Field Theory Methods for the Nonlinear Evolution of Large-Scale Cosmological Structures

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Abstract

In this thesis we use fields theory methods to study the nonlinear evolution of large-scale cosmological structures. We attack the problem in two steps. First, on the basis of the Vlasov equation in Newtonian dynamics, we formulate the Keldysh-Schwinger action for the evolution system. Then the Wilsonian renormalization theory is introduced to deal with the ultraviolet divergences. We find that our renormalized approach shows cut-off dependence, and that a continuous stochastic source arises due to the renormalization, which are expected. Second, different from the convention theory that the physical variables are density and velocity perturbations, we describe the system as a perturbed matter field evolving in curved spacetime. The prediction of the power spectrum in the linear regime matches with the standard result. We expect that the divergence of the one-loop effect could be reduced by the dimensional regularization, and we plan to find out the nonlinear limit k_{NL} of our theory, however, we do not have enough time to cover these discussions in this thesis.

Contents

Chapter 1

Introduction

The theoretical study of the nonlinear evolution of cosmic large-scale structures (LSS) is becoming a major research activity within the cosmological community. This is motivated in part by the existing and upcoming cosmic surveys [1, 2, 3] which will map galaxy positions and redshifts for a large fraction of the observable universe. More fundamentally, it offers us a great opportunity to understand the physics of the very early universe, dark energy, dark matter and inflation.

The analysis of these observations requires tools and methods to understand how the small primordial perturbations imprinted from inflation evolve to the structures that our telescopes measure. In the early period, we only had observations on very large scale (today of the order of 10 Gpc). Assuming the cold dark matter (CDM) model [4, 5, 6], the linear perturbation theory [7, 8, 9] is very accurate. Currently, as observations go to smaller large-scale structures, the linear perturbation theory breaks down.

The main problem is that nonlinear effects can not be ignored at scales smaller than 100 Mpc. The traditional approach to study the development of LSS is to make use of numerical simulations [10, 11]. Although they provide very valuable insights into the late time evolution of gravitational instabilities, they do not incorporate large dynamic scale, and are difficult to adopt to the variety of cosmological models available today beyond ΛCDM. Analytical tools and methods should be introduced to do calculations from first principles. In this thesis we try to use field theory methods to study the nonlinear evolution of large-scale cosmological structure.

The basic approach [12, 13] that has been used so far is based on the study of the Vlasov equation of the large-scale structure system with only gravitational interaction. The Vlasov equation gives the nonlinear conservation and Euler equations. Standard perturbation theory has been first introduced to study the nonlinear evolution [14], but it has ill-defined expansion parameter and ultraviolet (UV) divergences. Later renormalized Perturbation theory [15, 16] has been developed to improve calculations of power spectra, however, it has the cut-off dependence and careful considerations are required to fix this cut-off scale to some physical scale. Concurrently, symmetries and consistency relations [17] have been applied to understand the mathematical structure better. Finally, the effective field theory of large-scale structure (EFToLSS) [18] is formulated in terms of an infrared (IR) effective fluid characterized by several parameters, such as speed of sound and viscosity. These corrective terms are obtained by integrating out the UV physics, and are complementary to symmetries and consistency relations. More importantly, the predictions of the effective field theory are found to be in much better agreement with observations than standard cosmological perturbation theory, already reaching percent precision at relatively short scale $k \simeq 0.24h$ Mpc^{-1}. However, current EFToLSS has one short-coming: the parameters of the corrective terms need to be read off from simulations.

In my thesis, I have attempted to fix the short-comings of standard perturbation theory and avoid the short-coming of the EFT. In order to do this, I have used two approaches to describe the system. The motivation of the first approach is to study the behaviour of the renormalized theory. On the basis of Newton dynamics, I formulate the theory in terms of the Keldysh-Schwinger action by introducing auxiliary fields, and then I use Wilsonian renormalization to get the effective action. It turns out that our renormalized approach shows cut-off dependence, and that a continuous stochastic source arises due to the renormalization, which are expected. Then I move on to the second approach, which is to explore a completely new formalism to describe the system. Conventionally, the system is described in terms of nonlinear equations for density and velocity perturbations with Gaussian initial conditions. I will describe the system in terms of the effective action of the perturbed matter field with non-Gaussian initial conditions, with the hope that (a) the dependence on the UV cutoff is under control, and that (b) the evolution in the UV regime can be understood analytically. First I consider a scalar field evolving in curved spacetime. The system has been formulated in different gauges, and we have found that it is easier to describe it in the ADM formalism. It turned out that our formalism has the correct scale dependence at the beginning of the linear regime. The one-loop perturbation is expected to give reasonable predictions in the non-linear regime, and the cut-off dependence may be cancelled by dimensional regularization. Due to limited time, we could not cover the one-loop effect and the formalism for the fermion field.

The content of the thesis is organized as follows: in chapter 2 we review how to describe the evolution of LSS with only gravitational interaction in Newtonian dynamics, and how to simplify the evolution equations by ignoring the anistropical pressure and the curl modes of the velocity. Then we rewrite the evolution equations in Fourier form and in matrix form, for later discussion. In the end of the chapter, we shortly discuss the linear evolution.

In chapter 3, we show that the evolution of LSS should be treated as a stochastic process with the initial Gaussian power spectrum. Then on the basis of the matrix form of the evolution equations in chapter 2 and the stochastic mechnism, we formulate the Schwinger-Keldysh action and the corresponding Feynman rules to calculate the power spectrum. For the one-loop power spectrum, there is ultraviolet divergences, which motivates us to do the Wilsonian renormalization in chapter 4. We find the terms generated by the renormalization flow, and check whether these terms obey the Galilean invariance, and then calculate the predicted power spectrum to compare our theory with the effective field theory.

In chapter 5, we start our second formalism. First, we give an introduction to general relativity and gauge invariance. We write down the action for the scalar field in the curved spacetime in comoving gauge, and expand the action into the zeroth, first ad second orders. The zeroth and first order actions would give the evolution equation of the background spacetime, and the second action gives the constrain equation for the metric perturbation in terms of the perturbed matter field. Our purpose is to insert the solution of the metric perturbation back into the action of the perturbed matter field, and then we obtain the effective action for the perturbed matter field. Based on this effective action, we can give predictions for the power spectrum, and see its behaviour of ultraviolet divergences. However, it turns out that the constraints are not easy to solve, which motivates us to use a different formalism in the next chapter.

In chapter 6, we change to the zero-curvature gauge and use the ADM formalism to describe the system. The detail of how to obtain the Hilbert-Einstein action is shown in the Appendix A. Our purpose is the same, i.e. to obtain the effective action for the perturbed matter field. First we show an example of this method by repeating some existing work. Then we get the effective action for our system. By using the canonical quantization, we go to the frame of quantum field theory, and use field theory methods to calculate the power spectrum. A toy initial condition is considered first. Then we consider the real initial condition, which is non-Gaussian. It turned out that our formalism has the correct scale dependence at the beginning of the linear regime. We end this thesis by the summary in chapter 7.

Chapter 2

From the Vlasov equation to a set of basic equations

In this chapter, we will build Vlasov equation for our system. This theoretical structure has been used in plenty of studies [19, 12, 13]. Then, based on the Vlasov equation, mass conversation and Euler equations can be derived. Later we will use some appropriate approximations to simplify these equations and write them in a way that is more convenient for our later discussion. At the end of this section, we will make a short discussion about the linear regime.

2.1 Mass conservation and the Euler equations

Assuming the distances of interest are small compared to the curvature radius of the universe, we could apply the Newtonian dynamical laws to the evolution of gravitational perturbations as long as the local gravitational perturbations are small. The universe is described as full of dust particles whose only interaction is gravity, see [20, 21] for the original idea. For simplicity, it is assumed that their masses are all the same,m.

We could introduce the phase space distribution function, $f(\mathbf{x}, \mathbf{p})\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{p}$, which is the number of particles per volume element $\mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{p}$, where the position $\mathbf{x}$ of the particles is expressed in co-moving coordinates and the particle conjugate momentum $\mathbf{p}$ is given by

$$\mathbf{p} = \mathbf{u}ma. \tag{2.1}$$

In Eq. (2.1) a is the expansion factor, $\mathbf{u}$ is the peculiar velocity, i.e. the physical velocity in the background of the Hubble expansion. The Liouville theorem [22] states that the density of system points in the vicinity of a given system point traveling through phase-space is constant with time, which implies that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial}{\partial t} f(\mathbf{x}, \mathbf{p}, t) + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \cdot \frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{p}, t) + \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \cdot \frac{\partial}{\partial \mathbf{p}} f(\mathbf{x}, \mathbf{p}, t) = 0. \tag{2.2}$$

This is the Vlasov equation. We have the following dynamical relations of the position and momentum

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \frac{\mathbf{p}}{ma^2} \tag{2.3}$$

and

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -m\nabla_{\mathbf{x}}\Phi(\mathbf{x}, t), \tag{2.4}$$

where Φ is the gravitational potential. In the context of metric perturbation in an expanding universe the potential $\Phi(\mathbf{x})$ is sourced by the density contrast. So we have

$$\nabla^2\Phi(\mathbf{x}) = \frac{4\pi G m}{a}\left(\int \mathrm{d}^3\mathbf{p} f(\mathbf{x}, \mathbf{p}, t) - \frac{\bar{n}}{a^3}\right), \tag{2.5}$$

where $\bar{n}$ is the spatial average of $\int f(\mathbf{x}, \mathbf{p})\mathrm{d}^3\mathbf{p}$. Inserting Eq. (2.3) and (2.4) into the Vlasov equation, we then have

$$\frac{\partial}{\partial t} f(\mathbf{x}, \mathbf{p}, t) + \frac{\mathbf{p}}{ma^2} \cdot \frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}, \mathbf{p}, t) - m\nabla_{\mathbf{x}}\Phi(\mathbf{x}, t) \cdot \frac{\partial}{\partial \mathbf{p}} f(\mathbf{x}, \mathbf{p}, t) = 0. \tag{2.6}$$

We can now derive the basic conservation equations we are going to use from the first two moments of the Vlasov equation. The zeroth moment is the density field per volume $\mathrm{d}^3\mathbf{r}$ given by

$$\rho(\mathbf{x}, t) = \frac{m}{a^3}\int \mathrm{d}^3\mathbf{p} f(\mathbf{x}, \mathbf{p}, t). \tag{2.7}$$

It can be decomposed in a homogeneous part and an inhomogeneous part,

$$\rho(\mathbf{x},t) = \bar{\rho}(t)(1+\delta(\mathbf{x},t)). \tag{2.8}$$

The first moment of the phase space distribution is the mean velocity flow, (for each component) defined as

$$u_i(\mathbf{x},t) = \frac{1}{\int \mathrm{d}^3\mathbf{p} f(\mathbf{x},\mathbf{p},t)} \int \mathrm{d}^3\mathbf{p} \frac{p_i}{ma} f(\mathbf{x},\mathbf{p},t). \tag{2.9}$$

and the second moment defines the velocity dispersion $\sigma_{ij}(\mathbf{x},t)$, which is related to the internal contribution to the total pressure [23],

$$u_i(\mathbf{x},t)u_j(\mathbf{x},t) + \sigma_{ij}(\mathbf{x},t) = \frac{1}{\int \mathrm{d}^3\mathbf{p} f(\mathbf{x},\mathbf{p},t)} \int \mathrm{d}^3\mathbf{p} \frac{p_i}{ma}\frac{p_j}{ma} f(\mathbf{x},\mathbf{p},t). \tag{2.10}$$

The first two moments of the Vlasov equation give then the conservation and Euler equations, respectively,

$$\frac{\partial \delta(\mathbf{x},t)}{\partial t} + \frac{1}{a}[(1+\delta(\mathbf{x},t))u_i(\mathbf{x},t)]_{,i} = 0 \tag{2.11}$$

and

$$\frac{\partial u_i(\mathbf{x},t)}{\partial t} + \frac{\dot{a}}{a}u_i(\mathbf{x},t) + \frac{1}{a}u_j(\mathbf{x},t)u_i(\mathbf{x},t)_{,j} = -\frac{1}{a}\Phi(\mathbf{x},t)_{,i} - \frac{(\rho(\mathbf{x},t)\sigma_{ij}(\mathbf{x},t))_{,j}}{\rho(\mathbf{x},t)a}, \tag{2.12}$$

where $(,j)$ denotes the derivative with respect to the spatial coordinate x_j. The first equation is obtained by multiplying Eq. (2.6) by $\int \mathrm{d}^3\mathbf{p}$ and doing partial integration. The second one is obtained by multiplying Eq. (2.6) by $\int \mathrm{d}^3\mathbf{p}\frac{p_i}{ma}$ and also using partial integration.

The first term on the right hand side of Eq. (2.12) is the gravitational force, and the second is due to the pressure force, which in general can be anisotropic. There are subsequent equations that can be written for the whole hierarchy of the velocity moments and depending on the physical situations, the hierarchy can be truncated if micro-physics dictates a relation between the pressure tensor and the local density. In our later discussion about the nonlinear case of the equations, the second term is ignored, due to the single-flow approximation [13], which states that the early stages of the gravitational instabilities are characterized by a negligible velocity dispersion, and that the velocity dispersion is much smaller than the velocity gradients induced by the density fluctuations. It is worth noting that there are 4 degrees of freedom in the equations: one density and three velocity. In the next section, we could see that the number of degrees of freedom will be reduced to 2: one density and one velocity divergence.

2.2 The curl modes

Using the single flow approximation, one can see that the source term of the Euler equation is potential, i.e. it cannot generated any curl mode in the velocity field. Generally, one can decompose any three-dimensional field in a gradient part and a curl part

$$u_i(\mathbf{x}) = \phi(\mathbf{x})_{,i} + \mathbf{w}_i(\mathbf{x}) \tag{2.13}$$

where $\mathbf{w}_{i,i} = 0$. Defining the local vorticity as

$$w_k(\mathbf{x}) = \epsilon^{ijk}\mathbf{u}_i(\mathbf{x})_{,j} \tag{2.14}$$

where ϵ^{ijk} is the totally anti-symmetric Levi-Civita tensor, one can show that

$$w_k(\mathbf{x}) = \epsilon^{ijk}\mathbf{w}_i(\mathbf{x})_{,j} \tag{2.15}$$

and applying the operator $\epsilon^{ijk}\nabla_j$ to Eq. (2.12) one gets

$$\frac{\partial}{\partial t}w_k + \frac{\dot{a}}{a}w_k - \epsilon^{ijk}\epsilon^{lmi}(\mathbf{u}_l w_m)_{,j} = 0. \tag{2.16}$$

This equation simply expresses the fact that the vorticity is conserved throughout the expansion, i.e. no extra source for the vorticity. In the linear regime when the last term of this equation is dropped, it means that the vorticity scales like $1/a$. In the subsequent stage of the dynamics the vorticity can only grow in contracting regions, and it can not be created out of potential modes, since gravitational force is gradient and is independent from the curl modes of the velocity. As a consequence, from now on curl modes in the vector field will be ignored, and this is a reasonable approximation which has been demonstrated in various studies [24, 25, 26, 27]. So now the three velocity quantities are reduced to one quantity, the divergence of the velocity $\mathbf{u}_i(\mathbf{x})_{,i}$. This simplifies our equations a lot, as is shown in the following sections.

2.3 Equations in Fourier form

Introducing the conformal time $\tau = \int \mathrm{d}t/a$ and the conformal expansion rate $\mathcal{H} = \mathrm{d}\ln a/\mathrm{d}\tau$, one can rewrite Eq. (2.11) and (2.12) as

$$\frac{\partial\delta(\mathbf{x},\tau)}{\partial\tau} + u_i(\mathbf{x},\tau)_{,i} = (\delta(\mathbf{x},\tau)u_i(\mathbf{x},\tau))_{,i} \tag{2.17}$$

and

$$\frac{\partial u_i(\mathbf{x},\tau)}{\partial\tau} + \mathcal{H}u_i(\mathbf{x},\tau) + u_j(\mathbf{x},\tau)u_i(\mathbf{x},\tau)_{,j} = -\Phi(\mathbf{x},\tau)_{,i} - \frac{(\rho(\mathbf{x},\tau)\sigma_{ij}(\mathbf{x},\tau))_{,j}}{\rho(\mathbf{x},\tau)}. \tag{2.18}$$

As we can see, there are nonlinear coupling terms in the equations. To find out how the coupling terms contribute to the evolution of large-scale cosmological structure, it is more convenient to work in Fourier space. Our convention for the Fourier transform of a field $A(\mathbf{x},\tau)$ is

$$A(\mathbf{k},\tau) = \int \mathrm{d}^3\mathbf{x}\exp(-i\mathbf{k}\cdot\mathbf{x})A(\mathbf{x},\tau), \tag{2.19}$$

and its inverse transform is

$$A(\mathbf{x},\tau) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\exp(i\mathbf{k}\cdot\mathbf{x})A(\mathbf{k},\tau). \tag{2.20}$$

More specifically we have,

$$[\delta(\mathbf{x})u_i(\mathbf{x})]_{,i} = \int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\delta(\mathbf{k}_1)\theta(\mathbf{k}_2)\frac{\mathbf{k}_2\cdot(\mathbf{k}_1+\mathbf{k}_2)}{k_2^2}\exp(i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}), \tag{2.21}$$

and

$$[u_j(\mathbf{x},t)u_i(\mathbf{x},t)_{,j}]_{,i} = \int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\theta(\mathbf{k}_1)\theta(\mathbf{k}_2)\frac{\mathbf{k}_1\cdot\mathbf{k}_2\mathbf{k}_1\cdot(\mathbf{k}_1+\mathbf{k}_2)}{k_1^2k_2^2}\exp(i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}), \tag{2.22}$$

where $\theta(\mathbf{x})$ is the velocity divergence, defined as $\theta(\mathbf{x}) = u_i(\mathbf{x})_{,i}$. Taking the divergence of Eq. (2.18), plugging these expressions in the resulting equations and Fourier transforming them, one gets

$$\frac{\partial\delta(\mathbf{k},\tau)}{\partial\tau} + \theta(\mathbf{k},\tau) = -\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\delta_{\mathrm{D}}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\alpha(\mathbf{k}_1,\mathbf{k}_2)\delta(\mathbf{k}_1,\tau)\theta(\mathbf{k}_2,\tau), \tag{2.23}$$

and

$$\frac{\partial\theta(\mathbf{k},\tau)}{\partial\tau}+\mathcal{H}\theta(\mathbf{k},\tau)+\frac{3}{2}\Omega_m\mathcal{H}^2\delta(\mathbf{k},\tau) = -\int\frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\delta_{\mathrm{D}}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\beta(\mathbf{k}_1,\mathbf{k}_2)\theta(\mathbf{k}_1,\tau)\theta(\mathbf{k}_2,\tau), \tag{2.24}$$

where δ_D denotes the Dirac delta function. Here we have used the Friedman equation $\frac{\mathcal{H}^2}{a^2} = \frac{8\pi G}{3}\rho_c(t)$ with the definition $\Omega_m = \bar{\rho}(t)/\rho_c(t)$.

The nonlinear interactions on the right-hand side are responsible for the coupling between different Fourier modes, and are given by

$$\alpha(\mathbf{k}_1,\mathbf{k}_2) \equiv \frac{\mathbf{k}_1\cdot(\mathbf{k}_1+\mathbf{k}_2)}{k_1^2},\ \beta(\mathbf{k}_1,\mathbf{k}_2) \equiv \frac{|\mathbf{k}_1+\mathbf{k}_2|^2(\mathbf{k}_1\cdot\mathbf{k}_2)}{2k_1^2k_2^2}. \tag{2.25}$$

2.4 Equations in matrix form

The equations of motion can be rewritten in a more symmetric form by defining a two-component "vector"

$$\Psi_a(\mathbf{k},\eta) \equiv (\delta(\mathbf{k},\eta),\ -\theta(\mathbf{k},\eta)/\mathcal{H}), \tag{2.26}$$

where the index $a = 1, 2$ and we have introduced a new time variable,

$$\eta \equiv \ln a(\tau). \tag{2.27}$$

From now on the cosmological model we are considering is $\Omega_m = 1$.

Using $\Psi_a(\mathbf{k},\eta)$ one can rewrite Eq. (2.24) and (2.25) as [1]

$$\partial_\eta\Psi_a(\mathbf{k},\eta) + \Omega_{ab}\Psi_b(\mathbf{k},\eta) = \gamma^{(s)}_{abc}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\Psi_b(\mathbf{k}_1,\eta)\Psi_c(\mathbf{k}_2,\eta), \tag{2.28}$$

where

$$\Omega_{ab} = \begin{pmatrix} 0 & -1 \\ -3/2 & 1/2 \end{pmatrix},$$

and $\gamma^{(s)}_{abc}$ is the symmetric vertex tensor given by

$$\gamma^{(s)}_{121}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = \delta_{\mathrm{D}}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\alpha(\mathbf{k}_1,\mathbf{k}_2)/2, \tag{2.29}$$

[1] Here we use the convention that repeated Fourier arguments are integrated over

$$\gamma^{(s)}_{222}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = \delta_{\rm D}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\beta(\mathbf{k}_1,\mathbf{k}_2)/2, \tag{2.30}$$

$$\gamma^{(s)}_{abc}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = \gamma^{(s)}_{acb}(\mathbf{k},\mathbf{k}_2,\mathbf{k}_1), \tag{2.31}$$

and zero otherwise. For convenience, from now on we denote $\gamma^{(s)}_{abc}$ by γ_{abc}.

An implicit integral solution to Eq. (2.28) can be found by a Laplace transform in the variable η,

$$\Psi_b(\mathbf{k},\omega) = \phi_a(\mathbf{k}) + \gamma_{abc}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\oint \frac{d\omega_1}{2\pi i}\Psi_b(\mathbf{k}_1,\omega_1)\Psi_c(\mathbf{k}_2,\omega-\omega_1), \tag{2.32}$$

where $\phi_a(\mathbf{k}) \equiv \Psi_a(\mathbf{k},\eta=0)$ denotes the initial conditions, and $\sigma^{-1}_{ab} = \omega\delta_{ab} + \Omega_{ab}$. Multiplying by the matrix

$$\sigma_{ab} = \frac{1}{(2\omega+3)(\omega-1)}\begin{pmatrix} 2\omega+1 & 2 \\ 3 & 2\omega \end{pmatrix},$$

and performing the inverse Laplace transform, we get the following formal solution

$$\Psi_a(\mathbf{k},\eta) = g_{ab}(\eta)\phi_b(\mathbf{k}) + \int_0^\eta d\eta' g_{ab}(\eta-\eta')\gamma_{bcd}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\Psi_c(\mathbf{k}_1,\eta')\Psi_d(\mathbf{k}_2,\eta'), \tag{2.33}$$

where the linear propagator $g_{ab}(\eta)$ is defined as

$$g_{ab}(\eta) = \oint_{c-i\infty}^{c+i\infty} \frac{d\omega}{2\pi i}\sigma_{ab}(\omega)e^{\omega\eta} = \frac{e^\eta}{5}\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} - \frac{e^{-3\eta/2}}{5}\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix}$$

for $\eta \geq 0$, whereas $g_{ab}(\eta) = 0$ for $\eta \leq 0$, and $g_{ab}(\eta) \to \delta_{ab}$ for $\eta \to 0^+$.

From a physical viewpoint, the most interesting initial conditions are when $\delta(\mathbf{k},\eta=0)$and $\theta(\mathbf{k},\eta=0)$ are random fields and proportional to each other, in which case we can write

$$\phi_a(\mathbf{k}) = u_a\delta(\mathbf{k},\eta=0), \tag{2.34}$$

where u_a is a two component "vector". For the growing mode initial conditions, $u_a = (1,\ 1)$, and the growth factor is $e^\eta = a$. For the decaying mode initial conditions, $u_a = (2/3,\ -1)$. For simplicity, we will only consider the case $\phi_a(\mathbf{k}) = u_a\delta(\mathbf{k},\eta=0)$ in this thesis.

2.4.1 The linear theory

In the linear regime, we have the solution

$$\Psi_a(\mathbf{k}, \eta) = g_{ab}(\eta)\phi_b(\mathbf{k}), \tag{2.35}$$

which simply means that each momentum mode evolve independently and the growth factors of different modes are the same. This is a very important property of the linear evolution. We will come back to this later. We should point out that the linear solution obtained here is limited to the cosmological mode $\Omega_m = 1$. The linear case has been studied in a general cosmological background, and this property still holds. The details of the linear case in a general cosmological background can be found in [12][13].

Chapter 3

The nonlinear solutions: field representations and power spectrum

For the nonlinear case, it is typical to study it by solving the "vector" and find the field representation to calculate the power spectrum [12][13], which is known as the standard perturbation theory. Here we will show a different formalism. First, we will introduce the origin of stochasticity in our dynamical system. The dynamic evolution can be treated as a stochastic process. Based on the stochastic equation, a Schwinger-Keldysh action can be formulated. Then we could find the corresponding Feynman rules and use Feynman diagrams to calculate the power spectrum, which is a statistical quantity related to observation. At the end of this section we will compare this approach with the approach of the standard perturbation theory.

3.1 Initial conditions

Here we will provide some arguments why the system is stochastic. Then we will define the power spectrum and choose a Gaussian power spectrum as the initial condition. With the Gaussian initial spectrum, we will formulate the evolution of LSS as a stochastic system.

3.1.1 The origin of stochasticity

Our interest about the large-scale structures of the universe is to understand how the present distribution of matter on cosmological scales results from the growth of primordial fluctuations on a homogeneous universe with gravitational interaction. However, we do not have any direct observational access to primordial fluctuations, i.e. we could not find definite initial conditions for the deterministic evolution equations. In addition, the time scale for cosmological evolution is extremely large and we can not make observations over the whole time scale, which means it is impossible to follow the evolution of a single system by observations. In other words, what we observe through the past light cone is different objects at different times of their evolution. Due to these reasons, tests of cosmological theories which characterize these primordial fluctuations are not deterministic in nature but statistical.

