# Exotic Differential Structures on the 7-sphere

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# Contents

# Introduction

In elementary point set topology one has the concept of two manifolds being homeomorphic. Then in differential geometry/topology there is the concept of two (smooth) manifolds being diffeomorphic. While the latter is clearly stronger, it is hard to come up with any examples of manifolds homeomorphic but not diffeomorphic to one another; almost any two homeomorphic smooth manifolds we can think of also turn out to be diffeomorphic. In fact, it has been proven [Moi77] that any two homeomorphic 1, 2 or 3 dimensional smooth manifolds are also diffeomorphic.

In 1956 John Milnor published a paper showing that there are so-called *exotic spheres* that are homeomorphic, but not diffeomorphic to the 7-sphere $S^7$. These exotic spheres provide the first counter example showing that homeomorphism is not the same as diffeomorphism for smooth manifolds. As is usual in topology, proving that these exotic spheres are not diffeomorphic to $S^7$ proceeds by introducing some mathematical objects that are invariant under diffeomorphisms and then show that these invariants are not the same for the two manifolds in question. We will spend the bulk of this thesis defining so-called *characteristic classes*, which will be used to define the promised invariants required for our main theorem. The entire discussion will furthermore be completely from a differentiable viewpoint. While most literature on the topic of exotic spheres uses singular cohomology with integer coefficients, we chose to only use de Rham cohomology. This offers greater geometrical intuition to be used, and simplifies some of the discussion.

This thesis is divided into four parts. In the first part we will introduce a variety of concepts in cohomology theory required for the introduction of characteristic classes. In the second part we will use the so-called Thom class to introduce the Euler class, from which the characteristic classes are defined. In the third part we will define the Chern and Pontryagin characteristic classes and prove some of their properties. Finally in the fourth part we will use these characteristic classes to introduce the Hirzebruch signature theorem, which will be the most important ingredient in the proof of existence of exotic spheres. We will then go on to introduce the exotic spheres and show that they are indeed homeomorphic but not diffeomorphic to $S^7$.

We assume the reader is familiar with most basic concepts of differential geometry. In particular, the reader should be familiar with tangent spaces, differential forms, Stokes' theorem and de Rham cohomology. For an excellent introduction to the topic of differential geometry consult [Lee13]. We will also use elementary results from point-set topology, group theory and ring theory. Some familiarity with the terminology of algebraic topology will be helpful.

# 1 De Rham cohomology

In this first part we will introduce the de Rham cohomology and prove some elementary properties of the cohomology theory such as the Poincaré lemmas and Poincaré duality. We assume the reader is familiar with the basic definition of de Rham cohomology, and we will build upon this foundation. If the reader is already familiar with Poincaré duality, this part can be skipped. Unless stated otherwise the contents of this part are based on [BT82].

## 1.1 Preliminary definitions

In this section we will introduce de Rham cohomology and introduce some of its properties. The de Rham cohomology is the cohomology theory that naturally arises when considering the exterior derivative $d$ on the space of differential forms $\Omega^q(M)$ on a differentiable manifold $M$. For the sake of convenience we assume any manifold to be both connected and smooth unless stated otherwise . First we define the (graded) algebra of differential forms:

**Definition 1.1.1:** Let $\Omega^q(M)$ be the set of degree $q$ forms on $M$. We define the *graded algebra of differential forms* as $\Omega^*(M) = \bigoplus_{q=0}^{\infty} \Omega^q(M)$. ▲

We know that $d : \Omega^q(M) \to \Omega^{q+1}(M)$ has the property $d^2 = 0$. Hence we can use it to define a cohomology theory:

**Definition 1.1.2:** We define the *de Rham cohomology* $H^*(M)$ of a manifold $M$ as the graded algebra $\bigoplus_{q=0}^{\infty} H^q(M)$ where $H^q(M)$ is given by the quotient

$$H^q(M) = \frac{\ker d : \Omega^q(M) \to \Omega^{q+1}(M)}{\operatorname{im} d : \Omega^{q-1}(M) \to \Omega^q(M)}.$$

Alternatively we can say that $H^*(M)$ is the *cohomology of the cochain complex* $\Omega^*(M)$. ▲

This definition is interesting because it tells us when an element is in the kernel of $d$. But since any form in the image of $d$ is trivially also in the kernel, we simply take the quotient of $\ker d$ by $\operatorname{im} d$ to give us more information. This quotient turns out to be a finite real vector space under mild conditions. A closed form $\omega$ does not have to be in the image of $d$, but if we restrict $\omega$ to a small enough open set then we can always write it as a form in the image of $d$; de Rham cohomology provides us a way of measuring the obstruction of extending local properties of forms to global ones.

A very useful tool when working with de Rham cohomology is the Mayer-Vietoris sequence. Let $M = U \cup V$ with $U$ and $V$ open, and let $U \sqcup V$ denote the disjoint union of $U$ and $V$. We get a series of inclusions $U \cap V \xrightarrow{i_U, i_V} U \sqcup V \to M$, where $i_U$ and $i_V$ are the inclusions of $U \cap V$ into respectively $U \subset U \sqcup V$ and $V \subset U \sqcup V$. This induces the following exact sequence on forms

$$0 \to \Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{i_U^* - i_V^*} \Omega^*(U \cap V) \to 0.$$

By an elementary construction we can extend this short exact sequence on forms to a long exact sequence on cohomology called the *Mayer-Vietoris sequence* (see for example [Hat01, p. 117]):

$$\begin{tikzcd} \cdots \arrow[r] & H^q(M) \arrow[r] & H^q(U) \oplus H^q(V) \arrow[r] & H^q(U \cap V) \arrow[dll, "d^*", rounded corners] \\ & H^{q+1}(M) \arrow[r] & H^{q+1}(U) \oplus H^{q+1}(V) \arrow[r] & H^{q+1}(U \cap V) \arrow[r] & \cdots \end{tikzcd}$$

**Proposition 1.1.3:** Recall that a partition of unity subordinate to a cover $\{U_i\}$ of $M$ is a set of smooth functions $\{\rho_i\}$ such that $\operatorname{Supp}\rho_i \subset U_i$ for each i, and furthermore each $x \in M$ has a neighborhood in which only a finite amount of $\rho_i$ are non-zero, and finally $\Sigma_i \rho_i = 1$ on the whole manifold. It is a standard result that such a partition of unity exists subordinate to any locally finite open cover of $M$. ▲

There is an explicit formula for the coboundary operator obtained through a proof entirely analogous to that of Lemma 1.1.6 on the next page, therefore we will just cite the result [BT82, p.23].

**Lemma 1.1.4:** Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the cover $\{U, V\}$ then we have for $\omega \in H^*(U \cap V)$ that

$$d^*\omega = \begin{cases} -d(\rho_V)\omega & \text{on } U, \\ d(\rho_U)\omega & \text{on } V. \end{cases}$$ ▲

We will now proceed to define compactly supported de Rham cohomology.

**Definition 1.1.5:** Let $\Omega_c^*(M) \subset \Omega^*(M)$ be the graded algebra of forms on $M$ with compact support, that is each $\omega \in \Omega_c^*(M)$ has support within a compact subset of $M$. Then with the same definition of the exterior derivative $d$ as for the globally defined forms we define the *compactly supported de Rham cohomology*, or *compact cohomology* for short, as $H_c^*(M) = \bigoplus_{q=0}^{\infty} H_c^q(M)$ where $H_c^q(M)$ is given by the quotient

$$H_c^q(M) = \frac{\ker d : \Omega_c^q(M) \to \Omega_c^{q+1}(M)}{\operatorname{im} d : \Omega_c^{q-1}(M) \to \Omega^q(M)}.$$ ▲

With globally defined forms we were interested in the maps on cohomology induced by pullbacks of maps between manifolds. In the compact case we consider instead the extension by zero under inclusions of open sets. Let $j : U \hookrightarrow M$ be an inclusion with $U$ open. This induces a map $j_* : \Omega_c^*(U) \to \Omega_c^*(M)$ given by extending forms $\omega$ on $U$ by zero on the complement of $U$ in $M$. The support of $j_*\omega$ is then still compact. Furthermore $j_*$ evidently commutes with exterior derivation and hence also induces a well-defined map on cohomology.

Suppose we have a series of inclusions $U \cap V \xrightarrow{i_U, i_V} U \sqcup V \xrightarrow{j_U, j_V} M$. Consider the maps $\delta : \omega \mapsto (-i_{U*}\omega, i_{V*}\omega)$ and $\sigma : (\omega, \tau) \mapsto j_{U*}\omega + j_{V*}\tau$, in other words signed inclusion, and the sum of the inclusions respectively. With these maps we define the short exact sequence,

$$0 \leftarrow \Omega_c^*(M) \xleftarrow{\sigma} \Omega_c^*(U) \oplus \Omega_c^*(V) \xleftarrow{\delta} \Omega_c^*(U \cap V) \leftarrow 0.$$

From this sequence we get a corresponding sequence in cohomology called the *Mayer-Vietoris sequence* for compact cohomology. Note the reversal of arrows compared to the original Mayer-Vietoris sequence:

$$\begin{tikzcd}
\cdots & H_c^{q+1}(M) \arrow[l] & H_c^{q+1}(U)\oplus H_c^{q+1}(V) \arrow[l] & H_c^{q+1}(U\cap V) \arrow[l] & \\
& H_c^{q}(M) \arrow[urr, "d_*"] & H_c^{q}(U)\oplus H_c^{q}(V) \arrow[l] & H_c^{q}(U\cap V) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

The action of $d_*$ is derived in the following lemma:

**Lemma 1.1.6:** For $\omega \in H_c^{q+1}(M)$ we have $(-i_{U*}d_*\omega, i_{V*}d_*\omega) = (d(\rho_U)\omega, d(\rho_V)\omega)$ where $\rho$ is a partition of unity subordinate to the cover $\{U, V\}$. ▲

Proof: Consider the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 & \Omega_c^*(M) \arrow[l] \arrow[d, "d"] & \Omega_c^*(U)\oplus\Omega_c^*(V) \arrow[l, "\sigma"'] \arrow[d, "d"] & \Omega_c^*(U\cap V) \arrow[l, "\delta"'] \arrow[d, "d"] & 0 \arrow[l] \\
0 & \Omega_c^{*+1}(M) \arrow[l] & \Omega_c^{*+1}(U)\oplus\Omega_c^{*+1}(V) \arrow[l, "\sigma"'] & \Omega_c^{*+1}(U\cap V) \arrow[l, "\delta"'] & 0 \arrow[l]
\end{tikzcd}$$

The connecting homomorphism $d_*$ is defined as follows: Let $\omega \in H_c^*(M)$, then since $\sigma$ is surjective, there is a $\tau$ such that $\sigma(\tau) = \omega$. Now we have $0 = d\omega = \sigma(d\tau)$ and by exactness there is an $\alpha$ such that $\delta\alpha = d\tau$. Now we define $d_*\omega = \alpha$. It is a standard result that this makes $d_*$ well-defined. In this case we have, $\sigma(\rho_U, \rho_V) = \omega$ with $\rho$ a partition of unity subordinate to $U$, $V$. Now $\delta d_*\omega = (d(\rho_U)\omega, d(\rho_V)\omega)$, proving the result after applying the definition of $\delta$. □

Because it is useful to introduce the notion as soon as possible, we will cover fiber and vector bundles in the next section.

## 1.2 Fiber bundles

In this section we will give a concise review of the theory of fiber and vector bundles. We will recall the formal definition of a fiber bundle and introduce relevant terminology. We will then state the definition of vector bundle and provide a definition of some of the operations on vector bundles we will use later.

**Definition 1.2.1:** Let $M$ be a manifold (not necessarily smooth). We call $E$ together with projection $\pi : E \to M$ a *fiber bundle* over a base space $M$ with fiber $F$ if $M$ has an open cover $\{U_\alpha\}$ together homeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ such that for each $\alpha$ the following diagram commutes:

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha"] \arrow[d, "\pi"] & U_\alpha \times F \arrow[dl, "p"] \\
U_\alpha \subset M &
\end{tikzcd}$$

where $p : U_\alpha \times F \to U_\alpha$ is the projection to the first factor. We call the set of all such pairs $(U_\alpha, \varphi_\alpha)$ a *local trivialization* of $E$. ▲

**Definition 1.2.2:** Let $(U_\alpha, \varphi_\alpha)$ be a local trivialization of a fiber bundle $E$. We define the transition functions $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$ of $E$ by $g_{\alpha\beta} = \varphi_\alpha \varphi_\beta^{-1}$. Conversely, given a cover $U_i$ and a set of transition functions $g_{\alpha\beta}$ defined on overlaps $U_\alpha \cap U_\beta$ satisfying the cocycle condition

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma},$$

we can construct a fiber bundle that has $g_{\alpha\beta}$ as its transition functions [BT82, p.48]. We call a set of maps $\{g_{\alpha\beta}\}$ satisfying the cocycle condition a *cocycle*, and we will sometimes refer to the set of transition functions as the cocycle of $E$. ▲

We can also look at maps between fiber bundles. Of course, these maps should be compatible with the projection map and preserve the structure of the fiber bundle. This can be made precise as follows:

**Definition 1.2.3:** Let $\pi : E \to M$ and $\rho : F \to N$ be two fiber bundles. Let $f : E \to F$ and $g : M \to N$ be continuous maps. We call the pair $(f, g)$ a *bundle morphism* if the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & F \arrow[d, "\rho"] \\ M \arrow[r, "g"'] & N \end{tikzcd}$$

We also say that $f$ covers the map $g$. If $f : E \to F$ is a map of fiber bundles over the same base space we typically require that $f$ covers the identity map for it to be a morphism. We also call a bundle morphism $f : E \to F$ covering the identity map a bundle *isomorphism* if it is a homeomorphism, and we then call the two bundles $E$ and $F$ isomorphic. ▲

We say $E$ is a *trivial bundle* if $E$ is isomorphic to the trivial product bundle $M \times F$.

**Definition 1.2.4:** (Pullback bundle) Suppose we have a fiber bundle $\pi :\ E \to M$ and a map $f :\ M \to N$. Then $f$ induces a fiber bundle $f^*E$ over $N$ called the *pullback bundle* of $E$ by $f$. We define $f^*E$ as a subset of $N \times E$:

$$f^*E = \{(n, e) \in N \times E \,|\, f(n) = \pi(e)\}.$$

In other words it is the maximal set that makes the following diagram commutative,

$$\begin{tikzcd} N \times E \supset f^*E \arrow[r, "\tilde{f}"] \arrow[d, "\rho"'] & E \arrow[d, "\pi"] \\ N \arrow[r, "f"] & M \end{tikzcd}$$

where $\rho$, $\tilde{f}$ are the projections of $N \times E$ to respectively $N$ and $E$. The fiber at $y \in N$ is then given by $\pi^{-1} f(y)$ and $f^*E$ has $\rho$ as its projection. Since the pullback of a product bundle is clearly a product bundle, the local trivialization is pulled back to a local trivialization and $f^*E$ is also a fiber bundle. Furthermore the transition functions $g_{\alpha\beta}$ of $E$ induce transition functions $f^*g_{\alpha\beta}$ on $f^*E$. ▲

**Definition 1.2.5:** A section $s : M \to E$ of a fiber bundle $\pi : E \to M$ is a continuous map such that

$\pi \circ s = \mathrm{Id}$. That is, it assigns to every point in the manifold a specific element of the fiber over that point. ▲

We mainly study smooth manifolds, and we will require some extra compatibility with the smooth structure for our fiber bundles.

**Definition 1.2.6:** A *rank-$k$ smooth vector bundle* $\pi : E \to M$ over a smooth manifold $M$ is a fiber bundle that is itself a smooth manifold and for which $\pi$ is smooth and has local trivializations that consist of diffeomorphisms. We also require the fiber of $E$ to be the vector space $\mathbb{R}^k$ and the transition functions to give linear isomorphisms when restricted to each fiber. A rank-$k$ *complex vector bundle* has fiber $\mathbb{C}^k$. Throughout this text we will assume all vector bundles to be smooth. ▲

Let $\pi : E \to M$ be a rank-$k$ vector bundle with cocycle $g_{\alpha\beta}$. If we restrict $g_{\alpha\beta}$ to some point $x \in U_\alpha \cap U_\beta$ we obtain an isomorphism $\mathbb{R}^k \to \mathbb{R}^k$. Therefore we can say that $g_{\alpha\beta}|_x$ is an element of $GL(k, \mathbb{R})$. Furthermore we have a map $U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ given by $x \mapsto g_{\alpha\beta}|_x$.

**Definition 1.2.7:** The image of this map may very well lie in some subgroup $H$ of $GL(k, \mathbb{R})$ for all $\alpha$, $\beta$. If this is the case we say that the *structure group of $E$ is reduced to $H$*. Furthermore if there exists a vector bundle $E'$ isomorphic to $E$ with structure group reduced to some subgroup $H$ of $GL(k, \mathbb{R})$, then we say that the structure group of $E$ can be reduced to $H$. Here $H$ does not necessarily have to be the smallest subgroup with this property. ▲

**Definition 1.2.8:** We can always reduce the structure group of a vector bundle to the orthogonal group $O(k)$ [BT82, p. 55]. We therefore call a rank-$k$ vector bundle *orientable* if its structure group can be reduced to the special orthogonal group $SO(k)$. ▲

**Definition 1.2.9:** We can restrict a vector bundle $\pi : E \to M$ to a subset of $U$ to get a new vector bundle $E|_U$ defined by $E|_U = \pi^{-1}(U)$ whose fiber at a point $p \in U$ is simply the same as that of the original bundle, i.e. $(E|_U)_p = E_p$ with the same vector space structure. ▲

We will now introduce several operations on vector bundles. Most of these are constructed by taking some operation on vector spaces and applying it fiberwise to the bundles. Refer to [MS74, p. 32] for a summary of the following definitions as well as a proof that such fiberwise definitions produce well-defined vector bundles.

**Definition 1.2.10:** The *product bundle* $\pi \times \rho : E \times F \to M \times N$ of two vector bundles $\pi : E \to M$ and $\rho : F \to N$ is defined as the fiberwise product of the two vector bundles. That is, its fiber at $(p, q) \in M \times N$ is $\pi^{-1}(p) \times \rho^{-1}(q)$. ▲

**Definition 1.2.11:** The *direct sum bundle* $E \oplus F$ (also referred to as the *Whitney sum*) of two vector bundles $\pi : E \to M$ and $\rho : F \to M$ over the same base space is defined by taking the direct sum fiberwise. That is, it has fiber $\pi^{-1}(p) \oplus \rho^{-1}(p)$. More precisely, if $d$ is the diagonal embedding of $M$ into $M \times M$, then $E \oplus F$ is defined as $d^*(E \times F)$. ▲

**Definition 1.2.12:** The *tensor product* $E \otimes F$ of two vector bundles $\pi : E \to M$ and $\rho : F \to M$ is defined by fiberwise taking the tensor product (refer to [Lee13, p.305] for a definition of the tensor product of two vector spaces). It has projection $\pi \otimes \rho$ and furthermore if $E$ and $F$ have respectively transition functions $g_{\alpha\beta}$ and $f_{\gamma\delta}$ then $E \otimes F$ has transition functions $g_{\alpha\beta} \otimes f_{\gamma\delta}$. ▲

Several other operations on vector bundles will be defined throughout the text when necessary. In the next section we shall derive an important result for the computation of these cohomology theories.

## 1.3 The Poincaré lemmas

Suppose we have a manifold with a good cover. That is, a cover for which each element of the cover, as well as any finite intersection of elements in the cover, is diffeomorphic to $\mathbb{R}^n$. As we will prove in this section, the cohomology for spaces diffeomorphic to $\mathbb{R}^n$ is always that of a point. Suppose we have a space $M$ that has a good cover with 2 elements, then if we look at the Mayer-Vietoris sequence associated to this cover we can actually deduce the cohomology of $M$. Now by applying the Mayer-Vietoris sequence inductively to the cardinality of a good cover it is possible to obtain the cohomology for any manifold admitting a good cover given we know what the cohomology is for $\mathbb{R}^n$. This is true for both the global and the compact supported case. In this section we will therefore compute both $H_c^*(\mathbb{R}^n)$ and $H^*(\mathbb{R}^n)$. This section is very technical, and understanding of the proofs is not required for understanding later sections.

**Lemma 1.3.1:** (Poincaré lemma) The de Rham cohomology of $\mathbb{R}^n$ is isomorphic to that of a point for all $n$, that is $H^q(\mathbb{R}^n) = 0$ for $q > 0$ and $H^0(\mathbb{R}^n) = \mathbb{R}$ ▲

Proof: We will proceed by explicitly constructing the isomorphism. These isomorphisms are based on considering $\mathbb{R}^{n+1}$ as a trivial vector bundle $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ over $\mathbb{R}^n$ with zero section $s$. The projection and section then induce maps on the space of forms as shown in the following diagram:

$$\begin{tikzcd} \mathbb{R}^n \times \mathbb{R} \arrow[d, shift left, "\pi"] \\ \mathbb{R}^n \arrow[u, shift left, "s"] \end{tikzcd} \qquad \begin{tikzcd} \Omega^*(\mathbb{R}^n \times \mathbb{R}) \arrow[d, shift right, "s^*"'] \\ \Omega^*(\mathbb{R}^n) \arrow[u, shift right, "\pi^*"'] \end{tikzcd}$$

We want to show that the projection and zero section induce inverse isomorphisms in cohomology. Note that this is not true on the level of forms. While we have that $\pi \circ s = 1$ (and hence $s^* \circ \pi^* = 1$), we do not have $s \circ \pi = 1$ nor $\pi^* \circ s^* = 1$, however this identity does hold on cohomology. We want to show that $1 - \pi^* \circ s^* = \pm(dK \pm Kd)$ for some map $K : \Omega^q(\mathbb{R}^n \times \mathbb{R}) \to \Omega^{q-1}(\mathbb{R}^n \times \mathbb{R})$. This is sufficient since $(dK \pm Kd)\omega = dK\omega$ for closed forms, hence $1 - \pi^* \circ s^*$ sends closed forms to exact forms and is a trivial map on cohomology. We call a map $K$ satisfying such a relationship a *homotopy operator*.

Since $\pi$ is surjective we can write any form $\omega \in \Omega^*(\mathbb{R}^n \times \mathbb{R})$ as

$$\omega = (\pi^*\phi) f(x,t) + (\pi^*\theta) g(x,t) dt, \tag{1.1}$$

where $\phi$ and $\theta$ are forms on $\mathbb{R}^n$, $t$ is a coordinate function of the fiber $\mathbb{R}$, and $f$, $g$ are smooth functions on $\mathbb{R}^{n+1}$. Using the same notation we then define $K$ by

$$K\omega = \pi^*\theta \int_t^0 g(x,t)\, dt.$$

That is, we integrate the form over the fiber $\mathbb{R}$. To simplify notation we will occasionally omit function arguments and use the Einstein summation convention. We will first show $1 - \pi^* \circ s^* = \pm(dK \pm Kd)$ for forms without the $dt$ term, that is, let $\omega = (\pi^*\phi)f(x,t)$. Then,

$$(1-\pi^*\circ s^*)\omega = (1-\pi^*\circ s^*)(\pi^*\phi)f(x,t) = (\pi^*\phi)f(x,t) - (\pi^*\phi)f(x,0). \tag{1.2}$$

Whereas on the other side we have

$$\begin{aligned}(dK-Kd)\omega = -Kd\omega &= -K\left[(d\pi^*\phi)f + (-1)^q(\pi^*\phi)\left(\frac{\partial f}{\partial x^i}dx^i + \frac{\partial f}{\partial t}dt\right)\right]\\ &= (-1)^{q+1}K\left((\pi^*\phi)\frac{\partial f}{\partial t}dt\right)\\ &= (-1)^q(\pi^*\phi)\int_0^t \frac{\partial f}{\partial t}\,dt = (-1)^q(\pi^*\phi)(f(x,t)-f(x,0)),\end{aligned}$$

where in the step from the first to the second line we omitted the terms without a $dt$ since they get mapped to 0 by $K$. The last line is the same as (1.2) up to sign, as required. Now for $\omega = (\pi^*\theta)g(x,t)dt$ we get $(1-\pi^*\circ s^*)\omega = \omega$ since $s^*(dt) = 0$. Now on the other side we get:

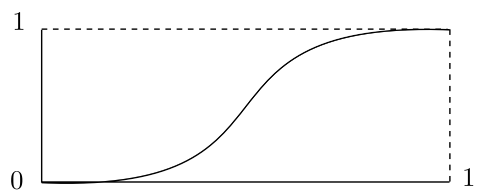
$$\begin{aligned}(dK-Kd)\omega &= d\left[(\pi^*\theta)\int g\,dt\right] - K\left[(d\pi^*\theta)g + (-1)^{q-1}(\pi^*\theta)\left(\frac{\partial g}{\partial x^i}dx^i dt\right)\right]\\ &= (d\pi^*\theta)\int g\,dt + (-1)^{q-1}(\pi^*\theta)\left[dx^i\int\frac{\partial g}{\partial x^i}dt + dt\int\frac{\partial g}{\partial t}dt\right] -\\ &\quad - (d\pi^*\theta)\int g\,dt - (-1)^{q-1}(\pi^*\theta)\left[dx^i\int\frac{\partial g}{\partial x^i}dt\right]\\ &= (-1)^{q-1}(\pi^*\theta)dt\int\frac{\partial g}{\partial t}dt = \omega.\end{aligned}$$

And hence $1-\pi^*\circ s^* = (-1)^q(dK-Kd)$, from which we conclude that $\pi^*$ and $s^*$ are inverse isomorphisms. We then get $H^q(\mathbb{R}^n) = H^q(\mathbb{R}^{n+1})$, and since for $\mathbb{R}^0$ we have $H^0(\mathbb{R}^0) = \mathbb{R}$ and $H^q(\mathbb{R}^0) = 0$ for $q > 0$, we conclude the Poincaré lemma by induction. □

Note that if $\{U_\alpha, \phi_\alpha\}_{\alpha\in I}$ is an atlas for $M$, then $\{U_\alpha\times\mathbb{R}, (\phi_\alpha, \mathrm{Id})\}_{\alpha\in I}$ is an atlas for $M\times\mathbb{R}$. And furthermore the definition of the operator $K$ does not depend on the choice of atlas on $M$, from this we conclude the following corollary:

**Corollary 1.3.2:** $H^*(M) = H^*(M\times\mathbb{R})$ and hence also $H^*(M) = H^*(M\times\mathbb{R}^n)$. ▲

This corollary provides with the tools for a concise proof of the homotopy invariance of de Rham cohomology. Let $F : M\times I \to N$ be a smooth homotopy (that is, $F$ is a smooth map and also a homotopy) between maps $f$ and $g$ such that $f = F\circ s_1$ and $g = F\circ s_0$ with $s_t$ the trivial section sending $x\mapsto(x,t)$. We can extend $F$ to $M\times\mathbb{R}$ by $F(\cdot,t) = f$ for $t > 1$ and $F(\cdot,t) = g$ for $t < 0$. Note that this does not necessarily give a smooth map, but we can take a different parametrization of $I$ (shown below) by a smooth map which has derivative 0 at either end of the unit interval to make $F$ smooth again.



Since $s_t^*$ and $\pi^*$ are inverse isomorphisms on cohomology, we have that $s_t^*$ induces the same map on cohomology for all $t$. Note that smoothness of $F$ is required to have a well-defined map $F^*$. Now we have

$$f^* = (F \circ s_1)^* = s_1^* \circ F^* = s_0^* \circ F^* = g^*.$$

And hence we conclude the homotopy invariance of de Rham cohomology:

**Theorem 1.3.3:** (Smoothly) homotopic maps induce the same maps on de Rham cohomology. In particular, smoothly homotopy equivalent spaces have isomorphic de Rham cohomology. ▲

This also immediately gives another interpretation of the Poincaré lemma. Recall that a space is contractible if it's homotopic to a point. Therefore the Poincaré lemma essentially states that the cohomology of contractible spaces is the same as that of a point.

