Macroscopic and Microscopic Description of 5-dimensional and 4-dimensional Black Holes in String Theory

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June 30, 2015

Master's thesis
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Abstract

We study how to describe 5-dimensional and 4-dimensional black holes by D-branes in string theory. In the 5-dimensional case we arrive at the Tangherlini black hole, while we find the extremal Reissner-Nordström black hole in four dimensions. For both black holes we compute the entropy both macroscopically and microscopically and we check that they agree. The results above are known in literature. Here we work out some examples in detail. Moreover, in this thesis we have computed the black hole charges. In the 4-dimensional case we try to set the magnetic charges to zero by using electromagnetic duality in order to obtain a simpler black hole description. We conclude that this is not possible since two new magnetic charges arise. We interpret how this electromagnetic duality transformation in $d = 4$ influences the D-branes in $d = 10$. We find that the set of D-branes has been changed to another set of D-branes, which describes the same black hole metric.

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1 Introduction

General relativity is an effective field theory that can only be trusted up to some cutoff scale: the Planck scale, $\ell_p \sim 10^{-35}m$. How to combine gravity with quantum mechanics on such small scales is still an open problem. Singularity theorems indicate that the spacetime curvature is too high to be properly described by general relativity in two cases: at the big bang and in the center of black holes.

A promising candidate to describe this theory of quantum gravity and thereby unify all fundamental forces in physics, is string theory. In string theory, matter is not described by point particles but rather by tiny loops of strings.

If we do not consider the big bang but restrict ourselves to black holes, there are two famous problems that a true theory of quantum gravity should be able to solve. Firstly, Hawking derived semiclassically that black holes emit particles under the influence of quantum mechanics [1]. We see this as pure thermal radiation that contains no information about the current state of a black hole; let alone its history. This implies that when something has fallen into a black hole, we can only observe a mass gain. We can never be sure, though, what kind of object or particle has fallen in since its information is lost. We call this the information paradox (see [2] and [3] and references therein).

Secondly, Bekenstein and Hawking derived that the entropy of a black holes is proportional to the area of the event horizon [1] [4],

$$S_{BH} = \frac{A}{4G_N}.$$

From statistical mechanics we are used to an entropy that is built up by counting the number of possible microstates, using the Boltzmann law

$$S_{stat} = k_B \log[\Omega].$$

The nonzero Bekenstein-Hawking entropy above suggests that a black hole also has certain microstates and in that case we should be able to count them microscopically.

Thus a true theory of quantum gravity should be able to solve the black hole information paradox and explain the entropy microscopically. We review the latter in this thesis. This counting of black hole microstates has been done by Strominger and Vafa in 1996 for extremal 5-dimensional black holes [5] and subsequently many others have tried to compute the microscopic entropy of other black holes and to check that this agrees with the macroscopic Bekenstein-Hawking result. For instance, this has been done in $d = 5$ for near-extremal charged black holes [6] [7], for charged rotating black holes [8] and in $d = 4$ for extremal charged black holes [9]. For black holes that are not charged or that are far from extremal the computation of the microscopic entropy is still an open problem. Moreover, it is not even known how to describe the Schwarzschild black hole in string theory as all objects we have at hand in string theory carry charges.

In this thesis, we study how to describe 5-dimensional and 4-dimensional extremal charged black holes in string theory. Namely, we will use certain combinations of D-branes that give exactly the required black hole metric after compactification from ten to five or four dimensions. Then we check that the macroscopic and microscopic entropy of the resulting black holes agree up to leading order. These main results are known in literature. Here we work out some examples in more detail.

In addition, our aim is to compute the electric and magnetic charges of the black hole and act with an electromagnetic duality transformation on them. In this way, we try to set the magnetic charge to zero in order to obtain a simpler black hole description. Moreover, we consider how the microscopic perspective of 10-dimensional D-branes changes under this transformation. We will see that the set of D-branes we started with has been changed to another set of D-branes, which describes the same black hole but carries different charges.

We will derive all results in Chapter 6. In the preceding chapters we develop the necessary framework of black holes in general relativity, string theory, D-branes and compactification.

In Section 1.1 we give an outline in more detail and in Section 1.2 we comment on our conventions.

1.1 Outline

We start with refreshing the basics of black holes from general relativity in Chapter 2. We consider the Schwarzschild and the Reissner-Nordström black hole, of which the latter can have electric and magnetic charge. We introduce Hodge duality and electromagnetic duality, which we use to rotate these electric and magnetic charges into each other later on in Chapter 6.

However, first our goal is to create a 4-dimensional black hole from 10-dimensional string theory. Therefore, we briefly discuss the very basics of string theory in Chapter 3. From all five superstring theories that exist, we restrict ourselves to type IIA and type IIB theory in this thesis. We will see that type II theories split into four sectors, of which we are mostly interested in the so called NS-NS sector and R-R sector. Moreover, we especially focus on D-branes or Dp-branes, which are p-dimensional membranes on which strings can end.

In Chapter 4 we look into D-branes in more detail. We see how their metric looks like and that they are electrically or magnetically charged with respect to potentials in the R-R sector.

After this introduction to black holes, string theory and D-branes, we are ready to go from 10-dimensional string theory to the 4-dimensional reality of general relativity. This can be done by the process of dimensional reduction or compactification which is explained in Chapter 5.

In Chapter 6 all previous chapters come together when we perform compactification and derive all major results. We start with a certain configuration of multiple D-branes in type II superstring theory. First we go from $D = 10$ to $d = 5$ to find a 5-dimensional black hole in Section 6.3. We also derive the entropy of the resulting black hole both macroscopically and microscopically and check whether these two agree. The result in $d = 5$ is

$$S = 2\pi\sqrt{N_1 N_5 N_W},$$

where N_1 and N_5 are the number of D1 and D5 branes and N_W is an integer. Moreover, we compute the electric and magnetic charges of the black hole.

We repeat the compactification and all calculations of the entropy and charges in $d = 4$ in Section 6.4, where we start from another set of D-branes that will give exactly the extremal Reissner-Nordström black hole. In $d = 4$ we find this black hole

$$S = 2\pi\sqrt{N_2 N_{NS5} N_6 N_W},$$

where N_2, N_{NS5}, N_6 are the number of D2, NS5 and D6 branes. As a final result we try to set the magnetic charges of the 4-dimensional black hole to zero in Section 6.5 by using electromagnetic duality.

We draw our conclusions and we give an outlook on further research in Chapter 7.

1.2 Conventions

In this thesis we use the following conventions.

We use the spacetime signature $\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1)$ and we adapt natural units, $\hbar = c = k_B = 1$. We do keep the Newton constant G_N explicitly in all equations, though.

When we go to spherical coordinates then θ is the polar angle running from 0 to π and ϕ_i are one or more azimuthal angles (in $[0, 2\pi]$).

We use $\tilde{\epsilon}_{\mu_1 \ldots \mu_n}$ for the curved space epsilon tensor and $\epsilon_{\mu_1 \ldots \mu_n}$ for the flat space epsilon tensor, where we define $\epsilon_{12\ldots n} = +1$. Note that the latter convention is opposite to what is used in [10].

In Table 1.2 we give an overview of other relevant conventions and how these differ between this thesis (in the column on the right) and four sets of lecture notes or books that have been the main sources to write this thesis.

	Johnson [11]	Maldacena [12]	Mohaupt [13]	Tong [14]	**This thesis**
spacetime metric (Einstein frame)	$\tilde{G}_{\mu\nu}$	$\tilde{g}_E$	$g_{\mu\nu}$	$\tilde{G}_{\mu\nu}$	$g_{\mu\nu}$
spacetime metric (string frame)	$g_{\mu\nu}$	$G_{\mu\nu}$	$G_{\mu\nu}$	$G_{\mu\nu}$	$G_{\mu\nu}$
worldsheet metric	h_{ab}	-	-	$g_{\alpha\beta}$	h_{ab}
auxiliary metric	γ_{ab}	-	-	$\gamma_{\alpha\beta}$	γ_{ab}
(total) dilaton	Φ	ϕ	ϕ	Φ	ϕ
dilaton (constant/varying part)	Φ_0 / $\Phi - \Phi_0$	ϕ_∞ / $\phi - \phi_\infty$	$\langle\phi\rangle$ / $\phi - \langle\phi\rangle$	Φ_0 / $\tilde{\Phi}$	$\langle\phi\rangle$ / $\tilde{\phi}$
n-form R-R gauge fields	$C_{(n)}$	A	A_n	C_n	A_n
string coupling constant	g_s	g	g_S	g_s	g_s
normalization of charges (6.36)	-	$1/(4\pi)$	$1/(4\pi)$	1	$1/(4\pi)$

2 Black Holes

In this chapter we recall some basic knowledge about black holes from general relativity. In Section 2.1 we consider how a black hole can be formed in nature. In sections 2.2, 2.3 and 2.4 we consider the Schwarzschild, Reissner-Nordström and Tangherlini black holes. In case of the Reissner-Nordström black hole in Section 2.3 we discuss the extremal limit. In the rest of this thesis we are mostly interested in the extremal Reissner-Nordström black hole. Moreover, we introduce Hodge duality and electromagnetic duality, which we will use in Chapter 6 to try to get rid of all magnetic charge. In addition, we look at the entropy of black holes in Section 2.5.

2.1 Formation of Black Holes

When you are at a birthday party or in a bar talking about the rather mysterious subject of black holes, often the first question that arises is: *"Do black holes really exist?"* (yes, they do). After the first shock, listeners usually ask you how black holes are created. To be prepared for this situation, let us discuss this more practical question before we study the theory of black holes in more detail.

A star at the end of its lifetime evolves to a black hole under certain conditions. Namely, if the mass of the star is large enough then it will implode to form a black hole. In the following, we discuss briefly why this star will collapse and what the mass limit is for this process. In this discussion we follow [15] and references therein.

During the active period of a star protons in the core are fused to a bound state. This bound state has lower energy than the unbound state of separate protons. Therefore, some energy must be released and this happens by highly energetic photons. The photons are emitted constantly and perform an outward pressure: the radiation pressure. This outward force counteracts the inward gravitational force that the star performs on itself, i.e. the (heavy) core performs a gravitational force on the outer shells.

As long as a star is still active, radiation pressure is generated preventing it from collapse to a black hole. However, when a star runs out of fuel the fusion processes decrease and not enough radiation pressure is generated to withstand the gravitational pull. Therefore, the star will collapse to a smaller object. Multiple objects or remnants are possible; the outcome depends on the mass of the original or progenitor star (PS), i.e. the star before it was at the end of its lifetime.

We have made a simplified flowchart in Figure 2.1 that indicates whether a star will collapse to a black hole or not. Depending on the mass of the PS it will collapse to either a white dwarf or a neutron star. A white dwarf remnant will cool down slowly since the fusion process has stopped.

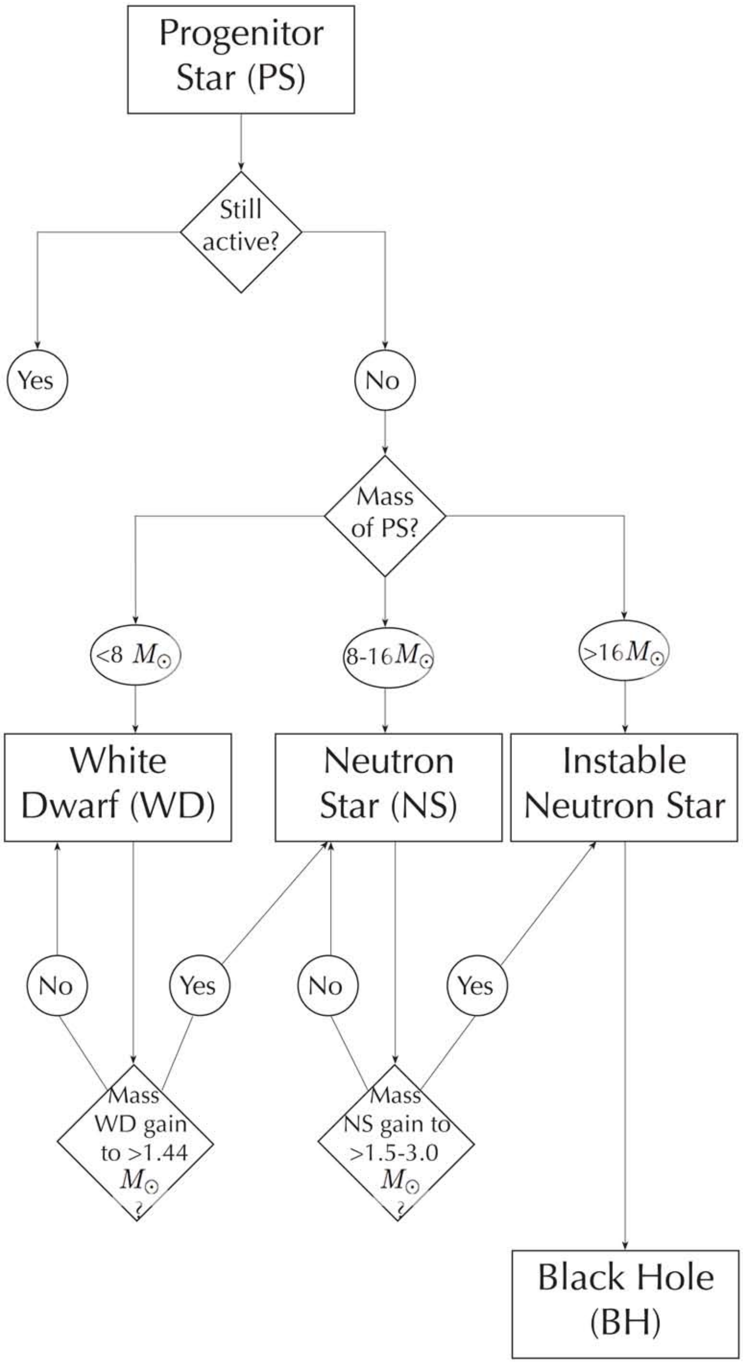

Figure 2.1: This flowchart indicates whether a star will collapse to a black hole or not at the end of its lifetime. The term progenitor star (PS) is used to refer to the original star, i.e. before the star reached the end of its lifetime and $1M_\odot$ means one solar mass. The mass limit in the left down diamond is called the Chandrasekhar limit and the limit in the right down diamond is the Tolman-Oppenheimer-Volkoff (TOV) limit.

Instead of radiation pressure a white dwarf has an outward force coming from electron degeneracy at the core by the Pauli exclusion principle. On the other hand, if the PS has more than eight times the solar mass $M_{\odot}$, then the protons in the core fuse with electrons and become neutrons. A huge amount of gravitational potential energy is released and blows the star's outer layers away. We see this explosion as a supernova. A static neutron star remains, which has an outward force now exerted by neutron degeneracy (neutrons are fermions). If the mass of the PS was even larger than $16M_{\odot}$ then the resulting neutron star is unstable: the gravitational force is still larger than the neutron degeneracy force. Another collapse follows immediately: to a black hole.

In addition, if a white dwarf gains mass to more than $1.44M_{\odot}$ it will collapse to a stable neutron star. Mass gain may happen if comets beat in or if two stars collide. This limit has been derived by [16]. Moreover, if the mass of a stable neutron star exceeds $1.5 - 3.0M_{\odot}$ at some point it becomes unstable and collapses to a black hole. We call this mass limit the Tolman-Oppenheimer-Volkoff (TOV) limit [17] [18] [19]. More information on stellar evolution can be found in [20].

2.2 The Schwarzschild Metric

In this and the following sections we briefly recall some black hole metrics from Einstein's general relativity. General relativity sets space and time on an equal footing and calls such a combination a spacetime, which is described by a metric. Gravitational forces are seen as inherent characteristics of a spacetime, while all other forces are seen as external forces. Once matter is present, it transforms a spacetime such that it is not flat anymore (as in special relativity) but becomes curved. Nearly everything you want to know about general relativity can be found in [10].

The simplest description of a black hole is the Schwarzschild black hole. This is the unique spherical solution of the Einstein-Hilbert action in vacuum [10],

$$S_{EH} = \int d^D x \sqrt{-g}\, R, \tag{2.1}$$

where $R = R(g)$ is the Ricci scalar, and the Schwarzschild metric is given by

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2,$$

where the metric on a unit two-sphere is

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

Asymptotically far away from the Schwarzschild black hole, i.e. in the limit that $r \to \infty$, we recover the Minkowski metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$$

and therefore we call this metric asymptotically flat. The horizon is located when $g^{rr} = 0$, i.e. at a distance $r = 2G_N M$ from the center of the black hole.

2.3 The Reissner-Nordström Metric

More generally, a black hole can have a certain electric and magnetic charge. Such a charged black hole is described by the Reissner-Nordström metric, which is a solution of the Einstein-Hilbert action plus a Maxwell term $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and is given by

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega_2^2, \tag{2.2}$$

where we have defined

$$\Delta = 1 - \frac{2G_N M}{r} + \frac{G_N(Q^2 + P^2)}{r^2}$$

with Q the electric charge and P the magnetic charge of the black hole. In addition to the metric, the Reissner-Nordström black hole has a gauge field $F_{\mu\nu}$ with nonzero components given by

$$\begin{cases} F_{rt} = -\dfrac{Q}{r^2} \\ F_{\theta\phi} = P\sin(\theta). \end{cases}$$

Note that the metric above is consistent since we obtain the Schwarzschild solution if we set both charges Q and P to zero. Often a new charge $q := \sqrt{Q^2 + P^2}$ is defined or the magnetic charge P is omitted to simplify the expressions. In this thesis, though, we are interested in both electric and magnetic charge and for this purpose we keep Q and P explicitly.

2.3.1 Extremal Black Holes

An interesting property of the Reissner-Nordström metric is that it has either zero, one or two event horizons - depending on the mass and charges. Namely, solving $g^{rr}(r) = \Delta(r) = 0$, we find that the event horizons are located at

$$r_\pm = G_N M \pm \sqrt{G_N^2 M^2 - G_N(Q^2 + P^2)}.$$

We distinguish three cases:

1. If $G_N M^2 < Q^2 + P^2$: no event horizon.
   In this case the expression $g^{rr}(r) = \Delta(r)$ has no real roots as $\Delta(r)$ is always positive. Therefore, there is no event horizon at all and we say that we have a naked black hole or a naked singularity. However, as stated in [10], *"The nakedness of the singularity offends our sense of decency, as well as the cosmic censorship conjecture."* The cosmic censorship conjecture states that naked black holes are unphysical and will not occur in nature. Heuristically, we can see this from the condition $G_N M^2 < Q^2 + P^2$ which implies that the energy of charged particles is larger than the total energy such that at some point in time the mass of the black hole should have been negative in order to create the charge carrying particles.
2. If $G_N M^2 > Q^2 + P^2$: two event horizons.
3. If $G_N M^2 = Q^2 + P^2$: one event horizon.
   In this case the total energy is exactly equal to the energy in the electromagnetic field and we say that we are dealing with the extreme Reissner-Nordström solution or with an extremal black hole. Note that all equations will be much simpler than in the other cases as the square root drops out.

Extremal black holes can be described within the framework of string theory, where we call them BPS states. However, it is still an open problem how to describe non-extremal ones. So far, it is only possible to describe black holes that are near-extremal, i.e. by making a perturbation around the extremal case. In this way we use extremal black holes as a theoretical tool since they are not realistic to occur in our universe. Namely, they cannot be formed by a gravitational collapse and they appear to be unstable. That is, if one adds some matter then the black hole goes to the first case and forms a naked black hole (which is not allowed). Moreover, in the extremal case it is remarkable that the Hawking temperature of the black hole becomes zero and therefore it cannot radiate, violating the second law of thermodynamics [21].

Despite all these drawbacks, we will focus on extremal black holes in this thesis as these are the only black holes that we can describe really well in string theory. From now on this focus will be implicit, i.e. if we say 'black hole' we mean 'extremal black hole'.

Thus for a black hole, i.e. an extremal black hole, we can simplify the metric (2.2) by substituting $G_N M^2 = Q^2 + P^2$ and recognizing a square,

$$ds^2 = -\left(1 - \frac{G_N M}{r}\right)^2 dt^2 + \left(1 - \frac{G_N M}{r}\right)^{-2} dr^2 + r^2 d\Omega_2^2.$$

Let us now replace r by the isotropic coordinate

$$\rho = r - G_N M$$

such that

$$ds^2 = -\left(\frac{\rho}{\rho + G_N M}\right)^2 dt^2 + \left(\frac{\rho}{\rho + G_N M}\right)^{-2} dr^2 + r^2 d\Omega_2^2. \qquad (2.3)$$

Note that the event horizon is now located at $\rho = 0$. We can rewrite (2.3) in the nice form

$$ds^2 = -H^{-2}(\rho)dt^2 + H^2(\rho)\left[d\rho^2 + \rho^2 d{\Omega_2}^2\right], \qquad (2.4)$$

where we defined

$$H(\rho) = 1 + \frac{G_N M}{\rho},$$

which is a harmonic function with respect to the spacelike coordinates, i.e. the Laplace operator acting on such a function gives zero,

$$\nabla^2 H(t, x^1, x^2, x^3) = \left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_3}^2}\right) H(t, x^1, x^2, x^3) = 0. \quad (2.5)$$

Throughout the following we use the isotropic metric in terms of harmonic functions (2.4). This is useful since we will see in Chapter 4 that a metric of D-branes can also be written in terms of harmonic functions. Afterwards, in Chapter 6 we look for a set of D-branes for which the metric is exactly equal to the Reissner-Nordström metric. Then we would have a black hole described by D-branes.