The observable universe is thus treated as a stochastic realization of a statistical ensemble of possibilities, which is the standard model and can be found in Peacock's book [28]. The reasonable goal is to make statistical predictions, which in turn depend on the statistical properties of the primordial perturbations. In the following section we will introduce power spectra to describe the statistical properties.

3.1.2 Observable quantity–power spectrum

The observable quantities of interest are actually the statistical properties of the density field. The basic statistical quantity is the correlation functions. According to the Cosmological Principle, i.e the assumption that the Universe is statistically isotropic and homogeneous, the correlator $\langle\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle$ is a function of the distance r only. This defines the two-point correlation function,

$$\xi(r) = \langle\delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle. \tag{3.1}$$

In Fourier space, according to the Fourier transformation of the field,

$$\delta(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^2}\, \delta(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}), \tag{3.2}$$

the two-point correlator of the Fourier modes takes the following form,

$$\begin{aligned}\langle\delta(\mathbf{k})\delta(\mathbf{k}')\rangle &= \int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{r}\xi(r)\exp[-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{r}] \\ &= \delta_{\mathrm{D}}(\mathbf{k}+\mathbf{k}')\int \mathrm{d}^3\mathbf{r}\xi(r)\exp(i\mathbf{k}\cdot\mathbf{r}) \\ &\equiv \delta_{\mathrm{D}}(\mathbf{k}+\mathbf{k}')P(k),\end{aligned} \tag{3.3}$$

where $P(k)$ is the power spectrum of the density field.

3.1.3 Primordial Gaussianity

There are mainly two types of theoretical models which provide a theoretical explanation of the origin of stochasticity. One is the class of inflation models, in which the stochastic properties of the fields originate from quantum fluctuations of a scalar field, the inflaton. The inflation models generically give birth to adiabatic Gaussian initial fluctuations. The other is the class of topological defect models. In this case the origin of stochasticity comes from thermal fluctuations of a field that undergoes a phase transition as the universe cools, and can be Gaussian or non-Gaussian. There are two types of observations about the distribution properties of matter, cosmic microwave background (CMB) and LSS. Observations from the CMB, the latest PLANCK results for example, give f_{NL}^{loc}[1] $= 2.5 \pm 5.7$ (at 68% statistical confidence level) from observations of temperature [29]. Observations from LSS (for example [30]) are expected to provide better constraints on primordial non-Gaussianity in the coming years. For simplicity, in our discussion, we will in general assume a Gaussian initial condition.

Here we just give a brief introduction to the single-field inflation mechanism. The complete discussion about it can be found in plenty of articles (for example, in [31, 32]).

During the inflationary phase, the energy density of the universe is dominated by the density stored in the inflation field ϕ. This field has quantum fluctuations of the following form,

$$\delta\phi = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}[a_{\mathbf{k}}\varphi_k(t)\exp(i\mathbf{k}\cdot\mathbf{x}) + a^\dagger_{\mathbf{k}}\varphi^*_k(t)\exp(-i\mathbf{k}\cdot\mathbf{x})], \tag{3.4}$$

[1] f_{NL}^{loc} means local non-Gaussianity, and $f_{NL}^{loc} = 0$ for Gaussian distribution.

where $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ are the creation and annihilation operators for a wave mode $\mathbf{k}$. The operators obey the standard commutation relation,

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}}] = (2\pi)^3 \delta_{\mathrm{D}}(\mathbf{k} + \mathbf{k}'), \tag{3.5}$$

and the mode functions $\varphi_k(t)$ are obtained from the Klein-Gordon equation for ϕ in an expanding Universe. In de-Sitter cosmological model, it has the following form,

$$\varphi_k(t) = \frac{H}{(2k)^{1/2}k}\left(1 + \frac{k}{aH}\right)\exp(\frac{ik}{aH}), \tag{3.6}$$

where a is the expansion factor and H is the Hubble constant. When the modes exit the Hubble radius, $k/(aH) \ll 1$, we could see that the dominant mode is

$$\phi_{\mathbf{k}} \approx \frac{iH}{(2k)^{1/2}k}(\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}}),\ \delta\phi = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.7}$$

Since these modes are all proportional to $(\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{-\mathbf{k}})$ and any combinations of $\phi_{\mathbf{k}}$ commute with each other, the quantum nature of the fluctuations has disappeared. Thus the field ϕ can be seen as a classic stochastic field, where ensemble averages are identified with vacuum expectation values.

These modes re-enter the Hubble radius after the inflationary phase. They imprint the properties of their energy fluctuations on the gravitational potential, which is how initial Gaussian stochasticity in our system is produced.

3.1.4 Initial conditions for the nonlinear evolution

We have discussed that in the linear regime, there is no interaction between different momentum modes and the growth factors of different modes are the same, so the initial Gaussian statistical property is kept and becomes the initial condition for the nonlinear regime. In the nonlinear regime, we could consider the nonlinear evolution as a stochastic process by treating the initial Gaussian power spectrum as a stochastic term. Introducing the stochastic term, the "vector" equation (2.28) can be written into the following,

$$\partial_\eta \Psi_a(\mathbf{k}, \eta) + \Omega_{ab}\Psi_b(\mathbf{k}, \eta) - \gamma^{(s)}_{abc}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2)\Psi_b(\mathbf{k}_1, \eta)\Psi_c(\mathbf{k}_2, \eta) = J_a(\mathbf{k}, \eta) \tag{3.8}$$

with

$$\langle J_a(\mathbf{k},\eta)J_b(\mathbf{k}',\eta')\rangle = \mathcal{N}(\mathbf{k},\eta)(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\delta(\eta-\eta')\delta_{ab}, \qquad (3.9)$$

where

$$\mathcal{N}(\mathbf{k},\eta) = P_{in}(k)\delta(\eta-\eta_{in}) + \Delta(k,\eta). \qquad (3.10)$$

$P_{in}(k)\delta(\eta-\eta_{in})$ is the stochastic contribution from the initial power spectrum, and we denote $\Delta_0 \equiv P_{in}(k)\delta(\eta-\eta_{in})$. It is worth noting that the stochastic equation is more general than the mechanism mentioned before. The second term $\Delta(k,\eta)$ means there could be a stochastic contribution which acts not only at the initial time, but also continuously. This continuous contribution could come from the nonlinear evolution of the system itself. We will start with only the initial contribution $\mathcal{N}(\mathbf{k},\eta) = \Delta_0$ and in the next section we will see how a continuous contribution $\Delta(k,\eta)$ arises.

3.2 The Schwinger-Keldysh action

The Schwinger-Keldysh [33, 34, 35], $in-in$ or closed-time-path (CTP), formalism is constructed to study the evolution of observables in quantum systems. Contrary to the $in-out$ formalism, the standard action formalism in particle field theory, which is used to study the transition amplitude between an initial and a final state with specific evolution, the Schwinger-Keldysh formalism can be used to study the evolution of some operator from an initial state as $t = t_0$ to a final state at $t > t_0$ without knowing what the suitable late time states are. Evolving an operator from t_0 to t requires a time-ordered evolution from t_0 to t and the corresponding anti-time-ordered evolution, which can be thought of as the evolution from t back to t_0, the so-called $in-in$ formalism. It can also be seen as an evolution along a closed time contour, i.e. the closed-time-path (CTP). This formalism has been applied in non-equilibrium process in many areas, such as in cosmology, condensed matter physics and heavy ion collisions.(some citations here) The corresponding effective action, or two-particle irreducible action can be obtained in the standard way, which is via a double Legendre transformation of the classical actions, one of which is with respect to the field and the other is about the two-point correlation functions. More details of the $in-in$ formalism can be found in this review [36].

In our case, the system is described by the field equation (3.8) and the equation (3.9) of the stochastic quantity. The expectation value of any observable $\mathcal{O}[\Psi]$ with respect to the system can be written as

$$\langle \mathcal{O}[\Psi] \rangle = \int DJ_a \mathrm{e}^{-\frac{1}{2}\int J_a \cdot \prime \dashv \lfloor J_b} \int D\Psi_a$$
$$\delta_{\mathrm{D}}(\partial_\eta \Psi_a(\mathbf{k},\eta) + \Omega_{ab}\Psi_b(\mathbf{k},\eta) - \gamma^{(s)}_{abc}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\Psi_b(\mathbf{k}_1,\eta)\Psi_c(\mathbf{k}_2,\eta) - J_a(\mathbf{k},\eta)). \tag{3.11}$$

Similar to the approach in [37], by expressing the delta functional as a functional "Fourier transform" with the aid of an auxiliary field χ and performing the Gaussian J integral we have

$$\langle \mathcal{O}[\Psi] \rangle = \int D\chi \int D\Psi \mathcal{O}[\Psi] \mathrm{e}^{\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}[i\chi_a(\partial_\eta \Psi_a + \Omega_{ab}\Psi_b - \gamma^{(s)}_{abc}\Psi_b\Psi_c) - \frac{1}{2}\chi_a \Delta_{0ab}\chi_b]}. \tag{3.12}$$

To proceed, it is more convenient to write the action in the symmetrical form. The action corresponding to our stochastic system is the following,

$$S = \int \mathrm{d}\eta \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\frac{1}{2}\begin{pmatrix} \Psi_a , \chi_a \end{pmatrix} \begin{pmatrix} 0 & -\partial_\eta \delta_{ab} + \Omega_{ba} \\ -\partial_\eta \delta_{ab} + \Omega_{ab} & -i\Delta_0 \end{pmatrix} \begin{pmatrix} \Psi_b \\ \chi_b \end{pmatrix} - \gamma_{abc}\chi_a \Psi_b \Psi_c \right] \tag{3.13}$$

The action can also be obtained from the generating functional formalism, as in [16]. From this action, it is straightforward to get the corresponding Feynman rules. The free correlation functions are determined as the elements of the functional and matrix inverse of the matrix operator in Eq. (3.13) which is found to be:

$$\begin{pmatrix} \langle \Psi_a(\eta,)\Psi_b(\eta')\rangle & \langle \Psi_a(\eta,)\chi_b(\eta')\rangle \\ \langle \chi_a(\eta,)\Psi_b(\eta')\rangle & \langle \chi_a(\eta,)\chi_b(\eta')\rangle \end{pmatrix} \equiv -\mathrm{i}\begin{pmatrix} 0 & -\partial_\eta \delta_{ab} + \Omega_{ba} \\ -\partial_\eta \delta_{ab} + \Omega_{ab} & -i\Delta_{0ab} \end{pmatrix}^{-1} \delta(\eta - \eta')$$
$$= \begin{pmatrix} F_{ab} & -iG^R_{ab}(\eta,\eta') \\ -iG^A_{ab}(\eta,\eta') & 0 \end{pmatrix}. \tag{3.14}$$

Here $G^{(R,A)}_{ab}(\eta,\eta')$ are the retarded and advanced Green functions for the operator $(-\partial_\eta \delta_{ab} + \Omega_{ba})$,

$$G^R_{ab}(\eta,\eta') = G^A_{ab}(\eta',\eta) = g_{ab}(\eta' - \eta). \tag{3.15}$$

$F_{ab}(\eta,\eta',k)$ is the free two-point correlation function,

$$F_{ab}(\eta,\eta',k) = g_{ac}(\eta)u_c P_0(k) u_d g_{bd}(\eta'). \tag{3.16}$$

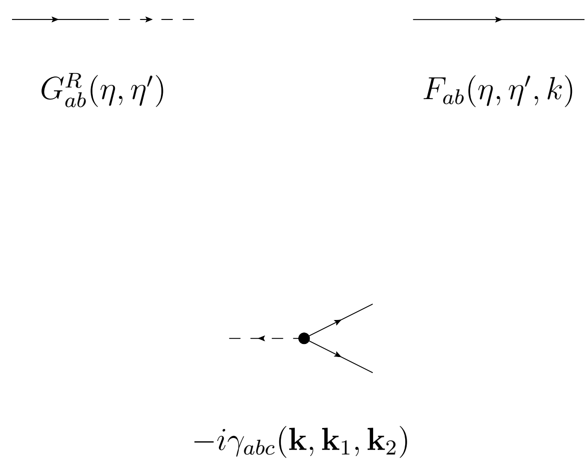


Figure 3.1: The Feynman rules corresponding to the action (3.13).

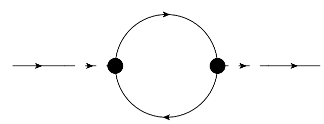
The Feynman rules are shown in fig. (1). The propagator $G^R_{ab}(\eta, \eta')$ represents linear evolution from η to η'. The index $a = 1$ ($a = 2$) corresponds to the density (velocity divergence) field. The vertex represents all the nonlinear interactions between momentum modes. The vertex obeys wavenumber conservation, with two incoming solid lines with wavenumber $\mathbf{k}_1$ and $\mathbf{k}_2$, and one outgoing dotted line with wavenumber $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$. It should be noted that internal indices are summed over and interaction times are integrated over the full interval $[0, \eta]$.

With the Feynman rules at hand, we could calculate physical quantities using the standard perturbation theory in field theory.

3.2.1 The one-loop power spectrum

The power spectrum in our dynamical system is defined as follows,

$$\langle \Psi_a(\mathbf{k}, \eta,)\Psi_b(\mathbf{k}', \eta')\rangle \equiv \delta_{\mathrm{D}}(\mathbf{k} + \mathbf{k}')P_{ab}(\mathbf{k}, \eta). \tag{3.17}$$



$$\langle \Psi_a(\mathbf{k}, \eta,)\Psi_b(\mathbf{k}', \eta')\rangle$$

Figure 3.2: A Feynman diagram for power spectrum at one-loop order.

As we can see, the zeroth order of the power spectrum is simply the free propagator F_{ab}. Due to the interactions between different modes, namely the vertex term, the power spectrum has higher-order contributions. The easiest way to see it is to draw down the Feynman diagrams corresponding to this two-point correlator. For example, the second-order expansion of the vertex term gives two vertices, which could form a one-loop diagram, shown in fig. (2).

Using Feynman rules, we can obtain the corresponding expression,

$$P_{ab}^{(1)}(k) = 4\int_0^{\eta} \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} g_{ac}(\eta - s_1)\gamma_{cde}(\mathbf{k}, \mathbf{q}, \mathbf{k}-\mathbf{q}) g_{df}(s_1) u_f P_0(q) g_{hj}(s_2) u_j \\ g_{gn}(s_2) u_n P_0(|\mathbf{k}-\mathbf{q}|) u_m g_{em}(s_1) \gamma_{ihg}(-\mathbf{k}, -\mathbf{q}, \mathbf{q}-\mathbf{k}) g_{ib}(s_2). \tag{3.18}$$

This expression is difficult to evaluate. We can assume a power law initial spectrum $P_0(k) = Ak^n$ to see the divergence. In some papers [12, 38], it has been found that for $n \leq -1$ an infrared divergence arises because of terms like $\int P_0(q)\mathrm{d}^3q/q^2$. As $n > -1$ an ultraviolet divergence is caused by the integration of the internal momentum to infinity.

The power spectrum could also be calculated in the approach of the standard perturbation theory, as we have mentioned at the beginning of this section. However, we could notice that the Schwinger-Keldysh formalism has some advantages over the standard perturbation theory.

One is that it is easier to obtain all contributing diagrams at a certain perturbation order of power spectra in the Schwinger-Keldysh formalism. I have tried to get all the diagrams for power spectra up to two-loop order, in total 29 diagrams. These 29 diagrams match with the 29 diagrams mentioned in [14] with a set of matching rules of the convention about how to draw the diagrams. In the action formalism, I could just find all the diagrams systematically. However, in the standard perturbation theory, one should start with the diagrams of the series perturbation expansion of Ψ, and it is rather cumbersome.

Another is that we could calculate power spectrum at a certain order more directly. Generally, diagrams in the standard perturbation theory are calculated with kernel functions of the perturbation solution of Ψ. These kernel functions have to be calculated order by order, and it is not straightforward to write down the expression for a certain diagram.

Last but not least, in the standard perturbation theory, there are two main problems. First, inhomogeneities are large at small scales, i.e. $\delta > 1$, which means that it's not reasonable to choose δ as the expansion parameter any more. Second, for a power-law initial condition, loop integrals are normally divergent, but no counter terms appear in the theory to cancel them, which means that the predictions depend on the cutoff and are hence unphysical. However, the Schwinger-Keldysh formalism offers a great chance to avoid these problems with useful techniques from standard field theory. That is what we will show in the next chapter.

Chapter 4

Renormalized Field Theory for non-linear perturbations

With the action and the Feynman rules, it is possible to see how the higher-order diagrams contribute to the power spectrum. In this section we will borrow techniques from quantum field theory, and try to compute the Wilsonian renormalization group flow to one loop in order to study the type of terms generated by the flow and examine the renormalizability of the theory.

4.1 Wilsonian renormalization approach

We have shown that the power spectrum could have ultraviolet divergences. To study it, we will use the Wilsonian renormalization approach here. An introduction about this approach can be found in Chapter 12 in Peskin's book [39]. Wilson's method is based on the functional integral approach to field theory, in which the degrees of freedom of a quantum field are variables of integration. In this approach, one can study the origin of ultraviolet divergences by isolating the dependence of the functional integral on the short-wavelength degrees of freedom of the field. We will apply this method to the action obtained in the last section.

From the action, Eq. (3.13), we have

$$
\begin{aligned}
Z &= \int D\chi \int D\Psi \mathrm{e}^{iS} \\
&= \int D\chi \int D\Psi \mathrm{e}^{i\int \mathrm{d}\eta \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\frac{1}{2}(\,\Psi_a\,,\,\chi_a\,)\begin{pmatrix} 0 & -\partial_\eta\delta_{ab}+\Omega_{ba} \\ -\partial_\eta\delta_{ab}+\Omega_{ab} & -i\Delta_{0ab}\end{pmatrix}\begin{pmatrix}\Psi_b \\ \chi_b\end{pmatrix}-\gamma_{abc}\chi_a\Psi_b\Psi_c\right]}.
\end{aligned}
\tag{4.1}
$$

To carry out the integration over the high momentum degrees of freedom of Ψ and χ, some definitions should be introduced. First, we impose a sharp ultraviolet cutoff Λ, which means we integrate field variables with $|\mathbf{k}| \leq \Lambda$. Then we divide the integration variables Ψ and χ into two groups, the short-wavelength group and the long-wavelength group. Choose a fraction $b < 1$. The variables $\Psi(\mathbf{k})$ and $\chi(\mathbf{k})$ with $b\Lambda \leq |\mathbf{k}| < \Lambda$ are the high-momentum degrees of freedom that we will integrate over. We define $\Psi \equiv \Psi^L + \Psi^S$, $\Psi^L(\mathbf{k})$ with $|\mathbf{k}| < b\Lambda$ and $\Psi^S(\mathbf{k})$ with $b\Lambda \leq |\mathbf{k}| < \Lambda$. We do the same to χ. Now we could insert $\Psi \equiv \Psi^L + \Psi^S$ and $\chi \equiv \chi^L + \chi^S$ into Eq. (4.1) and rewrite it as

$$
\begin{aligned}
Z &= \int [D\chi]_\Lambda \int [D\Psi]_\Lambda \\
&\quad \mathrm{e}^{i\int \mathrm{d}\eta \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\frac{1}{2}\left(\,\Psi_a^L+\Psi_a^S\,,\,\chi_a^L+\chi_a^S\,\right)\begin{pmatrix} 0 & -\partial_\eta\delta_{ab}+\Omega_{ba} \\ -\partial_\eta\delta_{ab}+\Omega_{ab} & -i\Delta_0\end{pmatrix}\begin{pmatrix}\Psi_b^L+\Psi_b^S \\ \chi_b^L+\chi_b^S\end{pmatrix}-\gamma_{abc}(\chi_a^L+\chi_a^S)(\Psi_b^L+\Psi_b^S)(\Psi_c^L+\Psi_c^S)\right]} \\
&\equiv \int [D\chi]_\Lambda \int [D\Psi]_\Lambda \mathrm{e}^{i\int \mathrm{d}\eta \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\mathcal{L}_1}.
\end{aligned}
\tag{4.2}
$$

$\mathcal{L}_1$ is introduced to make it easier to represent these short-wavelength and long-wavelength parts and their coupling terms. It is not the usual Lagrangian, and is defined as the following,

$$\mathcal{L}_1 \equiv \frac{1}{2} \left(\Psi_a^L , \chi_a^L \right) \begin{pmatrix} 0 & -\partial_\eta \delta_{ab} + \Omega_{ba} \\ -\partial_\eta \delta_{ab} + \Omega_{ab} & -i\Delta_0 \end{pmatrix} \begin{pmatrix} \Psi_b^L \\ \chi_b^L \end{pmatrix} - \gamma_{abc} \chi_a^L \Psi_b^L \Psi_c^L \tag{4.3}$$

$$+ \frac{1}{2} \left(\Psi_a^S , \chi_a^S \right) \begin{pmatrix} 0 & -\partial_\eta \delta_{ab} + \Omega_{ba} \\ -\partial_\eta \delta_{ab} + \Omega_{ab} & -i\Delta_0 \end{pmatrix} \begin{pmatrix} \Psi_b^S \\ \chi_b^S \end{pmatrix} - \gamma_{abc} \chi_a^S \Psi_b^S \Psi_c^S \tag{4.4}$$

$$- \gamma_{abc} [\chi_a^L (\Psi_b^L \Psi_c^S + \Psi_b^S \Psi_c^L + \Psi_b^S \Psi_c^S) + \chi_a^S (\Psi_b^L \Psi_c^L + \Psi_b^L \Psi_c^S + \Psi_b^S \Psi_c^L)] \tag{4.5}$$

$$+ \frac{1}{2} \left(\Psi_a^S , \chi_a^S \right) \begin{pmatrix} 0 & -\partial_\eta \delta_{ab} + \Omega_{ba} \\ -\partial_\eta \delta_{ab} + \Omega_{ab} & -i\Delta_0 \end{pmatrix} \begin{pmatrix} \Psi_b^L \\ \chi_b^L \end{pmatrix} + \frac{1}{2} \left(\Psi_a^L , \chi_a^L \right) \begin{pmatrix} 0 & -\partial_\eta \delta_{ab} + \Omega_{ba} \\ -\partial_\eta \delta_{ab} + \Omega_{ab} & -i\Delta_0 \end{pmatrix} \begin{pmatrix} \Psi_b^S \\ \chi_b^S \end{pmatrix} . \tag{4.6}$$

As we can see, expression (4.3) is the long-wavelength part, and expression (4.4) is the short-wavelength part. Expression (4.5) is the coupling vertex part, and expression (4.6) contains the coupling matrix terms.

The next step is to perform the integral over $\Psi^S(\mathbf{k})$ and $\chi^S(\mathbf{k})$ with $b\Lambda \leq |\mathbf{k}| < \Lambda$. This integration will transform Eq. (4.2) into an expression of the form

$$Z = \int [D\chi]_{b\Lambda} \int [D\Psi]_{b\Lambda} \ \exp(iS_{eff}). \tag{4.7}$$

where the effective action S_{eff} contains the original terms and also terms generated by integrating over $\Psi^S(\mathbf{k})$ and $\chi^S(\mathbf{k})$ with $b\Lambda \leq |\mathbf{k}| < \Lambda$. We will show how to get the effective action in the next section.

4.1.1 The effective action

Carrying out the integrals over $\Psi^S(\mathbf{k})$ and $\chi^S(\mathbf{k})$ can only be done perturbatively. We will treat expression (4.4) as the leading-order term, which is

$$\mathcal{L}_{12} = \left(\Psi_a^S , \chi_a^S \right) \begin{pmatrix} 0 & -\partial_\eta \delta_{ab} + \Omega_{ba} \\ -\partial_\eta \delta_{ab} + \Omega_{ab} & -i\Delta_0 \end{pmatrix} \begin{pmatrix} \Psi_b^S \\ \chi_b^S \end{pmatrix} - \gamma_{abc} \chi_a^S \Psi_b^S \Psi_c^S . \tag{4.8}$$

This will give the Feynman rules of the short-wavelength part. As we can see, Eq. (4.8) has the same structure as expression (4.3). So it will give the same Feynman rules as the Feynman rules we have derived from the original action Eq. (3.13).

The other terms which need to be integrated, i.e. the vertex terms (4.5) and the matrix terms (4.6), will be treated as perturbations. We will expand the exponential and evaluate the various contributions from these perturbations by using Wick's theorem with the Feynman rules given by Eq. (4.8).

We first evaluate the terms that result from expanding to one power of the matrix terms in the exponent. They are terms like $\Psi^S\chi^L$, and they are simply zero. The same result applies to the terms that result from expanding to one power of the vertex terms in the exponent. That is to say, we need to expand all the vertex terms and matrix terms at least to second order in the exponent.

Checking the second order of the matrix terms, we find out terms like $\chi^S\chi^L\chi^S\chi^L$, $\Psi^S\chi^L\chi^S\chi^L$, $\Psi^S\chi^L\Psi^S\chi^L$, $\Psi^S\Psi^L\Psi^S\chi^L$ and $\Psi^S\Psi^L\Psi^S\Psi^L$. Using Wick's theorem and drawing the corresponding Feynman diagrams, we find that all the diagrams are disconnected. This is also very easy to see from the expression. Since there is no vertex in the expression, short-wavelength fields can only be connected to short-wavelength fields and form a bubble. It has been proven in a standard textbook [39] that these disconnected diagrams have no contribution to the effective action.

The first order of the matrix terms combined with the first order of the vertex terms will also form a second order contribution. For example, $\Psi^S\chi^L$ combined with $\gamma_{abc}\chi_a^L\Psi_b^L\Psi_c^S$ will contribute terms like $\gamma_{abc}\chi_a^L\Psi_b^L\chi_c^L$ once the short-wavelength field is integrated out. However, we will leave these generated vertex terms for later discussion.

