Universiteit Utrecht

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Bachelor thesis

Boolean-Valued Models of Set Theory

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1 Introduction

Set theory is the foundation of modern mathematics and (at the naive level) the first thing any mathematics student learns at the university. Although the concept of a set was first introduced by Georg Cantor in the 19th century, the axiomatic system that defines set theory today was introduced by Zermelo and Fraenkel in the early 20th century. This axiomatic system, called ZF, has both its strengths and its weaknesses. On one hand it allows us to perform ordinary naive set theory in other fields of mathematics without having to worry about such problems as Russel's paradox. On the other hand, ZF is an extension of Peano arithmetic and therefore, by the famous Gödel incompleteness theorems, we cannot prove that ZF is consistent. For we would need some ordinary mathematics to do this, but ZF is our system of ordinary mathematics, hence ZF would prove it's own consistency. However, we can perform *relative* consistency proofs in set theory, meaning that we can prove the implication 'if ZF is consistent, then ZF+ϕ is consistent', with ϕ a certain theorem of ZF. This allows us to perform independence proofs in set theory, for if we would prove this implication for both ϕ and $\neg\phi$ we would have derived that ϕ is independent of ZF. The best way of giving these relative consistency proofs is by giving a method of creating different models of ZF, in order to make it easier to find suitable models.
In this thesis we will lay out the concept of *Boolean-valued models of set theory.* This method of creating models of set theory was first developed by Dana Scott in an unpublished paper in 1967, and draws heavily on the concept of *Boolean algebra's*, named after George Boole (1815-1864). Most of the material in this thesis is taken from [1], which is by far the most clear and complete source on the subject.
In section 2, we will will briefly refresh the concept of ZF, give an introduction on Boolean algebra's, and prove a number of properties of Boolean algebra's which will be freely used in the rest of the thesis. In section 3 we will define a Boolean-valued model of set theory, and prove some properties and theorem's about these models. We will also prove that such a Boolean-valued model is in fact a structure in first order predicate logic satisfying all the axioms of ZFC (ZF+ the axiom of choice). In section 4, we will briefly discuss the application of Boolean-valued models in independence proofs.
We assume the reader to be familiar with some basic results in model theory, set theory and topology. For instance: such notions as a language of first order logic, a structure, the concept of ordinal numbers or the concept of a topological space will not be explained here.

2 Preliminaries

2.1 Set theory and formal logic

We will start with a few concepts in logic (especially set theory) which may not be so well-known. The first is the notion of *definable sets* in a model. Suppose we are given a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$. We call a set $A \subseteq \mathcal{M}$ *definable* in parameters from $\mathcal{M}$ if there is an $\mathcal{L}$-formula $\phi(x_1, ..., x_{k+1})$ such that for some $m_1, ..., m_k \in \mathcal{M}$:

$$A = \{m \in \mathcal{M} \mid \mathcal{M} \models \phi(m_1, ..., m_k, m)\}$$

Since this thesis will focus on creating models for ZF, we remind the reader that ZF is a theory in the language $\mathcal{L} = \{\in\}$, where $\in$ is a binary relationship symbol representing *elementhood*. ZF is given by the following axioms:

1. *Axiom of Extensionality*
   $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$
2. *Axiom of Pairing*
   $\forall x \forall y \exists z \forall w(w \in z \leftrightarrow (w = x \vee w = y))$
3. *Axiom scheme of separation*
   $\forall x \exists y \forall z(z \in y \leftrightarrow (z \in x \wedge \phi(z)))$
   Where y is not free in ϕ
4. *Axiom of Union*
   $\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(z \in w))$
5. *Axiom of Power Set*
   $\forall x \exists y \forall z(z \in y \leftrightarrow \forall w \in z(w \in x))$
6. *Axiom of Infinity*
   $\exists x(\emptyset \in x \wedge \forall y \in x \exists z \in x(y \in z)))$
7. *Axiom Scheme of Replacement*
   $\forall u(\forall x \in u \exists y \phi(x, y) \rightarrow \exists v \forall x \in u \exists y \in v \phi(x, y))$
   Where v is not free in $\phi(x, y)$
8. *Axiom of Regularity*
   $\forall x(\forall y \in x \phi(y) \rightarrow \phi(x)) \rightarrow \forall x \phi(x)$
   Where y is not free in $\phi(x)$

Note that we do not actually need the axiom of pairing, for it can be proved by means of the other axioms.

Theorem 2.1. *The axiom of pairing can be proved from the other axioms of ZF.*

Proof. Suppose x and y are sets. Note that the existence of a set (like x) implies the existence of the empty set, for $\emptyset = \{z \in x \mid z \neq z\}$ by separation, and this means that $\mathcal{P}(\emptyset)$ and $\mathcal{P}(\mathcal{P}(\emptyset))$ are also sets. We now define $\phi(x, y, u, v)$ to be the formula $(u = \emptyset \wedge v = x) \vee (u = \{\emptyset\} \wedge v = y)$. By the axiom of replacement we know that $\{v \mid \exists u \in \mathcal{P}(\mathcal{P}(\emptyset))\phi(x, y, u, v)\}$ is a set, and this is exactly the set $\{x, y\}$. So if x and y are sets then $\{x, y\}$ is a set. □

Therefore, if we need to prove that for a certain structure $\mathcal{M}$ we have $\mathcal{M} \models$ ZF, we only have to prove that all the axioms given above *except for the second one* hold in $\mathcal{M}$.
The axiom system of ZF is usually augmented with the axiom of choice, which is the assertion that every surjective function $f : X \to Y$ has a *section*, that is a function $s : Y \to X$ such that for all $y \in Y$: $f(s(y)) = y$. Note that this is also a theorem in the language of ZF because we can express the concept of a function in the language of ZF. The system of ZF with the axiom of choice (denoted by AC) is called ZFC.
Before we can move to the concept of Boolean algebra's we need one more important set-theoretical notion, namely the 'Von Neumann universe', V. This is the class of all sets, and can be constructed as follows. Let $V_0 = \emptyset$, if α is an ordinal and V_α is defined then $V_{\alpha+1} = \mathcal{P}(V_\alpha)$. If λ is a limit ordinal and V_α is defined for all ordinals $\alpha < \lambda$ then $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$. By this recursive definition V_α is defined for all ordinals α. For example: $V_1 = \{\emptyset\}$ and $V_2 = \{\emptyset, \{\emptyset\}\}$. Now we set $V = \bigcup_{\alpha \in ORD} V_\alpha$, where ORD is the class of all ordinals. Because $\emptyset$ is a set we see by the axiom of power set and the axiom of union that V_α is a set for every ordinal α. One can now ask how Russel's paradox is avoided, for isn't V now defined by taking the union of sets, and therefore itself a set? The answer is that we are taking the union of sets over a class (namely ORD), and not a set. And therefore the axiom of union can not be used in this situation, meaning that V is not a set itself.
We can prove that $\forall x \exists \alpha (x \in V_\alpha)$ holds by proving that the V_α are transitive sets, and then use the axiom of regularity. It is easily seen that if x is a transitive set, so $x \subset \mathcal{P}(x)$, then $\mathcal{P}(x)$ is transitive as well. For suppose $a \in \mathcal{P}(x)$, then $a \subseteq x \subset \mathcal{P}(x)$, so $a \subset \mathcal{P}(x)$ so $a \in \mathcal{P}(\mathcal{P}(x))$, which means that $\mathcal{P}(x) \subset \mathcal{P}(\mathcal{P}(x))$, so $\mathcal{P}(x)$ is transitive. Because $V_0 \subset V_1$ and for every limit ordinal λ we have $V_\alpha \subseteq \bigcup_{\beta<\lambda} V_\beta$ if $\alpha < \lambda$ it follows by induction on ordinals that for every $\alpha, \beta \in ORD$ we have $\alpha < \beta$ implies $V_\alpha \subseteq V_\beta$. We are now ready to prove the following theorem:

Theorem 2.2. *For every set x, there is an ordinal α such that $x \in V_\alpha$. Meaning that ZF $\vdash \forall x \exists \alpha (x \in V_\alpha)$.*

Proof. We will use the axiom of regularity. Let $\phi(u)$ be the formula $\exists \alpha (u \in V_\alpha)$. Now suppose x is a set and ZF $\vdash \forall y \in x \phi(y)$. According to the axiom of replacement we now have ZF $\vdash \exists \beta \forall y \in x \exists \alpha \in \beta (y \in V_\beta)$ and therefore ZF $\vdash \exists \beta (x \subset V_\beta)$ so ZF $\vdash \exists \beta (x \in V_{\beta+1})$. Therefore ZF $\vdash \phi(x)$ so ZF $\vdash \forall x (\forall y \in x \phi(y) \to \phi(x))$ so according to the axiom of regularity ZF $\vdash \forall x \phi(x)$, so ZF $\vdash \forall x \exists \alpha (x \in V_\alpha)$. □

Note that the formula $\exists \alpha (x \in V_\alpha)$ can be written in the language of ZF because there is a formula $\psi(\alpha)$ in the language of ZF stating 'α is an ordinal', and therefore the V_α are formally defined in the language of ZF.
We can now define a function 'rank' from V to ORD by stating that rank(x) is the least α such that $x \in V_{\alpha+1}$. In the future we can therefore prove a property of all sets by using

induction on rank.

Remark: It turns out that we can check all the axioms of ZFC in V, making V into a model of ZFC. Of course this is not a proof of consistency of ZFC, because we are working inside the system. It is also possible to construct an important submodel of V, called the constructible universe L, which is also a model of ZFC. We do this by defining $L_0 = \emptyset$ and for λ a limit ordinal $L_\lambda = \bigcup_{\alpha\in\lambda} L_\alpha$, which is the same as in the definition of V. However, instead of letting $L_{\alpha+1} = \mathcal{P}(L_\alpha)$ as in the definition of V, we let $L_{\alpha+1}$ be the set of all subsets of L_α which are definable in parameters from L_α. And now we define, as usual, L to be $L = \bigcup_{\alpha\in ORD} L_\alpha$. We call a set *constructible* if it is an element of L. The statement that every set is constructible is called the *axiom of constructibility*, and denoted by $V = L$. In fact, this statement turns out to be independent of ZFC.

2.2 Boolean algebra's

In this section, we will give the definition and a few elementary yet very useful properties of Boolean algebra's, which will later be needed in order to define a Boolean-valued model.

Defintion 2.3. *A* Boolean algebra *is a structure* $\langle B, \vee, \wedge, \neg, 0, 1\rangle$ *where* B *is called the* universe *of this algebra,* 0 *and* 1 *are two distinct elements of* B*,* $\vee$ *and* $\wedge$ *are two binary operations on* B *and* $\neg$ *is a unary operation on* B *and we have* $\forall a, b, c \in B$*:*

$$\begin{array}{ll} a \vee b = b \vee a & a \wedge b = b \wedge a \\ a \vee (b \vee c) = (a \vee b) \vee c & a \wedge (b \wedge c) = (a \wedge b) \wedge c \\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) & a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \\ 0 \vee a = a & 1 \wedge a = a \\ a \vee \neg a = 1 & a \wedge \neg a = 0 \end{array}$$

We shall denote a Boolean algebra $\langle B, \vee, \wedge, \neg, 0, 1\rangle$ by its universe B. In fact, there are several different possible notations in literature for a Boolean algebra. We could for example have chosen to denote $\vee$ and $\wedge$ with $+$ and $\cdot$ respectively. Note that with this notation, a Boolean algebra is in some ways just the same as a ring. For example, it has two binary operations which are associative and commutative. Also, there is the notion of an inverse. However, our notation has the heuristic advantage that we know the symbols $\wedge$ and $\vee$ from propositional logic, and the laws of a Boolean algebra are all consistent with those of classical first order logic. In fact, suppose we are given a language $\mathcal{L}$ of first order logic. Define an equivalence relation $\sim$ between sentences in this language by $\phi \sim \psi$ iff $\vdash \phi \leftrightarrow \psi$. The *Lindenbaum-Tarski algebra* of this language is the Boolean algebra having the class of all equivalence classes as its universe, and where $\wedge$, $\vee$ and $\neg$ are regular conjunction, disjunction and negation respectively. In this algebra the element 0 is given by $\perp$ and 1 by $\neg\perp$. This interpretation makes our notation very natural in a sense, where as the notation of a ring would yield $\forall a, b, c \in B(a + bc = (a + b)(a + c))$, which is very different from behavior in an ordinary ring. We will have to be careful only to use the laws of a Boolean algebra in proving theorems, and not accidentally use usual inference rules from classical first order logic though. We will also use the following abbreviaton: suppose $a, b \in B$, then $(a \Rightarrow b) \in B$ is the element $\neg a \vee b$.

Boolean algebra's have a lot of useful properties which are mostly easy to prove.

Theorem 2.4. *Suppose we are given a Boolean algebra B, then we have, for all $a, b \in B$:*
i) $a \vee a = a$ *and* $a \wedge a = a$.
ii) $a \vee (a \wedge b) = a$ *and* $a \wedge (a \vee b) = a$.
iii) $\neg 0 = 1$ *and* $\neg 1 = 0$.
iv) $1 \vee a = 1$ *and* $0 \wedge a = 0$.
v) if $(a \vee b) = 1$ *and* $a \wedge b = 0$ *then* $b = \neg a$.
vi) $\neg\neg a = a$.
vii) $\neg(a \vee b) = \neg a \wedge \neg b$ *and* $\neg(a \wedge b) = \neg a \vee \neg b$.

Proof. For i), we see that $a \vee a = (a \vee a) \wedge 1 = (a \vee a) \wedge (a \vee \neg a) = a \vee (a \wedge \neg a) = a \vee 0 = a$ and $a \wedge a = (a \wedge a) \vee 0 = (a \wedge a) \vee (a \wedge \neg a) = a \wedge (a \vee \neg a) = a \wedge 1 = a$.
For ii), we notice first that $a \vee (a \wedge b) = (a \vee a) \wedge (a \vee b) = a \wedge (a \vee b)$, so we only have to prove the first of the two statements. To do this, we notice that $a \vee (a \wedge b) = (a \wedge 1) \vee (a \wedge b) = a \wedge (1 \vee b) = a \wedge (\neg b \vee b \vee b) = a \wedge (\neg b \vee b) = a \wedge 1 = a$, where we have used property i).
For iii), we see that $\neg 0 = \neg 0 \vee 0 = 1$ and $\neg 1 = 1 \wedge \neg 1 = 0$.
For iv) we use i) and see: $a \vee 1 = a \vee a \vee \neg a = a \vee \neg a = 1$ and $a \wedge 0 = a \wedge a \wedge \neg a = a \wedge \neg a = 0$.
The proof of v) is slightly more tricky then the ones above. Suppose $a \vee b = 1$ and $a \wedge b = 0$. Then we see that $b \wedge \neg a = (b \wedge \neg a) \vee 0 = (b \wedge \neg a) \vee (a \wedge b) = b \wedge (a \vee \neg a) = b \wedge 1 = b$. However, we also see that $b \wedge \neg a = (b \wedge \neg a) \vee 0 = (b \wedge \neg a) \vee (a \wedge \neg a) = \neg a \wedge (a \vee b) = \neg a \wedge 1 = \neg a$. This proves that $b = \neg a$.
vi) follows from v), since $a \vee \neg a = 1$ and $a \wedge \neg a = 0$ we see that $a = \neg(\neg a)$.
To prove vii) (De Morgan's laws), we will use v). To prove the first of the laws, we have to prove, according to v), that $((a \vee b) \vee (\neg a \wedge \neg b)) = 1$ and $((a \vee b) \wedge (\neg a \wedge \neg b)) = 0$. We see:
$(a \vee b) \vee (\neg a \wedge \neg b) = a \vee (b \vee (\neg a \wedge \neg b)) = a \vee ((b \vee \neg a) \wedge (b \vee \neg b)) = a \vee ((b \vee \neg a) \wedge 1) = a \vee (b \vee \neg a) = 1 \vee b = 1$. And:
$(a \vee b) \wedge (\neg a \wedge \neg b) = \neg b \wedge (\neg a \wedge (a \vee b)) = \neg b \wedge ((\neg a \wedge a) \vee (\neg a \wedge b)) = \neg b \wedge (0 \vee (\neg a \wedge b)) = \neg b \wedge \neg a \wedge b = 0 \wedge \neg a = 0$. This proves the first De Morgan law.
To prove the second, we can use the first and vi). We see:
$\neg(a \wedge b) = \neg(\neg\neg a \wedge \neg\neg b) = \neg(\neg(\neg a \vee \neg b)) = \neg a \vee \neg b$. □

We can define a natural ordering $\leq$ on a Boolean algebra B by stating that $\forall a, b \in B$: $a \leq b$ if $a = a \wedge b$.

Theorem 2.5. *The ordering* $a \leq b$ *if* $a = a \wedge b$ *is a partial ordering.*

Proof. Because $a = a \wedge a$ we see that $a \leq a$ for all $a \in B$. Suppose $a, b, c \in B$ and $a \leq b$ and $b \leq c$. Then $a = a \wedge b$ and $b = b \wedge c$, so $a = a \wedge b = a \wedge (b \wedge c) = (a \wedge b) \wedge c = a \wedge c$, so $a \leq c$. Suppose $a \leq b$ and $b \leq a$, so $a = a \wedge b$ and $b = a \wedge b$, then of course $a = b$. □

Theorem 2.6. *Suppose a and b are in some Boolean algebra B, then* $a \leq b$ *iff* $a \Rightarrow b = 1$.

Proof. Suppose $a \leq b$, then $a = a \wedge b$ so $\neg a \vee b = \neg a \vee \neg b \vee b = \neg a \vee 1 = 1$, so $a \Rightarrow b = 1$. Suppose $a \Rightarrow b = 1$, then $\neg a \vee b = 1$ so $a = a \wedge 1 = a \wedge (\neg a \vee b) = 1 \vee (a \wedge b) = a \wedge b$ so $a \leq b$. □

We can easily prove (by induction on n) that if $\{x_1, ..., x_n\} \subseteq B$, then $x_1 \wedge ... \wedge x_n$ is the greatest lower bound (infimum) of this set, and $x_1 \vee ... \vee x_n$ the smallest upper bound (supremum). In general, if I is some index set, it is possible (but not always the case!) that $\{x_i \mid i \in I\}$ has an infimum, denoted by $\bigwedge_{i \in I} x_i$, and/or a supremum, denoted by $\bigvee_{i \in I} x_i$. We call a Boolean algebra B *complete* if every $A \subseteq B$ has an infimum and a supremum. Since in this thesis we will only be working with complete Boolean algebra's it is useful to prove some properties which are unique to this particular class of Boolean algebra's. The first is a generalized version of De Morgan's law.

