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Bachelor Thesis

Ricci flow on surfaces

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Introduction

About the Ricci flow

The main object of study in this thesis is the Ricci flow. The Ricci flow was introduced in 1982 by Richard Hamilton in his paper "Three-manifolds with positive Ricci curvature" (see [Ham82]). This method of Ricci flow was proposed as a strategy for proving the geometrization conjecture as put forth by William Thurston. The conjecture is about the classification of compact three-dimensional manifolds and has as a corollary the Poincaré conjecture.
The idea of the Ricci flow is to introduce a evolution equation for the Riemannian metric:

$$\frac{\partial g(t)}{\partial t} = -2Ric[g(t)].$$

This evolution equation is inspired by the classical heat equation from physics (although this is not directly clear from the form of the equation as given here). It evolves the Riemannian metric on the manifold towards metric that is more even.
In 2002 and 2003 the Russian mathematician Grigori Perelman published a series of papers ([Per02, Per03b, Per03a]) in which he proved the geometrization conjecture. In these papers he used the Ricci flow techniques as proposed by Hamilton together with several innovations of his own.
In other applications the Ricci flow has been successfully employed as well. An example is the proof of the differentiable sphere theorem by Simon Brendle and Richard Schoen (see [BS08]).

About this thesis

We will discuss the Ricci flow in the simplest setting, on two dimensional Riemannian manifolds. The result central in our discussion of the Ricci flow is the uniformization theorem. This is a result from complex analysis and it classifies Riemann surfaces (complex one-dimensional manifolds). It is a classical result first proved by Poincaré. As is turns out a connection exists between this result and the Ricci flow on two dimensional Riemannian manifolds. Actually in a somewhat restricted setting the Ricci flow can be used to give a proof of this theorem. In this thesis we will give a discussion of this proof.

In the first chapter we give a very brief discussion of some preliminary notions which are needed to discuss the Ricci flow. The reader is assumed to be familiar with the concept of smooth manifolds. We mainly cover concepts related to Riemannian manifolds. In the second chapter we introduce the concept of the Ricci flow and study how several quantities related to the metric evolve under the flow. We also introduce the normalised Ricci flow and look at the case of surfaces. In the final chapter most of the work is done. We discuss the connection between the Ricci flow and the uniformization theorem. Furthermore we discuss the proof of the uniformization theorem using Ricci flow. We will not treat the whole proof but focus mainly on the parts about long time existence and convergence of the Ricci flow. The discussion of these topics will be more in-depth. See Section 3.1 for a more complete overview of what is discussed in this final chapter.

This thesis is based on several texts about the Ricci flow (most importantly [AH11] and [CK04]). The aim of the author is to provide a text about the Ricci flow on surfaces that is as complete as possible on the topic it covers. To this end the author has combined results from several sources and has provided some proofs of his own.

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I would also like to thank my family and friends for all the help they have offered.

Chapter 1

Preliminaries

In this chapter we will give a very brief discussion of some definitions and notation needed for our treatment of the Ricci flow. The reader is assumed to be familiar with smooth manifolds. We will introduce the concept of a Riemannian manifold but our treatment will be very succinct. For a complete introduction we refer to [Lee97].

In what follows we will always assume the Einstein summation convention and denote $\partial_i = \frac{\partial}{\partial x^i}$. Furthermore we assume manifolds and maps to be smooth unless otherwise specified.

1.1 Tensor fields

We first discuss several conventions regarding tensor fields. From now on let M be a manifold.

Definition 1.1:
We define the (k, l) tensor bundle to be $T^{(k,l)}M = \otimes^k T^*M \otimes \otimes^l TM$. A section $F \in \Gamma(T^{(k,l)}M)$ of this bundle is called a (k, l) tensor field.

We will denote $\mathfrak{X}(M) = \Gamma(TM)$ and $\Omega^1(M) = \Gamma(T^*M)$. It will turn out to be useful to characterize tensor fields as multilinear functions. Given finite dimensional vectorspaces V_i and W we denote the vector space of all multilinear functions $V_1 \times ... \times V_n \to W$ by $\mathrm{Lin}(V_1, ..., V_n; W)$.

Proposition 1.2 (Tensor field characterization)**:**
Consider the map

$$\Phi \colon \Gamma(T^{(k,l)}M) \to \mathrm{Lin}(\underbrace{\mathfrak{X}(M), ..., \mathfrak{X}(M)}_{k \text{ copies}}, \underbrace{\Omega^1(M), ..., \Omega^1(M)}_{l \text{ copies}}; C^\infty(M))$$

given by

$$\Phi(\omega_1 \otimes ... \otimes \omega_k \otimes Y_1 \otimes ... \otimes Y_l)(X_1, ..., X_k, \alpha_1, ..., \alpha_l) = \omega_1(X_1) \cdots \omega_k(X_k) \cdot \alpha_1(Y_1) \cdots \alpha_l(Y_l)$$

for vector fields X_i, Y_j and one-forms α_s and ω_n. This map is a (canonical) isomorphism. This shows that

$$\Gamma(T^{(k,l)}M) \cong \mathrm{Lin}(\underbrace{\mathfrak{X}(M), ..., \mathfrak{X}(M)}_{k \text{ copies}}, \underbrace{\Omega^1(M), ..., \Omega^1(M)}_{l \text{ copies}}; C^\infty(M)).$$

For a proof see [Lee13, p. 318]. Using this result we can define tensor fields on a manifold using $C^\infty(M)$ multilinear maps.

1.2 Riemannian manifolds

Our main object of study will be Riemannian manifolds. A Riemannian manifold is a manifold together with an extra structure called the Riemannian metric.

Definition 1.3 (Riemannian manifold)**:**
On a manifold M a Riemannian metric g is a symmetric $(2,0)$ tensor field that is positive definite i.e. $g(X,X) > 0$ if $X \in TM$ is nonzero. The pair (M,g) is called a Riemannian manifold.

The Riemannian metric defines an inner product on all fibers T_pM so we will sometimes use the notation $\langle X,Y\rangle_g := g(X,Y)$. The length of a tangent vector $X \in TM$ is defined as $\|X\|_g := \sqrt{\langle X,X\rangle_g}$.

Example 1.4:
An example of a Riemannian manifold is $\mathbb{R}^n$ equipped with the Euclidean metric $g_{can} = \sum_{i=1}^n dx^i \otimes dx^i$.

1.2.1 Raising and lowering operators

The vector spaces T_pM and T_p^*M are isomorphic because they both have the same dimension. Using the metric we can actually define an explicit bundle isomorphism between TM and T^*M.

Proposition 1.5:
The map $\flat\colon TM \to T^*M\colon X \mapsto X^\flat$ defined by $(X^\flat)(Y) = g(X,Y)$ is a bundle isomorphism.

For a proof see [Lee13, p. 341]. We write $\sharp\colon T^*M \to TM\colon \omega \mapsto \omega^\sharp$ for the inverse of $\flat$. The map $\flat$ is sometimes called the lowering operator and $\sharp$ the raising operator. In local coordinates we have $g = g_{ij}\,dx^i \otimes dx^j$. Now $X^\flat$ is given by $X^\flat = g_{ij}\,X^i dx^j$. We denote by $g^{ij} = (g^{-1})_{ij}$ the components of the inverse of the matrix g_{ij}. In coordinates we now have $\omega^\sharp = g^{ij}\,\omega_i\partial_j$.

Definition 1.6 (Induced metrics)**:**
On the dual bundle T^*M we define an induced metric by $\langle\omega,\nu\rangle_g := \langle\omega^\sharp,\nu^\sharp\rangle_g$.
On the tensor bundle $T^{(k,l)}M$ we define an induced metric which in local coordinates is given by

$$\langle F,G\rangle_g = g^{j_1s_1}...g^{j_ks_k}\,g_{i_1r_1}...g_{i_lr_l}\,F^{i_1...i_l}_{j_1...j_k}G^{r_1...r_l}_{s_1...s_k}.$$

Definition 1.7 (Contraction and metric contraction)**:**
The contraction operation is the natural pairing of the space TM with its dual T^*M. In the simplest case we define $\mathrm{tr}\colon T^{(1,1)}M \to \mathbb{R}$ for a vector field X and a one-form ω as

$$\mathrm{tr}(\omega \otimes X) = \omega(X).$$

For a general tensor field of mixed type we need to specify which factors we want to pair. We define for indices $0 \le a \le k+1$ and $0 \le b \le l+1$ the contraction operation $\mathrm{tr}_{ab}\colon T^{(k+1,l+1)}M \to T^{(k,l)}M$ as

$$\mathrm{tr}_{ab}(\omega_1 \otimes ... \otimes \omega_{l+1} \otimes Y_1 \otimes ... \otimes Y_{k+1}) = \omega_a(Y_b)\,\omega_1 \otimes ... \otimes \widehat{\omega_a} \otimes ... \otimes \omega_{l+1} \otimes Y_1 \otimes ... \otimes \widehat{Y_b} \otimes ... \otimes Y_{k+1}$$

for vector fields Y_i and one-forms ω_j. Using the lowering and raising operators we can extend this definition to tensor fields which are not of mixed type. For $(k,0)$ and $(0,l)$ tensor fields we define a metric contraction, denoted by tr_g, as

$$\mathrm{tr}_{g,ab}(\omega_1 \otimes ... \otimes \omega_k) = \omega_a(\omega_b^\sharp)\,\omega_1 \otimes ... \otimes \widehat{\omega_a} \otimes ... \otimes \widehat{\omega_b} \otimes ... \otimes \omega_k$$
$$\mathrm{tr}_{g,ab}(Y_1 \otimes ... \otimes Y_l) = (Y_a)^\flat(Y_b)\,Y_1 \otimes ... \otimes \widehat{Y_a} \otimes ... \otimes \widehat{Y_b} \otimes ... \otimes Y_l.$$

We will write $\mathrm{tr}_{ab,cd}$ to denote two contractions being performed at once.

1.2.2 Riemannian volume form

Another application of the Riemannian metric is that it allows us to integrate scalar functions over the manifold. Given an oriented Riemannian manifold we can intrinsically define a differential n-form which we call the Riemannian volume form and denote by vol_n.

Definition 1.8 (Riemannian volume form)**:**
On an oriented Riemannian manifold the Riemannian volume form, denoted by vol_n*, is the unique* n*-form such that* $\mathrm{vol}_n(e_1, ..., e_n) = 1$ *for every positively oriented orthonormal basis* $(e_i)_{i=1}^n$ *of a tangent space.*

An integral of an arbitrary function $f \in C^\infty(M)$ can now be defined as

$$\int_M f \ \mathrm{vol}_n \,.$$

The volume of the manifold is defined as

$$\mathrm{Vol}(M) = \int_M \mathrm{vol}_n \,.$$

If (x^i) is a positively oriented coordinate system then vol_n is given in local coordinates by

$$\mathrm{vol}_n = \sqrt{\det(g_{ij})}\, dx^1 \wedge ... \wedge dx^n.$$

For a proof that the Riemannian volume form exists and for a proof of its coordinate expression see [Lee13, p. 389].

1.2.3 Isometries

Isometries provide a notion of equivalence between two Riemannian manifolds.

Definition 1.9 (Isometry)**:**
An isometry between two Riemannian manifolds is defined to be a diffeomorphism $\phi\colon M \to M'$ *that preserves the metric structure i.e.* $g = \phi^* g'$*.*

A somewhat weaker notion of equivalence between Riemannian manifolds is that of conformal equivalence.

Definition 1.10 (Conformal metrics)**:**
We call two metrics g *and* g' *on a manifold conformal to each other if there exists a function* $f\colon M \to \mathbb{R}_{>0}$ *such that* $g = f \cdot g'$*. We call two Riemannian manifolds* (M, g) *and* (M', g') *conformally equivalent if there is a diffeomorphism* $\phi\colon M \to M'$ *such that if* g *is conformal to* $\phi^* g'$*.*

1.3 Connections

The concept of a connection on a vector bundle provides a notion of a directional derivative in the more general context of arbitrary vector bundles. In this section we let $\pi\colon E \to M$ be an arbitrary vector bundle over M.

Definition 1.11 (Connection)**:**
An affine connection is a map $\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ *denoted as* $(X, s) \mapsto \nabla_X s$ *that satisfies the following properties:*

1. *It is $C^\infty(M)$ linear in the first argument:*
$$\nabla_{fX+gY}s = f\nabla_X s + g\nabla_Y s \text{ for } f,g \in C^\infty(M).$$
2. *It is linear in the second argument:*
$$\nabla_X(as+bs') = a\nabla_X s + b\nabla_X s' \text{ for } a,b \in \mathbb{R}.$$
3. *It acts as a derivation on the second argument:*
$$\nabla_X(fs) = X(f)s + f\nabla_X s \text{ for } f \in C^\infty(M).$$

The expression $\nabla_X s$ is called the covariant derivative of s in the direction X. Using the connection on E we also define an associated connection on the dual bundle E^*.

Proposition 1.12 (Connection on dual bundle)**:**
Given a connection ∇ on E then there is a unique connection ∇ on E^* such that
$$X(\alpha(s)) = \alpha(\nabla_X s) + (\nabla_X a)(s)$$
for all $s \in \Gamma(E)$ and $\alpha \in \Gamma(E^*)$.

For $\alpha \in \Gamma(E^*)$ this connection is simply given by $(\nabla_X\alpha)(s) = X(\alpha(s)) - \alpha(\nabla_X s)$ for all $s \in \Gamma(E)$ (see for example [Lee97, p. 54]). Using this extension of the connection to the dual bundle we can also define an associated connection on the vector bundle $\otimes^k E^* \otimes \otimes^l E$

Proposition 1.13:
Given a connection ∇ on E there is a unique connection ∇ on $\otimes^k E^* \otimes \otimes^l E$ such that the following properties hold:

1. *On $\otimes^0 E^* \otimes \otimes^0 E = C^\infty(M)$ we have $\nabla_X f = X(f)$.*
2. *On E and E^* the connection coincided with the previously described connections.*
3. *The connection obeys the product rule:*
$$\nabla_X(F \otimes G) = (\nabla_X T) \otimes G + F \otimes (\nabla_X G).$$
4. *The connection commutes with the contraction operation:*
$$\nabla_X(\operatorname{tr}_{ab} F) = \operatorname{tr}_{ab} \nabla_X F.$$

For all vector fields X and indices a and b.

This connection on $\otimes^k E^* \otimes \otimes^l E$ is given by
$$(\nabla_X F)(s_1, ..., s_k, \alpha_1, ..., \alpha_l) = X(F(s_1, ..., s_k, \alpha_1, ..., \alpha_l))$$
$$-\sum_{i=1}^{k} F(s_1, ..., \nabla_X s_i, ..., s_k, \alpha_1, ..., \alpha_l) - \sum_{j=1}^{l} F(s_1, ..., s_k, \alpha_1, ..., \nabla_X \alpha_j, ..., \alpha_l)$$
for $F \in \Gamma(\otimes^k E^* \otimes \otimes^l E)$ and sections $s_1, ..., s_k \in \Gamma(E)$ and $\alpha_1, ..., \alpha_l \in \Gamma(E^*)$ (again see [Lee97, p. 54]).

Remark 1.14 From now on we will use the shorthand notation ∇_i for ∇_{∂_i}.

Remark 1.15 In what follows we will often use Proposition 1.13 to extend a given connection to the vector bundle $\otimes^k E^* \otimes \otimes^l E$ without explicitly mentioning it.

1.3.1 Levi-Civita Connection

Of special interest is the case of connections on TM. In this case a connection is a mapping of the form $\nabla\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$. To give a coordinate expression for a given connection we introduce the Christoffel symbol: Γ^k_{ij}. It is defined by $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$. In local coordinates we find

$$\nabla_X Y = \nabla_X(Y^j\partial_j) = X(Y^j)\partial_j + Y^j\nabla_{X^i\partial_i}\partial_j = X(Y^j)\partial_j + X^iY^j\Gamma^k_{ij}\partial_k. \tag{1.1}$$

While in general a manifold admits many connections it turns out that there is one uniquely determined connection on a Riemannian manifold that is in some way compatible with the metric. This connection is called the Levi-Civita connection.

Proposition 1.16 (The Levi-Civita connection)**:**
Let M be a Riemannian manifold. There exists a unique connection on TM that satisfies the following two properties:

1. *Symmetry:* $\nabla_XY - \nabla_YX = [X,Y]$
2. *Compatibility with the metric:* $X(g(Y,Z)) = g(\nabla_XY,Z) + g(Y,\nabla_YZ)$

for all vector fields $X,Y,Z\in\mathfrak{X}(M)$.

For a proof see [Lee97, p. 68]. For the Christoffel symbol of the Levi-Civita connection we have the following expression:

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{1.2}$$

Example 1.17:
From (1.2) we see that on the Riemannian manifold $(\mathbb{R}^n, g_{can})$ we have $\Gamma^k_{ij}=0$. This implies that on in this case we simply have $\nabla_XY = X(Y^j)\partial_j$.

Remark 1.18 The properties of the connection imply that for vector fields X, Y and Z we have $(\nabla_Xg)(Y,Z) = X(g(Y,Z)) - g(\nabla_XY,Z) - g(Y,\nabla_XZ)$. Hence the compatibility with the metric property of the Levi-Civita connection implies that $\nabla_Xg = 0$ for all vector fields X. This means the metric is covariantly constant with respect to the Levi-Civita connection. As consequence of this property we have that the Levi-Civita connection commutes with the metric contraction:

Proposition 1.19:
Let F be a either a $(k,0)$ tensor or a $(0,l)$ tensor on a Riemannian manifold. Then we have for any vector field X and indices a,b that

$$\nabla_X(\operatorname{tr}_{g,ab}F) = \operatorname{tr}_{g,ab}(\nabla_XF).$$

Proof. First let F be a $(0,l)$ tensor field with $l\geq 2$. We first notice that $\operatorname{tr}_{g,ab}F = \operatorname{tr}_{1a,2b}(g\otimes F)$ (to verify this identity one can write both sides out in coordinates). Using that $\nabla_Xg=0$ gives $\nabla_X(g\otimes F) = (\nabla_Xg)\otimes F + g\otimes(\nabla_XF) = g\otimes(\nabla_XF)$. Combining these calculations yields

$$\nabla_X(\operatorname{tr}_{g,ab}F) = \nabla_X(\operatorname{tr}_{1a,2b}(g\otimes F)) = \operatorname{tr}_{1a,2b}(\nabla_X(g\otimes F)) = \operatorname{tr}_{1a,2b}(g\otimes(\nabla_XF)) = \operatorname{tr}_{g,12}(\nabla_XF).$$

A calculation using the properties of the connection reveals that for the dual metric we also have $\nabla_X(\langle\alpha,\beta\rangle_g) = \langle\nabla_X\alpha,\beta\rangle_g + \langle\alpha,\nabla_X\beta\rangle_g$ for any one-forms α and β. Using this fact a similar proof can be given to show that the Levi-Civita connection also commutes with the metric contraction for $(k,0)$ tensor fields. □

We will almost exclusively use the Levi-Civita connection. So unless otherwise stated we assume a connection on $T^{(k,l)}(M)$ to be the Levi-Civita connection and Γ^k_{ij} to be the corresponding Christoffel symbol.

