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Bachelor's thesis

Scott's Model for the Intuitionistic Continuum

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Abstract

Various models have been introduced in the study of intuitionistic mathematics. One such model is Scott's model for the intuitionistic continuum. In his 1968 article *Extending the Topological Interpretation to Intuitionistic Analysis*, he represents the intuitionistic reals as continuous functions from a topological space into the classical reals. The presentation of this model including added examples and extended proofs is the main topic of this thesis. However, we also provide the reader with additional background material. We start out by a brief introduction to intuitionism and its logic. Futhermore, this thesis contains a discussion of some of the topological properties of the *Baire space*, since they are used in the model.

Contents

1 Introduction

At the beginning of the twentieth century, the Dutch mathematician L.E.J. Brouwer introduced a completely new philosophy of mathematics known as *intuitionism*. Although it stirred up much debate about the foundations of mathematics soon after its introduction [8, pp. 1], intuitionism has only gained a small following. Today, intuitionism is mostly studied in relation to classical mathematics [13, pp. 273].

In the study of intuitionism, various models in classical mathematics have been proposed. In constructing such a model, one first finds some set X of classical mathematical structures. The elements of X represent the 'values' of formulas. Futhermore, one 'translates' the symbols of intuitionistic logic as operations on the elements of X. Finally, one assigns some specific elements as representing valid formulas. An important advantage of this approach is that one can reason about X in a classical manner, providing a familiar framework for the classical mathematician [9].

In this thesis, we will take X to be the set of opens of a given topological space T. The logical $\wedge$ symbol for example, will be interpretated as taking the intersection of two opens. In particular, T itself will represent a valid formula. This interpretation is known as the *topological interpretation*.

Guided by Dana Scott's 1968 article *Extending the Topological Interpretation to Intuitionistic Analysis*, we use this interpretation to construct a classical model for the intuitionistic continuum. We do so by representing the intuitionistic reals as continuous functions from a given topological space into the classical reals.

Before introducing the model, we start in section 2 with a brief introduction to intuitionism and its philosophy. We present the so-called *weak counterexamples* which Brouwer used to argue that the principle of the excluded middle was invalid for mathematics. After explaining the intuitionistic interpretation of logical connectives, we go over some of the basic order properties of the intuitionistic continuum.

These properties are then used to construct the above-mentioned model in section 3.1. Subsequently, we present some independence results regarding open formulae in two or three variables in section 3.2. Finally, in section 3.3, we extend our model to include other relations and operations and use it to compare the intuitionistic continuum to the classical continuum. We end section 3.3 by some remarks on Brouwer's Continuity Theorem.

Throughout section 3.2 – 3.3, we use some topological properties of the *Baire space*. The appendix may be consulted in case the reader is unfamiliar with the Baire space or the concept of homogeneous spaces.

Theorems, definitions, remarks, etc. are numbered as follows: Theorem $x.y.z$ can be found in section $x.y$, after Definition $x.y.(z-1)$ and before Remark $x.y.(z+1)$. The appendix is numbered independently and its theorems, definitions, etc. are numbered as $A.z$, where A indicates references to the appendix.

2 What is intuitionism?

Intuitionism is a philosophy of mathematics founded by the Dutch mathematician L.E.J. Brouwer at the beginning of the twentieth century [8, pp. 1]. In this thesis, we will study a model of intuitionistic logic. While this model is interesting by itself, this chapter will provide the reader with some (philosophical) background, placing the model in a broader context. One may view it as a motivation for studying the model, although intuitionistic logic certainly has its applications in mathematics [8, section 4.2].

According to Brouwer, mathematics deals with mental constructions and is not concerned with discovering (absolute) truths that exist independently of us [14, chapter 1, section 1.6]. Because we only deal with mental constructions, the existence of mathematical objects depends on our ability to construct them. As Heyting notes in [3, pp. 2]:

> If "to exist" does not mean "to be constructed", it must have some metaphysical meaning. It cannot be the task of mathematics to investigate this meaning or to decide whether it is tenable or not. [..] Brouwer's program entails that we study mathematics as something simpler, more immediate than metaphysics. In the study of mental mathematical constructions "to exist" must be synonymous with "to be constructed".

Therefore, "a mathematical statement is true if we have proof of it and false if we can show that the assumption that there is a proof for the statement leads to a contradiction" [14, section 1.1.6].

According to Brouwer, we only use a mathematical language in order to convey our constructions to others and because, unlike the ideal mathematician, we do not have perfect recall [14, section 1.1.6]. Logic is a way to encode our mathematical constructions. However, we can never be fully sure that any formal system validly represents our mathematical constructions, even if it is free of contradictions. For example, Brouwer thought that the principle of the excluded middle ($\varphi \vee \neg\varphi$ for any statement φ) was invalid for infinite structures, making classical logic unfit for mathematics [8, pp. 2]. We will get back to this in the upcoming sections. Before we do so, we answer an important question: what are Brouwer's constructions based on?

The answer is that they are based on our intuition of time, which is an a priori notion, according to Brouwer. The concept of the natural numbers arises from our ability to consider a single unit and in our minds indefinitely add copies of this single unit. When we think of the natural numbers as steps in time, it becomes clear that we count backwards, producing the numbers 0, -1, -2,. . . . Furthermore, for every two moments in time, we can conceive of a moment in between these two moments. Moreover, we realise that we can (in principle) extend the timeline forwards and backwards infinitely, although we will never 'reach infinity'. This provides us with an intuition of the continuum [13, pp. 274].

To Brouwer, the reals are not points on the continuum, but rather a real number is the process which at each step consists of choosing two ‘moments’ on the timeline, so that these ‘moments’ lie closer to each other at every step of the process[1] [13, pp. 276]. Fittingly, we may use Cauchy sequences of rational numbers as representatives of real numbers [3, section 3.2.2].

2.1 An invalid construction

In this section we will clarify what Brouwer means by mathematical construction. Let

$$x = \begin{cases} 1 & \text{if Goldbach's conjecture is true;} \\ 0 & \text{if Goldbach's conjecture is false.} \end{cases}$$

This may seem like a valid definition of some integer x. Brouwer however, does not consider this to be valid. For, if x was an integer, we would have *proof* that it was, meaning we could specify some $n \in \mathbb{N}$ such that x equals n. Being able to do so, would imply a decision about the validity of Goldbach's conjecture, which intuitionistically means we have proof of Goldbach's conjecture or we could derive a contradiction from the assumption that there is a proof of Goldbach's conjecture. However, Goldbach's conjecture is, as yet, an undecided mathematical proposition, so this is (for now) impossible.

We see that $\varphi \vee \neg\varphi$ cannot be true for arbitrary statements φ, because it would imply the availability of a universal method for obtaining a proof of φ or a contradiction from a supposed proof of φ. Clearly, if φ expresses Goldbach's conjecture, then no such method is available to us (at least at the time of writing).

The following section provides some more interesting counterexamples to the principle of the excluded middle.

2.2 Weak counterexamples

In this section we will present two construction which Brouwer used to challenge the principle of the excluded middle.

Example 2.2.1 ([8, pp. 3])**.** Define a sequence $(a_k)_{k\in\mathbb{N}}$ by the following:

$$a_k = \begin{cases} 2^{-k_0} & \text{if } k \geq k_0 \text{ and } k_0 \text{ is the first} \\ & \text{counterexample to Goldbach's conjecture;} \\ 2^{-k} & \text{else.} \end{cases}$$

Note that every a_k can be calculated in a finite amount of time, so that, in principle, an arbitrary number of a_k's may be computed. Furthermore, notice that $(a_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, so that it defines a real number α.

[1] For this reason, Brouwer introduced choice sequences in intuitionistic mathematics [14, section 4.6.1].

Intuitionistically it makes no sense to assert that $\alpha = 0$ or $\alpha > 0$, because this would mean that one had a proof of either $\alpha = 0$ or $\alpha > 0$, implying a proof of or an counterexample to the Goldbach conjecture. Furthermore, the claim $\alpha = 0 \vee \alpha \neq 0$ does not hold, so we see once more that $\varphi \vee \neg\varphi$ is not intuitionistically valid for an arbitrary statement φ. The real number α is called a 'floating number'.

Example 2.2.2 ([2, pp. 277])**.** Let α be the real number defined by the Cauchy sequence $(a_k)_{k\in\mathbb{N}}$, where

$$a_k = \begin{cases} (-2)^{-k_0} & \text{if } k \geq k_0 \text{ and } k_0 \text{ is the first integer } n \text{ such that when } \pi \text{ is} \\ & \text{expanded into } n \text{ decimals 99 consecutive 9's can be found;} \\ (-2)^{-k} & \text{else.} \end{cases}$$

Note that the statement "there are is no sequence of 99 consecutive 9's in the decimal expansion of π" is an undecided mathematical statement. Similar to the previous example, it does not make sense to claim that $\alpha < 0 \vee \alpha = 0 \vee \alpha > 0$. Note that this implies that $\forall x \in \mathbb{R}\,(x < 0 \vee x = 0 \vee x > 0)$ is not intuitionistically valid.

Let A be the statement "α is rational". Then $\neg A$ expresses that α is irrational. Suppose that α is irrational, then there would be no sequence of 99 consecutive 9's in the decimal expansion of π. Hence, $\alpha = 0$. But this contradicts our assumption that α is irrational. We see that we derive a contradiction from the assumption (that there is a proof of) $\neg A$. Thus, we have shown that $\neg\neg A$, viz. α cannot be irrational. However, we cannot claim that α is rational, because this claim means that we could find some $p, q \in \mathbb{Z}$ such that $\alpha = p/q$, which would imply a decision on the decimal expansion of π. We see that $\neg\neg A \to A$ fails.

2.3 Intuitionistic logic

In this thesis we will deal with intuitionistic logic. Therefore, it should be clear how one should intuitionistically interpret the basic logical connectives. The following paragraph is based on [8, pp. 5] and [14, section 1.3.1].

Heyting formulated an interpretation of the symbols $\vee, \wedge, \neg, \to$ and $\perp$ (a special symbol that stands for "absurdity" or a contradiction). This interpretation is known as the *Brouwer-Heyting-Kolmogorov (BHK) interpretation.* This interpretation defines what it means to give an intuitionistic proof of a propositional formula φ in terms of its constituents.

- A proof of $\varphi \wedge \psi$ is given by presenting a proof of φ and a proof of ψ.
- A proof of $\varphi \vee \psi$ is given by presenting either a proof of φ or a proof of ψ.
- A proof of $\varphi \to \psi$ is a construction which allows us to transform any proof of φ into a proof of ψ.

- Absurdity $\bot$ has no proof; a proof of $\neg\varphi$ is a construction which allows us to transform any hypothetical proof of φ into a proof of contradiction.

Now that we know how to intuitionistically interpret the logical symbols, we may present some intuitionistic logical laws.

In the previous section we saw that $\neg\neg\varphi \to \varphi$ is not intuitionistically valid for arbitrary statements φ. However, we can prove $\varphi \to \neg\neg\varphi$.

Proof. For suppose that we have a proof of φ. We need to show that a proof of $\neg\varphi$ leads to a contradiction, but this is immediate if we assume φ, because $\varphi \wedge \neg\varphi \to \bot$. ■

Now that we have shown that $\varphi \to \neg\neg\varphi$, we can prove $\neg\neg\neg\varphi \to \neg\varphi$.

Proof. Suppose we have a proof of $\neg\neg\neg\varphi$. We need to show that a proof of φ leads to a contradiction. Using the previous proof we can obtain a proof of $\neg\neg\varphi$ from a proof of φ, which contradicts $\neg\neg\neg\varphi$. ■

In section 2.2 we have only shown that the principle of the excluded middle is unacceptable as a intuitionistic logical law, but we have not refuted it, i.e., we have not shown how to derive an actual contradiction from the assumption that the principle of the excluded middle is valid. In fact, we are not able to find a statement φ such that $\neg(\varphi \vee \neg\varphi)$ holds, because $\neg\neg(\varphi \vee \neg\varphi)$ *is* valid. We may prove this as follows.

Proof. We need to show that a proof of $\neg(\varphi \vee \neg\varphi)$ leads to a contradiction. Having a proof of $\neg(\varphi \vee \neg\varphi)$ means we have a proof of $\varphi \vee \neg\varphi \to \bot$. Hence, we also have a proof of $\varphi \to \bot$ and thus of $\neg\varphi$. We see that a proof of $\neg(\varphi \vee \neg\varphi)$ implies a proof of $\neg\varphi$, but a proof of $\neg\varphi$ can clearly be transformed into a proof of $\varphi \vee \neg\varphi$, so that $\neg(\varphi \vee \neg\varphi) \to \varphi \vee \neg\varphi$. Since $\neg(\varphi \vee \neg\varphi) \to \neg(\varphi \vee \neg\varphi)$ is obviously valid, we obtain $\neg(\varphi \vee \neg\varphi) \to (\varphi \vee \neg\varphi) \wedge \neg(\varphi \vee \neg\varphi)$ and hence the desired $\neg(\varphi \vee \neg\varphi) \to \bot$. ■

The above is still quite informal and only deals with propositional logic. Heyting has formalised intuitionistic predicate logic in a system that is known as *Heyting's predicate calculus*.

