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Translating Theories

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Preface

As a prelude to the body of my bachelor thesis in completion of the BA Philosophy at Utrecht University, I'd like to share some personal thoughts.

During my first meeting with my supervisor, Prof. Dr. Albert Visser, we discussed the topic of my thesis project and from ensuing discussions our focus changed from its original approach to the topic at hand: translating a theory into a language using different primitive notions, such that the result is the same theory, even though they most probably have very different axioms.

I am greatly indebted to my supervisor for sharing his theoretical work with me and allowing me to expand beyond the theoretical framework provided. I thank Albert for his intensive guidance and enthusiasm that made this possible.

1. Introduction

When constructing any axiomatic system, one has to start somewhere. There will always be undefined notions, in terms of which the other notions are defined. Attempting to define these so-called *primitive notions* in terms of others will inevitably lead to an infinite regress. Primitive notions can be roughly divided into two categories: objects, and relations. For instance, in ZFC the primitive objects are *sets* and the primitive relation is $\in$.

In some sense, the choice of primitive notions is arbitrary. Instead of *sets*, we could be talking about *beer bottles* and given the same axioms, we could prove for example the Schröder–Bernstein theorem about *beer bottles.* That does, however, not mean that it actually makes sense to talk about injective function between beer bottles *in reality.*[1]

In another sense, the choice of primitive notions is crucial. When choosing primitive notions, we have some intuitive idea about what the notions are about. On the basis of this idea we formulate the axioms. Take for example the primitive notion of a *line*. It is only because we have an intuitive idea about what a line *is*, that we are able to state axioms about *lines*, such as "two lines intersect if and only if they are not parallel".

In this way, a different choice in primitive notions can result in two quite different axiomatic systems, even though they might be axiomatizations of the same thing. Take for example two ways of axiomatizing arithmetic. On the one hand, there is the well-known theory PA, which has *numbers* as its primitive objects. On the other hand it is also possible to axiomatize arithmetic with *sets* as primitive elements, as is shown in [KW07].

In this paper we will study precisely that phenomenon of theories which are the same, but are formulated in terms of different primitive objects. Doing so, we will present a way of translating a theory into a language using different primitive notions, such that the result is the same theory, even though they most probably have very different axioms.

As far as we know this is a new topic of research. However, it is related to the active body of research in the axiomatization and sameness of theories. In for example [Vis06] the question "when are two theories the same?" is raised. From there, the question "can we generate a theory such that it is the same as a given theory?" seems not far away.

For this we do need a notion of sameness of theories. From a few possible notions we choose that of *bi-interpretability*, which is presented in Section 2. In Section 3 we will prove our main Theorem, which will serve as a tool for translating theories. In Section 4 we will demonstrate this tool in two simple example cases, one in which it is can be used meaningfully and one in which it can't. In Section 5 we will present a more complex example and take the first steps to translate Tarski's axiomatization of geometry in terms of points, into geometry in terms of lines. This section provides a further connection to previous research. In, for instance, [Pam05] the sameness of geometric theories is studied as well, although for a different notion of sameness, and for different geometric theories. Finally, in Section 6, we will recap what we have found and what more there is to explore.

[1] There is a famous anecdote in which Hilbert, when talking about his axiomatization of geometry, is quoted as saying "One must be able to say at all times - instead of points, straight lines, and planes - tables, chairs, and beer mugs".

2. Bi-interpretability

There are several notions for saying two theories are the same. In this section we present the notion that we will use throughout this paper. The definitions are reformulations of those in Appendix A. of [FV14].[2]

As the name suggests, *bi-interpretability* has something to do with being able to interpret two theories into each other. Before we go on to define the notion of *interpretation*, which is between theories, we first have to define the notion of *translation*, which is between languages.

Definition 2.1. Let Σ and Θ be signatures. An *n-dimensional translation* $\tau : \Sigma \to \Theta$ is a triple $\langle n, \delta, F\rangle$, where δ is a Θ-formula with n free variables and F is a mapping that associates to each m-ary relation symbol R of Σ, a Θ-formula with $n \times m$ free variables such that

$$F(R)(x_1, \ldots, x_n) \to (\delta(x_1) \wedge \ldots \wedge \delta(x_n)).$$

Given a translation $\tau : \Sigma \to \Theta$, we can translate Σ-formulas to Θ-formulas as follows.

- $(R(x_1, \ldots, x_n))^\tau := F(R)(\vec{x}_1, \ldots, \vec{x}_n)$;
- $(\cdot)^\tau$ commutes with the propositional connectives;
- $(\forall x A)^\tau := \forall \vec{x}\, (\delta(x) \to A^\tau)$;
- $(\exists A)^\tau := \exists \vec{x}\, (\delta(x) \wedge A^\tau)$.

We write δ_τ for 'the δ of τ' and R_τ for '$F(R)$ for the F of τ'.

Furthermore, we define the following operations on translations.

Definition 2.2. The *identity translation* on Σ is the translation $\mathsf{id}_\Sigma : \Sigma \to \Sigma$ defined by:

- $\delta_{\mathsf{id}_\Sigma}(x) :\leftrightarrow (x = x)$,
- $R_{\mathsf{id}_\Sigma}(\vec{x}) :\leftrightarrow R(\vec{x})$.

Since every element is in the domain, and every relation symbol gets translated to itself, the idenity translation translates any formula to itself.

Suppose we have a translation $\tau : \Sigma \to \Theta$ and a translation $\upsilon : \Theta \to \Xi$. Then we can translate Σ-sentences into Ξ-sentences by first translating them under τ and then under υ. These kind of translations are called composite translations and are defined as follows.

Definition 2.3. Let $\tau : \Sigma \to \Theta$ be an n-dimensional translation and $\upsilon : \Theta \to \Xi$ an m-dimensional translation. The *composite translation of υ after τ* is the $n \times m$-dimensional translation $\upsilon \circ \tau : \Sigma \to \Xi$, given by:

- $\delta_{\upsilon\circ\tau}(\vec{x}_1, \ldots, \vec{x}_n) :\to (\delta_\upsilon(\vec{x}_1) \wedge \ldots \wedge \delta_\upsilon(\vec{x}_n) \wedge (\delta_\tau(x_1, \ldots, x_n))^\upsilon)$,
- $R_{\upsilon\circ\tau}(\vec{x}_{1,1}, \ldots, \vec{x}_{n,k}) :\leftrightarrow (\delta_\upsilon(\vec{x}_{1,1}) \wedge \ldots \wedge \delta_\upsilon \vec{x}_{n,k} \wedge (R_\tau(x_{1,1}, \ldots, x_{n,k}))^\upsilon)$.

For each k-place relation symbol R of Σ.

Now that we have all we need about translations, we can specify under what condition they support an interpretation.