We now turn to the second order of the vertex terms. The expression has the following structure,

$$\begin{aligned}\mathcal{L}_{12}^2 = \gamma_{abc}[\chi_a^L(\Psi_b^L\Psi_c^S + \Psi_b^S\Psi_c^L + \Psi_b^S\Psi_c^S) + \chi_a^S(\Psi_b^L\Psi_c^L + \Psi_b^L\Psi_c^S + \Psi_b^S\Psi_c^L)] \\ \cdot \gamma_{def}[\chi_d^L(\Psi_e^L\Psi_f^S + \Psi_e^S\Psi_f^L + \Psi_e^S\Psi_f^S) + \chi_d^S(\Psi_e^L\Psi_f^L + \Psi_e^L\Psi_f^S + \Psi_e^S\Psi_f^L)].\end{aligned} \tag{4.9}$$

Before we start to evaluate, we go back to the original action Eq. (3.13).

(A) (B)

Figure 4.1: Generated terms

There are $\Psi^L\chi^L$, $\chi^L\chi^L$ and vertex terms. We would expect the generated terms give the same form and modify the coefficients in the long-wavelength action.

Similarly, using Wick's theorem, we find that once the short-wavelength part has been integrated out, the contributing terms are like $\chi^L\chi^L$, $\Psi^L\chi^L$, $\chi^L\Psi^L\Psi^L$ and $\chi^L\Psi^L\Psi^L\Psi^L$. We will ignore the $\chi^L\Psi^L\Psi^L$ and $\chi^L\Psi^L\Psi^L\Psi^L$ terms and keep the vertex at the original tree-level and focus on renormalizing of the Green functions, using the same approximation as in [16].

In this approximation, we only need to evaluate the expressions in $\mathcal{L}_{12}^2$ which would generate terms like $\Psi^L\chi^L$ and $\chi^L\chi^L$.

The expression which contributes to $\Psi^L\chi^L$ is $\gamma_{abc}\chi_a^L\Psi_b^S\Psi_c^S\cdot\gamma_{def}\chi_d^S(\Psi_e^L\Psi_f^S+\Psi_e^S\Psi_f^L)$. Now we could find the corresponding Feynman diagrams and use Feynman rules to evaluate them. It should be noted that we have taken the coefficient i in front of $\mathcal{L}$ in Z into consideration when we start to evaluate the expression.

With Feynman rules, we could write down the contribution from the diagram (A) in fig. (4.1).

$$\chi^L_a(\eta)A_{ab}\Psi^L_b(s_2) = -4i\int_{b\Lambda}^{\Lambda}\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\chi^L_a(\eta)\gamma_{ade}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})g_{df}(\eta)u_f g_{eg}(\eta-\eta^{'})$$
$$\gamma_{ghb}(\mathbf{k}-\mathbf{q},-\mathbf{q},\mathbf{k})g_{hj}(\eta^{'})u_j P_0(q)\Psi^L_b(\eta^{'}). \tag{4.10}$$

The factor 4 comes from the facts that there are two contribution terms and the diagram has a symmetry factor 2. The coefficient (-i) comes from the convention of the retarded and advanced Green functions, based on which the propagators are $-iG^{(A,R)}_{ab}$.

The expression which contributes to $\chi^L\chi^L$ is $\gamma_{abc}\chi^S_a\Psi^L_b\Psi^L_c\cdot\gamma_{def}\chi^S_d\Psi^L_e\Psi^L_f$. We will evaluate this in the similar way as for $\Psi^L\chi^L$.
The contribution from the diagram (B) in fig. (3) is

$$\chi^L_a(\eta)B_{ab}\chi^L_b(\eta^{'}) = 4\int_{b\Lambda}^{\Lambda}\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\chi^L_a(\eta)\gamma_{ade}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})g_{df}(\eta)u_f P_0(q)g_{hj}(\eta^{'})u_j$$
$$g_{gn}(\eta^{'})u_n P_0(|\mathbf{k}-\mathbf{q}|)u_m g_{em}(\eta)\gamma_{bhg}(-\mathbf{k},-\mathbf{q},\mathbf{q}-\mathbf{k})\chi^L_b(\eta^{'}). \tag{4.11}$$

The factor 4 comes from the fact that the diagram has a symmetry factor 4.

Before calculating, we should notice some approximations in our system. First, the internal momentum $\mathbf{q}$ is much larger than the external momentum $\mathbf{k}$, i.e. $\mathbf{q}\gg\mathbf{k}$, which is natural since $b\Lambda\leq|\mathbf{q}|<\Lambda$. The second is that we will consider only the fastest growing mode, which is the dominate contribution. Since we consider only the fastest growing mode, we have $g_{ab}(\eta)u_b = e^{\eta}u_a$ with the growing mode $u_a = (1,1)$. In the approximation $|\mathbf{k}|\ll|\mathbf{q}|$, the vertex could also be simplified. We denote $\gamma_{abc}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})u_b = \tilde{\gamma}_{ac}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q})$, and

$$\tilde{\gamma}_{ac}(\mathbf{k},\mathbf{q},\mathbf{k}-\mathbf{q}) = \begin{pmatrix} \frac{k}{2q}x & \frac{k}{2q}x \\ 0 & -\frac{k^2}{2q^2} \end{pmatrix},$$

where $x\equiv\cos<\mathbf{k},\mathbf{q}>$, and $<\mathbf{k},\mathbf{q}>$ denotes the angle between these two vectors **k** and **q**. The expression for A_{ab} can be rewritten as

$$A_{ab} = -4i \int_{b\Lambda}^{\Lambda} \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} e^{(\eta+\eta')} \tilde{\gamma}_{ae}(\mathbf{k}, \mathbf{q}, \mathbf{k}-\mathbf{q}) g_{ef}(\eta - \eta')$$
$$\tilde{\gamma}_{gb}(\mathbf{k}-\mathbf{q}, -\mathbf{q}, \mathbf{k}) P_0(|\mathbf{k}-\mathbf{q}|)$$
$$= -\frac{ie^{2\eta}}{5\pi^2} \int_{b\Lambda}^{\Lambda} \mathrm{d}q \int_{-1}^{1} \mathrm{d}x q^2 P_0(q) \begin{pmatrix} \frac{3kx}{2q} - \frac{3k^2x}{2q^2} & \frac{3kx}{2q} - \frac{3k^2x}{2q^2} - x^2 \\ -\frac{1}{2}(\frac{3kx}{2q} - \frac{3k^2x}{2q^2}) & -\frac{1}{2}(\frac{3kx}{2q} - \frac{3k^2x}{2q^2} - x^2) \end{pmatrix}$$
$$= -\frac{ie^{2\eta}}{5\pi^2} \int_{b\Lambda}^{\Lambda} \mathrm{d}q q^2 P_0(q) \begin{pmatrix} 0 & -\frac{2}{3} \\ 0 & -\frac{2}{3} \end{pmatrix}. \tag{4.12}$$

Now we turn to calculate B_{ab}.

$$B_{ab} = 4 \int_{b\Lambda}^{\Lambda} \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} e^{2(\eta+\eta')} \tilde{\gamma}_{ae}(\mathbf{k}, \mathbf{q}, \mathbf{k}-\mathbf{q}) u_e P_0(q)$$
$$u_g \tilde{\gamma}_{bg}(\mathbf{k}, \mathbf{q}, \mathbf{k}-\mathbf{q}) P_0(|\mathbf{k}-\mathbf{q}|)$$
$$= \frac{e^{2(\eta+\eta')}}{\pi^2} \int_{b\Lambda}^{\Lambda} \mathrm{d}q \int_{-1}^{1} \mathrm{d}x 2\pi q^2 P_0(q) P_0(|\mathbf{k}-\mathbf{q}|) \begin{pmatrix} \frac{2kx}{q}(\frac{1}{2} - \frac{k}{2q}) & -\frac{1}{2}x^2 \\ \frac{-k^2}{q^2}(\frac{1}{2} - \frac{k}{2q}) & \frac{kx}{4q} \end{pmatrix}$$
$$= \frac{e^{2(\eta+\eta')}}{\pi^2} \int_{b\Lambda}^{\Lambda} \mathrm{d}q q^2 P_0(q) P_0(|\mathbf{k}-\mathbf{q}|) \begin{pmatrix} 0 & -\frac{1}{3} \\ \frac{-k^2}{q^2} & 0 \end{pmatrix}. \tag{4.13}$$

Finally we come to the form of the effective action, which is the following

$$S_{eff} = \int \mathrm{d}\eta \int_0^{\eta} \mathrm{d}\eta' \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[\frac{1}{2} \left(\Psi_a^L(\eta), \chi_a^L(\eta) \right) M \begin{pmatrix} \Psi_b^L(\eta') \\ \chi_b^L(\eta') \end{pmatrix} - \gamma_{abc} \chi_a^L \Psi_b^L \Psi_c^L \right], \tag{4.14}$$

$$M \equiv \begin{pmatrix} 0 & \delta(\eta-\eta')(-\partial_{\eta'}\delta_{ab}+\Omega_{ba})-iA_{ab} \\ \delta(\eta-\eta')(-\partial_{\eta'}\delta_{ab}+\Omega_{ab})-iA_{ab} & -i\delta(\eta-\eta')\Delta_{0ab}-iB_{ab} \end{pmatrix}. \tag{4.15}$$

4.2 Comparison with effective field theory

4.2.1 Introduction to the effective field theory

Here we give a short introduction to the effective field theory (EFT), and details can be found in the paper [40].

The EFT is a cosmological fluid description for cold dark matter, and by extension all matter including baryons which trace the dark matter. We have shown that the renormalized perturbation theory tries to solve non-linear equations for a pressureless fluid. The EFT works in a different way. It tries to solve non-linear equations for a different fluid with anisotropic pressure and viscosity by matching to N-body simulations. The prediction of the EFT is in percent agreement with the full nonlinear spectrum as obtained by CAMB up to $k \approx 0.24h\mathrm{Mpc}^{-1}$. The main idea of EFT is shown in the following:

- Similar to what we have shown in 2.1, they introduce a smoothing function on a length scale Λ^{-1}, and obtain the Euler equations with an effective stress-tensor $[\tau^{ij}]_\Lambda$ that is sourced by the short-modes δ_s.
- The effective stress-tensor can be defined as a function of the long wavelength fluctuations, including parameters c_s^2 c_v^2 defined by proper correlation functions of short wavelength and long wavelength fluctuations. These parameters can be evaluated from N-body simulations.
- Then start the perturbation theory with the EFT. For example, for the one-loop perturbation theory, the nonlinear equations of motion are,

$$\frac{\partial \delta_l}{\partial \tau} + \theta_l = -\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3} \delta_{\mathrm{D}}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\alpha(\mathbf{k}_1,\mathbf{k}_2)\delta_l(\mathbf{k}_1,\tau)\theta_l(\mathbf{k}_2,\tau), \tag{4.16}$$

and

$$\frac{\partial \theta_l}{\partial \tau} + \mathcal{H}\theta_l + \frac{3}{2}\Omega_m \mathcal{H}_0^2 \delta_l - c_s^2 k^2 \delta_l + \frac{c_v^2 k^2}{\mathcal{H}}\theta_l = -\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3} \delta_{\mathrm{D}}(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\beta(\mathbf{k}_1,\mathbf{k}_2)\theta_l(\mathbf{k}_1,\tau)\theta_l(\mathbf{k}_2,\tau), \tag{4.17}$$

where index "l" means long wavelength modes, and index "0" for $\mathcal{H}$ means that it is evaluated at present time. We can see that it has extra terms $(-c_s^2 k^2 \delta_l + \frac{c_v^2 k^2}{\mathcal{H}}\theta_l)$ compared with Eq. (2.23) and (2.24) in our early introduction.

Generated terms and Galilean invariance

Since our system is described in terms of Newtonian dynamics, it should keep Galilean invariance. Galilean invariance states that the laws of motion are the same in all inertial frames. So in principle, terms generated by the renormalization flow should obey Galilean invariance. Galilean invariance of the evolution of large-scale structures has been explored recently. Ref. [41]

shows how to do Galilean transformations for different systems. Ref. [17] states the implications of the Galilean invariance. The first implication is the corresponding Ward identity, which further implies the consistency relation between the power spectrum and the bispectrum. The second is that this principle provides a way to construct an effective field theory of large-scale structure, which is shown in Ref. [42] in detail.

4.2.2 Comparison with EFT

Now we go back to the effective action, Eq. (4.14), we find that

- $\chi_a^L(\eta) A_{ab} \Psi_b^L(s_2)$ gives $\Lambda^n \theta$ in the evolution equation.
- $\chi_a^L(\eta) B_{ab} \chi_b^L(\eta^{'})$ means a continuous stochastic source.

Comparing with current theories, we find that,

- Renormalized theory: the renormalized propagator is different, and has cut-off dependent.
- EFT: there are terms $k^2\delta$ and $k^2\theta$ in the evolution equation.

In conclusions, our renormalized approach shows cut-off dependence, and that a continuous stochastic source arises due to the renormalization, which are expected. The sad thing is that our effective action is different from the published renormalization theory and EFT, and we expect that the prediction of the power spectrum would not be satisfying. In our work, the question still remains: why our result is different from the published work about renormalization? We will try to answer it in the future.

Chapter 5

The evolution of a scalar field in curved spacetime–comoving gauge

Conventionally, the system is described in terms of nonlinear equations for density and velocity perturbations with Gaussian initial conditions. Standard perturbation theory has ambiguous expansion parameter and cut-off dependence. The predictions of EFT match with observations and simulations very well, but it needs to read parameters from simulation. Here we will explore a completely new formalism to describe the system, with the hope that we could control the cut-off dependence by dimensional regulariztion. We want to describe the system in terms of the effective action of the perturbed matter field with non-Gaussian initial conditions, which might reduce the nonlinear problem. We will study the evolution of LSS as perturbed matter fields evolving in curved spacetime. For simplicity, we will first consider a scalar field, try to solve the constraints of the spacetime metric to get the action for the perturbed matter field, and then see the predictions of our formalism. Some introduction to general relativity and gauge invariance is given at the beginning of this chapter.

5.1 Recap General Relativity

In this section, we will recap the theory of general relativity. Details can be found in many textbooks (for example, [43, 44]). Einstein's equivalence principle states that an observer cannot perform a local experiment, based on which he or she would be able to conclude whether he or she is placed in an accelerating or a gravitating system. This principle implies the general covariance principle when a gravitational system is described in terms of spacetime coordinates x^μ and the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. According to the general covariance principle, any physical observable in a metric theory of gravitation should be invariant under general coordinate transformations $x^\mu \to \tilde{x}^\mu(x)$. The metric $g_{\mu\nu}$ transforms as a tensor. The covariance principle requires the general derivative ∂_μ to be promoted to the covariant derivative D_μ. This introduces the Christoffel connection $\Gamma^\lambda_{\mu\nu}$, defined as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right). \tag{5.1}$$

And the covariant derivative of a covariant vector A_μ is, $D_\nu A_\mu \equiv \partial_\nu A_\mu - \Gamma^\lambda_{\mu\nu}A_\lambda$

Because space-time is curved, repeated covariant differentiation does not commute. For a vector A^μ,

$$D_\alpha(D_\beta A^\mu) - D_\beta(D_\alpha A^\mu) = R^\mu_{\nu\alpha\beta}A^\nu, \tag{5.2}$$

where the curvature tensor is defined by

$$R^\mu_{\nu\alpha\beta} \equiv \partial_\alpha\Gamma^\mu_{\nu\beta} - \partial_\beta\Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\nu\alpha}. \tag{5.3}$$

From the curvature tensor, we can define the Ricci tensor $R_{\mu\nu} \equiv R^\lambda_{\mu\lambda\nu}$, the curvature scalar $R \equiv R^\mu_\mu$ and the Einstein curvature tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.

The Einstein field equation for the classical theory of gravitation is

$$G_{\mu\nu} - \frac{\Lambda}{c^2}g_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}, \tag{5.4}$$

where Λ is the cosmological constant term, $T_{\mu\nu}$ is the energy-momentum tensor, and G_N is the Newton gravitational constant. The equation can be obtained by postulating the principle of general covariance, by requiring that in the weak field non-relativistic limit one recovers the Newton theory of gravitation, and by requiring that the equation of motion contains at most two time derivatives. Here we will show how to derive it from the Hilbert-Einstein action

$$S = S_{HE} + S_{matter}, \tag{5.5}$$

$$S_{HE} = -\int \mathrm{d}^4x\sqrt{-g}\frac{c^4}{16\pi G_N}\left(R + 2\frac{\Lambda}{c^2}\right), \tag{5.6}$$

$$S_{matter} = \int \mathrm{d}^4x\sqrt{-g}\mathcal{L}_{matter}, \tag{5.7}$$

where $g = \det[g_{\mu\nu}]$ is the determinant of the metric tensor, Λ is assumed to be zero in our case, and $\sqrt{g}\mathcal{L}_{matter}$ is the matter field Lagrangian.
The energy-momentum tensor $T_{\mu\nu}$ for the matter field is defined as

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}}. \tag{5.8}$$

Varying the action S in terms of the metric tensor $g^{\mu\nu}$, we have the following form,

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\delta g^{\mu\nu}\left[-\frac{c^4}{16\pi G_N}G_{\mu\nu} + \frac{1}{2}T_{\mu\nu}\right], \tag{5.9}$$

which gives the Einstein equation. For convenience, we will use the convention $c = 1$ and denote $8\pi G_N = M_p^2$, where M_p is called the Planck mass.

Now we apply the theory of general relativity to the evolution of cosmology. The cosmological principle states that the universe is spatially isotropic and homogeneous on large scales. Accordingly, the line element of the most general homogeneous spacetime is of the form,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d^2\phi)\right], \tag{5.10}$$

where $a(t)$ is the scale factor, and r, θ and ϕ are spherical coordinates, and $k = (-1; 0; 1)$ determines if the universe has constant negative, zero, or positive spatial curvature. The metric is called the Friedmann-Lemaître-Robertson-Walker (FLRW) metric.

For a spatially flat universe, $k = 0$, in Cartesian coordinates, the metric is $g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2)$ and the Einstein curvature tensor is the following,

$$G_{00} = 3\frac{\dot{a}^2}{a^2}, \tag{5.11}$$

$$G_{0i} = 0, \tag{5.12}$$

$$G_{ij} = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) g_{ij}. \tag{5.13}$$

For an ideal fluid, its energy-momentum tensor is of the form,

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu}, \tag{5.14}$$

where u_μ, ρ and P denote the four-velocity, energy density and pressure of the fluid, respectively. In a rest frame, $u_\mu = \delta^0_\mu$, and the energy-momentum tensor reduces to,

$$T_{00} = \rho,\ T_{ij} = P g_{ij}. \tag{5.15}$$

The (00) and (ij) components of the Einstein equation can be recast as the following FLRW equations,

$$\frac{\dot{a}^2}{a^2} = \frac{1}{M_p^2}\frac{\rho}{3}, \tag{5.16}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_p^2}(\rho + 3P). \tag{5.17}$$

These equations govern the evolution of the Universe in the Standard Cosmological Model.

Here the FLRW equations are derived from homogeneous spacetime. For the system of large-scale structure, we should consider inhomogeneous spacetime. It is worth noting that these equations can also be derived from inhomogeneous spacetime, which will be shown later.

5.1.1 Gauge invariance

In this section, we will give a short introduction to the gauge invariance of cosmological perturbations. We should point out that our formalism will be developed in a fixed gauge, however, some arguments in our formalism are related to the gauge invariance.

According to the general covariance principle, physics remains the same under coordinate transformations. For the unperturbed universe, there are preferred coordinates, which are of the form $\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$. There are some advantages of this coordinate system. First, the time coordinate is just proper time along the world-line, and co-moving observers can be defined to describe the expansion of the universe. Secondly, there is no mixing of space and time in the metric, and the spatial coordinates, when expressed in terms of spherical coordinates, show that the universe is homogeneous. For the perturbed universe, however, there is no uniquely preferred choice. The only constraints are that the coordinates must reduce to the standard unperturbed universe in the limit where the perturbations vanish, and that physics is the same in different gauges. Choosing a specific gauge means choose a slicing and threading of space-time, which is explained in detail in the appendices A about the $3+1$ formalism. Different gauges have different advantages in discussing perturbations. The detail of different gauges and how to do gauge transformations can be found in [9, 45].

5.2 The Hilbert-Einstein action

We choose $g_{\mu\nu} = \mathrm{diag}(-N^2(t,\mathbf{x}), a^2 e^{-2\Psi_N(t,\mathbf{X})}, a^2 e^{-2\Psi_N(t,\mathbf{X})}, a^2 e^{-2\Psi_N(t,\mathbf{X})})$ as the perturbed metric. We can define $N(t,\mathbf{x}) \equiv \bar{N}(t) e^{\Phi_N(t,\mathbf{X})}$, where $\bar{N}(t)$ is called the elapse function, and $\Phi_N(t,\mathbf{x})$ is the Newtonian potential. If we set $\bar{N}(t) = 1$ and expand the exponent to first order, we could recover the conformal Newtonian gauge.

The corresponding Hilbert-Einstein action $S_{HE} = \int \mathrm{d}^4x \frac{M_p^2}{2}\sqrt{-g}R$ is the following,

$$\begin{aligned} \frac{1}{2}\sqrt{-g}R = & -ae^{-\Psi_N}[-\partial_i N \partial_i \Psi_N + \nabla^2 N + N((\nabla\Psi_N)^2 - 2\nabla^2\Psi_N)] \\ & + \frac{3ae^{-3\Psi_N}}{N^2}\left\{a\partial_0 N(-a^{'} + a\partial_0\Psi_N) + N\left[a^{'2} - 4aa^{'}\partial_0\Psi_N + a(a^{''} + a(2(\partial_0\Psi_N)^2 - \partial_0^2\Psi_N))\right]\right\} \end{aligned} \tag{5.18}$$

where $a^{'} = \frac{\mathrm{d}a}{\mathrm{d}t}$.
We can do partial integration to the first term in the second part of Eq.

(5.18), and we have

$$
\begin{aligned}
A \equiv & \frac{3ae^{-3\Psi_N}}{N^2}[a\partial_0 N(-a^{'}+a\partial_0\Psi_N)] \\
= & -3a^2e^{-3\Psi_N}(-a^{'}+a\partial_0\Psi_N)\partial_0(\frac{1}{N}) \\
= & -\partial_0\left[3a^2e^{-3\Psi_N}(-a^{'}+a\partial_0\Psi_N)\frac{1}{N}\right]+\frac{3}{N}\partial_0\left[a^2e^{-3\Psi_N}(-a^{'}+a\partial_0\Psi_N)\right] \\
= & -\partial_0\left[3a^2e^{-3\Psi_N}(-a^{'}+a\partial_0\Psi_N)\frac{1}{N}\right]+\frac{3a}{N}e^{-3\Psi_N}\left[-(2a^{'2}+aa^{''})+6aa^{'}\partial_0\Psi_N\right] \\
+ & \frac{3a}{N}e^{-3\Psi_N}\left[a^2(-3(\partial_0\Psi_N)^2+\partial_0^2\Psi_N)\right]. \qquad (5.19)
\end{aligned}
$$

Inserting the result form the partial integration back into $-\frac{1}{2}\sqrt{-g}R$, we arrive at the following expression,

$$
\begin{aligned}
\frac{1}{2}\sqrt{-g}R = & -ae^{-\Psi_N}\left[-\partial_i N\partial_i\Psi_N+\nabla^2N+N((\nabla\Psi_N)^2-2\nabla^2\Psi_N)\right] \\
& +\frac{3ae^{-3\Psi_N}}{N}\left[-a^{'2}+2aa^{'}\partial_0\Psi_N-a^2(\partial_0\Psi_N)^2\right]-\partial_0\left[\frac{3a^2}{N}e^{-3\Psi_N}(-a^{'}+a\partial_0\Psi_N)\right].
\end{aligned}
\qquad (5.20)
$$

Ignoring the boundary term, we can rewrite it into the following form,

$$
\begin{aligned}
\frac{1}{2}\sqrt{-g}R = & -ae^{-\Psi_N}[-\bar{N}e^{\Phi_N}\partial_i\Phi_N\partial_i\Psi_N+\bar{N}e^{\Phi_N}((\nabla\Phi_N)^2+\nabla^2\Phi_N)+\bar{N}e^{\Phi_N}((\nabla\Psi_N)^2-2\nabla^2\Psi_N)] \\
& +\frac{3ae^{-3\Psi_N-\Phi_N}}{\bar{N}}[-a^{'2}+2aa^{'}\partial_0\Psi_N-a^2(\partial_0\Psi_N)^2] \\
= & -a\bar{N}e^{-\Psi_N+\Phi_N}[-\partial_i\Phi_N\partial_i\Psi_N+(\nabla\Phi_N)^2+\nabla^2\Phi_N+(\nabla\Psi_N)^2-2\nabla^2\Psi_N] \\
& +\frac{3ae^{-3\Psi_N-\Phi_N}}{\bar{N}}[-a^{'2}+2aa^{'}\partial_0\Psi_N-a^2(\partial_0\Psi_N)^2]. \qquad (5.21)
\end{aligned}
$$

Setting $\Phi_N(t,\mathbf{x})=\Psi_N(t,\mathbf{x})=0$, we obtain the background Hilbert-Einstein action S_{0HE},

$$
S_{0HE} = -\int \mathrm{d}^4x M_p^2\frac{3aa^{'2}}{\bar{N}}. \qquad (5.22)
$$

Expanding the Hilbert-Einstein action in terms of $\Phi_N(t,\mathbf{x})$ and $\Psi_N(t,\mathbf{x})$, we have the first order action S_{1HE} and the second order action S_{2HE} in the following,

$$S_{1HE} = \int \mathrm{d}^4x M_p^2 [\frac{3aa^{'2}}{\bar{N}}(3\Psi_N + \Phi_N) + \frac{6a^2a^{'}}{\bar{N}}\partial_0\Psi_N - a\bar{N}(\nabla^2\Phi_N - 2\nabla^2\Psi_N)], \tag{5.23}$$

$$S_{2HE} = \int \mathrm{d}^4x M_p^2[-a\bar{N}[-\partial_i\Phi_N\partial_i\Psi_N + (\nabla\Phi_N)^2 + (\nabla\Psi_N)^2 + (-\Psi_N + \Phi_N)(\nabla^2\Phi_N - 2\nabla^2\Psi_N)]]$$
$$\int \mathrm{d}^4x M_p^2[-\frac{3a}{\bar{N}}[2(3\Psi_N + \Phi_N)aa^{'}\partial_0\Psi_N + a^2(\partial_0\Psi_N)^2] - \frac{3aa^{'2}}{\bar{N}}\frac{(3\Psi_N + \Phi_N)^2}{2}]. \tag{5.24}$$

We consider the scalar field $\Phi(t, \mathbf{x}) = \Phi_0(t) + \varphi(t, \mathbf{x})$ as the matter field. $\Phi_0(t)$ is the homogeneous background field, which depends only on time and drives the evolution of the scale factor, and $\psi(t, \mathbf{x})$ is the perturbation field, which contributes to the Newtonian potentials. The action for the matter field is the following,

$$\begin{aligned} S_{matter} &= \int \mathrm{d}^4x \sqrt{-g}\mathcal{L}_{matter} \\ &= \int \mathrm{d}^4x \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi)\right] \\ &= \int \mathrm{d}^4x\, ae^{-3\Psi_N}\left(\frac{a^2}{2N}\Phi^{'2} - e^{2\Psi_N}\frac{N}{2}(\nabla\Phi)^2 - a^2NV(\Phi)\right) \\ &= \int \mathrm{d}^4x\, a\bar{N}e^{-3\Psi_N+\Phi_N}\left(\frac{a^2\dot{\Phi}^2}{2} - e^{2\Psi_N}\frac{(\nabla\Phi)^2}{2} - a^2V(\Phi)\right). \end{aligned} \tag{5.25}$$

The last line of the equation (5.25) is obtained by defining a new time derivative $\mathrm{d}\tau \equiv \bar{N}\mathrm{d}t$, writing $\frac{\mathrm{d}A}{\mathrm{d}\tau} \equiv \dot{A}$ and inserting the resulting identity $\dot{\Phi} = \frac{\Phi^{'}}{\bar{N}}$.