We will now spend the rest of the section proving the Poincaré lemma counterpart for compactly supported cohomology. The proof analogously also starts with defining a homotopy operator. The key difference is that we now want an isomorphism $H_c^{q+1}(\mathbb{R}^n \times \mathbb{R}) \to H_c^q(\mathbb{R}^n)$. Furthermore instead of working with pull backs we want to work with push-forward maps. Let as before $\omega \in \Omega_c^*(\mathbb{R}^{n+1})$ be of the form

$$\omega = (\pi^*\phi)f(x,t) + (\pi^*\theta)g(x,t)dt,$$

where we again see $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ as a vector bundle. We define integration along the fiber $\pi_* : \Omega_c^*(M \times \mathbb{R}) \to \Omega_c^{*-1}(M)$ by

$$\pi_*\omega = \theta \int_{-\infty}^{\infty} g(x,t)dt.$$

**Proposition 1.3.4:** $\pi_*$ commutes with $d$ and hence induces a map on cohomology ▲

We have to prove that $\pi_* d = d\pi_*$. By linearity of both operators it's enough to show commutativity for $\omega = (\pi^*\phi)f(x,t)$ and $\omega = (\pi^*\phi)f(x,t)dt$ separately. First let $\omega = (\pi^*\phi)f(x,t)$. We get

$$\begin{aligned}\pi_* d\omega &= \pi_* \left[(d\pi^*\phi)f + (-1)^{q-1}(\pi^*\phi)\left(\frac{\partial f}{\partial x^i}dx^i + \frac{\partial f}{\partial t}dt\right)\right] \\ &= (-1)^{q-1}(\pi^*\phi)dt\left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial t}dt\right),\end{aligned}$$

which is zero by compact support of $f$. On the other hand $d\pi_*\omega = 0$, so we get commutativity. Now for $\omega = (\pi^*\phi)f(x,t)dt$ we have,

$$d\pi_*\omega = d\left[\phi \int f dt\right].$$

Whereas,

$$\begin{aligned}
\pi_* d\omega &= \pi_* \left[ (d\pi^*\phi) f + (-1)^{q-1} (\pi^*\phi) \left( \frac{\partial f}{\partial x^i} dx^i dt \right) \right] \\
&= (d\phi) \int f dt + (-1)^{q-1} \phi dx^i \int \frac{\partial f}{\partial x^i} dt \\
&= d \left[ \phi \int f dt \right].
\end{aligned}$$

Which proves commutativity of $\pi_*$ and $d$. □

To show that $\pi_*$ is an isomorphism, we will now construct its inverse. Let $e = e(t)dt$ be a compactly supported 1-form on $\mathbb{R}$ with total integral 1. Then let $e_* : \Omega_c^*(M) \to \Omega_c^{*+1}(M \times \mathbb{R})$ be the map defined by $\phi \mapsto (\pi^*\phi) \wedge e$. Since $de = 0$ we have that $e_*$ commutes with $d$ and hence $e_*$ induces a map on cohomology.

**Proposition 1.3.5:** $e_*$ and $\pi_*$ are inverse isomorphisms. ▲

First of all we have that $\pi_* e_* \omega = \pi_*((\pi^*\omega) \wedge e(t)dt) = \omega \int e(t)dt = \omega$. For the inverse relationship we will show $1 - e_*\pi_* = (-1)^{q-1}(dK - Kd)$ for some $K$. Let again $\omega$ be given by

$$\omega = (\pi^*\phi) f(x,t) + (\pi^*\theta) g(x,t) dt.$$

From this we define $K$ by,

$$K\omega = \pi^*\theta \int_{-\infty}^{t} g(x,t)dt - \pi^*\theta A(t) \int_{-\infty}^{\infty} g(x,t)dt,$$

with $A(t) = \int_{-\infty}^{t} e(t)dt$.

**Lemma 1.3.6:** With the definition above we have $1 - e_*\pi_* = (-1)^{q-1}(dK - Kd)$. ▲

PROOF: Let first $\omega = \pi^*\phi f(x,t)$. Then $(1 - e_*\pi_*)\omega = \omega$. We have,

$$\begin{aligned}
(dK - Kd)\omega = -Kd\omega &= -K \left[ (d\pi^*\phi) f + (-1)^{q-1} (\pi^*\phi) \left( \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial t} dt \right) \right] \\
&= (-1)^{q-1} \left[ \pi^*\phi \int_{-\infty}^{t} \frac{\partial f}{\partial t} dt - \pi^*\phi A(t) \int_{-\infty}^{\infty} \frac{\partial f}{\partial t} dt \right] \\
&= (-1)^{q-1} \pi^*\phi f = (-1)^{q-1}\omega.
\end{aligned}$$

Assume on the other hand that $\omega = \pi^*\phi f(x,t)dt$, we then get,

$$(1 - e_*\pi_*)\omega = \omega - e_*\phi \int_{-\infty}^{\infty} f(x,t)dt = \omega - \pi^*\phi \int_{-\infty}^{\infty} f(x,t)dt \wedge e. \tag{1.3}$$

Now we will separately calculate $Kd$ and $dK$. First we get,

$$\begin{aligned}
Kd\omega &= K \left[ (d\pi^*\phi) f + (-1)^{q-1} (\pi^*\phi) \left( \frac{\partial f}{\partial x^i} dx^i dt \right) \right] \\
&= d\pi^*\phi \int_{-\infty}^{t} f dt - d\pi^*\phi A(t) \int_{-\infty}^{\infty} f dt + (-1)^{q-1} \left[ \pi^*\phi \int_{-\infty}^{t} \frac{\partial f}{\partial x^i} dx^i dt - \pi^*\phi A(t) \int_{-\infty}^{\infty} \frac{\partial f}{\partial x^i} dx^i dt \right].
\end{aligned}$$

And one the other hand we have

$$\begin{aligned} dK\omega &= d\left[\pi^*\phi \int_{-\infty}^{t} f dt - \pi^*\phi A(t) \int_{-\infty}^{\infty} f dt\right] \\ &= d\pi^*\phi \int_{-\infty}^{t} f dt + (-1)^{q-1}\pi^*\phi \int_{-\infty}^{t} \frac{\partial f}{\partial x^i} dx^i dt + (-1)^{q-1}\pi^*\phi dt \int_{-\infty}^{t} \frac{\partial f}{\partial t} dt - \\ &\quad - d\pi^*\phi A(t) \int_{-\infty}^{\infty} f dt - \pi^*\phi dA(t) \int_{-\infty}^{\infty} f dt - (-1)^{q-1}\pi^*\phi A(t) \int_{-\infty}^{\infty} \frac{\partial f}{\partial x^i} dx^i dt - \\ &\quad - (-1)^{q-1}\pi^*\phi A(t) dt \int_{-\infty}^{\infty} \frac{\partial f}{\partial t} dt. \end{aligned}$$

Now noting that $dA(t) = e$ and taking the difference we get

$$(dK - Kd)\omega = (-1)^{q-1}\left[\pi^* f dt - \pi^*\phi \left(\int_{-\infty}^{\infty} f(x,t)\right) e\right].$$

Comparing this to (1.3) we conclude that indeed $1 - e_*\pi_* = (-1)^{q-1}(dK - Kd)$. □

With this lemma we have immediately also proven Proposition 1.3.5. We then conclude the Poincaré lemma for compact cohomology by noting that in dimension zero we have $H_c^0(\mathbb{R}^0) = \mathbb{R}$, and $H_c^q(\mathbb{R}^0) = 0$ for $q > 0$.

**Lemma 1.3.7:** (Poincaré lemma for compact cohomology) We have that $H_c^q(\mathbb{R}^n) = 0$ for $q \neq n$ and $H_c^n(\mathbb{R}^n) = \mathbb{R}$. ▲

**Remark 1.3.8:** By the Poincaré lemma for compact cohomology we clearly see that compact cohomology is unlike its global counterpart not homotopy invariant. ▲

## 1.4 Poincaré duality

We will now go on to prove another important result called Poincaré duality. This gives a relationship between the compact and the global de Rham cohomology. Normally, e.g. for singular cohomology, Poincaré duality would give a relationship between cohomology and homology, but since de Rham cohomology does not have an easily defined corresponding homology, we stick to only using cohomology. The result is as follows:

**Theorem 1.4.1:** (Poincaré duality) Let $M$ be an orientable manifold of dimension $n$. Then $H^q(M) \simeq (H_c^{n-q}(M))^*$, where the asterisk denotes the dual space. ▲

We already have this result for $\mathbb{R}^n$ due to the Poincaré lemmas by noting that $\mathbb{R}^n$ is isomorphic to its own dual. The idea is to use the Mayer-Vietoris sequence of both the compact and global cohomology to prove this result for any manifold with a finite good cover by induction to the cardinality of the cover. The result for any smooth manifold then follows by a generalization of the argument we give here [BT82, p. 46]. We will first construct an explicit isomorphism $S : H^q(M) \to (H_c^{n-q}(M))^*$ given by integration:

$$S : \omega \mapsto \int_M \omega \wedge \cdot\,,$$

where $\displaystyle\int_M \omega \wedge \cdot : \tau \mapsto \int_M \omega \wedge \tau.$

Note that $S(\omega)$ is indeed an element of $(H_c^{n-q}(M))^*$ since it sends elements of $H_c^{n-q}(M)$ to $\mathbb{R}$. By compact support $S(\omega)(\tau)$ is always defined, and clearly $S$ is also bilinear.

**Lemma 1.4.2:** $S$ is an isomorphism $H^*(\mathbb{R}^n) \to (H_c^{n-*}(\mathbb{R}^n))^*$ for all $n$. ▲

Proof: For $q \neq 0$ we have $0 = H^q(\mathbb{R}^n) = H_c^{n-q}(\mathbb{R}^n) = (H_c^{n-q}(\mathbb{R}^n))^*$ hence any linear map is an isomorphism. Therefore let $q = 0$. We then have $1 = \dim H^0(\mathbb{R}^n) = \dim H_c^n(\mathbb{R}^n) = \dim(H_c^n(\mathbb{R}^n))^*$, so $S$ is an isomorphism if and only if it is an injection. Let $\omega = f(x^1, \ldots, x^n) \in H^0(\mathbb{R}^n)$ and $\tau = g(x^1, \ldots, x^n) dx^1 \wedge \cdots \wedge dx^n \in H_c^n(\mathbb{R}^n)$, now suppose $S(\omega) = 0$ or equivalently $S(\omega)(\tau) = 0$ for all $\tau \in H_c^{n-q}(\mathbb{R}^n)$, then we have

$$\int_{\mathbb{R}^n} f \cdot g \, dx^1 \wedge \cdots \wedge dx^n = 0, \quad \text{for all compactly supported } g.$$

This means that the set $\{f \neq 0\}$ has measure zero on all compact sets. By continuity $f = 0$ on all compact sets and hence $f = 0$ on $\mathbb{R}^n$ proving $S$ is injective, and hence an isomorphism. □

Now consider the following diagram induced by the Mayer-Vietoris sequences for a good cover $\{U, V\}$:

$$\begin{tikzcd}
{} \arrow[r] & H^q(U \cup V) \arrow[r] \arrow[d, "S"] & H^q(U) \oplus H^q(V) \arrow[r] \arrow[d, "S"] & H^q(U \cap V) \arrow[r, "d^*"] \arrow[d, "S"] & {} \\
{} \arrow[r] & (H_c^{n-q}(U \cup V))^* \arrow[r] & (H_c^{n-q}(U))^* \oplus (H_c^{n-q}(V))^* \arrow[r] & (H_c^{n-q}(U \cap V))^* \arrow[r, "(d_*)^*"] & {}
\end{tikzcd} \tag{1.4}$$

Here the rows are exact. This is clear for the top row, and exactness of the bottom row follows from the fact that short exact sequences remain exact when taking duals; only with the directions of the maps reversed (granted all the relevant groups are freely generated, as is the case since we are working with $\mathbb{R}$, see [Jac09, p.149]). We wish to show that $S$ is an isomorphism for $H^q(U \cup V) \to (H_c^{n-q}(U \cup V))^*$. We know that the two maps left of $H^q(U \cup V)$ are isomorphisms and also the two to the right, since $U \cap V$, $U$ and $V$ are all diffeomorphic to $\mathbb{R}^n$. This situation calls for the five lemma:

**Lemma 1.4.3:** (Five lemma) Consider the following commutative diagram of Abelian groups, where the rows are exact:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "\alpha"'] & B \arrow[r, "b"] \arrow[d, "\beta"'] & C \arrow[r, "c"] \arrow[d, "\gamma"'] & D \arrow[r, "d"] \arrow[d, "\delta"'] & E \arrow[d, "\varepsilon"'] \\
F \arrow[r, "f"] & G \arrow[r, "g"] & H \arrow[r, "h"] & I \arrow[r, "i"] & J
\end{tikzcd}$$

If $\alpha$, $\beta$, $\delta$, $\varepsilon$ are isomorphisms, then so is $\gamma$. ▲

The proof proceeds by straightforward diagram chasing (see [Hat01, p. 129]). This lemma applies precisely to our situation; we only need to show that the diagram (1.4) is commutative to show that $S : H^q(U \cup V) \to (H_c^{n-q}(U \cup V))^*$ is an isomorphism.

**Lemma 1.4.4:** The diagram (1.4) above is commutative. ▲

Proof: In the first square we have

$$\begin{array}{ccc} H^q(U\cup V) & \xrightarrow{\quad\Sigma\quad} & H^q(U)\oplus H^q(V) \\ \big\downarrow S & & \big\downarrow S \\ (H_c^{n-q}(U\cup V))^* & \xrightarrow{\sigma^*} & (H_c^{n-q}(U))^*\oplus (H_c^{n-q}(V))^* \end{array}$$

here the top horizontal map is restriction of forms, and the bottom horizontal map is the pullback of summing the two forms. We have that

$$(\sigma^* S(\omega))(\alpha,\beta) = S(\omega)(\alpha+\beta) = S(\omega,\omega)(\alpha,\beta) = S(\Sigma(\omega))(\alpha,\beta).$$

And hence the square commutes. In the second square we have

$$\begin{array}{ccc} H^q(U)\oplus H^q(V) & \xrightarrow{\quad\Delta\quad} & H^q(U\cap V) \\ \big\downarrow S & & \big\downarrow S \\ (H_c^{n-q}(U))^*\oplus (H_c^{n-q}(V))^* & \xrightarrow{\delta} & (H_c^{n-q}(U\cap V))^* \end{array}$$

Where the top map is the difference and the bottom map is the pullback of signed inclusion. Writing both compositions out we get

$$\delta^*(S(\omega,\tau))(\alpha) = S(\omega,\tau)(-\alpha,\alpha) = S(\tau)(\alpha) - S(\omega)(\alpha) = S(\Delta(\omega,\tau))(\alpha),$$

showing that the second square commutes as well. Now finally we have the square

$$\begin{array}{ccc} H^q(U\cap V) & \xrightarrow{\quad d^*\quad} & H^{q+1}(U\cup V) \\ \big\downarrow S & & \big\downarrow S \\ (H_c^{n-q}(U\cap V))^* & \xrightarrow{(d_*)^*} & (H_c^{n-q+1}(U\cup V))^* \end{array}$$

We have to prove that $S(d^*(\omega))(\tau) = (d_*)^* S(\omega)(\tau) = S(\omega)(d_*\tau)$, in other words:

$$\int_{U\cap V} \omega\wedge d_*\tau = \int_{U\cup V} d^*\omega\wedge\tau.$$

From Lemma's 1.1.4 and 1.1.6 we know what the maps $d_*$ and $d^*$ do, and applying this we obtain

$$\int_{U\cap V} \omega\wedge d_*\tau = \int_{U\cap V} \omega\wedge (d\rho_V)\tau = (-1)^q \int_{U\cap V} (d\rho_V)\omega\wedge\tau.$$

Whereas, since $d^*\omega$ has support in $U\cap V$, we get:

$$\int_{U\cup V} d^*\omega\wedge\tau = -\int_{U\cap V} (d\rho_V)\omega\wedge\tau.$$

Proving commutativity of the diagram *up to sign*. However we can simply change the sign of either connecting homomorphism appropriately while retaining exactness. This proves commutativity for a sign-corrected version of the diagram. □

By applying the five lemma we have proven that $S$ is an isomorphism $H^q(U\cup V)\to (H_c^{n-q}(U\cup V))^*$. Now this generalizes to any manifold with a finite good cover by induction; suppose $S: H^*(M)\to (H_c^{n-*}(M))^*$

is an isomorphism for all orientable manifolds with a good cover of cardinality $n$. Then suppose $M$ is orientable and has a good cover $\{U_k\}_{k=0}^n$ of cardinality $n+1$. Now $\bigcup_{k=1}^n U_k$ admits a good cover of cardinality $n$ and hence $S$ is an isomorphism. Furthermore by goodness of the cover for $M$ we have that $(\bigcup_{k=1}^n U_k) \cap U_0$ also admits a good cover of cardinality $n$, and hence has $S$ acting as an isomorphism. The proof above of commutativity above still holds for $\{U, V\} = \{\bigcup_{k=1}^n U_k, U_0\}$ so by the five lemma we have that $S$ is also an isomorphism for $M$. Now by induction Poincaré duality holds for any orientable manifold admitting a finite good cover. □

In addition we also have a certain sense of naturality of the Poincaré isomorphism $S$.

**Corollary 1.4.5:** (Naturality of the Poincaré isomorphism) Let $M$ and $N$ be respectively $m$ and $n$ dimensional manifolds. Suppose we have an embedding $f : M \hookrightarrow N$, then the following diagram commutes:

$$\begin{tikzcd} H^*(N) \arrow[r, "f^*"] \arrow[d, "S"'] & H^*(M) \arrow[d, "S"] \\ (H_c^{n-*}(N))^* \arrow[r, "(f_*)^*"] & (H_c^{m-*}(M))^* \end{tikzcd} \tag{1.5}$$

That is, for $\alpha \in H_c^*(M)$ we have

$$\int_M f^*\omega \wedge \alpha = \int_N \omega \wedge f_*\alpha. \tag{1.6}$$

▲

Proof: From the definition of $(f_*)^*$ it follows immediately that the commutativity of the diagram (1.5) is indeed equivalent with (1.6). Equality of these integrals then follows from the fact that $f_*\alpha$ is given by extension by zero of $\alpha \circ f^{-1}$ to the whole of $N$. Hence we have,

$$\int_N \omega \wedge f_*\alpha = \int_{f(M)} \omega \wedge f_*\alpha.$$

But since $f$ is a diffeomorphism unto its image we have,

$$\int_{f(M)} \omega \wedge f_*\alpha = \int_M f^*(\omega \wedge f_*\alpha).$$

Now by noting that $f^*f_*(\alpha) = \alpha \circ (f^{-1}f) = \alpha$ on $f(M)$. We conclude that,

$$\int_M f^*\omega \wedge \alpha = \int_N \omega \wedge f_*\alpha,$$

and hence that the diagram is indeed commutative, proving naturality for the Poincaré isomorphism. □

## 1.5 Leray-Hirsch theorem

Another important theorem we will prove is the Leray-Hirsch theorem. The proof is similar to that of Poincaré duality.

**Theorem 1.5.1:** (Leray-Hirsch) Let $\pi : E \to M$ be a fiber bundle with fiber $F$, such that $H^*(F)$ is finite-dimensional. Suppose there are global cohomology classes $e_1, \ldots, e_k \in H^*(E)$ whose restriction to

$F$ freely generate $H^*(F)$ at every point. Then $H^*(E)$ is a free module over $H^*(M)$ with basis $e_1, \dots e_k$. We write $H^*(E) \simeq H^*(M) \otimes \mathbb{R}\{e_1, \dots, e_k\}$. That is, any element in $H^*(E)$ can be written as a linear combination of elements of form $\omega \wedge e_{i_1} \wedge \dots \wedge, e_{i_r}$ with $\{i_j\} \subset \{1, \dots, k\}$ some (possibly empty) index set and $\omega \in H^*(M)$. ▲

Proof: As in the proof of Poincaré duality we will construct some isomorphism, which is trivially an isomorphism for Euclidean spaces, and then use the Mayer-Vietoris sequence and the five lemma to get an isomorphism for any base manifold admitting a finite good cover. The result is actually also true for the case without a finite good cover (so long as $F$ still has finite dimensional cohomology). The proof of that result requires a generalization of the Mayer-Vietoris based argument we use here, but this generalization requires several pages to introduce (see for example [BT82, p.106-108]).

We define a map $\psi : H^*(M) \otimes \mathbb{R}\{e_1, \dots, e_k\} \to H^*(E)$ by,

$$\omega \otimes (e_{i_1} \cdots e_{i_r}) \mapsto \pi^*\omega \wedge e_{i_1} \wedge \dots \wedge e_{i_r}.$$

Note that $\psi$ is an isomorphism for $M = \mathbb{R}^n$ due to the Poincaré lemma as in Corollary 1.3.2. Let $\{U, V\}$ be a good cover with two elements and consider the Mayer-Vietoris sequence,

$$\cdots \longrightarrow H^p(U \cup V) \longrightarrow H^p(U) \oplus H^p(V) \longrightarrow H^p(U \cap V) \longrightarrow \cdots$$

Let $k$ be a fixed integer. If we tensor the sequence with $\mathbb{R}^k = \mathbb{R}\{e_1, \dots, e_k\}$ at every point we retain exactness due to the following proposition [Jac09, p. 154]:

**Proposition 1.5.2:** Let $V$ be a vector space. Let $A_1 \to A_2 \to A_3 \to \cdots$ be an exact sequence of modules. Then the sequence,

$$A_1 \otimes V \to A_2 \otimes V \to A_3 \otimes V \to \cdots$$

is also an exact sequence when the maps are tensored with the identity on $V$ at every point. ▲

By applying this proposition we conclude that the following sequence is also exact:

$$\cdots \longrightarrow H^p(U \cup V) \otimes \mathbb{R}^k \longrightarrow \left(H^p(U) \otimes \mathbb{R}^k\right) \oplus \left(H^p(V) \otimes \mathbb{R}^k\right) \longrightarrow H^p(U \cap V) \otimes \mathbb{R}^k \longrightarrow \cdots$$

Now summing over $p$ at every step we get the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_{p=0}^n H^p(U \cup V) \otimes \mathbb{R}^k \arrow[r] \arrow[d, "\psi"] & \bigoplus_{p=0}^n \begin{matrix} H^p(U) \otimes \mathbb{R}^k \\ \oplus H^p(V) \otimes \mathbb{R}^k \end{matrix} \arrow[r] \arrow[d, "\psi"] & \bigoplus_{p=0}^n H^p(U \cap V) \otimes \mathbb{R}^k \arrow[r, "d^* \otimes \mathrm{Id}"] \arrow[d, "\psi"] & \cdots \\
\cdots \arrow[r] & H^n(E|_{U \cup V}) \arrow[r] & H^n(E|_U) \oplus H^n(E|_V) \arrow[r] & H^n(E|_{U \cap V}) \arrow[r, "d^*"] & \cdots
\end{tikzcd}$$

Where the connecting homomorphism is given by $d^* \otimes \mathrm{Id}$ which maps $H^p(U \cap V)$ onto $H^{p+1}(U \cup V)$ for each $p$, and acts as the identity on $\mathbb{R}^k$. Showing commutativity is straightforward and we will only show commutativity for the non-trivial square:

$$\begin{tikzcd} \bigoplus_{p=0}^{n} H^p(U\cap V)\otimes\mathbb{R}^k \arrow[r, "d^*\otimes\mathrm{Id}"] \arrow[d, "\psi"'] & \bigoplus_{p=0}^{n+1} H^p(U\cup V)\otimes\mathbb{R}^k \arrow[d, "\psi"] \\ H^n(E|_{U\cap V}) \arrow[r, "d^*"] & H^{n+1}(E|_{U\cup V}) \end{tikzcd}$$

By linearity we only have to check commutativity for elements of form $\omega\otimes\phi\in H^p(U\cap V)\otimes\mathbb{R}^k$. We get,

$$\psi(d^*\otimes\mathrm{Id}(\omega\otimes\phi)) = (\pi^* d^*\omega)\wedge\phi.$$

Let $\{\rho_U,\,\rho_V\}$ be a partition of unity subordinate to $\{U,\,V\}$. Recall from Lemma 1.1.4 that on $U\cap V$ we have that $d^*\omega$ is given by $d(\rho_U\omega)$. Also note that $\{\pi^*\rho_U^*,\pi^*\rho_V^*\}$ is still a partition of unity. Therefore,

$$d^*\,\psi(\omega\wedge\phi) = d^*(\pi^*\omega\wedge\phi) = d(\pi^*\rho_U)\,\pi^*\omega\wedge\phi = \pi^*(d\rho_U\omega)\wedge\phi = (\pi^* d^*\omega)\wedge\phi.$$

and hence we conclude commutativity. Now by the five-lemma we conclude that $\psi$ is an isomorphism for the union $U\cup V$ of two open sets diffeomorphic to $\mathbb{R}^n$. Then by the same argument as in the proof of Poincaré duality we get by induction on the cardinality of a good cover the required result for any manifold admitting a good cover. □

Let $M\times N$ be the trivial bundle over $M$. Then if $N$ has finite dimensional cohomology, there is clearly a finite set of cohomology classes on $M\times N$ that generate $H^*(N)$. This proves the the Künneth formula:

**Theorem 1.5.3:** (Künneth formula) The cohomology of a product of two spaces is given by

$$H^*(M\times N) = H^*(M)\otimes H^*(N),$$

if at least one of the two spaces has a finite cohomology. ▲

**Remark 1.5.4:** The fact that the $e_i$'s generate the fiber cohomology is necessary for the Leray-Hirsch theorem to hold. We don't in general have $H^*(E) = H^*(M)\otimes H^*(F)$. If we consider $\rho : E\to F$ as a fiber bundle, the most immediate definition of $\psi : H^*(M)\otimes H^*(F)\to H^*(E)$ would be $\psi : \omega\otimes\phi\to\pi^*\omega\otimes\rho^*\phi$. But this map is not necessarily injective, since $\rho^*\phi$ could easily be exact. For example consider the Hopf fibration $S^3\to S^2$, this is a fiber bundle over $S^2$ with fiber $S^1$. Clearly $H^*(S^3)\neq H^*(S^2)\otimes H^*(S^1)$. ▲

With the Künneth formula and Leray-Hirsch theorem we conclude this introductory part of this thesis. In the next two parts we will introduce characteristic classes, and with that the bulk of this thesis.

# 2 Thom and Euler classes

In this part we shall define the Euler class, which will then be used to define the Chern class in the next part by applying the Leray-Hirsch theorem. In order to define the Euler class we first need to introduce the Thom isomorphism and Thom class, which is based on a generalization of the proof of the Poincaré lemmas. Sections 2.1 and 2.2 are based on [BT82], and section 2.3 is based on [MS74].

## 2.1 Thom isomorhpism

Recall that in the proof of the Poincaré lemma for compact cohomology we constructed isomorphisms $H_c^{n+1}(\mathbb{R}^n \times \mathbb{R}) \to H_c^n(\mathbb{R}^n)$. In this section we wish to generalize this result to an isomorphism of some kind of cohomology of a vector bundle $E \to M$ to the de Rham cohomology of a lower degree in $M$. This isomorphism of interest shall be the *Thom isomorphism.* First we shall define a generalization of the compactly supported de Rham cohomology to vector bundles.

**Definition 2.1.1:** (Compact vertical cohomology) Let $\pi : E \to M$ be a vector bundle over $M$. We define $\Omega^*_{cv}(M)$ as the set of smooth forms $\omega$ which have *compact support in the vertical direction.* That is for any $K \subset M$ compact, $\pi^{-1}(K) \cap \operatorname{Supp}\omega$ is compact (see fig. 2.1). The cohomology $H^*_{cv}(M)$ of this complex is called the *compact vertical cohomology.* ▲

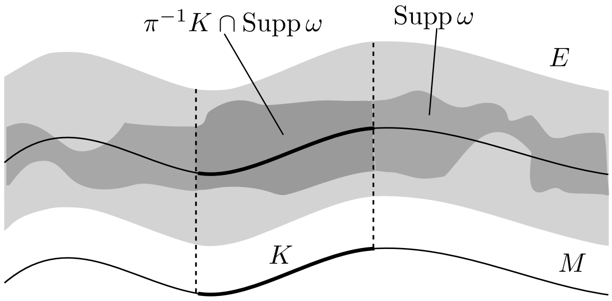



Figure 2.1: The support of $\omega$ is compact when restricted to a compact subset of the manifold. Furthermore the restriction to each fiber also has compact support.

Later in this section we will prove the following theorem:

**Theorem 2.1.2:** (Thom isomorphism). Let $\pi : E \to M$ be an orientable vector bundle of rank $n$ over a manifold $M$ admitting a finite good cover. Then $H^*_{cv}(E) \simeq H^{*-n}(M)$. ▲

Just like in the proof of the Poincaré lemma for compactly supported cohomology, we are interested in the map induced on cohomology given by integrating along the fibers.