Usually we replace ρ by r in (2.4) again such that the event horizon is located at $r = 0$.

2.3.2 Hodge Duality and Electromagnetic Duality

We are interested in the electric and magnetic charges of an (extremal Reissner-Nordström) black hole. When we are looking at a specific black hole we may ask: *Is there any magnetic charge?* Or: *Is it possible to set this magnetic charge to zero in order to obtain a simpler black hole description?* This might be achieved by using electromagnetic duality: a transformation that changes magnetic charge into electric charge and vice versa. Before we move on to this interesting transformation, some knowledge of Hodge duality is needed.

Hodge Duality

Hodge duality is an operation on differential forms on an n-dimensional manifold, for instance on a 4-dimensional spacetime. This operation is defined by the Hodge star operator $*$ which sends a p-form on an n-dimensional manifold to an $(n-p)$-form on the same manifold. The resulting $(n-p)$-form is called 'the dual' of the original p-form. The Hodge star $*$ on an n-dimensional manifold is defined by

$$({}^*A)_{\mu_1\cdots\mu_{n-p}} = \frac{1}{p!}\tilde{\epsilon}^{\nu_1\cdots\nu_p}{}_{\mu_1\cdots\mu_{n-p}}A_{\nu_1\cdots\nu_p}, \tag{2.6}$$

where A is a p-form and $\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n}$ is the curved space Levi-Civita or epsilon tensor

$$\tilde{\epsilon}_{\mu_1\mu_2\cdots\mu_n} = \begin{cases} \sqrt{-g} & \text{if } \mu_1\mu_2\cdots\mu_n \text{ even permutation of } 12\ldots n. \\ -\sqrt{-g} & \text{if } \mu_1\mu_2\cdots\mu_n \text{ odd permutation of } 12\ldots n. \end{cases}$$

Recall that we can write the p-form A and the $(n-p)$-form ${}^*A$ called 'A dual' in the language of differential forms as

$$A = A_p = A_{\nu_1\cdots\nu_p}dx^{\nu_1}\wedge dx^{\nu_2}\wedge\cdots\wedge dx^{\nu_p}$$

and

$${}^*A = ({}^*A)_{n-p} = ({}^*A)_{\mu_1\cdots\mu_{n-p}}dx^{\mu_1}\wedge dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_{n-p}},$$

respectively. Throughout this thesis we use the notation $A = A_p$ to denote a p-form.

For example, a 2-form on a 5-dimensional spacetime is sent to a 3-form with some factor in front. Note that in Minkowski spacetime the epsilon tensor is equal to ± 1 and then the factor is $\pm\frac{1}{p!} = \pm\frac{1}{2!}$. On the other hand, the same 2-form on a 4-dimensional spacetime is sent to another 2-form. This special property is used to define electromagnetic duality, which is only valid in four dimensions.

As we will compute explicitly in (6.39) in Chapter 6, taking the dual twice gives a factor of minus one in Lorentzian spacetimes, ${}^{**} = -1$, while in Euclidean spacetimes ${}^{**} = +1$.

Electromagnetic Duality

Let us now focus on electromagnetic duality which rotates electric charge into magnetic charge and vice versa. In this section we follow section 2.3 of [13]. In general electromagnetic duality is defined as a transformation on a 2-form field strength and its dual, given by

$$\begin{pmatrix} F_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} F_{\mu\nu} \\ {}^*F_{\mu\nu} \end{pmatrix}, \tag{2.7}$$

where the transformation matrix is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})$$

with $GL(2,\mathbb{R})$ the group of invertible (2x2)-matrices with real coefficients, i.e. with non-zero determinant.

However, for a Lorentzian spacetime we have to take a particular subgroup of $GL(2,\mathbb{R})$. We claim that this should be the symplectic group $Sp(2,\mathbb{R}) \subset GL(2,\mathbb{R})$ that is defined by $S \in Sp(2,\mathbb{R})$ if

$$S^T \Omega S = \Omega,$$

where the matrix Ω is given by

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This can be seen from the following. Suppose we want to interchange $F_{\mu\nu}$ with its dual and we use electromagnetic duality (2.7) with a matrix in $GL(2,\mathbb{R})$. Writing $G_{\mu\nu} = {}^*F_{\mu\nu}$, this gives

$$\begin{pmatrix} \tilde{F} \\ \tilde{G} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} G \\ F \end{pmatrix},$$

where we dropped the indices. There is a problem, though, with this equation. It follows that

$$\begin{cases} {}^*(\tilde{F}) = & {}^{**}F \\ \widetilde{({}^*F)} = & F \end{cases}$$

and we certainly want that ${}^*(\tilde{F}) = \widetilde{({}^*F)}$ for consistency. However, we know that ${}^{**} = -1$ in our Lorentzian spacetime, which implies that $F = -F$. We can solve this problem by replacing the matrix above by a matrix in $Sp(2,\mathbb{R})$. In this case we use:

$$\begin{pmatrix} F \\ {}^*F \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F \\ {}^*F \end{pmatrix} = \begin{pmatrix} -{}^*F \\ F \end{pmatrix}.$$

Thus if we take the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

as transformation matrix then $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$ completely interchange (up to a minus sign). On the other hand, if we take the identity matrix then clearly $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$ remain unchanged. Of course, any transformation between these two extreme situations is possible.

One might ask whether such a transformation is allowed in the first place. Is the physics afterwards not completely different? We will show in the following that at least Maxwell's equations without source terms are invariant under electromagnetic duality transformations. Namely, the Einstein-Maxwell action without sources is given by

$$S = \int d^4x \, \sqrt{-g} \left(\frac{1}{16\pi G_N} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right).$$

This theory gives rise to the **Maxwell equations** in curved spacetime,

$$\begin{cases} \nabla_\mu F^{\mu\nu} = & 0 \\ \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = & 0, \end{cases}$$

where the first equation displays the equations of motion and the second equation is the Bianchi identity. Using the dual field ${}^*F$ given by

$${}^*F_{\mu\nu} = \frac{1}{2} \tilde{\epsilon}^{\rho\sigma}{}_{\mu\nu} F_{\rho\sigma} = \frac{1}{2} \tilde{\epsilon}_{\mu\nu\rho\sigma} F^{\rho\sigma},$$

we can rewrite this set of equations in a nicer way as

$$\begin{cases} \nabla_\mu F^{\mu\nu} = & 0 \\ \nabla_\mu {}^*F^{\mu\nu} = & 0. \end{cases} \tag{2.8}$$

In this way we see that the two Maxwell equations have exactly the same form for $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$. Therefore, if the Maxwell equations (2.8) are satisfied then any linear combination of $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$ satisfies this equation as well. Moreover, the dual of such a linear combination will also satisfy this equation. Hence a linear transformation such as electromagnetic duality (2.7) leaves the Maxwell equations without source terms (2.8) invariant. Note that such a transformation does change the action itself. Moreover, in general Maxwell's equations with source terms are not invariant. However, in the case where both electric and magnetic charge are present, there can still be an invariance for transformations in a discrete subgroup of $GL(2, \mathbb{R})$.

Now an important question to address is: *What happens to the charges after an electromagnetic duality transformation?* Therefore, it is crucial to define what charge is. Namely, we define magnetic charge P as

$$P = \frac{1}{4\pi} \oint F \tag{2.9}$$

and electric charge Q as

$$Q = \frac{1}{4\pi} \oint {}^*F, \tag{2.10}$$

where the integration surfaces must surround the sources. It follows that electromagnetic duality (2.7) can be rewritten as a transformation of Q, P:

$$\begin{pmatrix} P \\ Q \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}. \tag{2.11}$$

Therefore, interchanging the field strength and its dual is the same as interchanging the magnetic and electric charge. Yet another way to see this is that we effectively interchange the electric field $\vec{E}$ and magnetic field $\vec{B}$.

2.4 The Tangherlini Metric

So far we only considered black holes in 4-dimensional spacetimes. Although the framework of general relativity has been set up in four dimensions, there is nothing that keeps us from constructing black holes living in a higher number of dimensions. In this section we introduce a metric for a 5-dimensional black hole that forms a generalization of the Reissner-Nordström metric: the Tangherlini metric. It turns out that 5-dimensional black holes are easier to describe within string theory than their 4-dimensional brothers. That is why we use the Tangherlini metric in Chapter 6 before we repeat that analysis for 4-dimensional black holes.

Written in terms of harmonic functions, it is given by [13]

$$ds^2 = -H^{-2}dt^2 + H(dr^2 + r^2 d\Omega_3^2), \tag{2.12}$$

where

$$H = H(t, x^1, x^2, x^3, x^4) = 1 + \frac{Q}{r^2},$$

is a harmonic function with respect to the four spacelike coordinates, i.e. the Laplacian acting on H gives zero, analogous to (2.5). Note that in five dimensions we now have r to the power minus two instead of minus one in order to be a harmonic function. As usual, r is given by $r^2 = {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2$ and it is an isotropic coordinate. In addition, the Tangherlini black hole has a gauge field $F_{\mu\nu}$ of which the only nonzero component is given by

$$F_{rt} = -\frac{Q}{r^3}.$$

2.5 Entropy of Black Holes

The entropy of a black hole depends on the area A of its event horizon. This dependence is linear and the factor is set by the famous Bekenstein-Hawking entropy

$$S_{BH} = \frac{1}{G_N}\frac{A}{4}, \tag{2.13}$$

where $G_N = G_N^{(D)}$ is Newton's constant in D spacetime dimensions. This formula is also known as the area law. When we restore the units we can already see that the entropy of a black hole is a very large number. Namely, then the entropy divided by Boltzmann constant, which is dimensionless, is equal to

$$\frac{S_{BH}}{k_B} = \frac{c^3}{4G_N\hbar}A = \frac{(3.0\cdot 10^8)^3}{4(6.7\cdot 10^{-11})\cdot(1.1\cdot 10^{-34})}A \approx 9.2\cdot 10^{68}A. \quad (2.14)$$

We can for instance compute the entropy for a black hole that has one solar mass (if this were possible) and then we have a lower bound for the black hole entropy. The Schwarzschild radius of the sun is

$$r = \frac{2G_NM}{c^2} = \frac{2(6.7\cdot 10^{-11})\cdot(2.0\cdot 10^{30})}{(3.0\cdot 10^8)^2} \approx 3.0\cdot 10^3 m$$

and since $A = 4\pi r^2$ we find that for one solar mass

$$\frac{S_{BH}}{k_B} \approx 1.0\cdot 10^{77}.$$

One of the main goals of this thesis is to compute the entropy of a black hole both macroscopically and microscopically, which we will do in Chapter 6. In the macroscopic derivation we use the Bekenstein-Hawking entropy.

Without actually computing it, we can already see how the entropy scales for Schwarzschild and for extremal Reissner-Nordström black holes. Namely, the entropy is linear in the area which is given by $4\pi r^2$ in $d = 4$. Moreover, the event horizon is located at $r = 2G_NM$ for Schwarzschild and half this distance for extremal Reissner-Nordström. Thus for both black holes the entropy goes like $S \sim M^2$, which we can replace by $S \sim (Q^2 + P^2)$ in the extremal Reissner-Nordström case.

3 String Theory in a Nutshell

In this chapter we briefly recall some subjects from string theory that we need in later chapters, especially the concept of D-branes. Some basic knowledge of string theory (e.g. a one semester course) is assumed as this chapter only aims to recall and state results. Good sources to learn more about string theory are:

- the lecture notes of Tong [14] on string theory, which are very famous among students for the clear explanations,
- the two volumes [22] and [23] 'String theory' by Polchinski, which have a somewhat more mathematical approach and are very complete,
- and in particular 'D-branes' by Johnson [11], which is a very pedagogical book on string theory and D-branes, which also contains a lot of information about black holes.

In Section 3.1 we recall bosonic string theory and the concept of D-branes. We consider superstring theories in Section 3.2, where both fermions and bosons are present. We concentrate on type IIA and IIB theory and we take a look at the various fields (the spectrum) of its R-R and NS-NS sectors. Moreover, we explain the distinction between metrics written in Einstein frame and in string frame in Section 3.2.2. In addition, we introduce S-duality in Section 3.3.

3.1 Bosonic String Theory

As formulated in [14]: "*The premise of string theory is that, at the fundamental level, matter does not consist of point-particles but rather of tiny loops of string.*" This change of perspective is possible since string theory lives in more than three spacelike dimensions in contrast to Einstein's theory of general relativity.

In this section we consider bosonic string theory, which describes only physical particles that are bosons and live in 26 spacetime dimensions. In the next section superstring theory is considered, which is a more complete theory in the sense that it contains both bosons and fermions. Superstring theory lives in 10 dimensions and therefore we will mostly be in 10 dimensions in this thesis.

It is worthwhile mentioning that one can also consider M-theory which is an extension of superstring theories and takes place in 11 dimensions. However, M-theory is outside the scope of this thesis.

3.1.1 The String Worldsheet

A string is a one-dimensional object propagating through space and time. In this way it sweeps out a 2-dimensional surface that we call its worldsheet.

Note that the worldsheet is analogous to the worldline in relativity; only generalized to one extra spacelike dimension. We denote the worldsheet as $X^\mu = X^\mu(\tau, \sigma)$ and it is parametrized by the coordinates σ (spacelike) and τ (timelike; the proper time). Often these are written as $\vec{\tau} = \tau = (\tau, \sigma)$ or as τ^a with $\tau^1 = \tau$ and $\tau^2 = \sigma$. In the rest of this chapter Latin indices a, b, m, n are used for coordinates on the worldsheet (induced coordinates) and Greek indices α, β, μ, ν for spacetime coordinates. In both cases indices run over spacelike as well as timelike coordinates.

In general there are two kinds of strings: open strings which have two endpoints and closed strings whose endpoints are attached by periodic boundary conditions. The endpoints are located at $\sigma = 0$ and $\sigma = \bar{\sigma}$, where we pick the convention that $\bar{\sigma} = \pi$ for open strings and $\bar{\sigma} = 2\pi$ for closed strings.

An important parameter of a string is its tension T: the mass per unit length of the string. It is given by

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{2\pi l_s^2},$$

where $l_s = \sqrt{\alpha'}$ is the characteristic length scale of the string or string scale.

Let us start with strings propagating in a flat spacetime background. When the Minkowski metric $\eta_{\mu\nu}$ is restricted to the worldsheet surface it induces a metric

$$h_{ab} = \partial_a X^\mu \cdot \partial_b X^\nu \eta_{\mu\nu},$$

where $\partial_a = \partial / \partial \tau^a$. This metric h_{ab} is called the induced metric.

The simplest meaningful action to consider comes from minimizing the area of the worldsheet. This Polyakov action yields

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \; (-\gamma)^{1/2} \gamma^{ab} h_{ab}, \tag{3.1}$$

where we introduced an independent auxiliary metric on the worldsheet $\gamma_{ab}(\sigma, \tau)$. The derivation can be found in any text book on string theory.

3.1.2 Neumann and Dirichlet Boundary Conditions

Let us derive the equations of motion by varying the Polyakov action. If one varies (3.1) with respect to γ^{ab} one finds the Virasoro constraints: the requirement that the energy-momentum tensor vanishes, $T_{ab} = 0$ [22]. In addition, we vary with respect to X^μ. Namely, we see that $S \to S + \delta S$ under the infinitesimal transformation $X^\mu \to X^\mu + \delta X^\mu$ with

$$\delta S = -\frac{1}{2\pi\alpha'} \int d^2\tau \; (-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \cdot \partial_b (\delta X_\mu).$$

Writing out the summation over b and performing partial integration with respect to $b = \sigma$ and $b = \tau$, respectively, we obtain

$$\begin{aligned} \delta S = & -\frac{1}{2\pi\alpha'} \int d\tau \; (-\gamma)^{1/2} \gamma^{a\sigma} \partial_a X_\mu \cdot \delta X^\mu \Big|_{\sigma=0}^{\sigma=\bar{\sigma}} + \\ & -\frac{1}{2\pi\alpha'} \int d\sigma \; (-\gamma)^{1/2} \gamma^{a\tau} \partial_a X_\mu \cdot \delta X^\mu \Big|_{\tau=\tau_i}^{\tau=\tau_f} + \\ & +\frac{1}{2\pi\alpha'} \int d^2\tau \; \partial_b \left((-\gamma)^{1/2} \gamma^{ab} \partial_a X^\mu \right) \delta X_\mu . \end{aligned} \tag{3.2}$$

To derive the equations of motion we have to impose that $\delta X^\mu(\sigma, \tau) = 0$ at the initial time $\tau = \tau_i$ and at the final time $\tau = \tau_f$ such that momentum cannot leak away. Therefore, the second term is zero. Now we impose boundary conditions such that the first term is zero as well such that we end up with the equation of motion

$$\partial_a \left((-\gamma)^{1/2} \gamma^{ab} \partial_b X^\mu \right) = 0 \tag{3.3}$$

in addition to the Virasoro constraints. What boundary conditions can we have? First of all, we can make the strings **closed:**

$$\begin{cases} X^\mu(\tau, \sigma + 2\pi) = & X^\mu(\tau, \sigma) \\ \partial_\sigma X^\mu(\tau, \sigma + 2\pi) = & \partial_\sigma X^\mu(\tau, \sigma) \end{cases}$$

and the first term in (3.2) vanishes immediately since $\bar{\sigma} = 2\pi$. On the other hand, multiple boundary conditions are possible for open strings. Namely, **Neumann boundary conditions:**

$$\begin{cases} \partial_\sigma X^\mu(\tau, 0) = & 0 \\ \partial_\sigma X^\mu(\tau, \pi) = & 0 \end{cases} \tag{3.4}$$

or **Dirichlet boundary conditions:**

$$\begin{cases} \delta X^\mu(\tau, 0) = & 0 \\ \delta X^\mu(\tau, \pi) = & 0, \end{cases} \tag{3.5}$$

which are equivalent to

$$\begin{cases} \partial_\tau X^\mu(\tau, 0) = & 0 \\ \partial_\tau X^\mu(\tau, \pi) = & 0. \end{cases}$$

In the case of Dirichlet boundary conditions the endpoints of the string lie at a fixed position in space. On the other hand, with Neumann boundary conditions the endpoints move freely through space at the speed of light [14].

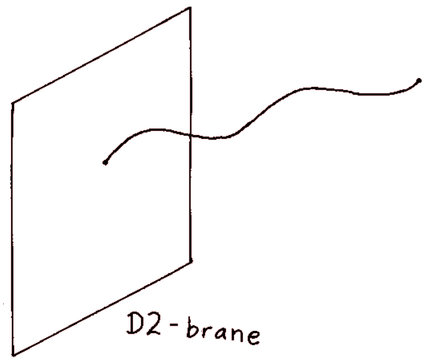

Figure 3.1: An open string ending on a D2-brane.
Source: http://jefferywinkler.com/beyondstandardmodel6.html

3.1.3 D-branes

Each spacelike dimension can have either Neumann or Dirichlet conditions. For instance, if 20 dimensions have Dirichlet conditions and 5 have Neumann conditions then the end points are constrained to move on a 5-dimensional hypersurface, which we call a D-brane or more specifically a Dp-brane where D stands for Dirichlet and p is the number of spatial dimensions of the hypersurface. In this case we have a D5-brane. Thus a Dp-brane is a hypersurface on which open strings can end. In general a D0-brane is a point particle, a D1-brane is a string itself, a D2-brane is a membrane (as drawn in Figure 3.1) and so on.

As we will see in Chapter 6, combining multiple D-branes in a particular way gives the same spacetime metric as a black hole. But first we will look at D-branes in more detail in Chapter 4.

To summarize, we have seen the string worldsheet $X^\mu(\tau, \sigma)$, the Polyakov action (3.1) and its equations of motion (3.3). In order to satisfy these equations of motion open strings need either Neumann boundary conditions (3.4) or Dirichlet boundary conditions (3.5) or a combination of the two. In the latter case D-branes emerge.

