COSMOLOGICAL SINGULARITY AND BOUNCE IN EINSTEIN-CARTAN-KIBBLE-SCHIAMA GRAVITY

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ABSTRACT

In this thesis we consider a generalised Einstein-Cartan theory, and the effects of including space-time torsion in the description of gravity. We add the most general covariant dimension four operators to general relativity coupling torsion with fermionic fields, with arbitrary strength. If the gravitational action is taken to be he Einstein-Hilbert action, torsion is local and non dynamical and can be integrated out to yield to an effective four-fermion interaction. In this theory we study the dynamics of a collapsing universe that begins in a thermal state and find that – instead of a big crunch singularity – the Universe with torsion undergoes a *bounce*. We solve the dynamical equations (a) classically (without particle production); (b) including the production of fermions in a fixed background in the Hartree-Fock approximation and (c) including the quantum backreaction of fermions onto the background space-time. In the first and last cases the Universe undergoes a bounce. The production of fermions due to the coupling to a contracting homogeneous background speeds up the bounce, implying that the quantum contributions from fermions is negative. When compared with former works on the subject, our treatment is fully microscopic (namely, we treat fermions by solving the corresponding Dirac equations), and quantum (in the sense that we include fermionic loop contributions).

CONTENTS

1 INTRODUCTION

Einstein's theory of general relativity was published nearly 100 years ago. In all this time, we can claim we have understood gravity on macroscopic scales extremely well. We tested the dynamical and geometrical properties of space-time in our solar system, we observed gravitational lensing near galaxies, and indirectly detected black holes.

Such picture of the gravitational interaction, however, is still far from being complete. A first evidence of this is found in cosmological data: nearly 95% of the universe visible through gravitational interaction, is made of *dark matter* and *dark energy*. If we trust the standard model of particle physics to be complete, then gravity must be modified on large scales, and the dark components of the universe be the result of geometrical corrections to the Einstein's equations. On the other hand, if we assume that gravity is the complete theory, then we need extra particles and exotic fields to be added to the standard model's catalogue.

The second main source of sleepless nights for theoreticians, are the complications arising when the microscopic behaviour of gravity is explored. Near black holes singularities, gravity obeys quantum mechanics, and should be described by an unified theory of quantum gravity. Constructing such a theory, however, has proven to be quite a difficult task, and we are still far from a concrete answer.

To improve our understanding, we must search for modifications of gravity that leave unaltered its predictions on solar system's scales, but bring it closer to the concepts at play in the quantum world. Elementary particles, in the standard model, are modelled by irreducible representations of the Poincaré group. Each representation is classified by mass and spin: mass is connected to the translational part of the group, and spin with the rotational part [13]. Since general relativity couples the energy-momentum tensor, which characterise a macroscopic distribution of mass, to the geometrical quantity representing the curvature, it is natural to ask whether also spin could be coupled to the geometry of space-time. The geometrical quantity of interest is, in this case, the torsion tensor, which is the antisymmetric part of the connection, $S^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{[\mu\nu]}$. Varying the matter action with respect to torsion, leads to the spin density of matter,

$$\Pi_{\lambda[\mu\nu]} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta S^{\lambda[\mu\nu]}} \,,$$

which, analogously to the energy-momentum tensor, describes a macroscopic distribution of particles with spin. The resulting theory was proposed for the first time by E. Cartan [5, 6] and in subsequent works [17, 34], and goes by the name of Einstein-Cartan-Kibble-Sciama (ECKS) gravity.

A caveat here is that one can think of spin in two ways: classically or according to quantum mechanics. For example, if planets with spin would couple to torsion we should have observed its consequences on solar system's scales. Since it has not been the case, we should think about torsion more like an operator which couples to the intrinsic spinorial structure of quantum theories. In a sense that will become clearer later, torsion can be seen as a local and topological modification of space-time, that modifies the behaviour of gravity in particular at high energy densities.

Inclusion of torsion in gravitational theories can be understood as follows: Einstein's General Relativity, in first order formalism, has two dynamical variables, the metric $g_{\mu\nu}$ and the connection $\Gamma^{\lambda}{}_{\mu\nu}$. These are sufficient to fully describe the space-time manifold, once dimension and metric signature are specified. Postulating that the gravitational action is proportional to the Ricci scalar, leads to the equations of motion, 10 for the metric and 64 for the connection.

Assuming that the connection is symmetric, and neglecting the 24 torsion components, leads to the equations of motion for the connection whose solutions are the Christoffel symbols. However, without this assumption, Einstein's theory will contain torsion which must then be treated as an independent variable. Cartan went as far as arguing that a twisted coordinate transformation can produce non vanishing torsion [5, 12].

Acknowledging these arguments, leads to considering torsion as a dynamical variable. One must therefore identify its sources and construct the corresponding torsion-matter interaction terms. This can be done in two ways: by including translational symmetry in the theory, next to diffeomorphisms invariance, or by considering only the coordinate transformation symmetry. In this paper we consider the torsion's source to be fermionic matter, which yields to two possibile interactions operators of energy dimension 4[1]

$$\sqrt{-g}S_{\lambda\mu\nu}\epsilon^{\lambda\mu\nu\sigma}\left(\xi\bar{\psi}\gamma^5\gamma_\sigma\psi+\xi'\bar{\psi}\gamma_\sigma\psi\right), \tag{1}$$

that respect Lorentz invariance. The particular choice $\xi = 1$, $\xi' = 0$ corresponds to the Einstein-Cartan gravity, and can be deduced from the above by imposing translational invariance [12]. However we do not know whether translational symmetry is a fundamental component of nature, and we therefore choose to study the most general case (1). Furthermore an interaction term such as (1) might follow from the UV completion of gravity[2] and is of interest because it might lead to a classical theory of gravity devoid of singularities.

1 Here $S^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{[\mu\nu]}$: note the positions and ordering of indices.

2 In Quantum Loop Gravity, for instance, such an interaction does arise with $\xi' = 0$, but $\xi \neq 1$.

We also would like to point out that in neither of the interaction terms (1) torsion couples to the fermions spin, since spin, according to the classification in [25], is given by the spatial part of the tensor matrices of the Clifford basis $\Sigma_i = \frac{i}{4}\epsilon_{ijk}\gamma^{[j}\gamma^{k]} = -\frac{i}{2}\gamma^5\vec{\gamma}\gamma^0$. Constructing a covariant version of spin-torsion coupling leads to operators of dimension five, which couple torsion to the tensorial bilinear

$$\eta S_{\lambda\mu\nu}\nabla^\lambda\bar{\psi}\gamma^{[\mu}\gamma^{\nu]}\psi + \eta'\bar{\psi}\gamma^{[\mu}\gamma^{\nu]}\psi\nabla_\lambda S^\lambda{}_{\mu\nu}\cdots .$$

Such interactions would be more complicated to deal with, and we will not analyse them here. In our theory, torsion couples to the fermionic vector and pseudo-vector bilinears, according to Eq. (1).

Since torsion couples to vector currents, its contribution vanishes when averaged on a spatially isotropic distribution of matter, and therefore the effects of its interactions are important on small scales. For this reason, ECKS gravity is essentially indistinguishable from general relativity on all scales where the latter has been tested, and significantly differs from it only at high energies and at small scales. Prominent examples where one can probe high energies and/or small scales are cosmology and black holes, the former being the subject of the current study. One should keep in mind that, because Cartan theory reduces to general relativity at large scales, no experiment so far has been able to disprove Cartan theory [2, 20, 21, 22], which therefore remains a viable microscopic theory of gravity. It is unlikely that torsion can change the divergence structure of gravity, thereby gravity with torsion remains non-renormalizable and the question of the ultraviolet completion of gravity remains open. This is because torsion, as it turns out, is not a dynamical field, but a Lagrange multiplier, and therefore cannot be quantised.

The literature contains several efforts of making predictions using Einstein-Cartan theory, however what all those references have in common is their use of classical spin fluid as a source of torsion [28, 29, 35].

Classical spin fluid of Weyssenhoof and Raabe induces a canonical spin tensor density, $s^\rho_{\mu\nu} = s_{\mu\nu}u^\rho$, where $s_{\mu\nu}$ is the spin density and $s_{\mu\nu}u^\nu = 0$. Hehl et al [13] point out that "unfortunately, there seems to be no satisfactory Lagrangian for this distribution, and therefore no unambiguous road to a minimally coupled theory".

This classical description is not satisfactory from a field theoretical perspective: in [3, 4] the torsion tensor in the fluid rest frame is given by $S^\lambda{}_{\mu\nu} = 8\pi G s_{\mu\nu}u^\lambda = 2\pi G\delta^\lambda_0\epsilon_{ijk}\bar{\psi}\gamma^5\gamma^k\psi$. This form of the torsion tensor neglects the zeroth component of the axial current, which is, as it will become clearer later, the most important contribution to the stress-energy tensor.

The main purpose of this thesis is to extend the existing classical analysis starting from a microscopic theory, in which no assumptions on the spin fluid are made. We consider the interaction terms (1) setting $\xi' = 0$, which is effectively a generalisation of ECKS theory[3]. We then apply this theory to an homogeneous and isotropic universe, initially in a thermal state, undergoing a gravitational collapse and we show that the contribution induced by torsion coupling prevent the formation of singularities. Instead, the collapsing universe undergoes a bounce. This conclusion holds both when fermions are treated classically –*i.e.* when fermion production due to coupling to gravity is neglected – and quantum field theoretically, when particle creation due to the fermion coupling to a contracting gravitational background is accounted for.

3 The converse case is described in the Appendix E

2 NOTATIONS AND CONVENTIONS

- Unless specified otherwise, we work in units $\hbar = c = 1$
- Indices: greek indices denote space-time coordinates $\mu, \nu, \cdots$; the *first* half of the latin alphabet, $a, b, \cdots$ labels indices in the tangent space; the *second* half of the latin alphabet, $i, j, \cdots$ labels spatial components of vectors and tensor; when the fields spinorial indices are required, we us the *first* half of the greek alphabet, i. e.$\psi \equiv \psi_\alpha$, $\alpha, \beta, \cdots$.
- Index summation notation: when we write indices up and down we mean
$$T_{ab\cdots}Q^{ac\cdots} = \sum_a T_{ab\cdots}Q^{ac\cdots} \,,$$
- Indices raising and lowering, relations between tensors in flat space and in curved space:
$$\begin{aligned}
\eta_{ab} &= \mathrm{diag}(+1,-1,-1,-1), \\
U_{aa_1\cdots} &= \eta_{ab}U^b{}_{a_1\cdots}, \\
U_{\mu\mu_1\cdots} &= g_{\mu\nu}U^\nu{}_{\mu_1\cdots}, \\
T^{\nu_1\nu_2\cdots}_{\mu_1\mu_2\cdots} &= e^{a_1}_{\mu_1}e^{a_2}_{\mu_2}\cdots e^{\nu_1}_{b_1}e^{\nu_2}_{b_2}\cdots T^{b_1b_2\cdots}_{a_1a_2\cdots}, \\
\nabla_\mu T^{aa_1\cdots}_{bb_1\cdots} &= \partial_\mu T^{aa_1\cdots}_{bb_1\cdots} + \omega^a{}_{c\mu}T^{ca_1\cdots}_{bb_1\cdots} + \omega^{a_1}{}_{c\mu}T^{ac\cdots}_{bb_1\cdots} + \cdots \\
&\quad -\omega^c{}_{b\mu}T^{aa_1\cdots}_{cb_1\cdots} - \omega^c{}_{b_1\mu}T^{aa_1\cdots}_{bc\cdots} - \cdots \,, \\
\nabla_\sigma T^{\mu\mu_1\cdots}_{\nu\nu_1\cdots} &= \partial_\mu T^{\mu\mu_1\cdots}_{\nu\nu_1\cdots} + \Gamma^\mu{}_{\lambda\sigma}T^{\lambda\mu_1\cdots}_{\nu\nu_1\cdots} + \Gamma^{\mu_1}{}_{\lambda\sigma}T^{\mu\lambda\cdots}_{\nu\nu_1\cdots} + \cdots \\
&\quad -\Gamma^\lambda{}_{\nu\sigma}T^{\mu\mu_1\cdots}_{\lambda\nu_1\cdots} - \Gamma^\lambda{}_{\nu_1\sigma}T^{\mu\mu_1\cdots}_{\nu\lambda\cdots} - \cdots \,.
\end{aligned}$$
- Determinants
$$\begin{aligned}
\mathrm{Det}(e^a_\mu) &\equiv e \\
\mathrm{Det}(g_{\mu\nu}) &\equiv g \\
\sqrt{-g} &= e
\end{aligned}$$
- Symmetrisation and anti symmetrisation:
$$\begin{aligned}
T_{(ab)} &= \frac{T_{ab}+T_{ba}}{2}, \\
T_{[ab]} &= \frac{T_{ab}-T_{ba}}{2}.
\end{aligned}$$
- Gauge symmetry:
$$\mathcal{D}_a T^b_\mu = \partial_a T^b_\mu + e^\sigma_a \omega^b{}_{c\sigma}T^c_\mu.$$