**Definition 2.1.3:** Let $\pi : E \to M$ be an orientable vector bundle of rank $n$ with trivializing open cover $\{U_\alpha\}$. For each $U_\alpha$ we have a set of fiber coordinates $t_1, \dots, t_n$ on $E|_{U_\alpha}$. We call a form $\omega \in \Omega^{*-n}_{cv}$ of *top degree on fibers* on $U_\alpha$ if the form $\omega_\alpha = \omega|_{\pi^{-1}U_\alpha}$ can be written as

$$\omega_\alpha = (\pi^*\phi) f(x,t) dt_1 \cdots dt_n. \tag{2.1}$$

For some $\phi \in \Omega^*(M)$ and $f \in C^\infty(E|_{U_\alpha})$, with $x$ local manifold coordinates and $t$ local fiber coordinates.▲

Just like how we were only interested in integration on forms with a $dt$ term for integration along fibers of compactly supported forms, in this case we are only interested in integrating forms that are of top degree on fibers. Note that in any chart we can always split a form in a part that is of lower-than-top degree on fibers and a part that is of top degree on fibers.

**Definition 2.1.4:** Let $\tau \in H*_{cv}(E)$ and let $\{U_\alpha\}$ be a trivializing open cover. Then let $\omega|_{U_\alpha} = \omega_\alpha$ be of top degree on fibers as in equation (2.1). Then we locally define integration on fibers $\pi_* : \Omega^*_{cv}(E|_{U_\alpha}) \to \Omega^{*-n}_{cv}(U_\alpha)$ by

$$(\pi_*\tau)|_{U_\alpha} = \pi_*\omega_\alpha = \phi \int_{\mathbb{R}^n} f(x,t)dt_1 \cdots dt_n. \qquad \blacktriangle$$

Now $\pi_*\tau$ defines a global form $\sum_\alpha \rho_\alpha\omega_\alpha$ on $\Omega^*_{cv}(E)$, with $\{\rho_\alpha\}$ a partition of unity. This is because on an overlap $U_\alpha \cap U_\beta$ we have

$$\pi_*\omega_\beta = \pi_*\omega_\alpha \circ g_{\alpha\beta} = \phi \int_{\mathbb{R}^n} (\det g_{\alpha\beta}) f(x,t)dt_1 \cdots dt_n = \pi_*\omega_\alpha,$$

where we have used that $g_{\alpha\beta} : U_\alpha \cap U_\beta \to SO(n)$, which is asserted by the orientability of $E$. The fact that we have $\det(g_{\alpha\beta}) = 1$ is precisely why we defined the orientation on $E$ as we did in Definition 1.2.8. As with any map of forms we want $\pi_*$ to commute with $d$ so that it induces a well-defined map on cohomology.

**Proposition 2.1.5:** Integration on fibers commutes with the exterior derivative; $\pi_* d = d\pi_*$. $\blacktriangle$

Proof: Since $\pi_*$ is defined locally, we may assume that $E = M \times \mathbb{R}^n$ without loss of generality. As before we consider the case where $\omega \in \Omega^*_{cv}$ is of top degree on fibers and the case where it is not, which is sufficient by linearity. Let $\omega = (\pi^*\phi)f(x,t)dt_1 \cdots dt_n$. Write $dt = dt_1 \cdots dt_n$ then,

$$\pi_* d\omega = \pi_* \left( (\pi^* d\phi) f \, dt + (-1)^{\deg\phi} \pi^*\phi \frac{\partial f}{\partial x^i} dx^i dt \right) = d\phi \int f \, dt + (-1)^{\deg\phi} dx^i \phi \int \frac{\partial f}{\partial x^i} \, dt.$$

Whereas,

$$d\pi_*\omega = d\left( \phi \int f \, dt \right) = d\phi \int f \, dt + (-1)^{\deg\phi} dx^i \phi \int \frac{\partial f}{\partial x^i} \, dt,$$

hence $\pi_* d\omega = d\pi_*\omega$. Now let $\omega = (\pi^*\phi)f(x,t)\psi$ with $\psi = dt_{i_1} \cdots dt_{i_k}$ for some index set $\{i_j\}$ such that $\psi$ is not of top degree. We then have $d\pi_*\omega = 0$, and furthermore

$$\pi_* d\omega = \pi_* \left( (\pi^* d\phi) f\psi + (-1)^{\deg\phi} \pi^*\phi \left[ \frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial t^j} dt^j \right] \psi \right),$$

which is trivially zero unless $dt_j\psi = \pm dt_1 \cdots dt_n$ for some $j$. In that case we get,

$$\pi_* d\omega = \pm \pi^*\phi \int \frac{\partial f}{\partial t_j} dt = 0$$

by compact support in the vertical direction. Hence we conclude that $\pi_*$ commutes with $d$. $\square$

Now for the Thom isomorphism we need one more fact:

**Lemma 2.1.6:** (Projection formula) Let $\pi : E \to M$ be an oriented rank-$n$ vector bundle, $\tau \in \Omega^*(M)$

and $\omega \in \Omega^*_{cv}(E)$ then $\pi_*((\pi^*\tau)\omega) = \tau(\pi_*\omega)$. ▲

Proof: Without loss of generality we can assume $E = M \times \mathbb{R}^n$ and restrict our attention to a single chart. Unless $\omega$ is of top degree on fibers both sides are trivially zero. Therefore assume that $\omega = (\pi^*\phi) f dt_1 \cdots dt_n$ for some $\phi$, $f$. Then we have,

$$\pi_*((\pi^*\tau)\omega) = \pi_*((\pi^*\tau\phi)f dt = \tau\phi \int f\, dt = \tau\pi_*\omega,$$

and hence we get the required result. □

As promised we will now prove the Thom isomorphism:

**Theorem 2.1.7:** (Thom isomorphism). Let $\pi : E \to M$ be an orientable vector bundle of rank-$n$ over a manifold $M$, then

$$H^*_{cv}(E) \simeq H^{*-n}(M)$$

where the isomorphism is given by $\pi_*$. ▲

Proof: We will only prove this for manifolds with finite good cover, which makes the proof substantially shorter. The proof for a not necessarily finite good cover proceeds by a generalization of the argument given here [BT82, p. 131]. We will first prove the result for product bundles, and then use induction on the cardinality of a good cover together with the Mayer-Vietoris sequence to generalize the result. The proof for product bundles is the same as that for the Poincaré lemma for compact cohomology only with a different homotopy operator and inverse homomorphism. Let $dt$ denote $dt_1 \cdots dt_n$ and let $e = e(t)dt$ be a top degree form on $\mathbb{R}^n$. Then define $e_* : \Omega^*_{cv}(M) \to \Omega^*_{cv}(M \times \mathbb{R}^n)$ by $e_*\omega = \omega \wedge e$. Now we define $K$ on forms of top degree on fibers (i.e. $\omega = (\pi^*\phi)f(x,t)dt$) as,

$$\begin{aligned} K\omega &= (\pi^*\phi) \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_n} f(x,t)dt - (\pi^*\phi)A(t) \int_{\mathbb{R}^n} f(x,t)dt, \\ \text{with} \quad A(t) &= \int_{-\infty}^{t_1} \cdots \int_{-\infty}^{t_n} e(t)dt \end{aligned}$$

With these definitions the proof of the Poincaré lemma for compact cohomology carries over mutatis mutandis to show that

$$H^*_{cv}(M \times \mathbb{R}^n) \simeq H^{*-n}(M).$$

Now for $U$, $V \subset M$ open we have the following exact sequence:

$$0 \longrightarrow \Omega^*_{cv}(E|_{U\cup V}) \longrightarrow \Omega^*_{cv}(E|_U) \oplus \Omega^*_{cv}(E|_V) \longrightarrow \Omega^*_{cv}(E|_{U\cap V}) \longrightarrow 0.$$

This naturally induces a long exact Mayer-Vietoris sequence in vertically compact cohomology. We again couple this with the Mayer-Vietoris sequence of $M$ for the same $U$ and $V$:

$$\begin{tikzcd}
\cdots \arrow[r] & H^*_{cv}(E|_{U\cup V}) \arrow[r] \arrow[d, "\pi_*"] & H^*_{cv}(E|_U) \oplus H^*_{cv}(E|_V) \arrow[r] \arrow[d, "\pi_*"] & H^*_{cv}(E|_{U\cap V}) \arrow[r, "d^*"] \arrow[d, "\pi_*"] & H^{*+1}_{cv}(E|_{U\cup V}) \arrow[r] \arrow[d, "\pi_*"] & \cdots \\
\cdots \arrow[r] & H^{*-n}(U\cup V) \arrow[r] & H^{*-n}(U) \oplus H^{*-n}(V) \arrow[r] & H^{*-n}(U\cap V) \arrow[r, "d^*"] & H^{*+1-n}(U\cup V) \arrow[r] & \cdots
\end{tikzcd}$$

Note that for $U$ contractible we have that $E|_U \simeq U \times \mathbb{R}^n$ is trivial [BT82, p.59]. Hence $\pi_* : H^*_{cv}(E|_U) \to H^{*-n}(U)$ is an isomorphism. If we can show commutativity of this diagram, then by the five lemma we have that for $\{U, V\}$ a good cover, $\pi_* : H^*_{cv}(E|_{U\cup V}) \to H^{*-n}(U \cup V)$ is an isomorphism. Then by the same induction argument on the cardinality of a good cover as in the proof of Poincaré duality we have proven the Thom isomorphism for $M$ with a finite good cover.

Now commutativity follows trivially for square on the left and in the middle by writing out the definitions. We will only show commutativity for the square,

$$\begin{tikzcd}
H^*_{cv}(E|_{U\cap V}) \arrow[r, "d^*"] \arrow[d, "\pi_*"] & H^{*+1}_{cv}(E|_{U\cup V}) \arrow[d, "\pi_*"] \\
H^{*-n}(U\cap V) \arrow[r, "d^*"] & H^{*+1-n}(U\cup V)
\end{tikzcd}$$

Let $[\omega] \in H^*_{cv}(E|_{U\cap V})$. We have,

$$\pi_* d^*\omega = \pi_*((d\pi^*\rho_U)\omega) = (d\rho_U)\pi_*\omega = d^*\pi_*\omega,$$

where in the first and last step we applied Proposition 1.1.4 for an expression of $d^*$, and in the middle step we used the projection formula 2.1.6. This shows commutativity, and by the five lemma we conclude the Thom isomorphism for a good cover with two elements. By the same induction argument on the cardinality of a good cover as in the proof of Poincaré duality we conclude the Thom isomorphism for any manifold admitting a finite good cover. ☐

**Definition 2.1.8:** (Thom class) We call the inverse image of $\pi_*$ the Thom isomorphism $\mathscr{T} : H^*(M) \to H^{*+n}_{cv}(E)$. Note that $H^0(M) = \mathbb{R}$ is generated by the constant function 1 on $M$. We call $\Phi_E = \mathscr{T}(1)$ the *Thom class* on the oriented vector bundle $E$. ▲

The Thom class determines the Thom isomorphism uniquely, since by the projection formula we have,

$$\pi_*(\pi^*\omega \wedge \Phi_E) = \omega \wedge \pi_*\Phi_E = \omega \wedge 1 = \omega = \pi_*\mathscr{T}(\omega),$$

hence $\mathscr{T}(\omega) = \pi^*\omega \wedge \Phi_E$. Since $\pi_*\Phi_E = 1$, the Thom class restricts to each fiber as a compactly supported 1-form with total integral 1, which is also a generator for the compact cohomology $H^n_c(\pi^{-1}(x))$ on each fiber. Conversely the class in $\Phi_E \in H^k_{cv}(E)$ that restricts to the generator of $H^n_c(\pi^{-1}(x))$ such that $\pi_*\Phi = 1$ is clearly unique. From this we conclude:

**Proposition 2.1.9:** The Thom class is uniquely characterized as the form in $H^k_{cv}$ whose restriction to fibers integrates to 1. ▲

An important property of the Thom class is that it behaves naturally with respect to pullbacks in the following sense:

**Proposition 2.1.10:** (Naturality of the Thom class) Let $M \to N$ be a smooth map and let $\pi : E \to M$ be a vector bundle as above. Then the Thom class $\Phi_{f^*E}$ of $f^*E$ is determined by $\Phi_{f^*E} = \tilde{f}^*\Phi_E$, where $\tilde{f}$ is as in Definition 1.2.4. That is, the Thom class of the pullback is the pullback of the Thom class. ▲

Proof: The Thom class is characterized by the fact that it integrates to 1 on fibers. Therefore we just have to show that $\tilde{f}^*\Phi_E$ integrates to 1 on $f^*E|_p$. We can identify $f^*E|_p$ with $\pi^{-1}(f(p))$ via $\tilde{f}$. By the diffeomorphism invariance of integration we have,

$$\int_{f^*E|_p} \tilde{f}^*\Phi_E = \int_{\pi^{-1}(f(p))} \Phi_E = 1.$$

Hence $\tilde{f}^*\Phi_E$ is the Thom class of $f^*E$ and we conclude naturality. □

Then finally there is another property of the Thom class which makes it behave well with operations on vector bundles. This property will be very important later on to deduce a similar property for the Euler class.

**Proposition 2.1.11:** The Thom class of a direct sum bundle $E \oplus F$ is given by $\Phi_{E\oplus F} = \pi_E^*\Phi_E \wedge \pi_F^*\Phi_F$ where $\pi_E$, $\pi_F$ are the projections of $E \oplus F$ to respectively $E$ and $F$. ▲

Proof: This is immediate since $\pi_1^*\Phi_E \wedge \pi_2^*\Phi_F$ is a class in $H^{m+n}_{cv}(E \oplus F)$ which integrates to 1 on the fibers (where $m$, $n$ are the respective ranks of the two bundles).

## 2.2 Poincaré duality and the Thom class

Using an elementary construction we can associate a class in the de Rham cohomology to every closed submanifold $S$ of our manifold $M$. It turns out that this class is the same as the Thom class of a naturally defined vector bundle over $S$. This equivalence is very interesting but only a very small amount of this section is actually used in the rest of this thesis. Therefore this section can be skipped without loss of continuity.

Let $S$ be a closed oriented submanifold of dimension $k$ and let $\omega \in H^k_c(M)$ be a $k$-form with compact support on $M$. Then if $i : S \hookrightarrow M$ is the inclusion, $i^*\omega$ has compact support as well. Now $\omega \mapsto \int_S i^*\omega$ is a linear functional and hence an element of $(H^k_c(M))^*$. But by Poincaré duality we have that $(H^k_c(M))^* \simeq H^{n-k}(M)$, where by the explicit isomorphism used in the proof of Poincaré duality we find that there is an $\eta_S$ such that

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S.$$

**Definition 2.2.1:** With the same notation as above we define $\eta_S \in H^{n-k}(M)$ the *Poincaré dual* of the submanifold $S$ of $M$. ▲

In other words the Poincaré dual $\eta_S$ of a submanifold $S$ is the element in cohomology such that integrating $\omega$ over $S$ leads to the same result as integrating $\omega \wedge \eta_S$ over $M$. We want to relate this to the Thom class of some vector bundle. The vector bundle of interest turns out to be the normal bundle of $S$.

**Definition 2.2.2:** (Normal bundle) Let $S \subset M$ be an oriented closed $k$-dimensional submanifold of $M$ where $M$ has dimension $n$. We can see the tangent space $T_S$ of $S$ as a subspace of the tangent space $T_M|_S$ of $M$ restricted to $S$. We now define the rank-$(n-k)$ vector bundle called the *normal bundle* $N$ as the subspace of $T_M|_S$ obtained by taking the pointwise quotient of $T_M|_S$ by $T_S$; that is, $N$ is the pointwise orthogonal complement of $T_S$ in $T_M|S$. Or more geometrically it the set of vectors in $T_M|_S$ that are normal to the submanifold $S$. ▲

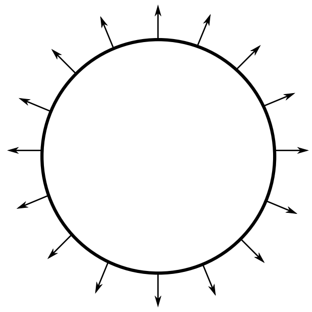


Figure 2.2: Visualization of the normal bundle of $S^1$ as a closed submanifold of $\mathbb{R}^2$.

Now it turns out the normal bundle is always diffeomorphic to a submanifold $T \subset M$ called the *tubular neighborhood of $S$ in $M$* [BT82, p. 66]. In general a tubular neighborhood is a neighborhood of $S$ in $M$ that is diffeomorphic to a rank-$(n-k)$ vector bundle over $S$ such that $S$ is diffeomorphic to the zero section of this bundle. In the example above such a tubular neighborhood could be $S^1 \times \mathbb{R}$ embedded as an (open) annulus in $\mathbb{R}^2$. Or if we see $S^1$ as a submanifold of $\mathbb{R}^3$, then its tubular neighborhood could be the (open) filled torus $S^1 \times \mathbb{R}^2$.

From the definition of the tubular neighborhood $T$ we get the following sequence of maps by the Thom isomorphism:

$$H^*(S) \xrightarrow{\mathscr{T}} H_{cv}^{*+n-k}(T) \xrightarrow{j_*} H^{*+n-k}(M).$$

Where $j_*$ is the map induced by the inclusion $j : T \hookrightarrow M$ by extending to zero on $M \setminus T$. It turns out that the Thom class $\Phi_N$ of the normal bundle $N$ of $S$ is precisely the Poincaré dual of $S$:

**Theorem 2.2.3:** Let $S \subset M$ be an oriented closed $k$-dimensional submanifold of $M$ where $M$ has dimension $n$. The Poincaré dual $\eta_S \in H^{n-k}(M)$ of $S$ of is then represented by the Thom class $\Phi_N$ of the normal bundle of $S$. That is,

$$\eta_S = j_* \Phi_N.$$ ▲

Proof: We have to show that $j_*\Phi_N$ fulfills the condition for being the Poincaré dual of $S$, that is,

$$\int_M \omega \wedge j_*\Phi_N = \int_S i^*\omega$$

for $i : S \hookrightarrow M$ the inclusion. Let $h : S \hookrightarrow T$ be the inclusion of $S$ into its tubular neighborhood $T$, then

$$\int_S i^*\omega = \int_S h^*\omega,$$

since both maps agree on $T$. Now if we consider $\pi : T \to S$ the deformation retract of $T$ onto $S$ induced by the projection map of the normal bundle, we have that $i^*$ and $\pi^*$ are inverse isomorphisms in cohomology by homotopy invariance of de Rham cohomology. Hence we have that $\omega = \pi^* i^* \omega + d\tau$ for some $\tau$. Consequently we get,

$$\int_M \omega \wedge j_* \Phi_N = \int_T \omega \wedge \Phi_N = \int_T (\pi^* i^* \omega + d\tau) \wedge \Phi_N = \int_T (\pi^* i^* \omega) \wedge \Phi_N.$$

Here the last step follows by the fact that $\Phi_N$ is closed and then applying Stokes' theorem to $\int_T d(\tau \wedge \Phi_N)$. The first step follows by the fact that $\Phi_N$ has support in $T$. We consider $S$ as the zero section of the normal bundle $N$ diffeomorphic to $T$. In this way we can also first integrate over fibers by applying $\pi_*$ and then integrate over $S$ instead:

$$\int_T (\pi^* i^* \omega) \wedge \Phi_N = \int_S \pi_* (\pi^* i^* \omega \wedge \Phi_N) = \int_S i^* \omega \wedge \pi_* \Phi_N,$$

where in the last step we applied the projection formula 2.1.6. Note that $\pi_* \Phi_N = 1$ per definition, hence $\int_M \omega \wedge j_* \Phi_N = \int_S i^* \omega$ and $\eta_S = j_* \Phi_N$ as required. □

We can use this theorem to derive a result which will be useful later on. We wish to prove that under certain conditions if we have two submanifolds $R, S \subset M$ then,

$$\eta_{R \cap S} = \eta_R \wedge \eta_S.$$

Now the condition is that $R$ and $S$ intersect transversally:

**Definition 2.2.4:** Two submanifolds $R, S \subset M$ *intersect transversally* if for every $p \in R \cap S$ we have that,

$$T_p R \oplus T_p S = T_p M,$$

which can geometrically be interpreted as the spaces being nowhere tangent to each other in the intersection. ▲

**Proposition 2.2.5:** Suppose two submanifolds $R, S$ intersect transversally, then the Poincaré dual of $R \cap S$ is given by,

$$\eta_{R \cap S} = \eta_R \wedge \eta_S,$$

where $\eta_R$ and $\eta_S$ denote the respective Poincaré duals of $R$ and $S$. ▲

Proof: Consider the normal bundle $N_{R \cap S}$ of the intersection. Any vector that is normal to $TR$ or $TS$ is also normal to $T(R \cap S)$ hence we see that $N_R \oplus N_S \subseteq N_{R \cap S}$. Now because of transversality we have that $\operatorname{codim} R \cap S = \operatorname{codim} R + \operatorname{codim} S$ [GP74, p.30]. Therefore we conclude that $N_{R \cap S} = N_R \oplus N_S$. Now by Proposition 2.1.11 and Theorem 2.2.3 we have,

$$\eta_{R \cap S} = \Phi(N_{R \cap S}) = \Phi(N_R \oplus N_S) = \Phi(N_R) \wedge \Phi(N_S) = \eta_R \wedge \eta_S$$

from which we conclude the proposition. □

## 2.3 Euler class

Using the Thom isomorphism and Poincaré duality we can introduce an important invariant of vector bundles called the Euler class. This class will later be used to introduce Chern and subsequently Pontryagin classes, our most important mathematical objects of study.

We want the Euler class to be a class in cohomology that behaves nicely with operations on vector bundles such as pullbacks and direct sums. Furthermore we want the Euler class to detect some form of triviality of the bundle, e.g. the Euler class should be zero for product bundles and bundles admitting global non-vanishing sections. In this manner it would provide us a tool to measure how 'twisted' a vector bundle is, for example. The following definition has precisely these properties, as we will show during the remainder of this section.

**Definition 2.3.1:** (Euler class) Let $\pi : E \to M$ be a rank-$k$ vector bundle. Consider the Thom class $\Phi_E$ of $E$. The Euler class $e(E) \in H^k(M)$ is defined to be the pullback $s^*\Phi_E$ by the zero section of the Thom class. That is, the Euler class is obtained by restricting the Thom class to the base manifold. ▲

**Proposition 2.3.2:** Let $E \to M$ be a rank-$k$ vector bundle with $k$ odd, then $e(E) = 0$. ▲

Proof: Consider $\mathscr{T}(e(E))$, since $e(E) = s^*\Phi_E$ this is,

$$\mathscr{T}(e(E)) = (\pi^* s^* \Phi_E) \wedge \Phi_E,$$

Since $\pi^* s^* = 1$ on cohomology and $\pi_* \mathscr{T} = 1$, we get $e(E) = \pi_*(\Phi_E \wedge \Phi_E)$, but if $k$ is odd then $\Phi_E \wedge \Phi_E = -\Phi_E \wedge \Phi_E$ and hence $e(E)$ is exact. □

**Proposition 2.3.3:** The Euler class is natural with respect to pullbacks. That is, let $E \to M$ be a vector bundle and let $f : M \to N$ be a smooth map, then $e(f^*E) = f^*e(E)$. ▲

Proof: Consider the following (commuting) diagram:

$$\begin{tikzcd}
N \times E \supset f^*E \arrow[r, "\tilde{f}"] \arrow[d, bend left, "\rho"] & E \arrow[d, bend left, "\pi"] \\
N \arrow[u, bend left, "s_N"] \arrow[r, "f"] & M \arrow[u, bend left, "s_M"]
\end{tikzcd}$$

where $s_N$, $s_M$ are the respective zero sections, and $\rho$, $\pi$ the projections. The most important point is that $\tilde{f} \circ s_N = s_M \circ f$. This is because both maps send a point $x \in N$ to $0 \in E|_{f(x)}$. Hence

$$e(f^*E) = s_N^* \Phi_{f^*E} = s_N^* \tilde{f}^* \Phi_E = f^* s_M^* \Phi_E = f^* e(E),$$

by naturality of the Thom class. Hence we conclude naturality of the Euler class. □

Recall that two vector bundles $\pi : E \to M$ and $\rho : F \to M$ are isomorphic if there exist a bundle homomorphism $f$ that has an inverse $f^{-1}$ that is also an homomorphism, i.e. the following diagram commutes:

$$\begin{tikzcd}
E \arrow[rr, bend left, "f"] \arrow[dr, "\pi"'] & & F \arrow[ll, bend left, "f^{-1}"] \arrow[dl, "\rho"] \\
& M &
\end{tikzcd}$$

On vector bundles we have that the construction of the pullback bundle is invariant under homotopy by a result of [Hus93, Theorem 4.7].

**Proposition 2.3.4:** Let $M$, $N$ be manifolds and $E$ be a vector bundle of $N$. If $f, g : N \to M$ are homotopic maps then $f^*E$ and $g^*E$ are isomorphic vector bundles. ▲

**Proposition 2.3.5:** Isomorphic vector bundles have the same Euler class if they have the same orientation, and differ by a sign if they have reversed orientation. ▲

PROOF: Let $f : E \to F$ be an isomorphism of vector bundles over the same base space, and let $s_E$ and $s_F$ be the respective zero sections. Then $s_F = f \circ s_E$, hence $s_F^* = s_E^* f^*$ and we get,

$$e(F) = s_F^* \Phi_F = s_E^* f^* \Phi_F = \pm s_E^* \Phi_E = \pm e(E)$$

where the step $f^*\Phi_F = \pm\Phi_E$ follows because $f^*\Phi_F$ integrates to $\pm 1$ on fibers since $f$ acts as a diffeomorphism $E_p \to F_p$ on fibers. □

**Proposition 2.3.6:** From the definition of the Thom class we see that orientation reversal induces a sign change in the Thom class, and hence also the Euler class changes sign under orientation reversal. ▲

**Proposition 2.3.7:** (Whitney product formula) Consider the direct sum of two bundles $E \oplus F$. Then the Euler class of this direct sum is the product of the two Euler class, i.e. $e(E \oplus F) = e(E) \wedge e(F)$. ▲

PROOF: Let $\pi_E, \pi_F$ be the projections of $E \oplus F$ on respectively $E$ and $F$. Recall that $\Phi_{E\oplus F} = \pi_E^*\Phi_E \wedge \pi_F^*\Phi_F$. Now if $s$ is the zero section of $E \oplus F$ then $\pi_E \circ s$ and $\pi_F \circ s$ are the zero sections of $E$ and $F$ respectively. Hence we get,

$$e(E \oplus F) = s^*\Phi_{E\oplus F} = s^*(\pi_E^*\Phi_E \wedge \pi_F^*\Phi_F) = e(E) \wedge e(F),$$

as required. □

**Definition 2.3.8:** The Euler class of a smooth manifold is that of its tangent space, that is $e(M) := e(TM)$. ▲

Earlier we mentioned that the Euler class measures how 'twisted' a vector bundle is. We will make this statement precise:

**Proposition 2.3.9:** If a vector bundle $\pi : E \to M$ admits a non-vanishing global section $s : M \to E$, then its Euler class vanishes. ▲

PROOF: Let $\xi$ be the trivial line bundle over $M$ spanned by the the section $s$. We then have $E = \xi \oplus \xi^\perp$. Since $\xi$ is of rank one, it has in particular odd rank and hence vanishing Euler class. Now by the Whitney product formula,

$$e(E) = e(\xi) \wedge e(\xi^{\perp}) = 0,$$

completing the proof. □

Another property of the Euler class that we are interested in is its integral, $\int_M e(E)$. Since a diffeomorphism $f : M \to N$ sends $e(TM)$ to $f^*e(TN)$ we see that $\int_M e(M)$ is a diffeomorphism invariant. We call this invariant the Euler number of $M$. This turns out to be an integer, in fact we have by [BT82, p. 128] that:

**Theorem 2.3.10:** The Euler number $\int_M e(E)$ is equal to the Euler characteristic

$$\chi(M) = \sum_q (-1)^q \dim H^q(M).$$ ▲

An even more important result for us is that the integral over the Euler class is integer for any vector bundle that admits a section with finitely many zeros. We have by [BT82, p.124] that:

**Theorem 2.3.11:** Let $s$ be a section of $E$ with finitely many zeros. Then $\int_M e(E)$ is the sum of the local degree of $s$ at its zeros. ▲

The local degree is defined as follows. Let $M$ be an $n$-dimensional manifold and let $E \to M$ be a rank-$n$ vector bundle. If the rank of $E$ would not be $n$ then $\int_M e(E)$ would vanish. Consider a section $s$ of $E$ with zero $x \in M$. Take a small enough neighborhood $U$ of $x$ such that $E|_U$ is diffeomorphic to $\mathbb{R}^n \times \mathbb{R}^n$. Then the map $u = s/||s||$ induces a map $S^{n-1} \to S^{n-1}$. The local degree of $s$ at $x$ (also known as the *index* of $s$ at $x$) is then defined as the degree of this map $u$. That is, consider the map induced by $u$ on homology $u_* : H_{n-1}(S^{n-1}; \mathbb{Z}) \to H_{n-1}(S^{n-1}; \mathbb{Z})$. This map is of form $u_*\gamma = \alpha\gamma$ with $\alpha \in \mathbb{Z}$ the degree of $u$. This degree is a homotopy invariant, and this construction is hence well-defined.