1.3.2 Total covariant derivative

Because $X \mapsto \nabla_X F$ is $C^\infty(M)$ linear we can define a tensor field ∇F called the total covariant derivative of F. This total covariant derivative is an analogue of the total derivative from multivariable analysis.

Definition 1.20 (Total covariant derivative)**:**
Given a (k,l) tensor field we define a $(k+1,l)$ tensor field ∇F called the total covariant derivative of F. On vector fields X, Y_i and one-forms α_j this tensor fields acts as follows:

$$(\nabla F)(X, Y_1, ..., Y_k, \alpha_1, ..., \alpha_l) = (\nabla_X F)(Y_1, ..., Y_k, \alpha_1, ..., \alpha_k).$$

Higher covariant derivatives can be defined inductively.

Definition 1.21 (Higher covariant derivatives)**:**
Given a (k,l) tensor fields F we define a $(k+p,l)$ tensor field $\nabla^p F$ that acts on vector field X_i, Y_j and one-forms α_n as

$$(\nabla^p F)(X_1, ..., X_p, Y_1, ..., Y_k, \alpha_1, ..., \alpha_l) = [\nabla_{X_1}(\nabla^{p-1}F)](X_2, ..., X_p, Y_1, ..., Y_k, \alpha_1, ..., \alpha_l).$$

We will use the notation $\nabla^2_{X,Y} F$ for $\mathrm{tr}_{13,24}(X \otimes Y \otimes \nabla^2 F)$. It is important to note that this is distinct from the notation $\nabla_X \nabla_Y F$. In the latter case the operator ∇_X works on the (k,l) tensor field $\nabla_Y F$. Where in the former case we have that ∇_X works on the $(k+1,l)$ tensor field ∇F. To find a expression for $\nabla^2_{X,Y}$ we use the properties of the connection and definition of the higher covariant derivative to find:

$$\begin{aligned}
\nabla_X \nabla_Y F &= \nabla_X[\mathrm{tr}_{12}(Y \otimes \nabla F)] = \mathrm{tr}_{12}[\nabla_X(Y \otimes \nabla F)] \\
&= \mathrm{tr}_{12}[(\nabla_X Y) \otimes \nabla F] + \mathrm{tr}_{12}[Y \otimes (\nabla_X \nabla F)] \\
&= \nabla_{(\nabla_X Y)} F + \mathrm{tr}_{13,24}[X \otimes Y \otimes \nabla^2 F] \\
&= \nabla_{(\nabla_X Y)} F + \nabla^2_{X,Y} F
\end{aligned}$$

for every tensor F. From this we find that

$$\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{(\nabla_X Y)}. \tag{1.3}$$

1.4 The Laplace-Beltrami operator

We now introduce the Laplace-Beltrami operator. This operator is the analogue of the Laplace operator.

Definition 1.22 (The Laplace-Beltrami operator)**:**
The Laplace-Beltrami operator $\Delta\colon T^{(k,l)}M \to T^{(k,l)}M$ acts on tensor fields F as follows:

$$\Delta F = \mathrm{tr}_{g,12}(\nabla^2 F).$$

We will refer to this operator simply as the Laplacian.

Example 1.23:
On the Riemannian manifold $(\mathbb{R}^n, g_{can})$ the Laplace-Beltrami operator acting on scalar functions coincides with the classical Laplacian

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x^i \partial x^i}.$$

We define harmonic functions as follows:

Definition 1.24 (Harmonic functions)**:**
A function $f \in C^\infty(M)$ is called harmonic if $\Delta f = 0$ everywhere on the manifold.

Proposition 1.25:
A harmonic function on a compact and oriented Riemannian manifold is constant.

For a proof see [Lee13]. Another common way to define the Laplacian on $C^\infty(M)$ is by using the gradient and divergence operators. The gradient operator $\operatorname{grad}\colon C^\infty(M) \to \mathfrak{X}(M)$ is defined by $\operatorname{grad}(f) = (df)^\sharp$. The operator $\operatorname{div}\colon \mathfrak{X}(M) \to C^\infty(M)$ is defined such that

$$\mathcal{L}_X \operatorname{vol}_n = d(\iota_X \operatorname{vol}_n) = \operatorname{div}(X) \operatorname{vol}_n$$

holds for all vector fields $X \in \mathfrak{X}(M)$. Now the Laplacian on $C^\infty(M)$ can also be defined as $\Delta = \operatorname{div} \circ \operatorname{grad}$. See [AH11, p. 46] for a proof that these definitions coincide.

1.5 Geodesics and the exponential map

Another notion that we can define using the metric is that of curve length and geodesics.

Definition 1.26 (Curve length)**:**
Given a curve $\gamma\colon I \to M$ we define the length of this curve as

$$L(\gamma) = \int_I \|\dot\gamma(t)\|_g \, dt.$$

On a connected manifold we can define a distance function $d\colon M \times M \to \mathbb{R}$ as

$$d(p,q) = \inf\{L(\gamma) \mid \gamma \text{ a curve connecting } p \text{ and } q\}.$$

Proposition 1.27 (Distance function)**:**
The function $d\colon M \times M \to [0,\infty)$ as defined above is indeed a distance function and the topology induced by this distance coincides with the topology of M.

For a proof see [Lee13, p. 339]. Looking at this definition we can ask ourselves which curves have minimal length. To answer this we define geodesics.

Definition 1.28 (Geodesics)**:**
A curve $\gamma\colon I \to M$ is called a geodesic if

$$\ddot\gamma^k + \Gamma^k_{ij}\dot\gamma_i\dot\gamma_j = 0 \quad \forall k \in \{1, ..., n\}$$

in local coordinates.

Example 1.29:
On the Riemannian manifold $(\mathbb{R}^n, g_{can})$ we have $\Gamma^k_{ij} = 0$. The geodesic equation simply becomes $\ddot\gamma^k = 0$ for all $1 \le k \le n$. This means geodesics in $\mathbb{R}^n$ are straight lines. In $\mathbb{R}^n$ we know that straight lines are curves of minimal length.

It turns out that these geodesics are (at least locally) curves of minimal length (see [LVdB, Th. 4.7]). Standard results from the theory of ordinary differentials give us the following result:

Proposition 1.30 (Short time existence of geodesics)**:**
For every $p \in M$ and $X \in T_pM$ there exists a unique geodesic γ_X such that

$$\gamma_X(0) = p \quad \textit{and} \ \dot{\gamma}_X(0) = X.$$

This geodesic is defined on a maximal open interval I_X which contains 0.

See [Lee97, p. 58]. Using this result we can introduce a special set of local coordinates. We do this via the exponential map.

Proposition 1.31 (The exponential map)**:**
The set $\Omega = \{X \in TM \mid [0,1] \subset I_X\}$ is an open subset of TM and the map $\exp\colon \Omega \to M$ given by $\exp(X) = \gamma_X(1)$ is smooth. Furthermore for every $p \in M$ there is a neighbourhood $U \subset T_pM$ of 0 such that $\exp$ restricts to a diffeomorphism on U.

See Proposition 5.7 and 5.10 of [Lee97].

Definition 1.32 (Normal coordinates)**:**
Let $p \in M$ and let U be a neighbourhood of 0 in T_pM on which $\exp_p$ restricts to a diffeomorphism. Let (e_i) be an orthonormal basis of T_pM. This gives the isomorphism $E\colon \mathbb{R}^n \to T_pM$, $E(x^1, ...x^n) = x^i e_i$. We define $\phi\colon \exp_p(U) \to \mathbb{R}^n$ by $\phi = E^{-1} \circ \exp_p^{-1}$ and $V = \exp_p(U)$. Now the pair (V, ϕ) defines a coordinate chart. This coordinate chart is called a normal coordinate chart centred at p.

There is a one-to-one correspondence between orthonormal bases of T_pM and normal coordinates centred at p. Normal coordinates have the following properties:

Proposition 1.33 (Properties of the normal coordinates)**:**
Let $p \in M$ and $(U, (x^1, ..., x^n))$ be a normal coordinate chart centred at p. The following properties hold:

1. *In coordinates we have that $p = (0, ..., 0)$.*
2. *At the point p we have $g_{ij}|_p = \delta_{ij}$.*
3. *The Christoffel symbols vanish at p: $\Gamma^k_{ij}|_p = 0$.*
4. *The derivatives of the metric at p also vanish: $\partial_k g_{ij}|_p = 0$.*

For a proof see [Lee97, p. 78].

Remark 1.34 In a normal coordinate system we have $\Gamma^k_{ij}|_p = 0$ hence $\nabla_i \partial_j|_p = 0$. This means that in view of Proposition 1.13 that for a tensor field F we find $(\nabla_s F)^{i_1,...i_l}_{j_1,...,j_k}|_p = (\partial_s F^{i_1,...i_l}_{j_1,...,j_k})|_p$. We also see from (1.3) that $\nabla^2_{i,j}|_p = \nabla_i \nabla_j|_p$. These facts simplify tensor calculations in normal coordinates considerably.

1.6 Curvature

A tool that can be used to compare Riemannian manifolds is curvature. Curvature measures how much the manifold deviates from being flat.

Definition 1.35 (Riemann curvature endomorphism)**:**
We define the Riemann curvature endomorphism to be the map $R\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$:

$$R(X,Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z.$$

By using (1.3) and the symmetry property of the Levi-Civita connection we find

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{(\nabla_X Y - \nabla_Y X)}Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \quad (1.4)$$

We can regard R both as a bundle homomorphism $R\colon \otimes^2 TM \to \mathrm{End}(TM)$ or when using the identification $\mathrm{End}(V) \cong V \otimes V^*$ as a $(3,1)$ tensor field. In the latter case we can write R in local coordinates as

$$R = R^l_{ijk} dx^i \otimes dx^j \otimes dx^k \otimes \partial_l \text{ with } R^l_{ijk} \text{ such that } R(\partial_i, \partial_j)\partial_k = R^l_{ijk}\partial_l.$$

The coordinate expression for R^l_{ijk} is given by

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js}. \quad (1.5)$$

Remark 1.36 Using that $\Gamma^k_{ij} = 0$ for the Riemannian manifold $(\mathbb{R}^n, g_{can})$ we find that $R = 0$. So the Euclidean space is flat. In fact a converse statement also holds: if for a manifold $R = 0$ then it is locally isometric to $(\mathbb{R}^n, g_{can})$ (see [Lee97]).

Remark 1.37 Using Proposition 1.13 we can extend R to work on tensor bundles and interpret R as a bundle homomorphism $R\colon \otimes^2 TM \to \mathrm{End}(T^{(k,l)}M)$. It satisfies the following properties: $R(X,Y)(\mathrm{tr}_{ab} F) = \mathrm{tr}_{ab}(R(X,Y)F)$ and $R(X,Y)(F \otimes G) = (R(X,Y)F) \otimes G + F \otimes (R(X,Y)G)$. For more details see [AH11, p. 31].

Definition 1.38 (Riemann curvature tensor)**:**
We define the Riemann curvature tensor Rm as the $(4,0)$ tensor field obtained by lowering the last index of the tensor field R described above. Thus $Rm = R^\flat$ which acts on vector fields as

$$Rm(X,Y,Z,W) = g(R(X,Y)Z,W).$$

Proposition 1.39 (Symmetries of the Riemann curvature tensor)**:**
The Riemann curvature tensor satisfies the following symmetries:

1. $Rm(X,Y,Z,W) = -Rm(Y,X,Z,W)$.
2. $Rm(X,Y,Z,W) = -Rm(X,Y,W,Z)$.
3. $Rm(X,Y,Z,W) = Rm(Z,W,X,Y)$.
4. *The first Bianchi identity:*

$$Rm(X,Y,Z,W) + Rm(Y,Z,X,W) + Rm(Z,X,Y,W) = 0.$$

See [Lee97, p. 121]. In local coordinates these symmetries can be condensed to

$$Rm_{ijkl} = -Rm_{jikl} = -Rm_{ijlk} = Rm_{klij}$$
$$Rm_{ijkl} + Rm_{jkil} + Rm_{kijl} = 0.$$

The symmetry properties imply that the Riemann curvature tensor Rm can be seen as a section of the bundle $\mathrm{Sym}^2(\wedge^2 TM)$. Because a $(4,0)$ tensor field is somewhat unwieldy we will define some lower rank tensor fields which carry much of the same information.

Definition 1.40 (Ricci curvature)**:**
We define the $(2,0)$ *tensor field Ric, called the Ricci curvature as follows:*

$$Ric = \operatorname{tr}_{11} R.$$

In coordinates we have $Ric_{ij} = R^k_{kij}$ and using the symmetries of R we find that Ric is a symmetric tensor field.

Definition 1.41 (Scalar curvature)**:**
We define the scalar curvature S *as the metric contraction of Ric:*

$$S = \operatorname{tr}_{g,12} Ric.$$

In coordinates we find $S = g^{ij} R^k_{kij}$.

There is one final notion of curvature we will define. Suppose $\Pi \subset T_pM$ is a 2 dimensional linear subspace of T_pM. For this argument we denote by Ω the restriction of Rm to Π. We notice that $\Omega \in \operatorname{Sym}^2(\wedge^2\Pi)$ which is of dimension 1. The 4-form ω is a basis element of this space if defined as follows:

$$\omega(X,Y,Z,W) = g(X,Z)g(Y,W) - g(X,W)g(Y,Z).$$

It is easy to check that this element ω has the same symmetries hence $\omega \in \operatorname{Sym}^2(\wedge^2\Pi)$. It is also nonzero because $\omega(e_1,e_2,e_1,e_2) = 1$ for any orthonormal basis (e_1,e_2). This means that for a certain scalar value K we have $\Omega = -K\omega$. By definition, this K is called the sectional curvature.

Definition 1.42 (Sectional curvature)**:**
Let $\Pi \subset T_pM$ *be a 2 dimensional subspace. We define the sectional curvature of this subspace as*

$$K[\Pi] = \frac{-Rm(X,Y,X,Y)}{g(X,X)g(Y,Y) - (g(X,Y))^2}$$

for any pair (X,Y) *that spans* Π.

By our previous discussion we see that $K[\Pi]$ is independent of the basis chosen for Π. In particular we have for an orthonormal basis that $K[\Pi] = Rm(e_1,e_2,e_1,e_2)$.

For a general symmetric bilinear map $\beta\colon V \times V \to \mathbb{R}$ we can look at its quadratic form $\alpha(v) = \beta(v,v)$. This quadratic form determines β via the relation $\beta(v,w) = \frac{1}{2}(\alpha(v+w) - \alpha(v) - \alpha(w))$. We see that $X \wedge Y \mapsto Rm(X,Y,X,Y)$ is the quadratic form of Rm hence this map determines Rm. Furthermore $Rm(X,Y,X,Y)$ is fully determined by $K[\Pi]$ (with Π the span of X and Y). This implies that the sectional curvatures $K[\Pi]$ completely determine Rm.

1.6.1 Surfaces

In the case of surfaces (M^2, g) some simplifications can be made. There is only one sectional curvature which we will denote simply as K. On two-dimensional Riemannian manifolds the sectional curvature is also called the Gaussian curvature. From our previous discussion we know that this sectional curvature completely determines the Riemann curvature tensor. More concretely we have in local coordinates:

$$Rm_{ijkl} = -K[g_{ik}g_{jl} - g_{il}g_{jk}].$$

By applying a contraction to this identity we find that

$$Ric_{ij} = g^{kl} Rm_{kijl} = -K[g^{kl} g_{kj} g_{il} - g^{kl} g_{kl} g_{ij}] = -K[\delta^l_j g_{il} - (\mathrm{tr}_{g,12}\, g) g_{ij}] = -K[1-2] g_{ij} = K g_{ij}.$$

So we find $Ric = Kg$. Contracting once more gives us $S = \mathrm{tr}_{g,12}\, Ric = K\, \mathrm{tr}_{g,12}\, g = 2K$, so the scalar curvature is twice the sectional curvature.

An important result regarding curvature on a two dimensional Riemannian manifold is the Gauss-Bonnet theorem. It shows that globally the curvature is determined by the topology of the surface.

Theorem 1.43 (Gauss-Bonnet theorem)**:**
Let (M^2, g) be a compact and oriented Riemannian manifold of dimension two, then

$$\int_M K \,\mathrm{vol}_2 = 2\pi\chi(M).$$

Here $\chi(M)$ denotes the Euler characteristic of M.

See [Lee97, p. 167].

Chapter 2

Ricci flow on surfaces

In this chapter we will begin our discussion of the Ricci flow on surfaces. First we will introduce the Ricci flow on Riemannian manifolds. In this general case we derive evolution equations for the curvature. Then we will introduce a useful modification to the Ricci flow and specialise to the case of Riemannian surfaces.

2.1 Ricci flow

Ricci flow as introduced by Hamilton is defined as follows:

Definition 2.1 (Ricci flow)**:**
Ricci flow is the evolution of the metric $g(t)$ by the following differential equation:

$$\begin{aligned} \frac{\partial g(t)}{\partial t} &= -2Ric[g(t)] \\ g(0) &= g_0. \end{aligned} \tag{2.1}$$

To make sense of this definition we consider $g(t)$ to be a one-parameter family of metrics on the fixed manifold M. We will refer to the parameter t simply as "time" throughout this text. The time derivative of the metric as used in the differential equation is defined as follows:

Definition 2.2 (Time derivative of a one-parameter family of tensor fields)**:**
Assume $T(t)$ is a smooth one-parameter family of (k, l) tensor fields. The time derivative $\partial T/\partial t$ is a tensor field of the same type such that

$$(\frac{\partial T}{\partial t})(Y_1, ..., Y_k, \omega_1, ..., \omega_l) = \frac{\partial}{\partial t}(T(t)(Y_1, ..., Y_k, \omega_1, ..., \omega_l))$$

for all vector fields Y_i and one-forms ω_j that are independent of time.