Definition 2.4. Let U be a Σ-theory, V a Θ-theory and $\tau : \Sigma \to \Theta$ an n-dimensional translation. Then $K := \langle U, \tau, V\rangle$ is an *n-dimensional interpretation of U in V* if

$$U \vdash A \Rightarrow V \vdash A^\tau$$

[2] There are some delicate matters in dealing with variable symbols that are ignored here.

for all Σ-sentences A. ∎

We write τ_K for the second component of K and R_K for R_{τ_K}. Analogous to translations we define the *identity interpretation* of a Σ-theory U as $\mathsf{id}_U := \langle U, \mathsf{id}_\Sigma, U\rangle$ and the *composite interpretation* $L \circ K : U \to W$ of $L : V \to W$ after $K : U \to V$ as $\langle U, \tau_L \circ \tau_K, W\rangle$.

Given a model $\mathcal{M}$ of V, an interpretation $K : U \to V$ gives rise to an internal model $\widetilde{K}(\mathcal{M})$ of U. For an n-dimensional interpretation K, the internal model $\widetilde{K}(\mathcal{M})$ contains those elements $\vec{x} \in \mathcal{M}^n$ such that $\delta_K(\vec{x})$.

Now that we have rigorously defined what it means to interpret one theory into another, we can use this notion to determine a notion for when two theories are considered the same.

One might think that it is sufficient to say that two theories are the same if they are *mutually interpretable*, i.e. if they can be interpreted into each other. However, as is shown in [Vis06], mutual interpretability fails to preserve some essential properties such as decidability.

It is clear that we need a stricter notion of sameness of theories if we want to make certain that essential properties, such as decidability, are preserved. One way to obtain this is by considering the composite interpretations. When given interpretations $K : U \to V$ and $L : V \to U$ we can compare the composite interpretations $L \circ K : U \to U$ and $K \circ L : V \to V$ with the respective identity interpretations $\mathsf{id}_U : U \to U$ and $\mathsf{id}_V : V \to V$. We can then say that the theories are the same precisely when the respective composite and identity interpretations are the same.

For this, we need a notion of sameness of interpretations. Suppose we say that any two interpretations $K, K' : U \to V$ are the same. Then the mere existence of interpretations $K : U \to V$ and $L : V \to U$ is sufficient to guarantee that the composite interpretations are equal to the identity interpretations. In that case we end up with the same notion of mutual interpretability of which we have seen that it is not sufficiently strict.

We can make this notion of sameness of theories stricter by making our notion of sameness of interpretations stricter. There are several notions for the sameness of interpretations, which in turn lead to different notions of sameness of theories. The notion that seems to most respect intuition, in the sense that it preserves most essential properties without being *too* strict, is *i-isomorphism* for the sameness of interpretations, which leads to *bi-interpretability* for the sameness of theories. Syntactically, we define *i-isomorphism* as follows.

Definition 2.5. Let $K : U \to V$ be an n-dimensional and $L : U \to V$ an m-dimensional interpretation and let F be a V-definable formula with $n + m$ free variables. Then F is an *i-isomorphism* between K and L if V proves that F has the following properties.

- $V \vdash \vec{x}F\vec{y} \to (\delta_K(\vec{x}) \wedge \delta_L(\vec{y}))$.
- $V \vdash (\vec{x} =_K \vec{z} \wedge \vec{y} =_L \vec{w} \wedge \vec{u}F\vec{v}) \to \vec{x}F\vec{y}$.
- $V \vdash \forall \vec{x}(\delta_K(\vec{x}) \to \exists \vec{y}(\delta_L(\vec{y}) \wedge \vec{x}F\vec{y}))$.
- $V \vdash (\vec{x}F\vec{y} \wedge \vec{x}F\vec{z}) \to y =_L z$.
- $V \vdash (\vec{x}_1F\vec{y}_1 \wedge \ldots \wedge \vec{x}_kF\vec{y}_k \wedge R_L(\vec{x}_1, \ldots, \vec{x}_n)) \to R_K(\vec{y}_1, \ldots, \vec{y}_n)$.
- $V \vdash \forall \vec{y}(\delta_L(\vec{y}) \to \exists \vec{x}(\delta_K(\vec{x}) \wedge \vec{x}F\vec{y}))$.
- $V \vdash (\vec{x}F\vec{y} \wedge \vec{z}F\vec{y}) \to x =_K z$.
- $V \vdash (\vec{x}_1F\vec{y}_1 \wedge \ldots \wedge \vec{x}_kF\vec{y}_k \wedge R_K(\vec{y}_1, \ldots, \vec{y}_n)) \to R_L(\vec{x}_1, \ldots, \vec{x}_n)$. ∎

The previous definition in entirely syntactic. We can also give a semantic definition of i-isomorphisms, in terms of the internal models generated by interpretations. It is equivalent to Definition 2.5 to say that $K : U \to V$ and $L : U \to V$ are i-isomorphic if there is a function F such that for all models $\mathcal{M}$ of V, we have that F is an isomorphism between the internal models $\widetilde{K}(\mathcal{M})$ and $\widetilde{L}(\mathcal{M})$.

For theories of finite signature there is also a third equivalent way of defining the notion of i-isomorphism. It is stated — without proof — in [Vis06]. For the sake of completeness we will prove it here.

Theorem 2.6 ([Vis06, A.1])**.** *Suppose the signature of U is finite and consider interpretations $K, L : U \to V$. Suppose that, for every model $\mathcal{M}$ of V, there is an $\mathcal{M}$-definable isomorphism between $\widetilde{K}(\mathcal{M})$ and $\widetilde{L}(\mathcal{M})$. Then, we have that K and L are i-isomorphic.*

Proof. Let V^* be the theory consisting of V, together with the axiom

$$\text{“}F\text{ is not an isomorphism”},$$

for all V-formulas that could possibly serve as an i-isomorphism between K and L. That is, all V-formulas with $n \times m$ free variables, given that K is an n-dimensional interpretation and L is an m-dimensional interpretation. Such an axiom is precisely the disjunction of the negations of the axioms of Definition 2.5 for some V-definable formula F.

Since, by hypothesis, we have that for any model $\mathcal{M}$ of V there is an isomorphism between $\widetilde{K}(\mathcal{M})$ and $\widetilde{L}(\mathcal{M})$, we know that V^* has no models. By the compactness theorem we have that there is a finite subtheory V' of V^* that has no models.

Note that V' is a subtheory of V together with “F is not an isomorphism” for a *finite* number of i-isomorphisms. Since all models $\mathcal{M}$ of V are *not* models of V', the total number of isomorphisms between internal models generated by K and L is finite.

