The edge adjacency matrix of a graph

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Bachelor Thesis

August 6, 2015

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Bachelor programme: Mathematics
ECTS: 7.5

Contents

Introduction

The main object of study of this thesis is graphs, which are certain combinatorial objects. These graphs give rise to certain algebraic objects, the so called edge adjacency matrices. These come in two flavours, the first of which we will call the extended edge adjacency matrix, and the other we will call the (regular) edge adjacency matrix. Graphs and various concepts related to them are introduced in the first chapter. This will take some effort, but this effort will make the introduction of the adjacency matrices quite easy. This introduction also takes place in the first chapter.

Two natural questions then arise. Can we give (necessary and sufficient) criteria for an arbitrary matrix to be the edge adjacency matrix of a graph? And does the edge adjacency matrix of a graph contain all the information about this graph, that is, can we determine a graph from its edge adjacency matrix? Both question receive an answer in the second chapter, first in the extended case, and then in the regular case. We will also give various equivalent criteria for the first question.

The third chapter is then devoted to the study of the edge adjacency matrices themselves. Notations developed in the second chapter will prove useful for this purpose. First, we investigate when an edge adjacency matrix is invertible. We then turn to the spectrum of edge adjacency matrices, and determine this spectrum for a matrix quite similar to the edge adjacency matrix. Finding the spectrum of the edge adjacency matrices themselves is an unsolved problem, which is discussed shortly at the end of the third chapter.

The new results from this thesis are the criteria for being an edge adjacency matrix (in the second chapter) and the spectrum of the aforementioned variant of the edge adjacency matrix (in the third chapter). The other theorems stated were already known. These two results can also be called the main results of this thesis.

1 Graphs

The goal of this section is to introduce the most important object of study for this thesis: graphs. While the concept of a graph can easily be grasped intuitively, it shall be convenient for our purposes to have an actual definition of a graph as well. This definition will be slightly formal, bit it makes graphs susceptible to algebraic methods. In particular, we will apply methods of linear algebra to the study of graphs. However, while working with graphs, one should never forget the intuitive concept.

1.1 Graphs informally

Let us explain what a graph is using an example.

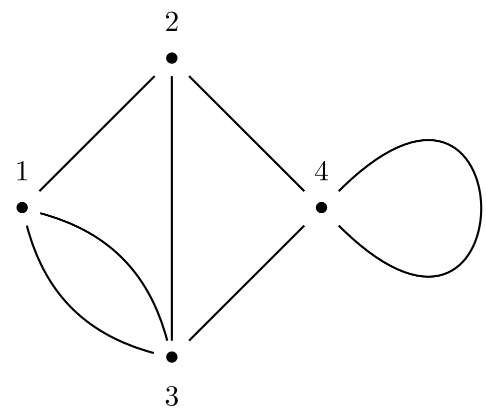


Figure 1: A graph

As one can see, a graphs consists of nodes, that we have labeled 1 up until 4, and lines between them. The former are called *vertices*, and the latter are called *edges*. Each edge has two end points that are vertices. It may occur that these two vertices coincide. We then speak of a *loop* rather than an edge. In the above example, there is a loop attached to the vertex with label 4. Another possibility is that there are multiple edges having the same pair of end vertices. In figure 1, there is a double edge between the vertices labeled 1 and 3. While these two edges *are* different, they are not really distinguishable as far as their end vertices are concerned. If a graph contains no loops and no multiple edges, we call it *simple*.[1] Graphs that do contain multiple edges and/or loops are sometimes referred to as multigraphs or pseudographs. However, in our terminology, these are simply called graphs. Finally, notice that there does not necessarily exist an edge between two given vertices. In figure 1, there are no edges between the vertices labeled 1 and 4, and also no loops attached to the vertices with labels 1, 2 and 3.

[1] In some other sources, only simple graphs are acknowledged as graphs. This is, however, not the definition we shall employ here.

While the above example contains four vertices, the number of vertices of a graph may in general be any non-zero cardinality. In particular, this number may be infinite. There is also little restriction on the edges: the number of edges between two given (possibly coinciding) vertices can be any cardinality, possibly zero as we saw earlier. If the number of vertices and the total number of edges are both finite, then we say of the graph itself that it is finite. Informally, finite graphs are those graphs that can be drawn on a (possibly very large) piece of paper. This also shows the restriction of the informal concept of a graph: it can be rather hard to picture an infinite graph. With the formal definition of graphs, which will be in terms of sets, there is no difficulty here. This point is of minor importance, however, as we will be dealing exclusively with finite graphs in this thesis.

The discussion above should provide the reader with a fairly good idea of what a typical graph looks like. There is, however, a second kind of graph, namely *directed* graphs. To avoid confusion, we shall rename the graphs discussed above to *undirected* graphs. Let us give an example of a directed graph.

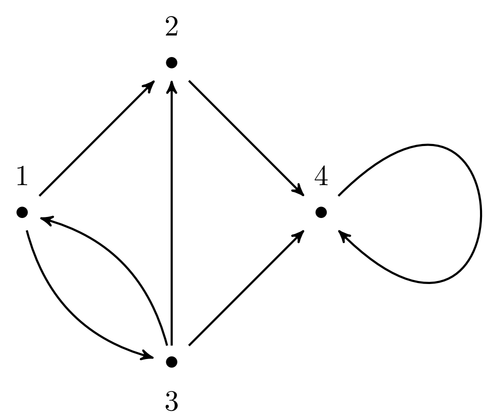


Figure 2: A directed graph

Rather than being a set of nodes with lines between them, a directed graph is a set of nodes with arrows between then. The nodes are still referred to as vertices, while the arrows are called *directed edges*. This terminology reflects the fact that the difference between undirected and directed graphs does not lie in their vertex sets, but in the nature of the edges. Thus, there is no such thing as a 'directed vertex'. To an (undirected) edge, one can only assign a pair of not necessarily distinct vertices, but when dealing with a directed edge, these end vertices are distinguishable. There is a vertex from which the directed edge *originates*, and one in which it *terminates*, the latter being the vertex where the arrowhead is located. If these two vertices coincide, we speak of a *directed loop*. If there are two or more directed edges originating from the same vertex and terminating in the same vertex, we say that the graph contains multiple directed edges. Notice that the graph in figure 2 does *not* contain multiple directed edges. Indeed, while there are two arrows located between the vertices labeled 1 and 3, they go in different directions, making them distinguishable by their original and terminal vertices. It does still contain a loop, at the vertex with label 4. The definition of finiteness, like those of loops and multiple edges, is similar to the undirected case.

Directed and undirected graphs are by no means unrelated. One thing that can already be noticed in the above examples is that removing all arrowheads from figure 2 yields figure 1. We will investigate such relations further once we have given a more precise definition of graphs.

1.2 Graphs formally

Before we can get to the definition of graphs, we need to introduce a small piece of notation.

Definition 1.2.1. For a set S and a cardinal number κ, we define $\binom{S}{\kappa} = \{S' \subset S \mid |S'| = \kappa\}$, the set of all subsets of S with cardinality α.

Now let us finally define graphs.

Definition 1.2.2 (Graph)**.** Graphs come in two versions.

- An undirected graph G is a triple (V, E, r), where V and E are sets, V is non-empty, and r is a function $E \to \binom{V}{1} \cup \binom{V}{2}$.
- A directed graph $\vec{G}$ is a triple $\left(V, \vec{E}, \vec{r}\right)$, where V and $\vec{E}$ are sets, V is non-empty, and $\vec{r}$ is a function $\vec{E} \to V \times V$.

The members of V are called vertices, the elements of E ($\vec{E}$) are called edges (directed edges), and r ($\vec{r}$) is called the *location function*. We say that G ($\vec{G}$) is finite if both V and E ($\vec{E}$) are.

Let us explain how this quite abstract definition relates to the examples from the previous section, starting with figure 1. For each vertex, we need an element in the vertex set V of G. For simplicity, let us take $V = \{1, 2, 3, 4\}$. Also, for each line, we need an element in E. There are seven lines, E should have seven elements; let us take $\{a, b, c, d, e, f, g\}$. Finally, we need to specify for each line what its end vertices are, and this is what the function r does. If the end vertices of a line are i and j, we let its image under r be $\{i, j\}$. Notice that $\{i, j\} \in \binom{V}{2}$ if $i \neq j$ and $\{i, j\} \in \binom{V}{1}$ if $i = j$, so this makes sense. So a possible function r would be

$$r : a \mapsto \{1, 2\}, b \mapsto \{1, 3\}, c \mapsto \{1, 3\}, d \mapsto \{2, 3\}, e \mapsto \{2, 4\}, f \mapsto \{3, 4\}, g \mapsto \{4\}.$$

For the directed graph in figure 2, we can take $V = \{1, 2, 3, 4\}$ and $\vec{E} = \{a, b, c, d, e, f, g\}$, as there again are four nodes and seven arrows. Now we need to specify for each arrow the (distinguishable!) vertices from which it originates, and in which it terminates. We do this by assigning $(i, j) \in V \times V$ if an arrow originates in i and terminates in j. So a possible function $\vec{r}$ would be

$$\vec{r} : a \mapsto (1, 2), b \mapsto (1, 3), c \mapsto (3, 1), d \mapsto (3, 2), e \mapsto (2, 4), f \mapsto (3, 4), g \mapsto (4, 4).$$

Conversely, one can make a picture of a given finite graph $G = (V, E, r)$ or $\vec{G} = \left(V, \vec{E}, \vec{r}\right)$. For each v in the vertex set V, draw a node n_v. In the undirected case, draw a line between n_v and n_w for each element of $r^{-1}(\{v, w\})$ if $v \neq w$; and attach a loop to n_v for each each element of $r^{-1}(\{v\})$. In the directed case, draw an arrow from n_v to n_w for each element of $\vec{r}^{-1}((v, w))$. Now let us state the following convention, that hopefully increases readability and reduces redundancy.

Convention. Unless stated otherwise, an undirected graph G is assumed to be the triple (V, E, r), while a directed graph $\vec{G}$ is assumed to be the triple $\left(V, \vec{E}, \vec{r}\right)$. In particular, when graphs G and $\vec{G}$ are being discussed simultaneously, they are assumed to have the same vertex set.

Intuitively, the location function tells us where a certain (directed) edge is located. It also enables us to capture the intuition behind multiple edges. While two edges having the same end vertices really *are* different, they are indistinguishable if one is only interested in vertices. Our definition of graphs permits edges to be different in E, but to be projected by r onto the same element in $\binom{V}{1} \cup \binom{V}{2}$. Something similar holds in the directed case. This is also the motivation behind the following definition, which formalizes the concept of having multiple edges.

Definition 1.2.3 (Multiple edges)**.**

- Let G be an undirected graph. We say that G contains multiple edges if r is not injective.
- Let $\vec{G}$ be a directed graph. We say that $\vec{G}$ contains multiple directed edges if $\vec{r}$ is not injective.

Obviously, if G is an undirected graph, then $v \in V$ is an end vertex of $e \in E$ precisely when $v \in r(e)$. In the directed case, we can be a bit more specific.

Definition 1.2.4 (Original vertex, terminal vertex)**.** Let $\vec{G}$ be a directed graph and define functions $\pi_1, \pi_2 : V \times V \to V$ by $\pi_1((v, w)) = v$ and $\pi_2((v, w)) = w$ for $v, w \in V$. Now we define functions $o, t : \vec{E} \to V$ by $o(\vec{e}) = \pi_1\left(\vec{r}(\vec{e})\right)$ and $t(\vec{e}) = \pi_2\left(\vec{r}(\vec{e})\right)$ for $\vec{e} \in \vec{E}$. We call $o(\vec{e})$ the *original vertex* of $\vec{e}$, and $t(\vec{e})$ the *terminal vertex* of $\vec{e}$.

We can also give a precise and elegant definition of what we mean by a loop.

Definition 1.2.5 (Loop)**.**

- Let G be an undirected graph. We say that $e \in E$ is a loop if $r(e) \in \binom{V}{1}$
- Let $\vec{G}$ be a directed graph. We say that $\vec{e} \in \vec{E}$ is a directed loop if $o(\vec{e}) = t(\vec{e})$.

As a final example of formalizing our concept from section 1.1, let us define the procedure of 'removing arrowheads' from a directed graph.

Definition 1.2.6. Let G be an undirected graph and $\vec{G}$ be a directed graph with the same vertex set.

- Define the function $u : V \times V \to \binom{V}{1} \cup \binom{V}{2}$ by $u((v, w)) = \{v, w\}$ for $v, w \in V$. We shall call an $e \in E$ an *undirected version* of an $\vec{e} \in E$ if $r(e) = u(\vec{r}(\vec{e}))$. We also say that $\vec{e}$ is a directed version of e.

- A graph G with the same vertex set as $\vec{G}$ is an undirected version of $\vec{G}$ if there is a bijection $f : \vec{E} \to E$ which sends a directed edge in $\vec{E}$ to an undirected version in E. More precisely, f satisfies $r \circ f = u \circ \vec{r}$. We also say that $\vec{G}$ is a directed version of G.