2.4 Order in the continuum

Heyting has interpreted the intuitionistic continuum in [3]. We will treat the order relation on the continuum as an abstract relation satisfying some fundamental properties, which are justified by Heyting. These properties are then used to construct a model for the intuitionistic continuum in *classical* mathematics. The content of this section is due to [11, chapter 1].

2.4.1 Definitions

The relations needed for the construction of our model are quite basic and we will start with the $<$ relation. In this section all the properties discussed are understood to hold for any real numbers x, y, z. The following properties hold for the $<$ relation:

$$\neg\,(x < y \wedge y < x)\,; \tag{2.4.1}$$
$$x < y \rightarrow x < z \vee z < y. \tag{2.4.2}$$

Note that transitivity of $<$ follows from the properties above. For if we have real numbers x, y, z such that $x < y$ and $y < z$ then it follows from (2.4.2) that $x < z$ or $z < y$. If $z < y$ was true, then we would obtain a contradiction with the assumption $y < z$ using (2.4.1). So we see that $x < z$, hence:

$$x < y \wedge y < z \rightarrow x < z. \tag{2.4.3}$$

We define the "apartness" relation for which we will write $\#$ by:

$$x \mathrel{\#} y \leftrightarrow x < y \vee y < x. \tag{2.4.4}$$

Furthermore, we define:

$$x \leq y \leftrightarrow \neg y < x; \tag{2.4.5}$$
$$x = y \leftrightarrow \neg x \mathrel{\#} y. \tag{2.4.6}$$

We also define $\geq$ and $>$ with the obvious meanings.

2.4.2 Properties of the order relation

Please note that $x \leq y \rightarrow x < y \vee x = y$ is not intuitionistically valid. Consider example 2.2.1 again. Clearly, $0 \leq \alpha$, but $0 < \alpha \vee \alpha = 0$ does not hold.

However, we can prove:

$$x < y \leftrightarrow x \leq y \wedge x \mathrel{\#} y. \tag{2.4.7}$$

Proof. Suppose $x < y$, then clearly $x \mathrel{\#} y$ by (2.4.4). Furthermore, we could obtain a contradiction from the assumption $y < x$ using (2.4.1), so $\neg y < x$ holds. Hence, $x < y \rightarrow x \leq y \wedge x \mathrel{\#} y$ by (2.4.5). For the converse, assume $x \leq y \wedge x \mathrel{\#} y$. Then $\neg y < x$ and $x < y \vee y < x$ by (2.4.4) and (2.4.5). We see that $x < y$, concluding our proof. ■

Similarly, we may prove:

$$x = y \leftrightarrow x \leq y \wedge y \leq x. \tag{2.4.8}$$

Furthermore, if we set $y = x$ in (2.4.1), we see that $<$ is irreflexive and by (2.4.5) the relation $\leq$ is reflexive:

$$x \leq x. \tag{2.4.9}$$

Moreover, we may show:

$$x \leq y \wedge y \leq z \to x \leq z; \tag{2.4.10}$$
$$x \leq y \wedge y < z \to x < z; \tag{2.4.11}$$
$$x < y \wedge y \leq z \to x < z. \tag{2.4.12}$$

Proof. We will prove the first statement here; the others are proved similarly. Suppose that $x \leq y \wedge y \leq z$. To show that $x \leq z$ holds we have to show (by (2.4.4)) that the assumption $z < x$ leads to a contradiction. Suppose that $z < x$. Then by (2.4.2), $z < y \vee y < x$. Note that we can obtain a contradiction from both $z < y$ and $y < x$ by (2.4.5) and our assumptions $x \leq y$ and $y \leq z$, finishing our proof. ■

Using (2.4.9) - (2.4.12) we may show that $=$ is indeed an equality relation:

$$x = x; \tag{2.4.13}$$
$$x = y \wedge A(x) \to A(y), \tag{2.4.14}$$

where A is any first-order formula involving only $<, \leq, =$ and $\#$.

Proof. Note that (2.4.13) follows directly from (2.4.8) and (2.4.9). To prove (2.4.14), note that it suffices to show that $x = y \wedge x < z \to y < z$ and $x = y \wedge z < x \to z < y$, because all relations can be defined in terms of the $<$ relation. Suppose $x = y$ and $x < z$. Formula (2.4.8) gives us $y \leq x$. We may now invoke (2.4.11) to conclude $x < z$. The formula $x = y \wedge z < x \to z < y$ is proved similarly. ■

2.4.3 Rationals in the continuum

We now extend our language by adding variables q, r, s ranging over the rational numbers and rational constants where necessary. We may assume:

$$q < r \vee \neg q < r,$$

because of the constructive nature of the rationals. The theories of the rationals in intuitionistic and classical logic coincide. The rationals are dense in the continuum in the following sense:

$$\forall x\, \exists q, r\, (q < x \wedge x < r); \tag{2.4.15}$$
$$\forall x, y\ (x < y \to \exists q\, (x < q \wedge q < y))\,. \tag{2.4.16}$$

We can now show:

$$\text{The following are equivalent :} \tag{2.4.17}$$

$$\begin{array}{rl} \text{(i)} & x < q; \\ \text{(ii)} & \exists r\,(r < q \wedge x < r); \\ \text{(iii)} & \exists r\,(r < q \wedge x \leq r). \end{array}$$

A similar result (2.4.17′) holds with $<$ and $\leq$ replaced by $>$ and $\geq$ respectively.

Proof. The proof of (i) $\Rightarrow$ (ii) is immediate by (2.4.16). We may use (2.4.7) to show (ii) $\Rightarrow$ (iii). Finally, we can prove (iii) $\Rightarrow$ (i) using (2.4.11). ■

We also have:

$$\text{The following are equivalent:}$$

$$\begin{array}{rl} \text{(i)} & x \leq r; \\ \text{(ii)} & \forall s\,(s > r \rightarrow x < s); \\ \text{(iii)} & \forall s\,(s > r \rightarrow x \leq s). \end{array}$$

Proof. The proof of (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is easy by (2.4.10) and (2.4.11). We will prove (iii) $\Rightarrow$ (i). By (2.4.5), we have to obtain a contradiction from the assumption that $r < x$. Assuming $x > r$, we use (2.4.17′)(ii) to find an s such that $s > r$ and $x > s$. By (iii), $x \leq s$, which contradicts $x > s$ because of (2.4.5). ■

3 Scott's model

In this section we will construct a model for the intuitionistic continuum in *classical* mathematics based on the properties of the continuum as seen in the previous section.

This chapter is based on Dana Scott's article *Extending the topological interpretation to intuitionistic analysis* [11]. I have added some proofs and examples and corrected some (minor) mistakes in the original article.

The chapter is built up as follows. In the first section we will construct a model motivated by section 2.4. The second section provides us with some independence results and gives us a decision method for open formulae in two or three variables. Finally, the third section we extend our model to include other relations and operations and use it to compare the intuitionistic to the classical continuum.

3.1 The topological interpretation

We start out by associating to each formula A an *open* subset $[\![A]\!]$ of a given topological space T, so that the following rules are satisfied:

$$[\![A \wedge B]\!] = [\![A]\!] \cap [\![B]\!]; \tag{3.1.1}$$

$$[\![A \vee B]\!] = [\![A]\!] \cup [\![B]\!]; \tag{3.1.2}$$

$$[\![\neg A]\!] = ([\![A]\!]^c)^\circ; \tag{3.1.3}$$

$$[\![A \to B]\!] = ([\![A]\!]^c \cup [\![B]\!])^\circ; \tag{3.1.4}$$

where U^c denotes the complement of U in T and U° is the interior of U in T for every subset U of T. Furthermore, the variables are interpreted as ranging over the elements ξ in some given domain $\mathscr{R}$ (we use "$\mathscr{R}$" here because we are going to construct a model for the reals). Thus for quantified formulae we have:

$$[\![\exists x\, A(x)]\!] = \bigcup_{\xi \in \mathscr{R}} [\![A(\xi)]\!]; \tag{3.1.5}$$

$$[\![\forall x\, A(x)]\!] = \Big(\bigcap_{\xi \in \mathscr{R}} [\![A(\xi)]\!] \Big)^\circ. \tag{3.1.6}$$

(Note that we really should write $A(\ulcorner \xi \urcorner)$ where $\ulcorner \xi \urcorner$ is the name of the object ξ in the formal language.)

As is proved in [10], all formulae A provable in Heyting's predicate calculus are valid in this interpretation in the sense that $[\![A]\!] = T$.

Remark 3.1.1. Note that if we set $T = (0, 2) \subseteq \mathbb{R}$ and $[\![A]\!] = T \setminus \{1\}$, then

$$\begin{aligned} [\![\neg A \vee A]\!] &= [\![\neg A]\!] \cup [\![A]\!] \\ &= ([\![A]\!]^c)^\circ \cup [\![A]\!] \\ &= \{1\}^\circ \cup (T \setminus \{1\}) \\ &= T \setminus \{1\} \neq T; \end{aligned}$$

so that $[\![\neg A \vee A]\!]$ does not necessarily equal T, as one would expect from a model for intuitionistic predicate logic.

Furthermore, note that if $A \to B$ is valid, i.e. $[\![A \to B]\!] = T$, then

$$t \in [\![A]\!] \Rightarrow t \in [\![B]\!].$$

We now wish to interpret the $<$, $\#$, $=$ and $\leq$ relations in this topological interpretation. Let us assume that (2.4.1) - (2.4.6) and (2.4.15) - (2.4.16) are valid in this particular interpretation, where $\mathscr{R}$ is assumed to include $\mathbb{Q}$.

We start by noting that for each $t \in T$ an element $\xi \in \mathscr{R}$ determines a cut in $\mathbb{Q}$ where the upper part of the cut is the set $\{r \in \mathbb{Q} \mid t \in [\![\xi < r]\!]\}$. This suggests associating with $\xi \in \mathscr{R}$ a function $\xi\colon T \to \mathbb{R}$ defined as

$$\xi(t) = \inf\{r \in \mathbb{Q} \mid t \in [\![\xi < r]\!]\}.$$

Note that $\xi(t)$ has constant value q if $\xi = q \in \mathbb{Q}$. In general we have the following theorem.

Theorem 3.1.2. *The function $\xi\colon T \to \mathbb{R}$ is continuous for all $\xi \in \mathscr{R}$.*

Proof. We first show:

$$\{t \in T \mid \xi(t) < q\} = [\![\xi < q]\!]. \tag{3.1.7}$$

For if $\xi(t) < q$, then by definition of the function ξ, we have $t \in [\![\xi < r]\!]$ for some $r < q$. But then (2.4.3) tells us that $t \in [\![\xi < q]\!]$. For the converse, assume $t \in [\![\xi < q]\!]$. By (2.4.17), we have $t \in [\![\exists r\,(r < q \wedge \xi < r)]\!]$. We use (3.1.5) to obtain $t \in [\![\xi < r]\!]$ for some $r < q$. Thus, $\xi(t) \leq r < q$, completing the proof of our claim.

We then show that

$$\{t \in T \mid q < \xi(t)\} = [\![q < \xi]\!]. \tag{3.1.8}$$

Assume that $q < \xi(t)$. Then, by definition, there exists an $r > q$ such that $t \notin [\![\xi < s]\!]$ for all $s < r$. Take any s with $q < s < r$. Using (2.4.3) we find that the following formulae are valid:

$$s < \xi \wedge \xi < q \to s < q;$$
$$q < s \wedge s < \xi \to q < \xi;$$
$$\xi < q \wedge q < s \to \xi < s;$$

and thus (see (3.1.4)):

$$(([\![s < \xi]\!] \cap [\![\xi < q]\!])^c \cup [\![s < q]\!])^\circ = T, \text{ hence } ([\![s < \xi]\!] \cap [\![\xi < q]\!])^c \cup [\![s < q]\!] = T;$$
$$(([\![q < s]\!] \cap [\![s < \xi]\!])^c \cup [\![q < \xi]\!])^\circ = T, \text{ hence } ([\![q < s]\!] \cap [\![s < \xi]\!])^c \cup [\![q < \xi]\!] = T;$$
$$(([\![\xi < q]\!] \cap [\![q < s]\!])^c \cup [\![\xi < s]\!])^\circ = T, \text{ hence } ([\![\xi < q]\!] \cap [\![q < s]\!])^c \cup [\![\xi < s]\!] = T.$$

Using De Morgan's law and the fact that $[\![q < s]\!] = T$ (and thus $[\![s < q]\!] = \emptyset$ by (2.4.1)), we have:

$$\begin{aligned} [\![s < \xi]\!]^c \cup [\![\xi < q]\!]^c &= T; \\ [\![s < \xi]\!]^c \cup [\![q < \xi]\!] &= T; \\ [\![\xi < q]\!]^c \cup [\![\xi < s]\!] &= T; \end{aligned}$$

which allows us to claim $[\![q < \xi]\!] \cup [\![\xi < s]\!] = T$. Since, $t \notin [\![\xi < s]\!]$, we must have: $t \in [\![q < \xi]\!]$, so that $\{t \in T \mid q < \xi(t)\} \subseteq [\![q < \xi]\!]$.