We should point out that the action for the matter field contains two types of perturbed quantities, the perturbed metric $\Phi_N(t, \mathbf{x})$ and $\Psi_N(t, \mathbf{x})$ and the perturbed field. Later we will see how these two types of perturbed terms contribute to the action differently.

Setting $\Phi_N(t, \mathbf{x}) = \Psi_N(t, \mathbf{x}) = 0$, we obtain the action for the background matter field,

$$S_{0matter} = \int \mathrm{d}^4x\, a\bar{N}\left(\frac{a^2\Phi^{'2}}{2\bar{N}^2} - \frac{(\nabla\Phi)^2}{2} - a^2V(\Phi_0)\right). \tag{5.26}$$

We will ignore the back-reaction from the perturbation field, and only consider the homogeneous background field $\Phi_0(t)$ as the source for driving the evolution of the scale factor. Since $\Phi_0(t)$ is homogeneous, $\nabla\Phi_0(t) = 0$. So the action for the background matter field reduces to

$$S_{0matter} = \int \mathrm{d}^4x \; a\bar{N}\left(\frac{a^2{\Phi_0}'^2}{2\bar{N}^2} - a^2 V(\Phi_0)\right). \tag{5.27}$$

The corresponding energy-momentum tensor is $(\rho + P)u_\mu u_\nu + P g_{\mu\nu}$, with

$$\rho = \frac{{\Phi_0}'^2}{2\bar{N}^2} + V(\Phi_0),\; P = \frac{{\Phi_0}'^2}{2\bar{N}^2} - V(\Phi_0). \tag{5.28}$$

When $\Phi_N(t,\mathbf{x})$ and $\Psi_N(t,\mathbf{x})$ do not vanish, we could expand the action for the background in terms of them to first and second order. The corresponding actions as the following,

$$S_{1matter} = \int \mathrm{d}^4x \; a\bar{N}\left(-(3\Psi_N + \Phi_N)\frac{a^2{\Phi_0}'^2}{2\bar{N}^2} - (-3\Psi_N + \Phi_N)a^2 V\right), \tag{5.29}$$

$$S_{2matter} = \int \mathrm{d}^4x \; a\bar{N}\left(\frac{(3\Psi_N + \Phi_N)^2}{2}\frac{a^2{\Phi_0}'^2}{2\bar{N}^2} - a^2 V\frac{(-3\Psi_N + \Phi_N)^2}{2}\right). \tag{5.30}$$

It is worth noticing that here we only expand the action for the background field in terms of the perturbed metric, and we will consider the perturbed matter field later.

5.2.1 The zeroth order–the FLRW equations

For the background, the total action is

$$S_0 = S_{0HE} + S_{0matter} = -\int \mathrm{d}^4x \; M_p^2 \frac{3a{a'}^2}{\bar{N}} + \int \mathrm{d}^4x a\bar{N}\left(\frac{a^2{\Phi_0'}^2}{2\bar{N}^2} - a^2 V(\Phi_0)\right). \tag{5.31}$$

Varying the action S_0 with respect to $\bar{N}$ and a respectively, we can recover the FLRW equations,

$$\frac{\delta S_0}{\delta \bar{N}} = 0 = M_p^2 \frac{3a{a'}^2}{\bar{N}^2} + a\left(-\frac{a^2{\Phi_0}'^2}{2\bar{N}^2} - a^2 V(\Phi_0)\right), \tag{5.32}$$

$$\frac{\delta S_0}{\delta a} = 0 = M_p^2 \left[\frac{3a'^2}{\bar{N}} - 2(\frac{3aa'}{\bar{N}})' \right] + 3a^2 \bar{N} \left(-\frac{\Phi_0'^2}{2\bar{N}^2} - V(\Phi_0) \right). \quad (5.33)$$

We can define a new time derivative $\mathrm{d}\tau \equiv \bar{N}\mathrm{d}t$, write $\frac{\mathrm{d}A}{\mathrm{d}\tau} \equiv \dot{A}$ and insert the resulted identity $\dot{\Phi} = \frac{\Phi'}{\bar{N}}$ into Eq. (5.32) and Eq. (5.33), and we arrive at the following equations,

$$\frac{\dot{a}^2}{a^2} = \frac{1}{3M_p^2} \left(\frac{\dot{\Phi}_0^2}{2} + V(\Phi_0) \right), \quad (5.34)$$

$$\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} = -\frac{1}{M_p^2} \left(\frac{\dot{\Phi}_0^2}{2} - V(\Phi_0) \right). \quad (5.35)$$

Inserting Eq. (5.34) into Eq. (5.35), we get the following equation,

$$\frac{\ddot{a}}{a} = -\frac{1}{3M_p^2} \left(\dot{\Phi}_0^2 - V \right). \quad (5.36)$$

From the explicit expression of the energy-momentum tensor Eq. (5.28), we can see that they are the FLRW equations.

5.2.2 The first order

For the first order, combining Eq. (5.23) and Eq. (5.29), we have the first-order total action $S_1 = S_{1HE} + S_{1matter}$,

$$\begin{aligned} S_1 = &\int \mathrm{d}^4x \, \frac{1}{M_p^2} [\frac{3aa'^2}{\bar{N}}(3\Psi_N + \Phi_N) + \frac{6a^2 a'}{\bar{N}} \partial_0 \Psi_N] \\ &+ \int \mathrm{d}^4x \, a\bar{N} \left(-(3\Psi_N + \Phi_N)\frac{a^2 \Phi_0'^2}{2\bar{N}^2} - (-\Psi_N + \Phi_N)\frac{(\nabla\Phi_0)^2}{2} - (-3\Psi_N + \Phi_N)a^2 V \right) \end{aligned} \quad (5.37)$$

Varying the action S_1 with respect to Φ_N and Ψ_N respectively, we have

$$\frac{1}{\bar{N}} \frac{\delta S_1}{\delta \Phi_N} = 0 = M_p^2 \frac{3aa'^2}{\bar{N}^2} + a \left(-\frac{a^2 \Phi_0'^2}{2\bar{N}^2} - a^2 V \right), \quad (5.38)$$

$$\frac{1}{\bar{N}}\frac{\delta S_1}{\delta \Psi_N} = 0 = M_p^2\left[\frac{9aa'^2}{\bar{N}^2} - \frac{6}{\bar{N}}(\frac{a^2a'}{\bar{N}})'\right] + 3a^3\bar{N}\left(-\frac{\Phi_0'^2}{2\bar{N}^2} + V\right). \qquad (5.39)$$

After simplifying them, we can see that they are exactly the same as the FLRW equations (5.34) and (5.35), hence the first order action is linear proportional to the background action, thus can be set to zero.

5.2.3 The second order

For the second order of the Hilbert-Einstein action, the matter fields are the second-order background field and the perturbed field. The total second-order action is $S_2 = S_{2HE} + S_{2matter} + S_{Pmatter}$, where $S_{Pmatter}$ is of the following form,

$$\begin{aligned} S_{Pmatter} &= \int \mathrm{d}^4x\,\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}m^2\varphi^2\right] \\ &= \int \mathrm{d}^4x\; a\bar{N}e^{-3\Psi_N+\Phi_N}\left(\frac{a^2}{2\bar{N}^2}e^{-2\Phi_N}\varphi'^2 - e^{2\Psi_N}\frac{(\nabla\varphi)^2}{2} - \frac{1}{2}a^2m^2\varphi^2\right), \end{aligned} \qquad (5.40)$$

where we have used $V'' = m^2$.

Varying the action S_2 with respect to Φ_N and Ψ_N respectively, we have

$$\begin{aligned} 0 &= \frac{\delta S_2}{\delta\Phi_N}\frac{1}{2a^3\bar{N}} \\ &= M_p^2[\frac{\nabla^2\Phi_N}{a^2} - \frac{3\dot{a}^2}{2a^2}(3\Psi_N + \Phi_N) - \frac{3\dot{a}}{a}\dot{\Psi}_N] \\ &+ \frac{1}{2}\left[(3\Psi_N + \Phi_N)\frac{\dot{\Phi}_0^2}{2} - V_0(-3\Psi_N + \Phi_N)\right] \\ &+ \frac{1}{2}\left[-\frac{\dot{\varphi}^2}{2} - \frac{(\nabla\varphi)^2}{2a^2} - \frac{1}{2}m^2\varphi^2\right], \end{aligned} \qquad (5.41)$$

and

$$
\begin{aligned}
0 &= \frac{\delta S_2}{\delta \Psi_N}\frac{1}{2a^3\bar{N}} \\
&= -\frac{1}{2}\left[(3\Psi_N+\Phi_N)\frac{3\dot{\Phi}_0^2}{2}+3V_0(-3\Psi_N+\Phi_N)\right]-\frac{1}{2}\left[-\frac{3\dot{\varphi}^2}{2}+\frac{(\nabla\varphi)^2}{2a^2}-\frac{3}{2}m^2\varphi^2\right] \\
&+ M_p^2\left[\frac{\nabla^2(\Psi_N-\Phi_N)}{a^2}-\frac{3}{a^3\bar{N}}\partial_0(a^3\dot{\Psi}_N)-\frac{3}{a^3\bar{N}}\partial_0(a^2\dot{a}(3\Psi_N+\Phi_N))+\frac{9\dot{a}}{a}\dot{\Psi}_N\right] \\
&+ M_p^2\left[\frac{9\dot{a}^2}{a^2}(3\Psi_N+\Phi_N)\right]. \qquad (5.42)
\end{aligned}
$$

We can replace the terms of background fields with the FLRW equations, and finally arrive at the following equations,

$$
\frac{\nabla^2\Phi_N}{a^2}-3H^2\Phi_N-3H\dot{\Psi}_N-\dot{H}\Phi_N=\frac{1}{2M_p^2}\left[\frac{\dot{\varphi}^2}{2}-\frac{(\nabla\varphi)^2}{2a^2}+\frac{1}{2}m^2\varphi^2\right], \quad (5.43)
$$

and

$$
\begin{aligned}
0 &= \frac{\nabla^2(\Phi_N-\Psi_N)}{3a^2}+\ddot{\Psi}_N+(3\dot{\Psi}_N+\dot{\Phi}_N)H+\dot{H}\Phi_N-(3\Psi_N-2\Phi_N)H^2 \\
&+ \frac{1}{2M_p^2}\left[-\frac{\dot{\varphi}^2}{2}+\frac{(\nabla\varphi)^2}{6a^2}+\frac{1}{2}m^2\varphi^2\right]. \qquad (5.44)
\end{aligned}
$$

These equations are very complicated. Our approach is similar to the approach in [9], and the main difference is that they have used the gauge invariant form. In the gauge invariant form, doing variation with respect to the unfixed gauge quantities B_i and E_{ij} could give two additional constraint equations, one of which is $\Phi_N = \Psi_N$. The constraint equations could be used to simplify the above equations a lot. However, here we are using a fixed gauge, so we could not obtain the constraint equations in the same way.

Then the question arises: why are these equations so complicated? We should notice that the gauge invariant form of the scalar perturbation contains two parts, the perturbed field φ and the perturbed metric Ψ_N. So even when we start from a fixed gauge, the scalar perturbation is not gauge-fixed. This explains why we have got these complicates equations.

To make things simpler, we want to fix the scalar perturbation with $\Psi_N = 0$, which motivates us to start with a different fixed gauge in a different formalism.

Chapter 6

The evolution of a scalar field in curved spacetime–zero-curvature gauge

Recall that our idea is to consider a scalar field, try to solve the constraints of the spacetime metric to get the action for the perturbed matter field, and then see the predictions of our formalism. In the last chapter, we have shown that it is difficult to realize the idea. In this chapter, we will try to realize this idea in a different gauge–the zero-curvature gauge, and we will use the ADM formalism for the Hilbert-Einstein action.

6.1 The ADM formalism

The ADM formalism [46] has been introduced to resolve the problem caused by the general coordinate invariance underlying Einstein's gravitational theory. In Newton's gravitational theory, once the initial values of the field amplitudes and their first time derivatives are specified, the time development of a field viewed as a dynamical quantity will be determined. In general relativity, however, the metric field $g_{\mu\nu}$ could be modified at any later time simply by carrying out a general coordinate transformation, which does not effect the observable quantities. Thus it is necessary to separate the metric field into the parts carrying the dynamical information and those characterizing the coordinate system. In the ADM formalism, by making use of the Hilbert-Einstein action and the $3+1$ formalism, the theory can be cast into canonical form which determines the independent dynamical modes of the gravitational field. In our approach, we will focus on the scalar perturbation and ignore the tensor perturbation, i.e, the gravitational wave.

In this section we will introduce the ADM formalism, and reproduce part of Maldacena's work [31]. In the next chapter, we will adapt this approach to the evolution of large-scale structures.

In the ADM formalism, the metric takes the form

$$ds^2 = -(Ndt)^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{6.1}$$

and the Hilbert-Einstein action becomes

$$S_{HE} = \int \mathrm{d}^4x \frac{M_p^2}{2}\sqrt{h}\left[N^{(3)}R + N^{-1}(E_{ij}E^{ij} - E^2)\right], \tag{6.2}$$

where $E_{ij} = Nk_{ij}$, and K_{ij} is the extrinsic curvature defined in Eq. (A.19) and (A.27).

The details of the derivation are shown in our appendices A. It should be pointed out that in the ADM formalism, we use the $(-,+,+,+)$ convention for the metric. The ADM formulation of the Hilbert-Einstein explicitly shows that spatial coordinates are re-parametrization invariant. Later we will show that the system is also time re-parametrization invariant.

We describe the matter field as a scalar field Φ, which can be written in terms of the background field and the perturbed field. So the total action is the following,

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{h}\left[M_p^2[N^{(3)}R + N^{-1}(E_{ij}E^{ij} - E^2)] - 2NV + N^{-1}(\partial_0\Phi - N^i\partial_i\Phi)^2 - Nh^{ij}\partial_i\Phi\partial_j\Phi\right], \tag{6.3}$$

where h_{ij} and Φ are dynamical variables while N and N^i are constraint fields. It is worth noticing that we should use h_{ij} and h^{ij} to lower and raise the indices.

6.1.1 Solving the constraints

The following approach is similar to Maldacena's approach [31]. We will choose a gauge for h_{ij} and Φ that fix spatial re-parametrization. A convenient gauge is

$$\delta\Phi \equiv \varphi(t,x),\ h_{ij} = a^2(\delta_{ij} + \gamma_{ij}),\ \partial_i\gamma_{ij} = 0,\ \gamma_{ii} = 0, \tag{6.4}$$

where $a(t)$ is the scale factor, φ is the fluctuation of the scalar field (and is a first order quantity), and γ_{ij} is the gravitational wave, which will be ignored in our case. In this gauge, ${}^{(3)}R = 0$. To be clear, we will denote Φ_0 as the background value of the scalar field and φ as the perturbation field. The background field Φ_0 only depends on time, and the potential can also be splitted into the background part and the perturbation part,

$$V = V_0 + V^{'}|_{\varphi=0}\varphi + V^{''}|_{\varphi=0}\frac{\varphi^2}{2} + V^{'''}|_{\varphi=0}\frac{\varphi^3}{6} + \cdots, \tag{6.5}$$

where the dots mean higher derivatives and will be ignored.

We can rewrite the action into

$$\begin{aligned} S &= \frac{1}{2}\int \mathrm{d}^4x\sqrt{h}\left[M_p^2 N^{-1}(E_{ij}E^{ij} - E^2) - 2N(V_0 + V^{'}\varphi + V^{''}\frac{\varphi^2}{2} + V^{'''}\frac{\varphi^3}{6})\right] \\ &+ \frac{1}{2}\int \mathrm{d}^4x\sqrt{h}\left[N^{-1}(\partial_0\Phi_0 + \partial_0\varphi - N^i\partial_i\varphi)^2 - N\frac{(\nabla\varphi)^2}{a^2}\right]. \end{aligned} \tag{6.6}$$

To find the action for the perturbation field, we need to solve for N and N^i through their equations of motion and insert the results back into the action. Doing variation with respect to N^i and N, we arrive at the following momentum and Hamiltonian constraints,

$$2\nabla_i[N^{-1}(E^i{}_j - \delta^i{}_jE)] - 2M_p^{-2}N^{-1}\partial_j\varphi(\partial_0\Phi_0 + \partial_0\varphi - N^i\partial_i\varphi) = 0, \tag{6.7}$$

$$\begin{aligned} 0 &= -M_p^{-2}\left[2(V_0 + V^{'}\varphi + V^{''}\frac{\varphi^2}{2} + V^{'''}\frac{\varphi^3}{6}) + \frac{(\nabla\varphi)^2}{a^2} + N^{-2}(\partial_0\Phi_0 + \partial_0\varphi - N^i\partial_i\varphi)^2\right] \\ &- N^{-2}(E_{ij}E^{ij} - E^2). \end{aligned} \tag{6.8}$$

We can solve these equations to first order by introducing the following parametrization,

$$N = \bar{N}(t)(1 + n(t,x)),\ N^i = a^{-1}\bar{N}(a^{-1}\partial_i s + n_i{}^T), \tag{6.9}$$

where $\partial_i n_i{}^T = 0$. We define $\bar{N}^{-1}\partial_0\Phi_0 \equiv \dot{\Phi}_0$.

Similar to what we have done in the last section, the background FLRW equations and field equation can be obtained by a variation of the background action with respect to $\bar{N}$, a and Φ_0. These equations are, respectively,

$$\begin{aligned} H^2 &= \frac{1}{3M_p^2}(\frac{\dot{\Phi}_0{}^2}{2} + V_0), \\ \dot{H} &= -\frac{\dot{\Phi}_0{}^2}{2M_p^2}, \\ 0 &= \ddot{\Phi}_0 + 3H\dot{\Phi}_0 + V', \end{aligned} \tag{6.10}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter.

To first order, the constraints are

$$\partial_j[(1-n)(-4H)] - a^{-1}\partial_i{}^2 n_j{}^T - 2M_p^{-2}\partial_j\varphi\dot{\Phi}_0 = 0, \tag{6.11}$$

$$H(-6H + \frac{4}{a^2}\nabla^2 s) = M_p^{-2}[-2(V_0 + V'\varphi)(1+2n) - (\dot{\Phi}_0{}^2 + 2\dot{\Phi}_0\dot{\varphi})]. \tag{6.12}$$

In Eq. (6.11), n and $n_i{}^T$ are independent, and thus we have

$$\partial_j[(1-n)(-4H)] - 2M_p^{-2}\partial_j\varphi\dot{\Phi}_0 = 0,\ \partial_i{}^2 n_j{}^T = 0, \tag{6.13}$$

which give the results,

$$n = \frac{\dot{\Phi}_0}{2M_p^2 H}\varphi,\ n_i{}^T = 0. \tag{6.14}$$

Making use of the solutions for n and $n_i{}^T$, the background FLRW equations and the field equation, we obtain the following result,

$$\nabla^2 s = \frac{a^2\dot{\Phi}_0{}^2}{2M_p^2 H^2}\bar{N}^{-1}\partial_0(-\frac{H}{\dot{\Phi}_0}\varphi). \tag{6.15}$$

6.1.2 The effective action

Now we can insert the solutions for N and N^i back into the action and expand the action up to the third order. To do this, it is enough to compute N and N^i to first order. The reason is that the second order terms in N will multiply the first order constraint equation which vanishes, and the third order term in N will multiply the zeroth order constraint equation which again vanishes. The argument for N^i is similar. The detail of this proof can be found in Appendix B of the paper [47].

After some lenthy calculation, we obtain the following actions,

$$S_0 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3\left[-V_0 - \frac{2\dot{\Phi}_0}{H}V_0\varphi - 2V^{'}\varphi\right], \tag{6.16}$$

$$S_2 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3\left[-\frac{\dot{\Phi_0}^2}{M_p^4H^2}V_0\varphi^2 - \frac{2\dot{\Phi}_0}{M_p^2H}V^{'}\varphi^2 - V^{''}\varphi^2 + \dot{\varphi}^2 - \frac{(\nabla\varphi)^2}{a^2}\right], \tag{6.17}$$

$$\begin{aligned} S_3 &= \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-2nV^{''}\frac{\varphi^2}{2} - V^{'''}\frac{\varphi^3}{3} - n^3(-6H^2) - n^3(\dot{\Phi}_0)^2] \\ &+ \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-n\dot{\varphi}^2 - na^{-2}(\nabla\varphi)^2] \\ &+ \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-n(a^{-4}(\partial_j\partial_k s)^2 - a^{-4}(\nabla^2 s)^2)] \\ &+ \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[n^2\frac{2H\nabla^2 s}{a^2} + 2n^2\dot{\Phi}_0\dot{\varphi} - 2\dot{\varphi}a^{-2}\partial_i s\partial_i\varphi], \end{aligned} \tag{6.18}$$

where S_2 and S_3 mean the second and third order actions respectively. The detail of the calculation is presented in the appendices B.2.

Another convenient gauge is

$$\delta\Phi = 0,\ h_{ij} = a^2((1+2\zeta)\delta_{ij} + \gamma_{ij}),\ \partial_i\gamma_{ij} = 0,\ \gamma_{ii} = 0 \tag{6.19}$$

where ζ is a first order quantity and parametrizes the scalar fluctuation, and γ is the same as in the gauge (6.4). In this gauge, the second order action is the following,

$$S_2 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3\frac{\dot{\Phi_0}^2}{H^2}\left[\dot{\zeta}^2 - \frac{(\nabla\zeta)^2}{a^2}\right]. \tag{6.20}$$

In the gauge invariant form, a gauge invariant variable can be defined as $\omega = \zeta - \frac{H}{\dot{\Phi}_0}\varphi$. The second order action is

$$S_2 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3\frac{\dot{\Phi}_0^{\,2}}{H^2}\left[\dot{\omega}^2 - \frac{(\nabla\omega)^2}{a^2}\right]. \tag{6.21}$$

In Appendix B.3 we have shown that in the gauge (6.4) the second order action can be recast into the same form.

6.2 LSS in the ADM formalism

In this section, we will show how to use the ADM formalism to describe the evolution of LSS. Ultimately, the evolution of dark matter is determined by the Hilbert-Einstein action and the matter action for some field. Dark matter can be fermonic or bosonic, and it is probably massive, such that at late times it can be approximated by non-relativistic matter[1].

6.2.1 The effective action for LSS

Notice that the action, Eq. (6.6), contain the first-order perturbation, which comes from the coupling of the metric perturbation and background matter field. As we have shown in subsubsection 5.2.2, the first-order action gives only the FLRW equations, which means that only higher order actions are important for the study of the scalar perturbation. To delete the first-order perturbation, we can use the assumption that $\dot{\Phi}_0 = 0$, which implies $V' = 0$ from the equation of motion for Φ_0. In this case, the dominant parts of the solutions for the constraints n and s are of second order, which are sufficient for us to obtain the effective action up to the fourth order. Later we will also show that the higher order solutions are suppressed. This further implies that the 1-loop perturbation of the power spectrum should not be divergent.