Directing an undirected graph amounts to adding an arrowhead to each edge at one arbitrarily chosen end. Intuitively, the undirected version of a directed graph is unique, while an undirected graph G may have more than one directed version. So we could ask ourselves whether we can actually prove this. It then becomes clear that we do not yet have a notion of sameness for graphs yet. To close this section, let us define graph isomorphisms; as one would expect, these are structure preserving bijections. The structure that is to be preserved is given by the location function: end vertices in the undirected case, and original and terminal vertices in the directed case.

Definition 1.2.7 (Graph isomorphism)**.**

- Let G and $H = (W, F, s)$ be undirected graphs. An *isomorphism* between G and H is a pair (g_1, g_2), where $g_1 : V \to W$ and $g_2 : E \to F$ are bijections that respect the location functions in the sense that $g_1(v) \in s(g_2(e))$ if and only if $v \in r(e)$, for all $v \in V$ and $e \in E$
- Let $\vec{G}$ and $\vec{H} = \left(W, \vec{F}, \vec{s}\right)$ be directed graphs. An *isomorphism* between $\vec{G}$ and $\vec{H}$ is a pair (g_1, g_2), where $g_1 : V \to W$ and $g_2 : \vec{E} \to \vec{F}$ are bijections that respect the location functions in the sense that the diagram

$$\begin{array}{ccc} \vec{E} & \xrightarrow{g_2} & \vec{F} \\ {\scriptstyle o}\downarrow & & \downarrow{\scriptstyle o} \\ V & \xrightarrow{g_1} & W \end{array}$$

  commutes, and similarly for t.

We say that two (un)directed graphs are *isomorphic* if there is an isomorphism between them.

It is a tedious, yet standard, exercise to check that isomorphism is an equivalence relation that preserves relevant properties, such as being finite or containing loops. With the above definition, one may also show that the undirected version of a directed graph $\vec{G}$ is unique up to isomorphism, while the directed version of an undirected graph G is not necessarily so.

Intuitively, finite isomorphic graphs can be drawn in the same way. This expresses the feeling that the specific nature of vertices and edges is of no importance when we consider graphs only up to isomorphism. However, pictures of graphs can also obscure isomorphisms. The two graphs pictured in figure 3 below are, although differently drawn, in fact isomorphic.

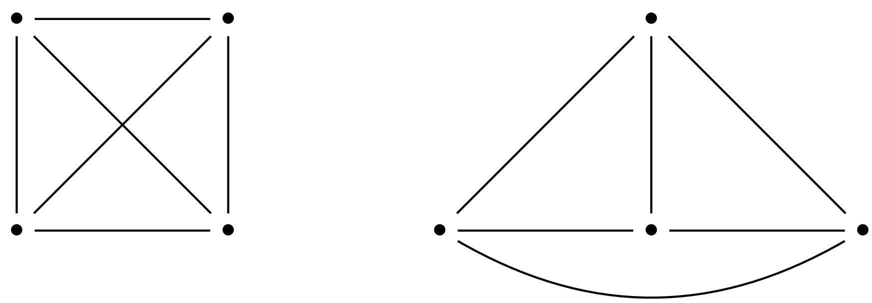


Figure 3: Two isomorphic graphs

1.3 Degrees

In the previous section, we have gone to some length to obtain a definition of graphs and fundamental objects related to them. In this section, we will introduce one more concept that we will need later: the degree of a vertex. Informally, the degree of a vertex is the number of (directed) edges attached to it. For example, every vertex of the graph shown in figure 3 has degree 3. In the directed case, we can make a further distinction. We can count the number of directed edges originating from it, and the number of directed edges terminating in it. These are called the outdegree and the indegree respectively. Let us make this precise. To avoid technicalities, we will only consider finite graphs.

Definition 1.3.1 (Degree)**.**

- Let G be a finite undirected graph and let $v \in V$. The *degree* of v, denoted by $\deg v$, is defined as $2\left|r^{-1}(\{v\})\right| + \sum_{\substack{w \in V \\ v \neq w}} \left|r^{-1}(\{v, w\})\right|$.
- Let $\vec{G}$ be a finite directed graph and let $v \in V$. The *indegree* of v, denoted by $\deg_{\text{in}}(v)$, is defined as $\left|t^{-1}(v)\right|$. Similarly, the *outdegree* $\deg_{\text{out}}(v)$ of v is defined as $\left|o^{-1}(v)\right|$. The *total degree*, or simply the degree, of v is defined as $\deg_{\text{in}}(v) + \deg_{\text{out}}(v)$, and will be denoted by $\deg v$.

The definition in the undirected case is somewhat opaque, but it simply says that, for all $w \neq v$, we count the number of edges between v and w, and that loops attached to v are counted twice. This makes sense, because intuitively, loops contribute 2 to the degree of v. A central result concerning degrees is the following.

Theorem 1.3.1 (Degree sum formula)**.** *Let G be a finite undirected graph. Then we have $\sum_{v \in V} \deg v = 2|E|$. Similarly, if $\vec{G}$ is a finite directed graph, then $\sum_{v \in V} \deg v = 2\left|\vec{E}\right|$.*

Proof. Intuitively, this is quite clear. Indeed, the sum on the left hand side counts each edge twice: in the undirected case, once for each of its two end vertices, and in the directed case, once for its original vertex and once for its terminal vertex. Let us make this precise.

We will start with the directed case. We have

$$\begin{aligned}
2\left|\vec{E}\right| &= \left|\vec{E}\right| + \left|\vec{E}\right| \\
&= \sum_{v\in V}\left|o^{-1}(v)\right| + \sum_{v\in V}\left|t^{-1}(v)\right| \\
&= \sum_{v\in V}\deg_{\text{out}}(v) + \sum_{v\in V}\deg_{\text{in}}(v) \\
&= \sum_{v\in V}(\deg_{\text{out}}(v) + \deg_{\text{in}}(v)) \\
&= \sum_{v\in V}\deg v.
\end{aligned}$$

For the undirected case, we also have to do some careful counting, but this is a bit more complicated. First of all, notice that every $x \in \binom{V}{2}$ can be written in exactly two ways as $\{v, w\}$ with $v, w \in V$ and $v \neq w$. Therefore,

$$\begin{aligned}
2|E| &= 2\sum_{x\in\binom{V}{1}\cup\binom{V}{2}}\left|r^{-1}(x)\right| \\
&= 2\sum_{x\in\binom{V}{1}}\left|r^{-1}(x)\right| + 2\sum_{x\in\binom{V}{2}}\left|r^{-1}(x)\right| \\
&= 2\sum_{v\in V}\left|r^{-1}(\{v\})\right| + \sum_{\substack{v,w\in V\\ v\neq w}}\left|r^{-1}(\{v,w\})\right| \\
&= \sum_{v\in V}2\left|r^{-1}(\{v\})\right| + \sum_{v\in V}\sum_{\substack{w\in V\\ v\neq w}}\left|r^{-1}(\{v,w\})\right| \\
&= \sum_{v\in V}\left(2\left|r^{-1}(\{v\})\right| + \sum_{\substack{w\in V\\ v\neq w}}\left|r^{-1}(\{v,w\})\right|\right) \\
&= \sum_{v\in V}\deg v.
\end{aligned}$$

□

We shall also need the following definitions, which are easy to state in terms of degrees.

Definition 1.3.2. Let G be a (directed or undirected) graph with vertex set V.

- We call $v \in V$ an *isolated vertex* if $\deg v = 0$.
- We call $v \in V$ a *pendant vertex* if $\deg v = 1$.

1.4 Adjacency matrices

As we mentioned earlier, the specific nature of vertices and edges is not relevant when we consider graphs only up to isomorphism. In fact, all information one could wish to have about a specific graph, is contained in the following algebraic object.

Definition 1.4.1 (Vertex adjacency matrix)**.** Let G be a finite undirected graph and write $V = \{v_1, \ldots, v_n\}$ with $|V| = n$. The *vertex adjacency matrix* A_G of G is the $n \times n$-matrix given by

$$(A_G)_{i,j} = \begin{cases} \text{the number of edges between } v_i \text{ and } v_j & \text{if } i \neq j; \\ \text{twice the number of loops attached to } v_i & \text{if } i = j, \end{cases}$$

for $i, j \in \{1, \ldots, n\}$. Or more precisely,

$$(A_G)_{i,j} = \begin{cases} |r^{-1}(\{v_i, v_j\})| & \text{if } i \neq j; \\ 2 \cdot |r^{-1}(\{v_i\})| & \text{if } i = j. \end{cases}$$

It should be noticed that A_G depends on the specific labeling we choose for the vertices of G. Choosing a different labeling will result in a permutation of the rows and the columns of A_G. Let us consider an example.

Example 1.4.1. Let G be the graph from figure 1, and label its vertices as indicated in the picture (that is, the vertex with label 1 is v_1, etcetera). Then

$$A_G = \begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Notice that the loop attached to v_4 is counted twice: the coefficient at position $(4, 4)$ is equal to 2, rather than 1. ♢

The sum of the elements in row i is, by definition, equal to the degree of v_i. This means that the sum of *all* coefficients in A_G must be equal to $\sum_{i=1}^{n} \deg v_i = \sum_{v \in V} \deg v = 2|E|$, so the number of edges of G can be read off from its vertex adjacency matrix. Obviously, the number of vertices can also be determined, since it is by definition the dimension of A_G. In fact, given the matrix A_G, we can wholly reconstruct the graph G. Notice that such a reconstruction will be *up to isomorphism*. That is, if undirected graphs G and H have the same vertex adjacency matrix, then they are isomorphic. We will not prove this here, but instead we will try to define edge-analogues of the matrix defined above. For this purpose, it turns out to be convenient to make G into a directed graph first. We start with a preliminary definition.

Definition 1.4.2 (Reverse edge)**.** Let $\vec{G}$ be a directed graph. We say that $\vec{e}_1, \vec{e}_2 \in \vec{E}$ are *reverses* of each other if $o(\vec{e}_1) = t(\vec{e}_2)$ and $t(\vec{e}_1) = o(\vec{e}_2)$. More down to earth, this means that $\vec{e}_1$ and $\vec{e}_2$ are arrows between the same pair of vertices but pointing in different directions.

Now we are ready to make an undirected graph G into a directed graph. We will *not* do this by the 'adding arrowheads' procedure from definition 1.2.6. As we noticed, this procedure carries a certain arbitrariness, that the following definition will in fact remove.

Definition 1.4.3 (Doubly directed version)**.** Let G be a finite undirected graph. A *doubly directed version* of G is a pair $\left(\vec{G}, p\right)$, where $\vec{G}$ is a directed graph on the same vertex set as G, and p is a function $\vec{E} \to E$ that satisfies the following properties.

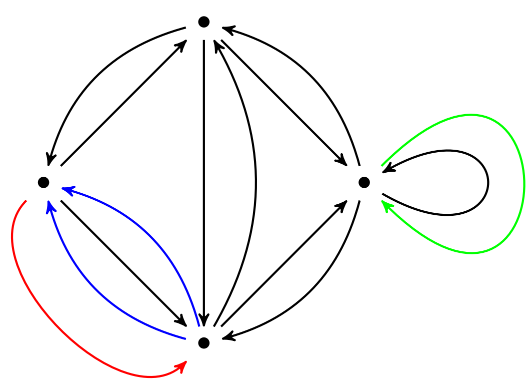
1. $|p^{-1}(e)| = 2$ for all $e \in E$. (In particular, p is surjective.)
2. If $\vec{e}_1 \neq \vec{e}_2$ are in $p^{-1}(e)$, then $\vec{e}_1$ and $\vec{e}_2$ reverses of each other, and e is an undirected version of both.

If $\vec{e}_1 \neq \vec{e}_2$ are in $p^{-1}(e)$, we will call them each others *converse*.

Informally, $\vec{G}$ is a directed graph that contains two directed edges for each edge of G. These two directed edges are located between the same two vertices as e, and point in opposite directions. The function p projects these two directed edges back onto e itself.

To avoid possible confusion, we stress the difference between reverses and converses. A directed edge $\vec{e}$ may have many reverses in $\vec{G}$. However, only one of those is singled out by p as *the* converse of $\vec{e}$. This is why it is important to view p as part of the doubly directed version. We might also have stated an alternative definition: "$\vec{G}$ is a doubly directed version of G if there exists a function p such that ...". Then, however, we would no longer be able to speak of converses. Let us consider the example from figure 1 again.

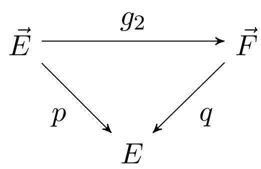
Example 1.4.2. Let G again be the graph from figure 1. Knowing that a doubly directed version of G contains two reverse edges for each edge of G, we deduce that $\vec{G}$ must look like this:



This, however, does not yet determine a doubly directed version of G. Indeed, the red edge has two reverses in $\vec{G}$, indicated in blue, both of which could be chosen to be its converse. Notice that the green edge *also* has two reverses in $\vec{G}$, namely itself and the other loop in $\vec{G}$. However, since converse edges are always *different*, we do not have a choice here. So there correspond exactly two possible doubly directed versions of G to the directed graph pictured above. ◊

As we mentioned earlier, a doubly directed version is less arbitrary than a regular directed version. While for the latter we have to choose where to put the arrowhead, the former just adds both possibilities. Therefore one expects any two doubly directed version of a given graph to be isomorphic. Let us make this precise.