Conversely, assume that $t \in [\![q < \xi]\!]$. Making use of (2.4.17′) and (3.1.5) we find that $t \in [\![r < \xi]\!]$ for some $r > q$. If $s < r$, then $t \in [\![s < \xi]\!]$ because of (2.4.3). Hence we may use (2.4.1) to conclude: for all $s < r$ we have $t \notin [\![\xi < s]\!]$. Thus, $r \leq \xi(t)$ and so $q < \xi(t)$, as desired.

We can now finish the proof swiftly. Since any open of $\mathbb{R}$ can be written as the union of intervals with rational endpoints, it suffices to show that $\xi^{-1}((q, q'))$ is open in T for any $q, q' \in \mathbb{Q}$ with $q' > q$. Note that:

$$\begin{aligned} \xi^{-1}((q, q')) &= \{t \in T \mid q < \xi(t)\} \cap \{t \in T \mid \xi(t) < q'\} \\ &= [\![q < \xi]\!] \cap [\![\xi < q']\!] \quad \text{(by (3.1.7) and (3.1.8))}; \end{aligned}$$

which is open in T, completing our proof. ■

Having proved this, we can now also prove the following theorem.

Theorem 3.1.3. *The following equalities hold:*

$$\begin{aligned} [\![\xi < \eta]\!] &= \{t \in T \mid \xi(t) < \eta(t)\}; \\ [\![\xi \# \eta]\!] &= \{t \in T \mid \xi(t) \neq \eta(t)\}; \\ [\![\xi \leq \eta]\!] &= \{t \in T \mid \xi(t) \leq \eta(t)\}^\circ; \\ [\![\xi = \eta]\!] &= \{t \in T \mid \xi(t) = \eta(t)\}^\circ. \end{aligned}$$

Proof. We will only proof the first equality. The others follow directly from (2.4.4) - (2.4.6) and the first equality. We start by showing that $[\![\xi < \eta]\!] \subseteq \{t \in T \mid \xi(t) < \eta(t)\}$. Let $t \in [\![\xi < \eta]\!]$. By (2.4.16), $t \in [\![\exists q\, (\xi < q \wedge q < \eta)]\!]$, hence $t \in [\![\xi < q \wedge q < \eta]\!]$ for some $q \in \mathbb{Q}$ by (3.1.5). But then, by (3.1.1), $t \in [\![\xi < q]\!] \cap [\![q < \eta]\!]$. Thus, $t \in \{t \in T \mid \xi(t) < \eta(t)\}$ by (3.1.7) and (3.1.8).

For the converse, let $t \in T$ such that $\xi(t) < \eta(t)$. Since the rationals are dense in the classical reals, there exists some $q \in \mathbb{Q}$ with $\xi(t) < q < \eta(t)$. So, $t \in \{t \in T \mid \xi(t) < q\} \cap \{t \in T \mid q < \eta(t)\}$. Hence, by (3.1.7), (3.1.8) and (3.1.1), $t \in [\![\xi < q \wedge q < \eta]\!]$. Using (2.4.3), we obtain: $t \in [\![\xi < \eta]\!]$, as desired. ■

These facts suggest that a classical model for the intuitionistic continuum is to let $\mathscr{R}$ be the collection of *all* continuous functions $\xi\colon T \to \mathbb{R}$ and to use rules (3.1.1) - (3.1.6) for evaluating formulae. The reason for using all the continuous functions is that $\mathscr{R}$ should be complete, but we will not go into this argument (neither does Scott).

3.2 Some independence results

In this section we will present a decision method for open formulae in two or three variables. We will start with open formulae in two variables.

3.2.1 Two variables

Consider an open formula in two variables. In view of (2.4.4) - (2.4.6), there is a formula $A(P, Q)$ of propositional calculus such that the given formula is equivalent to $A(x < y, y < x)$. We have the following theorem.

Theorem 3.2.1. *The formula*

$$A(x < y, y < x)$$

is provable from (2.4.1)–(2.4.2) *if and only if the propositional formula*

$$\neg(P \wedge Q) \to A(P, Q)$$

is provable in Heyting's predicate calculus (see the end of section 2.3).

Proof. Note that if the propositional formula is provable, then $A(x < y, y < x)$ is easily seen to be valid, since we can substitute $x < y$ for P and $y < x$ for Q and use (2.4.1) to obtain $A(x < y, y < x)$ for any x and y.

For the converse, assume that the propositional formula is not provable in Heyting's predicate calculus. We have to show that $A(x < y, y < x)$ is not provable from (2.4.1)–(2.4.2). We will do so by showing that there are $\xi, \eta \in \mathscr{R}$ such that $[\![A(\xi < \eta, \eta < \xi)]\!] \neq T$.

We will use some results stated in [10, pp. 385–395]: there is a *metric* topological space T and there are assignments of open subsets $[\![P]\!]$ and $[\![Q]\!]$ to P and Q such that:

$$[\![\neg(P \wedge Q)]\!] \not\subseteq [\![A(P, Q)]\!].$$

If we set $T' = [\![\neg(P \wedge Q)]\!] \subseteq T$, then T' is also a metric topological space. Furthermore, $[\![P]\!]' = [\![P]\!] \cap T'$ and $[\![Q]\!]' = [\![Q]\!] \cap T'$ are disjoint, since:

$$\begin{aligned} [\![P]\!]' \cap [\![Q]\!]' &= [\![P]\!] \cap [\![Q]\!] \cap [\![\neg(P \wedge Q)]\!] \\ &= [\![P]\!] \cap [\![Q]\!] \cap (T \setminus [\![P \wedge Q]\!])^\circ \\ &= ([\![P]\!] \cap [\![Q]\!]) \cap (T \setminus ([\![P]\!] \cap [\![Q]\!]))^\circ \\ &= \emptyset. \end{aligned}$$

We relativize to a subspace and set $T = [\![\neg(P \wedge Q)]\!]$, so that $[\![P]\!]$ and $[\![Q]\!]$ are disjoint open sets.

This is where our model comes into play. We define:

$$\begin{aligned} \pi_P \colon T &\to \mathbb{R} \\ t &\mapsto \inf\{d(t, t') \mid t' \in [\![P]\!]^c\}, \end{aligned}$$

where d is the metric on T; similarly for π_Q. These functions are clearly non-negative. Moreover, these functions are continuous. To prove this, we need to show that $\pi_P^{-1}((a,b))$ is open in T for any open (a,b) of $\mathbb{R}$. Suppose $t \in \pi_P^{-1}((a,b))$. We need to prove that there exists some $\epsilon > 0$ such that $B_\epsilon(t) = \{u \in T \mid d(u,t) < \epsilon\} \subseteq \pi_P^{-1}((a,b))$. That is, there exists an $\epsilon > 0$ with $\pi_P(t') \in (a,b)$ for all $t' \in B_\epsilon(t)$. Take $\epsilon = \min\{b - \pi_P(t), \pi_P(t) - a\}$ and $t' \in B_\epsilon(t)$. We then have:

$$\begin{aligned}
\pi_P(t') &= \inf\{d(t',u) \mid u \in [\![P]\!]^c\} \\
&\leq \inf\{d(t,t') + d(t,u) \mid u \in [\![P]\!]^c\} \quad \text{(triangle inequality)} \\
&= d(t,t') + \pi_P(t) \\
&< \epsilon + \pi_P(t) \quad (\text{since } t' \in B_\epsilon(t)) \\
&\leq b - \pi_P(t) + \pi_P(t) = b.
\end{aligned}$$

Similarly, $\pi_P(t') > a$, so $\pi_P(t') \in (a,b)$, as desired.

Now that we know that π_P and π_Q are continuous we may use Theorem (3.1.3) to obtain:

$$[\![P]\!] = \{t \in T \mid \pi_P(t) > 0\} = [\![\pi_P > 0]\!]; \tag{3.2.1}$$

similarly for π_Q. (Note that the first equality follows from the fact that $[\![P]\!]^c$ is closed.) Define continuous functions $\xi, \eta \colon T \to \mathbb{R}$ by:

$$\begin{aligned}
\xi(t) &= 0; \\
\eta(t) &= \pi_P(t) - \pi_Q(t).
\end{aligned}$$

Since $[\![P]\!]$ and $[\![Q]\!]$ are disjoint and $\pi_P(t), \pi_Q(t) \geq 0$ for all $t \in T$, we have in view of (3.2.1):

$$\begin{aligned}
\pi_P(t) > 0 &\Rightarrow \pi_Q(t) = 0; \\
\pi_Q(t) > 0 &\Rightarrow \pi_P(t) = 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
[\![\xi < \eta]\!] &= [\![\pi_Q < \pi_P]\!] = [\![P]\!]; \\
[\![\eta < \xi]\!] &= [\![\pi_P < \pi_Q]\!] = [\![Q]\!];
\end{aligned}$$

which implies that

$$[\![A(\xi < \eta, \eta < \xi)]\!] = [\![A(P,Q)]\!].$$

Since our choice of T, $[\![P]\!]$ and $[\![Q]\!]$ ensured that $[\![A(P,Q)]\!] \neq T$; we have found $\xi, \eta \in \mathscr{R}$ such that $[\![A(\xi < \eta, \eta < \xi)]\!] \neq T$, completing our proof. ■

Remark 3.2.2. Note that the model was constructed depending on the formula. However, if we take the Baire space $\mathbb{N}^{\mathbb{N}}$ (see Definition A.5) as topological space T, then this space can be used for all counterexamples by Theorem 4.1 in [10, pp. 130–131], thus fixing the model.

For this reason, we take the topological space T to be Baire space $\mathbb{N}^{\mathbb{N}}$ (see Definition A.5) from now on.

If case of invalidity of $A(x<y,y<x)$, we have the following result.

Theorem 3.2.3. *If the formula*

$$A(x<y,y<x)$$

is not provable from (2.4.1)–(2.4.2)*, then*

$$\forall x \neg \forall y\, A(x<y,y<x)$$

is valid in our model.

Proof. Define a function $\xi\colon T\to\mathbb{R}$ by $\xi(t)=0$ for all $t\in T$. Suppose that the formula $A(x<y,y<x)$ is not provable. By the previous proof, we have a function $\eta\colon T\to\mathbb{R}$ such that

$$[\![A(\xi<\eta,\eta<\xi)]\!]\neq T. \tag{3.2.2}$$

Notice that $\{\eta'+\xi' \mid \eta'\in\mathscr{R}\}=\mathscr{R}$ for any $\xi'\in\mathscr{R}$. Therefore,

$$\bigcap_{\zeta\in\mathscr{R}}[\![A(\xi'<\zeta)]\!]=\bigcap_{\eta'\in\mathscr{R}}[\![A(\xi'<\eta'+\xi')]\!]=\bigcap_{\eta'\in\mathscr{R}}[\![A(\xi<\eta')]\!];$$

hence for any $\xi'\in\mathscr{R}$,

$$[\![\forall y\, A(\xi'<y,y<\xi')]\!]=[\![\forall y\, A(\xi<y,y<\xi)]\!]. \tag{3.2.3}$$

Let $f\colon T\to T$ be any autohomeomorphism of T. Then, $\xi\circ f$ and $\eta\circ f$ are continuous functions, for which we have:

$$\begin{aligned} t\in[\![\xi\circ f<\eta\circ f]\!]&=\{u\in T\mid \xi(f(u))<\eta(f(u))\} \text{ if and only if}\\ f(t)\in[\![\xi<\eta]\!]&=\{u\in T\mid \xi(u)<\eta(u)\}.\end{aligned}$$

Thus,

$$f(t)\in[\![A(\xi<\eta,\eta<\xi)]\!]\Leftrightarrow t\in[\![A(\xi\circ f<\eta\circ f,\eta\circ f<\xi\circ f)]\!];$$

and therefore,

$$\begin{aligned}[\![A(\xi\circ f<\eta\circ f,\eta\circ f<\xi\circ f)]\!]&=\{t\in T\mid f(t)\in[\![A(\xi<\eta,\eta<\xi)]\!]\}\\ &=f^{-1}\left([\![A(\xi<\eta,\eta<\xi)]\!]\right). \end{aligned} \tag{3.2.4}$$