We see that the function $(Y_1, ..., Y_k, \omega_1, ..., \omega_l) \mapsto (\partial T/\partial t)(Y_1, ..., Y_k, \omega_1, ..., \omega_l)$ is $C^\infty(M)$ multilinear. Using the characterization of tensor fields as multilinear maps (as in Proposition 1.2) we see that $\partial T/\partial t$ does indeed define a (k, l) tensor field. In local coordinates we have that

$$\frac{\partial T}{\partial t} = (\frac{\partial}{\partial t} T^{i_1, ..., i_l}_{j_1, ..., j_k})\, dx^{j_1} \otimes ... \otimes dx^{j_k} \otimes \partial_{i_1} \otimes ... \otimes \partial_{i_l}.$$

Armed with this definition of a time derivative we can now interpret (2.1) as a differential equation. In local coordinates a solution of the Ricci flow must obey the following differential equation:

$$\frac{\partial g_{ij}}{\partial t} = -2Ric_{ij}[g(t)] \quad \text{and} \quad g_{ij}(0) = (g_0)_{ij}.$$

2.2 Evolution of curvature

In order to study the Ricci flow we need to investigate how the various quantities related to curvature, R, Ric and S, vary under an evolving metric. First we look at the general case of a smoothly evolving metric and then specialise the results where needed to the case that $g(t)$ evolves under the Ricci flow.

In this section we will take $g(t)$ to be a smooth one-parameter family of metrics and we denote $h(t) = \partial g(t)/\partial t$. We now investigate how several quantities related to g vary. All proofs in this section can also be found in [AH11].

Proposition 2.3 (Time derivative of the metric inverse)**:**
For the components of the metric inverse ($g^{ij} = (g^{-1})_{ij}$) the time derivative is given by

$$\frac{\partial}{\partial t} g^{ij} = -g^{ik} g^{jl} h_{kl}.$$

Proof. We begin by noting that $g_{kl} g^{lj} = \delta^j_k$. Taking the time derivative yields

$$(\frac{\partial}{\partial t} g_{kl}) g^{lj} + g_{kl}(\frac{\partial}{\partial t} g^{lj}) = 0.$$

Multiplying by g^{ik} gives

$$g^{ik} g^{jl}(\frac{\partial}{\partial t} g_{kl}) + g^{ik} g_{kl}(\frac{\partial}{\partial t} g^{lj}) = g^{ik} g^{jl} h_{kl} + \delta^i_l(\frac{\partial}{\partial t} g^{lj}) = g^{ik} g^{jl} h_{kl} + (\frac{\partial}{\partial t} g^{ij})$$

and from this we find $\partial g^{ij}/\partial t = -g^{ik} g^{jl} h_{kl}$. □

The time dependence of the metric also means that the Levi-Civita connection, $\nabla = \nabla_{g(t)}$, becomes time dependent. Although the operator $\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is not a tensor field we can define its time derivative in the same way as we did for tensor fields:

$$(\frac{\partial}{\partial t}\nabla)_X Y := \frac{\partial}{\partial t}(\nabla_X Y).$$

It is interesting to note that $(\partial/\partial t)\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is $C^\infty(M)$ bilinear because

$$\begin{aligned} (\frac{\partial}{\partial t}\nabla)_{fX} Y &= \frac{\partial}{\partial t}(f\nabla_X Y) = f(\frac{\partial}{\partial t}\nabla)_X Y \\ (\frac{\partial}{\partial t}\nabla)_X (fY) &= \frac{\partial}{\partial t}(X(f) + f\nabla_X Y) = 0 + f(\frac{\partial}{\partial t}\nabla)_X Y. \end{aligned}$$

Using the characterization of tensor fields as multilinear maps (as in Proposition 1.2) we find that $(\partial/\partial t)\nabla$ defines a $(2,1)$ tensor field on M. In coordinates we have $(\partial/\partial t)\nabla = (\partial\Gamma^k_{ij}/\partial t)\ dx^i \otimes dx^j \otimes \partial_k$.

Proposition 2.4 (Time derivative of the Levi-Civita connection)**:**
The time derivative of the Christoffel symbol Γ_{ij}^k of the Levi-Civita connection is given by

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left\{(\nabla_i h)_{jl} + (\nabla_j h)_{il} - (\nabla_l h)_{ij}\right\}.$$

Here we used $(\nabla_a h)_{bc}$ to denote $(\nabla h)(\partial_a, \partial_b, \partial_c)$.

Proof. Recall from (1.2) that in coordinates Γ_{ij}^k is given by

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

Let $p \in M$ be arbitrary. In order to find the time derivative of $\Gamma_{ij}^k|_p$ at a certain time $t = t_0$ we choose to work in a normal coordinate system with respect to the metric $g(t_0)$ that is centred at p. We now take the time derivative to find

$$\frac{\partial}{\partial t}\Gamma_{ij}^k|_p = \frac{1}{2}(\frac{\partial}{\partial t}g^{kl})(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})|_p + \frac{1}{2}g^{kl}(\partial_i(\frac{\partial}{\partial t}g_{jl}) + \partial_j(\frac{\partial}{\partial t}g_{il}) - \partial_l(\frac{\partial}{\partial t}g_{ij}))|_p =$$
$$0 + \frac{1}{2}g^{kl}(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij})|_p.$$

Here we found that the first term is zero because $\partial_a g_{bc}|_p = 0$ in the chosen coordinate system. Now using the facts from Remark 1.34 we find that

$$\frac{\partial}{\partial t}\Gamma_{ij}^k|_p = \frac{1}{2}g^{kl}\left\{(\nabla_i h)_{jl} + (\nabla_j h)_{il} - (\nabla_l h)_{ij}\right\}|_p.$$

Now we notice that the left-hand side of this identity is the coordinate expression for the tensor field $\frac{\partial}{\partial t}\nabla$. The right-hand side is also a coordinate expression for a tensor field. From this we conclude that the identity holds true in any coordinate system. $\square$

Proposition 2.5 (Time derivative of Riemann curvature)**:**
The time derivative of the Riemann curvature R is given by

$$\frac{\partial}{\partial t}R_{ijk}^l = \frac{1}{2}g^{ls}\left\{\begin{array}{c} (\nabla_{i,j}^2 h)_{ks} + (\nabla_{i,k}^2 h)_{js} - (\nabla_{i,s}^2 h)_{jk} \\ -(\nabla_{j,i}^2 h)_{ks} - (\nabla_{j,k}^2 h)_{is} + (\nabla_{j,s}^2 h)_{ik} \end{array}\right\}.$$

Here we used $(\nabla_{a,b}^2 h)_{cd}$ to denote $(\nabla^2 h)(\partial_a, \partial_b, \partial_c, \partial_d)$.

Proof. Recall from (1.5) that R_{ijk}^l is given by

$$R_{ijk}^l = \partial_i\Gamma_{jk}^l - \partial_j\Gamma_{ik}^l + \Gamma_{jk}^s\Gamma_{is}^l - \Gamma_{ik}^s\Gamma_{js}^l.$$

Let $p \in M$ be arbitrary. Again we will work with normal coordinates with respect to $g(t_0)$ centred at p. Taking the time derivative yields

$$\frac{\partial}{\partial t}R_{ijk}^l = \partial_i(\frac{\partial}{\partial t}\Gamma_{jk}^l) - \partial_j(\frac{\partial}{\partial t}\Gamma_{ik}^l) + (\frac{\partial}{\partial t}\Gamma_{jk}^s)\Gamma_{is}^l + \Gamma_{jk}^s(\frac{\partial}{\partial t}\Gamma_{is}^l) - (\frac{\partial}{\partial t}\Gamma_{ik}^s)\Gamma_{js}^l - \Gamma_{ik}^s(\frac{\partial}{\partial t}\Gamma_{js}^l).$$

Using that $\Gamma_{ab}^c|_p = 0$ and using the result from Proposition 2.4 we find that at p we have

$$\frac{\partial}{\partial t}R_{ijk}^l = \frac{1}{2}\partial_i(g^{ls}\left\{(\nabla_j h)_{ks} + (\nabla_k h)_{js} - (\nabla_s h)_{jk}\right\}) - \frac{1}{2}\partial_j(g^{ls}\left\{(\nabla_i h)_{ks} + (\nabla_k h)_{is} - (\nabla_s h)_{ik}\right\}).$$

We are working in a normal coordinate system hence $\partial_a(\nabla_b h)_{cd} = (\nabla^2_{a,b}h)_{cd}$ by Remark 1.34. Substituting this gives

$$\frac{\partial}{\partial t}R^l_{ijk} = \frac{1}{2}g^{ls}\left\{ \begin{array}{c} (\nabla^2_{i,j}h)_{ks} + (\nabla^2_{i,k}h)_{js} - (\nabla^2_{i,s}h)_{jk} \\ -(\nabla^2_{j,i}h)_{ks} - (\nabla^2_{j,k}h)_{is} + (\nabla^2_{j,s}h)_{ik} \end{array} \right\}.$$

As was the case in the previous proof both sides of this equation are local coordinate expressions of tensor fields hence the identity must hold in every coordinate system. □

Proposition 2.6 (Time derivative of the Ricci curvature)**:**
The time derivative of the Ricci curvature is given by

$$\frac{\partial}{\partial t}Ric_{ij} = \frac{1}{2}g^{ks}\left\{(\nabla^2_{k,i}h)_{js} + (\nabla^2_{k,j}h)_{is} - (\nabla^2_{k,s}h)_{ij} - (\nabla^2_{i,j}h)_{ks}\right\}.$$

Proof. By contracting the result of Proposition 2.5 on $i = l$ we find that

$$\begin{aligned} \frac{\partial}{\partial t}Ric_{jk} = \frac{\partial}{\partial t}R^i_{ijk} &= \frac{1}{2}g^{is}\left\{(\nabla^2_{i,j}h)_{ks} + (\nabla^2_{i,k}h)_{js} - (\nabla^2_{i,s}h)_{jk} - (\nabla^2_{j,k}h)_{is}\right\} \\ &+ \frac{1}{2}g^{is}\left\{(\nabla^2_{j,s}h)_{ik} - (\nabla^2_{j,i}h)_{ks}\right\}. \end{aligned} \tag{$*$}$$

We now claim that $g^{is}\left\{(\nabla^2_{j,s}h)_{ik} - (\nabla^2_{j,i}h)_{ks}\right\} = 0$. To see that this is true notice that

$$g^{is}(\nabla^2_{j,s}h)_{ik} - g^{is}(\nabla^2_{j,i}h)_{ks} = g^{is}(\nabla^2_{j,s}h)_{ik} - g^{si}(\nabla^2_{j,s}h)_{ik} = 0.$$

Which is true because g^{si} is symmetric. Renaming indices in $(*)$ gives us the stated result. □

Proposition 2.7 (Time derivative of the scalar curvature)**:**
The time derivative of the scalar curvature is given by

$$\frac{\partial}{\partial t}S = -\Delta(\operatorname{tr}_{g,12} h) + \operatorname{tr}_{g,13,24}(\nabla^2 h) - \langle h, Ric\rangle_g.$$

Proof. We apply the time derivative to the coordinate expression of S and find that

$$\frac{\partial}{\partial t}S = \frac{\partial}{\partial t}(g^{ij}Ric_{ij}) = (\frac{\partial}{\partial t}g^{ij})Ric_{ij} + g^{ij}(\frac{\partial}{\partial t}Ric_{ij}).$$

For the first term we find by using Proposition 2.3 that

$$(\frac{\partial}{\partial t}g^{ij})Ric_{ij} = -g^{ik}g^{jl}h_{kl}Ric_{ij}.$$

For the second term we use Proposition 2.6 and find by renaming indices that

$$\begin{aligned} g^{ij}(\frac{\partial}{\partial t}Ric_{ij}) &= \frac{1}{2}g^{ij}g^{ks}\left\{(\nabla^2_{k,i}h)_{js} + (\nabla^2_{k,j}h)_{is} - (\nabla^2_{k,s}h)_{ij} - (\nabla^2_{i,j}h)_{ks}\right\} \\ &= g^{ij}g^{ks}\left\{(\nabla^2_{k,i}h)_{js} - (\nabla^2_{i,j}h)_{ks}\right\} = g^{ik}g^{jl}\left\{(\nabla^2_{j,i}h)_{kl} - (\nabla^2_{i,k}h)_{jl}\right\}. \end{aligned}$$

By combining these identities we find

$$\frac{\partial}{\partial t}S = g^{ik}g^{jl}\left\{(\nabla^2_{j,i}h)_{kl} - (\nabla^2_{i,k}h)_{jl} - h_{kl}Ric_{ij}\right\}.$$

The first term in this expression is equal to $\operatorname{tr}_{g,14,23}(\nabla^2 h) = \operatorname{tr}_{g,13,24}(\nabla^2 h)$ (here we used that h is symmetric). And the second term is equal to

$$-\operatorname{tr}_{g,12,34} \nabla^2 h = -\operatorname{tr}_{g,12} \nabla^2(\operatorname{tr}_{g,12} h) = -\Delta(\operatorname{tr}_{g,12} h)$$

because the metric contraction commutes with ∇. The last term is clearly equal to $-\langle h, Ric\rangle$. This proves that

$$\frac{\partial}{\partial t} S = -\Delta_g(\operatorname{tr}_{g,12} h) + \operatorname{tr}_{g,13,24}(\nabla^2 h) - \langle h, Ric\rangle_g.$$

□

Proposition 2.8 (Time derivative of the volume form)**:**
The time derivative of the volume form is given by

$$\frac{\partial}{\partial t} \operatorname{vol}_n = \frac{1}{2}(\operatorname{tr}_{g,12} h) \operatorname{vol}_n .$$

Proof. In any positively oriented coordinate system vol_n is given by

$$\operatorname{vol}_n = \sqrt{\det(g_{ij})}\; dx^1 \wedge \ldots \wedge dx^n.$$

We will use the Jacobi formula for the derivative of a determinant which states that for a differentiable family of invertible matrices $A(t)$ we have

$$\frac{d}{dt} \det A(t) = (\det A(t)) \operatorname{tr}(A^{-1} \frac{dA}{dt}).$$

Using this we find

$$\frac{\partial}{\partial t}\sqrt{\det(g_{ij})} = \frac{1}{2}\frac{1}{\sqrt{\det(g_{ij})}}\frac{\partial}{\partial t}\det(g_{ij}) = \frac{1}{2}\frac{1}{\sqrt{\det(g_{ij})}}\det(g_{ij})\operatorname{tr}(g^{-1}h) = \frac{1}{2}(\operatorname{tr}_{g,12} h)\sqrt{\det(g_{ij})}.$$

Substituting this in the coordinate expression for vol_n yields

$$\frac{\partial}{\partial t} \operatorname{vol}_n = \frac{1}{2}(\operatorname{tr}_{g,12} h) \operatorname{vol}_n .$$

□

2.3 Normalised Ricci flow

When the metric evolves under the Ricci flow we have that $h = -2Ric$. Now Proposition 2.8 shows that if the manifold M is orientable its volume evolves as follows:

$$\frac{\partial}{\partial t}\operatorname{Vol}(M) = \int_M \frac{\partial}{\partial t} \operatorname{vol}_n = -\int_M (\operatorname{tr}_{g,12} Ric) \operatorname{vol}_n = -\int_M S \operatorname{vol}_n = -4\pi\chi(M).$$

Here the last equality follows from the Gauss-Bonnet theorem. We find that if $\chi(M) \neq 0$ the manifold either collapses to a point in finite time or expands infinitely. As the underlying manifold doesn't change by Ricci flow we mean by collapsing to a point that for a certain time we have $g = 0$. This then means that all points are distance zero apart from the perspective of the Riemannian distance function as defined in Proposition 1.27. It will turn out to be useful to look at a modified flow which has the property that $\operatorname{Vol}(M)$ remains constant during evolution.

Definition 2.9 (Normalised Ricci flow)**:**
On an orientable manifold the normalised Ricci flow is the evolution of the metric $g(t)$ under the differential equation

$$\begin{aligned}\frac{\partial g(t)}{\partial t} &= \frac{2s}{n}g(t) - 2Ric[g(t)]\\ g(0) &= g_0.\end{aligned}$$

Here s is defined to be the average scalar curvature

$$s = \frac{\int_M S\,\mathrm{vol}_n}{\int_M \mathrm{vol}_n} = \frac{4\pi\chi(M)}{\mathrm{Vol}(M)}.$$

Remark 2.10 We will sometimes refer to the Ricci flow as RF and to the normalised Ricci flow as NRF.

Under the normalised Ricci flow we have $h = \frac{2s}{n}g - 2Ric$ and the variation of the volume is now given by

$$\frac{\partial}{\partial t}\mathrm{Vol}(M) = \int_M \frac{1}{n}(\mathrm{tr}_{g,12}\, g)s\,\mathrm{vol}_n - \int_M (\mathrm{tr}_{g,12}\, Ric)\,\mathrm{vol}_n = s\cdot\mathrm{Vol}(M) - \int_M S\,\mathrm{vol}_n = 0.$$

This fact implies that the quantity s, the average scalar curvature, remains constant during the evolution of the NRF.

2.4 Ricci flow on surfaces

We now specialise our study of the Ricci flow to the case of surfaces. As discussed in Section 1.6.1 we have that $Ric = Kg = \frac{1}{2}Sg$ so the NRF equation becomes

$$\begin{aligned}\frac{\partial g(t)}{\partial t} &= (s - S(t))g(t)\\ g(0) &= g_0.\end{aligned} \tag{2.2}$$

Remark 2.11 If a Riemannian surface has constant scalar curvature then $S = s$ and the metric is a fixed point of the normalised Ricci flow. Conversely if a solution to the normalised Ricci flow is a fixed point we find that we must have $S = s$ hence the scalar curvature is constant. In conclusion: A Riemannian metric on a surface is a fixed point of the normalised Ricci flow if and only if it has constant scalar curvature.