**Example 2.3.12:** We can use the local degree of sections to extend our definition to Euler numbers of non-orientable bundles. With the definition given in this section orientability is necessary to define the Euler class. This is because we cannot define the Thom class on non-orientable bundles, since the construction of the Thom class relies heavily on the integration. Furthermore we can also construct an Euler number that is non-vanishing for odd rank bundles. Consider for example the Möbius strip $M$ as a vector bundle over $S^1$ with a section $s$ as shown below. This section has precisely one zero, namely at

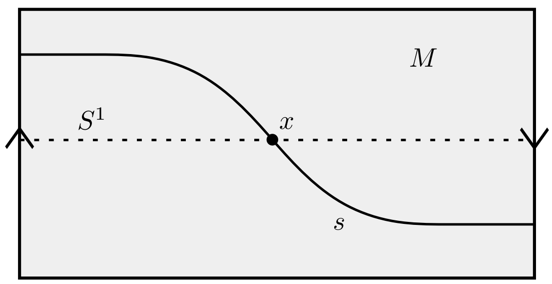



Figure 2.3: Example of a section of a Möbius strip seen as a line bundle over $S^1$.

$x$. Furthermore it should be clear that any section of $M$ with a finite amount of zeros would have an odd amount of zeros counting multiplicities. In this case consider the behavior of $s$ in a small neighborhood of $x$. Clearly the map $s/||s|| : S^0 \to S^0$ sends $-1 \in S^0$ to $1 \in S^0$ and visa-versa, therefore its degree is $-1$. Hence we could say the Euler number of the Möbius strip is $-1$. Note however that the section $-s$ would have given us degree $+1$ at $x$, therefore from this discussion we can conclude that the sign of the Euler number of the Möbius strip is not well-defined, but otherwise Theorem 2.3.11 does provide us with a crude way of computing such a number. This gives us a definition of the Euler number in $\mathbb{Z}_2$ which can still detect the non-triviality of bundles like the Möbius strip. ▲

As a less trivial example we will consider the canonical line bundle $\pi : \gamma \to \mathbb{C}P^k$ which assigns to every line $\ell \in \mathbb{C}P^k$ the complex line in $\mathbb{C}^{k+1}$ of all the points contained in the line. That is, for $\ell \in \mathbb{C}P^k$ we have $\pi^{-1}(\ell) = \{x \in \ell \,|\, x \in \mathbb{C}^{k+1}\}$. Note that $\gamma$ is a complex line bundle, but we want to see it as a real line bundle instead so that we can compute its Euler number. To do this we identify $\mathbb{C}$ with $\mathbb{R}^2$ fiberwise, and consequently the fact that $\gamma$ is a complex vector bundle merely becomes a minor technicality. We now compute the Euler number of the canonical line bundle over $\mathbb{C}P^1$ specifically.

**Proposition 2.3.13:** The Euler number $\int_{\mathbb{C}P^1} e(\gamma)$ of the canonical line bundle $\gamma \to \mathbb{C}P^1$ is $-1$. ▲

Proof: The proof proceeds by applying Theorem 2.3.11. Thus we first need to find a suitable section of $\gamma$. Let $[x_0, x_1] \in \mathbb{C}P^1$ for $(x_0, x_1) \neq (0, 0)$ denote the unique complex line in $\mathbb{C}^2$ containing $(x_0, x_1)$. Now we assume that $|x_0|^2 + |x_1|^2 = 1$, and we define a section $s$ by

$$s([x_0, x_1]) = (x_0\overline{x_0}, x_1\overline{x_0})\,.$$

This is well-defined because if take any other point $(\lambda x_0, \lambda x_1)$ with unit norm on the line $[x_0, x_1]$, then clearly since $|\lambda| = 1$ the value of this section doesn't change. Furthermore we note that this section only vanishes at the single point $[0, 1] \in \mathbb{C}P^1$. Now to compute its Euler number we only need to know the local degree at this point. The way were are going to compute this degree is by explicitly looking at what the section looks like in a parametrization around the point $[0, 1]$. For example we can parametrize $\mathbb{C}P^1 \setminus \{[1, 0]\}$ by mapping each $u \in \mathbb{C}$ to the line $[u, 1]$. The fiber at $[u, 1]$ is then given by the set $\{(\lambda u, \lambda), \lambda \in \mathbb{C}\}$, therefore in this chart we have trivialized the bundle to a space diffeomorphic to $\mathbb{C}^2$. Now the section $s$ sends a line $[u, 1]$ to

$$s([u, 1]) = \left(\frac{|u|^2}{1 + |u|^2}, \frac{\overline{u}}{1 + |u|^2}\right) = \frac{\overline{u}}{1 + |u|^2}(u, 1).$$

From this we see that $s$ induces a map $\mathbb{C} \to \mathbb{C}$ given by $u \mapsto \overline{u}/(1 + |u|^2)$. This map has degree $-1$ since it would for example take the unit circle in $\mathbb{C}$ parametrized counter clockwise to a circle in $\mathbb{C}$ parametrized clockwise. From this we conclude that the section $s$ has local degree $-1$ at the line $[0, 1]$. And hence the Euler number is $-1$ as well. □

In conclusion we saw that integration on fibers produced a natural isomorphism between $H^*(M)$ and $H^*_{cv}(E)$, we then looked at the inverse of this isomorphism; the Thom isomorphism. This Thom isomorphism could then be written as the wedge product with the Thom class in bundle cohomology. This Thom class then behaved naturally under operations on vector bundles. We then looked at the Euler class, which arises as a counterpart to the Thom class in the base cohomology. The Euler class also turned out to be very compatible with operations on vector bundles, and had some other interesting properties as well. In the next part we will use the Euler class to define some more classes that behave naturally with respect to vector bundle operations, but give more information about our space.

# 3 Chern and Pontryagin classes

In this part we will introduce the Chern and Pontryagin classes of a vector bundle. We will prove a number of important properties about the Chern classes related to its behavior with respect to vector bundle operations. Additionally we will provide some results to aid computation of these Chern classes. After that we will introduce the Pontryagin classes and Pontryagin numbers. These will be used together with the Hirzebruch signature formula to construct an invariant used to distinguish exotic spheres from regular spheres. And then finally we will consider a theorem characterizing vector bundles by homotopy classes of maps into a Grassmannian. But first we will make a short digression on the ring structure of cohomology. Unless stated otherwise everything in this part is based on [BT82].

## 3.1 Ring structure of cohomology

Up to this point we always considered $H^*(M)$ as a graded algebra. However the wedge product on forms provides us with a method of multiplying elements in cohomology. Together with addition provided by the group structure this makes $H^*(M)$ into a ring. Its multiplication is *graded commutative*, that is $\alpha \wedge \beta = (-1)^{(\deg\alpha)(\deg\beta)}\beta \wedge \alpha$.

The ring structure can be of interest in the following way. Suppose we have some non-trivial $\alpha \in H^k(M)$ for some $n$-dimensional $M$ (so that $H^k(M) = 0$ for $k > n$). Now if $k$ is odd we have $\alpha \wedge \alpha = -\alpha \wedge \alpha$, hence $\alpha^2 = 0$. But if $k$ is even this is not necessarily true. However $\deg \alpha^m = mk$ assures that $\alpha^m = 0$ if $mk > n$, though there could of course be an $m \leq n/k$ for which $\alpha^m = 0$. Knowing the ring structure of $H^*(M)$ will for example tell us exactly when $\alpha^m = 0$ and when it's not.

Our main example of a space whose cohomology has an interesting ring structure will be the complex projective space $\mathbb{C}P^n$. We will denote with $\mathbb{R}[x]$ the ring of polynomials with real coefficients, and with $\mathbb{R}[x]/x^{n+1}$ the ring generated by the polynomials under the relation $x^{n+1} = 0$, in other words the ring of polynomials with degree $\leq n$. This can also be seen as the ring $\mathbb{R}[x]$ modulo the ideal generated by $x^{n+1}$. We have the following result:

**Theorem 3.1.1:** The ring structure of $H^*(\mathbb{C}P^n)$ is given by:

$$H^*(\mathbb{C}P^n) = \frac{\mathbb{R}[\alpha]}{\alpha^{n+1}},$$

with $\alpha \in H^2(\mathbb{C}P^n)$ a generator of the cohomology in degree 2. ▲

PROOF: Since $H^{2i}(\mathbb{C}P^n) = \mathbb{R}$ for all $i$, we just have to check that $\alpha^n \in H^{2n}(\mathbb{C}P^n)$ is non-trivial, with $\alpha \in H^2(\mathbb{C}P^n)$ a generator. We will prove this by showing $\int_{\mathbb{C}P^n} \alpha^n \neq 0$ by induction on $n$.

Note that the inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induces an isomorphism on cohomology on degree $\leq 2n-2$. This fact is apparent if we for example consider the way $\mathbb{C}P^n$ is constructed as a CW complex (see for example [Hat01, p.6]). Suppose $\alpha^{n-1} \in H^{2n-2}(\mathbb{C}P^{n-1})$ is a generator, then because the inclusion distributes with wedge products we have that $\alpha^i$ generates $H^{2i}(\mathbb{C}P^{2n})$ for all $i < n$. Let $\beta \in H^2_c(\mathbb{C}P^n)$ be the dual of $\alpha^{n-1}$ under Poincaré duality. Since $\mathbb{C}P^n$ is compact, all forms are compactly supported and hence

$\beta \in H^2(\mathbb{C}P^n)$. Furthermore we have per definition that,

$$\int_{\mathbb{C}P^n} \beta \wedge \alpha^{n-1} = 1.$$

Now if $\beta = d\phi$ is exact we have by the fact that $\alpha$ is closed that

$$\int_{\mathbb{C}P^n} d\phi \wedge \alpha^{n-1} = \int_{\mathbb{C}P^n} d(\phi \wedge \alpha^{n-1}) = 0.$$

Hence $\beta$ is a non-trivial element in $H^2(\mathbb{C}P^n) \simeq \mathbb{R}$ which means there must be some $0 \neq \lambda \in \mathbb{R}$ such that $\beta = \lambda\alpha$. Now since $\beta \wedge \alpha^{n-1}$ is non-trivial, so is $\alpha^n$. By induction any element in $H^*(\mathbb{C}P^n)$ can now be written as a linear combination of $\alpha^i$'s, proving the theorem. □

We can refine this result slightly by finding a canonical generator for which not only $\int_{\mathbb{C}P^n} \alpha^n \neq 0$ but actually $\int_{\mathbb{C}P^n} \alpha^n = 1$. This result is actually only important much later on when we will prove the Hirzebruch signature theorem.

**Proposition 3.1.2:** Define $\alpha = -e(\gamma) \in H^2(\mathbb{C}P^n)$, with $\gamma \to \mathbb{C}P^n$ the canonical line bundle as defined in Proposition 2.3.13. Then $H^*(\mathbb{C}P^n) \simeq \mathbb{R}[\alpha]/\alpha^{n+1}$ and furthermore we have

$$\int_{\mathbb{C}P^n} \alpha^n = 1.$$ ▲

Proof: First let $\gamma^j$ be the canonical line bundle over $\mathbb{C}P^j$ with $j < n$. We can naturally embed $\mathbb{C}P^j$ into $\mathbb{C}P^n$ via a map $i$. Then consider the pullback bundle $i^*\gamma^n$ over $\mathbb{C}P^j$. Its fiber at a point $\ell \in \mathbb{C}P^j$ is simply the set of points in the line $\ell \subset \mathbb{C}^j \subset \mathbb{C}^{n+1}$, therefore we have that $i^*\gamma^n = \gamma^j$. By naturality of the Euler class we then have,

$$e(\gamma^j) = e(i^*\gamma^n) = i^*e(\gamma^n).$$

Since $i^*$ is actually an isomorphism on cohomology, we can naturally identify $\alpha \in H^2(\mathbb{C}P^j)$ with $\alpha \in H^2(\mathbb{C}P^n)$ for any $j, n$.

We will proceed by induction on $n$. For the base step pick $\alpha \in H^2(\mathbb{C}P^1)$ such that $\int_{\mathbb{C}P^1} \alpha = 1$. Then suppose that $\int_{\mathbb{C}P^{n-1}} \alpha^{n-1} = 1$. We will show that $\alpha$ is the same as the the Poincaré dual $\eta_{\mathbb{C}P^{n-1}}$ of the closed submanifold $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$. Suppose for a moment that these two classes are indeed the same, then we would get

$$\int_{\mathbb{C}P^n} \alpha^n = \int_{\mathbb{C}P^n} \alpha^{n-1} \wedge \eta_{\mathbb{C}P^{n-1}} = \int_{\mathbb{C}P^{n-1}} \alpha^{n-1} = 1,$$

which would complete the induction step. Now since $H^2(\mathbb{C}P^n) \simeq \mathbb{R}$ we have that if two forms integrate to the same number, then they define the same cohomology class. From Proposition 2.3.13 we know that

$$\int_{\mathbb{C}P^1} \alpha = 1.$$

Therefore consider the integral

$$\int_{\mathbb{C}P^1} \eta_{\mathbb{C}P^{n-1}}.$$

In both cases we see $\mathbb{C}P^1$ as a subset of $\mathbb{C}P^n$. To be more precise we can see $\mathbb{C}P^1$ as the complex projective space spanned by the line $[x_1, x_2, 0, \dots, 0]$, and we embed $\mathbb{C}P^{n-1}$ as the space spanned by $[x_1, 0, x_3, \dots, x_n]$. Consider now the Poincaré dual $\eta_{\mathbb{C}P^1}$. We get,

$$\int_{\mathbb{C}P^1} \eta_{\mathbb{C}P^{n-1}} = \int_{\mathbb{C}P^n} \eta_{\mathbb{C}P^{n-1}} \wedge \eta_{\mathbb{C}P^1}.$$

Now by Proposition 2.2.5 we have that if $\mathbb{C}P^1$ and $\mathbb{C}P^{n-1}$ intersect transversally then,

$$\eta_{\mathbb{C}P^{n-1}} \wedge \eta_{\mathbb{C}P^1} = \eta_{\mathbb{C}P^{n-1} \cap \mathbb{C}P^1} = \eta_{[1,0,\dots,0]},$$

where $\eta_{[1,0,\dots,0]}$ is the Poincaré dual of the point $[1, 0, \dots, 0] \in \mathbb{C}P^n$. Thus we would get that,

$$\int_{\mathbb{C}P^1} \eta_{\mathbb{C}P^{n-1}} = \int_{\mathbb{C}P^n} \eta_{[1,0,\dots,0]} = \int_{[1,0,\dots,0]} 1 = 1.$$

Unfortunately to show that $\mathbb{C}P^1$ and $\mathbb{C}P^{n-1}$ intersect transversally we need some more terminology, and we will prove this result in Corollary 3.3.8. Now if we assume $\mathbb{C}P^1$ and $\mathbb{C}P^{n-1}$ do intersect transversally, then we get $\alpha = \eta_{\mathbb{C}P^{n-1}}$ and hence $\int_{\mathbb{C}P^n} \alpha^n = \int_{\mathbb{C}P^{n-1}} \alpha^{n-1} = 1$ by induction. Furthermore by the proof of Theorem 3.1.1 we also conclude that $H^*(\mathbb{C}P^n) = \mathbb{R}[\alpha]/\alpha^{n+1}$, completing the proof. □

## 3.2 Chern classes

In this section we consider a complex vector bundle $\rho : E \to M$ of rank-$k$, that is, a fiber bundle with fibers in $\mathbb{C}^k$. Let $g_{\alpha\beta} : U\alpha \cap U_\beta \to GL(n, \mathbb{C})$ be its transition functions. Every rank-$k$ complex vector bundle $E$ has an associated rank-$2k$ real vector bundle $E_\mathbb{R}$, called the *realification* of $E$, which is determined by the identification $\mathbb{C} = \mathbb{R}^2$.

**Definition 3.2.1:** We define the *projectivication* $\pi : P(E) \to M$ of the complex vector bundle $E$ to be the fiber bundle with fibers $P(E)|_p = P(E_p)$ the projective space of the fibers of $E$. That is, we replace $\mathbb{C}^k$ with $\mathbb{C}P^{k-1}$ fiberwise. ▲

With this definition $P(E)$ has transition functions $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to PGL(n, \mathbb{C})$ induced from $g_{\alpha\beta}$, where $PGL(n, \mathbb{C})$ is the projective general linear group. The projective linear group is defined by noting that scalar matrices $\lambda I$ act trivially on the projective space for $\lambda \in \mathbb{C}$ nonzero. Therefore when considering the action of $GL(n, \mathbb{C})$ on $P(\mathbb{C}^n)$ it is sufficient to consider them modulo the scalar matrices: $PGL(n, \mathbb{C}) = GL(n, \mathbb{C})/\{\text{scalar matrices}\}$.

To $P(E)$ we can associate several fiber bundles. Recall for example $\pi^*E$, the pullback bundle over $P(E)$ whose fiber at $\ell_p$ is $\pi^{-1}(p) = E_p$, as in Definition 1.2.4. Furthermore we have the universal subbundle, which is a generalization of the canonical line bundle.

**Definition 3.2.2:** The *universal subbundle* $S$ of $P(E)$ is the fiber bundle obtained as the set

$$S = \{(\ell_p, v) \in \pi^{-1}E \,|\, v \in \ell_p\}.$$

Then the fiber of $S$ at $\ell_p$ is given by the set of points in $\ell_p$ considered as a line through $E_p$. ▲

The constructions of the projectivication and the universal subbundle are visualized in Figure 3.1.

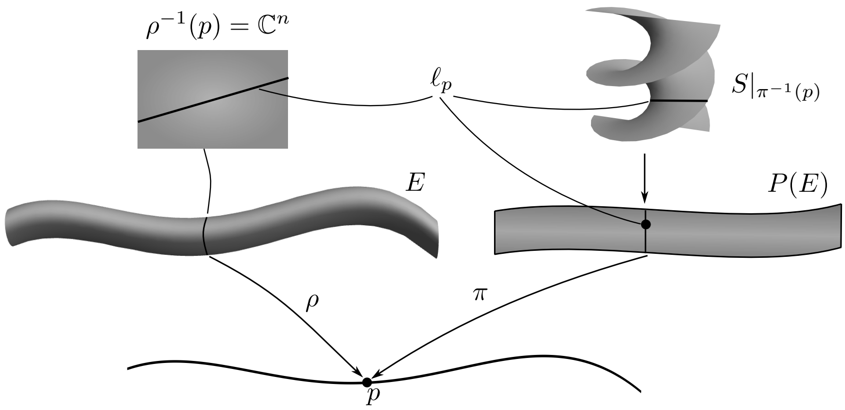



Figure 3.1: Visualization of the definition of the projectivication $P(E)$ of a complex vector bundle $E$ and its universal subbundle $S$. Note that $S|_{\pi^{-1}(p)}$ is shown as a helicoid, this is based on the interpretation of $\mathbb{R}P^2$ as a portion of helicoid with top and bottom identified.

**Definition 3.2.3:** We define the *universal quotient bundle* $Q$ by the short exact sequence

$$0 \longrightarrow S \longrightarrow \pi^* E \longrightarrow Q \longrightarrow 0.$$

Its fiber at $\ell_p$ is the orthogonal complement $\ell_p^\perp$ of the line $\ell_p$ as a subspace of $\mathbb{C}^k$. That is, we have $\pi^* E = S \oplus Q$. The sequence above is often called the *tautological sequence* of $E$. ▲

Now we will start with the construction of Chern classes. We first of all denote $x = -e(S_\mathbb{R})$ and note that $x$ is an element in $H^2(P(E))$ since $S_\mathbb{R}$ is a vector bundle over $P(E)$. The restriction of $x$ to each fiber hence defines a class in $H^2(P(E)_{\ell_p})$. Now each fiber of $P(E)$ is a copy of $\mathbb{C}P^{n-1}$, which has as its cohomology linear combinations of $\alpha^i$ for $i \leq n-1$ with $\alpha \in H^2(\mathbb{C}P^{n-1})$ a generator. Hence the cohomology classes $1, x, \ldots, x^{n-1}$ are global forms that freely generate the fiber cohomology, therefore by the Leray-Hirsch Theorem 1.5.1 we conclude that $H^*(P(E))$ is a free module over $H^*(M)$ with basis $\{1, x, \ldots, x^{n-1}\}$;

$$H^*(P(E)) \simeq H^*(M) \otimes H^*(\mathbb{C}P^{n-1}).$$

Now while $x^n$ is trivial restricted to each fiber, it is not necessarily trivial in $H^n(P(E))$. But by Leray-Hirsch it can be written as a linear combination of $x^i$'s. In other words there are classes $c_i(E) \in H^{2i}(M)$ such that,

$$x^n + \pi^* c_1(E) x^{n-1} + \cdots + \pi^* c_n(E) = 0. \tag{3.1}$$

In other words,

$$H^*(P(E)) = \frac{\mathbb{R}[x]}{I},$$

with $I$ the ideal generated by $x^n + \pi^* c_1(E) x^{n-1} + \cdots + \pi^* c_n(E)$. Note that the $\pi^*$ makes sure all the terms are in $H^*(P(E))$, but usually for notational brevity we omit the $\pi^*$'s and simply write $x^n + c_1(E)x^{n-1} + \cdots + c_n(E) = 0$. We define $c_i(E)$ to be the $i$-th *Chern class of $E$* and we define the *total Chern class*

$c(E)$ as

$$c(E) = 1 + c_1(E) + \cdots + c_n(E).$$

**Proposition 3.2.4:** The first Chern class of a complex line bundle $L \to M$ is $e(L_{\mathbb{R}})$. ▲

Proof: We have $P(L) = M$, since the projectivication of a line is a point. The universal subbundle of $P(L)$ is then $L$ itself, giving $x = -e(L_{\mathbb{R}})$ and hence that $c_1(L) = e(L_{\mathbb{R}})$. □

If we let $E = M \times \mathbb{C}^n$ be a trivial bundle we have $P(E) = M \times \mathbb{C}P^{n-1}$, which means $x^n = 0$ by the ring structure of $\mathbb{C}P^{n-1}$. Hence all the Chern classes of a trivial bundle are zero. This gives rise to the interpretation of the Chern classes being a measure of how the bundle is twisted. Now just like the Euler class the Chern classes behave naturally with respect to smooth maps:

**Proposition 3.2.5:** (Naturality) Let $f : M \to N$ be a map between manifolds and $E$ be a complex vector bundle over $M$. Then $c_i(f^*E) = f^*c_i(E)$ for all $i$. ▲

Proof: Note that the projectivication and universal subbundle are defined fiberwise. Then since the fiber of $f^*E$ at $p$ is simply $E|_{f(p)}$, we see that $P(f^*E) = f^*PE$ and similarly the universal subbundle of $f^*PE$ is simply $f^*S$, where $S$ is the universal subbundle of $PE$. Therefore if we take $x_E$ and $x_{f^*E}$ the generators of the respective fiber cohomology rings we have that

$$x_{f^*E} = -c_1(f^*S) = -e((f^*S)_{\mathbb{R}}) = -f^*e(S_{\mathbb{R}}) = f^*x_E,$$

where the fact that $(f^*S)_{\mathbb{R}} = f^*(S_{\mathbb{R}})$ follows naturally from the definition of the pullback bundle. Hence if we apply $f^*$ to equation (3.1) we get,

$$x^n_{f^*E} + f^*c_1(E)x^{n-1}_{f^*E} + \cdots + f^*c_n(E) = 0.$$

This means per definition that $c_i(f^*E) = f^*c_i(E)$. □

Recall from Proposition 2.3.5 that isomorphic vector bundles have up to orientation reversal the same Euler class. There is a similar result for Chern classes:

**Proposition 3.2.6:** Let $E$ and $F$ be two isomorphic rank-$n$ orientable complex vector bundles over $M$. If $E$ and $F$ have the same orientation then $c_i(E) = c_i(F)$, if they have reversed orientations then $c_i(E) = (-1)^{n-i}c_i(F)$. ▲

Proof: Let $f : E \to F$ be a bundle isomorphism. This bundle isomorphism then induces a bundle isomorphism $\tilde{f} : S_E \to S_F$ on the universal subbundles of $\pi : PE \to M$ and $\rho : PF \to M$. Hence we have that the generators of $x \in H^2(PE)$ and $y \in H^2(PF)$ are related by,

$$f^*y = -e(f^*(S_F)_{\mathbb{R}}) = \mp e((S_E)_{\mathbb{R}}) = \pm x.$$

Here the $\pm$ is understood to be $+1$ for $f$ orientation preserving, and $-1$ for $f$ orientation reversing. We furthermore have that $\pi = \rho \circ f$. Hence

$$\begin{aligned} 0 &= y^n + \rho^* c_1(F) y^{n-1} + \cdots + \rho^* c_n(F) \\ f^*(0) &= f^* y^n + f^*(\rho^* c_1(F) y^{n-1}) + \cdots + f^*(\rho^* c_n(F)) \\ &= x^n + \pi^* c_1(F)(\pm x)^{n-1} + \pi^* c_2(F)(\pm x)^{n-2} + \cdots + \pi^* c_n(F). \end{aligned}$$

From which we conclude that $c_i(E) = (\pm 1)^{n-i} c_i(F)$ as required. □

## 3.3 The splitting principle

In this section we will give a construction that makes it easier to prove polynomial identities of Chern classes. Consider for example the Whitney product formula:

$$c(A \oplus B) = c(A)c(B).$$

The proof of this identity proceeds by finding a bundle $F(E) \to M$ together with a map $\sigma : F(E) \to M$ such that $\sigma^* E$ is a direct sum of line bundles, and $\sigma^* : H^*(M) \to H^*(F(E))$ is injective. This is useful because suppose we have such a $\sigma$, then

$$\sigma^* c(A \oplus B) = c(\sigma^*(A \oplus B)) = c(L_1 \oplus \cdots \oplus L_n).$$

with $L_i$ line bundles. Then by injectivity of $\sigma^*$ it suffices to show the identity only for direct sums of line bundles, which may be substantially easier than for the general case.

**Proposition 3.3.1:** (Splitting principle) For a given complex vector bundle $\rho : E \to M$ such a $\sigma$ and $F(E)$ always exist. The space $F(E)$ is called a *split manifold* of $E$. ▲

Proof: The proof proceeds by induction on the rank of $E$. If $E$ has rank 1 then the result is trivial. Suppose $E$ has rank-2 then $\sigma : P(E) \to M$ is a split manifold since there is the tautological exact sequence:

$$0 \longrightarrow S(E) \longrightarrow \sigma^* E \longrightarrow Q(E) \longrightarrow 0, \tag{3.2}$$

where $S(E)$ and $Q(E)$ are respectively the universal subbundle and the quotient bundle of $P(E)$, and $\sigma^* E$ is the pullback bundle. By the splitting Lemma 3.3.2 below we then have that $\sigma^* E = S(E) \oplus Q(E)$. Note that $S_E$ has rank 1 by definition and $\sigma^* E$ has $E_p$ as fibers and is hence rank 2, hence $Q_E$ has rank 1 and $\sigma^* E$ decomposes into line bundles.

