# Matrix Model Cosmology

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## Contents


1 Introduction . . . 5

2 Matrix Model of the One Dimensional String . . . 7

2.1 Triangulations . . . 7

2.1.1 Green's Functions . . . 9

2.1.2 Approximation to Continuous Geometry . . . 9

2.2 Free Fermions . . . 10

2.2.1 Diagonalising the Theory . . . 10

2.2.2 Free Fermions and the Double Scaling Limit . . . 12

3 Conformal String Theories . . . 14

3.1 Conformal Invariance of String Theories . . . 14

3.2 More Conformal Strings - The Linear Dilaton . . . 15

3.3 Physics in the Linear Dilaton Background . . . 16

3.3.1 The String Coupling Constant . . . 16

3.3.2 The Liouville Wall . . . 17

4 The Toy Cosmological Model . . . 18

4.1 Matrix Model/Liouville Theory Correspondence . . . 18

4.2 Topology of One Dimensional String Theory . . . 18

4.3 Orbifolds . . . 19

4.3.1 Some Definitions . . . 19

4.3.2 Fixed Points of an Orbifold . . . 19

4.3.3 The Orbifold $\mathcal{S}^1/\mathbb{Z}_2$ . . . 20

4.4 String Theory on Orbifolds . . . 20

4.4.1 Strings on Circles . . . 21

4.4.2 Strings on $\mathcal{S}^1/\mathbb{Z}_2$ . . . 21

4.5 Defining the Toy Model . . . 22

4.6 Some Issues Surrounding the Toy Model . . . 22

4.6.1 String Coupling . . . 22

4.6.2 The Role of the Twisted States . . . 23

4.6.3 Interpretation of $\mathcal{S}^1/\mathbb{Z}_2$ as time . . . 23

5 Partition Functions . . . 24

5.1 Heat Kernel Methods . . . 24

5.1.1 Some Scalar Identities . . . 24

5.1.2 Generalising to Matrices . . . 24

5.1.3 Application to Partition Functions . . . 25

5.1.4 Modification of Traces by Identifications . . . 26

5.2 Twisted Partition Functions . . . 26

5.2.1 Partition Function in a Representation . . . 26

5.2.2 Twists by Gauge Fields . . . 27

6 The Circle Partition Function . . . 28

6.1 Embedding Triangulations into $\mathcal{S}^1$ . . . 28

6.2 The Matrix Model for $\mathcal{S}^1$ . . . 28

6.2.1 The Heat Kernel Operator . . . 28

6.2.2 Applying the Heat Kernel Method . . . 30

6.2.3 Step 1 - Evaluating the Propagator Term . . . 30

6.2.4 Step 2 - Poisson Resummation . . . 31

6.2.5 Step 3 - Renormalising the Integral . . . 32

6.2.6 Step 4 - Summing Over $n$ and Integrating Over $\omega^2$ . . . 32

6.2.7 Step 5 - Some Algebraic Manipulations . . . 33

6.2.8 Determining the Twisted Partition Function from the Heat Kernel Trace 33

7 The Matrix Model Partition Function on the Orbifold . . . 34

7.1 Some Motivation . . . 34

7.2 Holonomies on Orbifolds . . . 35

7.3 $\mathcal{S}^1/\mathbb{Z}_2$ as a Quotient of $\mathbb{R}$ . . . 36

7.4 Calculating the Holonomies . . . 37

7.5 Calculating the Operator . . . 38

7.6 The Heat Kernel Trace . . . 39

7.7 Evaluating the Propagator Term . . . 40

7.7.1 Case A - Step 1 - Poisson Resummation . . . 41

7.7.2 Case A - Step 2 - Renormalising the Integral . . . 41

7.7.3 Case A - Step 3 - Summing Over $n$ and Integrating Over $\omega^2$ . . . . . . . . 42

7.7.4 Case A - Step 4 - Some Algebraic Manipulations . . . . . . . . . . . . . . 42

7.7.5 Case B - Step 1 - Gaussian Integration . . . . . . . . . . . . . . . . . . . . 43

7.7.6 Case B - Part 2 - Renormalisation of the Integral . . . . . . . . . . . . . . 44

7.8 The Orbifold Twisted Partition Function . . . . . . . . . . . . . . . . . . . . . . 44

7.8.1 The $n = \frac{N}{2}$ Representation . . . . . . . . . . . . . . . . . . . . . . . . . 45

8 Extracting Conclusions About The Toy Model . . . . . . . . . . . . . . . . . . . . 46

8.0.2 String Coupling . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 46

8.0.3 The role of Twisted States . . . . . . . . . . . . . . . . . . . . . . . . . . 46

8.0.4 Interpretation of $\mathcal{S}^1/\mathbb{Z}_2$ as Time . . . . . . . . . . . . . . . . . . . . . . 47

8.0.5 Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 47

# 1 Introduction

String theory is arguably the best known candidate for a theory of quantum gravity. String theory replaces the point particles of quantum field theory with one dimensional strings that trace out two dimensional worldsheets in spacetime. The vibrational modes of this string are the various particles that propagate in spacetime. The theory describes gravity because it is found that the spectrum of string modes must contain a mode that appears to be a massless spin 2 boson at low energies. This excitation can be identified as the graviton. The low energy effective field theory of this excitation matches Einstein's theory of relativity.

String theory is also a quantum mechanical theory. It avoids the problems of quantum gravity because the finite size of the string means that the string is insensitive to high energy spacetime fluctuations that manifest over short distance. It appears that this finite size is enough to shield string theory from the problems with renormalisability that beset attempts to directly quantise gravity. therefore, string theory appears to combine general relativity and quantum mechanics into a consistent theory of gravity.

If string theory is truly a consistent theory of quantum gravity it should offer new insights into gravity in the presence of singularities such as black holes where classical general relativity is not strictly well defined. However, string theory is still difficult to work with in its most popular form, the 10 dimensional superstring. However, rather than dealing with the full, intractible theory one often has the option of working with toy models instead.

A good toy model should represent enough of a simplification of the theory that it becomes tractable. On the other hand, the theory should not be so oversimplified as to become trivial. One of the ways of constructing toy models that was fruitful in the past was to look at lower dimensional systems.

The toy model that will be considered in this thesis utilises a relation between a the quantum mechanics of hermitian matrices (a matrix model) and Euclidean string theory in a particular two dimensional background. The matrix model is believed to reproduce all the predictions of the two dimensional string theory. However, the matrix model has a significant advantage over the string theory in that it can be solved exactly to all orders in perturbation theory. Therefore the matrix model represents an opportunity to study the string theory to arbitrary accuracy.

In the string theory, one of the two dimensions has a fixed topology, whereas the topology of the other can be chosen. The correspondence between the string an the matrix model was first established when this "free dimension" had the topology of the line. Later both the free dimension and the matrix model were defined on a circle and it was shown that the correspondence still holds and the matrix model remains solvable.

The toy model will be constructed by considering the two dimensional string theory where the free dimension is a type of singular space known as an "orbifold". The orbifold in question is know as $\mathcal{S}^1/\mathbb{Z}_2$ and is roughly like a line with a singularity at either end. It is conjectured that the matrix model defined on $\mathcal{S}^1/\mathbb{Z}_2$ continues to reproduce the string theory on $\mathcal{S}^1/\mathbb{Z}_2$.

If this is true then this toy model may have a cosmological interpretation. The idea is to Wick rotate the $\mathcal{S}^1/\mathbb{Z}_2$ Euclidean target space becomes Lorentzian and the orbifold plays the role of the time coordinate. If this works then the spacetime obtained in this way will have a singularity in the far past and far future. This spacetime may admit an interpretation as a universe that starts from a big bang and eventually collapses into a big crunch.

Given that the matrix quantum mechanics reproduces the string theory but is calculable to all orders what will be obtained at the end of this procedure is a model of string propagation between cosmological singularities in which everything can be calculated to all orders in perturbation theory. The toy model will provide a complete picture of a simple big bang/ big crunch universe in string theory.

Of course, the procedure by which the toy model can be obtained can be described but this does not mean that something won't go wrong when this is actually attempted. The ultimate aim of this thesis will be to begin the investigation of the toy model by attempting to calculate the matrix model partition function on the orbifold.

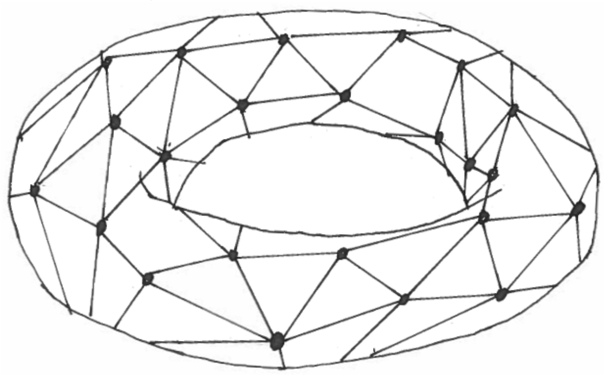
The structure of the thesis is as follows. In the first two sections the correspondence between the matrix model and the two dimensional string theory will be reviewed by showing that both theories are equivalent to a non critical string theory in one dimension, though the precise nature of the equivalence will be different for the matrix model and the two dimensional string theory.

The third section will draw upon the first two sections to discuss the toy model in greater detail. String propagation on orbifolds will be reviewed and some expected features of the toy model will be enumerated.

In the remainder of the thesis attention will shift towards calculation of the matrix model partition function. In section 4 some mathematical techniques will be introduced and in section 5 these will be applied to review the calculation of the matrix model partition function on the orbifold.

In section 6 the method of section 5 will be applied to the calculation of orbifold partition function. Unfortunately the calculation will not be completed as the final integral is currently intractable. However, despite the fact that the integral cannot be completed there is still some evidence that the toy model is indeed well defined.

Fig. 1: A tesselation of a torus.



## 2 Matrix Model of the One Dimensional String

In this section Euclidean string worldsheets in a one dimensional target space will be studied by approximating the worldsheets by imagining it to be composed of many flat polygons joined together [1] [2]. It will turn out that these approximations are described by the quantum mechanics of $N \times N$ Hermitian matrices. This theory will turn out to be much more tractable than string theory.

The starting point will be to consider the embedding of Riemann surfaces of all topologies with a cosmological constant $\mu$ into a single dimension. The partition function for the statistical sum of these embeddings is

$$Z = \sum_{\text{Topologies}} \int \mathcal{D}g\mathcal{D}X e^{-\frac{1}{4\pi}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X\partial_b X+\mu+R^{(2)}}$$

The cosmological constant has the effect of weighting each contributing worldsheet according to its size. This breaks conformal invariance. Therefore this string theory does not really make sense as a string theory, due to the lack of conformal invariance. However, it will later be seen Section 3 that this string theory can be reinterpreted as a well defined conformal string theory.

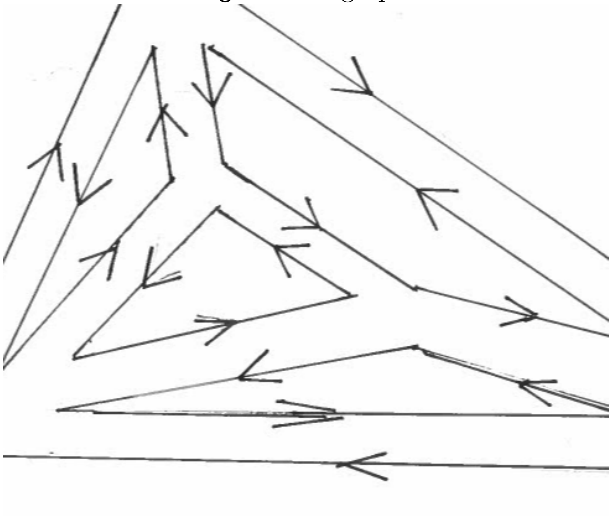
### 2.1 Triangulations

The idea is to take a Riemann surface and approximate it stitching together flat polygons. To be a bit more precise, The aim is to construct approximations to Riemann surfaces out of the basic topological ingredients of vertices, edges and faces, as shown in Figure 1. Each Riemann surface can be represented like this in numerous ways: One could use only triangles or pentagons, or the approximation could contain more or less polygons. However, the ultimate aim will be to find a limit in which the size and number of the polygons becomes very small so that a smooth worldsheet geometry can be reproduced. It will turn out that details about the specific shapes used to approximate the Riemann surface will be irrelevant.

The first necessary ingredient in constructing these approximations is to replace the lines representing the edges in Figure 1 with a pair of parallel lines, as shown in Figure 2. This is not purely cosmetic: The “fatgraph” obtained in this way can be interpreted as a Feynman diagram obtained from the quantum mechanics of $N \times N$ Hermitian matrices. If the only triangles are used to approximate the worldsheet then an appropriate path integral for the matrix quantum mechanics is

$$\mathcal{Z} = \int \mathcal{D}M e^{-N\int_0^\beta\int_0^\beta dtdt'Tr\left(\frac{1}{2}M(t')G^{-1}(t-t')M(t)\right)+N\int dt\left(Tr\lambda M^3(t)\right)}$$

Fig. 2: A fatgraph.



For the moment $G^{-1}(t-t')$ will not be specified to keep the theory reasonably general. The theory contains an $M^3$ interactions. The relation between this matrix quantum mechanics and string theory can be justified simply by looking at the Feynman rules of the theory.

1. The theory contains cubic vertices. Each vertex comes with a factor of $N\lambda$, due to the overall factor of $N$ in the action. $M^3$ interactions also mean that means that 3 propagators will meet at each interaction vertex: any Feynman diagram will look like it is made up of triangles.
2. The propagator of the theory is given by $\frac{1}{N}G(t-t')$.
3. Each closed loop in a Feynman diagram of the theory corresponds to a trace over matrix indices. Each closed loop should therefore also contribute an extra factor of $N$ as $N$ matrix entries are summed in the trace.

Now, consider the free energy of this theory. The free energy $\mathcal{F} = -ln\mathcal{Z}$ gives the sum over connected, irreducible Feynman diagrams. With a bit of topological reasoning the form of $\mathcal{F}$ can be deduced.

The contribution of a graph with $V$ vertices, $E$ propagators and $F$ closed loops should be

$$N^{F-E+V}\lambda^V \int \prod_{i=1}^{V} dt_i \prod_{\langle i,j\rangle} G(t_i - t_j)$$

This was obtained by writing down the factors of $\lambda$ and $N$ that are known to be present and then connecting all the vertices with propagators.

The free energy is should be the sum of all these contributions:

$$\mathcal{F} = \sum_h \sum_V N^{2-2h}\lambda^V \int \prod_{i=1}^{V} dt_i \prod_{\langle i,j\rangle} G(t_i - t_j) \tag{1}$$

In this expression the sum $F - E + V$ has been replaced by $2 - 2h$. This is because if the propagators are interpreted as edges of a triangulation and the traces are interpreted as the triangulations faces, $F - E + V = \chi$, the Euler number. As the aim is to approximate Riemann surfaces, and a Riemann surface has $\chi = 2-2h$, where $h$ is the number of handles in the surface, the free energy has taken the form of a sum over topologies of the Riemann surfaces that the matrix Feynman diagram approximates.

Compare this expression for the free energy to the partition function that is obtained by directly discretising string theory embedded in one dimension [3]:

$$Z_{\text{Discrete String}} = \sum_h g_s^{2h-2} \sum \kappa^V \int dt_1 \cdots \int dt_V \prod_{\langle i,j \rangle} G_{String}(t_i - t_j) \tag{2}$$

Here $g_S$ is the string coupling constant and $\kappa = e^{\frac{A_\Lambda}{4\pi}\mu}$ , where $\mu$ is the worldsheet cosmological constant. There is clearly a striking similarity between the two expressions. The formal identifications can be made:

$$g_s \cong \frac{1}{N} \qquad \kappa \cong \lambda \qquad \mathcal{F} \cong Z_{\text{Discrete String}}$$

Already it can be seen that some limit must be taken for the perturbative string theory to match the matrix model: The condition that the string coupling $g_S$ is small is equivalent to the size $N$ of the matrices being large. This comparison also allows the discrete cosmological constant of the matrix model to be identified as $\mu_{discrete} = -ln\lambda$.

However, the identification has some gaps. No choice has been made about $G(t-t')$ in matrix quantum mechanics, it could in principle be very different from $G_{String}(t_i - t_j)$. As well as that, there is no principle restricting the diagrams that contribute to (1). In principle, Feynman diagrams that are bad approximations to Riemannian surfaces may have a nontrivial contribution or even dominate. Both problems can be solved.

### 2.1.1 Green's Functions

The two dimensional discrete propagator that arises from the Polyakov action is [2]

$$G_{String}(t-t') = e^{(-\frac{(t-t')^2}{2}}$$

The discrete Green's function of the one dimensional matrix model can be taken to be the same. However, this choice of Green's function does not lead to the standard kinetic term for a one dimensional theory. The Green's function leading to the standard kinetic term is

$$G_{Matrix}(t-t') = e^{(-|t-t'|)}$$

In fact, it turns out that in the particular case that is being considered here the two propagators are in the same universality class. Near a critical point of the theory it won't matter which propagator is chosen.

