunately, the same result can be derived within the framework of Newtonian physics, and we shall do so for accessibility.

Newtonian physics is ill-suited for a dynamical space, but suppose we have a 3D solid ball living in an otherwise empty 3D flat space. If the sphere has a time-dependent radius $R(t)$ and a constant mass M, we can evolve via Newton's laws. We can then associate the ball with the universe itself. It is important to understand that the empty space outside the ball has no physical relevance and is merely an artifact of the description—in fact, it is absent in general relativity, as we can evolve space itself without the need of a static ambient.

A test particle by definition is one whose back-reaction, that is, its gravitational effect on the sphere, is negligible.

Now put a test particle of mass m on the surface of the ball. It will experience a Newtonian gravitational force:

$$F = -\frac{GMm}{R(t)^2} \tag{2.35}$$

Using Newton's second law, we obtain:

$$\begin{aligned} F &= m\ddot{R}(t) && (2.36) \\ \implies \ddot{R}(t) &= -\frac{GM}{R(t)^2} && (2.37) \\ \ddot{R}(t)\dot{R}(t) &= -\frac{GM}{R(t)^2}\dot{R}(t) && (2.38) \\ \int dt\, \ddot{R}(t)\dot{R}(t) &= -\int dt\, \frac{GM}{R(t)^2}\dot{R}(t) && (2.39) \\ \frac{1}{2}\dot{R}(t)^2 &= \frac{GM}{R(t)} + E && (2.40) \end{aligned}$$

where we have called the constant of integration E. Essentially, this equation describes conservation of energy, with $K = \frac{1}{2}\dot{R}(t)$ being the kinetic energy of space, $U = -\frac{GM}{R(t)}$ the potential energy, and $E = K + U$ the total energy energy of space, which might not be zero.

The mass of the ball is constant, so if its radius changes, so too does its density:

$$M = \frac{4\pi}{3}\rho(t)R(t) \qquad (2.41)$$

where we have assumed that the sphere is homogeneous, i.e., the density ρ doesn't depend on space. Similarly, by assuming the sphere is isotropic—it looks the same even after a rotation—we may write:

$$R(t) = a(t)r \qquad (2.42)$$

where r is constant. Here we have introduced perhaps the most important quantity in cosmology, the scale factor $a(t)$. To say the universe expands is to say a increases. If the scale factor doubles in size, then all distances are doubled. Thus complete knowledge of the scale factor fully determines the universe—at least the ideal homogeneous and isotropic universe, without silly impurities such as galaxies and people.

Assuming no other forces. If the scale factor doubles, distances inside tightly bound systems–the atoms in your body, the planets in a star system, the stars in a galaxy–will not change.

Inserting Equations 2.41 and 2.42 into 2.40, we get:

$$\begin{aligned} \frac{1}{2}r^2\dot{a}(t)^2 &= \frac{4\pi G r^3 a(t)^3 \rho(t)}{3ra(t)} + E && (2.43) \\ \left(\frac{\dot{a}(t)}{a(t)}\right)^2 &= \frac{8\pi G}{3}\rho(t) + \frac{2E}{r^2 a^2(t)} && (2.44) \end{aligned}$$

Equation 2.44 is strictly identical to the Friedmann equation one obtains using general relativity, as long as we understand the following:

- The mass density $\rho(t)$ should replaced by an *energy* density $\varepsilon(t)$.
- The intrinsic distance r is the *radius of curvature* of space (*not* space*time*).

- The intrinsic energy E is given by $-k/2$, where k is the sign of the curvature of space (for example, -1 for a hyperboloid, 0 for a flat space, and +1 for a sphere).
- We add a cosmological constant term $\Lambda/3$. This term has no Newtonian counterpart but is permissible by general relativity.

The quantity $\frac{\dot{a}(t)}{a(t)}$ is in fact so important we give a name to it—the Hubble parameter $H(t)$.

With these considerations in mind, the Friedmann equation is:

$$H^2 = \frac{8\pi G}{3}\varepsilon(t) + \frac{\Lambda}{3} - \frac{k}{r^2 a^2} \tag{2.45}$$

As we have said, in this universe, any spatial distance $d\vec{x}$ scales over time as $a(t)d\vec{x}$ though time intervals dt do not. From this it follows that the spacetime interval should be:

$$ds^2 = dt^2 - a(t)^2 d_\Sigma \vec{x}^2 \tag{2.46}$$

where $d_\Sigma \vec{x}$ is a three-dimensional metric for a spatial surface of constant curvature. If the universe is flat ($k = 0$) then $d_\Sigma \vec{x}^2 = d\vec{x}^2 = dx_1^2 + dx_2^2 + dx_3^2$.

However, we could consider a different set of coordinates where time intervals do scale with time. The so-called conformal time τ is constructed simply by setting $dt = a d\tau$. Then:

$$ds^2 = a(\tau)^2 (d\tau^2 - d_\Sigma \vec{x}^2) \tag{2.47}$$

2.2.2 *Fluid equation*

The Friedmann equation cannot be solved as is, because knowledge of $\varepsilon(t)$ is lacking. But, once again using the ball of radius $R(t) = a(t)r$, we notice that any small heat flow from or into the ball to the ambient space is:

$$q = dE + PdV \tag{2.48}$$

This is simply the first law of thermodynamics. But $q = 0$; the ambient space has no physical meaning and thus heat cannot leak into it. In other words, the expansion of the universe must be adiabatic. Then it follows that:

$$\dot{E} + P\dot{V} = 0 \tag{2.49}$$

But we know the volume of the ball:

$$V = \frac{4\pi}{3} r^3 a^3 \implies \dot{V} = 3V\frac{\dot{a}}{a} = 3VH \tag{2.50}$$

Component	w
Radiation (relativistic)	1/3
Matter (non-relativistic)	0
Dark energy	any <-1/3
Vacuum	-1

Table 2.1: Examples of the most commonly cited components of the cosmological fluid and their corresponding state parameter w.

And obviously the total energy can be given in terms of the energy density and the volume:

$$E = \varepsilon V \implies \dot{E} = V\dot{\varepsilon} + \varepsilon\dot{V} = V(\dot{\varepsilon} + 3\varepsilon H) \tag{2.51}$$

Thus giving the cosmological fluid equation:

$$\dot{\varepsilon} + 3H(\varepsilon + P) = 0 \tag{2.52}$$

A negative pressure is equivalent to a positive tension.

In general, P and ε are related via some equation of state $P = w\varepsilon$ where w is usually constant. Values of w for common fluid components are given in Table 2.1. Knowledge of the equation of state allows us to solve the fluid and Friedmann equations together to obtain the scale factor of the universe. Note that, from Equation 2.52, if $P = -\varepsilon$, then ε must be a constant—possibly zero, but not necessarily so. This is the equation of state of the vacuum, of space devoid of any fluid.

2.3 QUANTUM FIELD THEORY

We set Planck's constant $\hbar = 1$ from now on.

All the fluids mentioned in the previous section, whether photons or dark energy, should be described by fields, functions associating one or more values to each point $x = (t, \vec{x})$ of spacetime. For simplicity, we consider only scalar fields $\phi(x)$, which associate a single numerical value. A field theory is specified once we construct an action $S[\phi]$. In the same way the field $\phi(x)$ returns a value to each possible point in spacetime, the action $S[\phi]$ returns a value to each possible field configuration $\phi(x)$ in the space of field configurations:

$$S[\phi] = \int d^4x \sqrt{-\det g}\, \mathcal{L}(\phi(x)) \tag{2.53}$$

In general relativity, $\det g < 0$, so a minus sign is needed under the root.

where the Lagrangian $\mathcal{L}$ is some function defining the physics of the theory. Recall from calculus that under a change of coordinates with Jacobian J, we have $d^4x \to \det J\, d^4x$, while from Equation 2.18, $\det g \to \det(J^{-1})^2 \det g$; thus, $d^4x\sqrt{-\det g} \to d^4x\sqrt{-\det g}$. This means that the action will have the same mathematical form in whatever coordinates it is written on, if the Lagrangian also has this property.

Classically, there is only one configuration in which $\phi(x)$ is allowed to exist, the one for which $\frac{\delta S}{\delta \phi} = 0$; this corresponds to the usual Lagrange equations of classical physics. Quantum mechanically, this is not the case. The field has the right to exist in a quantum superposition of all possible and conceivable field configurations, with the caveat that not all configurations are equally likely. Rather, each is weighted by the amplitude $e^{iS[\phi]}$. Thus the path integral is born:

$$Z = \int \mathcal{D}\phi \, e^{iS[\phi]} \tag{2.54}$$

where $\int \mathcal{D}\phi$ should be informally understood as a sum over all field configurations.

Correlators form the basic observables in any quantum field theory, as they showcase how the values of the fields at different points are related. For instance, the two-point correlator $\langle \phi(x)\phi(y) \rangle$ is the amplitude for a certain perturbation in the field, i.e., a particle, to propagate from x to y and is thus connected to the mass of the particle. The correlator, or n-point function, is defined as:

$$\langle \phi(x_1)\phi(x_2)...\phi(x_n) \rangle = \int \mathcal{D}\phi \, \phi(x_1)\phi(x_2)...\phi(x_n) \, e^{iS[\phi]} \tag{2.55}$$

2.3.1 *Field transformations*

Consider a field $\phi(x)$ and suppose we perform a transformation of spacetime consisting of a simple translation: $x \to x + \varepsilon a$. Then:

$$\phi(x) \to \phi(x)' = \phi(x + \varepsilon a) \tag{2.56}$$

If ε is infinitesimally small, we expand $\phi(x + \varepsilon a) = \phi(x) + \varepsilon a^\mu \partial_\mu \phi(x)$. The infinitesimal variation of the field ϕ under this transformation therefore is:

We use the notation $\partial_\mu = \frac{\partial}{\partial x^\mu}$.

$$\delta_\varepsilon \phi = \phi' - \phi = \varepsilon^\mu a^\mu \partial_\mu \phi \tag{2.57}$$

We notice we can write $\delta_\varepsilon \phi = \varepsilon a \cdot \partial \phi$. In general, for any transformation, its generator G is defined as the linear operator that produces the infinitesimal variation of the field under this transformation:

$$\delta_\varepsilon \phi = \varepsilon G \phi \tag{2.58}$$

In this case, the generator for a translation in the a direction is thus $a \cdot P = a^\mu \partial_\mu$.

