On the existence of symplectic forms on open manifolds

Maarten Mol

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supervisor: Dr. Fabian Ziltener

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Abstract

We prove a theorem by Gromov which says that on every smooth, even-dimensional, open and compact manifold that admits a non-degenerate 2-form, there exists a symplectic form. The two main ingredients for the proof are Morse functions and Gromov's telescope construction.

Acknowledgments

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Contents

1 Introduction

1.1 Motivation and main result

A symplectic form on a smooth manifold M is a closed and non-degenerate differential 2-form on M.[1] Symplectic forms on manifolds originated from classical Hamiltonian mechanics. In classical Hamiltonian mechanics the state of a particle is described by its position on a given manifold X and by its generalized momentum at a given position. By physicists, the space of such pairs of positions and momenta is called phase space. Mathematically this is the cotangent bundle T^*X, which naturally comes equipped with a symplectic form.[2] Using this symplectic form, Hamilton's equation can be derived for a specific Hamiltonian function. This equation fully determines a particle's trajectory through phase space in time.

So symplectic geometry is the natural mathematical setting to describe classical Hamiltonian mechanics. Symplectic geometry has also been applied in other areas within mathematics such as representation theory, partial differential equations, dynamical systems and algebraic geometry. Gromov-Witten invariants are an example of an application in algebraic geometry. [3]

This bachelor thesis is concerned with the following natural question.

Question 1.1. *Under which conditions does there exist a symplectic form on a given smooth manifold?*

To begin with, symplectic forms can only exist on orientable, even-dimensional manifolds. This is not sufficient, since they do for example not exist on S^{2n} for $n \geq 2$.[4] A surprising answer was given by Mikhail Gromov in his doctoral thesis [Gro69]. This is the main result of this thesis.

Theorem 1.2 (Gromov)**.** *Let M be a smooth even-dimensional manifold that is open (its connected components are non-compact or have non-empty boundary). Given a non-degenerate 2-form $\tau \in \Omega^2(M)$ and any $[a] \in H^2_{dR}(M;\mathbb{R})$ the following hold:*
(i) There exists a smooth homotopy of non-degenerate 2-forms between τ and a symplectic form ω on M which represents the class $[a]$.
(ii) If there exists a smooth homotopy of non-degenerate 2-forms between two symplectic forms ω_0 and ω_1 on M that both represent the class $[a]$, then there exists a smooth homotopy of symplectic forms on M between ω_0 and ω_1, that represents the class $[a]$ at every time.

For open, even-dimensional, smooth manifolds it reduces Question 1.1 to the question of existence of a non-degenerate 2-form. We will only prove part (i) of Theorem 1.2 in the

[1] The precise definition of a symplectic form on a smooth manifold is given in Section 2.1.

[2] For more details on this symplectic form see for example [dS01, Chapter 2].

[3] For a more elaborate discussion on the origin of symplectic geometry and its application in other fields in mathematics, see [Wei81].

[4] The argumentation for these statements is given in Section 2.1.

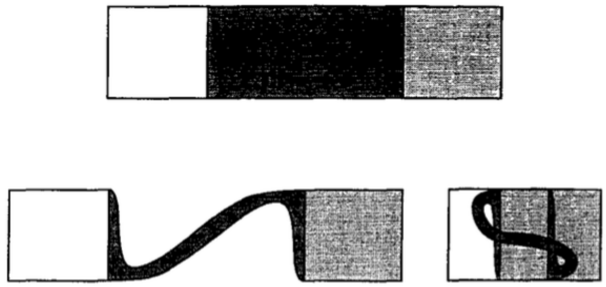

Figure 1: A zig-zag immersion

case that M is compact. The proof is mainly based on the theory of Morse functions and on Gromov's telescope construction. A key ingredient in the telescope construction is the smooth immersion depicted in two steps in Figure 1. The immersion is a composition of a map from the upper rectangle to the lower left figure, and a map from the lower left figure, to the lower right figure.

From Gromov's theorem the following corollary is immediate.[5] Let M be an open, smooth, even-dimensional manifold and let $[a] \in H^2_{dR}(M;\mathbb{R})$. Denote by $\Omega^2_{nd}(M)$ the space of non-degenerate 2-forms, by $\mathbb{S}_a$ the space of symplectic forms on M representing $[a]$ and by $\pi_0(X)$ the set of path components of a topological space X.

Corollary 1.3 (Correspondence of path-components)**.** *The map $\pi_0(\mathbb{S}_a) \to \pi_0(\Omega^2_{nd}(M))$ induced by the inclusion map $\mathbb{S}_a \hookrightarrow \Omega^2_{nd}(M)$ is a bijection.*

In particular $\pi_0(S_a)$ and $\pi_0(\Omega^2_{nd}(M))$ are isomorphic. Are $\mathbb{S}_a$ and $\Omega^2_{nd}(M)$ even homotopy equivalent? In fact, Gromov proved that the inclusion map $\mathbb{S}_a \hookrightarrow \Omega^2_{nd}(M)$ is a homotopy equivalence. The proof of this is beyond the scope of this text. The interested reader is referred to [EM02, Theorem 10.2.2].

1.2 Organization

In Section 2 we develop some basic definitions and theorems regarding symplectic geometry and non-degenerate 2-forms. Furthermore, we show that Theorem 1.2 does not hold if we do not assume that the manifold is open, by means of a counterexample. In Section 3 the theory of Morse functions is developed, which is one of the two main ingredients for the proof of Gromov's theorem. The second main ingredient is based on Gromov's telescope construction. Section 4 is devoted to this. Finally, in Section 5, Theorem 1.2(i) will be proved for compact manifolds. For this proof we follow that of [MS99, Theorem 7.34].

[5](i) Implies surjectivity and (ii) implies injectivity.

2 Preliminaries

In this section some basic concepts regarding symplectic geometry and non-degenerate forms are explained. After this section the reader should understand the statement of Gromov's theorem. Although we do not follow a treatment of one specific source, most results in this section can be found in [MS99, Chapter 2].

Remark 2.1. Throughout this thesis by a manifold we will mean a smooth, real manifold with (possibly empty) boundary, unless explicitly stated otherwise. We will sometimes work with manifolds with corners. The definition of manifold with corners can be found in [Lee12, Chapter 16]. Furthermore, by a vector space we will always mean a finite-dimensional, real vector space.

2.1 Symplectic manifolds and vector spaces

The main theorem that will be proved is the symplectic analogue to the Gram-Schmidt theorem. We will also show that the conditions posed on the manifold M are necessary for Gromov's theorem to hold.

Let us start by defining symplectic vector spaces and manifolds.

Definition 2.2. Let V be a vector space and $\omega : V \times V \to \mathbb{R}$ a bilinear map. ω is called symplectic if:

(i) ω is anti-symmetric, i.e. $\omega(v, u) = -\omega(u, v)$ for all $u, v \in V$

(ii) ω is non-degenerate, i.e. if $\omega(v, u) = 0$ for all $u \in V$ then $v = 0$

The pair (V, ω) is called a symplectic vector space.

Definition 2.3. A differential 2-form ω on a manifold M is called symplectic if:

(i) ω is closed, i.e. $d\omega = 0$, where d denotes the exterior derivative

(ii) ω is non-degenerate, i.e. the bilinear map $\omega(x) : T_xM \times T_xM \to \mathbb{R}$ is non-degenerate for all $x \in M$

The pair (M, ω) is called a symplectic manifold.

Remark 2.4. (i) Note first that for every symplectic vector space (V, ω), the pair $(V, \tilde{\omega})$ is a symplectic manifold, where $\tilde{\omega}(x) = \omega$ for all $x \in V$. Here we identify T_xV and V canonically. Closedness of $\tilde{\omega}$ follows since its component functions in any global chart for V are constant. The other requirements for $\tilde{\omega}$ to be a symplectic form are obviously satisfied.
(ii) Secondly, note that if (M, ω) is a symplectic manifold, then for each $x \in M$ the pair $(T_xM, \omega(x))$ is a symplectic vector space. The converse statement does not hold, for a differential 2-form can be non-degenerate and not closed, as is shown in the following example.

Remark 2.5. By $\{\partial_1|_x, ..., \partial_n|_x\}$ we will denote the standard basis for $T_x\mathbb{R}^n$ and by δ_{ij} we denote the Kronecker delta. Moreover, for a smooth function f from an open $U \subset \mathbb{R}^n$ into an open $V \subset \mathbb{R}^m$ we denote by $D_i f$ the partial derivative of f with respect to its i^{th} variable.

Example 2.6. Let $f \in C^\infty(\mathbb{R}^4)$ be such that $f(x) \neq 0$ for all $x \in \mathbb{R}^4$ and $df \neq 0$.[6] On the smooth manifold $\mathbb{R}^4$ with its standard smooth structure, define $\omega \in \Omega^2(\mathbb{R}^4)$ by:

$$\omega = f \cdot dx_1 \wedge dx_2 + f \cdot dx_3 \wedge dx_4$$

Then for every $x \in \mathbb{R}^4$, we have:

$$\omega(x)(\sum_{i=1}^{4} v_i \, \partial_i|_x, \partial_j|_x) = f(x)(v_1\delta_{j2} - v_2\delta_{j1}) + f(x)(v_3\delta_{j4} - v_4\delta_{j3})$$

from which it follows that ω is non-degenerate. Moreover:

$$d\omega = \sum_{i=3}^{4} D_i f dx_i \wedge dx_1 \wedge dx_2 + \sum_{i=1}^{2} D_i f dx_i \wedge dx_3 \wedge dx_4 \neq 0$$

since $df \neq 0$. So ω is not closed.

Example 2.7. On the vector space $\mathbb{R}^{2n}$ we define $\omega_0 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ by:

$$\omega_0(v, u) = \sum_{i=1}^{n} v_{2i-1}u_{2i} - v_{2i}u_{2i-1}$$

Here v_j and u_j denote the j^{th} coordinate for v respectively u with respect to the standard basis for $\mathbb{R}^{2n}$. It is clear from its definition that ω_0 is anti-symmetric and bilinear. Non-degeneracy follows by the observation that:

$$\omega_0(v, e_{2j}) = v_{2j-1} \text{ and } \omega_0(v, e_{2j-1}) = -v_{2j}$$

Thus ω_0 is symplectic. As in Remark 2.4(i) $(\mathbb{R}^{2n}, \tilde{\omega})$ is a symplectic manifold. We denote the symplectic form $\tilde{\omega}$ by ω_0, like we did for the initial bilinear map, and will refer to it as the standard symplectic form on $\mathbb{R}^{2n}$. Note that in the standard smooth chart for $\mathbb{R}^{2n}$ we have:

$$\omega_0 = \sum_{i=1}^{n} dx_{2i-1} \wedge dx_{2i}$$

In particular, we see that ω_0 is exact.

[6] $f(x) = e^{x_1}$ would be an example.

By Remark 2.4(i) it becomes clear that symplectic vector spaces can be considered as a special, simpler class of symplectic manifolds. Moreover, by Remark 2.4(ii) it is plausible that results about symplectic vector spaces will imply results about symplectic manifolds. For this reason we'll first focus our attention on symplectic vector spaces.

Definition 2.8. Let (V, ω) be a symplectic vector space and W a linear subspace of V. We define the symplectic complement of W in (V, ω) to be:

$$W^\omega = \{v \in V |\ \omega(v, u) = 0 \text{ for all } u \in W\}.$$

Proposition 2.9. *Let (V, ω) and W be as in Definition 2.8. Then:*
(i) W^ω is a linear subspace of V and dim(W)+dim(W^ω)=dim$V$
(ii) $\omega_{|W\times W}$ is symplectic if and only if $W \cap W^\omega = \{0\}$. In this case we have $W \oplus W^\omega = V$.

Proof. Define $\iota_\omega : V \to V^*$ by $\iota_\omega(v)(u) = \omega(v, u)$. ι_ω is well-defined since ω is linear in its second component and linear since ω is linear in its first component. By non-degeneracy of ω it is immediate that ι_ω is injective. Since $\dim(V) = \dim(V^*)$ it follows by the rank-nullity theorem that ι_ω is a linear isomorphism. Now define $T : V \to W^*$ by $T(v) = \iota_\omega(v)|_W$. Then T is linear since ι_ω is, $\ker(T) = W^\omega$ (in particular W^ω is a linear subspace of V) and $\mathrm{im}(T) = W^*$ since ι_ω is surjective. So by the rank-nullity theorem we have:

$$\dim(V) = \dim(W^*) + \dim(W^\omega) = \dim(W) + \dim(W^\omega)$$

The if and only if part of (ii) is immediate from the definition of non-degeneracy. To see the last implication note that if (i) and $W \cap W^\omega = \{0\}$ hold, then by basic linear algebra it follows that $W \oplus W^\omega = V$. □

Remark 2.10. A linear subspace such that $W \cap W^\omega = \{0\}$ is called a symplectic subspace. By Proposition 2.9(ii), $(W, \omega_{|W\times W})$ is a symplectic vector space if and only if W is a symplectic subspace of (V, ω).

Using Proposition 2.9 we will prove an important classification result about symplectic vector spaces. In order to give meaning to this theorem's statement, we need the following definition.

Definition 2.11. Let (V, ω) and (V', ω') be symplectic vector spaces. A linear map $\varphi : V \to V'$ is called linear symplectic if $\varphi^*\omega' := \omega'(\varphi\cdot, \varphi\cdot) = \omega$. If a linear symplectic map is also a linear isomorphism, then it's called a linear symplectomorphism. If a linear symplectomorphism from (V, ω) to (V', ω') exists, then (V, ω) is called linearly symplectomorphic to (V', ω').

Remark 2.12. It's straightforward to check that one symplectic vector space being linearly symplectomorphic to another is an equivalence relation. In particular it's a symmetric relation, hence it makes sense to say that two spaces are linearly symplectomorphic, without specifying direction.

Theorem 2.13 (Classification of symplectic vector spaces). *Let (V, ω) be a symplectic vector space. Then there is an $n \in \mathbb{N}$ and a basis $\{v_1, ..., v_n, u_1, ..., u_n\}$ of V, such that $w(v_i, u_j) = \delta_{ij}$ and $w(v_i, v_j) = w(u_i, u_j) = 0$ for all i and j. Moreover, (V, ω) and $(\mathbb{R}^{2n}, \omega_0)$ are linearly symplectomorphic. As a direct consequence, all symplectic vector spaces of equal dimension are linearly symplectomorphic.*

Proof. Note first that for any symplectic subspace W, W^ω is symplectic too. Too see this apply Proposition 2.9(i) to both W and W^ω to find that $\dim(W)$=$\dim((W^\omega)^\omega)$. Since moreover $W \subset (W^\omega)^\omega$ is clear, equality follows. Hence we have:

$$(W^\omega)^\omega \cap W^\omega = W \cap W^\omega$$

so W^ω is symplectic if W is.
We now prove the first statement by induction on the dimension of V. If $\dim(V) = 0$, the statement is trivially true.
Now suppose $\dim(V) = k \geq 1$ and suppose that the statement is true for symplectic vector spaces of dimension $< k$. Since $\dim(V) \geq 1$ there is a $0 \neq v \in V$ and since ω is non-degenerate, there is a $0 \neq u \in V$ such that $\omega(v, u) \neq 0$, hence $\omega(v_1, u_1) = 1$ for $v_1 = v$ and $u_1 = \frac{u}{\omega(v,u)}$. We define $V_1 = \text{span}(v_1, u_1)$ and $W = V_1^\omega$. Note that $\omega(z, z) = 0$ for every $z \in V$ by anti-symmetry, hence by bilinearity we have:

$$\omega(\lambda v_1 + \mu u_1, v_1) = -\mu \text{ and } \omega(\lambda v_1 + \mu u_1, u_1) = \lambda$$

for all $\lambda, \mu \in \mathbb{R}$. It follows that u_1 and v_1 are linearly independent and that V_1 is symplectic, thus W is symplectic too. Therefore $(W, \omega_{|W \times W})$ is a symplectic vector space of dimension $k-2$ so by the induction hypothesis there is a basis $\{v_2, ..., v_k, u_2, ..., u_k\}$ of W such that $w(v_i, u_j) = \delta_{ij}$ and $w(v_i, v_j) = w(u_i, u_j) = 0$ for all $i, j \geq 2$.
By Proposition 2.9(ii) it follows that $\{v_1, ..., v_k, u_1, ..., u_k\}$ is a basis of V and by construction it holds that $w(v_i, u_j) = \delta_{ij}$ and $w(v_i, v_j) = w(u_i, u_j) = 0$ for all i and j.
To see the second statement, define $\varphi : V \to \mathbb{R}^{2n}$ by:

$$\varphi(\sum_{i=1}^n \lambda_i v_i + \mu_i u_i) = \sum_{i=1}^n \lambda_i e_{2i-1} + \mu_i e_{2i}$$

φ is a well-defined linear isomorphism since $\{v_1, ..., v_n, u_1, ..., u_n\}$ and $\{e_1, ..., e_{2n}\}$ are bases for V respectively $\mathbb{R}^{2n}$. Moreover, we find that:

$$\begin{aligned}
\omega(\sum_{i=1}^n \lambda_i v_i + \mu_i u_i, \sum_{j=1}^n \lambda'_j v_j + \mu'_j u_j) &= \sum_{i=1}^n \lambda_i \mu'_i - \lambda'_i \mu_i \\
&= \omega_0(\sum_{i=1}^n \lambda_i e_{2i-1} + \mu_i e_{2i}, \sum_{j=1}^n \lambda'_j e_{2j-1} + \mu'_j e_{2j}) \\
&= \omega_0(\varphi(\sum_{i=1}^n \lambda_i v_i + \mu_i u_i), \varphi(\sum_{j=1}^n \lambda'_j v_j + \mu'_j u_j))
\end{aligned}$$

so $\varphi^* \omega_0 = \omega$, which shows that φ is a linear symplectomorphism. $\square$

As a first consequence, Theorem 2.13 shows that all symplectic vector spaces are even-dimensional. By Remark 2.4(ii) this implies that all symplectic manifolds must have even manifold-dimension, since the manifold-dimension is equal to the vector space dimension of the tangent space at each point in the manifold. This explains the condition of even dimension in Gromov's theorem. Another consequence of this theorem, which will be used to prove Gromov's theorem in Section 5, is the following.

Consequence 2.14. *Let M be a smooth, $2n$-manifold and let $\tau \in \Omega^2(M)$ be non-degenerate. For any $p \in int(M)$ there exists a smooth chart (U, φ) around p such that $(\varphi^*\omega_0)(p) = \tau(p)$.*[7]

Proof. It is immediate from the definition of a non-degenerate differential 2-form that $\tau(p) : T_pM \times T_pM \to \mathbb{R}$ is a symplectic bilinear map. Hence by Theorem 2.13 it follows that there is a linear symplectomorphism $\psi : (T_pM, \tau(p)) \to (\mathbb{R}^{2n}, \omega_0)$. Now let (U, φ_0) be a smooth chart around p, and let $\iota : \mathbb{R}^{2n} \to T_{\varphi_0(p)}\mathbb{R}^{2n}$ denote the canonical linear isomorphism. Note that

$$\psi \circ d(\varphi_0^{-1})_{\varphi_0(p)} \circ \iota : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$$

is a linear isomorphism, being a composition of linear isomorphisms. Thus it is a diffeomorphism, and so (U, φ), where we define

$$\varphi = \psi \circ d(\varphi_0^{-1})_{\varphi_0(p)} \circ \iota \circ \varphi_0$$

is a smooth chart for M around p. Moreover, we have:

$$d\varphi_p = d(\psi \circ d(\varphi_0^{-1})_{\varphi_0(p)} \circ \iota)_{\varphi_0(p)} \circ d(\varphi_0)_p = \iota \circ (\psi \circ d(\varphi_0^{-1})_{\varphi_0(p)} \circ \iota) \circ \iota^{-1} \circ d(\varphi_0)_p = \iota \circ \psi$$

where in the second step we use that the differential of a linear map is the map itself when tangent space and the underlying vector space are identified via the canonical linear isomorphism. From this we find:

$$(\varphi^*\omega_0)(p) = (\iota \circ \psi)^*(\omega_0(\varphi(p))) = \psi^*\omega_0 = \tau(p)$$

as desired. Here the first and second ω_0 denote the form and the last denotes the bilinear map. □

The second part of Theorem 2.13 has an extension which will be useful to us in the proof of the telescope construction in Section 4.

Proposition 2.15. *Let M be a smooth manifold (possibly with corners) and let $\omega : M \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth map such that ω_p is a symplectic bilinear map for every $p \in M$.*[8] *Then there is a smooth map $\varphi : M \to Aut(\mathbb{R}^{2n})$ such that $\omega_p = \varphi(p)^*\omega_0$ for every $p \in M$.*

[7] By $\mathrm{int}(M)$ we denote the manifold interior of a manifold M.

[8] Here we denote $\omega_p := \omega(p, \cdot, \cdot)$.

Proof. We can explicitly construct a basis $\{v_{1,p}, ..., v_{n,p}, u_{1,p}, ..., u_{n,p}\}$ of $\mathbb{R}^{2n}$, such that $v_{i,p}$ and $u_{i,p}$ depend smoothly on p for every $i = 1, ..., n$ and such that $\omega_p(v_{i,p}, u_{j,p}) = \delta_{i,j}$ and $\omega_p(u_{i,p}, u_{j,p}) = \omega_p(v_{i,p}, v_{j,p}) = 0$ for all $i, j = 1, ..., n$ and all $p \in M$. For the details of this, see [MS99, Exercise 2.11, 2.12].