### 2.1.2 Approximation to Continuous Geometry

The aim is to find some limit in which the discrete partition function of the matrix model can be considered equivalent to the continuous string theory partition function. It has already been established that $g_s = \frac{1}{N}$, so the . String theory as a sum over worldsheet topologies is only valid when the coupling constant $g_s$ is small. This model will therefore only be related to the sum over topologies when $N \to \infty$, i.e. the limit that the matrices become very large.

However, it is also clear that this alone is not sufficient. As it stands, the sum over all possible numbers of vertices means that poor approximations to Riemann surfaces with few vertices may contribute to the partition function as well as good approximations with many vertices. The way around this is to look for a limit in which the good approximations dominate the partition function, while the bad approximations are suppressed.

This can be understood in the following way: The term $\mu_{discrete} = ln\lambda$ appears in the free energy expansion (1). This term essentially describes how much energy adding a new vertex to the theory costs, because it appears in 1) as $\lambda^V = e^{V\mu_{discrete}}$. Because of the presence of a cosmological constant, larger worldsheets are energetically suppressed.

On the other hand, consider the entropy of a triangulation with many vertices. A triangulation with a large number of vertices clearly has many more ways those vertices could be arranged than a triangulation with a small number of vertices. Therefore, large surfaces has a higher entropy than small ones. This means that large surfaces contribute more to the free energy simply by virtue of there being more states corresponding to large vertices.

The key then is to look for the point where the discrete cosmological constant balances against the entropy in such a way that the triangulations wiht a large number of vertices begin to dominate the the free energy expansion. In this case the typical triangulation that contributes to (1) is a good approximation to a Riemann surface.

It turns out that the way to do this is to take the so called "double scaling limit":

$$N \to \infty, \qquad \lambda \to 0, \qquad N \cdot g_3 \sim \text{Remains constant.}$$

It is much more instructive to show this after reducing the system to $N$ free, nonrelativistic fermions, which will be covered in the next section.

## 2.2 Free Fermions

Part of the reason why the matrix model is easier to work with than the string theory is that when the model is defined on a simple enough space the matrix model can be reduced to a system of free fermions. The way to see this is to diagonalise the matrix qunatum mechanics.

### 2.2.1 Diagonalising the Theory

The first step in constructing the free fermion action is to make another modification of the action. As all physical results are obtained in the double scaling limit in which the geometry becomes continuous it is possible move from the discrete matrix action to the continuous one given by

$$\int_0^\beta dt \left(\frac{dM(t)}{dt}\right)^2 + M(t)^2 - \lambda M^3(t)$$

The partition function of the matrix quantum mechanics theory is given by

$$Z = \int \mathcal{D}M exp\left[-N\int_{-\frac{T}{2}}^{\frac{T}{2}} dt \quad Tr\left(\frac{1}{2}\frac{dM}{dt}^2 + \frac{1}{2}M^2 - \lambda g_3 M^3\right)\right]$$

The action is globally invariant under $U(N)$ rotations, where the field $M$ is in the adjoint representation of the gauge group. At each value of the time $t$, $M(t)$ is a Hermitian matrix and is thus a unitary transform of a diagonal matrix. As this is true at every point, the matrix can be decomposed as

$$M(t) = \Omega(t)\Lambda(t)\Omega^\dagger(t)$$

Where $\Lambda(t)$ is the diagonal matrix of the eigenvalues $\lambda_i = \Lambda_i i$ and $\Omega(t)$ is a unitary matrix.

Rewrite the action in terms of this decomposition of the matrix field. The terms polynomial in the matrix field are easy to deal with, due to the cyclicity of the trace:

$$Tr\left[M^n\right] = Tr\left[\left(\Omega\Lambda\omega^\dagger\right)^n\right] = Tr\left[\Lambda^n\right] = \sum_{i=1}^{N} \lambda_i^n$$

The terms involving derivatives of $M$ are a bit more complicated to deal with.

$$Tr\left[\frac{dM}{dt}\right] = Tr\left[\frac{d(\Omega\Lambda\Omega^\dagger}{dt}\right]$$

$$= Tr\left[\dot{\Lambda} + \Omega^\dagger\dot{\Omega}\Lambda + \Lambda\dot{\Omega}^\dagger\Omega\right]$$

$$= Tr\left[\dot{\Lambda} + [\Omega^\dagger\dot{\Omega}, \Lambda]\right]$$

Where in the last line the identity $\Omega^\dagger\dot{\Omega} = -\dot{\Omega}^\dagger\Omega$ has been used, which is just the derivative of $\Omega^\dagger\Omega = \mathbb{1}$.

The kinetic term of the theory then works out as

$$Tr\left[\frac{1}{2}\frac{dM}{dt}^2\right]$$

$$= Tr\left[\frac{1}{2}(\dot{\Lambda} + [\Omega^\dagger\dot{\Omega}, \Lambda])(\dot{\Lambda} + [\Omega^\dagger\dot{\Omega}, \Lambda])\right]$$

$$= Tr\left[\frac{1}{2}\dot{\Lambda}^2 + \frac{1}{2}[\Omega^\dagger\dot{\Omega}, \Lambda]^2\right]$$

In arriving at the final line, it has been used that

$$Tr\left[\dot{\Lambda}[\Omega^\dagger\dot{\Omega}, \Lambda] + [\Omega^\dagger\dot{\Omega}, \Lambda]\dot{\Lambda}\right]$$

$$= Tr\left[\dot{\Lambda}\Omega^\dagger\dot{\Omega}\Lambda - \dot{\Lambda}\Lambda\Omega^\dagger\dot{\Omega} + \Omega^\dagger\dot{\Omega}\Lambda\dot{\Lambda} - \Lambda\Omega^\dagger\dot{\Omega}\dot{\Lambda}\right]$$

$$= Tr\left[2\Omega^\dagger\dot{\Omega}[\Lambda, \dot{\Lambda}]\right] = 0$$

Because both $\Lambda$ and $\dot{\Lambda}$ are diagonal and hence commute. Thus the when $Tr\left[(\dot{\Lambda} + [\Omega^\dagger\dot{\Omega}, \Lambda])^2\right]$ is expanded the cross terms vanish.

Thus the diagonalised Lagrangian for the theory is

$$\mathcal{L}_{Diag} = Tr\left[\frac{1}{2}\dot{\Lambda}^2 + \frac{1}{2}[\Omega^\dagger\dot{\Omega}, \Lambda]^2 + \lambda\Lambda^3\right]$$

The Hamiltonian of the theory is [2].

$$H = \sum_i \frac{-\beta}{2}\frac{1}{\Delta}\frac{\partial}{\partial\lambda_i}\Delta + \mathcal{V}(M) + \frac{1}{2\beta}\sum_{i<j}\frac{\hat{\Pi}_{ij}\hat{\Pi}_{ji}}{(\lambda_i - \lambda_j)^2}$$

Here the $\{\lambda_i\}$ are the eigenvalues of $M$, $\hat{\Pi}_{ji}$ are the momentum operators corresponding to angular rotations of the field $M$ and $\Delta(\Lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ is the Vandermonde determinant, which is the Jacobian of the change of variables from $M \to \Omega\Lambda\Omega^\dagger$. $\mathcal{V}(M)$ is the matrix field potential.

If this Hamiltonian acts on a state of the theory in the singlet representation $\Psi_{singlet}(\lambda)$ then the by definition the state will be annihilated by the angular momenta $\hat{\Pi}_{ij}$. On such a state the Schrodinger equation is

$$H\Psi_{singlet}(\lambda) = \sum_i \frac{-\beta}{2}\frac{1}{\Delta}\frac{\partial}{\partial\lambda_i}\Delta + \mathcal{V}(M)\Psi_{singlet}(\lambda) \tag{3}$$

Now, $\Psi_{singlet}(\lambda)$ must also be completely symmetric under any permutation of the eigenvalues $\lambda_i$. This is because such a permutation just amounts to a unitary transform of the matrix $\Lambda$, and being in the singlet $\Psi_{singlet}(\lambda)$ doesn't know anything about such transformations. The Vandermonde determinant $\Delta$ is completely antisymmetric under such permutations. After the decomposition $\Psi_{singlet}(\lambda) = \Delta^{-1}\tilde{\Psi}(\lambda)$ the new wavefunction $\tilde{\Psi}(\lambda)$ must be completely antisymmetric under permutations of the $\lambda_i$. Inserting this decomposition into (3) one finds

$$H\Delta^{-1}\tilde{\Psi}(\lambda) = \frac{1}{\Delta}\sum_i \frac{-\beta}{2}\frac{1}{\Delta}\frac{\partial}{\partial\lambda_i} + \mathcal{V}(M)\tilde{\Psi}(\lambda)$$

From this expression it can be seen that the eigenvalues behave as particles in the potential $\mathcal{V}(M)$, with wavefunction $\tilde{\Psi}(\lambda)$. As the wavefunction is completely antisymmetric this justifies the claim that the eigenvalues of the matrix behave as fermions.

In all the cases that are considered in this thesis there will be good reasons to restrict the states of matrix quantum mechanics to be in the singlet state.

### 2.2.2 Free Fermions and the Double Scaling Limit

The picture of matrix quantum mechanics as free fermions gives useful insight into the continuum limit of the theory.

The potential that has been used so far is somewhat problematic. The potential is

$$\mathcal{V}(M) = Tr(\frac{N}{2}M^2 - N\lambda M^3)$$

This potential is not bounded from below, as shown in Figure 3. However, ignoring tunneling for the moment, stable states do exist in the theory. Decreasing $\lambda$, the cubic coupling, in this model will lower the energy difference between the energy levels of this model. One can imagine reducing the cubic coupling $\lambda$ while simulataneously increasing $N$. As long as $\lambda N$ remains fixed the effect of this will be to slowly push the Fermi sea of the eigenvalues up to the tip of the quadratic peak while simultaneously reducing the space between eignevalues. In the limit

$$\lambda \to 0 \qquad N \to \infty \qquad \lambda N \sim \text{constant}$$

There will be an infinite number of eigenvalues in the well while the Fermi sea will just begin to flow over the top of the quadratic maximum. This is precisely the kind of critical behaviour that was discussed in Section 2.1.2.

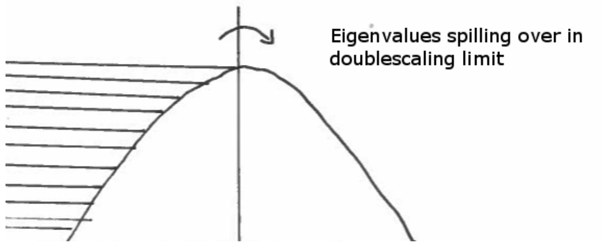
An important aspect of the double scaling limit is that only the physics near the quadratic maximum matters. Make the rescaling $NM \to M$. The action of the theory transforms as

$$\mathcal{S} = N \quad Tr\int_0^T dt\frac{1}{2}\left(\frac{dM}{dt}\right)^2 + \frac{1}{2}M^2 - \lambda M^3$$

$$\mathcal{S} = \quad Tr\int_0^T dt\frac{1}{2}\left(\frac{dM}{dt}\right)^2 + \frac{1}{2}M^2 - \frac{\lambda}{\sqrt{N}}M^3$$

After this reparameterisation it is easy to see that in the double scaling limit the contribution of the $M^3$ term in the potential becomes negligible. The essential physics is captured by the physics

Fig. 3: Eigenvalues just about to spill over the quadratic maximum in the double scaling limit.




of the inverse harmonic oscillator, as shown in Figure 3 The idea is to Finally, in the double scaling limit there is no need to worry about tunneling of eigenvalues through the quadratic maximum into the area where physics is ill defined. In the double scaling limit these tunnelling effects are suppressed.

Tunnelling through the maximum is important in some circumstances. The matrix model can be viewed as a nonperturbative definition of the string theory. If one wants to study nonperturbative effects, the matrix model can be made well defined by some choice of potential with a lower bound. In this picture, tunnelling through the barrier corresponds to nonperturbative effects [4].

# 3 Conformal String Theories

In order for string theories to be well defined, they must be invariant under conformal transformations. A string theory that is invariant under conformal transformations is called "critical", while one that is not is called "non-critical". To define the matrix model, the closed string theory embedded in one dimension with a worldsheet cosmological constant $\mu$ was approximated by triangulations. Such a string theory is an example of a non critical string theory. As it is a non critical string theory, the theory is not a well defined string theory. However, this does not matter as it will be shown in this section that a $D$ dimensional non critical string theory can be interpreted as a $D+1$ dimensional critical string theory in a particular background. In particular the 1 dimensional string of the previous section can be shown to be equivalent to a two dimensional string theory.

## 3.1 Conformal Invariance of String Theories

For a closed string theory, the partition function for a string theory in $D$ dimensions is given by [5]

$$Z = \int \sum_{\text{topologies}} \mathcal{D}X\mathcal{D}g e^{-\frac{1}{4\pi}\int d^2\sigma\sqrt{g}g^{ab}\partial_a X_\mu \partial_b X_\nu G^{\mu\nu} - \int d^2\sigma\sqrt{g}R^{(2)}} \tag{4}$$

This is a sum over embeddings of Riemann surfaces of definite topology into a $D$ dimensional target spacetime. The embedding of the surface is defined by the maps $X^\mu(\sigma)$. The partition function is defined in Euclidean space.

At the classical level this defines a theory that is invariant under conformal transformations $g_{ab} \to e^{2\omega}g_{ab}$ because the action in the partition function is classically conformally invariant. However, string theory is a quantum mechanical theory and classical invariance is no guarantee of quantum mechanical invariance. In general, the conformal invariance of the theory defined by (4) is anomalous. To see why, consider a variation of partition function:

$$\begin{aligned}
\delta Z &= \int \sum_{\text{topologies}} \mathcal{D}X\mathcal{D}g\delta e^{-S} \\
&= \int d^2\sigma \quad -\delta S e^{-S} \\
&= \int d^2\sigma \quad -\delta g_{ab}\frac{\delta S}{\delta g_{ab}} e^{-S}
\end{aligned}$$

The energy momentum tensor is defined $T_{ab} = \frac{-4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{ab}}$. For a conformal transformation $\delta g_{ab} = 2\delta\omega e^{2\omega}g_{ab}$. Applying these identities gives

$$\begin{aligned}
\delta Z &= \int d^2\sigma\sqrt{g} \quad \frac{1}{2\pi}\delta\omega e^{2\omega}g_{ab}T^{ab}e^{-S} \\
&= \int d^2\sigma\sqrt{g} \quad \frac{1}{2\pi}\delta\omega e^{2\omega}T^a_a e^{-S} = \int d^2\sigma\sqrt{g} \quad \frac{1}{2\pi}\delta\omega e^{2\omega}\langle T^a_a\rangle
\end{aligned}$$

On a two dimensional manifold like the string worldsheet the Ricci scalar $R^{(2)}$ is the only independent scalar that contains at most two derivatives. Therefore $\langle T^a_a\rangle = a_1\langle R^{(2)}\rangle$ for some constant of proprtionality $a_1$. The constant can be determined by using conformal conformal field theory techniques. It turns out to be $a_1 = -\frac{c}{12}$, where $c$ is the central charge of conformal field theory on the worldsheet. In the present case, $c = D - 26$. Then

$$\delta Z = \int d^2\sigma\sqrt{g} \quad \frac{26-D}{24\pi}\delta\omega e^{2\omega}\langle R^{(2)}\rangle \tag{5}$$

This vanishes for $D = 26$. The critical string theory in $D = 26$ is the standard bosonic string theory.

One thing that can be done to the variation (5) is to use the non-anomalous invariance under volume preserving diffeomorphisms to fix the metric to be $g_{ab} = e^{2\omega}\delta_{ab}$. In this conformal gauge the Ricci scalar is given by $R^{(2)} = -2e^{-2\omega}\delta^{ab}\partial_a\partial b\omega$. Inserting this gives

$$\delta Z = \int d^2\sigma\sqrt{g} \quad \frac{D-26}{24\pi}\langle\delta\omega\delta^{ab}\partial_a\partial_b\omega\rangle \tag{6}$$

Outside the critical dimension $D = 26$ the variation of the partition function can still be made to vanish if a term is added to the action such that its variation (6) will cancel. Therefore a quantum conformally invariant theory is defined by the action

$$S_{conformal} = S + \frac{26-D}{24\pi}\int d^2\sigma\sqrt{g}\delta^{ab}\partial_a\phi\partial_b\phi$$

The new field $\phi$ is interpreted as a new dimension. Under conformal transformations it has the transformation property $\phi \to \phi - \omega$.

This procedure can be repeated for a $D$ dimensional string theory with a cosmological constant $\mu$ on the worldsheet. Carrying out this procedure in a similar way gives the action [5]

$$S_{conformal}(\mu) = \int d^2\sigma\sqrt{\hat{g}} \quad \hat{g}^{ab}\partial_a X^\mu\partial_b X_\mu + \mu e^{2\phi} + \frac{26-D}{3}\left(\hat{g}^{ab}\partial_a\phi\partial_b + R^{(2)}\phi\right) \tag{7}$$

Here the notation $\hat{g}_{ab}$ indicates that the metric for this action is taken at a scale factor appropriate to the value of the field embedding field $\phi$.