Field transformations do not need to be solely spacetime transformations, though. It is perfectly permissible to mix different fields together, as is the case of gauge transformations or supersymmetry. If some function f (such as a correlator) is symmetric under the transformation, we should have $\delta_\varepsilon f = \varepsilon G f = 0$. As a simple example, consider a two-point function and let us find its variation under a symmetry of the field:

$$\begin{aligned}
\delta_\varepsilon\langle\phi(x)\phi(y)\rangle &= 0 & (2.59)\\
\langle\, (\delta_\varepsilon\phi(x))\phi(y)\,\rangle + \langle\, \phi(x)(\delta_\varepsilon\phi(y))\,\rangle &= 0 & (2.60)\\
\varepsilon\left[\langle\, G(x)\phi(x)\phi(y)\,\rangle + \langle\, \phi(x)G(y)\phi(y)\,\rangle\right] &= 0 & (2.61)\\
\left[(G(x) + G(y)\right]\langle\phi(x)\phi(y)\rangle &= 0 & (2.62)
\end{aligned}$$

where the generator $G(x)$ acts only on $\phi(x)$ and similarly for $G(y)$. In short, for an n-point correlator, the sum of each individual generator acting on each individual field must annihilate the correlator. As the generators are typically differential operators, this provides us with a differential equation that enforces a certain shape on the correlator, only from the symmetries of the field, without doing any quantum field theory proper.

Let us once again consider the example of translations. If the fields are invariant under them, our two-point function should satisfy:

$$\begin{aligned}
\phi(x)\phi(y)\rangle &= F(x,y) & (2.63)\\
\text{and}\;\; \left[a\cdot P_x + a\cdot P_y\right]\langle\phi(x)\phi(y)\rangle &= 0 & (2.64)\\
\Longrightarrow\;\; \left[a\cdot P_x + a\cdot P_y\right]F(x,y) &= 0 & (2.65)\\
a^\mu\left[\frac{\partial F}{\partial x^\mu} + \frac{\partial F}{\partial y^\mu}\right] &= 0 & (2.66)\\
\frac{\partial F}{\partial x^\mu} &= -\frac{\partial F}{\partial y^\mu} & (2.67)
\end{aligned}$$

with solution $F(x,y) = F(x-y)$. In other words, the two-point function can only depend on the vector connecting the two points, not on the points themselves. If we were to now repeat this procedure with rotational symmetry, we would've seen that $F(x-y) = F(|x-y|)$, i.e., it depends only on the distance between the points. While we would need to actually calculate the path integral to obtain the expression for F, symmetries allows us to quickly ascertain what form correlators should have, even when the path integral cannot be performed.

In general, any isometry of spacetime will be a symmetry of the field if the field's vacuum state (i.e., with no particles present) is invariant under that isometry. This is not a trivial assumption, as the phenomenon of spontaneous symmetry breaking by a non-symmetric vacuum state is not at all uncommon in field theory

There is one final concept regarding transformations we will make use of, that of an *integral curve*. The generators produce infinitesimal transformations, but what about the finite transformations? We can imagine performing an infinite number of infinitesimal transformations. If a parameter λ describes "for how long" we perform this procedure, then $\phi(t,\vec{x})(\lambda)$ is a collection of different field configurations $\phi(t,\vec{x})$, one for each λ describing the "amount" of transformation. Then we have that:

$$\frac{d}{d\lambda}\phi(\lambda) = \delta_\varepsilon \phi = \varepsilon G \phi \tag{2.68}$$

This is called an integral curve because we are essentially integrating the generator by solving the differential equation above.

2.4 INFLATION

We now have the machinery in place to describe cosmic inflation. The players of the game will be the dynamical metric $g_{\mu\nu}$ and a scalar field ϕ called the inflaton, which is coupled to gravity.

2.4.1 *Slow-roll background*

Let us assume that the metric takes the form of a Friedmann metric like Equation 2.46, but the evolution of the scale factor $a(t)$ remains unknown. Let us focus only on the background evolution of ϕ, thought to be a homogeneous perfect fluid. But the energy density and pressure of a fluid with a potential V are known:

$$\varepsilon(t) = \frac{1}{2}\dot{\phi}(t) + V(\phi) \tag{2.69}$$

$$P(t) = \frac{1}{2}\dot{\phi}(t) - V(\phi) \tag{2.70}$$

so the equation of state is $P = -\varepsilon + \dot{\phi}$. It is almost the equation of state of the vacuum, $P = -\varepsilon$. In any case, plugging this energy and pressure into the fluid Equation 2.1 yields:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{2.71}$$

where we have used $\frac{d}{dt}V(\phi) = \frac{d\phi}{dt}\frac{\partial}{\partial\phi}V(\phi) = \dot{\phi}V'(\phi)$. Together with the Friedmann Equation 2.44:

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi} + V(\phi)\right) \tag{2.72}$$

it will determine $a(t)$. Notice that the inflaton ϕ is the only one playing a role in the dynamics of the metric; we are treating the contributions by the cosmological constant or any curvature as negligible. The slow-roll approximation consists of neglecting $\ddot{\phi}$ compared to $V'(\phi)$ in the fluid Equation 2.71 and $\dot{\phi}$ to $V(\phi)$ in the Friedmann Equation 2.72. These conditions are equivalent to saying that:

$$\left|\frac{V'(\phi)}{V(\phi)}\right| \ll 1 \quad \left|\frac{V''(\phi)}{V(\phi)}\right| \ll 1 \tag{2.73}$$

from where the name "slow roll" comes from: The field slowly rolls down the potential. If this holds, the fluid and Friedmann equations simplify to:

$$\dot{\phi} = -\frac{V'(\phi)}{3H} \tag{2.74}$$
$$H^2 = \frac{8\pi G}{3} V(\phi) \tag{2.75}$$

But the time derivative of Equation 2.75 is $2\dot{H}H = \frac{8\pi G}{3}\dot{\phi}V'(\phi)$ and using Equation 2.74 to eliminate $\dot{\phi}$ results in:

$$\dot{H} = -\frac{4\pi G}{9H^2}[V'(\phi)]^2 \tag{2.76}$$
$$\frac{\dot{H}}{H^2} = -\frac{4\pi G}{9H^4}[V'(\phi)]^2 \tag{2.77}$$

Now we use Equation 2.75 once again to fully eliminate H^4 from the RHS:

$$\frac{\dot{H}}{H^2} = -\frac{1}{16\pi G}\left[\frac{V'(\phi)}{V(\phi)}\right]^2 = -\epsilon \tag{2.78}$$

where ϵ is the slow roll parameter. Thus, one of the conditions for slow-roll inflation is that $\epsilon \ll 1$.

Knowledge of ϵ—which depends uniquely on the inflaton potential—fixes the background dynamics of the expanding spacetime. For example, suppose $\epsilon = 0$. Then $\dot{H} = 0$. So we have:

$$\frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = 0 \tag{2.79}$$
$$\frac{\ddot{a}a - \dot{a}^2}{a^2} = 0 \tag{2.80}$$
$$\ddot{a}a - \dot{a}^2 = 0 \tag{2.81}$$
$$\implies a = a_0 e^{Ht} \tag{2.82}$$

where we have ignored the collapsing e^{-Ht} solution. Then using $dt = ad\tau$, we get $\tau = -\frac{1}{H}e^{-Ht}$ and can put the metric in the conformal form described by Equation 2.47:

$$ds^2 = \frac{d\tau^2 - d\vec{\chi}^2}{(H\tau)^2} \tag{2.83}$$

which we have already seen before; it is the metric of Equation 2.31 with three dimensions of space.

2.4.2 *Primordial fluctuations*

The solution discussed in Section 2.4.1 is only for the background of the inflaton. In reality, the full inflaton takes the form:

$$\phi(\tau, \vec{\chi}) = \phi_0(\tau) + \pi(\tau, \vec{\chi}) \tag{2.84}$$

where ϕ_0 is the background and π a perturbation. This is necessary because the universe is not, in fact, perfectly homogeneous. As such, the Friedmann metric will no longer work. However, at low amplitudes of π, we may employ the perturbed Friedmann metric. The procedure is highly technical but the metric we will consider takes the form [9]:

$$ds^2 = a(\tau)^2 \left(d\tau^2 + e^{-2H\pi(\vec{\chi})} d\vec{\chi}^2 \right) \tag{2.85}$$

in the limit of $\tau \to 0$. In this limit, it is known $\pi(\tau, \vec{\chi})$ freezes out, that is, stops evolving in time and becomes a function of space only.

For those familiar with the curvature perturbation ζ, we have $\zeta \approx -H\pi$ [10].

The objective then is to calculate correlators of the inflaton perturbations, such as $\langle \pi(\vec{\chi}_1)\pi(\vec{\chi}_2) \rangle$. For three-point functions, this was first done in [11] using the in-in formalism. There is also hope that we might use a dS/CFT duality to relate gravitational string theories in de Sitter space (the space of the metric of Equation 2.83) with conformal field theories on the $\tau \to 0$ boundary (see Section 4.2). This has been greatly successful with the AdS/CFT correspondence. Unfortunately, no solutions to string theory are known in de Sitter space, which limits this method for the time being.