Lemma 1.4.1. *Let G be a finite undirected graph and let $\left(\vec{G}, p\right)$ and $\left(\vec{H}, q\right)$, where $\vec{H} = \left(V, \vec{F}, \vec{s}\right)$, be two doubly directed versions of G. Then $\vec{G}$ and $\vec{H}$ are isomorphic by a graph isomorphism (g_1, g_2) that respects p and q in the sense that*



commutes.

Proof. We pick $g_1 = \mathrm{id}_V$. Now consider an $e \in E$. Suppose $\vec{e}_1, \vec{e}_2 \in \vec{E}$ are the two distinct elements of $p^{-1}(e)$, and $\vec{f}_1, \vec{f}_2 \in \vec{F}$ are the two distinct elements of $q^{-1}(e)$. Write $r(e) = \{v, w\}$ with $v, w \in V$. Now we consider two cases.

- $v \neq w$. By the definition of an undirected version, we must have $\{o(\vec{e}_1), t(\vec{e}_1)\} = \{v, w\} = \{o(\vec{f}_1), t(\vec{f}_1)\}$. So we have two mutually exclusive cases.
  - $o(\vec{e}_1) = o(\vec{f}_1)$ and $t(\vec{e}_1) = t(\vec{f}_1)$. Then by the definition of reverses, we have

    $$o(\vec{e}_2) = t(\vec{e}_1) = t(\vec{f}_1) = o(\vec{f}_2) \text{ and } t(\vec{e}_2) = o(\vec{e}_1) = o(\vec{f}_1) = t(\vec{f}_2)$$

    Now set $g_2(\vec{e}_1) = \vec{f}_1$ and $g_2(\vec{e}_2) = \vec{f}_2$.
  - $o(\vec{e}_1) = t(\vec{f}_1)$ and $t(\vec{e}_1) = o(\vec{f}_1)$. Then we have

    $$o(\vec{e}_1) = t(\vec{f}_1) = o(\vec{f}_2) \text{ and } t(\vec{e}_1) = o(\vec{f}_1) = t(\vec{f}_2),$$

    so we may put $g_2(\vec{e}_1) = \vec{f}_2$. Similarly,

    $$o(\vec{e}_2) = t(\vec{e}_1) = o(\vec{f}_1) \text{ and } t(\vec{e}_2) = o(\vec{e}_1) = t(\vec{f}_1),$$

    so we also set $g_2(\vec{e}_2) = \vec{f}_1$.
- $v = w$. Then $\{o(\vec{e}_i), t(\vec{e}_i)\}$ and $\{o(\vec{f}_i), t(\vec{f}_i)\}$ must be equal to $\{v, w\} = \{v\}$ for $i = 1, 2$. So $\vec{e}_1$, $\vec{e}_2$, $\vec{f}_1$ and $\vec{f}_2$ all originate and terminate in v. Now set $g_2(\vec{e}_1) = \vec{f}_1$ and $g_2(\vec{e}_2) = \vec{f}_2$.

Since p is surjective, this defines g_2 on all of $\vec{E}$. From the way we defined g_2, it is immediately clear that the above diagram commutes. Furthermore, by the observations we made above it is also clear that g_2 respects original and terminal vertices. It remains to show that g_2 is a bijection. But g_2 is bijective on each $p^{-1}(e)$, so this is the case. □

Consider a finite undirected graph G and write $|E| = m$. Then any doubly directed version $\vec{G}$ of G has exactly $2m$ directed edges. We will now consider the vector space S_G consisting of all complex linear combinations of these directed edges. Of course, this space is isomorphic to $\mathbb{C}^{2m}$ and has $\vec{E}$ as a basis. The promised edge-analogues of the vertex adjacency matrix will be matrices of certain linear maps from S_G to itself with respect to the basis $\vec{E}$. Of course, in order to do this, we first need to label the elements of $\vec{E}$. Now it should be remembered that the projection function p is also part of the information that makes $\vec{G}$ into a doubly directed version of G. The labeling of $\vec{E}$ will therefore not be wholly arbitrary.

Convention. Let G be a finite undirected graph with m edges and let $\left(\vec{G}, p\right)$ be a doubly directed version. We label the edges of G arbitrarily: $E = \{e_1, \ldots, e_m\}$. Now for $1 \leq i \leq m$, we write $p^{-1}(e_i) = \{\vec{e}_i, \vec{e}_{i+m}\}$. Therefore, $\vec{E}$ consists of the directed edges $\vec{e}_1, \ldots, \vec{e}_{2m}$. This will be our ordered standard basis for S_G.

In other words, we demand the converse of $\vec{e}_i$ to be $\vec{e}_{i+m}$. The fact the converses are reverses of each other, can now be expressed as:

$$o(\vec{e}_{i+m}) = t(\vec{e}_i) \text{ and } t(\vec{e}_{i+m}) = o(\vec{e}_i) \tag{1}$$

for all $i \in \{1, \ldots, m\}$. If we write $I = \{1, \ldots, 2m\}$ and read indices modulo $2m$, then property (1) above holds for *all* $i \in I$. Now we can finally define our edge-analogues of the vertex adjacency matrix for a graph.

Definition 1.4.4 (Edge adjacency matrices)**.** Let G be a finite undirected graph and $\vec{G}$ be a doubly directed version of G. The *extended edge adjacency matrix* $\tilde{T}_G$ of G is the $(2m) \times (2m)$-matrix given by:

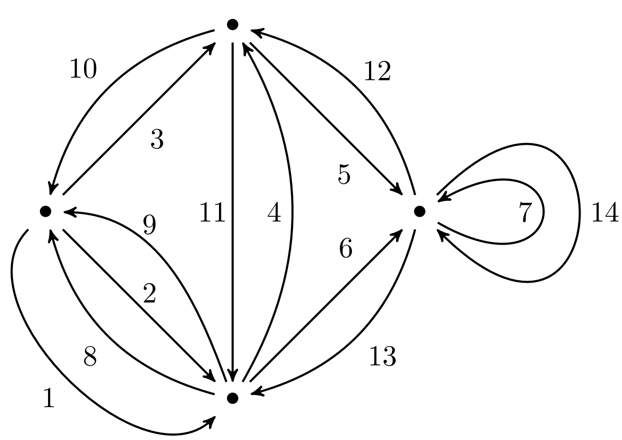
$$\text{for all } i, j \in I : (\tilde{T}_G)_{i,j} = \begin{cases} 1 \text{ if } t(\vec{e}_i) = o(\vec{e}_j); \\ 0 \text{ otherwise.} \end{cases}$$

The *edge adjacency matrix* T_G is the $(2m) \times (2m)$-matrix given by:

$$\text{for all } i, j \in I : (T_G)_{i,j} = \begin{cases} 1 \text{ if } t(\vec{e}_i) = o(\vec{e}_j) \text{ and } j \not\equiv i + m \mod 2m; \\ 0 \text{ otherwise.} \end{cases}$$

Informally, $\tilde{T}_G$ has a 1 on position (i, j) if and only if $\vec{e}_i$ flows into $\vec{e}_j$. The same holds for T_G, with the convention that a directed edge does not flow into its converse edge. Let us continue our example from figure 1.

Example 1.4.3. Let G once more be the graph from figure 1. We choose the following labeling for the edges of $\vec{G}$:



Notice that this labeling also implies a choice for p. Now we have

$$\tilde{T}_G = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix},$$

while

$$T_G = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}.$$

Notice that from the point of view of T_G, the directed edges labeled 7 and 14 do not flow into each other, but *do* flow into themselves. $\Diamond$

Let us investigate how $\tilde{T}_G$ and T_G act on S_G. For $i \in I$, we have

$$\tilde{T}_G(\vec{e}_i) = \sum_{\substack{j \in I \\ T_{j,i}=1}} \vec{e}_j = \sum_{\substack{j \in I \\ t(\vec{e}_j)=o(\vec{e}_i)}} \vec{e}_j,$$

so in other words, $\tilde{T}_G(\vec{e}_i)$ is the sum of all edges flowing into $\vec{e}_i$. Similarly, $T_G(\vec{e}_i)$ is the sum of all edges flowing into $\vec{e}_i$, with the exception of $\vec{e}_{i+m}$. This also means that $\tilde{T}_G^t(\vec{e}_i)$ is equal to the sum of all edges that $\vec{e}_i$ flows into, while for T_G^t, the similar statement holds where converse edges again are ruled out.

The edge adjacency matrix T_G was first studied in [1] and [4] and is often used for counting walks. Informally, if you are walking along the edge e_i in the direction of $\vec{e}_i$, then T_G^t tells you where you can go next *without walking back*. Indeed, as we already mentioned above, $T_G^t \vec{e}_i$ is the sum of all edges that $\vec{e}_i$ flows into, except the converse edge $\vec{e}_{i+m}$, which is the 'walking back' route. By applying the matrix T_G^t to itself repeatedly, we can count in how many ways we can walk through the graph G, without ever walking back. Such walks are called *non-backtracking*. Similarly, applying $\tilde{T}_G$ to itself repeatedly counts walks that *are* allowed to backtrack. However, as non-backtracking walks are often more interesting than walks with backtracks, the matrix T_G is studied more often than $\tilde{T}_G$ in the literature.

At this point, we should also spend a few words on the well-definedness of the edge adjacency matrices. To a graph G, there may correspond multiple matrices $\tilde{T}_G$ and T_G, because we also require a labeling of the edges. This is no different from the vertex case, where we needed to label the vertices. The above makes it clear, however, that *as operators on* S_G, the maps $\tilde{T}_G$ and T_G only depend on the doubly directed version $\left(\vec{G}, p\right)$, which is itself determined by G (all these statements hold, of course, up to isomorphism). In fact, one needs the information provided by p only for T_G, where converses are of importance. So in example 1.4.2, we can already determine $\tilde{T}_G$ from the picture, while for T_G, we really need to fix p.

2 Characterization of adjacency matrices

As we mentioned in the previous chapter, a graph G is completely determined by its vertex adjacency matrix A_G. That is, to each A_G there corresponds *at most one* graph G, up to isomorphism. A natural question would now be whether there is always *exactly one* graph for every matrix. Here we should first clarify what we mean by 'matrix'. Obviously, a non-square matrix, or a matrix containing a coefficient π, will not be the vertex adjacency matrix of anything. So let us consider only square matrices with nonnegative integer coefficients. But even then, vertex adjacency matrices have some properties that general matrices do not have: A_G is always symmetric, and its diagonal coefficients must be even. These properties turn out to characterise vertex adjacency matrices completely. That is to say: a square matrix with nonnegative integer coefficients is of the form A_G if and only if it is symmetric and has even diagonal coefficients. We will not prove this here, even though the proof is not difficult, but we will instead try to find an analogous result for our edge adjacency matrices.

2.1 The extended edge adjacency matrix

We will start by considering $\tilde{T}_G$.

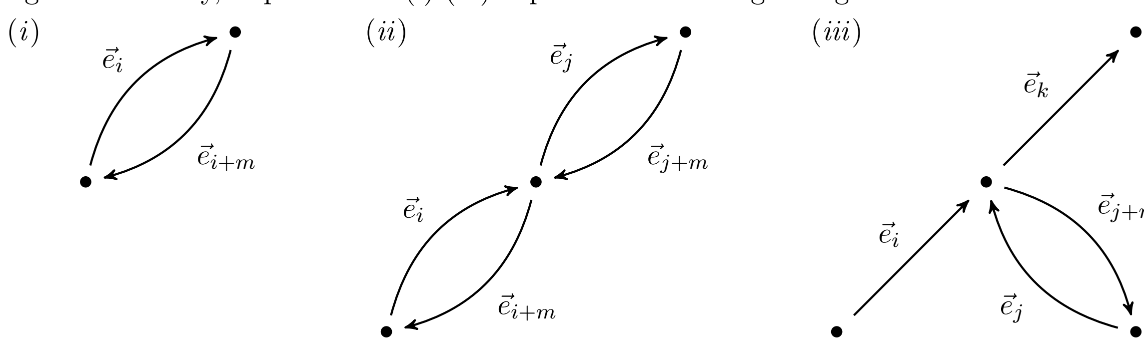
Theorem 2.1.1. *Let $m \in \mathbb{Z}_{>0}$ and let $\tilde{T}$ be a $(2m) \times (2m)$-matrix with entries in $\{0, 1\}$. Then $\tilde{T}$ is the extended edge adjacency matrix of some finite undirected graph G with m edges if, and only if, the following conditions are satisfied:*

(i) $\tilde{T}_{i,i+m} = 1$ *for all* $i \in I$;

(ii) $\tilde{T}_{i,j} = \tilde{T}_{j+m,i+m}$ *for all* $i, j \in I$;

(iii) for all $i, j \in I$, *we have: if* $\tilde{T}_{i,k} = \tilde{T}_{j,k} = 1$ *for some* $k \in I$, *then* $\tilde{T}_{i,j+m} = 1$.

Proof. First, suppose $\tilde{T}$ is the extended edge adjacency matrix $\tilde{T}_G$ of the graph G with m edges. Intuitively, requirements (i)-(iii) express the following configurations in G.