3.2 Superstring Theory

So far only bosonic fields were present in our theory, i.e. fields that obey the canonical commutation relations. If we want to add fermions to the system as well, we should add fermionic fields $\psi^\mu(\tau, \sigma)$ to the Polyakov action, which satisfy canonical anti-commutation relations. As a result, the action becomes supersymmetric and therefore the resulting theory is called superstring theory.

The spectrum of bosonic string theory contains tachyons that travel faster than the speed of light such that the theory is inconsistent. Superstring theory does not contain tachyons anymore and resolves this problem.

A few discrete choices are to be made when adding fermions to the worldsheet and as a result there are multiple superstring theories. We do not go into detail regarding this procedure; we only quote the results. Namely, five superstring theories exist:

- Type I
  contains closed superstrings with both left- and right-moving fermions and $\mathcal{N} = 2$ supersymmetry. It also contains open superstrings with Neumann boundary conditions.
- Type IIA & Type IIB
  contain closed superstrings with both left- and right-moving fermions and $\mathcal{N} = 2$ supersymmetry. They also contain open strings with Neumann and Dirichlet boundary conditions such that D-branes are present.
- Heterotic $SO(32)$ & Heterotic $E_8 \times E_8$
  contain closed superstrings with only right-moving fermions and $\mathcal{N} = 1$ supersymmetry.

These five string theories are all limited cases of one larger theory called M-theory, which lives in 11 instead of 10 dimensions.

3.2.1 The Neveu-Schwarz and Ramond Sector

Throughout this thesis we will only work in type IIA and IIB string theories. It is instructive, though, to study type I theory before introducing type II theories. We will see that type I theory splits in two sectors and as a result type II theories split in four sectors.

Type I Superstring Theory

Closed superstrings can have right-moving fermionic modes $\psi^\mu(\tau, \sigma)$ and left-moving fermionic modes $\tilde{\psi}^\mu(\tau, \sigma)$. Recall that a fermion can be represented by a spinor, which can behave either periodically or anti-periodically. That is, when a fermion goes around the cylinder $\sigma \to \sigma + 2\pi$ it can have two possible boundary conditions. Only these solutions obey the equations of motion for fermionic fields on the worldsheet and preserve Lorentz invariance. If the fermion obeys the periodic conditions [23]

$$\begin{cases} \psi^\mu(\tau, \sigma + 2\pi) = & \psi^\mu(\tau, \sigma) \\ \tilde{\psi}^\mu(\tau, \sigma + 2\pi) = & \tilde{\psi}^\mu(\tau, \sigma), \end{cases}$$

we say that the fermion is **in the Ramond (R) sector**. If it has the anti-periodic conditions

$$\begin{cases} \psi^\mu(\tau, \sigma + 2\pi) = & -\psi^\mu(\tau, \sigma) \\ \tilde{\psi}^\mu(\tau, \sigma + 2\pi) = & -\tilde{\psi}^\mu(\tau, \sigma), \end{cases}$$

then it is **in the Neveu-Schwarz (NS) sector**. This is all we need to know about type I theory.

Type IIA and IIB Superstring Theory

Type IIA and IIB theory contain closed strings with right-moving and left-moving modes as well. From two open strings living in type I theory we can make a closed string by joining the endpoints. As we saw before, an open string lives either in the Ramond (R) or Neveu-Schwarz (NS) sector. Thus, if we combine two strings and keep track of their order, then there are four possibilities for a closed string to be in: the R-R, NS-R, R-NS and NS-NS sector. Another way to see this is that for fermions either periodic (R) or anti-periodic (NS) boundary conditions can be chosen for the left- and right-moving parts independently.

It turns out that the ground state of the R sector (in type I) obeys the Clifford algebra and so it is represented by a spinor. The NS sector is represented by a vector. Therefore, the combinations R-R and NS-NS (in type II) contain bosons represented by vectors while the R-NS and NS-R sectors contain fermions represented by spinors.

Now let us consider the massless spectra of type IIA and type IIB theory to see what the differences are. We quote the results [13]. In the NS-NS sector the spectra are exactly the same for type IIA and IIB:

$$\text{NS-NS}: \qquad G_{\mu\nu}, B_{\mu\nu}, \phi, \tag{3.6}$$

where $G_{\mu\nu}$ is the graviton (not to be confused with the Einstein tensor), $B_{\mu\nu}$ is an antisymmetric tensor and the scalar field ϕ is the dilaton. In addition, the spectrum of the R-R sector is

$$\text{R-R:} \qquad \begin{cases} \text{IIA}: & A_1, A_3 \\ \text{IIB}: & A_0, A_2, A_4, \end{cases} \tag{3.7}$$

where all A_n are n-forms

$$A_n = A_{\mu_1 \ldots \mu_n}\, dx^{\mu_1} \ldots dx^{\mu_n}.$$

Both fermionic sectors, NS-R and R-NS, contain two gravitini ψ^μ and two scalar dilatini ψ:

$$\text{NS-R/R-NS:} \qquad \begin{cases} \text{IIA}: & \psi_+^{(1)\mu}, \psi_-^{(2)\mu}, \psi_+^{(1)}, \psi_-^{(2)} \\ \text{IIB}: & \psi_+^{(1)\mu}, \psi_+^{(2)\mu}, \psi_+^{(1)}, \psi_+^{(2)}, \end{cases}$$

where in the IIB case the theory is chiral since all particles have positive chirality.

3.2.2 Einstein Frame and String Frame

We would like to know the spacetime action corresponding to type IIA and IIB theory such that we can place our strings and D-branes in this background. Earlier we only placed them in Minkowski space (according to the $\eta_{\mu\nu}$ in the induced metric of (3.1)). Before we write down this action in the next section, we need to choose a parametrization. Namely, there are two common parametrizations for actions and metrics in string theory: string frame and Einstein frame. Einstein frame is the parametrization we are used to in general relativity. The terms in Einstein frame are canonically normalized, i.e. the action looks like

$$S = \frac{1}{2\kappa_{D,phys}^2} S_{EH} + S_M,$$

where S_M is the matter action, S_{EH} is the Einstein-Hilbert action (2.1) and the constant

$$\kappa_{D,phys} = \sqrt{8\pi G_N^{(D)}} \tag{3.8}$$

is the gravitational coupling in D spacetime dimensions. We will see in Section 5.1 how the Newton constant $G_N^{(D)}$ depends D. The matter action S_M can contain all kinds of interactions. For instance a kinetic term in canonical form looks like

$$S_M = -\int d^D x\ \frac{1}{6}\sqrt{-g}\ \partial_\mu\phi\partial^\mu\phi$$

On the other hand, if we are in string frame, the dilaton ϕ is present as an exponent to some power: $e^{\#\phi}$. We denote the metric in Einstein frame as $g_{\mu\nu}$ and in string frame as $G_{\mu\nu}$. We can go from string frame to Einstein frame and vice versa by

$$g_{\mu\nu} = G_{\mu\nu}\ e^{-\frac{4(\phi-\langle\phi\rangle)}{D-2}} = G_{\mu\nu}\ e^{-\frac{4\tilde{\phi}}{D-2}}, \tag{3.9}$$

where D is the number of spacetime dimensions and $\langle\phi\rangle$ is the vacuum expectation value (vev) of the dilaton. Thus $\langle\phi\rangle$ is the constant part and $\tilde{\phi} = \phi - \langle\phi\rangle$ is the varying part of the dilaton. One can see this equation as the definition of string frame.

Thus if you see a factor $e^{\#\phi}$ in front of the Einstein-Hilbert term $\sqrt{-g}R$ then you are in string frame and otherwise you are in Einstein frame. However, if the value of the dilaton is known and constant, it can be filled in and it is not easy to see in which frame you are anymore. We will encounter this problem later on. The solution is to look up the starting point of its derivation.

3.2.3 The NS-NS and R-R Supergravity Action

Let us consider type IIA and IIB theory in more detail and write down the corresponding spacetime actions. We are only interested in the bosonic part and therefore we only look at the NS-NS and R-R sectors. Moreover, we consider the low energy effective action. Namely, string theory in the limit where the energy is low is called supergravity (sugra). In this limit we can write down a low energy effective action or supergravity action. While supergravity is a simplified case of string theory, it is the supersymmetric extension of Einstein gravity.

We start with the NS-NS sector which is the same for type IIA and IIB, as we have seen in (3.6). The bosonic part of the supergravity action in string frame is given by (see e.g. [13])

$$S_{\text{NS-NS}} = \frac{1}{2{\kappa_D}^2}\int d^Dx\,\sqrt{-G}\,e^{-2\phi}\Bigg(R(G)+4\partial_\mu\phi\partial^\mu\phi+ \\ -\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\Bigg), \tag{3.10}$$

where $H = dB$ is the field strength of the NS-NS antisymmetric tensor $B_{\mu\nu}$ so H is a 3-form. Also, κ_D has the dimension of a D-dimensional gravitational coupling but is not yet normalized correctly to be the physical one. Actually, we should set the dimension D equal to 10 as the NS-NS sector only exists in superstring theory. However, we want to keep the discussion general for later use.

Note that exactly the three fields from the spectrum of the NS-NS sector appear in the action and no more than these. Here $G_{\mu\nu}$ is the string metric but it also represents the graviton.

Now let us rewrite (3.10) in Einstein frame using (3.9). Multiplying the metric by a factor λ gives a factor λ^D in the determinant and it follows that

$$\sqrt{-G} = \sqrt{-g}\;e^{\frac{2D\tilde{\phi}}{D-2}}. \tag{3.11}$$

In order to use this equation we split the dilaton in a varying part and a constant part, $\phi = (\phi - \langle\phi\rangle) + \langle\phi\rangle = \tilde{\phi} + \langle\phi\rangle$. We use the constant part to define the string coupling constant:

$$g_s = e^{\langle\phi\rangle}.$$

Then we absorbe the constant part in the gravitational coupling by a redefinition

$$\kappa_{D,phys} = e^{\langle\phi\rangle}\kappa_D = g_s\kappa_D. \tag{3.12}$$

Now the gravitational coupling $\kappa_{D,phys}$ is canonically normalized and corresponds to the physical one (3.8).

An important point is that the Ricci scalar changes as well under a transformation of the metric. From (3.9) we see that we have a conformal transformation from the string metric $G_{\mu\nu}$ to the Einstein metric $g_{\mu\nu}$:

$$G_{\mu\nu} \to g_{\mu\nu} = e^{2\omega} G_{\mu\nu}$$

with $\omega = -2\tilde{\phi}/(D-2)$. Then one can check by the definition of the Ricci scalar and a lot of bookkeeping, that the Ricci scalar transforms as

$$R \to \tilde{R} = e^{-2\omega} \left(R - 2(D-1)\nabla^2\omega - (D-2)(D-1)\partial_\mu\omega\partial^\mu\omega . \right)$$

In addition, H is invariant as it has nothing to do with the metric. Now we are looking for an action that goes like $\tilde{S} = \sqrt{-g}(\tilde{R}(g) + \dots)$, i.e. without the dilaton in front. This can be found by converting this $\tilde{S}$ to string frame and comparing with (3.10) to see which terms should still be added or subtracted. In addition, using (3.11) and substituting our choice of ω yields the Einstein frame action

$$S_{\text{NS-NS}} = \frac{1}{2\kappa^2_{D,phys}} \int d^D x \, \sqrt{-g} \left(\tilde{R}(g) - \frac{1}{12} e^{4\tilde{\phi}/(D-2)} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \right.$$
$$\left. - \frac{4}{D-2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \right).$$

Finally we go to the correct number of dimensions $D = 10$,

$$S_{\text{NS-NS}} = \frac{1}{2\kappa^2_{10,phys}} \int d^{10} x \, \sqrt{-g} \left(\tilde{R}(g) - \frac{1}{12} e^{\tilde{\phi}/2} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{2} \partial_\mu \tilde{\phi} \partial^\mu \tilde{\phi} \right).$$

Now we turn our attention to the R-R sector. Recall from (3.7) that this sector contains only gauge fields A_n for certain n. A kinetic term in the R-R sector looks like

$$S_{\text{R-R}} \simeq \sum_{\text{some } \{n\}} \int_D F_{n+1} \wedge^* F_{n+1}, \tag{3.13}$$

where $F_{n+1} = dA_n$ and we use the symbol $\simeq$ to indicate that we do not keep track of constant factors. Here the summation runs over $n = 1, 3$ for type IIA and $n = 0, 2, 4$ for type IIB. In fact, the full action contains more terms, for instance Chern-Simons terms. To keep things simple we ignore this here. The complete action can be found in [23].

3.3 S-duality

We consider a symmetry of type IIB supergravity that we call S-duality or string duality. We need this symmetry for the calculation of the D-brane mass in Section 6.1.1.

S-duality is a symmetry under $SL(2,\mathbb{R})$. Recall that $U \in SL(2,\mathbb{R})$ if and only if U is a real (2×2)-matrix such that $\det(U) = +1$. We define the complex scalar field

$$\tau = A_0 + ie^{-\phi},$$

where A_0 is the 0-form R-R potential and ϕ is the dilaton. We also define

$$H = \begin{pmatrix} dB_2 \\ dA_2 \end{pmatrix},$$

where B_2 is the NS-NS antisymmetric tensor and A_2 is the R-R 2-form. Then we apply an $SL(2,\mathbb{R})$ transformation

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}),$$

i.e. we define a group action of $SL(2,\mathbb{R})$ on the set $\{H,\tau\}$ by

$$\begin{cases} H \to UH \\ \tau \to \dfrac{a\tau+b}{c\tau+d}. \end{cases} \tag{3.14}$$

Note that the set $\{H,\tau\}$ contains all fields in the IIB spectrum except $G_{\mu\nu}$ and A_4 (see (3.6) and (3.7)). We define the string metric and A_4 to be invariant under S-duality or self-dual, i.e. they are invariant under the group action above.

In Section 4.5 we will come back to S-duality and see how it acts on D-branes. However, first we introduce p-branes and we consider D-branes in more detail in Chapter 4.

4 p-branes and D-branes

In Section 3.1.3 we encountered Dp-branes or D-branes for the first time. Since we need D-branes in order to build black holes, we will consider them in more detail in this chapter. First we look at p-branes in Section 4.1, which form a more general class of p-dimensional objects. In Section 4.2 we explain that they couple to $(p+1)$-form potentials. As we will see in Section 4.3, Dp-branes are an example of p-branes and turn out to correspond to the extremal case of the Reissner-Nordström black hole. In addition, we consider another kind of p-branes in Section 4.4: NS5-branes. Then we apply S-duality to D-branes in Section 4.5 to see what happens.

4.1 p-branes

Before we move on to D-branes, we introduce the concept of p-branes, which are more general objects. As we shall see, the difference is simple: p-branes are p-dimensional surfaces and Dp-branes are p-dimensional surfaces on which strings can end [24].

Our starting point is the bosonic part of the supergravity actions of type IIA and IIB superstring theory, which forms the background where our strings and branes live. We already encountered the most interesting terms in (3.10), with $D = 10$, and (3.13). Note that the only difference between type IIA and type IIB is that n in A_n is odd or even, respectively. For completeness, we quote the full bosonic action [23] [25]. For type IIA we have (in string frame)

$$\begin{aligned} S_{\mathrm{IIA}} = \frac{1}{2\kappa_D{}^2}\int d^{10}x\sqrt{-G}\Bigg\{ & e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}\left(H_3\right)^2\right) + \\ & - \frac{1}{4}\left(F_2\right)^2 - \frac{1}{48}\left(F_4\right)^2\Bigg\} - \frac{1}{4\kappa_D{}^2}\int B_2\wedge dA_3\wedge dA_3 \end{aligned}$$

and for type IIB

$$\begin{aligned} S_{\mathrm{IIB}} = \frac{1}{2\kappa_D{}^2}\int d^{10}x\sqrt{-G}\Bigg\{ & e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}\left(H_3\right)^2\right) + \\ & - \frac{1}{12}\left[F_3 + A_0\wedge H_3\right]^2 - \frac{1}{2}\left(dA_0\right)^2 - \frac{1}{480}\left(F_5\right)^2\Bigg\} + \\ & + \frac{1}{4\kappa_D{}^2}\int\left[A_4 + \frac{1}{2}B_2\wedge A_2\right]F_3\wedge H_3, \end{aligned}$$

where the R-R field strengths are $F_2 = dA_1$, $F_3 = dA_2$, $F_4 = dA_3 + H_3\wedge A_1$, $F_5 = dA_4 + H_3\wedge A_2$ and A_0 is the R-R scalar field. As we saw before, $G_{\mu\nu}$

is the string metric or graviton, ϕ is the dilaton and $H_3 = dB_2$ is the field strength of the NS-NS 2-form. Note that the terms in brackets $(\dots)$ are exactly equal to the NS-NS action (3.10).

The equations of motion of the actions above can be determined and one can search for solutions. An interesting set of solutions are p-dimensional extended objects, which we call p-branes. They source the graviton, the dilaton and the R-R potentials A_n. In contrast to the complexity of the action, the defining equations of p-branes look relatively simple [26] [27]:

$$\begin{cases} ds^2 &= H_p^{-1/2}(r)\left(-K(r)dt^2 + \sum_{i=1}^{p} dx_i{}^2\right) + \\ & \quad + H_p^{1/2}(r)\left(\frac{dr^2}{K(r)} + r^2 d\Omega_{8-p}{}^2\right) \\ e^{\phi} &= g_s H_p^{\frac{3-p}{4}}(r) \\ A_{0\dots p} &= \frac{1}{2}\left(H_p^{-1}(r) - 1\right), \end{cases} \tag{4.1}$$

where the coordinates x^i lie parallel to the brane, $g_s = e^{\langle\phi\rangle}$ is the string coupling constant, $d\Omega_{8-p}{}^2$ is the metric on a unit $(8-p)$-sphere and

$$H_p(r) = 1 + \frac{\alpha_p c_p N_p}{r^{7-p}}$$
$$K(r) = 1 - \left(\frac{r_H}{r}\right)^{7-p}$$

with the constants

$$\begin{aligned} c_p &= 2^{5-p}\pi^{(5-p)/2}\Gamma\left(\frac{7-p}{2}\right) g_s \alpha'^{(7-p)/2} \\ \alpha_p &= \sqrt{1 + \left(\frac{r_H^{7-p}}{2c_pN_p}\right)^2} - \frac{r_H^{7-p}}{2c_pN_p}, \end{aligned} \tag{4.2}$$

where r_H is the horizon parameter and N_p is the number of p-branes. We will see in Section 5.5 that we may need an array of multiple parallel p-branes. This is all in string frame (in Einstein frame it looks even worse). We see that the p-brane (4.1) has a singularity at $r = 0$ and a horizon at $K(r) = 0$ thus at $r = r_H$.

Let us take the coefficient c_p for granted for the moment. However, it is important to understand where these coefficients come from as they connect our 10-dimensional theory of branes with the 5-dimensional and 4-dimensional black holes we want to describe. That is why we will derive them in detail in Section 6.1.2.

Let us pause for a moment to see what this metric presents, except for all the constants. How can we see that this is indeed a p-dimensional surface?

As we are in 9 spatial dimensions, a p-dimensional membrane is localized, i.e. pointlike, in the $9-p$ dimensions that are transverse to the membrane. Therefore, we have rotational symmetry in these $9-p$ directions and may use spherical coordinates on a $(8-p)$-sphere. Thus we can use coordinates $(t, x^1, x^2, \ldots, x^p, r, \phi_1, \ldots, \phi_{8-p})$, exactly as is done in the metric (4.1). The directions parallel to the brane x_i are given in the first part of the metric and the transverse directions in the 'spherical' second part.

Note that this division is already very similar to the Reissner-Nordström black hole (in isotropic coordinates) given in (2.4), which we repeat here:

$$ds^2 = -H^{-2}dt^2 + H^2 \left[dr^2 + r^2 d\Omega_2{}^2\right], \tag{4.3}$$

where

$$H = H(r) = 1 + \frac{G_N M}{r}.$$

However, an important difference is that r is defined differently in (4.1) and (4.3). In the Reissner-Nordström case we look at a pointlike object ($p = 0$ if you wish) so spherical symmetry holds in all directions. As a consequence no parallel directions x_i are present in (4.3).

On the other hand, one could suggest to take $p = 6$ such that the rotational symmetry is on a 2-sphere in both cases. To create a black hole metric, one must then get rid of the additional 6 dimensions x_i of the 6-brane in some way. This can be done by the procedure of compactification, which we introduce in Chapter 5. Moreover, it turns out that one 6-brane is not enough to create a stable black hole solution that is dressed with a horizon: we need to combine it with other branes. We will explore this in Chapter 6.