In fact, the procedure outlined here is equivalent to fixing the conformal symmetry of the non-critical string by the Faddeev Popov method. The non-critical and critical strings should be considered to be the same theory.

The for $D = 1$ the $2D$ field theory that is obtained by introducing the mode $\phi$ is known as the "Liouville Field Theory" and describes a model of two dimensional gravity [6]. Accordingly, the coordinate $\phi$ will be known as the "Liouville mode".

The general takeaway of this section should be that while $D$ dimensional noncritical string theories are not proper string theories, they can be interpreted as $D + 1$ dimensional critical string theories. In particular, the 1 dimensional string theory with cosmological constant $\mu$ which the matrix model was constructed from con be interpreted as the critical, two dimensional string theory defined by the action (7) for $D = 1$.

## 3.2 More Conformal Strings - The Linear Dilaton

In order to interpret the matrix model calculations it will be necessary to understand the theory with action (7) at $D = 1$ in a bit more depth. The aim will be to understand what kind of spacetime it describes.

To this end, introduce another way of getting confromally invariant string theories. The action of a string theory necessarily includes some spacetime background. In the action of (4) the only background was the metric field $G_{\mu\nu}$, implictly present through the contraction of the indices of the embeddings $X^\mu$. In general a string theory could propagate in a background described by any number of fields on the target spacetime [1]. In particular, As well as the metric

[1] These fields can be identified with modes of the string, it is not the case that the quantum field theories of the background fields are independent of the string theory.

the background is often taken to include a Dilaton field $\Phi$ and an antisymmetric tensor field $B_{\mu\nu}$.

$\beta$ functions can be derived for the background fields. If the $\beta$ functions for the background fields all vanish then the string theory on this background enjoys conformal symmetry. For the fields $G_{\mu\nu}, B_{\mu,\nu}$ and $\Phi$ the $\beta$ functions are [5][2]

$$\beta_\Phi = \frac{D-26}{6} - \frac{\alpha}{2}\nabla^2\Phi + \alpha\nabla_\rho\Phi\nabla^\rho\Phi - \frac{\alpha}{24}\Phi - \frac{\alpha}{24}H_{\mu\nu\rho}H^{\mu\nu\rho}$$

$$\beta^G_{\mu\nu} = \alpha R_{\mu\nu} + 2\alpha\nabla_\mu\nabla_\nu\Phi - \frac{\alpha}{4}H_{\mu\rho\sigma}H^{\rho\sigma}_\nu$$

$$\beta^B_{\mu\nu} = -\frac{\alpha}{2}\nabla^\rho H_{\rho\mu\nu} + \alpha\nabla^\rho\Phi H_{\rho\mu\nu}$$

Here $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$ is the field strength associated with $B_{\mu\nu}$.

A solution to these equations in any spacetime dimension $d$ is the linear dilaton background, given by

$$G_{\mu\nu} = \eta_{\mu\nu} \qquad B_{\mu\nu} = 0 \qquad \Phi = V_\mu X^\mu$$

Where $V_\mu$ is a vector satisfying $V_\mu V^\mu = \frac{26-d}{6}$. For definiteness, take $V^\mu = \sqrt{\frac{26-d}{6}}\delta^\mu_1$.

The appropriate worldsheet action for strings propagating in the linear dilaton spacetime is

$$S_{LinearDilaton} = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}g^{ab}\eta_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + R^{(2)}\sqrt{\frac{26-d}{6}}X^1 + T_0 e^{\left(\left(\frac{26-d}{6}\right)^{\frac{1}{2}} - \left(\frac{2-d}{6}\right)^{\frac{1}{2}}\right)X^1}$$

Compare the the linear dilaton action for $d = 2$ to (7) for $D = 1$:

$$S_{LinearDilaton} = \frac{1}{4\pi}\int d^2\sigma\sqrt{g}g^{ab}\eta_{\mu\nu}\partial_a X^\mu\partial_b X^\nu + R^{(2)}2X^1 + T_0 e^{2X^1}$$

$$S_{conformal}(\mu) = \int d^2\sigma\sqrt{\hat{g}} \quad \hat{g}^{ab}\partial_a X^\mu\partial_b X_\mu + \mu e^{2\phi} + \frac{25}{3}\left(\hat{g}^{ab}\partial_a\phi\partial_b + R^{(2)}\phi\right)$$

The two theories are certainly very similar. There isn't a rescaling that makes every factor agree, but quantum effects [6] in the transformation of the measure in the partition function allow one to transform one theory into the other.

## 3.3 Physics in the Linear Dilaton Background

Understanding the $d = 2$ linear dilaton background will provide understanding of the spacetime that will be studied through the matrix model. The two important effects that will be explained are the coordinate dependence of the string coupling constant and the tendency of the system to remain in a region where the theory is weakly coupled.

### 3.3.1 The String Coupling Constant

Consider the partition function for the $d = 2$ linear dilaton background.

$$Z = \int \sum_{\text{topologies}} \mathcal{D}X\mathcal{D}e^{-S}$$

[2] Technically these are the $\beta$ functions to first order in the string tension $\alpha$. This parameter has been set to 1 thus far, for simplicity.

If the action is written as $S = S_0 + \frac{1}{4\pi}\int d^2\sigma\sqrt{g}R^{(2)}2X^1$

On a 2 dimensional manifold the integral of the Ricci scalar just gives the Euler characteristic of the manifold. Therefore, near a given embedding coordinate $\hat{X}^1$ on the target space,

$$e^{-\frac{1}{4\pi}\int d^2\sigma\sqrt{g}R^{(2)}2X^1} \sim e^{(2h-2)2\hat{X}^1}$$

From this, the string coupling constant $g_s$ in the region of an embedding coordinate $\hat{X}^1$ can be seen to be

$$g_s \sim e^{2\hat{X}^1}$$

Therefore the string coupling constant depends on the location of the string along the dilaton direction. The coordinate dependence of the string coupling constant leads to modifications of the mass spectrum of the theory, among other effects [7]

### 3.3.2 The Liouville Wall

The terms $T_0 e^{2X^1}$ and $\mu e^{2\phi}$ in the linear dilaton and Lioville theories both lead to much the same effect. Both terms provide an exponentially increasing potential as $X^1$ or $\phi$ increases.

The large potential keeps the string at small values of $X^1$ or $\phi$. As discussion of the previous section implies that the string coupling will be small when $X^1$ or $\phi$ are small. Therefore it seems that the physics of the Liouville/linear dilaton field theory conspires to keep the string at weak coupling.

# 4 The Toy Cosmological Model

## 4.1 Matrix Model/Liouville Theory Correspondence

It has now been shown that the string theory with a worldsheet cosmological constant can be both considered as the critical Liouville string theory and computed by considering the matrix model. The conclusion is therefore that information about the Liouville string theory can be obtained by studying the matrix model.

In particular, the partition function of the Liouville string theory is $Z_{Liouville} = -lnZ_{MatrixModel}\big|_{Doublescalinglimit}$. Therefore, if the partition function for the matrix model is known in the double scaling limit then the partition function for the Lioville string theory is easy to work out.

## 4.2 Topology of One Dimensional String Theory

Consider again the one dimensional non critical string with cosmological constant. It is embedded into some target space by the function $X$. How does the topology of the target space of this theory affect the identifications made in the previous sections?

The argument that the $D$ dimensional noncritical string theory is equivalent to to a $D+1$ dimensional critical string theory depended only on the transformations of fields on the string wordsheet under worldsheet conformal transformations. The topology of the target space does not enter into the argument given in section 3.1. therefore the conclusion that the 1 dimensional string theory with cosmological constant can be identified with the two dimensional Liouville string theory holds no matter what the topology of the target space.

The argument in Section 2.1 establishing a relationship between the 1 dimensional theory and the matrix model only depends on the topology of the target space through the definition of the Green's functions for the matrix model and the discretised string theory. If the string and triangulation Green's functions of a given topology are in the same universality class then the argument still holds.

It therefore seems that there is a fair bit of leeway in choosing a target space for the 1D map $X$. If a one dimensional space $\mathcal{M}$ is chosen as the target space of the 1 dimensional theory then this modifies both the matrix model and the Liouville string theory.

For the matrix model, the topology of $\mathcal{M}$ will manifest itself in the definition of the Green's functions for the triangulation. They must obey the same constraints as the string worldsheet Green's functions, otherwise the identification of the matrix free energy (1) and the discretised worldsheet action (2) cannot be made.

In the Louville string theory the embedding $X$ is just one of the embedding maps of the Liouville string theory. The string theory target space is just $\mathcal{M} \times$ Liouville.

The conclusion is that it is relatively easy to change the topology of the target space of the matrix model and its associated Liouville string theory. Therefore, it is possible that the matrix model and its assosiated string theory could be placed on a singular space. This would allow the matrix model to be used as a tool to study the propagation of the Liouville string theory on a singular space. In particular the theories could be placed on an Orbifold.

## 4.3 Orbifolds

Before attempting to define a matrix model on an orbifold some facts about orbifolds are reviewed [5], [8], [9]. An orbifold is obtained by taking the quotient of a manifold by a suitably defined group action. In the case that the group action is nontrivial everywhere then the resulting space will be a manifold. If on the other hand there are points where the action of the group is trivial then the resulting spacetime is singular at these points and is an orbifold.

### 4.3.1 Some Definitions

Given some group $\mathcal{G}$ and a manifold $\mathcal{M}$, an action of the group on the manifold is given by a map

$$\varphi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$$

such that

1. $\varphi(\mathbb{1}, x) = x$
2. $\varphi(g \cdot h, x) = \varphi(g, \varphi(h, x))$

These conditions mean that the identity of the group leaves all points alone and that the group action reproduces the compatibility of the underlying group multiplication operation. Condition 2. also allows a simplification of notation. The map $\varphi$ can be dropped and the map can simply be written $\varphi(g, x) = g \cdot x$.

The isotropy subgroup at a point $x$ on the manifold is the subgroup of $\mathcal{G}$ such that $gx = x$. It is denoted $\mathcal{G}_x$. When the isotropy subgroup only contains $\mathbb{1}$ then the group action is called free at $x$. When the isotropy subgroup is trivial everywhere then the group action is called free.

At a given point $x$ the set of all points $y \in \mathcal{M}$ such that $y = g \cdot x$ for some $g \in \mathcal{G}$ is called the orbit of $x$. It is denoted $\mathcal{G}(x)$.

An orbifold can then be defined as $\mathcal{M}/\mathcal{G}$, where all points in the same group orbit are identified. Any point where the isotropy subgroup is nontrivial is called a fixed point of the orbifold.

### 4.3.2 Fixed Points of an Orbifold

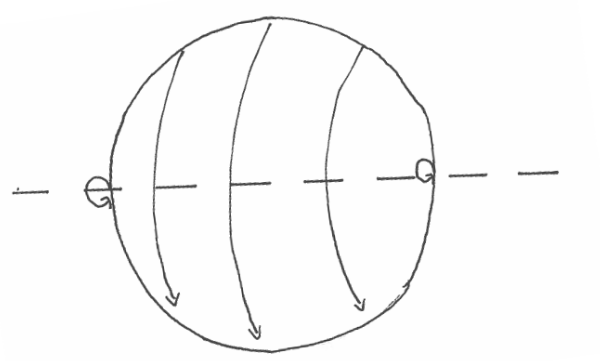
Fixed points of an orbifold are physically interesting because they correspond to singularities. To see why this is, consider the complex plane $\mathbb{C}$. Elements of $\mathbb{C}$ can be written in the form $z = re^{i\theta}$.

To define a group action on $\mathbb{C}$ use the group $\mathbb{Z}_n$. This is the group of integers mod $n$, where the group operation is addition and any numbers that differ from each other by an integer multiple of $n$ are identified. The elements of this group can be taken to be the integers between 0 and $n-1$. 0 is the identity of the group.

Define the group action for an element $k \in \mathbb{Z}_n$ to be $k \cdot re^{i\theta} = re^{i\theta + i2\pi\frac{k}{n}\pi}$. This satisfies the two conditions for a group action as

1. $0 \cdot re^{i\theta} = re^{i\theta+0} = re^{i\theta}$
2. $(k+l) \cdot re^{i\theta} = re^{i\theta + i2\pi\frac{k}{n} + i2\pi\frac{l}{n}} = k \cdot re^{i\theta + i2\pi\frac{l}{n}} = k \cdot \left(l \cdot re^{i\theta}\right)$

Fig. 4: The group action of $\mathbb{Z}_2$



The orbifold formed by quotienting $\mathbb{C}$ by the orbits of this action is a cone. A point labelled by angle $\theta$ is identified with all points $\theta + \frac{2\pi k}{n}$, for $k \in \mathbb{Z}_n$. There is a single point where the group action is not free at the origin of the complex plane, where $r = 0$.

$$k \cdot 0 = 0 \quad e^{i2\pi \frac{k}{n}} = 0$$

Now consider travelling around the orbifold singularity by increasing $\theta$. Due to the orbifolding procedure, $\theta$ is the same angular point as $\theta + \frac{2\pi i}{n}$. Therefore, after travelling an angular distance of $\frac{2\pi}{n}$ with fixed $r$, one arrives back at the starting position. This means that the orbifold fixed point is a singularity of curvature providing an angular deficit of $\Delta\phi = \frac{n-1}{n}2\pi$.

### 4.3.3 The Orbifold $\mathcal{S}^1/\mathbb{Z}_2$

A canonical example of an orbifold is $\mathcal{S}^1/\mathbb{Z}_2$. As the goal of this thesis is to investigate the matrix model defined on this orbifold it will be worthwhile to enumerate some of its properties here.

The group $\mathbb{Z}_2$ consists of two elements, the identity $\mathbb{1}$ and an element $z$. The only nontrivial group product is $z \cdot z = \mathbb{1}$. An action of this group on the circle $\mathcal{S}^1$ is given by $z \cdot x = -x$, where $-x$ is the reflection of $x$ across an axis of symmetry bisecting the circleE. To give an expression for the group action in coordinates, consider the open set containing all points of the circle but one. A coordinate chart for this open set is given by mapping it onto the interval $(-\pi R, \pi R)$. In this coordinate chart the group action of $z$ is just $z \cdot x = -x$. [3]

Quotienting $\mathcal{S}^1$ by the orbits of this group action then corresponds to squashing the circle into a line. There are two fixed points of the action where the axis of symmetry bisects the circle, as at these points $z \cdot x = -x = x$, the point is its own reflection. The orbifold $\mathcal{S}^1/\mathbb{Z}_2$ therefore looks like a line interval with a singularity at either end.

## 4.4 String Theory on Orbifolds

Theories of point particles are not well defined on singular spaces. For example, the Schwarzschild solution of general relativity contains a singularity at the origin. Even at the classical level the Riemann tensor diverges at the singularity meaning that this is not really a well defined point of the spacetime. This leads to the solution being geodesically incomplete: Geodesics which lead

[3] Note that the triviality of this statement is only a consequence of the choice of chart. If the chart had instead mapped onto the interval $(0, 2\pi R$ then the group action in these coordinates would be $z \cdot x = 2\pi R - x$

into the singularity end abruptly after a finite amount of proper time, which means that nothing can really be said about what happens to a particle that meets the singularity.

It turns out that string theory is not ill defined on orbifolds. The singularities of an orbifold space present no impediment to a well defined theory.

### 4.4.1 Strings on Circles

First analyse the string embedded into a circle. As a starting point, consider the standard mode expansion of the embedding function $X^\mu(\sigma)$ of the closed string into its worldsheet modes.

$$X^\mu = x^\mu + p^\mu \tau + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n} e^{-in(\tau-\sigma)} + \frac{\tilde{\alpha}_n^\mu}{n} e^{-in(\tau+\sigma)}$$

This mode expansion is obtained by demanding periodicity in the coordinate $\sigma$, $X^\mu(\sigma + 2\pi R) = X^\mu(\sigma)$, as appropriate for a closed string.

If the $p$th dimension of spacetime is taken to be a circle of radius $R$ then the string needs only satisfy periodic boundary conditions up to a translation of the embedding field by $2\pi m R$, $m \in \mathbb{Z}$, as this is an equivalent embedding. Furthermore, for embedding into a circle the $p$th momentum $p^p$ will be quantised, $p^p = \frac{n}{R}$, $n \in \mathbb{Z}$. The appropriate mode expansion for the $p$th mode will then be

$$X^p = x^p + \frac{n}{R}\tau + mR\sigma + \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{\alpha_n^p}{n} e^{-in(\tau-\sigma)} + \frac{\tilde{\alpha}_n^p}{n} e^{-in(\tau+\sigma)}$$

The term proportional to $\sigma$ leads to the new boundary conditions. $m$ is called the winding number and indicates describes how many times the string wraps around the compact dimension. The key thing to note is that the topology of the embedding dimension has manifested itself in a change of boundary conditions for the embedding field.

### 4.4.2 Strings on $\mathcal{S}^1/\mathbb{Z}_2$

Now suppose the $p$th mode is embedded into $\mathcal{S}^1/\mathbb{Z}_2$. There are two effects. Firstly, the spectrum of the theory will be restricted to states that are invariant under the action of $z$. If this were not the case then the state would not have a well defined value on the orbifold.