Theorem 2.7. ***(De Morgan's law)*** *Suppose B is a complete Boolean algebra, I is some index set and $\{x_i \mid i \in I\} \subseteq B$. Then*

$$\neg \bigwedge_{i \in I} x_i = \bigvee_{i \in I} \neg x_i \quad \& \quad \neg \bigvee_{i \in I} x_i = \bigwedge_{i \in I} \neg x_i$$

Proof. Suppose first that $a, b \in B$ and $a \leq b$. So $a = a \wedge b$ and by De Morgan: $\neg a = \neg a \vee \neg b$. So $\neg b \wedge \neg a = \neg b \wedge (\neg a \vee \neg b) = \neg b$ so $\neg b \leq \neg a$. We are now going to use this result to prove theorem 2.7. We know that $\bigwedge_{i \in I} x_i \leq x_j$ for all $j \in I$, so $\neg x_j \leq \neg \bigwedge_{i \in I} x_i$ for all $j \in I$. Also, suppose $\neg x_j \leq a$ for all $j \in I$ and some $a \in B$. Then $\neg a \leq x_j$ for all $j \in I$ so $\neg a \leq \bigwedge_{i \in I} x_i$ and so $\neg \bigwedge_{i \in I} x_i \leq a$. So $\neg \bigwedge_{i \in I} x_i = \bigvee_{i \in I} \neg x_i$. The proof that $\neg \bigvee_{i \in I} x_i = \bigwedge_{i \in I} \neg x_i$ is identical to the one given, and shall therefore be omitted. □

Theorem 2.8. *Suppose B is a Boolean algebra and $a, b, c \in B$, and suppose $a \leq b$. Then $a \wedge c \leq b \wedge c$ and $a \vee c \leq b \vee c$.*

Proof. $a = a \wedge b$ so $a \wedge c = a \wedge b \wedge c = a \wedge b \wedge c \wedge c$ because $c \wedge c = c$, so $a \wedge c \leq b \wedge c$. Also: $a \vee c = (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ so $a \vee c \leq b \vee c$. □

Corollary 2.9. *Suppose B is a complete Boolean algebra, $y \in B$, I is some index set and $\{x_i \mid i \in I\} \subseteq B$. Then*

$$y \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (y \wedge x_i) \quad \& \quad y \vee \bigwedge_{i \in I} x_i = \bigwedge_{i \in I} (y \vee x_i)$$

In order to clarify the concepts introduced in this section we will give a few example's of Boolean algebra's.

Example 1. Suppose A is any non-empty set, then $\langle \mathcal{P}(A), \cup, \cap, -^c, \emptyset, A \rangle$ is a Boolean algebra, where $-^c$ means that the inverse of a set $X \subseteq A$ is the complement of X in A. It is easy to check that these operations on sets satisfy all the properties of a Boolean algebra. The ordering is defined by $X \leq Y$ iff $X \subseteq Y$, and we see that this is a complete Boolean algebra where the supremum of $\{X_i \subseteq A \mid i \in I\}$ with I some index set is $\bigcup_{i \in I} X_i$ and the infimum is $\bigcap_{i \in I} X_i$. If A consists of only 1 element then $\mathcal{P}(A)$ consists of only 2 elements, which are by definition the constants 0 and 1. This Boolean algebra is called the *trivial* Boolean algebra.

Example 2. In this example we will treat the Boolean algebra of *regular opens* of a topological space. This example is far from trivial, but it will be very important in proving the independence of the generalized continuum hypothesis (GCH) from ZFC, and therefore, we will treat it in detail. Suppose $(X, \mathcal{T})$ is some topological space. We call a set $A \in \mathcal{T}$ a *regular open* set if $\mathring{\overline{A}} = A$. Note that this is the same as saying that $A^{\perp\perp} = A$, where $A^{\perp} = \overline{A}^c$ is the complement of the closure of A. This is the definition of regular open we will use from here on. The collection of regular open sets in $\mathcal{T}$ is denoted by $\mathrm{RO}(X)$.

Theorem 2.10. *$\langle RO(X), (- \cup -)^{\perp\perp}, \cap, -^{\perp}, \emptyset, X \rangle$ is a Boolean algebra.*

In order to prove this we will need some basic topological facts, such as $(A \cup B)^\perp = A^\perp \cap B^\perp$ and $A \subseteq B$ implies $B^\perp \subseteq A^\perp$ for all opens A and B. On top of this we will need a sequence of lemma's.

Lemma 2.1. *If A is an open then $A \subseteq A^{\perp\perp}$.*

Proof. $A \subseteq \overline{A}$ so $\overline{A}^c \subseteq A^c$. Since A is open we know that A^c is closed so $\overline{A^c} = A^c$. This yields $\overline{\overline{A}^c} \subseteq \overline{A^c} = A^c$ so $A^{cc} \subseteq \overline{\overline{A}^c}^c$ so $A \subseteq A^{\perp\perp}$. □

This shows us that an open is regular open iff $A^{\perp\perp} \subseteq A$.

Lemma 2.2. *If A is an open then $A^\perp = A^{\perp\perp\perp}$.*

Proof. From the previous lemma we know that $A \subseteq A^{\perp\perp}$ so $A^{\perp\perp\perp} \subseteq A^\perp$. On the other hand, we know that $A^\perp$ is open because it is the complement of a closed set. This shows us that $A^\perp \subseteq (A^\perp)^{\perp\perp}$, so $A^\perp \subseteq A^{\perp\perp\perp}$. We conclude that $A^\perp = A^{\perp\perp\perp}$. □

So if A is an open set then $A^\perp$ is always a regular open set.

Lemma 2.3. *If A and B are opens then $(A \cap B)^{\perp\perp} = A^{\perp\perp} \cap B^{\perp\perp}$.*

Proof. First of all, we know that $A \cap B \subseteq A$ so $A^\perp \subseteq (A \cap B)^\perp$ so $(A \cap B)^{\perp\perp} \subseteq A^{\perp\perp}$. By symmetry we see that $(A \cap B)^{\perp\perp} \subseteq B^{\perp\perp}$ so $(A \cap B)^{\perp\perp} \subseteq A^{\perp\perp} \cap B^{\perp\perp}$.
For the converse we use that if A is an open then $A \cap \overline{B} \subseteq \overline{A \cap B}$. By taking complements we see that $(A \cap B)^\perp \subseteq A^c \cup B^\perp$ and therefore $\overline{(A \cap B)^\perp} \subseteq \overline{A^c \cup B^\perp}$. By taking complements we find $\overline{A^c \cup B^\perp}^c \subseteq (A \cap B)^{\perp\perp}$. Because taking closure is distributive over unions and because A is open and therefore A^c is closed we see that we have $A \cap B^{\perp\perp} \subseteq (A \cap B)^{\perp\perp}$ whenever A is open. So if A and B are open then so is $A^{\perp\perp}$, and therefore we find $A^{\perp\perp} \cap B^{\perp\perp} \subseteq (A^{\perp\perp} \cap B)^{\perp\perp}$. Also, because B is open we find $A^{\perp\perp} \cap B \subseteq (A \cap B)^{\perp\perp}$, and therefore $(A^{\perp\perp} \cap B)^{\perp\perp} \subseteq (A \cap B)^{\perp\perp\perp\perp}$ by lemma 2.1. Combining this with lemma 2.2 gives us

$$A^{\perp\perp} \cap B^{\perp\perp} \subseteq (A^{\perp\perp} \cap B)^{\perp\perp} \subseteq (A \cap B)^{\perp\perp}$$

And therefore lemma 2.3 is proven. □

We are now ready to prove theorem 2.10.

Proof. We first fix three elements $A, B, C \in \mathrm{RO}(X)$. We note that proving commutativity of the binary operations is trivial, and so is associativity of $\cap$. In order to prove associativity of $(- \cup -)^{\perp\perp}$ we use $(A \cup B)^\perp = A^\perp \cap B^\perp$ and note that for all opens A, B and C we have:

$$((A \cup B)^{\perp\perp} \cup C)^{\perp\perp} = ((A \cup B)^\perp \cap C^\perp)^\perp = (A^\perp \cap B^\perp \cap C^\perp)^\perp$$

Since this expression is symmetrical in A, B and C it is equal to $(A \cup (B \cup C)^{\perp\perp})^{\perp\perp}$.
We now need to prove distributivity. We first notice that because $A \cup B$ and $A \cup C$ are opens whenever A, B and C are opens we can use lemma 2.3 and see that

$$(A \cup (B \cap C))^{\perp\perp} = ((A \cup B) \cap (A \cup C))^{\perp\perp} = (A \cup B)^{\perp\perp} \cap (A \cup C)^{\perp\perp}$$

In order to prove that $A \cap (B \cup C)^{\perp\perp} = ((A \cap B) \cup (A \cap C))^{\perp\perp}$ we again use lemma 2.3 and see

$$\begin{aligned} A \cap (B \cup C)^{\perp\perp} &= A^{\perp\perp} \cap (B \cup C)^{\perp\perp} \\ &= (A \cap (B \cup C))^{\perp\perp} \\ &= ((A \cap B) \cup (A \cap C))^{\perp\perp} \end{aligned}$$

Because we are working with regular open sets we see that $(\emptyset \cup A)^{\perp\perp} = A^{\perp\perp} = A$. Also: $X \cap A = A$.
Finally, we see that $A \cap A^{\perp} = \emptyset$ and

$$(A \cup A^{\perp})^{\perp\perp} = (A^{\perp} \cap A^{\perp\perp})^{\perp} = (A^{\perp} \cap A)^{\perp} = \emptyset^{\perp} = A$$

which completes the proof that $\langle \mathrm{RO}(X), (- \cup -)^{\perp\perp}, \cap, -^{\perp}, \emptyset, X \rangle$ is a Boolean algebra. $\square$

3 The Boolean-valued model $V^{(B)}$

3.1 $V^{(B)}$ and the Boolean truth value

In this section, we will use our knowledge of Boolean algebra's to construct models of ZFC. In fact, we can construct a model $V^{(B)}$ of ZFC for every complete Boolean algebra B. It is therefore only logical that different Boolean algebra's will give rise to different models. This will allow us to prove the independence of several theorems (like the continuum hypothesis, CH) from ZFC. All we would have to do is choose two suitable Boolean algebra's to create two different models of ZFC. One in which CH is true and one in which it is not true. In this section, we will use a fixed complete Boolean algebra, B. We will also assume that $B \in V$, so B is a set.

The *Boolean universe* $V^{(B)}$ (or universe of B-valued sets) consists of functions from $V^{(B)}$ itself to B. It can be constructed as follows. Define for every ordinal α:

$$V_\alpha^{(B)} = \{x \mid x \text{ is a function with values in } B,\ \exists\gamma < \alpha(dom(x) \subseteq V_\gamma^{(B)})\}$$

So $V_\alpha^{(B)}$ is defined by recursion for all ordinals α. We see that $V_0^{(B)} = \emptyset$, $V_1^{(B)}$ is a set consisting of just 1 element, being the empty function e. $V_2^{(B)} = \{e\} \cup \{\langle e, b\rangle \mid b \in B\}$ etc. We notice that for every ordinal α we have $V_\alpha^{(B)} \subseteq V_{\alpha+1}^{(B)}$, because if $x \in V_\alpha^{(B)}$ then x is a function with values in B and $\mathrm{dom}(x) \subseteq V_\gamma^{(B)}$ with $\gamma < \alpha$, but since $\gamma < \alpha + 1$ it is by definition also an element of $V_{\alpha+1}^{(B)}$. So we have a sequence $V_0^{(B)} \subseteq V_1^{(B)} \subseteq ... \subseteq V_\omega^{(B)} \subseteq ....$ We now define $V^{(B)}$ to be $V^{(B)} = \{x \mid \exists\alpha(x \in V_\alpha^{(B)})\}$. Note that by induction on ORD it is easy to prove that for every Boolean algebra B the class $V_\alpha^{(B)}$ is in fact a set, while $V^{(B)}$ is not a set because we take the union over all ordinals. This justifies the name 'universe of B-valued sets'.

In order to work with $V^{(B)}$ we will define the first-order language $\mathcal{L}^{(B)}$, which consists of $\in$ and a constant for every element of $V^{(B)}$. We can use an induction principle in $V^{(B)}$ in much the same way that we can in V. Let ϕ be a formula, then

$$\forall x \in V^{(B)}(\forall y \in \mathrm{dom}(x)\phi(y) \to \phi(x)) \to \forall x \in V^{(B)}\phi(x)$$

Now that $V^{(B)}$ is defined, we can define a map $\|\cdot\|$ from the class of all $\mathcal{L}^{(B)}$-sentences to B. This map will assign to every $\mathcal{L}^{(B)}$-sentence ϕ the *Boolean truth value* of ϕ. We say that a sentence ϕ is *true* in $V^{(B)}$ if $\|\phi\| = 1$. Therefore, in the definition of this map, we will have to keep in mind that we want all the axioms of ZFC to be true in $V^{(B)}$.

Since it will turn out that defining the Boolean truth value of atomic formulas is the hardest, we will first suppose that this has already been done. So suppose that ϕ and ψ are two $\mathcal{L}^{(B)}$-sentences and that $\|\phi\|$ and $\|\psi\|$ are already defined, then we set $\|\phi \wedge \psi\| = \|\phi\| \wedge \|\psi\|$ and $\|\neg\phi\| = \neg\|\phi\|$. From this it also follows that $\|\phi \vee \psi\| = \|\phi\| \vee \|\psi\|$ and $\|\phi \to \psi\| = \|\phi\| \Rightarrow \|\psi\|$. If $\phi(x)$ is a formula with one free variable in $\mathcal{L}^{(B)}$ and $\|\phi(u)\|$ has been defined for all $u \in V^{(B)}$, then $\{\|\phi(u)\| \mid u \in V^{(B)}\}$ is a subset of B and we can define $\|\exists x\phi(x)\| = \bigvee_{u \in V^{(B)}} \|\phi(u)\|$ and $\|\forall x\phi(x)\| = \bigwedge_{u \in V^{(B)}} \|\phi(u)\|$. Now denote by $u(x)$ the function u which sends x to 1 if $x \in u$ and to 0 if $x \notin u$. Then we want $\|\exists x \in u\phi(x)\| = \bigvee_{x \in \mathrm{dom}(u)}(u(x) \wedge \|\phi(x)\|)$ and $\|\forall x \in u\phi(x)\| = \bigwedge_{x \in \mathrm{dom}(u)}(u(x) \Rightarrow \|\phi(x)\|)$ to be true. We will keep this in mind in constructing the truth value of atomic formulas, but prove them later from the official definition we will give. We also want the axiom of extensionality to hold, so we want $\|u = v\| = \|\forall x \in u(x \in v) \wedge \forall y \in v(y \in u)\|$. And we also want the axiom $u \in v \leftrightarrow \exists x \in v(x = u)$ to hold, so we want $\|u \in v\| = \|\exists x \in v(x = u)\|$. If we combine this with our earlier results we find for all $u, v \in V^{(B)}$:

$$\|u \in v\| = \bigvee_{x \in \mathrm{dom}(v)} (v(x) \wedge \|u = x\|) \text{ and}$$

$$\|u = v\| = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow \|x \in v\|) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow \|y \in u\|)$$

With this explanation in mind we can now give the preceding results in a single coherent definition.

Defintion 3.1. *$\| \cdot \|$ is a map from the class of all $\mathcal{L}^{(B)}$-sentences to B which assigns to a sentence ϕ the Boolean truth value of ϕ, and which satisfies the following properties, for all $u, v \in V^{(B)}$ and for all $\mathcal{L}^{(B)}$-sentences ϕ, ψ:*

$$\|u \in v\| = \bigvee_{x \in dom(v)} (v(x) \wedge \|u = x\|)$$

$$\|u = v\| = \bigwedge_{x \in dom(u)} (u(x) \Rightarrow \|x \in v\|) \wedge \bigwedge_{y \in dom(v)} (v(y) \Rightarrow \|y \in u\|)$$

$$\|\exists x \phi(x)\| = \bigvee_{u \in V^{(B)}} \|\phi(u)\| \quad \textit{and} \quad \|\forall x \phi(x)\| = \bigwedge_{u \in V^{(B)}} \|\phi(u)\|$$

$$\|\phi \wedge \psi\| = \|\phi\| \wedge \|\psi\| \quad \textit{and} \quad \|\neg \phi\| = \neg \|\phi\|$$

We say that an $\mathcal{L}^{(B)}$-sentence ϕ is *true* in $V^{(B)}$ if $\|\phi\| = 1$. In this case, we write $V^{(B)} \models \phi$. A rule of inference is valid in $V^{(B)}$ if it preserves truth. So if we have a first order predicate logical rule of inference 'from ϕ we can deduce ψ', then this rule is valid in $V^{(B)}$ if $\|\psi\| = 1$ whenever $\|\phi\| = 1$. These definitions give us the following easy yet important theorem, which is crucial in being able to prove that $V^{(B)}$ actually is a model of ZFC. In fact, the theorem states that $V^{(B)}$ with the given interpretation of truth is a structure in first order predicate logic.