As a final remark, the practical reader might be interested in knowing how to relate the calculation of inflaton correlators to an actual measurement. One of the best source of experimental data in cosmology is the cosmic microwave background (CMB), a sea of low-frequency photons that permeates the entirety of space. The CMB is extremely homogeneous, but not perfectly so. At any given point $\vec{\chi}$ in the sky, we associate a temperature contrast $[T(\vec{\chi}) - \bar{T}]/\bar{T}$ where $T(\vec{\chi})$ is the measured temperature and $\bar{T}$ the average temperature of the entire sky. Working in spherical coordinates (θ, φ), we can expand the contrast in terms of spherical harmonics, to work with the linearly independent modes $a_{\ell m}$:

Think of the CMB temperature in a certain region as associated with the average energy of the photons in it.

$$\frac{T(\theta, \varphi) - \bar{T}}{\bar{T}} = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{+\ell} a_{\ell m} Y^{\ell m}(\theta, \varphi) \tag{2.86}$$

where the functions $Y^{\ell m}$ are known.

Intuitively, the multipole ℓ reflects the scale of the $a_{\ell m}$ component, with small ℓ being large scale features, and m the orientation of these features.

We can define the CMB power spectrum as the average:

$$C_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{+\ell} |a_{\ell m}| \tag{2.87}$$

Very little information is lost by averaging over m because the CMB is isotropic, i.e., it looks the same in all directions. In momentum space, we can denote the two-point correlator $\langle \pi(\vec{k}_1)\pi(\vec{k}_2)\rangle = P(k)$ due to translational and rotational symmeties (recall Section 2.3.1). The function $P(k)$ is called the power spectrum of the primordial fluctuations. Then we can relate C_ℓ to $P(k)$ via the following convolution [12]:

$$C_\ell = \frac{2}{\pi} \int k^2 dk\, P(k) \Delta_\ell(k)^2 \tag{2.88}$$

where the so-called transfer function $\Delta(k)$ is typically found numerically. Now it is a simple matter of deconvolving the above expression to obtain the primordial power spectrum $P(k)$ directly from C_ℓ. This gives the powerful conclusion that quantum fluctuations of the early universe seed the statistical fluctuations of the CMB we observe today. Thus, to look at the CMB is to indirectly look at the inflaton.

If C_ℓ contains all the information you need to recreate the sky we observe, we say it is Gaussian. If that is not the case, it is non-Gaussian. The search for non-Gaussianities remains one of the most important goals of observational cosmology, because if they exist they will relate to higher inflaton correlators such as $\langle \pi\pi\pi \rangle$. This would shed further light into the type of interactions the inflaton undergoes, which is crucial knowledge if we wish to one day discern its fundamental nature.

Part II

DE SITTER

We will derive the finite isometries of de Sitter space in the so-called flat slicing coordinates, showing explicitly how they are inherited from the isometries of Minkowski space. This part starts with a 2D de Sitter space embedded in a 3D Minkowski space; we then extend the procedure to the full 4D de Sitter.

3 ISOMETRIES OF 2D DS

3.1 GLOBAL COORDINATES

We denote Minkowski space in three dimensions as $\mathbb{M}^3$ and the three corresponding coordinates by (x, y, z) where z is timelike; the metric signature is $(- - +)$. Then the two-dimensional de Sitter space is a submanifold of $\mathbb{M}^3$. It is characterized by the following embedding condition:

In general, $||x||^2 = -\frac{1}{H^2}$ for an arbitrary Hubble scale.

$$||x||^2 = -1 \implies -x^2 - y^2 + z^2 = -1 \tag{3.1}$$

Figure 3.1: The de Sitter hyperboloid embedded in Minkowski space, with the $x + z = 0$ plane cutting diagonally through it.

Geometrically, the embedding equation is that of a hyperboloid, though $||x||^2 = -1$ is by definition a sphere.

Because we embedded it in a three dimensional Minkowski space, our de Sitter space will inherit all $\mathbb{M}^3$ isometries that preserve the embedding condition $||x||^2 = -1$. So from the full Poincaré group, it is clear we lose the three translations of spacetime, but the Lorentz group $SO(1, 2)$ will persist as an isometry group, consisting of the zx-boost, the zy-boost and the xy-rotation. From this we conclude 2D de Sitter space possesses three isometries; for a two-dimensional manifold this means it must be maximally symmetry.

A maximally symmetric manifold of dimension n has $\frac{n(n+1)}{2}$ isometries.

There is still the issue of the coordinate system used to represent dS. The global coordinates x, y, z are inadequate for most purposes. To see this, we eliminate z in favor of x and y using the embedding condition:

$$z^2 = x^2 + y^2 - 1 \tag{3.2}$$

From this, recalculating the metric results in:

$$ds^2 = dz^2 - dx^2 - dy^2 = \frac{(xdx + ydy)^2}{x^2 + y^2 - 1} - dx^2 - dy^2 \tag{3.3}$$

The metric is not diagonal. It is convenient to parametrize x and y into a more favorable coordinate system that shall diagonalize it.

3.2 FLAT SLICING COORDINATES

There are three useful parametrizations of the global coordinates that will produce the desired diagonal metric, each corresponding to one of the three possible spatial curvatures signs k we saw in the Friedmann Equation 2.44. In this thesis, we concern ourselves only with the so-called flat slicing chart, the one with $k = 0$. They are the coordinates (t, χ) given by:

$$x = \cosh(t) - \frac{1}{2}\exp(t)\chi^2 \qquad y = \exp(t)\chi \tag{3.4}$$

and they cover only the part of the hyperboloid for which $x + z \geq 0$. In this new coordinate system, the metric becomes:

$$ds^2 = dt^2 - \left(e^t\right)^2 d\chi^2 \tag{3.5}$$

which is a diagonal and flat Friedmann metric with scale factor $a(t) = e^t$. The time t here is cosmic; we can further simplify the metric by going to conformal time τ. Using $a(t) = e^t$ and $dt = ad\tau$, we get $\tau = -e^{-t}$, or $t = -\ln(-\tau)$ from which it immediately follows that:

$$ds^2 = \frac{d\tau^2 - d\chi^2}{\tau^2} \tag{3.6}$$

for $\tau < 0$. Naturally, we have seen this already—it is the metric of Equation 2.83. Thus the de Sitter space is simply an inflationary background with slow roll parameter $\epsilon = 0$. We already know two isometries: the dS translation $\chi \to \chi + a$ and the dS dilation $(\tau, \chi) \to \Lambda(\tau, \chi)$. The symmetry that remains hidden we shall call a dS boost, for reasons that will later become apparent.

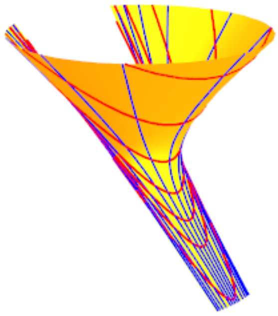

Figure 3.2: The portion of dS space covered in flat slicing coordinates. The slightly thicker lines (red in the digital version of this thesis) are lines of equal τ, whereas the thinner (blue) lines are of equal χ.

Meanwhile, in conformal coordinates, the chart simplifies to:

$$x = \frac{1}{2}\left(\frac{-1-\tau^2+\chi^2}{\tau}\right) \qquad y = -\frac{\chi}{\tau} \tag{3.7}$$

Also of importance is the z coordinate:

$$z = -\frac{1}{2}\left(\frac{1-\tau^2+\chi^2}{\tau}\right) \tag{3.8}$$

To study how the dS isometries arise from Minkowski transformations acting on (x, y, z), we must invert the above equations to express τ and χ in terms of x, y and z. This gives the result:

There are in fact two possible solutions, but only one satisfies $\tau > 0$ and $x + z \geq 0$.

$$\tau = -\frac{1}{x+z} \qquad \chi = \frac{y}{x+z} \tag{3.9}$$

Importantly, the dS isometries in flat coordinates are not in an one-to-one correspondence to the Minkowski isometries, so it will not follow that we can find each dS isometry simply by applying the three generators of $SO(1,2)$ on (x, y, z) and then using the above equations to write τ and χ after each transformation.

3.2.1 *dS dilation*

One special lucky case, however, is that the dS dilation corresponds exactly to a zx-boost in Minkowski.

Explicitly, the Lorentz transformation for this boost with a rapidity φ is:

$$z' = \cosh(\varphi)z + \sinh(\varphi)x \tag{3.10}$$
$$x' = \sinh(\varphi)z + \cosh(\varphi)x \tag{3.11}$$
$$y' = y \tag{3.12}$$

Then the factor $1/(x+z)$ will transform like:

$$x' + z' = [\cosh(\varphi) + \sinh(\varphi)]\,(x+z) = \Lambda^{-1}(x+z) \tag{3.13}$$

where $\Lambda^{-1} = \cosh(\varphi) + \sinh(\varphi)$. Thus the coordinates (τ, χ) transform like:

$$\begin{aligned} \tau' &= \Lambda\tau && (3.14)\\ \chi' &= \Lambda\chi && (3.15) \end{aligned}$$

Since $\Lambda \in]0, +\infty[$ for $\varphi \in]-\infty, +\infty[$, we have indeed obtained the dS dilation.