A property of the normalised Ricci flow in two dimensions is that the solution stays conformal to the initial metric during evolution. We will prove this in Section 3.4. This means that if a solution to the NRF exists on the interval $[0, T)$ we can write it as $g(x,t) = u(x,t)\cdot g_0(x)$ for some $u\colon M\times[0,T)\to\mathbb{R}$. A fact we will prove later is that on compact surfaces the metric $g(t)$ also remains equivalent to $g(0)$ (see for example Proposition 3.22). This implies that we must have $u > 0$. Hence we can write $u(x,t) = e^{v(x,t)}$ for some $v\colon M\times[0,T)\to\mathbb{R}$. Substituting $g(t) = e^{v(t)}g_0$ into (2.2) immediately gives us that $\partial v/\partial t = s - S$.

Remark 2.12 Throughout this text we will sometimes suppress the position and time dependence of various quantities from our notation to improve readability.

We now look for a differential equation for u such that $u \cdot g_0$ is a solution to the NRF.

Proposition 2.13 (Scalar PDE for NRF on surfaces)**:**
Suppose that $g(x,t) = u(x,t) \cdot g_0(x)$ is a solution to the NRF on surfaces. Then u satisfies the following evolution equation:

$$\frac{\partial u}{\partial t} = \Delta_{g_0} \log u + su - S_{g_0} \quad \textit{with} \quad u(0) = 1. \tag{2.3}$$

Here Δ_{g_0} is the Laplace-Beltrami operator associated with the initial metric g_0. Furthermore S_{g_0} is the scalar curvature of the initial metric. Conversely if u satisfies this evolution equation then $g(x,t) = u(x,t) \cdot g_0(x)$ is a solution to the NRF on surfaces.

Proof. Let $g(t) = u(t)g_0$ be a solution to the NRF on surfaces. We use the following transformation identity: if $g = u \cdot h$ then $S_g = \frac{1}{u}(S_h - \Delta_h \log u)$ (see Corollary A.5). Substituting this in (2.2) gives us

$$\frac{\partial}{\partial t} g(t) = (\frac{\partial u}{\partial t})g_0 = (s - S(t))g(t) = (s - \frac{1}{u}(S_{g_0} - \Delta_{g_0} \log u))ug_0 = (\Delta_{g_0} \log u + su - S_{g_0})g_0.$$

This means that for $g(t) = u(t)g_0$ to be a solution to the NRF we have that u must evolve by (2.3). Conversely if we define $g(t) = u(t)g_0$ with $u(t)$ evolving by (2.3) then it is clear that $g(t)$ is a solution to the NRF. □

It will turn out to be important to be able to determine how the scalar curvature changes under the NRF. The following two proofs are taken from [AH11].

Proposition 2.14:
Under the normalised Ricci flow on surfaces the scalar curvature evolves in the following way:

$$\frac{\partial S}{\partial t} = \Delta_{g(t)} S + S(S - s).$$

Proof. For NRF on surfaces we have $h = (s - S)g$. From Proposition 2.7 we know that

$$\frac{\partial}{\partial t} S = -\Delta_g(\operatorname{tr}_{g,12} h) + \operatorname{tr}_{g,13,24}(\nabla^2 h) - \langle h, Ric \rangle_g. \tag{$*$}$$

For the first term we have $\operatorname{tr}_{g,12} h = 2(s - S)$ so $-\Delta_g(\operatorname{tr}_{g,12} h) = 2\Delta_g S$. For the last term we substitute that $Ric = \frac{1}{2}Sg$ so $-\langle h, Ric \rangle_g = \frac{1}{2}S(S - s)\langle g, g \rangle_g = S(S - s)$. Finally we look at the coordinate expression of the remaining term to find that

$$\operatorname{tr}_{g,13,24}(\nabla^2 h) = g^{ij} g^{kl} (\nabla^2_{i,k}(s - S)g)_{jl} = -g^{ij} g^{kl} g_{jl} \nabla^2_{i,k} S = -g^{ij} \delta^k_j \nabla^2_{i,k} S = -\Delta_g S.$$

Here we used that $\nabla g = 0$. Substituting these results into $(*)$ gives us the required result. □

Finally substituting $h = (s - S)g$ into the result from Proposition 2.4 gives us the following:

Proposition 2.15:
Under the normalised Ricci flow on surfaces we have

$$\frac{\partial}{\partial t} \Gamma^k_{ij} = -\frac{1}{2}\delta^k_j \nabla_i S - \frac{1}{2}\delta^k_i \nabla_j S + \frac{1}{2} g_{ij} g^{kl} \nabla_l S.$$

Chapter 3

The uniformization theorem

In this chapter we will discuss the uniformization theorem and study how the Ricci flow relates to this result.

The uniformization theorem is a result from complex analysis and provides classification of one-dimensional complex manifolds. In this text we will not study this theorem in its full generality but restrict ourselves to a particular consequence of this theorem. A strong connection exists between this consequence and the Ricci flow. We will now state the result we are interested in:

Theorem 3.1 (The uniformization theorem)**:**
Every compact and orientable Riemannian manifold (M^2, g) *of dimension two admits a metric conformal to* g *which has constant scalar curvature.*

Although the actual result is more general we will still refer to this result as "the uniformization theorem" throughout this text.

3.1 Overview

As mentioned above a connection exists between the uniformization theorem and the Ricci flow. We will discuss this connection and show how the Ricci flow can be used to prove Theorem 3.1.

In Remark 2.11 we observed that the fixed points of the normalised Ricci flow are precisely those metrics that have constant scalar curvature. This leads us to hope that these metrics of constant curvature act as attractors of the Ricci flow. A metric evolving under the Ricci flow would then converge towards a metric of constant scalar curvature.

We will discuss the following result which shows that metrics of constant curvature indeed act as attractors to the Ricci flow. In addition this result also implies Theorem 3.1.

Theorem 3.2 (Convergence of the normalised Ricci flow)**:**
Assume that (M, g_0) *is a two dimensional Riemannian manifold that is compact and orientable. Then a unique solution* $g(t)$ *of the normalised Ricci flow as given by (2.2) exists on* $[0, \infty)$*. Furthermore this solution converges to a smooth metric* g_∞ *which is conformal to* g_0 *and has constant scalar curvature.*

It is clear that this theorem immediately implies the uniformization theorem because the limiting metric has all the required properties. It is interesting that in two dimensions Ricci flow is so well behaved because in higher dimensions a result like Theorem 3.2 does not hold in general (see for example [CK04, Ch. 1]).

The strategy for proving this theorem is as follows:

1. First short time existence and uniqueness is proved for the NRF.
2. Then it is shown that the NRF exists for all times if we have a priori bounds on the scalar curvature.
3. These a priori bounds on the scalar curvature are derived thus proving long time existence.
4. Three different cases are distinguished $\chi(M) < 0$, $\chi(M) = 0$ and $\chi(M) > 0$. Convergence to a limiting metric is proved for each case individually.

A complete treatment of this proof is not within the scope of this text. We intend to prove the first three steps and prove step 4 for the cases of $\chi(M) < 0$ and $\chi(M) = 0$. For the final case of $\chi(M) > 0$ we will provide references to a complete proof. The first step of the proof will be discussed in Section 3.4, the second step in Section 3.5 and the third will be discussed in Section 3.6. Finally a discussion of the last step is given in Sections 3.7, 3.8 and 3.9.

The original proof of this theorem was given by Hamilton in [Ham88] (save for a part later proved by Chow in [Cho91]). In this text however we follow the treatment found in [CK04, Ch. 5]. The proof strategy followed in this text is taken from this book. However some proofs have been modified and several gaps have been filled in. Most notably the author deviates from [CK04] in the Sections 3.4 and 3.5 to ensure that the results better fit the case of Ricci flow on surfaces. In each section a short discussion about the origin of the proofs can be found.

3.2 The maximum principle

In the following proofs we will repeatedly make use of a technical tool called the maximum principle. It allows us to estimate the solutions of certain differential equations. We begin with some terminology.

Definition 3.3 (Supersolutions)**:**
Consider the differential equation

$$\frac{\partial v}{\partial t} = \Delta v + \Phi(v)$$

with $\Phi\colon \mathbb{R} \to \mathbb{R}$ *some function. Now we call* u *a supersolution to this differential equation if we have that*

$$\frac{\partial u}{\partial t} \geq \Delta u + \Phi(u)$$

everywhere on the domain of u*.*

Remark 3.4 The notion of a subsolution is defined similarly but with the "$\geq$" sign replaced by "$\leq$".

The term "maximum principle" can refer to many different results but we will only discuss and use the following result:

Proposition 3.5 (The maximum principle)**:**
Let M be a compact manifold and assume $g(t)$ is a smooth one-parameter family of metrics on the interval $[0,T)$. Let $\Phi\colon \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. Assume that $u\colon M \times [0,T) \to \mathbb{R}$ is a C^2 supersolution such that

$$\frac{\partial u(x,t)}{\partial t} \geq \Delta_{g(t)} u(x,t) + \Phi(u(x,t))$$

holds for all $(x,t) \in M \times [0,T)$. Let $\phi\colon [0,T) \to \mathbb{R}$ be a differentiable function such that

$$\frac{\partial \phi(t)}{\partial t} = \Phi(\phi(t)).$$

If at $t = 0$ we have $u(x,0) \geq \phi(0)$ for all $x \in M$ then $u(x,t) \geq \phi(t)$ for all $(x,t) \in M \times [0,T)$.

Remark 3.6 A similar statement is true for subsolutions and this result is called the minimum principle. In this text we will for simplicity refer to this result as the maximum principle as well.

For a proof see [CK04, p. 96].

3.3 Some matters of notation

The proofs in the following sections contain several lengthy expressions. In order to improve readability we introduce some extra notation.

Given two functions $f, g\colon \mathbb{R} \to \mathbb{R}$ we write $f \ll g$ if there exists a constant $C > 0$ such that $f(t) \leq C \cdot g(t)$ for all $t \in \mathbb{R}$. If the functions are not defined on the whole of $\mathbb{R}$ we mean that the inequality should hold on the common domain of the two functions. As an example consider a function f that is bounded by some $C > 0$ (thus $|f| \leq C$), in this case we can write $|f| \ll 1$.

Several times we will estimate quantities by looking at their evolution equation. For the purpose of estimating the exact structure of these equations is less important. We introduce the following notation: for tensor fields A and B we refer by $A * B$ to a tensor field which is a finite linear combination of contractions and metric contractions of the tensor product $A \otimes B$. To give an example of this notation assume that both A and B are $(2,0)$ tensors. Then we can write

$$(\operatorname{tr}_{g,12} A) \otimes B + \operatorname{tr}_{g,14}(A \otimes B) + A \otimes (\operatorname{tr}_{g,12} B) = A * B.$$

One useful thing to notice is that the Cauchy-Schwarz inequality gives us that $|\operatorname{tr}_{11}(\omega \otimes X)| = \left|\langle \omega^\sharp, X\rangle_g\right| \leq \left\|\omega^\sharp\right\|_g \|X\|_g = \|\omega\|_g \|X\|_g$. From this we conclude that in general $\|A * B\|_g \ll \|A\|_g \|B\|_g$.

We have also postponed the proofs of several identities used throughout this text to appendix A.

3.4 Short time existence

We now begin with the actual proof of Theorem 3.2. In this section we prove that the normalised Ricci flow enjoys short time existence. We will also prove that the solution stays conformal to the initial metric. In this section we aim to prove the following result:

Proposition 3.7 (Short time existence)**:**
Let (M^2, g_0) be a compact Riemannian surface. Then there exists a unique and smooth solution to the normalised Ricci flow with $g(0) = g_0$ which is defined on the interval $[0, \epsilon)$ for some $\epsilon > 0$. Furthermore the solution $g(t)$ is conformal to g_0 for all $t \in [0, \epsilon)$.

This result follows mainly from PDE theory on manifold. It is not within the scope of this text to give a full discussion of this PDE theory. Hence we will only reference important results. References treating the short time existence of the NRF in two dimensions are somewhat scarce hence the proofs given in this section are by the author.

The short time existence and uniqueness for the Ricci flow have been proved by Hamilton in [Ham82]. In this paper he proved the following result:

Proposition 3.8:
If (M^n, g_0) is a compact Riemannian manifold then there exists a unique solution $g(t)$ to the normalised Ricci flow defined on some positive time interval $[0, \epsilon)$ such that $g(0) = g_0$.

See [CK04, Thm. 3.13] and [Ham82]. This result implies almost all the statements of Proposition 3.7. What remains to be proved is that this solution remains conformal to g_0. To show this we first prove independently of the result of Proposition 3.8 that a conformal solution to the NRF exists.

Proposition 3.9:
Let (M^2, g_0) be a compact Riemannian surface. Then a constant $\epsilon > 0$ and a function $u\colon M \times [0, \epsilon) \to \mathbb{R}$ exist such that $g(x, t) = u(x, t)g_0(x)$ is a smooth solution to the normalised Ricci flow with $g(0) = g_0$.

This result is a consequence from the theory of partial differential equations. In particular it will follow from the following result:

Proposition 3.10:
Let (M, g) be a compact Riemannian manifold. Consider the differential equation for a scalar function u:

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= a^{ij}(x, u(x,t))\nabla^2_{i,j} u + f(x, u(x,t), \nabla u(x,t)) \\ u(x, 0) &= \phi(x)\end{aligned}$$

Here a^{ij} are the components of a smooth symmetric $(0, 2)$ tensor field. We assume that a^{ij} are, in local coordinates, smooth functions of their arguments. Furthermore f is a smooth scalar function which depends smoothly on its arguments. Finally we have $\phi \in C^\infty(M)$.
If the tensor field $a^{ij}(x, \phi(x))$ is positive definite everywhere then a smooth and unique solution to the differential equation exists on a interval $[0, \epsilon)$ for some $\epsilon > 0$.

For a discussion of this result we refer to [Aub98, Thm 4.51]. Now it is a matter of applying this result to the differential equation given by (2.3).

Proof of Proposition 3.9. From Proposition 2.13 we know that $g(t) = u(t)g_0$ is a solution to the NRF if and only if

$$\frac{\partial u}{\partial t} = \Delta_{g_0} \log u + su - S_{g_0} \quad \text{with} \quad u(0) = 1. \tag{$*$}$$

We now want to write this equation in a form to which we can apply Proposition 3.10. Hence we seek another expression for the quantity $\Delta_0 \log u$. We find that for every $f > 0$ we have

$$\operatorname{grad}(\log f) = (d(\log f))^\sharp = (\frac{1}{f}df)^\sharp = \frac{1}{f}\operatorname{grad} f.$$

For every $f \in C^\infty(M)$ and vector field X we have

$$\operatorname{div}(fX)\operatorname{vol}_n = d(\iota_{fX}(\operatorname{vol}_n)) = d(f \cdot \iota_X(\operatorname{vol}_n)) =$$
$$df \wedge \iota_X(\operatorname{vol}_n) + f d(\iota_X(\operatorname{vol}_n)) = X(f)\operatorname{vol}_n + f\operatorname{div}(X)\operatorname{vol}_n$$

hence $\operatorname{div}(fX) = X(f) + f\operatorname{div}(X)$. Combining these results yields that for $\Delta_{g_0}\log u$ we have

$$\Delta_{g_0}(\log u) = (\operatorname{div}_{g_0} \circ \operatorname{grad}_{g_0})(\log u) = \operatorname{div}_{g_0}(\frac{1}{u}\operatorname{grad}_{g_0} u) = (\operatorname{grad}_{g_0} u)(\frac{1}{u}) + \frac{1}{u}\operatorname{div}_{g_0}(\operatorname{grad}_{g_0} u) =$$
$$-\frac{1}{u^2}(\operatorname{grad}_{g_0} u)(u) + \frac{1}{u}\Delta_{g_0} u = -\frac{1}{u^2}\left\|\operatorname{grad}_{g_0} u\right\|_{g_0}^2 + \frac{1}{u}\Delta_{g_0} u$$

Here we used that $(\operatorname{grad}_{g_0} u)(u) = du(\operatorname{grad}_{g_0}) = \langle \operatorname{grad}_{g_0} u, \operatorname{grad}_{g_0} u\rangle_{g_0} = \left\|\operatorname{grad}_{g_0} u\right\|_{g_0}^2$. Substituting this result into $(*)$ gives

$$\frac{\partial u}{\partial t} = \frac{1}{u}\Delta_{g_0} u - \frac{1}{u^2}\left\|\operatorname{grad}_{g_0} u\right\|_{g_0}^2 + su - S_{g_0}.$$

This equation takes the form of the differential equation in Proposition 3.10 if we define

$$a^{ij}(x,u) = \frac{g_0^{ij}}{u} \quad \text{and} \quad f(x,u,\nabla u) = -\frac{1}{u^2}\left\|\operatorname{grad}_{g_0} u\right\|_{g_0}^2 + su - S_{g_0}.$$

Because g_0 is symmetric and positive definite it is clear that a^{ij} is a symmetric tensor field and that $a^{ij}(x,\phi)$ is positive definite everywhere. Applying Proposition 3.10 yields that a smooth solution $u(t)$ to $(*)$ exists on an interval $[0,\epsilon)$ for some $\epsilon > 0$. We conclude that $g(x,t) = u(x,t)g_0(x)$ defines a smooth solution to the NRF on surfaces on the interval $[0,\epsilon)$. This proves the proposition. □

Now we can combine the short time existence result from Proposition 3.9 with the uniqueness property of Proposition 3.8. This yields the following proof of Proposition 3.7:

Proof of Proposition 3.7. By Proposition 3.8 we know a solution $g(t)$ to the normalised Ricci flow on surfaces exists on a interval $[0,\epsilon)$ for some $\epsilon > 0$. Possibly by shrinking ϵ we know by Proposition 3.9 that on this interval also a smooth and conformal solution $\widetilde{g}(x,t) = u(x,t)g_0(x)$ to the NRF exists. However by the uniqueness clause of Proposition 3.8 we know these solutions must coincide on $[0,\epsilon)$. We conclude that a smooth and conformal solution to the NRF exists on the interval $[0,\epsilon)$ and that this solution is unique. This proves the result. □

3.5 Long time existence

In this section we will prove that appropriate bounds on the scalar curvature imply that a solution to the normalised Ricci flow exists for all times. From the result discussed in the previous section we know the normalised Ricci flow enjoys short time existence and uniqueness. This means that we can define for some $T \leq \infty$ a maximal interval $[0,T)$ on which the conformal solution $g(t)$ exists. It is maximal in the sense that if $T < \infty$ then no solution to the NRF exists on the interval $[0, T + \epsilon)$ for any $\epsilon > 0$.