4.2 A p-brane couples to a (p+1)-form potential

In the previous section we encountered p-branes, of which Dp-branes are an example. How do p-branes interact with the rest of string theory? In the string worldsheet actions of IIA and IIB supergravity certain couplings arise. We do not present the worldsheet actions here but we refer to [23]. The important thing is that they contain terms like

$$\mu_p \int_{W_{p+1}} A_{p+1}, \tag{4.4}$$

where μ_p is a constant and the integration is over W_{p+1}, the worldvolume of a p-brane (a generalization of the worldsheet area of a string). We conclude that the R-R gauge field A_{p+1} couples to the worldvolume W_{p+1}. Or: we say that the potential A_{p+1} couples to a p-brane. One can see this as a

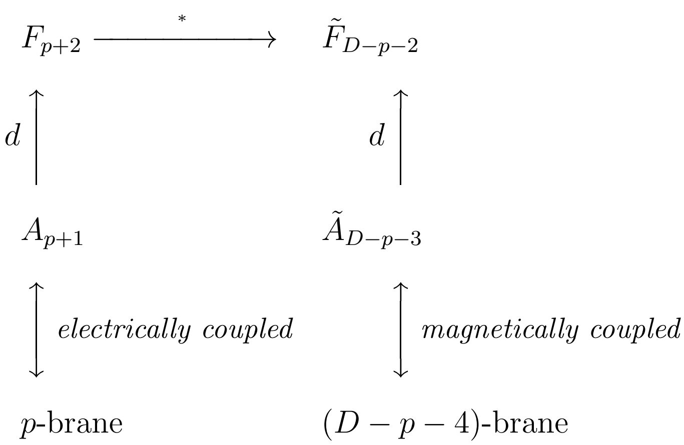

Figure 4.1: This diagram explains why the 'dual' of a p-brane is a $\tilde{p}$-brane with $\tilde{p} = D - p - 4$. We use that a p-brane couples to a potential that is a $(p+1)$-form. Moreover, the field strength (or d operator) of a potential n-form is an $(n+1)$-form. The * on top is the Hodge star operator, which sends an n-form in D spacetime dimensions to its Hodge dual: a $(D-n)$-form.

generalization of the coupling of a point source j to a current A_μ in four dimensions, $\int jA_\mu dx^\mu$.

Moreover, we call (4.4) electric coupling. This is just a definition in order to make a distinction between electric and magnetic coupling. Namely, since a p-brane couples to the field A_{p+1} it is also related to the field strength $F_{p+2} = dA_{p+1}$. Now we can take the Hodge dual of F (see Section 2.3.2) and we obtain some form $\tilde{F}$, which we can see as another field strength. Another $\tilde{p}$-brane will be related to this new field strength and this is the relation that we call magnetic. This procedure is illustrated in Figure 4.1 and looking at this figure it becomes clear that $\tilde{p} = D - p - 4$.

Inspired by [28], we summarize this as

$$F_{p+2} \text{ couples } \begin{cases} \text{electrically to:} & p\text{-branes} \\ \text{magnetically to:} & (p-6)\text{-branes.} \end{cases} \tag{4.5}$$

Now we write out explicitly all brane couplings that emerge in type IIA and IIB theory using (4.5) and $\tilde{p} = D - p - 4$. Namely, in the **NS-NS sector (in both type IIA and IIB)** we have:

$$H_3 \text{ couples } \begin{cases} \text{electrically to:} & 1\text{-branes} & \Rightarrow & \text{F1-string} \\ \text{magnetically to:} & 5\text{-branes} & \Rightarrow & \text{NS5-brane,} \end{cases}$$

where F1 is called the fundamental string. We will not need the fundamental string in the following but we will comment on the NS5-brane or solitonic 5-brane in Section 4.4.

The **R-R sector in type IIA** contains:

$$F_2 \text{ couples } \begin{cases} \text{electrically to:} & \text{0-branes} \quad \Rightarrow \quad \text{D0-brane} \\ \text{magnetically to:} & \text{6-branes} \quad \Rightarrow \quad \text{D6-brane} \end{cases}$$

$$F_4 \text{ couples } \begin{cases} \text{electrically to:} & \text{2-branes} \quad \Rightarrow \quad \text{D2-brane} \\ \text{magnetically to:} & \text{4-branes} \quad \Rightarrow \quad \text{D4-brane.} \end{cases}$$

Note that type IIA theory only contains Dp-branes with p even. This is logical since IIA has only gauge fields A_n with n odd.

On the other hand, we have for the **R-R sector in type IIB**:

$$F_1 \text{ couples } \begin{cases} \text{electrically to:} & (-1)\text{-branes} \quad \Rightarrow \quad \text{D(-1)-brane} \\ \text{magnetically to:} & \text{7-branes} \quad \Rightarrow \quad \text{D7-brane} \end{cases}$$

$$F_3 \text{ couples } \begin{cases} \text{electrically to:} & \text{1-branes} \quad \Rightarrow \quad \text{D1-brane} \\ \text{magnetically to:} & \text{5-branes} \quad \Rightarrow \quad \text{D5-brane} \end{cases}$$

$$F_5 \text{ couples } \begin{cases} \text{electrically to:} & \text{3-branes} \quad \Rightarrow \quad \text{D3-brane} \\ \text{magnetically to:} & \text{3-branes} \quad \Rightarrow \quad \text{D3-brane,} \end{cases}$$

where the D(-1)-brane and D7-brane are special objects, which we do not consider. Note that in the type IIB case only Dp-branes occur with p odd, as expected.

4.3 From p-branes to D-branes

In the previous we just claimed that D-branes form a subset of p-branes. In this section we make this more rigorous. First we look at the extremal limit of p-branes and then we will see how these extremal p-branes are related to D-branes.

Extremal p-branes

Since we only look at extremal Reissner-Nordström black holes, it is useful to take our p-branes to the extremal limit as well. Recall from Section 2.3.1 that in the Reissner-Nordström case the extremal limit is $G_N M^2 = Q^2 + P^2$. We claim that this corresponds for p-branes to $\alpha_p = 1$ in (4.2) [11]. This can be checked by computing the mass and charges of the p-branes in this limit and checking that the above equality holds. We will not present this calculation here but we do introduce formulae to compute the mass and charges in Chapter 6.

Let us substitute $\alpha_p = 1$ in (4.1) to obtain extremal p-branes. Defining

$$\beta = \frac{{r_H}^{7-p}}{2c_p N_p},$$

we see that $\alpha_p = 1$ is equivalent to $\beta = 0$ so $r_H = 0$. As a result, $K(r) = 1$ for all r and the metric (4.1) takes the simple form

$$\begin{cases} ds^2 = H_p^{-1/2}\left(-dt^2 + {dx_1}^2 + \cdots + {dx_p}^2\right) + H_p^{1/2}\left(dr^2 + r^2 {d\Omega_{8-p}}^2\right) \\ e^{\phi} = g_s H_p^{\frac{3-p}{4}} \\ A_{0\ldots p} = \frac{1}{2}\left(H_p^{-1} - 1\right), \end{cases} \tag{4.6}$$

where we dropped the r-dependence and

$$H_p = 1 + \frac{Q_p}{r^{7-p}} = 1 + \frac{c_p N_p}{r^{7-p}} \tag{4.7}$$

with the constant c_p to be derived later on.

Correspondence

In fact, extremal p-branes that are charged with respect to the R-R fields A_n and D-branes are two descriptions of the same objects [29] [30]. While p-branes are extended supergravity solutions, D-branes are solutions of perturbative string theory. However, all known properties of extremal p-branes and D-branes coincide and thus we can identify them.

Examples of such properties are given in the two references above. Namely, extremal p-branes and D-branes:

- carry the same charges (loosely formulated).
- preserve half of the supersymmetries.
- can be located at any point in transverse space.
- low energy scattering, emission and absorption of various string states by the branes agree.
- have the same translational symmetries.

Let us rephrase the first property. It turns out that one unit c_p of the (R-R) charge of a p-brane corresponds exactly to the central charge of one Dp-brane. Therefore, to equal a p-brane charge of $N_p c_p$, we have to identify it with a combination of N_p Dp-branes. We conclude that a Dp-brane is the same as an R-R charged extremal p-brane. Therefore, the **Dp-brane**

metric is equal to (4.6), which we write as

$$\begin{cases} ds^2 & = H_p^{-1/2}\left(-dt^2 + dx_1{}^2 + \cdots + dx_p{}^2\right) + \\ & \quad + H_p^{1/2}\left(dx_{p+1}{}^2 + \cdots + dx_9{}^2\right) \\ e^{\phi} & = g_s H_p^{\frac{3-p}{4}} \\ A_{0\ldots p} & = \frac{1}{2}\left(H_p^{-1} - 1\right), \end{cases} \tag{4.8}$$

in order to make a clear distinction between the coordinates parallel to the brane $x^1, \ldots, x^p$ and the transverse coordinates $x^{p+1}, \ldots, x^9$, which lie perpendicular to the brane. The harmonic function $H(r)$ only depends on the directions perpendicular to the brane (where rotational symmetry is present) via $r^2 = x_{p+1}{}^2 + \cdots + x_9{}^2$.

4.4 The NS5-brane

We will see in Chapter 6 that in order to make a stable 5-dimensional black hole it will suffice to take a combination of Dp-branes. However, it turns out that to make a 4-dimensional black hole, which corresponds better to reality, we need to add the NS5-brane or solitonic 5-brane. We quote the solution of the NS5-brane, which is magnetically charged with respect to the NS-NS field $B_{\mu\nu}$. The metric of an **NS5-brane** lying in the $x^1, \ldots, x^5$ directions is

$$\begin{cases} ds^2 = -dt^2 + dx_1{}^2 + \cdots + dx_5{}^2 + H_5\left(dx_6{}^2 + \cdots + dx_9{}^2\right) \\ e^{\phi} = g_s H_5^{1/2} \\ H_{ijk} = \frac{1}{2}\epsilon_{ijkl}\partial_l H_5, \end{cases} \tag{4.9}$$

where

$$H_5 = 1 + \frac{Q_5}{r^{d-3}} \tag{4.10}$$

with $Q_5 = c_{NS5} N_{NS5}$ the magnetic charge with respect to $B_{\mu\nu}$ and

$$r^2 = x_1{}^2 + \cdots + x_5{}^2.$$

4.5 Branes and S-duality

In Section 3.3 we introduced S-duality. Let us now apply this to p-branes. Recall that S-duality is only a symmetry of type IIB supergravity, which contains even forms A_n. Thus it only makes sense to look at the action of S-duality on p-branes with p odd. These are: F1, D1, D3, D5 and NS5. As we know from Section 4.2, they are respectively coupled to the fields $B_{\mu\nu}$

(electrically), A_2 (electrically), A_4 (magnetically), A_6 (magnetically) and $B_{\mu\nu}$ (magnetically) [13].

After quantization the charges of D-branes are quantized and have to stay quantized. Therefore, we cannot multiply the fields in H and τ by continuous parameters a, b, c, d anymore. The solution is to replace $SL(2,\mathbb{R})$ by its subgroup $SL(2,\mathbb{Z})$.

Moreover, it is common to consider the case that A_0 is set to zero such that $\tau = ie^{-\phi}$. Let us now look at

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2,\mathbb{Z}),$$

which we call the S-generator. By using (3.14) we see that under the S-generator

$$H = \begin{pmatrix} \partial B_2 \\ \partial A_2 \end{pmatrix} \to - \begin{pmatrix} \partial A_2 \\ \partial B_2 \end{pmatrix}$$

and

$$\tau \to -\frac{1}{\tau} \qquad \text{or} \qquad ie^{-\phi} \to ie^{\phi} \qquad \text{or} \qquad g_s \to \frac{1}{g_s}. \tag{4.11}$$

We conclude that the S-generator interchanges A_2 and B_2 and flips the sign of ϕ. This results in

$$\text{D1} \longleftrightarrow \text{F1} \qquad \text{and} \qquad \text{D5} \longleftrightarrow \text{NS5}$$

by comparing the actions and fields of the corresponding branes. Moreover, D3 is invariant. We say that S-duality sends D1 to F1 and D5 to NS5 (and vice versa).

In addition to (4.11) S-duality also changes the radii R_i if we are in a theory that has been compactified over one or more circles S^1, with radii R_i (see Chapter 5). Namely, then the action of S-duality is given by [12]:

$$\begin{cases} g_s \to & g_s' = \dfrac{1}{g_s} \\ R_i \to & R_i' = \dfrac{R_i}{\sqrt{g_s}}. \end{cases} \tag{4.12}$$

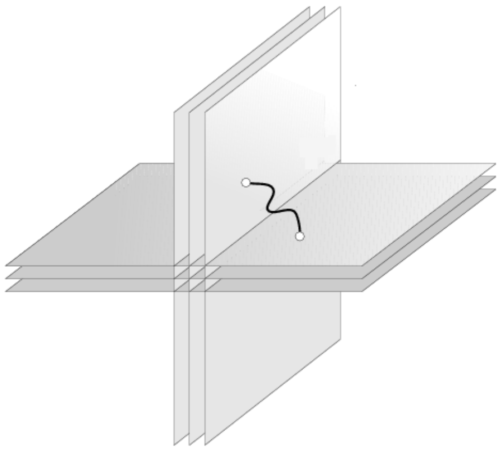

Figure 5.1: An intuitive picture of a combination of intersecting smeared D-branes. Open strings may be present with the endpoints attached to different branes or the same branes. *Source:* [31]

5 Toroidal Compactification

In this chapter we explain the mechanism to go from 10-dimensional superstring theory to the 4-dimensional real world. This procedure is called compactification or dimensional reduction. One must always specify on which manifold one compactifies. The simplest manifold to choose is a circle S^1 (if reducing only 1 dimension). This is represented in Figure 5.1. Namely, if one wraps a 2-dimensional plane over a circle, it becomes a cylinder. If you then make the radius of this cylinder very small or look at it from a large distance the cylinder looks like a line and effectively you have gone from two dimensions to one dimension.

More generally, we can compactify over a torus $T^n = S^1 \times \cdots \times S^1$ (if reducing n dimensions). Unless stated otherwise, we always compactify over tori and this is called toroidal compactification.

In the following we introduce our conventions in Section 5.1 and we see how the Newton constant and dilaton change during compactification in our conventions. Then we consider the method of Kaluza-Klein reduction in Section 5.2. In Section 5.3 we see that momentum along a compactified direction becomes a quantized charge. Moreover, we look at T-duality and how it acts on branes in Section 5.4 and we consider arraying multiple parallel D-branes in Section 5.5.

5.1 Unit Conventions

An important aspect of dimensional reduction is keeping track of the right constants and numerical factors. One should be aware that every book and paper has its own combination of conventions. In this section we look at the numerical value of the Newton constant in ten dimensions and we see how it changes in d dimensions. We also consider how the dilaton depends on the number of dimensions.

Newton Constant in d Dimensions

The Newton constant in 10 dimensions is related to the gravitational coupling $\kappa_{10,phys}$ (see (3.8)). Moreover, they are related to α' and the string coupling g_s in the following way [13]:

$$2{\kappa_{10,phys}}^2 = 16\pi G_N^{(10)} = (2\pi)^7 {g_s}^2 \alpha'^4. \tag{5.1}$$

This is often written in terms of the string scale $l_s = \sqrt{\alpha'}$ but we stick to α'. As suggested by the notation $G_N^{(D)}$, the Newton constant depends on the spacetime dimension D and its value changes when we compactify from 10 to d dimensions. This can be seen from the integration measure. We write this out explicitly in Einstein frame. For now, we assume that the 10-dimensional metric $g_{\mu\nu}^{(10)}$ is diagonal. As we will see, then the reduced metric $g_{\mu\nu}^{(d)}$ will also be diagonal. Using

$$\sqrt{-g^{(10)}} = \sqrt{-g^{(d)}}\sqrt{g_{dd}^{(10)} \cdot \ldots \cdot g_{99}^{(10)}}$$

we then factorize the integration measure as

$$\begin{aligned}
&\frac{1}{G_N^{(10)}} \int d^{10}x \, \sqrt{-g^{(10)}} \ldots = \\
&\qquad = \frac{1}{G_N^{(10)}} \left(\int d^{10-d}x \, \sqrt{g_{dd}^{(10)} \cdot \ldots \cdot g_{99}^{(10)}} \right) \left(\int d^d x \, \sqrt{-g^{(d)}} \ldots \right) \\
&\qquad = \frac{1}{G_N^{(10)}} \, (2\pi)^{10-d} V_{10-d} \int d^d x \, \sqrt{-g^{(d)}} \ldots ,
\end{aligned}$$

where $(2\pi)^{10-d}V_{10-d}$ is the volume of the manifold we compactify over: in our case a $(10-d)$-dimensional torus T^{10-d}. Another convention is to absorb the power of 2π in V_{10-d}. However, we prefer to keep it explicit because it will cancel out later to the power of 2π in the Newton constant (5.1).

In order to have consistency between the 10-dimensional and compactified theory we demand that

$$\frac{1}{G_N^{(10)}} \int d^{10}x \, \sqrt{-g^{(10)}} \ldots = \frac{1}{G_N^{(d)}} \int d^d x \, \sqrt{-g^{(d)}} \ldots . \tag{5.2}$$

We conclude that the d-dimensional Newton constant must obey

$$G_N^{(d)} = \frac{G_N^{(10)}}{(2\pi)^{10-d} V_{10-d}}. \tag{5.3}$$

Note that the factorization of the integral is only possible if the action does not depend on the $10-d$ internal coordinates. Otherwise it does not make sense to compactify.

Dilaton in d Dimensions

In the previous we considered how the Newton constant changes under dimensional reduction by looking at how the measure factorizes in Einstein frame. If we go to string frame, though, there is also a dilaton present in the action.

In Einstein frame we saw that the change in Newton constant exactly compensates for the volume form $\sqrt{g_{dd}^{(10)} \cdot \ldots \cdot g_{99}^{(10)}}$. In string frame we replace $\sqrt{-g}$ by $e^{-2\phi}\sqrt{-G}$ (in $D=10$) and now we compensate this change of the Newton constant not by a changing string metric but by a changing dilaton instead. In other words, we define the dilaton in d dimensions to be

$$e^{-2\phi_d} = \sqrt{G_{dd}^{10} \cdot \ldots \cdot G_{99}^{10}}\; e^{-2\phi_{10}} \tag{5.4}$$

such that (5.2) holds in string frame as well.

5.2 Kaluza-Klein Reduction

Finally we consider how to go from a 10-dimensional metric to a d-dimensional one. First, one needs to clarify over which manifold the compactification is done. We always compactify over a torus T^n, unless stated otherwise. Then, if the metric is diagonal, the answer is simple. We just throw away the metric components corresponding to the coordinates we want to compactify. For instance, if we want to compactify the coordinate x^9 of 10-dimensional Minkowski space over a circle S^1, then we simply replace the metric

$$ds_{10}{}^2 = -dt^2 + dx_1{}^2 + \cdots + dx_8{}^2 + dx_9{}^2$$

by

$$ds_9{}^2 = -dt^2 + dx_1{}^2 + \cdots + dx_8{}^2.$$

Similarly, compactifying the coordinates $x^4, \ldots, x^9$ over the torus T^6 we end up with the 4-dimensional metric

$$ds_4{}^2 = -dt^2 + dx_1{}^2 + dx_2{}^2 + dx_3{}^2.$$

Recall that it must be the case that none of the remaining metric components depends on the internal coordinate(s).

Now what happens if there are off-diagonal components? Then we use the Kaluza-Klein procedure for dimensional reduction [32]. Consider the reduction of D to $(D-1)$ dimensions where we compactify the coordinate y on a circle S^1, i.e. we make the identification $y \simeq y + 2\pi R$, where R is the radius of the circle. We denote the D-dimensional coordinate system without hats and the $(D-1)$ system with hats. Namely, we write $\{x^\mu\} = \{x^{\hat{\mu}}, y\}$, where $\mu = 0, \ldots, D$ and $\hat{\mu} = 0, \ldots, (D-1)$.

Given a 10-dimensional metric $G_{\mu\nu}$, the $(D-1)$-dimensional metric $G_{\hat{\mu}\hat{\nu}}$ is given by Kaluza-Klein decomposition:

$$G_{\mu\nu} = \begin{pmatrix} G_{\hat{\mu}\hat{\nu}} + e^{2\sigma} A_{\hat{\mu}} A_{\hat{\nu}} & e^{2\sigma} A_{\hat{\mu}} \\ e^{2\sigma} A_{\hat{\nu}} & e^{2\sigma} \end{pmatrix}, \tag{5.5}$$

where A_μ is the Kaluza-Klein gauge field and σ is the Kaluza-Klein scalar. As an example we work out the case of a pp-wave in detail, following [13].