Using this basis, for every $p \in M$ we define $\varphi(p)$ such that $\omega_p = \varphi(p)^*\omega_0$ as in the proof of Theorem 2.13. From its definition it is clear that $\varphi(p)^{-1}$ is smoothly dependent on p, so since inversion : $\mathrm{Aut}(\mathbb{R}^n) \to \mathrm{Aut}(\mathbb{R}^n)$ is smooth, it follows that $\varphi(p)$ depends smoothly on p. □

2.2 Properties of non-degenerate 2-forms

In this subsection we will state and prove some properties of non-degenerate 2-forms. Using these, we will give a counterexample which shows that Gromov's theorem does not hold if we remove the condition that M is an open manifold.

Proposition 2.16. *Let V be a $2n$-dimensional vector space and $\omega : V \times V \to \mathbb{R}$ bilinear and anti-symmetric. Then ω is non-degenerate if and only if $\omega^{\wedge n} \neq 0$.*[9]

Proof. Suppose first that ω is non-degenerate. Then (V, ω) is a symplectic vector space, so by Theorem 2.13 there is a basis $\{v_1, ..., v_n, u_1, ..., u_n\}$ for V such that $w(v_i, u_j) = \delta_{ij}$ and $w(v_i, v_j) = w(u_i, u_j) = 0$ for all i and j. Denote by $\{v^1, ..., v^n, u^1, ..., u^n\}$ the basis dual to the one above. Now note that $\omega = \sum_{i=1}^n v^i \wedge u^i$, since both sides are bilinear maps which are equal on the basis $\{v_1, ..., v_n, u_1, ..., u_n\}$. From this it follows that:

$$\omega^{\wedge n} = \sum_{i_1=1}^{n} ... \sum_{i_n=1}^{n} (v^{i_1} \wedge u^{i_1} \wedge ... \wedge v^{i_n} \wedge u^{i_n})$$

Any term with $i_j = i_k$ for some $j \neq k$ equals 0. Moreover, since interchanging any two pairs $v^{i_k} \wedge u^{i_k}$ and $v^{i_j} \wedge u^{i_j}$ is an even permutation we have that any term with $i_j \neq i_k$ if $j \neq k$ is equal to $v^1 \wedge u^1 \wedge ... \wedge v^n \wedge u^n$. Since the latter case occurs $n!$ times, it follows that:

$$\omega^{\wedge n} = n! v^1 \wedge u^1 \wedge ... \wedge v^n \wedge u^n \neq 0$$

For the converse statement, suppose ω is degenerate, i.e. there is a non-zero $v \in V$ such that $\omega(v, u) = 0$ for all $u \in V$. Choose a basis $\{v_1, ..., v_{2n}\}$ for V such that $v_1 = v$ and denote by $\{v^1, ..., v^{2n}\}$ its dual basis. Denote by $\Lambda^{2n}(V^*)$ the vector space of $2n$-linear anti-symmetric maps $V^{2n} \to \mathbb{R}$.
Then since $\dim(V) = 2n$ we have $\dim(\Lambda^n(V^*)) = 1$, so since clearly $0 \neq v^1 \wedge ... \wedge v^{2n} \in \Lambda^{2n}(V^*)$ it follows that $\omega^{\wedge n} = \lambda v^1 \wedge ... \wedge v^{2n}$ for some $\lambda \in \mathbb{R}$. Note that by definition of the wedgeproduct $\omega^{\wedge n}(v_1, ..., v_{2n})$ is a sum of products, where in each product there is a factor $\omega(v_1, v_k)$ for some $k \neq 1$ or a factor $\omega(v_k, v_1)$ for some $k \neq 1$. So since $\omega(v_1, u) = 0$ for all $u \in V$ this implies that $\omega^{\wedge n}(v_1, ..., v_{2n}) = 0$. But $v^1 \wedge ... \wedge v^{2n}(v_1, ..., v_{2n}) = 1$, so $\lambda = 0$ and so $\omega^{\wedge n} = 0$. □

[9] Here by $\omega^{\wedge n} = \omega \wedge ... \wedge \omega$ we mean the n-fold wedgeproduct.

Remark 2.17. From Proposition 2.16 it is immediate that for any symplectic form ω on a smooth $2n$-manifold M, $\omega^{\wedge n}$ is nowhere-vanishing. Recall that if a smooth n-manifold M admits a nowhere-vanishing n-form, then M is orientable. Hence any symplectic manifold is orientable.
Now suppose (M, ω) is a symplectic manifold and suppose that M is closed (compact without boundary). We'll show that ω cannot be exact. Let M have the orientation induced by $\omega^{\wedge n}$. Then ω is positively oriented, hence $\int_M \omega^{\wedge n} > 0$. Now suppose ω is exact, then $\omega = d\sigma$ for some $\sigma \in \Omega^1(M)$. So:

$$\omega^{\wedge n} = d\sigma \wedge \omega^{\wedge n-1} = d(\sigma \wedge \omega^{\wedge n-1})$$

where we use that $d\omega = 0$. This would imply by Stokes' theorem that $\int_M \omega^{\wedge n} = 0$, since $\partial M = \emptyset$, which is a contradiction.

Definition 2.18. Let M be a manifold. A smooth bundle homomorphism $J : TM \to TM$ such that $J|_{T_pM} \circ J|_{T_pM} = -\mathrm{Id}_{T_pM}$ for all $p \in M$ is called an almost complex structure on M. The pair (M, J) is called an almost complex manifold.

Proposition 2.19. *If a manifold admits an almost complex structure, then it admits a non-degenerate 2-form.*

In the proof of this proposition, we choose a Riemannian metric. Recall that a Riemannian metric g on a smooth manifold M is a smooth section of $L^2(M)$ such that $g_p : T_pM \times T_pM \to \mathbb{R}$ is an inner product, for every $p \in M$. Here $L^2(M)$ denotes the vector bundle with fibres the vector spaces of bilinear maps $T_pM \times T_pM \to \mathbb{R}$. Further, recall that any smooth manifold M admits a Riemannian metric.

Proof of Proposition 2.19. Let (M, J) be an almost complex manifold. Choose a Riemannian metric g on M, and define $\omega \in \Omega^2(M)$ by:

$$\omega_p(u, v) = g_p(v, J_p u) - g_p(u, J_p v)$$

Here by J_p we mean $J|_{T_pM}$. Bilinearity of ω_p follows by bilinearity of g_p and anti-symmetry of ω_p follows by symmetry of g_p. That ω is indeed smooth follows by smoothness of g and J. Moreover, non-degeneracy of ω follows by the observation that for every $p \in M$ and every non-zero $v \in T_pM$:

$$\omega_p(v, J_p v) = g_p(J_p v, J_p v) - g_p(v, J_p(J_p v)) = g_p(J_p v, J_p v) + g_p(v, v) > 0$$

since g_p is positive definite. □

We will now give the promised counterexample.

Example 2.20. S^6 is an even-dimensional closed manifold which admits an almost complex structure. A proof of this can be found in [MS99, Example 4.4]. Thus by Proposition 2.19 it admits a non-degenerate 2-form. On the other hand, we have that $H^2_{dR}(S^n) = \{0\}$

for $n > 2$, so every closed 2-form on S^n must be exact for $n > 2$.[10] Therefore, by Remark 2.17, S^{2n} does not admit a symplectic form for $n > 1$ and in particular S^6 does not. We conclude that the conclusions of Gromov's theorem cannot hold for S^6. Thus the condition for M to be open is necessary.

In the remainder of this subsection, we will focus on some topological properties of non-degenerate 2-forms. These properties will be applied in the proof of the main lemma and main result in Section 4 and 5.

Proposition 2.21. *Let V be a vector space, $\{e_1, ..., e_n\}$ a basis for V and $\omega : V \times V \to \mathbb{R}$ bilinear. Then ω is non-degenerate if and only if the $n \times n$-matrix with $(i, j)^{th}$ coefficient $\omega(e_i, e_j)$ is invertible.*

Proof. Define $\iota_\omega : V \to V^*$ by $\iota_\omega(v)(u) = \omega(v, u)$, as in the proof of Proposition 2.9. As in that proof, ι_ω is a linear isomorphism if ω is non-degenerate. The converse is immediately clear. Now note that ι_ω is represented by the matrix with $(i, j)^{th}$ coefficient $-\omega(e_i, e_j)$, with respect to the basis $\{e_1, ..., e_{2n}\}$ for V and its dual basis for V^*. So the statement follows. $\square$

Remark 2.22. Before stating the next proposition we introduce some notation.
(i) Let M be a manifold, $p \in M$, $f \in C^\infty(M)$ and let (U, φ) be a chart around p. We will denote:

$$\partial_i^\varphi|_p := d(\varphi^{-1})_{\varphi(p)}(\partial_i|_{\varphi(p)}) \quad \text{and} \quad f_\varphi := f \circ \varphi^{-1}$$

Recall that $\left\{ \partial_1^\varphi|_p, ..., \partial_n^\varphi|_p \right\}$ is a basis of T_pM. By

$$\left\{ dx_1^\varphi|_p, ..., dx_n^\varphi|_p \right\}$$

we will denote its dual basis.
(ii) Let M be a manifold and $k \in \mathbb{N}$. By $\Lambda^k(M) \to M$ we denote the smooth vector bundle with fibres the vector spaces $\Lambda^k(T_p^*M)$ of anti-symmetric k-linear maps from the k-fold cartesian product $T_pM \times ... \times T_pM$ to $\mathbb{R}$. Moreover, we denote:

$$\Lambda^2_{nd}(M) = \left\{ (p, \omega) \in \Lambda^2(M) | \ \omega \text{ is non-degenerate} \right\}$$

Proposition 2.23. *Let M be a smooth $2n$-manifold. Then $\Lambda^2_{nd}(M)$ is open in $\Lambda^2(M)$.*

Proof of Proposition 2.23. Let $(p', \omega') \in \Lambda^2_{nd}(M)$, let (U, φ) a chart for M around p' and let $\pi : \Lambda^2(M) \to M$ denote the bundle projection. Define:

$$f : \pi^{-1}(U) \to \mathrm{Mat}(n \times n, \mathbb{R}) \text{ by } f(p, \omega)_{i,j} = \omega(\partial_i^\varphi|_p, \partial_j^\varphi|_p)$$

By Proposition 2.21, we have that:

$$(p', \omega') \in f^{-1}(\mathrm{GL}(n, \mathbb{R})) \subset \Lambda^2_{nd}(M)$$

[10] For a proof of this see [Lee12, Theorem 17.21].

We will show that f is smooth, so that by continuity of f it follows that $f^{-1}(\mathrm{GL}(n,\mathbb{R}))$ is an open neighbourhood of (p',ω') in $\pi^{-1}(U)$ and so in $\Lambda^2(M)$. Since we found such a neighbourhood for arbitrary $(p',\omega')\in\Lambda^2_{nd}(M)$ it follows that $\Lambda^2_{nd}(M)$ is a union of opens in $\Lambda^2(M)$ and so is open itself. Thus it remains to prove that f is smooth.

Since φ induces a local frame for $\Lambda^2(M)$, there is a local trivialization $\Phi:\pi^{-1}(U)\to U\times\mathbb{R}^{\binom{n}{2}}$ such that:

$$\Phi(p,\sum_{i_1<i_2}\omega_{i_1,i_2}\, dx^{\varphi}_{i_1}\big|_p\wedge dx^{\varphi}_{i_2}\big|_p)=(p,\sum_{i_1<i_2}\omega_{i_1,i_2}e_{i_1,i_2})$$

Now note that for $i<j$ we have : $f_{i,j}\circ\Phi^{-1}(p,\sum_{i_1<i_2}\omega_{i_1,i_2}e_{i_1,i_2})=\omega_{i,j}$. So $f_{i,j}\circ\Phi^{-1}$ is smooth and so is $f_{i,j}$ since Φ is a diffeomorphism. By anti-symmetry we find that $f_{i,i}=0$ for all i and $f_{i,j}=-f_{j,i}$ for all i and j, and so $f_{i,j}$ is smooth for all i and j, hence f is smooth. $\square$

Remark 2.24. Note that Proposition 2.23 can also be derived from Proposition 2.16 using a similar strategy, by showing that $\omega^{\wedge n}\neq 0$ is an open condition.

We will also need the following topological fact.

Proposition 2.25. *Let X, Y and Z be topological spaces, let $S\subset X$ and let $H:X\times Y\to Z$ be continuous. Further, suppose that O is open in Z, that Y is compact and that:*

$$H(x,y)\in O \text{ for all } x\in S \text{ and for all } y\in Y$$

Then there is an open neighbourhood U of S in X such that $H(x,y)\in O$ for all $(x,y)\in U\times Y$.

Proof. By continuity of H, $H^{-1}(O)$ is an open subset of $X\times Y$, which contains $\{x\}\times Y$, for every $x\in S$. So since Y is compact, by the Tube Lemma we find that for every $x\in S$ there is an open neighbourhood U_x in X of x, such that $U_x\times Y\subset H^{-1}(O)$. Hence $U:=\cup_{x\in S}U_x$ is the desired open neighbourhood of S in X. $\square$

Consequence 2.26. *Let M be a smooth manifold (possibly with corners), $S\subset M$ and let $\omega,\tau\in\Omega^2(M)$ be non-degenerate 2-forms such that $\tau(p)=\omega(p)$ for all $p\in S$. Define $\omega_t=(1-t)\tau+t\omega$. Then there is an open neighbourhood U of S on which ω_t is non-degenerate for all $t\in[0,1]$.*

Proof. Define $H:[0,1]\times M\to\Lambda^2(M)$ by:

$$H(t,x)=\omega_t(x)$$

Then H is smooth, thus continuous. Moreover, $H(t,p)=\tau(p)\in\Lambda^2_{nd}(M)$ for all $t\in[0,1]$ and all $p\in S$. So by Proposition 2.23 and 2.25 we find the desired open neighbourhood U of S. $\square$

2.3 Homotopies of non-degenerate 2-forms

In this subsection we define homotopies of non-degenerate 2-forms and we prove some facts about them, which will be applied in the proof of the main lemma and main result in Section 4 and 5.

Definition 2.27. Let M and N be smooth manifolds (possibly with corners), let $B \subset C^\infty(M, N)$ and let $a < b$. A smooth B-homotopy is a smooth map $\varphi : [a, b] \times M \to N$ such that $\varphi_t \in B$ for every $t \in [a, b]$.[11]

Remark 2.28. (i) There are a few special cases of Definition 2.27. For example, take $B = \Omega^2_{nd}(M)$, i.e. B is the set of non-degenerate 2-forms on M, and $N = \Lambda^2_{nd}(M)$. Then we also call φ a smooth homotopy of non-degenerate 2-forms, instead of a smooth $\Omega^2_{nd}(M)$-homotopy. Similarly, if B is the set of smooth immersions $M \to N$, we also call φ a smooth homotopy of immersions. The only exception to this rule of thumb is when B is the set of diffeomorphisms. In this case we also call φ a smooth isotopy.

(ii) Let M and N be smooth manifolds (possibly with corners) and let $S \subset M$. We will say that a homotopy $\varphi : [a, b] \times M \to N$ is constant in time on S, if $\varphi_t(p) = \varphi_0(p)$ for all $t \in [a, b]$ and all $p \in S$.

Smooth homotopies of non-degenerate forms can be concatenated, as follows.

Proposition 2.29. *Let M be a smooth manifold (possibly with corners) and let $f, g : [0, 1] \times M \to \Lambda^2(M)$ be smooth homotopies of non-degenerate 2-forms such that $f_1 = g_0$. Then there is a smooth homotopy of non-degenerate 2-forms $h : [0, 1] \times M \to \Lambda^2(M)$ such that $h_0 = f_0$ and $h_1 = g_1$.*
If in addition there is a subset $S \subset M$ such that f and g are constant in time on S, then h can be chosen to be constant in time on S.

Proof. By smoothness of f_t and g_t we can choose $\varepsilon > 0$ and smooth extensions $\tilde{f}_t, \tilde{g}_t :]-\varepsilon, 1+\varepsilon[\times M \to \Lambda^2(M)$. Choose a smooth bump function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta = 1$ on $[\frac{1}{4}, \infty[$ and is supported in $]0, \infty[$. Define $\alpha : \mathbb{R} \to [0, 2]$ by:

$$\alpha(t) = \beta(t) + \beta\left(t - \frac{3}{4}\right)$$

Now define $\tilde{h} :]\varepsilon, 1+\varepsilon[\times M \to \Lambda^2(M)$ by:

$$\tilde{h}_t(x) = \begin{cases} \tilde{f}_{\alpha(t)}(x) & \text{if } t \in]-\varepsilon, \frac{3}{4}[\\ \tilde{g}_{\alpha(t)-1}(x) & \text{if } t \in]\frac{1}{4}, 1+\varepsilon[\end{cases}$$

By our choice of α and the fact that $\tilde{f}_1 = \tilde{g}_0$, this is well-defined. $\tilde{h}$ is smooth since it is smooth on the opens $]-\varepsilon, \frac{3}{4}[\times M$ and $]\frac{1}{4}, 1+\varepsilon[\times M$, which form an open cover of

[11] We denote $\varphi_t := \varphi(t, \cdot)$. Moreover, by smoothness of a map $f : [a, b] \times M \to N$ we mean that there is an $\varepsilon > 0$ and a smooth extension $\tilde{f} :]a - \varepsilon, b + \varepsilon[\times M \to N$ of f.

$]-\varepsilon, 1+\varepsilon[\times M$. Moreover, $\tilde{h}_0 = f_0$ and $\tilde{h}_1 = g_1$. Since f and g are homotopies of non-degenerate 2-forms, it now follows that $h := \tilde{h}_{|[0,1]\times M}$ is the desired smooth homotopy of non-degenerate 2-forms. In case of the additional statement it follows that h also suffices by the observation that $g_0(x) = f_1(x) = f_0(x)$ for all $x \in S$. □

The next proposition will also be used in the proof of Gromov's theorem.

Remark 2.30. Before stating the next proposition we introduce some more notation.
(i) We will allow ourselves an abuse of notation when dealing with differential forms. Suppose that M is a smooth manifold and that $S \subset M$ is a submanifold of codimension 0. By definition of submanifold, the inclusion $\iota : S \to M$ is a smooth embedding, so in particular ι is an immersion. By the rank-nullity theorem, ι is a submersion too, so $d\iota_p$ is a linear isomorphism for every $p \in S$. Hence $d\iota : TS \to TM|_S$ is a canonical smooth bundle isomorphism. This induces a canonical linear isomorphism:

$$\varphi : \Omega^k(S) \to \left\{\tau \in C^\infty(S, \Lambda^k(M)) \middle|\ \tau_p \in \Lambda^k(T_p^*M) \text{ for all } p \in S\right\}$$

Explicitly, we have $\varphi(\tau)(p) = ((d\iota_p)^{-1})^*(\tau(p))$ and $\varphi^{-1}(\tau)(p) = (d\iota_p)^*(\tau(p))$. To avoid such notation, we will just write τ for $\varphi(\tau)$ and $\varphi^{-1}(\tau)$ in the respective cases.
(ii) Let M and N be smooth manifolds, let S be a submanifold of M and of N and let $\tau \in \Omega^k(M)$ and $\tau' \in \Omega^k(N)$. Denote by $\iota_M : S \to M$ and $\iota_N : S \to N$ inclusion. By saying that $\tau = \tau'$ on S, we mean that $\iota_M^*\tau = \iota_N^*\tau'$.

Proposition 2.31. *Let M be a smooth manifold, let $f \in C^\infty(M)$ and let c be a regular value of f such that $f^{-1}(]-\infty, c]) \subset int(M)$.*[12] *Let $\varepsilon > 0$ such that $c - \varepsilon$ is a regular value of f too. Moreover, let*

$$h : [0,1] \times f^{-1}(]-\infty, c]) \to \Lambda^2(f^{-1}(]-\infty, c]))$$

be a smooth homotopy of non-degenerate forms and let $\tau \in \Omega^2_{nd}(M)$ such that $h_0 = \tau$ on $f^{-1}(]-\infty, c])$.
Then there is a smooth homotopy of non-degenerate forms

$$\tilde{h} : [0,1] \times M \to \Lambda^2(M)$$

such that $\tilde{h}_0 = \tau$, and $\tilde{h}_1 = h_1$ on $f^{-1}(]-\infty, c-\varepsilon])$.

Proof of Proposition 2.31. Choose a smooth cutoff function $\beta : \mathbb{R} \to [0,1]$ such that $\beta = 1$ on $]-\infty, c - \frac{3\varepsilon}{4}]$ and $\beta = 0$ on $[c - \frac{\varepsilon}{4}, \infty[$. Now define $\tilde{h} : [0,1] \times M \to \Lambda^2(M)$ by:

$$\tilde{h}_t(x) = \begin{cases} h_t(x) & \text{if } f(x) < c - \frac{3\varepsilon}{4} \\ h_{t\beta(f(x))}(x) & \text{if } c - \varepsilon < f(x) < c \\ \tau(x) & \text{if } f(x) > c - \frac{\varepsilon}{4} \end{cases}$$

[12] It is a consequence of the rank theorem that $f^{-1}(]-\infty, c])$ is a submanifold of the manifold without boundary int(M) (i.e. that inclusion is a smooth embedding), if c is a regular value of f. It follows that $f^{-1}(]-\infty, c])$ is also a submanifold of M.