We denote these isomorphims $F_1, \ldots, F_n$. Note that for any model $\mathcal{M}$ of V it holds that one of $F_1, \ldots, F_n$ is an isomorphism between $\widetilde{K}(\mathcal{M})$ and $\widetilde{L}(\mathcal{M})$. Now we can combine these formulas into one formula

$$F^* = \begin{cases} F_1 \text{ if } F_1 \text{ is an isomorphism,} \\ F_2 \text{ if } F_2 \text{ is an isomorphism and } F_1 \text{ is not an isomorphism,} \\ \cdots \\ F_n \text{ if } F_n \text{ is an isomorphism and } F_1, \ldots, F_{n-1} \text{ are not isomorphisms.} \end{cases}$$

which is an isomorphism between $\widetilde{K}(\mathcal{M})$ and $\widetilde{L}(\mathcal{M})$ for all models $\mathcal{M}$ of V. By the semantic definition K and L are i-isomorphic. □

It is nice to see that there is a connection between the syntactic and the semantic meaning of i-isomorphisms, since the theory that we will develop in the following chapters hinges on this notion. Therefore, even though we will formulate our theory on a syntactic level, the connection with a semantic meaning will always be present.

The notion of i-isomorphism for the sameness of interpretations leads to the following definition for the sameness of theories.

Definition 2.7. Let $K : U \to V$ and $L : V \to U$ be interpretations. Furthermore, let F be a U-definable formula and G a V-definable formula. Then U and V are *bi-interpretable* if U proves that F is an i-isomorphism between $L \circ K$ and id_U and V proves that G is an i-isomorphism between $K \circ L$ and id_V. ∎

3. Bi-interpretability bases and the main theorem

Suppose we have a theory U in a signature Σ. Can we use translations back and forth to form a Θ-theory V with which it is bi-interpretable? It turns out that, using the machinery of the definition of bi-interpretability, we can. In this section we will show in an abstract setting why this is the case.

Consider our new theory V. What axioms must V have for it to be bi-interpretable with U? Since we need interpretations between the two theories, we know that V must prove the axioms of U under some translation.

This, however, is not sufficient. We also need both theories to prove that there are i-isomorphisms between the respective composite interpretations and identity interpretations. These axioms are given in Definition 2.5. And again the translations of these axioms need also to be included in order to guarantee that our translations still support interpretations.

The set of axioms described in the previous paragraph depend only on the respective languages and isomorphisms and not on the theories and interpretations. Therefore, we can formulate these axioms — which we know our new theory V must satisfy — without yet having determined V. This is done in the following definition.

Definition 3.1. Let Σ, Θ be signatures, $\tau : \Sigma \to \Theta$ an n-dimensional translation, $\upsilon : \Theta \to \Sigma$ an m-dimensional translation, F a Σ-formula with $(n \times m) + 1$ free variables and G a Θ-formula with $(m \times n) + 1$ free variables. The *bi-interpretability basis* of τ, υ, F, G, denoted $\mathsf{BB}_{\tau,\upsilon,F,G}$, is the Σ-theory with the following axioms.

$\mathsf{BB}_{\tau,\upsilon,F,G}1.$ $\vdash \mathcal{I}_{=}$,

$\mathsf{BB}_{\tau,\upsilon,F,G}2.$ $\vdash \exists x(x = x)$,

$\mathsf{BB}_{\tau,\upsilon,F,G}3.$ $\vdash \vec{x}Fa \to \delta_{\upsilon\circ\tau}(\vec{x})$,

$\mathsf{BB}_{\tau,\upsilon,F,G}4.$ $\vdash (\vec{x} =_{\upsilon\circ\tau} \vec{y} \land \vec{x}Fa) \to \vec{y}Fa$,

$\mathsf{BB}_{\tau,\upsilon,F,G}5.$ $\vdash \delta_{\upsilon\circ\tau}(\vec{x}) \to \exists a(\vec{x}Fa)$,

$\mathsf{BB}_{\tau,\upsilon,F,G}6.$ $\vdash (\vec{x}Fa \land \vec{x}Fb) \to a = b$,

$\mathsf{BB}_{\tau,\upsilon,F,G}7.$ $\vdash (\vec{x}_1Fa_1 \land \cdots \land \vec{x}_nFa_n \land R_{\upsilon\circ\tau}(\vec{x}_1, \cdots, \vec{x}_n)) \to R(a_1, \cdots, a_n)$,

$\mathsf{BB}_{\tau,\upsilon,F,G}8.$ $\vdash \forall a\exists\vec{x}(\delta_{\upsilon\circ\tau}(\vec{x}) \land \vec{x}Fa)$,

$\mathsf{BB}_{\tau,\upsilon,F,G}9.$ $\vdash (\vec{x}Fa \land \vec{y}Fa) \to \vec{x} =_{\upsilon\circ\tau} \vec{y}$,

$\mathsf{BB}_{\tau,\upsilon,F,G}10.$ $\vdash (\vec{x}_1Fa_1 \land \cdots \land \vec{x}_nFa_n \land R(a_1, \cdots, a_n)) \to R_{\upsilon\circ\tau}(\vec{x}_1, \cdots, \vec{x}_n)$,

$\mathsf{BB}_{\tau,\upsilon,F,G}11.$ $\vdash (\mathsf{BB}_{\upsilon,\tau,G,F}1.)^{\upsilon}$,

…

$\mathsf{BB}_{\tau,\upsilon,F,G}20.$ $\vdash (\mathsf{BB}_{\upsilon,\tau,G,F}11.)^{\upsilon}$.

In other words, $\mathsf{BB}_{\tau,\upsilon,F,G}$ is the set of axioms that prove that F is an isomorphism between interpretations supported by $\upsilon \circ \tau$ and id_Σ, together with the translations under υ of the axioms that prove that G is an isomorphism between interpretations supported by $\tau \circ \upsilon$ and id_Θ. ∎

The first axiom of the bi-interpretability bases is the set of equality axioms as defined in Appendix A. We need them because, as is custom when dealing with interpretations, we treat equality as a regular relation symbol. The 2^{nd} axiom of the bi-interpretability basis definition might seem trivial, but when translated in the 12^{th} axiom, it guarantees us that δ_υ is non-empty.

By definition, the bi-interpretability bases contain the axioms *necessary* for the two functions to witness that two theories in the respective languages are bi-interpretable under the interpretations supported by the two languages.

The reason that they are so useful is that they are also *sufficient* to prove that two theories are bi-interpretable. This fact is proven in the following proposition and theorem. We will later see that we can use that property to translate a theory into another language. That is, to construct a theory that is bi-interpretable with a given theory in another language.

Consider two bi-interpretability bases with 'flipped' translations and formulas, e.g. $\mathsf{BB}(\tau, \upsilon, F, G)$ and $\mathsf{BB}(\upsilon, \tau, G, F)$. By definition these theories are interpretable in each other, since they contain the respective translations of each others axioms. In the following proposition we show that they are not only mutually interpretable, but also bi-interpretable.