3.2.2 *dS translation*

Because this isometry is manifest, we know its form:

$$\begin{aligned} \tau' &= \tau && (3.16)\\ \chi' &= \chi + a && (3.17) \end{aligned}$$

Recalling that

$$\tau = -\frac{1}{x+z} \qquad \chi = \frac{y}{x+z} \tag{3.18}$$

we see that to obtain the translation, we must have

$$\begin{aligned} x' + z' &= x' + z' && (3.19)\\ y' &= y + a(x+z) && (3.20) \end{aligned}$$

A generic Lorentz transformation in $\mathbb{M}^3$ can be written in terms of the zx-boost B_x, zy-boost B_y and xy-rotation R:

This can be done in any order, but since the transformations don't commute, the order does change the angles.

$$M = B_x(\varphi)B_y(\psi)R(\theta) \tag{3.21}$$

where φ, ψ, θ are the rapidities and angles of the three Lorentz matrices:

$$B_x(\varphi) = \begin{bmatrix} \cosh\varphi & 0 & \sinh\varphi \\ 0 & 1 & 0 \\ \sinh\varphi & 0 & \cosh\varphi \end{bmatrix} \quad B_y(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\psi & \sinh\psi \\ 0 & \sinh\psi & \cosh\psi \end{bmatrix}$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.22}$$

We now must simply apply M on (x, y, z) and demand that the result be identical to Equations 3.19 and 3.20 for any x, y, z. This will solve for the three angles, giving the following result:

$$\varphi = \log\left(\frac{1}{\sqrt{1+a^2}}\right) \tag{3.23}$$
$$\psi = \operatorname{arcsinh}(a) \tag{3.24}$$
$$\theta = \arctan(a) \tag{3.25}$$

3.2.3 *dS boost*

This is the most complicated. To figure out the finite form of this isometry, we must start with the infinitesimal generator (Appendix A):

$$K = 2\chi\tau\partial_\tau + (\tau^2 + \chi^2)\partial_\chi \tag{3.26}$$

The finite transformation is described by the integral curve of the infinitesimal generator (see Section 2.3.1). The integral curve can be seen as the trajectory of a point in the manifold, indicating to where the point moves to as the transformation is continuously applied, which is described by the parameter λ. The equation of motion for $\tau(\lambda)$ is obtained by applying the generator to τ, i.e., $\tau'(\lambda) = \delta_\varepsilon\tau = \varepsilon K\tau$; ditto for χ. Thus:

$$\tau' = 2\varepsilon\chi\tau \tag{3.27}$$
$$\chi' = \varepsilon(\tau^2 + \chi^2) \tag{3.28}$$

Going to x, y, z coordinates, we get the equivalent system of differential equations:

$$(z+x)' = -2\varepsilon y \tag{3.29}$$
$$y'(z+x) - (z+x)'y = \varepsilon(1+y^2) \tag{3.30}$$

Solving these (Appendix B), we obtain the integral curves, resulting in the finite form of the transformation in terms of a parameter b:

$$z + x \to z + x + 2by + b^2(z-x) \tag{3.31}$$
$$y \to y + b(z-x) \tag{3.32}$$

Going back to τ, χ coordinates, the transformations are:

$$\tau \rightarrow \mathcal{B}_b(\tau, \chi)\, \tau \tag{3.33}$$
$$\chi \rightarrow \chi + b(\chi^2 - \tau^2) \tag{3.34}$$

where the boost factor is:

$$\mathcal{B}_b(\tau, \chi) = \frac{1}{1 + 2b\chi + b^2(\chi^2 - \tau^2)} \tag{3.35}$$

thus giving the finite form of the dS boost in flat slicing coordinates.

We still want to know how this transformation is inherited from the Minkowski isometries, as it will be important in the 4D case. To do this, we once again enforce that a certain Lorentz transformation of the form $M = B_x(\varphi)B_y(\psi)R(\theta)$ produces the above transformation. We thus get:

$$\varphi = \log(\sqrt{1+b^2}) \tag{3.36}$$
$$\psi = \operatorname{arcsinh}(b) \tag{3.37}$$
$$\theta = \arccos\left(\frac{1}{\sqrt{1+b^2}}\right) \tag{3.38}$$

Because M acts on each x, y and z, we can now write down the transformation for the x and the z individually, not just $x + z$, something which was not possible from the integral curve alone. Simply applying M we obtain the full transformation:

$$x \rightarrow x + by - \frac{1}{2}b^2 x \tag{3.39}$$
$$y \rightarrow y + b(z - x) \tag{3.40}$$
$$z \rightarrow z + by + \frac{1}{2}b^2 z \tag{3.41}$$

4 ISOMETRIES OF 4D DS

The procedure outlined in the previous chapter generalizes straightforwardly to any number dimensions; we will consider the usual four-dimensional spacetime here. Given Minkowski $\mathbb{M}^5$ with $(x_0, x_1, x_2, x_3, x_4)$ and signature $(+----)$, with x_0 being timelike, de Sitter space is characterized by the embedding:

$$||x||^2 = -1 \implies -x_0^2 + x_1^2 + x_3^2 + x_4^2 + x_5^2 = 1 \tag{4.1}$$

Again, the isometry group will be the subgroup of $\mathbb{M}^5$ that leaves the embedding invariant—the Lorentz group $SO(1,4)$. We will now have three boosts and three rotations. But first we must diagonalize the metric once again.

4.1 FLAT SLICING COORDINATES

The flat slicing coordinates are now given by the following chart, already using conformal time:

$$x_4 = \frac{1}{2}\left(\frac{-1-\tau^2+\chi^2}{\tau}\right) \tag{4.2}$$

$$x_i = -\frac{\chi_i}{\tau}, \qquad \text{for } i = 1,2,3 \tag{4.3}$$

with $\chi^2 = |\vec{\chi}|^2$. The metric is now, unexcitingly:

$$ds^2 = \frac{d\tau^2 - d\vec{\chi}^2}{\tau^2} \tag{4.4}$$

The parametrization for x_0 follows from the embedding condition and is:

$$x_0 = -\frac{1}{2}\left(\frac{1-\tau^2+\chi^2}{\tau}\right) \tag{4.5}$$

It should come as no surprise that the inversion of the coordinates is:

$$\tau = \frac{-1}{x_0 + x_4} \tag{4.6}$$

$$\chi_i = \frac{x_i}{x_0 + x_4} \qquad \text{for } i = 1,2,3 \tag{4.7}$$

Because each x_i is in a one-to-one correspondence with each χ_i, the transformations in 3D space follow immediately from the 1D transformations.

4.1.1 dS dilation, translation and rotation

Previously, the dS dilation was realized by the Lorentz boost B_x. Not much has changed except x is now called x_4. The dilation $(\tau, \vec{\chi}) \to \Lambda(\tau, \vec{\chi})$ is thus realized by the Lorentz boost B_{x_4}.

As for the translation, we saw it was a result of the Lorentz transformation $B_x(\varphi)B_y(\psi)R(\theta)$. Now, when one χ_i translates, the others remain invariant, and since we have a one-to-one correspondence between the χ_i and the x_i for $i = 1, 2, 3$, we keep the same Lorentz transformation as before. Therefore, the χ_i translation will be realized by $B_{x_4}(\varphi)B_{x_i}(\psi)R_{x_i x_4}(\theta)$ with the same expression for the rapidities and angles described by Equation 3.23.

Finally, we have the rotation. This one is new—we did not have rotations in 1D space. However, it is immediately obvious that the rotations $R_{x_1x_2}$, $R_{x_2x_3}$ and $R_{x_3x_1}$ in Minkowski space match the corresponding rotations for χ_1,χ_2 and χ_3 of de Sitter one-to-one.

4.1.2 dS boost

We could, of course, solve the integral curve once again to obtain the 4D dS boost, but the differential equations now become laborious to solve. Now that we know how the boost is inherited via the Lorentz transformations, though, there is no need for the integral curves. The argument is similar to that of the translations—each dS boost is realized by the Lorentz transformation $B_{x_4}(\varphi)B_{x_i}(\psi)R_{x_i x_4}(\theta)$ for $i = 1, 2, 3$ with the same angles as given by Equation 3.38. Simply do the Lorentz transformation then convert to the flat slicing chart. So a boost along χ_1 is given by:

$$\tau \to \frac{\tau}{1 + 2b\chi_1 + b^2(\chi^2 - \tau^2)} \tag{4.8}$$

$$\chi_1 \to \frac{\chi_1 + b(\chi^2 - \tau^2)}{1 + 2b\chi_1 + b^2(\chi^2 - \tau^2)} \tag{4.9}$$

$$\chi_2 \to \frac{\chi_2}{1 + 2b\chi_1 + b^2(\chi^2 - \tau^2)} \tag{4.10}$$

$$\chi_3 \to \frac{\chi_3}{1 + 2b\chi_1 + b^2(\chi^2 - \tau^2)} \tag{4.11}$$

with similar expressions for the boosts along χ_2 and χ_3. Since a boost in a spatial direction leaves the others unchanged except for a multiplicative factor, this justifies the name "boost" we gave before.

If we choose $\vec{b} = (b, 0, 0)$, the expression for the boost can be compactly written as:

$$\tau \to \mathcal{B}_{\vec{b}}(\tau, \chi)\, \tau \tag{4.12}$$

$$\vec{\chi} \to \mathcal{B}_{\vec{b}}(\tau, \chi)\, [\vec{\chi} + \vec{b}(\chi^2 - \tau^2)] \tag{4.13}$$

where the boost factor in 3D is now:

$$\mathcal{B}_{\vec{b}}(\tau,\vec{\chi}) = \frac{1}{1+2\vec{b}\cdot\vec{\chi}+b^2(\chi^2-\tau^2)} \tag{4.14}$$

This expression is invariant under rotations, which are another isometry. So we extend the above transformation rule for any generic boost along any direction.