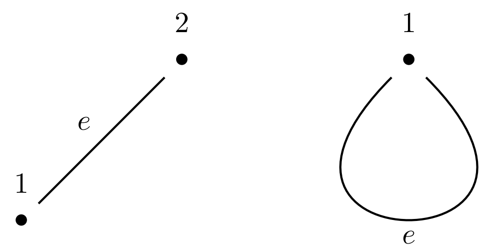


More formally, we have the following. For all $i \in I$, we have by definition that $t(\vec{e}_i) = o(\vec{e}_{i+m})$, so $\tilde{T}_{i,i+m}$ must indeed be equal to 1, i.e. (i) holds. If $\tilde{T}_{i,j} = 1$ for some $i, j \in I$, then we get $t(\vec{e}_{j+m}) = o(\vec{e}_j) = t(\vec{e}_i) = o(\vec{e}_{i+m})$, and therefore $\tilde{T}_{j+m,i+m} = 1$. Conversely, if $\tilde{T}_{j+m,i+m} = 1$, then by the same argument, $\tilde{T}_{i,j} = \tilde{T}_{i+2m,j+2m} = 1$. So (ii) holds. For (iii), suppose that $i, j, k \in I$ satisfy $\tilde{T}_{i,k} = \tilde{T}_{j,k} = 1$. Then $t(\vec{e}_i) = o(\vec{e}_k) = t(\vec{e}_j) = o(\vec{e}_{j+m})$, and we may indeed conclude that $\tilde{T}_{i,j+m} = 1$.

Before we prove the converse direction, we'll first derive another property from (i), (ii) and (iii), namely:

$$\text{for all } i, j, k \in I, \text{ we have: if } \tilde{T}_{i,j+m} = 1, \text{ then } \tilde{T}_{i,k} = \tilde{T}_{j,k}. \qquad (*)$$

Suppose $\tilde{T}_{i,j+m} = 1$. If $\tilde{T}_{i,k} = 1$, then by (ii), we get $\tilde{T}_{j,i+m} = \tilde{T}_{j+2m,i+m} = \tilde{T}_{i,j+m} = 1$ and $\tilde{T}_{k+m,i+m} = \tilde{T}_{i,k} = 1$. Now (iii) yields $1 = \tilde{T}_{j,k+2m} = \tilde{T}_{j,k}$. Since $\tilde{T}_{i,j+m} = 1$ implies $\tilde{T}_{j,i+m} = 1$ by (ii), we can show with the same argument that $\tilde{T}_{j,k} = 1 \Longrightarrow \tilde{T}_{i,k} = 1$. So $(*)$ indeed holds.
For the converse direction of the theorem, we'll show by induction on m that any $(2m) \times (2m)$-matrix $\tilde{T}$ with coefficients in $\{0, 1\}$ and satisfying (i), (ii) and (iii), is the extended edge adjacency matrix of some graph G.
Basis. For $m = 1$, property (i) gives $\tilde{T}_{1,2} = \tilde{T}_{2,1} = 1$, whereas (ii) gives $\tilde{T}_{1,1} = \tilde{T}_{2,2}$. So $\tilde{T}$ can be $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. In the first case, we can take $G = (\{1, 2\}, \{e\}, r)$, where $r(e) = \{1, 2\}$, while in the second case, we can take $G = (\{1\}, \{e\}, r)$, where $r(e) = \{1\}$. Or, more visually:



Step. Suppose we have proven the statement for $m - 1$ for a certain $m \geq 2$. Let $\tilde{T}$ be a $(2m) \times (2m)$-matrix with coefficients in $\{0, 1\}$ satisfying (i)-(iii) and let T^- be the matrix obtained from $\tilde{T}$ by removing the m^{th} and $(2m)^{\text{th}}$ rows and columns. Then it is easy to check that T^- satisfies (i), (ii) and (iii) with $m - 1$ instead of m, so by the induction hypothesis, it is the extended edge adjacency matrix of some undirected graph G^- with edges $e_1, \ldots, e_{m-1}$. Consider a doubly directed version $\vec{G}^-$ with edges $\vec{e}_1, \ldots, \vec{e}_{2m-2}$. First, relabel the edges $\vec{e}_i$ to $\vec{e}_{i+1}$ for $m \leq i \leq 2m - 2$. We then get a directed graph with edges $\vec{e}_1, \ldots, \vec{e}_{m-1}, \vec{e}_{m+1}, \ldots, \vec{e}_{2m-1}$ such that, when indices are read modulo $2m$, the converse of $\vec{e}_i$ is $\vec{e}_{i+m}$ (rather than $\vec{e}_{i+m-1}$ when indices are read modulo $2m - 2$). We will construct a directed graph $\vec{G}$ from $\vec{G}^-$ by adding new vertices and adding edges $\vec{e}_m$ and $\vec{e}_{2m}$. Consider the following cases.

1. There is some $\alpha \neq m, 2m$ such that $\tilde{T}_{\alpha,m} = 1$. Then set $o(\vec{e}_m) = t(\vec{e}_\alpha)$.
2. There is no such α. Then add a new vertex v to G^- and set $o(\vec{e}_m) = v$.

For $t(\vec{e}_m)$, we consider the following cases.

1. There is some $\beta \neq m, 2m$ such that $\tilde{T}_{m,\beta} = 1$. Then set $t(\vec{e}_m) = o(\vec{e}_\beta)$.
2. There is no such β. Then consider $\tilde{T}_{m,m}$.
   (a) $\tilde{T}_{m,m} = 1$. Then set $t(\vec{e}_m) = o(\vec{e}_m)$.
   (b) $\tilde{T}_{m,m} = 0$. Then add a new vertex $w \neq o(\vec{e}_m)$ to G^- and set $t(\vec{e}_m) = w$.

Of course, we set $o(\vec{e}_{2m}) = t(\vec{e}_m)$ and $t(\vec{e}_{2m}) = o(\vec{e}_m)$. Now let G that has the same vertex set as $\vec{G}$ and has all edges of G^- along with an extra edge e_m such that $r(e_m) = \{o(\vec{e}_m), t(\vec{e}_m)\}$.

Extend p^- to a function p on the edge set of $\vec{G}$ by postulating $p(\vec{e}_m) = p(\vec{e}_{2m}) = e_m$. Then $\left(\vec{G}, p\right)$ is a doubly directed version of G. We claim that $\tilde{T}$ is the extended edge adjacency matrix of G (via the chosen labeling of $\vec{G}$).
Since we haven't changed anything about the edges that already were in $\vec{G}^-$, we need only consider the entries $\tilde{T}_{i,j}$ with $\{i,j\} \cap \{m, 2m\} \neq \emptyset$. As $\tilde{T}$ satisfies (ii) and because of the way we constructed $\vec{e}_{2m}$, we actually only have to consider $\tilde{T}_{i,j}$ with i or j (or both) equal to m. Suppose $j = m$ and $i \neq m$. If $i = 2m$, then we have $\tilde{T}_{2m,m} = 1$, and indeed $t(\vec{e}_{2m}) = o(\vec{e}_m)$. So suppose $i \neq 2m$. We consider two cases:

1. There is some $\alpha \neq m, 2m$ such that $\tilde{T}_{\alpha,m} = 1$. Then if $\tilde{T}_{i,m} = 1$, we get by (iii) that $\tilde{T}_{i,\alpha+m} = 1$, so $t(\vec{e}_i) = o(\vec{e}_{\alpha+m}) = t(\vec{e}_\alpha) = o(\vec{e}_m)$.

   Conversely, suppose $t(\vec{e}_i) = o(\vec{e}_m)$. Then we get $t(\vec{e}_i) = o(\vec{e}_m) = t(\vec{e}_\alpha) = o(\vec{e}_{\alpha+m})$, implying $\tilde{T}_{i,\alpha+m} = 1$. Now $(*)$ gives $\tilde{T}_{i,m} = \tilde{T}_{\alpha,m} = 1$. So $\tilde{T}_{i,m}$ is correct.

2. There is no such α, then $\tilde{T}_{i,m} = 0$. Also, since $o(\vec{e}_m)$ is a fresh vertex, we have $t(\vec{e}_i) \neq o(\vec{e}_m)$. So $\tilde{T}_{i,m}$ is correct.

The case where $i = m$ and $j \neq m$ is very similar, but with reversed directions, so we omit it. We are then left with checking $\tilde{T}_{m,m}$. We consider the same two cases.

1. There is some $\alpha \neq m, 2m$ such that $\tilde{T}_{\alpha,m} = 1$. Suppose $\tilde{T}_{m,m} = 1$, then (iii) gives $\tilde{T}_{m,\alpha+m} = 1$. So there is a β with $\tilde{T}_{m,\beta} = 1$. Now we know $o(\vec{e}_m) = t(\vec{e}_\alpha)$ and $t(\vec{e}_m) = o(\vec{e}_\beta)$. Since $\tilde{T}_{m,\alpha+m} = 1$, we have $\tilde{T}_{\alpha,\beta} = \tilde{T}_{m,\beta} = 1$ by $(*)$, so $t(\vec{e}_\alpha) = o(\vec{e}_\beta)$. Combining these, we get $o(\vec{e}_m) = t(\vec{e}_\alpha) = o(\vec{e}_\beta) = t(\vec{e}_m)$.

   Conversely, suppose now that $o(\vec{e}_m) = t(\vec{e}_m)$. Then we didn't introduce a new vertex to be $t(\vec{e}_m)$, so there must be a $\beta \neq m, 2m$ such that $\tilde{T}_{m,\beta} = 1$. Now we have $t(\vec{e}_\alpha) = o(\vec{e}_m) = t(\vec{e}_m) = o(\vec{e}_\beta)$, so $\tilde{T}_{\alpha,\beta} = 1$. Now combining $\tilde{T}_{\alpha,\beta} = \tilde{T}_{m,\beta} = 1$ with (iii) gives $\tilde{T}_{\alpha,2m} = 1$. Now $\tilde{T}_{m,\alpha+m} = \tilde{T}_{\alpha,2m} = 1 = \tilde{T}_{\alpha,m} = \tilde{T}_{2m,\alpha+m}$ and again (iii) gives $\tilde{T}_{m,m} = 1$.

2. There is no such α. Suppose $\tilde{T}_{m,m} = 1$. If there is a β such that $\tilde{T}_{m,\beta} = 1$. Then $\tilde{T}_{\beta+m,2m} = \tilde{T}_{m,\beta} = 1 = \tilde{T}_{m,m} = \tilde{T}_{2m,2m}$, so (iii) gives $\tilde{T}_{\beta+m,m} = 1$. But this is a contradiction with our assumption, because then we could take $\alpha = \beta + m \neq m, 2m$. So there is no such β. As $\tilde{T}_{m,m} = 1$, we automatically get $t(\vec{e}_m) = o(\vec{e}_m)$.

   Conversely, suppose $t(\vec{e}_m) = o(\vec{e}_m) = v$. Then $t(\vec{e}_m)$ is a fresh vertex, so there is no $\beta \neq m, 2m$ with $\tilde{T}_{m,\beta} = 1$. Now, if $\tilde{T}_{m,m}$ were equal to 0, we'd have $t(\vec{e}_m) \neq o(\vec{e}_m)$, which cannot occur. So $\tilde{T}_{m,m} = 1$.

This concludes the induction. □

The above theorem allows us to determine whether a graph is the extended edge adjacency matrix of some graphs without having to actually find such a graph. It removes the combinatorics from the question and characterizes the class of extended edge adjacency matrices in terms of only the matrix coefficients.

Now, the reader may have noticed that we have not yet addressed the other natural question concerning adjacency matrices: is G uniquely determined by $\tilde{T}_G$? Perhaps surprisingly, the answer is no. The reason for this, however, is rather trivial. Suppose we have a graph G with extended edge adjacency matrix $\tilde{T}_G$ and let G^+ be a graph that we obtain by adding an isolated vertex to G. Then it is easy to check that $\tilde{T}_G = \tilde{T}_{G^+}$. This should not come as a surprise: the extended edge adjacency matrix lays out the edge structure of a graph, and isolated vertices have very little to do with edges. Therefore, our question becomes far more interesting if we prohibit the occurrence of isolated vertices, and then the answer becomes affirmative. First, we need to observe the following relation between degrees in G and degrees in $\vec{G}$.

Lemma 2.1.2. *Let G be a finite undirected graph, v be a vertex and $\left(\vec{G}, p\right)$ be a doubly directed version of G. Then indegree and outdegree of v in $\vec{G}$ are both equal to the degree of v in G.*

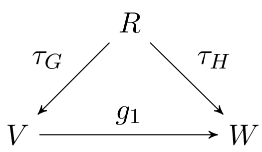
Proof. This is a matter of careful counting. First of all, suppose $\vec{e}$ is a directed edge of $\vec{G}$ terminating in v, and write $e = p(\vec{e})$. Then v must be in $r(e)$, since $e = p(\vec{e})$ is an undirected version of $\vec{e}$. So all directed edges $\vec{e}$ terminating in v are in $p^{-1}(e)$ for some edge e with $v \in r(e)$.
Let $w \neq v$ be a vertex and consider an edge e with $r(e) = \{v, w\}$. Then $p^{-1}(e)$ contains exactly one directed edge terminating in v (the other one terminating in $w \neq v$). Furthermore, if e is an edge such that $r(e) = \{v\}$, then both elements of $p^{-1}(e)$ must terminate in v. We conclude that the indegree of v in $\vec{G}$ must be equal to $2\left|r^{-1}(\{v\})\right| + \sum_{\substack{w \in V \\ v \neq w}} \left|r^{-1}(\{v, w\})\right|$, which is exactly the degree of v in G. The statement for the outdegree is proven similarly. □

In particular, a vertex is isolated in G if and only if it is isolated in $\vec{G}$, a fact that we will need for our uniqueness result. The following theorem can be found in [2], while the proof is due to the author.