Theorem 3.2. *All the axioms of first order predicate logic with equality are true in $V^{(B)}$, and all the inference rules of first order predicate logic are valid in $V^{(B)}$.*

Since there are quit a lot of axioms and inference rules and the proofs look a lot alike or are trivial we will not prove all of them. We will instead only prove the following:

1. $\|\phi \rightarrow (\psi \rightarrow \phi)\| = 1$
2. $\|\phi \wedge \psi \rightarrow \phi\| = 1$
3. $\|(\phi \rightarrow \psi) \rightarrow ((\phi \rightarrow \neg\psi) \rightarrow \neg\phi)\| = 1$

4. $\|\neg\phi \to (\phi \to \psi)\| = 1$

5. If $\|\phi\| = 1$ and $\|\phi \to \psi\| = 1$ then $\|\psi\| = 1$ (modus ponens)

6. $\|u = u\| = 1$

7. $u(x) \leq \|x \in u\|$ for $x \in \mathrm{dom}(u)$

8. $\|u = v\| = \|v = u\|$

9. $\|u = v\| \wedge \|v = w\| \leq \|u = w\|$

10. $\|u = v\| \wedge \|u \in w\| \leq \|v \in w\|$

11. $\|v = w\| \wedge \|u \in v\| \leq \|u \in w\|$

12. $\|u = v\| \wedge \|\phi(u)\| \leq \|\phi(v)\|$ for any $\mathcal{L}^{(B)}$-formula ϕ

Proof. 1.

$$\begin{aligned}\|\phi \to (\psi \to \phi)\| &= \|\phi\| \Rightarrow (\|\psi\| \Rightarrow \|\phi\|)\\ &= \neg\|\phi\| \vee (\neg\|\psi\| \vee \|\phi\|)\\ &= \neg\|\psi\| \vee 1\\ &= 1\end{aligned}$$

2.

$$\begin{aligned}\|\phi \wedge \psi \to \phi\| &= \neg\|\phi \wedge \psi\| \vee \|\phi\|\\ &= \neg\|\phi\| \vee \neg\|\psi\| \vee \|\psi\|\\ &= \neg\|\psi\| \vee 1\\ &= 1\end{aligned}$$

3.

$$\begin{aligned}\|(\phi \to \psi) \to ((\phi \to \neg\psi) \to \neg\phi)\| &= \neg(\|\phi\| \Rightarrow \|\psi\|) \vee (\neg(\|\phi\| \Rightarrow \neg\|\psi\|) \vee \neg\|\phi\|)\\ &= (\|\phi\| \wedge \neg\|\psi\|) \vee \neg\|\phi\| \vee (\|\phi\| \wedge \|\psi\|)\\ &= \neg\|\phi\| \vee (\|\phi\| \wedge (\|\psi\| \vee \neg\|\psi\|))\\ &= \neg\|\phi\| \vee (\|\phi\| \wedge 1)\\ &= \|\phi\| \vee \neg\|\phi\|\\ &= 1\end{aligned}$$

4.

$$\begin{aligned}\|\neg\phi \to (\phi \to \psi)\| &= \|\phi\| \vee \|\phi \to \psi\|\\ &= \|\phi\| \vee (\neg\|\phi\| \vee \|\psi\|)\\ &= \|\psi\| \vee (\|\phi\| \vee \neg\|\phi)\\ &= \|\psi\| \vee 1\\ &= 1\end{aligned}$$

5. Suppose $\|\phi\| = 1$ and $\|\phi \to \psi\| = 1$. Then $\|\psi\| = \|\psi\| \vee 0 = \|\psi\| \vee \neg 1$ and $\|\phi\| = 1$ so this becomes $\|\psi\| \vee \neg\|\phi\| = \|\phi \to \psi\| = 1$.

6. First, we know that for all $a, b \in B$ we have that $a \leq a \vee b$ because $a = a \wedge (a \vee b)$. We can now prove our statement using the induction principle in $V^{(B)}$. Suppose that $\|v = v\| = 1$ for all $v \in \mathrm{dom}(u)$, with $u \in V^{(B)}$. Then for all $v \in \mathrm{dom}(u)$ we have $\|v \in u\| = \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge \|v = x\|) = (u(v) \wedge \|v = v\|) \vee \bigvee_{x \in \mathrm{dom}(u)} (u(x) \wedge \|v = x\|) \geq (u(v) \wedge \|v = v\|) = u(v)$. This gives us

$$\begin{aligned}
\|u = u\| &= \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow \|x \in u\|) \wedge \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow \|x \in u\|) \\
&= \bigwedge_{x \in \mathrm{dom}(u)} (\neg u(x) \vee \|x \in u\|) \\
&= \bigwedge_{x \in \mathrm{dom}(u)} \left(\neg u(x) \vee \bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge \|y = x\|) \right) \\
&\geq \bigwedge_{x \in \mathrm{dom}(u)} (\neg u(x) \vee u(x)) \\
&= 1
\end{aligned}$$

So because $\|u = u\| \leq 1$ we find $\|u = u\| = 1$, and thus by induction $\|u = u\| = 1$ for all $u \in V^{(B)}$.

7. We can use the previous result and a similar technique to prove that $u(x) \leq \|x \in u\|$ for $x \in \mathrm{dom}(u)$. We see:

$$\begin{aligned}
\|x \in u\| &= \bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge \|y = x\|) \\
&= (u(x) \wedge \|x = x\|) \vee \bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge \|y = x\|) \\
&\geq u(x) \wedge \|x = x\| \\
&= u(x)
\end{aligned}$$

8. The fact that $\|u = v\| = \|v = u\|$ is a direct result from the symmetry in the expression of $\|u = v\|$.

9. We will use the induction principle in $V^{(B)}$. Suppose as induction hypothesis that

$$\forall v, w \in V^{(B)} (\|x = v\| \wedge \|v = w\| \leq \|x = w\|)$$

for all x in the domain of some $u \in V^{(B)}$. Now suppose that $v, w \in V^{(B)}$ and $x \in \mathrm{dom}(u)$, $y \in \mathrm{dom}(v)$ and $z \in \mathrm{dom}(w)$. Then according to the induction hypothesis we have $\|x = y\| \wedge \|y = z\| \leq \|x = z\|$ so $\|x = y\| \wedge \|y = z\| \wedge w(z) \leq \|x = z\| \wedge w(z)$. Taking the supremum over all the elements of $\mathrm{dom}(w)$ of both sides of this inequality yields

$$\bigvee_{z \in \mathrm{dom}(w)} (\|x = y\| \wedge \|y = z\| \wedge w(z)) \leq \bigvee_{z \in \mathrm{dom}(w)} (\|x = z\| \wedge w(z))$$

According to the definition of $\|\cdot \in \cdot\|$ this says that $\|x=y\| \wedge \|y \in w\| \leq \|x \in w\|$. Our next step is to prove that $\|v=w\| \wedge v(y) \leq \|y \in w\|$. We see by definition that

$$\|v=w\| = \bigwedge_{a \in \mathrm{dom}(v)} (v(a) \Rightarrow \|a \in w\|) \wedge \bigwedge_{b \in \mathrm{dom}(w)} (w(b) \Rightarrow \|b \in v\|)$$

so $\|v=w\| \leq \neg v(y) \vee \|y \in w\|$ because $y \in \mathrm{dom}(v)$. This will yield that $\|v=w\| \wedge v(y) \leq \|y \in w\|$. Combining our two results so far gives us

$$\|x=y\| \wedge \|v=w\| \wedge v(y) \leq \|x=y\| \wedge \|y \in w\| \leq \|x \in w\|$$

Taking the supremum over all the elements of $\mathrm{dom}(v)$ of both sides of this inequality yields by definition $\|x \in v\| \wedge \|v=w\| \leq \|x \in w\|$.
Next, we see that $u(x) \wedge \|u=v\| \leq \|x \in v\|$, the proof is identical to that of $\|v=w\| \wedge v(y) \leq \|y \in w\|$. We use this to see that

$$\|u=v\| \wedge \|v=w\| \wedge u(x) \leq \|x \in v\| \wedge \|v=w\| \leq \|x \in w\|$$

If we now joint both sides of the inequality with $\neg u(x)$ we find $\neg u(x) \vee (\|u=v\| \wedge \|v=w\|) \leq u(x) \Rightarrow \|x \in w\|$ which yields $\|u=v\| \wedge \|v=w\| \leq u(x) \Rightarrow \|x \in w\|$. We now take the supremum of both sides of this inequality over all the elements in $\mathrm{dom}(u)$, and find

$$\|u=v\| \wedge \|v=w\| \leq \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow \|x \in w\|)$$

So from the induction hypothesis we obtain the inequality above. We now observe that due to 8. our induction hypothesis is equal to

$$\forall v, w \in V^{(B)} (\|w=v\| \wedge \|v=x\| \leq \|w=x\|)$$

which yields

$$\|w=v\| \wedge \|v=u\| \leq \bigwedge_{x \in \mathrm{dom}(w)} (w(x) \Rightarrow \|x \in u\|)$$

By joining these two inequalities we find

$$\|u=v\| \wedge \|v=w\| \wedge \|w=v\| \wedge \|v=u\| \leq \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow \|x \in w\|) \wedge \bigwedge_{x \in \mathrm{dom}(w)} (w(x) \Rightarrow \|x \in u\|)$$

$$\|u=v\| \wedge \|v=w\| \leq \|u=w\|$$

10. Suppose $z \in \mathrm{dom}(w)$, then 9. gives us that $\|v=u\| \wedge \|u=z\| \wedge w(z) \leq \|v=z\| \wedge w(z)$ so by taking the supremum over all the elements of $\mathrm{dom}(w)$ of both sides of this inequality we find

$$\bigvee_{z \in \mathrm{dom}(w)} (\|v=u\| \wedge \|u=z\| \wedge w(z)) \leq \bigvee_{z \in \mathrm{dom}(w)} (\|v=z\| \wedge w(z))$$

and this yields $\|u=v\| \wedge \|u \in w\| \leq \|v \in w\|$.

11. Suppose $y \in \mathrm{dom}(v)$. We know by definition that

$$\|v = w\| = \bigwedge_{a \in \mathrm{dom}(v)} (v(a) \Rightarrow \|a \in w\|) \wedge \bigwedge_{b \in \mathrm{dom}(w)} (w(b) \Rightarrow \|b \in v\|)$$

And because $y \in \mathrm{dom}(v)$ we find that $\|v = w\| \leq \neg v(y) \vee \|y \in w\|$ which yields $\|v = w\| \wedge v(y) \leq \|y \in w\|$. So by using 10 we see that

$$\|v = w\| \wedge v(y) \wedge \|u = y\| \leq \|y \in w\| \wedge \|u = y\| \leq \|u \in w\|$$

By taking the supremum over all the elements of $\mathrm{dom}(v)$ of both sides of this inequality yields $\|v = w\| \wedge \bigvee_{a \in \mathrm{dom}(v)} (v(a) \wedge \|u = a\|) \leq \|u \in w\|$ so $\|v = w\| \wedge \|u \in v\| \leq \|u \in w\|$.

12. We will proof this by induction on the complexity of ϕ. Suppose $\phi(u)$ is an atomic formula, then $\phi(u)$ will be of the form $u = y$ with $y \in V^{(B)}$, or $u \in y$ with $y \in V^{(B)}$ or $y \in u$ with $y \in V^{(B)}$. We have proved these cases in 9, 10 and 11 respectively. Suppose that $\phi(u)$ is of the form $\psi(u) \wedge \chi(u)$, and suppose $\|u = v\| \wedge \|\psi(u)\| \leq \|\psi(v)\|$ and $\|u = v\| \wedge \|\chi(u)\| \leq \|\chi(v)\|$. Then we see that

$$\begin{aligned} \|u = v\| \wedge \|\phi(u)\| &= \|u = v\| \wedge \|u = v\| \wedge \|\psi(u)\| \wedge \|\chi(u)\| \\ &\leq \|\psi(v)\| \wedge \|\chi(v)\| \\ &= \|\phi(v)\| \end{aligned}$$

Suppose that $\phi(u)$ is of the form $\neg\psi(u)$, and suppose $\|u = v\| \wedge \|\psi(u)\| \leq \|\psi(v)\|$. Because of this we notice that $\|u = v\| \wedge \neg\|\psi(u)\| \wedge \|\psi(v)\| \leq \|\psi(u)\| \wedge \neg\|\psi(u)\| = 0$, so $\|u = v\| \wedge \neg\|\psi(u)\| \wedge \|\psi(v)\| = 0$. Now we see that

$$\begin{aligned} \|u = v\| \wedge \neg\|\psi(u)\| &= \|u = v\| \wedge \neg\|\psi(u)\| \wedge 1 \\ &= \|u = v\| \wedge \neg\|\psi(u)\| \wedge (\|\psi(v)\| \vee \neg\|\psi(v)\|) \\ &= (\|u = v\| \wedge \neg\|\psi(u)\| \wedge \|\psi(v)\|) \vee (\|u = v\| \wedge \neg\|\psi(u)\| \wedge \neg\|\psi(v)\|) \\ &= 0 \vee (\|u = v\| \wedge \neg\|\psi(u)\| \wedge \neg\|\psi(v)\|) \\ &= \|u = v\| \wedge \neg\|\psi(u)\| \wedge \neg\|\psi(v)\| \end{aligned}$$

So $\|u = v\| \wedge \neg\|\psi(u)\| = \|u = v\| \wedge \neg\|\psi(u)\| \wedge \neg\|\psi(v)\|$ so $\|u = v\| \wedge \neg\|\psi(u)\| \leq \neg\|\psi(v)\|$. Suppose that $\phi(u)$ is of the form $\exists y \psi(y, u)$, and for all $z \in V^{(B)}$ we have that $\|u = v\| \wedge \|\psi(z, u)\| \leq \|\psi(z, v\|$. Then

$$\begin{aligned} \|u = v\| \wedge \|\phi(u)\| &= \|u = v\| \wedge \bigvee_{z \in V^{(B)}} \|\psi(z, u)\| \\ &= \bigvee_{z \in V^{(B)}} (\|u = v\| \wedge \|\psi(z, u)\|) \\ &\leq \bigvee_{z \in V^{(B)}} (\|\psi(z, v)\|) \\ &= \|\exists y \psi(y, v)\| \\ &= \|\phi(v)\| \end{aligned}$$

Therefore, the result holds for every $\mathcal{L}^{(B)}$-formula ϕ.

□

3.2 Standard elements of $V^{(B)}$

The previous section concluded with the proof that $V^{(B)}$ is indeed a structure in first order predicate logic. The goal of next section will be to prove that it is a model of ZF. The proof of this will be quite long, since we will have to prove the truth of all the axioms of ZF in $V^{(B)}$ one by one. Before this can be done we will need to do some preliminary work, which is done in this section. In fact, the results of this section will only be used in proving the truth of the axiom of infinity in $V^{(B)}$. However, the results are interesting by themselves, and give us a better understanding of Boolean valued models.

The first notion we need is that of a *complete Boolean subalgebra*. Suppose we are given a Boolean algebra B. We call $B' \subseteq B$ a subalgebra of B if B' is nonempty and closed under $\wedge$, $\vee$ and $\neg$. We see that if $B' \subseteq B$ is a subalgebra then $\{0,1\} \subseteq B'$, because if $x \in B'$ then $x \wedge \neg x = 0 \in B'$ and $x \vee \neg x = 1 \in B'$. Therefore, B' is itself a Boolean algebra with the same Boolean operations and the same constants as B. If $B' \subseteq B$ is a Boolean subalgebra then we call B' a complete Boolean subalgebra if it is complete and for every index set I and $\{x_i \mid i \in I\} \subseteq B'$ we have $\bigvee_{i \in I} x_i$ and $\bigwedge_{i \in I} x_i$ formed in B' are the same as $\bigvee_{i \in I} x_i$ and $\bigwedge_{i \in I} x_i$ formed in B. When we are dealing with subalgebras, we need to work with two Boolean truth values, one for each distinct Boolean algebra. Whenever B' is a subalgebra of B we will denote their truth values by $\|\cdot\|^{B'}$ and $\|\cdot\|^{B}$ respectively.

Theorem 3.3. *Suppose B' is a complete Boolean subalgebra of B, then $V^{(B')} \subseteq V^{(B)}$ and for every $u, v \in V^{(B')}$ we have $\|u = v\|^{B'} = \|u = v\|^{B}$ and $\|u \in v\|^{B'} = \|u \in v\|^{B}$.*

Proof. We first prove $V_\alpha^{(B')} \subseteq V_\alpha^{(B)}$ for all ordinals α by induction on ORD. We see that $V_0^{(B')} = V_0^{(B)} = \emptyset$. Suppose that λ is an ordinal such that $V_\alpha^{(B')} \subseteq V_\alpha^{(B)}$ for all ordinals $\alpha < \lambda$, then $V_\lambda^{(B')}$ consists of functions with values in $B' \subseteq B$ with domain in $V_\alpha^{(B')} \subseteq V_\alpha^{(B)}$ for some $\alpha < \lambda$, and therefore it consists of elements of $V_\lambda^{(B)}$. So we have $V_\alpha^{(B')} \subseteq V_\alpha^{(B)}$ for all ordinals α.

We will prove the next two assertions simultaneously by induction on the relation $y \in \mathrm{dom}(v)$. Suppose that $v \in V^{(B')}$ such that for all $y \in \mathrm{dom}(v)$ we have for all $x \in V^{(B')}$:
$\|x \in y\|^{B'} = \|x \in y\|^{B}$, $\|x = y\|^{B'} = \|x = y\|^{B}$ and $\|y \in x\|^{B'} = \|y \in x\|^{B}$. For $x \in V^{(B')}$ we find:

$$\begin{aligned}
\|x \in v\|^{B'} &= \bigvee_{y \in \mathrm{dom}(v)} (v(y) \wedge \|x = y\|^{B'}) \\
&= \bigvee_{y \in \mathrm{dom}(v)} (v(y) \wedge \|x = y\|^{B}) \\
&= \|x \in v\|^{B}
\end{aligned}$$

In the same way we find that $\|v \in x\|^{B'} = \|v \in x\|^{B}$. We also notice that

$$\|x = v\|^{B'} = \bigwedge_{z \in \mathrm{dom}(v)} (x(z) \Rightarrow \|z \in v\|^{B'}) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow \|y \in x\|^{B'})$$

Since $\|z \in v\|^{B'} = \|z \in v\|^{B}$ according to our previous results and $\|y \in x\|^{B'} = \|y \in x\|^{B}$ according to our induction hypothesis and the fact that $y \in \mathrm{dom}(v)$ we see that this expression is equal to $\|x = v\|^{B}$. □

Actually, theorem 3.3 can be seen as the induction basis of a very natural theorem which is proven by induction on the complexity of a formula (we have treated the case of atomic formulas above). We call a formula ϕ *restricted* if all quantifiers in ϕ are bounded, that is of the form $\forall x \in y$ or $\exists x \in y$. On a side note, we call a formula ψ a Σ_1-formula (and sometimes write $\psi \in \Sigma_1$) if ψ is of the form $\exists x_1 ... x_n \phi$ with ϕ a restricted formula. We first need a theorem on the truth value of restricted formulas. In fact, we already mentioned these in constructing the truth value, but we will now prove that they actually have the form that we wanted them to have.