In general let $E \to M$ be a rank-$k$ complex vector bundle. Then consider again $\sigma : P(E) \to M$ with tautological short exact sequence (3.2). Then since $S(E)$ has rank 1 and $\sigma^* E$ has rank $k$, $Q(E)$ has rank $k-1$. Now consider $\sigma : P(Q(E)) \to P(E)$ with exact sequence

$$0 \longrightarrow S(Q(E)) \longrightarrow \sigma^*(Q(E)) \longrightarrow Q^2(E) \longrightarrow 0\ .$$

Since $\sigma^* E = S(E) \oplus Q(E)$ we have,

$$(\sigma^*)^2 E = \sigma^*(S(E) \oplus Q(E)) = Q^2(E) \oplus S(Q(E)) \oplus \sigma^* S(E).$$

Here $(\sigma^*)^2 E$ has rank $k$, $Q^2(E)$ has rank $k-2$ and $S(Q(E))$ and $\sigma^* S(E)$ both have rank 1. This procedure then continues with induction until we hit a $Q^{k-1}(E)$ of rank 1. That is, consider the following diagram:

$$\begin{tikzcd}
E \arrow[d] & Q(E) \arrow[d] & Q^2(E) \arrow[d] & & Q^{k-2}(E) \arrow[d] & Q^{k-1}(E) \arrow[d] \\
M & P(E) \arrow[l, "\sigma"'] & P(Q(E)) \arrow[l, "\sigma"'] & \cdots \arrow[l, "\sigma"'] & P(Q^{k-3}(E)) \arrow[l, "\sigma"'] & P(Q^{k-2}(E)) \arrow[l, "\sigma"'] \\
 & \sigma^* E \arrow[u] & (\sigma^*)^2 E \arrow[u] & & (\sigma^*)^{k-3} E \arrow[u] & (\sigma^*)^{k-2} E \arrow[u]
\end{tikzcd}$$

By the short exact sequence

$$0 \longrightarrow S(Q^p(E)) \longrightarrow \sigma^*(Q^p(E)) \longrightarrow Q^{p+1}(E) \longrightarrow 0,$$

we have that

$$\sigma^* Q^p(E) = Q^{p+1}(E) \oplus S(Q^{p-1}(E)).$$

This inductively leads to,

$$(\sigma^*)^{k-2} E = Q^{k-1}(E) \bigoplus_{p=0}^{k-2} (\sigma^*)^p S(Q^p(E))$$

which is a direct sum of line bundles, hence $\sigma^k : P(Q^{k-2}(E)) \to M$ is a split manifold. The fact that $(\sigma^k)^* : H^*(M) \to H^*(P(Q^{k-2}(E)))$ is an injection follows immediately from the fact that $\sigma$ is a projection map, and we conclude the proof. □

**Lemma 3.3.2:** (Splitting lemma for vector bundles) Suppose we have a short exact sequence of real or complex vector bundles of finite rank,

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ j\ } C \longrightarrow 0,$$

then $B \simeq A \oplus C$. ▲

Proof: It is a standard result that by the existence of partitions of unity we can endow the vector bundles with a metric (Riemannian or Hermitian). Since we assume the bundles to be of finite rank we can define the orthogonal complement $(\operatorname{im} i)^\perp$ of $\operatorname{im} i$. This gives,

$$B = \operatorname{im} i \oplus (\operatorname{im} i)^\perp = \operatorname{im} i \oplus (\ker j)^\perp = \operatorname{im} i \oplus \frac{B}{\ker j} \simeq A \oplus C,$$

where the second equality follows from exactness and the last equality follows from surjectivity of $j$ and injectivity of $i$. □

As promised we will now go on to prove the Whitney product formula.

**Proposition 3.3.3:** (Whitney product formula) Let $A$ and $B$ be two complex vector bundles. Then,

$$c(A \oplus B) = c(A)c(B)$$

In other words, the total Chern class of the direct sum bundle is the product of the total Chern classes. ▲

Proof: By Proposition 3.3.1 and the remark at the beginning of this section it suffices to prove the identity for direct sums of line bundles. Therefore let $E = L_1 \oplus \cdots \oplus L_n$ be a vector bundle over $M$ for some line bundles $L_i$ and consider $S(E)$. For a line bundle $L_i \to M$ we have $P(L_i)|_p = \{\ell_{i,\,p}\}$ and hence $S(L_i) \simeq L_i$, with $\ell_{i,\,p}$ a line in $L_i|_p$. This way we obtain projections $s_i : S(E) \to L_i$. Since for every point in $P(E)$ this gives a map $S(E) \to L_i$ we can see each $s_i$ as a section of $\mathrm{Hom}(S(E),\, L_i) = S(E)^* \otimes L$ with $S(E)^*$ the fiberwise dual bundle of $S(E)$. Define

$$U_i = \{y \in P(E)|s_i(y) \neq 0\}.$$

Then restricted to $U_i$ the bundle $S^* \otimes L_i$ admits a non-vanishing section, and consequently has a vanishing Euler class. Since $S^* \otimes L_i$ is a line bundle, we then have $c_1(S^* \otimes L_i) = e(S^* \otimes L_i) = 0$. Also note that all the $s_i$ can not be zero at the same time, so $\{U_i\}$ is actually an open cover of $P(E)$. Now the crux comes in considering the product

$$\prod_{i=1}^{n} c_1(S^* \otimes L_i).$$

Since at least one $c_1(S^* \otimes L_i)$ is exact on each point in $P(E)$ by the fact that $\{U_i\}$ is a cover, one would expect the product to become trivial as well. However this is not enough for the product to be exact, since the product of a closed and exact form is not necessarily exact. Therefore we want $c_1(S^* \otimes L_i)$ to vanish identically (i.e. on the level of forms) on $V_i$ where $\{V_i\}$ is an open cover. In that case we do get that the product is identically zero.

We will make this a bit more precise. We have that $c_1(S^* \otimes L_i)|_{U_i} = d\omega_i$ for some $d\omega_i \in H^2(U_i)$. Now we wish to extend $d\omega_i$ to the whole of $P(E)$ so that the difference $c_1(S^* \otimes L_i) - d\omega_i$ vanishes on $U_i$. For this extension we need the following construction:

**Lemma 3.3.4:** Let $\{U_i\}$ be an open cover of a manifold $M$. Then there exists an open cover $\{V_i\}$ of $M$ such that $\overline{V_i} \subset U_i$ for each $i$, and smooth functions $\rho_i : M \to [0,1]$ for which $\mathrm{Supp}\, \rho_i \subset U_i$ and $\rho_i|_{V_i} = 1$. ▲

Proof: The statement about existence of such $V_i$ is exactly the shrinking lemma as proven in [Cra14, Lemma 5.17]. For the second statement we note that $\{U_i,\, M \setminus \overline{V_i}\}$ is an open cover for any fixed $i$. Let $\{\rho_i,\, \psi_i\}$ be a partition of unity subordinate to this cover. Then $\mathrm{Supp}\, \rho_i \subset U_i$ and $\rho_i|_{V_i} = 1$, since $\rho_i + \psi_i = 1$ and $\psi_i = 0$ on $V_i$ per definition. Hence $\rho_i$ satisfies the requirements. □

Using these $\rho_i$ we can take $c_1(S^* \otimes L_i) - d\rho_i\omega_i$ which is the same in cohomology a $c_1(S^* \otimes L_i)$ but vanishes identically on $V_i$. Furthermore we have for $L_1$, $L_2$ line bundles, $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ [Hat09, p.86]. Hence $c_1(S^* \otimes L_i) = c_1(S^*) + c_1(L_i)$. Since the identity map is a nowhere vanishing section of

$\mathrm{Hom}(S, S) = S^* \otimes S$ we have $0 = c_1(S^* \otimes S) = c_1(S^*) + c_1(S)$ and hence $c_1(S^*) = -c_1(S) = x$. Finally since $c_1(S^* \otimes L_i) - d\rho_i\omega_i$ and $c_1(S^* \otimes L_i)$ represent the same form in cohomology we have,

$$0 = \prod_{i=1}^{n} (c_1(S^* \otimes L_i) - d\rho_i\omega_i) = \prod_{i=1}^{n} c_1(S^* \otimes L_i) = \prod_{i=1}^{n} (c_1(S^*) + c_1(L_i)) = \prod_{i=1}^{n} (x + c_1(L_i)). \quad (3.3)$$

The right hand side is now a polynomial of degree $n$ and the coefficients of $x^i$ are precisely the Chern classes $c_i(E)$. Furthermore the total Chern class of each $L_i$ is per definition $1 + c_1(L_i)$ and the total Chern class $c(E)$ is obtained by taking $x \mapsto 1$ in the polynomial above. Hence we get,

$$c(E) = \prod_{i=1}^{n} (1 + c_1(L_i)) = \prod_{i=1}^{n} c(L_i).$$

Proving the Whitney sum formula for (direct sums of) line bundles, and by the splitting principle also for arbitrary complex vector bundles. □

**Corollary 3.3.5:** Each Chern class $c_i(E)$ of a rank-$k$ bundle $E$ is a $\mathbb{Z}$-linear combination of wedge products of Euler classes of line bundles. More specifically, if $E = f^*(L_1 \oplus \cdots \oplus L_k)$, then

$$c_i(E) = \sigma_i(e(L_1), \ldots, e(L_n)).$$

Where $\sigma_i$ is the $i$-th symmetrical polynomial in $n$ arguments. ▲

Proof: We have that,

$$0 = \prod_{i=1}^{n} (x + c_1(L_i)) = \prod_{i=1}^{n} (x + e(L_i)).$$

which per definition of $\sigma_i$ means that the coefficient of $x^i$ is precisely given by $\sigma_i(e(L_1), \ldots, e(L_n))$. □

**Corollary 3.3.6:** Let $E$ and $F$ be respectively rank-$m$ and $n$ complex vector bundles. If we look at Equation (3.3) we not only have $c(E \oplus F) = c(E)c(F)$ but we actually have equality in every degree for the equation:

$$x^{m+n} + c_1(E \oplus F)x^{m+n-1} + \cdots + c_{m+n}(E \oplus F) = (x^m + c_1(E)x^{m-1} + \cdots + c_m(E)) \cdot (x^n + c_1(F)x^{n-1} + \cdots + c_n(F))$$

Hence we obtain

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E)c_j(F)$$ ▲

As an important application of the Whitney product formula we will compute the total Chern class of the projective space.

**Proposition 3.3.7:** The total Chern class of the complex projective space is given by $c(\mathbb{C}P^n) = (1+x)^{n+1}$ with $x = -c_1(\gamma^n)$, where $\gamma^n$ is the canonical line bundle over $\mathbb{C}P^n$ as defined in Proposition 2.3.13. ▲

Proof: The most important ingredient in the proof will be to show that the holomorphic tangent bundle $T = T\mathbb{C}P^n$ is isomorphic to $Q \otimes \gamma^{n*}$. Here we let $Q$ denote the vector bundle obtained as the quotient of $\mathbb{C}^{n+1}$ by $\gamma^n$, which is analogous to the construction of quotient bundle of a projectivication. We will proceed to show the identity fiberwise. Let $\ell \in \mathbb{C}P^n$, then $\gamma^n_\ell = \ell \subset \mathbb{C}^{n+1}$ and $Q_\ell = \ell^\perp \subset \mathbb{C}^{n+1}$, meaning $Q_\ell \otimes \gamma^{n*}_\ell$ boils down to $\mathrm{Hom}(\ell, \ell^\perp)$ the set of linear maps from $\ell$ to its orthogonal complement. To see that this is the same as the holomorphic tangent space $T_\ell$ at $\ell$ we take a closer look at the definition of $T_\ell$. Per definition $T_\ell$ is the vector space of derivations of holomorphic functions at $\ell$. Let $h$ be a holomorphic function on $\mathbb{C}P^n$, then $h$ induces a holomorphic function on $\mathbb{C}^{n+1}$ by,

$$h(x) = \begin{cases} 0 & x = 0, \\ ||x|| h(\ell_x) & \text{otherwise.} \end{cases}$$

where $\ell_x$ is the line through $x$ and the origin. Now $T_x\mathbb{C}^{n+1}$ is simply given by the set of derivations $s$ that for a given $v \in \mathbb{C}^{n+1}$ send a holomorphic function $f$ to,

$$s(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(x + vt)\Big|_{t=0}.$$

Suppose that $v \in \ell$, then for a holomorphic function $f$ on $\mathbb{C}^{n+1}$ induced from one on $\mathbb{C}P^n$ as above we have,

$$s(f) = \frac{\mathrm{d}}{\mathrm{d}t} f(x + vt)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} ||x + vt|| f(\ell)\Big|_{t=0} = 0.$$

Hence only the component of $v$ orthogonal to $\ell$ determines the action of the derivative. Furthermore any derivation $s \in T_x\mathbb{C}^{n+1}$ determines a derivation $\tilde{s}$ in $T_{\ell_x}$ by $\tilde{s}(h) = s(h)$ where on the right side we have the induced function on $\mathbb{C}^{n+1}$. Now suppose we have a homomorphism $L : \ell \to \ell^\perp$, then for each $p \in \ell$ the vector $L(p)$ defines a derivation $s_{L(p)}$, and each of these derivations then determine the same derivation $s_L$ in $T_\ell$. We can see that this correspondence gives an isomorphism $\mathrm{Hom}(\ell, \ell^\perp) \simeq T_\ell$ as required. From this we indeed see that $Q \otimes \gamma^{n*} \simeq T$.

Over $\mathbb{C}P^n$ we have the following tautological exact sequence,

$$0 \longrightarrow \gamma^n \longrightarrow \mathbb{C}^{n+1} \longrightarrow Q \longrightarrow 0\ .$$

If we tensor all the elements with $\gamma^{n*}$ we get sequence,

$$0 \longrightarrow \gamma^{n*} \otimes \gamma^n \longrightarrow \gamma^{n*} \otimes \mathbb{C}^{n+1} \longrightarrow \gamma^{n*} \otimes Q \longrightarrow 0\ ,$$

which is exact by Proposition 1.5.2. Now $\gamma^{n*} \otimes \gamma^n$ admits a section given by the identity map, hence $c(\gamma^{n*} \otimes \gamma^n) = 1$. Furthermore we have $c(\gamma^{n*} \otimes \mathbb{C}^{n+1}) = c(\gamma^{n*} \otimes \gamma^n)c(\gamma^{n*} \otimes Q) = c(\gamma^{n*} \otimes Q)$ by the Whitney product formula. But $\gamma^{n*} \otimes \mathbb{C}^{n+1} = \bigoplus_{i=1}^{n+1} \gamma^{n*}$ and $\gamma^{n*}$ has total Chern class $1 + x$. If we combine this we get,

$$c(\mathbb{C}P^n) = c(T) = c(\gamma^{n*} \otimes Q) = (1 + x)^{n+1}$$

giving the required result. □

Using the interpretation of $T\mathbb{C}P^n$ provided in this proof we can finally prove the transversality of the intersection of $\mathbb{C}P^{n-1}$ and $\mathbb{C}P^1$ embedded in $\mathbb{C}P^n$ as was needed in the proof of Proposition 3.1.2.

**Corollary 3.3.8:** If we embed $\mathbb{C}P^1$ into $\mathbb{C}P^n$ as the space spanned by $[x_1, x_2, 0, \dots, 0]$, and embed $\mathbb{C}P^{n-1}$ into $\mathbb{C}P^n$ as the space spanned by $[x_1, 0, x_3, x_4, \dots, x_n]$, then $\mathbb{C}P^1$ and $\mathbb{C}P^{n-1}$ intersect transversally at the point $[1, 0, \dots, 0]$. ▲

Proof: We have to show that $T\mathbb{C}P^1 \oplus T\mathbb{C}P^{n-1}|_\ell = T\mathbb{C}P^n|_\ell$ with $\ell = [1, 0, \dots, 0]$. We know that the latter is given by $\ell^\perp \otimes \ell^* = \langle 0, x_1, x_2, \dots, x_n \rangle \otimes \ell^*$ by the proof of the previous proposition, where we use $\langle \cdot \rangle$ to denote the span with a bit of abuse of notation. On the other hand $T\mathbb{C}P^1$ has tangent space $\langle 0, 0, x_3, \dots, x_n \rangle \otimes \ell^*$ by the same argument. Similarly $T\mathbb{C}P^{n-1}$ has tangent space $\langle 0, x_2, \dots, 0 \rangle \otimes \ell^*$. If we then take the direct sum of $T\mathbb{C}P^1$ and $T\mathbb{C}P^{n-1}$ we get the required result. □

## 3.4 Pontryagin classes

The Chern classes are useful tools for studying complex vector bundles, but when dealing with (real) smooth manifolds we may be more interested in real vector bundles and their invariants. For example, the tangent space of a smooth manifold is a very important real vector bundle. In this section we will define a real counterpart of Chern classes called the *Pontryagin classes* $p_i(E)$ of a (real) vector bundle $E$.

We obtain $p_i(E)$ from constructing a complex vector bundle out of $E$ called the complexification $E^\mathbb{C}$ of $E$. The complexification is nothing more than formally extending the scalar multiplication to include complex numbers. In other words we take,

$$E^\mathbb{C} = E \otimes_\mathbb{R} \mathbb{C}.$$

If $E$ has transition functions $g_{\alpha\beta}$ then $E^\mathbb{C}$ has transition functions $g_{\alpha\beta} \otimes 1_\mathbb{C}$ and similarly a map $f : E \to F$ induces a map $f^\mathbb{C} : E^\mathbb{C} \to F^\mathbb{C}$ given by $f^\mathbb{C}(v \otimes z) = f(v) \otimes z$.

From a complex vector space $V$ we can construct its conjugate vector space $\overline{V}$ by taking $zv \mapsto \overline{z}v$, that is, taking the conjugate of scalar multiplication. With this we really mean that for a basis $\{e_i\}$ and $z_i \in \mathbb{C}$ we have $z_1 e_1 + \cdots + z_n e_n \mapsto \overline{z_1} e_1 + \cdots + \overline{z_n} e_n$.

If we take the conjugate fiberwise we can naturally construct the conjugate bundle $\overline{E}$ of $E$. The transition functions $g_{\alpha\beta}$ on $E$ then become $\overline{g_{\alpha\beta}}$ on $\overline{E}$. Introducing a Hermitian structure on a complex vector bundle $E$ we can reduce its structure group to the group of unitary matrices [BT82, p.267]. For unitary matrices we have $(g_{\alpha\beta}^t)^{-1} = \overline{g_{\alpha\beta}}$ where the left side denotes the inverse of the transpose. Coincidentally the dual bundle $E^*$ has transition functions $(g_{\alpha\beta}^t)^{-1}$ [BT82, p.56], and we conclude that $\overline{E} \simeq E^*$.

Note that elements of $V^\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ are of form $v_1 \otimes 1 + v_2 \otimes i$ with $v_i \in V$. This gives the natural isomorphism $V^\mathbb{C} \to \overline{V^\mathbb{C}}$ given by

$$v_1 \otimes 1 + v_2 \otimes i \mapsto v_1 \otimes 1 - v_2 \otimes i.$$

In other words $V^\mathbb{C}$ is isomorphic to its conjugate, and hence also to its dual. This has an important implication on the Chern classes of a complexified real vector bundle $E^\mathbb{C}$. This isomorphism preserves the transition functions $g_{\alpha\beta} \otimes \mathbb{C}$ since $g_{\alpha\beta}$ is real. Hence $E^\mathbb{C}$ is isomorphic to $\overline{E}^\mathbb{C}$ in an orientation preserving manner. Recall that $c_1(S(E^\mathbb{C})) = -c_1(S((E^\mathbb{C})^*))$ and hence in the polynomial definition the Chern classes we get $x \mapsto -x$ under conjugation. This then means that $c_i(E^\mathbb{C}) = (-1)^i c_i(\overline{E^\mathbb{C}})$, since this isomorphism is orientation preserving we also have $c_i(E^{\mathbb{C}}) = c_i(\overline{E^{\mathbb{C}}})$, hence $c_i(E^{\mathbb{C}}) = 0$ for $i$ odd. This leads us to make the following definition:

**Definition 3.4.1:** The $i$-th Pontryagin class $p_i(E) \in H^{4i}(M)$ of a real vector bundle $E \to M$ is given by

$$p_i(E) = (-1)^i c_{2i}(E^{\mathbb{C}}).$$

That is, the $2i$-th Chern class of the complexification of $E$. The *total Pontryagin class* is then given by

$$p(E) = 1 + p_1(E) + \cdots p_n(E) = 1 - c_2(E^{\mathbb{C}}) + c_4(E^{\mathbb{C}}) - c_6(E^{\mathbb{C}}) + \cdots + (-1)^n c_{2n}(E^{\mathbb{C}})$$

The $(-1)^i$ is to make some formulas related to the Pontryagin class look simpler, and some authors (like [BT82]) don't use this convention. ▲

**Proposition 3.4.2:** (Whitney product formula) The Whitney product formula also holds for Pontryagin classes. That is, for $E$ and $F$ real vector bundles we have

$$p(E \oplus F) = p(E)p(F).$$ ▲

PROOF: We will follow [MS74, p. 175]. Let $E$ and $F$ be respectively rank-$m$ and $n$ real vector bundles. Note that $(E \oplus F)^{\mathbb{C}} = E^{\mathbb{C}} \oplus F^{\mathbb{C}}$. From Corollary 3.3.6 and the fact that all the odd Chern classes vanish we have that

$$c_{2k}(E^{\mathbb{C}} \oplus F^{\mathbb{C}}) = \sum_{i+j=k} c_{2i}(E^{\mathbb{C}}) c_{2j}(F^{\mathbb{C}}).$$

Now multiplying both sides with $(-1)^k = (-1)^i(-1)^j$ we obtain

$$(-1)^k c_{2k}(E^{\mathbb{C}} \oplus F^{\mathbb{C}}) = \sum_{i+j=k} (-1)^i c_{2i}(E^{\mathbb{C}})\ (-1)^j c_{2j}(F^{\mathbb{C}}),$$

which precisely leads to $p(E \oplus F) = p(E)p(F)$. □

We define the Pontryagin class of a manifold $M$ to be that of its tangent bundle. Note that for a given real manifold $M$ of dimension $4k$ the tangent bundle is of rank $4k$ and hence the $i$-th Pontryagin class $p_i(M)$ is a $4i$ form. Hence if we have a set of positive integers $I = \{i_1, \ldots, i_r\}$ with $\sum_j i_j = k$ then the wedge product $\bigwedge_j p_{i_j}(M)$ defines a form of degree $4k$. This leads to the following definition.

**Definition 3.4.3:** Let $I = \{i_1, \ldots, i_r\}$ be a set of positive integers with sum $\sum_j i_j = k$. We call $I$ a *partition of* $k$. Given a $4k$-dimensional manifold $M$ we call the $I$-th Pontryagin number $p_I(M)$ of $M$ the real number given by,

$$p_I(M) = \int_M p_{i_1}(M) \wedge \cdots \wedge p_{i_r}(M).$$ ▲

**Proposition 3.4.4:** All the Pontryagin numbers of a manifold $M$ are integer. ▲

PROOF: By Corollary 3.3.5 and the splitting principle we have that each $p_i(E)$ can be written as a $\mathbb{Z}$ linear combination of wedge products of Euler classes of line bundles. If $M$ is $n$-dimensional and

$I = \{i_1, \dots, i_r\}$ is a partition of $n$, then $p_{i_1} \wedge \dots \wedge p_{i_r}$ is a linear combination of Euler classes of some set of line bundles $L_{j_i}$:

$$p_{i_1} \wedge \cdots \wedge p_{i_r} = \sum_j n_j e(L_{j_1}) \wedge \cdots \wedge e(L_{j_n}) = \sum_j n_j e(L_{j_1} \oplus \cdots \oplus L_{j_n}),$$

where $n_j \in \mathbb{Z}$. The last step follows from the Whitney product formula for Euler classes. Each $e(L_{j_1} \oplus \cdots \oplus L_{j_n})$ then integrates to an integer by Theorem 2.3.11, and hence the Pontryagin number is integer for each partition of $n$. □

**Remark 3.4.5:** We can also define the Pontryagin class of a complex vector bundle $E$ with fiber $V$. This is done by first taking the realification $V_{\mathbb{R}}$ of the fibers, i.e. by taking the bundle obtained by 'throwing away' the complex structure under the isomorphism $\mathbb{C} \simeq \mathbb{R}^2$ sending a basis $z_i = x_i + iy_i$ to a basis $(x_i, y_i)$. Then consider the complexification $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ of $V_{\mathbb{R}}$. The map given by multiplication by $i$ on $V$ gives a linear transformation $J$ on $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ with $J^2 = -1$. Now $J$ has eigenvalues $\pm i$ and decomposes into a direct sum of of the $i$-eigenspace and $(-i)$-eigenspace. Note that the $i$-eigenspace is contained in $V$ and the $(-i)$-eigenspace is contained in the conjugate vector space $\overline{V}$. Since $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ has twice the dimension of $V$ we conclude,

$$V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = V \oplus \overline{V},$$

and hence also $E_{\mathbb{R}}^{\mathbb{C}} = E \oplus \overline{E}$. ▲

**Example 3.4.6:** The $n$-sphere $S^n$ has trivial Pontryagin classes, i.e. $p(S^n) = 1$. ▲

Proof: Embed $S^n$ in $\mathbb{R}^{n+1}$. We have the following exact sequence of vector bundles

$$0 \longrightarrow TS^n \longrightarrow T\mathbb{R}^{n+1}|_{S^n} \longrightarrow N \longrightarrow 0\ .$$

Here $N$ is the normal bundle obtained by taking the orthogonal complement of $TS^n$ in $T\mathbb{R}^{n+1}|_{S^n}$. From geometrical reasons we see $N \simeq S^n \times \mathbb{R}$, whereas $T\mathbb{R}^{n+1}|_{S^n}$ is also trivial. Hence these two bundles have trivial Chern class. By the Whitney product formula we now have

$$c(TS^{n,\mathbb{C}}) = c(N)c(TS^{n,\mathbb{C}}) = c(T\mathbb{R}^{n+1}|_{S^n}) = 1,$$

from which we conclude $p(S^n) = 1$. □

**Proposition 3.4.7:** The total Pontryagin class of the complex projective space $\mathbb{C}P^n$ is given by

$$p(\mathbb{C}P^n) = (1 + x^2)^{n+1},$$

where $x = -c_1(S)$ is minus the Euler class of the universal subbundle of $\mathbb{C}P^n$. ▲

Proof: By Proposition 3.3.7 we know $c(\mathbb{C}P^n) = c(T\mathbb{C}P^n) = (1 + x)^{n+1}$. We have,

$$(1 - x^2)^{n+1} = c(\mathbb{C}P^n)c(\overline{\mathbb{C}P^n}) = c((\mathbb{C}P^n)_{\mathbb{R}}^{\mathbb{C}}) = \sum_{i=0}^{n} c_{2i}((\mathbb{C}P^n)_{\mathbb{R}}^{\mathbb{C}}) = \sum_{i=0}^{n} (-1)^i p_i(\mathbb{C}P^n).$$

Since we get equality in all degrees of cohomology on the left and right side, we can replace all terms $\omega$

of order $4i$ in cohomology by $(-1)^i\omega$ and retain equality. Since $x^2$ is of degree 4, and $p_i(\mathbb{C}P^n)$ of degree $4i$ we get,

$$(1+x^2)^{n+1} = \sum_{i=0}^{n} p_i(\mathbb{C}P^n) = p(\mathbb{C}P^n)$$

as required. □

## 3.5 Flag manifolds

Next to the projectivication of a vector bundle, another interesting construction we can do on vector bundles is constructing its flag manifold. In order to define this construction we first need some preliminary definitions.

**Definition 3.5.1:** Let $V$ be a vector space. Then a *flag* in $V$ is a sequence of (vector) subspaces $A_1 \subset A_2 \subset \cdots \subset A_n = V$ where $\dim A_i = i$. The *flag manifold* $Fl(V)$ of $V$ is then the set of all flags in $V$. ▲

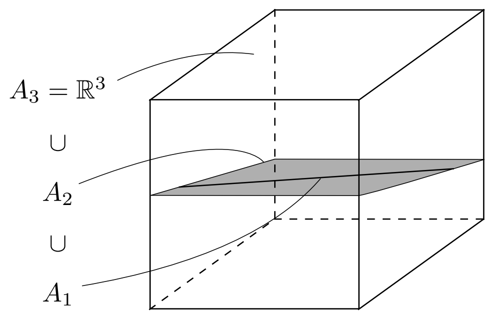



Figure 3.2: An example of a flag in $\mathbb{R}^3$. Note that all the subspaces must also contain the origin.