Theorem 2.1.3. *If $\tilde{T}$ is the extended edge adjacency matrix of some finite undirected graph G, then there exists, up to isomorphism, only one such graph G containing no isolated vertices.*

Proof. Consider two finite and undirected graphs G and $H = (W, F, s)$ with (labeled) doubly directed versions $\left(\vec{G}, p\right)$ and $\left(\vec{H}, q\right)$, where $\vec{H} = \left(W, \vec{F}, \vec{s}\right)$. Suppose furthermore that G and H contain no isolated vertices and that $\tilde{T}_G = \tilde{T}_H$. We will construct an isomorphism (g_1, g_2) between G and H.
Let us write $2m$ with $m \geq 1$ for the dimension of $\tilde{T}_G = \tilde{T}_H$, then G and H both contain m edges. Now notice that in order to speak about $\tilde{T}_G$ and $\tilde{T}_H$, we must have fixed a labeling of the edges of both graphs. So we have $E = \{e_1, \ldots, e_m\}$ and $F = \{f_1, \ldots, f_m\}$. Now define $g_2 : E \to F$ by $g_2(e_i) = f_i$ for $1 \leq i \leq m$. Obviously, this is a bijection.
Write $\mathbf{r}_i$ for the row vector that is the i^{th} row of $\tilde{T}_G$. In other words, $(\mathbf{r}_i)_j = (\tilde{T}_G)_{i,j}$ for all $i, j \in I$. We define $R = \{\mathbf{r}_1, \ldots, \mathbf{r}_{2m}\}$ as the set of all *different* rows of $\tilde{T}_G$. Our goal is to define a bijection $\tau_G : R \to V$. We take $\tau_G(\mathbf{r}_i) = t(\vec{e}_i)$.
First of all, we need to check that this map is well-defined. So suppose that $\mathbf{r}_i = \mathbf{r}_{i'}$ for some $i, i' \in I$. We always have $(\mathbf{r}_i)_{i+m} = (\tilde{T}_G)_{i,i+m} = 1$, so we get $(\mathbf{r}_{i'})_{i+m} = 1$, implying that $t(\vec{e}_{i'}) = o(\vec{e}_{i+m}) = t(\vec{e}_i)$. That is $\tau_G(i) = \tau_G(i')$, and τ_G is indeed well-defined. Conversely, suppose that $\tau_G(i) = \tau_G(i')$, that is, $t(\vec{e}_i) = t(\vec{e}_{i'})$. Then obviously, we have $t(\vec{e}_i) = o(\vec{e}_j)$ if and only if $t(\vec{e}_{i'}) = o(\vec{e}_j)$ for all $j \in I$. That is to say, $(\tilde{T}_G)_{i,j} = (\tilde{T}_G)_{i',j}$ for all $j \in I$, which means exactly that $\mathbf{r}_i = \mathbf{r}_{i'}$. So τ_G is injective. Finally, consider a vertex $v \in V$. Since v is not isolated, it has degree at least 1 in G. By lemma 2.1.2, the indegree of v in $\vec{G}$ must be

at least 1 as well. That is, there exists an $i \in I$ such that $\tau_G(\mathbf{r}_i) = t(\vec{e}_i) = v$, and we see that τ_G is surjective.
Since $\tilde{T}_H = \tilde{T}_G$, we can analogously define a bijection $\tau_H : R \to W$ by $\tau_H(\mathbf{r}_i) = t(\vec{f}_i)$. Now define $g_1 : V \to W$ as $\tau_H \tau_G^{-1}$. This is a bijection, as both τ_G and τ_H are. To visualize g_1, notice that the diagram



commutes.
Now suppose that $v \in r(e_i)$ for some $v \in V$ and $e_i \in E$. As e_i is an undirected version of $\vec{e}_i$, we have $v = t(\vec{e}_i)$ or $v = o(\vec{e}_i)$. First, suppose that $v = t(\vec{e}_i)$, then $\tau_G^{-1}(v)$ must be equal to $\mathbf{r}_i$. So $g_1(v) = \tau_H(\tau_G^{-1}(v)) = \tau_H(\mathbf{r}_i) = t(\vec{f}_i)$. Since f_i is an undirected version of $\vec{f}_i$, this means that $g_1(v) \in s(f_i) = s(g_2(e_i))$. Now suppose that $v = o(\vec{e}_i)$. Then $v = t(\vec{e}_{i+m})$, and we derive similarly that $g_1(v) = t(\vec{f}_{i+m})$, that is, $g_1(v) = o(\vec{f}_i)$. Now we again have $g_1(v) \in s(f_i) = s(g_2(e_i))$.
Conversely, suppose that $g_1(v) \in s(g_2(e_i))$ for some $v \in V$ and $e_i \in E$. That is, we have $g_1(v) \in s(f_i)$, and since f_i is an undirected version of $\vec{f}_i$, we get $g_1(v) = t(\vec{f}_i)$ or $g_1(v) = o(\vec{f}_i)$. First, suppose that $g_1(v) = t(\vec{f}_i)$. Then we get $\tau_H(\tau_G^{-1}(v)) = g_1(v) = t(\vec{f}_i) = \tau_H(\mathbf{r}_i)$, so as τ_H is invertible, this gives $\tau_G^{-1}(v) = \mathbf{r}_i$. We obtain $v = \tau_G(\mathbf{r}_i) = t(\vec{e}_i)$, and since e_i is an undirected version of $\vec{e}_i$, this means that $v \in r(e_i)$. Now suppose that $g_1(v) = o(\vec{f}_i)$. Then $g_1(v) = t(\vec{f}_{i+m})$, and we derive similarly that $v = t(\vec{e}_{i+m})$, that is, $v = o(\vec{e}_i)$. This again yields $v \in r(e_i)$.
We conclude that (g_1, g_2) preserves structure, and is therefore an isomorphism of graphs. □

2.2 The edge adjacency matrix

We would like to have results like theorems 2.1.1 and 2.1.3 for (regular) edge adjacency matrices as well. In order to obtain such results, we first need the following definition, that will turn out to be of central importance not only in this section, but further on as well.

Definition 2.2.1. For $m \in \mathbb{Z}_{>0}$, we define J_m as the $(2m) \times (2m)$-matrix $\begin{pmatrix} O_m & I_m \\ I_m & O_m \end{pmatrix}$, where I_m is the $m \times m$ identity matrix and O_m is the $m \times m$ zero matrix. In other words, $(J_m)_{i,j} = 1$ if $j \equiv i + m \mod 2m$; otherwise it is 0.

The matrix J_m links $\tilde{T}_G$ and T_G in the following sense.

Proposition 2.2.1. *For any graph G with m edges, we have $\tilde{T}_G = T_G + J_m$.*

Proof. Notice that we have $t(\vec{e}_i) = o(\vec{e}_{i+m})$ for all $i \in I$. So the difference between $\tilde{T}_G$ and T_G is that the former has a 1 on each position $(i, i+m)$, while the latter has a 0 on those positions. On all other positions, $\tilde{T}_G$ and T_G agree. But we can express this relation elegantly as $\tilde{T}_G = T_G + J_m$. □

This newly discovered link enables us to find the characterisation of edge adjacency matrices.

Theorem 2.2.2. *Let $m \in \mathbb{Z}_{>0}$ and let T be a $(2m) \times (2m)$-matrix with entries in $\{0,1\}$. Then T is the extended edge adjacency matrix of some finite undirected graph G with m edges if, and only if, the following conditions are satisfied:*

(i)' $T_{i,i+m} = 0$ *for all* $i \in I$*;*

(ii)' $T_{i,j} = T_{j+m,i+m}$ *for all* $i, j \in I$*;*

(iii)' for all $i, j \in I$*, we have: if* $i \not\equiv j \mod 2m$ *and* $T_{i,k} = T_{j,k} = 1$ *for some* $k \in I$*, then* $T_{i,j+m} = 1$*.*

Proof. By proposition 2.2.1 and theorem 2.1.1, the matrix T is the edge adjacency matrix of some finite graph G if, and only if, it satisfies the following conditions:

(i) $T_{i,i+m} + (J_m)_{i,i+m} = 1$ for all $i \in I$;

(ii) $T_{i,j} + (J_m)_{i,j} = T_{j+m,i+m} + (J_m)_{j+m,i+m}$ for all $i, j \in I$;

(iii) for all $i, j \in I$, we have: if $T_{i,k} + (J_m)_{i,k} = T_{j,k} + (J_m)_{j,k} = 1$ for some $k \in I$, then $T_{i,j+m} + (J_m)_{i,j+m} = 1$.

We have $J_{i,i+m} = 1$ for all $i \in I$, so (i) above is equivalent to (i)'. Also,

$$\begin{aligned}(J_m)_{i,j} = 1 &\Leftrightarrow i \equiv j + m \mod 2m \\ &\Leftrightarrow j + m \equiv (i+m) + m \mod 2m \\ &\Leftrightarrow (J_m)_{j+m,i+m} = 1, \end{aligned} \tag{2}$$

for all $i, j \in I$, so (ii) above is equivalent to (ii)'. So it remains to show that (iii) above and (iii)' are equivalent in the presence of (i)' and (ii)'.
From now on, we suppose that (i)' and (ii)' hold. First of all, notice that if $T_{i,j} = 1$, then $j \not\equiv i + m \mod 2m$, so $(J_m)_{i,j} = 0$. From this it also follows that if $(J_m)_{i,j} = 1$, then $T_{i,j}$ cannot be equal to 1, so we must have $T_{i,j} = 0$.
Suppose (iii) holds and consider $i, j, k \in I$ such that $i \neq j \mod 2m$ and $T_{i,k} = T_{j,k} = 1$. Then $(J_m)_{i,k} = (J_m)_{j,k} = 0$ and we have $T_{i,k} + (J_m)_{i,k} = T_{j,k} + (J_m)_{j,k} = 1$. Then by (iii), we get $T_{i,j+m} + (J_m)_{i,j+m} = 1$. As $j + m \not\equiv i + m \mod 2m$, we have that $(J_m)_{i,j+m} = 0$, so $T_{i,j+m} = 1$, that is, (iii)' holds.
Now suppose that (iii)' holds and consider $i, j, k \in I$ such that $T_{i,k} + (J_m)_{i,k} = T_{j,k} + (J_m)_{j,k} = 1$. If $i \equiv j \mod 2m$, then also $j + m \equiv i + m \mod 2m$, so $(J_m)_{i,j+m} = 1$. This means that $T_{i,j+m} = 0$, so we indeed have $T_{i,j+m} + (J_m)_{i,j+m} = 1$. From now on, assume that $i \not\equiv j \mod 2m$. We consider the following cases.

1. $(J_m)_{i,k} = 1$. Then $k \equiv i + m \mod 2m$, so we must have $T_{j,i+m} + (J_m)_{j,i+m} = T_{j,k} + (J_m)_{j,k} = 1$. Now notice that $T_{i,j+m} = T_{j+2m,i+m} = T_{j,i+m}$ and by (2), we have $(J_m)_{i,j+m} = (J_m)_{j+2m,i+m} = (J_m)_{j,i+m}$. So we indeed have $T_{i,j+m} + (J_m)_{i,j+m} = 1$.

2. $(J_m)_{j,k} = 1$. Then $k \equiv j + m \mod 2m$, so we must have $T_{i,j+m} + (J_m)_{i,j+m} = T_{i,k} + (J_m)_{i,k} = 1$.

3. $(J_m)_{i,k} = 0$ and $(J_m)_{i,k} = 0$. Then we must have $T_{i,k} = T_{j,k} = 1$, so (iii)' gives us that $T_{i,j+m} = 1$. This also means that $(J_m)_{i,j+m} = 0$ and we may conclude again that $T_{i,j+m} + (J_m)_{i,j+m} = 1$.

So (iii) indeed holds, and this concludes the proof. $\square$

Uniqueness can also be established.

Theorem 2.2.3. *If T is the extended edge adjacency matrix of some finite undirected graph G, then there exists, up to isomorphism, only one such graph G containing no isolated vertices.*

Proof. Suppose there were two such non-isomorphic graphs G and G'. Then by proposition 2.2.1, $T + J_m$ would be the extended edge adjacency matrix of both G and G', which contradicts theorem 2.1.3. $\square$

2.3 A more elegant characterization

The conditions in theorems 2.1.1 and 2.2.2 do not look quite easy to work with. In the vertex case, we can express the symmetry of the vertex adjacency matrix A_G simply by $A_G^t = A_G$. Therefore the question rises whether we can express properties (i)-(iii) or (i)'-(iii)' by some decent equations. It turns out that, to a certain extent, we can, but in order to do this, we'll need to do some more work.

First of all, let us examine the matrix J_m a bit closer. It has the following properties.