Theorem 3.4. *Let $\phi(x)$ be a formula with free variable x, and let $u \in V^{(B)}$. Then*

$$\|\exists x \in u\phi(x)\| = \bigvee_{y \in dom(u)} (u(y) \wedge \|\phi(y)\|) \quad \text{and} \quad \|\forall x \in u\phi(x)\| = \bigwedge_{y \in dom(u)} (u(y) \Rightarrow \|\phi(y)\|)$$

Proof. Since the proofs of these theorems work virtually identical we will only proof the first statement. We see the following:

$$\begin{aligned}
\|\exists x \in u\phi(x)\| &= \|\exists x(x \in u \wedge \phi(x))\| \\
&= \bigvee_{v \in V^{(B)}} \|v \in u \wedge \phi(v)\| \\
&= \bigvee_{v \in V^{(B)}} \left(\bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge \|v = y\|) \wedge \|\phi(v)\| \right) \\
&= \bigvee_{y \in \mathrm{dom}(u)} \left(u(y) \wedge \bigvee_{v \in V^{(B)}} (\|v = y \wedge \phi(v)\|) \right) \\
&= \bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge \|\exists x(x = y \wedge \phi(x))\|) \\
&= \bigvee_{y \in \mathrm{dom}(u)} (u(y) \wedge \|\phi(y)\|)
\end{aligned}$$

□

We now have the following theorem:

Theorem 3.5. *Suppose $B' \subseteq B$ is a complete Boolean subalgebra of B and $\phi(x_1, ..., x_n)$ a restricted formula. Then for any $u_1, ..., u_n \in V^{(B')}$ we have $\|\phi(u_1, ..., u_n)\|^{B'} = \|\phi(u_1, ..., u_n)\|^B$.*

Proof. We proof this by induction on the complexity of ϕ. We treated the case of an atomic formula ϕ in theorem 3.3. Suppose that ϕ is of the form $\psi \wedge \chi$, and suppose that $\|\psi\|^{B'} = \|\psi\|^B$ and $\|\chi\|^{B'} = \|\chi\|^B$, then our result follows immediately from the fact that $\|\psi \wedge \chi\| = \|\psi\| \wedge \|\chi\|$. The case that ϕ is of the form $\neg\|\psi\|$ is treated in the same way. Suppose that ϕ is of the form $\exists y \in x\psi(y, x_1, ..., x_n)$ and suppose that for any $u_0, u_1, ..., u_n \in V^{(B')}$ we have

$\|\psi(u_0, u_1, ..., u_n)\|^{B'} = \|\psi(u_0, u_1, ..., u_n)\|^{B}$. Then we find

$$\begin{aligned}\|\exists y \in v\psi(y, u_1, ..., u_n)\|^{B'} &= \bigvee_{u \in \mathrm{dom}(v)} (v(u) \wedge \|\psi(u, u_1, ..., u_n)\|^{B'}) \\ &= \bigvee_{u \in \mathrm{dom}(v)} (v(u) \wedge \|\psi(u, u_1, ..., u_n)\|^{B}) \\ &= \|\exists y \in v\psi(y, u_1, ..., u_n)\|^{B}\end{aligned}$$

It follows that $\|\phi(u_1, ..., u_n)\|^{B'} = \|\phi(u_1, ..., u_n)\|^{B}$ for every restricted formula ϕ. □

This theorem can be interpreted as saying that $V^{(B')}$ is in a natural sense a *submodel* of $V^{(B)}$. We now note that the trivial Boolean algebra $2 = \{0, 1\}$ is a subalgebra of every Boolean algebra B, meaning that $V^{(2)}$ is in this sense a submodel of every $V^{(B)}$. This gives us the opportunity to designate each member of V a representative in $V^{(B)}$ for every Boolean algebra B, by showing that there is in a natural sense an isomorphism between V and $V^{(2)}$. To that end we give the following definition:

Defintion 3.6. *For every $x \in V$: $\hat{x} = \{\langle \hat{y}, 1\rangle \mid y \in x\}$.*

We easily see by induction on the rank of x that $\hat{x} \in V^{(2)}$ for every $x \in V$, for if $\hat{y} \in V^{(2)}$ for every $y \in x$, then $\hat{x}$ consists of ordered pairs with a value in $2 = \{0, 1\}$ and input a member of $V^{(2)}$, and therefore $\hat{x} \in V^{(2)}$. Obviously, $\hat{x}$ is our representative of x, and elements of $V^{(B)}$ that are of the form $\hat{x}$ for some $x \in V$ are called *standard*.
We can now prove that there is a bijection between V and $V^{(2)}$, and we can prove certain properties of those standard elements.

Theorem 3.7.

1. *For every $x \in V$ and $u \in V^{(B)}$ we have $\|u \in \hat{x}\| = \bigvee_{y \in x} \|u = \hat{y}\|$*
2. *For every $x, y \in V$: $x \in y \leftrightarrow V^{(B)} \models \hat{x} \in \hat{y}$ and $x = y \leftrightarrow V^{(B)} \models \hat{x} = \hat{y}$*
3. *The map $x \mapsto \hat{x}$ from V into $V^{(2)}$ is injective*
4. *For every $u \in V^{(2)}$ there is an $x \in V$ such that $V^{(B)} \models u = \hat{x}$, so in $V^{(B)}$ the map $x \mapsto \hat{x}$ is surjective*
5. *For every formula $\phi(u_1, ..., u_n)$ and $x_1, ..., x_n \in V$: $\phi(x_1, ..., x_n) \leftrightarrow V^{(2)} \models \phi(\hat{x}_1, ..., \hat{x}_n)$*

Proof. For the proof of 1, suppose $x \in V$ and $u \in V^{(B)}$, then

$$\begin{aligned}\|u \in \hat{x}\| &= \bigvee_{y \in \mathrm{dom}(\hat{x})} (\hat{x}(y) \wedge \|u = y\|) \\ &= \bigvee_{y \in x} (\hat{x}(\hat{y}) \wedge \|u = \hat{y}\|) \\ &= \bigvee_{y \in x} \|u = \hat{y}\|\end{aligned}$$

Where this last step is due to the fact that $\hat{x}(\hat{y})$ is always 1 if $y \in x$ because of definition 3.6. We prove the assertions made in 2 simultaneously by induction on the rank of y. Suppose that $y \in V$ and for every $z \in V$ with $\text{rank}(z) < \text{rank}(y)$: $\forall x(x \in z \leftrightarrow V^{(B)} \models \hat{x} \in \hat{z})$ and $\forall x(z \in x \leftrightarrow V^{(B)} \models \hat{z} \in \hat{x})$ and $\forall x(x = z \leftrightarrow V^{(B)} \models \hat{x} = \hat{z})$. Then we see:

$$\begin{aligned} x \in y &\leftrightarrow \exists u \in y(x = u) \\ &\leftrightarrow \exists u \in y(\|\hat{x} = \hat{u}\| = 1) \\ &\leftrightarrow \bigvee_{u \in y} (\|\hat{x} = \hat{u}\|) = 1 \\ &\leftrightarrow \|\hat{x} \in \hat{y}\| = 1 \end{aligned}$$

Where the last step is due to the result of part 1 of this theorem.
Next, we see that by the definition of the truth value we have

$$\|\hat{x} = \hat{y}\| = \bigwedge_{u \in x} \|\hat{u} \in \hat{y}\| \wedge \bigwedge_{v \in y} \|\hat{v} \in \hat{x}\|$$

So $\|\hat{x} = \hat{y}\| = 1$ is equivalent to saying that $\bigwedge_{u \in x} \|\hat{u} \in \hat{y}\| = 1$ and $\bigwedge_{v \in y} \|\hat{v} \in \hat{x}\| = 1$. The first statement is equivalent to saying that $\|\hat{u} \in \hat{y}\| = 1$ for all $u \in x$. According to our result above this is equivalent to saying that $u \in y$ for all $u \in x$. The second statement is equivalent to saying that $v \in x$ for all $v \in y$ because of the fact that $\text{rank}(v) < \text{rank}(y)$ if $v \in y$ and our induction hypothesis. So by the axiom of extensionality:

$$\|\hat{x} = \hat{y}\| = 1 \leftrightarrow \forall u \in x(u \in y) \wedge \forall v \in y(v \in x) \leftrightarrow x = y$$

For the third part of the induction step we again use the above results and see:

$$\begin{aligned} y \in x &\leftrightarrow \exists u \in x(\|\hat{y} = \hat{u}\| = 1) \\ &\leftrightarrow \bigvee_{u \in x} \|\hat{y} = \hat{u}\| = 1 \\ &\leftrightarrow \|\hat{y} \in \hat{x}\| = 1 \end{aligned}$$

For the proof of 3 we notice that if $V^{(2)} \models \hat{x} = \hat{y}$ then $V^{(B)} \models \hat{x} = \hat{y}$ by theorem 3.5. The result then follows from 2.
We prove 4 by induction on the relation $v \in \text{dom}(u)$. Suppose $u \in V^{(2)}$ and $\forall v \in \text{dom}(u) \exists y \in V(\|v = \hat{y}\| = 1)$. We need to prove that there exists an $x \in V$ such that $\|u = \hat{x}\| = 1$. We define x by

$$x = \{y \in V \mid \exists v \in \text{dom}(u)(\|v = \hat{y}\| \wedge u(v) = 1)\}$$

We notice that $\text{dom}(u)$ is a set because $\exists \alpha \in \text{ORD}(\text{dom}(u) \in V_\alpha^{(2)})$. It follows from our result in 2 and the axiom of replacement that x is a set. We wish to prove that $\|u = \hat{x}\| = 1$. So we have to prove that

$$\bigwedge_{v \in \text{dom}(u)} (u(v) \Rightarrow \|v \in \hat{x}\|) \wedge \bigwedge_{z \in x} \|\hat{z} \in u\| = 1$$

So we have to prove that (i): if $v \in \text{dom}(u)$ then $u(v) \leq \|v \in \hat{x}\|$ and (ii): if $z \in x$ then $\|\hat{z} \in u\| = 1$. Since (ii) is the shortest we will prove this first. Suppose $z \in x$, then by definition of x there exists a $v \in \text{dom}(u)$ such that $\|\hat{z} = v\| = 1$ and $u(v) = 1$, so

$1 = \|\hat{z} = v\| \wedge \|v \in u\| \leq \|\hat{z} \in u\|$. For the proof of (i): suppose that $v \in \text{dom}(u)$, then according to the induction hypothesis there exists $y \in V$ such that $\|v = \hat{y}\| = 1$, and by definition of x we see that $y \in x$. This is equivalent to saying that $\|\hat{y} \in \hat{x}\| = 1$ by 2, and therefore we have $1 = \|v = \hat{y}\| \wedge \|\hat{y} \in \hat{x}\| \leq \|v \in \hat{x}\|$, so $u(v) \leq \|v \in \hat{x}\|$ because $\|v \in \hat{x}\| = 1$. This completes the proof of 4.
We prove 5 by induction on the complexity of ϕ. The case of atomic formulas has been treated in 2. If ϕ is equivalent to $\psi_1 \wedge \psi_2$ where ψ_1 and ψ_2 satisfy the statement in 5, then it is easy to see that ϕ also satisfies this statement. If ϕ is equivalent to $\neg\psi$ then this is also trivial. So we are left with just one case, which is slightly harder to prove. Suppose $x_1, ..., x_n \in V$ and $\phi(x_1, ..., x_n)$ is equivalent to $\exists x \psi(x, x_1, ..., x_n)$ and $\psi(x, x_1, ..., x_n) \leftrightarrow V^{(2)} \models \psi(\hat{x}, \hat{x}_1, ..., \hat{x}_n)$ for every $x \in V$. Suppose that $\phi(x_1, ..., x_n)$ is true, then there exists an $x \in V$ such that $\psi(x, x_1, ..., x_n)$ is true. This means that $\|\psi(\hat{x}, \hat{x}_1, ..., \hat{x}_n)\| = 1$ and because $\hat{x} \in V^{(2)}$ we see that

$$\bigvee_{u \in V^{(2)}} \|\psi(u, \hat{x}_1, ..., \hat{x}_n)\| = \|\phi(\hat{x}_1, ..., \hat{x}_n)\| = 1$$

Suppose on the other hand that $\|\phi(\hat{x}_1, ..., \hat{x}_n)\| = 1$, then $\bigvee_{u \in V^{(2)}} \|\psi(u, \hat{x}_1, ..., \hat{x}_n)\| = 1$ so there exists $u \in V^{(2)}$ such that $\|\psi(u, \hat{x}_1, ..., \hat{x}_n)\| = 1$. Because if this is not the case then because we work in the trivial Boolean algebra we see that $\|\psi(u, \hat{x}_1, ..., \hat{x}_n)\| = 0$ for all $u \in V^{(2)}$, but this would mean that $\bigvee_{u \in V^{(2)}} \|\psi(u, \hat{x}_1, ..., \hat{x}_n)\| = 0$. So we must have $\|\psi(u, \hat{x}_1, ..., \hat{x}_n)\| = 1$ for some $u \in V^{(2)}$. So because of 4 we know that there exists $x \in V$ such that $\|u = \hat{x}\| = 1$ and therefore by 12 of theorem 3.2 we see that $\|\psi(\hat{x}, \hat{x}_1, ..., \hat{x}_n)\| = 1$, so by the induction hypothesis $\psi(x, x_1, ..., x_n)$ is true. This completes the proof of 5. □

3.3 Proof that $V^{(B)}$ satisfies the axioms of ZF

In the previous subsection we have developed the concept of standard elements in $V^{(B)}$. In this subsection we will prove that $V^{(B)}$ is a model for ZF for every complete Boolean algebra B (hence the name 'Boolean-valued model of set theory' will be justified). It will turn out that the axiom of choice is also generally true in $V^{(B)}$, but in order to prove this we will need some more tools, which are developed in the next section. Since ZF consists of 7 axioms (we mentioned before that the axiom of pairing can be omitted) we will need to proof that the truth value of every one of these 7 axioms is equal to 1. Therefore, this subsection will simply consist of 7 proofs.

Theorem 3.8. *The axiom of extensionality, $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$, is true in $V^{(B)}$.*

Proof. By the definition of the truth value we have to prove that

$$\bigwedge_{z \in V^{(B)}} (\|z \in x \leftrightarrow z \in y\| \Rightarrow \|x = y\|) = 1$$

For every $x, y \in V^{(B)}$. We easily see that

$$\begin{aligned}
\|x = y\| &= \bigwedge_{z \in \mathrm{dom}(x)} (x(z) \Rightarrow \|z \in y\|) \wedge \bigwedge_{z \in \mathrm{dom}(y)} (y(z) \Rightarrow \|z \in x\|) \\
&= \|\forall z(z \in x \to z \in y)\| \wedge \|\forall z(z \in y \to z \in x)\| \\
&= \bigwedge_{z \in V^{(B)}} \|z \in x \leftrightarrow z \in y\|
\end{aligned}$$

and that immediately proofs the truth of the axiom of extensionality in $V^{(B)}$. □

Theorem 3.9. *The axiom scheme of separation, $\forall x \exists y \forall z(z \in y \leftrightarrow (z \in x \wedge \phi(z)))$ where y is not free in ϕ, is true in $V^{(B)}$.*

Proof. The goal is to define for a given $x \in V^{(B)}$ an element $y \in V^{(B)}$ such that $\bigwedge_{z \in V^{(B)}} \|z \in y \leftrightarrow (z \in x \wedge \phi(z))\| = 1$. We define y by $\mathrm{dom}(y) = \mathrm{dom}(x)$ and for $z \in V^{(B)}$: $y(z) = x(z) \wedge \|\phi(z)\|$. We must prove that (i): $\|\forall z \in y(z \in x \wedge \phi(z))\| = 1$ and (ii): $\|\forall z \in x(\phi(z) \to z \in y)\| = 1$. For (i) we notice that we must prove that $\bigwedge_{z \in \mathrm{dom}(y)}(y(z) \Rightarrow (z \in x \wedge \|\phi(z)\|)) = 1$ due to theorem 3.4. If $z \in \mathrm{dom}(y)$ then $y(z) = x(z) \wedge \|\phi(z)\|$ and since $x(z) \leq \|z \in x\|$ according to theorem 3.2 we find that $x(z) \wedge \|\phi(z)\| \Rightarrow \|z \in x\| \wedge \|\phi(z)\| = 1$, so $y(z) \Rightarrow \|z \in x\| \wedge \|\phi(z)\| = 1$, which proofs (i). For (ii) we must prove that if $z \in \mathrm{dom}(x)$ then $x(z) \Rightarrow \|\phi(z) \to z \in y\| = 1$. Using the definitions of $y(z)$ and $\Rightarrow$ this can be written as having to proof that $\neg y(z) \vee \|z \in y\| = 1$. This means that we have to prove that $y(z) \Rightarrow \|z \in y\| = 1$ which is true because $y(z) \leq \|z \in y\|$ according to theorem 3.2. And that completes the proof that the axiom of separation is true in $V^{(B)}$. □

Theorem 3.10. *The axiom of union, $\forall x \exists y \forall z(z \in y \leftrightarrow \exists w \in x(z \in w))$, is true in $V^{(B)}$.*

Proof. Suppose that $x \in V^{(B)}$. We need to find an element $y \in V^{(B)}$ that identifies with the union of all the elements in x. We define this y in the following way. $\mathrm{dom}(y) = \bigcup\{\mathrm{dom}(w) \mid w \in \mathrm{dom}(x)\}$ and for $z \in \mathrm{dom}(y)$: $y(z) = \|\exists w \in x(z \in w)\|$. In order to prove that in $V^{(B)}$ this y identifies with the union of the elements of x we must prove two things. First we have to prove that this y is in the union of the elements of x. So we have to prove: $\|\forall z \in y \exists w \in x(z \in w)\| = 1$. We see:

$$\begin{aligned}
\|\forall z \in y \exists w \in x(z \in w\| &= \bigwedge_{z \in \mathrm{dom}(y)} (y(z) \Rightarrow \|\exists w \in x(z \in w)\|) \\
&= \bigwedge_{z \in \mathrm{dom}(y)} (\|\exists w \in x(z \in w)\| \Rightarrow \|\exists w \in x(z \in w)\|) \\
&= 1
\end{aligned}$$

Next we have to prove that the union of the elements of x is in y, so we have to prove: $\|\forall w \in x \forall z \in w(z \in y)\| = 1$. We notice that

$$\|\forall w \in x \forall z \in w(z \in y)\| = \bigwedge_{w \in \mathrm{dom}(x)} \bigwedge_{z \in \mathrm{dom}(w)} (x(w) \wedge w(z) \Rightarrow \|z \in y\|)$$

If $w \in \text{dom}(x)$ and $z \in \text{dom}(w)$ then

$$\begin{aligned} x(w) \wedge w(z) &\leq x(w) \wedge \|z \in w\| \\ &= \bigvee_{w \in \text{dom}(x)} (x(w) \wedge \|z \in w\|) \\ &= \|\exists w \in x(z \in w)\| \\ &= y(z) \end{aligned}$$

We also notice that by the definition of $\text{dom}(y)$, if $w \in \text{dom}(x)$ and $z \in \text{dom}(w)$ then $z \in \text{dom}(y)$ so we know that $y(z) \leq \|z \in y\|$. So we have that for $w \in \text{dom}(x)$ and $z \in \text{dom}(w)$: $x(w) \wedge w(z) \leq y(z) \leq \|z \in y\|$ which shows that

$$\bigwedge_{w \in \text{dom}(x)} \bigwedge_{z \in \text{dom}(w)} (x(w) \wedge w(z) \Rightarrow \|z \in y\|) = 1$$

So the element y does indeed function as the union of the elements of x, so the axiom of union holds in $V^{(B)}$. □

Theorem 3.11. *The axiom of power set,* $\forall x \exists y \forall z (z \in y \leftrightarrow \forall w \in z(w \in x))$*, is true in* $V^{(B)}$.