We use hydrodynamic quantities to express the background energy tensor, $T_\mu^\nu = \mathrm{diag}(\rho_c, -P_c, -P_c, -P_c)$, and $P_c = w(t)\rho_c$. In terms of the scalar field, $P_c = \frac{\dot{\Phi}^2}{2} - V$ and $\rho_c = \frac{\dot{\Phi}^2}{2} + V$. The background fluid action can be written

[1] Relativistic correction can be included, and in general the correction is small.

in terms of pressure as in[48],

$$S_{0m} = \int \mathrm{d}^4 x a^3 \bar{N} P_c(\bar{N}), \tag{6.22}$$

where $\frac{\partial}{\partial \bar{N}} P_c(\bar{N}) = -\frac{1}{\bar{N}}(\rho_c + P_c)$. In the Hilbert-Einstein action, the background part is the following,

$$S_{0HE} = \int \mathrm{d}^4 x M_p^2 a^3 \bar{N}(-3H^2). \tag{6.23}$$

So the total background action is $S_0 = S_{0m} + S_{0HE}$. Varying S_0 with respect to $\bar{N}$ and a, we obtain the FLRW equations,

$$\frac{\delta S_0}{\delta \bar{N}} = 0 = -a^3 \rho_c + 3M_p^2 a^3 H^2, \tag{6.24}$$

$$\frac{\delta S_0}{\delta a} = 0 = 3a^2 \bar{N} P_c + 3M_p^2 a^2 \bar{N} \left(2\dot{H} + 3H^2\right), \tag{6.25}$$

which can be simplified as,

$$H^2 = \frac{1}{3M_p^2} \rho_c, \tag{6.26}$$

$$\dot{H} = -\frac{1}{2M_p^2}(\rho_c + P_c). \tag{6.27}$$

The action of the perturbed matter field φ is

$$S_{pmatter} = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{h} \left[-NV'' \varphi^2 + N^{-1}(\partial_0 \varphi - N^i \partial_i \varphi)^2 - N \frac{(\nabla \varphi)^2}{a^2} \right] \tag{6.28}$$

where The total action is $S = S_{HE} + S_0 + S_{pmatter}$, then,

$$\begin{aligned} S &= \frac{1}{2} \int \mathrm{d}^4 x \sqrt{h} \left[-NV'' \varphi^2 + N^{-1}(\partial_0 \varphi - N^i \partial_i \varphi)^2 - N \frac{(\nabla \varphi)^2}{a^2} + \bar{N} P_c \right] \\ &+ \frac{1}{2} \int \mathrm{d}^4 x \sqrt{h} \left[M_p^2 [N^{(3)} R + N^{-1}(E_{ij} E^{ij} - E^2)] \right]. \end{aligned} \tag{6.29}$$

Next, we need to get the solutions of the constraints again, and the leading order is second order. The constrain equations are the following,

$$2\nabla_i [N^{-1}(E^i{}_j - \delta^i{}_j E)] - 2M_p^{-2} N^{-1} \partial_j \varphi \partial_0 \varphi = 0, \tag{6.30}$$

$$-N^{-2}(E_{ij}E^{ij}-E^2)-M_p^{-2}\left[2V''\frac{\varphi^2}{2}+\frac{(\nabla\varphi)^2}{a^2}+N^{-2}(\partial_0\varphi)^2\right]-\frac{2}{M_p^2}\rho_c=0. \tag{6.31}$$

More explicitly, we have

$$\partial_j[(1-n)(-4H)]-a^{-1}\partial_i{}^2n_j{}^T-2M_p^{-2}N^{-1}\partial_j\varphi\partial_0\varphi=0, \tag{6.32}$$

$$-H(-6H+\frac{4}{a^2}\nabla^2 s)-M_p^{-2}\left[2V''\frac{\varphi^2}{2}+\frac{(\nabla\varphi)^2}{a^2}+\dot{\varphi}^2+2(1+2n)\rho_c\right]=0, \tag{6.33}$$

where $-6H^2+\frac{2}{M_p^2}\rho_c=0$. Thus we obtain the leading order solutions,

$$n=\frac{1}{2M_p^2H}[\varphi\dot{\varphi}+\frac{1}{\nabla^2}(\nabla\varphi)\cdot(\nabla\dot{\varphi})],\ n_i{}^T=0, \tag{6.34}$$

$$\frac{\nabla^2 s}{a^2}=-3nH-\frac{1}{4M_p^2H}\left[V''\varphi^2+\frac{(\nabla\varphi)^2}{a^2}+\dot{\varphi}^2\right]. \tag{6.35}$$

Inserting these solutions back into the action Eq. (6.29), we get the effective action. Expanding the effective action, we obtain the second, third and fourth order actions as follows,

$$S_2=\frac{1}{2}\int d^4x\bar{N}a^3\left[-V''\varphi^2+\dot{\varphi}^2-\frac{(\nabla\varphi)^2}{a^2}+2n\rho_c\right], \tag{6.36}$$

$$S_3=\frac{1}{2}\int d^4x\bar{N}a^3\left[-V'''\frac{\varphi^3}{3}\right], \tag{6.37}$$

$$S_4=\frac{1}{2}\int d^4x\bar{N}a^3\left[-V''''\frac{\varphi^4}{12}-nV''\varphi^2-n\dot{\varphi}^2-na^{-2}(\nabla\varphi)^2-2\dot{\varphi}a^{-2}\partial_i s\partial_i\varphi\right]. \tag{6.38}$$

In the appendices C.2, we have shown that the last term in S_2 is simply a boundary term. Thus the second order action is

$$S_2=\frac{1}{2}\int d^4x\bar{N}a^3\left[-V''\varphi^2+\dot{\varphi}^2-\frac{(\nabla\varphi)^2}{a^2}\right]. \tag{6.39}$$

To calculate the power spectrum, we will make use the quantum field theory methods. In the following, we will show how to quantize the field and how to generalize the theory to D dimension for the dimensional regularization.

6.2.2 Canonical quantization of the scalar field

Setting $\bar{N} = a$, we can write down the quadratic Lagrangian density,

$$\mathcal{L} = \frac{1}{2}a^2\big[(\varphi')^2 - (\nabla\varphi)^2 - m^2a^2\varphi^2\big], \tag{6.40}$$

where the prime represents taking the derivative with respect to conformal time η, and $\mathrm{d}\eta = a\mathrm{d}t$.

The field equation

The Lagrangian density of a scalar field in a general curved spacetime [49] is,

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}\big[g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) - (m^2 + \xi R)\phi^2\big], \tag{6.41}$$

where $\phi(x)$ is the scalar field and m is the mass of the field quanta. $\xi R\phi^2$ denotes the coupling between the scalar field and the gravitation field with coupling constant ξ. In our discussion, we are only interested in the case when $\xi = 0$, which means that the scalar field minimally couples with the gravitational field. Varying the Lagrangian density (6.41) with respect to the field ϕ, we get the equation of motion for the scalar field,

$$\frac{1}{\sqrt{-g}}\partial_\alpha\big(\sqrt{-g}g^{\alpha\beta}\partial_\beta\phi\big) + m^2\phi = 0. \tag{6.42}$$

This equation is valid in any background spacetime. In our case, Eq. (6.40) is the Lagrangian density of the scalar field φ in the homogeneous and isotropic Robertson-Walker(RW) spacetime without spatial curvature. The line element is,

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^idx^j = a^2(\eta)(d\eta^2 - \delta_{ij}dx^idx^j). \tag{6.43}$$

The relation between them is $dt = a(\eta)d\eta$. Substituting the metric $g_{\alpha\beta} = \mathrm{diag}\{-1, a^2, a^2, a^2\}$ into the Klein-Gordan equation(6.42), one has

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} - \frac{1}{a^2}\nabla^2\varphi + m^2\varphi = 0, \tag{6.44}$$

where the dot represents taking the derivative with respect to the physical time t. In conformal time, where $g_{\alpha\beta} = \mathrm{diag}\{-a^2, a^2, a^2, a^2\}$, the Klein-Gordon equation is

$$\varphi'' + 2\frac{a'}{a}\varphi' - \nabla^2\varphi + m^2a^2\varphi = 0, \tag{6.45}$$

where the prime represents derivation with respect to the conformal time η. We can rescale the field to kill the first order time derivative of the field in the field equation. Define the new field function as $\chi = a\varphi$, the Lagrangian density can be rewritten as

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\sqrt{-g}\big[g^{\alpha\beta}(\partial_\alpha\varphi)(\partial_\beta\varphi) - m^2\varphi^2\big] \\
&= \frac{1}{2}a^2\big[(\varphi')^2 - (\nabla\varphi)^2 - m^2a^2\varphi^2\big] \\
&= \frac{1}{2}\left[(\chi')^2 - (\nabla\chi)^2 - m^2a^2\chi^2 + \left(\frac{a'}{a}\right)^2\chi^2 - 2\frac{a'}{a}\chi\chi'\chi^2\right] \\
&= \frac{1}{2}\left[(\chi')^2 - (\nabla\chi)^2 - m^2a^2\chi^2 + \frac{a''}{a}\chi^2 - \left(\frac{a'}{a}\chi^2\right)'\right] \\
&= \frac{1}{2}\left[(\chi')^2 - (\nabla\chi)^2 - m^2a^2\chi^2 + \frac{a''}{a}\chi^2\right], \qquad (6.46)
\end{aligned}$$

where we have dropped the total time derivative term $(\frac{a'}{a}\chi^2)'$. The conjugate field $\pi(\mathbf{x}, \eta)$ is defined as

$$\pi = \frac{\partial\mathcal{L}}{\partial\chi'} = \chi'. \qquad (6.47)$$

The Hamiltonian density $\mathcal{H}$ can be obtained via the Legendre transformation,

$$\mathcal{H} = \pi\chi' - \mathcal{L} = \frac{1}{2}\big[\pi^2 + (\nabla\chi)^2 + m^2a^2\phi^2 - \frac{a''}{a}\chi^2\big]. \qquad (6.48)$$

The field equation for χ is then,

$$\chi'' - \nabla^2\chi + (m^2a^2 - \frac{a''}{a})\chi = 0. \qquad (6.49)$$

The term $\nabla^2\chi$ in the field equation (6.49) suggests us to do a Fourier transform,

$$\chi(\mathbf{x}, \eta) = \int \frac{d\mathbf{k}}{(2\pi)^3}\chi_{\mathbf{k}}(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}. \qquad (6.50)$$

Substituting this into Eq.(6.49), one gets the equation for $\chi_{\mathbf{k}}(\eta)$,

$$\chi''_{\mathbf{k}}(\eta) + \omega_k^2\chi_{\mathbf{k}}(\eta) = 0, \qquad (6.51)$$

where ω_k^2 is defined as

$$\omega_k^2 = k^2 + m^2a^2 - \frac{a''}{a}. \qquad (6.52)$$

Since Eq.(6.51) is a second order differential equation, it has two independent fundamental solutions. Suppose that the complex function $u_k(\eta)$ is one of the fundamental solutions, then its complex conjugate, $u_k^*(\eta)$, is the other fundamental solution. The general solution $\chi_{\mathbf{k}}(\eta)$ is the linear combination of $u_k(\eta)$ and $u_k^*(\eta)$,

$$\chi_{\mathbf{k}}(\eta) = a_{\mathbf{k}} u_k(\eta) + b_{\mathbf{k}} u_k^*(\eta), \tag{6.53}$$

where $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ are stochastic variables which are dependent on the wave number $\mathbf{k}$. Since $\chi(\mathbf{x}, \eta)$ is a real scalar field, its Fourier modes satisfy

$$\chi_{\mathbf{k}}^*(\eta) = \chi_{-\mathbf{k}}(\eta).$$

This leads to a constraint on $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$,

$$a_{\mathbf{k}}^* = b_{-\mathbf{k}},$$

which implies that the general solution $\chi_{\mathbf{k}}(\eta)$ should be

$$\chi_{\mathbf{k}}(\eta) = a_{\mathbf{k}} u_k(\eta) + a_{-\mathbf{k}}^* u_k^*(\eta). \tag{6.54}$$

Substituting Eq.(6.54) into the Fourier transform of the scalar field $\chi(\mathbf{x}, \eta)$, one gets

$$\chi(\mathbf{x}, \eta) = \int \frac{d\mathbf{k}}{(2\pi)^3} \left[a_{\mathbf{k}} u_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^* u_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right]. \tag{6.55}$$

Here, we have assumed that $u_k(\eta)$ is the positive frequency mode function and $u_k^*(\eta)$ is the negative frequency mode function.

Quantization

To quantize the scalar field $\hat{\chi}(\mathbf{x}, \eta)$, we can simply promote $\hat{\chi}(\mathbf{x}, \eta), \hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^\dagger$ from ordinary functions to annihilation and creation operators, with the commutation relations,

$$[\hat{a}_{\mathbf{k}},\, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}'), \quad [\hat{a}_{\mathbf{k}},\, \hat{a}_{\mathbf{k}'}] = 0, \quad [\hat{a}_{\mathbf{k}}^\dagger,\, \hat{a}_{\mathbf{k}'}^\dagger] = 0. \tag{6.56}$$

The field and the conjugate field are, respectively,

$$\hat{\chi}(\mathbf{x}, \eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[\hat{a}_{\mathbf{k}} u_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}_{\mathbf{k}}^\dagger u_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right], \tag{6.57}$$

and

$$\pi(\mathbf{x},\eta) = \chi'(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\big[\hat{a}_{\mathbf{k}} u'_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{k}} u^{*\prime}_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}\big]. \tag{6.58}$$

Now we can calculate the equal-time commutation relation between $\chi(\mathbf{x},\eta)$ and $\pi(\mathbf{x},\eta)$.

$$\begin{aligned}
[\chi(\mathbf{x},\eta),\pi(\mathbf{x}',\eta)] &= \left[\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\big(\hat{a}_{\mathbf{k}} u_k e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{k}} u^*_k e^{-i\mathbf{k}\cdot\mathbf{x}}\big), \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\big(\hat{a}_{\mathbf{k}'} u'_{k'} e^{i\mathbf{k}'\cdot\mathbf{x}'} + \hat{a}^\dagger_{\mathbf{k}'} u^{*}_{k'}{}' e^{-i\mathbf{k}'\cdot\mathbf{x}'}\big)\right] \\
&= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^6}\big\{[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}] u_k u^{*}_{k'}{}' e^{i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{x}'} + [\hat{a}^\dagger_{\mathbf{k}},\hat{a}_{\mathbf{k}'}] u^*_k u'_{k'} e^{-i\mathbf{k}\cdot\mathbf{x}+i\mathbf{k}'\cdot\mathbf{x}'}\big\} \\
&= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\big\{u_k u^{*}_k{}' e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} - u^*_k u'_k e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\big\} \\
&= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}(u_k u^{*\prime}_k - u^*_k u'_k) e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \\
&= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} W[u_k,u^*_k] e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')},
\end{aligned} \tag{6.59}$$

where $W[u_k,u^*_k] \equiv u_k u^{*\prime}_k - u^*_k u'_k$ is called the Wronskian, which is an integration constant from Eq. (6.51). The equal-time commutation relation,

$$[\chi(\mathbf{x},\eta),\pi(\mathbf{x}',\eta)] = i\delta(\mathbf{x}-\mathbf{x}'), \tag{6.60}$$

requires $W[u_k,u^*_k] = i$. We have solved for the mode functions in the appendices D.2 by using the WKB approximation, and the solution is

$$u = \frac{1}{\sqrt{2W}}\exp\left(\pm i\int_0^\eta W\mathrm{d}\eta\right), \tag{6.61}$$

where $W^2 = \omega^2 - \frac{1}{2}(\frac{W''}{W} - \frac{3W'^2}{2W^2})$.

6.2.3 The effective action in D-dimensional spacetime

Now we generalize our derivation to D-dimensional spacetime, which is useful for dimensional regularization. The whole approach is similar to what we have done in 4-dimensional spacetime. The action is the following,

$$S = \frac{1}{2}\int \mathrm{d}^D x\sqrt{h}\left[N^{(3)}R + N^{-1}(E_{ij}E^{ij} - E^2) - 2NV + N^{-1}(\partial_0\varphi - N^i\partial_i\varphi)^2 - N\frac{(\nabla\varphi)^2}{a^2} + 2\bar{N}P_c\right], \tag{6.62}$$

The constrain equations are

$$2\nabla_i[N^{-1}(E^i{}_j - \delta^i{}_j E)] - 2M_p^{-2}N^{-1}\partial_j\varphi\partial_0\varphi = 0, \tag{6.63}$$

$$-N^{-2}(E_{ij}E^{ij} - E^2) - M_p^{-2}\left[2V''\frac{\varphi^2}{2} + \frac{(\nabla\varphi)^2}{a^2} + N^{-2}(\partial_0\varphi)^2 + 2\rho_c\right] = 0. \tag{6.64}$$

where the values of $(E^i{}_j - \delta^i{}_j E)$ and $(E_{ij}E^{ij} - E^2)$ in D-dimensional spacetime are shown in the appendices B.1. The background equations can be obtained in the same way as in subsubsection (6.2.1),

$$\begin{aligned} H^2 &= \frac{2}{(D-1)(D-2)M_p^2}\rho_c, \\ \dot{H} &= -\frac{1}{(D-2)M_p^2}(\rho_c + P_c). \end{aligned} \tag{6.65}$$

The leading order solutions are

$$n = \frac{1}{(D-2)M_p^2 H}[\varphi\dot{\varphi} + \frac{1}{\nabla^2}(\nabla\varphi)\cdot(\nabla\dot{\varphi})], \; n_i{}^T = 0, \tag{6.66}$$

and

$$\frac{\nabla^2 s}{a^2} = -\frac{6nH}{D-2} - \frac{1}{2(D-2)M_p^2 H}\left[V''\varphi^2 + \frac{(\nabla\varphi)^2}{a^2} + \dot{\varphi}^2\right]. \tag{6.67}$$

The second, third and fourth order actions are the following,

$$S_2 = \frac{1}{2}\int \mathrm{d}^D x \bar{N} a^{(D-1)}\left[-V''\varphi^2 + \dot{\varphi}^2 - \frac{(\nabla\varphi)^2}{a^2} + 2n\rho_c\right], \tag{6.68}$$

$$S_3 = \frac{1}{2}\int \mathrm{d}^D x \bar{N} a^{(D-1)}\left[-V'''\frac{\varphi^3}{3}\right], \tag{6.69}$$

$$S_4 = \frac{1}{2}\int \mathrm{d}^D x \bar{N} a^{(D-1)}\left[-V''''\frac{\varphi^4}{12} - nV''\varphi^2 - n\dot{\varphi}^2 - na^{-2}(\nabla\varphi)^2 - 2\dot{\varphi}a^{-2}\partial_i s\partial_i\varphi\right]. \tag{6.70}$$

Similarly, the last term in S_2 is just a boundary term, which will be dropped. The canonical quantization based on the quadratic action is shown in the

Appendix D.2.2. With the assumption that $V''' = V'''' = 0$, the cubic action simply vanishes. The quatic action can be rewritten as

$$
\begin{aligned}
S_4 &= \frac{1}{2}\int \mathrm{d}^D x \bar{N} a^{(D-1)} \left[\frac{12H}{(D-2)}\dot{\varphi}\partial_i\varphi\partial_i(\frac{1}{\nabla^2}n) - \frac{2}{(D-2)M_p^2 H}\frac{1}{\nabla^2}\left[\partial_i(\dot{\varphi}\partial_i\varphi)\delta\rho\right]\right] \\
&+ \frac{1}{2}\int \mathrm{d}^D x \bar{N} a^{(D-1)} \frac{2}{(D-2)M_p^2 H}\partial_i\left[\dot{\varphi}\partial_i\varphi\frac{1}{\nabla^2}\delta\rho\right], \qquad (6.71)
\end{aligned}
$$

where the last term is a boundary term, and $\delta\rho \equiv \frac{1}{2}\left[V''\varphi^2 + \dot{\varphi}^2 + a^{-2}(\nabla\varphi)^2\right]$.

With the action at hand, our next step is to calculate power spectra.

6.3 The power spectrum

In this section, first we will see the evolution of the power spectrum from the second order action, with Gaussian and non-Gaussian initial conditions respectively. Then we will see how the 1-loop perturbations from the higher order actions contribute to the power spectrum, including the dimensional regularization. We hope that this one-loop effect has better convergence than standard perturbation theory.

The most direct approach is that we can get the evolution equation for the perturbed field φ, solve it perturbatively with an initial Gaussian power spectrum, and then calculate the 1-loop perturbation. Varying the action $(S_2 + S_3 + S_4)$ with respect to φ, we have the evolution equation,

$$\frac{\delta(S_2 + S_3 + S_4)}{\delta\varphi}\frac{1}{\bar{N}a^3} = J, \qquad (6.72)$$

where J is the stochastic term, and similar to Eq. (3.9),

$$\langle J(\mathbf{k},\eta)J(\mathbf{k}',\eta')\rangle = \mathcal{N}(\mathbf{k},\eta)(2\pi)^3\delta(\mathbf{k}+\mathbf{k}')\delta(\eta-\eta'). \qquad (6.73)$$

In principle, we could solve this equation perturbatively and then calculate the power spectra. However, we have a more convenient way to calculate the power spectra. In the next section, we will construct the effective action with an initial density operator, and try to understand how the effective action runs with the scale.

6.3.1 The Gaussian initial condition

We choose the initial density matrix to be Gaussian, which is

$$\hat{\rho} = N \exp\left[-\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\varphi(\mathbf{x})A(\mathbf{x},\mathbf{y},t_0)\varphi(\mathbf{y})\right], \tag{6.74}$$

where N is the normalization constant, and $A(\mathbf{x},\mathbf{y},t_0)$ is related to the two-point correlation function of the field. Discussion about the more general Gaussian distribution can be found in [35, 50]. According to the Cosmological Principle, i.e the assumption that the Universe is statistically isotropic and homogeneous, $A(\mathbf{x},\mathbf{y},t_0)$ is a function of the distance ($||\mathbf{x}-\mathbf{y}||$) only. In Fourier space,

$$\begin{aligned}
\hat{\rho} &= N \exp\left[-\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}e^{i\mathbf{k}_1\cdot\mathbf{x}}\varphi(\mathbf{k}_1)A(||\mathbf{x}-\mathbf{y}||,t_0)\int \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}e^{i\mathbf{k}_2\cdot\mathbf{y}}\varphi(\mathbf{k}_2)\right] \\
&= N \exp\left[-\frac{1}{2}\int \mathrm{d}^3(\mathbf{x}-\mathbf{y})\int \mathrm{d}^3\mathbf{y}\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}e^{i\mathbf{k}_1\cdot\mathbf{x}}\varphi(\mathbf{k}_1)A(||\mathbf{x}-\mathbf{y}||,t_0)\int \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}e^{i\mathbf{k}_2\cdot\mathbf{y}}\varphi(\mathbf{k}_2)\right] \\
&= N \exp\left[-\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}e^{i\mathbf{k}_1\cdot(\mathbf{x}+\mathbf{y})}\varphi(\mathbf{k}_1)A(||\mathbf{x}||,t_0)e^{i\mathbf{k}_2\cdot\mathbf{y}}\varphi(\mathbf{k}_2)\right] \\
&= N \exp\left[-\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{k}_2}{(2\pi)^3}\delta(\mathbf{k}_1+\mathbf{k}_2)e^{i\mathbf{k}_1\cdot\mathbf{x}}\varphi(\mathbf{k}_1)A(||\mathbf{x}||,t_0)\varphi(\mathbf{k}_2)\right] \\
&= N \exp\left[-\frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\varphi(-\mathbf{k})A(k,t_0)\varphi(\mathbf{k})\right],
\end{aligned} \tag{6.75}$$

where we have defined $A(k,t_0) \equiv \int \mathrm{d}^3\mathbf{x}A(||\mathbf{x}||,t)\exp(-i\mathbf{k}\cdot\mathbf{x})$.