Compactification of a pp-wave

We consider a gravitational wave that is planar fronted and has parallel rays. This is abbreviated to a planar-fronted parallel wave or pp-wave or simply W. A D-dimensional pp-wave with $D > 4$ is given by the metric (in Einstein frame):

$$ds^2 = (-1 + K)dt^2 + d\vec{x}^2 + (1 + K)\, dy^2 - 2K dy dt, \tag{5.6}$$

where

$$K(r) = \frac{Q}{r^{D-4}}$$

is a harmonic function. We compactify the y coordinate on a circle. Namely, we use (5.5) in three steps to find the unknowns σ, $A_{\hat{\mu}}$ and $G_{\hat{\mu}\hat{\nu}}$. Since there is one cross term between y and t and we compactify over y, the reduced metric will only have a different $G_{\hat{t}\hat{t}}$ component. Therefore, the only nonzero component of the Kaluza-Klein gauge field is $A_{\hat{t}}$.

1. The metric component G_{yy} corresponds to $e^{2\sigma}$ so
$$e^{2\sigma} = 1 + K.$$
2. The cross term G_{ty} corresponds to the two off-diagonal elements of (5.5). Thus $e^{2\sigma} A_{\hat{t}} = -K$ such that
$$A_{\hat{t}} = -\frac{K}{1+K}.$$
3. Now we have the equation
$$G_{tt} = -1 + K = G_{\hat{t}\hat{t}} + e^{2\sigma} \left(A_{\hat{t}}\right)^2.$$
Substituting the results above gives
$$G_{\hat{t}\hat{t}} = -\frac{1}{1+K}.$$

We conclude that the metric in $(D-1)$ dimensions is

$$ds^2 = -\frac{1}{1+K}d\hat{t}^2 + d\vec{\hat{x}}^2.$$

As a final remark, we rewrite this result by replacing K with a more common harmonic function,

$$H = H_W = 1 + K = 1 + \frac{Q_W}{r^{D-4}}.$$

This yields

$$\begin{cases} ds^2 = -H^{-1}d\hat{t}^2 + d\vec{\hat{x}}^2 \\ e^{2\sigma} = H \\ A_{\hat{t}} = H^{-1} - 1. \end{cases} \tag{5.7}$$

5.3 Quantization

After dimensional reduction the hidden dimension obeys a periodicity $x^9 \simeq x^9 + 2\pi R$. Therefore, we can perform a Fourier expansion and this gives massive modes that are related to the radius R. Namely, when one compactifies a string X^μ the momentum in the x^9-direction, P_9, becomes quantized:

$$P_9 = \frac{n}{R}, \qquad n \in \mathbb{Z}.$$

Now we call the integer n the momentum of the string. This is the famous result of Kaluza and Klein: after dimensional reduction a momentum becomes a quantized charge $Q \sim n$. We call this a charge since the Kaluza-Klein field, A_μ in (5.5), acts as its potential. Originally, this was used to combine general relativity and electromagnetism by starting with GR in 5 dimensions and compactifying to 4 dimensions. Nowadays Kaluza-Klein theory is used as the basis for all compactifications of string theory.

In addition, a string can be winded around the internal direction and then

$$X^9 \to X^9 + 2\pi R m, \qquad m \in \mathbb{Z}.$$

We say that m is the winding number of the string. Now the duo of winding number and quantized momentum (m, n) describes a string. The Virasoro constraints can now be written as [12]

$$E^2 = \vec{P}^2 + \left(\frac{n}{R} - \frac{mR}{\alpha'}\right)^2 + \frac{4}{\alpha'}N_L$$

and exactly the same equation with N_R instead of N_L. Here $N_{L,R}$ are the total net oscillator levels of the string, i.e. the total number of excitations, and $\vec{P} = (P_1, \ldots, P_8)^T$. This equation makes sense since we have only rewritten the on-shell condition $M^2 = P^2$ and denoted the momenta in the internal direction (P_9: the term in brackets) and in the other directions ($\vec{P}$) separately. From this equation we read off the mass M of the string,

$$M^2 = \left(\frac{n}{R} - \frac{mR}{\alpha'}\right)^2 . \tag{5.8}$$

5.4 T-duality

In Section 3.3 we encountered S-duality. In this section we comment briefly on another symmetry called T-duality.

T-duality is only defined in an environment that has already been compactified over a radius R. Namely, then one can see that the spectrum (5.8) is invariant under a transformation $R \to \alpha'/R$ if we simultaneously interchange winding number m and momentum n. This is called T-duality and it turns out to be a symmetry of the whole string theory. It is even a symmetry of the interactions if we require an extra change in the string coupling g_s, i.e. such that T-duality is given by [12]

$$\begin{cases} R \to \quad R' = \dfrac{\alpha'}{R} \\ g_s \to \quad g_s' = \dfrac{g_s\sqrt{\alpha'}}{R}. \end{cases} \tag{5.9}$$

If for instance the coordinate x^9 is compactified, we say that T-duality is done in the x^9-direction and we replace R in the above set of equations by R_9 for clarity.

As a result, all fields $G_{\mu\nu}$, $B_{\mu\nu}$, ϕ, A_n have some coordinates that change under T-duality. They are twisted into each other, if you wish. We do not quote the formulae that describe these changes since they are quite numerous; we refer to equations (5.4) and (8.2) of [11].

An interesting remark is that for open strings T-duality interchanges Neumann and Dirichlet boundary conditions. This is an extra motivation to introduce D-branes: even if all strings have Neumann conditions, then after applying the symmetry of T-duality D-branes naturally arise in the theory.

5.4.1 T-duality on D-branes

Now let us apply T-duality to a Dp-brane. T-duality can either be applied in a direction parallel or transverse to the Dp-brane. Using the formulae describing how T-duality changes the fields, one can show that T-dualising in a direction transverse to the Dp-brane gives exactly a D(p−1)-brane, i.e. with the metric, dilaton etcetera given by (4.8) for $p-1$. On the other hand, T-dualising in a direction parallel to the Dp-brane gives exactly a D(p+1)-brane. For these calculations we refer to section 10.4 of [11].

However, in the latter case there is a subtlety. A parallel direction of a Dp-brane is not an isometry direction such that periodic boundary conditions $x^9 \simeq x^9 + 2\pi R_9$ are not satisfied. This can be solved by 'smearing' the Dp-brane in this direction, i.e. one places an array of parallel Dp-branes at a regular distance in this direction in order to fulfill the periodic boundary conditions.

5.5 Smearing

Smearing is not only important for T-duality but also for dimensional reduction as a whole. Namely, a solution in ten dimensions will only be a solution of the compactified theory if it obeys periodic boundary conditions $x_i \simeq x_i + 2\pi R_i$. Therefore we have to make an array of p-branes in its transverse directions with a distance $2\pi R_i$ between neighbours in the x^i direction. Together these form a lattice and we replace the single-centered harmonic function

$$H_p = 1 + \frac{Q_p}{r^{7-p}}$$

by a multi-centered harmonic function that accounts for the whole array of p-branes:

$$H_p = 1 + \sum_{\vec{n}\in\text{lattice}} \frac{c_p N_p}{(\vec{r} - 2\pi R_i \vec{n})^{7-p}}, \tag{5.10}$$

where c_p is the corresponding constant. Note that the lattice is $(10-d-p)$-dimensional as we only fill the $10-d$ dimensions that will be compactified and p directions are already parallel to the branes. We could also have imposed this harmonic function from the beginning since it is a more general solution to the Laplace equation by linearity. Physically, this solution makes sense since the parallel extremal p-branes in the lattice are in static equilibrium with each other. That is, the repulsive forces (electric and magnetic, exerted by the gauge fields) exactly cancel to the attractive forces (gravitational and dilatonic) [13]. This is called the no-force theorem.

If we are far away relative to the compactification scale, we can replace the sum in (5.10) by an integral over the $10-d-p$ dimensions of the lattice. Each time that we integrate one dimension, the power of r increases by one yielding

$$\int \frac{1}{r^{7-p}}\, d^{10-d-p}x \simeq \frac{1}{r^{(7-p)-(10-d-p)}} = \frac{1}{r^{d-3}}.$$

Thus we can rewrite the harmonic function (5.10) as

$$H_p = 1 + \frac{Q_p}{r^{d-3}} = 1 + \frac{c_p^{(d)} N_p}{r^{d-3}}, \tag{5.11}$$

where $c_p^{(d)}$ is some constant depending on d and p, which we call the coefficient of the harmonic function. Observe that the power of r does not depend on p anymore and is the correct one for the spherically symmetric solution of the Laplace equation in $d-1$ spatial dimensions. For example, recall that in $d = 4$ spacetime dimensions the Poisson equation $\nabla^2 f = 0$ is solved by $f \sim 1/r$.

From now on we write all harmonic functions of branes in the form (5.11) and we will compute the constants $c_p^{(d)}$ explicitly in Section 6.1.2. More information about 'smearing' or 'arraying' can be found in section 3.2 of [33].

6 Black Holes from D-branes

In this chapter we describe black holes by a certain combination of p-branes. After we have found the Reissner-Nordström black hole our goal is twofold. First we compute the entropy of the black hole both macroscopically and microscopically. Then we want to compute the electric and magnetic charges of the black hole and see whether we can set the magnetic charge to zero by making use of electromagnetic duality.

In section 6.1 we calculate the coefficients $c_p^{(d)}$ that occur in the harmonic functions. Then we discuss in section 6.2 why and how we can combine multiple D-branes to make a black hole. In section 6.3 we concentrate on the 5-dimensional case as this turns out to be easier than the 4-dimensional one. We derive the black hole metric, which corresponds to Tangherlini. Then we find the macroscopic and microscopic entropy and the gauge fields. Moreover, we compute the electric and magnetic charges with respect to these fields. In section 6.4 we repeat all these derivations for another combination of p-branes in $d = 4$ such that we arrive at the Reissner-Nordström black hole. As a final result, we look whether we can describe $d = 4$ black holes in a simpler way by getting rid of the magnetic charges using electromagnetic duality in section 6.5. Moreover, we investigate what happens to the microscopic theory of branes after the electromagnetic duality transformation.

6.1 Preliminary Calculations

In order to compute electric and magnetic charge we need to find the coefficients $c_p^{(d)}$ in d dimensions. And in order to find these we need the mass of p-branes. We study these two subjects in this section, following [12].

6.1.1 Mass of D-branes

Let us start with the mass of a D1-brane or D-string. First we consider the mass of a 'normal' string, which is given by (5.8) and depends on the winding number m and the quantized momentum n. In particular, we consider a winding string with minimal mass, i.e. $m = 1, n = 0$ such that

$$M = \frac{R_9}{\alpha'}, \tag{6.1}$$

where x^9 is the compactified direction. We claim that the fundamental string F1 has this property [12].

On the other hand, the mass for momentum states is minimal if $m = 0$, $n = 1$:

$$M = \frac{1}{R_9}. \tag{6.2}$$

This momentum state corresponds to the pp-wave W we encountered in section 5.2.

In the following we use S-duality and T-duality to go from F1 to Dp-branes. First, it is important to realize that two slightly different definitions of the Einstein metric $g_{\mu\nu}$ are used in literature. In our definition we have absorbed a factor of $g_s = e^{\langle\phi\rangle}$ in the gravitational coupling constant κ (see (3.12)). This corresponds to a relation between string frame and (our) Einstein frame as given by (3.9). This version of the Einstein metric is appropriate if one uses the standard equations of general relativity to calculate mass and entropy.

On the other hand, it is possible to define the Einstein metric with the factor $\sqrt{g_s}$ absorbed in the metric instead of κ. In this text we call this version the adapted Einstein metric $\tilde{g}_{\mu\nu}$. The two versions are related by

$$\tilde{g}_{\mu\nu} = {g_s}^{-1/2} g_{\mu\nu}. \tag{6.3}$$

One can recognize that a text uses the adapted Einstein metric if it gives the relation between Einstein frame $\tilde{g}_{\mu\nu}$ and string frame as

$$\tilde{g}_{\mu\nu} = G_{\mu\nu}\; e^{-\frac{4\phi}{D-2}},$$

i.e. with $\phi = \tilde{\phi} + \langle\phi\rangle$ instead of $\tilde{\phi}$. The adapted Einstein frame is useful in the case of IIB S-duality since $\tilde{g}_{\mu\nu}$ is invariant under S-duality (it corresponds to the string metric $G_{\mu\nu}$ at infinity as $\langle\phi\rangle = \phi_\infty$). More specifically, $\tilde{g}_{\mu\nu}$ is invariant as long as we make a redefinition

$$\tilde{\alpha}' = g_s \alpha', \tag{6.4}$$

where $\alpha' \to \tilde{\alpha}'$ under S-duality.

Let us now find the mass of D1 from the F1 mass (6.1) using S-duality (see sections 3.3 and 4.5). We do this using the relation (6.3) to go to the adapted Einstein frame such that we can use its invariance. Recall that the Einstein mass M (6.1) comes from the on-shell condition

$$M^2 = P^2 = P_\mu P^\mu = g^{\mu\nu} P_\mu P_\nu.$$

It follows that

$$M^2 = {g_s}^{-1/2}\; \tilde{g}^{\mu\nu} P_\mu P_\nu = {g_s}^{-1/2} \tilde{M}^2,$$

for masses $\tilde{M}$ measured with respect to $\tilde{g}_{\mu\nu}$. Thus masses measured with respect to adapted Einstein frame and Einstein frame differ by

$$\tilde{M} = {g_s}^{1/4} M.$$

Since $\tilde{M}$ is invariant under S-duality we see that S-duality acts on the F1 mass M as

$$ {g_s}^{1/4}\frac{R_9}{\tilde{\alpha}'} = \tilde{M}' = \tilde{M} = {g_s}^{1/4}M = {g_s}^{1/4}\frac{R_9}{\alpha'}. $$

Note that the mass $\tilde{M}$ is indeed invariant under (4.12) and (6.4). Hence

$$ M' = \frac{R_9}{\tilde{\alpha}'} = \frac{R_9}{g_s\alpha'}. $$

We know that S-duality sends the fundamental string F1 to a D1-brane. Thus this is equal to the mass of a D1-brane

$$ M^{D1} = \frac{R_9}{g_s\alpha'} \tag{6.5} $$

that is lying in the x^9 direction. Now we apply T-duality in the direction x^8, which is perpendicular to the brane. This results in a D2-brane (see section 5.4). Thus using (5.9) we have

$$ M^{D2} = \frac{R_8R_9}{g_s\alpha'^{3/2}}. \tag{6.6} $$

Now we can apply T-duality to the transverse direction x^7 to obtain the mass of a D3-brane, and so on. We find

$$ M^{Dp} = \frac{R_{10-p}\cdots R_9}{g_s\alpha'^{(p+1)/2}}, \tag{6.7} $$

where the Dp-brane lies in the directions $x^{10-p},\ldots,x^9$. As a final result, we apply S-duality to go from the D5-brane to the NS5-brane. This gives again a factor of $1/g_s$:

$$ M^{NS5} = \frac{R_5\cdots R_9}{{g_s}^2\alpha'^3}. \tag{6.8} $$

6.1.2 Coefficients of the Harmonic Functions

The aim of this section is to obtain the correct coefficients of the harmonic functions corresponding to the branes. Namely, what is the value of $c_p^{(d)}$ in (5.11) for Dp-branes and what is the value of c_{NS5} in (4.10)?

To compute these coefficients we need the mass of the branes given in the previous section. Moreover, we need the value of the Newton constant in ten dimensions that we know from (5.1):

$$ G_N^{(10)} = 8\pi^6 g_s^2\alpha'^4. \tag{6.9} $$

Then the mass is computed from the value of the metric component g_{00} at infinity, which goes like [12]

$$g_{00} \sim \frac{16\pi G_N^{(d)} M}{(d-2)\omega_{d-2}} \frac{1}{r^{d-3}} = \frac{d-3}{d-2} \frac{c^{(d)}}{r^{d-3}},$$

where the volume of the unit sphere S^d in $d+1$ dimensions is

$$\omega_d = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)}.$$

This results in the coefficient

$$c_p^{(d)} = \frac{16\pi G_N^{(d)}}{(d-3)\omega_{d-2}} M_p, \tag{6.10}$$

where M_p is the mass of the p-brane under consideration.

Now let us work out all coefficients that we need later. How nice that you are reading my thesis! Please send me an e-mail at jeemijnscheen [at] gmail [dot] com and I will make sure you will receive a home-baked surprise. We start in $d = 5$ dimensions with the D5-brane coefficient $c_5^{(5)}$. Substituting the D5-brane mass (6.7) gives

$$c_5^{(5)} = \frac{8\pi G_N^{(5)}}{\omega_3} \frac{R_5 \cdots R_9}{g_s \alpha'^3}.$$

In addition, we have

$$\omega_3 = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2$$

using $\Gamma(n) = (n-1)!$. Moreover, using (6.9) we find

$$G_N^{(5)} = \frac{G_N^{(10)}}{(2\pi)^5 R_5 \cdots R_9} = \frac{\pi {g_s}^2 \alpha'^4}{4 R_5 \cdots R_9}$$

such that

$$c_5^{(5)} = g_s \alpha'. \tag{6.11}$$

Observe that the dependence on all radii cancels out because we are in the special case $d = 10 - p$.

We stay in five dimensions to compute the D1-brane coefficient $c_1^{(5)}$ as well. Namely, starting from (6.10) and using $\omega_3 = 2\pi^2$ again, together with the D1-brane mass (6.5), we conclude that

$$c_1^{(5)} = \frac{4 G_N^{(5)} R_9}{\pi g_s \alpha'}. \tag{6.12}$$

Moreover, the coefficient of a pp-wave W in the x^9 direction is given by

$$c_W^{(5)} = \frac{4G_N^{(5)}}{\pi R_9}, \tag{6.13}$$

using the pp-wave mass (6.2).

Now we turn to the relevant coefficients in 4 dimensions. In this case we have

$$\omega_2 = \frac{2\pi^{3/2}}{\Gamma\left(\frac{3}{2}\right)} = 4\pi,$$

where we used $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Then the D2-brane coefficient $c_2^{(4)}$ is

$$c_2^{(4)} = \frac{4G_N^{(4)} R_8 R_9}{g_s \alpha'^{3/2}} \tag{6.14}$$

by using the mass (6.6). Using the masses of NS5-branes (6.8) and D6-branes (6.7), we find the coefficients

$$c_{NS5}^{(4)} = 4G_N^{(4)} \frac{R_5 \cdots R_9}{{g_s}^2 \alpha'^3}$$

and

$$c_6^{(4)} = 4G_N^{(4)} \frac{R_4 \ldots R_9}{g_s \alpha'^{7/2}}.$$

We can simplify the two coefficients above by observing that

$$G_N^{(4)} = \frac{G_N^{(10)}}{(2\pi)^6 R_4 \ldots R_9} = \frac{{g_s}^2 \alpha'^4}{8 R_4 \ldots R_9}.$$

It follows that

$$\begin{aligned} c_{NS5}^{(4)} &= \frac{\alpha'}{2R_4} \\ c_6^{(4)} &= \frac{g_s \sqrt{\alpha'}}{2}, \end{aligned} \tag{6.15}$$

where all radii dropped out in the latter case since again $d = 10 - p$.

Finally, we note that the coefficient of W in 4 dimensions becomes

$$c_W^{(4)} = \frac{4G_N^{(4)}}{R_9}. \tag{6.16}$$

Observe that we have used the following notation: we went from 10 to 5 dimensions by compactifying $x^5, \ldots, x^9$ over a torus $T^5 = S^1 \times \cdots \times S^1$, where the circles have radii $R_5, \ldots, R_9$. Afterwards, we compactified the x^4 direction as well on a circle with radius R_4.

6.2 Multiple Intersecting D-branes

In this section we discuss why we need multiple D-branes at all. Moreover, we will see how to combine their actions.

As a first attempt to describe a $d = 5$ black hole from branes one could take a D5-brane in type IIB superstring theory (where odd branes live) and wrap it on T^5. Afterwards the resulting object is pointlike in $d = 5$ dimensions and its metric turns out to correspond to the type II analogs of Sen's 5-dimensional black holes [12]. However, the Bekenstein-Hawking entropy (2.14) of the resulting metric is zero. In other words, the area of the horizon is zero and we have a naked black hole. Thus one single D5-brane does not suffice to describe a black hole with a horizon and we may try a combination of branes.

We refer to section 17.3.1 of [11] for a pedagogical example to find a combination of branes that works. Here one starts with a D5-brane and compares it with the Tangherlini metric (2.12). Then one adds certain branes and pp-waves until the metric and fields equal the Tangherlini metric. The result is a combination of a D1-brane, a D5-brane and a pp-wave W. We discuss this particular combination in detail in the following section. It is important to realize that there are a lot more possible configurations that yield the same black hole metric and the same holds for the $d = 4$ case.