The infinitesimal transformation, taken by considering the limit of an infinitesimal vector $\vec{b}$, is to first order:

$$\begin{aligned} \tau &\to \tau + 2(\vec{b}\cdot\vec{\chi})\,\tau \\ \vec{\chi} &\to \vec{\chi} - 2(\vec{b}\cdot\vec{\chi})\,\vec{\chi} + (\chi^2-\tau^2)\,\vec{b} \end{aligned} \tag{4.15}$$

which agrees with results found in literature [13].

4.2 DS/CFT

It was advertised that a field theory living in the curved spacetime of 4D de Sitter should correspond to a conformal theory in its 3D flat boundary. Let us discuss this fact.

We take the boundary of dS to correspond to the limit of infinite future $\tau \to 0$. Then we notice that all ten dS isometries preserve this condition, that is, they map $\tau = 0$ to $\tau = 0$.

Now let us assume that we have a field theory of one or more fields living in dS. For simplicity we take these fields to be scalars $\phi(\tau,\vec{\chi})$. On the boundary, they will become $\phi_0(\vec{\chi}) = \phi(0,\chi)$. As explained in Section 2.3.1, if the fields' vacua are symmetric under the metric isometries—and we shall assume they are—then said isometries will be symmetries of the fields as well. Our fields, whether ϕ in the 4D bulk or ϕ_0 int the 3D boundary, are therefore symmetric under translations, rotations, dilations and boosts.

The boundary is purely spatial, i.e., ϕ_0 is simply a function of space. Thus ϕ_0 is symmetric under three spatial translations and three spatial rotations, which are precisely the isometries of flat Euclidean space in three dimensions.

Meanwhile, the dS boost takes a very special form in the boundary. Setting $\tau = 0$ gives:

$$\vec{\chi} \to \frac{\vec{\chi}+\vec{b}\chi^2}{1+2\vec{b}\cdot\vec{\chi}+b^2\chi^2} \tag{4.16}$$

which, as we have seen in Section 2.1.4, is the SCT.

So, to recap, our field ϕ has translational and rotational symmetry and, on the boundary, ϕ_0 preserves these symmetries. But since these are precisely the isometries of flat space, we might see ϕ_0 as being in flat space. But even under this view, ϕ has two other remaining symmetries: dilations and the SCT. These are *not* isometries of flat space, but they *do* correspond to its conformal transformations. Therefore, field theories in de Sitter space should be equivalent to field theories living on the $\tau \to 0$ boundary, where they inhabit a flat space but also possess the conformal symmetries of this space. In other words, they are conformal field theories. Do note that the conformal scaling dimension Δ of the boundary field should depend on the τ-dependence of the bulk field [14].

This equivalence is called the dS/CFT correspondence.

Part III

QUASI-DE SITTER

We will now apply the symmetries of the previous part to the quasi-de Sitter space which corresponds to slow roll inflation. Some symmetries will be preserved, others will be broken, and some will reemerge in a restricted form. Finally, we discuss an application of these results.

5 ISOMETRIES OF QUASI-DS

5.1 CONSTRUCTION OF QUASI-DS

Describing the inflationary background of our universe as the de Sitter universe, while convenient, consigns inflation to never end—this is because dS is a non-evolving space, with constant curvature everywhere and at all times. Any physically relevant model of inflation, however, has to end. Perhaps the most famous of such models is the quasi-de Sitter space. By breaking the time symmetries of de Sitter by a small amount $\epsilon \ll 1$, we obtain a space that evolves in time while still being homogeneous. We have already seen this slow roll parameter in Section 2.4.1, where it was defined as:

In flat slicing coordinates $ds^2 = \frac{d\tau^2 - d\vec{\chi}^2}{\tau^2}$, even the boundary of infinite future $\tau \to 0$ has constant curvature; it is not a gravitational singularity.

$$\epsilon = -\frac{\dot{H}}{H^2} \tag{5.1}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Then, using $dt = a\,d\tau$:

$$\frac{d}{d\tau}\left(\frac{1}{aH}\right) = \frac{dt}{d\tau}\frac{d}{dt}\left(\frac{1}{aH}\right) \tag{5.2}$$

$$= a\,\frac{-\dot{a}H - \dot{H}a}{a^2H^2} \tag{5.3}$$

$$= -\frac{\dot{H}}{H^2} - \frac{\dot{a}}{aH} \tag{5.4}$$

$$= \epsilon - 1 \tag{5.5}$$

So far this is exact. The slow roll approximation consists, first, in setting ϵ to a nonzero constant. Then we can solve the above equation straightforwardly:

$$\frac{1}{aH} = (\epsilon - 1)\tau \tag{5.6}$$

Further assume that $\epsilon \ll 1$ but nonzero. As we saw, with $\epsilon = 0$ we retrieve pure dS, hence the name quasi-dS for small ϵ. In any case, we get:

$$aH = \frac{-(1+\epsilon)}{\tau} \tag{5.7}$$

From the definition of the Hubble parameter, we obtain:

$$H = \frac{1}{a}\frac{da}{dt} = \frac{1}{a}\frac{da}{d\tau}\frac{d\tau}{dt} = \frac{1}{a^2}\frac{da}{d\tau} \tag{5.8}$$

From Equations 5.7 and 5.8, this solvable differential equation follows:

$$\frac{a'}{a} = -(1+\epsilon)\tau \tag{5.9}$$

$$a = \left(\frac{C}{\tau}\right)^{1+\epsilon} \tag{5.10}$$

where C is a dimensionful constant of integration. The scale factor is dimensionless, so by dimensional consistency and comparing it to the $\epsilon = 0$ case of Equation 2.31, we must have $C = \frac{1}{H_*}$ where H_* is some fixed Hubble scale. As it will not be necessary for the calculations that follow, we set $H_* = 1$; it can be retrieved via dimensional analysis. The Friedmann metric is now found, again in the $(\tau, \vec{\chi})$ coordinates:

$$ds^2 = a(\tau)^2(d\tau^2 - d\vec{\chi}^2) \tag{5.11}$$

$$= \frac{d\tau^2 - d\vec{\chi}^2}{\tau^{2+2\epsilon}} \tag{5.12}$$

As it was advertised, quasi-dS remains homogeneous: Any purely spatial isometry inherited from exact dS remains an isometry. Isometries involving time, however, are now clearly broken, but only slightly so, as characterized by the small ϵ.

5.2 APPROXIMATE ISOMETRIES

Space translations and rotations remain exact isometries of quasi-dS. As for dilations and boosts, they will be slightly broken as they involve a transformation of time. Nonetheless, we can still calculate how the metric changes under the transformations.

5.2.1 *quasi-dS dilation*

Recall that the dilation, which was an isometry of our original pure de Sitter space, took the form:

$$\tau \to \Lambda\tau \tag{5.13}$$

$$\vec{\chi} \to \Lambda\vec{\chi} \tag{5.14}$$

Obviously, then $d\tau \to \Lambda d\tau$ and $d\vec{\chi} \to \Lambda d\vec{\chi}$ under a dilation. We quickly observe that the spacetime distance indeed is not invariant:

$$ds^2 \to \frac{\Lambda^2 d\tau^2 - \Lambda^2 d\vec{\chi}^2}{\Lambda^{2+2\epsilon}\tau^{2+2\epsilon}} \tag{5.15}$$

$$ds \to \frac{1}{\Lambda^{\epsilon}} ds \tag{5.16}$$

From this we can also see that the dilation is what is called a quasi-isometry, because the distance between any two points cannot grow arbitrarily large.

5.2.2 *quasi-dS boost*

Meanwhile, the exact dS boost isometry by some vector $\vec{b}$ took the form we derived in Equation 4.13:

$$\tau \rightarrow \mathcal{B}_{\vec{b}}(\tau, \vec{\chi})\, \tau \tag{5.17}$$

$$\vec{\chi} \rightarrow \mathcal{B}_{\vec{b}}(\tau, \vec{\chi})\, [\vec{\chi} + \vec{b}(\chi^2 - \tau^2)] \tag{5.18}$$

where the boost factor is:

$$\mathcal{B}_{\vec{b}}(\tau, \chi) = \frac{1}{1 + 2\vec{b} \cdot \vec{\chi} + b^2(\chi^2 - \tau^2)} \tag{5.19}$$

We now have to calculate how the spacetime forms $d\tau$ and $d\vec{\chi}$ transform under the boost. As one may imagine, this is fairly laborious, but otherwise a trivial task. For simplicity, we shall only quote the final result for the transformation of the interval ds under the boost isometry:

$$ds \rightarrow \frac{1}{\mathcal{B}_{\vec{b}}(\tau, \chi)^{\epsilon}} ds \tag{5.20}$$

Curiously, the symmetry breaking is spacetime-depended; the distance between two points can grow arbitrarily large depending on their respective spacetime coordinates. It becomes obvious at this point that a full understanding of the boost factor is necessary.

5.2.2.1 *Boost factor*

Using rotational invariance, let us set $\vec{\chi} = (\chi, 0, 0)$ and pick $\vec{b} = (b, 0, 0)$ so that we may plot $\mathcal{B}_{\vec{b}}(\tau, \vec{\chi})$ for some value of b (Figure 5.1).