A flag can be thought of as an ordered basis of our vector space. That is, every $A_i$ is spanned by $A_{i-1}$ plus a single vector $e_i \in A_{i-1}^{\perp}$, and we see that $(e_1, \ldots, e_n)$ is then a basis for $V$, where $\dim V = n$. Now the fact that $Fl(V)$ is a manifold is clear when we consider that we can transform any flag into any other flag by an element of $GL(n, \mathbb{C})$ (if we assume $V = \mathbb{C}^n$, the real case is entirely analogous). In other words $GL(n, \mathbb{C})$ acts transitively on $Fl(V)$. If we fix a flag $A$ in $V$, we can define the surjective map

$$GL(n, \mathbb{C}) \xrightarrow{I} Fl(V): \qquad \varphi \longmapsto \varphi(A).$$

We now have by the first isomorphism theorem that

$$Fl(V) \simeq \frac{Gl(n, \mathbb{C})}{\ker I}.$$

Now by [War83, p. 120] the quotient of a Lie group by a closed topological subgroup is a Lie group, making $Fl(V)$ in particular into a manifold. The *flag bundle* of a vector bundle is then obtained by taking the flag manifold of the fibers.

**Proposition 3.5.2:** The flag bundle of a complex vector bundle $Fl(E)$ is precisely the *split manifold* $F(E)$. ▲

Proof: Suppose $E$ is rank-2, then on each fiber we would just have flags $\ell_p \subset E_p$, making $P(E) = Fl(E)$. But for rank-2 bundles $P(E)$ is precisely the split manifold, as we noted in the proof of Proposition 3.3.1.

Suppose now in general we have a rank-$n$ vector bundle $E$ and note that the projectivication $P(E)$ is on each fiber precisely the collection of all one-dimensional subspaces $\ell_p$ of $E_p$. For our two dimensional subspaces of our flag we look at $P(Q(E))$. Recall that $Q(E)$ is defined by the tautological exact sequence,

$$0 \longrightarrow S(E) \longrightarrow \sigma^* E \longrightarrow Q(E) \longrightarrow 0,$$

where $\sigma : P(E) \to M$ is the projection. For $\ell_1 \in P(E)|_p$, the fiber of $S(E)$ at $\ell_1$ is the set of points in $\ell_1$ seen as a line in $E_p$. The fiber at $\ell_1$ in $\sigma^* E$ is then the whole of $E_p \simeq \mathbb{C}^n$. Now $Q(E)|_{\ell_1} = \sigma^* E_p / S(E)|_{\ell_p} = \mathbb{C}^n/\ell_1$, in other words $Q(E)|_{\ell_1}$ is exactly the orthogonal complement of the line $\ell_1$. Note that now $P(Q(E))|_{\ell_1}$ is the set of all the lines orthogonal to $\ell_1$. Let $\ell_2 \in P(Q(E))|_{\ell_1}$, then we get the sequence $\ell_1 \subset (\ell_1, \ell_2) \subset E_p$ where with $(\ell_1, \ell_2)$ we denote the subspace of $E_p$ spanned by the two lines. In this way $P(Q(E))|_{\ell_1}$ defines all the two dimensional subspaces in which $\ell_1$ is contained, and hence $P(Q(E))$ is the set of all such *partial* flags that end at dimension 2 (i.e. sequences $A_1 \subset A_2 \subset E_p$).

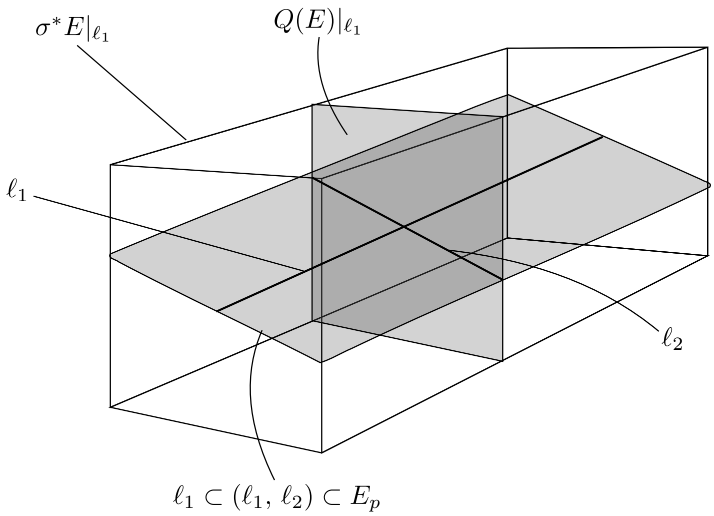



Figure 3.3: The schematic summary of the construction above.

This construction then generalizes easily to $P(Q^k(E))$ for $k < n-1$. For all $i < k$ choose some $\ell_{i+1} \in P(Q^{i-1}(E))|_{\ell_i}$ so that we get a sequence,

$$\ell_1 \subset (\ell_1, \ell_2) \subset \cdots \subset (\ell_1, \dots, \ell_k) \subset E_p.$$

By the exact sequence

$$0 \longrightarrow S(Q^{k-1}(E))|_{\ell_k} \longrightarrow \sigma^*(Q^{k-1}(E))|_{\ell_k} \longrightarrow Q^k(E)|_{\ell_k} \longrightarrow 0,$$

and the fact that $S(Q^{k-1}(E))|_{\ell_k} = \ell_k$ and $\sigma^*(Q^{k-1}(E))|_{\ell_k} = Q^{k-1}(E)|_{\ell_k}$ we have that $Q^k(E)|_{\ell_k}$ is the orthogonal complement of $\ell_k$ in $Q^{k-1}(E)|_{\ell_k}$. Therefore if we take some $\ell_{k+1} \in P(Q^k(E))|_{\ell_k}$ we immediately get that $\ell_{k+1}$ is orthogonal to all $\ell_i$ for $i \leq k$. Hence we can extend the sequence we got above,

$$\ell_1 \subset (\ell_1, \ell_2) \subset \cdots \subset (\ell_1, \ldots, \ell_{k+1}) \subset E_p.$$

It should be clear that $P(Q^k(E))|_{\ell_k}$ contains all such $\ell_{k+1}$ which would extend the sequence. Hence we conclude by induction that $P(Q^k(E))$ contains all such partial flags of length $k+1$, and if we extend the construction we conclude by induction that the flag bundle is the split manifold of $E$, that is $Fl(E) = P(Q^{n-2}(E)) = F(E)$. □

## 3.6 The universal bundle

Recall that under a pullback to the flag bundle a vector bundle becomes a sum of line bundles. Furthermore if we have $c_i(E) = 0$ then by naturality also $c_i(f^*E) = 0$ for any $f$. Furthermore the pullback of $E$ by a constant map is a trivial bundle. All of these are examples of the idea that taking pullbacks simplifies or 'untwists' the bundle. Recall that in the theory of covering spaces we have under mild conditions that a universal cover always exists, that is a cover which has the property that any other cover is covered by the universal cover. By a similar analog we will prove that there is a complex vector bundle called the *universal bundle* such that any bundle is the pullback of the universal bundle. To construct the universal bundle we will first construct a generalization of the projectivication of a vector bundle called the Grassmannian:

**Definition 3.6.1:** Let $V$ be a complex vector space of rank $n$. The *Grassmannian* $G_k(V)$ of $V$ is the set of all $(n-k)$-dimensional subspaces of $V$. The Grassmannian $G_k(E)$ of a complex vector bundle is then taken fiberwise. Note that $G_{n-1}(V)$ is the set of one-dimensional subspaces of $V$, which is exactly the projectivication $P(V)$ of $V$. ▲

We will use a similar argument as in the construction of $Fl(V)$ to determine the manifold structure of $G_k(V)$. Note that the group $U(n)$ of unitary matrices acts transitively on $G_k(V)$. Fix an $(n-k)$-dimensional subspace $A \subset V$. Any matrix in $U(n)$ that leaves $A$ invariant also fixes $A^\perp$. Furthermore the subgroup of unitary matrices under which $A$ is fixed is $U(n-k)$, whereas the subgroup that leaves $A^\perp$ fixed is $U(k)$. Now the same argument as we used for $Fl(V)$ we have that,

$$G_k(V) = \frac{U(n)}{U(n-k) \times U(k)}.$$

Hence $G_k(V)$ is a manifold by the fact that the quotient of a Lie group by a closed subgroup is a manifold [War83, p.120]. We can also associate a generalization of the universal subbundle $S(V)$ and universal quotient bundle $Q(V)$ to the Grassmannian. The fiber of $S_k(V) \to G_k(V)$ at a point $\Lambda \in G_k(V)$ is the set of points in the $(n-k)$-dimensional subspace $\Lambda$. The universal quotient bundle $Q_k(V)$ is then defined by the short exact sequence,

$$0 \longrightarrow S_k(V) \longrightarrow G_k(V) \times V \longrightarrow Q_k(V) \longrightarrow 0.$$

As a bundle over $G_k(V)$, the product bundle $G_k(V) \times V$ has rank $n$, hence we see that $S_k(V)$ has rank $n-k$ and $Q_k(V)$ has rank $k$. These notions are then also naturally extended fiberwise to the Grassmannians of complex vector bundles.

We will then construct the universal bundle by using the universal quotient bundle. First we need the following lemma:

**Lemma 3.6.2:** Let $E \to M$ be a rank-$k$ complex vector bundle. If $M$ admits a finite good cover then there is a finite set of smooth sections which span the fiber of $E$ at every point. ▲

PROOF: Let $\{U_i\}$ be a finite good cover of $M$. The restriction $E|_{U_i}$ is trivial by the fact that $U_i$ is contractible. Hence $E|_{U_i}$ admits sections $\{s_{i,1}, \ldots, s_{i,k}\}$ that span the fibers of $E|_{U_i}$. By Lemma 3.3.4 there exists a good cover $V_i$ with $\overline{V_i} \subset U_i$ and functions $f_i : M \to [0,1]$ with $\operatorname{Supp} f \subset U_i$ and $f_i|_{V_i} = 1$. Hence $\{f_i s_{i,1}, \ldots, f_i s_{i,k}\}$ are globally defined sections that span the fibers of $E|_{V_i}$, and therefore $\bigcup_i \{f_i s_{i,1}, \ldots, f_i s_{i,k}\}$ is a set of smooth sections satisfying the requirements of this lemma by the fact that $V_i$ is a cover. □

**Proposition 3.6.3:** Let $E \to M$ be a rank-$k$ complex vector bundle, and let $M$ admit a finite good cover and fiber spanning set of sections $\{s_1, \ldots, s_n\}$. Then there exists a map $f$ from $M$ to some Grassmannian $G_k(\mathbb{C}^n)$ such that $E$ is the pullback under $f$ of $Q_k(\mathbb{C}^n)$, that is, $E = f^* Q_k(\mathbb{C}^n)$. ▲

PROOF: Let $V$ be the complex vector space spanned by $\{s_1, \ldots, s_n\}$. Consider for $p \in M$ the evaluation map $\mathrm{ev}_p : V \to E_p$ given by $\mathrm{ev}_p(s_i) = s_i(p)$. This map is clearly surjective, hence $\operatorname{im} \mathrm{ev}_p = E_p$ meaning $\ker \mathrm{ev}_p = V/\operatorname{im} \mathrm{ev}_p$. Furthermore,

$$Q_k(V)|_{\ker \mathrm{ev}_p} = \frac{(G_k(V) \times V)|_{\ker \mathrm{ev}_p}}{S_k(V)|_{\ker \mathrm{ev}_p}} = \frac{V}{\ker \mathrm{ev}_p} = E_p.$$

We define a map $f : M \to G_k(V)$ given by $p \mapsto \ker \mathrm{ev}_p$;

$$\begin{tikzcd} E \arrow[d] & Q_k(V) \arrow[d] \\ M \arrow[r, "f"] & G_k(V) \end{tikzcd}$$

Now this $f$ pulls $Q_k(V)$ back to $E$ since $(f^* Q_k(V))_p = Q_k(V)_{f(p)} = E_p$. Hence $E = f^* Q_k(V)$ as required. Furthermore by the way the sections $\{s_1, \ldots, s_n\}$ are constructed in Lemma 3.6.2, we can for any rank-$k$ vector bundle associate the just constructed $V$ with $\mathbb{C}^n$ where $n = m \cdot k$ with $m$ the cardinality of the good cover. Hence we can actually find a map $\tilde{f} : M \to G_k(\mathbb{C}^n)$ such that $\tilde{f}^* Q_k(\mathbb{C}^n) = E$, proving the proposition. □

This result can be used to show that the isomorphism class of a complex vector bundle is completely determined by its Chern classes. Recall first that the pullbacks of two homotopic maps are isomorphic by Proposition 2.3.4. The converse statement; that maps whose pullbacks are isomorphic are also homotopic holds by [Hus75, 7.6], albeit with some restrictions:

**Lemma 3.6.4:** Let $M$ be a manifold of dimension $m$. Then for any $n \geq k + m/2$ and two maps $f, g : M \to G_k(\mathbb{C}^n)$, we have that if $f^* Q_k(\mathbb{C}^n) \simeq g^* Q_k(\mathbb{C}^n)$ then $f$ and $g$ are homotopic. ▲

We now write $\mathrm{Vect}_k(M, \mathbb{C})$ for the set of isomorphism classes of rank-$k$ complex vector bundles over $M$. We furthermore use $[M, G_k(\mathbb{C}^n)]$ to denote the set of homotopy classes of functions $M \to G_k(\mathbb{C}^n)$. We then have the following theorem:

**Theorem 3.6.5:** Let $M$ admit a finite good cover. Then for all $k$ there is an $n$ sufficiently large such that there is a bijective correspondence

$$\mathrm{Vect}_k(M,\,\mathbb{C}) \simeq [M,\,G_k(\mathbb{C}^n)]$$

between isomorphism classes of rank-$k$ vector bundles and homotopy classes of functions from $M$ into the complex Grassmannian $G_k(\mathbb{C}^n)$. ▲

Proof: First let $n \geq k + m/2$ as in Lemma 3.6.4. Let $[f] \in [M,\,G_k(\mathbb{C}^n)]$ be a homotopy class of functions. Then we define the map $\alpha : [M,\,G_k(\mathbb{C}^n)] \to \mathrm{Vect}_k(M,\,\mathbb{C})$ by $[f] \mapsto f^*Q_k(\mathbb{C}^n)$. This map is well-defined because for $[f] = [g]$ we have $f^*Q_k(\mathbb{C}^n) \simeq g^*Q_k(\mathbb{C}^n)$ by Lemma 2.3.4. The map is also injective for suppose $f^*Q_k(\mathbb{C}^n) = g^*Q_k(\mathbb{C}^n)$, then Lemma 3.6.4 asserts that the two maps are homotopic.

For the inverse map $\beta : \mathrm{Vect}_k(M,\,\mathbb{C}) \to [M,\,G_k(\mathbb{C}^n)]$ we will by Proposition 3.6.3 have for every $[E] \in \mathrm{Vect}_k(M,\,\mathbb{C})$ a map $f : M \to G_k(\mathbb{C}^n)$ such that $f^*Q_k(\mathbb{C}^n) \simeq E$. We then take $\beta[E] = [f]$. This map is furthermore well-defined because for $E \simeq F$ we have $f^*Q_k(\mathbb{C}^n) \simeq F \simeq E$, and if $f$ is homotopic to $g$ then $f^*Q_k(\mathbb{C}^n) \simeq g^*Q_k(\mathbb{C}^n)$ making the map injective as well. Finally per construction the two maps are clearly inverse to each other. We hence conclude the theorem. □

We can use this theorem to show that Chern classes characterize the vector bundle in the following sense. Suppose we have a transformation $T : \mathrm{Vect}_k(\,\cdot\,,\,\mathbb{C}) \to H^*(\,\cdot\,)$. Furthermore assume $T$ is natural in the sense that it respects naturality of both objects, i.e. $T(f^*E) = f^*T(E)$. Clearly taking any polynomial in Chern classes is such a transformation. In fact, we have the following theorem,

**Theorem 3.6.6:** Let $T$ be as above, then $T : \mathrm{Vect}_k(M,\,\mathbb{C}) \to H^*(M)$ can be written as the same polynomial in Chern classes for any $M$ admitting a good cover. ▲

Proof: Let $E$ be a rank-$k$ complex vector bundle, and let $f$ be such that $f^*Q_k(\mathbb{C}^n) = E$ for some $n$. We have that the cohomology of the Grassmannian is generated by the Chern classes of $Q_k(\mathbb{C}^n)$ [BT82, p.293]. We have,

$$T(E) = T(f^*Q_k(\mathbb{C}^n)) = f^*T(Q_k(\mathbb{C}^n)).$$

But now $T(Q_k(\mathbb{C}^n)) \in H^*(G_k(\mathbb{C}^n)$, and hence it is a polynomial in the Chern classes,

$$T(Q_k(\mathbb{C}^n)) = P_T(c_1(Q_k(\mathbb{C}^n)), \ldots,\, c_k(Q_k(\mathbb{C}^n))).$$

By naturality we now have,

$$T(E) = f^*T(Q_k(\mathbb{C}^n)) = f^*P_T(c_1(Q_k(\mathbb{C}^n)), \ldots,\, c_k(Q_k(\mathbb{C}^n))) = P_T(c_1(E), \ldots,\, c_k(E)).$$

The polynomial $P_T$ is clearly independent of $E$, proving the theorem. □

At the beginning of this part we introduced Chern classes as a somewhat more general object than the Euler class. We then went on to prove a wealth of properties of Chern classes and introduced the very useful splitting principle. Then we introduced the Pontryagin classes and numbers, which will turn out to be very important in the proof of existence of exotic spheres. Although its not required for the next part, we then finally introduced flag bundles and introduced the universal bundle. These two objects give a clear demonstration of the power of Chern classes when proving identities related to vector bundles.

# 4 Exotic spheres

In this part we will consider an application of the characteristic classes introduced in the previous section. Namely, we will prove the existence of an 'exotic' differential structure on $S^7$. That is, we will construct a space homeomorphic but not diffeomorphic to $S^7$. For this we first need to introduce some new formalisms. We will mainly follow Kreck's exposition of Milnor's original proof, as well as the proof provided by Milnor in his original paper (see [Kre10] and [Mil56] respectively). Most of the literature on this topic uses singular cohomology with $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ coefficients. We have chosen to use de Rham cohomology, which required some of the proofs to be adjusted.

## 4.1 Cobordism

In this section we will define a new equivalence relations on manifolds called cobordism. The contents of this section are mainly based on [MS74]. From Stokes' theorem we know that a manifold $\partial M$ being the boundary of another manifold $M$ has important consequences for integrating over $\partial M$. Namely if we restrict a on $M$ globally defined form $\omega$ to $\partial M$, then

$$\int_{\partial M} \omega = \int_M d\omega.$$

If $\omega$ is then closed, the integral becomes zero. For this reason and others one can wonder when a manifold is the boundary of another manifold. This property is characterized by the equivalence class known as cobordism:

**Definition 4.1.1:** We call two compact smooth manifolds (without border) $M$, $N$ *cobordant* if there exists a compact smooth manifold with boundary $T$ such that $\partial T = M \sqcup -N$, where $-N$ denotes $N$ with reversed orientation. We call $T$ the *cobordism* of $M$ and $N$. ▲

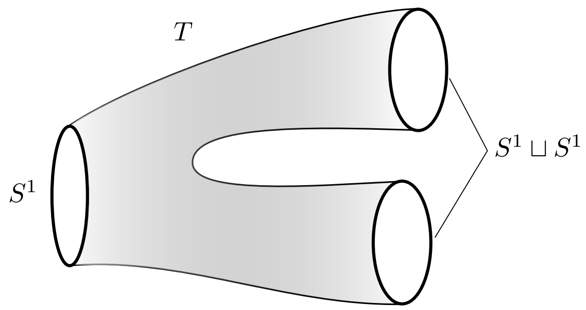



Figure 4.1: A cobordism $T$ between $S^1$ and $S^1 \sqcup S^1$

It is easy to see that cobordism is an equivalence relation. For example $M$ is cobordant to $M$ by cobordism $M \times [0, 1]$. Reflexivity is also trivial, and transitivity can be understood from Figure 4.2, where one can argue [MS74, p. 201] that the 'kink' at the middle ring can be smoothened out.

Note that the compactness requirement is essential, or we should at least require the spaces to be closed when embedded into $\mathbb{R}^n$ relative to the subspace topology, for otherwise the space $[0, 1) \times M$ would always make any manifold $M$ cobordant to the empty set, making cobordism a trivial relation.

Now consider the set $\Omega_n$ of all cobordism classes. If we take disjoint unions as addition and $\emptyset$ as the zero

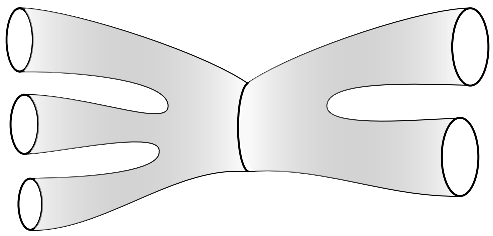
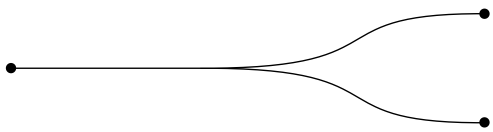


Figure 4.2: Diagram showing reflexivity of cobordism with $S^1 \sqcup S^1 \sqcup S^1 \sim S^1 \sim S^1 \sqcup S^1$ as example.

element, then $\Omega_n$ becomes an Abelian group. That is, an $n$-dimensional manifold is null-cobordant if it is the boundary of some $(n+1)$-dimensional smooth manifold with boundary. Disjoint union is a well-defined group operation because suppose $A_1$, $A_2$ are cobordant to respectively $B_1$, $B_2$ with cobordisms $T_1$, $T_2$ then $A_1 \sqcup A_2$ is cobordant to $B_1 \sqcup B_2$ with cobordism $T_1 \sqcup T_2$. Furthermore inversion is given by orientation reversal, since $M \sqcup -M$ is the boundary of $M \times [0,1]$.

We can also consider the Cartesian product $\times : \Omega_n \times \Omega_m \to \Omega_{n+m}$ sending $M, N \mapsto M \times N$. If $M, M'$ and $N', N$ are cobordant, then $M \times N$ is cobordant to $M \times N'$, which is then cobordant to $M' \times N'$. Thus the Cartesian product makes $\Omega_* = \oplus_i \Omega_i$ a graded ring (n.b. we don't require a ring to have a multiplicative identity). We call the ring $\Omega_*$ the cobordism ring. The orientation on any $n$-dimensional manifold is determined by a non-vanishing $n$-form. Therefore if we have $M$, $N$ respectively $m$ and $n$ dimensional with respective orientation forms $\omega$, $\tau$, we can define an orientation on $M \times N$ by $\omega \wedge \tau = (-1)^{mn} \tau \wedge \omega$. This also means that $M \times N = (-1)^{nm} N \times M$, which makes $\times$ a graded commutative product on $\Omega_*$, just like the wedge product on the space of forms $\Omega^*(M)$.

From Figures 4.1 and 4.2 above we can more or less immediately see that any disjoint union of copies of $S^1$ is cobordant to any other disjoint union of copies of $S^1$. This is actually true because $S^1$ is the boundary of the disk $D^2$. More generally it turns out that $\Omega_1 = 0$, and even $\Omega_2 = \Omega_3 = 0$ [MS74, p. 203]. On the other hand not all cobordism groups are trivial, for example $\Omega_0 = \mathbb{Z}$, which can be intuitively understood by noting that a cobordism between a set of one point and a set of two points would have to look something like the one below, but this is clearly not a smooth manifold with boundary. And since the set of one point evidently generates $\Omega_0$ this shows that $\Omega_0 = \mathbb{Z}$.



In general we have that $\Omega_{4k}$ is cyclic for any $k$ due to the the following theorem. Refer to [MS74, §18] for a proof.

**Theorem 4.1.2:** The cobordism ring $\Omega_* \otimes \mathbb{Q}$ (with torsion removed) is independently generated by $\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \ldots$. That is set of all the products,

$$\mathbb{C}P^{2i_1} \times \cdots \times \mathbb{C}P^{2i_r},$$

where $\{i_1, \ldots, i_r\}$ is a partition of $k$, form a basis for $\Omega_{4k}$. The groups $\Omega_k$ with $k \neq 0 \mod 4$ are finite and hence have $\Omega_k \otimes \mathbb{Q} = 0$. ▲

We will now show a connection between cobordism and the Pontryagin numbers as defined in Definition 3.4.3.

**Proposition 4.1.3:** The Pontryagin numbers are cobordism invariant, i.e. the have the same value for cobordant manifolds. Furthermore the Pontryagin numbers give for each partition $I$ of $k$ a group homomorphism $\Omega_{4k} \to \mathbb{R}$ for each $k$. ▲

Proof: We first note that the Pontryagin numbers flip sign under orientation reversal. This is because the Pontryagin *classes* remain invariant under orientation reversal. The Euler class flips sign under orientation reversal, but recall from Proposition 3.2.6 that under orientation reversal we have $c_i(TM) \mapsto (-1)^{n-i}c_i(TM)$. Since the dimension of our manifold is even, and we only consider the even Chern classes, we conclude invariance of the Pontryagin classes under orientation reversal. Then under disjoint union of two manifolds $M \sqcup N$ we simply get,

$$\begin{aligned} p_I(M \sqcup N) &= \int_{M \sqcup N} p_{i_1}(M \sqcup N) \wedge \cdots \wedge p_{i_r}(M \sqcup N) \\ &= \int_M p_{i_1}(M) \wedge \cdots \wedge p_{i_r}(M) + \int_N p_{i_1}(N) \wedge \cdots \wedge p_{i_r}(N) = p_I(M) + p_I(N). \end{aligned}$$

Since $p_q(M \sqcup N)|_M = p_q(M)$ by naturality for all $q$. The cobordism invariance is then a result of the naturality of Pontryagin classes (which is deduced trivially from naturality of Chern classes). We will show that $p_I(\partial M) = 0$ for any smooth manifold $M$. Let $j$ be the inclusion $\partial M \hookrightarrow M$, then $j^*p_i(M) = j^*p_i(TM) = p_i(j^*TM)$. Now by the collar neighborhood theorem there is a neighborhood $V \subset M$ diffeomorphic to $\partial M \times [0, 1)$ [MS74, p. 200]:

**Theorem 4.1.4:** (Collar neighborhood) Let $M$ be a smooth manifold with boundary. There exists an open neighborhood $V$ of $\partial M$ in $M$ which is diffeomorphic to $\partial M \times [0, 1)$. This neighborhood $V$ is called the collar neighborhood. This construction is shown in Figure 4.3. ▲

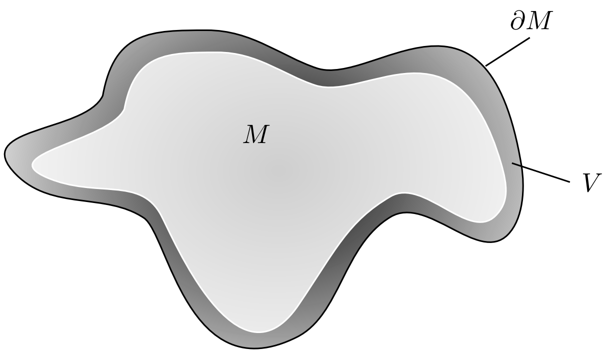



Figure 4.3: Visualization of a collar neighborhood. Here $V \subset M$ is a neighborhood of $\partial M \times [0, 1)$.

Let $f$ be this diffeomorphism. Since clearly the image of $j^*$ lies in $V$ we have $p_i(j^*TM) = p_i(j^*TV) = p_i(j^*f^*T(\partial M \times [0, 1))$. But $T(\partial M \times [0, 1)) = T(\partial M) \oplus \mathbb{R}$, hence $j^*(p_i(TM)) = j^*f^*p_i(T\partial M)$ by the Whitney product formula. Since integration is diffeomorphism invariant up to sign we now get,

$$0 = \int_M \bigwedge_k dp_{i_k}(TM) = \int_{j(\partial M)} \bigwedge_k p_{i_k}(TM) = \int_{\partial M} \bigwedge_k j^*f^*p_{i_k}(T\partial M) = j^*f^*p_I(\partial M).$$

Where the first part is due to Stokes' theorem and the fact that the Pontryagin class is closed, and the $\bigwedge$ is short hand for the multiple wedging of forms. Now since $j$ and $f$ are diffeomorphisms (onto their images), we conclude $p_I(\partial M) = 0$ and hence the Pontryagin numbers are cobordism invariant. By the remarks at the beginning of this proof we conclude that the map $\Omega_{4k} \to \mathbb{R}$ induced by $M \mapsto p_I(M)$ is a homomorphism. □

## 4.2 The signature

Consider a $4k$-dimensional oriented manifold $M$. By integrating over forms we get a bilinear pairing $S(M)\colon H_c^{2k}(M) \times H_c^{2k}(M) \to \mathbb{R}$ called the *intersection form*. It is explicitly given by,

$$(\alpha, \beta) \mapsto \int_M \alpha \wedge \beta.$$

By the even degree of both forms we have $\alpha \wedge \beta = \beta \wedge \alpha$ making $S(M)$ symmetric. Note that reversing orientation on $M$ changes the sign of the intersection form.