This is well-defined by our choice of β and the fact that $\tau = h_0$ on $f^{-1}(]-\infty, c])$. $\tilde{h}$ is smooth since it is smooth on $[0,1] \times f^{-1}(]-\infty, c-\frac{3\varepsilon}{4}[)$, $[0,1] \times f^{-1}(]c-\varepsilon, c[)$ and $[0,1] \times f^{-1}(]c-\frac{\varepsilon}{4}, \infty[)$, which form an open cover of $[0,1] \times M$. Moreover, $\tilde{h}$ is a homotopy of non-degenerate 2-forms since h is and since τ is non-degenerate. Finally, we have that $\tilde{h}_0 = \tau$ since $\tau = h_0$ on $f^{-1}(]-\infty, c])$ and we have $\tilde{h}_1 = h_1$ on $f^{-1}(]-\infty, c-\varepsilon])$ by definition of $\tilde{h}$. $\square$

3 Morse functions

Morse functions form one of the main tools in the proof of Gromov's theorem. In this section we develop the theory about Morse functions that will be used. Since this theory is only used in Section 5, the reader could safely decide to read Section 4 first.
In subsections 3.1 and 3.2 we mostly follow the treatment of [Mil63, Part 1.2], respectively [Mil63, Part 1.3]. In subsection 3.3 we mainly follow that of [Mil65, Section 2].

3.1 Basic definitions and Morse' Lemma

In this subsection we define Morse functions and prove Morse' Lemma.

We will first define the Hessian of a smooth function at a critical point. Let M be a smooth manifold and $f \in C^\infty(M)$. Recall that a point $p \in M$ is called a critical point of f if $df_p = 0$. Let $\sigma_f \in \Omega^1(M)$ be defined by $\sigma_f(q) = (q, df_q)$ and let p be a critical point of f. Then $\sigma_f(p) = (p, 0)$ and so its differential at p is a linear map $d\sigma_f|_p : T_pM \to T_{(p,0)}(T^*M)$. We also have:

Proposition 3.1. *Let $E \xrightarrow{\pi} M$ be a smooth vector bundle of rank k and $p \in M$. Then there is a canonical linear isomorphism $\tau : T_pM \times E_p \to T_{(p,0)}E$.*

We postpone the proof of Proposition 3.1. Let $\mathrm{pr}_2 : T_pM \times T_p^*M \to T_p^*M$ denote the projection.

Definition 3.2. Let M be a manifold, $f \in C^\infty(M)$ and let $p \in M$ be a critical point of f. In the notation of the above discussion, we define the Hessian of f at p to be the linear map:

$$H_pf : T_pM \to T_p^*M, \quad H_pf = pr_2 \circ \tau^{-1} \circ d\sigma_f|_p$$

Proof of Proposition 3.1. Define $\tau : T_pM \times E_p \to T_{(p,0)}E$ by:

$$\tau(\delta_M, v)[f] = \delta_M[f(\cdot, 0)] + D_v f(p, \cdot)(0)$$

for $f \in C^\infty(E)$. Here the last term denotes the directional derivative of $f(p, \cdot)$ at the point 0 in the direction v. It is easily checked that $f(\cdot, 0) \in C^\infty(M)$, $f(p, \cdot) \in C^\infty(E_p)$ and by using standard product rules for differentiation that $\tau(\delta_M, v)$ is a derivation at $(p, 0)$. Thus τ is well-defined. Further, τ is clearly linear. So since

$$\dim(T_{(p,0)}E) = \dim(M) + \mathrm{rank}(E) = \dim(T_pM \times E_p)$$

it suffices to show that τ is injective, to show that τ is a linear isomorphism.

To this end, suppose $\tau(\delta_M, v) = 0$. First, let $g \in C^\infty(M)$. Define $f : E \to \mathbb{R}$ by $f(q, v) = g(q)$. Clearly, $f \in C^\infty(E)$, and so:

$$0 = \tau(\delta_M, v)[f] = \delta_M[g] + 0 = \delta_M[g]$$

Since $g \in C^\infty(M)$ was arbitrary, $\delta_M = 0$. Now choose a local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ such that $p \in U$. Denote by $\pi_i : U \times \mathbb{R}^k \to \mathbb{R}$ the projection onto the $i + 1^{th}$ component. By a standard use of the extension lemma for smooth functions on manifolds applied to $\pi_i \circ \Phi$ for $i = 1, ..., k$, there is an open neighbourhood $\tilde{U}$ of p in M such that $\tilde{U} \subset U$ and there are functions $h_1, ..., h_k \in C^\infty(E)$ such that $h_i|_{\pi^{-1}(\tilde{U})} = \pi_i \circ \Phi$ for all i. By the latter property it follows that $h_i(p, \cdot) = \pi_i \circ \Phi(p, \cdot)$ and so we have:

$$0 = \tau(\delta_M, v)[h_i] = D_v(\pi_i \circ \Phi(p, \cdot))(0) = \pi_i(\Phi(p, v))$$

where we use that $\pi_i \circ \Phi(p, \cdot)$ is linear. This holds for all $i = 1, ..., k$ so $\Phi(p, v) = (p, 0)$ and $\Phi(p, \cdot)$ is an isomorphism so $v = 0$, which proves that τ is injective. □

We are now ready to define Morse functions.

Definition 3.3. A critical point p of a function $f \in C^\infty(M)$ is called non-degenerate if $H_p f$ is a linear isomorphism. A function $f \in C^\infty(M)$ is called a Morse function if every critical point of f is non-degenerate.

The following proposition can be used to compute the Hessian and to check whether a smooth function is Morse.

Proposition 3.4. *Let M be a smooth n-manifold and $f \in C^\infty(M)$. The following statements hold:*
(i) $p \in M$ is a critical point of f if and only if $D_i(f_\varphi)(\varphi(p)) = 0$ for all $i = 1, ..., n$, for some (or equivalently every) chart (U, φ) around p.
(ii) If (U, φ) is a chart around a critical point p of f then:

$$H_p f(\partial_i^\varphi|_p) = \sum_{j=1}^{n} D_i D_j(f_\varphi)(\varphi(p))\, dx_j^\varphi\big|_p$$

(iii) For a critical point p of f it holds that: $H_p f$ is a linear isomorphism if and only if the matrix with $(i, j)^{th}$ coefficient $D_i D_j(f_\varphi)(\varphi(p))$ is invertible for some (or equivalently every) chart (U, φ) around p.

Proof. Statement (i) follows since for every chart (U, φ) around p we have:

$$df_p(\partial_i^\varphi|_p) = D_i(f_\varphi)(\varphi(p))$$

and $\left\{\partial_1^\varphi|_p, ..., \partial_n^\varphi|_p\right\}$ is a basis of T_pM.
For (ii), let (U, φ) be a chart around p. Let $v \in T_pM$, then in the notation of the above discussion we have $H_p f(v) = \mathrm{pr}_2(\tau^{-1}(d\sigma_f|_p(v)))$. Write $\tau^{-1}(d\sigma_f|_p(v)) = (\delta_M, \tilde{v}) \in T_pM \times T_p^*M$, then $H_p f(v) = \tilde{v}$. First we derive an expression for $\tilde{v}$. To this end, let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ denote the local trivialisation corresponding to the local frame induced by φ, so that:

$$\Phi(q, \sum_{i=1}^{n} \lambda_i\, dx_i^\varphi|_q) = (q, \sum_{i=1}^{n} \lambda_i e_i)$$

Define π_i and h_i (for this Φ) as in the proof of Proposition 3.1. Then $\tilde{v} = \sum_{i=1}^{n} \pi_i(\Phi(p, \tilde{v}))\, dx_i^{\varphi}|_p$. Further, note that since

$$h_i|_{\pi^{-1}(\tilde{U})}(\cdot, 0) = \pi_i \circ \Phi(\cdot, 0) = 0$$

and $\pi^{-1}(\tilde{U})$ is an open neighbourhood of p it follows that $\delta_M[h_i(\cdot, 0)] = 0$. So as in the proof of Proposition 3.1 we have $\tau(\delta_M, \tilde{v})[h_i] = \pi_i(\Phi(p, \tilde{v}))$, hence:

$$\tilde{v} = \sum_{i=1}^{n} \tau(\delta_M, \tilde{v})[h_i]\, dx_i^{\varphi}|_p = \sum_{i=1}^{n} d\sigma_f|_p\,(v)[h_i]\, dx_i^{\varphi}|_p = \sum_{i=1}^{n} v[h_i \circ \sigma_f]\, dx_i^{\varphi}|_p$$

which is the promised expression. Combining all of this it follows that:

$$H_p f(\partial_i^{\varphi}|_p) = \sum_{j=1}^{n} \partial_i^{\varphi}|_p\,[h_j \circ \sigma_f]\, dx_j^{\varphi}\big|_p = \sum_{j=1}^{n} D_i(h_j \circ \sigma_f \circ \varphi^{-1})(\varphi(p))\, dx_j^{\varphi}\big|_p$$

Now note that for all $q \in \tilde{U}$ we have that

$$\sigma_f(q) = (q, df_q) = (q, \sum_{i=1}^{n} D_i(f_\varphi)(\varphi(q))\, dx_i^{\varphi}|_q)$$

and so that

$$h_j \circ \sigma_f \circ \varphi^{-1}(x) = \pi_j \circ \Phi \circ \sigma_f \circ \varphi^{-1}(x) = D_j(f_\varphi)(x)$$

for all $x \in \varphi(\tilde{U})$. Since $\varphi(\tilde{U})$ is an open neighbourhood of $\varphi(p)$, statement (ii) now follows.

To see statement (iii), note that from statement (ii) it follows that this matrix is the matrix representation of $H_p f$ with respect to the bases $\left\{\partial_1^{\varphi}|_p, ..., \partial_n^{\varphi}|_p\right\}$ for T_pM and $\left\{dx_1^{\varphi}|_p, ..., dx_n^{\varphi}|_p\right\}$ for T_p^*M. Hence (iii) follows. $\square$

Recall that we have defined the Hessian at a critical point p of a smooth function as a linear map $H_p f : T_pM \to T_p^*M$. We can also view $H_p f$ as a bilinear map, by the next proposition.

Proposition 3.5. *Let V be a vector space. The vector space $L(V;V^*)$ of linear maps $V \to V^*$ is canonically isomorphic to the vector space $L^2(V)$ of bilinear maps $V \times V \to \mathbb{R}$.*

Proof. Define the maps:

$$\begin{aligned} \varphi : L(V; V^*) \to L^2(V) \quad &\text{by} \quad \varphi(\alpha)(v, w) = (\alpha(v))(w) \\ \psi : L^2(V) \to L(V; V^*) \quad &\text{by} \quad (\psi(\beta)(v))(w) = \beta(v, w) \end{aligned}$$

It is easily checked that these maps indeed map into their prescribed codomain and that they're linear and mutually inverse. $\square$

Next we will define the index of a critical point of a Morse function.

Definition 3.6. Let V be a vector space and $\beta : V \times V \to \mathbb{R}$ bilinear. The index of β is defined as the maximum of the dimensions of subspaces W of V such that $\beta(v, v) < 0$ for all $v \in W \setminus \{0\}$.
The (Morse) index of a critical point p of a smooth function f is defined as the index of $H_p f$, viewed as a bilinear map. It will be denoted by $I(p)$.

The following propositions allow us to compute the Morse index of a critical point.

Proposition 3.7. *Let V be a vector space and $\beta : V \times V \to \mathbb{R}$ a symmetric bilinear map. Let $\{b_1, ..., b_n\}$ be a basis of V and let $L : V \to V$ be the linear map represented by the matrix with $(i, j)^{th}$ coefficient $\beta(b_i, b_j)$, with respect to this basis. Then the index of β is equal to the dimension of the negative eigenspace of L.*

Proof. Denote by I the index of β and by $\langle \cdot, \cdot \rangle$ the standard inner product for V with respect to the basis $\{b_1, ..., b_n\}$. Note first that M is symmetric, since β is. So there is an orthonormal basis $\{v_1, ..., v_n\}$ for V with respect to $\langle \cdot, \cdot \rangle$ of eigenvectors of L. Denote the eigenvalue of v_i by λ_i and the dimension of the negative eigenspace of L (i.e. $\text{span}\{v_i | \lambda_i < 0\}$) by D^-. Bilinearity of β shows that: $\beta(v, w) = \langle v, Lw \rangle$ for all $v, w \in V$. Thus:

$$\beta(v_i, v_j) = \lambda_j \langle v_i, v_j \rangle = \lambda_j \delta_{i,j}$$

for all i and j, hence from bilinearity of β it follows that:

$$\beta(\sum_{i=1}^{n} \mu_i v_i, \sum_{j=1}^{n} \mu_j v_j) = \sum_{i=1}^{n} \lambda_i \mu_i^2 \tag{3.1}$$

From (3.1) it follows that $\beta(v, v) < 0$ for any non-zero v in the negative eigenspace of L, hence $I \geq D^-$.
On the other hand let W be a subspace of V such that $\beta(v, v) < 0$ for every non-zero $v \in W$. Note that $\beta(v, v) \geq 0$ for every non-zero $v \in \text{span}\{v_i | \lambda_i \geq 0\}$ by (3.1), hence

$$W \cap \text{span}\{v_i | \lambda_i \geq 0\} = \{0\}$$

and so $W \oplus \text{span}\{v_i | \lambda_i \geq 0\}$ is a linear subspace of V of dimension

$$\dim(W) + \dim(V) - D^-$$

Thus it follows that $\dim(W) \leq D^-$. This shows that $I \leq D^-$, which finishes the proof. $\square$

Proposition 3.8. *Let p be a critical point of a Morse function f on a smooth n-manifold M and let (U, φ) be a chart around p. Then $I(p)$ is the dimension of the negative eigenspace of the linear map :$T_pM \to T_pM$ represented by the matrix defined in Proposition 3.4(iii), with respect to $\left\{ \partial_1^{\varphi}|_p, ..., \partial_n^{\varphi}|_p \right\}$.*

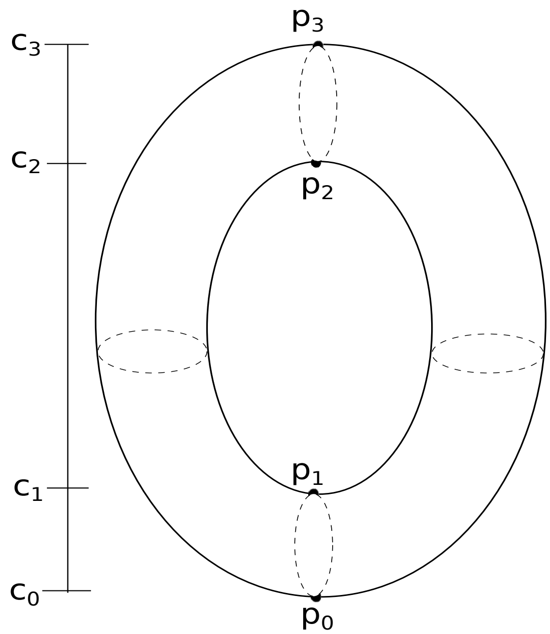

Figure 2: The height function is a Morse function on the torus

Proof. In this proof, we view H_pf as a bilinear map. Note that from Proposition 3.4(ii) it follows that:

$$H_pf(\partial_i^\varphi|_p, \partial_j^\varphi|_p) = D_iD_j(f_\varphi)(\varphi(p))$$

for all $i, j = 1, ..., n$. Since $\left\{\partial_1^\varphi|_p, ..., \partial_n^\varphi|_p\right\}$ is a basis for T_pM, this implies that H_pf is symmetric. Applying Proposition 3.7 to this basis yields the desired. □

Example 3.9. Let the torus be embedded in $\mathbb{R}^3$ as in Figure 2. By applying Proposition 3.4 it is straightforward to compute that the height function is a Morse function with 4 critical points one at each of the critical values $c_0 < c_1 < c_2 < c_3$. Using Proposition 3.8 one can compute that the lowest critical point has index 0, the highest has index 2 and the others have index 1.

Next, we shall prove an important lemma often referred to as Morse' lemma. It describes the behaviour of Morse functions at critical points in terms of the Morse index.

Lemma 3.10 (Morse)**.** *Let f be a Morse function on a smooth n-manifold M and let p be a critical point of f in int(M). Then there is chart (U, φ) around p such that $\varphi(p) = 0$ and:*

$$f_\varphi(x_1, ..., x_n) = f(p) - x_1^2 - ... - x_{I(p)}^2 + x_{I(p)+1}^2 + ... + x_n^2$$

Proof. To begin with, note that if there is a chart (U, φ) around p such that $\varphi(p) = 0$ and $f_\varphi(x_1, ..., x_n) = f(p) \pm x_1^2 \pm ... \pm x_n^2$ where the number of minus signs is λ, then by Proposition 3.8 this λ must be the Morse index $I(p)$. A reordering of the component functions of φ would then give the desired chart, hence it suffices to prove that such (U, φ) exists.

We will prove by induction that for every $k = 0, ..., n$ there is a chart (U_k, φ_k) such that $\varphi_k(p) = 0$ and:

$$f_{\varphi_k}(x_1, ..., x_n) = f(p) + \sum_{i=1}^{k} \pm x_i^2 + \sum_{i,j>k} x_i x_j H_{ij}^k(x_1, ..., x_n)$$

for some smooth map $H^k : \varphi_k(U_k) \to \mathrm{Sym}(n, \mathbb{R})$ such that $H^k(0)$ is diagonal and $H_{i,i}^k(0) = 0$ if $i \leq k$ and $H_{i,i}^k(0) \neq 0$ if $i > k$. In the case $k = n$, f_{φ_k} is of the desired form and so this would prove the lemma.

For the case $k = 0$ choose (U, φ) to be a chart around p such that $\varphi(p) = 0$ and $\varphi(U) = B_1^n(0)$. By Taylor's theorem we have:[13]

$$f_\varphi(x_1, ..., x_n) = f(p) + \sum_{i,j>0} x_i x_j \int_0^1 (1-t) D_i D_j f_\varphi(tx_1, ..., tx_n) dt$$

using that $D_i(f_\varphi)(0) = 0$ for all i, by Proposition 3.4(i). Set

$$H_{ij}(x_1, ..., x_n) = \int_0^1 (1-t) D_i D_j (f_\varphi)(tx_1, ..., tx_n) dt$$

Then $H_{ij}(0) = \frac{1}{2} D_i D_j (f_\varphi)(0)$, so $H(0)$ is invertible by Proposition 3.4(iii). Moreover, H is smooth, and symmetric at every point. From this we'll construct a new chart and H^0, so that $H^0(0)$ is diagonal and only has non-zero diagonal elements.

To this end, note first that since $H(0)$ is a symmetric $n \times n$ matrix, there is a basis of orthonormal eigenvectors $\{v_1, ..., v_n\}$ in $\mathbb{R}^n$. Define the linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(e_i) = v_i$, where $\{e_1, ..., e_n\}$ is the standard basis for $\mathbb{R}^n$. Then since $\{v_1, ..., v_n\}$ is a basis for $\mathbb{R}^n$ too, T is a linear isomorphism, hence a diffeomorphism, and so $(U_0, \varphi_0) := (U, T^{-1} \circ \varphi)$ is a smooth chart around p such that $\varphi_0(p) = 0$. Next, define

$$H_{ij}^0 := v_i^T (H \circ T) v_j$$

by matrix multiplication. Then H^0 is smooth and symmetric on $\varphi_0(U_0)$, since H is so on $\varphi(U)$. Further, note that since $H(0)$ is invertible it only has non-zero eigenvalues, and note that $H_{ij}^0(0) = \lambda_i \delta_{ij}$, where λ_i denotes the eigenvalue for v_i. From this it follows that $H^0(0)$ is diagonal and only has non-zero diagonal elements. A straightforward computation

[13] Here we use that $\varphi(U)$ is convex and contains 0. See [Lee12, Theorem C.15] for a proof of Taylor's theorem.

shows that the pair (U_0, φ_0) and H^0 satisfies the remaining requirement.
Now suppose that the statement is true for some $k \in \{0, ..., n-1\}$ and let (U, φ) be a chart as in the induction hypothesis. Then by continuity of H^k and by possibly shrinking U we may assume that $H^k_{k+1,k+1}(x) \neq 0$ for all $x \in \varphi(U)$. Hence $\sqrt{|H^k_{k+1,k+1}|}$ is smooth and nonzero on $\varphi(U)$. Now define $\psi : \varphi(U) \to \mathbb{R}^n$ by $\psi(x)_i = x_i$ if $i \neq k+1$ and:

$$\psi(x)_{k+1} = \frac{x_{k+1}}{\sqrt{|H^k_{k+1,k+1}(x)|}} - \sum_{i>k+1} x_i \frac{H^k_{i,k+1}(x)}{H^k_{k+1,k+1}(x)}$$

Then ψ is smooth. Further, since $\mathrm{J}_\psi(0)_{ij}$=0 for $i > j$ we find that:[14]

$$\mathrm{Det}(\mathrm{J}_\psi(0)) = D_{k+1}\psi_{k+1}(0) = \frac{1}{\sqrt{|H^k_{k+1,k+1}(0)|}} \neq 0$$

so by the inverse function theorem there are open neighbourhoods U' and V of 0 such that $\psi : U' \to V$ is a diffeomorphism. Thus $(U_{k+1}, \varphi_{k+1}) = (\varphi^{-1}(V), \psi^{-1} \circ \varphi)$ is a smooth chart around p such that $\varphi_{k+1}(p) = 0$. We set:

$$H^{k+1}_{ij} = (H^k_{ij} - \frac{H^k_{i,k+1} H^k_{j,k+1}}{H^k_{k+1,k+1}}) \circ \psi$$

A straightforward computation shows $f \circ \varphi^{-1}_{k+1}$ has the desired form, and that H^{k+1} has the desired properties. This proves the induction step. $\square$

Remark 3.11. (i) Let M be a manifold and f a Morse function on M. Note that from Morse' Lemma and Proposition 3.4(i) it follows that the critical points of f in $\mathrm{int}(M)$ are isolated. Since the set $\{p \in M | df_p \neq 0\}$ is open in M, it follows that if M is compact and f has no critical points on ∂M, then f must have finitely many critical points.