The second effect is that the string boundary conditions change. For the circle, the topology of the space meant that the closed string need only be periodic up to a translation by $2\pi m R$, as both $X^p$ and $X^p + 2\pi m R$ are the same point in the target space. These states were labelled by the winding number $m$. Now, because $-X^p$ is identified with $X^p$ the closed string need only be periodic up to some combination of translations by $2\pi R$ and reflections. There should be states such that

$$X^p(\sigma + 2\pi) = -X^p(\sigma) + 2\pi m R$$

These states are called the "twisted states".

The mode expansion for a twisted state is

$$X^p = x^p + \frac{i}{\sqrt{2}} \sum_{n} \frac{\alpha^p_{\frac{2n+1}{2}}}{\frac{2n+1}{2}} e^{-i\frac{2n+1}{2}(\tau-\sigma)} + \frac{\tilde{\alpha}^p_{\frac{2n+1}{2}}}{\frac{2n+1}{2}} e^{-i\frac{2n+1}{2}(\tau+\sigma)}$$

The constant mode $x^p$ may only be one of the fixed points where $x^p = -x^p$. Thus the twisted states provide a new family of half integer modes that are pinned to the fixed points under the orbifold action.

Despite the singularities of the target space, all the embedding functions remain well defined. Therefore the theory can still be said to be well defined even on the singular space.

## 4.5 Defining the Toy Model

As matrix models are easy to deal with they make good candidates for toy models. For example, a matrix model black hole was studied in [3]. The proposed toy cosmological model is the Liouville string theory with the target space of $X$ as $\mathcal{S}^1/\mathbb{Z}_2$. The orbifold singularities are conjectured to become cosmological singularities. A big bang/ big crunch universe was constructed for field theory in [10] and thge propagation of strings in the presence of an orbifold singularity interpreted as a big bang was studied in [7], [11].

For this to be a useful toy model the theory needs to have a corresponding matrix model. In Section 2.1 it was argued that the only sensitivity of the triangulations on the topology of the target space of $X$ was through matching the boundary conditions of the propagators in (1) and (2). The discrete propagators on the circle are [12]

$$G_{String;circle}(t-t') = \sum_{m=-\infty}^{m=\infty} e^{-(t-t'+m2\pi R)^2} \qquad G_{Triangulation;circle}(t-t') = \sum_{m=-\infty}^{m=\infty} e^{-|t-t'+m2\pi R|}$$

On the orbifold the points $t$ and $-t$ are identified. Hence the discrete propagator must be invariant under this identification. Therefore, the discrete propagators for the orbifold should be like

$$G_{String;orbifold}(t,t') \sim G_{String}(t-t') + G_{String}(t+t')$$

$$G_{Triangulation;orbifold}(t,t') \sim G_{Triangulation;circle}(t-t') + G_{Triangulation;circle}(t+t')$$

As these propagators are just sums of propagators in the same universality class it follows that the orbifold propagators are in the same universality class. Therefore there is a matrix model corresponding to the toy model.

## 4.6 Some Issues Surrounding the Toy Model

As stated in the introduction the idea is to use this the Liouville string theory as a low dimensional prototype of a big bang to big crunch universe. The hope is that the matrix model will allow string amplitudes to be calculated to all orders. However, there are some potential pitfalls that could stop this being achieved.

### 4.6.1 String Coupling

If the strings become strongly coupled near the orbifold singularity this complicates the model. The matrix model in the double scaling limit reproduces only perturbative string theory, which breaks down when the strings become strongly coupled. However, there is reason to suspect that this will not be an issue. In the Louville string theory on ordinary spaces the string coupling goes like $g_s \sim e^{2\phi}$. The Liouville wall keeps the string in the region where the string is weakly coupled. As already observed the Liouville mode does not appear to be particularly sensitive to the topology of the target space of $X$. It appears reasonable to guess that on the orbifold the string coupling will still go like $g_s \sim e^{2\phi}$. Thus even at the orbifold singularities the Liouville wall keeps the string weakly coupled [7].

If this is not the case, evidence that the string is becoming strongly coupled may be found when considering matrix model calculations. If these calculations diverge this suggests that the perturbation theory has become invalid and the string coupling has become strong.

If this turns out to be the case it may be possible that the Liouville string theory can be investigated using the matrix model. As remarked in Section 2.2 non perturbatively valid matrix models can be defined where nonperturbative effects corresond eigenvalue tunneling.

### 4.6.2 The Role of the Twisted States

As noted in Section 4.4.2 there are twisted states that do not move far from the orbifold singularities on $\mathcal{S}^1/\mathbb{Z}_2$. It would be interesting to know what states these correspond to in the matrix model. The matrix model could be used to shed light on how the twisted states resolve the singularity.

### 4.6.3 Interpretation of $\mathcal{S}^1/\mathbb{Z}_2$ as time

The toy model is currently defined as a Euclidean string theory. In order for an identification of the orbifold singularities to be made as cosmological singularities it will be necessary perform a Wick rotation so that the signature of the target space is Lorentzian. This allows the orbifold to be interpreted as the time of the theory. One singularity will then correspond to the big bang in the past, another will correspond to the big crunch in the future. Another worry is that there may be closed timelike curves present in the toy model. It will be necessary to check that this is not the case.

The remainder of this thesis will be spent attempting to calculate the matrix model partiton function on the orbifold. At the end of the thesis whether the obtained results shed any light on the questions raised here will be discussed.

# 5 Partition Functions

In this section some mathematical techniques for the evaluation of partition functions are introduced. They will be used to calculate matrix model partition functions in later sections.

## 5.1 Heat Kernel Methods

An important technique for evaluating the matrix model partition function on nontrivial topologies is the heat kernel method.

### 5.1.1 Some Scalar Identities

Begin with some identities for scalar functions. The heat kernel method relies on generalising these identities so that they also apply to matrices.

First, consider the representation for the inverse of $x$.

$$x^{-1} = \frac{1}{x} = \int_0^\infty d\tau e^{-x\tau} \tag{8}$$

By integrating this expression an integral representation for the logarithm is obtained.

$$\begin{aligned} ln(x) &= \int dx \frac{1}{x} = \int dx \int_0^\infty d\tau e^{-x\tau} \\ &= -\int_\epsilon^\infty \frac{d\tau}{\tau} e^{-x\tau} \end{aligned} \tag{9}$$

The lower limit of the integration has been changed to $\epsilon$ to indicate that care must be taken to ensure this function is well defined. If the lower limit of the $\tau$ integral was left at 0 then the expression would not be well defined owing to the singular point of the integrand here. Some prescription must be decided on to take care of this infinity.

### 5.1.2 Generalising to Matrices

First, generalise expression (8). Consider the same integral, but now take $x \to M$, $M$ is an invertible square matrix.

$$\begin{aligned} \int_0^\infty d\tau e^{-\tau M} &= \int_0^\infty d\tau \sum_{n=0}^\infty \frac{(-\tau M)^n}{n!} \\ &= -\sum_{n=0}^\infty M^{-1} \frac{(-\tau M)^{n+1}}{(n+1)n!} \Big|_0^\infty + 1 - 1 \\ &= -M^{-1} \sum_{n=0}^\infty \frac{(-\tau M)^n}{n!} \Big|_0^\infty = -M^{-1} \left( e^{-\tau M} \Big|_0^\infty \right) \\ &= M^{-1} \end{aligned}$$

Going from the second line to the third line $+1/-1$ is included at the upper/lower evaluation point and the summation variable is shifted to include it as the $n = 0$ term.

The generalisation of the second equation (9) is:

$$ln(|M|) = -Tr \int_\epsilon^\infty \frac{d\tau}{\tau} e^{-\tau M}$$

Where $M$ is a Hermitian, positive definite matrix with eigenvalues $m_i$. To prove this, use that M can be diagonalised by a unitary transformation and the cyclicity of the trace to get

$$-Tr\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}e^{-\tau M}$$

$$= -Tr\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}e^{-\tau U M_{diagonal}U^{\dagger}}$$

$$= -Tr\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}Ue^{-\tau M_{diagonal}}U^{\dagger}$$

$$= -\sum_i\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}e^{-\tau m_i}$$

Where in the last line the cyclicity of the trace was used to remove the unitary matrices and the trace was expressed as a sum. Now, at this point expression (9) can be used to get

$$\sum_i\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}e^{-\tau m_i} = \sum_i ln(m_i) = ln(\prod_i m_i) = ln(|M|) \tag{10}$$

So that $ln(|M|) = Tr\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}e^{-\tau M}$ as required.

### 5.1.3 Application to Partition Functions

Suppose we have a Gaussian theory of some fields, denoted $F(\mathbf{n})$, where $\mathbf{n}$ denotes the independent quantum numbers of the theory. The partition function for the Gaussian theory can be written as

$$Z = \int \mathcal{D}F \quad exp\left(-\sum_{\mathbf{n},\mathbf{m}}F^{\dagger}(\mathbf{n})\hat{\mathcal{O}}(\mathbf{n},\mathbf{m})F(\mathbf{m})\right)$$

This integral is well defined when $\hat{\mathcal{O}}(\mathbf{n},\mathbf{m})$ is Hermitian and positive definite. When the field $F$ is Hermitian and valued in $\mathbb{C}$ the result of the functional integral is

$$Z = |\hat{\mathcal{O}}|^{-\frac{1}{2}}$$

Where the determinant is considered as the determinant of $\hat{\mathcal{O}}(\mathbf{n},\mathbf{m})$ considered as a linear operator on the Hilbert space of states labelled by the quantum numbers of the field, i.e. states $|\mathbf{n}\rangle$.

(10) can be immediately applied to obtain that

$$-\frac{1}{2}ln(\hat{\mathcal{O}}) = Tr\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}e^{-\tau\hat{\mathcal{O}}}$$

$$= \int_{\epsilon}^{\infty}\frac{d\tau}{\tau}\sum_{\mathbf{n}}\langle\mathbf{n}|\, e^{-\tau\hat{\mathcal{O}}}\,|\mathbf{n}\rangle$$

Where the trace has been commuted with the integral. This useful because $\langle\mathbf{m}|\, e^{-\tau\hat{\mathcal{O}}}\,|\mathbf{n}\rangle$ is the propagator. It describes the evolution of a quantum state $|\mathbf{n}\rangle$ to the state $|\mathbf{m}\rangle$ after Euclidean time $\tau$. The Hamiltonian is $\hat{\mathcal{O}}$.

### 5.1.4 Modification of Traces by Identifications

An important complication arises when evaluating traces on topologically nontrivial spaces. The trace of an operator is defined as

$$Tr\mathcal{O} = \sum_{\{q\}} \langle q| \mathcal{O} |q\rangle$$

Where $\{q\}$ is the set of all quantum numbers. However, it may be the case that some topological condition means that some of the quantum numbers are identified. A concrete example of this is when the one looks at states labelled by a position $x \in \mathcal{S}^1$. Then the identification $x \approx x + 2\pi R$ must be accounted for. This means that the trace of an operator over these states should be

$$Tr\mathcal{O} = \sum_{n=-\infty}^{n=\infty} \int_0^{2\pi R} dx \, \langle x| \mathcal{O} |x + 2\pi nR\rangle$$

This modification is necessary to ensure that every way in which the quantum numbers on the left match the quantum numbers on the right is included.

## 5.2 Twisted Partition Functions

### 5.2.1 Partition Function in a Representation

The twisted partition function is a modification of the partition function for a field that transforms under some gauge group $\mathcal{G}$. The idea is to "twist" the final state in the partition function by some matrix $\Omega \in \mathcal{G}$. For a matrix valued field, as in MQM, the precise definition of the twisted partition function is [13]

$$Z(\Omega) = Tr\left(e^{-\beta H}\hat{\Theta(\Omega)}\right)$$

$\hat{\Theta(\Omega)}$ is the twist operator: Its action on state of the matrix valued field is

$$\hat{\Theta(\Omega)}\, |M\rangle = \left|\Omega M \Omega^\dagger\right\rangle$$

The twisted partition function is important because it can be used to calculate the contribution to the full partition function from states $|M\rangle$ in a particular representation of the gauge group [4]. This works because the twisted partition function can be decomposed as

$$Z(\Omega) = \sum_R d_R Z_R \chi_R(\Omega)$$

Here $R$ labels representations, $d_R$ is a numerical factor representing the size of the particular representation, $\chi_R(\Omega)$ is the character of the representation of the twist matrix $\Omega$ in the representation R, i.e. the trace of the representation of $\Omega$. In particular, note that $Z(\mathbb{1}) = \sum_R d_R Z_R \chi_R(\mathbb{1})$ is just an expression for the standard partition function in terms of the partition functions in each representation.

The characters satisfy the orthogonality property

$$\int d\Omega \chi_{R_1}(\Omega^\dagger)\chi_{R_2}(\Omega \cdot \omega) = \delta_{R_1 R_2}\chi_{R_1}(\omega)$$

Note also that $\chi_R(\Omega^\dagger) = \overline{\chi_R(\Omega)}$.

[4] Note that the state $|M\rangle$ need not be in the same representation of the gauge group as the actual field $M$ is.

Using the orthogonality property

$$Z_R = \frac{1}{d_R} \int d\Omega \overline{\chi_R}(\Omega) Z(\Omega)$$

So if the twisted partition function is known then the partition function in any particular representation can be obtained.

### 5.2.2 Twists by Gauge Fields

In the previous section the twisted partition function was introduced, which is the partition function for a field which obeys some twisted boundary conditions after propagation by by $\beta$, i.e. $M(t+\beta) = \Omega M(t) \Omega^\dagger$. However, in the case of a the circle (or more generally any manifold of nontrivial topology) it was argued that the effect of a flat gauge field on nontrivial topology can be regarded as modifying the boundary conditions of a gauge covariant field.

Suppose then that the twist $\Omega$ is written in the form $\Omega = e^{\oint_{S^1} A}$, where $A$ is a constant, anti-Hermitian matrix. Then the effect of the twist is precisely the same as if the field $M$ was propagating in the background described by gauge field $A$.

For a given twist $\Omega = e^{\oint_{S^1} A}$:

$$Z(e^{\oint_{S^1} A}) = Tr\left(e^{-\beta H} \Theta(e^{\hat{\oint}_{S^1} A})\right)$$
$$= Tr\left(e^{-\beta H(A)}\right)$$

Where $H(A)$ is the Hamiltonian coupled to the gauge field $A$.

For a constant gauge field the integral over $U$ can also be converted into an integral over $A$.

$$U = e^{\beta A} \Rightarrow dU = ||\beta e^{\beta A}|| dA$$

As U is a unitary matrix, the absolute value of the Jacobian of this transformation is $\beta^N$. The $\beta^N$ can just be absorbed into the normalisation of the measure $dA$ and doing this gives $dU = dA$. Therefore, to evaluate a twisted partition function it is possible to remove the twist $\Omega$ by coupling the problem to a gauge field $A$ such that $\Omega = e^{\oint A}$. the integral over the twists $U$ can then be replaced by in integral over the possible gauge fields $A$. The expression $e^{\oint A}$ is known as the "holonomy" of the gauge connection $A$. The integral is therefore over holonomies.

# 6 The Circle Partition Function

Now the mathematical tools that were developed in the previous sections can be applied in order to calculate the partition function for the matrix model on the circle.

## 6.1 Embedding Triangulations into $\mathcal{S}^1$

One tricky issue about the embedding of the matrix model into $\mathcal{S}^1$ is the potential presence of vortices on the triangulation. The reason these arise is that the Green's function of the discrete theory on the circle now becomes [12]

$$G(t-t') = \sum_{m=-\infty}^{\infty} e^{|t-t'+2\pi Rm|} \tag{11}$$

This new form is invariant under $t \to t+2\pi R$. However, consider travelling around a face in some triangulation and looking at how the vertex coordinates change. The new propagator (11) implies that as each edge is traversed the coordinate picks up an additional $2\pi m_i R$ at the $i$th edge, i.e. after travelling around a loop the vertex coordinate $x_0$ shifts by $x_0 \to x_0 + 2\pi \left(\sum_i m_i\right) R$. This indicates that there is vorticity in the theory.

This vorticity should be removed from the theory. This is because string theory defined on a circle has the property of T duality. This means the partition function is invariant under the interchange $R \to \frac{1}{R}$. Vortex excitations explicitly break $T$ duality and so if the matrix model is to accurately reproduce the string theory the vortex excitations should be removed.

The vortex excitations should correspond states of the matrix model that are in representations of the gauge group other than the singlet. In [13] the energy cost for non singlet excitations was estimated and found to be very high.

The conclusion is that only the contribution of the singlet sector to the partition function should be included in the matrix model.

## 6.2 The Matrix Model for $\mathcal{S}^1$

The matrix model for$\mathcal{S}^1$ is discussed in [13].

As discussed in Section 2.2.2, in the double scaling limit all terms in the matrix model potential proportional to $M^n$, $n \geq 3$ are irrelevant in the double scaling limit. The action for the matrix model is therefore

$$\mathcal{S} = \int_0^{2\pi R} dt \frac{1}{2}\left(\frac{dM}{dt}\right)^2 + \frac{1}{2}M^2$$

Periodic boundary conditions now apply to the field: $M(x) = M(x + 2\pi R)$.