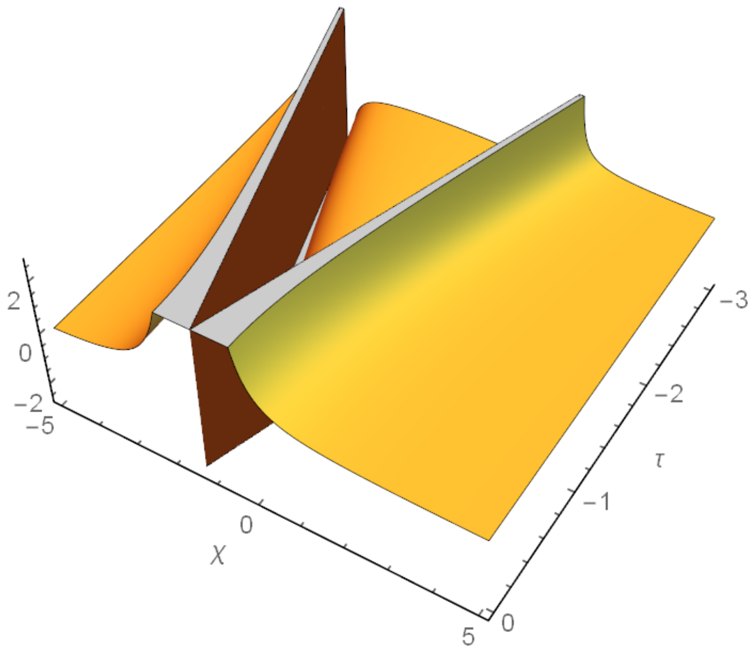

Figure 5.1: Plot of the boost factor $\mathcal{B}_{\vec{b}}(\tau,\vec{\chi})$ for $\vec{\chi}=\chi\vec{u}_1$ and $\vec{b}=0.7\vec{u}_1$.

As we can see, there are two lines forming a "V" shaped figure—these correspond to $\mathcal{B}_{\vec{b}}(\tau,\vec{\chi})=\pm\infty$. Their equations are given by

$$1+2\vec{b}\cdot\vec{\chi}+b^2(\chi^2-\tau^2) = 0 \quad (5.21)$$

$$\tau^2 = \frac{1}{b^2}(1+2\vec{b}\cdot\vec{\chi}+b^2\chi^2) \quad (5.22)$$

In the case of the plot of Figure 5.1, this reduces to $\chi=\pm\tau-\frac{1}{b}$. Note that because $ds\to\frac{1}{\mathcal{B}_{\vec{b}}(\tau,\chi)^{\epsilon}}ds$ and $\epsilon\ll 1$, the metric isn't immediately degenerate for $\mathcal{B}_{\vec{b}}(\tau,\vec{\chi})\gg 1$.

5.2.2.2 *Perfect submanifold*

Let us now focus on $\mathcal{B}_{\vec{b}}(\tau,\vec{\chi})=1$. This equation defines a submanifold of quasi-dS in which $ds\to ds$, i.e., where the boost becomes an exact isometry once again, for any value of ϵ. We thus see that:

While the equation for the perfect submanifold embedding appears similar to the embedding of dS in $\mathbb{M}^4$, it is not itself a dS, because the ambient space is quasi-dS, not $\mathbb{M}^4$.

$$1+2\vec{b}\cdot\vec{\chi}+b^2(\chi^2-\tau^2) = 1 \quad (5.23)$$

$$b^2\tau^2-b^2\chi^2-2\vec{b}\cdot\vec{\chi} = 0 \quad (5.24)$$

$$\tau^2-\chi^2-2\frac{\vec{b}}{b^2}\cdot\vec{\chi}-\frac{1}{b^2} = -\frac{1}{b^2} \quad (5.25)$$

$$\tau^2-\left(\vec{\chi}+\frac{1}{\vec{b}}\right)^2 = -\frac{1}{b^2} \quad (5.26)$$

where $\frac{1}{\vec{b}}=\frac{\vec{b}}{b^2}$, i.e., the inversion of $\vec{b}$.This establishes the perfect submanifold, that is, the submanifold of quasi-de Sitter space in which

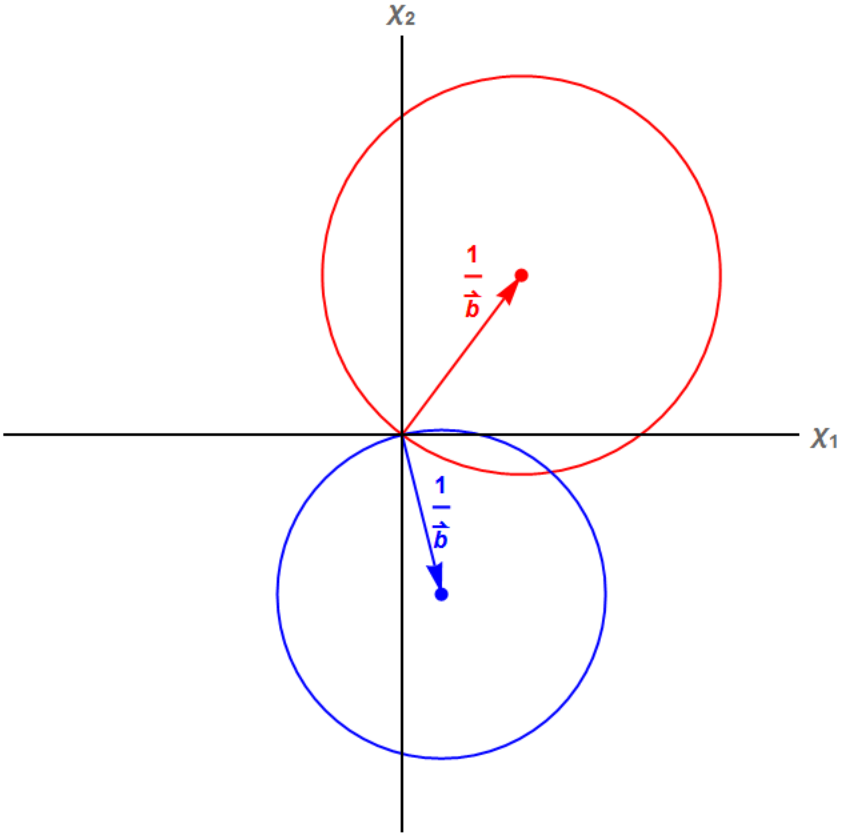

Figure 5.2: For every vector $\vec{b}$, there exists a circle of radius $\frac{1}{b}$ centered on $\frac{1}{\vec{b}}$ in which the SCT is an exact isometry.

the dS boost is an exact, not approximate, symmetry of the metric. Do note that we have one such manifold for each vector $\vec{b}$.

Of particular interest is the case where the boost becomes the special conformal transformation, that is, on the boundary of infinite future $\tau \to 0$. Then the submanifold equation is simply:

$$\left(\vec{\chi} + \frac{1}{\vec{b}}\right)^2 = \frac{1}{b^2} \tag{5.27}$$

i.e., that of spheres of radii $\frac{1}{b}$ centered on $\frac{1}{\vec{b}}$. Since translations are still exact isometries, we could transform $\vec{\chi} \to \vec{\chi} - \frac{1}{\vec{b}}$ for a specific vector $\vec{b}$ to center the corresponding sphere at the origin.

6 CONSISTENCY RELATIONS

Here we discuss how the isometries of de Sitter space or quasi-de Sitter space we found can be non-linearly realized for the inflaton field and possible applications of this result.

Symmetries of the metric are not automatically symmetries of our field. *If* the field's background is symmetric, then the field will evolve in a symmetric manner. For the inflaton it *cannot* be the case. This is because the homogeneous inflaton background $\phi_0(t)$ has to be a clock to keep track of when inflation ends. In fact, with proper change of coordinates and field redefinitions, we can write:

We return to the use of cosmic time t rather than conformal time $\tau = -\frac{1}{H}e^{-Ht}$.

$$\phi(t,\vec{x}) = \phi_0(t) + \pi(t,\vec{x}) \tag{6.1}$$

$$\phi_0(t) = t \tag{6.2}$$

where ϕ_0 is the homogeneous background and π a perturbation. Clearly, the background is not symmetric under any de Sitter isometry that involves time. However, it is still possible to realize these isometries if ϕ possesses additional symmetries. Let us see this procedure in action. The time dilation $\tau \to \Lambda\tau$ in cosmic time is:

$$t \to t - \frac{\log\Lambda}{H} \tag{6.3}$$

Suppose now ϕ is shift-symmetric, i.e., $\phi \to \phi + c$ for constant c is a symmetry of the action. The current effective models for inflation do predict the inflaton has this property in dS and approximately in quasi-dS [15]. Then a dilation will add the $-\log\Lambda/H$ term, but a shift of $c = \log\Lambda/H$ will kill it. In this way, the background ϕ_0 will remain invariant under the combined dilation-shift and we associate the shift of the inflaton to a shift in the perturbation. But from the metric of Equation 2.85 in the infinite future,

$$ds^2 = dt^2 - a(t)^2 e^{-2H\pi(\vec{x})} d\vec{x}^2 \tag{6.4}$$

a shift $\pi \to \pi - \log\Lambda/H$ realizes a *spatial* dilation by Λ. So a constant shift to π corresponds to a dilation.

Similarly, boosts will transform time as follows, to first order in $\vec{b}$:

$$t \to t + 2\vec{b}\cdot\vec{x} \tag{6.5}$$

Thus, for this symmetry to be linearly realized, the inflaton needs to have not a shift symmetry, but a Galilean symmetry $\phi \to \phi + \vec{b}\cdot\vec{x}$. In any event, this will introduce a gradient to π, which is realized by a first order SCT [13].

Even if the inflaton does not possess the required symmetries to non-linearly realize the de Sitter isometries, we may still obtain consistency relations related to the breaking of this symmetry, but only if there is a single inflaton. If multiple fields contribute to inflation, the situation becomes far more complex, as there are then multiple inflation clocks.

6.1 DERIVATION OF CONSISTENCY RELATIONS

Consider the correlator between one long mode $\pi_L(\vec{\chi})$ and two short, $\pi_S(\vec{\chi}_1)$ and $\pi_S(\vec{\chi}_2)$. A long mode in this context means that the long mode is predominantly composed of Fourier components of small wavelength in momentum space.

A consistency relation is also called a soft relation.