(ii) Let M be a smooth manifold and $f \in C^\infty(M)$. Note that from Morse' Lemma and Proposition 3.8 it follows that a point $p \in \mathrm{int}(M)$ is a local minimum of f if and only of $I(p) = 0$ and a local maximum of f if and only if $I(p) = n$.

3.2 Gradient flow

Having developed the definitions and most basic properties of Morse functions, an outlook on how this theory will help us prove Gromov's theorem seems in order. Roughly speaking, a Morse function keeps track of the homotopy type of a smooth manifold. Moreover, they allow an exhaustion of a manifold by submanifolds of the form $f^{-1}(-\infty, a]$. This will allow us to construct the homotopy in Gromov's theorem step by step. In this section we will prove a theorem by means of which we can extend the homotopy between critical level sets of f.

[14]Here $\mathrm{J}_\psi(0)$ denotes the Jacobian of ψ at 0.

Remark 3.12. Given a smooth manifold M and $f \in C^\infty(M)$ we will use the notation:

$$M^a = f^{-1}(]-\infty, a])$$

It is a consequence of the rank theorem that M^a is a submanifold of M, if a is a regular value of f and $M^a \subset \mathrm{int}(M)$.

Theorem 3.13. *Let M be a smooth manifold without boundary and $f \in C^\infty(M)$. Let $a < b$ such that $f^{-1}([a, b])$ is compact and contains no critical points of f, and such that $M^a \neq \emptyset$. Then there is a smooth isotopy $\varphi : [0, 1] \times M \to M$ such that $\varphi_0 = Id_M$ and φ_1 maps M^b onto M^a. In particular, the submanifolds M^a and M^b are diffeomorphic.*

To prove this we need a Riemannian metric again. Given a smooth Riemannian manifold (M, g), we define:

$$\tilde{g} : TM \to T^*M \text{ by } \tilde{g}(p, v) = g_p(v, \cdot)$$

Since g_p is positive definite, it follows that $\tilde{g}$ restricts to a linear isomorphism $T_pM \to T_p^*M$ for every $p \in M$. Using this, it follows that $\tilde{g}$ is a smooth bundle isomorphism.

Definition 3.14. Let (M, g) be a smooth Riemannian manifold and $f \in C^\infty(M)$. The gradient of f with respect to g is defined as the smooth vector field $\mathrm{grad}(f)_p = \tilde{g}^{-1}(df_p)$.

Remark 3.15. Since $\mathrm{grad}(f)$ is a composition of smooth maps it is indeed smooth. Further, note that $\mathrm{grad}(f)$ is the unique vector field X such that $df_p = g_p(X_p, \cdot)$ for all $p \in M$.

We are now ready to prove Theorem 3.13.

Proof of Theorem 3.13. The isotopy will be constructed using the flow of a vector field. Note first that since the set $\{p \in M | df_p \neq 0\}$ is open in M and $f^{-1}([a, b]) \subset \{p \in M | df_p \neq 0\}$, for every $p \in f^{-1}([a, b])$ there is an open neighbourhood U_p of p such that f has no critical points on U_p. Since M has a basis for its topology of pre-compact sets, we can choose these U_p to be pre-compact, and by compactness of $f^{-1}([a, b])$ we can cover $f^{-1}([a, b])$ by finitely many such U_p. Define U to be the union of these U_p. Then U is an open neighbourhood of $f^{-1}([a, b])$ and it's pre-compact being a finite union of pre-compact sets.
Next, choose a Riemannian metric g on M. Note that since $\tilde{g}$ restricts to a linear isomorphism on the fibres of TM, $\mathrm{grad}(f)_p \neq 0$ for all $p \in U$. Thus since g_p is positive definite it follows that $g_p(\mathrm{grad}(f)_p, \mathrm{grad}(f)_p) > 0$ for all $p \in U$. Therefore, the vector field $\tilde{X}$ on U defined by

$$\tilde{X}_p = \frac{\mathrm{grad}(f)_p}{g_p(\mathrm{grad}(f)_p, \mathrm{grad}(f)_p)}$$

is well-defined and smooth on U. Now define X to be a smooth vector field on M such that $X_p = \tilde{X}_p$ for all $p \in f^{-1}([a, b])$ that is supported in U. This exists by a standard extension lemma for vector fields, since $f^{-1}([a, b])$ is closed in M. Let φ^X be the flow of

X, which is a global flow since X is compactly supported, as U is pre-compact. We call this the normalized **gradient flow** of f with respect to g. Now we define:

$$\varphi : [0,1] \times M \to M \text{ by } \varphi_t(p) = \varphi^X_{t(a-b)}(p)$$

Then φ is a smooth isotopy since φ^X is a global flow and $\varphi_0 = \varphi^X_0 = \mathrm{Id}_M$. For the last property, note that on one hand:

$$X_{\varphi^X(s,p)}[f] = \left.\frac{d}{dt}\right|_{t=s} (f \circ \varphi^X_t(p)) \text{ for all } (s,p) \in \mathbb{R} \times M$$

since φ^X is the flow of X. On the other hand we have:

$$\mathrm{grad}(f)_p[f] = df_p(\mathrm{grad}(f)_p) = g_p(\mathrm{grad}(f)_p, \mathrm{grad}(f)_p)$$

for all $p \in M$ by Remark 3.15, and so $X_p[f] = 1$ for all $p \in f^{-1}([a,b])$. So it follows that:

$$f \circ \varphi^X_t(p) = f(\varphi^X_0(p)) + t = f(p) + t \tag{3.2}$$

for all (t,p) such that $\varphi^X_s(p) \in f^{-1}([a,b])$ for all s between (and including) 0 and t. Using this it is not hard to show that for every $p \in f^{-1}([a,b])$:

$$\max\left\{t \in \mathbb{R}_{\geq 0} | \ \varphi^X_s(p) \in f^{-1}([a,b]) \text{ for all } s \in [0,t]\right\} = b - f(p)$$
$$\min\left\{t \in \mathbb{R}_{\leq 0} | \ \varphi^X_s(p) \in f^{-1}([a,b]) \text{ for all } s \in [t,0]\right\} = a - f(p)$$

It follows that if $p \in f^{-1}([a,b])$, then $\varphi^X_s(p) \in f^{-1}([a,b])$ for all $s \in [a - f(p), b - f(p)]$. Thus if $p \in f^{-1}([a,b])$, then (3.2) holds for all $t \in [a - f(p), b - f(p)]$.
We will now show that φ^X_{b-a} maps M^a into M^b and that φ^X_{a-b} maps M^b into M^a. Noting that φ^X_{b-a} and φ^X_{a-b} are mutually inverse, the last property follows.
For the first statement, let $p \in M^a$. Suppose that $f \circ \varphi^X_{b-a}(p) > b$. By the intermediate-value theorem there is a $t \in [0, b-a[$ such that $f \circ \varphi^X_t(p) = a$. It now follows by (3.2) that:

$$f \circ \varphi^X_{b-a}(p) = f \circ \varphi^X_{b-a-t}(\varphi^X_t(a)) = a + b - a - t \leq b$$

which is a contradiction. Thus $\varphi^X_{b-a}(p) \in M^b$.
For the second statement, we claim that if $f(p) \leq a$ then $f \circ \varphi^X_t(p) \leq a$ for all $t \leq 0$. Suppose $f(p) \leq b$. If $f(p) \leq a$ then by the claim it follows directly that $f \circ \varphi^X_{a-b}(p) \leq a$. If $a < f(p) \leq b$, then $f(p) - b \leq 0$ and by (3.2) we have $f \circ \varphi^X_{a-f(p)}(p) = a$. So by the claim it follows that:

$$f \circ \varphi^X_{a-b}(p) = f \circ \varphi^X_{f(p)-b}(\varphi^X_{a-f(p)}(p)) \leq a$$

as desired.
It remains to prove the claim. To this end, suppose $f(p) \leq a$ and define:

$$T = \left\{t \leq 0 | \ f \circ \varphi^X_s(p) \leq a \text{ for all } s \in [t,0]\right\}$$

Now suppose T is bounded from below. Since $0 \in T$ it then follows that $s := \inf(T)$ is finite. Define $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(t) = f \circ \varphi_t^X(p)$. By continuity of γ it follows that $\gamma(s) \leq a$. If $\gamma(s) = a$ then by (3.2) we find that $\frac{d\gamma}{dt}(s) = 1$ and so we find an $\varepsilon > 0$ such that $\gamma(t) \leq a$ for all $t \in]s - \varepsilon, s]$. If $\gamma(s) < a$ we find such an ε by continuity of γ. In any case, this is in contradiction with $s = \inf(T)$. We conclude that T is not bounded from below, which shows the claim. □

Theorem 3.13 has the following useful extensions to manifolds with boundary.

Theorem 3.16. *Let M be a smooth manifold and $f \in C^\infty(M)$. Let $a < b$ such that $f^{-1}([a, b])$ is compact and contains no critical points of f, and such that $M^a \neq \emptyset$.*
(i) If $f^{-1}([a, b]) \subset int(M)$, then there is a smooth isotopy $\varphi : [0, 1] \times M \to M$ such that $\varphi_0 = Id_M$ and φ_1 maps M^b onto M^a. In particular M^b is diffeomorphic to M^a.
(ii) If $\partial M = f^{-1}(\{b\})$ and b is the maximum of f, then there is a smooth homotopy of injective immersions $\varphi : [0, 1] \times M \to M$ such that $\varphi_0 = Id_M$ and φ_1 maps M onto M^a. In particular, M is diffeomorphic to M^a.

Proof. For (i), we apply Theorem 3.13 to $\text{int}(M)$ to find a smooth isotopy $\psi : [0, 1] \times \text{int}(M) \to \text{int}(M)$ such that $\psi_0 = \text{Id}_{\text{int}(M)}$ and ψ_1 maps M^b onto M^a. Further, from the proof of Theorem 3.13 it is clear that $\psi_t(x) = x$ for all x outside the compact set $\text{supp}(X)$ and all $t \in [0, 1]$, where X is the vector field on M that was constructed in that proof. By construction we have $\partial M \subset M \backslash \text{supp}(X)$. Now define $\varphi : [0, 1] \times M \to M$ by:

$$\varphi_t(x) = \begin{cases} \psi_t(x) & \text{if } x \in \text{int}(M) \\ x & \text{if } x \in M \backslash \text{supp}(X) \end{cases}$$

This is well-defined since both maps agree on the overlap, by the above. Further, φ is smooth on the opens $[0, 1] \times \text{int}(M)$ and $[0, 1] \times (M \backslash \text{supp}(X))$, which cover $[0, 1] \times M$, thus φ is smooth on $[0, 1] \times M$. Note that for every t, the inverse of φ_t is given by:

$$\varphi_t^{-1}(x) = \begin{cases} \psi_t^{-1}(x) & \text{if } x \in \text{int}(M) \\ x & \text{if } x \in M \backslash \text{supp}(X) \end{cases}$$

which is well-defined and smooth by an analogous argument. Thus φ is indeed a smooth isotopy.

To prove part (ii), we define a vector field X on M as in the proof of Theorem 3.13. Note that $M = M^b$. The main difference with the situation in Theorem 3.13 is that since M has non-empty boundary, X need not have a global flow. However, note that X is compactly supported and X_p is outward-pointing for all $p \in \partial M$.[15] We postpone the prove of the latter fact. By similar techniques as used in [Lee12, Theorem 9.12-9.16] to

[15] We use the convention that boundary charts map into $\mathbb{H}^n = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$ and so $v \in T_pM$ is outward pointing if and only if its n^{th} component in some (or equivalently every) chart is strictly negative.

prove that a compactly supported vector field on a smooth manifold without boundary has a global flow, it can be shown that a compactly supported vector field X such that X_p is outward-pointing for all $p \in \partial M$, has a flow φ^X such that:

(i) for every $p \in M$, the maximal integral curve through p is defined on $\mathbb{R}$ and $\varphi_t^X(p) \in \operatorname{int}(M)$ for all $t \in \mathbb{R}$, or there is a unique $t_p \geq 0$ such that the maximal integral curve through p is defined on $]-\infty, t_p]$ and such that $\varphi_{t_p}^X(p) \in \partial M$ and $\varphi_t^X(p) \in \operatorname{int}(M)$ for all $t < t_p$.

(ii) $\varphi_t^X : M \to M$ is a smooth injective immersion for every $t \leq 0$.

Hence we can define φ as in the proof of Theorem 3.13, to obtain a smooth homotopy of injective immersions $\varphi : [0,1] \times M \to M$. Then $\varphi_0 = \varphi_0^X = \mathrm{Id}_M$ and it remains to show that $\varphi_1(M) = M^a$ or equivalently that $\varphi_{a-b}^X(M) = M^a$.

As in the proof of Theorem 3.13, it can be shown that (3.2) holds for all (t,p) such that $t \leq t_p$ and such that $\varphi_s^X(p) \in f^{-1}([a,b])$ for all s between (and including) 0 and t. Using this it is not hard to show that for every $p \in f^{-1}([a,b])$:

$$\max\left\{t \in \mathbb{R}_{\geq 0} \middle|\ t \leq t_p \text{ and } \varphi_s^X(p) \in f^{-1}([a,b]) \text{ for all } s \in [0,t]\right\} = t_p$$
$$\min\left\{t \in \mathbb{R}_{\leq 0} \middle|\ \varphi_s^X(p) \in f^{-1}([a,b]) \text{ for all } s \in [t,0]\right\} = a - f(p)$$

It follows that if $p \in f^{-1}([a,b])$, then $\varphi_s^X(p) \in f^{-1}([a,b])$ for all $s \in [a - f(p), t_p]$. So if $p \in f^{-1}([a,b])$, then (3.2) holds for all $t \in [a - f(p), t_p]$. In particular it follows that: if $p \in f^{-1}([a,b])$, then $b = f \circ \varphi_{t_p}^X(p) = f(p) + t_p$, so $t_p = b - f(p)$. Just as in the proof of Theorem 3.13, it now follows that $\varphi_{a-b}^X(M) \subset M^a$. To show the other inclusion, it suffices to show that $\varphi_{b-a}^X(p)$ is defined for all $p \in M^a$.

To this end, let $p \in M^a$. If $\varphi_t^X(p)$ is defined for all $t \in \mathbb{R}$ there is nothing to prove so suppose otherwise. Since $f \circ \varphi_{t_p}^X(p) = b$ it follows by the intermediate-value theorem that there is a $t_0 \in [0, t_p[$ such that $f \circ \varphi_{t_0}^X(p) = a$. Then by the above the maximal integral curve through $\varphi_{t_0}^X(p)$ is defined on $]-\infty, b-a]$ and so the maximal integral curve through p is defined on $]-\infty, b-a+t_0]$. So $\varphi_{b-a}^X(p)$ is defined since $t_0 \geq 0$.

It now remains to prove that X_p is outward pointing for all $p \in \partial M$. To this end, let $p \in \partial M$ and let (U, φ) be a chart around p. Write $X_p = \sum_{i=1}^n v_i(p)\, \partial_i^\varphi|_p$. Then:

$$1 = df_p(X_p) = X_p[f] = \sum_{i=1}^n v_i(p) D_i(f_\varphi)(\varphi(p)) = v_n(p) \cdot \lim_{t \downarrow 0} \frac{f_\varphi(te_n + \varphi(p)) - b}{t}$$

Here in the first step we used that $p \in f^{-1}([a,b])$ and in the fourth step we used that f is constant on ∂M so that $D_i(f_\varphi)(\varphi(p)) = 0$ for all $i = 1, ..., n-1$. Now since b is the maximum of f, the limit on the right is ≤ 0. Thus $v_n(p) < 0$ as required. This finishes the proof of the theorem. $\square$

3.3 Existence of a Morse function

Of course, to use the theorems about Morse functions on a given manifold M, we need one to exist on M. This is the content of this subsection and of the following theorem.

Theorem 3.17. *Let M be a compact, connected, smooth n-manifold with non-empty boundary and suppose $n \geq 2$. Then there exists Morse function $f \in C^\infty(M)$ that has the following properties:*

(i) $\partial M = f^{-1}(\{c\})$ for some $c \in \mathbb{R}$
(ii) If p and q are different critical points of f then $f(p) \neq f(q)$
(iii) f has no critical points of index n
(iv) f has 1 critical point of index 0
(v) The critical points of f lie in int(M).

To prove Theorem 3.17 we need the following.

Definition 3.18. Let W be a smooth, compact n-manifold with boundary $\partial W = V_0 \sqcup V_1$, where V_0 and V_1 are disjoint, and both open and closed submanifolds of ∂W. Then the triple (W, V_0, V_1) is called an n-triad.
Moreover, a Morse function on W with the properties:

(i) $f(W) \subset [a, b]$, $V_0 = f^{-1}(\{a\})$ and $V_1 = f^{-1}(\{b\})$ for some $a < b$
(ii) All critical points of f lie in $\mathrm{int}(W)$

is called a triad Morse function on (W, V_0, V_1).

Theorem 3.19. *On every n-triad (W, V_0, V_1) there exists a triad Morse function.*

Theorem 3.20. *Let (W, V_0, V_1) be an n-triad such that $V_i \neq \emptyset$ for $i = 0$ and $i = 1$. Given a triad Morse function f on (W, V_0, V_1) there is a triad Morse function $\tilde{f}$ on (W, V_0, V_1) which agrees with f on an open neighbourhood of ∂W and has the same critical points with the same Morse index as for f, and is such that:*

$$\tilde{f}(p) = \tilde{f}(q) \text{ if } I(p) = I(q) \text{ and } \tilde{f}(p) < \tilde{f}(q) \text{ if } I(p) < I(q)$$

for all critical points p and q of $\tilde{f}$.

Theorem 3.21. *Let (W, V_0, V_1) be an n-triad such that $V_i \neq \emptyset$ for $i = 0$ and $i = 1$ and $H_0(W, V_0) = \{0\}$. Let f be a triad Morse function on (W, V_0, V_1) with the property that:*

$$f(p) = f(q) \text{ if } I(p) = I(q) \text{ and } f(p) < f(q) \text{ if } I(p) < I(q)$$

and let k_i denote the number of critical points of index i of f.
Then there is a triad Morse function $\tilde{f}$ on (W, V_0, V_1) which agrees with f on an open

neighbourhood of ∂W, such that $\tilde{f}$ has the same critical points with the same Morse index as for f, except for the critical points of f of index 0, 1 and 2. For such critical points we can choose $\tilde{f}$ to be as in one of the following three cases:

(i) *If $k_0 \leq k_1$, we can choose $\tilde{f}$ so that $\tilde{f}$ has no critical points of index* 0, *$k_1 - k_0$ critical points of index* 1 *and k_2 critical points of index* 2.

(ii) *If $k_0 > k_1$, we can choose $\tilde{f}$ so that $\tilde{f}$ has no critical points of index* 0 *and* 1, *and has $k_2 + k_0 - k_1$ critical points of index* 2.

(iii) *If $k_0 > k_1$, we can also choose $\tilde{f}$ so that $\tilde{f}$ has $k_0 - k_1$ critical points of index* 0, *no critical points of index 1 and k_2 critical points of index* 2.

Lemma 3.22. *Let f be a triad Morse function on an n-triad (W, V_0, V_1). Then there is a triad Morse function $\tilde{f}$ on $(W, V_0.V_1)$, with the same critical points and Morse indices such that $\tilde{f}(p) \neq \tilde{f}(q)$ for all different critical points p and q of $\tilde{f}$.*

Remark 3.23. In the statement of Theorem 3.21, by $H_0(W, V_0)$ we mean relative singular homology. Recall that $H_0(W, V_0) = \{0\}$ if and only if V_0 intersects all path-components of W.

Unfortunately, the proofs of Theorem 3.20 and 3.21 are beyond the scope of this thesis. They can be found in [Mil65, Theorem 4.8], [Mil65, Theorem 8.1] and [Mil65, Lemma 8.3].[16] We will however prove Theorem 3.19 and Lemma 3.22. Before we do this, we prove Theorem 3.17.

Proof of Theorem 3.17. Note first that from the proofs of Theorem 3.19 and Lemma 3.22, it is clear that these statements hold in the case $V_0 = \emptyset$ and $V_1 = \partial W$. Hence we find a Morse function f on M such that all critical points of f lie in $\text{int}(M)$, $f(M) \subset]a, b]$, $f^{-1}(\{b\}) = \partial M$ and such that f attains different critical values on distinct critical points.
By Morse' Lemma it is seen that f is not constant. By compactness of M, f attains a global minimum c_0 in some $p_0 \in M$. Since f is not constant, p_0 cannot lie in ∂W, hence $p_0 \in \text{int}(W)$ is a critical point of index 0, by Remark 3.11(ii).
Because p_0 is the only critical point with critical value c_0 and c_0 is the global minimum of f and f has finitely critical points (Remark 3.11(i)), we can choose $\varepsilon > 0$ small enough such that p_0 is the only critical point of f that is contained in $f^{-1}(]-\infty, c_0 + \varepsilon])$ and such that $c_0 + \varepsilon < b$.

Claim 1. We can choose $\varepsilon > 0$ small enough such that in addition $f^{-1}([c_0 + \varepsilon, b])$ is path-connected and $f^{-1}(\{c_0 + \varepsilon\}) \neq \emptyset$.

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[16] These theorems are not formulated precisely as in [Mil65], but they can be deduced from the proofs of the theorems in [Mil65].