Let us now address the question how to combine the metrics and fields of multiple branes. All Dp-brane metric components are written in terms of harmonic functions H. Now two Dp-brane metrics can be combined by the harmonic function rule, i.e. just multiply the corresponding metric components [34]. For example,

$$G_{\mu\nu,\mathrm{D2\,+\,D4}} = G_{\mu\nu,\mathrm{D2}} \cdot G_{\mu\nu,\mathrm{D4}} \tag{6.17}$$

for the $\mu\nu$ component, with some abuse of notation. The same rule applies for the dilaton ϕ, which is given in terms of harmonic functions as well. Moreover, additional fields remain unchanged. For example, the D2-brane has a field A_{012} and the D4-brane has a field $A_{0\ldots4}$. The combination D2 + D4 has both these potentials.

Before we move on to building black holes, we make a final comment on the charges of D-branes. We saw in section 5.3 that momenta become quantized after compactification. Moreover, the charges with respect to gauge fields that are already present in 10 dimensions become quantized as well. Namely, they obey a generalized Dirac quantization condition [13]

$$PQ \simeq n, \qquad \text{with } n \in \mathbb{Z}.$$

The factor in $\simeq$ depends on the normalization of electric (6.36) and magnetic charge (6.37). In our case we normalized by $1/(4\pi)$ and then

$$PQ = n, \qquad \text{with } n \in \mathbb{Z}.$$

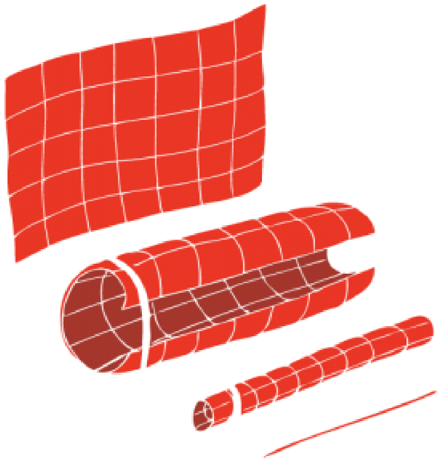

Figure 6.1: Wrapping a 2-dimensional plane over a circle S^1 gives a cylinder, which looks like a 1-dimensional line from a large distance.
Source: http://whystringtheory.com/toolbox/d-branes/

Another convention that is often used is normalizing the charges by $1/\sqrt{2\pi}$ such that $PQ = 2\pi n$.

6.3 5-dimensional Black Holes

In this section we follow [12] and we study the configuration of a D1-brane, D5-brane and a pp-wave W. Although we say "a D1-brane" we assume that it has been smeared out as an array of D1-branes in the perpendicular directions, as explained in section 5.5. Such a configuration of smeared D-branes may look like Figure 6.1. After compactification to five dimensions this configuration will yield a Tangherlini black hole.

Let us first specify in which directions our branes lie exactly. We start in ten dimensions in IIB supergravity (where odd branes live) and we choose the notation where $x^1 \dots x^4$ are the extended directions, i.e. are the 'normal' uncompactified coordinates in $d = 5$. We summarize our configuration as

	0	1	2	3	4	5	6	7	8	9
D1	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\perp$	$\perp$	$\perp$	$\perp$	$-$
D5	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-$	$-$	$-$	$-$	$-$
W	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\perp$	$\perp$	$\perp$	$\perp$	$\rightarrow$,

(6.18)

where $-$ denotes a direction parallel to the brane and $\perp$ is a transverse direction. Moreover, we use $\rightarrow$ for the direction in which the pp-wave travels and $\cdot$ means a pointlike direction.

It follows from the procedure of smearing that the harmonic functions in a metric of combined branes only depends on the overall transverse coordinates. That is, we have seen in the Dp-metric that the harmonic function $H_p(r)$ depends on the coordinates that are perpendicular to the brane. For instance, in the case of the D1-brane above these are $x^1, \dots, x^8$. However, if we smear the D1-brane in the $x^5, \dots, x^8$ directions and combine it with the D5-brane above, then the metric of D1+D5 only depends on the coordinates that are perpendicular to both branes [33]. The result is that we have harmonic functions that only depend on $r^2 = {x_1}^2 + \dots + {x_4}^2$. We need this in the following as we can only compactify over the internal coordinates $x^5, \dots, x^9$ if their metric components do not depend on these internal coordinates.

We compactify over a 5-dimensional torus that we write as $T^5 = T^4 \times S^1$. Namely, first we compactify four coordinates on a 4-torus and we compute the intermediate result, which is called a black string. Afterwards we compactify a fifth coordinate on a circle. Schematically:

$$
\begin{array}{ll}
D = 10 & \\
\Big\downarrow {\scriptstyle x^5,\dots,x^8 \text{ on } T^4} & \\
D = 6 & \text{black string} \\
\Big\downarrow {\scriptstyle x^9 \text{ on } S^1} & \\
D = 5 & \text{black hole.}
\end{array}
$$

Compare this with the configuration (6.18).

Now we combine the metrics of a D1-brane (4.8), D5-brane (4.8) and pp-wave W (5.6) by using the harmonic function rule (6.17) (in string frame):

$$
\left\{
\begin{aligned}
{ds_{10}}^2 &= {H_1}^{-1/2}{H_5}^{-1/2}\left[-dt^2 + {dx_9}^2 + K(dt - dx_9)^2\right] \\
&\quad + {H_1}^{1/2}{H_5}^{1/2}\left({dx_1}^2 + {dx_2}^2 + {dx_3}^2 + {dx_4}^2\right) \\
&\quad + {H_1}^{1/2}{H_5}^{-1/2}\left({dx_5}^2 + {dx_6}^2 + {dx_7}^2 + {dx_8}^2\right) \\
e^{\phi} &= g_s H_1^{1/2} H_5^{-1/2} \\
A_{09} &= \frac{1}{2}\left({H_1}^{-1} - 1\right) \\
\bar{A}_{056789} &= \frac{1}{2}\left({H_5}^{-1} - 1\right),
\end{aligned}
\right. \tag{6.19}
$$

where we dropped the r-dependence and

$$
H_p(r) = 1 + \frac{Q_p}{r^2} = 1 + \frac{c_p^{(5)} N_p}{r^2}
$$

with $r^2 = {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2$. We use bars to make clear that the potentials A, $\bar{A}$ etcetera are completely different and originate from other branes. The potential $A = A_2 = A_{09}\ dt \wedge dx^9$ is coupled electrically to the D1-brane, which lies in the x^9 direction. Similarly, the D5-brane, that lies in the $x^5, \ldots, x^9$ directions, is electrically charged with respect to $\bar{A}$. We will see in section 6.3.3 what happens to these fields when compactifying to five dimensions. In this section we consider how the metric and the dilaton change.

Let us compactify $x^5, \ldots, x^8$ over T^4. All corresponding metric components are diagonal such that we can discard them:

$$\begin{aligned} {ds_6}^2 = {} & {H_1}^{-1/2}{H_5}^{-1/2}\left[-dt^2 + {dx_9}^2 + K(dt - dx_9)^2\right] \\ & + {H_1}^{1/2}{H_5}^{1/2}\left({dx_1}^2 + {dx_2}^2 + {dx_3}^2 + {dx_4}^2\right). \end{aligned}$$

The next step is to compactify this black string over a circle to get rid of the x^9 direction. Note that there is a cross term between x^9 and t such that we have to use the Kaluza-Klein procedure. Luckily, we already performed the Kaluza-Klein reduction of W from D to $D-1$ dimensions. Using the result (5.7) gives (still in string frame):

$$\begin{aligned} {ds_5}^2 = {} & -\,{H_1}^{-1/2}{H_5}^{-1/2}\,(1+K)^{-1}\,dt^2 \\ & + {H_1}^{1/2}{H_5}^{1/2}\left({dx_1}^2 + {dx_2}^2 + {dx_3}^2 + {dx_4}^2\right) \end{aligned}$$

with the additional Kaluza-Klein gauge field

$$\bar{\bar{A}}_t = {H_W}^{-1} - 1 = (1+K)^{-1} - 1. \tag{6.20}$$

Now our objective is to go to Einstein frame to see whether we have indeed arrived at a black hole. In five dimensions (3.9) implies that

$$g_{\mu\nu} = G_{\mu\nu} e^{-4\tilde{\phi}_5/3}$$

and we see from (5.4) that

$$\begin{aligned} e^{-2\phi_5} &= \sqrt{G_{55}\cdot \ldots \cdot G_{99}} \cdot e^{-2\phi_{10}} \\ &= {g_s}^{-2}\left({H_1}^{3/4}{H_5}^{-5/4}(1+K)^{1/2}\right)\cdot {H_1}^{-1}H_5 \\ &= {g_s}^{-2}{H_1}^{-1/4}{H_5}^{-1/4}(1+K)^{1/2}. \end{aligned}$$

Thus

$$e^{-4\tilde{\phi}_5/3} = {H_1}^{-1/6}{H_5}^{-1/6}(1+K)^{1/3}$$

such that the metric in Einstein frame becomes

$$\begin{cases} {ds_5}^2 = -\,[H_1H_5(1+K)]^{-2/3}\,dt^2 \\ \qquad\quad + [H_1H_5(1+K)]^{1/3}\left({dx_1}^2 + {dx_2}^2 + {dx_3}^2 + {dx_4}^2\right) \\ e^{\phi_5} = g_s H_1^{1/8} H_5^{1/8}(1+K)^{-1/4}. \end{cases} \tag{6.21}$$

We claimed that we would arrive at the Tangherlini black hole. Let us check this. Namely, if we choose the charges such that

$$Q := Q_1 = Q_5 = Q_W \tag{6.22}$$

then all harmonic functions, including $1 + K$, are given by

$$H = 1 + \frac{Q}{r^2}.$$

Now the metric in Einstein frame can be written as

$$\begin{cases} ds_5{}^2 = & -H^{-2}dt^2 + H\left(dx_1{}^2 + dx_2{}^2 + dx_3{}^2 + dx_4{}^2\right) \\ e^{\phi_5} = & g_s, \end{cases}$$

which indeed equals the Tangherlini metric (2.12).

6.3.1 Macroscopic Entropy

Let us calculate the entropy of the black hole macroscopically, i.e. using the Bekenstein-Hawking formula (2.14).

Note that the outcome is independent of the number of dimensions. Namely, using the dimensional dependence of the Newton constant (5.3),

$$S_{BH} = \frac{1}{G_N^{(10)}}\frac{A^{(10)}}{4} = \frac{1}{G_N^{(d)}(2\pi)^{10-d}V_{10-d}}\frac{A^{(d)}(2\pi)^{10-d}V_{10-d}}{4} = \frac{1}{G_N^{(d)}}\frac{A^{(d)}}{4}.$$

We compute the entropy in five dimensions from the Einstein metric (6.21). Since we are still in isotropic coordinates, the area of the event horizon is

$$\begin{aligned} A^{(5)} &= \oint_{S^3} \sqrt{g_{22}g_{33}g_{44}}\,\Bigg|_{r=0} \\ &= \sqrt{H_1 H_5 (1+K)}r^3 2\pi^2\,\Big|_{r=0} \\ &= 2\pi^2\sqrt{c_1 N_1 c_5 N_5 c_W N_W}. \end{aligned}$$

Using the coefficients (6.11), (6.12) and (6.13) we obtain

$$c_1 c_5 c_W = \frac{16\left(G_N^{(5)}\right)^2}{\pi^2}$$

such that the entropy of the black hole is

$$S = 2\pi\sqrt{N_1 N_5 N_W}, \tag{6.23}$$

where N_1 and N_5 are the number of D1 and D5 branes and N_W is an integer depending on the momentum of W.

Let us now consider the case where all charges are equal, i.e. when the black hole is Tangherlini. The condition (6.22) implies that

$$N_1 N_5 N_W = \frac{Q^3}{c_1 c_5 c_W} = \frac{\pi^2 Q^3}{16\left(G_N^{(5)}\right)^2}$$

so the entropy of the Tangherlini black hole is

$$S = \frac{\pi^2}{2G_N^{(5)}} Q^{3/2}.$$

6.3.2 Microscopic Entropy

Strominger and Vafa [5] derived the black hole entropy microscopically by counting all possible states from the D-brane point of view. We summarize this procedure in this section and check that it corresponds to the Bekenstein-Hawking entropy (6.23).

For a system of D1 and D5 branes we expect as quantum numbers: the number of D1 branes, the number of D5 branes, momentum and angular momentum. The first two are given by N_1 and N_5 as before. However, it is quite unclear to what kind of momentum the quantum number N_W corresponds. Note that the system of smeared D1 and D5 branes has symmetry group

$$\begin{array}{lllll} SO(1,1) & \times & SO(4)_{\parallel} & \times & SO(4)_{\perp} \\ (t, x^9) & & (x^5, \ldots, x^8) & & (x^1, \ldots, x^4). \end{array}$$

As the branes preserve this symmetry, they cannot carry any momentum and we have to find another source for momentum. It turns out that momentum is carried by open strings ending on the D-branes.

Now we make the same simplification as was done in the original paper by Strominger and Vafa [5]:

$$\left[\mathrm{Vol}(T^4)\right]^{1/4} \ll R_9, \tag{6.24}$$

i.e. the volume of the torus is very small compared to the radius of the circle (recall that we compactify over $T^4 \times R_9$). Therefore, we effectively have a $(1+1)$-dimensional theory on the D-branes (only depending on x^9 and t) and the momentum is directed along x^9.

We quote the resulting number of open strings that end on the D-branes and together give rise to the momentum N_W. Namely, the open strings are oriented and can either span between D1 and D5 or begin and end on the same type of brane. That is, we can have 1-1, 1-5, 5-1 and 5-5 open strings. It turns out that the entropy is maximized when there are only 1-5 and

5-1 strings [13]. Moreover, the number of bosonic strings is $4N_1N_5$ and the number of fermionic strings is also $4N_1N_5$. This number comes from the fact that there are N_1N_5 duos of D1 and D5 branes between which 1-5 strings can be located. This is multiplied by the 2 possible orientations of the strings. Moreover, the strings are described by a 2-dimensional representation. This state counting is explained in more detail in [12].

In addition, we mention the central charge c of the system. Central charge is defined as a charge that commutes with all operators in the super Poincaré algebra, i.e. the algebra of extended supersymmetry, which is used in supergravity. More information on central charge can be found in [13]. Since a boson(ic string) carries central charge $c = 1$ and a fermion(ic string) has $c = 1/2$, we see that our system of $4N_1N_5$ bosonic and fermionic strings has central charge

$$c = 4N_1N_5\left(1 + \frac{1}{2}\right) = 6N_1N_5. \tag{6.25}$$

Now the question is: in how many ways can we distribute the integer momentum N_W over the $4N_1N_5$ bosonic and $4N_1N_5$ fermionic strings? Let us denote this degeneracy by $\Omega(N_W)$. It is legitimate to call this a degeneracy since all open strings end on branes that sit at the same point in the compactified spacetime, i.e. all endpoints are located in the center of the black hole. Therefore, the strings are massless states, which contribute the same amount of energy. In the following our aim is to compute $\Omega(N_W)$ in order to find the microscopic entropy $S = \ln(\Omega)$.

We observe that $\Omega(N_W)$ also appears in the partition function

$$Z = \sum_{x_i} e^{-\beta H(x_1, x_2, \dots)},$$

where the sum runs over all possible states. Namely, the energy of the system consists only of momentum since we are dealing with massless states. It is given by the $(1+1)$-dimensional Hamiltonian

$$H = \frac{N_W}{R}.$$

We define q such that

$$Z = \sum_{\text{all states}} e^{-\beta N_W/R} = \sum_{\text{all states}} q^{N_W}.$$

When multiple states have a certain integer momentum N_W then they contribute by the same amount. Therefore, we can rewrite the sum as

$$Z = \sum_{N_W=-n_0}^{\infty} \Omega(N_W) q^{N_W}$$

such that the partition function has a pole of order n_0. Then the coefficients $\Omega(N_W)$ of the expansion are given by

$$\Omega(N_W) = \frac{1}{2\pi i} \oint_\gamma \frac{Z(q)}{q^{N_W+1}}\, dq, \tag{6.26}$$

where γ is a counterclockwise contour in the complex plane enclosing the pole.

Performing a coordinate transformation from q to τ with

$$q = e^{2\pi i \tau},$$

we can rewrite (6.26) as

$$\Omega(N_W) = \int \frac{Z(\tau)}{e^{2\pi i \tau N_W}}\, d\tau.$$

Cardy explains that there is a modular invariance $Z(\tau) = Z(-1/\tau)$ and how this implies that we have for small τ [35] [36]

$$Z(\tau) \simeq e^{2\pi i n_0/\tau}.$$

Moreover, it turns out that the central charge c is such that $n_0 = c/24$. Then

$$\Omega(N_W) \simeq \int e^{2\pi i [c/24\tau - N_W \tau]}\, d\tau.$$

Now we estimate the integral using a saddle-point approximation. Observe that the exponent $f(\tau) = 2\pi i\,[c/24\tau - N_W\tau]$ has two extremal values of which one is a maximum, located at

$$\tau_0 = \sqrt{\frac{c}{24N_W}}\; i.$$

Approximating $f(\tau) \approx f(\tau_0) + 1/2\;(\tau-\tau_0)^2 f''(\tau_0)$ we find the Gaussian

$$\begin{aligned}\Omega(N_W) &\approx \int e^{2\pi\sqrt{\frac{cN_W}{6}}} e^{2\pi N_W \frac{24N_W}{c}(\tau-\tau_0)^2}\, d\tau \\ &= \frac{c^{1/4}}{\sqrt{2N_W}(24N_W)^{1/4}} e^{2\pi\sqrt{\frac{cN_W}{6}}} = A e^{2\pi\sqrt{\frac{cN_W}{6}}}.\end{aligned}$$

Finally, we can calculate the microscopic entropy:

$$S = \ln[\Omega(N_W)] \simeq 2\pi\sqrt{\frac{cN_W}{6}} + \ln(A). \tag{6.27}$$

Substituting the central charge (6.25) we find up to leading order

$$S = 2\pi\sqrt{N_1 N_5 N_W},$$

which corresponds nicely to the macroscopic entropy (6.23).

6.3.3 Gauge Fields and Charges

In the previous sections we saw how the metric and dilaton can be compactified. In this section we take a look at how the gauge fields are compactified. Then we will compute the corresponding charges in five dimensions.

Gauge Fields in $D = 10$

Recall from (6.19) that we have two gauge fields A and $\bar{A}$ in ten dimensions that correspond to D1 and D5, respectively:

$$\begin{cases} A_{09} = \frac{1}{2}\left({H_1}^{-1} - 1\right) \\ \bar{A}_{056789} = \frac{1}{2}\left({H_5}^{-1} - 1\right). \end{cases} \tag{6.28}$$

Note that we only write down the nonzero components of these fields. That is, we have a 2-form potential A with A_{09} as above and all other components equal to zero. And we have a 6-form $\bar{A}$ with as only nonzero component $\bar{A}_{056789}$.

It is more common to rewrite this 7-form field strength $\bar{F} = d\bar{A}$ to its Hodge dual $^*\bar{F}$ in 10 dimensions since that is a lower-dimensional form. Recall from Figure 4.1 that the D5-brane is then charged magnetically with respect to the 3-form $^*\bar{F}$. First we compute the field strength $\bar{F}$ of

$$\bar{A}_6 = \bar{A}_{056789}\ dx^0 \wedge dx^5 \wedge \cdots \wedge dx^9.$$

Namely,

$$\begin{aligned} \bar{F}_7 &= \partial_\mu \bar{A}_{056789}\ dx^\mu dx^0 \wedge dx^5 \wedge \cdots \wedge dx^9 \\ &= -\frac{1}{2}{H_5}^{-2}(\partial_\mu H_5)\ dx^\mu \wedge dx^0 \wedge dx^5 \wedge \cdots \wedge dx^9 \end{aligned}$$

and

$$H_5 = 1 + \frac{Q_5}{r^2},$$

where $r^2 = {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2$, such that only $\partial_i H_5$ with $i = 1, 2, 3, 4$ is nonzero. Hence the only nonzero components of $\bar{F}_7$ are $\bar{F}_{i056789}$. Now we take the Hodge dual using (2.6),

$$\bar{H}_{jkl} = (^*\bar{F})_{jkl} = \frac{1}{7!}\tilde{\epsilon}^{\mu_1\ldots\mu_7}{}_{jkl}\ \bar{F}_{\mu_1\ldots\mu_7} = \tilde{\epsilon}^{i056789}{}_{jkl}\ \bar{F}_{i056789}, \tag{6.29}$$

where $i, j, k, l = 1, 2, 3, 4$ and $\tilde{\epsilon}_{0123456789} = +1$ is the curved space epsilon tensor.