To apply the Heat Kernel method to the calculation of this partition function, one needs to work out what the operator $\mathcal{O}(\theta)$ should be when the partition function is written as

### 6.2.1 The Heat Kernel Operator

The action is

$$\mathcal{S} = Tr\left(\int dt \frac{1}{2} DMDM + \frac{\omega^2}{2}M^2\right)$$

For the adjoint representation, $DM = \frac{dM}{dx} + [A, M]$. Therefore, after diagonalising the gauge field the integral of the kinetic term will be

$$\int dx \sum_{i,j} \frac{1}{2}(DM)_ij(DM)_{ji} \tag{12}$$

$$= \int dx \sum_{i,j} \frac{1}{2} \left( \frac{dM_{ij}}{dx} + [\theta, M]_{ij} \right) \left( \frac{dM_{ji}}{dx} + [\theta, M]_{ji} \right)$$

The commutators can be dealt with as follows:

$$[A, M]_{i,j} = \sum_k A_{ik} M_{kj} - M_{ik} A_{kj}$$

$$= \sum_k i\theta_i \delta_{ik} M_{kj} - M_{ik} i\theta_k \delta_{kj}$$

$$= i\theta_i M_{ij} - i\theta_j M_{ij} = i\theta_{ij} M_{ij}$$

Where $\theta_{ij} = \theta_i - \theta_j$.

Using this in the expression for the kinetic term (12) gives

$$\int dx \sum_{i,j} \frac{1}{2} \left( \frac{dM_{ij}}{dx} + i\theta_{ij} M_{ij} \right) \left( \frac{dM_{ji}}{dx} + i\theta_{ji} M_{ji} \right)$$

$$= \int dx \sum_{i,j} \frac{1}{2} \left( \frac{dM_{ij}}{dx} \frac{dM_{ji}}{dx} + i\theta_{ij} M_{ij} \frac{dM_{ji}}{dx} + \frac{dM_{ij}}{dx} \theta_{ji} M_{ji} - \theta_{ij} M_{ij} \theta_{ji} M_{ji} \right)$$

Integrating by parts so that all the derivatives act on the right hand matrix $M$ gives

$$\int dx \sum_{i,j} \frac{1}{2} \left( -M_{ij} \frac{d^2}{dx^2} M_{ji} + i\theta_{ij} M_{ij} \frac{dM_{ji}}{dx} - M_{ij} i\theta_{ji} \frac{dM_{ji}}{dx} - \theta_{ij} M_{ij} \theta_{ji} M_{ji} \right)$$

$$= \int dx \sum_{i,j} \frac{1}{2} \left( -M_{ij} \frac{d^2}{dx^2} M_{ji} + 2i\theta_{ij} M_{ij} \frac{dM_{ji}}{dx} + \theta_{ij}^2 M_{ij} M_{ji} \right)$$

$$= \int dx \sum_{i,j} -M_{ij} \frac{1}{2} \left( \frac{d}{dx} - i\theta_{ij} \right)^2 M_{ji}$$

$$= \sum_{i,j,k,l} \int dx \int dx' M_{ij} \left[ -\delta_{il} \delta_{jk} \frac{1}{2} \left( \frac{d}{dx} - i\theta_{ij} \right)^2 \right] M_{kl}$$

From the last expression it can be concluded that the operator that is required for the heat kernel method is

$$\mathcal{O}_{ij;kl} = -\delta_{il} \delta_{jk} \left( \frac{1}{2} \left( \frac{d}{dx} - i\theta_{ij} \right)^2 - \frac{1}{2} \omega^2 \right) \tag{13}$$

Where the $\omega^2$ term is the contribution from the $\frac{\omega^2}{2}$ term in the Lagrangian.

### 6.2.2 Applying the Heat Kernel Method

Given the operator (13) the partition function for the matrix model on the circle can now be evaluated using the methods of Section 5.1. The expression that needs to be evaluated is

$$\int_{\epsilon}^{\infty} \frac{d\tau}{\tau} Tre^{-\tau\mathcal{O}}$$

As mentioned in Section 5.1 this expression must be renormalised in order to avoid the divergence of the $\tau$ integral. The prescription for doing this will be to take a derivative with respect to $\omega^2$, which brings down a factor of $\tau$ to make the integrand nonsingular. The $\tau$ integral will be evaluated and the limit $\epsilon \to 0$ will be taken. The result will be integrated with respect to $\omega^2$. This potentially introduces an undefined additive constant. This does not matter because ultimately the logarithm of this expression will be taken to obtain a result for the partition function(c.f. (1)). This means that the additive constant corresponds only to an irrelevant overall multiplicative factor, so it will just be ignored.

Writing out the trace in full gives

$$\int_{\epsilon}^{\infty} \frac{d\tau}{\tau} \sum_{i,j} \int_{0}^{2\pi R} dx \sum_{n=-\infty}^{n=\infty} \langle ji, x| \, e^{-\tau\mathcal{O}_{ij;ji}} \, |ij, x + 2\pi nR\rangle$$

$$= \int_{\epsilon}^{\infty} \frac{d\tau}{\tau} \sum_{i,j} \int_{0}^{2\pi R} dx \sum_{n=-\infty}^{n=\infty} \langle ij, x| \, e^{\tau\left(\frac{1}{2}\left(\frac{d}{dx} - i\theta_{ij}\right)^2 - \frac{1}{2}\omega^2\right)} \, |ji, x + 2\pi nR\rangle \tag{14}$$

The reversal of the indices on the bra and the ket arises as the convention has been chosen that $|ij, x\rangle \to \langle ji, x|$ under conjugation. This mimics the fact that the matrix entry conjugate to $M_{ij}$ is $M^*_{ji}$. The inner product is $\langle ij, x|kl, y\rangle = \delta_{il}\delta_{jk}\delta(x - y)$.

It will be easiest to break the calculation up into several steps.

### 6.2.3 Step 1 - Evaluating the Propagator Term

The first step in evaluating this is to calculate

$$\langle ij, x| \, e^{\tau\left(\frac{1}{2}\left(\frac{d}{dx} - i\theta_{ij}\right)^2 - \frac{1}{2}\omega^2\right)} \, |kl, y\rangle$$

Which is similar to a propagator. First note that the derivative operator $\frac{d}{dx}$ is the momentum operator conjugate to $x$ on the this Hilbert space of states. Therefore make the replacement $\frac{d}{dx} \to i\hat{p}$. This gives

$$\langle ij, x| \, e^{-\tau\left(\frac{1}{2}(\hat{p} - \theta_{ij})^2 + \frac{1}{2}\omega^2\right)} \, |kl, y\rangle$$

Next, insert the identity $\mathbb{1} = \sum \int \frac{dp}{2\pi} |mn, p\rangle \langle nm, p|$ to get

$$\sum_{m,n} \int \frac{dp}{2\pi} \langle ij, x| \, e^{-\tau\left(\frac{1}{2}(\hat{p} - \theta_{ij})^2 + \frac{1}{2}\omega^2\right)} \, |mn, p\rangle \, \langle nm, p|kl, y\rangle$$

Use that $\langle ij, x|kl, p\rangle = \delta_{il}\delta jke^{ipx}$ to get

$$\sum_{m,n} \int \frac{dp}{2\pi} \langle ij, x| \, e^{-\tau\left(\frac{1}{2}(p - \theta_{ij})^2 + \frac{1}{2}\omega^2\right)} \, |mn, p\rangle \, \delta_{nl}\delta_{mk}e^{-ipy}$$

$$= \sum_{m,n} \int \frac{dp}{2\pi} \delta_{in}\delta_{jm}e^{ipx} e^{-\tau\left(\frac{1}{2}(p - \theta_{ij})^2 + \frac{1}{2}\omega^2\right)} \delta_{nl}\delta_{mk}e^{-ipy}$$

$$= \int \frac{dp}{2\pi} \delta_{il}\delta_{jk} e^{-\frac{\tau}{2}\left(p^2 - 2p\theta_{ij} + \theta_{ij}^2 + \omega^2 + 2i\frac{(y-x)}{\tau}\right)}$$

The integral can be performed by completing the square in $p$ in the argument of the exponential. Doing this gives

$$p^2 - 2p\theta_{ij} + \theta_{ij}^2 + \omega^2 + 2i\frac{(y-x)}{\tau} \to \left(p - \left(\theta_{ij} + i\frac{x-y}{\tau}\right)\right)^2 + \frac{(x-y)^2}{\tau^2} - 2i\theta_{ij}\frac{x-y}{\tau} + \omega^2$$

The $\theta_{ij}^2$ terms cancel during the completion of the square. Putting this back in and performing the Gaussian integral over $p$ gives

$$\int \frac{dp}{2\pi} \delta_{il}\delta_{jk} e^{-\frac{\tau}{2}\left(\left(p - \left(\theta_{ij} + i\frac{x-y}{\tau}\right)\right)^2 + \frac{(x-y)^2}{\tau^2} - 2i\theta_{ij}\frac{x-y}{\tau} + \omega^2\right)}$$

$$= \delta_{il}\delta_{jk} \frac{1}{2\pi}\sqrt{\frac{2\pi}{\tau}} e^{-\frac{\tau}{2}\left(+\frac{(x-y)^2}{\tau^2} - 2i\theta_{ij}\frac{x-y}{\tau} + \omega^2\right)}$$

$$= \delta_{il}\delta_{jk} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(x-y)^2}{2\tau} + i\theta_{ij}(x-y) - \frac{\omega^2}{2}\tau} \tag{15}$$

### 6.2.4 Step 2 - Poisson Resummation

The next step is to insert (15) into the full expression for the trace (14). In doing this, set $l = i, k = j$ and $y = x + 2\pi Rn$ so that the various quantum numbers take on the appropriate values for the trace. This gives

$$\int_\epsilon^\infty \frac{d\tau}{\tau} \sum_{i,j} \int_0^{2\pi R} dx \sum_{n=-\infty}^{n=\infty} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(2\pi Rn)^2}{2\tau} - i\theta_{ij}(2\pi Rn) - \frac{\omega^2}{2}\tau}$$

Note that the $x$ dependence of the integrand drops out. In order to evaluate this, use the Poisson resummation formula [5]

$$\sum_{n=-\infty}^{\infty} e^{-\pi a n^2 + 2\pi i b n} = a^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{\frac{-\pi(n-b)^2}{a}}$$

The relevant values of $a$ and $b$ are

$$a = \frac{2\pi R^2}{\tau} \qquad b = -R\theta_{ij}$$

The trace can therefore be written

$$\int_\epsilon^\infty \frac{d\tau}{\tau} \sum_{i,j} \int_0^{2\pi R} dx \sum_{n=-\infty}^{n=\infty} \frac{1}{\sqrt{2\pi\tau}} \sqrt{\frac{\tau}{2\pi R^2}} e^{\frac{-\pi(n+R\theta_{ij})^2}{2\pi R^2}\tau - \frac{\omega^2}{2}\tau}$$

$$= \int_\epsilon^\infty \frac{d\tau}{\tau} \sum_{i,j} \int_0^{2\pi R} dx \sum_{n=-\infty}^{n=\infty} \frac{1}{2\pi R} e^{\frac{-(n+R\theta_{ij})^2}{2R^2}\tau - \frac{\omega^2}{2}\tau}$$

$$= \int_\epsilon^\infty \frac{d\tau}{\tau} \sum_{i,j} \sum_{n=-\infty}^{n=\infty} e^{\frac{-(n+R\theta_{ij})^2}{2R^2}\tau - \frac{\omega^2}{2}\tau} \tag{16}$$

Where in the last line the integral over $x$ has been performed.

#### 6.2.5 Step 3 - Renormalising the Integral

Next the procedure for regularising the integral, described at the beginning of this section, will be carried out. Taking the derivative of (16) with respect to $\omega^2$ gives

$$\frac{\partial}{\partial\omega^2}\int_\epsilon^\infty \frac{d\tau}{\tau}\sum_{i,j}\sum_{n=-\infty}^{n=\infty} e^{\frac{-(n+R\theta_{ij})^2}{2R^2}\tau-\frac{\omega^2}{2}\tau}$$

$$= -\frac{1}{2}\int_\epsilon^\infty d\tau\sum_{i,j}\sum_{n=-\infty}^{n=\infty} e^{\frac{-(n+R\theta_{ij})^2}{2R^2}\tau-\frac{\omega^2}{2}\tau}$$

The integral over $\tau$ can now be evaluated and gives

$$-\frac{1}{2}\sum_{i,j}\sum_{n=-\infty}^{n=\infty}\left[\frac{1}{\frac{-(n+R\theta_{ij})^2}{2R^2}-\frac{\omega^2}{2}}e^{\frac{-(n+R\theta_{ij})^2}{2R^2}\tau-\frac{\omega^2}{2}\tau}\right]_\epsilon^\infty$$

$$=\sum_{i,j}\sum_{n=-\infty}^{n=\infty}\left[\frac{1}{\frac{(n+R\theta_{ij})^2}{R^2}+\omega^2}e^{\frac{-(n+R\theta_{ij})^2}{2R^2}\tau-\frac{\omega^2}{2}\tau}\right]_\epsilon^\infty$$

$$=\sum_{i,j}\sum_{n=-\infty}^{n=\infty}\frac{-1}{\frac{(n+R\theta_{ij})^2}{R^2}+\omega^2}e^{\frac{-(n+R\theta_{ij})^2}{2R^2}\epsilon-\frac{\omega^2}{2}\epsilon}$$

Taking the limit $\epsilon\to 0$ gives

$$\sum_{i,j}\sum_{n=-\infty}^{n=\infty}\frac{-1}{\frac{(n+R\theta_{ij})^2}{R^2}+\omega^2} \tag{17}$$

The sum over $n$ is easier to do before evaluating the integral over $\omega^2$, so this will be done in the next step.

#### 6.2.6 Step 4 - Summing Over $n$ and Integrating Over $\omega^2$

Now evaluate the sum over $n$ by using the identity

$$\sum_{n=-\infty}^{\infty}\frac{-1}{a+b(n+c)^2}=\frac{-\pi}{\sqrt{ab}}\left[coth\left(\pi\sqrt{\frac{a}{b}}-i\pi c\right)+coth\left(\pi\sqrt{\frac{a}{b}}+i\pi c\right)\right] \tag{18}$$

This identity can be proven by finding a contour integral whose poles reproduce the sum on the left hand side and deforming the contour.

In the case at hand

$$a=\omega^2 \qquad b=\frac{1}{R^2} \qquad c=R\theta_{ij}$$

Thus using the identity in (17)

$$\sum_{i,j}\sum_{n=-\infty}^{n=\infty}\frac{-1}{\frac{(n+R\theta_{ij})^2}{R^2}+\omega^2}$$

$$=\sum_{i,j}\frac{-\pi R}{\omega}\left[coth\left(\pi\omega R-i\pi R\theta_{ij}\right)+coth\left(\pi\omega R+i\pi R\theta_{ij}\right)\right]$$

This expression should now be integrated over $\omega^2$. The measure of integration is $d\omega^2=2\omega d\omega$.

$$\int d\omega\sum_{i,j}-\pi R\left[\frac{cosh\left(\pi\omega R-i\pi R\theta_{ij}\right)}{sinh\left(\pi\omega R-i\pi R\theta_{ij}\right)}+\frac{cosh\left(\pi\omega R+i\pi R\theta_{ij}\right)}{sinh\left(\pi\omega R+i\pi R\theta_{ij}\right)}\right]$$

$$=\sum_{i,j}-ln\left(sinh\left(\pi\omega R-i\pi R\theta_{ij}\right)\right)-ln\left(sinh\left(\pi\omega R+i\pi R\theta_{ij}\right)\right)$$

### 6.2.7 Step 5 - Some Algebraic Manipulations

It is difficult to perform the final sum over $i$ and $j$ and in fact it is better not to perform it as there is a useful identity that will be applied to the sum later. At this point the best thing to do is to carry out some algebraic manipulations to rewrite the heat kernel trace. In the following, define $q = e^{-2\pi i\omega}$.

$$= \sum_{i,j} -ln\left(sinh\left(\pi\omega R - i\pi R\theta_{ij}\right)\right) - ln\left(sinh\left(\pi\omega R + i\pi R\theta_{ij}\right)\right)$$

$$= \sum_{i,j} -ln\left(\frac{e^{\pi\omega R - i\pi R\theta_{ij}} - e^{-\pi\omega R + i\pi R\theta_{ij}}}{2}\right) - ln\left(\frac{e^{\pi\omega R + i\pi R\theta_{ij}} - e^{-\pi\omega R - i\pi R\theta_{ij}}}{2}\right)$$

$$= \sum_{i,j} -ln\left(q^{-\frac{1}{2}}e^{-i\pi R\theta_{ij}} - q^{\frac{1}{2}}e^{i\pi R\theta_{ij}}\right) - ln\left(q^{-\frac{1}{2}}e^{i\pi R\theta_{ij}} - q^{\frac{1}{2}}e^{-i\pi R\theta_{ij}}\right) + ln(4)$$

$$= \sum_{i,j} ln\left(\frac{q^{-\frac{1}{2}}}{e^{-i\pi R\theta_{ij}} - qei\pi R\theta_{ij}}\right) + ln\left(\frac{q^{-\frac{1}{2}}}{e^{+i\pi R\theta_{ij}} - qe^{-i\pi R\theta_{ij}}}\right) + ln(4)$$

$$= \sum_{i,j} -ln\left(e^{-i\pi R\theta_{ij}}\right) + ln\left(\frac{q^{-\frac{1}{2}}}{1 - qe^{i2\pi R\theta_{ij}}}\right) - ln\left(e^{+i\pi R\theta_{ij}}\right) + ln\left(\frac{q^{-\frac{1}{2}}}{1 - qe^{-i2\pi R\theta_{ij}}}\right) + ln(4)$$

$$= \sum_{i,j} ln\left(\frac{q^{-\frac{1}{2}}}{1 - e^{i2\pi R\theta_{ij}}}\right) + ln\left(-\frac{q^{-\frac{1}{2}}}{1 - qe^{i2\pi R\theta_{ij}}}\right)$$

$$= \sum_{i,j} 2ln\left(\frac{q^{-\frac{1}{2}}}{1 - qe^{i2\pi R\theta_{ij}}}\right)$$

Where in the second to last line the term $ln4$ has been discarded, as it will just lead to an unimportant prefactor. Also, in the second term of that line the exchange has been made $\theta_{ij} \to -\theta_{ji}$. The summed indices of the second term can then be relabelled to bring the expression into its final form.