Proof. Suppose we have a given $x \in V^{(B)}$. Then define y by $\text{dom}(y) = B^{\text{dom}(x)}$ (the set of all functions from $\text{dom}(x)$ to B) and for every $u \in \text{dom}(y)$: $y(u) = \|u \subseteq x\|$ where $u \subseteq x$ is an abbreviation for $\forall w \in u(w \in x)$ (we will freely use this abbreviation from now on). We need to show that y identifies with the power set of x. So we have to prove that $\|\forall z(z \in y \leftrightarrow z \subseteq x)\| = 1$. The first part of this is to prove that $\bigwedge_{z \in \text{dom}(y)}(y(z) \Rightarrow \|z \subseteq x\|) = 1$, but this is immediate because of the definition of y. So we just have to prove that $\|\forall z(z \subseteq x \rightarrow z \in y)\| = 1$. We now fix an arbitrary $z \in V^{(B)}$, and have to prove that $\|z \subseteq x \rightarrow z \in y\| = 1$. Since this is hard to do for this z we will define an element z' in $V^{(B)}$ in such a way that this is easier to be done, and which is under the assumption that $z \subseteq x$ equal to z in $V^{(B)}$. So we will define z' by $\text{dom}(z') = \text{dom}(x)$ and for $w \in \text{dom}(z')$: $z'(w) = \|w \in z\|$. We notice that since $\text{dom}(y) = B^{\text{dom}(x)}$ we have that $z' \in \text{dom}(y)$. We now have to prove that (i): $\|z \subseteq x \rightarrow z = z'\| = 1$ and (ii): $\|z \subseteq x \rightarrow z' \in y\| = 1$.
To prove (i) we notice that for any $w \in V^{(B)}$:

$$\begin{aligned} \|w \in z'\| &= \bigvee_{v \in \text{dom}(z')} (z'(v) \wedge \|w = v\|) \\ &= \bigvee_{v \in \text{dom}(z')} (\|v \in z\| \wedge \|w = v\|) \\ &\leq \bigvee_{v \in \text{dom}(z')} (\|w \in z\|) \\ &= \|w \in z\| \end{aligned}$$

So $\|z' \subseteq z\| = 1$. For the reverse we notice that

$$\begin{aligned}\|w \in x \wedge w \in z\| &= \bigvee_{v \in \mathrm{dom}(x)} (x(v) \wedge \|w = v\| \wedge \|w \in z\|) \\ &\leq \bigvee_{v \in \mathrm{dom}(x)} (\|w = v\| \wedge \|v \in z\|) \\ &= \bigvee_{v \in \mathrm{dom}(z')} (\|w = v\| \wedge z'(v)\|) \\ &= \|w \in z'\|\end{aligned}$$

The second step of the calculation above requires some explanation. We know that for $v \in \mathrm{dom}(x)$: $x(v) \wedge \|w = v\| \wedge \|w \in z\| \leq \|w = v\|$ and since $\|w = v\| \wedge \|w \in z\| \leq \|v \in z\|$ we also see that $x(v) \wedge \|w = v\| \wedge \|w \in z\| \leq \|v \in z\|$. This combines to the result in step two of the calculation above.
We now have that $\|w \in x\| \wedge \|w \in z\| \leq \|w \in z'\|$ so $\|(z \subseteq x \wedge w \in z) \rightarrow w \in z'\| = 1$. Combining with the result that $\|z' \subseteq z\| = 1$ completes the proof of (i).
To prove (ii) we notice that

$$\begin{aligned}\|z \subseteq x\| &= \bigwedge_{w \in V^{(B)}} (\|w \in z\| \Rightarrow \|w \in x\|) \\ &\leq \bigwedge_{w \in \mathrm{dom}(z')} (z'(w) \Rightarrow \|w \in x\|) \\ &= \|\forall w \in z'(w \in x)\| \\ &= \|z' \subseteq x\| \\ &= y(z') \\ &\leq \|z' \in y\|\end{aligned}$$

Where the last step can be made because $z' \in \mathrm{dom}(y)$. This completes the proof of (ii), and with it the proof of the truth of the axiom of power set in $V^{(B)}$. □

Theorem 3.12. *The axiom of infinity, $\exists x(\emptyset \in x \wedge \forall y \in x \exists z \in x(y \in z))$, is true in $V^{(B)}$.*

Proof. We need to find an element $x \in V^{(B)}$ such that $\|\phi(x)\| = 1$, where $\phi(x)$ is the formula $\emptyset \in x \wedge \forall y \in x \exists z \in x(y \in z)$. First we notice that $\phi(x)$ is a restricted formula because all the quantifiers are bounded. Next we notice that in V, the formula $\phi(\omega)$ is true, where ω is the first infinite ordinal. Because $\emptyset \in \omega$ and if $y \in \omega$ then $y + 1 \in \omega$ and $y \in y + 1$. We can now use theorem 3.7 and theorem 3.5 to find $\phi(\omega) \leftrightarrow V^{(2)} \models \phi(\hat{\omega}) \leftrightarrow V^{(B)} \models \phi(\hat{\omega})$. So by putting $x = \hat{\omega}$ we find the truth of the axiom of infinity in $V^{(B)}$. □

Theorem 3.13. *The axiom scheme of replacement, $\forall u(\forall x \in u \exists y \phi(x, y) \rightarrow \exists v \forall x \in u \exists y \in v \phi(x, y))$ where v is not free in $\phi(x, y)$, is true in $V^{(B)}$.*

Proof. Suppose we have a given fixed $u \in V^{(B)}$. We know that $B \in V$ because we had assumed that B is a complete Boolean algebra with a set as universe. For a fixed $x \in \mathrm{dom}(u)$ we see that $A_x = \{\|\phi(x, y)\| \mid y \in V^{(B)}\} \subset B$ so A_x is a set. This shows that for every $b \in A_x$ there exists an ordinal α such that $b = \|\phi(x, y)\|$ and $y \in V_\alpha^{(B)}$. We now use the axiom of replacement in V to see that there exists an ordinal $\alpha(x)$ such that for all

$b \in A_x$ there exists an ordinal $\beta \in \alpha(x)$ such that $b = \|\phi(x,y)\|$ and $y \in V_\beta^{(B)}$. We now use the axiom of union in V to define $\alpha = \bigcup\{\alpha(x) \mid x \in \text{dom}(u)\}$. Then we see that $\{\|\phi(x,y)\| \mid x, y \in V^{(B)}\} = \{\|\phi(x,y)\| \mid x \in V^{(B)}, y \in V_\alpha^{(B)}\}$. We now define $v \in V^{(B)}$ by $\text{dom}(v) = V_\alpha^{(B)}$ and for $z \in \text{dom}(v)$: $y(z) = 1$. Then we see that

$$\begin{aligned}
\|\forall x \in u \exists y \phi(x,y)\| &= \bigwedge_{x \in \text{dom}(u)} \left(u(x) \Rightarrow \bigvee_{y \in V^{(B)}} \|\phi(x,y)\| \right) \\
&\leq \bigwedge_{x \in \text{dom}(u)} \left(u(x) \Rightarrow \bigvee_{y \in V_\alpha^{(B)}} \|\phi(x,y)\| \right) \\
&= \bigwedge_{x \in \text{dom}(u)} (u(x) \Rightarrow \|\exists y \in v \phi(x,y)\|) \\
&= \|\forall x \in u \exists y \in v \phi(x,y)\|
\end{aligned}$$

By taking the supremum over all $v \in V^{(B)}$ on both sides of the inequality above we find the truth of the axiom of replacement in $V^{(B)}$. □

Theorem 3.14. *The axiom of regularity,* $\forall x(\forall y \in x\phi(y) \to \phi(x)) \to \forall x\phi(x)$ *where* y *is not free in* $\phi(x)$*, is true in* $V^{(B)}$*.*

Proof. We have to prove that $\|\forall x(\forall y \in x\phi(y) \to \phi(x))\| \leq \|\forall x\phi(x)\|$. Now let $\|\forall x(\forall y \in x\phi(y) \to \phi(x))\| = b$, then it is enough to show that $b \leq \|\phi(x)\|$ for all $x \in V^{(B)}$. Because if we can prove this then we also know that b is smaller then or equal to the supremum of $\|\phi(x)\|$ over all $x \in V^{(B)}$. We are now going to use the induction principle in $V^{(B)}$. Suppose $x \in V^{(B)}$ and for all $y \in \text{dom}(x)$ we have that $b \leq \|\phi(y)\|$. Then $b \leq \bigwedge_{y \in \text{dom}(x)} \|\phi(y)\|$ and because $\|\phi(y)\| \leq (x(y) \Rightarrow \|\phi(y)\|)$ for all $y \in \text{dom}(x)$ we find that

$$\bigwedge_{y \in \text{dom}(x)} \|\phi(y)\| \leq \bigwedge_{y \in \text{dom}(x)} (x(y) \Rightarrow \|\phi(y)\|) = \|\forall y \in x\phi(y)\|$$

Because of the definition of b we see that

$$b = \bigwedge_{x \in V^{(B)}} (\|\forall y \in x\phi(y)\| \Rightarrow \|\phi(x)\|)$$

And this means that for our particular x we have that $b \leq (\|\forall y \in x\phi(y)\| \Rightarrow \|\phi(x)\|)$. Putting these findings together yields

$$b \leq \|(\forall y \in x\phi(y) \to \phi(x)) \wedge \forall y \in x\phi(y)\| = \|\phi(x)\|$$

and this completes the proof of the axiom of regularity in $V^{(B)}$. □

3.4 Proof that $V^{(B)} \models$ AC

In this section we will prove that the axiom of choice is true in $V^{(B)}$ for every complete Boolean algebra B. We will use the equivalence between AC and Zorn's lemma, and we are also going to need a theorem called 'the maximum principle'. In order to prove this theorem, we will need a concept called 'mixtures', so we will start this section by defining these mixtures.

Suppose I is some index set and $A = \{a_i \mid i \in I\} \subseteq B$ and $U = \{u_i \mid i \in I\} \subseteq V^{(B)}$. Then we define the *mixture* of U with respect to A to be the element $u \in V^{(B)}$ such that $\mathrm{dom}(u) = \bigcup_{i \in I} \mathrm{dom}(u_i)$ and for $z \in \mathrm{dom}(u)$: $u(z) = \bigvee_{i \in I}(a_i \wedge \|z \in u_i\|)$. The term mixture is explained by noticing that we can perceive of u as the element obtained by mixing the elements of U with respect to the elements of A. This concept is made clear by the following theorem about mixtures:

Theorem 3.15. ***(Mixing lemma)*** *Suppose I is some index set and $A = \{a_i \mid i \in I\} \subseteq B$ and $U = \{u_i \mid i \in I\} \subseteq V^{(B)}$, and u is the mixture of U with respect to A, and suppose $a_i \wedge a_j \leq \|u_i = u_j\|$ for all $i, j \in I$. Then for all $i \in I$: $a_i \leq \|u = u_i\|$.*

Proof. We know by the definition of the truth value that

$$\|u = u_i\| = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow \|x \in u_i\|) \wedge \bigwedge_{y \in \mathrm{dom}(u_i)} (u_i(y) \Rightarrow \|y \in u\|)$$

We first notice that by the definition of the mixture we have for every $x \in \mathrm{dom}(u)$:

$$\begin{aligned} a_i \wedge u(x) &= a_i \wedge \bigvee_{j \in I} (a_j \wedge \|x \in u_j\|) \\ &= \bigvee_{j \in I} (a_i \wedge a_j \wedge \|x \in u_j\|) \\ &\leq \bigvee_{j \in I} (\|u_i = u_j\| \wedge \|x \in u_j\|) \\ &= \|x \in u_i\| \end{aligned}$$

And therefore, $a_i \leq (u(x) \Rightarrow \|x \in u_i\|)$ for every $x \in \mathrm{dom}(u)$, which shows that $a_i \leq \bigwedge_{x \in \mathrm{dom}(u)}(u(x) \Rightarrow \|x \in u_i\|)$.

Now suppose $y \in \mathrm{dom}(u_i)$, then $a_i \wedge u_i(y) \leq a_i \wedge \|y \in u_i\| \leq u(y) \leq \|y \in u\|$ because $y \in \mathrm{dom}(u)$. This shows us that $a_i \leq \bigwedge_{y \in \mathrm{dom}(u_i)}(u_i(y) \Rightarrow \|y \in u\|)$, which completes the proof that $a_i \leq \|u = u_i\|$. □

In particular, the mixing lemma can be applied if A is an *antichain*, that is, if $a_i \wedge a_j = 0$ for every $i, j \in I$ with $i \neq j$. We can now use the mixing lemma to prove that $V^{(B)}$ has enough elements such that the supremum over every formula, $\bigvee_{u \in V^{(B)}} \|\phi(u)\|$, is actually attained by some element in $V^{(B)}$.

Theorem 3.16. ***(Maximum principle)*** *Let $\phi(x)$ be some Boolean formula, then there exists a $u \in V^{(B)}$ such that $\|\exists x \phi(x)\| = \|\phi(u)\|$.*

Proof. We have to prove that there exists a $u \in V^{(B)}$ such that $\|\phi(u)\| = \bigvee_{v \in V^{(B)}} \|\phi(v)\|$. Because B is a set we know that $\{\|\phi(u)\| \mid u \in V^{(B)}\} \subseteq B$ is also a set, and therefore we can use the axiom of choice in V to see that there is an ordinal α and a set $\{u_\lambda \mid \lambda < \alpha\} \subseteq V^{(B)}$ such that $\{\|\phi(u)\| \mid u \in V^{(B)}\} = \{\|\phi(u_\lambda)\| \mid \lambda < \alpha\}$. So we can now write $\|\exists x \phi(x)\| = \bigvee_{\lambda < \alpha} \|\phi(u_\lambda)\|$. Now define for every $\lambda < \alpha$ the element $a_\lambda = \|\phi(u_\lambda)\| \wedge \neg \bigvee_{\eta < \lambda} \|\phi(u_\eta)\|$.

Then we can prove that $A = \{a_\lambda \mid \lambda < \alpha\}$ is an antichain. To do this, suppose $\gamma < \theta < \alpha$, then

$$\begin{aligned} a_\gamma \wedge a_\theta &= \|\phi(u_\gamma)\| \wedge \neg \bigvee_{\lambda<\gamma} \|\phi(u_\lambda)\| \wedge \|\phi(u_\theta)\| \wedge \neg \bigvee_{\lambda<\theta} \|\phi(u_\lambda)\| \\ &= \|\phi(u_\gamma)\| \wedge \bigwedge_{\lambda<\gamma} \neg\|\phi(u_\lambda)\| \wedge \|\phi(u_\theta)\| \wedge \bigwedge_{\lambda<\theta} \neg\|\phi(u_\lambda)\| \\ &\leq \|\phi(u_\gamma)\| \wedge \neg\|\phi(u_\gamma)\| \\ &= 0 \end{aligned}$$

This shows that we can now use the mixing lemma to prove that there exists an $u \in V^{(B)}$ which satisfies $a_\lambda \leq \|u = u_\lambda\|$ for all $\lambda < \alpha$. For this u we notice that $\|\phi(u)\| \leq \bigvee_{x \in V^{(B)}} \|\phi(x)\| = \|\exists x \phi(x)\|$. Also, because $a_\lambda \leq \|\phi(u_\lambda)\|$ for every $\lambda < \alpha$ by the definition of a_λ, we see that $a_\lambda \leq \|\phi(u_\lambda)\| \wedge \|u = u_\lambda\| \leq \|\phi(u)\|$ for every $\lambda < \alpha$. So if we can prove that $\|\exists x \phi(x)\| \leq \bigvee_{\lambda<\alpha} a_\lambda$ we are done. So we have to prove that

$$\bigvee_{\lambda<\alpha} \|\phi(u_\lambda)\| \leq \bigvee_{\lambda<\alpha} a_\lambda = \bigvee_{\lambda<\alpha} \|\phi(u_\lambda)\| \wedge \bigvee_{\lambda<\alpha} \bigwedge_{\eta<\lambda} \neg\|\phi(u_\eta)\|$$

But the truth of this statement is clear since the ordinals are well-ordered by $\in$, so if we have the truth of $\phi(u_\lambda)$ for some $\lambda < \alpha$ then we know that there is a smallest $\lambda < \alpha$ such that $\phi(u_\lambda)$ is true. This completes the proof of the maximum principle. □

The maximum principle has two important corollaries, which will prove useful in the proof of Zorn's lemma.