All modes eventually freeze, but long modes do so much earlier. At the moment when the short modes freeze, the long mode has long since become a non-dynamical background field. Thus:

$$\langle \pi_L(\vec{\chi})\pi_S(\vec{\chi}_1)\pi_S(\vec{\chi}_2)\rangle = \Big\langle \pi_L(\vec{\chi})\, \langle \pi_S(\vec{\chi}_1)\pi_S(\vec{\chi}_2)\rangle_{\pi_L(\vec{\chi})} \Big\rangle$$

where a subscript in a correlator means it is evaluated in the corresponding background. Meanwhile, a non-dynamical mode simply acts on the dynamical ones as a spacetime transformation $\vec{\chi} \to \vec{\chi}'$ [13]. This means a power spectrum of short modes evaluated under the background of a long mode is the same power spectrum without the background but under a spacetime transformation:

$$\langle \pi_S(\vec{\chi}_1)\pi_S(\vec{\chi}_2)\rangle_{\pi_L(\vec{\chi})} = \langle \pi_S(\vec{\chi}'_1)\pi_S(\vec{\chi}'_2)\rangle \tag{6.6}$$

So we have removed the background, but information about the long mode is still encoded via the spacetime transformation. For visibility, we will omit the space variables and simple denote $\pi = \pi(\vec{\chi})$ and $\tilde{\pi} = \pi(\vec{\chi}')$. Multiply both sides by another copy of the long mode π_L and take the average:

$$\langle \pi_L \, \langle \pi_S \pi_S \rangle_{\pi_L} \rangle = \langle \pi_L \, \langle \tilde{\pi}_S \tilde{\pi}_S \rangle \rangle \tag{6.7}$$

But if the generator of the transformation is $G = G(\vec{\chi}_1) + G(\vec{\chi}_2)$ (recall Equation 2.62), we have:

$$\langle \tilde{\pi}_S \tilde{\pi}_S \rangle = \langle \pi_S \pi_S \rangle + G \langle \pi_S \pi_S \rangle \tag{6.8}$$

to first order in the transformation. Now:

$$\langle \pi_L \, \langle \pi_S \pi_S \rangle_{\pi_L} \rangle = \langle \pi_L \, G \langle \pi_S \pi_S \rangle \rangle \tag{6.9}$$

because $\langle \pi_L \, \langle \pi_S \pi_S \rangle \rangle = 0$. From this, we now get the consistency relation between a long mode and two short:

$$\langle \pi_L \pi_S \pi_S \rangle = \langle \pi_L \, G \langle \pi_S \pi_S \rangle \rangle \tag{6.10}$$

Notice that if G is a symmetry of the power spectrum $\langle \pi_S \pi_S \rangle$, then the result is zero. In other words, because the long mode is just a spacetime transformation, if said transformation leaves the power spectrum invariant, then of course there can be no correlation between long and short modes. But the equation is still true if the symmetry is broken, as it is in quasi-de Sitter, which gives us a consistency relation between the two correlators based on the breaking of the symmetry.

To understand which spacetime transformations correspond to the long mode we simply expand π_L to first order in the gradient around the origin (because space translations are still exact isometries, we can expand around it without loss of generality):

$$\pi_L(\vec{\chi}) = \pi_L(0) + \vec{\chi} \cdot \vec{\partial}_\chi \pi_L(0) \tag{6.11}$$

But as we have seen, a constant background corresponds to a dilation and a gradient to a first-order boost. This means we can use the associated generators for the spatial dilation and SCT to derive the consistency relations which are available in the literature [13].

Implications of this thesis's result includes the possibility of using the finite de Sitter boost we obtained to get the corresponding non-linearly realized symmetry to any order. This could be used for example to expand the field to higher order in the gradient, which would be relevant for consistency relations involving two long modes, or to derive the relations to higher order in the coordinates. This is an area of active research [16, 17].

Part IV

CONCLUSION

7 CONCLUSION AND OUTLOOK

Inflation remains the preferred solution to several problems encountered in physical cosmology. As we have seen, de Sitter space is a useful gravitational background for our models of inflation and the importance of this space's isometries was therefore highlighted. We have explicitly shown how the de Sitter isometries are inherited from the Minkowski embedding, which allowed us to construct the finite form of the de Sitter boost in flat coordinates, a result which we believe was not available in the literature before. The link between de Sitter space and conformal field theories (dS/CFT) was also established. This is a modern area of research, which could flourish if a better understanding of string theories in dS is achieved.

Second, we studied what role these isometries play in quasi-de Sitter space. This is important because often the limit of exact de Sitter is not acceptable and we must employ the slow roll approximation. We have derived how some of these isometries are broken and discussed the properties of the broken boost. Furthermore, even if the inflaton background is not symmetrical under these isometries (either in dS or quasi-dS), the non-linear realization of these transformations allows us to relate correlators to the amount of symmetry breaking. This means the full and exact isometry group of de Sitter space that we have obtained is of great importance even outside of dS.

A valid counterpoint that could be presented to this thesis is that we did not consider the other two coordinate system for dS, namely the closed and open charts. It is true that a full treatment of de Sitter isometries would repeat the procedure for these two, but we believe this is not an issue of great importance. The flat coordinate system is the most widely employed and physically convenient system of dS. If necessary, though, the procedures outlined in this thesis to obtain the exact isometries of dS generalizes to closed and open dS in a straightforward manner.

This thesis's results naturally extend to further study of non-linearly realized inflaton symmetries. With the exact expression of the boost, we can easily expand it to any order. This could then be employed to calculate to higher order the effects of one or more long inflaton modes with respect to short modes in a correlator. We hope this will lead to new insights into the consistency relations of cosmology, which are an essential experimental tool in the observation of CMB non-Gaussianities.

Part V

APPENDIX

A KILLING VECTORS OF DS

In the flat conformal chart, the metric of the 2D de Sitter space takes the form:

$$ds^2 = \frac{d\tau^2 - d\chi^2}{\tau^2} \tag{A.1}$$

What are the Killing vectors ξ of dS, written in this chart? Obviously the metric doesn't depend on χ, so one Killing vector should be $\xi^\mu = (0,1)$. Now, using that

$$g_{\mu\nu} = \frac{1}{\tau^2}\eta_{\mu\nu} \implies \partial_\rho g_{\mu\nu} = -\frac{2}{\tau^3}\delta^0_\rho \eta_{\mu\nu} \tag{A.2}$$

we can calculate:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{A.3}$$

$$= \frac{1}{\tau}\left(\eta^{\rho 0}\eta_{\mu\nu} - \delta_{0\nu}\delta^\rho_\mu - \delta_{0\mu}\delta^\rho_\nu\right) \tag{A.4}$$

The Killing equations are:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{A.5}$$

$$\implies \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\Gamma^\rho_{\mu\nu}\xi_\rho = 0 \tag{A.6}$$

giving three equations:

$$\partial_0 \xi_0 + \frac{1}{\tau}\xi_0 = 0 \tag{A.7}$$

$$\partial_1 \xi_1 + \frac{1}{\tau}\xi_0 = 0 \tag{A.8}$$

$$\partial_0 \xi_1 + \partial_1 \xi_0 + \frac{2}{\tau}\xi_1 = 0 \tag{A.9}$$

Equation A.7 is separable, i.e., $\frac{\partial \xi_0}{\xi_0} = -\frac{\partial \tau}{\tau}$ and thus readily admits the solution

$$\xi_0(\tau,\chi) = \frac{f(\chi)}{\tau} \tag{A.10}$$

where $f(\chi)$ is an unknown function. Let us write by ansatz that $f(\chi) = A\chi^B$. Then Equation A.8 becomes:

$$\frac{\partial \xi_1}{\partial \chi} + \frac{A\chi^B}{\tau^2} = 0 \tag{A.11}$$

which, being also separable, admits the solution:

$$\xi_1(\tau,\chi) = -\frac{A\chi^{B+1}}{(B+1)\tau^2} + g(\tau) \tag{A.12}$$

where $g(\tau)$ is another unknown function. It can be found via Equation A.9, which now reads:

$$\frac{AB\chi^{B-1}}{\tau} + \frac{2}{\tau}g(\tau) + g'(\tau) = 0 \tag{A.13}$$

and is solved by

$$g(\tau) = -\frac{1}{2}AB\chi^{B-1} + \frac{C}{\tau^2} \tag{A.14}$$

As $g(\tau)$ is not a function of χ, the above solution is patent nonsense unless the χ-dependency vanishes. This forces $A = 0$, $B = 0$ or $B = 1$. We can now write three independent Killing vectors:
When $A = 0$, $B = 0$ and $C = -1$, we have:

$$\xi_\mu = (0, -\frac{1}{\tau^2}) \tag{A.15}$$

$$\xi^\mu = (0, 1) \tag{A.16}$$

which is the spatial translation vector, as expected.
When $A = 1$, $B = 0$ and $C = 0$, we have:

$$\xi_\mu = (\frac{1}{\tau}, -\frac{\chi}{\tau^2}) \tag{A.17}$$

$$\xi^\mu = (\tau, \chi) \tag{A.18}$$

corresponding to dilations.
Finally, if $A = 2$, $B = 1$ and $C = 0$, we have:

$$\xi_\mu = (\frac{2\chi}{\tau}, -\frac{\chi^2+\tau^2}{\tau^2}) \tag{A.19}$$

$$\xi^\mu = (2\chi\tau, \chi^2+\tau^2) \tag{A.20}$$

which is a dS boost vector.

Incidentally, since we have found three Killing vectors and a maximally symmetric 2D manifold has three isometries, this establishes that dS is maximally symmetric.