Also we have not written it explicitly, but

$$f_{ab} = f_{ab}^{g_1 g_2 \cdot},$$

where g_i are internal group indices, summed with the representation $\varphi^{g_1 g_2 \cdots}$ they act upon.

- Torsion trace and torsion dual

$$\begin{aligned} S_\mu &= S^\lambda{}_{\mu\lambda} \\ S^\star{}_\mu &= \epsilon_{\mu\sigma\lambda\nu} S^{\sigma\lambda\nu} \end{aligned}$$

Spin density trace and dual

$$\begin{aligned} \Pi_\mu &= \Pi^\lambda{}_{\mu\lambda} \\ \Pi^\star{}_\mu &= \epsilon_{\mu\sigma\lambda\nu} \Pi^{\sigma\lambda\nu} \end{aligned}$$

- The $\circ$ above any quantity indicates that it is calculated according to the prescription

$$\Gamma^\lambda{}_{\mu\nu} = \left\{ {}^{\lambda}_{\mu\nu} \right\},$$

where the Christoffel symbols are

$$\left\{ {}^{\lambda}_{\mu\nu} \right\} = \frac{g^{\lambda\sigma}}{2} \left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma} \right),$$

That is: $\overset{\circ}{\mathcal{X}}$ indicates $\mathcal{X}$ as it is in General Relativity.

- Given two operators $\hat{A}$ and $\hat{B}$ we define the commutator and the anti commutator

$$\begin{aligned} \left[\hat{A}, \hat{B}\right] &= \hat{A}\hat{B} - \hat{B}\hat{A}, \\ \left\{\hat{A}, \hat{B}\right\} &= \hat{A}\hat{B} + \hat{B}\hat{A}. \end{aligned}$$

- Clifford algebra basis

$$\begin{aligned} &\mathbb{1} \\ \gamma^\mu &= e^\mu_a \gamma^a \\ \gamma^5 &= -\frac{i}{4!} \epsilon_{\mu\nu\sigma\lambda} \gamma^\mu \gamma^\nu \gamma^\sigma \gamma^\lambda \\ S^{\mu\nu} &= \frac{i}{2} [\gamma^\mu, \gamma^\nu] \end{aligned}$$

- Friedmann-Lemaître-Roberson-Walker geometry. We define the Hubble paramenter

$$H = \frac{\dot{a}}{a} = \frac{a'}{a^2},$$

where $\dot{f} = \frac{df}{dt}$, $f' = \frac{df}{d\eta}$.

- Short hand integral notation

$$\int_{\vec{x}} = \int d\vec{x} = \int \frac{d^3x}{(2\pi)^3} = \int r^2 \sin\theta \frac{drd\theta d\varphi}{(2\pi)^3} \,,$$

  and in D space-time dimensions we write the angular integration

$$d\Omega_{D-1} = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \,.$$

- A confusion might arise since k means both the momentum coordinate and $|\vec{k}|$. To avoid this confusion

$$F(x,k) \text{ or } F(k) \,,$$

  mean a function of the *four* momenta in Wigner representation and in ordinary Fourier representation. While

$$F(\eta,k) \text{ or } F(t,k) \,,$$

  means we are talking about $k = |\vec{k}|$. In most situations, it will be clear from the context which one is which.

- Since we are talking about both real and complex masses, we write the labels explicitly only when required and define

$$m = \sqrt{m_R^2 + m_I^2} \,.$$

- Particular functions

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty dt\, t^{z-1} e^{-t} \,, \\
\Gamma(n+1) &= n! \,, \\
\psi^{(n)}(z) &= \frac{d^{(n+1)}}{dz^{(n+1)}} \log\Gamma(z) \,, \\
\zeta(z) &= \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^z \,.
\end{aligned}$$

Part I

REVIEW OF THE THEORY

3 GEOMETRICAL INTRODUCTION

3.1 THE GEOMETRICAL INTERPRETATION OF TORSION

A geometrical manifold $\mathcal{M}$ is characterised by its local similarity to flat space: at each point on it we can construct the tangent space, which is flat and homeomorphic to the Minkowski space-time. Because we want to deal with fermions in next sections, we define the vierbein field:

$$e^a_\mu e^b_\nu \eta_{ab} \equiv g_{\mu\nu} \,, \tag{2}$$

as a linear map between the tangent space and the space-time manifold. We use latin indices, $a, b, c \cdots$, to refer to the tangent space, and greek indices $\mu, \nu, \lambda \cdots$ for the space-time manifold $\mathcal{M}$. As in general relativity, we want to be able to take covariant derivatives. For tensors and vector defined on $\mathcal{M}$ we use the connection $\Gamma^\lambda{}_{\mu\nu}$. For objects defined on the tangent space, we define the spinor connection $\omega^a{}_{b\mu}$. As an example on how to use the two, consider the covariant derivative of the flat tangent metric.

$$\nabla_\mu \eta_{ab} = \partial_\mu \eta_{ab} - \omega^c{}_{a\mu}\eta_{cb} - \omega^c{}_{b\mu}\eta_{ac} = -2\omega_{(ab)\mu} = 0.$$

We imposed the last equality as a consistency condition: the flat space-time metric should be always parallel transported. As a result, we have obtained that the spinor connection is antisymmetric in the first two indices.

A further consistency condition is that the vierbein field is covariantly conserved, a condition that implies the metric compatibility. With such an assumption, we can relate the spinor connection and the connection living on $\mathcal{M}$

$$\begin{aligned} \omega^a{}_{b\nu} =& e^a_\lambda \left(\partial_\nu e^\lambda_b + \Gamma^\lambda{}_{\nu\mu} e^\mu_b\right) \,, \\ \implies \Gamma^\lambda{}_{\nu\mu} =& e^\lambda_a \left(\partial_\nu e^a_\mu + \omega^a{}_{b\nu} e^b_\mu\right) \,. \end{aligned} \tag{3}$$

The assumption we just introduced, namely that the metric η_{ab} and the manifold metric $g_{\mu\nu}$ are covariantly conserved, is a physical one. η_{ab} describes the way a local observer perceives its surroundings, namely how he measures distances. It should not change under infinitesimal displacement. Analogously, the global metric $g_{\mu\nu}$ should be invariant under parallel transport. These two consistency conditions then imply that the vierbein field is conserved too. So far, we have not yet specified how the connection $\Gamma^\lambda{}_{\mu\nu}$ is structured, or if we can still express parts of it via the Christoffel symbols, as done in

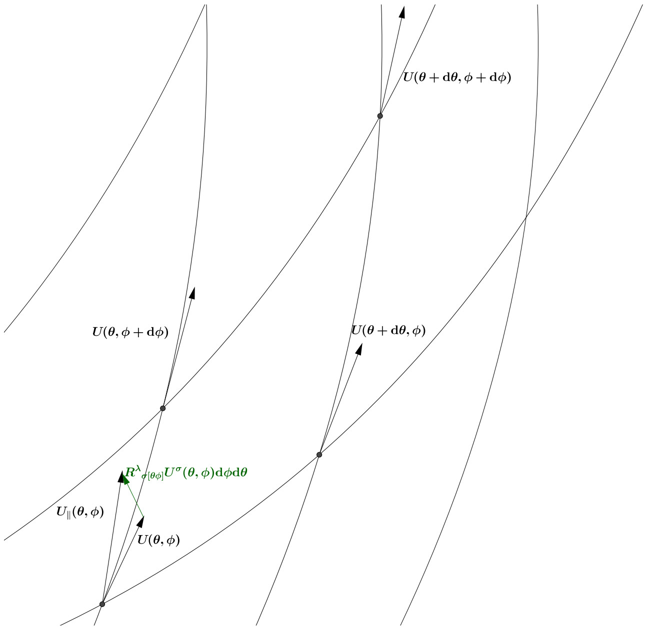


Figure 1: Two dimensional illustration of curvature: vectors parallel transported around a closed loop end up rotated, from U to $U_{\|}$. The Riemann tensor measures the difference $U_{\|} - U$.

General Relativity. To this purpose, we use the metric compatibility condition to rewrite 0 in the fancy way

$$\begin{aligned} 0 &= \nabla_{\nu} g_{\mu\lambda} + \nabla_{\mu} g_{\nu\lambda} - \nabla_{\lambda} g_{\mu\nu} = \\ &= 2\{_{\lambda\mu\nu}\} - 2\Gamma_{\lambda(\mu\nu)} + 4S_{(\mu\nu)\lambda} \,, \end{aligned}$$

where $\{^{\lambda}{}_{\mu\nu}\}$ is our notation for the Christoffel symbols. Such that we can express the most general connection respecting metric compatibility as

$$\Gamma^{\lambda}{}_{\mu\nu} = \{^{\lambda}{}_{\mu\nu}\} + 2S_{(\mu\nu)}{}^{\lambda} + S^{\lambda}{}_{\mu\nu}. \tag{4}$$

This procedure will allow us to separate the General Relativity contributions from whatever is coming from torsion.

We are going to refer to a space-time with torsion as U_4 and a curved, but torsion free as V_4. In V_4, curvature is characterised by the Riemann tensor. In a U_4 space-time, the same holds, except that the Riemann tensor is now a function of the metric and torsion. Since we are also dealing with the tangent space, our definition contains latin indices

$$\begin{aligned} R^{a}{}_{b\mu\nu} =& \partial_{\nu} \omega^{a}{}_{b\mu} - \partial_{\mu} \omega^{a}{}_{b\nu} + \omega^{a}{}_{c\nu} \omega^{c}{}_{b\mu} - \omega^{a}{}_{c\mu} \omega^{c}{}_{b\nu} = \\ =& e^{a}_{\lambda} e^{\sigma}_{b} R^{\lambda}{}_{\sigma\mu\nu} \,, \end{aligned} \tag{5}$$

where the last equality is a non trivial identity that connects the vierbein formulation with the metric one. As a matter of fact, because of this identity, one can choose to work with the full connection $\Gamma^{\lambda}{}_{\mu\nu}$, the spinor connection $\omega^{a}{}_{b\mu}$, or the torsion tensor $S^{\lambda}{}_{\mu\nu}$ as dynamical variables.