Corollary 3.17. *Let $\phi(x)$ be a Boolean formula such that $V^{(B)} \models \exists x \phi(x)$. Then for all $v \in V^{(B)}$ there exists a $u \in V^{(B)}$ such that $\|\phi(u)\| = 1$ and $\|\phi(v)\| = \|u = v\|$.*

Proof. Suppose we have a given $v \in V^{(B)}$. Since $\|\exists x \phi(x)\| = 1$ we can use the maximum principle to find a $w \in V^{(B)}$ such that $\|\phi(w)\| = 1$. Now choose $a = \|\phi(v)\|$ and let u be the mixture of $U = \{v, w\}$ with respect to $A = \{a, \neg a\}$. Because A is an antichain we can use the mixing lemma to see that $a \leq \|u = v\|$ and $\neg a \leq \|u = w\|$. additionally we know that $a \leq \|\phi(v)\|$ because we know the equality between the two, and we know that $\neg a \leq \|\phi(w)\|$ because $\|\phi(w)\| = 1$. This shows us that

$$1 = a \vee \neg a \leq \|\phi(v) \wedge u = v\| \vee \|\phi(w) \wedge u = w\| \leq \|\phi(u)\|$$

So we know that $\|\phi(u)\| = 1$, and this gives us that $\|u = v\| = \|\phi(u)\| \wedge \|u = v\| \leq \|\phi(v)\|$ and because we already knew that $\|\phi(v)\| = a \leq \|u = v\|$ we find that our u satisfies $\|\phi(u)\| = 1$ and $\|\phi(v)\| = \|u = v\|$. □

Corollary 3.18. *Let $\phi(x)$ and $\psi(x)$ be Boolean formulas such that $V^{(B)} \models \exists x \phi(x)$ and for all $u \in V^{(B)}$: if $V^{(B)} \models \phi(u)$ then $V^{(B)} \models \psi(u)$. Then $V^{(B)} \models \forall x(\phi(x) \to \psi(x))$.*

Proof. Suppose we are given a $v \in V^{(B)}$. Then we have to prove that $V^{(B)} \models \phi(v) \to \psi(v)$, meaning that we have to prove that $\|\phi(v)\| \leq \|\psi(v)\|$. We use corollary 3.17 to find a $u \in V^{(B)}$ such that $\|\phi(u)\| = 1$ and $\|\phi(v)\| = \|u = v\|$. Then we know that $\|\psi(u)\| = 1$ so $\|\phi(v)\| = \|u = v\| = \|u = v\| \wedge \|\psi(u)\| \leq \|\psi(v)\|$. □

The last notion we need in order to prove that $V^{(B)} \models \text{AC}$ is that of a *core* of an element of $V^{(B)}$. Suppose $u \in V^{(B)}$, then we call $v \subseteq V^{(B)}$ a core of u if for all $x \in v$ we have $\|x \in u\| = 1$ and for all $y \in V^{(B)}$ such that $\|y \in u\| = 1$ there exists a unique $x \in v$ such that $\|x = y\| = 1$. In fact it is possible to prove that every element of $V^{(B)}$ has a core, which is a very useful theorem, for it allows us to use the core of an element for every given element.

Theorem 3.19. *Every $u \in V^{(B)}$ has a core.*

Proof. Suppose $u \in V^{(B)}$. Define for every $x \in V^{(B)}$: $x' = \{(z, u(z) \wedge \|z = x\|) \mid z \in \text{dom}(u)\}$. Then by the axiom of replacement there exists a set $w \subseteq V^{(B)}$ such that for every $x \in V^{(B)}$ there exists $y \in w$ such that $x' = y'$. Now define the equivalence relation $\sim$ on $\{x \in w \mid \|x \in u\| = 1\}$ by $x \sim y \leftrightarrow \|x = y\| = 1$. Now let v be defined as the set containing exactly one element of every equivalence class in this equivalence relation. Then if $a \in v$ we know that $a \in \{x \in w \mid \|x \in u\| = 1\}$ and therefore $\|a \in u\| = 1$. Also, if $b \in V^{(B)}$ and $\|b \in u\| = 1$ then there exists $a \in w$ such that $a' = b'$, which yields $\|a = b\| = 1$, which means that $\|a \in u\| = 1$, and therefore $a \in \{x \in w \mid \|x \in u\| = 1\}$. Now let c be the element in v such that $\|a = c\| = 1$, then c is the unique element in v such that $\|b = c\| = 1$. This shows that v is a core of u. □

Another useful theorem about the concept of a core is the following:

Theorem 3.20. *Suppose $u \in V^{(B)}$ such that $V^{(B)} \models u \neq \emptyset$, and let v be a core of u. Then for every $x \in V^{(B)}$ there exists an element $z \in v$ such that $\|x = z\| = \|x \in u\|$.*

Proof. Let $\phi(z)$ be the formula $z \in u$, then we know that $V^{(B)} \models \exists z \phi(z)$ because $V^{(B)} \models u \neq \emptyset$. So by using corollary 3.17 we know that for every $x \in V^{(B)}$ there exists a $y \in V^{(B)}$ such that $\|y \in u\| = 1$ and $\|x = y\| = \|x \in u\|$. So by the definition of a core there exists a unique element $z \in v$ such that $\|y = z\| = 1$. So we have

$$\|x \in u\| = \|x = y\| = \|x = y\| \wedge \|y = z\| \leq \|x = z\|$$

and

$$\|x = z\| = \|x = z\| \wedge \|y = z\| \leq \|x = y\| = \|x \in u\|$$

Which shows that we have $\|x = z\| = \|x \in u\|$. □

We now posses all the tools we need to prove Zorn's lemma. Before we do this, remember that Zorn's lemma is the assertion that if every chain in a nonempty poset has an upper bound in this poset, then this poset contains a maximal element. Note that we can define a predicate in the language of ZF stating that $(x, \leq_x)$ is a poset. And also a predicate that states that an element is a chain in this poset, and one stating that an element is an upper bound or a maximal element. In other words, we can formally write Zorn's lemma in the language of ZF. We will give one example of such a predicate. If we want to define $\phi(y)$ in such a way that it expresses that y is a poset, then we would have to define it in the following way:

$$\begin{aligned}\exists x, \leq_x (y = (x, \leq_x) \wedge \leq_x \in \mathcal{P}(x \times x) \\ &\wedge \forall p \in x((p,p) \in \leq_x) \\ &\wedge \forall p, q, r \in x(((p,q) \in \leq_x \wedge (q,r) \in \leq_x) \to (p,r) \in \leq_x) \\ &\wedge \forall p, q \in x(((p,q) \in \leq_x \wedge (q,p) \in \leq_x) \to p = q))\end{aligned}$$

Note that we have already assumed that we have have defined the concepts of a power set, a cartesian product and an ordered pair, but that these are easy to define. A more detailed account of this is in the appendix. From now on we will write $p \leq_x q$ to indicate that for $p, q \in x$ we have $(p, q) \in \leq_x$.

Theorem 3.21. *Zorn's lemma, if every chain in a nonempty poset P has an upper bound in P then P contains a maximal element, is true in $V^{(B)}$.*

Proof. Let $\phi(y)$ be the Boolean formula stating that y is a nonempty poset in which every chain has an upper bound, and let $\psi(y)$ be the Boolean formula stating that y has a maximal element. Since Zorn's lemma is the assertion $\forall x(\phi(x) \to \psi(x))$ we can use corollary 3.18 to see that we only have to prove that $V^{(B)} \models \phi(x)$ implies $V^{(B)} \models \psi(x)$ for every $x \in V^{(B)}$. We will do this by first defining a core of this x, and then use Zorn's lemma in V to identify a maximal element of this core. We can then prove that in $V^{(B)}$ this is also a maximal element of x.
Let $x \in V^{(B)}$ such that $(x, \leq_x)$ is a poset in which every chain has an upper bound, and let y be a core of x (we know this is always possible because of theorem 3.19). Now define the ordering $\leq_y$ on y by $a \leq_y b \leftrightarrow \|a \leq_x b\| = 1$ for $a, b \in y$. Because $\leq_x$ is a partial ordering we easily see that $\leq_y$ is also a partial ordering. In fact we can prove that every chain in y has an upper bound in y. For let C be a chain in y and define $C' = C \times \{1\} \in V^{(B)}$. Then we see that $V^{(B)} \models C'$ *is a chain in* x. So we know by our assumption that in $V^{(B)}$ it is true that there exists a $z \in x$ such that z is an upper bound for C'. We now use the maximum principle to see that there exists a z such that in $V^{(B)}$ it is true that $z \in x$ and that z is an upper bound for C'. This means that $\|z \in x\| = 1$ so by the definition of y there exists a $w \in y$ such that $\|z = w\| = 1$. Now let $a \in C$, then $\|a \in C'\| = 1$ and $\|a \leq_x z\| = 1$ and thus we have $\|a \leq_x w\| = 1$. By the definition of $\leq_y$ this shows that $a \leq_y w$ so for every $a \in C$ we know that $a \leq_y w$, so w is an upper bound for C. So every chain in y has an upper bound in y, and since y is a set we can use Zorn's lemma in V to see that y has a maximal element c. Because $c \in y$ we know that $\|c \in x\| = 1$. In fact, we can prove that $V^{(B)} \models c$ *is a maximal element for* x.
Let $a \in V^{(B)}$, then we are going to prove that if $\|c \leq_x a \wedge a \in x\| \leq \|a = c\|$, which would prove that in $V^{(B)}$ it is indeed true that c is a maximal element for x. Use theorem 3.20 to find an element $b \in y$ such that $\|a = b\| = \|a \in x\|$. Then we see that

$$\|c \leq_x a \wedge a \in x\| = \|c \leq_x a \wedge a = b\| \leq \|c \leq_x b\|$$

Now let m be the mixture of $\{b, c\}$ with respect to $\{d, \neg d\}$ where $d = \|c \leq_x b\|$. Because $\{d, \neg d\}$ is an antichain we can use the mixing lemma to see that $\|c \leq_x b\| \leq \|b = m\|$ and $\neg\|c \leq_x b\| \leq \|c = m\|$. Since $b, c \in y$ we know that $\|b \in x\| = \|c \in x\| = 1$ and this shows us that

$$\begin{aligned} 1 &= \|c \leq_x b\| \vee \neg\|c \leq_x b\| \\ &\leq \|m = b\| \vee \|m = c\| \\ &= (\|m = b\| \wedge \|b \in x\|) \vee (\|m = c\| \wedge \|c \in x\|) \\ &\leq \|m \in x\| \vee \|m \in x\| \\ &= \|m \in x\| \end{aligned}$$

So we know that $\|m \in x\| = 1$ so by the definition of y there exists a $k \in y$ such that $\|m = k\| = 1$. We notice that because $\|c \leq_x b\| \leq \|m = b\|$ we have that $\|c \leq_x m\| = 1$, and thus $\|c \leq_x k\| = 1$, and because $c, k \in y$ we now have that $c \leq_y k$. Because c is a maximal element of y we now know that $c = k$. And now we finally see:

$$\begin{aligned}
\|c \leq_x a \wedge a \in x\| &= \|c \leq_x a \wedge a \in x\| \wedge \|a \in x\| \\
&\leq \|c \leq_x b\| \wedge \|a \in x\| \\
&\leq \|b = m\| \wedge \|a = b\| \\
&= \|b = m\| \wedge \|m = k\| \wedge \|a = b\| \\
&\leq \|b = k\| \wedge \|a = b\| \\
&= \|b = c\| \wedge \|a = b\| \\
&\leq \|a = c\|
\end{aligned}$$

And therefore we have that $\|c \leq_x a \wedge a \in x\| \leq \|a = c\|$ so $V^{(B)} \models \forall a \in x(c \leq_x a \rightarrow a = c)$, meaning that c is a maximal element for x in $V^{(B)}$. So the proof of Zorn's lemma in $V^{(B)}$ is complete. $\square$

4 Application in independence proofs

In this section we will give a brief introduction in how we can use Boolean-valued models of set theory to give independence proofs. In order to do this we first need to explore the famous method of *forcing*, which was developed by Paul Cohen in 1963 and adapted by Dana Scott for Boolean-valued models. We will also need some material about cardinals in $V^{(B)}$. However, this material will not be treated in detail. The interested reader will find a more detailed account of this material in [1]. Finally, we will use these concepts to prove that the generalized continuum hypothesis is independent of ZFC.

4.1 The forcing theorem

The idea of forcing is that we identify a poset $(P, \leq)$ inside a Boolean algebra B, and regard the elements of P as states of information. We then regard a formula as true in state p if we are *forced* to accept it given the information of p. These matters will be clarified in this section, but to that end, we first need some results on posets.
Let $(P, \leq)$ be a poset consisting of, as we mentioned before, states of information. We say that all the information of $q \in P$ is also contained in $p \in P$ if $p \leq q$. We call $p, q \in P$ *compatible* if there exists an element $r \in P$ such that $r \leq p$ and $r \leq q$. We write this as $\mathrm{Comp}(p, q)$, and can think of it as meaning that there exists a state which contains all the information of both p and q, which means that p and q are mutually consistent. We call P *refined* if $\forall p, q \in P(q \nleq p \to \exists p' \leq q \neg\mathrm{Comp}(p, p'))$. Now we define for every $p \in P$ the set $O_p = \{q \in P \mid q \leq p\}$. Then $\{O_p \mid p \in P\}$ is a basis for a topology called the *left order topology* on P. We see that a subset $X \subseteq P$ is open in this topology precisely when $\forall x, y((y \in X \wedge x \leq y) \to x \in X)$. It is easily verified that this is indeed a topology. By example 2 in section 2 we know that the set of regular opens of this topology, denoted by $RO(P)$, is a Boolean algebra. In fact, this Boolean algebra is complete, if $\{U_i \mid i \in I\}$ is a family of opens then $\bigvee_{i \in I} U_i$ is the interior of the closure of $\bigcup_{i \in I} U_i$ and $\bigwedge_{i \in I} U_i$ is the interior of $\bigcap_{i \in I} U_i$. We will want to look at what closures and interiors look like in this topology. If $X \subseteq P$ then $\overline{X}$ should at least contain X. Also, since the complement of $\overline{X}$ is to be open and as large as possible, this complement should contain every element that is not greater than some element of X. So we have $\overline{X} = \{p \in P \mid \exists q \in X(q \leq p)\}$. The interior of X should consist of every element p such that all elements smaller than p (which have to be in $\mathring{X}$ because we want $\mathring{X}$ to be open) are in X. So $\mathring{X} = \{p \in P \mid \forall q \leq p(q \in X)\}$. Combining these shows that

$$\mathring{\overline{X}} = \{p \in P \mid \forall q \leq p \exists r \in X(r \leq q)\}$$

Applying this to O_p shows that

$$\begin{aligned}\mathring{\overline{O_p}} &= \{q \in P \mid \forall r \leq q(\exists r'(r' \leq r \wedge r' \leq p))\} \\ &= \{q \in P \mid \forall r \leq q \mathrm{Comp}(p, r)\}\end{aligned}$$

We now have the following theorem:

Theorem 4.1. *P is refined iff $O_p \in RO(P)$ for all $p \in P$.*

Proof. We have to prove that P is refined iff $O_p = \mathring{\overline{O_p}}$. Suppose P is refined, and fix an arbitrary $p \in P$. We know that $O_p \subseteq \mathring{\overline{O_p}}$ because of lemma 2.1. So suppose $q \in \mathring{\overline{O_p}}$, then we have to prove that $q \in O_p$, so we have to prove that $q \leq p$. Suppose $q \nleq p$, then because P is refined we know that $\exists r \leq q \neg \mathrm{Comp}(p, r)$, but since $q \in \mathring{\overline{O_p}}$ we know that $\forall r \leq q \mathrm{Comp}(p, r)$. From this clear contradiction it follows that $q \leq p$, so $q \in O_p$.
Suppose on the other hand that $O_p = \mathring{\overline{O_p}}$. Then for all $p, q \in P$: $q \leq p \leftrightarrow \forall r \leq q \mathrm{Comp}(p, r)$, so suppose $p, q \in P$ and $q \nleq p$, then $\neg \forall r \leq q \mathrm{Comp}(p, r)$, which is equivalent to $\exists r \leq q \neg \mathrm{Comp}(p, r)$, so P is refined. □

We can now prove an important theorem on how we can view a poset inside a Boolean algebra. We call a subset X of a Boolean algebra B *dense* if $0 \notin X$ and for every $b \in B$ there is an $x \in X$ such that $x \leq b$. We call a map $\phi : P \to B$ an *order-isomorphism* if it is bijective and satisfies $\forall p, q \in P(p \leq q \to \phi(p) \leq \phi(q))$. If $\phi : P \to B$ is an order-isomorphism we call P *order-isomorphic* to $\phi(P)$.