Let $\mathrm{Aut}(T)$ be the autohomeomorphism group of T and let y be any element of $[\![A(\xi<\eta,\eta<\xi)]\!]^c$. (Such a y exists by (3.2.2).) Since $T=\mathbb{N}^{\mathbb{N}}$ is homogeneous (see

Theorem A.8), we can find for each element $a \in [\![A(\xi < \eta, \eta < \xi)]\!]$ an autohomeomorphism $g_a \in \mathrm{Aut}(T)$ such that $g_a(a) = y$, i.e. $g(a) \notin [\![A(\xi < \eta, \eta < \xi)]\!]$. Define $G = \{g_a \in \mathrm{Aut}(T) \mid a \in [\![A(\xi < \eta, \eta < \xi)]\!]\}$ and let $G' = G \cup \{\mathrm{id}_T\}$. We have:

$$\bigcap_{f \in \mathrm{Aut}(T)} f^{-1}([\![A[\xi < \eta, \eta < \xi)]\!]) \subseteq \bigcap_{g \in G'} g^{-1}([\![A(\xi < \eta, \eta < \xi)]\!]) = \emptyset.$$

In view of (3.2.4) we may conclude:

$$\bigcap_{f \in \mathrm{Aut}(T)} [\![A(\xi \circ f < \eta \circ f, \eta \circ f < \xi \circ f)]\!] = \emptyset.$$

Since $\xi = 0$, this means that:

$$\bigcap_{f \in \mathrm{Aut}(T)} [\![A(\xi < \eta \circ f, \eta \circ f < \xi)]\!] = \emptyset;$$

hence,

$$\bigcap_{\eta' \in \mathscr{R}} [\![A(\xi < \eta', \eta' < \xi)]\!] = \emptyset.$$

So we see:

$$[\![\forall y, A(\xi < y, y < \xi)]\!] = \emptyset.$$

We may finally finish our proof by using (3.2.3) to show that:

$$\bigcap_{\xi' \in \mathscr{R}} [\![\neg\forall y\, A(\xi' < y, y < \xi')]\!] = [\![\neg\forall y\, A(\xi < y, y < \xi)]\!] = T,$$

so that:

$$[\![\forall x \neg\forall y\, A(x < y, y < x)]\!] = T,$$

as we wished to show. ∎

Corollary 3.2.4. *If the propositional formula*

$$\neg(P \wedge Q) \rightarrow A(P, Q)$$

is unprovable in Heyting's predicate calculus, then

$$\forall x \neg\forall y\, A(x < y, y < x)$$

is valid in our model.

Proof. The proof is immediate by Theorems (3.2.1) and (3.2.3). ∎

Example 3.2.5. If view of Examples (2.2.1) and (2.2.2) we may use Theorem (3.2.3) to conclude that the following formulas are all valid:

$$\begin{aligned}
&\forall x \neg \forall y\, (x < y \vee x = y \vee x > y);\\
&\forall x \neg \forall y\, (x = y \vee \neg x = y);\\
&\forall x \neg \forall y\, (x \leq y \vee x \geq y).
\end{aligned}$$

Example 3.2.6. Let the propositional formula $A(P, Q)$ be defined by:

$$A(P, Q) = \neg\neg(P \vee Q) \to P \vee Q.$$

We will show that

$$\neg(P \wedge Q) \to A(P, Q)$$

is not intuitionistically provable by using the topological interpretation as defined at the beginning of section 3.1. Let T be the topological space $(0, 2] \subseteq \mathbb{R}$ and define open subsets of T by $[\![P]\!] = (1, 2)$ and $[\![Q]\!] = \emptyset$. We easily see that $[\![\neg(P \wedge Q)]\!] = T$, so that it suffices to show that $[\![A(P, Q)]\!] \neq T$. Note that $[\![P \vee Q]\!] = (1, 2)$ and $[\![\neg\neg(P \vee Q)]\!] = (1, 2]$. Therefore, $[\![A(P, Q)]\!] = ((1, 2]^c \cup (1, 2))^\circ = (0, 2) \neq T$.

From this, it also follows that

$$\neg(P \wedge Q) \to \neg\neg P \to P$$

is intuitionistically unprovable. Thus, we may use Corollary 3.2.4 to conclude that the following formulas are valid:

$$\begin{aligned}
&\forall x \neg \forall y\, (\neg\neg x < y \to x < y) \quad && (\text{use } A(P, Q) = \neg\neg P \to P);\\
&\forall x \neg \forall y\, (\neg x = y \to x \neq y) && (\text{use } A(P, Q) = \neg\neg(P \vee Q) \to P \vee Q).
\end{aligned}$$

3.2.2 Three variables

In this section we will prove similar results for open formulae with three variables.

Let $B(x_0, x_1, x_2)$ be any open formula in three variables x_0, x_1, x_2. In view of (2.4.4) - (2.4.6) there is a formula

$$A(x_0 < x_1, x_0 < x_2, x_1 < x_0, x_1 < x_2, x_2 < x_0, x_2 < x_1),$$

for which we shall write

$$A(x_i < x_j : i, j < 3)$$

such that $A(x_i < x_j : i, j < 3)$ is equivalent to $B(x_0, x_1, x_2)$.

We will use the propositional letter P_{ij} to denote $x_i < x_j$ for all $i, j \in \{0, 1, 2\}$. Furthermore, let us denote by the propositional formula $A(P)$ the result of replacing all $x_i < x_j$ by P_{ij} in $A(x_i < x_j : i, j < 3)$.

Lastly, let $B(P)$ be the conjuction of all formulae of the forms $\neg(P_{ij} \wedge P_{ji})$ and $P_{ij} \to P_{ik} \vee P_{kj}$ for $i, j, k \in \{0, 1, 2\}$ (compare this to (2.4.1) and (2.4.2)).

Theorem 3.2.7. *The formula*

$$A(x_i < x_j : i, j < 3)$$

is provable from (2.4.1)–(2.4.2) *if and only if the propositional formula*

$$B(P) \to A(P)$$

is provable in Heyting's predicate calculus.

Proof. As in the proof of Theorem (3.2.1), it is easily seen that if the propositional formula is provable, then $A(x_i < x_j : i, j < 3)$ is also provable.

For the converse, suppose that the propositional formula is unprovable. We have to show that $A(x_i < x_j : i, j < 3)$ is not provable from (2.4.1)–(2.4.2).

Just as we have done in the the proof of Theorem (3.2.1) we evaluate each P_{ij} with an open set $[\![P_{ij}]\!]$ so that:

$$[\![B(P)]\!] = T, \text{ while } [\![A(P)]\!] \neq T.$$

We define non-negative functions for all $i, j, k, l \in \{0, 1, 2\}$

$$\begin{aligned} \sigma_{ijkl} \colon T &\to \mathbb{R} \\ t &\mapsto \inf\{d(t, t') \mid t' \in [\![P_{ij} \wedge P_{kl}]\!]^c\}, \end{aligned}$$

where d is the metric on T. We may show just as before that since all sets $[\![P_{ij} \wedge P_{kl}]\!]^c$ are closed, the functions σ_{ijkl} are all continuous and we have:

$$[\![\sigma_{ijkl} > 0]\!] = [\![P_{ij}]\!] \cap [\![P_{kl}]\!]. \tag{3.2.5}$$

We introduce continuous functions

$$\begin{aligned} \pi_{ij} \colon T &\to \mathbb{R} \\ t &\mapsto \sigma_{ijik}(t) + \sigma_{ijkj}(t) - \sigma_{jijk}(t) - \sigma_{jiki}(t), \end{aligned}$$

for each $i, j \in \{0, 1, 2\}$. Because $[\![B(P)]\!] = T$, we have

$$[\![P_{ij}]\!] \subseteq [\![P_{ik}]\!] \cup [\![P_{kj}]\!]; \tag{3.2.6}$$

$$[\![P_{ij}]\!] \cap [\![P_{ji}]\!] = \emptyset; \tag{3.2.7}$$

by definition of $B(P)$. Since π_{ij} is continuous, Theorem 3.1.3 tells us:

$$\begin{aligned} [\![\pi_{ij} > 0]\!] &= \{t \in T \mid \pi_{ij}(t) > 0\} \\ &= \{t \in T \mid \sigma_{ijik}(t) + \sigma_{ijkj}(t) > \sigma_{jijk}(t) + \sigma_{jiki}(t)\}. \end{aligned} \tag{3.2.8}$$

We shall use these equations to prove the following equality:

$$[\![\pi_{ij} > 0]\!] = [\![P_{ij}]\!]. \tag{3.2.9}$$

Suppose $t \in [\![\pi_{ij} > 0]\!]$. Because of the non-negativity of the σ_{ijkl}'s we must have $\sigma_{ijik}(t) > 0$ or $\sigma_{ijkj}(t) > 0$ by equation (3.2.8). In view of (3.2.5) we have:

$$\begin{aligned} t &\in ([\![P_{ij}]\!] \cap [\![P_{ik}]\!]) \cup ([\![P_{ij}]\!] \cap [\![P_{kj}]\!]) \\ &= [\![P_{ij}]\!] \cap ([\![P_{ik}]\!] \cup [\![P_{kj}]\!]) \\ &= [\![P_{ij}]\!] \quad \text{(by (3.2.6))}. \end{aligned}$$

For the converse, assume that $t \in [\![P_{ij}]\!]$. By (3.2.5) and (3.2.6) we have $\sigma_{ijik}(t) > 0$ or $\sigma_{ijkj}(t) > 0$. Furthermore, equation (3.2.7) tells that $t \notin [\![P_{ji}]\!]$. Thus, we may use the non-negativity of the functions σ_{ijkl} and equation (3.2.5) to conclude that $\sigma_{jijk}(t) = \sigma_{jiki}(t) = 0$. Hence, $t \in [\![\pi_{ij} > 0]\!]$, as desired.

One may, using the fact that $\sigma_{ijkl} = \sigma_{klij}$, show that $\pi_{ij} + \pi_{jk} + \pi_{ki} = 0$ for all $i, k, j, l \in \{0, 1, 2\}$. If we fix $k = 0$, we see that:

$$\pi_{ij} = -\pi_{j0} - \pi_{0i} = \pi_{0j} - \pi_{0i}.$$

(The latter equality follows by writing out the definition of π_{ij}.) Setting $\xi_i = \pi_{0i}$ and $\xi_j = \pi_{0j}$, we have by (3.2.9):

$$[\![\xi_i < \xi_j]\!] = [\![\pi_{ij} > 0]\!] = [\![P_{ij}]\!],$$

so that

$$[\![A(\xi)]\!] = [\![A(P)]\!],$$

where $A(\xi)$ is the result of substituting the x_i and x_j by ξ_i and ξ_j respectively. Our choice of the $[\![P_{ij}]\!]$'s ensured that $[\![A(P)]\!] \neq T$. Hence, $[\![A(\xi)]\!] \neq T$, which completes our proof. ■

Of course we also have the following theorem which is analogous to Theorem 3.2.3.

Theorem 3.2.8. *If the the formula*

$$A(x_i < x_j : i, j < 3)$$

is not provable from (2.4.1)–(2.4.2)*, then*

$$\forall x_0 \neg \forall x_1, x_2\, A(x_i < x_j : i, j < 3)$$

is valid in our model.

Proof. The proof of Theorem 3.2.3 is seen to be generalisable to the case of three variables. (Note that $\xi_0 = \pi_{00} = 0$ would take up the role of ξ in such a proof.) ■

It seems like we should be able to come up with similar results for open formulae with more variables. In fact, this is precisely what Scott has done in his follow-up article *Extending the Topological Interpretation to Intuitionistic Analysis, II* [12, pp. 236–240].

3.3 Enlarging the model

Up to this point we have discussed the properties of only the $<, \#, \leq$ and $=$ relations. If \$ is any (classical) relation between real numbers, then we may include the new relation \$ in our model $\mathscr{R}$ by setting:

$$[\![\xi \ \$ \ \eta]\!] = \{t \in T \mid \xi(t) \ \$ \ \xi(t)\}^\circ.$$

If \$ happens to represent an open relation, then we can drop the $^\circ$ at the right-hand side, which is precisely what we did in the cases of the $<$ and $\#$ relations. In case \$ is a closed relation and \$′ is the complementary open relation, then $[\![\xi \ \$ \ \eta]\!] = [\![\neg\xi \ \$' \ \eta]\!]$, which is how we defined the $\leq$ and $=$ relations.

The following theorem compares the classical to the intuitionistic continuum.