From now on we assume M to be a compact and orientable two dimensional manifold and $g(t)$ be to be evolving by the normalised Ricci flow.

The proofs in this section are inspired by the treatment found in [CK04, Ch. 6]. The proof of Lemma 3.13 is a slight modification of the proof of [CK04, Lemma 6.49]. The idea of estimating derivatives of the metric is taken from the proof of [CK04, Prop. 6.48] but the rest of the proof of Lemma 3.15 is by the author. The proof of Lemma 3.16 is also by the author.

We will prove the following result:

Proposition 3.11 (Long time existence)**:**
Suppose $[0, T)$ is the maximal interval of existence for the normalised Ricci flow. Assume that $S(t)$ is bounded on every finite subinterval $[0, \tau) \subset [0, T)$. More concretely this means that for every $0 < \tau < \infty$ with $\tau \leq T$ a constant $C_\tau > 0$ exists such that

$$\sup_{x \in M} |S(x,t)| \leq C_\tau \text{ for all } t \in [0, \tau).$$

If this is the case then we must have $T = \infty$, i.e. the solution to the Ricci flow exists for all time.

Remark 3.12 We can interpret the result from this proposition as follows: a solution to the Ricci flow always exists unless a singularity in S occurs at $t = T < \infty$. As we will later prove that on compact surfaces such a singularity can not occur.

We begin with several general results about convergence of the NRF.

Lemma 3.13:
Suppose the solution to the normalised Ricci flow exists on $[0, T)$ for some $0 < T \leq \infty$. We write the solution $g(t)$ as $g(x,t) = e^{v(x,t)} g(x,0)$. If

$$\int_0^T \left| \frac{\partial}{\partial t} v(x,t) \right| dt \leq C \quad \text{for all } x \in M$$

for some constant $C > 0$ then $g(t)$ converges to a continuous metric $g(T)$ which is conformal to g_0. We also have the uniform bounds

$$e^{-(C+1)} g(0) \leq g(t) \leq e^{C+1} g(0) \tag{3.1}$$

for all $t \in [0, T]$.

Remark 3.14 For two symmetric $(0,2)$ tensors A and B we mean by $A \leq B$ that $B - A$ is a non-negative definite quadratic form i.e. for all $V \in TM$ we have $(B - A)(V, V) \geq 0$. In the case of the bounds in (3.1) this implies in particular that $\|V\|_{g(0)} \ll \|V\|_{g(t)} \ll \|V\|_{g(0)}$ for all $V \in TM$.

Proof. We define $v(x,T)$ as

$$v(x,T) = v(x,0) + \int_0^T \frac{\partial}{\partial t} v(x,t)\, dt = 1 + \int_0^T \frac{\partial}{\partial t} v(x,t)\, dt. \tag{3.2}$$

This is well defined by our assumption that $\int_0^T |\partial v/\partial t| \; dt < \infty$. Using this assumption we also find that

$$|v(x,T) - v(x,t)| = \left| \int_t^T \frac{\partial}{\partial t} v(x,\tau) \, d\tau \right| \leq \int_t^T \left| \frac{\partial}{\partial t} v(x,\tau) \right| \, d\tau \xrightarrow{t \to T} 0.$$

Because M is compact this convergence is uniform. This means that $v(t)$ converges uniformly to $v(T)$ hence $v(x,T)$ defines a continuous function on M. From (3.2) we get the estimate $|v(x,t)| \leq 1 + C$. Exponentiating yields $e^{-(C+1)} \leq e^{v(x,t)} \leq e^{C+1}$ and from this the uniform bounds as in (3.1) follow immediately. Now the limiting metric $g(x,T)$ can be defined as $g(x,T) = e^{v(x,T)} g(x,0)$. It is easy to see that $g(T)$ defines a continuous metric on M which is conformal to g_0. □

We remember that for the NRF $\partial v(x,t)/\partial t = s - S(x,t)$ so the hypothesis from the previous lemma can be written as

$$\int_0^T |s - S(x,t)| \; dt \leq C \;\text{ for all } x \in M.$$

This result can be improved in the following way:

Lemma 3.15:
Assume we have in addition to the hypotheses from Lemma 3.13 also that for all $x \in M$ and every $k \geq 1$

$$\int_0^T \left\| \nabla^k S(x,t) \right\| \; dt < \infty.$$

Then $g(t)$ converges to a smooth limiting metric $g(T)$ with respect to the C^p norm for every $p \geq 0$.

Proof. Let (x^i) be any coordinate system. We will show that for every multi-index α the derivative $\partial^{|\alpha|} v(t)/\partial x^\alpha$ converges when $t \to T$. For this we first establish some notations which will make the bookkeeping easier.

We let D^k denote an arbitrary k-th order derivative $\partial^{|\alpha|} v/\partial x^\alpha$. For the purpose of this proof we will deviate slightly from our earlier definition of $A * B$ (as in Section 3.3). In this proof we allow only contractions and not metric contractions when writing $A * B$. In several instances the use of Leibniz's rule and the chain rule will produce tensor products of a tensor field and its derivatives. We introduce a notation to denote such a product:

$$P(A, DA, ..., D^j A) = A^{c_1} * (DA)^{c_2} * ... * (D^j A)^{c_j}$$

with c some multi-index. Another simplification in notation we will make is that sometimes we suppress the indices when writing the Christoffel symbol and simply write Γ.

We now claim that for $k \geq 1$

$$D^k S = \nabla^k S + \sum_{j=1}^{k-1} P(\Gamma, ..., D^{j-1}\Gamma) * \nabla^{k-j} S. \tag{$*$}$$

For $k = 1$ we have $DS = \nabla S$. Using that $DP(\Gamma, ..., D^{j-1}\Gamma) = P(\Gamma, ..., D^j \Gamma)$ and by the definition of the higher covariant derivative $D\nabla^k S = \nabla^{k+1} S + \Gamma * \nabla^k S$ we can easily check the inductive step:

$$DD^kS = D\nabla^kS + \sum_{j=1}^{k-1}[DP(\Gamma,...,D^{j-1}\Gamma) * \nabla^{k-j}S + P(\Gamma,...,D^{j-1}\Gamma) * D\nabla^{k-j}S] =$$

$$= \nabla^{k+1}S + \sum_{j=1}^{k} P(\Gamma,...,D^{j-1}\Gamma) * \nabla^{(k+1)-j}S.$$

Using induction we will prove the following claim:
Claim: For all $k \geq 1$ we have

$$\int_0^T \left|D^kS\right| \, dt < \infty.$$

For $k = 1$ we have $|DS| \ll \|\nabla S\|_{g(0)}$ and by Lemma 3.13 we have $\|\nabla S\|_{g(0)} \ll \|\nabla S\|_{g(t)}$ hence $|DS| \ll \|\nabla S\|_{g(t)}$. Integrating and using the hypothesis of the lemma gives us the required result. Now suppose our claim is true for $1 \leq q \leq k-1$. By inspecting $(*)$ we see that we need to estimate $D^p\Gamma_{ij}^l$ for $0 \leq p \leq k-2$. Using Proposition 2.15 we see that

$$\frac{\partial}{\partial t}D^p\Gamma_{ij}^l = D^p\frac{\partial}{\partial t}\Gamma_{ij}^l = D^p(-\delta_i^l\partial_jS - \delta_j^l\partial_iS + g_{ij}g^{ll}\partial_lS) = -\delta_i^lD^{p+1}S - \delta_j^lD^{p+1}S + D^p[g_{ij}g^{ls}\partial_sS].$$

We will now use the fact that $\partial_lg^{ij} = -g^{ii'}g^{jj'}(\partial_lg_{i'j'})$ (the proof of this is similar to the proof of Proposition 2.3). We now find that we can write the last term as

$$D^p(g_{ij}g^{ls}\partial_sS) = \sum_{n=0}^{p} P(g,...,D^ng) * D^{p-n+1}S.$$

In order to estimate D^ng for $0 \leq n \leq k-2$ we remember that $g(t) = e^{v(t)}g_0$. By differentiating $\partial v/\partial t = s - S$ and using our induction hypothesis we have

$$|D^nv(t) - D^nv(0)| \leq \int_0^t \left|D^n\frac{\partial v}{\partial t}\right| \, dt \leq \int_0^T |D^nS| \, dt < \infty.$$

This means that $|D^nv(t)| \ll 1$ and from this follows that also $\left|D^ne^{v(t)}\right| \ll 1$. By differentiating $g(t) = e^{v(t)}g_0$ we find that

$$D^ng(t) = \sum_{m=0}^{n} D^m[e^{v(t)}] * D^{n-m}g_0.$$

Because M is compact we have that D^mg_0 is also bounded hence $|D^ng(t)| \ll 1$. Now we can finally put this all together by observing that the bounds we derived for $|D^ng(t)|$ imply that $\left|\partial D^p\Gamma_{ij}^l(t)/\partial t\right|$ is bounded by

$$\left|\frac{\partial}{\partial t}D^p\Gamma_{ij}^l(t)\right| \ll \sum_{m=1}^{p} |D^mS(t)|.$$

Now integrating and using the induction hypothesis (remember that $p \leq k-2$) gives us

$$\left|D^p\Gamma_{ij}^l(t) - D^p\Gamma_{ij}^l(0)\right| \ll \sum_{m=1}^{p}\int_0^t |D^mS| \, dt \leq \sum_{m=1}^{p}\int_0^T |D^mS| \, dt < \infty$$

hence $|D^p\Gamma(t)| \ll 1$. Using this we find from $(*)$ that

$$\left|D^k S(t)\right| \ll \sum_{j=1}^{k} \left\|\nabla^j S(t)\right\|_{g(0)} \ll \sum_{j=1}^{k} \left\|\nabla^j S(t)\right\|_{g(t)}.$$

Integrating this inequality gives that there is a $C > 0$ such that

$$\int_0^T \left|D^k S\right| \, dt \leq C \sum_{j=1}^{k} \int_0^T \left\|\nabla^j S\right\| \, dt < \infty.$$

This proves the claim. Now we find using this result that

$$\left|D^k v(x,T) - D^k(x,t)\right| = \left|\int_t^T \frac{\partial}{\partial t} D^k v(x,\tau) \, d\tau\right| \leq \int_t^T \left|D^k S(x,\tau)\right| \, d\tau \xrightarrow{t \to T} 0.$$

This convergence is uniform since M is compact. This implies for any multi-index α that $\partial^{|\alpha|} v/\partial x^\alpha$ converges uniformly to a limit when $t \to T$. We conclude that $v(t) \to v(T)$ in any C^k norm hence the limit $v(T)$ is smooth. From this follows that $g(T) = e^{v(T)} g_0$ is smooth. □

We now return to our long time existence result. First we prove that bounds on S imply that bounds exist on the derivatives of S.

Lemma 3.16:
Suppose $T < \infty$ and we have a constant $C > 0$ such that $\sup_{x \in M} |S(x,t)| \leq C$ for all $t \in [0,T)$. Then for every $k \geq 1$ there exists a constant $C_k > 0$ such that

$$\sup_{x \in M} \left\|\nabla^k S(x,t)\right\| \leq C_k$$

for all $t \in [0,T)$.

Before we begin this proof we introduce some new notation. We will study the evolution equation of the quantity $\left\|\nabla^k S\right\|^2$. This equation will contain several terms $\nabla^l S$ of lower order $(l < k)$. It is convenient to introduce a notation for a particular combination of these lower order terms:

$$\mathcal{R}^k = \sum_{j=1}^{\lfloor k/2 \rfloor} (\nabla^j S) * (\nabla^{k-j} S). \tag{3.3}$$

Notice that this is a linear combination of all terms $\nabla^i S * \nabla^j S$ such that $i + j = k$ and $0 < i, j < k$.

Proof. We proceed by induction. Our hypothesis states that the result is true for $k = 0$. Now suppose it is true for $0 \leq j \leq k-1$.

We will now study the evolution equation of $\left\|\nabla^k S\right\|^2$. For clarity the derivation of this evolution equation has been moved to Appendix A. Using the notation we have just introduced Proposition A.11 gives us that

$$\frac{\partial}{\partial t} \left\|\nabla^k S\right\|^2 = \Delta \left\|\nabla^k S\right\|^2 - 2\left\|\nabla^{k+1} S\right\|^2 - (k+2)s\left\|\nabla^k S\right\|^2 + 4S\left\|\nabla^k S\right\|^2 + \nabla^k S * \mathcal{R}^k.$$

By the induction hypothesis we find that there exists a constant $B > 0$ such that $\left\|\mathcal{R}^k\right\| \leq B$. We find that for suitable $A > 0$ we have

$$\begin{aligned} \frac{\partial}{\partial t}\left\|\nabla^k S\right\|^2 &\leq \Delta \left\|\nabla^k S\right\|^2 + [4S - (k+2)s] \cdot \left\|\nabla^k S\right\|^2 + B\left\|\nabla^k S\right\| \\ &\leq \Delta \left\|\nabla^k S\right\|^2 + A\left\|\nabla^k S\right\|^2 + B\left\|\nabla^k S\right\| \\ &\leq \Delta \left\|\nabla^k S\right\|^2 + (A+1)\left\|\nabla^k S\right\|^2 + B \end{aligned}$$

The compactness of M implies that $\left\|\nabla^k S\right\|^2$ is bounded on $t = 0$. This means that we can use the maximum principle to compare $\left\|\nabla^k S\right\|^2$ with the solution of the differential equation $\partial\phi/\partial t = (A+1)\phi + B$. This yields that there exists a $D > 0$ such that on $[0, T)$ we have

$$\left\|\nabla^k S\right\|^2 \leq \frac{1}{A+1}\left[(B+D)e^{(A+1)t} - B\right].$$

Because the right-hand side of this inequality is bounded on $[0, T]$ we find $\left\|\nabla^k S\right\|^2$ is bounded on $[0, T)$. This proves the result. ☐

We can now prove Proposition 3.11.

Proof of Proposition 3.11. Suppose that $T < \infty$. Then by our hypothesis we have that a $C > 0$ exists such that $\sup_{x\in M}|S(x,t)| \leq C$ for all $t \in [0, T)$. Now by Lemma 3.16 we know $\sup_{x\in M}\left\|\nabla^k S(x,t)\right\| \leq C_k$ for all $k \geq 1$ and suitable constants $C_k > 0$. This means that we have

$$\int_0^T \left|\frac{\partial v(x,t)}{\partial t}\right| dt = \int_0^T |s - S(x,t)|\, dt \leq C \cdot T < \infty \quad \text{for all } x \in M$$

and for every $k \geq 1$

$$\int_0^T \left\|\nabla^k (\frac{\partial v}{\partial t})\right\| dt = \int_0^T \left\|\nabla^k S\right\| dt \leq C_k \cdot T < \infty \quad \text{for all } x \in M.$$

Now using Lemma 3.13 and Lemma 3.15 we find that $g(t)$ converges to a smooth metric $g(T)$. However Proposition 3.7 now implies the solution to the NRF can be extended to an interval $[T, T+\epsilon)$. This contradicts the definition of T. We conclude our assumption was false hence we must have $T = \infty$. ☐

3.6 A priori bounds on S

We now prove there exist a priori bounds on S on the domain of definition of the solution to the NRF. We will derive these bounds by using the maximum principle. All proofs in this section can be found in [CK04, Chap. 5]. However the proof of Lemma 3.16 is slightly modified to be somewhat more rigorous and the proof of Proposition 3.21 has been added by the author.

Proposition 3.17 (A priori bounds on scalar curvature)**:**
Let $[0, T)$ (with $T \leq \infty$) be the maximal interval on which the solution to the normalised Ricci flow exists. Then a constant $C > 0$, depending only on the initial metric, exists such that

$$|S(t) - s| \leq Ce^{st} \quad \textit{for all } t \in [0, T).$$

For the proof of this proposition we introduce a new concept: the curvature potential function.

Definition 3.18 (The curvature potential function)**:**
The curvature potential function is defined to be a smooth one-parameter family $t \mapsto f(t) \in C^\infty(M)$ such that $\Delta_{g(t)} f(t) = S(t) - s$ for all $t \in [0, T)$. We also require that

$$\frac{\partial f}{\partial t} = \Delta f + sf.$$

We first prove such a one-parameter family exists.

Lemma 3.19:
The curvature potential function as defined in Definition 3.18 exists.

To prove this result we will use the following result:

Proposition 3.20:
Let M be a compact manifold, let J be an interval and let $p \in M$. Assume that $g(t)$, for $t \in J$, is a smooth family of metrics on M. Furthermore let $\phi(t) \in C^\infty(M)$, for $t \in J$, be a smooth one-parameter family of smooth functions on M such that

$$\int_M \phi(t) \ \mathrm{vol}_n[g(t)] = 0$$

for all $t \in J$. Then there exists a one-parameter family $f(t) \in C^\infty(M)$, for $t \in J$, of smooth functions on M such that $\Delta_{g(t)} f(t) = \phi(t)$ and $f(p, t) = 0$ for all $t \in J$. This one-parameter family is smooth on $M \times J$.

This result can be proved using the properties of the Laplacian and functional analytic principles.

Proof of Lemma 3.19. First we prove the existence of a smooth one-parameter family $t \mapsto \widetilde{f}(t) \in C^\infty(M)$ with $\Delta_{g(t)} f(t) = S(t) - s$ but with a different boundary condition.

We pick an arbitrary $m \in M$ and let $\phi(t) = S(t) - s$. By applying Proposition 3.20 we conclude a smooth one-parameter family $[0, T) \ni t \mapsto \widetilde{f}(t) \in C^\infty(M)$ exists such that $\Delta_{g(t)} \widetilde{f}(t) = S(t) - s$ and $\widetilde{f}(m, t) = 0$.