Theorem 4.2. *P is refined iff it is order-isomorphic to a dense subset of a complete Boolean algebra.*

Proof. Suppose first that P is refined. We will show that the map $p \mapsto O_p$ is an order-isomorphism and that $\{O_p \mid p \in P\}$ is dense in $RO(P)$. We know by theorem 4.1 that $O_p \in RO(P)$ for every $p \in P$. Now suppose $p \leq q$, we have to show that $O_p \leq O_q$. In $RO(P)$ this means that we have to show that $(O_p^\perp \cup O_q)^{\perp\perp} = P$. Since the interior of P is P itself we have to show that

$$\overline{O_p^\perp \cup O_q} = \{r \in P \mid \exists r' \in (O_p^\perp \cup O_q)(r' \leq r)\} = P$$

So suppose $r \in P$, if $r \in O_p^\perp$ then we see that $r' = r$ satisfies the formula above so $r \in \overline{O_p^\perp \cup O_q}$. So suppose $r \notin O_p^\perp = \overline{O_p}^c$. Then $r \in \overline{O_p}$, so $\exists r' \in O_p(r' \leq r)$, but since $p \leq q$ we have that $r' \leq p \leq q$ and P is a poset so $r' \leq q$, meaning that $r' \in O_q$. So r satisfies $\exists r' \in (O_p^\perp \cup O_q)(r' \leq r)$, meaning that $\overline{O_p^\perp \cup O_q} = P$. So this map is indeed an order-isomorphism. We next show that the image is dense in $RO(P)$. We first notice that $p \in O_p$ for all $p \in P$, so $\emptyset \notin \{O_p \mid p \in P\}$. Now suppose $X \in RO(P)$, $X \neq \emptyset$, then choose an arbitrary $p \in X$. We will show that $O_p \leq X$. In fact we can use the same strategy as above, since we have to show that

$$\overline{O_p^\perp \cup X} = \{r \in P \mid \exists q \in (O_p^\perp \cup X)(q \leq r)\} = P$$

If $r \in O_p^\perp$ we can choose $q = r$, and if $r \notin O_p^\perp$ then $\exists r' \in O_p(r' \leq r)$, but since $r' \leq p$ and $p \in X$ we have that $r' \in X$ because X is open. So every $r \in P$ satisfies the formula above, so $O_p \leq X$ as desired. So if P is refined then $p \mapsto O_p$ is an order-isomorphism from P to a dense subset of a complete Boolean algebra.
Suppose on the other hand that P is order-isomorphic to a dense subset of a complete Boolean algebra B. Then we can identify P with the image of this order-isomorphism. So suppose $p, q \in P$ and $q \nleq p$, then $q \wedge \neg p \neq 0$, and therefore there exists $r \in P$ such that $r \leq q \wedge \neg p$. We will prove that P is refined by proving that this r satisfies $r \leq q$ and $\neg \mathrm{Comp}(p, r)$. Since $r \leq q \wedge \neg p \leq q$ the first part of this is trivial. Now suppose $\mathrm{Comp}(p, r)$, then there exists $a \in P$ such that $a \leq p$ and $a \leq r$. So $a \leq r \leq \neg p$, so $a \leq \neg p$ and we also had $a \leq p$ so

$a \leq p \wedge \neg p = 0$, so $a = 0$. But P is dense in B, so $0 \notin P$, so this is not possible. It follows that $\neg\text{Comp}(p, r)$, so P is refined as desired. □

We call a pair (B, e) a *Boolean completion* of a poset P if B is a complete Boolean algebra and $e: P \to B$ is an order-isomorphism and $e(P)$ is dense in B. We will also call P a *basis for* B. The previous theorem tells us that every refined poset has a Boolean completion. In fact, this Boolean completion will turn out to be unique up to isomorphism, so that we can regard a refined poset P as a dense subset of it's unique Boolean completion B. We naturally define an isomorphism between two Boolean algebra's B and B' to be a bijective map f such that $f(x \vee y) = f(x) \vee f(y)$ and $f(\neg x) = \neg f(x)$ for all $x, y \in B$.

Theorem 4.3. *Suppose (B, e) and (B', e') are two Boolean completions of a poset P, then there exists an isomorphism $f: B \to B'$ such that $f(e(P)) = e'(P)$.*

Proof. Define for every $x \in B$ the set $P_x = \{p \in P \mid e(p) \leq x\}$, then $e(P_x) = \{e(p) \mid p \in P \wedge e(p) \leq x\}$. We notice that $x = \bigvee\{e(p) \mid p \in P \wedge e(p) \leq x\}$, because if this is not the case, then there exists an element $y \in B$ such that $y \leq x$ and $y \neq x$ and $\bigvee\{e(p) \mid p \in P \wedge e(p) \leq x\} \leq y$. We then look at the element $x \wedge \neg y$. If this element is 0 then because $\neg y \wedge x = 1$ we have that $x = y$, so $x \wedge \neg y \neq 0$, which means that there exists $q \in P$ such that $e(q) \leq x \wedge \neg y$, so $e(q) \in \{e(p) \mid p \in P \wedge e(p) \leq x\}$, so $e(q) \leq y$. However, we also have $e(q) \leq \neg y$ so $e(q) = 0$, which is impossible because $0 \notin e(P)$.
We now define the map $f: B \to B'$ by $f(x) = \bigvee e'(P_x)$. We see that this map must be bijective because $e(P)$ and $e'(P)$ are dense in B and B'. Also, $P_{x \vee y} = P_x \cup P_y$ so $f(x \vee y) = f(x) \vee f(y)$ and $\bigvee e'(P_{\neg x}) = \bigwedge \neg e'(P_x)$ so $f(\neg x) = \neg f(x)$. So f is an isomorphism. □

We are now ready to define the *forcing relation*. Let P be a basis of the complete Boolean algebra B. We perceive of P as a dense subset of B. For $p \in P$ and ϕ a Boolean sentence we now define that *p forces* ϕ (written as $p \Vdash \phi$) as following:

$$p \Vdash \phi \quad \leftrightarrow \quad p \leq \|\phi\|$$

With this relation we can make a statement about the truth of a sentence in $V^{(B)}$ by looking at the elements of the basis of B. The behavior of this relation is contained in the so-called *forcing theorem*. This theorem consists of 13 different parts, so we will only prove a few of them.

Theorem 4.4. ***(Forcing theorem)*** *Let ϕ and ψ be Boolean sentences and $\sigma(x)$ a Boolean formula. Then:*

1. $p \Vdash \neg\phi$ *iff* $\neg\exists q \leq p(q \Vdash \phi)$
2. $p \Vdash \phi \wedge \psi$ *iff* $p \Vdash \phi$ *and* $p \Vdash \psi$
3. $p \Vdash \phi \vee \psi$ *iff* $\forall q \leq p \exists r \leq q(r \Vdash \phi$ *or* $r \Vdash \psi)$
4. $p \Vdash \phi \to \psi$ *iff* $\forall q \leq p(q \Vdash \phi \ \to \ q \Vdash \psi)$
5. $p \Vdash \forall x \sigma(x)$ *iff* $\forall u \in V^{(B)}(p \Vdash \sigma(u))$
6. $p \Vdash \exists x \sigma(x)$ *iff* $\forall q \leq p \exists r \leq q \exists u \in V^{(B)}(r \Vdash \sigma(u))$

7. *For* $a \in V$*:* $p \Vdash \forall x \in \hat{a}\sigma(x)$ *iff* $\forall x \in a(p \Vdash \sigma(\hat{x}))$

8. *For* $a \in V$*:* $p \Vdash \exists x \in \hat{a}\sigma(x)$ *iff* $\forall q \leq p \exists r \leq q \exists x \in a(r \Vdash \sigma(\hat{x}))$

9. $\|\phi\| = 0$ *iff* $\neg\exists p(p \Vdash \phi)$

10. $\|\phi\| = 1$ *iff* $\forall p(p \Vdash \phi)$

11. $\forall p \exists q \leq p(q \Vdash \phi$ *or* $q \Vdash \neg\phi)$

12. $(p \Vdash \phi) \rightarrow \neg(p \Vdash \neg\phi)$

13. $(q \leq p$ *and* $p \Vdash \phi) \rightarrow q \Vdash \phi$

Proof. For 1, suppose $p \Vdash \neg\phi$, and suppose $q \leq p$ and $q \Vdash \phi$, then $q \leq p \leq \neg\|\phi\|$ and $q \leq \|\phi\|$, so $q \leq 0$, which is impossible, so $\neg\exists q \leq p(q \Vdash \phi)$. Suppose on the other hand that $\neg(p \Vdash \neg\phi)$, then $p \neq p \wedge \neg\|\phi\|$ so $p \wedge \|\phi\| \neq p \wedge \neg\|\phi\| \wedge \|\phi\| = 0$, so there exists a $q \in P$ such that $q \leq p \wedge \|\phi\|$ because P is dense in B, so $\exists q \leq p(q \Vdash \phi)$.
For 3, we use 1 and De Morgan's law and notice that

$$\begin{aligned} p \Vdash \phi \vee \psi &\leftrightarrow p \Vdash \neg(\neg\phi \wedge \neg\psi) \\ &\leftrightarrow \neg\exists q \leq p(q \Vdash \neg\phi \wedge \neg\psi) \\ &\leftrightarrow \neg\exists q \leq p(q \Vdash \phi \text{ and } q \Vdash \psi) \\ &\leftrightarrow \neg\exists q \leq p(\neg\exists r \leq q(r \Vdash \phi) \text{ and } \neg\exists r' \leq q(r' \Vdash \psi)) \\ &\leftrightarrow \forall q \leq p \exists r \leq q(r \Vdash \phi \text{ or } r \Vdash \psi) \end{aligned}$$

For 5, we notice that $p \Vdash \forall x\sigma(x)$ iff $p \leq \bigwedge_{u \in V^{(B)}} \|\sigma(u)\|$ iff $\forall u \in V^{(B)}(p \Vdash \sigma(u))$.
For 6 we copy the strategy of 3, but use the generalized De Morgan law instead of the usual one, and see:

$$\begin{aligned} p \Vdash \exists x\sigma(x) &\leftrightarrow p \leq \neg \bigwedge_{u \in V^{(B)}} \neg\|\sigma(u)\| \\ &\leftrightarrow \neg\exists q \leq p \forall u \in V^{(B)} \neg\exists r \leq q(r \Vdash \sigma(u)) \\ &\leftrightarrow \forall q \leq p \exists r \leq q \exists u \in V^{(B)}(r \Vdash \sigma(u)) \end{aligned}$$

For 7, we use theorem 3.7 to see that

$$\begin{aligned} p \Vdash \forall x \in \hat{a}\ \sigma(x) &\leftrightarrow p \leq \bigwedge_{x \in \mathrm{dom}(\hat{a})} (\hat{a}(x) \Rightarrow \|\sigma(x)\|) \\ &\leftrightarrow p \leq \bigwedge_{x \in a} (\hat{a}(\hat{x}) \Rightarrow \|\phi(\hat{x})\|) \\ &\leftrightarrow p \leq \bigwedge_{x \in a} (\|\phi(\hat{x})\|) \\ &\leftrightarrow \forall x \in a(p \Vdash \phi(\hat{x})) \end{aligned}$$

For 10 we easily see that if $\|\phi\| = 1$ then $\forall p(p \Vdash \phi)$. Suppose on the other hand that $\|\phi\| \neq 1$, then $\neg\|\phi\| \neq 0$ so there exists a $q \in P$ such that $q \Vdash \neg\phi$, so if $q \Vdash \phi$ then $q \Vdash \bot$ by 2, so $q \leq 0$, which is impossible. So we have $\exists p \neg(p \Vdash \phi)$ so $\neg\forall p(p \Vdash \phi)$. □

4.2 Cardinals in $V^{(B)}$

In this section we will say something about how we can perceive of cardinals in $V^{(B)}$, and how cardinals in $V^{(B)}$ behave. We will eventually need this in proving the independence of the continuum hypothesis from ZFC, but the material on itself is not incredibility relevant for this thesis, and therefore, we shall omit some of the proofs. The interested reader can find the proofs in [1].
We first notice that if $\phi(x_1, ..., x_n)$ is a Σ_1 formula then $\phi(x_1, ..., x_n) \to V^{(B)} \models \phi(\hat{x}_1, ..., \hat{x}_n)$. The proof is easy with theorem 3.5, part 5 of theorem 3.7 and the maximum principle in $V^{(2)}$. Since the formula $|x| = |y| \in \Sigma_1$ we find that $|x| = |y| \to V^{(B)} \models |\hat{x}| = |\hat{y}|$. The converse is not true in general, so we will need a few theorems to learn more about this.

Theorem 4.5. *For all $\alpha \in ORD$ we have $V^{(B)} \models \hat{\aleph}_\alpha \leq \aleph_{\hat{\alpha}}$. If $\alpha = 0$ we have equality.*

Proof. A proof of the general case is by induction on α, the proof can be found in [1], page 48. The case where $\alpha = 0$ follows from theorem 3.7 part 5, theorem 3.5 and the fact that the formula $x = \aleph_0$ is restricted. □

Let Card(α) be the formula stating that α is a cardinal, then we have the following theorem about Card:

Theorem 4.6. *$V^{(B)} \models Card(\hat{\alpha})$ for all $\alpha \leq \omega$, and if $V^{(B)} \models Card(\hat{\alpha})$ then $Card(\alpha)$ is true in V.*

Proof. The case where $\alpha = \omega$ has been treated in theorem 4.5. It is a well-known fact that in any model of ZF the following theorem must hold: $\forall \alpha(\alpha \in \omega \to \text{Card}(\alpha))$, and therefore this formula is also true in $V^{(B)}$. But since we also have the truth of $\omega = \hat{\omega}$ in $V^{(B)}$ we find that $V^{(B)} \models \forall \alpha(\alpha \in \hat{\omega} \to \text{Card}(\alpha))$, so we have $\bigwedge_{\alpha \in \omega} \|\text{Card}(\hat{\alpha})\| = 1$, so $V^{(B)} \models \text{Card}(\hat{\alpha})$ for all $\alpha \leq \omega$. For the second part of the theorem we notice that $\neg\text{Card}(\alpha)$ is a Σ_1 formula, because a cardinal α is an ordinal such that there is no surjective function from any $\beta < \alpha$ to α. And therefore our result about Σ_1 formula's at the beginning of this section implies the contrapositive of the second part of this theorem. □

The next theorem shows us that if we put a mild condition on our Boolean algebra, the behavior of cardinals in $V^{(B)}$ is a lot better than in the general case. We say that a Boolean algebra B satisfies *ccc* if every antichain in B is countable. *ccc* stands for '*countable chain condition*', which might feel as a strange name, since the definition talks about the countability of antichains instead of chains. In order to prove the independence of the continuum hypothesis we can use a Boolean algebra which satisfies this condition, so it will be rewarding to look at the properties of cardinals in $V^{(B)}$ if B satisfies *ccc*.

Theorem 4.7. *Let B be a Boolean algebra satisfying* ccc*, and let $x, y \in V$, then for all α we have:*

1. *$Card(\alpha) \to V^{(B)} \models Card(\hat{\alpha})$*
2. *$V^{(B)} \models \hat{\aleph}_\alpha = \aleph_{\hat{\alpha}}$*
3. *$|x| = |y| \leftrightarrow V^{(B)} \models |\hat{x}| = |\hat{y}|$*

Proof. To prove 1, we notice that we can assume that $\alpha > \omega$, for if $\alpha \leq \omega$ the result is immediate from theorem 4.6. So we assume Card(α) and we also assume $\alpha > \omega$. So in order to prove that $V^{(B)} \models \text{Card}(\hat{\alpha})$ we have to prove that

$$\|\text{fun}(f) \wedge \text{dom}(f) = \hat{\beta} \wedge \text{ran}(f) = \hat{\alpha}\| = 0$$

for every $\beta < \alpha$ and $f \in V^{(B)}$. Here fun(f) means that f is a function. Suppose on the contrary that there exists a $f \in V^{(B)}$ and $\beta < \alpha$ such that $\|\text{fun}(f) \wedge \text{dom}(f) = \hat{\beta} \wedge \text{ran}(f) = \hat{\alpha}\| = a \neq 0$, then $a \leq a$ and $a \leq \bigwedge_{\epsilon<\alpha} \bigvee_{\delta<\beta} \|f(\hat{\delta}) = \hat{\epsilon}\|$, because $a \leq \|\text{ran}(f) = \hat{\alpha}\|$. So we find that

$$0 \neq a \leq \bigwedge_{\epsilon<\alpha} \bigvee_{\delta<\beta} (\|f(\hat{\delta}) = \hat{\epsilon}\| \wedge a)$$

And this shows that we must have for every $\epsilon < \alpha$ a least $\delta_\epsilon < \beta$ such that $\|f(\hat{\delta}_\epsilon) = \hat{\epsilon}\| \wedge a \neq 0$. Now define for every $\gamma < \beta$ the set $X_\gamma = \{\epsilon < \alpha \mid \delta_\epsilon = \gamma\}$. We now see that every $\epsilon \in \alpha$ is in X_γ for some $\gamma < \beta$. So if every X_γ is countable then $\alpha \leq \beta \times \omega$, which is impossible because α is an uncountable cardinal and $\beta < \alpha$. So there exists a $\gamma < \beta$ such that X_γ is uncountable. Now look at the set $\{\|f(\hat{\gamma}) = \hat{\epsilon}\| \wedge a \mid \epsilon \in X_\gamma\}$. We see that this is a subset of B which does not contain 0. Also, if $\epsilon_1, \epsilon_2 \in X_\gamma$ such that $\epsilon_1 \neq \epsilon_2$, then

$$\|f(\hat{\gamma}) = \hat{\epsilon_1}\| \wedge a \wedge \|f(\hat{\gamma}) = \hat{\epsilon_2}\| \wedge a \leq \|\epsilon_1 = \epsilon_2\| \wedge a = 0$$

Because $\neg(x = \epsilon_2)$ is a restricted formula and $\epsilon_1 \neq \epsilon_2$. So every element of $\{\|f(\hat{\gamma}) = \hat{\epsilon}\| \wedge a \mid \epsilon \in X_\gamma\}$ is unequal to 0 and because it is an antichain there are no two equal elements in this set. Therefore, because X_γ is uncountable we have found an uncountable antichain in B, which is impossible because B satisfies the countable chain condition. It follows that $V^{(B)} \models \text{Card}(\hat{\alpha})$.
To prove 2, we use induction on α. So suppose $V^{(B)} \models \aleph_{\hat{\beta}} = \hat{\aleph}_\beta$ for every $\beta < \alpha$. By theorem 4.5 we only have to prove that $V^{(B)} \models \aleph_{\hat{\alpha}} \leq \hat{\aleph}_\alpha$. Because of 1 we know that $V^{(B)} \models \text{Card}(\hat{\aleph}_\alpha)$ and if $\beta < \alpha$ then $V^{(B)} \models \aleph_{\hat{\beta}} = \hat{\aleph}_\beta < \hat{\aleph}_\alpha$, so

$$1 = \|\text{Card}(\hat{\aleph}_\alpha) \wedge \forall \beta < \hat{\alpha}(\aleph_\beta < \hat{\aleph}_\alpha)\| \leq \|\aleph_{\hat{\alpha}} \leq \hat{\aleph}_\alpha\|$$