Theorem 3.3.1. *Let $A_1, A_2, \dots, A_n, B$ be atomic formulae of the kind just considered with the new relational symbols. If*

$$A_1 \wedge A_2 \wedge \cdots \wedge A_n \to B$$

is valid classically, it is also valid in our model. If $B_1, B_2, \dots B_m$ are atomic formulae with open *relations, then the same holds for*

$$A_1 \wedge A_2 \wedge \cdots \wedge A_n \to B_1 \vee B_2 \vee \cdots \vee B_m.$$

Proof. Define

$$]\!]A[\![= \{t \in T \mid A(t)\}$$

for all formulae A that are conjunctions or disjunctions of atomic formulae with the new relational symbols.

Suppose that $A_1 \wedge A_2 \wedge \cdots \wedge A_n \to B$ holds classically. This means that if $t \in]\!]A_1 \wedge \cdots \wedge A_n[\![$, then $t \in]\!]B[\![$. So we have,

$$]\!]A_1 \wedge \cdots \wedge A_n[\![\subseteq]\!]B[\![.$$

Using the topological equation $(X \cap Y)^\circ = X^\circ \cap Y^\circ$, we see that

$$\begin{aligned}
[\![A_1 \wedge \cdots \wedge A_n]\!] &= [\![A_1]\!] \cap \cdots \cap [\![A_n]\!] \\
&=]\!]A_1[\![^\circ \cap \cdots \cap]\!]A_n[\![^\circ \\
&= (]\!]A_1[\![\cap \cdots \cap]\!]A_n[\![)^\circ \\
&= (]\!]A_1 \wedge \cdots \wedge A_n[\![)^\circ \subseteq \ ]\!]B[\![^\circ = [\![B]\!].
\end{aligned}$$

Hence, $[\![A_1 \wedge \cdots \wedge A_n]\!]^c \cup [\![B]\!] = T$, which proves the first claim.

The second claim may be proved similarly. Suppose that $A_1 \wedge A_2 \wedge \cdots \wedge A_n \to B_1 \vee B_2 \vee \cdots \vee B_m$ is classically valid, viz.

$$]\!]A_1 \wedge \cdots \wedge A_n[\![\subseteq]\!]B_1 \vee \cdots \vee B_m[\![.$$

Hence, just like before,

$$[\![A_1 \wedge \cdots \wedge A_n]\!] \subseteq (]\!]B_1 \vee \cdots \vee B_m[\![)^\circ = (]\!]B_1[\![\cup \cdots \cup]\!]B_m[\![)^\circ.$$

Note that $]\!]B_i[\![$ and $[\![B_i]\!]$ coincide for all $i \in \{1, 2, \ldots, n\}$, since the B_i's are atomic formulae with open relations. Thus,

$$\begin{aligned}[\![A_1 \wedge \cdots \wedge A_n]\!] \subseteq (]\!]B_1[\![\cup \cdots \cup]\!]B_m[\![)^\circ &= [\![B_1]\!] \cup \cdots \cup [\![B_m]\!])^\circ \\ &= [\![B_1 \vee \cdots \vee B_m]\!],\end{aligned}$$

by the topological fact that unions of opens are open.

So, we may conclude $[\![A_1 \wedge \cdots \wedge A_n]\!]^c \cup [\![B_1 \vee \cdots \vee B_m]\!] = T$, which completes our proof. ■

For quantified formulae we have the following result.

Theorem 3.3.2. *Suppose that the formula $\forall x \exists y\,(x \;\$\; y)$ is classically valid because there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that the formula $\forall x\,(x \;\$\; f(x))$ is true. Then $\forall x \exists y\,(x \;\$\; y)$ is also valid in our model.*

In case \$ is an open relation, $\forall x \exists y\,(x \;\$\; y)$ is valid in our model if there is a family $\{f_i \mid i \in I\}$ of continuous functions such that $\forall x \exists i \in I\,(x \;\$\; f_i(x))$.

Proof. The first claim is easily proved. Suppose that such an f exists. Then since, $\forall x\,(x \;\$\; f(x))$ is classically true, we have $[\![\xi \;\$\; f \circ \xi]\!] = T$ for all $\xi \in \mathscr{R}$. Because f is continuous, $f \circ \xi \in \mathscr{R}$, so that $\bigcup_{\eta \in \mathscr{R}} [\![\xi \;\$\; \eta]\!] = T$ for all $\xi \in \mathscr{R}$. Hence, $\bigcap_{\xi \in \mathscr{R}} \left(\bigcup_{\eta \in \mathscr{R}} [\![\xi \;\$\; \eta]\!] \right) = T$, which proves that $[\![\forall x \exists y\,(x \;\$\; y)]\!] = T$, as desired.

The second claim is proved similarly. Suppose that such a family $\{f_i \mid i \in I\}$ of continuous functions exists. Then clearly, $\bigcup_{i \in I} \{t \in T \mid \xi(t) \;\$\; f_i(\xi(t))\} = T$ for any $\xi \in \mathscr{R}$, which implies that $\bigcup_{i \in I} [\![\xi \;\$\; f_i \circ \xi]\!] = T$ for any $\xi \in \mathscr{R}$, because \$ is an open relation. Like before, we may conclude $[\![\forall x \exists y\,(x \;\$\; y)]\!] = T$, completing our proof. ■

3.3.1 Functions in our model

In this section we will deal with functions, instead of relations, on the elements of our model. Let $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be any continuous function. For any $\xi, \eta \in \mathscr{R}$, we define

$$\zeta = f(\xi, \eta)$$

to mean that

$$\zeta(t) = f(\xi(t), \eta(t))$$

for all $t \in T$. Cleary, ζ is also an element of $\mathscr{R}$. We can apply this method to the arithmetic operations $+, \cdot, -$ and to functions like $\sin, \cos$ and $\exp$. Because the elements of $\mathscr{R}$ are real-valued functions, any equation satisfied by functions with values in $\mathbb{R}$ is also valid on $\mathscr{R}$.

Thus $\forall x, y, z\,(x + y = x + z \to y = z)$ is valid in the model. However, we can prove that $\forall x, y\,(x \cdot y = 0 \to x = 0 \vee y = 0)$ is not. To do so, we need to find functions $\xi, \eta \colon T \to \mathbb{R}$ such that $[\![\xi \cdot \eta = 0 \to \xi = 0 \vee \eta = 0]\!] \neq T$. It suffices to find functions $\xi, \eta \colon \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ such that $[\![\xi \cdot \eta = 0 \to \xi = 0 \vee \eta = 0]\!] \neq \mathbb{R} \setminus \mathbb{Q}$, because $T = \mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$ (see Theorem A.9).

We start with defining *continuous* functions $\tilde{\xi}, \tilde{\eta} \colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{\xi}(t) = \begin{cases} \sin t & \text{if } 0 \leq t \leq \pi; \\ 0 & \text{otherwise;} \end{cases}$$

$$\tilde{\eta}(t) = \begin{cases} \sin t & \text{if } \pi \leq t \leq 2\pi; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, define $\xi, \eta \colon \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ by $\xi = \tilde{\xi}_{|\mathbb{R}\setminus\mathbb{Q}}$ and $\eta = \tilde{\eta}_{|\mathbb{R}\setminus\mathbb{Q}}$; that is the restriction of ξ and η to the irrationals. Since $\mathbb{R} \setminus \mathbb{Q}$ is a subspace of $\mathbb{R}$, the functions ξ and η are continuous.

Clearly, $\xi(t) \cdot \eta(t) = 0$ for all $t \in \mathbb{R} \setminus \mathbb{Q}$, so that $[\![\xi \cdot \eta = 0]\!] = \mathbb{R} \setminus \mathbb{Q}$. However,

$$\begin{aligned} [\![\xi = 0 \vee \eta = 0]\!] &= \{t \in \mathbb{R} \setminus \mathbb{Q} \mid \xi(t) = 0\}^\circ \cup \{t \in \mathbb{R} \setminus \mathbb{Q} \mid \eta(t) = 0\}^\circ \\ &= ((0, \pi)^c \cap \mathbb{R} \setminus \mathbb{Q})^\circ \cup ((\pi, 2\pi)^c \cap \mathbb{R} \setminus \mathbb{Q})^\circ \\ &= (((-\infty, 0) \cup (\pi, \infty)) \cup ((-\infty, \pi) \cup (2\pi, \infty))) \cap \mathbb{R} \setminus \mathbb{Q} \\ &= (\mathbb{R} \setminus \mathbb{Q}) \setminus \{\pi\} \neq \mathbb{R} \setminus \mathbb{Q}, \end{aligned}$$

which shows that $[\![\xi \cdot \eta = 0 \to \xi = 0 \vee \eta = 0]\!]$ is invalid in our model.

In fact, $\neg\forall x, y\,(x \cdot y = 0 \to x = 0 \vee y = 0)$ is valid in our model. This last claim points up the following theorem.

Theorem 3.3.3. *If a sentence A has no parameters (i.e. it only has bounded variables and constants besides operations and relations), then either A or $\neg A$ is valid in our model.*

Before we prove this theorem, we first introduce two helpful lemmas which will aid us in proving Theorem 3.3.3.

Lemma 3.3.4. *If f is any autohomeomorphism on T and U any subset of T, then*

$$f(U) = U \Rightarrow f(U^\circ) = U^\circ.$$

Proof. Suppose $f(U) = U$, then $f(U^\circ) \subseteq f(U) = U$. We see that $f(U^\circ)$ is an open (because f is an autohomeomorphism) set fully contained in U. Since U° is the *largest* open set contained in U, we must have $f(U^\circ) \subseteq U^\circ$.

For the converse, note that it follows from $f(U) = U$ that $f^{-1}(U) = U$. Hence, $f^{-1}(U^\circ) \subseteq f^{-1}(U) = U$. We see that $f^{-1}(U^\circ)$ is an open (because f is an autohomeomorphism) set fully contained in U. Because U° is the *largest* open set contained in U, we have $f^{-1}(U^\circ) \subseteq U^\circ$, from which we derive $U^\circ \subseteq f(U^\circ)$. We may conclude $f(U^\circ) = U^\circ$, as we wished to prove. ■

We now use Lemma 3.3.4 to prove the following lemma.

Lemma 3.3.5. *If a sentence A has no parameters (i.e. it only has bounded variables and constants besides operations and relations), then its value is invariant under all autohomeomorphisms of T, i.e. $f(\llbracket A \rrbracket) = \llbracket A \rrbracket$ for any autohomeomorphism f.*

Proof. This may be proved by induction on the complexity of the sentence. Let the function f be any autohomeomorphism on T.

If A does not contain any variables or logical connectives, then A expresses some atomic truth or falsity and therefore, $\llbracket A \rrbracket$ is either T or $\emptyset$; so that $\llbracket A \rrbracket$ is clearly invariant under f.

One easily proves that $f(U \cup V) = f(U) \cup f(V)$ for any $U, V \subseteq T$. Hence, $f(\llbracket A \vee B \rrbracket) = f(\llbracket A \rrbracket) \cup f(\llbracket B \rrbracket) = \llbracket A \vee B \rrbracket$ if $f(\llbracket A \rrbracket) = \llbracket A \rrbracket$ and $f(\llbracket B \rrbracket) = \llbracket B \rrbracket$.

Using the injectivity of f we may also show that $f(U \cap V) = f(U) \cap f(V)$ for any $U, V \subseteq T$. Therefore, $f(\llbracket A \wedge B \rrbracket) = f(\llbracket A \rrbracket) \cap f(\llbracket B \rrbracket) = \llbracket A \wedge B \rrbracket$ if $f(\llbracket A \rrbracket) = \llbracket A \rrbracket$ and $f(\llbracket B \rrbracket) = \llbracket B \rrbracket$.

Because f is surjective, we can show that $f(U^c) = (f(U))^c$. When we combine this result with Lemma 3.3.4 we see that $f(\llbracket \neg A \rrbracket) = f((\llbracket A \rrbracket^c)^\circ) = \llbracket \neg A \rrbracket$ if $f(\llbracket A \rrbracket) = \llbracket A \rrbracket$.

Using $f(U^c) = (f(U))^c$, $f(U \cup V) = f(U) \cup f(V)$ and Lemma 3.3.4, we see that if A and B are sentences such that their value is invariant under f, then $f(\llbracket A \to B \rrbracket) = f((\llbracket A \rrbracket^c \cup B)^\circ) = \llbracket A \to B \rrbracket$.