Proposition 3.2. *The bi-interpretability bases* $\mathsf{BB}(\tau, \upsilon, F, G)$ *and* $\mathsf{BB}(\upsilon, \tau, G, F)$ *are bi-interpretable.*

Proof. First we want to show that τ and υ support interpretations of the respective theories. Since the cases are symmetric it suffices to show that the triple $\langle \mathsf{BB}_{\tau,\upsilon,F,G},\ \tau,\ \mathsf{BB}_{\upsilon,\tau,G,F} \rangle$ is an interpretation. We will do this by showing that for each axiom $\mathsf{BB}_{\tau,\upsilon,F,G}\mathrm{i}$, we have $\mathsf{BB}_{\upsilon,\tau,G,F} \vdash (\mathsf{BB}_{\tau,\upsilon,F,G}\mathrm{i})^\tau$.

By definition, we have $\mathsf{BB}_{\upsilon,\tau,G,F}(i+10) \vdash (\mathsf{BB}_{\tau,\upsilon,F,G}\mathrm{i})^\tau$ for $i = 1, \dots, 10$. For $i = 11, \dots, 20$, we notice that

$$\mathsf{BB}_{\upsilon,\tau,G,F}(i-10) \vdash (\mathsf{BB}_{\tau,\upsilon,F,G}\mathrm{i})^\tau \Leftrightarrow \mathsf{BB}_{\upsilon,\tau,G,F}(i-10) \vdash (\mathsf{BB}_{\upsilon,\tau,G,F}(i-10))^{\tau\circ\upsilon}.$$

The statement right of the equivalence is true because the first eight axioms of $\mathsf{BB}_{\upsilon,\tau,G,F}$ prove that G is an isomorphism between interpretations supported by id and $\tau \circ \upsilon$.

The respective theories prove that F and G are isomorphisms and are thus bi-interpretable. □

A generalization of the previous proposition is that any theory that proves a bi-interpretability basis, is in fact bi-interpretable with the theory consisting of the 'flipped' bi-interpretability basis, together with the translated axioms of the original theory. This is proven in the following theorem.

Theorem 3.3. *Let U be a Σ-theory such that $U \vdash \mathsf{BB}_{\tau,\upsilon,F,G}$. Then U is bi-interpretable with the Θ-theory $V = U^\tau \cup \mathsf{BB}_{\upsilon,\tau,G,F}$.*

Proof. Analogous to the proof of Proposition 3.2 we will first show that the triples $K = \langle U,\ \tau,\ V \rangle$ and $L = \langle V,\ \upsilon,\ U \rangle$ are interpretations.

By definition, $V \vdash A^\tau$ for all $A \in U$, so K is an interpretation. For $B \in V$, we have two cases from which we deduce that L is an interpretation.

Case 1. $B \in \mathsf{BB}_{\upsilon,\tau,G,F}$. Then, as seen in the proof of Proposition 3.2, we have $U \vdash B^\upsilon$.

Case 2. $B \in U^\tau$. Then $B^\upsilon \in U^{\upsilon\circ\tau}$, and since U proves that F is an isomorphism between interpretations supported by id_Σ and $\upsilon \circ \tau$ it also holds that $U \vdash B^\upsilon$.

Since $U \vdash \mathsf{BB}_{\tau,\upsilon,F,G}$, it holds that U proves that F is an isomorphism between K and id_Σ. Analogously, it holds that V proves that G is an isomorphism between L and id_Θ and thus U and V are bi-interpretable. □

With this theorem we have precisely what we need to translate a theory. Given a theory, we only have to find a bi-interpretability basis that it proves. Once we have done that, it is only a matter of computation to find the bi-interpretable theory in a different language.

Note that, while previous theorem *is* very useful for finding a bi-interpretable theory in a different signature, it does not turn it into a trivial matter. One still has to formulate translations and isomorphisms that are such that they generate a coherent bi-interpretability basis that is proven by U. In the next section we explore how this can be done, and what happens when the translations don't meet these requirements.

Before we do that, we show as a corollary to the previous theorem one of the properties that makes bi-interpretability such a good notion of sameness of theories, namely that it preserves finite axiomatizability.

Corollary 3.4. *Let U and V be bi-interpretable. Then U is finitely axiomatizable if and only if V is.*

Proof. Let $K = \langle U,\ \tau,\ V\rangle$ and let $L = \langle V,\ \upsilon,\ U\rangle$ be the interpretations and $F : L \circ K \to \mathsf{id}_U$ and $G : K \circ L \to \mathsf{id}_V$ the isomorphisms that witness that U and V are bi-interpretable.

Because of the bi-interpretability, we know that U proves that F is an isomorphism and V proves that G is an isomorphism. Since K and L are interpretations, we also know that U and V prove each others axioms under the respective translations. It follows that $U \vdash \mathsf{BB}_{\tau,\upsilon,F,G}$ and $V \vdash \mathsf{BB}_{\upsilon,\tau,G,F}$.

Now suppose that U is finitely axiomatizable. Then $V' = U^\tau \cup \mathsf{BB}_{\upsilon,\tau,G,F}$ is a finitely axiomatizable theory such that $V \vdash V'$. Let A be a V-sentence such that $V \vdash A$. Then $U \vdash A^\upsilon$ and thus $V' \vdash A^{\tau\circ\upsilon}$. Since V' proves that G is an isomorphism, we have $V' \vdash A$ and thus $V' \vdash V$. We conclude that $V = V'$ and is thus finitely axiomatizable. The other direction is symmetric. □

4. Two simple examples

In this section we will present two simple examples of the use and workings of bi-interpretability bases. In the first example we present a simple theory that we will translate into another language.

Example 4.1. Let U be the theory in the language Σ_U with only the identity relation, such that U states that there are at least two elements, i.e.

$U1.$ $\vdash \exists x \exists y (x \neq y)$.

We create translations between Σ_U and Σ_V that do nothing more than copy the domain of Σ_U. That is, we equip Σ_V with a one-place relation P and create translations back and forth such that for each element in U, we have precisely two elements in δ_υ, one for which P holds, and one for which P does not hold.

In order to be able to express in Σ_V that for every x such that $P(x)$, there is exactly one unique counterpart y such that $\neg P(y)$, we add a two-place relation symbol C. Precisely, we have xCy when $P(x)$, $\neg P(y)$ and y is the counterpart of x.

We define $\tau : \Sigma_U \to \Sigma_V$ as follows.

- $\delta_\tau(x) :\leftrightarrow P(x)$,
- $x =_\tau y :\leftrightarrow (\delta_\tau(x) \wedge \delta_\tau(y) \wedge x = y)$.

In the other direction, we define $\upsilon : \Sigma_V \to \Sigma_U$ as:

- $\delta_\upsilon(x, y) :\leftrightarrow (x = x \wedge y = y)$,
- $(x, y) =_\upsilon (z, w) :\leftrightarrow (\delta_\upsilon(x, y) \wedge \delta_\upsilon(z, w) \wedge ((x = y \wedge x = z \wedge x = w) \vee (x \neq y \wedge z \neq w \wedge x = z)))$,
- $P_\upsilon(x, y) :\leftrightarrow (\delta_\upsilon(x, y) \wedge x = y)$,
- $(x, y)C_\upsilon(z, w) :\leftrightarrow (\delta_\upsilon(x, y) \wedge \delta_\upsilon(z, w) \wedge x = y \wedge z \neq w \wedge z = x)$.