Now define $W = f^{-1}([c_0 + \varepsilon, b])$. Since $c_0 + \varepsilon$ is a regular value of f, (W, V_0, V_1) is an n-triad, such that $V_0 = f^{-1}(\{c_0 + \varepsilon\})$ and $V_1 = f^{-1}(\{b\})$, and $f|_W$ is a triad Morse function on W. Since W is path-connected, by Remark 3.23 it follows that:

$$H_0(W, V_0) = \{0\} = H_0(W, V_1)$$

Now note that $-f|_W$ is a triad Morse function on (W, V_0', V_1') such that $-f(W) \subset [-b, -(c_0 + \varepsilon)]$, $V_1' = V_0 = (-f)^{-1}(-\{c_0 + \varepsilon\})$ and $V_0' = V_1 = (-f)^{-1}(-\{b\})$. Hence combining Theorem 3.20 and 3.21 we obtain a triad Morse function f' on (W, V_0', V_1') which agrees with $-f$ on an open neighbourhood of ∂W and which has no critical points of index 0. Now $-f'$ is a triad Morse function on (W, V_0, V_1) which agrees with f on an open neighbourhood of ∂W. By Proposition 3.8 it follows that $p \in W$ is a critical point of f' of index i if and only if p is a critical points of $-f'$ of index $n - i$. Thus $-f'$ has no critical points of index n.

Now assume first that $n > 2$ or that $k_0 \leq k_1$ and $n = 2$, where k_i is the number of critical points of $-f'$ of index i. Again applying Theorem 3.20 and 3.21 to $-f'$, we obtain a triad Morse function $\tilde{f}$ which has no critical points of index n and no critical points of index 0, and which agrees with f on an open neighbourhood U of ∂W.
We choose V open in M such that $U = V \cap W$, and define $g : M \to \mathbb{R}$:

$$g(x) = \begin{cases} f(x) & \text{if } x \in f^{-1}(]-\infty, c_0 + \varepsilon[) \cup V \\ \tilde{f}(x) & \text{if } x \in f^{-1}(]c_0 + \varepsilon, \infty[) \end{cases}$$

Then g is well-defined by our choice of V. g is smooth since it is smooth on $f^{-1}(]-\infty, c_0 + \varepsilon[) \cup V$ and $f^{-1}(]c_0 + \varepsilon, \infty[)$ which form an open cover of M. Further, the critical points of g are p_0 and those of $\tilde{f}$. Thus g has one critical point of index 0 and none of index n. Since g agrees with f on an open neighbourhood of ∂M it follows that the critical points of g lie in $\mathrm{int}(M)$ and that $\partial M \subset g^{-1}(\{b\})$. By compactness of M, g attains a maximum on M, and since g has no critical points of index n this maximum must lie on ∂M, so it follows that $\partial M = g^{-1}(\{b\})$ and that b is the maximum value of g on M. Applying Lemma 3.22 again we may assume that g takes different values on distinct critical points. Thus g is as desired.

Now assume $n = 2$ and $k_0 > k_1$. Applying Theorem 3.20 and 3.21(iii), we obtain a triad Morse function with no critical points of index 1 and no critical points of index 2, which agrees with f on an open neighbourhood of ∂W. As in the first case, we define a Morse function g on M which agrees with f on an open neighbourhood of ∂M, and such that $\partial M = g^{-1}(\{b\})$ and b is the maximum value of g. In particular, g has no critical points on ∂M. On the contrary, in this case g only has critical points of index 0. By Lemma 3.22 we may again assume that g attains different values at distinct critical points. We will now show that g must have only 1 critical point.

Let p be the critical point of g with the highest critical value, say c. We choose a Morse chart (U, φ) around p, and choose $\delta > 0$ such that $B^n_\delta(0) \subset \varphi(U)$ and $c + \frac{\delta^2}{4} < b$. Define:

$$D = \varphi^{-1}(\overline{B^n_{\frac{\delta}{2}}}(0)) \text{ and } O = \varphi^{-1}(B^n_\delta(0))$$

By our choice of chart we have: $g(x) = c + |\varphi(x)|^2$ for all $x \in U$. From this it follows that:

$$g^{-1}(]-\infty, c + \frac{\delta^2}{4}]) = D \cup (g^{-1}(]-\infty, c + \frac{\delta^2}{4}]) \backslash O)$$

which is a union of two closed, disjoint subsets of $g^{-1}(]-\infty, c + \frac{\delta^2}{4}])$. By Theorem 3.16(ii) we have that $g^{-1}(]-\infty, c + \frac{\delta^2}{4}])$ is diffeomorphic to M, hence it is connected. Therefore we find $g^{-1}(]-\infty, c + \frac{\delta^2}{4}]) = D$. We conclude that g has only 1 critical point on $g^{-1}(]-\infty, c + \frac{\delta^2}{4}])$ and so on M. This shows that g is the desired function in this case, which proves the Theorem, up to our claim.

To prove the claim, we choose a Morse chart (U, φ) around p_0 and choose $\varepsilon > 0$ small enough such that $f^{-1}(]-\infty, c_0 + \varepsilon]) \subset U$ and $\overline{B^n_{\sqrt{\varepsilon}}}(0) \subset \varphi(U)$. Such ε can be chosen as follows. Since U is open, $M \backslash U$ is compact, hence f attains a minimum, say s, on $M \backslash U$. Since the global minimum c_0 is only attained by f in p_0, it follows that $s > c_0$. Now every $\varepsilon \in]0, s - c_0[$ satisfies $f^{-1}(]-\infty, c_0 + \varepsilon]) \subset U$. Since $\varphi(U)$ is an open neighbourhood of 0 we can choose $\varepsilon > 0$ as desired.

Since $f(x) = c_0 + |\varphi(x)|^2$ for all $x \in U$ it follows that:

$$V_0 = f^{-1}(\{c_0 + \varepsilon\}) = \varphi^{-1}(\partial B^n_{\sqrt{\varepsilon}}(0)) \neq \emptyset$$

which is path-connected since $n \geq 2$ and φ is a homeomorphism. Therefore, to show that W is path-connected, it suffices to show that every point in W is connected to a point in V_0 by a continuous path in W.

Let $p \in W$. M is connected, thus path-connected, so there is a continuous path $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) \in V_0$. We define:

$$T = \{t \in [0, 1] |\ f \circ \gamma(x) \geq c_0 + \varepsilon \text{ for all } x \in [0, t]\}$$

T is bounded above by 1 and $0 \in T$ thus $s = \sup(T)$ exists. Note that $s \in T$ by continuity of $f \circ \gamma$, so $\gamma(t) \in W$ for all $t \in [0, s]$. If $s < 1$ then by continuity of $f \circ \gamma$ it follows that $f \circ \gamma(s) = c_0 + \varepsilon$. Since the same holds if $s = 1$, we find that $\gamma|_{[0,s]}$ is continuous path in W from p to a point in V_0, which proves the claim. [17] $\square$

The remainder of this section will be devoted to the proof of Theorem 3.19 and Lemma 3.22. We will first prove Theorem 3.19, by using three lemmas.

Lemma 3.24. *On every n-triad (W, V_0, V_1) there exists a smooth function f with the following properties:*

(i) $f(W) \subset [0, 1]$, $V_0 = f^{-1}(\{0\})$ and $V_1 = f^{-1}(\{1\})$

(ii) All critical points of f lie in int(W)

[17] If $s = 0$, then $p = \gamma(s) \in V_0$ so there is nothing to prove in this case.

Proof. Since W is compact, it can be covered by finitely many charts $(U_1, \varphi_1), ..., (U_m, \varphi_m)$. Using that every open U is a union of the opens $U \cap (W \backslash V_0)$ and $U \cap (W \backslash V_1)$ we can choose the charts such that every chart domain intersects at most one of V_0 and V_1. Further by possibly shrinking the chart domains and composing with a translation and dilatation, we can choose the charts so that for every boundary chart (U, φ) we have $\varphi(U) = \mathbb{H}^n \cap B_1^n(0)$.
On each chart domain U_i define $f_i \in C^\infty(U_i)$ by:

$$f_i = \begin{cases} \varphi_i^n & \text{if } U_i \text{ intersects } V_0 \\ 1 - \varphi_i^n & \text{if } U_i \text{ intersects } V_1 \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Choose a smooth partition of unity $\{\psi_i\}$ subordinate to $\{U_1, ..., U_m\}$ and define:

$$f = \sum_{i=1}^m \psi_i f_i$$

Here, as usual, by $\psi_i f_i$ we mean its smooth extension to W by defining it to be 0 outside U_i. Then $f \in C^\infty(W)$.
Since $\varphi(U) = \mathbb{H}^n \cap B_1^n(0)$ for every boundary chart, it follows that $0 \leq f_i \leq 1$ for all $i = 1, ..., m$, from which it follows that $f(W) \subset [0, 1]$. Further, using that for every boundary chart $\varphi^n(x) = 0$ if and only if $x \in \partial W$, it is straightforward to check that $f(x) = 0$ if $x \in V_0$, $f(x) = 1$ if $x \in V_1$ and $0 < f(x) < 1$ if $x \in \text{int}(W)$. Hence $V_0 = f^{-1}(\{0\})$ and $V_1 = f^{-1}(\{1\})$. It remains to check (ii).
To this end, let $p \in \partial W$ and choose a boundary chart (U_i, φ_i) around p such that $\psi_i(p) > 0$. Note that:

$$D_n(f_{\varphi_i})(\varphi_i(p)) = \sum_{j: p \in \text{supp}(\psi_j)} D_n((f_j)_{\varphi_i})(\varphi_i(p))\psi_j(p) + f_j(p) D_n((\psi_j)_{\varphi_i})(\varphi_i(p))$$

For all j such that $p \in \text{supp}(\psi_j)$ we have $f_j(p) = 0$ if $p \in V_0$ and $f_j(p) = 1$ if $p \in V_1$, so in any case $f_j(p)$ is constant over such j. Therefore:

$$\sum_{j: p \in \text{supp}(\psi_j)} f_j(p) D_n((\psi_j)_{\varphi_i})(\varphi_i(p)) = f_j(p) D_n \left(\sum_{j: p \in \text{supp}(\psi_j)} (\psi_j)_{\varphi_i} \right) (\varphi_i(p)) = 0$$

where the last equality follows since $\sum_{j: p \in \text{supp}(\psi_j)} (\psi_j)_{\varphi_i} = 1$ on an open neighbourhood of $\varphi_i(p)$, because for every j such that $p \notin \text{supp}(\psi_j)$ there is an open neighbourhood of $\varphi_i(p)$ on which $(\psi_j)_{\varphi_i} = 0$ and the finite intersection of these is open.
Now note that if $p \in V_0$ then $D_n((f_i)_{\varphi_i})(\varphi_i(p)) = 1$ and if $p \in V_1$ then $D_n((f_i)_{\varphi_i})(\varphi_i(p)) = -1$. Further, if $p \in V_0$ then:

$$D_n((f_j)_{\varphi_i})(\varphi_i(p)) = \lim_{t \downarrow 0} \frac{\varphi_j^n(\varphi_i^{-1}(\varphi_i(p) + te_n))}{t} \geq 0$$

for all j such that $p \in \text{supp}(\psi_j)$. Analogously, $D_n((f_j)_{\varphi_i})(\varphi_i(p)) \leq 0$ for all j such that $p \in \text{supp}(\psi_j)$ if $p \in V_1$. Combining all this we find $D_n(f_{\varphi_i})(\varphi_i(p)) > 0$ if $p \in V_0$ and $D_n(f_{\varphi_i})(\varphi_i(p)) < 0$ if $p \in V_1$, so by Proposition 3.4(i) p is not a critical point of f. So (ii) follows. $\square$

For the next two lemmas we need two more definitions.

Definition 3.25. Let M be a manifold, let $K \subset M$ be compact and let (U, φ) be a chart for M. We will call such a pair (K, φ) a compact chart pair if $K \subset U$.

Definition 3.26. Let M be a compact, smooth n-manifold, $f, g \in C^\infty(M)$ and let $\mathcal{K} = \{(K_1, \varphi_1), ..., (K_m, \varphi_m)\}$ be a finite collection of compact chart pairs. We say that g is C^2_ε-close to f with respect to $\mathcal{K}$ if: [18]

$$|f_{\varphi_k} - g_{\varphi_k}|_{\varphi_k(K_k)} < \varepsilon, \quad |D_i(f_{\varphi_k}) - D_i(g_{\varphi_k})|_{\varphi_k(K_k)} < \varepsilon \quad \text{and} \quad |D_iD_j(f_{\varphi_k}) - D_iD_j(g_{\varphi_k})|_{\varphi_k(K_k)} < \varepsilon$$

for all $i, j = 1, ..., n$ and all $k = 1, ..., m$.

Lemma 3.27. *Let M be a smooth n-manifold and (K, φ) a compact chart pair. For every $f \in C^\infty(M)$ that is Morse on K,[19] there is a $\delta > 0$ such that all smooth functions that are C^2_δ-close to f with respect to (K, φ), are Morse on K too.*

Proof. Let U be a chart domain for φ such that $K \subset U$. For $g \in C^\infty(M)$ we define $M_g : U \to \text{Mat}(n \times n, \mathbb{R})$ by $(M_g(p))_{i,j} = D_iD_j(g_\varphi)(\varphi(p))$. The function:

$$|\nabla(f_\varphi) \circ \varphi| + |\text{Det}(M_f)|$$

is continuous and strictly positive on K since f is Morse on K (apply Proposition 3.4(i) and 3.4(iii)).[20] Hence it takes a minimum value $\varepsilon > 0$ on the compact set K.

Since f is smooth, M_f is continuous,[21] thus $M_f(K)$ is compact. In particular, there is an $R > 0$ such that $M_f(K) \subset B^{n\times n}_R(0)$. Since $\overline{B}^{n\times n}_{R+1}(0)$ is compact, Det is uniformly continuous on this set and so there is a $\delta_0 > 0$ such that if $A, B \in \overline{B}^{n\times n}_{R+1}(0)$ and $|A - B| < \delta_0$, then $|\text{Det}(A) - \text{Det}(B)| < \frac{\varepsilon}{2}$. Now choose $\delta = \min(\frac{\varepsilon}{2n}, \frac{\delta_0}{n^2}, \frac{1}{n^2})$.

If g is C^2_δ-close to f with respect to (K, φ), then the (reverse) triangle inequality shows that

$$||\nabla(f_\varphi)(\varphi(p))| - |\nabla(g_\varphi)(\varphi(p))|| < \frac{\varepsilon}{2}$$

for all $p \in K$ since $\delta \leq \frac{\varepsilon}{2n}$ and that $M_g(K) \subset \overline{B}^{n\times n}_{R+1}(0)$ since $\delta \leq \frac{1}{n^2}$. From the latter it follows that

$$||\text{Det}(M_f(p))| - |\text{Det}(M_g(p))|| < \frac{\varepsilon}{2}$$

[18] Here $|\cdot|_{\varphi_k}$ denotes the supremum norm on $C^\infty(\varphi_k(K_k))$.

[19] i.e. All its critical points $p \in K$ are non-degenerate.

[20] For an open $U \subset \mathbb{R}^n$ and $g \in C^\infty(U)$ we denote $\nabla g := (D_1 g, ..., D_n g) : U \to \mathbb{R}^n$.

[21] The topology on $\text{Mat}(n \times n, \mathbb{R})$ is induced by the Euclidean metric, by identifying $\text{Mat}(n \times n, \mathbb{R})$ with $\mathbb{R}^{n^2}$.

for all $p \in K$, since $\delta \leq \frac{\delta_0}{n^2}$. Therefore we find:

$$|\nabla(g_\varphi)(\varphi(p))| + |\mathrm{Det}(M_g(p))| > |\nabla(f_\varphi)(\varphi(p))| - \frac{\varepsilon}{2} + |\mathrm{Det}(M_f(p))| - \frac{\varepsilon}{2} \geq 0$$

for all $p \in K$. If $p \in K$ is a critical point of g then $|\nabla(g_\varphi)(\varphi(p))| = 0$ (Proposition 3.4(i)), so $\mathrm{Det}(M_g(p)) \neq 0$ by the above, thus $H_p g$ is an isomorphism by Proposition 3.4(iii). This proves the lemma. □

Lemma 3.28. *Let M be a smooth n-manifold, $f \in C^\infty(M)$ and let $\mathcal{K} = \{(K_1, \varphi_1), ..., (K_m, \varphi_m)\}$ be a collection of compact chart pairs with K_i contained in the chart (U_i, φ_i). For every $\varepsilon > 0$ and every $i = 1, ..., m$ such that $U_i \subset int(M)$, there is a $g \in C^\infty(M)$ which agrees with f outside U_i and a $\delta > 0$ such that every smooth function that is C^2_δ-close to g with respect to $\mathcal{K}$, is Morse on K_i and C^2_ε-close to f with respect to $\mathcal{K}$.*

Proof. Let $\varepsilon > 0$ and $i = 1, ..., m$ such that $U_i \subset \mathrm{int}(M)$. Since K_i is compact it is closed in M and since manifolds are normal there is an open neighbourhood V_i of K_i such that $\overline{V_i} \subset U_i$. Choose a smooth bump function $\beta : M \to [0, 1]$, such that $\beta = 1$ on $\overline{V_i}$ and is supported in U_i. We will prove that there is a linear map $L : \mathbb{R}^n \to \mathbb{R}$ such that $g = f + \beta \cdot (L \circ \varphi_i)$ is C^2_ε-close to f with respect to $\mathcal{K}$ and is Morse on K_i. Then by Lemma 3.27 we find a $\delta > 0$ such that every smooth function that is C^2_δ-close to g with respect to $\mathcal{K}$, is Morse on K_i. From Definition 3.26 it is clear that we can possibly choose this $\delta > 0$ smaller so that, in addition, the triangle inequality implies that every smooth function that is C^2_δ-close to g with respect to $\mathcal{K}$ is also C^2_ε-close to f with respect to $\mathcal{K}$.

For a linear map $L \in (\mathbb{R}^n)^*$ denote by $l_1, ..., l_n$ its coefficients with respect to the standard dual basis for $(\mathbb{R}^n)^*$ and set $l = (l_1, ..., l_n) \in \mathbb{R}^n$.

Claim 2. For every $r > 0$ there is an $L \in (\mathbb{R}^n)^*$ such that $|l| < r$ and $f + \beta \cdot (L \circ \varphi_i)$ is Morse on K_i.

We postpone the proof of Claim 2. Let $L \in (\mathbb{R}^n)^*$. First note that $\beta \cdot (L \circ \varphi_i) = 0$ on the open $M \backslash \mathrm{supp}(\beta)$. Secondly, for every $k = 1, ..., m$ and for all $x \in \varphi_k(U_i \cap U_k)$ we have:

$$(\beta \cdot (L \circ \varphi_i))_{\varphi_k}(x) = \beta_{\varphi_k}(x) \sum_{j=1}^{n} l_j(\varphi_i \circ \varphi_k^{-1}(x))_j$$

A straightforward calculation now shows that the partial derivatives of this are expressions that are polynomials in β_{φ_k}, $(\varphi_i \circ \varphi_k^{-1})_j$ and their first and second order partial derivatives, with coefficients in $\{l_1, ..., l_n\}$ and no constant term. Using this and the fact that β_{φ_k}, $(\varphi_i \circ \varphi_k^{-1})_j$ and their first and second order partial derivatives are continuous thus bounded on the compact set $\varphi_k(\mathrm{supp}(\beta) \cap K_k)$ it is clear that we can find $r > 0$ such that if $|l| < r$ then $f + \beta \cdot (L \circ \varphi_i)$ is C^2_ε-close to f with respect to $\mathcal{K}$. Combining this with Claim 2, we get the desired L.

To prove the claim we show that the set $\{l \in \mathbb{R}^n |\ f + (L \circ \varphi_i)$ is not Morse on $V_i\}$ has Lebesgue measure 0, from which the claim follows since a set of measure 0 cannot contain an open set and since $\beta = 1$ on V_i. To this end, note that by Proposition 3.4(i): $x \in V_i$ is a critical point of $f + (L \circ \varphi_i)$ if and only if $-\nabla(f_{\varphi_i})(\varphi_i(x)) = l$. Further note that by Proposition 3.4(iii): $f + (L \circ \varphi_i)$ is Morse on V_i if and only if $D(\nabla(f_{\varphi_i}))(\varphi_i(x))$ is surjective[22] for all critical points $x \in V_i$ of $f + (L \circ \varphi_i)$. Combining these two statements yields: $f + (L \circ \varphi_i)$ is Morse on V_i if and only if l is a regular value of $-\nabla(f_{\varphi_i})|_{\varphi_i(V_i)}$. Contraposition, the fact that $\varphi(V_i)$ is an open submanifold of $\mathbb{R}^n$ and Sard's theorem now imply that $\{l \in \mathbb{R}^n |\ f + (L \circ \varphi_i) \text{ is not Morse on } V_i\}$ has Lebesgue measure 0.[23] This proves the claim and the lemma. □

We can now prove Theorem 3.19.