We now show that $f\llbracket \exists x\, B(x) \rrbracket = \llbracket \exists x\, B(x) \rrbracket$ for any open formula in one variable $B(x)$. Suppose $t \in f(\llbracket \exists x\, B(x) \rrbracket)$, then there is a $u \in \llbracket \exists x\, B(x) \rrbracket$ such that $t = f(u)$. Since $u \in \llbracket \exists x\, B(x) \rrbracket = \bigcup_{\xi \in \mathscr{R}} \llbracket B(\xi) \rrbracket$, there exists $\eta \in \mathscr{R}$ such that $u \in \llbracket B(\eta) \rrbracket$. Note that $\eta \circ f^{-1}$ is an element of $\mathscr{R}$ for which we have: $f(u) \in \llbracket B(\eta \circ f^{-1}) \rrbracket$, so that $t = f(u) \in \llbracket \exists x\, B(x) \rrbracket$. Conversely, if $t \in \llbracket \exists x\, B(x) \rrbracket$, then there is some $\eta \in \mathscr{R}$ such that $t \in \llbracket B(\eta) \rrbracket$. Clearly then, $f^{-1}(t) \in \llbracket B(\eta \circ f) \rrbracket$, so that $f^{-1}(t) \in \llbracket \exists x\, B(x) \rrbracket$. Hence, $t \in \llbracket \exists x\, B(x) \rrbracket$, as we wished to show.

We conclude our proof by induction by showing that $f(\llbracket \forall x\, B(x) \rrbracket) = \llbracket \forall x\, B(x) \rrbracket$ for any open formula in one variable $B(x)$. We find it convenient to first prove:

$$f\Big(\bigcap_{\xi \in \mathscr{R}} \llbracket B(\xi) \rrbracket \Big) = \bigcap_{\xi \in \mathscr{R}} \llbracket B(\xi) \rrbracket;$$

and then use Lemma 3.3.4 to obtain the desired result.

Suppose $t \in f\Big(\bigcap_{\xi \in \mathscr{R}} \llbracket B(\xi) \rrbracket \Big)$, then there is a $u \in \bigcap_{\xi \in \mathscr{R}} \llbracket B(\xi) \rrbracket$ such that $t = f(u)$. We will show that $t = f(u) \in \bigcap_{\xi \in \mathscr{R}} \llbracket B(\xi) \rrbracket$ using a proof by contradiction. Suppose $f(u) \notin \bigcap_{\xi \in \mathscr{R}} \llbracket B(\xi) \rrbracket$. Then there exists some $\eta \in \mathscr{R}$ such that $f(u) \notin \llbracket B(\eta) \rrbracket$. But then, $u \notin \llbracket B(\eta \circ f) \rrbracket$; contradicting our assumption that $u \in \bigcap_{\xi \in \mathscr{R}} \llbracket B(\xi) \rrbracket$. For the converse, suppose $t \in \bigcap_{\xi \in \mathscr{R}} \llbracket B(\xi) \rrbracket$, but $f^{-1}(t) \notin \bigcap_{\xi \in \mathscr{R}} \llbracket B(\xi) \rrbracket$. We derive a contradiction. By $f^{-1}(t) \notin \bigcap_{\xi \in \mathscr{R}} \llbracket B(\xi) \rrbracket$, there exists some $\eta \in \mathscr{R}$ such that $f^{-1}(t) \notin \llbracket B(\eta) \rrbracket$. But then, $t \notin \llbracket B(\eta \circ f^{-1}) \rrbracket$; contradicting our assumption that

$t \in \bigcap_{\xi \in \mathscr{R}} [\![B(\xi)]\!]$, completing our proof of the claim. By Lemma 3.3.4 we see that $f([\![\forall x\, B(x)]\!]) = [\![\forall x\, B(x)]\!]$, as desired. ∎

Finally, we may prove Theorem 3.3.3.

Proof of Theorem 3.3.3. If A is such a sentence, then its value is invariant under all autohomeomorphisms of T by Lemma 3.3.5.

Since T is the *homogeneous* Baire space (see Theorem A.8), the only open sets so invariant are $\emptyset$ and T. Thus if the sentence A is not valid, then $\neg A$ is valid. ∎

Remark 3.3.6. In particular, the principle of the excluded middle holds for such sentences in the model. This feature is a consequence of our classical construction of the model, which may be regarded as undesirable. It does not seem too serious however, because most interesting sentences have parameters.

Although $\forall x, y\, (x \cdot y = 0 \rightarrow x = 0 \vee y = 0)$ was seen to be false; it is easily seen that

$$\forall x, y\, (x \cdot y = 0 \wedge x \# 0 \rightarrow y = 0) \tag{3.3.1}$$

is valid. However, one should not conclude that a disjunctive conclusion with closed relations is always invalid. The following is valid for example:

$$\forall x\, (x = x^2 \rightarrow x = 0 \vee x = 1).$$

Proof. Suppose $x = x^2$. This can obviously be rewritten as $x \cdot (1 - x) = 0$. By (2.4.2) we have $0 < x \vee x < 1$. If $0 < x$ is the case, then we derive $1 - x = 0$ by (3.3.1) and so $x = 1$. If $x < 1$ is the case, then $1 - x \# 0$, so by (3.3.1) we derive $x = 0$. ∎

We see that (2.4.2) gives us $x > 0 \vee x < \epsilon$ where ϵ may be any (small) positive rational number. This principle serves as a useful replacement for the following formula $x < 0 \vee x = 0 \vee x > 0$, which is not intuitionistically valid, as we have seen.

The use of certain continuous functions and of some of their properties sometimes gives us results about the pure theory of order which cannot be derived simply from (2.4.1) - (2.4.8). For example, the average of two real numbers x and y can be expressed as $(x + y)/2$ and clearly satisfies $x < y \rightarrow x < (x + y)/2 < y$, which we may use to derive

$$\forall x, y \exists z\, ((x < y \vee y < x) \rightarrow (x < z < y \vee x > z > y)).$$

Partial (i.e. not everywhere defined) functions are difficult to tackle, because one would only define x^{-1} for $x \# 0$. However, it seems difficult to make this procedure rigorous in a formal intuitionistic theory that allows the free formation of terms x^{-1} and not just the relation $y = x^{-1}$ (which is equivalent to $x \cdot y = 1$). We can, however, show that

$$\forall x\, (x \# 0 \rightarrow \exists y\, (x \cdot y = 1))$$

is valid in our model. For let $\xi \in \mathscr{R}$ and let t be an element of $[\![\xi \mathrel{\#} 0]\!]$. Then there is an integer $n \in \mathbb{N}_{>0}$ such that $|\xi(t)| > 1/n$, thus $t \in [\![\xi > 1/n]\!]$, so that $[\![\xi \mathrel{\#} 0]\!] \subseteq [\![\xi > 1/n]\!]$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be any continuous function such that $f(u) = u^{-1}$ for $|u| \geq 1/n$. We see that

$$[\![\xi \mathrel{\#} 0]\!] \subseteq [\![\xi > 1/n]\!] \subseteq [\![\xi \cdot f(\xi) = 1]\!] \subseteq [\![\exists y\,(\xi \cdot y = 1)]\!],$$

which proves our claim.

Instead of restricted variables such as the rational variables q, r, s we could introduce predicates such as $\xi \in \mathbb{Q}$ into our formal language. By definition,

$$[\![\xi \in \mathbb{Q}]\!] = \bigcup_{q \in \mathbb{Q}} [\![\xi = q]\!], \tag{3.3.2}$$

where on the right-hand side we have used q for the constant, rational function with value q.

Scott writes in [11, pp. 208]:

> *The reader should note that*
>
> $$[\![\xi \in \mathbb{Q}]\!] = T$$
>
> *if and only if the sets $[\![\xi = q]\!]$ are both open and closed in T.*

However, this statement is false. If we define $\xi\colon T \to \mathbb{R}$ as the embedding of the Baire space into the reals (see Theorem A.9), then $[\![\xi = q]\!] = \emptyset$ is open and closed for all constant and rational functions with value q, but $[\![\xi \in \mathbb{Q}]\!] = \emptyset \neq T$. The converse is still true and is proved easily, but we will not do so, as the result does not seem to be nearly as interesting when the equivalence is invalid.

We can show that the use of restricted variables has the same effect as the use of the predicates.

Theorem 3.3.7. *We have the following equalities:*

$$[\![\exists x \in \mathbb{Q}\, A(x)]\!] = \bigcup_{q \in \mathbb{Q}} [\![A(q)]\!];$$
$$[\![\forall x \in \mathbb{Q}\, A(x)]\!] = \Big(\bigcap_{q \in \mathbb{Q}} [\![A(q)]\!]\Big)^{\circ}.$$

Proof. We start out with the first equation. Note that

$$[\![\exists x \in \mathbb{Q}\, A(x)]\!] = [\![\exists x\,(x \in \mathbb{Q} \wedge A(x))]\!] = \bigcup_{\xi \in \mathscr{R}} [\![\xi \in \mathbb{Q} \wedge A(\xi)]\!]. \tag{3.3.3}$$

Suppose that $t \in \bigcup_{q \in \mathbb{Q}} [\![A(q)]\!]$, then there exists a constant function with some value $q \in \mathbb{Q}$ such that $t \in [\![A(q)]\!]$. Clearly, if we set $\xi = q$ (where q denotes the constant function), then $t \in [\![\xi \in \mathbb{Q} \wedge A(\xi)]\!]$, so that $[\![\exists x \in \mathbb{Q}\, A(x)]\!] \supseteq \bigcup_{q\in\mathbb{Q}} [\![A(q)]\!]$. Conversely, suppose that $t \in [\![\exists x \in \mathbb{Q}\, A(x)]\!]$, then by (3.3.3) and (3.3.2) there exists some constant function with value $q \in \mathbb{Q}$ such that $t \in [\![\xi = q]\!]$ and $t \in [\![A(\xi)]\!]$. Hence, $t \in [\![A(q)]\!]$, so that $[\![\exists x \in \mathbb{Q}\, A(x)]\!] \subseteq \bigcup_{q\in\mathbb{Q}} [\![A(q)]\!]$, completing the proof of the first claim.

For the second claim we will prove the slightly stronger claim

$$\bigcap_{\xi\in\mathscr{R}} [\![\xi \in \mathbb{Q} \to A(\xi)]\!] = \bigcap_{q\in\mathbb{Q}} [\![A(q)]\!],$$

which we may use to prove the second part of the theorem, since

$$[\![\forall x \in \mathbb{Q}\, A(x)]\!] = [\![\forall x\, (x \in \mathbb{Q} \to A(x))]\!] = \Big(\bigcap_{\xi\in\mathscr{R}} [\![\xi \in \mathbb{Q} \to A(\xi)]\!] \Big)^{\circ}.$$

First, assume $t \in \bigcap_{\xi\in\mathscr{R}} [\![\xi \in \mathbb{Q} \to A(\xi)]\!]$. We have for all $\xi \in \mathscr{R}$, if $t \in [\![\xi \in \mathbb{Q}]\!]$, then $t \in [\![A(\xi)]\!]$. Since $[\![q \in \mathbb{Q}]\!] = T$ for all constants functions q with values in $\mathbb{Q}$, we have $t \in [\![A(q)]\!]$ for all such q, so that $\bigcap_{\xi\in\mathscr{R}} [\![\xi \in \mathbb{Q} \to A(\xi)]\!] \subseteq \bigcap_{q\in\mathbb{Q}} [\![A(q)]\!]$.

For the converse, suppose $t \in \bigcap_{q\in\mathbb{Q}} [\![A(q)]\!]$. Let $\xi \in \mathscr{R}$ and assume $t \in [\![\xi \in \mathbb{Q}]\!]$. We have to show that $t \in [\![A(\xi)]\!]$. It follows from $t \in [\![\xi \in \mathbb{Q}]\!]$ by (3.3.2) that there exists some $q \in \mathbb{Q}$ such that $t \in [\![\xi = q]\!]$. Furthermore, $t \in [\![A(q)]\!]$ by our assumption that $t \in \bigcap_{q\in\mathbb{Q}} [\![A(q)]\!]$. Hence, $t \in [\![A(\xi)]\!]$, as we wished to show, since we may now conclude that $\bigcap_{\xi\in\mathscr{R}} [\![\xi \in \mathbb{Q} \to A(\xi)]\!] \supseteq \bigcap_{q\in\mathbb{Q}} [\![A(q)]\!]$. ∎

Another interesting predicate is defined by

$$[\![\xi \in \mathbb{D}]\!] = \bigcup_{a\in\mathbb{R}} [\![\xi = a]\!],$$

where a is used to denote a constant function. We can think of $\xi \in \mathbb{D}$ as meaning ξ is definite. Intuitionistic analysis allows for "indefinite" numbers, which is why a statement like $\forall x \exists y\, A(x, y)$ is so strong as the y must exist for definite and indefinite numbers x.