A toy model–Gaussian distribution for a harmonic oscillator

In the following we will present the techniques of calculations of a quantum mechanical Gaussian state [50]. The density is,

$$\hat{\rho} = \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y|x\rangle\rho\langle y|, \tag{6.76}$$

where $\rho(x,y;t) = \mathcal{N}(t)\exp[-a(t)x^2 - b(t)y^2 + 2c(t)xy]$. From $\hat{\rho}^\dagger = \hat{\rho}$, we have $b^* = a$ and $c^* = c$. The normalisation condition $\mathrm{Tr}[\hat{\rho}] = 1$ determines

$\mathcal{N} = \sqrt{\frac{2(a_R-c)}{\pi}}$, where a_R is the real part of a. We can derive the three correlators characterising the system:

$$\langle\hat{x}^2\rangle = \mathrm{Tr}[\hat{\rho}\hat{x}^2] = \int_{-\infty}^{\infty} \mathrm{d}x \langle x|\hat{\rho}\hat{x}^2|x\rangle = \frac{1}{4(a_R-c)}, \tag{6.77}$$

$$\langle\frac{1}{2}\{\hat{x},\hat{p}\}\rangle = -\frac{a_I}{2(a_R-c)}, \tag{6.78}$$

$$\langle\hat{p}^2\rangle = \frac{|a|^2-c^2}{a_R-c}, \tag{6.79}$$

where we have used $\hat{p}|x\rangle = i\hbar\partial_x|x\rangle$ with the convention $\hbar = 1$, $\langle\hat{x}\hat{p}\rangle = -i\frac{b-c}{2(a_R-c)}$ and $\langle\hat{p}\hat{x}\rangle = i\frac{a-c}{2(a_R-c)}$. The inverted relations are

$$a_I = -\frac{\langle\frac{1}{2}\{\hat{x},\hat{p}\}\rangle}{2\langle\hat{x}^2\rangle}, \tag{6.80}$$

$$a_R = \frac{\Delta^2+1}{8\langle\hat{x}^2\rangle}, \tag{6.81}$$

$$c = \frac{\Delta^2-1}{8\langle\hat{x}^2\rangle}, \tag{6.82}$$

where $\Delta^2 \equiv \frac{a_R+c}{a_R-c} \equiv 4\left[\langle\hat{x}^2\rangle\langle\hat{p}^2\rangle - \langle\frac{1}{2}\{\hat{x},\hat{p}\}\rangle^2\right]$. Here we consider the pure squeezed state, which has $\Delta^2 = 1$. For simplicity, we fix $\langle\{\hat{x},\hat{p}\}\rangle = -1$. From the inverted relations, we can see $c = 0$, $a_R = a_I$, so the distribution is reduced to a product state,

$$\rho(x,y;t) = \mathcal{N}(t)\exp[-a_R(x^2+y^2) - ia_R(x^2-y^2)]. \tag{6.83}$$

For a harmonic oscillator with m=1, the evolutions are

$$\hat{x}(t) = \hat{x}\cos(\omega t) + \frac{\hat{p}}{\omega}\sin(\omega t), \tag{6.84}$$

$$\hat{p}(t) = \hat{p}\cos(\omega t) - \omega\hat{x}\sin(\omega t). \tag{6.85}$$

We can calculate the two point correlation function,

$$\begin{aligned}
\langle\hat{x}(t)\hat{x}(t^{'})\rangle &= \langle\hat{x}^2\rangle\cos(\omega t)\cos(\omega t^{'}) + \langle\hat{x}\hat{p}\rangle\cos(\omega t)\sin(\omega t^{'}) \\
&+ \langle\hat{p}\hat{x}\rangle\sin(\omega t)\cos(\omega t^{'}) + \langle\hat{p}\hat{p}\rangle\sin(\omega t)\sin(\omega t^{'}) \\
&= \frac{1}{4a_R}\cos(\omega t)\cos(\omega t^{'}) - \frac{i+1}{2\omega}\cos(\omega t)\sin(\omega t^{'}) \\
&+ \frac{i-1}{2\omega}\sin(\omega t)\cos(\omega t^{'}) + \frac{2a_R}{\omega^2}\sin(\omega t)\sin(\omega t^{'}).
\end{aligned} \tag{6.86}$$

And

$$\langle[x(t),x(t^{'})]\rangle = -\frac{i}{\omega}\sin(\omega(t^{'}-t)), \tag{6.87}$$

which is the quantum contribution. The statistical part is

$$\langle\{x(t),x(t^{'})\}\rangle = \frac{1}{2a_R}\cos(\omega t)\cos(\omega t^{'}) - \frac{1}{\omega}\sin(\omega(t^{'}+t)) + \frac{4a_R}{\omega^2}\sin(\omega t)\sin(\omega t^{'}). \tag{6.88}$$

For a classical system with the initial momentum $p=0$, we have the probability distribution,

$$P(x,y;t) = \mathcal{N}(t)\exp[-2a_R x^2]. \tag{6.89}$$

The initial correlation function is

$$\langle x^2\rangle = \frac{1}{4a_R} \tag{6.90}$$

It evolves as

$$\langle x(t)x(t^{'})\rangle = \frac{1}{4a_R}\cos(\omega t)\cos(\omega t^{'}). \tag{6.91}$$

Gaussian distribution for the field

Recall that our field expansion is

$$\hat{\chi}(\mathbf{x},\eta) = \int\frac{d\mathbf{k}}{(2\pi)^3}\big[\hat{a}_{\mathbf{k}}u_k(\eta)e^{i\mathbf{k}\cdot\mathbf{x}} + \hat{a}^{\dagger}_{\mathbf{k}}u^*_k(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}\big]. \tag{6.92}$$

where $\hat{\chi}(\mathbf{x},\eta) = a\hat{\varphi}(\mathbf{x},\eta)$, and $u(\eta) = \frac{1}{\sqrt{2W}}\exp(\pm i\int_0^\eta W\mathrm{d}\eta)$. We define $\pi_u(\eta) \equiv u^{'}(\eta)$, and then we can rewrite the evolutions of the mode functions as the following,

$$u(\eta) = u\cos(\int_0^\eta W\mathrm{d}\eta) + \frac{\pi_u}{W}\sin(\int_0^\eta W\mathrm{d}\eta), \tag{6.93}$$

$$\pi_u(\eta) = \pi_u\cos(\int_0^\eta W\mathrm{d}\eta) - Wu\sin(\int_0^\eta W\mathrm{d}\eta). \tag{6.94}$$

where u and π_u are the initial values. Choosing the initial density matrix to be,

$$\rho(\chi,\chi_1;t) = \mathcal{N}(t)\exp[-(a_R+ia_R)|\chi|^2-(a_R-ia_R)|\chi_1|^2]. \tag{6.95}$$

We can choose coherent states to reduce the operators to classical quantities. For each mode,

$$\rho_k(u,u_1;t) = \mathcal{N}(t)\exp[-a_R(u^2+{u_1}^2)-ia_R(u^2-{u_1}^2)], \tag{6.96}$$

where u_1 is the mode function for $\hat{\chi}_1$, and $\rho(\chi,\chi_1;t)=\prod_k \rho_k(u,u_1;t)$.

The calculation of the correlation functions is similar to what we done for the harmonic oscillator. The results are the following,

$$\langle\hat{\chi}^\dagger\hat{\chi}\rangle = \frac{1}{4a_R}, \tag{6.97}$$

and the time evolution is,

$$\begin{aligned}\langle\hat{\chi}^\dagger(\eta,\mathbf{k})\hat{\chi}(\eta^{'},\mathbf{k})\rangle &= \frac{1}{4a_R}\cos(\int_0^\eta W\mathrm{d}\eta))\cos(\int_0^{\eta^{'}} W\mathrm{d}\eta)-\frac{i+1}{2W}\cos(\int_0^\eta W\mathrm{d}\eta)\sin(\int_0^{\eta^{'}} W\mathrm{d}\eta)\\ &+ \frac{i-1}{2W}\sin(\int_0^\eta W\mathrm{d}\eta)\cos(\int_0^{\eta^{'}} W\mathrm{d}\eta)\\ &+ \frac{2a_R}{W^2}\sin(\int_0^\eta W\mathrm{d}\eta)\sin(\int_0^{\eta^{'}} W\mathrm{d}\eta).\end{aligned} \tag{6.98}$$

The quantum part is,

$$\langle[\hat{\chi}^\dagger(\eta,\mathbf{k}),\hat{\chi}(\eta^{'},\mathbf{k})]\rangle = -\frac{i}{W}\sin(\int_\eta^{\eta^{'}} W\mathrm{d}\eta). \tag{6.99}$$

And the statistical part is

$$\begin{aligned}\langle\{\hat{\chi}^\dagger(\eta,\mathbf{k}),\hat{\chi}(\eta^{'},\mathbf{k})\}\rangle &= \frac{1}{2a_R}\cos(\int_0^\eta W\mathrm{d}\eta))\cos(\int_0^{\eta^{'}} W\mathrm{d}\eta)-\frac{1}{W}\sin(\int_0^\eta W\mathrm{d}\eta+\int_0^{\eta^{'}} W\mathrm{d}\eta)\\ &+ \frac{4a_R}{W^2}\sin(\int_0^\eta W\mathrm{d}\eta)\sin(\int_0^{\eta^{'}} W\mathrm{d}\eta).\end{aligned} \tag{6.100}$$

6.3.2 The Non-Gaussian initial condition

The observation result we have is the power spectrum for the curvature perturbation ψ_N. To find the initial condition for φ, we need to find the relation between curvature perturbations and density perturbations, which is expected to be the Poisson equation derived from the perturbed Einstein equation $\delta G^0_0 = \delta T^0_0$. Notice that we are using the zero-curvature gauge, i.e. $\psi_N = 0$, so we can not get the usual Poisson equation, $\nabla^2\psi_N(k,t) = 4\pi a^2 G_N \delta\rho(k,t)$. However, we can go to the gauge invariant form of the Poisson equation, which is derived as Eq. (6.40) in [9],

$$\nabla^2\Psi - 3\mathscr{H}\Psi^{'} - (\mathscr{H}^{'} + 2\mathscr{H}^2)\Psi = 4\pi G_N a^2(\delta\rho + \rho_0^{'}(B - E^{'})), \qquad (6.101)$$

where $\Psi = \psi_N - \frac{a^{'}}{a}(B - E^{'})$ is called the Bardeen potential, ρ_0 is the background density, $\delta\rho$ is the perturbed density and $\mathscr{H} \equiv aH = \frac{a^{'}}{a}$.

In the zero-curvature gauge, we choose $N_0 = a$, which gives $B = a\ s$, and the Poisson equation is reduced to

$$-3\mathscr{H}[n\mathscr{H} + (\mathscr{H}^2 - \mathscr{H}^{'})(a\ s)^{'}] - \frac{a^{'}}{a}\nabla^2(a\ s) = 4\pi G_N a^2(\delta\rho + \rho_0^{'} a\ s), \quad (6.102)$$

where $\delta\rho(\mathbf{k}, t_0)$ can be approximated by the quadratic part of the field perturbation,

$$\delta\rho(\mathbf{x},t) = (1/2)\dot{\varphi}(\mathbf{x},t)^2 + (1/2)(\frac{\nabla\varphi(\mathbf{x},t)}{a})^2 + (1/2)m^2\varphi(\mathbf{x},t)^2. \qquad (6.103)$$

In the co-moving gauge, the Poisson equation is of the following form,

$$\nabla^2\psi_N = 4\pi G_N a^2 \delta\rho. \qquad (6.104)$$

Since, $\frac{1}{t} \approx H \ll k/a \ll m$, $\delta\rho$ can be approximated by its dominant part, which is $\delta\rho(\mathbf{x},t) = (1/2)(\frac{\nabla\varphi(\mathbf{X},t)}{a})^2 + (1/2)m^2\varphi(\mathbf{x},t)^2$. In Fourier space, it is

$$-k^2\psi_N(\mathbf{k},t) = 2\pi G_N a^2 \int \frac{\mathrm{d}^3\mathbf{k}_1}{(2\pi)^3}\varphi(\mathbf{k}-\mathbf{k}_1)\varphi(\mathbf{k}_1)(m^2 - \frac{\mathbf{k}_1\cdot(\mathbf{k}-\mathbf{k}_1)}{a^2}), \quad (6.105)$$

where on the right hand side, it is the non-local term. Here we simplify it by the local term, and leave the non-local effect for future consideration. The the Poisson equation becomes,

$$\psi_N = -2\pi G_N \frac{a^2}{k^2}\varphi^2(\mathbf{k})\Omega^2. \qquad (6.106)$$

where we have defined $\Omega^2 = m^2 - \frac{k^2}{a^2}$.

Classical distribution

The probability distribution of ψ_N is chosen to be Gaussian, as we have discussed in subsubsection (3.1.3), which is

$$P(\psi_N(\mathbf{x}-\mathbf{y})) = \mathcal{N}\exp\left[-\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\psi_N(\mathbf{x})A(||\mathbf{x}-\mathbf{y}||,t)\psi_N(\mathbf{y})\right], \tag{6.107}$$

where $\mathcal{N}=\sqrt{\frac{A}{2\pi}}$ is the normalization constant, and $A(||\mathbf{x}-\mathbf{y}||,t)$ is related to the two-point correlation function. Doing Fourier transform, we have

$$P(\psi_N(\mathbf{k})) = \mathcal{N}\exp(-\frac{1}{2}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}|\psi_N|^2(\mathbf{k})A(k,t)), \tag{6.108}$$

and $\frac{1}{A(k)} = \langle|\psi_N(\mathbf{k})|^2\rangle \equiv \frac{2\pi^2}{k^3}P_{\psi_N}$. On sub-Hubble scales $P_{\psi_N}=\Delta_*(k/k_*)^2$, and on super-Hubble scales $P_{\psi_N}=\Delta_*(k/k_*)^{n_s-1}$.

Inverting Eq. (6.105) and making use of the above P_{ψ_N}, we obtain the initial distribution for φ

$$P(\varphi) = \mathcal{N}\exp\left(-\frac{\alpha}{2}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\varphi^4(\mathbf{k})\right), \tag{6.109}$$

where $\alpha \equiv A(k,t)(2\pi G_N\frac{a^2}{k^2}\Omega^2)^2$, and the normalization condition gives $\mathcal{N} = \prod_k \frac{(\frac{\alpha}{2})^{1/4}}{2\Gamma[\frac{5}{4}]}$. The classical correlation function is

$$\begin{aligned}\langle\varphi^2(k)\rangle &= \mathcal{N}\int \mathrm{d}\varphi\,\varphi^2(k)\exp\left(-\frac{\alpha}{2}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\varphi^4(\mathbf{k})\right)\\ &= \frac{\Gamma[\frac{3}{4}]}{4\Gamma[\frac{5}{4}]}(\frac{\alpha}{2})^{-1/2}.\end{aligned} \tag{6.110}$$

With the curvature perturbation $P_{\psi_N}=\Delta_*(k/k_*)^{n_s-1}$, we find that $\frac{\Gamma[\frac{3}{4}]}{2\Gamma[\frac{5}{4}]}(\frac{\alpha}{2})^{-1/2}\sim k^{1/2}$, which matches the standard result at the beginning of the linear reigme.

Quantum distribution

To see the quantum effect, we start with the pure squeezed state for ψ_N, with the density as Eq. (6.83). Then the density matrix for φ is

$$\rho(\varphi,\varphi_1,t) = \mathcal{N}\exp[-\frac{\alpha}{4}[(1+i)|\varphi|^4+(1-i)|\varphi_1|^4]]. \tag{6.111}$$

Similar to the procedures in the case of Gaussian initial condition, we can use coherent states to rewrite the density matrix in terms of the mode functions u and its conjugate momentum $\pi_u(\eta)$, and then use quantum mechanics to calculate the correlation functions. We obtain the following,

$$\begin{aligned} \langle\hat{\varphi}^\dagger(\eta,\mathbf{k})\hat{\varphi}(\eta^{'},\mathbf{k})\rangle &= \frac{\Gamma[\frac{3}{4}]}{4\Gamma[\frac{5}{4}]}(\frac{\alpha}{2})^{-1/2}\cos(\int_0^\eta W\mathrm{d}\eta))\cos(\int_0^{\eta^{'}} W\mathrm{d}\eta) \\ &- \frac{i+1}{2W}\cos(\int_0^\eta W\mathrm{d}\eta)\sin(\int_0^{\eta^{'}} W\mathrm{d}\eta) + \frac{i-1}{2W}\sin(\int_0^\eta W\mathrm{d}\eta)\cos(\int_0^{\eta^{'}} W\mathrm{d}\eta) \\ &+ \frac{\Gamma[\frac{7}{4}]}{W^2\Gamma[\frac{5}{4}]}(\frac{\alpha}{2})^{1/2}\sin(\int_0^\eta W\mathrm{d}\eta)\sin(\int_0^{\eta^{'}} W\mathrm{d}\eta). \end{aligned} \tag{6.112}$$

The quantum part is

$$\langle[\hat{\varphi}^\dagger(\eta,\mathbf{k}),\hat{\varphi}(\eta^{'},\mathbf{k})]\rangle = -\frac{i}{W}\sin(\int_\eta^{\eta^{'}} W\mathrm{d}\eta). \tag{6.113}$$

The statistical part is

$$\begin{aligned} \langle\{\hat{\varphi}^\dagger(\eta,\mathbf{k}),\hat{\varphi}(\eta^{'},\mathbf{k})\}\rangle &= \frac{\Gamma[\frac{3}{4}]}{2\Gamma[\frac{5}{4}]}(\frac{\alpha}{2})^{-1/2}\cos(\int_0^\eta W\mathrm{d}\eta))\cos(\int_0^{\eta^{'}} W\mathrm{d}\eta) \\ &- \frac{1}{W}\sin(\int_0^\eta W\mathrm{d}\eta + \int_0^{\eta^{'}} W\mathrm{d}\eta) \\ &+ \frac{2\Gamma[\frac{7}{4}]}{W^2\Gamma[\frac{5}{4}]}(\frac{\alpha}{2})^{1/2}\sin(\int_0^\eta W\mathrm{d}\eta)\sin(\int_0^{\eta^{'}} W\mathrm{d}\eta), \end{aligned} \tag{6.114}$$

which is the evolution of power spectrum. In the following we will try to include the one-loop effect for the power spectrum.

6.3.3 The 1-loop perturbations

The quartic action is

$$\begin{aligned} S_4 &= \frac{1}{2}\int \mathrm{d}^D x\bar{N}a^{(D-1)}\left[\frac{12}{(D-2)M_p^2}\dot{\varphi}\partial_i\varphi\partial_i(\frac{1}{\nabla^2}(\varphi\dot{\varphi}+\frac{1}{\nabla^2}(\nabla\varphi)\cdot(\nabla\dot{\varphi}))\right] \\ &- \frac{1}{2}\int \mathrm{d}^D x\bar{N}a^{(D-1)}\frac{2}{(D-2)M_p^2 H}\frac{1}{\nabla^2}\left[\partial_i(\dot{\varphi}\partial_i\varphi)\delta\rho\right] \\ &+ \frac{1}{2}\int \mathrm{d}^D x\bar{N}a^{(D-1)}\frac{2}{(D-2)M_p^2 H}\partial_i\left[\dot{\varphi}\partial_i\varphi\frac{1}{\nabla^2}\delta\rho\right], \end{aligned} \tag{6.115}$$

where the last term is a boundary term, and $\delta\rho \equiv \frac{1}{2}\left[V''\varphi^2 + \dot{\varphi}^2 + a^{-2}(\nabla\varphi)^2\right]$.

The steps to calculate the one-loop effect are varying the quartic action with respect to the field, calculating the expectation value of it, and then including it into the perturbed solution of the field. Then we could use the dimensional regularization to see how the divergences of the system behave. However, we could not cover it due to limited time.

Chapter 7

Summary

My thesis work contains two parts. The first part is trying to understand the renormalized theory in a new way. It turned out that our renormalized approach shows cut-off dependence, and that a continuous stochastic source arises due to the renormalization, which are expected. The sad thing is that our effective action is different from the published renormalization theory and EFT, and we expect that the prediction of the power spectrum would not be satisfying. In our work, the question still remains: why our result is different from the published work about renormalization? We will try to answer it in the future.

The second part is exploring a completely new formalism to describe the evolution of LSS. It turned out that our formalism has the correct scale dependence at the begining of the linear regime. The one-loop perturbation is expected to give reasonable predictions up to a nonlinear limit k_{NL}, and the cut-off dependence may be cancelled by dimensional regularization. Then we could compare with the nonlinear limit of standard perturbation theory $k_{NL} \sim 0.1\mathrm{Mpc}^{-1}$.

- If $k_{NL} > 0.1\mathrm{Mpc}^{-1}$, then this theory is good and could have some contribution about predicting the power spectrum.
- If $k_{NL} < 0.1\mathrm{Mpc}^{-1}$, use the Wilsonian renormalization to see the cut-off dependence.

However, due to limited time, I could not cover the one-loop effect in my thesis.

There are still some details to be improved: the higher order solutions of the constrains; more general states rather than a pure squeezed state while calculation the power spectrum; the proof that we could reduce the non-local term in the Poisson equation to be local.

Further research possibilities are: (a) including the one-loop effect in the prediction of the power spectrum; (b) continuing developing the formalism in terms of a fermion field, to see the vorticity of the velocity perturbation; (c) studying the bias model in this formalism.

Appendices

Appendix A

The $3 + 1$ formalism

The 3+1 formalism [46] is useful to solve Einstein's equations numerically and also to describe the gravitational system in terms of dynamical quantities. Here we introduce the $3 + 1$ formalism for the second advantage. The main derivations follow the textbook of Misner, Thorne and Wheeler[52]. First, we will introduce the notation of the splitted metric. Then, we will show how to relate the spacetime curvature to the curvature in the case of the splitted spacetime. In the end we will derive the ADM formalism of the Hilbert-Einstein action.

A.1 Splitting spacetime into space and time

Given a spacetime, choose the coordinate system $\{x^\mu\} = \{t, x, y, z\}$. The 4-dimension metric $g_{\mu\nu}$ contains all information about this spacetime. We now split the 4-dimensional spacetime as follow. $t = constant$ represents a 3-dimensional spacelike hypersurface Σ_t, which is part of the 4-dimensional spacetime. Choosing different values for $t = constant$, we can have a series of spacelike hypersurfaces, which is an alternative description of the entire 4-dimensional spacetime. In every single spacelike hypersurface Σ_t, or for $\mathrm{d}t = 0$, the infinitesimal spacetime distance between any two nearby points in Σ is

$$g_{ij}(t, x, y, z)dx^i dx^j \equiv h_{ij}(t, x, y, z)dx^i dx^j, \tag{A.1}$$

here h_{ij} is the metric of the 3-dimensional spacelike hypersurface, which is often called 3-metric or induced metric of 3-geometry. The induced metric h_{ij} can fully describe the intrinsic properties of the spacelike hypersurface

Σ_t. Consider another spacelike hypersurface Σ_{t+dt}, the induced metric is

$$g_{ij}(t+dt,x,y,z)dx^i dx^j \equiv h_{ij}(t+dt,x,y,z)dx^i dx^j. \tag{A.2}$$

To fully describe the 4-dimension spacetime, we also need to know the spacetime distance between the two successive hypersurfaces. We can build connectors between the lower hypersurface Σ_t and upper hypersurface Σ_{t+dt}. The connectors are based on a point (x,y,z) on the lower hypersurface Σ_t and perpendicular to the hypersurface Σ_t.

We define the length of the connector as "lapse function" in the following form,

$$\begin{pmatrix} lapse\ of \\ proper\ time \\ between\ lower \\ and\ upper \\ hypersurface \end{pmatrix} = \Big(lapse\ function \Big) dt = N(t,x,y,z)dt. \tag{A.3}$$

We also need to fix the position of connectors on the upper hypersurface Σ_{t+dt}. We define a "shift vector" N^i to represent the location of connectors in the upper hypersurface,

$$x^i_{upper} = x^i - N^i(t,x,y,z)dt. \tag{A.4}$$

Now the geometry between the two successive hypersurfaces is fixed with the lapse function N, the shift vector N^i and the induced metric h_{ij}. The spacetime distance between point (t,x^i) on the lower hypersurface and point $(t+dt,x^i+dx^i)$ on the upper hypersurface is

$$\begin{aligned} ds^2 &= h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) - (Ndt)^2 \\ &= h_{ij}N^i N^j dt^2 + h_{ij}N^i dt dx^j + h_{ij}N^j dt dx^i + h_{ij}dx^i dx^j - N^2 dt^2 \\ &= (h_{ij}N^i N^j - N^2)dt^2 + 2h_{ij}N^j dt dx^i + h_{ij}dx^i dx^j. \end{aligned} \tag{A.5}$$

Using the 4-metric of spacetime, we have

$$\begin{aligned} ds^2 &= g_{\alpha\beta}dx^\alpha dx^\beta \\ &= g_{00}dt^2 + 2g_{0i}dt dx^i + g_{ij}dx^i dx^j \end{aligned} \tag{A.6}$$

and

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0k} \\ g_{k0} & g_{ik} \end{pmatrix} = \begin{pmatrix} N_s N^s - N^2 & N_k \\ N_i & h_{ik} \end{pmatrix}, \tag{A.7}$$

where the quantities N^i are the components of the shift vector, while the $N_i = h_{ij}N^j$ are the covariant components.

Using the relation

$$g_{\alpha\beta}g^{\beta h} = \delta^h_\alpha, \tag{A.8}$$

one can calculate the reciprocal 4-metric $g^{\alpha\beta}$ in term of N, N^i and h^{ij}, which is the following,

$$\begin{pmatrix} g^{00} & g^{0k} \\ g^{i0} & g^{ik} \end{pmatrix} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^k}{N^2} \\ \frac{N^i}{N^2} & h^{ik} - \frac{N^i N^k}{N^2} \end{pmatrix}. \tag{A.9}$$

Define g and h as the determinant of $g_{\alpha\beta}$ and h_{ij} respectively, $g = \det \mid g_{\alpha\beta} \mid$ and $h = \det \mid h_{ij} \mid$. The volume element has the form

$$(-g)^{1/2}\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z = Nh^{1/2}\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z. \tag{A.10}$$

The unit normal vector field $\mathbf{n}$ of the spacelike hypersurface is of the following form,

$$n_\alpha = (-N, 0, 0, 0), \tag{A.11}$$

and

$$n^\alpha = \left(\frac{1}{N}, -\frac{N^i}{N}\right). \tag{A.12}$$

A.2 Intrinsic and extrinsic curvatures

The intrinsic curvature reflects the intrinsic curved properties of a hypersurface. In 4-dimensional spacetime, we have used the 4-metric $g_{\alpha\beta}$ to construct different curvature tensors, like ${}^{(4)}R^\alpha{}_{\beta h\delta}$, ${}^{(4)}R_{\alpha\beta}$ and so on. These curvature tensors are intrinsic curvatures of 4-dimensional spacetime. Similarly, we can use the 3-metric h_{ij} to construct $R^i{}_{jkl}$ et al. in 3-dimensional spacelike hypersurfaces. The extrinsic curvature is introduced when a lower dimensional hypersurface is embedded into a higher dimensional hypersurface. For a lower dimensional hypersurface, its intrinsic curvature is fixed, and its extrinsic curvature characterizes the way of embedding it into the higher dimensional hypersurface. The definition of the extrinsic curvature will be given later. It should be pointed out that in this section we use an upper index (4) to denote the 4-dimensional curvatures and connections, and the

3-dimensional quantities do not have such an index.

Consider a 3-dimensional spacelike hypersurface embedded in 4-dimensional spacetime. A tangent vector field $\mathbf{A}$ lies in the 3-dimensional hypersurface, so it can be expressed as

$$\mathbf{A} = A^i \mathbf{e}_i, \tag{A.13}$$

where $\mathbf{e}_i$ is the basis vectors of 3-dimensional hypersurface. And

$$A_j = \mathbf{A} \cdot \mathbf{e}_j = A^i \mathbf{e}_i \cdot \mathbf{e}_j = g_{ij} A^i = h_{ij} A^i, \tag{A.14}$$

where $\gamma_{ij} = g_{ij}$ is the metric of the 3-dimensional hypersurface. If we parallel transport $\mathbf{A}$ in the 4-dimensional spacetime along a curve that lies in the 3-dimensional spacelike hypersurface, $\mathbf{A}$ may not be tangent to the 3-dimensional hypersurface any more. Transporting $\mathbf{A}$ along the direction of the i-th coordinate direction is of the following form,

$${}^{(4)}\nabla_{\mathbf{e}_i} \mathbf{A} = {}^{(4)}\nabla_i \mathbf{A} = {}^{(4)}\nabla_i (A^j \mathbf{e}_j) = \frac{\partial A^j}{\partial x^i} \mathbf{e}_j + A^{j\,(4)}\Gamma^{\mu}_{ji} \mathbf{e}_\mu. \tag{A.15}$$

As a special case of (A.15),

$${}^{(4)}\nabla_{\mathbf{e}_i} \mathbf{e}_m = {}^{(4)}\Gamma^{\mu}_{im} \mathbf{e}_\mu, \tag{A.16}$$

which is the definition of connection coefficients ${}^{(4)}\Gamma^{\mu}_{im}$. From Eq. (A.15), we can see that $A^{j\,(4)}\Gamma^{0}_{ji}(\mathbf{e}_0 \cdot \mathbf{n})$, where $\mathbf{n}$ is the normal vector of 3-dimensional hypersurface, is the component which is out of the hypersurface.