We now look for a function $c\colon [0, T) \to \mathbb{R}$ depending only on time such that if we define $f(t) = \widetilde{f}(t) + c(t)$ we have

$$\frac{\partial f}{\partial t} = \Delta f + sf.$$

By differentiation the identity $\Delta \widetilde{f} = S - s$ with respect to t we find $\partial(\Delta \widetilde{f})/\partial t = \partial S/\partial t$. We find using $(\partial/\partial t)\Delta = (S - s)\Delta$ (see Proposition A.9) that

$$\frac{\partial}{\partial t}(\Delta \widetilde{f}) = (S - s)\Delta \widetilde{f} + \Delta \left(\frac{\partial}{\partial t} \widetilde{f} \right)$$

$$\frac{\partial S}{\partial t} = \Delta S + S(S - s) = \Delta\Delta \widetilde{f} + S\Delta \widetilde{f}.$$

Combining this yields

$$\Delta \left(\frac{\partial}{\partial t} \widetilde{f} \right) = \Delta\Delta \widetilde{f} + s\Delta \widetilde{f} = \Delta(\Delta \widetilde{f} + s\widetilde{f}).$$

From this we find that

$$\Delta\left(\frac{\partial}{\partial t}\widetilde{f}-\Delta\widetilde{f}+s\widetilde{f}\right)=0.$$

Because harmonic functions on compact manifolds are constant (see Proposition 1.25) we conclude there exists a function γ only depending on time such that

$$\frac{\partial}{\partial t}\widetilde{f}=\Delta\widetilde{f}+s\widetilde{f}+\gamma.$$

We see that this function is smooth and uniquely determined by $\widetilde{f}$. We let $c(t)$ be a solution to the differential equation $\partial c/\partial t = sc-\gamma$. An expression for c is $c(t)=-e^{st}\int_0^t e^{-s\tau}\gamma(\tau)\,d\tau$. Now it is easy to check that the function $f=\widetilde{f}+c(t)$ satisfies the properties $\Delta_{g(t)}f(t)=S(t)-s$ and

$$\frac{\partial f}{\partial t}=\Delta f+sf.$$

We conclude the curvature potential function exists. □

Now we can proof Proposition 3.17.

Proof of Proposition 3.17. The lower bound of S is the easiest to derive. We look at the quantity $\Phi=S-s$. Using the result from Proposition 2.14 we find that

$$\frac{\partial\Phi}{\partial t}=\frac{\partial S}{\partial t}=\Delta S+S(S-s)=\Delta(S-s)+(S-s)^2+s(S-s)\geq\Delta\Phi+s\Phi.$$

Because M is compact there exists a constant $C_1>0$ such that $\Phi(0)=S(0)-s\geq -C_1$. Appealing to the maximum principle we find that $\Phi(t)\geq -C_1e^{st}$ for all $t\in[0,T)$. The derivation of an upper bound requires considerably more work.

Here we make use of the curvature potential function introduced earlier. We look at the quantity $H=S-s+\|\nabla f\|^2$. It turns out H satisfies a particularly nice evolution equation to which we can apply the maximum principle.

Claim: The quantity H satisfies the following evolution:

$$\frac{\partial}{\partial t}H=\Delta H-2\|F\|^2+sH.$$

Here we denote by $F=\nabla^2 f-\frac{1}{2}(\Delta f)g$ the trace-free Hessian of f.

First we have that

$$\frac{\partial}{\partial t}(S-s)=\Delta S+S(S-s)=\Delta(S-s)+(\Delta f)^2+s(S-s).$$

Then we calculate using the identity $\Delta\nabla=\nabla\Delta+\frac{1}{2}S\nabla$ (see Proposition A.7) and Proposition A.10 that

$$\begin{gathered}\frac{\partial}{\partial t}(\|\nabla f\|^2)=(S-s)\|\nabla f\|^2+2\langle\nabla f,\frac{\partial}{\partial t}\nabla f\rangle_g\\ =(S-s)\|\nabla f\|^2+2\langle\nabla f,\nabla\Delta f+\nabla(sf)\rangle_g\\ =(S+s)\|\nabla f\|^2+2\langle\nabla f,\Delta\nabla f\rangle-S\|\nabla f\|_g^2=s\|\nabla f\|^2+2\langle\nabla f,\Delta\nabla f\rangle_g=\\ \Delta\|\nabla f\|^2-2\left\|\nabla^2 f\right\|+s\|\nabla f\|^2.\end{gathered}$$

In the last step we used the identity $\langle \nabla f, \Delta \nabla f \rangle_g = \frac{1}{2}\Delta \|\nabla f\|^2 - \|\nabla^2 f\|^2$ (see Proposition A.6). Combining these results gives

$$\frac{\partial}{\partial t} H = \Delta H + sH + (\Delta f)^2 - 2\left\|\nabla^2 f\right\|^2.$$

To see the stated identity holds we perform the following calculation:

$$\begin{aligned}\|F\|^2 = \langle F, F\rangle_g &= \left\|\nabla^2 f\right\|^2 - (\Delta f)\langle \nabla^2 f, g\rangle_g + \frac{1}{4}(\Delta f)^2\langle g, g\rangle_g \\ &= \left\|\nabla^2 f\right\|^2 - (\Delta f)^2 + \frac{1}{2}(\Delta f)^2 = \left\|\nabla^2 f\right\|^2 - \frac{1}{2}(\Delta f)^2.\end{aligned}$$

From this we find that we indeed have $\partial H/\partial t = \Delta H - 2\|F\|^2 + sH$. This proves our claim.

Now we notice that $\partial H/\partial t = \Delta H - 2\|F\|^2 + sH \leq \Delta H + sH$. Because M is compact there exists a $C_2 > 0$ such that $\Phi(0) \leq C_2$. Using the maximum principle we find that

$$S(t) - s \leq H(t) \leq C_2 e^{st} \text{ for all } t \in [0, T).$$

Combining this with our earlier estimate we conclude that for a certain $C > 0$ we have $|S(t) - s| \leq Ce^{st}$ on $[0, T)$. ☐

Now we can combine these bounds with the result from our earlier Proposition 3.11 to prove the following result:

Proposition 3.21 (Long time existence NRF)**:**
On a compact and orientable two dimensional Riemannian manifold the normalised Ricci flow given by

$$\begin{aligned}\frac{\partial g(t)}{\partial t} &= (s - S(t))g \\ g(0) &= g_0\end{aligned}$$

has a unique solution $g(t)$ which exists for all times $t \in [0, \infty)$.

Proof. Proposition 3.7 shows that for every smooth initial metric a unique solution to the NRF exists on an interval $[0, \epsilon)$ for some $\epsilon > 0$. This implies that we can, for some $T \leq \infty$, define a maximal interval $[0, T)$ on which a unique solution exists. Now using the bound $|S(t) - s| \leq Ce^{st}$ as in Proposition 3.17 we can show that the hypotheses of Proposition 3.11 are satisfied. To see this let $\tau < \infty$ and define the constant

$$C_\tau = |s| + \sup_{t \in [0, \tau]} Ce^{st}.$$

The bound $|S(t) - s| \leq Ce^{st}$ implies that $\sup_{x \in M} |S(x, t)| \leq C_\tau$ for all $t \in [0, \tau)$. Proposition 3.11 now states that $T = \infty$ thus the solution to the normalised Ricci flow exists for all times $t \in [0, \infty)$. ☐

Armed with this result we will from now on assume all solutions of the NRF to be defined on the whole of $[0, \infty)$ without explicitly mentioning this.

Using the curvature potential function we can also prove the following proposition:

Proposition 3.22 (Equivalence of metrics)**:**
Let $(M, g(t))$ be the solution of the normalised Ricci flow on a surface with $s \leq 0$. Then the metrics $g(t)$ are equivalent. More specifically there exists a $C \geq 1$ independent of time such that

$$\frac{1}{C} g(0) \leq g(t) \leq C g(0)$$

for all $t \in [0, \infty)$.

See Remark 3.14 for the interpretation of the inequalities between tensor fields as used in this proposition.

Proof. For f we have the following evolution equation $\partial f / \partial t = \Delta f + sf$. Because M is compact there is a constant $A > 0$ depending only on g_0 such that $|f(0)| \leq A$. Using the maximum principle gives us that $|f(t)| \leq A e^{st}$. Substituting $g(x,t) = u(x,t) g_0(x)$ into the differential equation for NRF (2.2) we find

$$\frac{\partial u(x)}{\partial t} = (s - S(x)) u(x) = (-\Delta f) u(x) = (sf - \frac{\partial f}{\partial t}) u(x).$$

Integrating this against time and using $|f(t)| \ll e^{st}$ yields

$$\left| \log \left(\frac{u(x,t)}{u(x,0)} \right) \right| = \left| s \int_0^t f(x, \tau)\, d\tau - f(x,t) + f(x,0) \right| \ll |s| \int_0^t e^{st}\, dt + e^{st} + 1 \ll e^{st} + 1 \ll 1.$$

In the last step we used $e^{st} \ll 1$ because $s \leq 0$. Exponentiating now gives us that there is a constant $c > 0$ such that $e^{-c} u(x,0) \leq u(x,t) \leq e^{c} u(x,0)$ for all times $t \in [0, \infty)$. We define $C = e^c$. This proves the proposition. □

3.7 Convergence for $\chi(M) < 0$

In the case that $\chi(M) < 0$ we have that $s < 0$ so the bounds

$$s - Ce^{st} \leq S \leq s + Ce^{st} \tag{3.4}$$

show that the scalar curvature converges uniformly to its average. What remains to be shown is that $g(t)$ converges to a smooth metric. The proof in this section is a slight modification of the proof found in [CK04, §5.5]. Changes have primarily been made to better fit results of Section 3.5 into the proof.

We will prove the following result:

Lemma 3.23:
Let $(M^2, g(t))$ be the solution to the normalised Ricci flow on a compact and orientable surface M with $\chi(M) < 0$. Then a unique smooth metric g_∞ on M exists such that $g(t)$ converges to g_∞. This convergence is with respect to the C^k norm for any $k \in \mathbb{N}$. Furthermore this limiting metric g_∞ is conformal to g_0 and has constant scalar curvature.

Proof. We begin this proof by showing that the smoothing properties of the Ricci flow imply that k-th order derivatives of S converge uniformly to zero.
Claim: For every $k \geq 1$ there exists a constant $C_k > 0$ such that

$$\left\| \nabla^k S(t) \right\|^2 \leq C_k e^{st/2}.$$

Our proof of this claim will be based on induction. The variation of the quantity $\left\|\nabla^k S\right\|^2$ under NRF is given by

$$\frac{\partial}{\partial t}\left\|\nabla^k S\right\|^2 = \Delta\left\|\nabla^k S\right\|^2 - 2\left\|\nabla^{k+1} S\right\|^2 - (k+2)s\left\|\nabla^k S\right\|^2 + 4S\left\|\nabla^k S\right\|^2 + \nabla^k S * \mathcal{R}^k$$

(this result is proved in appendix A see Proposition A.11). First we treat the case $k = 1$, looking at the definition of $\mathcal{R}^k$ we see that $\mathcal{R}^1 = 0$ and using the bounds on $|s - S|$ as in (3.4) we find that

$$\frac{\partial}{\partial t}\|\nabla S\|^2 = \Delta\|\nabla S\|^2 - \left\|\nabla^2 S\right\|^2 + 4(S-s)\|\nabla S\|^2 + s\|\nabla S\|^2 \leq \Delta\|\nabla S\|^2 + (s + 4Ce^{st})\|\nabla S\|^2.$$

For a certain time $t_0 > 0$ we have that if $t \geq t_0$ then $4Ce^{st} \leq -s/2$ (remember that $s < 0$). For such $t \geq t_0$ we have

$$\frac{\partial}{\partial t}\|\nabla S\|^2 \leq \Delta\|\nabla S\|^2 + \frac{s}{2}\|\nabla S\|^2.$$

Because M is compact there is a C_1 such that $\|\nabla S(0)\|^2 \leq C_1$. Applying the maximum principle we find $\|\nabla S(t)\|^2 \leq C_1 e^{st/2}$. This proves the case of $k = 1$. Now assume our statement is true for $1 \leq j \leq k-1$. For $t \to \infty$ we have $S \to s$. We have $s < 0$ so this implies there exists a $t_1 > 0$ such that if $t \geq t_1$ then $S(t) < 0$. The induction hypothesis yields that there is a constant $A > 0$ such that $\left\|\mathcal{R}^k\right\| \leq Ae^{st/2}$. We combine these two estimates to find for $t \geq t_1$ that

$$\frac{\partial}{\partial t}\left\|\nabla^k S\right\|^2 \leq \Delta\left\|\nabla^k S\right\|^2 - (k+2)s\left\|\nabla^k S\right\|^2 + Ae^{st/2}\left\|\nabla^k S\right\|.$$

The induction hypothesis also gives for a certain $A' > 0$ that

$$\frac{\partial}{\partial t}\left\|\nabla^{k-1} S\right\|^2 \leq \Delta\left\|\nabla^{k-1} S\right\|^2 - 2\left\|\nabla^k S\right\|^2 + A'e^{st/2}.$$

We now define the following quantity

$$\Phi = \left\|\nabla^k S\right\|^2 - (k+1)s\left\|\nabla^{k-1} S\right\|^2.$$

For suitable $A'', A''' > 0$ we now have

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi &\leq \Delta\Phi + ks\left\|\nabla^k S\right\|^2 + Ae^{st/2}\left\|\nabla^k S\right\| + A''e^{st/2} && \text{now using } x \geq 0 \implies x \leq 1 + x^2 \\
&\leq \Delta\Phi + (ks + Ae^{st/2})\left\|\nabla^k S\right\|^2 + (A + A'')e^{st/2} && \text{now using } (\left\|\nabla^k S\right\|^2 - \Phi) \ll e^{st/2} \\
&\leq \Delta\Phi + (ks + Ae^{st/2})\Phi + A'''e^{st/2}.
\end{aligned}$$

Because $Ae^{st/2} \to 0$ there exists a $t_2 \geq t_1$ such that $t \geq t_2$ implies $Ae^{st/2} \leq -ks/2$. For such times $t \geq t_2$ the following holds

$$\frac{\partial}{\partial t}\Phi \leq \Delta\Phi + \frac{ks}{2}\Phi + A'''e^{st/2}.$$

Because M is compact a constant $B > 0$ exists such that $\Phi(0) \leq B$. Using the maximum principle we can compare Φ with the solution of $\partial\phi/\partial t = ks/2\phi + C'''e^{st/2}$ with $\phi(0) = B$ and find $\Phi(t) \leq \phi(t)$. Using Lemma 3.24 we conclude that a constant $C_k > 0$ exists such that $\left\|\nabla^k S(t)\right\|^2 \leq \Phi(t) \leq C_k e^{st/2}$. This proves our claim.

Now we conclude using the bounds (3.4) that we have

$$\int_0^\infty |s - S| \, dt \leq C \int_0^\infty e^{st} \, dt < \infty.$$

Using the claim we have just proved we find that for any $k \geq 1$

$$\int_0^\infty \left\| \nabla^k S \right\|^2 \, dt \leq D \int_0^\infty e^{st/4} \, dt < \infty$$

for a constant $D > 0$ sufficiently large. Hence by Lemmas 3.13 and 3.15 we find that $g(t)$ converges in any C^k norm to a smooth limiting metric g_∞ which is conformal to g_0. From the bounds of (3.4) we immediately see that g_∞ must have constant scalar curvature. □

It remains to prove Lemma 3.24 which was used in the previous proof.

Lemma 3.24:
Let $\alpha \geq 1$, $s < 0$ and $A > 0$. Assume that $\phi(t)$ is a solution of the differential equation

$$\frac{\partial \phi}{\partial t} = \alpha s \phi + A e^{st/2}.$$

Then a constant $D > 0$ exists such that $\phi \leq D e^{st/2}$.

Proof. The solution to the differential equation is given by

$$\phi(t) = e^{\alpha s t} \left[\phi(0) + \frac{A}{s(\frac{1}{2} - \alpha)} (e^{(\frac{1}{2} - \alpha) st} - 1) \right].$$

It is easy to check that $\phi(t) e^{-st/2}$ converges to $\frac{A}{s(\frac{1}{2}-\alpha)}$ for $t \to \infty$. From this we conclude that there is a constant $D > 0$ such that $\phi(t) e^{-st/2} \leq D$. This proves the lemma. □

3.8 Convergence for $\chi(M) = 0$

We now treat the case of $\chi(M) = 0$. We have in this case that $s = 0$ and the bounds from Proposition 3.17 only show that S stays bounded for all times. This means that we must first prove that S actually converges to $s = 0$. The proofs of Lemmas 3.26 and 3.27 are from [CK04, §5.6]. The remainder of the proof of Lemma 3.25 is inspired by a discussion on the website "math.stackexchange.com" (see [Sta12]). Again modifications have been made so that the results of Section 3.5 fit better into the proof.

We will prove the following:

Lemma 3.25:
Let $(M^2, g(t))$ be the solution to the normalised Ricci flow on a compact and orientable surface M with $\chi(M) = 0$. Then a unique smooth metric g_∞ on M exists such that $g(t)$ converges to g_∞. This convergence is with respect to the C^k norm for any $k \in \mathbb{N}$. Furthermore this limiting metric g_∞ is conformal to g_0 and has constant scalar curvature.

We first prove S is bounded by a function which converges to zero.

Lemma 3.26:
Under the hypotheses of Lemma 3.25 we have that

$$|S(t)| \leq \frac{C}{1+t}$$

for a constant $C > 0$ sufficiently large.

Proof. The evolution equation for S is $\partial S/\partial t = \Delta S + S^2$ when $s = 0$. Because M is compact there is a $C_1 > 0$ such that $-C_1 \leq S(0)$. By using the maximum principle we can compare with the solution to $\partial\phi/\partial t = \phi^2$ with $\phi(0) = -C_1$. We find the lower bound of the form

$$\frac{-C_1}{1 + C_1 t} \leq S(t).$$

Finding a suitable upper bound requires more work. We will use the curvature potential function f as defined in the proof of Proposition 3.17.