With respect to any basis $x_i$ we can see $S(M)(x_i, x_j)$ as a matrix. Since this matrix is symmetric we can always diagonalize it by choosing some other basis $e_i$. We see that all $S(M)(e_i, e_i)$ are non-zero, for if $S(M)(e_i, e_i) = 0$ then for any $x$ we must have $S(M)(e_i, x) = 0$ meaning $e_i$ has to be exact by Poincaré duality. Hence we conclude that $S(M)$ is non-degenerate. Now by Sylvester's law of inertia [Syl52] we have that the difference in amount of positive and negative entries is invariant under choice of diagonalizing basis $e_i$. This leads to the definition of the signature $\tau(M)$ of $M$ as

$$\tau(M) = \sum_i \operatorname{sgn} S(M)(e_i, e_i),$$

where sgn denotes the sign function (we define $\operatorname{sgn}(0) = 0$). For manifolds with dimension not divisible by 4 we define $\tau(M) = 0$. Actually we call the signature of any bilinear form the difference in positive and negative The signature has several properties, most notably that it is cobordism invariant. In fact we shall prove the following proposition:

**Proposition 4.2.1:** Let $M$ be a compact smooth manifold, then the map $M \mapsto \tau(M)$ induces a ring homomorphism $\Omega_* \mapsto \mathbb{Z}$. That is, for any two compact smooth manifolds $M, N$ we have $\tau(\partial M) = 0$, $\tau(M \sqcup N) = \tau(M) + \tau(N)$ and $\tau(M \times N) = \tau(M)\tau(N)$. ▲

We will begin with cobordism invariance, the proof of which is based on [Kre10, p. 148-150]. Most of the work of the proof lies in the following lemma, which is essentially a result of Poincaré duality generalized to manifolds with boundary.

**Lemma 4.2.2:** Let $W$ be a $2k+1$-dimensional compact smooth manifold with boundary. By the collar neighborhood theorem we have an embedding $f : [0, 1) \times \partial W \to W$. This results in an embedding $j : \partial W \to \mathring{W}$ given by $j(x) = f(\frac{1}{2}, x)$. Consider the maps $j^* : H^k(\mathring{W}) \to H^k(\partial W)$, and $j_* : H_c^k(\partial W) \to H_c^k(\mathring{W})$. Then we have that $\ker j_* = \operatorname{im} j^*$ under the identification of $H^*(M)$ with $H_c^*(M)$ for compact manifolds $M$. ▲

See [Kre10, p. 149] for a proof. By naturality of Poincaré duality (Corollary 1.4.5), we have $j^* = (j_*)^*$ under the identification $H^k(W) = H_c^k(W)$. Recall that for any linear map $f : V \to W$ we have $V \simeq$

$\ker f \oplus \operatorname{im} f^*$. Hence,

$$\dim H_c^k(\partial W) = \dim(\ker j_* \oplus \operatorname{im}(j_*)^*) = 2\dim(\ker j_*) = 2\dim(\operatorname{im} j^*).$$

We will now need the following linear algebra lemma:

**Lemma 4.2.3:** Let $b : V \times V \to \mathbb{R}$ be a symmetric non-degenerate bilinear form on a finite-dimensional vector space $V$. Suppose there is a subspace $U \subset V$ with $2\dim U = \dim V$ such that for all $x, y \in U$ we have $b(x, y) = 0$, then the signature $\tau(b)$ of $b$ is 0. ▲

Proof: Let $\{e_i\}$ be a basis of $U$, and let $\{j_i\}$ be a basis for $U^\perp$ such that $b(f_i, e_j) = \delta_{ij}$, $b(e_i, e_j) = 0$ and $b(f_i, f_j) = 0$. Such a basis exists by the non-degeneracy of $b$ [Kre10, p.150]. We then have that $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ forms a basis for $V$. Now $\{e_i + f_i, e_i - f_i\}$ also forms a basis, and furthermore $b(e_i + f_i, e_j \pm f_j) = \delta_{ij} \pm \delta_{ij}$ and $b(e_i - f_i, e_j \pm f_j) = -\delta_{ij} \pm \delta_{ij}$. Now in this basis $b$ is diagonalized, and clearly has as many positive as negative entries on the diagonals making its signature vanish. □

Proof of Proposition 4.2.1: Recall that the intersection form $S(\partial M)$ is symmetric and non-degenerate. Suppose we take some $\alpha, \beta \in \operatorname{im} j^*$, i.e. let $\alpha = j^*\tilde{\alpha}$, $\beta = j^*\tilde{\beta}$ then,

$$S(\partial M)(\alpha, \beta) = \int_{\partial M} j^*\tilde{\alpha} \wedge j^*\tilde{\beta} = \int_{j(\partial M)} \tilde{\alpha} \wedge \tilde{\beta}.$$

On the other hand $\alpha, \beta \in \ker j_*$ by Lemma 4.2.2 hence,

$$0 = \int_{j(\partial M)} j_*\alpha \wedge j_*\beta = \int_{j(\partial M)} j_*j^*\tilde{\alpha} \wedge j_*j^*\tilde{\beta},$$

but since $j$ is a diffeomorphism onto its image, we have by the definitions of $j_*$ and $j^*$ that $j_*j^* = 1$ so that,

$$\int_{j(\partial M)} \tilde{\alpha} \wedge \tilde{\beta} = 0,$$

making in summary $S(\partial M)(\alpha, \beta) = 0$. Therefore $S$ is identically zero on a $1/2 \dim H_c^k(W)$-dimensional subspace, and by Lemma 4.2.3 we conclude that $\tau(\partial M) = 0$ for any smooth manifold $M$.

To prove that $\tau(M)$ is a ring homomorphism we first need to check that $\tau(M \sqcup N) = \tau(M) \sqcup \tau(N)$. This holds since clearly the signature form satisfies $S(M \sqcup N)(\alpha, \beta) = S(M)(\alpha|_M, \beta|_M) + S(N)(\alpha|_N, \beta|_N)$ and hence acts separately on bases for $H^{m/2}(M)$ and $H^{n/2}(N)$. Hence also the diagonalization can be done independently, and we get $\tau(M \sqcup N) = \tau(M) + \tau(N)$.

Finally we just need to check multiplicativity, i.e. $\tau(M \times N) = \tau(M)\tau(N)$. This requires a little bit more work. We will follow the proof of [Sto68, p. 220-221]. Let $P = M \times N$, and let these spaces have respective dimensions $p$, $m$, and $n$. Suppose $p \neq 0 \mod 4$ then at least one of $m$ and $n$ also has to be non-divisible by 4, meaning $\tau(M)\tau(N) = 0$, and multiplicativity holds. Now assume $p = m + n = 4k$ for some $k$. By the Künneth formula we have that,

$$H^{2k}(P) = \bigoplus_{s=0}^{2k} H^s(M) \otimes H^{2k-s}(N).$$

The form $S(P)$ acts separately on each of these $H^s(M) \otimes H^{2k-s}(N)$. Suppose therefore we have some $\omega \in H^s(M) \otimes H^{2k-s}(N)$ and $\tau \in H^t(M) \otimes H^{2k-t}(N)$ for some $s, t$. We see that $\omega \wedge \tau = 0$ if either $s + t > m$ or $(2k - s) + (2k - t) > n$. Then by the identity $4k = m + n$ we have $t = m - s$. Hence for $s < m/2$ the signature acts separately on subspaces of the form

$$V_s = H^s(M) \otimes H^{2k-s}(N) \oplus H^{m-s}(M) \otimes H^{2k+s-m}(N).$$

Whereas for $s = m/2$ these are of the form,

$$V_{m/2} = H^{m/2}(M) \otimes H^{n/2}(N).$$

Let $s < m/2$ and a choose basis $\{x_i\}$ for $H^s(M)$ and a basis $\{y_j\}$ for $H^{m-s}(N)$. By Poincaré duality and compactness we can identify $H^{m-s}(M)$ with $(H^s(M))^*$, for which we can choose a dual basis $\{x_i^*\}$ such that $\int_M x_i \wedge x_j^* = \delta_{ij}$, and similarly we get a basis $\{y_j^*\}$ such that $\int_N y_i \wedge y_j^* = \delta_{ij}$. Now $\{x_i \otimes y_j,\, x_i^* \otimes y_j^*\}_{ij}$ is a basis for $V_s$. On this basis we a have non- zero pairing $S(P)$ only for $S(P)(x_i \otimes y_j, x_i^* \otimes y_j^*)$ and $S(P)(x_i^* \otimes y_j^*, x_i \otimes y_j)$. If we order this basis as $\{x_1 \otimes y_1, x_1^* \otimes y_1^*, \dots x_a \otimes y_1, x_a^* \otimes y_1^*, x_1 \otimes y_2, x_1^* \otimes y_2^*, \dots, x_a \otimes y_b, x_a^* \otimes y_b^*\}$ for $a = \dim H^s(M)$ and $b = \dim H^{2k-s}(N)$ respectively, then the matrix of $S(P)$ takes the following form:

$$S(P) = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & \text{\Large 0} & \\ & & \ddots & & \\ & \text{\Large 0} & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}$$

Each block of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ has eigenvalues $1, -1$, therefore there are as many positive eigenvalues as there are negative eigenvalues, meaning the subspace $H^s(M) \otimes H^{2k-s}(N)$ has no net effect on the signature of $P$. Hence the signature is completely determined by the values of the intersection form on forms of type $\omega \wedge \tau$ with $\omega, \tau \in H^{m/2}(M) \otimes H^{n/2}(N)$. Assume $m, n = 0 \mod 4$ and let $\{x_i\}$ be a basis for $H^{m/2}(M)$ that diagonalizes $S(M)$ and let $\{y_j\}$ be a basis that diagonalizes $S(N)$. Then we have,

$$S(P)(x_i \otimes y_j, x_u \otimes y_v) = S(M)(x_i, x_u) S(N)(y_j, y_v).$$

We get,

$$\tau(P) = \sum_{i,j=0}^{a,b} \operatorname{sgn}\left[S(M)(x_i, x_i) S(N)(y_j, y_j)\right] = \tau(M)\tau(N),$$

where $a = \dim H^{m/2}(M)$ adn $b = \dim H^{n/2}(N)$. Thus we get multiplicativity for $m, n = 0$ mod 4. The only remaining possibility is that both $m, n = 2 \mod 4$. In that case we have that $m/2$ is odd, meaning forms $\omega, \tau \in H^{m/2}(M)$ are of odd degree and we get $\omega \wedge \tau = -\tau \wedge \omega$, hence the intersection forms $S(M)$ and $S(N)$ are both anti-symmetric. We can similarly see that $S(P)$ must have a anti-symmetric matrix representation. Let in any case $\{x_i\}$ be a basis for $H^{m/2}(M)$ with dual basis $\{x_i^*\}$, and similarly $\{y_j\}, \{y_j^*\}$ for $H^{n/2}(N)$. If we order the basis as $x_i \otimes y_j, x_i^* \otimes y_j, x_i \otimes y_j^*, x_i^* \otimes y_j^*$, then for each pair $i, j$

we get matrix,

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This matrix has eigenvalues $-1, -1, 1, 1$ and hence gives a vanishing signature. This way we see that $\tau(P)$ has to vanish as well. And hence $\tau(P) = \tau(M)\tau(N)$. We conclude that the signature indeed induces a ring homomorphism $\Omega_* \to \mathbb{Z}$. □

Our most important result in this section will be the Hirzebruch signature theorem. In order to state this theorem we first need to introduce the $L$-genus of a manifold, for which in turn we need to make a short digression on multiplicative sequences. Suppose we have a commutative graded algebra $A^*$ over some commutative ring $\Lambda$ (n.b. $A^*$ is not *skew/graded* commutative). Furthermore suppose we are given a sequence of polynomials,

$$K_1(x_1),\ K_2(x_1, x_2),\ K_3(x_1, x_2, x_3), \ldots$$

such that $x_i \in A^i$ is of degree $i$ and each $K_n$ is homogeneous of degree $n$. For the formal sum $x = 1 + x_1 + x_2 + \cdots$ we then define a new polynomial by,

$$K(x) = 1 + K_1(x_1) + K_2(x_1, x_2) + \cdots$$

**Definition 4.2.4:** The polynomials $K_n$ form a *multiplicative sequence* if we have,

$$K(xy) = K(x)K(y).$$

for any graded algebra $A^*$ over a commutative ring $\Lambda$ and $x = 1 + x_1 + x_2 + \cdots$, $y = 1 + y_1 + y_2 + \cdots$ with $x_i, y_i \in A^i$ of degree $i$. ▲

**Definition 4.2.5:** Suppose we have some formal power series $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \lambda_3 t^3 + \cdots$ with constant term equal to 1. Then we can define a multiplicative sequence $\{K_n\}$ from $f(t)$ by requiring the following identity to hold:

$$K(1 + t) = f(t)$$

for any element $t \in A^1$ of degree 1. See [MS74, p.222] for a proof of existence and uniqueness of a multiplicative sequence satisfying this property. We call the multiplicative sequence $\{K_n\}$ obtained this way the *multiplicative sequence belonging to the power series $f(t)$.* ▲

**Example 4.2.6:** The second element $K_2$ of the multiplicative sequence belonging to some power series $f(t) = 1 + \lambda_1 t + \cdots$ is given by,

$$K_2(a, b) = \lambda_2 a^2 + (\lambda_1^2 - 2\lambda_2)b.$$ ▲

Proof: We can expand the product $K(1 + x_1)K(1 + x_2)$ and only consider terms of degree at most 2.

On the one hand we get

$$K(1+x_1)K(1+x_2) = K(1+x_1+x_2+x_1x_2) = 1 + K_1(x_1+x_2) + K_2(x_1+x_2, x_1x_2) + \mathcal{O}(3),$$

where $\mathcal{O}(3)$ denotes all terms of degree 3 or higher. On the other hand,

$$\begin{aligned} K(1+x_1)K(1+x_2) &= (1+\lambda_1 x_1 + \lambda_2 x_1^2 + \cdots)(1+\lambda_1 x_2 + \lambda_2 x_2^2 + \cdots) \\ &= 1 + \lambda_1(x_1+x_2) + \lambda_1^2 x_1 x_2 + \lambda_2(x_1^2+x_2^2) + \mathcal{O}(3). \end{aligned}$$

Equating terms of degree 2 we get,

$$K_2(x_1+x_2, x_1x_2) = \lambda_1^2 x_1x_2 + \lambda_2(x_1^2+x_2^2).$$

Now defining $a = x_1 + x_2$ and $b = x_1x_2$ we obtain,

$$K_2(a,b) = \lambda_1^2 b + \lambda_2(a^2 - 2b).$$

By which we conclude the result. □

This procedure can be generalized to compute $K_n(x_1, \ldots, x_n)$ for any power series, see [Hir78, 1.2.2]. Another important construction related to multiplicative sequences is the $K$-genus:

**Definition 4.2.7:** Let $\{K_n\}$ be a multiplicative sequence. The $K$-genus $K[M]$ of a manifold $M$ of dimension $m$ is defined to be 0 if $m \neq 0 \mod 4$ and if $m = 4n$ it is defined by

$$K[M] = \int_M K(p(M)) = \int_M K_n(p_1(M), \ldots, p_n(M)).$$

Equality of second and last term follows from the fact that $K_n$ is the only term of $K$ of degree $4n$, and hence the only term with non-trivial integral. ▲

**Proposition 4.2.8:** The correspondence $M \mapsto K[M]$ gives a ring homomorphism $\Omega_* \to \mathbb{R}$. ▲

Proof: By the cobordism invariance of the Pontryagin numbers, we know that the $K$-genus is cobordism invariant. From the definition of the $K$-genus we also clearly have $K[M \sqcup N] = K[M] + K[N]$. Now $K[M \times N] = K[M]K[N]$ can be seen as follows. Let $M$ and $N$ have respectively degree $m$ and $n$. Then integration over $M \times N$ of forms with degree unequal to $m + n$ gives 0. Hence we have that

$$K[M \times N] = \int_{M \times N} K(p(M \times N)).$$

Since $T(M \times N) = TM \oplus TN$ [MS74, p.27] we have by the Whitney product formula 3.4.2 that

$$\int_{M \times N} K(p(M \times N)) = \int_{M \times N} K(p(M)p(N)) = \int_{M \times N} K(p(M)) \cdot K(p(N)).$$

Which is equal to $K[M]K[N]$ and we conclude that the correspondence $M \mapsto K[M]$ gives a ring homomorphism $\Omega_* \to \mathbb{R}$. □

There is one specific $K$-genus we are most interested in. Consider the power series

$$f(t) = \frac{\sqrt{z}}{\tanh\sqrt{z}} = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k} z^k}{(2k)!}, \tag{4.1}$$

with $B_i$ the $i$-th Bernoulli number. Then let $L_i$ be the multiplicative series belonging to $f(t)$. For example note that $L_1 = 2^2 B_2/2! p_1 = 1/3 p_1$ and from (4.2.6) we also have,

$$L_2 = \frac{1}{45}(7p_2 - p_1^2). \tag{4.2}$$

It turns out that there is a deep relationship between the $L$-genus and the signature:

**Theorem 4.2.9:** (Hirzebruch signature theorem) Let $M$ be an oriented compact smooth manifold. Then the signature $\tau(M)$ and the $L$-genus $L[M]$ are equal. ▲

PROOF: Since both the signature and the $L$-genus are ring homomorphisms $\Omega_* \to \mathbb{R}$ they also immediately give a ring homomorphism $\Omega_* \otimes \mathbb{Q} \to \mathbb{R}$. We hence just have to check that they agree on the set of generators of $\Omega_* \otimes \mathbb{Q}$. By Theorem 4.1.2 the generators for $\Omega_{4k}$ are given by $\mathbb{C}P^2, \mathbb{C}P^4, \ldots$, so we have to check that the theorem holds on $\mathbb{C}P^{2k}$ for all $k$.

From Theorem 3.1.2 we know that the cohomology $H^{2k}(\mathbb{C}P^{2k})$ is generated by a single element $x^k$, with $x \in H^2(\mathbb{C}P^{2k})$ given by minus the Euler class of the universal subbundle of $\mathbb{C}P^{2k}$. And furthermore we have that

$$S(\mathbb{C}P^{2k})(x^k, x^k) = \int_{\mathbb{C}P^{2k}} x^k \wedge x^k = 1,$$

making the signature $\tau(\mathbb{C}P^{2k}) = 1$ as well.

Now we need to prove that the $L$-genus of $\mathbb{C}P^{2k}$ is also equal to 1. Recall from Proposition 3.4.7 that the total Pontryagin class of $\mathbb{C}P^{2k}$ is given by $(1+x^2)^{2k+1}$, with $x$ defined the same as above. Per definition of $L$ we then have,

$$L(p(\mathbb{C}P^{2k})) = (L(1+x^2))^{2k+1} = \left(\frac{x}{\tanh(x)}\right)^{2k+1}. \tag{4.3}$$

Here we used in the first step the fact that $\{L_i\}$ is a multiplicative sequence. In the second step we used that $L$ is defined by a power series, where we see $x^2$ as an element of degree 1 over the algebra $\bigoplus_i H^{4i}(\mathbb{C}P^{2k})$. We will now compute $\int_{\mathbb{C}P^{2k}} L[\mathbb{C}P^{2k}](p(\mathbb{C}P^{2k}))$. To determine this we need to know the $x^{2k}$ coefficient $a_{2k}$ in the power series expansion of the expression in Equation (4.3), since this is the only part of (4.3) that does not have trivial integral. From complex analysis we know that if we take $x$ to be complex valued variable, the $x^{2k}$ term in the power series is given by dividing by $2\pi i x^{2k+1}$ and then integrating around the origin. In other words,

$$a_{2k} = \frac{1}{2\pi i} \oint \frac{x^{2k+1}}{x^{2k+1}(\tanh(x))^{2k+1}} = \frac{1}{2\pi i} \oint \frac{1}{(\tanh(x))^{2k+1}}.$$

Now we introduce the substitution $u = \tanh x$. We have,

$$\frac{du}{dx} = \frac{1}{\cosh^2 x} = (1 - \tanh^2 x) = 1 - u^2,$$

hence $dx = du/(1-u^2) = (1 + u^2 + u^4 + u^6 + \cdots)du$ and we get,

$$\frac{1}{2\pi i}\oint \frac{1}{(\tanh(x))^{2k+1}}dx = \frac{1}{2\pi i}\oint \frac{(1 + u^2 + u^4 + u^6 + \cdots)}{u^{2k+1}}du = \frac{1}{2\pi i}\oint \frac{u^{2k}}{u^{2k+1}}du = 1$$

by applying Cauchy's integral theorem. From this we can conclude that

$$L[\mathbb{C}P^{2k}](p(\mathbb{C}P^{2k})) = \int_{\mathbb{C}P^{2k}} x^{2k} = 1.$$

Showing the signature and $L$-genus are indeed the same for complex projective spaces, and since by Theorem 4.1.2 the complex projective spaces generate the cobordism ring, we conclude that the Hirzebruch signature theorem holds for arbitrary cobordism classes. □

## 4.3 Milnor manifolds

The exotic 7-spheres we will construct later this section turn out to be part of a class of manifolds we call *Milnor manifolds*. These manifolds are constructed as follows. Endow $S^3$ with a multiplicative group structure by seeing $S^3$ as a subgroup of $\mathbb{H}$ generated by the unit vectors in $\mathbb{H}$. We define for $k,\, \ell \in \mathbb{Z}$ a map $f_{k,\,\ell} : S^3 \times S^3 \to S^3 \times S^3$ given by,

$$(x,\, y) \mapsto (x,\, x^k y x^\ell).$$

Then we define the set of Milnor manifolds $M_{k,\,\ell}$ by,

$$M_{k,\,\ell} = D^4 \times S^3 \cup_{f_{k,\,\ell}} -D^4 \times S^3,$$

where the minus denotes reversal of orientation and $A \cup_f B$ is constructed as $A \sqcup B$ with the equivalence relation $a \sim b$ if $a \in \partial A$, $b \in \partial B$ and $b = f(a)$. In other words we use $f$ to glue the boundaries of the two spaces together. Note that in our case $f_{k,\,\ell}$ is a diffeomorphism since it has inverse $f_{-k,\,-\ell}$. Hence $f_{k,\,\ell}$ twists one of the boundaries before gluing them together.

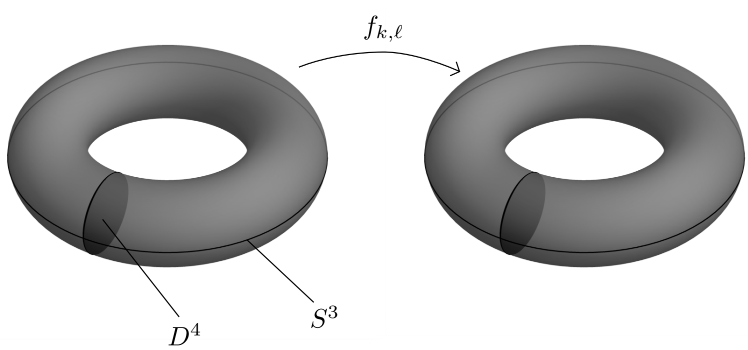



Figure 4.4: Visualization of the construction of $M_{k,\ell}$ out of two copies of $D^4 \times S^3$.

One can wonder how $S^7$ can be constructed as the union of two copies of $D^4 \times S^3$. It is easy to show that this is possible if we consider $S^7$ as the boundary of $D^8$. We then have,

$$\begin{aligned} S^7 &= \partial(D^8) = \partial(D^4 \times D^4) \\ &= (\partial D^4 \times D^4) \cup (D^4 \times \partial D^4) \\ &= D^4 \times S^3 \cup D^4 \times S^3. \end{aligned}$$

Therefore we conclude that with the gluing function obtained from the description above we can construct $S^7$ out of the union of two copies of $D^4 \times S^3$.

We will now spend the rest of this section computing the cohomology of the Milnor manifolds $M_{k,\ell}$. This computation is not necessary for subsequent sections, but it may provide the reader with a better intuition for these spaces. In order to compute the cohomology we will consider the Mayer-Vietoris sequence with $M_{k,\,\ell} = U \cup V$, where $U$ and $V$ are respectively either of the two copies of $D^4 \times S^3$ slightly extended across their boundary (in such a way that the set becomes open and $U \cup V$ covers $M_{k,\,\ell}$ while still being homotopic to the original space). We now wish to know the cohomology $H^*(U) \oplus H^*(V)$ and $H^*(U \cap V)$. The first is simply the direct sum of the cohomology of $S^3$ which is $\mathbb{R}^2$ in dimension 0, 3 and 0 elsewhere.

Note that $U \cap V$ is homotopic to $S^3 \times S^3$, which by the Künneth formula has cohomology,

$$H^q(S^3 \times S^3) = \begin{cases} \mathbb{R} & \text{for } q = 0, 6 \\ \mathbb{R}^2 & \text{for } q = 3 \\ 0 & \text{elsewhere.} \end{cases}$$

Now the only non-trivial parts in the Mayer-Vietoris sequence are,

$$0 \longrightarrow H^0(M_{k,\,\ell}) \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \longrightarrow 0 \tag{4.4}$$

$$0 \longrightarrow H^3(M_{k,\,\ell}) \xrightarrow{r^*} \mathbb{R}^2 \xrightarrow{i_U^* - i_V^*} \mathbb{R}^2 \xrightarrow{d^*} H^4(M_{k,\,\ell}) \longrightarrow 0 \tag{4.5}$$

$$0 \longrightarrow \mathbb{R} \longrightarrow H^7(M_{k,\,\ell}) \longrightarrow 0 \tag{4.6}$$

The first gives by the splitting Lemma 3.3.2 that $H^0(M_{k,\,\ell}) = \mathbb{R}$, and the third gives immediately $H^7(M_{k,\,\ell}) = \mathbb{R}$. We will now focus on the second non-trivial part. We require the following lemma:

**Lemma 4.3.1:** The map $i_U^* - i_V^* :\ H^3(U) \oplus H^3(V) \to H^3(U \cap V)$ has rank 1 if $k + \ell = 0$ and rank 2 otherwise. ▲

Proof: We first note that this is essentially the same as considering the map,

$$\Delta : H^3(D^4 \times S^3) \oplus H^3(D^4 \times S^3) \to H^3(S^3 \times S^3)$$

induced by the homotopy equivalences of these spaces to $U, V$ and $U \cap V$ respectively. Note that $\Delta$ is homotopic to the map $\Delta(\omega, \tau) = i^*\omega - j^*\tau$ where $i$ and $j$ are the inclusions $S^3 \times S^3 \hookrightarrow D^4 \times S^3$. Now by the way the gluing map $f_{k,\ell}^*$ works we see that $i \circ f_{k,\ell} = j$ (or $j \circ f_{k,\ell} = i$)

Now we have $\Delta(\omega, \tau) = i^*x - f_{k,\ell}^* i^* y$. Since $i^*$ is a non trivial linear map $\mathbb{R} \to \mathbb{R}^2$ it must be of form $x \mapsto (\mu x,\, \nu x)$. By a result of Kreck [Kre10, p. 116], we know what the map $f_{*k,\ell} : H_3(S^3 \times S^3) \to H_3(S^3 \times S^3)$

does on singular homology. Namely with respect to a certain basis it has matrix,

$$f_{*k,\ell} = \begin{pmatrix} 1 & 0 \\ k+\ell & 1 \end{pmatrix}$$

We can translate this result to de Rham cohomology as follows. Consider the de Rham isomorphism $S : H^*_{dR}(M) \to H^*_s(M)$ as defined in [Lee13, p.482]. Here the subscripts denote respectively de Rham cohomology and singular cohomology. Let $[\sigma] \in H^s_*(M)$ be a chain, and let $\omega$ be a form, then we have for a smooth map $f : M \to N$ that:

$$S(f^*\omega)[\sigma] = S(\omega)[f_*\sigma] = (f_*)^* S(\omega)[\sigma].$$

Where we see $S(\omega) \in H^*_s(M)$ as a functional $H^s_*(M) \to \mathbb{R}$. From this we conclude that in our specific case we have that the map $f^*_{k,\ell} : H^3(S^3 \times S^3) \to H^3(S^3 \times S^3)$ induced by $f_{k,\ell}$ is given by,

$$f^*_{k,\ell} = S^{-1} f^T_{*k\ell} S,$$

where $S$ is an isomorphism and $f^T_{*k,\ell}$ is the adjoint of $f_{*k,\ell}$. Now in summary if we apply this to $\Delta$ we get

$$\Delta(x,y) = i^*x - f^*_{k,\ell} i^* y = \begin{pmatrix} \mu x \\ \nu x \end{pmatrix} - f^*_{k,\ell} \begin{pmatrix} \mu y \\ \nu y \end{pmatrix}.$$

Now since $f^*_{k,\ell} = S^{-1} f^T_{*k,\ell} S$ and since by linearity $S(\mu y, \nu y) = (\widetilde{\mu} y, \widetilde{\nu} y)$ for some $\widetilde{\mu}, \widetilde{\nu}$, we get

$$\Delta(x,y) = S^{-1}(Si^*x - f^T_* S i^* y) = S^{-1} \left[ \begin{pmatrix} \widetilde{\mu} & -\widetilde{\mu} - (k+\ell)\widetilde{\nu} \\ \widetilde{\nu} & -\widetilde{\nu} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right]$$

The matrix between the square brackets has rank 1 if $k + \ell = 0$ and rank 2 otherwise. Since $S^{-1}$ is an isomorphism, we conclude that $\Delta$ has the same rank, and hence so does $i^*_U - i^*_V$. $\square$

Consider again the following part of the Mayer-Vietoris sequence:

$$0 \longrightarrow H^3(M_{k,\ell}) \xrightarrow{r^*} \mathbb{R}^2 \xrightarrow{i^*_U - i^*_V} \mathbb{R}^2 \xrightarrow{d^*} H^4(M_{k,\ell}) \longrightarrow 0.$$

Suppose $k + \ell \neq 0$, then by exactness and the lemma above we have $\operatorname{im} i^*_U - i^*_V = \mathbb{R}^2 = \ker d^*$, hence $\operatorname{im} d^* = 0$. However by exactness $d^*$ is surjective and hence $H^4(M_{k,\ell}) = 0$. By the splitting Lemma 3.3.2 we have $H^4(M_{k,\ell}) = \mathbb{R}$ for $k + \ell = 0$, but this result is not important since our main interest will be in the case $k + \ell \neq 0$. Similarly we have $0 = \ker i^*_U - i^*_V = \operatorname{im} r^*$, but by exactness $r^*$ is injective and hence $H^3(M_{k,\ell}) = 0$. This discussion can be summarized in the following proposition.