### 6.2.8 Determining the Twisted Partition Function from the Heat Kernel Trace

Now that the trace is evaluated the twisted partition function can be written down. The central equation of the heat kernel method is (10). Therefore

$$ln\left(Z\left(e^{i2\pi R\theta}\right)\right) = ln|\mathcal{O}|^{-\frac{1}{2}} = -\frac{1}{2}ln|\mathcal{O}| = \frac{1}{2}Tr\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}e^{-\tau\mathcal{O}}$$

$$\Rightarrow Z\left(e^{i2\pi R\theta}\right) = \prod_{i,j}\frac{q^{-\frac{1}{2}}}{1 - qe^{i2\pi R\theta_{ij}}} = \frac{q^{-\frac{N^2}{2}}}{\prod_{i,j} 1 - qe^{i2\pi R\theta_{ij}}}$$

The integral of the twisted partiiton function over twists is evaluated in [13]. The final result is that

$$Z_{singlet}(R) = \frac{q^{\frac{N^2}{2}}}{(1-q)(1-q^2)\cdots(1-q^N)}$$

This is the partition function of fermions on the circle. This is perhaps not unexpected given the analysis of Section 2.2.

# 7 The Matrix Model Partition Function on the Orbifold

To calculate the partition function on the orbifold the same method will be used as was used for the circle. The main steps will be

1. Work out the holonomies appropriate to possible $U(N)$ twists on the orbifold.
2. Restrict the matrix model so it is well defined on the orbifold.
3. Write the Lagrangian in the appropriate gauge background as $\sum M\mathcal{O}(A)M$.
4. Use the heat kernel method to calculate $|\mathcal{O}(A)|^{-\frac{1}{2}}$
5. Integrate over the possible twists.

## 7.1 Some Motivation

On the circle the purpose of projecting onto the singlet representation of the gauge group was to exclude the contribution of vortices from the partition function. This was because vortices would violate T duality. Looking at the string torus partition function indicated that T duality should be a symmetry of the full theory and therefore the matrix model should also respect T duality.

A similar story holds in the case of the orbifold. The torus modular partition function for a a coordinate embedded into $\mathcal{S}^1/\mathbb{Z}_2$ is [5]

$$Z_{orb}(R,\tau) = \frac{1}{2}Z_{tor}(R,\tau) + \left|\frac{\eta(\tau)}{\vartheta_{10}(0,\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_{01}(0,\tau)}\right| + \left|\frac{\eta(\tau)}{\vartheta_{00}(0,\tau)}\right|$$

Where

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) \qquad \vartheta_{00}(0,\tau) = \sum_{n=-\infty}^{\infty} q^{\frac{n^2}{2}}$$

$$\vartheta_{01}(0,\tau) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n^2}{2}} \qquad \vartheta_{10}(0,\tau) = \sum_{n=-\infty}^{\infty} q^{\frac{(n-\frac{1}{2})^2}{2}}$$

And $q = e^{2\pi i\tau}$.

The first term is just half the circle partition function on the torus. The other terms represent the contributions of the twisted states. The only $R$ dependence in the modular partition function arises is in the first term, which is already know to be T dual. The partition function on the orbifold is also expected to be T dual.

The upshot of this is that vortices should also be excluded from the matrix model partition function. Therefore the singlet contribution to the partition function should again be isolated by integrating the twisted partition function over twists.

A second issue is the presence of the orbifold singularities. While string theory is well defined on the orbifold it may be the case that the string becomes strongly coupled near the singularities leading to a breakdown of perturbative string theory. In this case the matrix model can still be applied, but the double scaling limit would not valid as non perturbative effects may become important.

One way to check whether this is an issue is simply to perform the calculation and see if there is any divergence of the matrix model in the double scaling limit or otherwise. If this is not the case it can be deduced that the string coupling remains small. However, one can offer arguments that this should indeed be the case.

The matrix model on the orbifold is dual to the two dimensional critical linear dilaton string theory embedded in $\mathcal{S}^1/\mathcal{Z}_2 \times$ Liouville. In the linear dilaton string theory the string coupling is goes like $e^{\phi}$, where $\phi$ is the Liouville direction. The effect of the large potential in the Liouville direction keeps the system in the small $\phi$ region and consequently keeps the string coupling small. This appears to hold independently of the topology of the other spacetime directions. Therefore, one can conclude that the size of the coupling constant remains small as the Liouville wall keeps the string at weak coupling, no matter what the topology of the other direction. If the orbifolded string theory admits a matrix model description then the orbifold singularities should not interfere with the matrix model partition function.

Finally, it may be the case that the matrix model is "larger" than the dual string theory. In the case of the circle, for example, the vortex contributions to the partition function had to be discarded because they represented states in the matrix model that had no analogue in the string theory. Similar considerations may arise in the matrix model on the orbifold. The most likely way this may manifest is through some gauge backgrounds not corresponding to states in the string theory. It may be the case that these backgrounds should be excluded from the final matrix model partition function.

## 7.2 Holonomies on Orbifolds

In order to apply the method Section 2.2.2 that was developed for calculating twisted partition functions on the circle, it will be necessary to work out how to find the holonomies on an orbifold. This question was investigated in [14].

The problem with applying the previous method is that it relied on the smooth structures, such as differentiation and integration, that can be defined on manifolds. The orbifold contains singular points where these smooth structures break down and so applying the same methods naively could lead to inconsistencies near the orbifold fixed points. While it would be possible to broaden the definitions of the various smooth structures so that they can be applied to orbifolds an alternative method will be taken here. The aim will be to recast the theory of holonomies in a way that does not depend on the smooth structures.

The starting point will be to use the result that inequivalent ground states of a gauge theory can be related to the fundamental group of a manifold by

$$\{\text{Inequivalent Ground States}\} = \frac{Hom(\pi_1(\mathcal{M}) \to \mathcal{G})}{\mathcal{G}} \tag{19}$$

Where $\pi_1(\mathcal{M})$ is the fundamental group of the manifold and $\mathcal{G}$ is the gauge group of the theory. The notation $Hom(\cdots)$ stands for homomorphisms. The inequivalent ground states are the representations of the fundamental group in the gauge group with representatives that are related by gauge transformations identified. The Inequivalent ground states are in 1 to 1 correspondence with the holonomies, so {Holonomies} can replace {Inequivalent Ground States} on the LHS of (19).

On the circle the inequivalent ground states of the theory are in one to one correspondence with the holonomies because each holonomy is described by $e^{\oint A}$. As $dA = 0$, the field strength associated with $A$ vanishes and so it describes a ground state of the theory. Each element of this set of representatives is therefore a holonomy of the theory.

The key fact that will be needed is that there exists an "exact sequence" relating the homotopy groups of a manifold and the homotopy groups of a manifold quotiented by a group action. An exact sequence is a series of vector spaces linked by a sequence of linear maps with the property that the image of each map is the kernel of the next. An example of an exact sequence is the differential forms on a manifold, with the map being supplied by the exterior derivative.

$$\cdots \xrightarrow{d} \Omega_p M \xrightarrow{d} \Omega_{p+1}(\mathcal{M}) \xrightarrow{d} \Omega_{p+2}(\mathcal{M}) \xrightarrow{d} \cdots$$

Because $d^2 = 0$ any $p$ form in the image of $d$ as a map from $p-1$ forms to $p$ forms is necessarily the kernel of $d$ as a map from $p$ forms to $p+1$ forms.

The relevant exact sequence of homotopy groups is

$$\cdots \to \pi_n(\mathcal{M}_0) \to \pi_n(\mathcal{M}_0/\mathcal{H}) \to \pi_{n-1}(\mathcal{H}) \to \pi_{n-1}(\mathcal{M}_0) \to \cdots$$

Here $\pi_n(\mathcal{M})$ is the $n$th homotopy group of a manifold $\mathcal{M}$.

Consider the case where $\mathcal{M}_0$ is a connected, simply connected manifold and $\mathcal{H}$ is a discrete group with an action on $\mathcal{M}_0$. Simply connected just means that $\pi_1(\mathcal{M}_0) = 0$, i.e. it is the trivial group. Connectedness implies that $\pi_0(\mathcal{M}_0) = 0$. In this case, part of the sequence (19) looks like

$$\cdots \to \pi_1(\mathcal{M}_0) = 0 \to \pi_1(\mathcal{M}_0/\mathcal{H}) \to \pi_0(\mathcal{H}) \to \pi_0(\mathcal{M}_0) = 0 \to \cdots$$

For a discrete group, $\pi_0(\mathcal{H} = H$. Therefore, it is the case that there is a sequence of maps

$$0 \to \pi_1(\mathcal{M}_0/\mathcal{H}) \to \mathcal{H} \to 0$$

The image of the first map in the sequence is just the identity, as linear maps always map the identity to the identity. Therefore, the kernel of the second map is just the identity and the second map is injective. The third map takes every element of $\mathcal{H}$ to 0 and hence the kernel is the entire space. However, the kernel of the third map must be the image of the second. Thus the image of the second map is the entire space meaning the second map is surjective. A map that is injective and surjective is a bijection, thus in this case $\pi_1(\mathcal{M}_0/\mathcal{H})$ is isomorphic to $\mathcal{H}$. They are the same group.

The conclusion of this is that if we can write the manifold $\mathcal{M}$ in (19) as the quotient of a connected, simply connected space by a discrete group $\mathcal{H}$, the relation (19) becomes

$$\{\text{Holonomies}\} = \frac{Hom(\mathcal{H} \to \mathcal{G})}{\mathcal{G}}$$

This definition of the holonomy is independent of the smooth structure of the manifold and is therefore suitable for application to orbifolds.

## 7.3 $\mathcal{S}^1/\mathbb{Z}_2$ as a Quotient of $\mathbb{R}$

In order to apply the results of the previous section to the orbifold, it is necessary to identify some simply connected space which $\mathcal{S}^1/\mathbb{Z}_2$ is a quotient of. To do this, consider the real line $\mathbb{R}$. This space can be identified with the circle of radius $R$ if the identifications are made $x \cong x + 2\pi R$. The circle can then be identified with the orbifold by identifying $x \cong -x$.

These two identifications can be understood as arising from two different group actions on $\mathbb{R}$. The first is an action of the group of integers under addition, $\mathbb{Z}$. This is an infinite cyclic group generated by the element 1 [5]. Denote this element $t$. For an element $n \in \mathbb{Z}$ an action of $n$ on $x \in \mathbb{R}$ is $n \cdot x = x + 2\pi n R$. This action is free everywhere and so produces no singular points. Thus the circle is $\mathbb{R}/\mathbb{Z}$.

$\mathbb{Z}_2$ also has an action on $\mathbb{R}$ given by $z \cdot x = -x$, i.e. reflection around $x = 0$. The fixed point is at $x = 0$. Combined together, these two group actions produce the orbifold. However, the

[5] Note that 1 is not the additive identity.

two groups must be combined appropriately. It is not the case that the orbifold is given by the quotient of $\mathbb{R}$ by the action of the group $\mathbb{Z} \times \mathbb{Z}_2$.

To see why, consider applying the transformation $z \cdot t \cdot z$ to $x$.

$$z \cdot t \cdot z \cdot x = z \cdot t \cdot -x = z \cdot -x + 2\pi R = x - 2\pi R = t^{-1} \cdot x$$

There is an additional structure on the group that acts on $\mathbb{R}$ to produce the orbifold, namely the relation $z \cdot t \cdot z = t^{-1}$. In fact, the appropriate group that $\mathbb{R}$ should be quotiented by is $\mathbb{Z} \rtimes \mathbb{Z}_2$, the semi-direct product of $\mathbb{Z}$ and $\mathbb{Z}_2$.

Writing $\mathcal{S}^1/\mathbb{Z}_2 = \mathbb{R}/\mathbb{Z} \rtimes \mathbb{Z}_2$ achieves the goal of writing the orbifold as the quotient of a connected, simply connected space by a discrete group. Therefore, for a gauge theory defined on $\mathcal{S}^1/\mathbb{Z}_2$:

$$\{\text{Holonomies}\} = \frac{Hom(\mathbb{Z} \rtimes \mathbb{Z}_2 \to \mathcal{G})}{\mathcal{G}}$$

## 7.4 Calculating the Holonomies

Applying the conclusions of Section 7.2, the holonomies of a $U(N)$ gauge theory on $\mathcal{S}^1/\mathbb{Z}_2$ are given by

$$\{\text{Holonomies}\} = \frac{Hom(\mathbb{Z} \rtimes \mathbb{Z}_2 \to U(N))}{U(N)}$$

Because all elements of $\mathbb{Z}$ are generated by the element $t$ and because $z$ is the only nontrivial element of $\mathbb{Z}_2$, only representatives of these elements need to be given.

Start with the element $z$. All elements of $U(N)$ are gauge equivalent to diagonal matrices, so the representative of $z$ can be taken to be diagonal. The element $z$ satisfies the property $z^2 = \mathbb{1}$ and so up to gauge transformations all the possible representatives of $z$ in $U(N)$ are given by

$$z_n = \begin{pmatrix} \mathbb{1}_{n\times n} & 0 \\ 0 & -\mathbb{1}_{(N-n)\times(N-n)} \end{pmatrix}$$

Where $0 \leq n \leq [\frac{N}{2}]$, with the square brackets indicating the integer part of $\frac{N}{2}$.

With the representations of $z$ fixed the appropriate restrictions of the fields of the matrix model can be made. As the matrix fields form a representation of the gauge group, in the background appropriate to $z_n$ the $\mathbb{Z}_2$ symmetry acts on them as $M(t) \to z_n * M(t) z_n^\dagger$. Here $*$ is the operator that acts as $*f(t) = f(-t)*$ and $** = \mathbb{1}$. It gives the $\mathbb{Z}_2$ action on the coordinate $t$.

Therefore the matrix field must satisfy the restriction

$$z_n * M(t) * z_2^\dagger = M(t)$$

$$\Rightarrow \begin{pmatrix} \mathbb{1}_{n\times n} & 0 \\ 0 & -\mathbb{1}_{(N-n)\times(N-n)} \end{pmatrix} * \begin{pmatrix} M_1(t) & \varphi(t) \\ \varphi^\dagger(t) & M_2(t) \end{pmatrix} * \begin{pmatrix} \mathbb{1}_{n\times n} & 0 \\ 0 & -\mathbb{1}_{(N-n)\times(N-n)} \end{pmatrix} = \begin{pmatrix} M_1(t) & \varphi(t) \\ \varphi^\dagger(t) & M_2(t) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} M_1(-t) & -\varphi(-t) \\ -\varphi^\dagger(-t) & M_2(-t) \end{pmatrix} = \begin{pmatrix} M_1(t) & \varphi(t) \\ \varphi^\dagger(t) & M_2(t) \end{pmatrix}$$

Where $M_1$ is an $n \times n$ matrix, $M_2$ is an $(N-n) \times (N-n)$ matrix and $\varphi$ is an $n \times (N-n)$ matrix. This restricts $M_1$ and $M_2$ to being even functions of $t$, while $\varphi$ must be an odd function of $t$.

A similar restriction applies to the gauge field:

$$z_n * A(t) * z_n^\dagger = -A(t)$$

The extra minus sign arises on the right hand side because $A(t)$ must transform in the same way as the derivative operator $\frac{d}{dt} \to -\frac{d}{dt}$. This ensures the covariant derivative has a well defined transformation under $\mathbb{Z}_2$. The restriction is that

$$i\begin{pmatrix} A_1(-t) & -B(-t) \\ -B^\dagger(-t) & A_2(-t) \end{pmatrix} = i\begin{pmatrix} -A_1(t) & -B(t) \\ -B^\dagger(t) & -A_2(t) \end{pmatrix}$$

Where $A_1$ is an $n \times n$ hermitian matrix, $A_2$ is an $(N-n) \times (N-n)$ hermitian matrix and $B$ is an $n \times (N-n)$ matrix[6]. In this case $A_1$ and $A_2$ are odd functions of $t$ and $B$ is an even function.