And this completes the induction step and with it the proof of 2.
3 follows immediately from 2. □

We have now arrived at the final theorem of this section. We first need to define *ordered pairs* in $V^{(B)}$. For $u, v \in V^{(B)}$ we define $\{u\}^{(B)} = \{\langle u, 1\rangle\}$, $\{u, v\}^{(B)} = \{u\}^{(B)} \cup \{v\}^{(B)}$ and $\langle u, v\rangle^{(B)} = \{\{u\}^{(B)}, \{u, v\}^{(B)}\}^{(B)}$. We easily see (by a proof identical to that of normal ordered pairs) that $V^{(B)} \models \forall u, v, x, y(\langle x, y\rangle^{(B)} = \langle u, v\rangle^{(B)} \leftrightarrow x = u \wedge y = v)$. These ordered pairs are used in the proof of the following theorem:

Theorem 4.8. *For every $u \in V^{(B)}$ we can find a $f \in V^{(B)}$ such that in $V^{(B)}$ this f is a function with $dom(u)\hat{}$ as domain and u is a subset of the range of f. Which shows that $V^{(B)} \models |u| \leq |dom(u)\hat{}|$.*

Proof. We define f by $f = \{\langle \hat{z}, z\rangle^{(B)} \mid z \in \text{dom}(u)\} \times \{1\}$. It is then easy to check that the conditions are met. An example is in [1] page 53. □

4.3 The independence of GCH from ZFC

In this final section we are going to show how we can use $V^{(B)}$ to prove the independence of GCH from ZFC. We will not prove the relative consistency of ZFC+GCH, but assume this to be done. This can be historically justified by noting that Kurt Gödel had already given a proof of this almost 30 years before the concept of Boolean-valued models was introduced. This proof is in [7]. In fact, we will need this result in order to prove the relative consistency of ZFC+¬GCH, because we will need the continuum hypothesis to make the desired construction. The fact that we can use GCH to obtain a valid proof of ZFC+¬GCH (which might sound like a paradox) follows from the following theorem:

Theorem 4.9. *Suppose T and T' are extensions of ZF such that Consis(ZF)→Consis(T') and in $\mathcal{L}_{ZF}$ we can define a constant B such that $T' \vdash B$* is a complete Boolean algebra *and for all $\phi \in T$: $T' \vdash \|\phi\| = 1$. Then Consis(ZF)→Consis(T).*

Proof. Suppose all the assumptions are true and suppose that T is inconsistent. Then by the compactness theorem T must have a finite inconsistent subtheory, so there must be $\phi_1, ..., \phi_n \in T$ such that $\phi_1 \wedge ... \wedge \phi_n \to \perp$. So we must have $T' \vdash \|\phi_1 \wedge ... \wedge \phi_n\| = 0$, but $T' \vdash \|\phi_1 \wedge ... \wedge \phi_n\| = 1$, so that $T' \vdash 0 = 1$, which is impossible. So T must be consistent. □

So if we take T' to be ZFC+GCH and T to be ZFC+¬GCH we see that all we have to do to prove the relative consistency of T is prove that we can use the continuum hypothesis to create a Boolean-valued model of set theory in which the continuum hypothesis fails.
We will now start working on the desired Boolean algebra. Suppose that I is some set and that 2 is assigned the discrete topology. Then 2^I with the product topology satisfies the condition that every family of disjoint opens is countable. From this it follows that $RO(2^I)$ satisfies the countable chain condition. Also, in general, if we have two nonempty sets x and y and $|y| \geq 2$, then we can partially order $C(x, y)$ (the set of all functions with values in y and domain a finite subset of x) by reverse inclusion, and obtain a refined partially ordered set. If we put $N(p) = \{f \in y^x \mid p \subseteq f\}$ for all $p \in C(x, y)$ we find that the $N(p)$ are a basis for the product topology on y^x where y is assigned the discrete topology. Also, this basis consists of clopen (closed and open) sets and is therefore contained in $RO(y^x)$. Also, we find that $C(x, y)$ is a refined partial ordering and that $p \mapsto N(p)$ is an order-isomorphism of $C(x, y)$ onto a dense subset of the Boolean algebra $RO(y^x)$, making $(RO(y^x), N)$ into a Boolean completion of $C(x, y)$, meaning that $C(x, y)$ is a basis for $RO(y^x)$. We will not prove these assertions here, more details on how this works are in [8]. We want an estimate of the cardinality of the Boolean algebra $RO(2^I)$, and it turns out that we can in fact give a very reasonable estimate. In order to prove this we first need the following slightly more general theorem.

Theorem 4.10. *Suppose X is a topological space such that every disjoint family of opens in X is countable. Let E be a basis for X and set $B = RO(X)$, the Boolean algebra of regular opens of X. Then $|B| \leq |E|^{\aleph_0}$.*

Proof. Suppose U is some element of B. Then we are going to show that U is uniquely determined by some disjoint subfamily of E. Let F be a maximal disjoint subfamily of $E \cap \mathcal{P}(U)$ (such an F exists because of Zorn's lemma). We will show that this F uniquely determines U. Set $G = \bigcup F$, then we can show that $G^{\perp\perp} = U$. First of all, we see that

$U \in RO(X)$ so $U^{\perp\perp} = U$. If $a \in G$ then there exists an $A \in F$ such that $a \in A$, and because $A \in E \cap \mathcal{P}(U)$ we have $A \subseteq U$ so $a \in U$, so $G \subseteq U$. Because every $A \in F$ is in E we see that G is a union of opens and therefore itself an open, so it follows from $G \subseteq U$ that $U^{\perp} \subseteq G^{\perp}$ and $G^{\perp\perp} \subseteq U^{\perp\perp} = U$. So we are left with the proof that $U \subseteq G^{\perp\perp}$. We therefore look at $U \backslash \overline{G} = U \cap G^{\perp}$. Because U and $G^{\perp}$ are open we see that $U \backslash \overline{G}$ is also open. Now suppose that it is nonempty, then there must be a set $A \in E$ such that $A \subseteq U \backslash \overline{G}$, because E is a basis of X. So because $A \subseteq U \cap G^{\perp}$ we find that $A \in E \cap \mathcal{P}(U)$ and A is disjoint from every member of F. But this goes against the maximality of F, so $U \backslash \overline{G}$ must be empty, so $U \subseteq \overline{G}$, and because U is open we find that $U \subseteq G^{\perp\perp}$ as desired. So $U = (\bigcup F)^{\perp\perp}$, so F uniquely determines U. And because any maximal disjoint subfamily of E is countable and therefore has cardinality of at most $\aleph_0$ we find that there are at most $|E|^{\aleph_0}$ disjoint subfamilies of E, so $|B| \leq |E|^{\aleph_0}$. □

We now go back to the Boolean algebra $RO(2^I)$. Suppose $|I| = \aleph_\alpha$, then $\{f \in 2^I \mid f(i_1) = a_1, ..., f(i_n) = a_n\}$ with $i_1, ..., i_n \in I$ and $a_1, ..., a_n \in 2$ is a base for 2^I and consists of sets which are all clopen. It follows that this base consists of elements of $RO(2^I)$ and because it has cardinality $\aleph_\alpha$ we find using theorem 4.10 the following estimate for $|B|$:

$$\aleph_\alpha \leq |RO(2^I)| \leq \aleph_\alpha^{\aleph_0}$$

We are now ready for the main theorem of this section:

Theorem 4.11. *Suppose $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$ and let $B = 2^{\omega \times \omega_\alpha}$. Then $V^{(B)} \models 2^{\aleph_0} = \aleph_{\hat{\alpha}}$.*

Proof. We first prove that $V^{(B)} \models 2^{\aleph_0} \leq \aleph_{\hat{\alpha}}$. Because $|\omega \times \omega_\alpha| = \aleph_\alpha$ we find that $|B| = \aleph_\alpha$. Define v to be the element with $\text{dom}(v) = B^{\text{dom}(\hat{\omega})}$ and for all $x \in \text{dom}(v)$: $v(x) = \|x \subseteq \hat{\omega}\|$. Then this v is the element which identifies with the power set of $\hat{\omega}$ (recall the proof of the axiom of power set in $V^{(B)}$). Because of theorem 4.8 we find that $V^{(B)} \models |v| \leq |B^{\text{dom}(\hat{\omega})}| = \hat{\aleph}_\alpha^{\hat{\aleph}_0} = \hat{\aleph}_\alpha$. So because of theorem 4.7 we now have that $V^{(B)} \models 2^{\aleph_0} \leq \aleph_{\hat{\alpha}}$.
We now have to prove that $V^{(B)} \models \aleph_{\hat{\alpha}} \leq 2^{\aleph_0}$. We do this by defining a function from $\aleph_{\hat{\alpha}}$ to $2^{\aleph_0}$ and show that in $V^{(B)}$ this function is injective. To this end, we first define for every $\gamma < \omega_\alpha$ the element $u_\gamma \in V^{(B)}$ by $\text{dom}(u_\gamma) = \text{dom}(\hat{\omega})$ and for $n \in \omega$: $u_\gamma(\hat{n}) = \{f \in 2^{\omega \times \omega_\alpha} \mid f(n, \gamma) = 1\}$. Then for every $\gamma < \omega_\alpha$ we find that

$$\|u_\gamma \subseteq \hat{\omega}\| = \bigwedge_{x \in \omega} (u_\gamma(\hat{x}) \Rightarrow \|\hat{x} \in \hat{\omega}\|) = 1$$

Because $x \in \omega$ iff $\|\hat{x} \in \hat{\omega}\| = 1$. So in $V^{(B)}$ every u_γ is an element of the power set of $\hat{\omega}$. We can perceive of $C(\omega \times \omega_\alpha, 2)$ as a dense subset of $RO(2^{\omega \times \omega_\alpha})$, ordered by reverse inclusion. Now suppose that $p \in C(\omega \times \omega_\alpha, 2)$, then $p \Vdash \hat{n} \in u_\gamma$ iff $p(n, \gamma) = 1$ and $p \Vdash \hat{n} \notin u_\gamma$ iff $p(n, \gamma) = 0$. Now suppose that $\gamma, \delta < \omega_\alpha$ and suppose $\gamma \neq \delta$. Then we can show that $\|u_\gamma = u_\delta\| = 0$. Because if it is not 0, then there exists a $p \in C(\omega \times \omega_\alpha, 2)$ such that $p \Vdash u_\gamma = u_\delta$. Now choose $n \in \omega$ such that for all $\xi < \omega_\alpha$: $(n, \xi) \notin \text{dom}(p)$. This is always possible because the domain of p is finite. Now define the element $q \in C(\omega \times \omega_\alpha, 2)$ by $q = p \cup \{((n, \gamma), 1)\} \cup \{((n, \delta), 0)\}$. Then $q(n, \gamma) = 1$ so $q \Vdash \hat{n} \in u_\gamma$, and $q(n, \delta) = 0$ so $q \Vdash \hat{n} \notin u_\delta$, so $q \Vdash u_\gamma \neq u_\delta$. But since $p \subseteq q$ we have $q \leq p$ (because the ordering is reverse inclusion), so $p \Vdash u_\gamma \neq u_\delta$. This is in clear contradiction with the assumption that $p \Vdash u_\gamma = u_\delta$, and therefore we must have $\|u_\gamma = u_\delta\| = 0$.

Now define the element $f \in V^{(B)}$ by

$$f = \{\langle \hat{\gamma}, u_\gamma \rangle^{(B)} \mid \gamma < \omega_\alpha\} \times \{1\}$$

Then we easily see that in $V^{(B)}$ this is a function from $\hat{\omega}_\alpha$ to $\mathcal{P}(\hat{\omega})$, because we had $\|u_\gamma \subseteq \hat{\omega}\| = 1$ for all $\gamma < \omega_\alpha$. Also, because $\|u_\gamma = u_\delta\| = 0$ whenever $\gamma \neq \delta$ we clearly see that $V^{(B)} \models f$ *is injective*. And because $\hat{\omega}_\alpha = \omega_{\hat{\alpha}}$ by theorem 4.7 it follows that $\aleph_{\hat{\alpha}} \leq 2^{\aleph_0}$, which completes the proof. □

Now if we assume GCH then $\aleph_2^{\aleph_0} = 2^{\aleph_1 \times \aleph_0} = \aleph_2$. So we can find a Boolean algebra B such that in $V^{(B)}$ we have the truth of $2^{\aleph_0} = \aleph_2$, which is in direct violation with GCH. It now follows from theorem 4.9 that Consis(ZF)→Consis(ZFC+¬GCH). And therefore, we have used our Boolean-valued models of set theory to prove that the continuum hypothesis is independent of ZFC.

A Appendix: Zorn's lemma in $\mathcal{L}_{\mathbf{ZF}}$

In this appendix we will explain how we can express Zorn's lemma in the language of ZF. Since Zorn's lemma is the assertion that for every poset we have that if every chain in this poset has an upper bound then the poset has a maximal element, we will need the concepts of a poset, a chain, an upper bound and a maximal element. We have already seen that

$$\begin{aligned}\exists x, \leq_x (y = (x, \leq_x) \wedge \leq_x \in \mathcal{P}(x \times x) \\ &\wedge \forall p \in x((p,p) \in \leq_x) \\ &\wedge \forall p,q,r \in x(((p,q) \in \leq_x \wedge (q,r) \in \leq_x) \to (p,r) \in \leq_x) \\ &\wedge \forall p,q \in x(((p,q) \in \leq_x \wedge (q,p) \in \leq_x) \to p = q))\end{aligned}$$

is a way of saying that P is a poset. In order to write that C is a chain in P we will have to write that C is a subset of x and that $\leq_x$ is total on C. This is done in the following way:

$$\forall y \in C(y \in x) \wedge \forall u, v \in C((u,v) \in \leq_x \vee (v,u) \in \leq_x)$$

The formula expressing that C has an upper bound in P is $\exists a \in x \forall y \in C((y,a) \in \leq_x)$. And we see that the following formula expresses that P has a maximal element: $\exists m \in x \neg \exists n \in x(\neg(m = n) \wedge (m,n) \in \leq_x)$.
We could now combine these formula's to get Zorn's lemma, but we see that we also need the concepts of a power set of a cartesian product and an ordered pair in the language of ZF. The statement that $\leq_x \in \mathcal{P}(x \times x)$ is $\forall y \in \leq_x \exists a, b \in x(y = (a,b))$. So the last thing we need is a formula expressing that y is the ordered pair (a,b). So we have to state that $y = \{\{a\},\{a,b\}\}$, and this is done in the following way:

$$\forall z(z \in y \leftrightarrow (a \in z \wedge \forall u \in z(\neg(u = a) \to u = b)))$$

Because we see that only the sets $\{a\}$ and $\{a,b\}$ satisfy this condition. We can now put all of this together to get the following formula for Zorn's lemma:

$$\begin{aligned}\forall P, x, \leq_x ((\forall z(z \in P \leftrightarrow (x \in z \wedge \forall u \in z(\neg(u = x) \to u = \leq_x))) \\ &\wedge \forall y \in \leq_x \exists a, b \in x(\forall z(z \in y \leftrightarrow (a \in z \wedge \forall u \in z(\neg(u = a) \to u = b)))) \\ &\wedge \forall p \in x \exists y \in \leq_x \forall z \in y(w \in z \leftrightarrow w = p) \\ &\wedge \forall p,q,r \in x(\exists y, z \in \leq_x (\forall a(a \in y \leftrightarrow (p \in a \wedge \forall u \in a(\neg(u = p) \to u = q))) \\ &\quad \wedge \forall a(a \in z \leftrightarrow (q \in a \wedge \forall u \in a(\neg(u = q) \to u = r)))) \\ &\quad \to \exists w \in \leq_x (\forall a(a \in w \leftrightarrow (p \in a \wedge \forall u \in a(\neg(u = p) \to u = r))))) \\ &\wedge \forall p,q \in x(\exists y, z \in \leq_x (\forall a(a \in y \leftrightarrow (p \in a \wedge \forall u \in a(\neg(u = p) \to u = q))) \\ &\quad \wedge \forall a(a \in z \leftrightarrow (q \in a \wedge \forall u \in a(\neg(u = q) \to u = p)))) \\ &\quad \to p = q) \\ &\wedge \forall C((\forall y \in C(y \in x) \\ &\quad \wedge \forall u, v \in C \exists y \in \leq_x (\forall a(a \in y \leftrightarrow (u \in a \wedge \forall w \in a(\neg(w = u) \to w = v))) \\ &\quad \vee \forall a(a \in y \leftrightarrow (v \in a \wedge \forall w \in a(\neg(w = v) \to w = u))))) \\ &\quad \to \exists a \in x \forall b \in C \exists y \in \leq_x (\forall z(z \in y \leftrightarrow (b \in z \wedge \forall u \in z(\neg(u = b) \to u = a)))))) \\ &\to \exists m \in x \neg \exists n \in x(\neg(m = n) \\ &\quad \wedge \exists y \in \leq_x (\forall a(a \in y \leftrightarrow (m \in a \wedge \forall u \in a(\neg(u = m) \to u = n))))))\end{aligned}$$

References

[1] John L. Bell, 2005
Set Theory, Boolean-Valued Models and Independence Proofs
Clarendon Press, Oxford, third edition

[2] Gaisi Takeuti and Wilson M. Zaring, 1973
Axiomatic Set Theory
Springer-Verlag, New-York, first edition

[3] Paul R. Halmos, 1967
Lectures on Boolean Algebras
Van Nostrand, New York, first edition

[4] Rowan Jacobs
Forcing
Unpublished paper

[5] J. van Oosten and I. Moerdijk, 2014
Sets, Models and Proofs
Unpublished lecture notes, revised edition

[6] Anatoly G. Kusraev and Semen S. Kutateladze 1990
Nonstandard Methods of Analysis
Kluwer Academic Publishers, Dordrecht, first edition (translation from original Russian edition)

[7] Kurt Gödel, 1940
The consistency of the continuum hypothesis
Ishi Press International, New York, 2009 edition

[8] John L. Kelley, 1955
general topology
Springer-Verlag, New-York, 1975 edition