**Proposition 4.3.2:** The cohomology of $M_{k,\ell}$ is given by,

$$H^q(M_{k,\ell}) = \begin{cases} \mathbb{R} & \text{for } q = 0,\, 7 \\ 0 & \text{otherwise.} \end{cases}$$

that is, the cohomology of $M_{k,\ell}$ is isomorphic to that of $S^7$. ▲

Since the cohomologies of $S^7$ and $M_{k,\ell}$ are isomorphic one could wonder whether the two spaces are also homotopic to each other, or even homeomorphic or diffeomorphic. By the generalized Poincaré conjecture [Sma61] we at least know that if $M_{k,\ell}$ is homotopic to $S^7$, it is also homeomorphic. It turns out that for specific values of $k$ and $\ell$ we can directly show that the Milnor manifold $M_{k,\ell}$ is indeed homeomorphic to $S^7$, but not diffeomorphic. In the next section we will prove it is not always diffeomorphic to $S^7$, and then in the section after that we will show it is homeomorphic to $S^7$.

## 4.4 Milnor manifolds are not all diffeomorphic to the 7-sphere

Since integration is diffeomorphism invariant up to sign, we see that the signature is invariant under orientation preserving diffeomorphisms. By naturality of the Pontryagin classes we also see that the Pontryagin numbers are invariant under orientation preserving diffeomorphisms. Furthermore both the signature and Pontryagin numbers evidently change sign under orientation reversing diffeomorphisms. The idea is then to use these invariants to show that $M_{k,\ell}$ cannot be diffeomorphic to $S^7$. To this end we first introduce another invariant. Because the cobordism group $\Omega_7$ is trivial [MS74, p. 203] we can for any 7-manifold find an 8-manifold that bounds it.

**Lemma 4.4.1:** (Milnor's invariant) Let $B$ be any 8-manifold with $\partial B = M$ a 7-manifold. Define

$$q(B) = \int_B p_1(B) \wedge p_1(B),$$

and let $\tau(B)$ be the signature of $B$. Then $\lambda(M) = 2q(B) - \tau(B)$ is mod 7 independent of choice of $B$. ▲

The idea is then to show that $\lambda(M_{k,\ell}) \neq \lambda(S^7) = 0$ for some $k, \ell$ and with some reasoning conclude that the spaces cannot be diffeomorphic. Note that for the construction of $\lambda(M_{k,\ell})$ we need the characteristic classes and signature of a manifold with boundary, and so far we have only introduced these for manifolds *without boundary*. The definitions and related theorems are however very much analogous, and we will refrain from going into detail about this. Consult [Kre10] for an exposition of how to define these objects for manifolds with boundary.

Proof of Lemma 4.4.1: Let $B$ and $B'$ be two 8-manifolds with boundary $M$. Let $C = B \sqcup_M -B'$ be the manifold obtained by gluing $B$ to $B'$ by identifying their common boundary. We will compute the signature of $C$. Applying the signature theorem to 8-manifolds we get by equation 4.2 that

$$\tau(C) = \int_C \frac{1}{45}(7p_2(C) - p_1^2(C)).$$

Then we have,

$$45\tau(C) + q(C) = \int_C 7p_2(C) - p_1^2(C) + p_1^2(C) = \int_C 7p_2(C) = 0 \mod 7,$$

since $\int_C p_2(C) \in \mathbb{Z}$ due to Proposition 3.4.4. Now we have,

$$0 = 90\tau(C) + 2q(C) = 2q(C) - \tau(C) \mod 7$$

Let $\omega \in H^8(C)$, and let $j$ be the inclusion $M \hookrightarrow C$ and $i_1, i_2$ the inclusions $B, B' \hookrightarrow C$ respectively. Note that $i_2$ is orientation reversing. We have,

$$\int_C \omega = \int_{i_1(B)\cup i_2(B')} \omega = \int_{i_1(B)} \omega + \int_{i_2(B')} \omega - \int_{i_1(B)\cap i_2(B')} \omega = \int_B i_1^*\omega - \int_{B'} i_2^*\omega - \int_M j^*\omega.$$

Note that the last term vanishes by applying Stokes' theorem. By naturality of the Pontryagin class we get $q(C) = q(B) - q(B')$ and by also applying the signature theorem we similarly get $\tau(C) = \tau(B) - \tau(B')$ and hence we conclude,

$$2q(B) - \tau(B) = 2q(B') - \tau(B') = 0 \mod 7,$$

proving the mod 7 invariance of $\lambda(M)$ under choice of bounding manifold $B$. □

**Corollary 4.4.2:** If $\lambda(M) \neq 0$ then $M$ is not diffeomorphic to $S^7$. ▲

Proof: Clearly if we reverse the orientation on $M$ then we get $\lambda(-M) = -\lambda(M)$. Now suppose $M$ admits an orientation reversing diffeomorphism $f : M \to M$, and let $M = \partial B$. Then $f$ induces an orientation preserving diffeomorphism $M \to \partial(-B)$, and hence we can identify $M$ with $\partial(-B)$. We conclude that if $M$ admits an orientation reversing diffeomorphism, then $\lambda(M) = \lambda(-M) = 0$. Note that $S^7$ admits an orientation reversing diffeomorphism, and by composition so does any manifold diffeomorphic to it. Therefore if $\lambda(M) \neq 0$, then $M$ is not diffeomorphic to $S^7$. □

Recall that we constructed the Milnor manifolds $M_{k,\ell}$ by taking $D^4 \times S^3 \sqcup -D^4 \times S^3$ and gluing the boundaries together using some $f_{k,\ell}$, where we see $S^3$ as the unit vector sphere in $\mathbb{H}$. We can see each half as an $S^3$ fiber bundle over $D^4$. By the gluing we then obtain an $S^3$ bundle $M_{k,\ell} \to S^4$. By extending the gluing map to the unit ball $D^4 \subset \mathbb{H}$ we can construct an analogous $D^4$ bundle $\pi : E_{k,\ell} \to S^4$ over $S^4$. This way we have $\partial E_{k,\ell} = M_{k,\ell}$, and we can use $E_{k,\ell}$ as bounding manifold to compute $\lambda(M_{k,\ell})$.

For the computation of $\lambda(M_{k,\ell})$ we need to know both $q(E_{k,\ell})$ and $\tau(E_{k,\ell})$. To compute the former we need a description of the tangent bundle $TE_{k,\ell}$. By [Mil56, p.403] we have,

$$TE_{k,\ell} = TS^4 \oplus \pi^*(M_{k,\ell}).$$

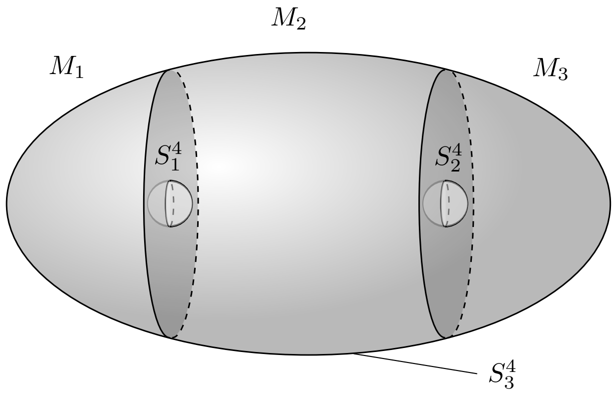
Since $p_1(TS^4) = 0$ we see by the Whitney product formula that $p_1(TE_{k,\ell}) = \pi^* p_1(M_{k,\ell})$, where we take the Pontryagin class of $M_{k,\ell}$ seen as a sphere bundle. Now we only defined Pontryagin classes for vector bundles, and there is a way to define characteristic classes for sphere bundles. However to avoid this discussion we note that in this case we can find a rank-4 complex vector bundle $L_{k,\ell} \to S^4$ such that taking the fiber wise unit sphere (that is replace $\mathbb{C}^4$ with $S^3$) we obtain $M_{k,\ell}$. Then the characteristic classes of $M_{k,\ell}$ are defined as those of $L_{k,\ell}$. To compute $p_1(M_{k,\ell})$ we will prove the following lemma:

**Lemma 4.4.3:** The map $(k,\ell) \mapsto \int_{S^4} p_1(M_{k,\ell})$ is a group homomorphism. In other words

$$\int_{S^4} p_1(M_{k+k',\ell+\ell'}) = \int_{S^4} p_1(M_{k,\ell}) + \int_{S^4} p_1(M_{k',\ell'}),$$

and $(0,0) \mapsto 0$. ▲

Proof: The proof proceeds by an analogous construction to [Kre10, p.157]. Consider the manifold $M$ given by $D^5$ with two copies of $D^5$ removed as shown below. We will slice $M$ into three pieces $M_1$, $M_2$ and $M_3$. We have $\partial M = S_3^4 - S_1^4 - S_2^4$ if we orient $S_i^4$ as in the diagram below.




Consider now the following fiber bundle $E$ over $M$. On $M_i$ we define it to be $M_i \times S^3$ and we will glue it together by using $f_{k,\ell}$ and $f_{k',\ell'}$ that is:

$$E = M_1 \times S^3 \cup_{f_{k,\ell}} M_2 \times S^3 \cup_{f_{k',\ell'}} M_3 \times S^3,$$

where as before with $\cup_{f_{k,\ell}}$ we mean taking disjoint union and using $f_{k,\ell}$ as gluing map along the boundaries. Note that $M_1 \cap S^4_1$ and $M_2 \cap S^4_1$ are both homeomorphic to $D^4$. Hence $E|_{S^4_1} = D^4 \times S^3 \cup_{f_{k,\ell}} D^4 \times S^3$. We consequently see that $E|_{S^4_1} = M_{k,\ell}$, and similarly $E|_{S^4_2} = M_{k',\ell'}$ and $E|_{S^4_3} = M_{k+k',\ell+\ell'}$, since $f_{k,\ell} \circ f_{k',\ell'} = f_{k+k',\ell+\ell'}$. That is, on $S^4_3$ we get contribution of both twisting at the $M_1$-$M_2$ border *and* at the $M_2$-$M_3$ border. Let $i_j$ be the inclusion $S^4_j \hookrightarrow \partial M$, then

$$0 = \int_M dp_1(E) = \int_{\partial M} p_1(E) = \int_{S^4_3} i_3^* p_1(E) - \int_{S^4_2} i_2^* p_1(E) - \int_{S^4_1} i_1^* p_1(E).$$

Applying naturality and rearranging the terms we get,

$$\int_{S^4_3} p_1(M_{k+k',\ell+\ell'}) = \int_{S^4_1} p_1(M_{k,\ell}) + \int_{S^4_2} p_1(M_{k',\ell'}),$$

as required. Now for $k = \ell = 0$ we just get the union of two copies of $D^4 \times S^3$ identified at the boundary. Such gluing leaves $S^3$ unchanged, but the boundary of the two $D^4$'s get identified, making an $S^4$, hence $M_{0,0} = S^4 \times S^3$. By Example 3.4.6 we know that the first Pontryagin class of $S^n$ vanishes, and by the fact that $T(S^3 \times S^4) = TS^3 \oplus TS^4$ [MS74, p.27] we then have

$$p(S^3 \times S^4) = p(S^3)p(S^4) = 1,$$

making $p_1(M_{0,0}) = 0$. Hence we conclude that $(k,\ell) \mapsto \int_{S^4} p_1(M_{k,\ell})$ is a group homomorphism, completing the proof. □

Consider the map $R : M_{k,\ell} \to M_{-\ell,-k}$ given on fibers by $(x,z) \mapsto (x, z^{-1})$, in other words taking the fiberwise multiplicative inverse. Evidently this map is an orientation reversing bundle isomorphism, giving $M_{k,\ell} = -M_{-\ell,-k}$ (NB: $k,\ell$ not only flip sign but also order). To see $R$ is indeed a well-defined map $M_{k,\ell} \to M_{-\ell,-k}$ we consider the gluing map $f_{k,\ell}$ and note

$$R(f_{k,\ell}(x,z)) = R(x, x^k z x^\ell) = (x, x^{-\ell} z^{-1} x^{-k}) = f_{-\ell,-k}(R(x,z)).$$

And hence it preserves the gluing on the boundaries used to define $M_{k,\ell}$. Now let $H^4(S^4)$ be generated by some $\alpha$ with $\int_{S^4} \alpha = 1$. Then since $(k,\ell) \mapsto \int_{S^4} p_1(M_{k,\ell})$ is a homomorphism we must have $p_1(M_{k,\ell}) = (ak+b\ell)\alpha$ for some coefficients $a$, $b$. But by applying the isomorphism $R$ we obtain $p_1(M_{k,\ell}) = p_1(M_{-\ell,-k})$. Hence $ak + b\ell = -a\ell - bk$, meaning we actually have for some $c \in \mathbb{Z}$:

$$p_1(M_{k,\ell}) = c(k-\ell)\alpha.$$

Since $M_{k,\ell}$ is a complex bundle, we have

$$p(M_{k,\ell}) = c(M_{k,\ell} \oplus \overline{M_{k,\ell}}) = c_1(M_{k,\ell})^2 - 2c_2(M_{k,\ell}) = -2c_2(M_{k,\ell}),$$

where the last step follows because $c_1(M_{k,\ell}) \in H^2(S^4) = 0$. The top Chern class is the same as the Euler class [BT82, p. 278], that is $c_2(M_{k,\ell}) = e(M_{k,\ell})$. By [Kre10, p. 159] we then have

$$\int_{S^4} e(M_{1,0}) = -1 = -\int_{S^4} \alpha.$$

From which we conclude that $p_1(M_{k,\ell}) = 2(k-\ell)\alpha$. Now finally since $p(S^4) = 1$ and $TE_{k,\ell} = TS^4 \oplus \pi^*(M_{k,\ell})$ we get,

$$p_1(E_{k,\ell}) = 2(k-\ell)\pi^*(\alpha).$$

From [Mil56, p. 403] we then have that,

$$\int_{E_{k,\ell}} \pi^*(\alpha)^2 = 1.$$

This gives $\tau(E_{k,\ell}) = 1$. Similarly we have

$$q(E_{k,\ell}) = \int_{E_{k,\ell}} p_1(E_{k,\ell}) \wedge p_1(E_{k,\ell}) = \int_{E_{k,\ell}} 4(k-\ell)^2\pi^*(\alpha)^2 = 4(k-\ell)^2,$$

and therefore,

$$\lambda(M_{k,\ell}) = 2q(E_{k,\ell}) - \tau(E_{k,\ell}) = 8(k-\ell)^2 - 1 = (k-\ell)^2 - 1 \mod 7.$$

By Corollary 4.4.2 we now conclude the following theorem:

**Theorem 4.4.4:** $M_{k,\ell}$ is not diffeomorphic to $S^7$ if $(k-\ell)^2 - 1 \neq 0 \mod 7$. ▲

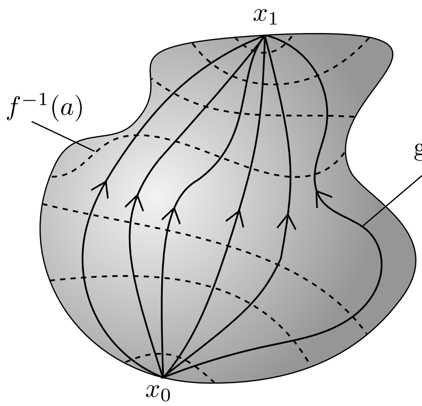
## 4.5 Milnor manifolds are homeomorphic to the 7-sphere

In this section we will prove that the Milnor manifolds $M_{k,\ell}$ defined in Section 4.3 are for certain values of $k$ and $\ell$ homeomorphic $S^7$. In the previous section we have shown that they are not diffeomorphic to $S^7$ for certain values of $k$ and $\ell$, therefore combining the results will give us a set of manifolds which are homeomorphic but not diffeomorphic to $S^7$. To prove $M_{k,\ell}$ is homeomorphic to $S^7$ we need the following theorem:

**Theorem 4.5.1:** (Reeb) Let $M$ be a compact $n$-manifold. Suppose there is a smooth function $f : M \to [0, 1]$ with exactly two critical points. Then $M$ is homeomorphic to $S^n$. ▲

Recall that the critical point of a function $f$ are the points $p$ such that $df(p) = 0$, i.e. the differential of the map is not surjective.

Proof: We will follow the proof presented in [Mil64]. Consider first the following diagram visualizing the situation. We can always normalize $f$ so that $f(x_0) = 0$, $f(x_1) = 1$, with $x_0$, $x_1$ the critical points.




We will show that $M - x_1$ is diffeomorphic to $\mathbb{R}^n$. Then $M$ would be the one-point compactification of $\mathbb{R}^n$, making it homeomorphic to $S^n$, since any two one-point compactifications of the same space are homeomorphic [Cra14, p. 80]. We require the following lemma [Mil64, p. 168]:

**Lemma 4.5.2:** (Brown, Stallings). Let $M$ be a smooth manifold such that every compact subset of $M$ is contained in a space diffeomorphic to $\mathbb{R}^n$. Then $M$ is itself diffeomorphic to $\mathbb{R}^n$. ▲

First of all $x_0$ has a neighborhood $U$ diffeomorphic to $\mathbb{R}^n$. By continuity, there must be an $\varepsilon$ such that $f^{-1}[0, \varepsilon]$ is contained in $U$. Now let $K$ be a compact subset of $M - x_1$. Again by continuity there must be an $\varepsilon'$ such that $K \subset f^{-1}[0, 1 - \varepsilon']$. The idea is to now show that there is a diffeomorphism $M - x_1 \to M - x_1$ sending $f^{-1}[0, \varepsilon]$ to $f^{-1}[0, 1 - \varepsilon']$. This diffeomorphism then sends $U$ to an open set diffeomorphic to $\mathbb{R}^n$ that covers $K$. Then by Lemma 4.5.2 we would conclude that $M$ is diffeomorphic to $\mathbb{R}^n$.

Equip $M$ with a Riemannian metric $\langle\cdot\rangle$. We will for $p \in M$ consider the vector field,

$$X = \rho(p) \, \frac{\mathrm{grad} f}{\langle \mathrm{grad} f, \mathrm{grad} f \rangle}$$

where $\rho(p)$ is a smooth function that is 1 on $f^{-1}[\varepsilon, 1 - \varepsilon']$ and vanishes outside a compact neighborhood of $f^{-1}[\varepsilon, 1 - \varepsilon']$. Recall that $\mathrm{grad} f = \sum_i \partial_i f \partial_i$ where $\{\partial_i\}$ is a local basis for the tangent space. Now let $\theta_t : M - x_1 \to M - x_1$ be the flow of $X$. We will show that this flow $\theta_t$ 'stretches' $f^{-1}[0, \varepsilon]$ diffeomorphically to $f^{-1}[0, \varepsilon + t]$. Consider the derivative,

$$\left.\frac{df(\theta_t(p))}{dt}\right|_{t=T} = (\partial_i f) \cdot \left(\frac{\theta_t(p)}{dt}\right)^i = \langle \mathrm{grad} f, X \rangle = \rho(p).$$

Since $\theta_0$ is the identity we have for any $f(p) = x$ and $t > 0$ that $f(\theta_r(p)) = x + \rho(p) \cdot t$ by integrating $df(\theta_t)/dt$ over $t$. Hence $\theta_{1-\varepsilon'-\varepsilon}$ sends $f^{-1}[0, \varepsilon]$ diffeomorphically to $f^{-1}[0, 1 - \varepsilon']$. Then $\theta_{1-\varepsilon'-\varepsilon}$ also sends $U$ diffeomorphically to some subset of $M - x_1$ that contains $K$. Hence by Lemma 4.5.2 we conclude that $M - x_1$ is diffeomorphic to $\mathbb{R}^n$. This makes $M$ a one-point compactification of $\mathbb{R}^n$ and we finally conclude that $M$ is homeomorphic to $S^n$. □

We will now prove Lemma 4.5.2, but first we require the following lemma (see [Pal60] for a proof):

**Lemma 4.5.3:** (Palais and Cerf) Let $\phi$ and $\psi$ be two smooth orientation preserving embeddings of $D^n$ into the interior of some connected manifold $M$. Then there exists a diffeomorphism $F : M \to M$ such that $\psi = F \circ \phi$. That is, we can extend the natural diffeomorphism of $\phi(D^n) \to \psi(D^n)$ to a function on the entire manifold. ▲

Proof of Lemma 4.5.2: We know that any paracompact manifold admits a sequence

$$W_1 \subset W_2 \subset W_3 \subset \cdots \subset M$$

of submanifolds with boundary such that each $W_i$ is diffeomorphic to a disk $D_i^n$, each $W_i$ is contained in the interior of $W_{i+1}$ and $\bigcup_i W_i = M$ [Mil64, p. 168]. For each $W_i$ we have some diffeomorphism to a disk $D_i^n$. Now consider the sequence of inclusions,

$$D_1^n \subset D_2^n \subset D_3^n \subset \cdots \subset \mathbb{R}^n,$$

and suppose we have a diffeomorphism $f_1 : D_1^n \to W_1^n$. This diffeomorphism can be extended into a map $f_2 : D_2^n \to W_2^n$ as follows. Let $g$ be an orientation preserving diffeomorphism $D_2^n \to W_2^n$. Now $f_1$ and $g$ are both orientation preserving diffeomorphisms of $D^n$ into the interior of $M$. Therefore there is a diffeomorphism $F$ such that $f_1 = F \circ g$ on $W_1$. Let $f_2 = F \circ g$, then if we apply the same procedure inductively to extend any diffeomorphism $f_i : D_i^n \to W_i$ to some $f_{i+1} : D_{i+1}^n \to W_{i+1}$, we get in the limit $i \to \infty$ a diffeomorphism $f : \mathbb{R}^n \to M$, completing the proof. □

**Proposition 4.5.4:** If $k + \ell = 1$ then $M_{k,\ell}$ admits a function with exactly two critical points. Hence by Reebs theorem $M_{k,1-k}$ is for any $k \in \mathbb{Z}$ homeomorphic to $S^7$. ▲

Proof: Recall that $M_{k,\ell}$ is constructed out of two copies of $D^4 \times S^3$. We get a homeomorphic space by taking two copies of $\mathbb{H} \times S^3$ where outside $\mathbb{H} \times S^3 \setminus \{0\}$ we identify the two spaces by $(u, v) \sim (u', v')$ if

$$(u', v') = \left( \frac{u}{||u||^2}, \frac{u^k v u^\ell}{||u||} \right).$$

We then define our required function by

$$g(u, v) = \frac{\Re(v)}{\sqrt{1 + ||u||^2}}$$

on one half and by

$$g(u', v') = \frac{\Re(u'')}{\sqrt{1 + ||u''||^2}}$$

on the other half, where $u'' = u'(v')^{-1}$. In both cases $\Re$ denotes the projection to the real part. We assume the reader to be familiar with quaternion arithmetic. We will first of all show that these functions agree on overlaps. First of all since $||v|| = 1$ we have,

$$||v'|| = \frac{||u^k v u^\ell||}{||u||} = ||u||^{k+\ell-1}.$$

We require $||v'|| = 1$, giving the restriction $k + \ell = 1$. That is, let $\ell = 1 - k$. We get,

$$\frac{1}{\sqrt{1+||u''||^2}} = \frac{1}{\sqrt{1+\frac{1}{||u||^2}}} = \frac{||u||}{\sqrt{1+||u||^2}}.$$

Hence the second definition of $g$ becomes,

$$g(u,v) = \frac{\Re(u'')||u||}{\sqrt{1+||u||^2}}.$$

This means we just have to show $\Re(u'')||u|| = \Re(v)$. For any $a \in \mathbb{H}$ we have the identity $2\Re(a) = a + a^*$ with $a^* = a^{-1}||a||^2$ the conjugate. Hence,

$$\begin{aligned} 2\Re(u'') &= u'' + ||u''||^2(u'')^{-1} = u'(v')^{-1} + \frac{1}{||u||^2}(u'(v')^{-1})^{-1} \\ &= \frac{u^k v^{-1} u^{-k}}{||u||} + \frac{u^{-k} v u^k}{||u||} && \text{since } ||u^{-k} v u^k|| = 1, \\ &= \frac{2\Re(u^{-k} v u^k)}{||u||} = \frac{2\Re(v)}{||u||}. \end{aligned}$$

where the last step follows from the fact that conjugation leaves the real part invariant. It is now straightforward to show that the gradient of both local expressions of $g$ only vanishes at the origin [Bog11, p. 15-17]. From this we conclude that $g$ has exactly two critical points. Which by Theorem 4.5.1 implies that $M_{k,1-k}$ is homeomorphic to $S^7$. □

Theorem 4.4.4 asserts that $M_{k,1-k}$ cannot be diffeomorphic to $S^7$ if

$$\lambda(M_{k,1-l}) = (2k-1)^2 - 1 \neq 0 \mod 7.$$

Expanding the square we get,

$$4k(k-1) \neq 0 \mod 7.$$

Since $\mathbb{Z}/7\mathbb{Z}$ is a field, it has no zero divisors and hence we conclude the main theorem of this thesis:

**Theorem 4.5.5:** (Existence of exotic spheres) $M_{k,1-k}$ is for any value of $k$ homeomorphic to $S^7$, however if $k \neq 0, 1 \mod 7$ then $M_{k,1-k}$ is not diffeomorphic to $S^7$. This shows that for smooth manifolds diffeomorphism of spaces is different from homeomorphism between spaces. ▲

For the entirety of the thesis we have only used de Rham cohomology and worked completely from a differential point of view. Yet many of the proofs especially in this last part have strong topological consequences. The characteristic classes we introduced in part 3 can actually also be defined using singular cohomology with integer coefficients, and are therefore not at all unique to smooth manifolds. The fact that all the classes we defined integrate to integer values perhaps already hinted that something more general was going on. Yet despite sticking to the less general differentiable viewpoint we were able to derive non-trivial results, which shows the power of the differential topological methods we used. Perhaps the strength of differentiable viewpoints lies in the fact that it appeals more to our intuition.

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