From the Killing vector fields, we get the infinitesimal generators for spatial translations, dilations and boosts, respectively:

$$P = \partial_\chi \tag{A.21}$$

$$D = \tau\partial_\tau + \chi\partial_\chi \tag{A.22}$$

$$K = 2\chi\tau\partial_\tau + (\chi^2+\tau^2)\partial_\chi \tag{A.23}$$

B THE DS BOOST INTEGRAL CURVE

Let us call $w_+ = z + x$ and $w_- = z - x$. Then we must solve the following system of differential equations:

$$\begin{aligned} w'_+ &= -2\varepsilon y && \text{(B.1)} \\ y'w_+ - w'_+ y &= \varepsilon(1+y^2) && \text{(B.2)} \end{aligned}$$

By applying the chain rule $y' = \frac{\mathrm{d}y}{\mathrm{d}\lambda} = \frac{\mathrm{d}y}{\mathrm{d}w_+}\frac{\mathrm{d}w_+}{\mathrm{d}\lambda}$ and using the first equation, the second one becomes:

$$\begin{aligned} -2\varepsilon y w_+ \frac{\mathrm{d}y}{\mathrm{d}w_+} + 2\varepsilon y^2 &= \varepsilon(1+y^2) && \text{(B.3)} \\ -2yw_+ \frac{\mathrm{d}y}{\mathrm{d}w_+} &= 1 - y^2 && \text{(B.4)} \\ \frac{-2y}{1-y^2}\mathrm{d}y &= \frac{\mathrm{d}w_+}{w_+} && \text{(B.5)} \\ \int_{y_0}^{y} \frac{-2\bar{y}}{1-\bar{y}^2}\mathrm{d}\bar{y} &= \int_{w_{0+}}^{w_+} \frac{\bar{w}_+}{\bar{w}_+} && \text{(B.6)} \\ \log\left(\frac{y^2-1}{y_0^2-1}\right) &= \log\left(\frac{w_+}{w_{0+}}\right) && \text{(B.7)} \\ \therefore y^2 &= \frac{w_+(y_0^2-1)}{w_{0+}} + 1 && \text{(B.8)} \end{aligned}$$

But recall the embedding condition:

$$\begin{aligned} z^2 &= x^2 + y^2 - 1 && \text{(B.9)} \\ z^2 - x^2 &= y^2 - 1 && \text{(B.10)} \\ w_+ w_- &= y^2 - 1 && \text{(B.11)} \end{aligned}$$

which means the initial conditions w_0 and y_0 are related via

$$\frac{y_0^2 - 1}{w_{0+}} = w_{0-} \qquad \text{(B.12)}$$

This simplifies our solution for y to a much nicer form:

$$y^2 = w_{0-} w_+ + 1 \qquad \text{(B.13)}$$

It remains to solve for w_+. Let us return to the first differential equation $w'_+ = -2\varepsilon y$. With the solution for y in hand, we now proceed:

$$
\begin{aligned}
w'_+ &= \pm 2\varepsilon\sqrt{w_{0-}w_+ + 1} && \text{(B.14)}\\
\frac{\mathrm{d}w_+}{\sqrt{w_{0-}w_+ + 1}} &= \pm 2\varepsilon \mathrm{d}\lambda && \text{(B.15)}\\
\int_{w_{0+}}^{w} \frac{\mathrm{d}\bar{w}_+}{\sqrt{w_{0-}\bar{w}_+ + 1}} &= \pm 2\varepsilon \int_0^{\lambda} \mathrm{d}\bar{\lambda} && \text{(B.16)}\\
\frac{2\sqrt{w_{0-}w_+ + 1}}{w_{0-}} - \frac{2\sqrt{w_{0-}w_{0+} + 1}}{w_{0-}} &= \pm 2\varepsilon\lambda && \text{(B.17)}
\end{aligned}
$$

But, once again, the embedding implies $w_{0-}w_{0+} + 1 = y_0^2$, from which we get:

$$
\begin{aligned}
\frac{2\sqrt{w_{0-}w_+ - 1}}{w_{0-}} - \frac{2y_0}{w_{0-}} &= \pm 2\varepsilon\lambda && \text{(B.18)}\\
\therefore w_+ &= w_{0+} \pm \varepsilon\lambda y_0 + (\varepsilon\lambda)^2 w_{0-} && \text{(B.19)}
\end{aligned}
$$

Coming back to y, we now have:

$$
\begin{aligned}
y^2 &= 1 + w_{0-}w_{0+} \pm 2\varepsilon\lambda w_{0-}y_0 + (\varepsilon\lambda)^2 w_{0-} && \text{(B.20)}\\
&= y_0^2 \pm 2\varepsilon\lambda w_{0-}y_0 + (\varepsilon\lambda)^2 w_{0-} && \text{(B.21)}\\
&= (y_0 \pm \varepsilon\lambda w_{0-})^2 && \text{(B.22)}\\
\therefore y &= y \pm \varepsilon\lambda w_{0-} && \text{(B.23)}
\end{aligned}
$$

Calling $b \doteqdot \pm\varepsilon\lambda$ a generic parameter, the transformation thus take the following form:

$$
\begin{aligned}
w_+ &\to w_+ + 2by + b^2 w_- && \text{(B.24)}\\
y &\to y + bw_- && \text{(B.25)}
\end{aligned}
$$

[1] Wikimedia Commons. Equirectangular projection, 2006. URL https://commons.wikimedia.org/wiki/File:Equirectangular-projection.jpg. (Cited on pages xi, 9, 11, 12, and 13.)

[2] Wikimedia Commons. Peirce quincuncial projection, 2006. URL https://commons.wikimedia.org/wiki/File:Peirce-quincuncial-projection.jpg. (Cited on pages xi, 9, 11, and 13.)

[3] E. Komatsu, K. M. Smith, J. Dunkley, C. L. Bennett, B. Gold, G. Hinshaw, N. Jarosik, D. Larson, M. R. Nolta, L. Page, D. N. Spergel, M. Halpern, R. S. Hill, A. Kogut, M. Limon, S. S. Meyer, N. Odegard, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright. Seven-year wilkinson microwave anisotropy probe (wmap) observations: Cosmological interpretation. 192: 18, February 2011. doi: 10.1088/0067-0049/192/2/18. (Cited on page 3.)

[4] Barbara Ryden. *Introduction to Cosmology*. Addison-Wesley, San Francisco, 2003. ISBN 0805389121. (Cited on page 4.)

[5] Javier Rubio. Higgs inflation and vacuum stability. *Journal of Physics: Conference Series*, 631(1):012032, 2015. URL http://stacks.iop.org/1742-6596/631/i=1/a=012032. (Cited on page 4.)

[6] Micheal Dine and Antonio Riotto. An inflaton candidate in gauge mediated supersymmetry breaking. *Phys. Rev. Lett.*, 79:2632–2635, Oct 1997. doi: 10.1103/PhysRevLett.79.2632. URL http://link.aps.org/doi/10.1103/PhysRevLett.79.2632. (Cited on page 4.)

[7] Andrew Strominger. The ds/cft correspondence. *Journal of High Energy Physics*, 2001(10):034, 2001. URL http://stacks.iop.org/1126-6708/2001/i=10/a=034. (Cited on page 5.)

[8] Alberto Zaffaroni. Introduction to the ads/cft correspondence. URL http://laces.web.cern.ch/Laces/LACES09/notes/dbranes/lezioniLosanna.pdf. (Cited on page 5.)

[9] Paolo Creminelli and Matias Zaldarriaga. A single-field consistency relation for the three-point function. *Journal of Cosmology and Astroparticle Physics*, 2004(10):006, 2004. URL http://stacks.iop.org/1475-7516/2004/i=10/a=006. (Cited on page 21.)

[10] Paolo Creminelli, Jorge Norena, and Marko Simonovic. Conformal consistency relations for single-field inflation. *Journal of Cosmology and Astroparticle Physics*, 2012(07):052, 2012. URL http://stacks.iop.org/1475-7516/2012/i=07/a=052. (Cited on page 21.)

[11] Juan Maldacena. Non-gaussian features of primordial fluctuations in single field inflationary models. *Journal of High Energy Physics*, 2003(05):013, 2003. URL http://stacks.iop.org/1126-6708/2003/i=05/a=013. (Cited on page 21.)

[12] Daniel Baumann. The physics of inflation. URL http://www.damtp.cam.ac.uk/user/db275/TEACHING/INFLATION/Lectures.pdf. (Cited on page 22.)

[13] Paolo Creminelli. Conformal invariance of scalar perturbations in inflation. *Phys. Rev. D*, 85:041302, Feb 2012. doi: 10.1103/PhysRevD.85.041302. URL http://link.aps.org/doi/10.1103/PhysRevD.85.041302. (Cited on pages 33, 43, 44, and 45.)

[14] Nima Arkani-Hamed and Juan Maldacena. Cosmological collider physics. 2015. URL http://arxiv.org/abs/1503.08043. (Cited on page 34.)

[15] Clifford Cheung, A. Liam Fitzpatrick, Jared Kaplan, Leonardo Senatore, and Paolo Creminelli. The effective field theory of inflation. *Journal of High Energy Physics*, 2008(03):014, 2008. URL http://stacks.iop.org/1126-6708/2008/i=03/a=014. (Cited on page 43.)

[16] Austin Joyce, Justin Khoury, and Marko Simonovic. Multiple soft limits of cosmological correlation functions. *Journal of Cosmology and Astroparticle Physics*, 2015(01):012, 2015. URL http://stacks.iop.org/1475-7516/2015/i=01/a=012. (Cited on page 45.)

[17] Mehrdad Mirbabayi and Matias Zaldarriaga. Double soft limits of cosmological correlations. *Journal of Cosmology and Astroparticle Physics*, 2015(03):025, 2015. URL http://stacks.iop.org/1475-7516/2015/i=03/a=025. (Cited on page 45.)