Proof of Theorem 3.19. Let (W, V_0, V_1) be an n-triad. Let $f \in C^\infty(W)$ be a function as obtained in Lemma 3.24. We will approximate f step by step to obtain the desired triad Morse function. Note that $O := \{p \in W |\ df_p \neq 0\}$ is an open neighbourhood of ∂W on which f has no critical points. Since manifolds are normal, we can choose an open neighbourhood V of ∂W such that $\overline{V} \subset O$. Now by compactness of W we can choose a finite open cover of chart domains $\mathcal{U}$ such that $U \subset O$ or $U \subset W \backslash \overline{V}$ for every $U \in \mathcal{U}$. Finally, by compactness of W we can choose a finite refinement $\{K_1, ..., K_m\}$ of $\mathcal{U}$, such that $K_i \subset O$ for all $i \leq k$ and $K_i \subset W \backslash \overline{V}$ for all $i > k$, for some k. Since $\{K_1, ..., K_m\}$ is a refinement of $\mathcal{U}$, we can choose charts $(U_1, \varphi_1), ..., (U_m, \varphi_m)$ such that $\mathcal{K} = \{(K_1, \varphi_1), ..., (K_m, \varphi_m)\}$ is a finite collection of compact chart pairs, $\{K_1, ..., K_m\}$ covers W and $U_i \subset W \backslash \overline{V}$ if $i > k$.

We will prove by induction that: for every $l = 0, ..., m - k$, there is a smooth function f_l on W such that $f_l|_V = f|_V$ and there is a $\delta_l > 0$ such that every smooth function g that is $C^2_{\delta_l}$-close to f_l with respect to $\mathcal{K}$ is Morse on $\cup_{i=1}^{k+l} K_i$ and satisfies $g(W \backslash V) \subset]0, 1[$.

For the base step, choose $f_0 = f$. By applying Lemma 3.27 k times to K_1,...,K_k we can find $\delta_0 > 0$ small enough such that all smooth functions that are $C^2_{\delta_0}$-close to f with respect to $\mathcal{K}$, are Morse on $\cup_{i=1}^{k} K_i$. Since f is bounded from above and below on the compact $W \backslash V$ and $f(\text{int}(W)) \subset]0, 1[$, we can choose $\delta_0 > 0$ even smaller such that in addition: all $g \in C^\infty(W)$ that are $C^2_{\delta_0}$-close to f with respect to $\mathcal{K}$ satisfy $g(W \backslash V) \subset]0, 1[$. This will be our choice of δ_0.

Now suppose f_l as above exists, for some $0 \leq l < m - k$. By Lemma 3.28, there is an f_{l+1} which agrees with f_l outside U_{k+l+1} and there is a $\delta_{l+1} > 0$ such that every smooth function that is $C^2_{\delta_{l+1}}$-close to f_{l+1} with respect to $\mathcal{K}$ is Morse on K_{k+l+1} and is $C^2_{\delta_l}$-close to f_l with respect to $\mathcal{K}$.

Note first that since f_l agrees with f on V and $U_{k+l+1} \subset W \backslash V$ it follows that f_{l+1} agrees with f on V. Moreover, if a smooth function g is $C^2_{\delta_{l+1}}$-close to f_{l+1} with respect to $\mathcal{K}$, it is $C^2_{\delta_l}$-close to f_l with respect to $\mathcal{K}$, so by our induction hypothesis it satisfies $g(W \backslash V) \subset]0, 1[$ and it is also Morse on $\cup_{i=1}^{k+l} K_i$. So g is Morse on $\cup_{i=1}^{k+l+1} K_i$. This finishes the inductive

[22] Recall that surjectivity and bijectivity are equivalent for linear maps between finite-dimensional vector spaces of equal dimension.

[23] For the statement and proof of Sard's theorem, see [Lee12, Theorem 6.10].

step.

Finally, we note that f_{m-k} is the desired triad Morse function. To see this, note first that f_{m-k} is certainly $C^2_{\delta_{m-k}}$-close to itself. So it is Morse on $\cup_{i=1}^m K_i = W$ and it satisfies $f_{m-k}(W \backslash V) \subset]0,1[$. Since f_{m-k} agrees with f on V it follows that f_{m-k} also has properties (i) and (ii), as desired. □

We end this section with a proof of Lemma 3.22.

Proof of Lemma 3.22. By Remark 3.11(i) f has finitely many critical points, say $p_1, ..., p_m$. We argue by induction over the number of critical points of f for which there is at least one different critical point for which f has the same value. Call this the agreement number of f. If this number is 0, then $f(p) \neq f(q)$ for all different critical points p and q of f, so we can choose $\tilde{f} = f$.

Now let $k \in \{1, ..., m\}$, suppose that the statement is true for all triad Morse functions on (W, V_0, V_1) with agreement number strictly smaller than k, and let f be a triad Morse function on (W, V_0, V_1) for which this number is k. Fix i such that $f(p_i) = f(p_j)$ for some $j \neq i$. Using Lemma 3.10 we can find a chart (U, φ) and an open V around p_i such that U is pre-compact and contains no critical points of f other than p_i, and such that $\overline{V} \subset U$ and $\overline{U} \subset \text{int}(W)$. The last property can be achieved since by assumption $p_i \in \text{int}(W)$. Choose a smooth bump function $\beta : W \to [0,1]$ supported in U, such that $\beta = 1$ on $\overline{V}$.

Note that $\text{supp}(\beta)$ is compact since it's contained in the compact $\overline{U}$. Hence by continuity $|\nabla(\beta_\varphi) \circ \varphi|$ attains a maximum value, say s, and $|\nabla(f_\varphi) \circ \varphi|$ attains a minimum value, say s', on $\text{supp}(\beta)$. Set $\delta_0 = \frac{s'}{s+1}$.

Since f has finitely many critical values we can find $\delta_1 > 0$ such that $[f(p_i) - \delta_1, f(p_i) + \delta_1]$ contains no critical values other than $f(p_i)$. Further, since $\overline{U}$ is compact f attains a maximum on it, say s^*, and since $\overline{U} \subset \text{int}(W)$ it follows that $s^* < 1$. Set $\delta_2 = \frac{1-s^*}{2}$.

Finally, set $\delta = \min(\delta_0, \delta_1, \delta_2)$ and define $g = f + \delta\beta$. By our choice of δ_2 it easily follows that $g^{-1}(0) = V_0$, $g^{-1}(1) = V_1$ and $g(W) \subset [0,1]$. By our choice of δ_1, $g(p_i) \neq g(p_j) = f(p_j)$ for all $j \neq i$.

Further, if p is a critical point of f, then $\beta = 0$ in the open neighbourhood $W \backslash \text{supp}(\beta)$ of p if $p \neq p_i$, and $\beta = 1$ in the open neighbourhood V of p if $p = p_i$. So it follows that p is a critical point of g and by applying Proposition 3.8 the Morse index of p for g is equal to that for f.

If $p \in W \backslash \text{supp}(\beta)$ is not a critical point of f, then f and g agree on the open neighbourhood $W \backslash \text{supp}(\beta)$ of p, so p is not a critical point of g by Proposition 3.4(i).

If $p \in \text{supp}(\beta)$ is not a critical point of f, then $|\nabla(f_\varphi)(\varphi(p))| > 0$ (Proposition 3.4(i)) and so by our choice of δ_0:

$$|\nabla(g_\varphi)(\varphi(p))| \geq |\nabla(f_\varphi)(\varphi(p))| - \delta |\nabla(\beta_\varphi)(\varphi(p))| > 0$$

thus p is not a critical point of g. We conclude that g is a triad Morse function on (W, V_0, V_1) with the same critical points and Morse indices as f, but with agreement number strictly smaller than k. The desired triad Morse function on (W, V_0, V_1) with agreement number 0 now exists by the induction hypothesis applied to g. This finishes the inductive step. $\square$

4 Main lemma

In this section we prove the main lemma that will be used to prove Gromov's theorem. The main ingredient will be the telescope construction. For the proofs of Lemma 4.4 and 4.8 we follow that of [MS99, Lemma 7.35]. For the remainder of this section, no particular source has been used.

4.1 Proof of the main lemma

In this subsection we prove the main lemma, up to the proof of Lemma 4.4, which will be proved in the next subsection.

Remark 4.1. (i) In the remainder of the text we will denote:

$$B^n_\varepsilon = \{x \in \mathbb{R}^n \,||x| < \varepsilon\} \quad \text{and} \quad D^n_\varepsilon = \overline{B^n_\varepsilon}$$

(ii) Let $\tau \in \Omega^k(M)$ be a k-form on a smooth manifold M, and let $S \subset M$ be a smooth manifold such that the inclusion $\iota : S \to M$ is smooth. By saying that τ is exact on S we mean that $\iota^*\tau \in \Omega^k(S)$ is exact.

Lemma 4.2 (Main Lemma)**.** *Let $r, s > 0$, let $1 \leq m < 2n$, let O be an open neighbourhood of $(\partial D^m_r) \times D^{2n-m}_s$ and let*

$$\tau \in \Omega^2(D^m_r \times D^{2n-m}_s)$$

be a non-degenerate 2-form that is exact on O. Furthermore, let $\sigma \in \Omega^1(O)$ such that $\tau = d\sigma$ on O. Then there exists a smooth homotopy

$$h : [0,1] \times D^m_r \times D^{2n-m}_s \to \Lambda^2(D^m_r \times D^{2n-m}_s)$$

of non-degenerate 2-forms, an open neighbourhood $O' \subset O$ of $(\partial D^m_r) \times D^{2n-m}_s$ and a $\sigma_1 \in \Omega^1(D^m_r \times D^{2n-m}_s)$ such that:

(i) $h_0 = \tau$

(ii) $h_1 = d\sigma_1$

(iii) $h_t = \tau$ on O' for all $t \in [0,1]$

(iv) $\sigma = \sigma_1$ on O'

Remark 4.3. (i) Let M and N be manifolds, $f : M \to N$ smooth, let S be a submanifold of N such that $f^{-1}(S)$ is a submanifold of M and let $\tau \in \Omega^k(S)$. If $f|_{f^{-1}(S)}$ is smooth when viewed as a map into S, then we allow ourselves an abuse of notation.[24] We will write $f^*\tau$ instead of the longer $(f|_{f^{-1}(S)})^*\tau$.

(ii) Let M and N be smooth manifolds of equal dimension, let $\tau \in \Omega^2(N)$ be a non-degenerate 2-form and let $\psi : M \to N$ be a smooth immersion or submersion. Then by the rank-nullity theorem, $d\psi_p$ is a linear isomorphism for every $p \in M$. From this it follows that $\psi^*\tau$ is a non-degenerate 2-form.

[24] This is true, for instance, if $S \subset \mathrm{int}(N)$ or if S is open in N.

The proof of this Lemma is based on the following.

Lemma 4.4. *Let $I = [-1, 1]$, let $1 \leq m < 2n$, let O be an open neighbourhood of $(\partial I^m) \times D_1^{2n-m}$ and let*

$$\tau \in \Omega^2(I^m \times D_1^{2n-m})$$

be a non-degenerate 2-form that is exact on O. Furthermore, let $\sigma \in \Omega^1(O)$ such that $\tau = d\sigma$ on O. Then there exists a smooth homotopy

$$h : [0, 1] \times I^m \times D_1^{2n-m} \to \Lambda^2(I^m \times D_1^{2n-m})$$

of non-degenerate 2-forms, an open neighbourhood $O' \subset O$ of $(\partial I^m) \times D_1^{2n-m}$ and a $\sigma_1 \in \Omega^1(I^m \times D_1^{2n-m})$ such that:

(i) $h_0 = \tau$
(ii) $h_1 = d\sigma_1$
(iii) $h_t = \tau$ *on* O' *for all* $t \in [0, 1]$
(iv) $\sigma = \sigma_1$ *on* O'

Proposition 4.5. *Let D be a smooth manifold diffeomorphic to D_1^n and denote by $p \in D$ the point corresponding to $0 \in \mathbb{R}^n$ by a given diffeomorphism. Let U be a neighbourhood of ∂D and let S be a submanifold (possibly with corners) of int(D) such that int(S) contains an open neighbourhood of p in int(D). Then there is an embedding $\psi : S \to$ int(D) such that $D \backslash \psi($int$(S)) \subset U$.*

We postpone the proofs of Lemma 4.4 and Proposition 4.5 and prove the main lemma first.

Proof of Lemma 4.2. Let τ as in the statement of the lemma be given, let O be the open neighbourhood of $(\partial D_r^m) \times D_s^{2n-m}$ on which τ is exact and let $\sigma \in \Omega^1(O)$ such that $\tau = d\sigma$ on O.
First we choose $r' \in]0, r[$ such that $A_{r',r} \times D_s^{2n-m} \subset O$, where $A_{r',r} = \{x \in \mathbb{R}^m | \ r' < |x| \leq r\}$ denotes the annulus, as follows. The function $f : D_r^m \times D_s^{2n-m} \to \mathbb{R}$ defined by $f(x, y) = |x|$ is continuous, hence by compactness of $(D_r^m \times D_s^{2n-m}) \backslash O$ it follows that f attains a maximum, say s, on $(D_r^m \times D_s^{2n-m}) \backslash O$. Since $(\partial D_r^m) \times D_s^{2n-m} \subset O$, it follows that $s < r$, hence we can choose $r' = \frac{s+r}{2}$.
By applying Proposition 4.5 we obtain a smooth embedding $\psi : I^m \to \text{int}(D_r^m)$ such that $D_r^m \backslash \psi(\text{int}(I^m)) \subset A_{r',r}$. Now note that

$$\Psi : I^m \times D_1^{2n-m} \to D_r^m \times D_s^{2n-m} \text{ defined by } \Psi(x, y) = (\psi(x), sy)$$

is a smooth embedding such that $D_r^m \times D_s^{2n-m} \backslash \Psi(\text{int}(I^m) \times D_1^{2n-m}) \subset O$. Hence $\Psi^{-1}(O)$ is an open neighbourhood of $(\partial I^m) \times D_1^{2n-m}$ and $\Psi^* \tau$ is a non-degenerate 2-form on $I^m \times D_1^{2n-m}$, which is exact on $\Psi^{-1}(O)$.

The latter statement can be seen as follows. Let

$$\iota : O \to D_r^m \times D_s^{2n-m} \text{ and } \iota_0 : \Psi^{-1}(O) \to I^m \times D_1^{2n-m}$$

denote inclusion. Further, let Ψ_0 be the map $\Psi \circ \iota_0$ viewed as a map into O. Then $\iota \circ \Psi_0 = \Psi \circ \iota_0$ and since O is open in $D_r^m \times D_s^{2n-m}$, Ψ_0 is smooth. Therefore:

$$\iota_0^* \Psi^* \tau = (\iota \circ \Psi_0)^* \tau = \Psi_0^* \iota^* \tau = d(\Psi_0^* \sigma)$$

as required.

Now we can apply Lemma 4.4 to obtain a smooth homotopy of non-degenerate 2-forms:

$$\tilde{h} : [0,1] \times I^m \times D_1^{2n-m} \to \Lambda^2(I^m \times D_1^{2n-m}),$$

an open neighbourhood $U \subset \Psi^{-1}(O)$ of $(\partial I^m) \times D_1^{2n-m}$ and a $\tilde{\sigma}_1 \in \Omega^1(I^m \times D_1^{2n-m})$ such that $\tilde{h}_0 = \Psi^* \tau$, $\tilde{h}_1 = d\tilde{\sigma}_1$, $\tilde{h}_t = \Psi^* \tau$ on U for every $t \in [0,1]$ and $\tilde{\sigma}_1 = \Psi_0^* \sigma$ on U.

Finally, we define $h : [0,1] \times D_r^m \times D_s^{2n-m} \to \Lambda^2(D_r^m \times D_s^{2n-m})$ by:

$$h_t(x) = \begin{cases} (\Psi^{-1})^* \tilde{h}_t(x) & \text{if } x \in \Psi(\text{int}(I^m) \times D_1^{2n-m}) \\ \tau(x) & \text{if } x \in D_r^m \times D_s^{2n-m} \setminus \Psi(I^m \times D_1^{2n-m} \setminus U) \end{cases}$$

It's straightforward to check that h is well-defined. Further, note that $\Psi(\text{int}(I^m) \times D_1^{2n-m})$ and $D_r^m \times D_s^{2n-m} \setminus \Psi(I^m \times D_1^{2n-m} \setminus U)$ form an open cover of $D_r^m \times D_s^{2n-m}$. They form a cover because $(\partial I^m) \times D_1^{2n-m} \subset U$. The latter set is open because $\Psi(I^m \times D_1^{2n-m} \setminus U)$ is closed in $\Psi(I^m \times D_1^{2n-m})$ which is compact and so is closed in $D_r^m \times D_s^{2n-m}$. To see that the first set is open, it suffices to see that $\psi(\text{int}(I^m))$ is open in $\text{int}(D_r^m)$, since:

$$\Psi(\text{int}(I^m) \times D_1^{2n-m}) = \psi(\text{int}(I^m)) \times D_s^{2n-m}$$

But this follows since ψ is an open map into $\text{int}(D_r^m)$, as it is a smooth immersion into $\text{int}(D_r^m)$.

Now it follows that h is smooth, since it is smooth on $[0,1] \times \Psi(\text{int}(I^m) \times D_1^{2n-m})$ and $[0,1] \times (D_r^m \times D_s^{2n-m} \setminus \Psi(I^m \times D_1^{2n-m} \setminus U))$ which form an open cover of $[0,1] \times D_r^m \times D_s^{2n-m}$. Moreover, since Ψ^{-1} is a diffeomorphism, it follows that h_t is non-degenerate for every $t \in [0,1]$. Properties (i) and (iii) are clearly satisfied, where for (iii) we take

$$O' := D_r^m \times D_s^{2n-m} \setminus \Psi(I^m \times D_1^{2n-m} \setminus U)$$

Note that $O' \subset O$. Thus it remains to check properties (ii) and (iv).

We define $\sigma_1 \in \Omega^1(D_r^m \times D_s^{2n-m})$ by:

$$\sigma_1(x) = \begin{cases} (\Psi^{-1})^* \tilde{\sigma}_1(x) & \text{if } x \in \Psi(\text{int}(I^m) \times D_1^{2n-m}) \\ \sigma(x) & \text{if } x \in D_r^m \times D_s^{2n-m} \setminus \Psi(I^m \times D_1^{2n-m} \setminus U) \end{cases}$$

From the fact that $\tilde{\sigma}_1 = \Psi_0^* \sigma$ on U, it follows that σ_1 is well-defined. By an argument analogous to that for h it follows that σ_1 is smooth. Moreover, since the exterior derivative acts locally, we see that $h_1 = d\sigma_1$. Finally, it is clear that $\sigma_1 = \sigma$ on O'. This proves (ii) and (iv). $\square$

Next, we prove Proposition 4.5.

Proof of Proposition 4.5. By the assumptions on D and S we can reduce to the case in which $D = D^n_1$ and S is a submanifold of $\text{int}(D)$ such that $\text{int}(S)$ contains an open neighbourhood of 0 in $\text{int}(D)$.
We will define a smooth embedding $\psi : S \to \text{int}(D)$, by means of the flow of a smooth vector field, as follows. Choose $\varepsilon \in]0, 1[$ such that $D^n_\varepsilon \subset \text{int}(S)$. As in the proof of Lemma 4.2, we can choose $r \in]\varepsilon, 1[$ such that $\{x \in \mathbb{R}^n | \ r < |x| \leq 1\} \subset U$. Next, choose $r_0, r_1 > 0$ such that $r < r_0 < r_1 < 1$ and choose a smooth cutoff function $\beta : \mathbb{R} \to [0, 1]$ such that $\beta = 1$ on $\mathbb{R}_{\leq r_0}$ and $\beta = 0$ on $\mathbb{R}_{\geq r_1}$. We define:

$$X_x = \beta(|x|) \sum_{i=1}^{n} x_i \, \partial_i|_x$$

It is clear that X defines a smooth, compactly supported vector field on $\text{int}(D)$, hence X has a global flow φ^X on $\text{int}(D)$. We define:

$$\psi = \varphi^X_{log(\frac{r}{\varepsilon})} \circ \iota$$

where $\iota : S \to \text{int}(D)$ denotes inclusion. ψ is a smooth embedding, being a composition of smooth embeddings. It remains to check the last property.
Note that, by our choice of r, it suffices to show that $\{x \in \mathbb{R}^n | \ |x| \leq r\} \subset \psi(\text{int}(S))$. To this end, let $p \in \text{int}(D)$ such that $|p| < r_0$. It is straightforward to check that

$$\gamma_p :] - \infty, \log\left(\frac{r_0}{|p|}\right)[\to \text{int}(D) \text{ defined by } \gamma_p(t) = e^t p$$

is an integral curve for X through p.
Now suppose $|p| \leq r$. Then $\frac{\varepsilon}{r}|p| < |p| < r_0$ and $\log(\frac{r_0}{\frac{\varepsilon}{r}|p|}) > \log(\frac{r}{\varepsilon})$, so:

$$\psi(\frac{\varepsilon}{r}p) = e^{\log(\frac{r}{\varepsilon})}\frac{\varepsilon}{r}p = p$$

Furthermore, note that $\frac{\varepsilon}{r}|p| \leq \varepsilon$. This shows that $\{x \in \mathbb{R}^n | \ |x| \leq r\} \subset \psi(D^n_\varepsilon)$, which implies the desired. $\square$

4.2 The telescope construction

This subsection will be devoted to the proof of Lemma 4.4. The main technique that we'll use is known as the telescope construction. We will need the following.

Proposition 4.6. *Let (X, d) be a compact metric space, let $A \subset X$ be closed and let U be an open neighbourhood of A. Then there is an $\varepsilon > 0$ such that $B^d_\varepsilon(x) \subset U$ for every $x \in A$.*

Proof. We define $f : X \to \mathbb{R}_{\geq 0}$ by:

$$f(x) = \inf \{d(x, y)|\ y \in A\}$$

Using the triangle inequality it follows that f is continuous. Since U is open in X and X is compact it follows that $X \backslash U$ is compact. Therefore f attains a minimum, say s, on $X \backslash U$. Since A is closed it follows that $f^{-1}(\{0\}) = A$. Hence $f^{-1}(\{0\}) \subset U$ and so $s > 0$. Now $\varepsilon = s$ is as required. $\square$

Remark 4.7. Let M_1,M_2 and M_3 be smooth manifolds (possibly with corners) and suppose that for every $p \in M_1$ we have a smooth map $f_p : M_2 \to M_3$. We will say that f_p is a smooth family over M_1, if the map $M_1 \times M_2 \to M_3$ given by $(p, q) \mapsto f_p(q)$ is smooth.