Proof. First note that

$$R^{\mu}_{\nu\sigma\lambda} = R^{a}_{b\sigma\lambda} e^{\mu}_{a} e^{b}_{\nu},$$

where the Riemann tensor in the tetraed formalism is given by Eq. (5). We rewrite the previous inserting Eq. (3) to get

$$\begin{aligned} R^{a}_{b\sigma\lambda} = \; & \Gamma^{a}{}_{b\lambda,\sigma} - \Gamma^{a}{}_{b\sigma,\lambda} + \Gamma^{a}{}_{c\sigma}\Gamma^{c}{}_{b\lambda} - \Gamma^{a}{}_{c\lambda}\Gamma^{c}{}_{b\sigma} \\ & + e^{a}_{\nu} e^{\nu}_{c,\sigma} \Gamma^{c}{}_{b\lambda} + e^{c}_{\nu} e^{\nu}_{b,\lambda} \Gamma^{a}{}_{c\sigma} - e^{a}_{\nu} e^{\nu}_{c,\lambda} \Gamma^{c}{}_{b\sigma} - e^{c}_{\nu} e^{\nu}_{b,\sigma} \Gamma^{a}{}_{c\lambda}. \end{aligned}$$

Now consider contraction with $e^{\mu}_{a} e^{b}_{\nu}$

$$\begin{aligned} R^{\mu}_{\nu\sigma\lambda} = \; & \Gamma^{a}{}_{b\lambda,\sigma} e^{\mu}_{a} e^{b}_{\nu} - \Gamma^{a}{}_{b\sigma,\lambda} e^{\mu}_{a} e^{b}_{\nu} + \Gamma^{\mu}{}_{\kappa\sigma}\Gamma^{\kappa}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\kappa\lambda}\Gamma^{\kappa}{}_{\nu\sigma} \\ & + e^{a}_{\kappa,\lambda} e^{\mu}_{a} \Gamma^{\kappa}{}_{\nu\sigma} - e^{a}_{\kappa,\sigma} e^{\mu}_{a} \Gamma^{\kappa}{}_{\nu\lambda} + e^{\kappa}_{b,\lambda} e^{b}_{\nu} \Gamma^{\mu}{}_{\kappa\sigma} - e^{\kappa}_{b,\sigma} e^{b}_{\nu} \Gamma^{\mu}{}_{\kappa\lambda}. \end{aligned}$$

$$\begin{aligned} (\textit{first line}): \quad & \Gamma^{\mu}{}_{\nu\lambda,\sigma} - \Gamma^{a}{}_{b\lambda} e^{\mu}_{a,\sigma} e^{b}_{\nu} - \Gamma^{a}{}_{b\lambda} e^{\mu}_{a} e^{b}_{\nu,\sigma} \\ & - \Gamma^{\mu}{}_{\nu\sigma,\lambda} + \Gamma^{a}{}_{b\sigma} e^{\mu}_{a,\lambda} e^{b}_{\nu} + \Gamma^{a}{}_{b\sigma} e^{\mu}_{a} e^{b}_{\nu,\lambda} \\ & + \Gamma^{\mu}{}_{\kappa\sigma}\Gamma^{\kappa}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\kappa\lambda}\Gamma^{\kappa}{}_{\nu\sigma}. \end{aligned}$$

Now we shall use that $\delta^{\mu}_{\nu,\lambda} = 0 = (e^{a}_{\nu} e^{\mu}_{a})_{,\lambda}$ to simplify

$$\begin{aligned} R^{\mu}_{\nu\sigma\lambda} = \; & \tilde{R}^{\mu}_{\nu\sigma\lambda} \\ & + \cancel{\Gamma^{a}{}_{\nu\sigma} e^{\mu}_{a,\lambda}} - \cancel{\Gamma^{a}{}_{\nu\lambda} e^{\mu}_{a,\sigma}} + \cancel{\Gamma^{\mu}{}_{b\sigma} e^{b}_{\nu,\lambda}} - \cancel{\Gamma^{\mu}{}_{b\lambda} e^{b}_{\nu,\sigma}} \\ & + \cancel{e^{a}_{\kappa,\lambda} e^{\mu}_{a} \Gamma^{\kappa}{}_{\nu\sigma}} - \cancel{e^{a}_{\kappa,\sigma} e^{\mu}_{a} \Gamma^{\kappa}{}_{\nu\lambda}} + \cancel{e^{\kappa}_{b,\lambda} e^{b}_{\nu} \Gamma^{\mu}{}_{\kappa\sigma}} - \cancel{e^{\mu}_{b,\sigma} e^{b}_{\nu} \Gamma^{\mu}{}_{\kappa\lambda}} \\ = \; & \tilde{R}^{\mu}_{\nu\sigma\lambda}. \end{aligned} \tag{6}$$

Where the tilde denotes the Riemann tensor calculated from the space-time connection.

□

In V_4 a vector parallel transported around a loop, changes accordingly to

$$\left([\nabla_{\mu}, \nabla_{\nu}] V^{\lambda} \right) dx^{\mu} dy^{\nu} = R^{\lambda}_{\sigma\mu\nu} V^{\sigma} dx^{\mu} dy^{\nu}, \tag{7}$$

as we can see in Figure 1, for a 2 dimensional space-time.

If we do the same sort of circuit, in U_4, we do get extra terms in equation (7)

$$\left([\nabla_{\mu}, \nabla_{\nu}] V^{\lambda} \right) dx^{\mu} dy^{\nu} = \left(R^{\lambda}_{\sigma\mu\nu} V^{\sigma} + 2 S^{\rho}{}_{\mu\nu} \nabla_{\rho} V^{\lambda} \right) dx^{\mu} dy^{\nu}. \tag{8}$$

Which also implies that in a flat space-time with torsion parallel transported vector still rotate.

The additional terms in Eq. (8) already show that a U_4 space-time is quite different from a V_4. Namely, in U_4 the Riemann tensor is not a complete description of the geometrical properties, since torsion affects parallel transport in a different way than curvature. To characterise more precisely what this "different way" might be, we look at half the circuit shown in Figure 1. Instead of considering a

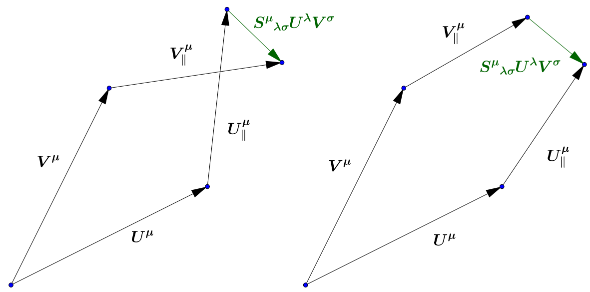


Figure 2: Geometrical interpretation of torsion tensor: parallelograms constructed by parallel transporting vectors do not close.

close, only consider parallel transporting two vectors along each others. This construction generates a parallelogram which, in U_4, does not close. More precisely, if U^{μ} is transported along V^{μ}, then the gap in the parallelogram will be

$$U^{\mu}_{\parallel} - V^{\mu}_{\parallel} = S^{\lambda}{}_{\mu\nu}U^{\mu}V^{\nu} \tag{9}$$

Figure 2 shows this concept graphically.

To conclude this chapter let's discuss some issues about the space-time U_4. In a space-time without torsion, the curvature can be locally set to zero by a coordinates transformation, which in practice means finding a coordinate frame where the Christoffel symbols vanish. In U_4, however, this seems to be not possible: since $S^{\lambda}{}_{\mu\nu}$ is a tensor[1], it cannot be set to zero by a coordinate transformation. In [11], however, it is argued that a frame constructed in terms of auto parallels will have the defining property that, in that frame, the covariant derivative reduces to an ordinary derivative. Locally, observers might not perceive curvature, nor torsion.

The reason for this property is that, in the vierbein formulation there are additional degrees of freedom which disappear from the metric theory we are used to. Consider for example a space-time where $g_{\mu\nu} = \eta_{\mu\nu}$. According to Eq. (3), we could still generate torsion[2]

$$\Gamma^{\lambda}{}_{[\mu\nu]} = e^{\lambda}_{a}\partial_{[\mu}e^{a}_{\nu]} \equiv S^{\lambda}{}_{\mu\nu}\,. \tag{10}$$

It is important to note that relation (10) preserves, at linear order, the structure $g_{\mu\nu} = \eta_{\mu\nu}$, but still gives non vanishing torsion. This happens because the vierbein formalism introduces new degrees of freedom that remain concealed in the metric [24, 32], "[$\cdots$]independent rotations not specified by the metric structure" [13]. The vierbein reference frame satisfying Eq. (10), can be understood as a frame twisting as it moves along auto parallels trajectories. In other words,

1 As the difference of two connections always transforms as a tensor.
2 We are doing this locally, so all expressions will contain positions dependence and, since we work infinitesimally, we stop at linear order in all expansions.

vectors transported along their own direction do not feel torsion, but when moved along any other direction they will precess. This consequence of torsion was compared by Cartan "to a [mechanical] medium having constant pressure and constant internal torque" [5]. A consequence of this is that the equivalence principle holds in U_4, since torsion effects can be nullified by a rotation of the vierbein field, that of Eq. (10).

3.2 physical interpretations of torsion

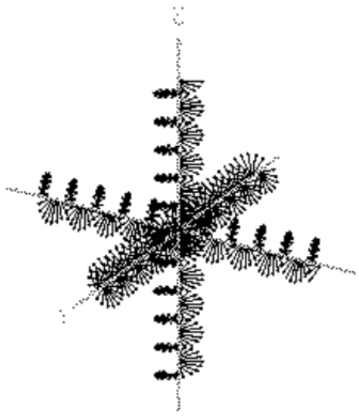


Figure 3: The so-called Cartan staircase: vectors parallel transported along their own direction do not change, but they precess if the direction of parallel transport has a component in their orthogonal space. This precession selects a direction (either left-handed or right-handed) which consequently breaks the parity symmetry(Image credit [12]).

The concepts we built in the last section can be developed further, and be interpreted from a field theoretical perspective. Fermions on curved space-times are described in the vierbein formalism ($\gamma^\mu = e^\mu_a \gamma^a$), so they couple to the vierbein field, rather than the metric. Accepting this subtlety leads to consider Eq. (10) rather seriously: the dynamics of fermions can generate torsion, by modifying the vierbein structure, which motivates the study of torsion from a field theoretical perspective. Since torsion selects a rotation direction (see Figure 3), it breaks parity and chirality symmetries. There is a clear evidence that microscopic violations of CP symmetry are allowed in Nature: for example, weak interaction violates parity, by preferring left handyness. The rotations of the vierbein mentioned in the last section might produce a similar effect. In U_4 space-times with torsion (but no curvature) vectors that are parallel transported rotate in the plane perpendicular to the motion. The direction of such a rotation can proceed either in the right hand direction or in the left one. This can be seen clearly from the Cartan staircase construction, Figure 3, a way of constructing a flat and Euclidean space-time with the torsion field given by Eq. (10).