Proposition 2.3.1. *Let $m \in \mathbb{Z}_{>0}$ and let M be an arbitrary $(2m) \times (2m)$-matrix. Then*

1. $J_m^t = J_m$;
2. $J_m^2 = I_{2m}$;
3. *for all $i, j \in I$, we have $(MJ_m)_{i,j} = M_{i,j+m}$ and $(J_m M)_{i,j} = M_{i+m,j}$. In particular, $(J_m M J_m)_{i,j} = M_{i+m,j+m}$.*

Proof. Immediate. $\square$

Definition 2.3.1. We define the function sign : $\mathbb{Z}_{\geq 0} \to \{0, 1\}$ by

$$\operatorname{sign}(x) = \begin{cases} 0 \text{ if } x = 0; \\ 1 \text{ if } x > 0. \end{cases}$$

Now we can state our result.

Theorem 2.3.2. *Let $m \in \mathbb{Z}_{>0}$ and let $\tilde{T}$ be a $(2m) \times (2m)$-matrix with entries in $\{0, 1\}$. Then $\tilde{T}$ is the extended edge adjacency matrix of some finite undirected graph G if, and only if, the following conditions are satisfied:*

(iv) $\operatorname{Tr}(\tilde{T} J_m) = 2m$;

(v) $\tilde{T} J_m = (\tilde{T} J_m)^t$;

(vi) $\operatorname{sign}\left((\tilde{T} J_m)^2\right) = \tilde{T} J_m$.

Proof. By theorem 2.1.1, it suffices to show that (iv), (v) and (vi) follow from (i), (ii) and (iii), and vice versa.
So first suppose that (i), (ii) and (iii) hold. Then for all $i \in I$, we have $(\tilde{T}J_m)_{i,i} = \tilde{T}_{i,i+m} = 1$, so all diagonal entries of $\tilde{T}J_m$ are equal to 1, giving a trace equal to $2m$. By (ii), we have for all $i, j \in I$ that

$$\left((\tilde{T}J_m)^t\right)_{i,j} = (\tilde{T}J_m)_{j,i} = \tilde{T}_{j,i+m} = \tilde{T}_{i+2m,j+m} = \tilde{T}_{i,j+m} = (\tilde{T}J_m)_{i,j},$$

so (v) holds. So we are left with (vi). Let $i, j \in I$ and notice that $(\tilde{T}J_m)^2 = (\tilde{T}J_m)(\tilde{T}J_m)^t = (\tilde{T}J_m)(J_m^t\tilde{T}^t) = \tilde{T}(J_m^2)\tilde{T}^t = \tilde{T}\tilde{T}^t$, so that $\left(\text{sign}\left((\tilde{T}J_m)^2\right)\right)_{i,j} = 1$ iff $\left((\tilde{T}J_m)^2\right)_{i,j} > 0$; iff $(\tilde{T}\tilde{T}^t)_{i,j} > 0$. Now $(\tilde{T}\tilde{T}^t)_{i,j}$ is equal to $\sum_{k\in I} \tilde{T}_{i,k}\tilde{T}^t_{k,j} = \sum_{k\in I} \tilde{T}_{i,k}\tilde{T}_{j,k}$. Since all the entries from $\tilde{T}$ are in $\{0, 1\}$, this expression is positive precisely if there is a $k \in I$ with $\tilde{T}_{i,k} = \tilde{T}_{j,k} = 1$. If such a k exists, then (iii) gives $\tilde{T}_{i,j+m} = 1$. Conversely, if $\tilde{T}_{i,j+m} = 1$, then by (i), $k = j+m$ satisfies $\tilde{T}_{i,k} = \tilde{T}_{j,k} = 1$. So there is a k with $\tilde{T}_{i,k} = \tilde{T}_{j,k} = 1$ iff $\tilde{T}_{i,j+m} = 1$; by item 3 of proposition 2.3.1, iff $(\tilde{T}J_m)_{i,j} = 1$. So (vi) indeed holds.
Now suppose that (iv), (v) and (vi) hold. As $\tilde{T}$ is a $\{0, 1\}$-matrix, its trace can only be equal to $2m$ if all diagonal entries are equal to 1. So for all $i \in I$, we get $(\tilde{T})_{i,i+m} = (\tilde{T}J_m)_{i,i} = 1$, i.e. (i) holds. By (v), we have for all $i, j \in I$ that

$$\tilde{T}_{i,j} = (\tilde{T}J_m)_{i,j+m} = \left((\tilde{T}J_m)^t\right)_{i,j+m} = (\tilde{T}J_m)_{j+m,i} = \tilde{T}_{j+m,i+m},$$

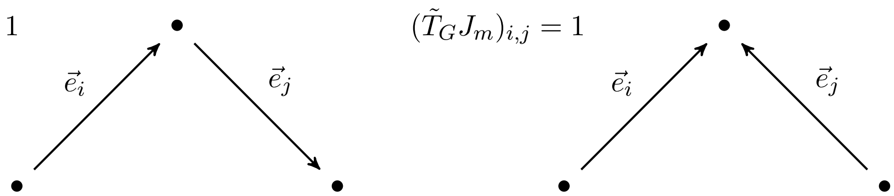
giving (ii). For (iii), suppose that $\tilde{T}_{i,k} = \tilde{T}_{j,k}$ for some $i, j, k \in I$. Then

$$(\tilde{T}\tilde{T}^t)_{i,j} = \sum_{\ell\in I} \tilde{T}_{i,\ell}\tilde{T}^t_{\ell,j} = \sum_{\ell\in I} \tilde{T}_{i,\ell}\tilde{T}_{j,\ell} \geq \tilde{T}_{i,k}\tilde{T}_{j,k} = 1 > 0.$$

By (v), we again have $\tilde{T}\tilde{T}^t = (\tilde{T}J_m)^2$, so $\left(\text{sign}\left((\tilde{T}J_m)^2\right)\right)_{i,j} = 1$. By (vi), we get $\tilde{T}_{i,j+m} = (\tilde{T}J_m)_{i,j} = \left(\text{sign}\left((\tilde{T}J_m)^2\right)\right)_{i,j} = 1$. □

One should notice that theorem 2.3.2 is really stating constraints on $\tilde{T}J_m$ rather than $\tilde{T}$. In certain ways, the former is much better behaved than the latter. This can be explained by considering the difference between $\tilde{T}_G$ and $\tilde{T}_GJ_m$ in terms of the graph G itself. Indeed, while $(\tilde{T}_G)_{i,j} = 1$ precisely if $t(\vec{e}_i) = o(\vec{e}_j)$, we have $(\tilde{T}_GJ_m)_{i,j} = 1$ iff $(\tilde{T}_G)_{i,j+m} = 1$; iff $t(\vec{e}_i) = o(e_{j+m})$; iff $t(\vec{e}_i) = t(\vec{e}_j)$.

$(\tilde{T}_G)_{i,j} = 1$ $(\tilde{T}_GJ_m)_{i,j} = 1$



This explains the symmetric behaviour of $\tilde{T}_GJ_m$. It also shows that, as a linear operator, $\tilde{T}_GJ_m$ is 'local'. That is, $(\tilde{T}_GJ_m)(\vec{e}_i)$ is composed entirely of basis vectors that have the same terminal vertex as $\vec{e}_i$. This is not the case for $\tilde{T}_G$. Indeed, $\tilde{T}_G$ counts walks, and walks are typically no 'local' object: the whole point of walking is to get somewhere else. This also shows that, for the purpose of counting walks through the graph, $\tilde{T}_G J_m$ is not interesting.

Let us state the analogous result for the (regular) edge adjacency matrix.

Theorem 2.3.3. *Let $m \in \mathbb{Z}_{>0}$ and let T be a $(2m) \times (2m)$-matrix with entries in $\{0, 1\}$. Then T is the edge adjacency matrix of some finite graph G if, and only if, the following conditions are satisfied:*

(iv)' $\mathrm{Tr}(TJ_m) = 0$*;*

(v)' $TJ_m = (TJ_m)^t$*;*

(vi)' $\mathrm{sign}\left((TJ_m + I_{2m})^2\right) = TJ_m + I_{2m}$.

Proof. Notice that $(T + J_m)J_m = TJ_m + J_m^2 = TJ_m + I_{2m}$. So by proposition 2.2.1 and theorem 2.3.2, T is the edge adjacency matrix of some finite graph G if and only if the following hold:

(iv) $\mathrm{Tr}(TJ_m + I_{2m}) = 2m$;

(v) $TJ_m + I_{2m} = (TJ_m + I_{2m})^t$;

(vi) $\mathrm{sign}\left((TJ_m + I_{2m})^2\right) = TJ_m + I_{2m}$.

We have that $\mathrm{Tr}(I_{2m}) = 2m$, so (iv) above is equivalent to (iv)', as the trace function is additive. Furthermore, we have $(TJ_m + I_{2m})^t = (TJ_m)^t + I_{2m}^t = (TJ_m)^t + I_{2m}$, so (v) above is equivalent to (v)'. Finally, (vi) above is just (vi)'. This concludes the proof. □

The third constraint in theorem 2.3.2 (or 2.3.3), although quite convenient for the purposes of calculation, still does not look too attractive. It says that a double application of $\tilde{T}J_m$ looks in a certain way like the operator $\tilde{T}J_m$ itself. We can uncover this similarity a bit further.

Definition 2.3.2. For $i \in I$, we define d_i as the degree in G of the vertex $t(\vec{e}_i)$, and define D_G^e as the $(2m) \times (2m)$ diagonal matrix $\mathrm{diag}(d_1, \ldots, d_{2m})$.

The diagonal matrix defined above turns out to be the missing link between $(\tilde{T}_G J_m)^2$ and $\tilde{T}_G J_m$. Recall from lemma 2.1.2 that the indegree of a certain vertex in $\vec{G}$ is equal to the degree of this vertex in G. So for $i \in I$, the number of directed edges in $\vec{G}$ terminating in $t(\vec{e}_i)$ is d_i. Now, if $i, j \in I$, then we have

$$\begin{aligned}
\left((\tilde{T}_G J_m)^2\right)_{i,j} &= \sum_{k \in I} (\tilde{T}_G J_m)_{i,k} (\tilde{T}_G J_m)_{k,j} \\
&= |\{k \in I \mid (\tilde{T}_G J_m)_{i,k} = (\tilde{T}_G J_m)_{k,j} = 1\}| \\
&= |\{k \in I \mid t(\vec{e}_i) = t(\vec{e}_k) = t(\vec{e}_j)\}| \\
&= \begin{cases} d_i & \text{if } t(\vec{e}_i) = t(\vec{e}_j) \\ 0 & \text{if } t(\vec{e}_i) \neq t(\vec{e}_j) \end{cases} \\
&= d_i (\tilde{T}_G J_m)_{i,j} = (D_G^e \tilde{T}_G J_m)_{i,j},
\end{aligned}$$

so $(\tilde{T}_G J_m)^2 = D_G^e \tilde{T}_G J_m$.

Conversely, if $\tilde{T}$ is a $(2m) \times (2m)$-matrix with coefficients in $\{0,1\}$, such that there is a diagonal matrix D with *nonzero* diagonal entries such that $(\tilde{T} J_m)^2 = D\tilde{T} J_m$, then clearly $\operatorname{sign}\left((\tilde{T} J_m)^2\right) = \tilde{T} J_m$. So we may replace (vi) in theorem 2.3.2 by

(vii) There exists a diagonal matrix D with nonzero diagonal entries such that $(\tilde{T} J_m)^2 = D\tilde{T} J_m$.

This constraint seems less desirable than (vi), because of the occurrence of an existential quantifier. We would like to be able to simply *calculate* whether a certain matrix is $\tilde{T}_G$ for some graph G, without having to *look for something* first. It turns out, however, that under the other hypotheses of theorem 2.3.2, a possible matrix D can be determined from $\tilde{T} J_m$ itself (so without having to refer to the degrees of vertices of a certain graph).