In the 3-dimensional hypersurface, the covariant derivative can be similarly defined,

$$A_{h|i} = \mathbf{e}_h \cdot \nabla_{\mathbf{e}_i} \mathbf{A} = \mathbf{e}_h \cdot \nabla_i \mathbf{A} = \frac{\partial A_h}{\partial x^i} - A^m \Gamma_{mhi}, \tag{A.17}$$

Γ_{mhi} is the connection coefficients constructed from the 3-metric h_{ij}. The relation between ${}^{(4)}\nabla \mathbf{e}_j$ and $\nabla \mathbf{e}_j$ is quite simple, project ${}^{(4)}\nabla \mathbf{A}$ orthogonally onto the hypersurface and one arrives at a parallel transport, or a covariant derivative that are intrinsic to the 3-dimensional hypersurface,

$${}^{(4)}\nabla_i \mathbf{e}_j = \nabla_i \mathbf{e}_j + {}^{(4)}\Gamma^{0}_{ji} (\mathbf{e}_0 \cdot \mathbf{n}) \frac{\mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}. \tag{A.18}$$

Now, we can define extrinsic curvature. When a vector on the 3-dimensional hypersurface, representing the direction of parallel transport, is mapped to another vector on the hypersurface, the extrinsic curvature reflects the difference between the local normal vector and the transported normal vector. The extrinsic curvature is defined as the following,

$$ {}^{(4)}\nabla_{\mathbf{e}_i}\mathbf{n} = -\mathbf{K}(\mathbf{e}_i) = -K_i^j \mathbf{e}_j, \tag{A.19} $$

where K_i^j are the components of the linear operator $\mathbf{K}$ in a coordinate representation. Taking into account the fact that $\mathbf{e}_i$ and $\mathbf{n}$ are orthogonal, i.e. $\mathbf{n} \cdot \mathbf{e}_i = 0$, one can prove that $\mathbf{K}$ is a symmetric tensor,

$$ \begin{aligned} K_{im} &= K_i^j \gamma_{jm} = K_i^j \mathbf{e}_j \cdot \mathbf{e}_m = \mathbf{e}_m \cdot (K_i^j \mathbf{e}_j) = -\mathbf{e}_m \cdot {}^{(4)}\nabla_{\mathbf{e}_i}\mathbf{n} \\ &= \mathbf{n} \cdot {}^{(4)}\nabla_{\mathbf{e}_i}\mathbf{e}_m = \mathbf{n} \cdot {}^{(4)}\Gamma^{\mu}_{mi}\mathbf{e}_\mu = \mathbf{n} \cdot {}^{(4)}\Gamma^{0}_{mi}\mathbf{e}_0 = \mathbf{n} \cdot {}^{(4)}\Gamma^{0}_{im}\mathbf{e}_0 \\ &= \mathbf{n} \cdot {}^{(4)}\nabla_{\mathbf{e}_m}\mathbf{e}_i = -\mathbf{e}_i \cdot {}^{(4)}\nabla_{\mathbf{e}_m}\mathbf{n} = K_m^j \mathbf{e}_j \cdot \mathbf{e}_i = K_m^j \gamma_{ji} \\ &= K_{mi}. \end{aligned} \tag{A.20} $$

Using the definition of extrinsic curvature, Eq. (A.19), one can relate the four dimensional covariant derivative with the three dimensional covariant derivative of vectors which are in the three dimensional hypersurface. Let's first derive the covariant derivative of basis vectors $\mathbf{e}_i$,

$$ {}^{(4)}\nabla_{\mathbf{e}_i}\mathbf{e}_j = {}^{(4)}\nabla_i \mathbf{e}_j = K_{ij}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + \nabla_i \mathbf{e}_j = K_{ij}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^h_{ij}\mathbf{e}_h. \tag{A.21} $$

For any vector $\mathbf{A}$ in the 3-dimensional hypersurface, we then have

$$ {}^{(4)}\nabla_i \mathbf{A} = A^j{}_{|i}\mathbf{e}_j + K_{ij}A^j \frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}, \tag{A.22} $$

where " | " represents the covariant derivative with respect to 3-dimensional metric h_{ij}. After splitting four dimensional spacetime as a 3+1 form, the metric $g_{\mu\nu}$ of 4-geometry is replaced by the lapse function N, the shift vector N^i and the reduced metric h_{ij} of 3-geometry. Now we will express the extrinsic curvature K_{ij} as the function of N, N^i and h_{ij}. As we have derived before, the normal vector $\mathbf{n}$ of 3-geometry is

$$ n_\alpha = (n_0, n_1, n_2, n_3) = (-N, 0, 0, 0). \tag{A.23} $$

According to the definition of extrinsic curvature, Eq. (A.19), it can be shown that

$$ K_{ik} = -\mathbf{e}_i \cdot {}^{(4)}\nabla_{\mathbf{e}_k}\mathbf{n} = -n_{i;k}. \tag{A.24} $$

Using Eq. (A.23), we have

$$\begin{aligned} n_{i;k} &= \frac{\partial n_i}{\partial x^k} - {}^{(4)}\Gamma^{\alpha}{}_{ik} n_{\alpha} \\ &= -{}^{(4)}\Gamma^{0}{}_{ik} n_0 \\ &= N {}^{(4)}\Gamma^{0}{}_{ik}. \end{aligned} \tag{A.25}$$

The connection coefficient of four dimensional spacetime ${}^{(4)}\Gamma^{0}{}_{ik}$ is the following,

$$\begin{aligned} {}^{(4)}\Gamma^{0}{}_{ik} &= \frac{1}{2} g^{0\alpha}(g_{\alpha i,k} + g_{\alpha k,i} - g_{ik,\alpha}) \\ &= \frac{1}{2} g^{00}(g_{0i,k} + g_{0k,i} - g_{ik,0}) + \frac{1}{2} g^{0j}(g_{ji,k} + g_{jk,i} - g_{ik,j}) \\ &= -\frac{1}{2N^2}[N_{i,k} + N_{k,i} - h_{ik,0} - N^j(h_{ji,k} + h_{jk,i} - h_{ik,j})] \\ &= -\frac{1}{2N^2}[N_{i,k} + N_{k,i} - h_{ik,0} - N_p h^{pj}(h_{ji,k} + h_{jk,i} - h_{ik,j})] \\ &= -\frac{1}{2N^2}[N_{i,k} + N_{k,i} - 2N_p \Gamma^p_{ik} - h_{ik,0}] \\ &= -\frac{1}{2N^2}[N_{i|k} + N_{k|i} - \frac{\partial h_{ik}}{\partial t}], \end{aligned} \tag{A.26}$$

where we have used the relation $h_{ij} = g_{ij}$. The final expression of K_{ik} is

$$K_{ik} = -n_{i;k} = \frac{1}{2N}[N_{i|k} + N_{k|i} - \frac{\partial h_{ik}}{\partial t}]. \tag{A.27}$$

Now we turn to find out how the Riemann curvature ${}^{(4)}R^{m}{}_{ijk}$ is related to the intrinsic curvature $R^{m}{}_{ijk}$ and the extrinsic curvature K_{ik}.

In the calculation, it is not convenient to use the coordinate basis,

$$\begin{aligned} &\text{basis vectors}, \ \mathbf{e}_0 = \partial_t, \ \mathbf{e}_i = \partial_i \\ &\text{basis } 1-\text{forms}, \ \omega^t = \mathbf{d}t, \ \omega^i = \mathbf{d}x^i, \end{aligned} \tag{A.28}$$

because ordinarily the basis vector $\mathbf{e}_0$ does not stand perpendicular to the hypersurface. Adopt a different basis but one that is still self-dual,

$$\begin{aligned} &\text{basis vectors}, \ \mathbf{e}_n \equiv \mathbf{n} = N^{-1}(\partial_t - N^m \partial_m), \ \mathbf{e}_i = \partial_i \\ &\text{basis } 1-\text{forms}, \ \omega^n = N\mathbf{d}t = (\mathbf{n}\cdot\mathbf{n})\mathbf{n}, \ \omega^i \equiv \mathbf{d}x^i + N^i \mathbf{d}t. \end{aligned} \tag{A.29}$$

Also use Greek labels $\bar{\alpha} = n, 1, 2, 3$, instead of Greek labels $\alpha = n, 1, 2, 3$, to denote the components.

Recall that the Riemann curvature is measured by the change in a vector on transport around a closed route, that is

$$\mathcal{R}(\mathbf{u}, \mathbf{v})\mathbf{w} = {}^{(4)}\nabla_{\mathbf{u}}{}^{(4)}\nabla_{\mathbf{v}}\mathbf{w} - {}^{(4)}\nabla_{\mathbf{v}}{}^{(4)}\nabla_{\mathbf{u}}\mathbf{w} - {}^{(4)}\nabla_{[\mathbf{u},\mathbf{v}]}\mathbf{w}. \qquad \text{(A.30)}$$

Choose a closed route in the 3-hypersurface Σ, then both vectors $\mathbf{u}$ and $\mathbf{v}$ in Eq. (A.30) lie in Σ. In particularly, let $\mathbf{u} = \mathbf{e}_j$, $\mathbf{v} = \mathbf{e}_k$, and taking into the fact that $\mathbf{e}_j$ and $\mathbf{e}_k$ commute, $[\mathbf{e}_j, \mathbf{e}_k] = 0$, Eq. (A.30) reduces to

$$\mathcal{R}(\mathbf{e}_j, \mathbf{e}_k)\mathbf{w} = {}^{(4)}\nabla_{\mathbf{e}_j}{}^{(4)}\nabla_{\mathbf{e}_k}\mathbf{w} - {}^{(4)}\nabla_{\mathbf{e}_k}{}^{(4)}\nabla_{\mathbf{e}_j}\mathbf{w} \qquad \text{(A.31)}$$

Further, let the vector $\mathbf{w}$ being transported is $\mathbf{e}_i$, then

$$\mathcal{R}(\mathbf{e}_j, \mathbf{e}_k)\mathbf{e}_i = {}^{(4)}\nabla_{\mathbf{e}_j}{}^{(4)}\nabla_{\mathbf{e}_k}\mathbf{e}_i - {}^{(4)}\nabla_{\mathbf{e}_k}{}^{(4)}\nabla_{\mathbf{e}_j}\mathbf{e}_i. \qquad \text{(A.32)}$$

Using the relation between the 4 dimensional covariant derivative and the 3 dimensional covariant derivative Eq. (A.21), we have

$$\begin{aligned}
{}^{(4)}\nabla_{\mathbf{e}_j}{}^{(4)}\nabla_{\mathbf{e}_k}\mathbf{e}_i &= {}^{(4)}\nabla_{\mathbf{e}_j}[K_{ik}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^m_{ik}\mathbf{e}_m] \\
&= \Gamma^m_{ik,j}\mathbf{e}_m + \Gamma^m_{ik}(K_{jm}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^s_{jm}\mathbf{e}_s) + K_{ik,j}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} - K_{ik}K^s_j\mathbf{e}_s\frac{1}{\mathbf{n}\cdot\mathbf{n}} \\
&= K_{ik,j}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} - K_{ik}K^s_j\mathbf{e}_s\frac{1}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^m_{ik,j}\mathbf{e}_m + \Gamma^m_{ik}(K_{jm}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^s_{jm}\mathbf{e}_s),
\end{aligned}$$

exchanging the indices j and k,

$$ {}^{(4)}\nabla_{\mathbf{e}_k}{}^{(4)}\nabla_{\mathbf{e}_j}\mathbf{e}_i = K_{ij,k}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} - K_{ij}K^s_k\mathbf{e}_s\frac{1}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^m_{ij,k}\mathbf{e}_m + \Gamma^m_{ij}(K_{km}\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + \Gamma^s_{km}\mathbf{e}_s),$$

then

$$\begin{aligned}
\mathcal{R}(\mathbf{e}_j, \mathbf{e}_k)\mathbf{e}_i &= {}^{(4)}\nabla_{\mathbf{e}_j}{}^{(4)}\nabla_{\mathbf{e}_k}\mathbf{e}_i - {}^{(4)}\nabla_{\mathbf{e}_k}{}^{(4)}\nabla_{\mathbf{e}_j}\mathbf{e}_i \\
&= [(K_{ik,j} - \Gamma^m_{ij}K_{mk}) - (K_{ij,k} - \Gamma^m_{ik}K_{mj})]\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + (\Gamma^m_{ik,j} - \Gamma^m_{ij,k} + \Gamma^s_{ik}\Gamma^m_{sj} - \Gamma^s_{ij}\Gamma^m_{sk})\mathbf{e}_m \\
&\quad + (K_{ij}K^m_k - K_{ik}K^m_j)\mathbf{e}_m\frac{1}{\mathbf{n}\cdot\mathbf{n}} \\
&= [(K_{ik,j} - \Gamma^m_{ij}K_{mk} - \Gamma^m_{kj}K_{mi}) - (K_{ij,k} - \Gamma^m_{ik}K_{mj} - \Gamma^m_{kj}K_{mi})]\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} \\
&\quad + (\Gamma^m_{ik,j} - \Gamma^m_{ij,k} + \Gamma^s_{ik}\Gamma^m_{sj} - \Gamma^s_{ij}\Gamma^m_{sk})\mathbf{e}_m + (K_{ij}K^m_k - K_{ik}K^m_j)\mathbf{e}_m\frac{1}{\mathbf{n}\cdot\mathbf{n}} \\
&= (K_{ik|j} - K_{ij|k})\frac{\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}} + [\frac{1}{\mathbf{n}\cdot\mathbf{n}}(K_{ij}K^m_k - K_{ik}K^m_j) + R^m{}_{ijk}]\mathbf{e}_m.
\end{aligned}$$

The components of the 4-dimensional Riemann curvature tensor are

$$ {}^{(4)}R^{m}{}_{ijk} = \omega^{m} \cdot [\mathcal{R}(\mathbf{e}_j, \mathbf{e}_k)\mathbf{e}_i] = R^{m}{}_{ijk} + (\mathbf{n}\cdot\mathbf{n})^{-1}(K_{ij}K^{m}_{k} - K_{ik}K^{m}_{j}), \quad \text{(A.33)} $$

and

$$ {}^{(4)}R^{n}{}_{ijk} = \omega^{n} \cdot [\mathcal{R}(\mathbf{e}_j, \mathbf{e}_k)\mathbf{e}_i] = K_{ij|k} - K_{ik|j}. \quad \text{(A.34)} $$

Eq. (A.33) and Eq. (A.34) are known as the equations of Gauss and Codazzi, and it is worth pointing out that m is the spatial index and n is comparable to the time index.

A.3 The ADM formalism of the Hilbert-Einstein action

Using the Gauss-Codazzi results, we can evaluate the ADM formalism of the Hilbert-Einstein action. To be consistent with the generally used notation, we use a upper index (3) to denote the 3-dimensional curvature.
From the definition of the Ricci tensor

$$ {}^{(4)}R_{\mu\nu} \equiv {}^{(4)}R^{\alpha}{}_{\mu\alpha\nu}, \quad \text{(A.35)} $$

the mixture components of Ricci curvature ${}^{(4)}R^{\mu}_{\nu}$ is

$$ {}^{(4)}R^{\mu}_{\nu} = {}^{(4)}R^{\alpha\mu}{}_{\alpha\nu}, \quad \text{(A.36)} $$

then

$$ {}^{(4)}R^{n}_{n} = {}^{(4)}R^{\alpha n}{}_{\alpha n} = {}^{(4)}R^{ni}{}_{ni}. \quad \text{(A.37)} $$

We have used the antisymmetric property of the Riemann curvature tensor in the last equality. And

$$ {}^{(4)}R^{i}_{i} = {}^{(4)}R^{\alpha i}{}_{\alpha i} = {}^{(4)}R^{ni}{}_{ni} + {}^{(4)}R^{ij}{}_{ij}. \quad \text{(A.38)} $$

So

$$ R = {}^{(4)}R^{n}_{n} + {}^{(4)}R^{i}_{i} = {}^{(4)}R^{ij}{}_{ij} + 2\,{}^{(4)}R^{ni}{}_{ni} = [{}^{(4)}R^{12}{}_{12} + {}^{(4)}R^{23}{}_{23} + {}^{(4)}R^{31}{}_{31}] + 2\,{}^{(4)}R^{ni}{}_{ni}. \quad \text{(A.39)} $$

It is very helpful to think of $\mathbf{e}_i, \mathbf{e}_j$ and $\mathbf{e}_k$ as being an orthonormal tetrad, $\mathbf{n}$ being itself already normalized and orthogonal to every vector in the hypersurface Σ. Then, using Eq. (A.33), one has

$$\begin{aligned} {}^{(4)}R^{ij}{}_{ij} &= 2({}^{(3)}R^{12}{}_{12} + {}^{(3)}R^{23}{}_{23} + {}^{(3)}R^{31}{}_{31}) \\ &+ 2(\mathbf{n}\cdot\mathbf{n})^{-1}[(K_1^2K_2^1 - K_2^2K_1^1) + (K_1^3K_3^1 - K_3^3K_1^1) + (K_3^2K_2^3 - K_2^2K_3^3)] \\ &= {}^{(3)}R - (\mathbf{n}\cdot\mathbf{n})^{-1}[(\mathrm{Tr}\mathbf{K})^2 - \mathrm{Tr}(\mathbf{K}^2)] \end{aligned} \tag{A.40}$$

Here R is the 3-dimensional scalar curvature and Tr stands for "trace of", thus,

$$\mathrm{Tr}\mathbf{K} = h^{ij}K_{ij} = h_{ij}K^{ij} = K_j^j, \tag{A.41}$$

and

$$\mathrm{Tr}\mathbf{K}^2 = K_j^m K_m^j = h_{js}K^{sm}h_{mi}K^{ij}. \tag{A.42}$$

The results, though obtained in an orthonormal tetrad, are covariant with respect to general coordinate transformations within the spacelike hypersurface. And it makes no explicit reference to any time coordinate. So we can directly translate the result to any non-orthonormal coordinate system $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k$.

Following the exercise in [52], one can show that

$${}^{(4)}R^i{}_{nin} = (Tr\mathbf{K})^2 - Tr\mathbf{K}^2 \text{ plus a covariant divergence.} \tag{A.43}$$

Thus inserting Eq. (A.40) and (A.43) into the Hilbert-Einstein action, we finally arrive at

$$S_{HE} = \int \mathrm{d}^4x \frac{1}{2}\sqrt{-g}R = \int \mathrm{d}^4x \frac{1}{2}N\sqrt{h}[{}^{(3)}R + (\mathbf{n}\cdot\mathbf{n})^{-1}[(Tr\mathbf{K})^2 - Tr(\mathbf{K}^2)]]. \tag{A.44}$$

Introducing $E_{ij} = NK_{ij}$, we can rewrite it into the following form,

$$S_{HE} = \int \mathrm{d}^4x \frac{1}{2}\sqrt{h}[N^{(3)}R + N^{-1}(E_{ij}E^{ij} - E^2)]. \tag{A.45}$$

Appendix B

The calculation of the effective action

In the section, we present details of calculations in subsection 6.1.

B.1 The explicit form of the action

The calculation of $(E_{ij}E^{ij} - E^2)$ is presented as the following.

$$\begin{aligned} E &= E_{ij}h^{ij} \\ &= \frac{1}{2}(\frac{\partial h_{ij}}{\partial t} - \nabla_i N_j - \nabla_j N_i)h^{ij} \\ &= \frac{1}{2}(\frac{\partial h_{ij}}{\partial t}h^{ij} - 2\nabla_i N^i) \\ &= 3H_0 - \nabla_i N^i, \end{aligned} \tag{B.1}$$

where we define $H_0 \equiv \bar{N}H$.

$$\begin{aligned} E^k{}_j &= E_{ij}h^{ik} \\ &= \frac{1}{2}(\frac{\partial h_{ij}}{\partial t} - \nabla_i N_j - \nabla_j N_i)h^{ik} \\ &= \frac{1}{2}(\frac{\partial h_{ij}}{\partial t}h^{ik} - h^{ik}\nabla_i(h_{jm}N^m) - \nabla_j N^k) \\ &= \frac{1}{2}(2H_0\delta^k{}_j - \nabla_k N^j - \nabla_j N^k). \end{aligned} \tag{B.2}$$

Inserting the above expressions into $(E_{ij}E^{ij} - E^2)$, we have

$$\begin{aligned}
E_{ij}E^{ij} - E^2 &= E^k{}_j E^j{}_k - E^2 \\
&= \frac{1}{4}(2H_0\delta^k{}_j - \nabla_k N^j - \nabla_j N^k)(2H_0\delta^j{}_k - \nabla_j N^k - \nabla_k N^j) \\
&- (3H_0 - \nabla_i N^i)^2 \\
&= 3{H_0}^2 + \frac{1}{4}(\nabla_j N^k + \nabla_k N^j)^2 - 2H_0\nabla_i N^i - (9{H_0}^2 + (\nabla_i N^i)^2 - 6H_0\nabla_i N^i) \\
&= -6{H_0}^2 + 4H_0\nabla_i N^i + \frac{1}{4}(\nabla_j N^k + \nabla_k N^j)^2 - (\nabla_i N^i)^2. \qquad \text{(B.3)}
\end{aligned}$$

We can generalize the calculations to D-dimension spacetime. Similarly, we have

$$E = (D-1)H_0 - \nabla_i N^i, \qquad \text{(B.4)}$$

$$E^k{}_j = \frac{1}{2}(2H_0\delta^k{}_j - \nabla_k N^j - \nabla_j N^k), \qquad \text{(B.5)}$$

$$E_{ij}E^{ij} - E^2 = -(D-1)(D-2){H_0}^2 + 2(D-2)H_0\nabla_i N^i + \frac{1}{4}(\nabla_j N^k + \nabla_k N^j)^2 - (\nabla_i N^i)^2. \qquad \text{(B.6)}$$

B.2 The expansion of the effective action

. Inserting the solutions for the constraints N and N_i, we have $S \equiv \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3L$, and

$$\begin{aligned}
L &= -2(1+n)V + (1-n+n^2-n^3+n^4)(-6H^2 + \frac{4H\nabla^2 s}{a^2} + L_2) \qquad \text{(B.7)} \\
&\quad + (1-n+n^2-n^3+n^4)(\dot{\phi} + \dot{\varphi} - a^{-2}\partial_i s\partial_i\varphi)^2 - (1+n)a^{-2}(\nabla\varphi)^2,
\end{aligned}$$

where L_2 denotes some more terms from $(E_{ij}E^{ij} - E^2)$ besides $(-6H^2 + \frac{4H\nabla^2 s}{a^2})$, and it is of the following form,

$$L_2 = a^{-4}(\partial_j\partial_k s)^2 - a^{-4}(\nabla^2 s)^2. \qquad \text{(B.8)}$$

We can show that $\frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3L_2 = 0$ by doing partial integration. Thus the action up to second order is the following,

$$\begin{aligned} S_{2nd} &= \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-2(1+n)(V_0+V^{'}\varphi) - 2V^{''}\frac{\varphi^2}{2} + (1-n+n^2)(-6H^2) + (1-n)\frac{4H\nabla^2 s}{a^2}] \\ &+ \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[(1-n+n^2)(\dot{\phi})^2 + (1-n)2\dot{\phi}\dot{\varphi} - 2\dot{\phi}a^{-2}\partial_i s\partial_i\varphi + \dot{\varphi}^2 - a^{-2}(\nabla\varphi)^2] \\ &= \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-2(2+n^2)V_0 - 2(1+n)V^{'}\varphi - 2V^{''}\frac{\varphi^2}{2} + \dot{\varphi}^2 - a^{-2}(\nabla\varphi)^2] \\ &+ \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[(1-n)\frac{4H\nabla^2 s}{a^2} - 2\dot{\phi}a^{-2}\partial_i s\partial_i\varphi + 2(1-n)\dot{\phi}\dot{\varphi}], \end{aligned} \quad (B.9)$$

Using the solutions of n and s, we find that

$$-n\frac{4H\nabla^2 s}{a^2} - 2\dot{\phi}a^{-2}\partial_i s\partial_i\varphi = -2a^{-2}\dot{\phi}\partial_i(\varphi\partial_i s), \quad (B.10)$$

which is the boundary term, and also

$$\frac{4H\nabla^2 s}{a^2} = -4nV_0 - 2V^{'}\varphi - 2\dot{\phi}\dot{\varphi}. \quad (B.11)$$

Using partial integration, we have

$$\frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-2n\dot{\phi}\dot{\varphi}] = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-2n^2V_0 - 2nV^{'}\varphi]. \quad (B.12)$$

Inserting these relations into Eq. (B.9), we arrive at

$$S_{2nd} = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-4(1+n+n^2)V_0 - 2V^{'}\varphi - 4nV^{'}\varphi - 2V^{''}\frac{\varphi^2}{2} + \dot{\varphi}^2 - a^{-2}(\nabla\varphi)^2], \quad (B.13)$$

which can be separated into the background part and the second order part,

$$S_0 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-4(1+n)V_0 - 2V^{'}\varphi], \quad (B.14)$$

and

$$S_2 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-4n^2V_0 - 4nV^{'}\varphi - 2V^{''}\frac{\varphi^2}{2} + \dot{\varphi}^2 - a^{-2}(\nabla\varphi)^2]. \quad (B.15)$$

They can be rewritten into the following form,

$$S_0 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-V_0 - \frac{2\dot{\phi}}{H}V_0\varphi - 2V'\varphi], \tag{B.16}$$

$$S_2 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-\frac{\dot{\phi}^2}{H^2}V_0\varphi^2 - \frac{2\dot{\phi}}{H}V'\varphi^2 - V''\varphi^2 + \dot{\varphi}^2 - \frac{(\nabla\varphi)^2}{a^2}], \tag{B.17}$$

For the third order action,

$$\begin{aligned} S_3 &= \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-2nV''\frac{\varphi^2}{2} - V'''\frac{\varphi^3}{3} - n^3(-6H^2) + n^2\frac{4H\nabla^2 s}{a^2} - n(a^{-4}(\partial_j\partial_k s)^2 - a^{-4}(\nabla^2 s)^2)] \\ &+ \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-n^3(\dot{\phi})^2 + 2n^2\dot{\phi}\dot{\varphi} - n\dot{\varphi}^2 + 2n\dot{\phi}a^{-2}\partial_i s\partial_i\varphi - 2\dot{\varphi}a^{-2}\partial_i s\partial_i\varphi - na^{-2}(\nabla\varphi)^2] \\ &= \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-2nV''\frac{\varphi^2}{2} - V'''\frac{\varphi^3}{3} - n^3(-6H^2) - n^3(\dot{\phi})^2] \\ &+ \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-n\dot{\varphi}^2 - na^{-2}(\nabla\varphi)^2] \\ &+ \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-n(a^{-4}(\partial_j\partial_k s)^2 - a^{-4}(\nabla^2 s)^2)] \\ &+ \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[n^2\frac{2H\nabla^2 s}{a^2} + 2n^2\dot{\phi}\dot{\varphi} - 2\dot{\varphi}a^{-2}\partial_i s\partial_i\varphi]. \end{aligned} \tag{B.18}$$

Notice that we have used partial integration to rewrite $(n^2\frac{4H\nabla^2 s}{a^2}+2n\dot{\phi}a^{-2}\partial_i s\partial_i\varphi)$ into $(n^2\frac{2H\nabla^2 s}{a^2})$. The cubic action Eq. (B.18) is the same as Eq. (3.8) in Maldacena's paper [31].