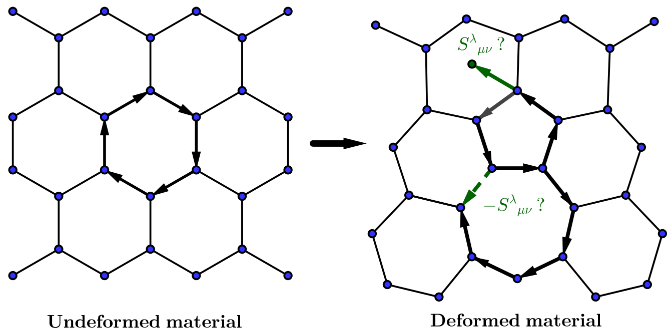


Figure 4: Mapping a graphene sample without structural defects into the same material with some local imperfections (a pentagon and a heptagon in place of hexagons). The undefected material is made of infinitesimal vectors, defined through the lattice points. In the continuum limit this corresponds to a flat space with no curvature nor torsion. To effectively describe the imperfection in the right figure, one can model this space as a flat Euclidean space with torsion.

An interesting analogy to point out is that effects described by the Cartan staircase construction are also found in another completely different area of physics: that of continuous material with structural defects. In Figure 4 we can see a representation of such a analogy. The geometrical properties of the imperfect material can be modelled effectively via a continuum geometrical description in a U_2 Euclidean space[3]. An extensive treatment can be found in [18].

In this description, built on Cosserat theory of continuum materials [31], a 3 dimensional body is pictured as a collection of oriented points. To model defects one introduces a translational (ϵ^a) and a rotational ($\Omega_{ab} = -\Omega_{ba}$) displacements. These quantities describe internal stresses and pressures, which modify the structure of the original lattice. To embed such discrete defects' in a continuum mechanics description, one can consider a flat Euclidean space, and apply an infinitesimal deformation of both connection and metric. Such

3 A 2 dimensional space with torsion and Euclidean signature.

deformations will be built of translations and rotations deformations of the vierbein and spinor connection [12]

$$\begin{aligned} \Delta e^a_\mu =& \overset{0}{\nabla}_\mu \epsilon^a - e^b_\mu \Omega^a_b\,, \\ \Delta \omega^a{}_{b\mu} =& \overset{0}{\nabla}_\mu \Omega^a_b\,, \end{aligned} \tag{11}$$

where Δ indicates taking the difference between deformed and regular material and 0 indicates that the quantity is calculated in the latter. From Eq. (11) we can see how defects in continuum materials can be interpreted: they produce curvature and torsion as a result of translational and rotational displacements.

A possible interpretation of the former analogy is that, if space-time is ultimately a discrete entity, made of points and lines, it might develop defects which, in the continuum limit of General Relativity, will be described by non vanishing torsion and curvature. Such gravitational defects could arise from a UV complete theory of quantum gravity, and are going to give a substantial contribution when the gravitational stresses that cause them become of the order of the Planck mass. In this sense, as anticipated in the introduction, a theory of gravity with torsion can be regarded as an effective theory of quantum gravity, where a particular microscopic structure of space-time is investigated.

Eqs. (11) can be regarded also from a different perspective: including torsion and curvature in the undeformed space, and applying the deformations ϵ^a and Ω_{ab} is analogous to performing an infinitesimal Poincaré gauge transformation. In a general U_4 space-time the Poincaré group generators[4],

$$\begin{aligned} \mathcal{D}_a, & \qquad \textit{for translations}, \\ f_{ab} = f_{[ab]}, & \qquad \textit{for rotations}, \end{aligned}$$

satisfy the generalised commutation relations [13, 14]

$$\left[f_{ab}, f_{cd}\right] = g_{c[a}f_{b]d} - g_{d[a}f_{b]c}\,, \tag{12a}$$

$$\left[f_{ab}, \mathcal{D}_c\right] = g_{c[a}\mathcal{D}_{b]}\,, \tag{12b}$$

$$\left[\mathcal{D}_a, \mathcal{D}_b\right] = e^\mu_a e^\nu_b \left(R^{cd}{}_{\mu\nu} f_{cd} + 2S^\lambda{}_{\mu\nu} e^c_\lambda \mathcal{D}_c\right)\,. \tag{12c}$$

Where the covariant derivative is defined

$$\mathcal{D}_a = e^\mu_a(\partial_\mu + \omega^{cd}{}_\mu f_{cd}) \equiv e^\mu_a \mathcal{D}_\mu\,, \tag{13}$$

and f_{ab} is specified by the representation the derivative acts on (for fermions, it would be $\frac{i}{2}\gamma_{[a}\gamma_{b]}$). From Eq. (12c), we can therefore identify the curvature and torsion as the gauge fields corresponding to rotations and translations. Torsion connects with translations, and the vierbein is the associated field, according to Eq. (10), while the

4 Where $\mathcal{D}_a$ is intended to act on tangent spaces indices only.

Riemann tensor is the field strength for the connection. Such a description is known as Poincarè gauge theory of gravity. However this formulation is not completely understood yet, namely since the gravitational geometry enters in the group structure equations, the global structure of the group cannot be determined by the local. Because a coordinate patch can cover only a part of the manifold, the Lie algebra generated in it will be restricted to such a domain.

Let us consider a arbitrary representation of the Poincaré group, φ. Under an infinitesimal translation, the variation of the covariant derivative is [26]

$$\mathcal{D}_\mu\varphi = e^a_\mu \mathcal{D}_a\varphi \to \left(e^a_\mu + \delta e^a_\mu\right)\mathcal{D}_a\varphi + \delta\omega^{ab}{}_\mu f_{ab}\varphi + \mathcal{D}_\mu\delta\varphi\,. \tag{14}$$

For all the variations to reciprocally cancel, we have to require minimal coupling. To see how, let us extend Eq. (10) to a curved spacetime, in non minimal coupling prescription. We would write

$$\omega^a{}_{b\mu} f^b_a = \omega^a{}_{(b\mu)} f^b_a + \xi\omega^a{}_{[b\mu]} f^b_a\,, \tag{15}$$

since the first part gives curvature and the second torsion (again, see Eq. (10)). When plugging Eq. (15) into Eq. (14), and imposing the gauge derivative to transform as a 1-form, forces $\xi = 1$ (and obviously other couplings are not allowed by gauge symmetry neither).

Proof. Indeed, we have, under the infinitesimal transformation $x^a \to x^a - \eta(x)^a$

$$\begin{aligned}
\delta e^a_\mu &= e^b_\mu \mathcal{D}_b\eta(x)^a + 2\eta^\lambda e^a_\sigma S^\sigma_{\mu\lambda}, && \text{(16a)}\\
\delta\omega^{ab}{}_\mu &= \eta^\lambda R^{ab}{}_{\lambda\mu}, && \text{(16b)}\\
\delta\varphi &= \eta^a\mathcal{D}_a\varphi, && \text{(16c)}
\end{aligned}$$

where we get the terms in Eq. (16a) as a consequence of Eq. (10). We then get, for the covariant derivative using Eqs. (12c - 16a - 16b)

$$\begin{aligned}
\mathcal{D}_a\varphi \to &\mathcal{D}_a\varphi + \eta^b\left[\mathcal{D}_b,\mathcal{D}_a\right]\varphi + \eta^b\mathcal{D}_a\mathcal{D}_b\varphi + (\xi-1)e^\mu_a\omega^a{}_{[b\mu]}f^b_a = \\
=&(\mathbb{1} + \eta^b\mathcal{D}_b)\mathcal{D}_a\varphi.
\end{aligned}$$

Obviously, the last equality only holds for $\xi = 1$, in which case the extra term on the left hand side cancels. This is due to the commutation between covariant derivatives: the geometry determines Eq. (8), from where the commutations relations are deduced. Minimal coupling is therefore required to retain the interpretation of torsion as the gauge field of translations. Non minimal coupling can be introduced, by losing such interpretation.

□

Now let us discuss the results obtained in this section. We learned that torsion has a natural interpretation as description of defects in continuum materials, and that such an analogy can be extended to the case in which space-time is quantised[5]. Even if one rejects this argument, it is possible to show that Einstein's theory of gravity leads to torsion as long as one considers all the 64 degrees of freedom of the connection. Torsion can also be a field, dynamically generated, and whose ultimate origin is a UV complete theory of gravity. In this case, as we will see in the next chapters, one can introduce the most general operator coupling torsion to fermions, with two unknown coupling constants.

If one wants to preserve the interpretation of torsion as the Gauge field for translations, then he must choose one specific operator, and the extra coupling constants will be automatically specified. We stress, however, that without non minimal coupling it is difficult to obtain substantial deviations from Einstein's theory at energies below the Planck scale. As we will demonstrate later on, the curvature scale at which torsion becomes significant reaches the Planck scale unless $\xi \gg 1$.

3.3 THE FIELD EQUATIONS

The decision on whether or not introduce non minimal coupling, we postpone to next sections, for now we focus on the gravitational sector and leave unspecified the matter action. It is possible that a UV finite gravitational Lagrangian differs from the low energy limit that Einstein's theory describes. However, we are going to assume that until the curvature scale remains lower than the Planck scale, the Einstein-Hilbert action describes the theory. The difference being that we are going to consider two dynamical objects:

- $g_{\mu\nu}$, the metric;
- $\Gamma^{\lambda}{}_{\mu\nu}$, the affine connection.

Considering the action to be the Einstein-Hilbert action, then we write (Palatini formalism)

$$S = \int d^4x\sqrt{-g}\left(-\frac{g^{\mu\nu}R_{\mu\nu}(\Gamma)}{16\pi G} + \mathcal{L}_m\right) , \tag{17}$$

where G is the gravitational coupling constant, and Γ is a general affine connection. It is pretty straightforward to show that variation with respect to the metric leads to the Einstein's equations

$$R_{\mu\nu}(\Gamma) - \frac{g_{\mu\nu}}{2}R(\Gamma) = \frac{8\pi G}{3}T_{\mu\nu} . \tag{18}$$

More interestingly variation with respect to the connection leads to

$$\Gamma^{\lambda}{}_{\mu\nu} = \{^{\lambda}{}_{\mu\nu}\} + 2S_{(\mu\nu)}{}^{\lambda} + S^{\lambda}{}_{[\mu\nu]} ,$$

5 As a further support to this claim, we cite the well known result that Loop Quantum Gravity leads to torsion like interactions.

where the position of the indices plays a key role. It is interesting to note how the same expression for the connection was found in Eq. (4) as following from the metric compatibility condition.