Having applied the appropriate restrictions, the representations of $t$ can be found. For a representation $n$ of $\mathbb{Z}$, they are given by

$$t_n(A) = exp\left(\oint i \begin{pmatrix} A_1 & B \\ B^\dagger & A_2 \end{pmatrix}\right)$$

These are representations of $t$ because [7]

$$t_n^k(A) t_n^l(A) = exp\left((k+l)\oint i \begin{pmatrix} A_1 & B \\ B^\dagger & A_2 \end{pmatrix}\right) = t_n^{k+l}(A)$$

The fact that the $A$ field comes from an orbifold of the circle means that $A(t)$ should also be a periodic function. Therefore $A(t)$ has the mode expansion

$$A(t) = i\begin{pmatrix} \sum_{n=1}^\infty A_{1,n} sin(nt) & \sum_{n=0}^\infty B_n cos(nt) \\ \sum_{n=0}^\infty B_n^\dagger cos(nt) & \sum_{n=1}^\infty A_{2,n} sin(nt) \end{pmatrix}$$

This means that

$$e^{\oint A} = exp\left(\int_0^{2\pi} dti \begin{pmatrix} \sum_{n=1}^\infty A_{1,n} sin(nt) & \sum_{n=0}^\infty B_n cos(nt) \\ \sum_{n=0}^\infty B_n^\dagger cos(nt) & \sum_{n=1}^\infty A_{2,n} sin(nt) \end{pmatrix}\right) = exp\left(i\begin{pmatrix} 0 & 2\pi R B_0 \\ 2\pi R B_0^\dagger & 0 \end{pmatrix}\right)$$

The holonomy therefore corresponds to a background field

$$A = i\begin{pmatrix} 0 & B_0 \\ B_0^\dagger & 0 \end{pmatrix} \tag{20}$$

Finally, it should be checked that a representation given by $z_n$ and $t_n(A)$ satisfy the relation $z_n t_n(A) z_n = t_n^{-1}(a)$.

$$\begin{aligned} z_n t_n(A) z_n &= \begin{pmatrix} \mathbb{1}_{n\times n} & 0 \\ 0 & -\mathbb{1}_{(N-n)\times(N-n)} \end{pmatrix} exp\left(i\begin{pmatrix} 0 & \oint B \\ \oint B^\dagger & 0 \end{pmatrix}\right)\begin{pmatrix} \mathbb{1}_{n\times n} & 0 \\ 0 & -\mathbb{1}_{(N-n)\times(N-n)} \end{pmatrix} \\ &= exp\left(i\begin{pmatrix} 0 & -\oint B \\ -\oint B^\dagger & 0 \end{pmatrix}\right) = t^{-1}(A) \end{aligned}$$

Where it was used that for $U$ a unitary matrix $Ue^M U^\dagger = e^{UMU^\dagger}$.

## 7.5 Calculating the Operator

The next step of the procedure is to rewrite the action of the gauged matrix model in the form

$$\mathcal{S}(A) = \sum_{i,j,k,l} \int_0^{2\pi R} dt \quad M_{ij} \mathcal{O}_{ij;kl} M_{kl}$$

[6] A factor of $i$ is in front of the matrix making $A$ antihermitian.

[7] Compare to the circle where any matrix $e^{\oint A}$ corresponded to a holonomy. The reasoning of Section 7.2 should apply equally well when the circle is viewed as $\mathbb{R}/\mathbb{Z}$ leading to the conclusion that the holonomies should also form representations of $t$ on the circle.

The action is

$$\mathcal{S}(A) = \sum_{i,j,k,l} \int_0^{2\pi R} dt \quad \frac{1}{2}(DM)_{ij}(DM)_{ji} + \frac{1}{2}\omega^2 M_{ij}M_{ji}$$

The kinetic term is

$$\sum_{i,j,k,l} \int_0^{2\pi R} dt (DM)_{ij}(DM)_{ji} = \sum_{i,j} \int_0^{2\pi R} dt \quad \left[\frac{d}{dt}M_{ij} + [A,M]_{ij}\right]\left[\frac{d}{dt}M_{ji} + [A,M]_{ji}\right]$$

$$= \sum_{i,j} \dot{M}_{ij}\dot{M}_{ji}$$

$$+\sum_{i,j,k} \int_0^{2\pi R} dt \quad \dot{M}_{ij}A_{jk}M_{ki} - \dot{M}_{ij}M_{jk}A_{ki} + A_{ij}M_{kl}\dot{M}_{li} - M_{ij}A_{jk}\dot{M}_{ki}$$

$$+\sum_{i,j,k,l} \int_0^{2\pi R} dt \quad A_{il}M_{lj}A_{jk}M_{ki} - A_{il}M_{lj}M_{jk}A_{ki} - M_{il}A_{lj}A_{jk}M_{ki} + M_{il}A_{lj}M_{jk}A_{ki}$$

$$= \sum_{i,j} \int_0^{2\pi R} dt \quad M_{ij} - \frac{d^2}{dt^2}M_{ji}$$

$$\sum_{i,j,k} \int_0^{2\pi R} dt \quad -2M_{ij}A_{jk}\frac{d}{dt}M_{ki} + 2M_{jk}A_{ij}\frac{d}{dt}_{ki}$$

$$\sum_{i,j,k,l} \int_0^{2\pi R} dt \quad + M_{lj}A_{il}A_{jk}M_{ki} - M_{lj}A_{il}A_{ki}M_{jk} - M_{il}A_{lj}A_{jk}M_{ki} + M_{il}A_{lj}A_{ki}M_{jk}$$

$$= \sum_{i,j,k,l} \int_0^{2\pi R} dt \quad M_{ij}\big[-\frac{1}{2}\delta_{il}\delta_{jk}\frac{d^2}{dt^2} + \delta_{il}\delta_{jk}\frac{\omega^2}{2} + (\delta_{jk}A_{li} - \delta_{il}A_{jk})\frac{d}{dt}$$

$$+A_{li}A_{jk} - \frac{1}{2}\sum_m (A_{lm}A_{mi}\delta_{jk} + A_{jm}A_{mk}\delta_{il})\big]M_{kl}$$

From this expression the operator $\mathcal{O}(A)$ can be identified:

$$\mathcal{O}(A)_{ij;kl} = -\frac{1}{2}\delta_{il}\delta_{jk}\frac{d^2}{dt^2} + \delta_{il}\delta_{jk}\frac{\omega^2}{2} + (\delta_{jk}A_{li} - \delta_{il}A_{jk})\frac{d}{dt}$$

$$+A_{li}A_{jk} - \frac{1}{2}\sum_m (A_{lm}A_{mi}\delta_{jk} + A_{jm}A_{mk}\delta_{il})$$

## 7.6 The Heat Kernel Trace

The next step is to evaluate the heat kernel trace of this operator. As before, the heat kernel method is to use the equality (10)

$$ln|\mathcal{O}| = -Tr\int_\epsilon^\infty \frac{d\tau}{\tau}e^{-\tau\mathcal{O}}$$

In this case the heat kernel trace is:

$$Tr\int_\epsilon^\infty \frac{d\tau}{\tau}e^{-\tau\mathcal{O}} = \int_\epsilon^\infty \frac{d\tau}{\tau}\sum_{i,j}\int_0^{2\pi R} dx \sum_{n=-\infty}^{n=\infty} \langle ij,x|\, e^{-\tau\mathcal{O}(A)_{ij;ji}}\Pi\, |ji, x+2\pi nR\rangle$$

A new feature of the trace on the orbifold is the presence of the projector

$$\Pi = \begin{pmatrix} \frac{1+*}{2} & \frac{1-*}{2} \\ \frac{1-*}{2} & \frac{1+*}{2} \end{pmatrix}$$

The purpose of this projector is to project the contribution of the indices corresponding to the blocks $M_1$ and $M_2$ of $M$ onto their symmetric parts. The indices corresponding to the off diagonal $\varphi$ blocks are projected onto their antisymmetric parts. This ensures the orbifold trace is over states that satisfy the correct symmetry properties under $\mathbb{Z}_2$. The resulting expression for heat kernel trace is:

$$\int_\epsilon^\infty \frac{d\tau}{\tau} \int_0^{2\pi R} dx \sum_{n=-\infty}^{n=\infty} \sum_{a,b} \frac{1}{2} \langle ab, x| e^{-\tau \mathcal{O}_{ab;ba}} |ba, x+2\pi nR\rangle + \frac{1}{2} \langle ab, x| e^{-\tau \mathcal{O}_{ab;ba}} |ba, -x+2\pi nR\rangle$$
$$+\sum_{\dot a,\dot b} \frac{1}{2} \left\langle \dot a\dot b, x\right| e^{-\tau \mathcal{O}_{\dot a\dot b;\dot b\dot a}} \left|\dot b\dot a, x+2\pi nR\right\rangle + \frac{1}{2} \left\langle \dot a\dot b, x\right| e^{-\tau \mathcal{O}_{\dot a\dot b;\dot b\dot a}} \left|\dot b\dot a, -x+2\pi nR\right\rangle$$
$$+\sum_{\dot a,b} \frac{1}{2} \langle \dot a b, x| e^{-\tau \mathcal{O}_{\dot a b;b\dot a}} |b\dot a, x+2\pi nR\rangle - \frac{1}{2} \langle \dot a b, x| e^{-\tau \mathcal{O}_{\dot a b;b\dot a}} |b\dot a, -x+2\pi nR\rangle$$
$$+\sum_{a,\dot b} \frac{1}{2} \left\langle a\dot b, x\right| e^{-\tau \mathcal{O}_{a\dot b;\dot b a}} \left|\dot b a, x+2\pi nR\right\rangle - \frac{1}{2} \left\langle a\dot b, x\right| e^{-\tau \mathcal{O}_{a\dot b;\dot b a}} \left|\dot b a, -x+2\pi nR\right\rangle \quad (21)$$

The dotted and undotted indices label the different blocks of the matrix $M$ corresponding to the representation $z_n$. The distribution of the different types of index can be seen by inspecting

$$M_{ij} = \begin{pmatrix} M_{1,ab} & \varphi_{a\dot b} \\ \varphi^\dagger_{\dot a b} & M_{2,\dot a\dot b} \end{pmatrix}$$

## 7.7 Evaluating the Propagator Term

As in the case of the circle partition function the first step is to evaluate the propagator term in the trace

$$\langle ij, x| e^{-\tau \mathcal{O}(A)_{ij;ji}} |ji, y\rangle$$
$$= \langle ij, x| exp - \tau\big( -\frac{1}{2}\delta_{ii}\delta_{jj}\frac{d^2}{dt^2} + \delta_{ii}\delta_{jj}\frac{\omega^2}{2} + (\delta_{jj}A_{ii} - \delta_{ii}A_{jj})\frac{d}{dt}$$
$$+A_{ii}A_{jj} - \frac{1}{2}\sum_m (A_{im}A_{mi}\delta_{jj} + A_{jm}A_{mj}\delta_{ii})\big) |ji, y\rangle$$

The background gauge field $A$ is of the form (20) and hence has no diagonal elements and is constant. Furthermore, the operator $\frac{d}{dt}$ should, as before, be replaced with the momentum operator according to $\frac{d}{dt} \to ip$. Doing this gives

$$\langle ij, x| exp - \tau \left(\frac{1}{2}\hat p^2 + \frac{\omega^2}{2} - \frac{1}{2}\sum_m (A_{im}A_{mi} + A_{jm}A_{mj})\right) |ji, y\rangle$$

Inserting the a factor of the identity $\mathbb{1} = \int \frac{dp}{2\pi} \sum_{k,l} |kl, p\rangle \langle lk, p|$ and using $\langle ij, x|kl, p\rangle = \delta_{il}\delta_{jk}e^{ipx}$ gives

$$\int \frac{dp}{2\pi} exp - \frac{\tau}{2}\left(p^2 + \omega^2 - \sum_m (A_{im}A_{mi} + A_{jm}A_{mj}) - \frac{2}{\tau}i(x-y)p\right)$$

Completing the square in the argument of the exponential gives

$$\int \frac{dp}{2\pi} exp - \frac{\tau}{2}\left(\left(p - i\frac{x-y}{\tau}\right)^2 + \left(\frac{x-y}{\tau}\right)^2 + \omega^2 - \sum_m (A_{im}A_{mi} + A_{jm}A_{mj})\right)$$

Performing the Gaussian integral over $p$ then gives

$$\frac{1}{\sqrt{2\pi\tau}}exp\left(-\frac{(x-y)^2}{2\tau}-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)$$

From (14 it can be seen that there are two cases:

A. $y = x + 2\pi nR$

B. $y = -x + 2\pi nR$

Both these contributions to the partition function trace will be treated in turn. The contribution from case A. works out very similarly to the circle partition function, while the contribution from case B. is novel.

### 7.7.1 Case A - Step 1 - Poisson Resummation

The trace for terms in case A. is

$$\int_\epsilon^\infty \frac{d\tau}{\tau}\int_0^{2\pi R}dx\sum_{n=-\infty}^{n=\infty}\sum_{i,j}\frac{1}{\sqrt{2\pi\tau}}exp\left(-\frac{(2\pi Rn)^2}{2\tau}-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)$$

The indices $i,j$ can be replaced by the the appropriate $a, b, \dot{a}\dot{b}$ indices for the desired sector at the end of the calculation.

The first step towards calculating this is to use the Poisson resummation formula [5]

$$\sum_{n=-\infty}^{\infty}e^{-\pi an^2+2\pi ibn}=a^{-\frac{1}{2}}\sum_{n=-\infty}^{\infty}e^{\frac{-\pi(n-b)^2}{a}}$$

In the current case

$$a=\frac{2\pi R^2}{\tau}\qquad b=0$$

The Poisson resummed trace is therefore

$$\int_\epsilon^\infty \frac{d\tau}{\tau}\int_0^{2\pi R}dx\sum_{n=-\infty}^{n=\infty}\sum_{i,j}\frac{1}{\sqrt{2\pi\tau}}\sqrt{\frac{\tau}{2\pi R^2}}exp\left(-\frac{n^2}{R^2}\frac{\tau}{2}-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)$$

As the expression is independent of $x$ the integral over $x$ can be performed leaving

$$\int_\epsilon^\infty \frac{d\tau}{\tau}\sum_{n=-\infty}^{n=\infty}\sum_{i,j}exp\left(-\frac{n^2}{R^2}\frac{\tau}{2}-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)$$

### 7.7.2 Case A - Step 2 - Renormalising the Integral

The integral over $\tau$ will be renormalised using the same method as was used for the circle partition function, namely by taking the derivative with respect to $\omega^2$, taking the limit $\epsilon \to 0$ and then later integrating with respect to $\omega^2$.