Proposition 2.3.4. *Let $\ell \in \mathbb{Z}_{>0}$ and let A be a symmetric $\ell \times \ell$-matrix with coefficients in $\{0,1\}$. Suppose furthermore that $\operatorname{Tr}(A) = \ell$. Then the following are equivalent.*

1. $\operatorname{sign}(A^2) = A$;
2. *there is a diagonal matrix D with nonzero diagonal coefficients such that $A^2 = DA$;*
3. *item 2 is satisfied by $D = \operatorname{diag}(d_1, \ldots, d_\ell)$, where $d_i = \sum_{i=1}^{\ell} A_{i,j}$.*

Proof. It is obvious that $3 \Longrightarrow 2 \Longrightarrow 1$, so we'll show that 1 implies 3. Suppose that $\operatorname{sign}(A^2) = A$ and write $A = (\mathbf{v}_1 \cdots \mathbf{v}_\ell)$, where the $\mathbf{v}_i$ are column vectors. First of all, notice that $A_{i,i}$ must be equal to 1 for all $i \in \{1, \ldots, \ell\}$, so $d_i \geq A_{i,i} = 1 > 0$. Since A is symmetric, we have $(A^2)_{i,j} = \mathbf{v}_i \cdot \mathbf{v}_j$ for $i, j \in \{1, \ldots, \ell\}$. Notice that $(DA)_{i,j} = d_i A_{i,j}$. So if $A_{i,j} = 0$, then $(DA)_{i,j} = 0$ and by assumption $\left(\operatorname{sign}(A^2)\right)_{i,j} = A_{i,j} = 0$, so $(A^2)_{i,j} = 0$. On the other hand, if $A_{i,j} = 1$, then $(DA)_{i,j} = d_i$, so it remains to show that $(A^2)_{i,j} = d_i$, i.e. $\mathbf{v}_i \cdot \mathbf{v}_j = d_i$. Suppose that for some $k \in \{1, \ldots, \ell\}$, we have $A_{k,i} = 1$. Then we also have $A_{i,k} = 0$, and therefore $(A^2)_{k,j} = \mathbf{v}_k \cdot \mathbf{v}_j = \sum_{l=1}^{\ell} A_{l,k} A_{l,j} \geq A_{i,k} A_{i,j} = 1 > 0$, so $A_{k,j} = 1$. Similarly, we have $A_{k,j} = 1 \Longrightarrow A_{k,i} = 1$. So $\mathbf{v}_i$ and $\mathbf{v}_j$ are equal, which means that $\mathbf{v}_i \cdot \mathbf{v}_j = \mathbf{v}_i \cdot \mathbf{v}_i = d_i$. □

Theorem 2.3.5. *Let $m \in \mathbb{Z}_{>0}$ and let $\tilde{T}$ be a $(2m) \times (2m)$-matrix with entries in $\{0,1\}$. Then $\tilde{T}$ is the extended edge adjacency matrix of some finite undirected graph G if, and only if, the following conditions are satisfied:*

(iv) $\operatorname{Tr}(\tilde{T} J_m) = 2m$;

(v) $\tilde{T} J_m = (\tilde{T} J_m)^t$;

(viii) $(\tilde{T} J_m)^2 = D\tilde{T} J_m$, *where* $D = \operatorname{diag}(d_1, \ldots, d_{2m})$ *and* $d_i = \sum_{j=1}^{2m} (\tilde{T} J_m)_{i,j}$.

Proof. By proposition 2.3.4, (vi) from theorem 2.3.2 is equivalent to (viii) in the presence of (iv) and (v). □

3 Further properties of adjacency matrices

In this chapter, we will focus less on the characterization of adjacency matrices, and more on the various properties we have discovered while formulating this characterisation. It turns out that the diagonal matrix D_G^e introduced in section 2.3 can be used to obtain some results quite elegantly. In particular, we will use it to study the invertibility and the spectrum of the edge adjacency matrices.

3.1 Invertibility

In order to formulate this result, we first need to introduce the following special graph.

Definition 3.1.1. For an integer $m \geq 1$, we define the graph mK_2 as

$$(\{v_1, \ldots, v_m, w_1, \ldots, w_m\}, \{e_1, \ldots e_m\}, r),$$

where $r(e_i) = \{v_i, w_i\}$ for $1 \leq i \leq m$.

Intuitively, mK_2 is the disjoint union of m segments, each segment consisting of two vertices with an edge between them.

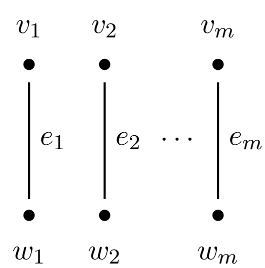


Figure 4: The graph mK_2

The following proposition is the reason why one should be interested in mK_2.

Proposition 3.1.1. *A finite undirected graph G consists solely of pendant vertices if and only if it is isomorphic to mK_2 for some integer $m \geq 1$.*

Proof. It is rather obvious that every vertex of mK_2 has degree 1. Now suppose that every vertex of G has degree 1 and let m be the number of edges of G. Then $|V| = \sum_{v \in V} 1 = \sum_{v \in V} \deg v = 2m$, by theorem 1.3.1. Now as there are no isolated vertices, every $v \in V$ must be in $r(e)$ for some $e \in E$. But $r(e)$ contains at most two vertices. Since the number of vertices is twice the number of edges, we deduce that $\{r(e) \mid e \in E\}$ must be a partition of V. Now the isomorphism with mK_2 is obvious. $\square$

In the following theorem, we do not allow the occurrence of isolated vertices, since these are of no interest to us.

Theorem 3.1.2. *Let G be a finite undirected graph with m edges that contains no isolated vertices.*

a) The matrix $\tilde{T}_G$ is invertible is and only if all its vertices are pendant. That is, if and only if G is isomorphic to mK_2. In this case, we have $\tilde{T}_G^{-1} = \tilde{T}_G = J_m$.

b) The matrix T_G is invertible if and only if G contains no pendant vertices. In this case, we have

$$T_G^{-1} = J_m(D_G^e - I_{2m})^{-1}(T_G J_m - D_G^e + 2I_{2m}). \tag{3}$$

Proof. Suppose $\tilde{T}_G$ is invertible. Since J_m is invertible as well, the relation $(\tilde{T}_G J_m)^2 = D_G^e \tilde{T}_G J_m$ gives us $\tilde{T} J_m = D_G^e$. Since there are no isolated vertices, the degree of every vertex occurs somewhere in D_G^e, so all degrees must be equal to 1. By proposition 3.1.1, this implies that G is isomorphic to mK_2. Conversely, if G is isomorphic to this graph, then, in whatever way we label the edges, $\tilde{T}_G = J_m$, which is its own inverse.
Now suppose that G has a vertex v of degree 1 and consider a doubly directed version $\left(\vec{G}, p\right)$ of G. By lemma 2.1.2, the indegree of v in $\vec{G}$ must also be equal to 1; let $\vec{e}_i$ be the unique directed edge terminating in v. Then for all $k \not\equiv i \mod 2m$, we have $t(\vec{e}_k) \neq v$, so for all $j \neq i + m \mod 2m$, we have $o(\vec{e}_j) = t(e_{j+m}) \neq v$. This means exactly that $(T_G)_{i,j} = 0$ for all $j \in I$. But then $\det T_G = 0$, so T_G cannot be invertible.
Finally, suppose that G does not have a vertex of degree 1. Then notice that $D_G^e - I_{2m}$ is a diagonal matrix with nonzero diagonal entries, therefore invertible. Since $T_G J_m = \tilde{T}_G J_m + I_{2m}$, we get

$$(T_G J_m)^2 + 2T_G J_m + I_{2m} = (T_G J_m + I_{2m})^2 = D_G^e (T_G J_m + I_{2m}) = D_G^e T_G J_m + D_G^e,$$

so $(T_G J_m - D_G^e - 2I_{2m}) T_G J_m = D_G^e - I_{2m}$. Since $D_G^e - I_{2m}$ is invertible, this gives

$$(D_G^e - I_{2m})^{-1}(T_G J_m - D_G^e - 2I_{2m}) T_G J_m = I_{2m}$$

and since $AB = I \implies BA = I$ for square matrices, we get

$$T_G J_m (D_G^e - I_{2m})^{-1}(T_G J_m - D_G^e - 2I_{2m}) = I_{2m}$$

and this proves the invertibility of T_G, along with (3). □

The criterion for the invertibility of T_G was already known from [1] and [4]. The above theorem, however, supplies additional information as it gives the inverse explicitly. Notice that this explicit inverse still requires us to calculate an inverse matrix. This matrix is, however, a diagonal matrix, so this is easy.

3.2 The spectrum of $T_G J_m$

In this section, we will consider the eigenvalues and eigenvectors of $T_G J_m$. We will to this by first considering a slightly different matrix, namely $\tilde{T}_G J_m$. First of all, notice that $\tilde{T}_G J_m$ is symmetric, so over $\mathbb{C}$, it has $2m$ real eigenvalues. Suppose that $\mathbf{v} \in \mathbb{C}^{2m}$ is a nonzero vector satisfying $\tilde{T}_G J_m \mathbf{v} = \lambda \mathbf{v}$ for some $\lambda \in \mathbb{R}$. Then we get

$$\lambda D^e_G \mathbf{v} = D^e_G \tilde{T}_G J_m \mathbf{v} = (\tilde{T}_G J_m)^2 \mathbf{v} = \lambda^2 \mathbf{v}. \tag{4}$$

If $\mathbf{v} = (v_1, \ldots, v_{2m})^t$ with $v_1, \ldots, v_{2m} \in \mathbb{C}$, then for some $i \in I$, we have $v_i \neq 0$. Comparing the i^{th} entries in (4) now gives $\lambda d_i v_i = \lambda^2 v_i$, and therefore $\lambda = 0$ or $\lambda = d_i$. In other words, every eigenvalue of $\tilde{T}_G J_m$ must be equal to 0 or the degree of some vertex of G.

Let us see whether we can actually find eigenvectors for these eigenvalues. Recall that $(\tilde{T}_G J_m)_{i,j}$ is equal to 1 if and only if $t(\vec{e}_i) = t(\vec{e}_j)$. So for $j \in I$, we see that $(\tilde{T}_G J_m)\vec{e}_j$ is equal to the sum of the elements in $\{\vec{e}_i \mid i \in I, t(\vec{e}_i) = t(\vec{e}_j)\}$. Now let v be a vertex of G; we assume that there are no isolated ones. Then by lemma 2.1.2, there are exactly $\deg(v) > 0$ values $j \in I$ such that $t(\vec{e}_j) = v$.

If we write $\mathbf{p}_v$ for $\sum_{t(\vec{e}_j)=v} \vec{e}_j$, then we get

$$\begin{aligned}
(\tilde{T}_G J_m)\mathbf{p}_v = (\tilde{T}_G J_m)\left(\sum_{t(\vec{e}_j)=v} \vec{e}_j\right) &= \sum_{t(\vec{e}_j)=v} (\tilde{T}_G J_m)\vec{e}_j = \sum_{t(\vec{e}_j)=v}\left(\sum_{t(\vec{e}_i)=t(\vec{e}_j)} \vec{e}_i\right) \\
&= \sum_{t(\vec{e}_i)=t(\vec{e}_j)=v} \vec{e}_i = \sum_{t(\vec{e}_j)=v}\left(\sum_{t(\vec{e}_i)=v} \vec{e}_i\right) = \deg v \sum_{t(\vec{e}_i)=v} \vec{e}_i = \deg v \cdot \mathbf{p}_v.
\end{aligned}$$

Therefore, $\mathbf{p}_v \neq 0$ is an eigenvector for the eigenvalue $\deg v$. It should be clear that, for vertices $v \neq w$, the vectors $\mathbf{p}_v$ and $\mathbf{p}_w$ are linearly independent, since they consist of different base vectors. So if we write $V = \{v_1, \ldots, v_n\}$, then the spectrum of $\tilde{T}_G J_m$ contains at least $\deg v_1, \ldots, \deg v_n$, *where we consider multiplicity.*

Can we have more positive eigenvalues in the spectrum of $\tilde{T}_G J_m$? We know that $\sum_{i=1}^n \deg v_i = 2m$. Since all eigenvalues of $\tilde{T} J_m$ are nonnegative, any other positive eigenvalues besides $\deg v_1, \ldots, \deg v_n$ in the spectrum of $\tilde{T}_G J_m$ would cause the trace of $\tilde{T}_G J_m$, which is the sum of the eigenvalues, to be *larger* than $2m$. But we already know that $\mathrm{Tr}(\tilde{T}_G J_m) = 2m$. So in addition to the eigenvalues $\deg v_1, \ldots, \deg v_n$, the eigenvalue 0 must occur with multiplicity $2m - n$. Let us state this as a theorem.

Theorem 3.2.1. *Let G be a finite undirected graph with n vertices, m edges and no isolated vertices. Write $V = \{v_1, \ldots, v_n\}$. Then the spectrum of $\tilde{T}_G J_m$ consists of*

$$\deg v_1, \ldots, \deg v_n, \underbrace{0, \ldots, 0}_{2m\text{-}n \text{ times}}.$$

In particular, notice that the eigenvalue 0 does not occur precisely if $n = 2m = \sum_{v \in V} \deg v$, which, as there are no isolated vertices, can occur only if all vertices are pendant. We have therefore rediscovered the first part of theorem 3.1.2.

We immediately state the following corollary.

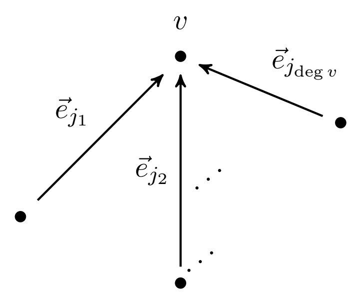
Corollary 3.2.2. *Let G be a finite undirected graph with n vertices, m edges and no isolated vertices. Write $V = \{v_1, \ldots, v_n\}$. Then the spectrum of $T_G J_m$ consists of*

$$\deg v_1 - 1, \ldots, \deg v_n - 1, \underbrace{-1, \ldots, -1}_{2m\text{-}n \text{ times}}.$$

Proof. Immediate, as $T_G J_m = (\tilde{T}_G - J_m)J_m = \tilde{T}_G J_m - J_m^2 = \tilde{T}_G J_m - I_{2m}$. □

In this case, the eigenvalue 0 does not occur precisely if $\deg v - 1$ is never equal to 0 for $v \in V$, that is, if there are no pendant vertices. We have therefore rediscovered the second part of theorem 3.1.2.