Thus we have arrived at $\bar{H}_3$, the dual of the field strength $\bar{F}_7 = d\bar{A}_6$. Now we rewrite our result to a simpler form by lowering all indices of the epsilon-tensor and replacing it by the flat space epsilon tensor. Writing the curved space tensor as $\tilde{\epsilon}$ and the flat space tensor as ϵ, they are related by

$$\tilde{\epsilon}_{\mu_1\ldots\mu_n} = \sqrt{-g}\ \epsilon_{\mu_1\ldots\mu_n}.$$

The determinant of the metric (6.19) is

$$\det(g) = {H_1}^3{H_5}^{-1}\left((-1+K)(1+K) - K^2\right) = -{H_1}^3{H_5}^{-1}. \qquad (6.30)$$

Moreover, we lower the indices

$$\tilde{\epsilon}^{i056789}{}_{jkl} = \tilde{\epsilon}_{\mu_1\mu_2\ldots\mu_7 jkl}\ g^{\mu_1 i}g^{\mu_2 0}g^{\mu_3 5}g^{\mu_4 6}\ldots g^{\mu_7 9}.$$

Since the only off-diagonal component is g_{09} this equals

$$\begin{aligned}\tilde{\epsilon}^{i056789}{}_{jkl} &= \tilde{\epsilon}_{i056789jkl}\ g^{ii}g^{00}g^{55}g^{66}\ldots g^{99}\\ &\quad + \tilde{\epsilon}_{i956780jkl}\ g^{ii}g^{90}g^{55}g^{66}\ldots g^{09}\\ &= \tilde{\epsilon}_{i056789jkl}\ g^{ii}g^{55}g^{66}\ldots g^{88}\ \left(g^{00}g^{99} - g^{90}g^{09}\right),\end{aligned}$$

where there is no summation over i and we used the antisymmetry of $\tilde{\epsilon}$. Now we need the inverse components of the metric. Note that $g^{11},\ldots,g^{88}$ are equal to $g_{11},\ldots,g_{88}$ to the power minus one. However, we need to invert the block

$$A = \begin{pmatrix} g_{00} & g_{09} \\ g_{90} & g_{99} \end{pmatrix} = {H_1}^{-1/2}{H_5}^{-1/2}\begin{pmatrix} -1+K & -K \\ -K & 1+K \end{pmatrix}$$

yielding

$$A^{-1} = \begin{pmatrix} g^{00} & g^{09} \\ g^{90} & g^{99} \end{pmatrix} = {H_1}^{1/2}{H_5}^{1/2}\begin{pmatrix} -1-K & -K \\ -K & 1-K \end{pmatrix}$$

such that

$$\tilde{\epsilon}^{i056789}{}_{jkl} = -\tilde{\epsilon}_{i056789jkl}\ {H_1}^{-3/2}{H_5}^{5/2}. \qquad (6.31)$$

Combining the results (6.30) and (6.31) we find

$$\tilde{\epsilon}^{i056789}{}_{jkl} = -\tilde{\epsilon}_{i056789jkl}\ {H_1}^{-3/2}{H_5}^{5/2} = -\epsilon_{i056789jkl}\ {H_5}^2$$

such that (6.29) becomes

$$\begin{aligned}\bar{H}_{jkl} = ({}^*\bar{F})_{jkl} &= -\ \epsilon_{i056789jkl}\ {H_5}^2\bar{F}_{i056789}\\ &= -\ \frac{1}{2}\epsilon_{i056789jkl}(\partial_i H_5),\end{aligned}$$

where summation over i is implied. Redefining

$$\epsilon_{ijkl} = \epsilon_{i056789jkl} = \epsilon_{056789ijkl}$$

we arrive at

$$\bar{H}_{jkl} = ({}^*\bar{F})_{jkl} = -\frac{1}{2}\epsilon_{ijkl}(\partial^i H_5), \tag{6.32}$$

where ϵ_{ijkl} is the flat space epsilon tensor. Note that the gauge field of D5 is equal to the gauge field of NS5 in (4.9). The only difference is in the constants: $Q_5 = c_{D5}N_{D5}$ versus $Q_5 = c_{NS5}, N_{NS5}$.

Thus we have rewritten the gauge fields in $D = 10$ (6.28) as a 2-form potential A and a 3-form field strength $\bar{H}$ with nonzero components

$$\begin{cases} A_{09} = \frac{1}{2}\left(H_1{}^{-1} - 1\right) \\ \bar{H}_{jkl} = -\frac{1}{2}\epsilon_{ijkl}(\partial^i H_5), \end{cases} \tag{6.33}$$

where $i, j, k, l = 1, 2, 3, 4$.

Gauge Fields in $d = 5$

Let us see what happens to the gauge fields (6.33) in $d = 5$. Since $\bar{H}_{ijk}$ only depends on the extended coordinates $x^1, \ldots, x^4$ it remains unchanged under compactification of the other coordinates. On the other hand, $A_2 = A_{09}\ dx^0 \wedge dx^9$ depends on x^9 so it may change.

Namely, let us write the coordinates in $d = 10$ with two hats and the coordinates in $d = 6$ with one hat. We write the resulting coordinates in $d = 5$ without hats. Now we split the tensor

$$A_2 = A_{\hat{\hat{\mu}}\hat{\hat{\nu}}}\ dx^{\hat{\hat{\mu}}} dx^{\hat{\hat{\nu}}}$$

in components in each compactification step:

$$\begin{array}{lcl|l}
D = \hat{\hat{10}}: & & d = \hat{6}: & d = 5: \\
A_{\hat{\hat{\mu}}\hat{\hat{\nu}}}\ \text{tensor} & \longrightarrow & \begin{cases} A_{\hat{\mu}\hat{\nu}}\ \text{tensor} \\ \hline A_{\hat{\mu}a}\ \text{vector} \\ \hline A_{ab}\ \text{scalar} \end{cases} & \longrightarrow \begin{cases} A_{\mu\nu}\ \ \text{tensor} \\ A_{\mu 9}\ \ \text{5-vector} \\ \hline A_{\mu a}\ \ \text{5-vector (4x)} \\ A_{9a}\ \ \text{scalar (4x)} \\ \hline A_{ab}\ \ \text{scalar (10x)}, \end{cases}
\end{array} \tag{6.34}$$

where $a, b = 5, 6, 7, 8$ are the coordinates on the torus. It is important to realize whether these objects are tensors, vectors or scalars in five dimensions with respect to the coordinates μ, ν in $d = 5$, as indicated above. We have written this division in general form. However, most components are zero: the only nonzero component in ten dimensions is $A_{\hat{\hat{\mu}}\hat{\hat{\nu}}} = A_{09}$. Therefore, in $d = 6$ only the tensor $A_{\hat{\mu}\hat{\nu}}$ is nonzero such that in $d = 5$ only the 5-vector

$A_{\mu 9}$ is nonzero. It is given by

$$A_{\mu 9} = \begin{pmatrix} \frac{1}{2}\left(H_1{}^{-1} - 1\right) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Analagously, we make a division for $\bar{H}_{\hat{\hat{\lambda}}\hat{\hat{\mu}}\hat{\hat{\nu}}}$. However, as we noted earlier, this field only depends on the extended coordinates such that the tensor $\bar{H}_{\lambda\mu\nu}$ remains unchanged while all other tensors, vectors and scalars in $d = 5$ are zero.

Thus the only difference between $D = 10$ and $d = 5$ is that the tensor $A_{\hat{\hat{\mu}}\hat{\hat{\nu}}} = A_{09}\ dx^0 dx^9$ is replaced by a 5-vector $A_{\mu 9}$, where the nonzero component has the same value as in $D = 10$. Moreover, we have an additional gauge field: the Kaluza-Klein field (6.20) originating from W. So the resulting fields in $d = 5$ are: the potential A_1, the field strength $\bar{H}_3$ and the potential $\bar{\bar{A}}_1$ given by

$$\begin{cases} (1) \qquad A_{09} & = \dfrac{1}{2}\left(H_1{}^{-1} - 1\right) \\ (2) \qquad \bar{H}_{jkl} & = -\dfrac{1}{2}\epsilon_{ijkl}(\partial^i H_5) \\ (3) \qquad \bar{\bar{A}}_t & = \left(H_W{}^{-1} - 1\right), \end{cases} \tag{6.35}$$

where all other components are zero.

Charges in $d = 5$

Let us now compute the charges of the black hole (6.21). Recall the definitions of electric charge Q (2.10) and magnetic charge P (2.9). We need to integrate an n-form over an n-dimensional sphere enclosing the source. Since a p-dimensional object or brane couples to a $(p+2)$-form field strength, this object carries electric charge

$$Q = \frac{1}{4\pi}\oint_{D-p-2} {}^*F_{p+2} \tag{6.36}$$

and magnetic charge

$$P = \frac{1}{4\pi}\oint_{p+2} F_{p+2}. \tag{6.37}$$

In five dimensions, we see that integrating $\tilde{F}_3 = {}^*F_2$ over S^3 gives electric charge. This works since S^3 does indeed enclose a point-like object in $d = 5$. On the other hand, magnetic charge would be obtained by integrating F_2

over S^2. However, the two-sphere cannot enclose a point-like object in $d = 5$ and therefore a 5-dimensional black hole can have no magnetic charge at all. It is worth mentioning that a (black) string can be enclosed by S^2 such that it can be magnetically charged in $d = 5$.

Given the fields (6.35) the black hole can be electrically charged with respect to three 2-form field strengths: (1) $F = dA$, (2) ${}^*\bar{H}$ and (3) $\bar{\bar{F}} = d\bar{\bar{A}}$. We denote these charges by $Q, \bar{Q}$ and $\bar{\bar{Q}}$, respectively, and we calculate them in the following.

(1) The field strength of (1) is

$$F = dA = \partial_\mu A_{09} dx^\mu \wedge dt = -\frac{1}{2} {H_1}^{-2} (\partial_\mu H_1)\ dx^\mu \wedge dt.$$

From

$$H_1 = 1 + \frac{c_1 N_1}{r^2}$$

we see that the only nonzero component of F is F_{rt}. To compute the electric charge (6.36) we integrate over the three-sphere. In radial coordinates this corresponds to

$$Q = \frac{1}{4\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\pi d\chi\ 3!({}^*F)_{\theta\phi\chi} r^3 \sin^2(\theta)\sin(\chi)$$

such that we only need the $\theta\phi\chi$-component of the dual field strength:

$$({}^*F)_{\theta\phi\chi} = \frac{1}{2!} \tilde{\epsilon}^{\ \mu\nu}{}_{\theta\phi\chi}\ F_{\mu\nu} = \tilde{\epsilon}^{\ rt}{}_{\theta\phi\chi}\ F_{rt}.$$

Using the metric (6.21),

$$({}^*F)_{\theta\phi\chi} = \sqrt{-g}\ \epsilon_{rt\theta\phi\chi} g^{rr} g^{tt} F_{rt} = [H_1 H_5 (1+K)]^{2/3} F_{rt},$$

where $\epsilon_{rt\theta\phi\chi} = -1$ cancels to the minus sign of g^{tt}. Thus

$$({}^*F)_{\theta\phi\chi} = -\frac{1}{2} [H_1 H_5 (1+K)]^{2/3} {H_1}^{-2} (\partial_r H_1)$$

and using the area of the unit three-sphere

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\pi d\chi\ \sin^2(\theta)\sin(\chi) = 2\pi^2$$

it follows that

$$\begin{aligned} Q &= -\frac{2\pi^2 \cdot 3!}{4\pi} {H_1}^{-4/3} [H_5(1+K)]^{2/3} \left(-\frac{c_1 Q_1}{r^3}\right) r^3 \\ &= 3\pi {H_1}^{-4/3} [H_5(1+K)]^{2/3} c_1 N_1. \end{aligned}$$

We still have to take the limit $r \to \infty$ such that all harmonic functions go to 1 yielding

$$Q = 3\pi c_1 N_1. \tag{6.38}$$

(2) Now we compute the charge $\bar{Q}$ with respect to the dual of (2), defining $\bar{F}_2 = {}^*\bar{H}_3$. Then the electric charge with respect to $\bar{F}$ is

$$\bar{Q} = \frac{1}{4\pi}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_0^{\pi} d\chi\ 3!({}^*\bar{F})_{\theta\phi\chi} r^3 \sin^2(\theta)\sin(\chi).$$

Effectively we take the dual twice. This should result in a factor of either plus one or minus one. Namely,

$$({}^*\bar{F})_{\theta\phi\chi} = \tilde{\epsilon}\ ^{\theta\phi\chi}{}_{rt} \cdot \tilde{\epsilon}\ ^{rt}{}_{\theta\phi\chi}\bar{H}_{\theta\phi\chi} = -\bar{H}_{\theta\phi\chi} \tag{6.39}$$

noting that lowering indices by the metric cancels to $(\sqrt{-g})^2$ from the curved space epsilon tensor. Now

$$\begin{aligned}\bar{Q} &= 3\pi\left(\frac{1}{2}\epsilon_{r\theta\phi\chi}\partial_r H_5\right) r^3 \\ &= -3\pi c_5 N_5.\end{aligned}$$

(3) The charge $\bar{\bar{Q}}$ with respect to $\bar{\bar{A}}_t$ is exactly analogous to the charge Q in (6.38) except for a factor 1/2 that is absent in the Kaluza-Klein field. Therefore, we have

$$\bar{\bar{Q}} = 6\pi c_W N_W.$$

We conclude that the 5-dimensional black hole has three electric charges:

$$\begin{cases} (1) & Q = 3\pi c_1^{(5)} N_1 \\ (2) & \bar{Q} = -3\pi c_5^{(5)} N_5 \\ (3) & \bar{\bar{Q}} = 6\pi c_W^{(5)} N_W \end{cases}$$

with respect to the fields (1), dual of (2) and (3) in (6.35), respectively. Moreover, we can substitute the coefficients from (6.11), (6.12) and (6.13):

$$\begin{cases} (1) & Q = \dfrac{12 G_N^{(5)} R_9}{g_s\alpha'} N_1 \\ (2) & \bar{Q} = -3\pi g_s \alpha' N_5 \\ (3) & \bar{\bar{Q}} = \dfrac{24 G_N^{(5)}}{R_9} N_W. \end{cases}$$

6.4 4-dimensional Black Holes

In this section we repeat the procedure of the previous section but now we compactify to $d = 4$ with a different configuration, following [33]. This time we start in type IIA supergravity.

We consider a D2-brane, a D6-brane and a pp-wave W. In addition, we have an NS5-brane. We let x^0, x^1, x^2, x^3 be the extended coordinates and we lay the branes in the following way:

$$
\begin{array}{lcccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{D2} & - & \cdot & \cdot & \cdot & - & \perp & \perp & \perp & \perp & - \\
\text{NS5} & - & \cdot & \cdot & \cdot & \perp & - & - & - & - & - \\
\text{D6} & - & \cdot & \cdot & \cdot & - & - & - & - & - & - \\
\text{W} & - & \cdot & \cdot & \cdot & \perp & \perp & \perp & \perp & \perp & \rightarrow .
\end{array}
\tag{6.40}
$$

This time we compactify on a 6-torus in two steps:

$$
\begin{array}{ll}
D = 10 & \\
\Big\downarrow {\scriptstyle x^4,\ldots,x^8 \text{ on } T^5} & \\
D = 5 & \text{black string} \\
\Big\downarrow {\scriptstyle x^9 \text{ on } S^1} & \\
D = 4 & \text{black hole}
\end{array}
$$

We use the harmonic function rule to combine the metrics of a D2-brane (4.8), D6-brane (4.8), NS5-brane (4.9) and pp-wave W (5.6), giving (in string frame):

$$
\left\{
\begin{aligned}
ds_{10}{}^2 &= H_2{}^{-1/2} H_6{}^{-1/2} \left[-dt^2 + dx_9{}^2 + K(dt - dx_9)^2 \right] \\
&\quad + H_2{}^{1/2} H_5 H_6{}^{1/2} \left(dx_1{}^2 + dx_2{}^2 + dx_3{}^2 \right) \\
&\quad + H_2{}^{-1/2} H_5 H_6{}^{-1/2} dx_4{}^2 \\
&\quad + H_2{}^{1/2} H_6{}^{-1/2} \left(dx_5{}^2 + dx_6{}^2 + dx_7{}^2 + dx_8{}^2 \right) \\
e^{\phi} &= g_s H_2^{1/4} H_5^{1/2} H_6^{-3/4} \\
A_{049} &= \frac{1}{2} \left(H_2{}^{-1} - 1 \right) \\
\bar{A}_{056789} &= \frac{1}{2} \left(H_5{}^{-1} - 1 \right) \\
\bar{\bar{A}}_{0456789} &= \frac{1}{2} \left(H_6{}^{-1} - 1 \right) ,
\end{aligned}
\right.
\tag{6.41}
$$

where we used that the NS5-brane field equals the D5-brane field (see the comment below (6.32)). Recall that the harmonic functions in $d = 4$ are given by

$$
H_p(r) = 1 + \frac{Q_p}{r} = 1 + \frac{c_p^{(4)} N_p}{r}
$$

with $r^2 = x_1{}^2 + x_2{}^2 + x_3{}^2$.

Compactifying $x^4, \dots, x^8$ over the 5-torus, we find the metric (still in string frame):

$$\begin{aligned} ds_5{}^2 = {} & H_2{}^{-1/2} H_6{}^{-1/2} \left[-dt^2 + dx_9{}^2 + K(dt - dx_9)^2 \right] \\ & + H_2{}^{1/2} H_5 H_6{}^{1/2} \left(dx_1{}^2 + dx_2{}^2 + dx_3{}^2 \right). \end{aligned}$$

Now we are left with compactifying x^9 over a circle. Again using the result (5.7) for W we find (in string frame):

$$\begin{aligned} ds_4{}^2 = {} & - H_2{}^{-1/2} H_6{}^{-1/2} (1+K)^{-1} dt^2 \\ & + H_2{}^{1/2} H_5 H_6{}^{1/2} \left(dx_1{}^2 + dx_2{}^2 + dx_3{}^2 \right). \end{aligned}$$

Going to Einstein frame we use (3.9) for $d = 4$:

$$g_{\mu\nu} = G_{\mu\nu} e^{-2\tilde{\phi}_4}$$

and we have

$$\begin{aligned} e^{-2\phi_4} &= \sqrt{G_{44} \cdot \ldots \cdot G_{99}} \cdot e^{-2\phi_{10}} \\ &= g_s{}^{-2} \left(H_2{}^{1/2} H_5{}^{1/2} H_6{}^{-3/2} (1+K)^{1/2} \right) \cdot H_2{}^{-1/2} H_5{}^{-1} H_6{}^{3/2} \\ &= g_s{}^{-2} H_5{}^{-1/2} (1+K)^{1/2}. \end{aligned}$$

Hence the metric in Einstein frame becomes

$$\left\{ \begin{aligned} ds_4{}^2 = {} & - \left[\sqrt{H_2(r) H_5(r) H_6(r) \left(1 + K(r)\right)} \right]^{-1} dt^2 \\ & + \sqrt{H_2(r) H_5(r) H_6(r) \left(1 + K(r)\right)} \left(dx_1{}^2 + dx_2{}^2 + dx_3{}^2 \right) \\ e^{\phi_4} = {} & g_s H_5{}^{1/4} (1+K)^{-1/4}. \end{aligned} \right. \tag{6.42}$$

Let us check that we have arrived at the extremal Reissner-Nordström black hole. Namely, when the charges are equal to each other,

$$Q := Q_2 = Q_5 = Q_6 = Q_W,$$

then all harmonic functions become equal to

$$H = 1 + \frac{Q}{r}$$

such that

$$\left\{ \begin{aligned} ds_4{}^2 &= - H^{-2} dt^2 + H^2 \left(dx_1{}^2 + dx_2{}^2 + dx_3{}^2 \right) \\ e^{\phi_4} &= g_s \end{aligned} \right.$$

and we have indeed found the Reissner-Nordström metric (2.4).

6.4.1 Macroscopic Entropy

Now we calculate the macroscopic entropy of the 4-dimensional black hole (6.42). The area of the event horizon is

$$\begin{aligned} A^{(4)} &= \oint_{S^2} \sqrt{g_{22}g_{33}} \Big|_{r=0} \\ &= \sqrt{H_2 H_{NS5} H_6 (1+K)} r^2 4\pi \Big|_{r=0} \\ &= 4\pi \sqrt{c_2 N_2 c_{NS5} N_{NS5} c_6 N_6 c_W N_W}. \end{aligned}$$

Note that the D2-brane lies in the directions x^4, x^9 instead of x^8, x^9 as was used to find the coefficient (6.14). Plugging in $c_2^{(4)}$ with this modification, together with (6.15) and (6.16) yields

$$c_2 c_{NS5} c_6 c_W = 4\left(G_N^{(4)}\right)^2$$

such that the entropy of the black hole is

$$S = 2\pi\sqrt{N_2 N_{NS5} N_6 N_W}, \tag{6.43}$$

where N_2, N_{NS5}, N_6 are the number of D2, NS5 and D6 branes and N_W is an integer that depends on the momentum of W.