Claim: We have $\|\nabla f(t)\|^2 \leq A/(1+t)$ for some constant $A > 0$.
Recall that the evolution equation for $\|\nabla f\|^2$ is

$$\frac{\partial}{\partial t}\|\nabla f\|^2 = \Delta\|\nabla f\|^2 - 2\left\|\nabla^2 f\right\|$$

(see the proof of Proposition 3.17). From this we immediately find that $(\partial\|\nabla f\|^2)/\partial t \leq \Delta\|\nabla f\|^2$. Because M is compact $\|\nabla f(0)\|^2$ is bounded so by the maximum principle we have $\|\nabla f(t)\|^2 \leq A_1$ for some $A_1 > 0$. We also find that

$$\frac{\partial}{\partial t} f^2 = 2f\Delta f = \Delta(f^2) - 2\|\nabla f\|^2.$$

We define the quantity $\Phi = t\|\nabla f\|^2 + f^2$ and calculate

$$\frac{\partial}{\partial t}\Phi = \Delta(t\|\nabla f\|^2) - 2t\left\|\nabla^2 f\right\|^2 + \|\nabla f\|^2 + \Delta(f^2) - 2\|\nabla f\|^2 \leq \Delta\Phi.$$

Because M is compact $\Phi(0)$ is bounded and using the maximum principle we find that $\Phi(t) \leq A_2$ for some $A_2 > 0$. Because $t\|\nabla f(t)\|^2 \leq \Phi(t)$ we conclude $\|\nabla f(t)\|^2 \leq A_2/t$. Combining these two estimates for $\|\nabla f\|^2$ we find that $\|\nabla f(t)\|^2 \leq A/(1+t)$ for $A > 0$ sufficiently large. This proves our claim.

To prove that $S(t) \ll 1/(1+t)$ we define the quantity $\Psi = S + 2\|\nabla f\|^2$. Because $S \leq \Psi$ we complete this proof by proving the following claim:

Claim: We have $\Psi \leq C_2/(1+t)$ for some constant $C_2 > 0$.
In the proof of Proposition 3.17 we calculated $0 \leq \|F\|^2 = \left\|\nabla^2 f\right\|^2 - \frac{1}{2}(\Delta f)^2$ hence $2\left\|\nabla^2 f\right\| \geq (\Delta f)^2 = S^2$. Using this we calculate that

$$\frac{\partial}{\partial t}\Psi = \Delta S + S^2 + 2\Delta\|\nabla f\|^2 - 4\left\|\nabla^2 f\right\|^2 \leq \Delta\Psi - S^2.$$

From this we find that $\frac{\partial}{\partial t}\Psi \leq \Delta\Psi$. Using that $\Psi(0)$ is bounded and appealing to the maximum principle gives $\Psi(t) \leq C_2$ for $C_2 > 0$ large enough. To improve this bound we look at $t\Psi$ and calculate

$$\frac{\partial}{\partial t} t\Psi = \Delta(t\Psi) - tS^2 + \Psi$$
$$= \Delta(t\Psi) - \frac{1}{2}tS^2 + S + 2\|\nabla f\|^2 - \frac{t}{2}\left(S + 2\|\nabla f\|^2\right)^2 + 2t\|\nabla f\|^2\left(S + \|\nabla f\|^2\right).$$

We now want to make an estimate at any point and time where $S \geq 0$. From our previous result we know that $t\|\nabla f\|^2 \leq A$. We define the constant $B := 2A > 0$. We have that if $t\Psi = t\left(S + 2\|\nabla f\|^2\right) \geq B$ then we must have $S \geq 0$. Now at any point and time where $t\Psi \geq B$ we have that

$$\begin{aligned}\frac{\partial}{\partial t} t\Psi &\leq \Delta(t\Psi) - \frac{1}{2}tS^2 - \frac{t}{2}\left(S + 2\|\nabla f\|^2\right)^2 + (1+2A)S + 2(1+A)\|\nabla f\|^2 \\ &\leq \Delta(t\Psi) - \frac{1}{2t}\left[t\left(S + 2\|\nabla f\|^2\right)\right]^2 - \left[\sqrt{\frac{t}{2}}S - \sqrt{\frac{1}{2t}}(1+2A)\right]^2 + \frac{A'}{t} \\ &\leq \Delta(t\Psi) - \frac{1}{2t}\left[t\left(S + 2\|\nabla f\|^2\right)\right]^2 + \frac{A'}{t}\end{aligned}$$

with $A' > 0$ large enough such that $2(1+A)\cdot t\|\nabla f\|^2 + \frac{1}{2}(1+2A)^2 \leq A'$. Now we can find a $D > B$ such that when $t(S + 2\|\nabla f\|^2) \geq D$ we have

$$\frac{\partial}{\partial t} t\Psi \leq \Delta(t\Psi) + \frac{1}{t}\left\{A' - \frac{1}{2}\left[t\left(S + \|\nabla f\|^2\right)\right]^2\right\} \leq \Delta(t\Psi).$$

We can use the maximum principle to conclude that $t\Psi \leq C_2$ possibly by increasing C_2. Combining these two estimates for Ψ we find $\Psi \leq C_2/(1+t)$. This proves our claim.

Now we have lower and upper bounds for S and we conclude that $|S(t)| \leq C/(1+t)$ for some $C > 0$ sufficiently large. □

We now prove the derivatives of S converge to zero as well.

Lemma 3.27:
Under the hypotheses of Lemma 3.25 we have for every $k \geq 1$ that a constant $C_k > 0$ exists such that

$$\left\|\nabla^k S(t)\right\|^2 \leq \frac{C_k}{(1+t)^{k+2}}.$$

Proof. We prove this result by induction. Lemma 3.26 proves the case $k = 0$. We now assume that it holds for $0 \leq j \leq k-1$. Proposition A.11 gives us that

$$\frac{\partial}{\partial t}\left\|\nabla^k S\right\|^2 = \Delta\left\|\nabla^k S\right\|^2 - 2\left\|\nabla^{k+1}S\right\|^2 + 4S\left\|\nabla^k S\right\|^2 + \nabla^k S * \mathcal{R}^k$$

As in the previous proof we define a quantity that satisfies a particular nice evolution equation. We define

$$\Phi = t^{k+3}\left\|\nabla^k S\right\|^2 + Nt^{k+2}\left\|\nabla^{k-1}S\right\|^2$$

with some positive constant N that we will determine later. We calculate

$$\frac{\partial}{\partial t} t^{k+3}\left\|\nabla^k S\right\|^2 \leq \Delta(t^{k+3}\left\|\nabla^k S\right\|^2) + 4t^{k+3}S\left\|\nabla^k S\right\|^2 + t^{k+3}\left\|\nabla^k S\right\|\left\|\mathcal{R}^k\right\| + (k+3)t^{k+2}\left\|\nabla^k S\right\|^2$$
$$\begin{aligned}\frac{\partial}{\partial t} t^{k+2}\left\|\nabla^{k-1}S\right\|^2 \leq{}& \Delta(t^{k+2}\left\|\nabla^{k-1}S\right\|^2) - 2t^{k+2}\left\|\nabla^k S\right\|^2 + 4t^{k+2}S\left\|\nabla^{k-1}S\right\|^2 \\ &+ t^{k+2}\left\|\nabla^{k-1}S\right\|\left\|\mathcal{R}^{k-1}\right\| + (k+2)t^{k+1}\left\|\nabla^{k-1}S\right\|^2.\end{aligned}$$

We use the induction hypothesis to derive the following estimates:

$$
\begin{aligned}
t^{k/2+2}\left\|\mathcal{R}^k(t)\right\| &\ll \sum_{j=1}^{\lfloor k/2\rfloor}\sqrt{t^{j+2}\left\|\nabla^j S(t)\right\|^2}\sqrt{t^{k-j+2}\left\|\nabla^{k-j} S(t)\right\|^2} \ll 1\\
t^{k+2}\left\|\nabla^{k-1}S(t)\right\|\left\|\mathcal{R}^{k-1}(t)\right\| &\ll \sum_{j=1}^{\lfloor (k-1)/2\rfloor}\sqrt{t^{k+1}\left\|\nabla^{k-1} S(t)\right\|^2}\sqrt{t^{j+2}\left\|\nabla^j S(t)\right\|^2}\sqrt{t^{k-j+1}\left\|\nabla^{k-j-1} S(t)\right\|^2} \ll 1\\
t^{k+2}S\left\|\nabla^{k-1}S(t)\right\|^2 &\ll 1\\
t^{k+1}\left\|\nabla^{k-1}S(t)\right\|^2 &\ll 1.
\end{aligned}
$$

Now there exist constants $A, B, C > 0$ such that

$$
\begin{aligned}
\frac{\partial}{\partial t}t^{k+3}\left\|\nabla^k S\right\|^2 &\le \Delta(t^{k+3}\left\|\nabla^k S\right\|^2) + At^{k+2}\left\|\nabla^k S\right\|^2 + B\sqrt{t^{k+2}\left\|\nabla^k S\right\|^2}\\
&\le \Delta(t^{k+3}\left\|\nabla^k S\right\|^2) + (A+B)t^{k+2}\left\|\nabla^k S\right\|^2 + B\\
\frac{\partial}{\partial t}t^{k+2}\left\|\nabla^{k-1} S\right\|^2 &\le \Delta(t^{k+2}\left\|\nabla^{k-1} S\right\|^2) - 2t^{k+2}\left\|\nabla^k S\right\|^2 + C.
\end{aligned}
$$

Now we take N to be a constant such that $2N > A+B$. Combining our previous estimates gives

$$
\frac{\partial}{\partial t}\Phi \le \Delta\Phi + (A+B-2N)t^{k+2}\left\|\nabla^k S\right\|^2 + (B + N\cdot C) \le \Delta\Phi + D.
$$

Here we denoted $D = B + N\cdot C$. At $t = 0$ we have $\Phi(0) = 0$. Now using the maximum principle yields that $\Phi(t) \le Dt$ for all $t \in [0,\infty)$. From this we find that $\left\|\nabla^k S(t)\right\|^2 \le D/t^{k+2}$. Because M is compact $\left\|\nabla^k S(x,t)\right\|^2$ is bounded on $M\times[0,1]$. We conclude a constant $C_k > 0$ exists such that $\left\|\nabla^k S(t)\right\|^2 \le C_k/(1+t)^{k+2}$. □

We will now prove Lemma 3.25 by applying the same strategy as in the proof of Lemma 3.23 i.e. applying Lemmas 3.13 and 3.15. However the bound found in Lemma 3.26 is not sufficient because $\int_0^\infty 1/(1+t)\,dt$ is not finite. We need to make some additional estimates first. This can be done by using the following two results:

Proposition 3.28 (A Sobolev inequality)**:**
Let (M^n, g) be a compact Riemannian manifold. For every $p \in \mathbb{R}$ with $p > n$ there is a $C = C(M,g,p) > 0$ such that

$$
\|u\|_\infty \le C\|u\|_{H_1^p}
$$

for any $u \in C^\infty(M)$.

Here $\|u\|_\infty$ denotes the supremum norm on M and $\|u\|_{H_1^p} = \|u\|_{L^p} + \|\nabla u\|_{g,L^p}$ a combination of L^p norms. For a proof see [Aub98].

Proposition 3.29 (A Poincaré-Sobolev inequality)**:**
Let (M^n, g) be a compact Riemannian manifold. Assume that $p, q \in \mathbb{R}$ such that $1 \le q < n$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ then there exists a $C = C(M,g,p) > 0$ such that for any $u \in C^\infty(M)$ we have

$$
\|u-\bar{u}\|_{L^p} \le C\|\nabla u\|_{L^q} \quad \textit{with } \bar{u} = \frac{1}{\mathit{Vol}(M)}\int_M u\,\mathrm{vol}_n\,.
$$

Here $H_1^q(M)$ denotes the first order Sobolev space on M. For a proof of this proposition see [Heb96].

Proof of Lemma 3.25. We apply the previous propositions on the Riemannian manifold (M, g_0). By Proposition 3.28 there is a $C = C(M, g_0, 3)$ such that

$$\sup_{x\in M} |S(x)| \leq C \|S\|_{g_0,H_1^3} = C(\|S\|_{L^3} + \|\nabla S\|_{g_0,L^3}).$$

The choice for $p = 3$ is somewhat arbitrary. Any $p > n$ will do as long as a corresponding $q < n$ exists. We now use that $\bar{S} = s = 0$ and apply Proposition 3.29 to find that there is a $C' = C'(M, g_0, 3)$ such that $\|S\|_{L^3} \leq C' \|\nabla S\|_{g_0,L^{6/5}}$. Combining these estimates gives

$$\sup_{x\in M} |S(x,t)| \ll \|\nabla S(t)\|_{g_0,L^3} + \|\nabla S(t)\|_{g_0,L^{6/5}}\,. \tag{$*$}$$

Now we use the equivalence of metrics $g(t)$ and g_0 (see Proposition 3.22) to estimate $\|\nabla S(t)\|_{g_0} \ll \|\nabla S(t)\|_{g(t)}$. By Lemma 3.27 we also have the estimate $\|\nabla S(t)\|_{g(t)} \ll 1/(1+t)^{3/2}$. This implies that for any $p \geq 1$ we can bound the quantity $\|\nabla S\|_{g_0,L^p}$ by

$$\|\nabla S\|_{g_0,L^p} = \left(\int_M (\|\nabla S\|_{g_0})^p \operatorname{vol}_n\right)^{1/p} \ll \frac{1}{(1+t)^{3/2}} \left(\int_M \operatorname{vol}_n\right)^{1/p} \ll \frac{1}{(1+t)^{3/2}}.$$

Substituting this bound into $(*)$ yields that $\sup_{x\in M} |S(x,t)| \ll 1/(1+t)^{3/2}$. Finally we can calculate

$$\int_0^\infty |S(x,t)|\, dt \ll \int_0^\infty \frac{1}{(1+t)^{3/2}}\, dt < \infty \quad \text{for all } x \in M$$

and for $k \geq 1$ that

$$\int_0^\infty \left\|\nabla^k S(x,t)\right\|\, dt \ll \int_0^\infty \frac{1}{(1+t)^{k/2+1}}\, dt < \infty \quad \text{for all } x \in M.$$

Hence by Lemmas 3.13 and 3.15 we conclude $g(t)$ converges in any C^k norm to a smooth limiting metric g_∞ which is conformal to g_0. The bounds on S imply this metric must have constant scalar curvature. □

Remark 3.30 It is possible to show that the convergence of $g(t)$ to g_∞ is actually exponential (see [Ham88]). However the proof of this fact presupposes the uniformization theorem so it can not be used for our purposes.

3.9 Convergence for $\chi(M) > 0$

When $\chi(M) > 0$ we have that $s > 0$. We see that in this case the bounds $-Ce^{st} \leq S - s \leq Ce^{st}$ deteriorate when $t \to \infty$. This is a sign that in this case the proof of Theorem 3.2 is most difficult. The proof for the case $\chi(M) \leq 0$ relies almost exclusively on the maximum principle. For $\chi(M) > 0$ the maximum principle is no longer enough and several other techniques are needed. As a result the proof for this case is not within the scope of this text. We will only state the result and provide references to a proof.

On compact surfaces with $\chi(M) > 0$ we do have the following result:

Lemma 3.31:
Let $(M^2, g(t))$ be the solution to the normalised Ricci flow on a compact and orientable surface M with $\chi(M) > 0$. Then a unique smooth metric g_∞ on M exists such that $g(t)$ converges to g_∞. This convergence is with respect to the C^k norm for any $k \in \mathbb{N}$. Furthermore this limiting metric g_∞ is conformal to g_0 and has constant scalar curvature.

This result was first proved by Hamilton for the case that the initial metric has $S(x, 0) > 0$ for all $x \in M$ (see [Ham88]). The proof of this result was completed by Chow in [Cho91]. He showed that for an arbitrary starting metric the scalar curvature eventually becomes positive everywhere when the metric evolves by the NRF. Then by restarting the flow at such a time we have reduced to the case Hamilton proved. A complete treatment of the case $\chi(M) > 0$ can be found in [CK04, §5.7]. It must be noted that the proofs as given in these references do not provide a proof for the uniformization theorem. This is because the result is used in the proofs. However it is possible to modify the arguments in order to circumvent the uniformization theorem (see [CLT06]). In this way a full proof using the Ricci flow of the uniformization theorem can still be achieved.

3.10 Conclusion

We have now discussed all the steps in the proof of the uniformization theorem based on Ricci flow. Combining the long time existence results of Proposition 3.21 with the convergence results from the previous three sections provides a full proof of Theorem 3.2.

Appendix A

Proofs of identities

A.1 Conformal metrics

In this appendix we present proofs of some of the identities used throughout Chapters 2 and 3.

Proposition A.1:
Let g and h be two metrics on a Riemannian surface related by $g = e^{2\phi}h$. Then S_g and S_h are related by

$$S_g = e^{-2\phi}(S_h - 2\Delta_h \phi).$$

Here we denote by Δ_h the Laplace-Beltrami operator associated to the metric h.

For the proof of this proposition we use the Cartan structural equations. We first introduce the connection matrix and curvature matrix:

Definition A.2 (Connection matrix)**:**
On a Riemannian manifold M let $(e_1, ..., e_n)$ be a local frame spanning TM. The connection matrix of ∇ relative to this frame is the matrix of one-forms $A = (A_i^j)$ such that $\nabla_X e_i = A_i^j(X)e_j$ for all $X \in \mathfrak{X}(M)$.

Definition A.3 (Curvature matrix)**:**
On a Riemannian manifold M let $(e_1, ..., e_n)$ be a local frame spanning TM. The curvature matrix of R relative to this frame is the matrix of two-forms $\Omega = (\Omega_i^j)$ such that $R(X,Y)e_i = \Omega_i^j(X,Y)e_j$ for all $X, Y \in \mathfrak{X}(M)$.