An example of an indefinite number in our model is easily found. Let $\xi\colon T \to \mathbb{R}$ be the embedding of $T = \mathbb{N}^{\mathbb{N}}$ into the reals (see Theorem A.9) . Notice that

$$\begin{aligned} [\![\xi \in \mathbb{D}]\!] &= \bigcup_{a\in\mathbb{R}} [\![\xi = a]\!] \\ &= \bigcup_{a\in\mathbb{R}} \{t \in T \mid \xi(t) = a\}^{\circ} \\ &= \bigcup_{a\in\mathbb{R}\setminus\mathbb{Q}} \{\xi^{-1}(a)\}^{\circ} = \emptyset; \end{aligned}$$

because singletons are not open in T (otherwise it would have the discrete topology). Hence, $[\![\xi \in \mathbb{D}]\!] \neq T$, so we see that ξ is an example of an indefinite number.

The predicates we just introduced might be called "non-standard" predicates to distinguish them from the kind of predicates $\$$ discussed at the beginning of this section. The non-standard notions have no counterpart in the classical theory. Although this remark may seem misleading in connection with $\mathbb{Q}$, one should note that we do *not* have the following equation for all $\xi \in \mathscr{R}$:

$$[\![\xi \in \mathbb{Q}]\!] = \{t \in T \mid \xi(t) \in \mathbb{Q}\}^\circ.$$

To see this, define a function $\xi \colon T \to \mathbb{R}$ by $\xi(t) = d(t, 0)$, where d is the metric on $T = \mathbb{N}^\mathbb{N}$ (see Proposition A.4) and 0 the *function* from $\mathbb{N}$ to $\mathbb{N}$ that maps any element of $\mathbb{N}$ to 0. We quickly see that ξ is continuous by using the $\epsilon - \delta$ definition. Pick any $x \in T$ and let ϵ be any strictly positive real number. Set $\delta = \epsilon$ and note that if y is an element of T such that $d(x, y) < \delta$, then by the reverse triangle inequality:

$$|d(x, 0) - d(y, 0)| = |d(x, 0) - d(0, y)| \leq d(x, y) < \delta = \epsilon.$$

By definition of the metric, $\{t \in T \mid \xi(t) \in \mathbb{Q}\} = T$. Notice that $\{0\} = \{t \in T \mid \xi(t) = d(t, 0) = 0\}$. Hence, $0 \notin [\![\xi \in \mathbb{Q}]\!] = \bigcup_{q \in \mathbb{Q}} \{t \in T \mid \xi(t) = q\}^\circ$, because $\{t \in T \mid \xi(t) = 0\}^\circ = \{0\}^\circ = \emptyset$. So we see that $[\![\xi \in \mathbb{Q}]\!] \neq \{t \in T \mid \xi(t) \in \mathbb{Q}\}^\circ$.

3.3.2 Brouwer's Continuity Theorem

Scott writes:

> It is seen that our model allows only for *continuous* functions on $\mathbb{R}$ to be automatically extended to $\mathscr{R}$. This limitation may indeed be essential. In view of Brouwer's Theorem on Continuity [. . .] it may be reasonable to conjecture that if $\forall x \exists! \, A(x, y)$ is valid in the model, then there exists a continuous function $f \colon \mathbb{R} \to \mathbb{R}$ such that $\forall x \, A(x, f(x))$ is also valid (at least if $A(x, y)$ has no additional parameters). It is clear that we cannot weaken $\exists!$ to $\exists$ for this model, because
>
> $$\forall x \exists y \, (y > 0 \to x \in \mathbb{Q})$$
>
> is valid, but there is no continuous function that can be used to obtain y even for all x where $x \in \mathbb{D}$ is valid.
>
> [11, emphasis as in the original]

(By $\exists! x$ we mean that there exists a *unique* x.)

In his follow-up article Scott proves his conjecture [12, pp. 246–247] and he also shows that Brouwer's Continuity Theorem is valid in the model [12, pp. 247–250]. It is beyond the scope of this thesis to incorporate this material here, so that we will end this section by some remarks on Brouwer's Continuity Theorem. In the following, $\mathcal{R}$ denotes the intuitionistic continuum.

Theorem 3.3.8. **(Brouwer's Continuity Theorem)** [3, pp. 47] *Any function $f(x)$ with values in $\mathcal{R}$ which is defined everywhere on a closed interval of $\mathcal{R}$ is uniformly continuous on that interval.*

The following paragraph intends to make the above theorem seem plausible to the reader. It is based on sections 4.6.1 and 1.3.6 from [14].

Recall our floating number α from example 2.2.1. Classically, the function defined by $f(x) = 0$ if $x = 0$ and $f(x) = 1$ if $x \neq 0$ is everywhere defined and is a standard example of a discontinuous function. However, intuitionistically it is not everywhere defined. For if it was, then we should be able to decide whether $f(\alpha) > 0$ or $f(\alpha) < 1$ by simply approximating $f(\alpha)$. However, such a decision implies that we are able to decide whether $\alpha = 0$ or $\alpha \neq 0$, which is impossible.

This illustrates our inability to construct a total (i.e. everywhere defined) discontinuous function from $\mathcal{R}$ to $\mathcal{R}$, which suggests that all total functions from $\mathcal{R}$ to $\mathcal{R}$ are continuous.

If we accept Brouwer's theorem, we can derive an interesting property of the intuitionistic continuum, namely that it is *indecomposable*. However, we first show that $\mathcal{R}$ is connected.

Lemma 3.3.9. *The intuitionistic continuum is connected, i.e. there are no non-empty opens $A, B \subseteq \mathcal{R}$ such that $A \cup B = \mathcal{R}$ and $A \cap B = \emptyset$.*

Proof. We will derive a contradiction from the assumption that such opens A and B exist. Pick $a_0 \in A$ and $b_0 \in B$. We may assume that $a_0 < b_0$. Recursively, define for every $j \in \mathbb{N}$:

$$(a_{j+1}, b_{j+1}) = \begin{cases} (a_j + \delta_j, b_j) & \text{if } a_j + \delta_j \in A; \\ (a_j, a_j + \delta_j) & \text{if } a_j + \delta_j \in B; \end{cases}$$

where $\delta_j = \frac{b_j - a_j}{2}$. Hence, we obtain a sequences of non-decreasing a_j's in A and non-increasing b_j's in B.

Notice that $a_j + \delta_j = b_j - \delta_j$, so that

$$\begin{aligned} \delta_j &= \frac{b_j - a_j}{2} \\ &= \frac{b_{j-1} - a_{j-1} - \delta_{j-1}}{2} \\ &= \frac{\frac{b_{j-1} - a_{j-1}}{2}}{2} \\ &= \frac{\delta_{j-1}}{2}. \end{aligned}$$

Hence,

$$\delta_j = \frac{\delta_0}{2^j} = \frac{b_0 - a_0}{2^j}.$$

Furthermore, note that

$$\begin{aligned} a_{j+1} - a_j &\leq (a_j + \delta_j) - a_j = \delta_j, \\ b_j - b_{j+1} &\leq b_j - (b_j - \delta_j) = \delta_j. \end{aligned}$$

So, $(a_j)_{j\in\mathbb{N}}$ and $(b_j)_{j\in\mathbb{N}}$ are Cauchy sequences, since $\delta_j \to 0$ for $j \to \infty$. Moreover, the sequences have a common limit x, because $b_j - a_j = 2\delta_j \to 0$ for $j \to \infty$.

We can now obtain our contradiction. Since $A \cup B = \mathcal{R}$, we must have $x \in A$ or $x \in B$. Suppose $x \in A$. The case that $x \in B$ is proved similarly. Then, by the fact that A is open, there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $(x, x+\epsilon) \subseteq A$. However, there also exists an $n \in \mathbb{N}$ such that $x < b_n < x + \epsilon$, implying that $b_n \in A$. This contradicts our assumption that $A \cap B = \emptyset$. ■

Proposition 3.3.10. [1] *The intuitionistic continuum is indecomposable. That is, if $\mathcal{R} = A \cup B$ and $A \cap B = \emptyset$, then either A or B is empty.*

Proof. If there were such non-empty A and B, then the function $f\colon \mathcal{R} \to \mathcal{R}$ defined by $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$ is a total function and hence continuous by Theorem 3.3.8. But then f would have to be constant (by Lemma 3.3.9), so that either A or B must be empty, contradicting our assumption. ■

Brouwer used this property to show that, when dealing with the intuitionistic continuum the statement $\forall x \in \mathcal{R}\,(x \in \mathbb{Q} \vee x \notin \mathbb{Q})$ is not only invalid, it can be refuted. For suppose that $\forall x \in \mathcal{R}\,(x \in \mathbb{Q} \vee x \notin \mathbb{Q})$ was true, then one could obtain a splitting of the continuum (into rationals and irrationals), which is impossible, as we have seen. Hence, $\neg\forall x \in \mathcal{R}\,(x \in \mathbb{Q} \vee x \notin \mathbb{Q})$ is valid.

A Appendix

We often used the Baire space (and some of its topological properties) in section 3. To prevent cropping up the main section, we moved the topological definitions and theorems to this appendix.

In this appendix, we define the Baire space. Furthermore, we will show that the Baire space is homogeneous. Lastly, we give some sources which may be used to find a proof for the fact that the Baire space is homeomorphic to the space of irrationals (with the subspace topology inherited from $\mathbb{R}$).

Definition A.1. [5, Definition 2.1] A *tree* on a set X is a collection T of finite sequences of elements of X such that every initial segment of a sequence in T also belongs to T.

We shall write $l(\sigma)$ for the length of a finite sequence σ and use $()$ to denote the empty sequence.

Lemma A.2. *The collection of all finite sequences of elements of a given set X is a tree. We denote this collection by $X^{<\omega}$.*

Proof. Pick any $\sigma \in X^{<\omega}$ and let τ be any initial segment of σ. Since $X^{<\omega}$ is the collection of *all* finite sequences of elements of X, we must have $\tau \in X^{<\omega}$. ■

Example A.3. Set $X = \{0, 1\}$ and let $T = X^{<\omega}$, then T is a tree by Lemma A.2. We can picture T as follows, motivating the term *tree*.

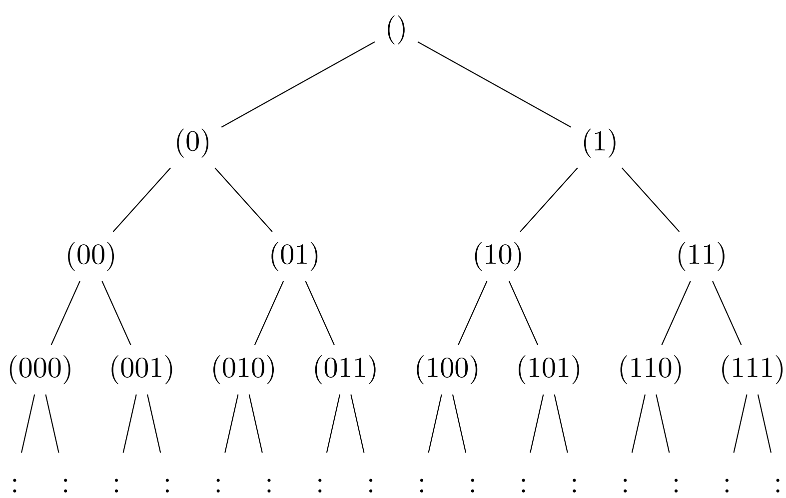

Figure 1: The collection $\{0, 1\}^{<\omega}$ is a tree. An edge is drawn between nodes σ and τ precisely when σ is a initial segment of τ or τ is a initial segment of σ and $|l(\sigma) - l(\tau)| = 1$.

By Lemma A.2 the collection $\mathbb{N}^{<\omega}$ is a tree, which we may picture as follows.

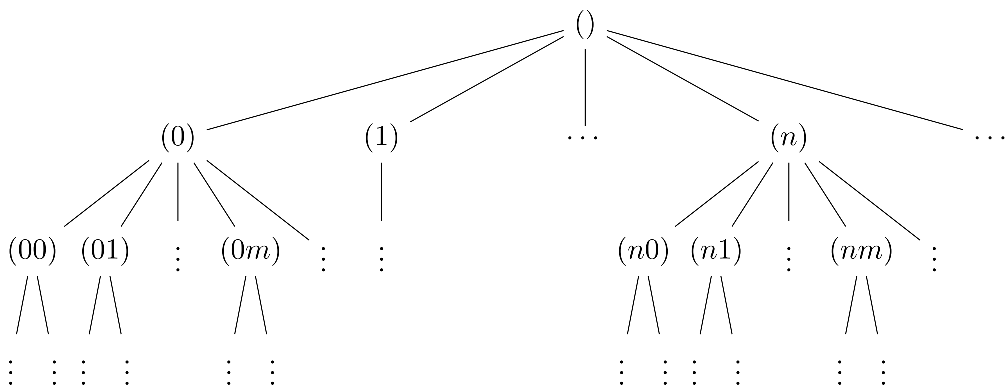

Figure 2: The collection $\mathbb{N}^{<\omega}$ is a tree. An edge is drawn between nodes σ and τ precisely when σ is a initial segment of τ or τ is a initial segment of σ and $|l(\sigma) - l(\tau)| = 1$.