In the case of equal charges we find

$$N_2 N_{NS5} N_6 N_W = \frac{Q^4}{c_2 c_{NS5} c_6 c_W} = \frac{Q^4}{4\left(G_N^{(4)}\right)^2}.$$

Thus the entropy of the Reissner-Nordström black hole is

$$S = \frac{\pi}{G_N^{(4)}} Q^2.$$

6.4.2 Microscopic Entropy

In section 6.3.2 we computed the microscopic entropy of the D1 + D5 system in the 5-dimensional case. Now that we are in four dimensions we have a D2+ NS5 + D6 system and the derivation goes completely analagous. We replace the simplification (6.24) by

$$\left[\mathrm{Vol}(T^5)\right]^{1/5} \ll R_9$$

such that we still have a $(1+1)$-dimensional theory. Moreover, the central charge is now different. Replacing $N_1 N_5$ by $N_2 N_{NS5} N_6$, we see that

$$c = 6 N_2 N_{NS5} N_6$$

implying that the leading order of the microscopic entropy (6.27) is now

$$S = 2\pi\sqrt{N_2 N_{NS5} N_6 N_W}.$$

This is in perfect agreement with the macroscopic result (6.43).

6.4.3 Gauge Fields and Charges

In this section we compute the charges of the 4-dimensional black hole.

Gauge Fields in $D = 10$

Again we rewrite the $D = 10$ gauge fields in (6.41) by taking the dual of $A_{01\ldots p}$ if $D-p-1 < p+1$. Analogous to the derivation of (6.32) we obtain a potential A_3 and two field strengths $\bar{H}_3$ and $\bar{\bar{H}}_2$ with nonzero components [12]

$$\begin{cases} A_{049} = \frac{1}{2}\left({H_2}^{-1} - 1\right) & \\ \bar{H}_{ijk} = \frac{1}{2}\epsilon_{ijkl}\partial^l H_5 & (i,j,k,l = 1,2,3,4) \\ \bar{\bar{H}}_{ij} = \frac{1}{2}\epsilon_{ijk}\partial^k H_6 & (i,j,k = 1,2,3). \end{cases}$$

An important difference with respect to the $d = 5$ case is that the coordinate x^4 is now not longer an extended coordinate, i.e. it will be compactified. Thus $\bar{H}$ may change under compactification while $\bar{\bar{H}}$ is invariant.

Gauge Fields in $d = 4$

We split the gauge fields when they go from 10 to 4 dimensions (see (6.34)). The 3-tensor $A_{\hat{\mu}\hat{\nu}\hat{\lambda}}$ splits into a 3-tensor $A_{\mu\nu\lambda}$ and a bunch of 2-tensors, vectors and scalars. As the only nonzero component of $A_{\hat{\mu}\hat{\nu}\hat{\lambda}}$ is A_{049}, just one of these tensors, vectors and scalars is nonzero: the vector $A_{\mu 49}$, which has as nonzero component A_{049}.

On the other hand, $\bar{H}$ splits in two parts when we compactify the coordinates $x^4 \ldots x^9$. To see this, we change notation from $i,j,k,l = 1,2,3,4$ to $i,j,k = 1,2,3$ and we keep 4 explicitly. Then $\bar{H}$ has independent components H_{ijk} and H_{ij4}, where the former is given by

$$\bar{H}_{ijk} = \frac{1}{2}\epsilon_{ijk4}\partial^4 H_5 = 0$$

and the latter is

$$\bar{H}_{ij4} = \frac{1}{2}\epsilon_{ij4l}\partial^l H_5 = -\frac{1}{2}\epsilon_{ijk}\partial^k H_5$$

by redefining $\epsilon_{123} = \epsilon_{1234} = +1$. Thus the 3-tensor $\bar{H}_{\hat{\mu}\hat{\nu}\hat{\lambda}}$ changes into a 2-tensor $\bar{H}_{\mu\nu 4}$ with nonzero components $\mu,\nu = i,j = 1,2,3$.

Moreover, the field $\bar{\bar{H}}$ has only nonzero components with respect to the extended coordinates so it remains unchanged. In addition, we have the Kaluza-Klein field. Thus we have in $d = 4$: the potentials A_1, $\bar{\bar{\bar{A}}}_1$ and the field strengths $\bar{H}_2$, $\bar{\bar{H}}_2$ with nonzero components given by

$$\begin{cases} (1) & A_{049} = \frac{1}{2}\left({H_2}^{-1} - 1\right) \\ (2) & \bar{H}_{ij4} = -\frac{1}{2}\epsilon_{ijk}\partial^k H_5 \\ (3) & \bar{\bar{H}}_{ij} = \frac{1}{2}\epsilon_{ijk}\partial^k H_6 \\ (4) & \bar{\bar{\bar{A}}}_t = \left({H_W}^{-1} - 1\right), \end{cases} \tag{6.44}$$

where now $i, j, k = 1, 2, 3$ in both cases.

Charges in $d = 4$

Now that we are in four dimensions the dual of a 2-form field strength is still a 2-form. Therefore, we need to integrate a 2-form field strength over S^2 for both electric and magnetic charge. Note that we can indeed enclose a point in four spacetime dimensions by S^2. Thus we can have both electric and magnetic charge with respect to the potentials (1), (4) and the field strengths (2), (3) in (6.44). However, (1) and (4) only have nonzero components involving the time coordinate such that magnetic charges with respect to them are zero. On the other hand, (2) and (3) are not charged electrically as they only have spatial components. We are left with computing the electric charges with respect to (1), (4) and the magnetic charges with respect to (2), (3):

(1) The electric charge with respect to A,

$$Q = \frac{1}{4\pi}\int_0^\pi d\theta \int_0^{2\pi} d\phi\ 2!({}^*F)_{\theta\phi} r^2 \sin(\theta),$$

is analogous to the first electric charge in five dimensions (6.38). The only differences are the factor of 2! instead of 3! and that the area of the unit two-sphere is 4π instead of $2\pi^2$ such that

$$Q = 4c_2 N_2.$$

(2) For magnetic charge with respect to $\bar{H}$ we have

$$\begin{aligned} \bar{P} &= \frac{1}{4\pi}\int_0^\pi d\theta \int_0^{2\pi} d\phi\ 2!\bar{H}_{\theta\phi} r^2 \sin(\theta) \\ &= -\epsilon_{\theta\phi r}(\partial_r H_5) r^2 \\ &= c_{NS5} N_{NS5}. \end{aligned}$$

(3) The magnetic charge with respect to $\bar{\bar{H}}$ goes analogously,

$$\bar{\bar{P}} = c_6 N_6.$$

(4) The electric charge of $\bar{\bar{\bar{A}}}$ is analogous to (1) except for the absence of a factor 1/2 in the potential. Thus

$$\bar{\bar{\bar{Q}}} = 8c_W N_W.$$

Summarizing the results, we have the charges

$$\begin{cases} (1) & Q &= 4c_2^{(4)} N_2 \\ (2) & \bar{P} &= c_{NS5}{}^{(4)} N_{NS5} \\ (3) & \bar{\bar{P}} &= c_6^{(4)} N_6 \\ (4) & \bar{\bar{\bar{Q}}} &= 8c_W^{(4)} N_W \end{cases}$$

with respect to the fields (1)-(4) in (6.44). Plugging in the coefficients (6.14), (6.15) and (6.16) yields

$$\begin{cases} (1) & Q &= \dfrac{16G_N^{(4)} R_8 R_9}{g_s \alpha'^{3/2}} N_2 \\ (2) & \bar{P} &= \dfrac{\alpha'}{2R_4} N_{NS5} \\ (3) & \bar{\bar{P}} &= \dfrac{g_s \sqrt{\alpha'}}{2} N_6 \\ (4) & \bar{\bar{\bar{Q}}} &= \dfrac{32G_N^{(4)}}{R_9} N_W. \end{cases}$$

6.5 Applying Electromagnetic Duality to 4-dimensional Black Holes

In the previous section we learned that the 4-dimensional extremal Reissner-Nordström black hole has four charges: two electric and two magnetic ones. Now we would like to set the magnetic charges of the black hole to zero in order to obtain a simpler black hole description. We will try to do so by an electromagnetic duality transformation (2.11) with $Sp(2,\mathbb{R})$ instead of $GL(2,\mathbb{R})$ (as we are in a Lorentzian spacetime). Namely, the black hole is charged with respect to the field (2) $\bar{H}_{ij4}$ as

$$\begin{pmatrix} \bar{P} \\ \bar{Q} \end{pmatrix} = \begin{pmatrix} c_{NS5} N_{NS5} \\ 0 \end{pmatrix}$$

and applying the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in Sp(2,\mathbb{R}) \tag{6.45}$$

the charge changes into

$$\begin{pmatrix} \bar{P} \\ \bar{Q} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_{NS5}N_{NS5} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ c_{NS5}N_{NS5} \end{pmatrix}.$$

Now there is only electric charge left with respect to $\bar{H}_{ij4}$. With the same transformation we change the charge with respect to (3) $\bar{\bar{H}}_{ij}$ to

$$\begin{pmatrix} \bar{\bar{P}} \\ \bar{\bar{Q}} \end{pmatrix} = \begin{pmatrix} 0 \\ c_6 N_6 \end{pmatrix}$$

such that the black hole does not have magnetic charge anymore as long as we do not touch the charges with respect to (1) A_{049} and (4) $\bar{\bar{\bar{A}}}_t$.

In the 10-dimensional action (3.13) all field strengths are located in separate terms. If this would still be the case in the 4-dimensional action then we may indeed apply different electromagnetic duality transformations to the different field strengths $F_{n+1} = dA_n$. Hence, in that case we can transform all magnetic charge to zero. However, it turns out that the action in $d = 4$ looks more complicated and is of the form

$$S \sim \mathcal{N}_{IJ} F_{\mu\nu}{}^{I} F^{\mu\nu J}, \tag{6.46}$$

where $\mathcal{N}_{IJ}$ is a non-trivial matrix containing scalar fields. The summation with respect to I, J runs over all field strengths $F_{\mu\nu}$ occurring in the theory. It follows that one can only act with the same matrix on all charges. In other words, we may not leave the charges with respect to (1) and (4) untouched.

Let us make this more rigorous. We define

$$F^{\pm}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} \pm i\,{}^{*}F_{\mu\nu}\right)$$

and we define another field strength $G_{\mu\nu\pm}$ where the $\pm$ index of $F^{\pm}_{\mu\nu}$ has been lowered by the symmetric matrix $\mathcal{N}_{IJ}$ (that acts as a metric):

$$\begin{cases} G^{\mu\nu}_{+I} &= \mathcal{N}_{IJ} F^{\mu\nu + J} \\ G^{\mu\nu}_{-I} &= \bar{\mathcal{N}}_{IJ} F^{\mu\nu - J}. \end{cases}$$

Then the field equations of (6.46) are given by [37]

$$\begin{cases} \partial^{\mu} \mathrm{Im}\left(F^{+I}_{\mu\nu}\right) = 0 \\ \partial_{\mu} \mathrm{Im}\left(G^{\mu\nu}_{+I}\right) = 0, \end{cases} \tag{6.47}$$

where the first line is the set of Bianchi equations and the second line gives the equations of motion. Compare this to the field equations (2.8) of a theory with decoupled field strengths. Naively, they look quite similar and one could try to define electromagnetic duality on the two-vector

$$\begin{pmatrix} F^{+I}_{\mu\nu} \\ G^{\mu\nu}_{+I} \end{pmatrix}$$

for each field strength I separately. This fails, though, since $G_{+I} = \mathcal{N}_{IJ} F^{+J}$ such that a vector for one fixed I still contains all field strengths $F^J_{\mu\nu}$. Hence one can only define a duality transformation that acts simultaneously on all field strengths such that the result still satisfies (6.47). This is a generalization of $Sp(2, \mathbb{R})$ to $Sp(2m, \mathbb{R})$, where m is the number of field strengths $(I, J = 1, \dots m)$. Namely, $S \in Sp(2m, \mathbb{R})$ if

$$S^T \Omega S = \Omega,$$

where

$$\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

Electromagnetic duality by a matrix $M \in Sp(2m, \mathbb{R})$ is now given by

$$\begin{pmatrix} \tilde{F}^+ \\ \tilde{G}_+ \end{pmatrix} = M \begin{pmatrix} \tilde{F}^+ \\ \tilde{G}_+ \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^+ \\ G_+ \end{pmatrix},$$

where A, B, C, D are $(m \times m)$-matrices and $F^+ = (F^{+1}_{\mu\nu}, \dots, F^{+m}_{\mu\nu})^T$.

We conclude that we have to act with the same matrix (6.45) on the charges (1)-(4). This makes not only the magnetic charges $\bar{P}, \bar{\bar{P}}$ purely electric but the electric charges $Q, \bar{\bar{\bar{Q}}}$ become purely magnetic as well. As a result we are always stuck with two magnetic charges.

Let us now take a look at what happens to the 10-dimensional theory of D-branes when electromagnetic duality is performed as above. Namely, we have the following relation between the 10- and 4-dimensional theories:

$$\begin{array}{ccccccc} S^{(10)} & \longrightarrow & \mathrm{EOM}^{(10)} & \longrightarrow & \mathrm{solution}_1^{(10)} & & \mathrm{solution}_2^{(10)} \\ \downarrow & & & & \downarrow & & \uparrow \\ S^{(4)} & \longrightarrow & \mathrm{EOM}^{(4)} & \longrightarrow & \mathrm{solution}_1^{(4)} & \xrightarrow{\text{EM duality}} & \mathrm{solution}_2^{(4)}, \end{array}$$

where $\mathrm{solution}_1^{(4)}$ is described by the $d = 4$ black hole metric while $\mathrm{solution}_1^{(10)}$ is given by the configuration of branes in 10 dimensions (6.40). Now we wonder what happens if we perform electromagnetic duality in $d = 4$ such that another $\mathrm{solution}_2^{(4)}$ is generated. How does the brane configuration $\mathrm{solution}_2^{(10)}$ look like?

We saw in section 2.3.2 that a transformation of $(Q, P)^T$ is equivalent to the same transformation of $(F_{\mu\nu}, {}^*F_{\mu\nu})^T$. Effectively we have interchanged $F_{\mu\nu}$ with its dual, i.e. we have performed Hodge duality on the field strength and its potential. According to Figure 4.1 a p-brane that is coupled to $A_{01\ldots p}$ changes into a $(D-p-4)$-brane coupled to $A_{0(p+1)\ldots 9}$. Therefore, if we transform all charges (1)-(4) then we end up with the setup solution$_2^{(10)}$:

	0	1	2	3	4	5	6	7	8	9
D0	$-$	$\cdot$	$\cdot$	$\cdot$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
F1	$-$	$\cdot$	$\cdot$	$\cdot$	$-$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$
D4	$-$	$\cdot$	$\cdot$	$\cdot$	$\perp$	$-$	$-$	$-$	$-$	$\perp$
W	$-$	$\cdot$	$\cdot$	$\cdot$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\rightarrow$.

We conclude that a combination of D0 and D4 branes, the fundamental string F1 and momentum W describes another black hole in four dimensions, which has the same black hole metric as our original combination (6.40) of D2, NS5, D6 and W but its charges are different.

7 Conclusion and Outlook

In this thesis we have studied how 5-dimensional and 4-dimensional extremal charged black holes can be described in string theory by D-branes. We compactified a combination of D1-branes, D5-branes and momentum W in type IIB supergravity over a torus T^5. This gave a 5-dimensional black hole that corresponds to the Tangherlini black hole metric (if one chooses the charges to be equal). Afterwards, we compactified a D2-brane, an NS5-brane, a D6-brane and momentum W in type IIA supergravity over T^6. This yields a 4-dimensional black hole that nicely corresponds to the extremal Reissner-Nordström black hole (if the charges are chosen equal).

We studied how the entropy of these 5-dimensional and 4-dimensional black holes can be computed. First we did this macroscopically using the area law of Bekenstein and Hawking. Then we repeated the calculation from a microscopic point of view by counting the number of states that the system of D-branes and open strings ending on the D-branes can be in, making use of the Cardy formula. We checked that the macroscopic and microscopic results for the entropy agree up to leading order.

So far, all these results are known. In addition to this, we calculated the electric and magnetic charges corresponding to the 5-dimensional and 4-dimensional black holes. We found that the 5-dimensional black hole has three electric charges while the 4-dimensional one has two electric and two magnetic charges. In the 4-dimensional case we investigated whether it is possible to set the magnetic charges to zero by making use of electromagnetic duality transformations in order to make the description simpler. In this way we could transform the magnetic charges to electric charge. On the other hand, the two electric charges changed into magnetic charge and therefore we are always stuck with magnetic charge. We conclude that it is not possible to achieve a simpler description in this way.

Last but not least, we looked at what has changed in the microscopic point of view of D-branes $D = 10$ after performing the electromagnetic duality transformation. The result is that we have gone from a D2 + NS5 + D6 + W system to a system of D0 + F1 + D4 + W. We conclude that this combination also describes a 4-dimensional extremal Reissner-Nordström black hole, i.e. it has the same black hole metric but may have other charges.

This final result makes sense since there are many more ways to describe a black hole in string theory than the ones we investigated in this thesis. In the following we comment on other possibilities and ongoing research.

Black Holes from Nothing but D-branes

We considered certain configurations of branes in this thesis but many more configurations are possible that yield similar black holes in $d = 5$ and $d = 4$. In the previous, we looked at a combination of D-branes and momentum W (and NS5 in the $d = 4$ case). It is also possible to describe a black hole by Dp-branes only; for a 4-dimensional black hole one then needs to combine four (or more) Dp-branes. An interesting paper by Balasubramanian and Larsen [38] argues that in order to create a regular extremal black hole after compactification over a six-torus, such a configuration must satisfy three conditions: both the dilaton and the moduli must be finite at the horizon (regularity) and some of the supersymmetry must be preserved (extremality). This restricts the number of possible configurations strongly. We quote all resulting configurations of four Dp-branes. Here we denote a Dp-brane by its parallel directions among brackets e.g. (1256) is a D4-brane lying in the directions x^1, x^2, x^5 and x^6. Then the possible configurations are, up to permutations of $12\ldots6$ [38]:

$$\begin{aligned}
&(123), (345), (146), (256)\\
&(1234), (3456), (1256), ()\\
&(1234), (3456), (16), (25)\\
&(12345), (126), (346), (5)\\
&(123456), (12), (34), (56)
\end{aligned}$$

where x^7, x^8, x^9 are now the extended directions in $d = 4$ and () is a D0-brane. Recall that T-duality in a perpendicular direction adds this direction to the brane and T-duality in a parallel direction removes this direction from the brane. Thus one can see that all configurations above are T-dual to each other. We conclude that there is in fact one unique regular extremal black hole that can be made from four Dp-branes in $d = 4$. This very interesting property only holds in four dimensions.

M-theory

In this thesis we only considered string theory and thus we used D-branes. However, one can also study 11-dimensional M-theory, where black holes can be made out of so-called M-branes. Namely, certain combinations of multiple M2 and M5 give a 4-dimensional black hole after compactification over T^7 [39]. The entropy of a black hole from M-branes can be computed microscopically and turns out to correspond to the macroscopic result [40]. More information on M-branes can be found in [41].

Compactification over Other Manifolds

Instead of compactifying over a torus one can also compactify over other manifolds. For instance over $K3$ or Calabi-Yau (CY) manifolds. More information about $K3$ is given in [11] and the appendices of [21], while an overview of Calabi-Yau compactification is presented in [23].

Multi-centered Black Holes

So far we only considered single-centered black holes and we have seen that they are described by extremal p-brane bound state configurations. However, some D-brane bound states give multi-centered black holes. From a large distance these may look just like single-centered black holes but in fact it are multiple black holes very close to each other, which can all carry their own charges [42]. Multi-centered black holes are extremal solutions of supergravity that are stationary but not static [43]. The exact solutions are derived in [44]. Currently, there is a lot of ongoing research on multi-centered black holes. Some lecture notes on multi-centered black holes are given in [45]. Many more information and open problems are listed in [46].

Further Research

In all cases above (configurations of D-branes only; M-theory; other manifolds; multi-centered black holes) it would be interesting to study the microscopic entropy and check whether this agrees with the macroscopic result. One may also compute its charges and investigate whether the magnetic charge can be set to zero this time (depending on the form of $\mathcal{N}_{IJ}$ in the action). Moreover, one can investigate the effect of electromagnetic duality on the higher 10-dimensional (or 11-dimensional) theory.

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