Proof. Clearly we can consider separate variations with respect to symmetric and antisymmetric parts by writing

$$\delta\Gamma^{\lambda}{}_{\mu\nu} = \delta\Gamma^{\lambda}{}_{(\mu\nu)} + \delta\Gamma^{\lambda}{}_{[\mu\nu]},$$

we then get

$$\begin{aligned}\sqrt{-g}g^{\sigma\nu}\delta R^{\lambda}{}_{\sigma\lambda\nu} &= \delta\Gamma^{\alpha}{}_{(\beta\gamma)}\left[\frac{1}{2}g^{\beta\gamma}g_{\lambda\sigma}\nabla_{\alpha}g^{\lambda\sigma} - \nabla_{\alpha}g^{\beta\gamma} + \right.\\ &\left. +\delta^{\gamma}_{\alpha}\left(\nabla_{\lambda}g^{\beta\lambda} - \frac{1}{2}g^{\beta\kappa}g_{\lambda\sigma}\nabla_{k}g^{\lambda\sigma}\right) + 2S^{(\gamma\beta)}{}_{\alpha}\right] + \\ &+2\delta\Gamma^{\alpha}{}_{[\beta\gamma]}\left(S^{[\gamma\beta]}{}_{\alpha} + 2(\delta^{\gamma}_{\alpha}S^{\beta} - \delta^{\beta}_{\alpha}S^{\gamma})\right).\end{aligned}$$

Which must then be set equal to $\delta\mathcal{L}_m$. This set of equations has the solution

$$\nabla_{\alpha}g^{\beta\gamma} = 0\,, \tag{19}$$

$$4g_{\lambda[\mu}S_{\nu]} + S_{\lambda[\mu\nu]} - 2S_{[\mu\nu]\lambda} = 8\pi G\Pi_{\lambda[\mu\nu]}\,, \tag{20}$$

where $\Pi_{\lambda[\mu\nu]} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta S^{\lambda\mu\nu}}$, as it can be seen from the solution to the metric compatibility condition Eq. (4), and rewriting $\delta\Gamma^{\lambda}{}_{\mu\nu} = 2\delta S_{(\mu\nu)}{}^{\lambda} + \delta S^{\lambda}{}_{[\mu\nu]}$.
□

The result contained in Eqs. (19 - 20) is not the most general solution to the equations in Palatini formalism. Indeed a geometry where the metric is not covariantly conserved is possible, which will provide additional equations for the non metricity tensor (i. e.$Q_{\alpha\beta\gamma} = \nabla_{\alpha}g_{\beta\gamma}$). However, as physics goes, there seem to be no really meaningful interpretation of non metric theories.

Consider this: the metric is used by an observer to measure distances and angles. Imposing covariance forces different observer to agree on what they measure, however non metricity implies that distances' measure depends on the *path*. A free falling observer will change the way he observes distances as he moves, and for two different observer to decide if they agree on a measurement will be necessary to know the local history of the trajectory. It is perhaps possible to conceive situations in which such an effect will be dynamically generated, but for the following we are disregard such example as unphysical and consider only the solution (19 - 20).

Now, let us examine the equation of motion for torsion in greater detail. We can invert Eq. (20) to obtain

$$S_{\mu\nu\lambda} = 8\pi G\left(\Pi_{[\lambda\nu]\mu} - \frac{1}{2}g_{\mu[\lambda}\Pi_{\nu]}\right). \tag{21}$$

Proof. Eq. (20) can be solved by finding S_μ in terms of Π, and solving for the symmetric and antisymmetric parts. First note that the left hand side can be written as

$$S_{\lambda[\mu\nu]} - 2S_{[\mu\nu]\lambda} + 4g_{\lambda[\mu}S_{\nu]} = 4g_{\lambda[\mu}S_{\nu]} - S_{\mu\nu\lambda} - 2S_{(\nu\lambda)\mu} = 8\pi G\Pi_{\lambda\mu\nu}.$$

Now consider the contraction with $g^{\nu\lambda}$ to find

$$S_\mu = -2\pi G\Pi_\mu.$$

And substitute back to get

$$2S_{(\nu\lambda)\mu} + S_{\mu\nu\lambda} = -8\pi G\left(\Pi_{\lambda\mu\nu} + g_{\lambda[\mu}\Pi_{\nu]}\right).$$

Just anti symmetrisation in $[\nu\lambda]$ gives

$$S_{\mu\nu\lambda} = 8\pi G\left(\Pi_{[\lambda\nu]\mu} - \frac{1}{2}g_{\mu[\lambda}\Pi_{\nu]}\right). \tag{22}$$

This result can be checked for consistency with the equation obtained by varying the action with respect to the spin connection, as commonly done in the the literature.
□

If the solution from Eqs. (19 - 20) is plugged into Eq. (18), the contributions coming from torsion will modify geometrically the left hand side, accordingly to

$$R = \overset{\circ}{R} - 4\overset{\circ}{\nabla}_\kappa S^\kappa - 4S_\kappa S^\kappa - (2S_{(\kappa\rho)\mu} + S_{\mu\kappa\rho})(2S^{(\rho\mu)\kappa} + S^{\kappa\rho\mu}). \tag{23}$$

However, since the solution in Eq. (21) allows us to completely express torsion in terms of its sources, we would rather derive an effective theory with the source $\Pi_{\lambda\mu\nu}$ in place of $S_{\lambda\mu\nu}$. Such a procedure will produce an effective field theory, where the torsion is replaced by interaction terms for its matter sources. Note that, since Eq. (21) does not contain any derivatives, such an effective field theory will be exact. In Heisenberg picture, Eq. (21) is to be intended as an exact operator identity, a linear relation between the torsion operator and the fields operators sourcing it. In a General Relativity sense, this procedure is equivalent to go to a reference frame where torsion is zero (for example by making use of Eq. (4)), but where the fields still perceive its influence, by effective covariant interactions. Finally, in the context of interpreting torsion from the quantum gravity perspective as in Section 3.2, from Eq. (21) torsion can be seen as a local operator describing in matter contributions from a UV complete theory. Inside matter, it is plausible that the quantum gravitational field equations change (as, for example, the Maxwell equations inside matter do). Such corrections will be linear in first approximation, and respect the low energy limit defined by the Einstein-Hilbert action. An argument can be formulated to explain why Eq. (21) does not contain propagating parts: corrections from quantum gravity should change the action (17) at order $\frac{1}{\kappa^2} = M_p^4$. Since the lower order in the gravitational coupling is already captured in the Einstein-Hilbert action, we can consider non dynamical torsion as a valid formulation until the Planck energy scale is reached.

If Eq. (22) is substituted into the action (17), the effective field theory action can be found in general

$$S = \int d^4x\sqrt{-g}\left(-\frac{\overset{\circ}{R}}{16\pi G} + \overset{\circ}{\mathcal{L}}_m - \frac{1}{2}\left(\overset{\circ}{\nabla}_\kappa \Pi^\kappa - 2\pi G \Pi_\kappa \Pi^\kappa - 8\pi G \Pi_{\rho\mu\kappa}\Pi^{\mu\kappa\rho}\right)\right) . \tag{24}$$

Note the interaction terms have energy dimension $[\Pi^2] = [E^6]$ this restricts the possible interactions terms.

Varying the action (24) leads to the modified Einstein's equations. The only surviving gravitational degree of freedom is now the metric $g_{\mu\nu}$

$$\begin{aligned} \overset{\circ}{G}_{\mu\nu} = & \overset{\circ}{R}_{\mu\nu} - \frac{g_{\mu\nu}}{2}\overset{\circ}{R} = 8\pi G\left[T_{\mu\nu} + \overset{\circ}{\nabla}_\kappa \Pi_{(\mu\nu)}{}^\kappa - \overset{\circ}{\nabla}_{(\mu}\Pi_{\nu)} + \right. \\ & \left. + \frac{g_{\mu\nu}}{2}\left(\overset{\circ}{\nabla}_\kappa \Pi^\kappa - (2\pi G)\Pi_\lambda \Pi^\lambda - (8\pi G)\Pi_{\rho\lambda\kappa}\Pi^{\lambda\kappa\rho}\right)\right] \\ & + (8\pi G)^2\left[\Pi^{\lambda\kappa}{}_\mu \Pi_{\kappa\lambda\nu} - \frac{1}{2}\Pi^\kappa \Pi_{(\mu\nu)\kappa} + \frac{1}{2}\Pi_\mu \Pi_\nu\right] . \end{aligned} \tag{25}$$

Note that the Einstein's equations (25) contain only symmetric contraction of the torsion source. It is not that the anti symmetric part of Eq. (25) does not contribute, but is is satisfied thanks to Lorentz invariance of the matter action (see Appendix A).

To make any further steps from Eq. (25) we would have to make assumptions on the fields in $\mathcal{L}_m$ and on the metric. The next chapter is dedicated to analysing the interactions that torsion produces, in case of fermionic matter.

Part II

MICROSCOPIC ANALYSIS OF TORSION INTERACTIONS

4 FERMIONS ON CURVED SPACE-TIMES

4.1 THE INTERACTION TERMS

We know [25] that the generator of lorentz transformations for spinors is provided by

$$f_{ab} = -\frac{1}{8}\left[\gamma_a, \gamma_b\right],$$

where γ_a are elements of the Clifford algebra, satisfying

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbb{1}.$$

According to the prescription from Eq. (13), we can define the covariant derivative

$$\mathcal{D}_a\psi = e^\mu_a\left(\partial_\mu - \frac{1}{8}\omega^{ab}{}_\mu\left[\gamma_a, \gamma_b\right]\right)\psi, \tag{26a}$$

$$\mathcal{D}_a\bar{\psi} = e^\mu_a\bar{\psi}\left(\overleftarrow{\partial}_\mu + \frac{1}{8}\omega^{ab}{}_\mu\left[\gamma_a, \gamma_b\right]\right). \tag{26b}$$

We can therefore construct the minimally coupled Lagrangian by the prescription $\partial_a \to \mathcal{D}_a$, we then would get the free Dirac action

$$S_\psi = \int d^4x\, e\left(\frac{i}{2}\left(\bar{\psi}\gamma^a e^\mu_a\mathcal{D}_\mu\psi - (\mathcal{D}_\mu\bar{\psi})e^\mu_a\gamma^a\psi\right) - \bar{\psi}\hat{M}\psi\right),$$

where we wrote the mass as a matrix $\hat{M} = m_R + im_I\gamma^5$ to allow for possible mass mixing or CP violating mass[1]. Naturally we can define $\gamma^\mu = e^\mu_a\gamma^a$, such that the commutations relations get generalised to

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1},$$

and we can rewrite the action in the form

$$S_\psi = \int d^4x\sqrt{-g}\left(\frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\{\gamma^\mu, \Gamma^\mu\}\psi\right) - \bar{\psi}\hat{M}\psi\right), \tag{27}$$

where we defined

$$\Gamma_\mu = -\frac{1}{8}\omega^{ab}{}_\mu\left[\gamma_a, \gamma_b\right].$$

Now, we note that we can write

$$\{\gamma^\mu, \Gamma_\mu\} = -\frac{1}{8}\omega_{ab\mu}\{\gamma^\mu, [\gamma^a, \gamma^b]\} = -\frac{1}{2}\omega_{ab\mu}\gamma^{[a}\gamma^b\gamma^{\mu]}, \tag{28}$$

1 Note that $m_{R,I}$ specify the most general mass that we can add: because of Lorentz invariance, $\hat{M}$ can only be function of the scalar elements of the Clifford algebra, namely $\mathbb{1}, \gamma^5$.

that is, the spinor connection times a totally antisymmetric product of γ's.