As desired, the domain of υ contains precisely twice as many elements as U, up to υ-equality. To each x we associate two elements after translations. The first is the pair (x, x). The second is the equivalence class of all (x, y) such that $x \neq y$, which are equal up to $=_\upsilon$. The translation defines the latter as the counterpart of the former, i.e. $(x, x)C_\upsilon(x, y)$ and $P_\upsilon(x, x)$ and $\neg P_\upsilon(x, y)$.

In order to find potential functions F and G that can serve as isomorphisms in our bi-interpretation, we calculate the respective compositions of our translations.

- $\delta_{\upsilon\circ\tau}(x, y) \leftrightarrow x = y$,
- $(x, y) =_{\upsilon\circ\tau} (z, w) \leftrightarrow (\delta_{\upsilon\circ\tau}(x, y) \wedge \delta_{\upsilon\circ\tau}(z, w) \wedge x = z)$.

We deduce that a suitable candidate is $(x, y)Fz :\leftrightarrow (\delta_{\upsilon\circ\tau}(x, y) \wedge x = z)$. This is not surprising, since we've seen that the first object of the pair determines the object that it is associated with.

Conversely,

- $\delta_{\tau\circ\upsilon}(x, y) \leftrightarrow (P(x) \wedge P(y))$,
- $(x, y) =_{\tau\circ\upsilon} (z, w) \leftrightarrow (\delta_{\tau\circ\upsilon}(x, y) \wedge \delta_{\tau\circ\upsilon}(z, w) \wedge ((x = y \wedge x = z \wedge x = w) \vee (x \neq y \wedge z \neq w \wedge x = z)))$,
- $P_{\tau\circ\upsilon}(x, y) \leftrightarrow (\delta_{\tau\circ\upsilon}(x, y) \wedge x = y)$,
- $(x, y)C_{\tau\circ\upsilon}(z, w) \leftrightarrow (\delta_{\tau\circ\upsilon}(x, y) \wedge \delta_{\tau\circ\upsilon}(z, w) \wedge x = y \wedge z \neq w \wedge z = x)$.

We find that we can define a potential isomorphism

$$(x, y)Gz :\leftrightarrow (\delta_{\tau\circ\upsilon}(x, y) \wedge ((x = y \wedge x = z) \vee (x \neq y \wedge xCz))).$$

The formula G associates to each pair (x, y), the object x when $x = y$, and the $\neg P$-counterpart of x when $x \neq y$. Since our domain only consists of P-objects and we defined our translation such that we have precisely as many P and $\neg P$ elements, it follows that G should be an isomorphism.

It can be somewhat tediously verified that the only non-tautological axiom of the bi-interpretability basis $\mathsf{BB}_{\tau,\upsilon,F,G}$ is the second one, that specifies that there is at least one element. Of course, that means that $U \vdash \mathsf{BB}_{\tau,\upsilon,F,G}$ and thus that, by Proposition 3.3, the original theory U is bi-interpretable with the Σ_V-theory $V = U^\tau \cup \mathsf{BB}_{\upsilon,\tau,G,F}$.

It might seem a bit magical that we concluded that U and V are bi-interpretable without writing down any of the axioms of V. In order to make this fact more clear, we will state some of the theorems that follow from the axioms of V.

- $V \vdash \exists x \exists y (P(x) \wedge P(y) \wedge x \neq y)$, (from U^τ)
- $V \vdash xCa \to P(x)$, (from $\mathsf{BB}_{\upsilon,\tau,G,F}3$)
- $V \vdash xCa \to \neg P(a)$, (from $\mathsf{BB}_{\upsilon,\tau,G,F}10$)
- $V \vdash P(x) \to \exists a(xCa)$, (from $\mathsf{BB}_{\upsilon,\tau,G,F}3$, $\mathsf{BB}_{\upsilon,\tau,G,F}5$, U^τ)

- $V \vdash \neg P(a) \to \exists x(xCa)$, (from $\mathsf{BB}_{\upsilon,\tau,G,F}8$)
- $V \vdash (xCa \wedge yCa) \to x = y$, (from $\mathsf{BB}_{\upsilon,\tau,G,F}9$)

We see that V proves that there are at least two elements with P, and that C is a bijection between elements with P and elements without P. It is clear now that V is nothing more than a doubling of U and that thus the theories are bi-interpretable. We see that by defining our translations with a certain idea in mind, the bi-interpretability bases turn into a theory that corresponds to that idea. ▲

In the previous example, the translations back and forth made sense; that is, there is an idea behind the translations that respects the axioms of the theory that we wanted to translate and the translations are in a sense each others inverses. It is only because of that, that our theory proves the bi-interpretability basis and that the resulting translated theory is such as we imagined. In the next example we investigate what happens when the translations *don't* make sense.

Example 4.2. Let Σ_U and Σ_V and $\tau : \Sigma_U \to \Sigma_V$ be as in Example 4.1. In the other direction we define the translation $\nu : \Sigma_U \to \Sigma_V$ as:

- $\delta_\nu(x) :\leftrightarrow x = x$,
- $x =_\nu y :\leftrightarrow (\delta_\nu(x) \wedge x = y)$,
- $P_\nu(x) :\leftrightarrow \delta_\nu(x)$,
- $xC_\nu y :\leftrightarrow (\delta_\nu(x) \wedge \delta_\nu(y))$.

Composing the translations gives us

- $\delta_{\nu\circ\tau}(x) \leftrightarrow x = x$,
- $x =_{\nu\circ\tau} y \leftrightarrow x = y$,

and

- $\delta_{\tau\circ\nu}(x) \leftrightarrow P(x)$,
- $x =_{\tau\circ\nu} y \leftrightarrow (\delta_{\tau\circ\nu}(x) \wedge \delta_{\tau\circ\nu}(y) \wedge x = y)$,
- $P_{\tau\circ\nu}(x) \leftrightarrow \delta_{\tau\circ\nu}(x)$,
- $xC_{\tau\circ\nu}y \leftrightarrow (\delta_{\tau\circ\nu}(y) \wedge \delta_{\nu\circ\tau}(y))$.

Now we define the following Σ_U-formula I and Σ_V-formula J that will serve as our isomorphisms.

- $xIy :\leftrightarrow x = y$,
- $xJy :\leftrightarrow xCy$.