Lemma 4.8. *Under the assumptions of Lemma 4.4, there is a smooth family*

$$\sigma_x \in \Omega^1(I^m \times D_1^{2n-m}) \textit{ over } I^m$$

and an open neighbourhood V of $(\partial I^m) \times D_1^{2n-m}$ such that $d\sigma_x(x, 0) = \tau(x, 0)$ for every $x \in I^m$ and $\sigma_x = \sigma$ on V, for all x in an open neighbourhood of ∂I^m. Moreover, there is an $\varepsilon > 0$ such that for every $x \in I^m$ it holds that:

$$t d\sigma_x + (1 - t)\tau$$

is non-degenerate on $((x + [0, 3\varepsilon]^m) \cap I^m) \times D_\varepsilon^{2n-m}$ for every $t \in [0, 1]$.

Proof. Since $\tau(\cdot, 0)$ is smooth, it follows by Proposition 2.12 that there is a smooth map $\varphi : I^m \to \mathrm{Aut}(\mathbb{R}^{2n})$ such that

$$((\varphi(x)|_{I^m \times D_1^{2n-m}})^* \omega_0)(x, 0) = \tau(x, 0)$$

for every $x \in I^m$. We choose $\sigma_0 \in \Omega^1(\mathbb{R}^{2n})$ such that $\omega_0 = d\sigma_0$, and for every $x \in I^m$ we define $\sigma'_x \in \Omega^1(I^m \times D^{2n-m})$ by:

$$\sigma'_x = (\varphi(x)|_{I^m \times D_1^{2n-m}})^* \sigma_0$$

Then $d\sigma'_x = (\varphi(x)|_{I^m \times D_1^{2n-m}})^* \omega_0$ for every $x \in I^m$, hence $d\sigma'_x(x, 0) = \tau(x, 0)$ for every $x \in I^m$. Moreover, by smoothness of φ, σ_0 and ω_0 it follows that the maps:

$$\begin{aligned} I^m \times I^m \times D_1^{2n-m} &\to \Lambda^1(I^m \times D_1^{2n-m}), \quad (x', x, y) \mapsto \sigma'_{x'}(x, y) \\ I^m \times I^m \times D_1^{2n-m} &\to \Lambda^2(I^m \times D_1^{2n-m}), \quad (x', x, y) \mapsto d\sigma'_{x'}(x, y) \end{aligned}$$

are smooth.

Next, we define:

$$V_\delta := \{x \in I^m|\ \text{there is an } i = 1, ..., m \text{ such that } |x_i| > 1 - \delta\}$$

We choose $\delta_0 > 0$ such that $\overline{V_{\delta_0}} \times D_1^{2n-m} \subset O$ and choose a smooth bump function $\beta : I^m \to [0,1]$ such that $\beta = 1$ on $\overline{V_{\frac{\delta_0}{2}}}$ and β is supported in V_{δ_0}. Furthermore, let $\tilde{\sigma} \in \Omega^1(I^m \times D_1^{2n-m})$ be such that $\tilde{\sigma} = \sigma$ on $\overline{V_{\delta_0}} \times D_1^{2n-m}$. Now for every $x \in I^m$ we define $\sigma_x \in \Omega^1(I^m \times D_1^{2n-m})$ by:

$$\sigma_x = \beta(x)\tilde{\sigma} + (1 - \beta(x))\sigma'_x$$

Then σ_x is indeed a smooth family over I^m, and $\sigma_x = \sigma$ on $V_{\delta_0} \times D_1^{2n-m}$, for all x in the open neighbourhood $V_{\frac{\delta_0}{2}}$ of ∂I^m. Note that for every $x \in I^m$ we have:

$$d\sigma_x = \beta(x)d\tilde{\sigma} + (1 - \beta(x))d\sigma'_x$$

Therefore, $d\sigma_x \in \Omega^2(I^m \times D_1^{2n-m})$ is a smooth family over I^m. Thus it follows that the map $g : [0,1] \times I^m \times I^m \times D_1^{2n-m}$, defined by:

$$g(t, x', x, y) = (1 - t)\tau(x, y) + td\sigma_{x'}(x, y)$$

is smooth and in particular continuous.

Note that $d\tilde{\sigma} = d\sigma = \tau$ on $V_{\delta_0} \times D_1^{2n-m}$, hence $d\sigma_x(x, 0) = \tau(x, 0)$ for all $x \in I^m$. Therefore, $g(t, x', x, y)$ is non-degenerate for all $t \in [0,1]$ and all $(x', x, y) \in S$, where S is defined as:

$$S = \left\{(x', x, y) \in I^m \times I^m \times D_1^{2n-m} |\ x = x' \text{ and } y = 0\right\}$$

Hence by Proposition 2.23 and 2.25 there is an open neighbourhood U of S in $I^m \times I^m \times D_1^{2n-m}$ such that $g(t, x', x, y)$ is non-degenerate for all $t \in [0,1]$ and all $(x', x, y) \in U$. Applying Proposition 4.6 we can find $\varepsilon > 0$ such that for every $x \in I^m$, $g(t, x', x'', y)$ is non-degenerate for all $t \in [0,1]$ and all $(x', x'', y) \in (x + [0, 3\varepsilon]^m) \times (x + [0, 3\varepsilon]^m) \times D_\varepsilon^{2n-m}$. Hence ε is as required. □

We are now ready to prove Lemma 4.4

Proof of Lemma 4.4. The proof goes by induction over m. We will prove the base case $m = 1$. For $m > 1$, the same arguments used to prove the case $m = 1$ can be used to reduce to the case $m - 1$, which would prove the induction step.

So suppose that $m = 1$. Let $\sigma_x \in \Omega^1(I^1 \times D_1^{2n-1})$ be a smooth family over I, let V be an open neigbourhood of $(\partial I) \times D_1^{2n-m}$ and let $\varepsilon > 0$ all be as obtained in Lemma 4.8. By possibly choosing $\varepsilon > 0$ even smaller, we may further assume that $\sigma_x = \sigma$ on V, for all $x \in [-1, -1 + 3\varepsilon] \cup [1 - 3\varepsilon, 1]$, and that $([-1, -1 + 3\varepsilon] \cup [1 - 3\varepsilon, 1]) \times D_1^{2n-m} \subset V$.

Now, let $\alpha : \mathbb{R} \to [0,1]$ be a smooth bump function such that $\alpha = 0$ on $]-\infty, 0]$ and $\alpha = 1$ on $[\varepsilon, \infty[$. We define $\beta : I \times I \times D_1^{2n-1} \to [0,1]$ by $\beta_t(x, y) := \beta(t, x, y) = \alpha(x - t)$ and $\sigma_{x,x'} \in \Omega^1(I \times D_1^{2n-1})$ by:

$$\sigma_{x,x'} = (1 - \beta_{x'})\sigma_x + \beta_{x'}\sigma_{x'}$$

where $x, x' \in I$. Note that:

$$d\sigma_{x,x'} = d\beta_{x'} \wedge (\sigma_{x'} - \sigma_x) + (1 - \beta_{x'})d\sigma_x + \beta_{x'}d\sigma_{x'}$$

Therefore, the map $[0,1] \times I \times I \times (I \times D_1^{2n-1}) \to \Omega^1(I \times D_1^{2n-1})$ defined by

$$(t, x, x', p) \mapsto td\sigma_{x,x'}(p) + (1-t)\tau(p)$$

is smooth and for every $x \in I$ we have that:

$$td\sigma_{x,x}(p) + (1-t)\tau(p) = td\sigma_x(p) + (1-t)\tau(p)$$

is non-degenerate for all $t \in [0,1]$ and all $p \in [x, x+3\varepsilon] \times D_\varepsilon^{2n-1}$. Hence, by applying Proposition 2.23, 2.25 and 4.6 in a similar way as in which we obtained ε in Lemma 4.7, we can find $\delta \in]0, \varepsilon[$ such that: if $x, x' \in I$ such that $x < x' < x + \delta$, then $td\sigma_{x,x'} + (1-t)\tau$ is non-degenerate on $[x, x+3\varepsilon] \times D_\varepsilon^{2n-1}$ for all $t \in [0,1]$.
By our choice of $\varepsilon > 0$, for every $x \in [-1, 1-3\varepsilon]$, τ is homotopic to the exact form $d\sigma_x$ by a homotopy which is non-degenerate on $[x, x+3\varepsilon] \times D_\varepsilon^{2n-1}$. We will show that by our choice of δ, we can patch the 1-forms σ_x together, to obtain a 1-form on $I \times D_\varepsilon^{2n-1}$. This is called the **telescope construction**.
The exterior derivative of this 1-form will then (by construction) be homotopic to τ by a homotopy of non-degenerate 2-forms on $I \times D_\varepsilon^{2n-1}$ which is constant in time on an open neighbourhood of $(\partial I) \times D_\varepsilon^{2n-1}$. Finally, we will show how to replace $I \times D_\varepsilon^{2n-1}$ by $I \times D_1^{2n-1}$.
To this end, we choose a sequence

$$-1 = x_0 < x_1 < ... < x_N = 1 - 3\varepsilon < x_{N+1} = 1 - \varepsilon$$

such that $x_{i+1} - x_i < \delta < \varepsilon$ for all $i \in \{0, ..., N-1\}$. We set $A = I \times D_\varepsilon^{2n-m}$ and $\tilde{A} = [0, (2N+3)\varepsilon] \times D_\varepsilon^{2n-1}$ for notational convenience, and we let $\psi : \tilde{A} \to A$ be the affine diffeomorphism defined by $\psi(x,y) = (\frac{2}{(2N+3)\varepsilon}x - 1, y)$. Next, for $i \in \{0, ..., N\}$ we define:

$$\varphi_i : [2i\varepsilon, (2i+3)\varepsilon] \times D_\varepsilon^{2n-1} \to [x_i, x_{i+1} + \varepsilon] \times D_\varepsilon^{2n-1}$$

to be a smooth immersion for which there is a $\delta' \in]0, \frac{\varepsilon}{2}[$ such that:

$$\varphi_i(x,y) = \begin{cases} (x - 2i\varepsilon + x_i, y) & \text{if } x \in [2i\varepsilon, (2i+1)\varepsilon + \delta'] \text{ and } i \geq 1 \\ (x - (2i+2)\varepsilon + x_{i+1}, y) & \text{if } x \in [(2i+2)\varepsilon - \delta', (2i+3)\varepsilon] \text{ and } i \geq 0 \end{cases}$$

$$\varphi_0(x,y) = \begin{cases} (x-1, y) & \text{if } x \in [\varepsilon - \delta', \varepsilon + \delta'] \\ \psi(x,y) & \text{if } x \in [0, \delta'] \end{cases}$$

$$\varphi_N(x,y) = \begin{cases} (x + 1 - (2N+3)\varepsilon, y) & \text{if } x \in [2N\varepsilon, (2N+1)\varepsilon + \delta'] \\ \psi(x,y) & \text{if } x \in [(2N+2)\varepsilon - \delta', (2N+3)\varepsilon] \end{cases}$$

Such an immersion as in the case $i < N$ can be constructed as in Figure 3, which depicts two stages of the immersion.[25] In Figure 3, the vertical dimension is a dimension in D_ε^{2n-1}.

[25]This figure was copied from [MS99, Lemma 7.35] .

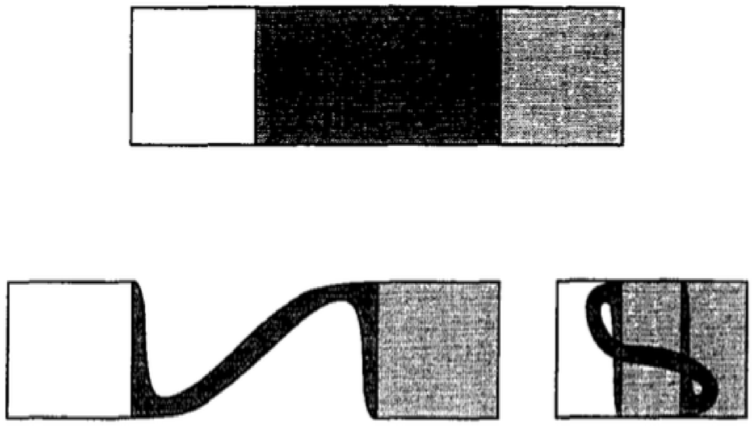

Figure 3: The immersion φ_i, for $i < N$.

This is where we use that $2n - m > 0$. For φ_i to coincide with ψ near $\{0, (2N+3)\varepsilon\}$ in the boundary cases $i = 1$ and $i = N$ we can use smooth cutoff functions.

Since φ_i and φ_{i+1} agree on $[(2i+2)\varepsilon, (2i+3)\varepsilon] \times D_\varepsilon^{2n-1}$, the maps $\varphi_0, ..., \varphi_N$ patch together to a smooth immersion $\varphi : \tilde{A} \to A$ such that $\varphi(x, y) = \varphi_i(x, y)$ if $x \in [2i\varepsilon, (2i+3)\varepsilon]$.

For notational convenience, we define $O_{\delta'} = ([0, \delta'[\cup](2N+2)\varepsilon - \delta', (2N+3)\varepsilon]) \times D_\varepsilon^{2n-1}$.

Now, there is a smooth homotopy of immersions $\Psi : [0,1] \times \tilde{A} \to A$ such that $\Psi_1 = \varphi$, $\Psi_0 = \psi$ and Ψ is constant in time on $O_{\delta'}$. It follows by Remark 4.3(ii) that the map:

$$f : [0,1] \times A \to \Lambda^2(A), \quad (t, p) \mapsto \Psi_t^* \tau(p)$$

is a smooth homotopy of non-degenerate 2-forms which is constant in time on $O_{\delta'}$.

We will construct a $\tilde{\sigma} \in \Omega^1(\tilde{A})$ such that $\tilde{\sigma} = \varphi^*\sigma$ on $O_{\delta'}$ and such that:

$$g : [0,1] \times \tilde{A} \to \Lambda^2(\tilde{A}), \quad (t, p) \mapsto t d\tilde{\sigma}(p) + (1-t)\varphi^*\tau(p)$$

is a smooth homotopy of non-degenerate 2-forms. Then g is constant in time on $O_{\delta'}$.

So by Proposition 2.25 we obtain a smooth homotopy of non-degenerate 2-forms $h : [0,1] \times \tilde{A} \to \Lambda^2(\tilde{A})$ such that $h_0 = \psi^*\tau$, $h_1 = d\tilde{\sigma}$ and h is constant in time on $O_{\delta'}$.

Then the map $[0,1] \times A \to \Lambda^2(A)$ defined by $(t, p) \mapsto (\psi^{-1})^* h_t(p)$ is the promised smooth homotopy of non-degenerate forms from τ to the exact form $d(\psi^{-1})^*\tilde{\sigma}$, on $A = I \times D_\varepsilon^{2n-1}$, which is constant in time on the open neighbourhood $\psi(O_{\delta'})$ of $(\partial I) \times D_\varepsilon^{2n-1}$.

We define $\tilde{\sigma}$ as follows:

$$\tilde{\sigma}(x, y) = \begin{cases} \varphi^*\sigma_{x_0}(x, y) & \text{if } x \in [0, 2\varepsilon[\\ \varphi^*\sigma_{x_{i-1}, x_i}(x, y) & \text{if } x \in]2i\varepsilon - \delta', (2i+1)\varepsilon + \delta'[\text{ and } i \geq 1 \\ \varphi^*\sigma_{x_i}(x, y) & \text{if } x \in](2i+1)\varepsilon, (2i+2)\varepsilon[\text{ and } 0 \leq i \leq N \\ \varphi^*\sigma_{x_N}(x, y) & \text{if } x \in](2N+1)\varepsilon, (2N+3)\varepsilon] \end{cases}$$

By our construction of φ and our definition on $\sigma_{x,x'}$, it follows that $\tilde{\sigma}$ is well-defined. Since every $p \in I \times D_{\varepsilon}^{2n-m}$ has an open neighbourhood on which $\tilde{\sigma}$ is smooth, $\tilde{\sigma}$ is smooth itself. By our choice of ε, δ and φ, it follows that g_t is indeed non-degenerate for every $t \in [0,1]$. Moreover, our choice of ε implies that $\tilde{\sigma} = \varphi^*\sigma_{x_0} = \varphi^*\sigma$ on $[0,\delta'] \times D_{\varepsilon}^{2n-m}$ and that $\tilde{\sigma} = \varphi^*\sigma_{x_N} = \varphi^*\sigma$ on $[(2N+2)\varepsilon - \delta', (2N+3)\varepsilon] \times D_{\varepsilon}^{2n-m}$, so indeed $\tilde{\sigma} = \varphi^*\sigma$ on $O_{\delta'}$.

It remains to replace $I \times D_{\varepsilon}^{2n-m}$ by $I \times D_1^{2n-m}$. To do so, we define $r \in C^\infty(I \times D_1^{2n-1})$ by $r(x,y) = |y|^2$. Although $r^{-1}(]-\infty, c])$ is not contained in $\operatorname{int}(I \times D_1^{2n-1})$, it does hold that $r^{-1}(]-\infty,c])$ is a submanifold of $I \times D_1^{2n-1}$ for every $c \in]0,1]$. Note that for $c = \varepsilon^2$, we find that $r^{-1}(]-\infty,c]) = I \times D_{\varepsilon}^{2n-1}$, and so by the same proof as for Proposition 2.31 it follows that there is a smooth homotopy of non-degenerate forms

$$\tilde{h} : [0,1] \times I \times D_1^{2n-1} \to \Omega^2(I \times D_1^{2n-1})$$

such that $\tilde{h}_0 = \tau$, and $\tilde{h}_1 = d(\psi^{-1})^*\tilde{\sigma}$ on $I \times D_{\frac{\varepsilon}{2}}^{2n-1}$. Moreover, note that by the construction of $\tilde{h}$ it follows that it is constant in time on an open neighbourhood of $(\partial I) \times D_1^{2n-1}$. Let $V_\delta := \left\{(x,y) \in I \times D_1^{2n-1} |\ |x| > 1-\delta\right\}$, and choose $\delta_0 > 0$ such that $\tilde{h}$ is constant in time on V_{δ_0} and such that $V_{\delta_0} \cap (I \times D_{\varepsilon}^{2n-1}) \subset \psi(O_{\delta'})$ and $V_{\delta_0} \subset O$. Note that $\tilde{h}_1 = d\sigma'$ on $V_{\delta_0} \cup (I \times B_{\frac{\varepsilon}{2}}^{2n-1})$, where $\sigma' \in \Omega^1(V_{\delta_0} \cup (I \times B_{\frac{\varepsilon}{2}}^{2n-1}))$ is defined by:

$$\sigma'(p) = \begin{cases} \sigma(p) & \text{if } p \in V_{\delta_0} \\ (\psi^{-1})^*\tilde{\sigma}(p) & \text{if } p \in I \times B_{\frac{\varepsilon}{2}}^{2n-1} \end{cases}$$

which is well-defined since $\tilde{\sigma} = \varphi^*\sigma$ on $O_{\delta'}$ and $\varphi|_{O_{\delta'}} = \psi|_{O_{\delta'}}$.

Finally, note that there is a smooth homotopy of injective immersions

$$\Phi : [0,1] \times I \times D_1^{2n-1} \to I \times D_1^{2n-1}$$

such that $\Phi_0 = \mathrm{Id}$, $\Phi_1(I \times D_1^{2n-1}) \subset V_{\delta_0} \cup (I \times B_{\frac{\varepsilon}{2}}^{2n-1})$ and $\Phi_t(p) = p$ for all $p \in V_{\frac{\delta_0}{2}}$ and all $t \in [0,1]$. Such a homotopy can for instance be constructed using the flow of the vector field:

$$X_{(x,y)} = \beta_0(x) \sum_{i=1}^{2n-1} -y_i\, \partial_{i+1}|_{(x,y)}$$

where $\beta_0 : \mathbb{R} \to [0,1]$ is a smooth bump function such that $\beta_0 = 1$ on $[-1 + \frac{3\delta_0}{4}, 1 - \frac{3\delta_0}{4}]$ and β_0 is supported in $]-1+\frac{\delta_0}{2}, 1 - \frac{\delta_0}{2}[$.

The map $[0,1] \times I \times D_1^{2n-m} \to \Omega^2(I \times D_1^{2n-m})$ defined by $(t,p) \mapsto \Phi_t^*\tilde{h}_t(p)$ is the required smooth homotopy of non-degenerate forms from τ to the exact form $\Phi_1^*\tilde{h}_1$. To see that $\Phi_1^*\tilde{h}_1$ is exact, let $\iota : V_{\delta_0} \cup (I \times B_{\frac{\varepsilon}{2}}^{2n-1}) \to I \times D_1^{2n-1}$ denote inclusion and let $\tilde{\Phi}$ denote the map Φ_1 viewed as a smooth map into the open submanifold $V_{\delta_0} \cup (I \times B_{\frac{\varepsilon}{2}}^{2n-1})$. Then we have:

$$\Phi_1^*\tilde{h}_1 = \tilde{\Phi}^*\iota^*\tilde{h}_1 = d(\tilde{\Phi}^*\sigma')$$

It is straightforward to check the remaining properties. This proves the lemma. $\square$

Remark 4.9. For the proof of Lemma 4.4 we followed that of [MS99, Lemma 7.35].[26] In [MS99, Lemma 7.35], the details of the inductive step are left to the reader and unfortunately I was unable to completely work out these details. The same goes for the smooth homotopy of immersions Ψ.