Proof. We need to prove the following identity

$$\{\gamma^a, [\gamma^b, \gamma^c]\} = 4\gamma^{[a}\gamma^b\gamma^{c]}.$$

Let us consider at first $(a \leftrightarrow b)$ exchanges, to show that the left hand side is antisymmetric under it

$$\begin{aligned}\{\gamma^a, [\gamma^b, \gamma^c]\} =& \frac{1}{3}(\gamma^a[\gamma^b, \gamma^c] + \gamma^b[\gamma^c, \gamma^a] + \gamma^c[\gamma^a, \gamma^b]) \\ &+ \frac{1}{3}([\gamma^b, \gamma^c]\gamma^a + [\gamma^c, \gamma^a]\gamma^b + [\gamma^a, \gamma^b]\gamma^c),\end{aligned}$$

which is manifestly anti symmetric in $(a \leftrightarrow b)$. The last step consist in noticing that

$$\gamma^{[a}\gamma^b\gamma^{c]} = \frac{1}{6}(\gamma^a[\gamma^b, \gamma^c] + \gamma^b[\gamma^c, \gamma^a] + \gamma^c[\gamma^a, \gamma^b]).$$

Equating the last two equations proves Eq. (28).
□

Armed with the identity Eq. (28) we can come back to the term

$$\begin{aligned}\{\gamma^\mu, \Gamma_\mu\} =& -\frac{1}{8}\omega_{ab\mu}\{\gamma^\mu, [\gamma^a, \gamma^b]\} = -\frac{1}{2}\omega_{ab\mu}\gamma^{[\mu}\gamma^a\gamma^{b]} = \\ =& -\frac{1}{2}\mathring{\omega}_{ab\mu}\gamma^{[\mu}\gamma^a\gamma^{b]} - \frac{\xi}{2}S_{\lambda\mu\nu}\gamma^{[\lambda}\gamma^\mu\gamma^{\nu]},\end{aligned}$$

where we have introduced a dimensionless coupling constant ξ, representing a non minimal coupling between fermions and torsion. From what we derived in Chapter 3, such a choice breaks translation invariance of the action. However, we want to keep this discussion general, as we do not know whether translational invariance is a fundamental feature of nature. To this end, we want to write the most general Lagrangian coupling torsion to the matter fields, without dynamical torsion and respecting Lorenz covariance. Before taking this step, however, let us rewrite the totally anti symmetric element of the Clifford algebra in Eq. (28) in a more convenient form

$$\gamma^{[\mu}\gamma^\nu\gamma^{\sigma]} = -i\epsilon^{\mu\nu\sigma\lambda}\gamma^5\gamma_\lambda,$$

where $\epsilon^{\mu\nu\sigma\lambda}$ is the totally anti symmetric tensor in four dimensions. Now we can write the most general coupling that does not contain derivatives as

$$\mathcal{L}_{int} = -S_{\lambda\mu\nu}\epsilon^{\mu\nu\sigma\lambda}\left(\xi\bar{\psi}\gamma^5\gamma_\lambda\psi + \xi'\bar{\psi}\gamma_\lambda\psi\right). \tag{29}$$

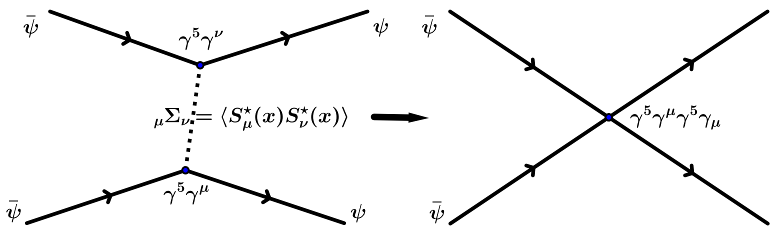


Figure 5: The interaction vertices in the complete theory (left) and in the effective field theory (right). Integrating out torsion leads to substitute its propagator by a point. Note that the spinor indices structure of the diagram on the right is non trivial, as we shall see later on in more details.

Before making further progress, note that Eq. (29) selects the most general dimension four operator coupling to the torsion tensor. One might ask: what about higher dimensions operators? Indeed it is possible to write interactions such as

$$\mathcal{L}_{int} = -S^{\lambda}{}_{\mu\nu}\left(\xi^{(2)}\overleftrightarrow{\nabla}_{\lambda}\bar{\psi}\gamma^{[\mu}\gamma^{\nu]}\psi\right). \tag{30}$$

For higher dimensions operators, such the ones appearing in Eq. (30), the interactions terms are not restricted that much by the symmetries as it happens for operators of dimension four. Also, the presence of the covariant derivative in Eq. (30) selects such interactions as higher energy exchange channels, so that we can neglect them in first approximation, and postulate that, if they exist, their contribution only matters at very high energies scale, when the full structure of a UV complete theory of gravity has to be applied.

From Eq. (29) it is clear what $\Pi_{\lambda\mu\nu}$ is going to be

$$\Pi^{\mu\nu\lambda} = -\frac{1}{4}\epsilon^{\mu\nu\lambda\sigma}\left(\xi\bar{\psi}\gamma^5\gamma_\sigma\psi + \xi'\bar{\psi}\gamma_\sigma\psi\right). \tag{31}$$

Which leads, using Eq. (24) to the effective matter action [15]

$$S^{eff}_{\psi} = \int d^4x\sqrt{-g}\Big[\frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\{\gamma^\mu,\Gamma^\mu\}\psi\right) - \bar{\psi}\hat{M}\psi \\ + \frac{3\pi G}{2}\left(\xi\bar{\psi}\gamma^5\gamma_\sigma\psi + \xi'\bar{\psi}\gamma_\sigma\psi\right)^2\Big]. \tag{32}$$

We have now integrated out torsion, but its influence on the matter fields has not been lost: the spin-$\frac{1}{2}$ fields feel the presence of torsion through an effective four fermions interaction.

Let us conclude this section by clarifying a common misconception. It is often claimed that torsion couples to elementary particles spin: as we can see from Eq. (90), rewritten in terms of the Poincaré group gauge derivative [13]

$$\mathcal{D}_\lambda\left(\sqrt{-g}\Pi_{[\mu\nu]}{}^{\lambda}\right) - T_{[\mu\nu]} = 0. \tag{33}$$

Since the energy momentum tensor is symmetric

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta g^{\nu\mu}},$$

Eq. (33) is a conservation law that identifies Π as a classical Noether current coming from rotational invariance. However, the elementary spin of fermions can be constructed from the Lorentz generator for spatial rotations, i. e. f_{ij}. According to [25] the fermionic spin operator is[2]

$$\Sigma_i = \frac{i}{8}\epsilon_{ijk}\left[\gamma^j, \gamma^k\right],$$

and has therefore connections with the tensor part of the Clifford algebra, and not the axial vector (which appears in Eq. (29) when $\xi = 1$, $\xi' = 0$). Coupling torsion to the particles elementary spin would yield to interactions between operators of higher dimension, such as Eq. (30).

4.2 DIRAC EQUATION ON FRIEDMANN-LEMAÎTRE-ROBERSON-WALKER SPACE-TIME

From the Lagrangian (32), to derive the Dirac equation, we just have to vary with respect to $\bar{\psi}$. For notation reasons we will eventually set $\xi' = 0$, since the essence of the interaction is captured by the axial current interaction alone. The main differences will be highlighted in Appendix E. We then find

$$\begin{aligned}\delta\bar{\psi}\sqrt{-g}\Big[& i\left(\gamma^\mu\mathcal{D}_\mu\psi + \Gamma_\mu\gamma^\mu\psi\right) - \hat{M}\psi + \\ & + 3\pi G\left(\xi\bar{\psi}\gamma^5\gamma_\sigma\psi + \xi'\bar{\psi}\gamma_\sigma\psi\right)\left(\xi\gamma^5\gamma_\sigma + \xi'\gamma_\sigma\right)\psi \\ & + \frac{i}{2}\partial_\mu(\sqrt{-g}\gamma^\mu\psi) + \sqrt{-g}\left(\Gamma_\mu\gamma^\mu - \gamma_\mu\Gamma^\mu\right)\Big].\end{aligned}$$

The last line will vanish, so that we find the Dirac equation to be

$$\left(i\gamma^\mu\mathcal{D}_\mu - \hat{M}\right)\psi = -3\pi G\left(\xi\bar{\psi}\gamma^5\gamma_\sigma\psi + \xi'\bar{\psi}\gamma_\sigma\psi\right)\left(\xi\gamma^5\gamma^\sigma + \xi'\gamma^\sigma\right)\psi. \tag{34}$$

Proof. We can write

$$\begin{aligned}&\partial_\mu(\sqrt{-g}\gamma^\mu) + \sqrt{-g}\left(\Gamma_\mu\gamma^\mu - \gamma^\mu\Gamma_\mu\right) = \\ =&\sqrt{-g}\left(\partial_\mu\gamma^\mu + \gamma^\mu\{_\mu{}^\sigma{}_\sigma\} + [\Gamma_\mu, \gamma^\mu]\right).\end{aligned} \tag{35}$$

Now let's look at

$$\begin{aligned}[\gamma^\nu, \Gamma_\mu] =& -\frac{1}{4}\omega_{[ab]\mu}[\gamma^\nu, \gamma^a\gamma^b] = -\frac{1}{4}\omega_{[ab]\mu}\left(\gamma^a[\gamma^\nu, \gamma^b] + [\gamma^\nu, \gamma^a]\gamma^b\right) = \\ =& -\frac{1}{4}\omega_{ab\mu}\left(\gamma^a\{\gamma^\nu, \gamma^b\} + 2\gamma^a\gamma^b\gamma^\nu + \{\gamma^\nu, \gamma^a\}\gamma^b + 2\gamma^a\gamma^\nu\gamma^b\right) = \\ =&\omega^b{}_{a\mu}\gamma^a e_b^\nu.\end{aligned}$$

2 Here i, j, k are spatial indices, see Section 2.

Such that Eq. (35) can now be written as

$$\nabla_\mu \gamma^\mu = \gamma^a \nabla_\mu e^\mu_a = 0 \,.$$

□

Now that we know how to deal with the Dirac theory in curved space-times, let us consider the specific example of Friedmann-Lemaître-Roberson-Walker (FLRW) metric

$$ds^2 = dt^2 - a(t)^2 d\vec{x}^2 \,,$$

where $d\vec{x}$ is the 3 dimensional space metric and $a(t)$ is the scale factor. The coordinate transformation $dt = a(\eta)d\eta$, where η is the conformal time, makes the metric proportional to the Minkowski's one,

$$g_{\mu\nu} = a(\eta)^2 \eta_{\mu\nu} \,.$$

In this coordinate system, we can prove the identity [19]

$$i\gamma^\mu \mathcal{D}_\mu \psi = a(\eta)^{-\frac{5}{2}} i\gamma^a \partial_a \left(a(\eta)^{\frac{3}{2}} \psi \right) \,. \tag{36}$$

Proof. A simple exercise is to derive that the Christoffel symbols are in the form

$$\{^0{}_{00}\} = -\{^0{}_{i0}\} = -\{^0{}_{0i}\} = \{^0{}_{ij}\} = H = \frac{1}{a^2}\frac{d}{d\eta}a = \frac{a'}{a^2} \,,$$

then we can write, using Eq. (3)

$$\Gamma_\mu = -\frac{1}{8} e^\nu_a (\partial_\mu e_{\nu b} - \Gamma^\sigma{}_{\mu\nu} e_{\sigma b})[\gamma^a, \gamma^b] = \begin{cases} \frac{H}{2}\gamma_i\gamma_0, \text{ If } \mu = i \,, \\ 0, \text{ If } \mu = 0 \,, \end{cases} \,.$$

Finally, writing

$$\begin{aligned} \gamma^\mu \mathcal{D}_\mu \psi =& \frac{1}{a^2} \left(\gamma^0 \partial_\eta - \vec{\gamma} \cdot \vec{\partial} - \gamma^i \Gamma_i \right) \psi = \\ =& \frac{1}{a^2} \left(\gamma^0 \partial_\eta - \vec{\gamma} \cdot \vec{\partial} - \frac{3}{2} H \gamma^0 \right) \psi = \frac{1}{a^{\frac{5}{2}}} \gamma^a \partial_a \left(a^{\frac{3}{2}} \psi \right) \,. \end{aligned}$$

□

Eq. (36) invites us to define the conformally rescaled fields

$$\chi = a^{\frac{3}{2}} \psi \,.$$

Other than this, we are going to expand the fermionic fields in their momentum representation, which, to fix conventions, reads

$$\chi(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \chi(\eta, \vec{k}) \,.$$

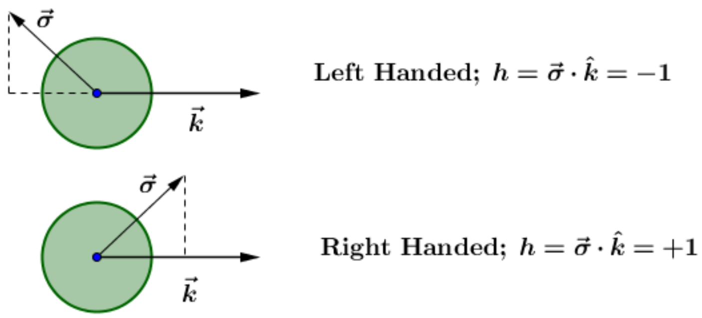


Figure 6: Graphical representation, in $\vec{\sigma}$-$\vec{k}$ plane of the helicity operator. Different helicity values exchange under $\vec{x} \to -\vec{x}$, the parity symmetry.