One might expect these nonsensical translations to generate inconsistent bi-interpretability bases. As long as both the bi-interpretability bases are inconsistent this would not contradict Proposition 3.2, since two inconsistent theories are trivially bi-interpretable. This is, however, not the case; the bi-interpretability bases *are* consistent, but something else interesting happens. We have

- $\mathsf{BB}_{\tau,\nu,I,J}1. \vdash \exists x(x = x)$,
- $\mathsf{BB}_{\tau,\nu,I,J}18. \vdash (\mathsf{BB}_{\nu,\tau,J,I}7)^{\upsilon} \vdash ((xCa \wedge xCb) \to a = b)^{\upsilon} \vdash \forall a \forall b(a = b)$.

We see that $\mathsf{BB}_{\tau,\nu,I,J}$ is the theory that proves that there is exactly one object.

For determining $\mathsf{BB}_{\nu,\tau,I,J}$, first note that it includes the following theorems.

(1) $\mathsf{BB}_{\nu,\tau,I,J} \vdash \exists x(x = x)$, (from $\mathsf{BB}_{\nu,\tau,I,J}2$)
(2) $\mathsf{BB}_{\nu,\tau,I,J} \vdash \forall a \exists x(P(x) \wedge xCa)$, (from $\mathsf{BB}_{\nu,\tau,I,J}8$)
(3) $\mathsf{BB}_{\nu,\tau,I,J} \vdash (xCa \wedge yCb \wedge P(x) \wedge P(y)) \to aCb$, (from $\mathsf{BB}_{\nu,\tau,I,J}7$)
(4) $\mathsf{BB}_{\nu,\tau,I,J} \vdash (xCa \wedge yCa) \to (P(x) \wedge P(y) \wedge x = y)$, (from $\mathsf{BB}_{\nu,\tau,I,J}9$)

Now suppose there are two objects x, y; we can do this because by (1) there is at least 1 object. Then by (2) we have

- $\mathsf{BB}_{\nu,\tau,I,J} \vdash \exists a(P(a) \wedge aCx)$,
- $\mathsf{BB}_{\nu,\tau,I,J} \vdash \exists b(P(b) \wedge bCy)$.

By an instance of (3) we get

- $\mathsf{BB}_{\nu,\tau,I,J} \vdash xCx$,
- $\mathsf{BB}_{\nu,\tau,I,J} \vdash yCx$.

Finally, using (4), we obtain

- $\mathsf{BB}_{\nu,\tau,I,J} \vdash P(x) \wedge x = y$.

So $\mathsf{BB}_{\nu,\tau,I,J}$ is the theory that proves that there is precisely one object x, and that for that x we have $P(x)$ and xCx.

We see that our bi-interpretability bases are indeed bi-interpretable, but yet they are not so useful. Our translation only makes sense for the very limited case that there is precisely one element. Therefore, we could only use it to translate theories that prove that there is precisely one element. If we want to translate richer theories, we have to use translations that respect the meaning of the relevant languages and theories. ▲

5. From points to lines: an application in geometry

There are several different axiomatizations of geometry. One of them, which we will consider in this section, is by Alfred Tarski and is presented clearly in [AG99].[3] This set of axioms, called *Tarski's Axioms*, has *points* as its primitive objects. We will consider a language of geometry with *lines* as primitive objects and will form translations between the languages, such that the tools of Section 3 can be used to translate *Tarski's Axioms* into this language. Unfortunately we do not have space to actually carry out this translation. However, the loss is not so great since it would be a purely computational matter, as will be discussed in Section 6.

In the literature, axiomatizations of geometry appear mostly in terms of points. However, we are not the first to discuss an axiomatization in terms of lines. In, for instance, [SS75] it shown that is possible to formulate *Tarski's Axioms* in a language with lines as primitive objects.[4] We go a step further and not only show that it is possible, but also take the first steps in constructing such an axiomatization.

Tarski's Axioms are formulated in the language Σ_{GP} with a three-place relation symbol B, for *betweenness* and a four-place relation symbol $\equiv$, for *congruence*, which are interpreted as follows.

$\mathsf{B}(x, y, z)$ whenever y is between x and z.

$(x, y) \equiv (z, w)$ whenever the lengths of the line segments xy and zw are equal.

We aim to create translations back and forth to the language Σ_{GL}, which has *lines* as its primitive objects. If the translations make sense with respect to geometry, we trust that there are isomorphisms such that *Tarski's Axioms* prove the

[3] To be complete: *Tarski's Axioms* do not axiomatize full geometry, but merely its elementary - i.e. first-order - part.

[4] This article was pointed out to me by Victor Pambuccian. It also includes a translation from the language (not the theory) of geometry of points into a language of geometry in terms of lines. Coincidentally our translations of betweenness are very similar.

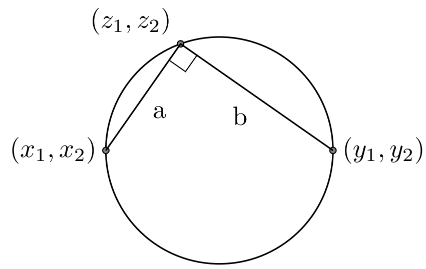

Figure 1. Thales' Theorem.

corresponding bi-interpretability basis. We will construct the language Σ_{GL} on the go, adding relation symbols whenever they are needed for our translation.

First we form a translation $\tau : \Sigma_{\mathsf{GP}} \to \Sigma_{\mathsf{GL}}$. Since every pair of intersecting lines corresponds to exactly one point, the domain of τ consists of precisely those lines. In order to be able to define a formula that specifies this domain, we add the following relation symbol to Σ_{GL}.

$$\mathsf{S}(x,y,z) \Leftrightarrow \begin{cases} x,y,z \text{ are distinct} & \text{and intersect in one point or,} \\ x = y \text{ and } x \neq z & \text{and } x,z \text{ intersect} \quad \text{or,} \\ y = z \text{ and } y \neq x & \text{and } y,x \text{ intersect.} \end{cases}$$

One might wonder why we chose to make S a three-place relation instead of just two-place. The reason is that the extra information that it gives us will come in handy when translating the relation symbols.

We are now ready to define the domain of τ:

$$\delta_\tau(x,y) :\leftrightarrow \mathsf{S}(x,y,y).$$

This new relation symbol can also be used for translating the equality relation:

$$(x,y) =_\tau (z,w) :\leftrightarrow (\delta_\tau(x,y) \wedge \delta_\tau(z,w) \wedge \mathsf{S}(x,y,z) \wedge \mathsf{S}(x,y,w)).$$

That is, the points represented by two pairs of lines are equal, precisely when the lines intersect at the same point.