Taking the derivative with respect to $\omega^2$:

$$\int_\epsilon^\infty \frac{d\tau}{\tau}\sum_{n=-\infty}^{n=\infty}\sum_{i,j}\frac{d}{d\omega^2}exp\left(-\frac{n^2}{R^2}\frac{\tau}{2}-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)$$

$$= \int_{\epsilon}^{\infty} \frac{d\tau}{\tau} \sum_{n=-\infty}^{n=\infty} \sum_{i,j} \quad \frac{-\tau}{2} \quad exp\left(-\frac{n^2}{R^2}\frac{\tau}{2} - \frac{\omega^2}{2}\tau + \frac{\tau}{2}\sum_m (A_{im}A_{mi} + A_{jm}A_{mj})\right)$$

$$= \int_{\epsilon}^{\infty} d\tau \sum_{n=-\infty}^{n=\infty} \sum_{i,j} \quad \frac{-1}{2} exp\left(-\frac{n^2}{R^2}\frac{\tau}{2} - \frac{\omega^2}{2}\tau + \frac{\tau}{2}\sum_m (A_{im}A_{mi} + A_{jm}A_{mj})\right)$$

It is now possible to evaluate the integral over $\tau$. Doing this gives

$$\sum_{n=-\infty}^{n=\infty} \sum_{i,j} \quad \frac{1}{\frac{n^2}{R^2} + \omega^2 - \sum_m (A_{im}A_{mi} + A_{jm}A_{mj})} \left[ exp\left(-\frac{n^2}{R^2}\frac{\tau}{2} - \frac{\omega^2}{2}\tau + \frac{\tau}{2}\sum_m (A_{im}A_{mi} + A_{jm}A_{mj})\right)\right]_{\epsilon}^{\infty}$$

Taking the limit $\epsilon \to 0$ of this expression leaves

$$\sum_{n=-\infty}^{n=\infty} \sum_{i,j} \quad \frac{-1}{\frac{n^2}{R^2} + \omega^2 - \sum_m (A_{im}A_{mi} + A_{jm}A_{mj})}$$

### 7.7.3 Case A - Step 3 - Summing Over $n$ and Integrating Over $\omega^2$

To evaluate the sum over $n$ expression (18) will be used again:

$$\sum_{n=-\infty}^{\infty} \frac{-1}{a + b(n+c)^2} = \frac{-\pi}{\sqrt{ab}} \left[ coth\left(\pi\sqrt{\frac{a}{b}} - i\pi c\right) + coth\left(\pi\sqrt{\frac{a}{b}} + i\pi c\right)\right]$$

To apply this to the case at hand, take

$$a = \omega^2 - \sum_m (A_{im}A_{mi} + A_{jm}A_{mj}) \qquad b = \frac{1}{R^2} \qquad c = 0$$

This gives

$$\sum_{i,j} \frac{-\pi R}{\sqrt{\omega^2 - \sum_m (A_{im}A_{mi} + A_{jm}A_{mj})}} \left[ coth\left(\pi R\sqrt{\omega^2 - \sum_m (A_{im}A_{mi} + A_{jm}A_{mj})}\right)\right]$$

Now the renormalisation must be finished by integrating the expression with respect to $\omega^2$. The appropriate measure for this integration is $d\omega^2 = 2\omega d\omega$. Doing this gives

$$\sum_{i,j} \int d\omega \frac{-2\pi\omega R}{\sqrt{\omega^2 - \sum_m (A_{im}A_{mi} + A_{jm}A_{mj})}} \left[ coth\left(\pi R\sqrt{\omega^2 - \sum_m (A_{im}A_{mi} + A_{jm}A_{mj})}\right)\right]$$

$$= \sum_{i,j} -2 \quad ln\left[ sinh\left(\pi R\sqrt{\omega^2 - \sum_m (A_{im}A_{mi} + A_{jm}A_{mj})}\right)\right] \tag{22}$$

### 7.7.4 Case A - Step 4 - Some Algebraic Manipulations

The last step is to put the result (22 into a manageable form. First define

$$\tilde{q}_{ij} = exp\left(-2\pi R\sqrt{\omega^2 - \sum_m (A_{im}A_{mi} + A_{jm}A_{mj})}\right)$$

Then

$$sinh\left(\pi R\sqrt{\omega^2-\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})}\right)=\frac{\tilde{q}_{ij}^{-\frac{1}{2}}-\tilde{q}_{ij}^{\frac{1}{2}}}{2}$$

$$\Rightarrow\sum_{i,j}-2\quad ln\left[sinh\left(\pi R\sqrt{\omega^2-\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})}\right)\right]=\sum_{i,j}-2\quad ln\left[\frac{\tilde{q}_{ij}^{-\frac{1}{2}}-\tilde{q}_{ij}^{\frac{1}{2}}}{2}\right]$$

Discarding an irrelevant additive constant contribution allows the final result to be written as

$$\int_\epsilon^\infty\frac{d\tau}{\tau}\int_0^{2\pi R}dx\sum_{n=-\infty}^{n=\infty}\sum_{i,j}\frac{1}{2}\left\langle ij,x\right|e^{-\tau\mathcal{O}_{ij;ji}}\left|ji,x+2\pi nR\right\rangle=\sum_{i,j}2ln\frac{\tilde{q}_{ij}^{\frac{1}{2}}}{1-\tilde{q}_{ij}}\tag{23}$$

#### 7.7.5 Case B - Step 1 - Gaussian Integration

Case B. contributions to the partition function are given by

$$\int_\epsilon^\infty\frac{d\tau}{\tau}\int_0^{2\pi R}dx\sum_{n=-\infty}^{n=\infty}\sum_{i,j}\frac{1}{2}\left\langle ij,x\right|e^{-\tau\mathcal{O}_{ij;ji}}\left|ji,-x+2\pi nR\right\rangle$$

$$=\int_\epsilon^\infty\frac{d\tau}{\tau}\int_0^{2\pi R}dx\sum_{n=-\infty}^{n=\infty}\sum_{i,j}\frac{1}{\sqrt{2\pi\tau}}exp\left(-\frac{(2x-2\pi nR)^2}{2\tau}-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)$$

The sum over $n$ and integral over $x$ can be combined in this expression to give a Gaussian integral over $x$. This is because

$$\sum_{n=-\infty}^{n=\infty}\int_0^{2\pi R}dxf(2x+2\pi nR)$$

$$=\sum_{n=-\infty}^{n=\infty}\int_{\pi nR}^{\pi(n+2)R}dxf(2x)$$

$$=\cdots\int_{\pi nR}^{\pi(n+2)R}dxf(2x)+\int_{\pi(n+1)R}^{\pi(n+3)R}dxf(2x)+\cdots$$

$$=\cdots\int_{\pi nR}^{\pi(n+1)R}dxf(2x)+\int_{\pi(n+1)R}^{\pi(n+2)R}dxf(2x)+\int_{\pi n+1R}^{\pi(n+2)R}dxf(2x)+\int_{\pi(n+2)R}^{\pi(n+3)R}dxf(2x)\cdots$$

$$=2\sum_{n=-\infty}^{n=\infty}\int_{\pi nR}^{\pi(n+1)R}dxf(2x)$$

$$=2\int_{-\infty}^{\infty}dxf(2x)$$

In the second line the change of variable $x+n\pi R\to x$ has been made. In the fourth line the sum has been written in a way that makes it clear that there are two integrals over each domain of integration, while in the last line all the integrals in the sum have been combined into a Gaussian integral. Therefore, the heat kernel trace is

$$\int_\epsilon^\infty\frac{d\tau}{\tau}\int_{-\infty}^{\infty}dx\sum_{i,j}\frac{2}{\sqrt{2\pi\tau}}exp\left(-\frac{2x^2}{\tau}-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)$$

Performing the Gaussian integral leads to

$$\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}\sum_{i,j}\frac{2}{\sqrt{2\pi\tau}}\sqrt{\frac{\pi\tau}{2}}exp\left(-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)$$

$$=\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}\sum_{i,j}exp\left(-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right) \tag{24}$$

### 7.7.6 Case B - Part 2 - Renormalisation of the Integral

This integral can be renormalised by exactly the same method that has been used in the cases treated so far. Acting on (24) with $\int d\omega 2\omega\frac{d}{d\omega^2}$ gives

$$\int d\omega 2\omega\frac{d}{d\omega^2}\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}\sum_{i,j}exp\left(-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)$$

$$=\int d\omega 2\omega\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}\sum_{i,j}\frac{-\tau}{2}exp\left(-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)$$

$$=\int_{\epsilon}^{\infty}d\tau\sum_{i,j}\int d\omega\quad -\omega exp\left(-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)$$

The $\tau$ integral gives

$$\sum_{i,j}\int d\omega\left[\frac{2\omega}{\omega^2-\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})}exp\left(-\frac{\omega^2}{2}\tau+\frac{\tau}{2}\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right)\right]_{\epsilon}^{\infty}$$

In the limit $\epsilon\to 0$ this is

$$\sum_{i,j}\int d\omega\frac{-2\omega}{\omega^2-\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})}$$

The integral over $\omega$ can be performed to give

$$\sum_{i,j}-ln\left[\omega^2-\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right]$$

Thus the case B. contributions give

$$\int_{\epsilon}^{\infty}\frac{d\tau}{\tau}\int_0^{2\pi R}dx\sum_{n=-\infty}^{n=\infty}\sum_{i,j}\frac{1}{2}\left\langle ij,x\right|e^{-\tau\mathcal{O}_{ij;ji}}\left|ji,-x+2\pi nR\right\rangle$$

$$\sum_{i,j}-ln\left[\omega^2-\sum_m(A_{im}A_{mi}+A_{jm}A_{mj})\right] \tag{25}$$

## 7.8 The Orbifold Twisted Partition Function

Now that the trace in the heat kernel equation has been calculated an expression for the twisted partition function can be written down.

$$ln|\mathcal{O}(A)|^{-\frac{1}{2}}=\frac{1}{2}Tr\int_{\epsilon}^{\infty}e^{-\tau\mathcal{O}(A)}$$

To find an expression for this, substitiute (23)and (25) to (14) to obtain

$$\frac{1}{4}\sum_{a,b} 2ln\frac{\tilde{q}_{ab}^{\frac{1}{2}}}{1-\tilde{q}_{ab}} + \frac{1}{4}\sum_{\dot{a},\dot{b}} 2ln\frac{\tilde{q}_{\dot{a}\dot{b}j}^{\frac{1}{2}}}{1-\tilde{q}_{\dot{a}\dot{b}}} + \frac{1}{4}\sum_{a,\dot{b}} 2ln\frac{\tilde{q}_{a\dot{b}}^{\frac{1}{2}}}{1-\tilde{q}_{a\dot{b}}} + \frac{1}{4}\sum_{\dot{a},b} 2ln\frac{\tilde{q}_{\dot{a}b}^{\frac{1}{2}}}{1-\tilde{q}_{\dot{a}b}}$$

$$-\sum_{a,b}\frac{1}{4}ln\left[\omega^2 - \sum_m (A_{am}A_{ma} + A_{bm}A_{mb})\right] - \sum_{\dot{a},\dot{b}}\frac{1}{4}ln\left[\omega^2 - \sum_m (A_{\dot{a}m}A_{m\dot{a}} + A_{\dot{b}m}A_{m\dot{b}})\right]$$

$$+\sum_{a,\dot{b}}\frac{1}{4}ln\left[\omega^2 - \sum_m (A_{am}A_{ma} + A_{\dot{b}m}A_{m\dot{b}})\right] + \sum_{\dot{a},b}\frac{1}{4}ln\left[\omega^2 - \sum_m (A_{\dot{a}m}A_{m\dot{a}} + A_{bm}A_{mb})\right]$$

The gauge twisted partition function can therefore be written as

$$Z(A) = |\mathcal{O}(A)|^{-\frac{1}{2}}$$

$$= \left[\prod_{i,j}\frac{\tilde{q}_{ij}^{\frac{1}{4}}}{(1-\tilde{q}_{ij})^{\frac{1}{2}}}\right] \frac{\prod_{a,\dot{b}}\left(\omega^2 - \sum_m (A_{am}A_{ma} + A_{\dot{b}m}A_{m\dot{b}})\right)^{\frac{1}{2}}}{\prod_{a,b}\left(\omega^2 - \sum_m (A_{am}A_{ma} + A_{bm}A_{mb})\right)^{\frac{1}{4}} \prod_{\dot{a},\dot{b}}\left(\omega^2 - \sum_m (A_{\dot{a}m}A_{m\dot{a}} + A_{\dot{b}m}A_{m\dot{b}})\right)^{\frac{1}{4}}} \tag{26}$$

Where the fact that

$$\prod_{a,\dot{b}}\omega^2 - \sum_m (A_{am}A_{ma} + A_{\dot{b}m}A_{m\dot{b}}) = \prod_{\dot{a},b}\omega^2 - \sum_m (A_{\dot{a}m}A_{m\dot{a}} + A_{bm}A_{mb})$$

has been used to combine the product over $a, \dot{b}$ with the product over $\dot{a}, b$..

Using the fact that

$$A = i\begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix}$$

Means that this can be written in terms of the matrices $B$ and $B^\dagger$.

First, note that

$$A \cdot A = i\begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} i\begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} = -\begin{pmatrix} BB^\dagger & 0 \\ 0 & B^\dagger B \end{pmatrix}$$

From this it can be concluded that $\sum_m A_{am}A_{ma} = -\sum_{\dot{b}} B_{a\dot{b}}B^\dagger_{\dot{b}a}$ and $\sum_m A_{\dot{a}m}A_{m\dot{a}} = -\sum_a B^\dagger_{\dot{a}a}B_{a\dot{a}}$, i.e. $\sum_m A_{am}A_{ma}$ is negative the magnitude squared of the $a$th column vector of $B^\dagger$, while $\sum_m A_{\dot{a}m}A_{m\dot{a}}$ is negative the magnitude squared of the $\dot{a}$th column vector of $B$

Rewriting (26) in terms of $B$ gives

$$Z(B) = \left[\prod_{i,j}\frac{\tilde{q}_{ij}^{\frac{1}{4}}}{(1-\tilde{q}_{ij})^{\frac{1}{2}}}\right] \frac{\prod_{a,\dot{b}}\left(\omega^2 + \sum_{\dot{c}} B_{a\dot{c}}B^\dagger_{\dot{c}a} + \sum_c B^\dagger_{\dot{b}c}B_{c\dot{b}}\right)^{\frac{1}{2}}}{\prod_{a,b}\left(\omega^2 + \sum_{\dot{c}} B_{a\dot{c}}B^\dagger_{\dot{c}a} + \sum_{\dot{c}} B_{b\dot{c}}B^\dagger_{\dot{c}b}\right)^{\frac{1}{4}} \prod_{\dot{a},\dot{b}}\left(\omega^2 + \sum_c B^\dagger_{\dot{a}c}B_{c\dot{a}} + \sum_c B^\dagger_{\dot{b}c}B_{c\dot{b}}\right)^{\frac{1}{4}}} \tag{27}$$

This only depends on the magnitudes of the various independent vectors in the matrix $B$.

### 7.8.1 The $n = \frac{N}{2}$ Representation

One representation where the partition function takes a particularly simple form is the $n = \frac{N}{2}$ representation when $N$ is even. Then the matrix $B$ is square. Decompose the matrix $B$ as $B =$

$V_2^\dagger D V_1^\dagger$, where $V_1$, $V_2$ are unitary, $D$ is diagonal. $B$ can be diagonalised by gauge transforming by

$$U = \begin{pmatrix} 0 & V_1 V_2 & 0 \end{pmatrix}$$

Under this transformation

$$\begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & V_1 \\ V_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} \begin{pmatrix} 0 & V_2^\dagger \\ V_1^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & V_1 B^\dagger V_2^\dagger \\ V_2 B V_1^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & D^\dagger \\ D & 0 \end{pmatrix}$$

In this case the magnitudes of the $\dot{a}$th column vector and $a$th row vector of $B$ are equal, thus the partition function reduces to

$$Z(B) = \left[ \prod_{i,j} \frac{\tilde{q}_{ij}^{\frac{1}{4}}}{(1-\tilde{q}_{ij})^{\frac{1}{2}}} \right]$$

# 8 Extracting Conclusions About The Toy Model

In this section the issues that were touched upon previously in Section 4.6 will be returned to draw conclusions about the matrix model.

In order to really obtain any insight into the toy model In order to project onto the singlet representation $Z(B)$ needs to be projected onto the singlet representation by integrating it over all gauge fields $B$. Even in the simple representation$\frac{N}{2}$ this is a highly nontrivial integral. Therefore the conclusions that can be drawn from (27) are somewhat limited.

Furthermore, it is not even guaranteed that this will give the full partition function for the matrix model. The derived expression is only the twisted partition function for a corresponding to particular representation of $\mathbb{Z}_2$. It may be the case that all representations of $\mathbb{Z}_2$ should be summed over for the final partition function. On inspection it may turn out that only particular values of $n$ are dual to the Liouville string theory. Perhaps certain values of $n$ should be excluded analogously to how $T$ duality violating vortices were excluded from the circle partition function.

### 8.0.2 String Coupling

However, some statements can be made. Firstly, the integral of (27)does not appear to diverge. This is an important indication that the string coupling does indeed remain small, providing some evidence to support the previous assertion that in the Liouville string theory the Liouville wall keeps the string at small coupling.

### 8.0.3 The role of Twisted States

Unfortunately not much can be gleaned about the twisted states from (27). There are some hints that the Liouville patition function will split up into normal string modes and twisted states however.

This is because the contribution to the partion function that arises due to the identification of $x$ and $-x$ is the term

$$\frac{\prod_{a,\dot{b}} \left(\omega^2 + \sum_{\dot{c}} B_{a\dot{c}} B^\dagger_{\dot{c}a} + \sum_c B^\dagger_{\dot{b}c} B_{c\dot{b}}\right)^{\frac{1}{2}}}{\prod_{a,b} \left(\omega^2 + \sum_{\dot{c}} B_{a\dot{c}} B^\dagger_{\dot{c}a} + \sum_{\dot{c}} B_{b\dot{c}} B^\dagger_{\dot{c}b}\right)^{\frac{1}{4}} \prod_{\dot{a},\dot{b}} \left(\omega^2 + \sum_c B^\dagger_{\dot{a}c} B_{c\dot{a}} + \sum_c B^\dagger_{\dot{b}c} B_{c\dot{b}}\right)^{\frac{1}{4}}}$$

The Liouville string partition function is obtained from the matrix model partition function by taking the logarithm. Very roughly then, it may be that these terms arising from the identification $x \sim -x$ may lead to a sector that can be identified as the twisted states in the Liouville string partition function. Note however that for this to work as described the evaluation of the integral over $B$ should leave the partition function in more or less its current form, which is quite unlikely.

### 8.0.4 Interpretation of $\mathcal{S}^1/\mathbb{Z}_2$ as Time

No conclusions can be drawn about the interpretation of the orbifold direction as time. The integral needs to be evaluated before one can think about analytic continuations.

### 8.0.5 Conclusion

The aim of this thesis was to define a 2 dimensional cosmological toy model with a big bang and a big crunch by putting the Liouville field theory on the orbifold $\mathcal{S}^1/\mathbb{Z}_2$. Unfortunately it is still unknown whether this produces a useful toy model. Calculational difficulties have meant that important questions about the model cannot yet be answered. Despite this, there is some evidence that perhaps the model will work as it is supposed to, namely that the theory does not appear to be strongly coupled anywhere.

In order to shine more light on the model it would be necessary to either surmount the calculational difficulties by finding some way to integrate equation 27 or to find an alternative way of calculating the matrix model's partition function that is more tractable.

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