Proposition A.4 (Cartan structural equations)**:**
Let M be a Riemannian manifold. Let (e_i) be a (local) orthonormal frame spanning TM and denote by (ϵ^i) its dual. Then we have that the following identities hold:

$$d\epsilon^j = \epsilon^i \wedge A_i^j \tag{A.1}$$

$$\Omega_i^j = dA_i^j - A_i^k \wedge A_k^j \tag{A.2}$$

Proof of Proposition A.4. To prove the first identity notice that we have

$$(d\epsilon^j)(e_k, e_l) = \underbrace{e_k(\epsilon^j(e_l))}_{e_k(\delta_l^j)=0} - \underbrace{e_l(\epsilon^j(e_k))}_{e_l(\delta_k^j)=0} - \epsilon^j([e_k, e_l]) = -\epsilon^j(\nabla_{e_k} e_l - \nabla_{e_l} e_k) =$$

$$-A_l^j(e_k) + A_k^j(e_l) = \delta_k^i A_i^j(e_l) - \delta_l^i A_i^j(e_k) = (\epsilon^i \wedge A_i^j)(e_k, e_l)$$

For the second identity we first calculate

$$\begin{aligned}\nabla^2_{X,Y}e_i &= \nabla_X\nabla_Y e_i - \nabla_{\nabla_X Y}e_i = \nabla_X(A_i^j(Y)e_j) - A_i^j(\nabla_X Y)e_j \\ = \underbrace{X(A_i^j(Y))e_j - A_i^j(\nabla_X Y)}_{=(\nabla_X A_i^j)(Y)e_j} &+ A_i^j(Y)A_j^k(Y)e_k = (\nabla_X A_i^j)(Y)e_j + A_i^k(Y)A_k^k(Y)e_j.\end{aligned}$$

Now we find that

$$R(X,Y)e_i = (\nabla^2_{X,Y} - \nabla^2_{Y,X})e_i = \underbrace{\left[(\nabla_X A_i^j)(Y) - (\nabla_Y A_i^j)(X)\right]}_{=(dA_i^j)(X,Y)} e_j + \underbrace{\left[A_i^k(Y)A_k^j(X) - A_i^k(X)A_k^j(Y)\right]}_{=-(A_i^k\wedge A_k^j)(X,Y)} e_j.$$

□

The following proof was taken from [CK04, §5.1].

Proof of Proposition A.1. Let (e_i) denote an orthonormal basis with respect to h and let (ϵ^i) be its dual. Then $\widetilde{e}_i = e^{-\phi}e_i$ and $\widetilde{\epsilon}^i = e^{\phi}\epsilon^i$ are orthonormal with respect to g. We will denote the connection matrix A_h as A and A_g as $\widetilde{A}$ also we denote the 2-form matrix Ω_h as Ω and Ω_g as $\widetilde{\Omega}$.

In two dimensions we have that

$$S_g = Ric_g(\widetilde{e}_1,\widetilde{e}_1) + Ric_g(\widetilde{e}_2,\widetilde{e}_2) = g(R_g(\widetilde{e}_2,\widetilde{e}_1)\widetilde{e}_1,\widetilde{e}_2) + g(R_g(\widetilde{e}_1,\widetilde{e}_2)\widetilde{e}_2,\widetilde{e}_1) = 2\widetilde{\Omega}_1^2(\widetilde{e}_2,\widetilde{e}_1).$$

So our task is to calculate $\widetilde{\Omega}_1^2$. First we notice that $\widetilde{A}_i^j$ is anti-symmetric by looking at $0 = X(\delta_{ij}) = X(g(e_i,e_j)) = g(\nabla_{g,X}e_i,e_j) + g(\nabla_{g,X}e_j,e_i) = \widetilde{A}_i^j(X) + \widetilde{A}_j^i(X)$. This means that $\widetilde{A}_1^k\wedge\widetilde{A}_k^2 = \widetilde{A}_1^1\otimes\widetilde{A}_1^2 + \widetilde{A}_1^2\otimes\widetilde{A}_2^2 = 0$. From this we conclude that $\widetilde{\Omega}_1^2 = d\widetilde{A}_1^2$ by (A.2).

From (A.1) follows that $d\widetilde{\epsilon}^1(\widetilde{e}_2,\widetilde{e}_1) = -\widetilde{A}_1^2(\widetilde{e}_1)$ and $d\widetilde{\epsilon}^2(\widetilde{e}_2,\widetilde{e}_1) = -\widetilde{A}_1^2(\widetilde{e}_2)$ hence

$$\widetilde{A}_1^2 = -(d\widetilde{\epsilon}^1)(\widetilde{e}_2,\widetilde{e}_1)\widetilde{\epsilon}^2 - (d\widetilde{\epsilon}^2)(\widetilde{e}_2,\widetilde{e}_1)\widetilde{\epsilon}^1. \qquad (*)$$

Now we calculate that

$$\begin{aligned}(d\widetilde{\epsilon}^1)(\widetilde{e}_2,\widetilde{e}_1) &= e^{-2\phi}d(e^{\phi}\epsilon^1)(e_2,e_1) = e^{-2\phi}\left[e^{\phi}d\phi\wedge\epsilon^1 + e^{\phi}d\epsilon^1\right](e_2,e_1) = e^{-\phi}\left[(d\epsilon^1)(e_2,e_1) + e_2(\phi)\right]\\ (d\widetilde{\epsilon}^2)(\widetilde{e}_2,\widetilde{e}_1) &= e^{-2\phi}d(e^{\phi}\epsilon^2)(e_2,e_1) = e^{-2\phi}\left[e^{\phi}d\phi\wedge\epsilon^2 + e^{\phi}d\epsilon^2\right](e_2,e_1) = e^{-\phi}\left[(d\epsilon^2)(e_2,e_1) - e_1(\phi)\right].\end{aligned}$$

Putting this into $(*)$ we find that

$$\widetilde{A}_1^2 = A_1^2 - e_2(\phi)\epsilon^1 + e_1(\phi)\epsilon^2$$

and from this we get

$$\widetilde{\Omega}_1^2 = d\widetilde{A}_1^2 = dA_1^2 - d(e_2(\phi))\wedge\epsilon^1 - e_2(\phi)d\epsilon^1 + d(e_1(\phi))\wedge\epsilon^2 + e_1(\phi)d\epsilon^2.$$

This now implies that

$$\begin{aligned}S_g &= 2\widetilde{\Omega}_1^2(\widetilde{e}_2,\widetilde{e}_1) = 2e^{-2\phi}\widetilde{\Omega}_1^2(e_2,e_1)\\ &= 2e^{-2\phi}[(dA_1^2)(e_2,e_1) - e_2e_2(\phi) - e_2(\phi)(d\epsilon^1)(e_2,e_1) - e_1e_1(\phi) + e_1(\phi)(d\epsilon^2)(e_2,e_1)].\end{aligned}$$

Finally by substituting that $d\epsilon^1 = (\epsilon^2 \wedge A_2^1)(e_2, e_1)$ and $d\epsilon^2 = (\epsilon^1 \wedge A_1^2)(e_2, e_1)$ we find

$$\begin{aligned}
&= 2e^{-2\phi}[\Omega_1^2(e_2, e_1) - e_1 e_1(\phi) - e_2 e_2(\phi) + e_2(\phi) A_1^2(e_1) - e_1(\phi) A_1^2(e_2)] \\
&= 2e^{-2\phi}[\frac{1}{2} S_h - \nabla_{e_1}\nabla_{e_1}\phi - \nabla_{e_2}\nabla_{e_2}\phi + \nabla_{\nabla_{e_1} e_1}\phi + \nabla_{\nabla_{e_2} e_2}\phi] \\
&= 2e^{-2\phi}[\frac{1}{2} S_h - (\nabla^2_{e_1,e_1} + \nabla^2_{e_2,e_2})\phi] = e^{-2\phi}[S_h - 2\Delta_h \phi].
\end{aligned}$$

This proves the proposition. □

A corollary of the previous proposition is the following result:

Corollary A.5:
Let g and h be two metrics on a Riemannian surface related by $g = u \cdot h$ with $u > 0$ then

$$S_g = \frac{1}{u}(S_h - \Delta_h \log u).$$

A.2 Several useful identities

Proposition A.6:
For any tensor field A on a Riemannian manifold we have

$$\Delta \|A\|^2 = 2 \|\nabla A\|^2 + 2\langle A, \Delta A\rangle_g.$$

Proof. We use the properties of the Levi-Civita connection to find

$$\begin{aligned}
\nabla_j \langle A, A\rangle_g &= 2\langle A, \nabla_j A\rangle_g \\
\nabla^2_{i,j}\langle A, A\rangle_g &= 2\langle \nabla_i A, \nabla_j A\rangle_g + 2\langle A, \nabla^2_{i,j} A\rangle_g.
\end{aligned}$$

Taking the metric contraction on $i = j$ yields the stated result. □

Proposition A.7:
On a two dimensional Riemannian manifold we have

$$\Delta\nabla S - \nabla\Delta S = \frac{1}{2} S \nabla S.$$

Proof. We use that $\nabla^2_{a,b} = \nabla^2_{b,a} + R(\partial_a, \partial_b)$ to find

$$\nabla^3_{i,j,k} S = \nabla^3_{i,k,j} S + \nabla_i R(\partial_j, \partial_k) S = \nabla^3_{k,i,j} S + R(\partial_i, \partial_k)\nabla_j S + \nabla_i R(\partial_j, \partial_k) S.$$

We have that $R(\partial_j, \partial_k) S = 0$ because S is a scalar function. To see that this is true we use (1.4) to find

$$R(X, Y)S = \nabla_X \nabla_Y S - \nabla_Y \nabla_X S - \nabla_{[X,Y]} S = X(YS) - Y(XS) - [X, Y]S = 0$$

for any vector fields X and Y. Now taking the metric contraction on $i = j$ we find

$$\Delta\nabla_k S - \nabla_k \Delta S = g^{ij} R(\partial_i, \partial_k)\nabla_j S.$$

Using the properties of R as in Remark 1.37 we find that $(R(X,Y)\nabla S)(Z) = -(\nabla S)(R(X,Y)Z)$ which gives us that

$$R(\partial_i,\partial_k)\nabla_j S = (R(\partial_i,\partial_k)\nabla S)(\partial_j) = -(\nabla S)(R(\partial_i,\partial_k)\partial_j) = -R^p_{ikj}\nabla_p S.$$

Now we use that in two dimensions we have $Ric = Kg$ to calculate that

$$-g^{ij}R^p_{ikj}\nabla_p S = g^{pq}Ric_{kq}\nabla_p S = Kg^{pq}g_{kq}\nabla_p S = \frac{1}{2}S\nabla_k S.$$

□

For the following results we recall the notation introduced in (3.3):

$$\mathcal{R}^k = \sum_{j=1}^{\lfloor k/2\rfloor}(\nabla^j S) * (\nabla^{k-j}S).$$

Proposition A.8:
On a Riemannian surface we the following identities:

1. $\Delta\nabla^k S - \nabla^k\Delta S = \frac{k}{2}S\nabla^k S + \mathcal{R}^k$
2. $\nabla^k(S^2) = 2S\nabla^k S + \mathcal{R}^k$

and if $g(t)$ evolves by the normalised Ricci flow:

3. $\frac{\partial}{\partial t}(\nabla^k S) = \nabla^k(\frac{\partial}{\partial t}S) + \mathcal{R}^k.$

Proof.
Proof of 1: We proceed by induction. We know the conclusion holds when $k = 1$ by Proposition A.7. Now suppose it is true for $k-1$ then

$$\nabla^k\Delta S = \nabla(\nabla^{k-1}\Delta S) = \nabla(\Delta\nabla^{k-1}S - \frac{k-1}{2}S\nabla^{k-1}S + \mathcal{R}^{k-1})$$
$$= \Delta\nabla^k S - \frac{1}{2}S\nabla^k S + \frac{k-1}{2}S\nabla^k S + \frac{k-1}{2}\underbrace{\nabla S\otimes\nabla^{k-1}S}_{=\mathcal{R}^k} + \mathcal{R}^k = \Delta\nabla^k S - \frac{k}{2}S\nabla^k S + \mathcal{R}^k.$$

This proves the first identity.
Proof of 2: Again we use induction. For $k = 1$ we calculate that $\nabla(S^2) = S(\nabla S) + (\nabla S)S = 2S\nabla S$. Now suppose the identity holds for $k-1$ then

$$\nabla^k(S^2) = \nabla(\nabla^{k-1}S^2) = \nabla(2S\nabla^{k-1}S + \mathcal{R}^{k-1}) = 2S\nabla^k S + \underbrace{2\nabla S\otimes\nabla^{k-1}S}_{=\mathcal{R}^k} + \mathcal{R}^k.$$

This proves the stated identity.
Proof of 3: We again proceed by induction. We have $\partial(\nabla S)/\partial t = \nabla(\partial S/\partial t)$ which proves the statement for $k = 1$. Now suppose it holds for $1 \le j \le k-1$. We calculate that

$$\frac{\partial}{\partial t}(\nabla^k S) = \nabla(\frac{\partial}{\partial t}\nabla^{k-1}S) + (\frac{\partial}{\partial t}\Gamma) * \nabla^{k-1}S = \nabla(\nabla^{k-1}(\frac{\partial}{\partial t}S) + \mathcal{R}^{k-1}) + (\frac{\partial}{\partial t}\Gamma) * \nabla^{k-1}S.$$

When we look at the expression for $\partial\Gamma/\partial t$ as given in Proposition 2.15 we see that $(\partial\Gamma/\partial t) * \nabla^{k-1}S = (\nabla S) * \nabla^{k-1}S = \mathcal{R}^k$. Putting this into the previous equation gives us that

$$\frac{\partial}{\partial t}(\nabla^k S)/\partial t = \nabla^k(\frac{\partial S}{\partial t}) + \mathcal{R}^k.$$

This proves the final identity. □

Proposition A.9 (Evolution of the Laplacian)**:**
Under normalised Ricci flow on surfaces the Laplacian evolves as

$$\frac{\partial}{\partial t}\Delta_{g(t)} = -(s-S)\Delta_{g(t)}.$$

This proof has been taken from [CK04, §5.1].

Proof. From Proposition 2.3 we find that $\partial g^{ij}/\partial t = -(s-S)g^{ij}$. Using this we calculate

$$\frac{\partial}{\partial t}\Delta_{g(t)} = \frac{\partial}{\partial t}\left[g^{ij}\nabla^2_{i,j}\right] = -(s-S)g^{ij}\nabla^2_{i,j}+g^{ij}\frac{\partial}{\partial t}\left[\nabla_i\nabla_j - \Gamma^k_{ij}\nabla_k\right] = -(s-S)\Delta_{g(t)}-g^{ij}(\frac{\partial}{\partial t}\Gamma^k_{ij})\nabla_k.$$

We now use Proposition 2.15 to find

$$g^{ij}\frac{\partial}{\partial t}\Gamma^k_{ij} = \frac{1}{2}g^{ij}\left\{-\delta^k_j\nabla_i S - \delta^k_i\nabla_j S + g_{ij}g^{kl}\nabla_l S\right\} =$$
$$-\frac{1}{2}g^{ik}\nabla_i S - \frac{1}{2}g^{jk}\nabla_j S + g^{kl}\nabla_l S = 0.$$

This proves the proposition. □

Proposition A.10 (Evolution of the norm)**:**
For a time dependent (k,l) tensor field A the norm $\|A\|^2$ evolves under the normalised Ricci flow on surfaces as

$$\frac{\partial}{\partial t}\|A\|^2_{g(t)} = (l-k)\cdot(s-S)\|A\|^2_{g(t)} + 2\langle A, \frac{\partial}{\partial t}A\rangle_{g(t)}.$$

Proof. Under the NRF in two dimensions we have $\partial g_{ij}/\partial t = (s-S)g_{ij}$ and by Proposition 2.3 that $\partial g^{ij}/\partial t = -(s-S)g^{ij}$. We use this to calculate that

$$\begin{aligned}\frac{\partial}{\partial t}\|A\|^2_{g(t)} &= \frac{\partial}{\partial t}(g^{j_1s_1}...g^{j_ks_k}\, g_{i_1r_1}...g_{i_lr_l}\, A^{i_1...i_l}_{j_1...j_k}A^{r_1...r_l}_{s_1...s_k})\\ &= (l-k)\cdot(s-S)\|A\|^2_{g(t)} + 2g^{j_1s_1}...g^{j_ks_k}\, g_{i_1r_1}...g_{i_lr_l}\,(\frac{\partial}{\partial t}A^{i_1...i_l}_{j_1...j_k})A^{r_1...r_l}_{s_1...s_k}\\ &= (l-k)\cdot(s-S)\|A\|^2_{g(t)} + 2\langle A, \frac{\partial}{\partial t}A\rangle_{g(t)}.\end{aligned}$$

□

Proposition A.11:
Under the normalised Ricci flow on surfaces the quantity $\left\|\nabla^k S\right\|^2$ evolves as

$$\frac{\partial}{\partial t}\left\|\nabla^k S\right\|^2 = \Delta\left\|\nabla^k S\right\|^2 - 2\left\|\nabla^{k+1}S\right\|^2 - (k+2)s\left\|\nabla^k S\right\|^2 + 4S\left\|\nabla^k S\right\|^2 + \nabla^k S * \mathcal{R}^k.$$

Proof. We use the identities of Proposition A.7 and the fact that $\partial S/\partial t = \Delta S + S^2 - sS$ to calculate that

$$\begin{aligned}\frac{\partial}{\partial t}(\nabla^k S) &= \nabla^k(\frac{\partial}{\partial t}S) + \mathcal{R}^k = \nabla^k(\Delta S) + \nabla^k(S^2) - s\nabla^k S + \mathcal{R}^k\\ &= \Delta\nabla^k S + (2-\frac{k}{2})S\nabla^k S - s\nabla^k S + \mathcal{R}^k.\end{aligned}$$

Now we find using Proposition A.10 that

$$\begin{aligned}\frac{\partial}{\partial t}\left\|\nabla^k S\right\|^2 &= k(S-s)\left\|\nabla^k S\right\|^2 + 2\langle \nabla^k S, \frac{\partial}{\partial t}\nabla^k S\rangle_g \\ &= -(k+2)s\left\|\nabla^k S\right\|^2 + 4S\left\|\nabla^k S\right\|^2 + \langle \nabla^k S, \Delta\nabla^k S\rangle_g + \nabla^k S * \mathcal{R}^k \\ &= \Delta\left\|\nabla^k S\right\|^2 - 2\left\|\nabla^{k+1} S\right\|^2 - (k+2)s\left\|\nabla^k S\right\|^2 + 4S\left\|\nabla^k S\right\|^2 + \nabla^k S * \mathcal{R}^k.\end{aligned}$$

□

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