Translating the other relation symbols of Σ_{GP} can be a bit tricky and requires some geometry, as well as a new relation symbol. In Σ_{GP}, the two relation symbols can be seen as capturing two distinct aspects of geometry. Specifically, betweenness captures the affine, and congruence the metric aspect. In Σ_{GL} there is not such a nice division of roles for the primitive relations. We will see that we need the relation S in both the translations of betweeness and of congruence. However, the relation S is not sufficient to capture the full language of geometry. It turns out that we can complete the language Σ_{GL} by adding the following primitive relation:

$$\mathsf{P}(x,y) \text{ whenever } x \text{ and } y \text{ are perpendicular.}$$

The reason this relation is sufficient is that it enables us to define which points lie on a circle with a certain diameter. By using Thales' theorem, we derive that given two points, defined by two pairs of lines (x_1,x_2) and (y_1,y_2), a point (z_1,z_2) lies on the circle with diameter $(x_1,x_2)(y_1,y_2)$ precisely when there are perpendicular lines a and b through (x_1,x_2) and (y_1,y_2) respectively, such that (z_1,z_2) lies on

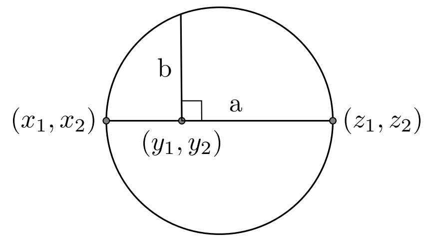

FIGURE 2. Betweenness in terms of lines.

the intersection of a and b. This idea is shown in Figure 1 and captured by the following abbreviation:

$$\mathsf{circle}((x_1,x_2),(y_1,y_2),(z_1,z_2)) :\leftrightarrow \exists a\, \exists b\, (\mathsf{S}(a,x_1,x_2) \wedge \mathsf{S}(a,z_1,z_2) \\ \wedge \mathsf{S}(b,y_1,y_2) \wedge \mathsf{S}(b,z_1,z_2) \wedge \mathsf{P}(a,b)),$$

which states that the point (z_1, z_2) lies on the circle with diameter $(x_1, x_2)(y_1, y_2)$.

We can use this abbreviation to translate betweenness. To see this, notice that whenever three points x, y, z lie on the same line, y is between x and z precisely when it lies within the circle with diameter xz. Precisely then, we can draw a line through that circle and through the point y, which is perpendicular to the line through x, y and z. As before, we illustrate this idea in Figure 2 and formalize it in the following translation:

$$\mathsf{B}_\tau((x_1,x_2),(y_1,y_2),(z_1,z_2)) :\leftrightarrow (\delta_\tau(x_1,x_2) \wedge \delta_\tau(y_1,y_2) \wedge \delta_\tau(z_1,z_2) \\ \wedge \exists a\, \exists b(\mathsf{S}(a,x_1,x_2) \wedge \mathsf{S}(a,y_1,y_2) \wedge \mathsf{S}(a,z_1,z_2) \\ \wedge \mathsf{S}(b,y_1,y_2) \wedge \exists c(\mathsf{circle}((x_1,x_2),(z_1,z_2),(b,c))) \\ \wedge \mathsf{P}(a,b))).$$

For translating congruence, we will also use circle. We know that any two line segments that generate the same circle must be of the same length. When checking if two lines are congruent, we can use this fact to reduce the problem to the case where the lines are parallel. Then it is just a matter of connecting them and seeing if they form a parallelogram, as pictured in Figure 3. Formally,

$$((x_1,x_2),(y_1,y_2)) \equiv_\tau ((z_1,z_2),(w_1,w_2)) :\leftrightarrow (\delta_\tau(x_1,x_2) \wedge \delta_\tau(x_1,x_2) \wedge \delta_\tau(y_1,y_2) \\ \wedge \delta_\tau(z_1,z_2) \wedge \delta_\tau(w_1,w_2) \leftrightarrow \exists a_1 \exists a_2 \exists b_1 \exists b_2(\\ \forall u_1 \forall u_2(\mathsf{circle}((a_1,a_2),(b_1,b_2),(u_1,u_2)) \leftrightarrow \mathsf{circle}((x_1,x_2),(y_1,y_2),(u_1,u_2))) \\ \wedge \exists g \exists h(\mathsf{S}(g,a_1,a_2) \wedge \mathsf{S}(g,b_1,b_2) \wedge \mathsf{S}(h,z_1,z_2) \wedge \mathsf{S}(h,w_1,w_2) \wedge \neg\mathsf{S}(g,h,h) \\ \wedge \exists j \exists k(\mathsf{S}(j,a_1,a_2) \wedge \mathsf{S}(j,z_1,z_2) \wedge \mathsf{S}(j,b_1,b_2) \wedge \mathsf{S}(j,w_1,w_2) \wedge \neg\mathsf{S}(j,k,k))).$$

Now that we have formulated all the primitive relations of Σ_{GL} and completed the translation τ, we can formulate a translation υ in the other direction.

To define our domain, we use the fact that any two non-equal points specify a line:

$$\delta_\upsilon(x,y) :\leftrightarrow x \neq y.$$

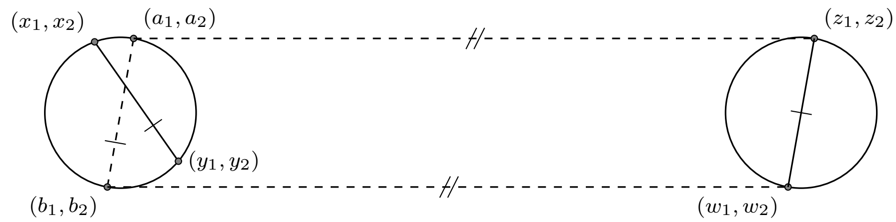

FIGURE 3. Congruence in terms of lines.

Before we go on to translate the relation symbols, we define the following abbreviation to express that three points lie on the same line:

$$\mathsf{line}(x, y, z) :\leftrightarrow (\mathsf{B}(x, y, z) \lor \mathsf{B}(z, x, y) \lor \mathsf{B}(y, z, x)).$$

Since two pairs of points specify the same line precisely when they both lie on that line, we translate the equality relation as follows.

$$(x, y) =_\upsilon (z, w) :\leftrightarrow (\delta_\upsilon(x, y) \land \delta_\upsilon(z, w) \land \mathsf{line}(x, y, z) \land \mathsf{line}(x, y, w)).$$

The relation $\mathsf{S}(x, y, z)$ holds precisely when there is a point through which all three lines go and those lines are not all equal. Formally,

$$\begin{aligned} S_\upsilon((x_1, x_2), (y_1, y_2), (z_1, z_2)) :\leftrightarrow (&\delta_\upsilon(x_1, x_2) \land \delta_\upsilon(y_1, y_2) \land \delta_\upsilon(z_1, z_2) \\ &\land \exists a(\mathsf{line}(a, x_1, x_2) \land \mathsf{line}(a, y_1, y_2) \land \mathsf{line}(a, z_1, z_2)) \\ &\land \neg(\mathsf{line}(x_1, y_1, y_2) \land \mathsf{line}(x_2, y_1, y_2) \land \mathsf{line}(x_1, z_1, z_2) \\ &\qquad \land \mathsf{line}(x_2, z_1, z_2))). \end{aligned}$$