Since FLRW space-time is spatially homogeneous and isotropic, we can make the ansatz that $\psi(\eta, \vec{k}) = \psi(\eta, k)$, that is, the fields are only functions of the absolute value of the momentum. Because of this reason, we will find that the helicity operator commutes with the Dirac hamiltonian. Therefore we can split the fermionic fields in their helicity eigenstates, by means of the helicity projector $\mathcal{P}_h$

$$\chi = \sum_h \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} \otimes \xi_h = \sum_h \mathcal{P}_h \chi = \sum_h \chi_h \,,$$

where ξ_h is the helicity eigenvector, i. e. $\hat{\mathcal{H}}\xi_h = h\xi_h$. The Dirac equation then will transform to

$$\begin{aligned}&\left(i\gamma^0\partial_\eta - hk\gamma^0\gamma^5 - a(\eta)\hat{M}\right)\chi_h(\eta, k) = \\ = &-\frac{3\pi G\xi^2}{a(\eta)^2}\left(\sum_{h'}\int\frac{d^3q}{(2\pi)^3}\bar{\chi}_{h'}(\eta, \left|\vec{q}-\vec{k}/2\right|)\gamma^5\gamma^\sigma\chi_{h'}(\eta, \left|\vec{q}+\vec{k}/2\right|)\right) \\ &\times\gamma^5\gamma_\sigma\chi_h(\eta, k)\,,\end{aligned} \tag{37}$$

where $k = |\vec{k}|$.

Proof. First let us derive the helicity operator. From Figure 6 we notice that

$$\hat{\mathcal{H}} = \frac{\vec{\Sigma}\cdot\vec{k}}{k} = \frac{\vec{k}^i}{8k}i\epsilon_{ijk}\gamma^j\gamma^k = \hat{k}\cdot\gamma^0\vec{\gamma}\gamma^5\,.$$

Now we can act on Eq. (34) from the left. The only non trivial terms are

$$\vec{k}\cdot\vec{\gamma} = hk\gamma^0\gamma^5\hat{\mathcal{H}}\,,$$

obviously commutes with $\hat{\mathcal{H}}$ and gives the second term in Eq. (37). The second non trivial term is

$$\left[\hat{\mathcal{H}}, \gamma^5\gamma^\sigma\right] = \hat{k}_i\gamma^0\left[\gamma^i, \gamma^j\right]\delta_j^\sigma\,,$$

which means that the non commuting part in Eq. (37) is given by

$$\left(\int \frac{d^3q}{(2\pi)^3}\bar{\chi}_{h'}(\eta,\vec{q}-\vec{k}/2)\gamma^5\gamma_j\chi_{h'}(\eta,\vec{q}+\vec{k}/2)\right)\hat{k}_i\gamma^0\left[\gamma^i,\gamma^j\right]\chi_h(\eta,k)\,.$$

The operator $\gamma^5\gamma_j$ in the brackets produce terms $\propto \vec{k}_j$, and terms $\propto \vec{q}_j$. The mixed products vanishing upon integration as a consequence of the problem's symmetries. Then all we are left with things such as

$$\hat{k}_j\hat{k}_i\gamma^0\left[\gamma^i,\gamma^j\right]\chi_h(\eta,k) = \hat{q}_j\hat{q}_i\gamma^0\left[\gamma^i,\gamma^j\right]\chi_h(\eta,q) = 0\,.$$

□

Note in Eq. (37) the η dependence in the mass and in the interaction strength, consequences of doing calculations on the dynamical FLRW space-time. Also, note that the interactions terms are integrated over the centre of mass momentum. This is a consequence of locality of torsion interaction: from solution (22) we know that the torsion propagator is linearly related to the fermionic fields propagator, and will be function to the centre of mass position of $\bar{\psi}\psi$ system. The fact that the interaction happens at the centre of mass position, because of Heisenberg uncertainty principle, makes the centre of mass momentum of the interaction completely undetermined (Figure 5).

We conclude this section with a remark: helicity is not a Lorentz invariant quantity, as boosts change $\vec{k}$ and h. Therefore one usually requires particles to be massless, such that we cannot change h by a simple boost. However, here we are stating only that helicity is a conserved quantum number, because the helicity operator commutes with the Dirac Hamiltonian in Eq. (37). In Heisemberg picture, this is evident by considering the evolution equation of an operator

$$\partial_\eta\hat{\mathcal{O}} = i\left[\hat{\mathcal{O}},\hat{H}\right]\,,$$

where $\hat{H}$ is the Hamiltonian evolution operator.

4.3 ENERGY MOMENTUM TENSOR RENORMALIZATION

In next chapter we are going to analyse the Dirac equation and the torsion interactions in greater detail, and rely on some approximations to solve it. Before turning to that, however, let us discuss renormalisation of quantum divergences, and how they can be removed from the energy momentum tensor. From the Lagrangian (32) we find [27]

$$T^{\psi}_{\mu\nu} = \frac{i}{2}\left(\bar{\psi}\gamma_\mu\mathcal{D}_\nu\psi - \mathcal{D}_\mu\bar{\psi}\gamma_\nu\psi\right) + \frac{3\pi G\xi^2}{2}g_{\mu\nu}\left(\bar{\psi}\gamma^5\gamma_\sigma\psi\bar{\psi}\gamma^5\gamma^\sigma\psi\right)\,,$$

which in FLRW space-times and in term of the fields χ becomes

$$T^{\psi}_{\mu\nu} = \delta^a_\mu\delta^b_\nu\left[\frac{i}{2a(\eta)^4}\left(\bar{\chi}\gamma_a\partial_b\chi - \partial_b\bar{\chi}\gamma_a\chi\right) + \frac{3\pi G\xi^2}{2a(\eta)^6}\eta_{ab}\left(\bar{\chi}\gamma^5\gamma_c\chi\bar{\chi}\gamma^5\gamma^c\chi\right)\right]\,. \tag{38}$$

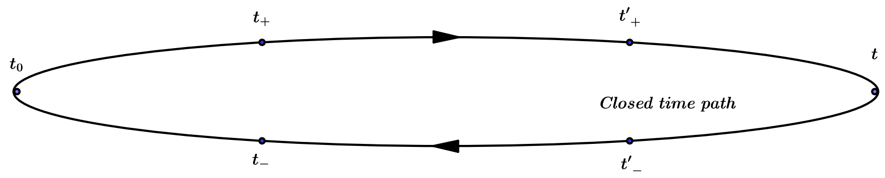


Figure 7: Closed time path in the propagator definition in the Schwinger-Kedysh formalism. The closed time contour implies that, when propagating from t to t', there are four possible choices for propagating. The propagators in the Feynman diagrams acquire a direction and for a given diagram containing non equal times propagator has to be summed over all propagators in Eqs. (39a - 39d) [16].

In solving the Dirac equation we are going to consider an initial thermal state for the field χ and evolve it accordingly to the Schwinger-Keldysh [7, 16, 33] formalism. The problem which such a procedure is that the initial thermal state contains quantum divergences that have to be removed. We will employ the scheme of dimensional regularisation, which means calculating divergent integrals in D dimensions, for the values of D in which the integrals are defined, and then analytically extend the result to $D = 4$. The useful aspect of dimensional regularisation is that it produces Lorentz invariant counter terms, as we will see.

The energy momentum tensor in Eq. (38) has to be averaged on the thermal state that we are considering,

$$\langle T_{\mu\nu}\rangle_\beta \,,$$

where $\beta = \frac{1}{k_B T}$ is the inverse temperature. Looking at Eq. (38) it is clear that we are going to need the thermal propagators, evaluated at coincidence, since the fields in the energy-momentum tensor are. To this end we define the 2-points Wightman functions [1, 30]

$$\begin{aligned}
iS^{++}_{\alpha\beta}(x;x') &= \langle \mathcal{T}\left[\psi_\alpha(x)\bar{\psi}_\beta(x')\right]\rangle_\beta \,, && \text{(39a)}\\
iS^{-+}_{\alpha\beta}(x;x') &= \langle \psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_\beta \,, && \text{(39b)}\\
iS^{+-}_{\alpha\beta}(x;x') &= -\langle \bar{\psi}_\alpha(x)\psi_\beta(x')\rangle_\beta \,, && \text{(39c)}\\
iS^{--}_{\alpha\beta}(x;x') &= \langle \bar{\mathcal{T}}\left[\psi_\alpha(x)\bar{\psi}_\beta(x')\right]\rangle_\beta \,, && \text{(39d)}
\end{aligned}$$

where $\mathcal{T}(\bar{\mathcal{T}})$ denote the (anti)time ordering operator.

In Schwinger-Kedysh formalism one considers a closed time path, in the complex plane, going from initial time t_0 to a finite time t(Figure 7). The 2-point functions in Eqs. (39a - 39d) can be connected with the propagator in this formalism, namely

$$S_{\alpha\beta}(x;x') = \langle \mathcal{T}_{\mathcal{C}}\left[\psi_\alpha(x)\bar{\psi}_\beta(x')\right]\rangle_\beta \,,$$

where $\mathcal{C}$ is the complex contour in Figure 7. Then points taken on the upper or lower branch of the contour ($t_\pm$, $t'_\pm$) will be ordered according to the standard time or anti time ordering, while points lying on different branches will be automatically ordered. From this we deduce the "$-$" sign in Eq. (39c), due to anti commutation relations for fermions.

The Schwinger-Keldysh formalism is particularly useful in describing non equilibrium situations: the analogue of the Boltzmann distribution function in classical mechanics can be constructed from S^{+-}, by means of a Wigner transformation(see Chapter 5 for more details). The different directions($+-$, $-+$) on the contour describe particles entering or leaving a definite state. For a system out of equilibrium, not satisfying detailed balance, this possibility plays a crucial role and in fact it is used to construct the analogue of the Boltzmann collision term.