Let f be any function from $\mathbb{N}$ to $\mathbb{N}$. We see that f determines a path in the tree $\mathbb{N}^{<\omega}$. The path consists of the nodes:

$$(f(0)); (f(0), f(1)); (f(0), f(1), f(2)); \ldots.$$

For example, the function $\mathrm{id}_{\mathbb{N}}$ gives us the path: $(0); (0,1); (0,1,2); \ldots.$

Define $d\colon \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ by

$$d(f,g) = \begin{cases} 2^{-n} & \text{if } n \text{ is the smallest positive integer} \\ & \text{such that } f(n) \neq g(n); \\ 0 & \text{if } f(n) = g(n) \text{ for all } n \in \mathbb{N}. \end{cases}$$

This definition is clearly not intuitionistically valid, since one cannot in general decide if two functions $f, g\colon \mathbb{N} \to \mathbb{N}$ are equal or not. For example, define $f\colon \mathbb{N} \to \mathbb{N}$ by

$$f(k) = \begin{cases} 1 & \text{if } k \text{ is a counterexample} \\ & \text{to Goldbach's conjecture;} \\ 0 & \text{otherwise.} \end{cases}$$

This is a valid definition, because we can (in principle) calculate $f(k)$ for every given $k \in \mathbb{N}$. If we define $g(k) = 0$ for all $k \in \mathbb{N}$, then it is intuitionistically impossible to claim $f = g$ or $f \neq g$, cf. section 2.2. However, we can give a intuitionistic definition of the function d. Define for every $n \in \mathbb{N}$:

$$d(f,g)_n = \begin{cases} 2^{-k} & \text{if } k < n \text{ and } k \text{ is the smallest positive} \\ & \text{integer such that } f(k) \neq g(k); \\ 2^{-n} & \text{otherwise.} \end{cases}$$

The sequence $(d(f,g)_n)_{n\in\mathbb{N}}$ is clearly a Cauchy sequence and we define $d(f,g)$ as its limit.

Proposition A.4. *The pair* $(\mathbb{N}^{\mathbb{N}}, d)$ *is a metric space.*

Proof. We immediately see that

$$\begin{aligned} &d(f,g) \geq 0 \\ &d(f,g) = 0 \Leftrightarrow f = g \\ &d(f,g) = d(g,f) \end{aligned}$$

for all $f, g \in \mathbb{N}^{\mathbb{N}}$. We still have to prove the triangle inequality, viz.

$$d(f,g) + d(g,h) \geq d(f,h)$$

for all $f, g, h \in \mathbb{N}^{\mathbb{N}}$. To do so, pick any $f, g, h \in \mathbb{N}^{\mathbb{N}}$. If $f = g$ or $g = h$, then the result is immediate. Therefore, suppose that $f \neq g$ and $g \neq h$, i.e. there exist $n_1, n_2 \in \mathbb{N}$ such that $d(f,g) = 2^{-n_1}$ and $d(g,h) = 2^{-n_2}$. Let $m = \min\{n_1, n_2\}$. Then, $f(k) = g(k) = h(k)$ for all $k < m$. Hence,

$$d(f,h) \leq 2^{-m}.$$

Furthermore,

$$\begin{aligned} d(f,g) + d(g,h) &= 2^{-n_1} + 2^{-n_2} \\ &= 2^{-m}(2^{-n_1+m} + 2^{-n_2+m}) \\ &\geq 2^{-m} \\ &\geq d(f,h), \end{aligned}$$

as we wished to show. ■

Definition A.5. We call the metric space $(\mathbb{N}^{\mathbb{N}}, d)$ the *Baire* space. We will denote it (just like the *set* $\mathbb{N}^{\mathbb{N}}$) by $\mathbb{N}^{\mathbb{N}}$.

Proposition A.6. *If* $\sigma = (a_0, a_1, \ldots, a_n) \in \mathbb{N}^{<\omega}$, *then we define*

$$U_\sigma = \{g \in \mathbb{N}^{\mathbb{N}} \mid g(0) = a_0, g(1) = a_1, \ldots, g(n) = a_n\}.$$

The basic opens of the Baire space are of this form.

Proof. Of course, if $\mathcal{T}_d$ is the topology induced by the metric d, then the pair $(\mathbb{N}^{\mathbb{N}}, \mathcal{T}_d)$ is a topological space, where the basic opens are of the form

$$B_\epsilon(f) = \{g \in \mathbb{N}^{\mathbb{N}} \mid d(f,g) < \epsilon\}$$

for any $f \in \mathbb{N}^{\mathbb{N}}$ and $\epsilon \in \mathbb{R}_{>0}$.

Let ϵ be an arbitrary element of $\mathbb{R}_{>0}$ and pick any $f \in \mathbb{N}^{\mathbb{N}}$. If $\epsilon > 1$, then $B_\epsilon(f) = \mathbb{N}^{\mathbb{N}}$. If $\epsilon \leq 1$, choose $n \in \mathbb{N}$ such that $2^{-(n+1)} < \epsilon \leq 2^{-n}$. Clearly, $B_{2^{-n}}(f) \supseteq B_\epsilon(f)$, because if $d(f,g) < \epsilon$, then $d(f,g) < 2^{-n}$. Moreover, if $g \in B_{2^{-n}}(f)$, then $f(k) = g(k)$ for all $k \in \{0, 1, \ldots, n\}$, so that $d(f,g) \leq 2^{-(n+1)} < \epsilon$. Hence, $B_{2^{-n}}(f) = B_\epsilon(f)$.

So we see that the basic opens are really of the form

$$B_{2^{-n}}(f) = \{g \in \mathbb{N}^{\mathbb{N}} \mid d(f,g) < 2^{-n}\}$$

for any $n \in \mathbb{N}$ and $f \in \mathbb{N}^{\mathbb{N}}$. Note that if $g \in B_{2^{-n}}(f)$, then $f(k) = g(k)$ for all $k \in \{0, 1, \ldots, n\}$. Therefore, if we set

$$\sigma = (a_0, \ldots, a_n) = (f(0), f(1), \ldots, f(n)),$$

then

$$U_\sigma = \{g \in \mathbb{N}^{\mathbb{N}} \mid g(0) = a_0, \ldots, g(n) = a_n\} = B_{2^{-n}}(f).$$

Conversely, given a sequence $\sigma = (a_0, \ldots, a_n) \in \mathbb{N}^{<\omega}$, we may pick any $f \in \mathbb{N}^{\mathbb{N}}$ with $f(k) = a_k$ for all $k \in \{0, 1, \ldots, n\}$ to obtain $U_\sigma = B_{2^{-n}}(f)$, completing our proof. ■

As we have seen, we may think of a function $f \in \mathbb{N}^{\mathbb{N}}$ as a path in the tree shown in figure 2. From this, we see that a basic open $U_{(a_0,a_1,\ldots a_n)}$ is the set of all paths in figure 2 that start with $(a_0); (a_0, a_1); \ldots; (a_0, a_1, \ldots, a_n)$.

Definition A.7. [6, pp. 112–113] A topological space T is called *homogeneous* if for every pair of points x, y of T, there exists an autohomeomorphism f of T such that $f(x) = y$. If a topological space is homogeneous, then all of its points are topologically equivalent.

Theorem A.8. *The Baire space is homogeneous.*

Proof. Pick any $f, g \in \mathbb{N}^{\mathbb{N}}$. We may assume that $f \neq g$. For if $f = g$, then the identity map on T is a suitable autohomeomorphism. For each $n \in \mathbb{N}$ define the function

$$\sigma_n \colon \mathbb{N} \to \mathbb{N}$$
$$k \mapsto \begin{cases} g(n) & \text{if } k = f(n); \\ f(n) & \text{if } k = g(n); \\ k & \text{otherwise.} \end{cases}$$

Note that σ_n is its own inverse.

Define the function

$$\xi \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$$

by

$$\xi(h)(n) = \sigma_n(h(n))$$

for all $n \in \mathbb{N}$.

We will prove that ξ is the desired autohomeomorphism. First of all, note that $\xi(f)(k) = \sigma_n(f(k)) = g(k)$ for any $n \in \mathbb{N}$, so that indeed $\xi(f) = g$. Moreover, notice that ξ is its own inverse, because we have for any $n \in \mathbb{N}$:

$$\begin{aligned}\xi(\xi(f))(n) &= \sigma_n(\xi(f)(n)) \\ &= \sigma_n(\sigma_n(f(n))) \\ &= f(n).\end{aligned}$$

Since ξ is invertible, it is a bijection. It remains to show that ξ and ξ^{-1} are continuous. Since ξ is its own inverse, it is enough to show that ξ is continuous. We have to show that $\xi^{-1}(U)$ is open in $\mathbb{N}^{\mathbb{N}}$ for any open $U \subseteq \mathbb{N}^{\mathbb{N}}$. It is sufficient to show this for the basic opens.

Let σ denote any finite sequence $(a_0, \ldots, a_k) \in \mathbb{N}^{<\omega}$. We will show that

$$\xi^{-1}(U_\sigma) = \{h \in \mathbb{N}^{\mathbb{N}} \mid \xi(h) \in U_\sigma\} = U_\tau,$$

where τ is the sequence $(\sigma_0(a_0), \sigma_1(a_1), \ldots, \sigma_k(a_k))$.

Suppose $\xi(h) \in U_\sigma$, viz. $\sigma_j(h(j)) = a_j$ for all $0 \leq j \leq k$. We apply σ_j to both sides of the equation and obtain $h(j) = \sigma_j(a_j)$ for all $0 \leq j \leq k$, because σ_j is its own inverse. We see that $h \in U_\tau$, so that $\xi^{-1}(U_\sigma) \subseteq U_\tau$.

For the converse, suppose $h \in U_\tau$, i.e. $h(j) = \sigma_j(a_j)$ for all $0 \leq j \leq k$. Applying σ_j to both sides of the equation gives us: $\sigma_j(h(j)) = a_j$. Hence, $\xi(h)(j) = a_j$ for all $0 \leq j \leq k$ and thus $\xi(h) \in U_\sigma$.

So we may conclude that $\xi^{-1}(U_\sigma) = U_\tau$, which shows that ξ is continuous, because U_τ is clearly open in $\mathbb{N}^{\mathbb{N}}$. This completes our proof. ■

Theorem A.9. *The Baire space is homeomorphic to the space of irrationals endowed with the subspace topology inherited from* $\mathbb{R}$.

Proof. One may proof this using continued fractions [4, pp. 42] or look at the proof given by Miller in [7, Theorem 1.1] ■

Bibliography

[1] Mark van Atten. Strong Counterexamples. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Summer 2011 edition, 2011. Available at: `http://plato.stanford.edu/entries/brouwer/strongcounterex`.

[2] Dirk van Dalen. Intuitionistic Logic. In Lou Gobble, editor, *The Blackwell Guide to Philosophical Logic*, pages 224–257. Blackwell Publishers, 2001.

[3] A. Heyting. *Intuitionism: An Introduction*. North-Holland Publishing Company, third edition, 1971.

[4] Thomas Jech. *Set Theory, The Third Millennium Edition, Revised and Expanded*. Springer Verlag, 2003.

[5] Alexander S. Kechris. *Classical Descriptive Set Theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, 1995.

[6] K. Kuratowski. *Topology, Volume 1*. Academic Press, 1966.

[7] Arnold W. Miller. *Descriptive Set Theory and Forcing: How to Prove Theorems about Borel Sets the Hard Way*, volume 4 of *Lecture Notes in Logic*. Springer-Verlag, 1995.

[8] Jaap van Oosten. Intuitionism. Lecture notes, available at: `http://www.staff.science.uu.nl/~ooste110/syllabi/intuitionism.pdf`, 1996.

[9] Jaap van Oosten. Studentenseminarium intuitionisme. Seminar introduction page, available at: `http://www.staff.science.uu.nl/~ooste110/int.html`, 2002.

[10] H. Rasiowa and R. Sikorski. *The Mathematics of Metamathematics*. Panstwowe Wydawnictwo Naukowe, 1963.

[11] Dana Scott. Extending the Topological Interpretation to Intuitionistic Analysis. *Compositio Mathematica*, 20:194–210, 1968.

[12] Dana Scott. Extending the Topological Interpretation to Intuitionistic Analysis, II. *Studies in Logic and the Foundations of Mathematics*, 60:235–255, 1970.

[13] Marco Swaen. Zwevende getallen. *Nieuw Archief voor Wiskunde*, 5/10(4):273–279, December 2009.

[14] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics: An Introduction, Volume 1*, volume 121 of *Studies in Logic and the Foundations of Mathematics*. Elsevier Science Publisher B.V., 1988.