[26] Although the statement of [MS99, Lemma 7.35] is not precisely the same as that of Lemma 4.4, their proofs are the same up to minor details.

5 Proof of the main result

Using the theory we have developed in the preceding sections, we will prove part (i) of Gromov's theorem in this section, in the case of a compact open manifold. We follow the proof of [MS99, Theorem 7.34].

Proof of Theorem 1.2(i). To begin with, note that we may reduce to the case that M is a connected, compact manifold with non-empty boundary. To see this, note that every connected component of M is closed in M and so is compact. Therefore, since M is an open manifold, every connected component is a connected, compact manifold with non-empty boundary. So, if the theorem holds for such manifolds, we can apply it to every connected component of M. The obtained homotopies for each connected component then clearly fit together to a smooth homotopy of non-degenerate 2-forms on all of M, hence the general case follows.

Thus assume that M is a connected, compact $2n$-manifold with non-empty boundary. By Theorem 3.17 there is a Morse function f on M and a $c \in \mathbb{R}$ with the following properties:

$$\partial M = f^{-1}(\{c\}) \tag{5.1}$$

$$f \text{ has only one critical point of index } 0 \tag{5.2}$$

$$f \text{ has no critical points of index } 2n \tag{5.3}$$

$$f \text{ has no critical points on } \partial M \tag{5.4}$$

$$f \text{ has different values at distinct critical points} \tag{5.5}$$

By Remark 3.4(i), f has finitely many, say m, critical points. We can order them as $p_1, ..., p_m$ with corresponding critical values: $c_1 < ... < c_m$.

Note first that by (5.3) and Remark 3.4(ii), f must attain its maximum on ∂M hence this maximum value is c by (5.1).[27] Therefore, since f is Morse thus not constant, by (5.2) and Remark 3.4(ii) it must attain its only local minimum in the critical point p_1. For convenience, we set $c_{m+1} := c$, so that we have:

$$c_1 < ... < c_{m+1} \quad \text{and} \quad M = f^{-1}([c_1, c_{m+1}])$$

We assume first that $[a] = 0 \in H^2_{dR}(M;\mathbb{R})$. We will proof by induction over k that for every $k \in \{0, ..., m\}$ there is a smooth homotopy of non-degenerate forms

$$h : [0,1] \times M \ \to \Lambda^2(M)$$

and a $b \in]c_k, c_{k+1}[$ such that $h_0 = \tau$, and h_1 is exact on M^b.

Suppose now that h and b are as in the case $k = n$, and let $\iota : M^b \to M$ be inclusion. Then $\iota^* h_1 = d\sigma_1$ for some $\sigma_1 \in \Omega^1(M^b)$. By Theorem 3.16(ii) there is a smooth homotopy of injective immersions $\psi : [0,1] \times M \to M$ such that $\psi_0 = \mathrm{Id}_M$ and $\psi_1(M) = M^b$. Denote

[27] M is compact so f attains a global minimum and maximum.

by $\tilde{\psi}$ the map ψ_1 viewed as a smooth map into the submanifold M^b of M.[28] Then we have:

$$\psi_1^* h_1 = \tilde{\psi}^* \iota^* h_1 = d(\tilde{\psi}^* \sigma) \quad \text{and} \quad \psi_0^* h_0 = \tau$$

Moreover, $\psi_t^* h_t$ is non-denegerate for every $t \in [0,1]$ by Remark 4.3(ii). Hence the smooth homotopy $[0,1] \times M \to \Lambda^2(M)$ defined by $(t,p) \mapsto \psi_t^* h_t(p)$ is a smooth homotopy of non-degenerate 2-forms from τ to an exact form on M, as desired. So this would prove the case $[a] = 0$.

Now we give the promised induction argument, starting with the base case. By Consequence 2.14 there is a chart (U, φ) around p_1 such that $(\varphi^* \omega_0)(p_1) = \tau(p_1)$. By Consequence 2.26 there is an open neighbourhood $\tilde{U}$ of p_1 such that $\tau(p) + t(\varphi^* \omega_0(p) - \tau(p))$ is non-degenerate for all $t \in [0,1]$ and $p \in \tilde{U}$. By taking intersections we may assume that $U \subset \tilde{U}$.
Since manifolds are normal there is an open neighbourhood V of p_1 such that $\overline{V} \subset U$. Choose a smooth bump function $\beta : M \to [0,1]$ such that $\beta = 1$ on $\overline{V}$ and β is supported in U, and define:

$$h : [0,1] \times M \to \Lambda^2(M) \quad \text{by} \quad h_t = \tau + t\beta \cdot (\varphi^* \omega_0 - \tau)$$

It is straightforward to check that this is a smooth homotopy from τ to h_1. If $p \in U$, then since $U \subset \tilde{U}$ and $0 \leq t\beta(p) \leq 1$ it follows that $h_t(p)$ is non-degenerate for all $t \in [0,1]$. If $p \notin U$ then $\beta(p) = 0$ hence $h_t(p) = \tau(p)$, which is non-degenerate for all $t \in [0,1]$. Thus h_t is non-degenerate for all $t \in [0,1]$.
Now, since ω_0 is exact, $\omega_0 = d\sigma_0$ for some $\sigma_0 \in \Omega^1(\mathbb{R}^{2n})$. Furthermore, we can choose $\varepsilon > 0$ such that $M^{c_1+\varepsilon} = f^{-1}(]-\infty, c_1+\varepsilon]) \subset V$. Denote by $\iota : M^{c_1+\varepsilon} \to M$ the inclusion, then we have:

$$\iota^* h_1 = \iota^* \tau + (\beta \circ \iota) \cdot (\iota^* \varphi^* \omega_0 - \iota^* \tau) = (\varphi \circ \iota)^* \omega_0 = d((\varphi \circ \iota)^* \sigma_0)$$

thus h_1 is exact on $M^{c_1+\varepsilon}$. So this would prove the base step. To choose such ε, note that $M \backslash V$ is closed in M and so is compact itself. Hence f attains a minimum, say s, on $M \backslash V$. Since c_1 is the global minimum of f on M, which is only attained in p_1, it must hold that $s > c_1$. Thus $\varepsilon = \frac{s-c_1}{2}$ has the desired property.
Now for the inductive step, suppose the induction statement is true for some $k \in \{0, ..., m-1\}$. Then there is some $b \in]c_k, c_{k+1}[$ and a smooth homotopy of non-degenerate forms

$$h : [0,1] \times M \to \Lambda^2(M)$$

such that $h_0 = \tau$ and h_1 is exact on M^b.
We will first show that this statement is then true for any $b' \in]b, c_{k+1}[$. For any such b' there is a smooth isotopy $\psi : [0,1] \times M \to M$ such that $\psi_0 = \mathrm{Id}_M$ and $\psi_1(M^{b'}) = M^b$, by Theorem 3.16(i). As before in an analogous argument, it follows that the map

[28] Although $\partial M \neq \emptyset$, $\tilde{\psi}$ is smooth since $M^b \subset \mathrm{int}(M)$.

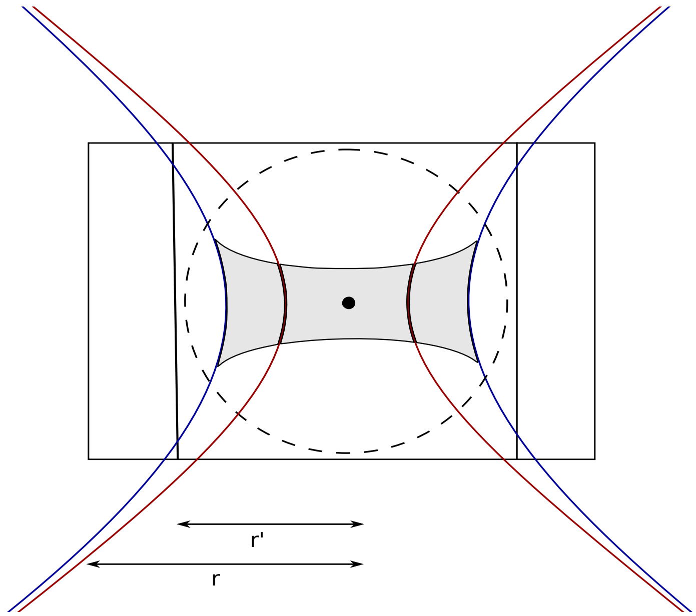

Figure 4: Neighbourhood of p_{k+1} in Morse chart for f.

$[0,1] \times M \to \Lambda^2(M)$ defined by $(t,p) \mapsto \psi_t^* h_t(p)$ is a smooth homotopy of non-degenerate forms from τ to the form $\psi_1^* h_1$ which is exact on $M^{b'}$. Thus we can choose $b' \in]b, c_{k+1}[$ arbitrarily close to c_{k+1}.

By using Lemma 4.2, we will show how to extend the homotopy to a non-degenerate 2-form which is exact on a submanifold that contains p_{k+1}. To this end, we proceed by choosing a Morse chart (U,φ) around p_{k+1}, as in Lemma 3.10. Let $m < 2n$ denote the Morse index of p_{k+1}. Then for $(x,y) \in (\mathbb{R}^m \times \mathbb{R}^{2n-m}) \cap \varphi(U)$, we have:

$$f_\varphi(x,y) = c_{k+1} + |y|^2 - |x|^2 \tag{5.6}$$

We choose $\varepsilon > 0$ such that $D^{2n}_{2\varepsilon} \subset \varphi(U)$ and $c_{k+1} + \varepsilon^2 < c_{k+2}$, and we define $F \in C^\infty(M)$ such that:

$$b < F(p_{k+1}) < f(p_{k+1}) = c_{k+1} \tag{5.7}$$

$$F \leq f \tag{5.8}$$

$$\operatorname{supp}(F - f) \subset \varphi^{-1}(B^{2n}_\varepsilon) \tag{5.9}$$

$$F \text{ has the same critical points as } f \text{ and } F(p_i) = f(p_i) \text{ for all } i \neq k+1 \tag{5.10}$$

F can be constructed as in the induction step in the proof of Lemma 3.22.
Next, we choose $r, s > 0$ such that $\sqrt{2}\varepsilon > r > s > \varepsilon$. Then:

$$D^m_r \times D^{2n-m}_s \subset \varphi(U) \quad \text{and} \quad \operatorname{supp}(F - f) \subset \varphi^{-1}(\operatorname{int}(D^m_r \times D^{2n-m}_s))$$

Further, we choose $c' < c_{k+1}$ such that $c' > c_{k+1} + s^2 - r^2$ and $c' > F(p_{k+1})$. Finally, we fix some $c'' \in]c', c_{k+1}[$ and some $r' > 0$ such that $\sqrt{c_{k+1} + s^2 - c''} < r' < \sqrt{c_{k+1} + s^2 - c'}$. Then in particular $r' < r$.

The constructed situation around p_{k+1} in the Morse chart for f is sketched in Figure 4, in the case $n = m = 1$. In this figure the dot is $\varphi(p_{k+1}) = 0$, the area left of the left, blue curve and right of the right, blue curve forms $f_\varphi^{-1}(] - \infty, c'])$. The area left of the left, red curve and right of the right, red curve forms $f_\varphi^{-1}(] - \infty, c''])$. The dashed circle is of radius ε and the grey area containing 0 depicts $F_\varphi^{-1}(] - \infty, c'])\backslash f_\varphi^{-1}(] - \infty, c'])$.

Now, as shown earlier we may assume that h_1 is exact on $M^{c''}$. Let $\sigma \in \Omega^1(M^{c''})$ such that $h_1 = d\sigma$ on $M^{c''}$. By our choice of r' we have:

$$\varphi^{-1}((\partial D^m_{r'}) \times D^{2n-m}_s) \subset f^{-1}(] - \infty, c''[)$$

So by Lemma 4.2 and Remark 4.3(ii) it follows that there is an open neighbourhood O of $\partial D^m_{r'} \times D^{2n-m}_s$, a smooth homotopy of non-degenerate 2-forms:

$$g : [0, 1] \times D^m_{r'} \times D^{2n-m}_s \to \Lambda^2(D^m_{r'} \times D^{2n-m}_s)$$

and a $\sigma' \in \Omega^1(D^m_{r'} \times D^{2n-m}_s)$ such that:

$$\begin{aligned}
&g_0 = (\varphi^{-1})^* h_1 \text{ on } D^m_{r'} \times D^{2n-m}_s \ ;\\
&g_t = (\varphi^{-1})^* h_1 \text{ on } O \text{ for every } t \in [0, 1] \ ;\\
&g_1 = d\sigma' \ ;\\
&\sigma' = (\varphi^{-1})^* \sigma \text{ on } O.
\end{aligned}$$

Now, let $A_{r',r} = \{x \in \mathbb{R}^m |\ r' < |x| \leq r\}$ denote the annulus. Note that $O' := O \cup (A_{r,r'} \times D^{2n-m}_s)$ is an open neighbourhood of $A_{r,r'} \times D^{2n-m}_s$ and that g and σ' extend to a smooth homotopy of non-degenerate forms:

$$\tilde{g} : [0, 1] \times D^m_r \times D^{2n-m}_s \to \Lambda^2(D^m_r \times D^{2n-m}_s)$$

respectively a 1-form $\tilde{\sigma} \in \Omega^1(D^m_r \times D^{2n-m}_s)$ by defining $\tilde{g}_t(p) = (\varphi^{-1})^* h_1(p)$ and $\tilde{\sigma}(p) = (\varphi^{-1})^* \sigma(p)$ for $p \in O'$. Finally, note that:

$$\tilde{g}_0 = (\varphi^{-1})^* h_1 \text{ on } D^m_r \times D^{2n-m}_s \ ; \tag{5.11}$$

$$\tilde{g}_t = (\varphi^{-1})^* h_1 \text{ on } O' \text{ for every } t \in [0, 1] \ ; \tag{5.12}$$

$$\tilde{g}_1 = d\tilde{\sigma} \ ; \tag{5.13}$$

$$\tilde{\sigma} = (\varphi^{-1})^* \sigma \text{ on } O'. \tag{5.14}$$

Now we define $\tilde{h} : [0, 1] \times F^{-1}(] - \infty, c']) \to \Lambda^2(F^{-1}(] - \infty, c']))$ by:

$$\tilde{h}_t(p) = \begin{cases} \varphi^* \tilde{g}_t(p) & \text{if } p \in F^{-1}(] - \infty, c']) \cap \varphi^{-1}(\text{int}(D^m_r \times D^{2n-m}_s)) \\ h_1(p) & \text{if } p \in F^{-1}(] - \infty, c'])\backslash \varphi^{-1}(D^m_{r'} \times D^{2n-m}_s) \end{cases}$$

That $\tilde{h}$ is well-defined follows from (5.12). Furthermore, note that:

$$F^{-1}(]-\infty,c']) \cap \varphi^{-1}(\mathrm{int}(D_r^m \times D_s^{2n-m})) \text{ and } F^{-1}(]-\infty,c'])\backslash\varphi^{-1}(D_{r'}^m \times D_s^{2n-m})$$

form an open cover of $F^{-1}(]-\infty,c'])$. To see that they form a cover, suppose $p \in F^{-1}(]-\infty,c'])$ and $p \in \varphi^{-1}(D_{r'}^m \times D_s^{2n-m})$. If $p \in \mathrm{supp}(F-f)$, then $p \in \varphi^{-1}(\mathrm{int}(D_r^m \times D_s^{2n-m}))$ by construction.

If $p \notin \mathrm{supp}(F-f)$, then $f(p) = F(p) \leq c'$ and, setting $(x,y) = \varphi(p)$, we have

$$f(p) \geq c_{k+1} - r'^2 + |y|^2$$

Thus $|y|^2 \leq c' - c_{k+1} + r'^2 < s^2$. Since further $|x| \leq r' < r$ it follows that $p \in \varphi^{-1}(\mathrm{int}(D_r^m \times D_s^{2n-m}))$, as desired.

It follows that $\tilde{h}$ is smooth since it is smooth on $F^{-1}(]-\infty,c']) \cap \varphi^{-1}(\mathrm{int}(D_r^m \times D_s^{2n-m}))$ and $F^{-1}(]-\infty,c'])\backslash\varphi^{-1}(D_{r'}^m \times D_s^{2n-m})$, which form an open cover of $[0,1] \times F^{-1}(]-\infty,c'])$. Since φ is a diffeomorphism, by Remark 4.3(ii) it follows that $\tilde{h}_t$ is non-degenerate for all $t \in [0,1]$.

Further, note that $\tilde{h}_0 = h_1$ on $F^{-1}(]-\infty,c'])$, and $\tilde{h}_1 = d\sigma_0$ where $\sigma_0 \in \Omega^1(F^{-1}(]-\infty,c']))$ is defined by:

$$\sigma_0(p) = \begin{cases} \varphi^*\tilde{\sigma}(p) & \text{if } p \in F^{-1}(]-\infty,c']) \cap \varphi^{-1}(\mathrm{int}(D_r^m \times D_s^{2n-m})) \\ \sigma(p) & \text{if } p \in F^{-1}(]-\infty,c'])\backslash\varphi^{-1}(D_{r'}^m \times D_s^{2n-m}) \end{cases}$$

That this is well-defined follows from the fact that

$$F^{-1}(]-\infty,c'])\backslash\varphi^{-1}(D_{r'}^m \times D_s^{2n-m}) \subset M^{c''}$$

and (5.14). Moreover, σ_0 is smooth by a reasoning analogous to that for $\tilde{h}$. Thus $\tilde{h}$ is a smooth homotopy of non-degenerate forms from h_1 to an exact form.

Now by applying Proposition 2.29 we obtain a smooth homotopy of non-degenerate 2-forms

$$h' : [0,1] \times F^{-1}(]-\infty,c']) \to \Lambda^2(F^{-1}(]-\infty,c']))$$

such that $h'_0 = \tau$ on $F^{-1}(]-\infty,c'])$ and $h'_1 = \tilde{h}_1$ is exact.

Next, fix some $\tilde{c} \in]F(p_{k+1}), c'[$ and apply Proposition 2.31 to obtain a smooth homotopy of non-degenerate 2-forms on M from τ to a 2-form which is exact on $F^{-1}(]-\infty,\tilde{c}])$.

Now note that:

$$f^{-1}(]-\infty,c_{k+1}+\varepsilon^2]) = F^{-1}(]-\infty,c_{k+1}+\varepsilon^2])$$

One inclusion is immediate from (5.8). The other follows by (5.9) and the fact that $f(p) \leq c_{k+1} + \varepsilon^2$ if $p \in \varphi^{-1}(B_\varepsilon^{2n})$ by (5.6).

Furthermore, F has the same critical points as f and the same critical values except at p_{k+1}. So since $F(p_{k+1}) < \tilde{c}$ and $c_{k+1} + \varepsilon^2 < c_{k+2}$, we find that $F^{-1}([\tilde{c}, c_{k+1}+\varepsilon^2])$ contains no critical points of F. Hence by Theorem 3.16(i), as before, we find that we can extend the homotopy that we obtained last, to a smooth homotopy of non-degenerate 2-forms on M from τ to τ_1, where τ_1 is exact on $F^{-1}(]-\infty, c+\varepsilon^2]) = f^{-1}(]-\infty, c+\varepsilon^2])$. This finishes the inductive step and so we have proved the theorem in the case $[a] = 0$.

For the case $[a] \neq 0$ we can use an analogous argument, with some minor changes. In this case we prove by induction: for every $k \in \{0, ..., m\}$ there is a smooth homotopy of non-degenerate forms $h : [0,1] \times M \to \Lambda^2(M)$ and a $b \in]c_k, c_{k+1}[$ such that $h_0 = \tau$, and $h_1 - a$ is exact on M^b. To prove this and the rest of the theorem we need only two extra observations.

Note first that whenever we use a smooth homotopy of injective immersions $\psi : [0,1] \times M \to M$ such that $\psi_0 = \mathrm{Id}_M$ and $\psi_1(M^{b'}) = M^b$ to replace a smooth homotopy of non-degenerate forms h by the smooth homotopy of non-degenerate forms $(t,p) \mapsto \psi_t^* h_t(p)$, it holds that if $h_1 - a$ is exact on M^b, then $\psi_1^* h_1 - a$ is exact on $M^{b'}$. Indeed, since ψ_1 is smoothly homotopic to $\psi_0 = \mathrm{Id}_M$ it follows that $\psi_1^* a - a$ is exact. Denote by $\iota_b : M^b \to M$, $\iota_{b'} : M^{b'} \to M$ the inclusions and by $\tilde{\psi}$ the map $\psi_1 \circ \iota_{b'}$ viewed as a smooth map into M^b. Then if $h_1 - a$ is exact on M^b,

$$\iota_{b'}^*(\psi_1^* h_1 - a) = \iota_{b'}^* \psi_1^*(h_1 - a) + \iota_{b'}^*(\psi_1^* a - a) = \tilde{\psi}^* \iota_b^*(h_1 - a) + \iota_{b'}^*(\psi_1^* a - a)$$

is exact being a sum of exact forms.

Secondly, for the other steps we also imitate the proof of the case $[a] = 0$, but with some small changes, for which we can use the extra fact that a closed form on a manifold M is exact on a contractible submanifold S of M. We leave it to the reader to check the details. This proves the theorem. □

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