Now let us turn back to theorem 3.2.1. We have already found eigenvectors corresponding to the nonzero eigenvalues. Can we also find all the eigenspace corresponding to the eigenvalue 0? Again, consider a vertex $v \in V$. Then, as we already noticed, there are exactly $\deg(v)$ values $j \in I$ such that $t(\vec{e}_j) = v$. Let us write $\{j \in I \mid t(\vec{e}_j) = v\} = \{j_1, \ldots, j_{\deg v}\}$. Then $\mathbf{p}_v = \sum_{\ell=1}^{\deg v} \vec{e}_{j_\ell}$.



Write $S_G|_v$ for the subspace of S_G spanned by $\{\vec{e}_{j_\ell} \mid 1 \leq \ell \leq \deg v\}$. Obviously, it has dimension $\deg v$. For $1 \leq \ell \leq \deg v$, we observe that $(\tilde{T}_G J_m)\vec{e}_{j_\ell} = \mathbf{p}_v \in S_G|_v$, so $(\tilde{T}_G J_m)$ is a well-defined operator on $S_G|_v$. Our observation also shows that $(\tilde{T}_G J_m)(S_G|_v) = \mathrm{Span}(\mathbf{p}_v)$, which has dimension 1. So the kernel of $\tilde{T}_G J_m$ restricted to $S_G|_v$ must have dimension $\deg v - 1$. That is, we find $\deg v - 1$ eigenvectors for $\tilde{T}_G J_m$ for the eigenvalue 0. Explicitly, we may take the following $\deg v - 1$ independent eigenvectors:

$$\vec{e}_{j_2} - \vec{e}_{j_1}, \vec{e}_{j_3} - \vec{e}_{j_1}, \ldots, \vec{e}_{j_{\deg v}} - \vec{e}_{j_1}.$$

For, if $1 < \ell \leq \deg v$, then $\vec{e}_{j_\ell} - \vec{e}_{j_1} \neq 0$ and

$$(\tilde{T}_G J_m)(\vec{e}_{j_\ell} - \vec{e}_{j_1}) = (\tilde{T}_G J_m)(\vec{e}_{j_\ell}) - (\tilde{T}_G J_m)(\vec{e}_{j_1}) = \mathbf{p}_v - \mathbf{p}_v = 0.$$

So for every vertex v, we find $\deg v - 1$ independent eigenvectors with eigenvalue 0. So we get

$$\sum_{i=1}^{n} (\deg v_i - 1) = \sum_{i=1}^{n} \deg v_i - \sum_{i=1}^{n} 1 = 2m - n$$

eigenvectors for the eigenvalue 0. Since S_G is the disjoint sum of all the $S_G|_{v_i}$, all these eigenvectors must be independent, as the eigenvectors from a fixed $S_G|_{v_i}$ are independent. This means that we have found ourselves a basis for the eigenspace corresponding to the eigenvalue 0. Notice that, in particular, we have also determined all eigenspaces for $T_G J_m$, since these are just the eigenspaces for $\tilde{T}_G J_m$.

3.3 The spectrum of T_G

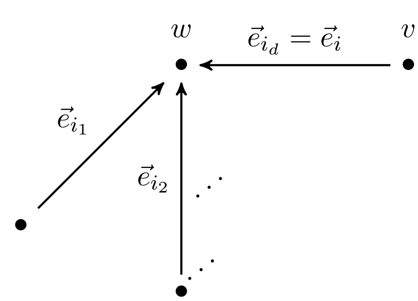
In this section, we turn to the spectrum of T_G itself rather than $T_G J_m$. For the latter, there was an obvious reason why there should be $2m$ eigenvalues: the matrix is symmetric. For T_G, however, matters are more complicated. Can this matrix always be diagonalized? The general answer to this question turns out to be no. First, we need the concept of a connected graph, which we will only need for the undirected case.

Definition 3.3.1. We say that a finite undirected graph G is *connected*, if for any $v, w \in V$, there is a sequence of vertices $v = v_0, v_1, \ldots, v_k = w$ for some $k \geq 0$, such that $r^{-1}(\{v_i, v_{i+1}\})$ is nonempty for $0 \leq i < k$. That is, there should exist an edge between v_i and v_{i+1} for $0 \leq i < k$.

Informally, a graph is connected if, for any pair of vertices, we can walk from the one vertex to the other over the edges of the graph. Now we have the following

Theorem 3.3.1. *Let G be a finite, undirected and connected graph that contains a pendant vertex and whose number of edges is larger than 1. Then T_G is not diagonalizable.*

Proof. We will show that the eigenvalue 0 has larger algebraic multiplicity than geometric multiplicity by finding an element of S_G that is in $\ker T_G^2$, but not in $\ker T_G$.
Let $v \in V$ be a pendant vertex. Then there is a unique edge, say e_i, such that $v \in r(e_i)$. Obviously, e_i is not a loop, since that would cause the degree of v to be at least 2. So there is a $w \neq v$ in $r(e_i)$. Now suppose that w also has degree 1. There v and w are not in $r(e)$ for any edge $e \neq e_i$. But by assumption, there must exist edges $e \neq e_i$, and these edges must have end vertices. So there is a vertex $u \in V$ different from v and w. By connectedness, we can find a path $v = v_0, v_1, \ldots, v_k = u$ such that $r^{-1}(\{v_i, v_{i+1}\})$ is nonempty for $0 \leq i < k$. Now suppose that $v_i = v$. Then there is only one vertex v_{i+1} such that $r^{-1}(\{v_i, v_{i+1}\})$ is nonempty, namely $v_{i+1} = w$. So $v_i = v \implies v_{i+1} = w$. Similarly, $v_i = w \implies v_{i+1} = v$. So $u = v_k$ must be equal to v or w, contradiction.
We conclude that $\deg w > 1$. We will write d instead of $\deg w$. Now let $\left(\vec{G}, p\right)$ be a doubly directed version of G, and assume without loss of generality that we have chosen the labeling in such a way that $t(e_i) = w$. By lemma 2.1.2, the indegree of w in $\vec{G}$ must be equal to d. Let $\vec{e}_{i_1}, \vec{e}_{i_2}, \ldots, \vec{e}_{i_d}$ be all directed edges originating from w, where $i_1, i_2, \ldots, i_d = i$ are different indices from I.



Now recall that T_G sends a directed edge to the sum of all edges flowing into it, with the exception of its converse edge. In the above picture, this means that

$$T_G(\vec{e}_{j_\ell+m}) = \sum_{j=1}^{d} \vec{e}_{i_j} - \vec{e}_{i_\ell} \text{ for } 1 \leq \ell \leq d. \tag{5}$$

Now we reason as follows (and this is where we use that $d > 1$): the d elements in (5) form a basis of $\mathrm{Span}(\vec{e}_{i_1}, \ldots, \vec{e}_{i_d})$, so we can find a linear combination of them equal to $\vec{e}_{i_d}$. Explicitly, we have

$$\begin{aligned}
&\sum_{\ell=1}^{d-1} (T_G\vec{e}_{i_\ell+m}) - (d-2)T_G\vec{e}_{i_d+m} \\
&= \sum_{\ell=1}^{d-1} \left(\sum_{j=1}^{d} (\vec{e}_{i_j}) - \vec{e}_{i_\ell} \right) - (d-2) \left(\sum_{j=1}^{d} (\vec{e}_{i_j}) - \vec{e}_{i_d} \right) \\
&= \sum_{\ell=1}^{d-1} \sum_{j=1}^{d} (\vec{e}_{i_j}) - \sum_{\ell=1}^{d-1} (\vec{e}_{i_\ell}) - (d-2) \sum_{j=1}^{d} (\vec{e}_{i_j}) + (d-2)\vec{e}_{i_d} \\
&= (d-1) \sum_{j=1}^{d} (\vec{e}_{i_j}) - \sum_{\ell=1}^{d} (\vec{e}_{i_\ell}) + \vec{e}_{i_d} - (d-2) \sum_{j=1}^{d} (\vec{e}_{i_j}) + (d-2)\vec{e}_{i_d} \\
&= (d-1)\vec{e}_{i_d}.
\end{aligned}$$

As T_G is linear, we get

$$\begin{aligned}
T_G \left(\frac{\sum_{\ell=1}^{d-1} (\vec{e}_{i_\ell+m}) - (d-2)\vec{e}_{i_d+m}}{d-1} \right) &= \frac{\sum_{\ell=1}^{d-1} (T_G\vec{e}_{i_\ell+m}) - (d-2)T_G\vec{e}_{i_d+m}}{d-1} \\
&= \frac{(d-1)\vec{e}_{i_d}}{d-1} = \vec{e}_{i_d} = \vec{e}_i.
\end{aligned}$$

Now notice that, as $\deg v = 1$, there is only one directed edge in $\vec{G}$ terminating in $v = o(\vec{e}_i)$, namely the converse of $\vec{e}_i$. So $T_G(\vec{e}_i) = 0$. This means that $\frac{\sum_{\ell=1}^{d-1} (\vec{e}_{i_\ell+m}) - (d-2)\vec{e}_{i_d+m}}{d-1} \in S_G$ is in $\ker T_G^2$, but not in $\ker T_G$. □

This result leads to the following question: if G is a finite undirected graph *containing no pendant vertices*, is T_G always diagonalizable? Unlike the eigenvalues of $\tilde{T}_G J_m$ and $T_G J_m$, the eigenvalues of T_G do not, in general, have a graph theoretical interpretation. In fact, they may very well be non-integral or even complex. Does this mean that we can say nothing about the spectrum of T_G?

Fortunately, this is not entirely true. We have the following result that can be found in [3], that we will not prove here.

Theorem 3.3.2. *Let G be a finite undirected graph with n vertices, m edges and no isolated vertices. If $n \neq m$, then T_G has the eigenvalue 1 that occurs with algebraic and geometric multiplicity $m - n + 1$.*

There exists a similar result for the eigenvalue -1. For this, we first need the following definition.

Definition 3.3.2. A finite undirected graph G is called *bipartite* if there is a partition of V into subsets V_1 and V_2 such that $r^{-1}(\{v_1, w_1\}) = \emptyset$ for all $v_1, w_1 \in V_1$ and $r^{-1}(\{v_2, w_2\}) = \emptyset$ for all $v_2, w_2 \in V_2$. In other words, edges are only allowed between an element of V_1 and an element of V_2.

Now we have the following result from [3].

Theorem 3.3.3. *Let G be a finite undirected graph with n vertices, m edges and no isolated vertices. If $n \neq m$ and G is not bipartite, then T_G has the eigenvalue -1 that occurs with algebraic and geometric multiplicity $m - n$. If G* is *bipartite, then both multiplicities are equal to $m - n + 1$ instead.*

Although the above two results identify some of the eigenvalues of T_G, we are still a few short. Counting algebraic multiplicities, we should have $2m$ complex eigenvalues. The above two theorems only account for $2m - 2n + 1$ (in the non-bipartite case) or $2m - 2n + 2$ (in the bipartite case) of them, leaving a gap of $2n - 1$ or $2n - 2$ respectively. These eigenvalues are more mysterious. We do have the following result, also from [3].

Theorem 3.3.4. *Let G be a finite undirected graph with n vertices, m edges and no isolated vertices. If $n \neq m$ and G contains no pendant vertices, then T_G is irreducible. By the Perron-Frobenius theorem, T_G has a real eigenvalue λ with algebraic and geometric multiplicity 1 and such that $|\lambda| > |\mu|$ for all other eigenvalues μ of T_G.*

Notice that the λ whose existence is guaranteed by the above theorem cannot be equal to 1 or -1. Indeed, 1 and -1 have the same absolute value, and the eigenvalue λ cannot have the same absolute value as any other eigenvalue. So theorem 3.3.4 tells us something about the 'mysterious' eigenvalues. However, little more is known, and the answer to the question whether a finite undirected graph G containing no pendant vertices always has a diagonalizable T_G, is unknown.

Conclusion

After introducing graphs and their various adjacency matrices, we considered two natural questions. First of all: when is a certain matrix the adjacency matrix of a graph? And secondly: does the adjacency matrix of a graph determine the graph uniquely? In the vertex case, the results were mentioned, while in both edge cases, full proofs were given. The answer to the second question was already known (for all cases), but the answer to the first question is new in both edge cases. While the regular edge adjacency matrix gets the most attention in the literature, it was convenient for our purposes to answer both questions for the extended edge adjacency matrix first. The results for the other case could then be derived quite easily.

Furthermore, we gave several equivalent criteria for a matrix to be the (extended) edge adjacency matrix of a graph. First, the theorem was stated in terms of coefficients, but later, we could formulate it in terms of matrix operations. The latter made it quite convenient to investigate the (extended) edge adjacency further. In particular, we studied its invertibility and its spectrum. The criteria for invertibility were already known. We completely determined the spectrum of a slight variation of the edge adjacency matrix, namely $T_G J_m$, which also is a new result. The analogous problem for the edge adjacency matrices of a graph stands unsolved. Some results concerning this problem were given, most of them without proofs.

To summarize, the two main results of this thesis are:

1. the criteria for a matrix to be the (extended) edge adjacency matrix of a graph;
2. finding the spectrum of $T_G J_m$.

The problem analogous to the latter result for the edge adjacency matrices themselves is the most important unsolved problem related to this thesis.

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