We complete our translations with perpendicularity. We use the fact that two lines are perpendicular, precisely when their intersection is the midpoint of the bottom of an isosceles triangle with endpoints on those two lines. This fact is illustrated in Figure 4. For two lines x, y, the following translation first checks if there is an intersection a, and then states that it must hold that any triangle with two endpoints on x, the segment of which has a as midpoint, and one endpoint on y, is an isosceles triangle. Precisely,

$$\begin{aligned} P_\upsilon((x_1, x_2), (y_1, y_2)) :\leftrightarrow (&\delta_\upsilon(x_1, x_2) \land \delta_\upsilon(y_1, y_2) \land \exists a(\mathsf{line}(a, x_1, x_2) \\ &\land \mathsf{line}(a, y_1, y_2) \land \forall b \forall c \forall d(\mathsf{line}(b, y_1, y_2) \land \mathsf{line}(c, x_1, x_2) \\ &\land \mathsf{line}(d, x_1, x_2) \land \mathsf{B}(c, a, d) \land ac \equiv ad) \to bc \equiv bd))). \end{aligned}$$

Again, we see what we have already seen in Section 4: translations can blow up into complicated, long formulas. Even though the above translations are certainly not the most efficient, calculating the composite translations and the bi-interpretability bases for *Tarski's Axioms* will always be an extremely tedious computational matter. In the next and final section we will discuss, among other things, the opportunities for automating this process.

What we *can* already do is construct formulas that will serve as our isomorphisms. For constructing the Σ_{GP}-formula F, we have to find precisely when two pairs of points $((x_1, x_2), (y_1, y_2))$ correspond to another point a. Firstly, the pairs of points should each specify a line. Secondly, those lines should specify a point a.

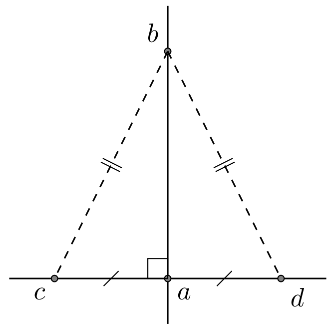

FIGURE 4. Perpendicularity in terms of points

We get:

$$((x_1, x_2), (y_1, y_2))Fa :\leftrightarrow (x_1 \neq x_2 \wedge y_1 \neq y_2 \\ \wedge \forall u(\mathsf{line}(x_1, x_2, u) \wedge \mathsf{line}(y_1, y_2, u)) \leftrightarrow u = a)).$$

In Σ_{GL}, we construct our isomorphism candidate G as follows. Of course the lines within each pair should intersect, so that they represent points. Those points should be non-equal, so that they specify a line a. Formally:

$$((x_1, x_2), (y_1, y_2))Ga :\leftrightarrow (\mathsf{S}(x_1, x_1, x_2) \wedge \mathsf{S}(y_1, y_1, y_2) \wedge \mathsf{S}(x_1, x_2, a) \\ \wedge \mathsf{S}(y_1, y_2, a) \wedge \neg(\mathsf{S}(x_1, x_2, y_1) \wedge \mathsf{S}(x_1, x_2, y_2)).$$

6. Conclusion

In this section we will discuss what we have found and what more there is to explore.

6.1. **What we have found.**

The most substantial results of this thesis are the definitions and proofs of Section 3. With these, we have shown that it is possible to give an axiomatization of the properties that a theory should have to be bi-interpretable with a theory in a different language. We called such axiomatizations *bi-interpretability bases*. An interesting thing is that a bi-interpretability basis can be formulated purely on the 'language side', independent of the theory that it is about. This is highlighted by the fact that bi-interpretability bases are generated by *translations*, and not by *interpretations*.

We have seen in Section 4, however, that for a bi-interpretability basis to be useful in translating a certain theory, the translations have to be such that they respect the meaning of that theory. This means that translating theories is not a trivial matter and one has to think carefully about how the translations can be formulated such that they make sense with respect to some theory.

Once the translations *do* make sense with respect to a certain theory, the real power of bi-interpretability bases shows. For, in that case, the theory proves the bi-interpretability basis and we can use Theorem 3.3 to translate the theory into a different language. We have done this in the simple case of Example 4.1 and, in Section 5, made the first steps to do this in the more extensive case of geometry.

6.2. **Further research.** An obvious next step is to complete the translation of geometry in terms of *points* into geometry in terms of *lines* that we started in Section 5. As we had already seen in Section 4, even in simple cases there is a lot of calculation involved in translating theories. We have to compute the composite translations as well as the bi-interpretability bases and the translations of the axioms of the original theory.

One interesting prospect for future work is doing these calculations computationally. With for instance the proof assistant Coq it is possible to derive equivalences in predicate logic automatically.

We imagine the following interaction between a researcher that aims to translate a theory and a computer program capable of calculation in predicate logic.

(1) The researcher comes up with translations back and forth between the original and the target language, such that they respect the theory that is to be translated. Furthermore, the researcher provides formulas the should be isomorphims between the respective composite and identity interpretations.[5]
(2) The computer calculates the bi-interpretability bases and verifies if the relevant one gets proven by the original theory. If it does, go to step (3). If it does not, go back to step (1) to try again.
(3) The computer calculates the translated theory using Theorem 3.3. In interaction with the researcher the computer proposes syntactically different, but semantically equivalent axiomatizations for the new theory. Once an axiomatization has been proposed with which the researcher is satisfied the process is done.

In fact, such a tool is the only thing we need to complete our translation of geometry. An alternative would be to try to simplify the translations such that the calculations could be easier done by a human. That seems to be the lesser option, since it would still be a lot of work to simplify the translations and even with simple translations quite some calculation is necessary.

Once such a tool is developed it can be used to translate all kinds of theories into languages with different primitive objects. It is interesting to see how such a small proof as the one in Section 3 can have such extensive implications.

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[5]One may doubt if it is always feasible to come up with formulas that can serve as isomorphisms without first calculating the composite translations. An alternative would be to let the computer first calculate the compositions, after which the researcher forms formulas to serve as isomorphisms.

Appendix A. Equality Axioms

For our definition of bi-interpretability basis, we need to have a finite axiomatization of equality. This is given in the following definition.

Definition A.1. Let Σ be a signature and $=$ a 2-place relation symbol of Σ. The *equality axioms of* $=$, denoted by $\mathcal{I}_=$, are given by:

$\mathcal{I}_= \vdash \forall x(x = x)$,

together with

$\mathcal{I}_= \vdash x_0 = x_1 \to (R(x_1, \ldots, x_n) \to R(x_0, \ldots, x_n))$,

$\ldots$

$\mathcal{I}_= \vdash x_0 = x_n \to (R(x_1, \ldots, x_n) \to R(x_1, \ldots, x_0))$,

for every n-place relation symbol R. ∎