For the fourth action, we could also write it down. However, we should point out that we have used the first order solutions of the constraints to get the effective action, and that we should get higher order solutions of the constraints to obtain the exact fourth order action. Here we just list the fourth order action in terms of the first order solutions of the constraints.

$$\begin{aligned} S_4 &= \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[-nV'''\frac{\varphi^3}{3} - 2V''''\frac{\varphi^4}{24} + n^4(-6H^2) - n^3\frac{4H\nabla^2 s}{a^2} + n^2(a^{-4}(\partial_j\partial_k s)^2 - a^{-4}(\nabla^2 s)^2) \\ &+ \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[n^4(\dot{\phi})^2 - 2n^3\dot{\phi}\dot{\varphi} + n^2\dot{\varphi}^2 - 2n^2\dot{\phi}a^{-2}\partial_i s\partial_i\varphi + 2n\dot{\varphi}a^{-2}\partial_i s\partial_i\varphi + (a^{-2}\partial_i s\partial_i\varphi)^2] \end{aligned} \tag{B.19}$$

B.3 The gauge invariant form

We define $\omega_1 \equiv -\frac{\varphi}{z}$, where $z \equiv \frac{\dot{\Phi}_0}{H}$. The action (Eq. (6.17)) can be written into [53]

$$S_2 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[\dot{\varphi}^2 - \frac{(\nabla\varphi)^2}{a^2} + (3H\frac{\dot{z}}{z} + \frac{\ddot{z}}{z})\varphi^2], \tag{B.20}$$

where we can prove that $3H\frac{\dot{z}}{z} + \frac{\ddot{z}}{z} = -\frac{\dot{\Phi_0}^2}{H^2}V_0 - \frac{2\dot{\Phi}_0}{H}V' - V''$. We can show that this action can be written into the form which is similar to the gauge invariant form,

$$\begin{aligned}
S_2 &= \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3z^2[\dot{\omega_1}^2 - \frac{(\nabla\omega_1)^2}{a^2}] \\
&= \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[z^2(\frac{\dot{\varphi}^2}{z^2} - \frac{2\dot{z}}{z^3}\varphi\dot{\varphi} + \frac{\dot{z}^2}{z^4}\varphi^2) - \frac{(\nabla\varphi)^2}{a^2}] \\
&= \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[\dot{\varphi}^2 + a^{-3}\varphi^2(\frac{\dot{z}}{z}a^3)^{\cdot} + \frac{\dot{z}^2}{z^2}\varphi^2 - \frac{(\nabla\varphi)^2}{a^2}] \\
&= \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3[\dot{\varphi}^2 - \frac{(\nabla\varphi)^2}{a^2} + (3H\frac{\dot{z}}{z} + \frac{\ddot{z}}{z})\varphi^2],
\end{aligned} \tag{B.21}$$

where we have used partial integration from the second step to the third step.

Appendix C

The effective action for LSS

In the section, we present details of calculations in subsection 6.2.1.

C.1 The expansion of the effective action

The total action for LSS is the following,

$$\begin{aligned} S &= \frac{1}{2}\int \mathrm{d}^4x\sqrt{h}\left[M_p^2[N^{(3)}R + N^{-1}(E_{ij}E^{ij} - E^2)]\right] \\ &+ \frac{1}{2}\int \mathrm{d}^4x\sqrt{h}\left[-NV''\varphi^2 + N^{-1}(\partial_0\varphi - N^i\partial_i\varphi)^2 - N\frac{(\nabla\varphi)^2}{a^2} + \bar{N}R\rho_c\right] \end{aligned} \tag{C.1}$$

And we have the leading order solutions,

$$n = \frac{1}{2M_p^2 H}[\varphi\dot{\varphi} + \frac{1}{\nabla^2}(\nabla\varphi)\cdot(\nabla\dot{\varphi})],\ n_i{}^T = 0, \tag{C.2}$$

$$\frac{\nabla^2 s}{a^2} = -3nH - \frac{1}{4M_p^2 H}\left[V''\varphi^2 + \frac{(\nabla\varphi)^2}{a^2} + \dot{\varphi}^2\right]. \tag{C.3}$$

Inserting these solutions back into the action Eq. (C.1), we can get the effective action. We can expand the effective action as we have done in Appendix B.2. The only differences are that we have $\dot{\phi} = 0$, $V_0 = V' = 0$ and the solutions of the constraints are of second order. So from Eq. (B.9), (B.18) and (B.19), we can read out the second, third and fourth order actions, which are the following,

$$S_2 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3\left[-V''\varphi^2 + \dot{\varphi}^2 - \frac{(\nabla\varphi)^2}{a^2} + 2n\rho_c\right], \tag{C.4}$$

$$S_3 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3\left[-V'''\frac{\varphi^3}{3}\right], \tag{C.5}$$

and

$$S_4 = \frac{1}{2}\int \mathrm{d}^4x\bar{N}a^3\left[-V''''\frac{\varphi^4}{12} - nV''\varphi^2 - n\dot{\varphi}^2 - na^{-2}(\nabla\varphi)^2 - 2\dot{\varphi}a^{-2}\partial_i s\partial_i\varphi\right]. \tag{C.6}$$

C.2 The quadratic action

In this section we will show that the term $2n\rho_c$ in the quadratic action is simply a boundary term. Since varying $2n\rho_c$ with respect to φ or $\dot{\varphi}$ would only give zero, it does not contribute to the field equation. In other words, $2n\rho_c$ is a boundary term. In the following we will calculate the variation of $2n\rho_c$ with respect to φ or $\dot{\varphi}$.

We can denote $\frac{1}{\nabla^2}$ by the Green's function $\frac{1}{4\pi||\mathbf{x}-\mathbf{x}'||}$, and we have

$$n^2M_p^2H = \varphi\dot{\varphi} + \int \mathrm{d}\mathbf{x}'\frac{1}{4\pi||\mathbf{x}-\mathbf{x}'||}(\nabla'\varphi)\cdot(\nabla\dot{\varphi}). \tag{C.7}$$

Varying it with respect to $\dot{\varphi}$, we have

$$\begin{aligned}
\frac{\partial(n^2M_p^2H)}{\partial\dot{\varphi}} &= \varphi(\mathbf{x}) - \nabla\int \mathrm{d}\mathbf{x}'\frac{1}{4\pi||\mathbf{x}-\mathbf{x}'||}\nabla'\varphi(\mathbf{x}') \\
&= \varphi(\mathbf{x}) + \int \mathrm{d}\mathbf{x}'\nabla'\frac{1}{4\pi||\mathbf{x}-\mathbf{x}'||}\nabla'\varphi(\mathbf{x}') \\
&= \varphi(\mathbf{x}) - \int \mathrm{d}\mathbf{x}'\nabla'^2\frac{1}{4\pi||\mathbf{x}-\mathbf{x}'||}\varphi(\mathbf{x}') \\
&= \varphi(\mathbf{x}) - \int \mathrm{d}\mathbf{x}'\delta(\mathbf{x}-\mathbf{x}')\varphi(\mathbf{x}') \\
&= 0,
\end{aligned} \tag{C.8}$$

where ∇' denots derivative with respect to $\mathbf{x}'$, and we have use partial integration and the equality that $\nabla\frac{1}{4\pi||\mathbf{x}-\mathbf{x}'||} = -\nabla'\frac{1}{4\pi||\mathbf{x}-\mathbf{x}'||}$. The variation of $2n\rho_c$ with respect to φ or $\dot{\varphi}$ can be calculated in the same way.

So ignoring the boundary term, we have the quadratic action,

$$S_2 = \frac{1}{2}\int \mathrm{d}^4x \bar{N} a^3 \left[-V''\varphi^2 + \dot{\varphi}^2 - \frac{(\nabla\varphi)^2}{a^2}\right]. \tag{C.9}$$

C.3 Higher order solutions of the constraints

The leading order solutions are obtained from the following constrain equations,

$$\partial_j[(1-n)(-4H)] - a^{-1}\partial_i{}^2 n_j{}^T - 2M_p^{-2}N^{-1}\partial_j\varphi\partial_0\varphi = 0, \tag{C.10}$$

and

$$-H(-6H + \frac{4}{a^2}\nabla^2 s) - M_p^{-2}\left[2V''\frac{\varphi^2}{2} + \frac{(\nabla\varphi)^2}{a^2} + \dot{\varphi}^2 + 2(1+2n)\rho_c\right] = 0. \tag{C.11}$$

To get the higher order solutions, we need to keep the higher order terms in these equations, and the equations are the following,

$$\partial_j[(1-n)(-4H)] - a^{-1}\partial_i{}^2 n_j{}^T - 2M_p^{-2}N^{-1}(\partial_j\varphi\partial_0\varphi - N^i\partial_j\varphi\partial_i\varphi) = 0, \tag{C.12}$$

and

$$\begin{aligned} 0 &= -H(-6H + \frac{4}{a^2}\nabla^2 s) - [(\frac{\partial_i\partial_j s}{a^2})^2 - (\frac{\nabla^2 s}{a^2})^2 + \frac{(\partial_i n_j{}^T + \partial_j n_i{}^T)^2}{4a^2}] \\ &- M_p^{-2}\left[2V''\frac{\varphi^2}{2} + \frac{(\nabla\varphi)^2}{a^2} + \dot{\varphi}^2 - 2(a^{-2}s_i + a^{-1}n_i{}^T)\partial_i\varphi\dot{\varphi} + 2(1+2n)\rho_c\right] \end{aligned} \tag{C.13}$$

We denote the leading order solutions by n_0, s_0 and $n_0{}^T$. Now the higher order solutions should obey,

$$\nabla^2 n_j{}^T = \frac{2}{M_p{}^2}\partial_j\varphi\partial_i\varphi n_i{}^T, \tag{C.14}$$

$$n = n_0 - \frac{1}{2M_p{}^2 H}\frac{\partial_i s}{a^2}\partial_i\varphi\varphi, \tag{C.15}$$

and

$$\frac{\nabla^2 s}{a^2} = \frac{\nabla^2 s_0}{a^2} - \frac{1}{4H}[(\frac{\partial_i\partial_j s_0}{a^2})^2 - (\frac{\nabla^2 s_0}{a^2})^2 + \frac{(\partial_i n_j{}^T + \partial_j n_i{}^T)^2}{4a^2}] + \frac{1}{2M_p{}^2 H}(a^{-2}s_{0i} + a^{-1}n_i{}^T)\partial_i\varphi\dot{\varphi}. \tag{C.16}$$

There are two questions to be answered for the higher order solutions:

1. whether they are suppressed or not.

2. whether ${n_i}^T$ should be included in the fourth order action.

Appendix D

On canonical quantization in curved spacetime

In this appendix, we will present a property of quantization in curved spacetime, namely that the annihilation and creation operators and the corresponding vacuum states are not unique, see [49, 54] for example. We will also show how to use the WKB-approximation [56] to solve for the mode functions.

D.1 The ambiguity of vacuum states

We have shown in 6.2.2 that the scalar field $\chi(\mathbf{x}, \eta)$ can be expanded using the mode functions u_k and u_k^*, and the corresponding annihilation and creation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}^{\dagger}$[1]. However, u_k and u_k^* are not the unique fundamental solutions to Eq.(6.51). Any linear combination of them forms a new set of solutions. We can construct two new independent fundamental solutions v_k and v_k^* via the Bogoliubov transformation [55],

$$v_k = \alpha_k u_k + \beta_k u_k^*, \quad v_k^* = \alpha_k^* u_k^* + \beta_k^* u_k, \tag{D.1}$$

where α_k and β_k are constants. The condition $W[v_k, v_k^*] = i$ requires that

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \tag{D.2}$$

[1] For convenience, from now on we will drop the hat on the operators and denote them by $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$.

The scalar field can be expanded in terms of the mode functions v_k and v_k^* and the corresponding annihilation and creation operators $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$,

$$\chi(\mathbf{x},\eta) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \big[b_{\mathbf{k}} v_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + b_{\mathbf{k}}^\dagger v_k^*(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \big]. \tag{D.3}$$

The annihilation and creation operators $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$ satisfy the standard commutation relations,

$$[b_{\mathbf{k}},\, b_{\mathbf{k}'}^\dagger] = \delta(\mathbf{k}-\mathbf{k}'), \quad [b_{\mathbf{k}},\, b_{\mathbf{k}'}] = 0, \quad [b_{\mathbf{k}}^\dagger,\, b_{\mathbf{k}'}^\dagger] = 0. \tag{D.4}$$

When we substitute the transformations between mode functions, Eq.(D.1), into Eq.(D.3), we can find the transformations between two sets of annihilation and creation operators.

$$\begin{aligned} \chi(\mathbf{x},\eta) &= \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \big[b_{\mathbf{k}}(\alpha_k u_k + \beta_k u_k^*) e^{i\mathbf{k}\cdot\mathbf{x}} + b_{\mathbf{k}}^\dagger (\alpha_k^* u_k^* + \beta_k^* u_k) e^{-i\mathbf{k}\cdot\mathbf{x}} \big] \\ &= \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \big[(\alpha_k b_{\mathbf{k}} + \beta_k b_{-\mathbf{k}}^\dagger) u_k e^{i\mathbf{k}\cdot\mathbf{x}} + (\alpha_k^* b_{\mathbf{k}}^\dagger + \beta_k^* b_{-\mathbf{k}}) u_k^* e^{-i\mathbf{k}\cdot\mathbf{x}} \big]. \end{aligned} \tag{D.5}$$

Comparing with Eq.(6.57), we find that

$$a_{\mathbf{k}} = \alpha_k b_{\mathbf{k}} + \beta_k b_{-\mathbf{k}}^\dagger, \quad a_{\mathbf{k}}^\dagger = \alpha_k^* b_{\mathbf{k}}^\dagger + \beta_k^* b_{-\mathbf{k}}. \tag{D.6}$$

The inverse transformations are,

$$b_{\mathbf{k}} = \alpha_k^* a_{\mathbf{k}} - \beta_k a_{-\mathbf{k}}^\dagger, \quad b_{\mathbf{k}}^\dagger = \alpha_k a_{\mathbf{k}}^\dagger - \beta_k^* a_{-\mathbf{k}}. \tag{D.7}$$

As we can see, there is no unique way to expand the field. There are infinite sets of mode functions and the corresponding annihilation and creation operators, which are connected by Bogoliubov transformations. For quantum fields in Minkowski spacetime, the annihilation and creation operators are used to define the vacuum state and the particle state. However, for quantum fields in curved spacetime, the annihilation and creation operators are not unique. Then the question arises: how should we choose the right vacuum? In the following, we will try to explain the problem.

For the two sets of annihilation and creation operators, the vacuum states are $|_{(a)}0\rangle$ and $|_{(b)}0\rangle$, respectively. And they are defined by

$$a_{\mathbf{k}}|_{(a)}0\rangle = 0, \quad b_{\mathbf{k}}|_{(b)}0\rangle = 0 \tag{D.8}$$

for all $\mathbf{k}$. The two different sets of particle states are generated by the creation operators $a^\dagger_{\mathbf{k}}$ and $b^\dagger_{\mathbf{k}}$ acting on the vacuum states respectively,

$$|_{(a)}m_{\mathbf{k}_1}, n_{\mathbf{k}_2}, ...\rangle \equiv \frac{1}{\sqrt{m!n!...}}\left[(a^\dagger_{\mathbf{k}_1})^m (a^\dagger_{\mathbf{k}_2})^n ...\right]|_{(a)}0\rangle \tag{D.9}$$

and

$$|_{(b)}m_{\mathbf{k}_1}, n_{\mathbf{k}_2}, ...\rangle \equiv \frac{1}{\sqrt{m!n!...}}\left[(b^\dagger_{\mathbf{k}_1})^m (b^\dagger_{\mathbf{k}_2})^n ...\right]|_{(b)}0\rangle. \tag{D.10}$$

The vacuum state $|_{(b)}0\rangle$ contains no "b-particles", but it may contain "a-particles". We can calculate the density of "a-particles" in the vacuum state $|_{(b)}0\rangle$,

$$\begin{aligned}
\langle_{(b)}0|N^{(a)}_{\mathbf{k}}|_{(b)}0\rangle &= \langle_{(b)}0|a^\dagger_{\mathbf{k}} a_{\mathbf{k}}|_{(b)}0\rangle \\
&= \langle_{(b)}0|(\alpha^*_k b^\dagger_{\mathbf{k}} + \beta^*_k b_{-\mathbf{k}})(\alpha_k b_{\mathbf{k}} + \beta_k b^\dagger_{-\mathbf{k}})|_{(b)}0\rangle \\
&= \langle_{(b)}0|\beta^*_k \beta_k b_{-\mathbf{k}} b^\dagger_{-\mathbf{k}}|_{(b)}0\rangle \\
&= |\beta_k|^2.
\end{aligned} \tag{D.11}$$

Therefore, the density of "a-particles" in the vacuum state $|_{(b)}0\rangle$ is $|\beta_k|^2$, which is nonzero if the Bogoliubov transformation is not an identity transformation. The other way around, one can also calculate the density of "b-particles" in the vacuum state $|_{(a)}0\rangle$.

When $\alpha_k = 1$ and $\beta_k = 0$, the vacuum is called BunchDavies vacuum, or the adiabatic vacuum. For this vacuum, in the infinity past, the high energy modes are in the lowest energy state. The details of the adiabatic vacuum can be found in [49]. We choose the Bunch-Davies vacuum in our case.

D.2 Solving for the mode functions

In this section, we will show how to use the WKB-approximation [56] to solve the mode functions, both in 4-dimensional spacetime and in D-dimensional spacetime.

D.2.1 In 4-dimensional spacetime

The equation for the mode functions is the following,

$$u''_{\mathbf{k}}(\eta) + \omega_k^2 u_{\mathbf{k}}(\eta) = 0, \tag{D.12}$$

where ω_k^2 is defined as

$$\omega_k^2 \equiv k^2 + m^2 a^2 - \frac{a''}{a}. \tag{D.13}$$

Since $\frac{\partial \omega_k^2}{\partial \eta} \ll \omega^3$ is valid [2], ω_k is slowly varying in time.

We can use the WKB-approximation to solve Eq. (D.12). First, make the ansatz

$$u = \rho \exp(\pm i \int_0^\eta W \mathrm{d}\eta). \tag{D.14}$$

Inserting this ansatz back into the equation, we have

$$\pm\, iW^{'} + (\ln \rho)'' + (\pm i W + \ln \rho)^2 + \omega^2 = 0. \tag{D.15}$$

The imaginary and real parts give

$$\pm\, W^{'} \pm 2W (\ln \rho)^{'} = 0 \tag{D.16}$$

and

$$(\ln \rho)^{''} + (\ln \rho)^2 - W^2 + \omega^2 = 0. \tag{D.17}$$

Eq. (D.16) has the solution $\rho = \frac{\alpha}{\sqrt{W}}$, where $\alpha = \frac{1}{\sqrt{2}}$ required by the condition that $W[u_k, u_k^*] = i$.

Inserting $\rho = \frac{1}{\sqrt{2W}}$ into Eq. (D.17), we have

$$W^2 = \omega^2 - \frac{1}{2}\left(\frac{W''}{W} - \frac{3{W'}^2}{2W^2}\right). \tag{D.18}$$

This equation can be solved perturbatively. The zeroth order solution is simply $W^{(0)} = \omega$, and the second order solution is

$${W^{(2)}}^2 = \omega^2 - \frac{1}{2}\left(\frac{\omega''}{\omega} - \frac{3{\omega'}^2}{2\omega^2}\right). \tag{D.19}$$

Further discussions about this approximation can be found in [49, 57].

[2] This condition requires $m \gg H$, which is valid in our case since we are dealing with sub-horizon physics.

D.2.2 In D-dimensional spacetime

In D-dimensional spacetime, setting $\bar{N} = a$, we can write down the quadratic Lagrangian density,

$$\mathcal{L} = \frac{1}{2} a^{(D-2)} \left[(\varphi')^2 - (\nabla\varphi)^2 - m^2 a^2 \varphi^2 \right], \tag{D.20}$$

where the prime represents taking the derivative with respect to the conformal time η, and $\mathrm{d}\eta = a\mathrm{d}t$.

Similar to the case in 4-dimensional spacetime, defining the new field function $\chi = a^{\frac{D-2}{2}}\varphi$, the Lagrangian density can be rewritten as

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2} a^{(D-2)} \left[(\varphi')^2 - (\nabla\varphi)^2 - m^2 a^2 \varphi^2 \right] \\
&= \frac{1}{2} \left[(\chi')^2 - (\nabla\chi)^2 - m^2 a^2 \chi^2 + \left(\frac{(D-2)a'}{2a} \right)^2 \chi^2 - (D-2)\frac{a'}{a}\chi\chi' \right] \\
&= \frac{1}{2} \left[(\chi')^2 - (\nabla\chi)^2 - m^2 a^2 \chi^2 + \frac{D-2}{2} \left(\frac{(D-4)}{2} (\frac{a'}{a})^2 + \frac{a''}{a} \right) \chi^2 - \left(\frac{(D-2)a'}{2a}\chi^2 \right)' \right] \\
&= \frac{1}{2} \left[(\chi')^2 - (\nabla\chi)^2 - m^2 a^2 \chi^2 + \frac{D-2}{2} \left(\frac{(D-4)}{2} (\frac{a'}{a})^2 + \frac{a''}{a} \right) \chi^2 \right],
\end{aligned} \tag{D.21}$$

where we have dropped the total time derivative term $\left(\frac{(D-2)a'}{2a}\chi^2 \right)'$. The field equation for χ is

$$\chi'' - \nabla^2\chi + \left(m^2 a^2 - \frac{(D-2)}{2} \left(\frac{(D-4)}{2} \left(\frac{a'}{a} \right)^2 + \frac{a''}{a} \right) \right) \chi = 0. \tag{D.22}$$

In Fourier modes, for $\chi_{\mathbf{k}}(\eta)$,

$$\chi''_{\mathbf{k}}(\eta) + \omega_k^2 \chi_{\mathbf{k}}(\eta) = 0, \tag{D.23}$$

where ω_k^2 is defined as

$$\omega_k^2 = k^2 + m^2 a^2 - \frac{(D-2)}{2} \left(\frac{(D-4)}{2} \left(\frac{a'}{a} \right)^2 + \frac{a''}{a} \right). \tag{D.24}$$

Eq.(D.23) can be solved with WKB-approximation, the same as we have solved Eq. (6.51). So the general solution $\chi_{\mathbf{k}}(\eta)$ is the linear combination of $u_k(\eta)$ and $u_k^*(\eta)$,

$$\chi_{\mathbf{k}}(\eta) = a_{\mathbf{k}} u_k(\eta) + b_{\mathbf{k}} u_k^*(\eta). \tag{D.25}$$

Similar to the case in 4-dimensional spacetime, we have

$$u = \frac{1}{\sqrt{2W}} \exp(\pm i \int_0^\eta W \mathrm{d}\eta), \tag{D.26}$$

where $W^2 = \omega^2 - \frac{1}{2}\left(\frac{W''}{W} - \frac{3W'^2}{2W^2}\right)$. Similarly, W can be solved perturbatively, and

$$W^{(0)} = \omega,\ W^{(2)^2} = \omega^2 - \frac{1}{2}\left(\frac{\omega''}{\omega} - \frac{3\omega'^2}{2\omega^2}\right). \tag{D.27}$$

The only difference is that in D-dimensional spacetime, there is an extra term proportional $(\frac{a'}{a})^2$ in ω_k^2.

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