To describe the fermionic system at thermal equilibrium, which will be our initial state, the whole machinery described above, however, is not necessary. The equilibrium distribution functions are given by the Fermi-Dirac and Bose-Einstein functions. To describe this initial situation, we follow the steps described in [1], and impose the Kubo-Martin-Schwinger (KMS) condition

$$S^{+-}(k) = -e^{-\beta k_0} S^{-+}(k)\,, \tag{40}$$

where k is the Fourier transform in the coordinate $r = x - x'$. Imposing the KMS relation corresponds to requiring detailed balance to be satisfied. From [1] we find that the KMS relation implies

$$\begin{aligned} S^{+-}(k) =& -\left(\not{k} - \hat{M}\right)\delta\left(k^2 - |m|^2\right) 2\pi \mathrm{sign}(k_0) f(k_0) = \\ &= \left(\not{k} - \hat{M}\right)\left(\theta(k_0) - f(|k_0|)\right) 2\pi\delta(k^2 - m^2)\,, \\ f(k_0) =& \frac{1}{e^{\beta k_0} + 1}\,. \end{aligned} \tag{41}$$

The term proportional to $\theta(k_0) = \frac{1}{2}(1 + \mathrm{sign} k_0)$ in Eq. (41) is the quantum divergence that we want to remove. Before calculating what the counter term might be, we notice that the opposite polarity of the propagator, $-+$, leads to the same expression in Eq. (41). Therefore their expressions in position representation at coincidence are the same.

The first term we calculate is the kinetic part of the energy momentum tensor, that is the one loop contribution. This leads to

$$\begin{aligned} \langle T^{div,1L}_{\mu\nu} \rangle =& \int \frac{d^D k}{(2\pi)^D} k_{(\mu} \mathrm{Tr}\left(\gamma_{\nu)} S^{+-}(k)\right) = \\ =& -\frac{g_{\mu\nu}}{D-1} \frac{\pi^{1-\frac{D}{2}}(D+1)}{2^D} \mu^{\frac{D-4}{2}} m^4 \left[-\frac{1}{D-4} - \frac{1}{2}\log\left(\frac{m^2}{\mu^2}\right) + \left(\frac{3}{2} - \gamma_E\right)\right] \\ \equiv& g_{\mu\nu} \Delta\Lambda\,, \end{aligned} \tag{42}$$

where the counter term in the Lagrangian to balance the divergences is given by a renormalisation of the cosmological constant. Note also that such a counter term does not change[3] when the proper scaling in the physical mass $m_{Phy} = a(\eta)m$ and $\frac{1}{a(\eta)^4}$, coming from the conformal rescaling, are inserted. Possible corrections of order $m'_{Phy} = Hm_{Phy}$ might have to be included to complete the procedure highlighted here.

Proof. Starting from the first Eq. (38) and using Eq. (41) we find

$$\begin{aligned}
T^{div,1L}_{00} &= \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{\sqrt{k^2+m^2}}{2}\,, \\
T^{div,1L}_{0i} &= 0\,, \\
T^{div,1L}_{ij} &= -\int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{k_i k_j}{2\sqrt{k^2+m^2}}\,.
\end{aligned}$$

Let's consider the vacuum energy density first:

$$\begin{aligned}
T^{div,1L}_{00} &= \frac{2\pi^{\frac{D-1}{2}}}{(2\pi)^{D-1}\Gamma\left(\frac{D-1}{2}\right)} \int_0^\infty dk\, k^{D-2}\sqrt{k^2+m^2} = \\
&= \frac{2\pi^{\frac{D-1}{2}}}{(2\pi)^{D-1}\Gamma\left(\frac{D-1}{2}\right)} \left(\int_0^\mu + \int_\mu^\infty\right) dk\, k^{D-2}\sqrt{k^2+m^2}\,.
\end{aligned}$$

Where we split the integral because the integrand is not convergent in any dimensions on $[0,+\infty]$, but it is in $[0,+\mu]$ and $[\mu,+\infty]$, respectively for $D > -1$ and $D < -1$. Then we integrate by parts to get

$$\begin{aligned}
&\left(\int_0^\mu + \int_\mu^\infty\right) dk\, k^{D-2}\sqrt{k^2+m^2} = \\
&= \frac{k^{D-1}}{D-1}\sqrt{k^2+m^2}\bigg|_0^\mu + \frac{k^{D-1}}{D-1}\sqrt{k^2+m^2}\bigg|_\mu^\infty - \left(\int_0^\mu + \int_\mu^\infty\right) dk\, \frac{k^D}{(D-1)\sqrt{k^2+m^2}} = \\
&= -\int_0^\infty dk \frac{k^D}{(D-1)\sqrt{k^2+m^2}} = \left(|m|^2\right)^{\frac{D}{2}} \frac{D+1}{4\sqrt{\pi}(D-1)}\Gamma\left(\frac{D-1}{2}\right)\Gamma\left(-\frac{D}{2}\right) \simeq \\
&\simeq -\frac{D+1}{2\sqrt{\pi}(D-1)}\Gamma\left(\frac{D-1}{2}\right) m^4 \mu^{\frac{D-4}{2}} \left(1 + \log\left(\frac{m^2}{\mu^2}\right)\frac{D-4}{2}\right) \\
&\quad \times \left(-\frac{1}{D-4} + \left(\frac{3}{4} - \frac{\gamma_E}{2}\right) + \mathcal{O}\left(D-4\right)\right)\,.
\end{aligned}$$

Then we consider $T^{div,1L}_{ij}$, and note that

$$-\int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{k_i k_j}{2\sqrt{k^2+m^2}} = \frac{2\pi^{\frac{D-1}{2}}}{(2\pi)^{D-1}\Gamma\left(\frac{D-1}{2}\right)} \frac{g_{ij}}{D-1} \int_0^\infty dk \frac{k^D}{\sqrt{k^2+m^2}}\,.$$

□

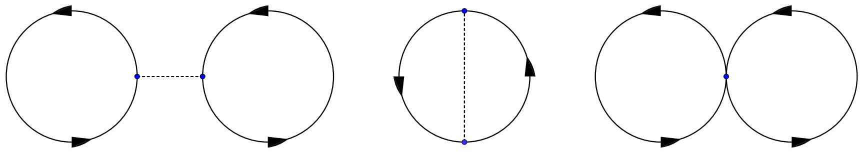


Figure 8: Two loop diagrams contributing to the energy-momentum tensor. As in Figure 5, the dashed line representing torsion is represented by a point in the effective theory.

Next we proceed to calculate the two loops corrections, that we can see in figure 8. To this purpose, we need to evaluate the fermionic four point function

$$\langle\bar{\psi}_\alpha\psi_\beta\bar{\psi}_\gamma\psi_\delta\rangle = \langle\bar{\psi}_\alpha\psi_\beta\rangle\langle\bar{\psi}_\gamma\psi_\delta\rangle - \langle\bar{\psi}_\alpha\psi_\delta\rangle\langle\bar{\psi}_\gamma\psi_\beta\rangle\,, \tag{43}$$

where we used the Wick theorem. Such a simplification, which is only possible for a gaussian initial state, corresponds to selecting the 2-loops contributions from Figure 8 as the only contributions to the energy momentum tensor (and later to the Dirac equation).

We refer to the first contribution in Figure 8 as Hartree term, and to the second as Fock term[4]. They correspond to the 2-loops contributions to the energy momentum tensor, and will be given by

$$\begin{aligned} T^{2L}_{\mu\nu} &= 3\pi G\xi^2 g_{\mu\nu}\left(\bar{\psi}\gamma^5\gamma_\sigma\psi\bar{\psi}\gamma^5\gamma^\sigma\psi\right) = \\ &=3\pi G\xi^2 g_{\mu\nu}\Big[\mathrm{Tr}\left(\gamma^5\gamma_\sigma S^{+-}(q)\right)\mathrm{Tr}\left(\gamma^5\gamma^\sigma S^{+-}(k)\right) - \\ &\quad - \mathrm{Tr}\left(\gamma^5\gamma_\sigma S^{+-}(q)\gamma^5\gamma^\sigma S^{+-}(k)\right)\Big]\,. \end{aligned} \tag{44}$$

The first term in Eq. (44) does not contribute, since $\mathrm{Tr}\left(\gamma^5\gamma^\mu\right) = \mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) = 0$. Therefore, we only have to consider the second term. We would find

$$\begin{aligned} T^{2L,div}_{\mu\nu} =& -24\pi G\xi^2 g_{\mu\nu}\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\left(k_\sigma q^\sigma + 2m^2\right) \\ &(\theta(k_0) - f(|k_0|))\,2\pi\delta(k^2-m^2)\times(\theta(q_0)-f(|q_0|))\,2\pi\delta(q^2-m^2)\,, \end{aligned} \tag{45}$$

3 Up to logarithmic corrections.
4 As we will see in next chapter, the reason for such a name is that what we are doing is in some sense analogous to the Hartree-Fock approximation for the interaction terms.

which yields to

$$\begin{aligned}
T^{2L,div}_{\mu\nu} =& -24\pi G\xi^2 g_{\mu\nu}\frac{m^6}{(4\pi)^D}\left(\mu^2\right)^{D-4}\times\\
&\times\left[\frac{4}{(D-4)^2}+\frac{4}{D-4}\left(\log\left(\frac{m^2}{\mu^2}\right)+(\gamma_E-4)\right)+\right.\\
&\left.+4(\gamma_E-1)\log\left(\frac{m^2}{\mu^2}\right)+\left(\frac{\pi^2}{6}+2\gamma_E^2-4\gamma_E+3\right)\right]\equiv g_{\mu\nu}\delta\Lambda^{2L}\,.
\end{aligned} \tag{46}$$

Again, notice that the mass scaling and the scaling coming from conformally rescaling the fields simplify up to logarithmic corrections. Then this term becomes a constant and renormalises the cosmological constant.

The careful reader might have noticed that Eq. (45) contains a term of the form

$$\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\theta(k_0)f(|q_0|)\left(k_\sigma q^\sigma+2m^2\right)2\pi\delta(k^2-m^2)2\pi\delta(q^2-m^2)\,,$$

which contains temperature dependence when evaluated at thermal equilibrium. Such contribution is problematic when we try to renormalise it, since it has an explicit dependence on the fermionic fluid we later want to evolve. However, taking a look at the original theory, rather the effective without torsion, it becomes clear that there is no need for such a counter term. In fact, even Eq. (46) is superfluous: in the original theory, the energy momentum tensor will contain terms in the form

$$\langle S^\star_\mu\bar\psi\gamma^5\gamma^\mu\psi\rangle\propto S^\star_\mu\mathrm{Tr}\left(\gamma^5\gamma^\mu S^{+-}\right)\,,$$

which vanish for the same reason as the Hartree term, because tracing $\gamma^5\gamma^\mu S^{+-}$ gives zero. The fact that the regularisation procedure produces temperature dependent counter terms, is a consequence of trying to renormalise the operator in the effective theory, rather than in the initial one. In the effective theory, the four fermion operator appearing in the energy momentum tensor, is a composite operator. In the original theory the coupling constant between torsion and the fermionic fields is dimensionless, which means the theory is renormalisable. However, this feature is lost in the effective theory, and the coupling constant carries a energy dimension (that of the Planck mass).

For the temperature dependent term we find

$$\begin{aligned}
T^{2L,div(2)}_{\mu\nu} =& 72\pi G\xi^2 g_{\mu\nu}m^3(k_BT)^3\mu^{\frac{D-4}{2}}\left[\frac{2}{D-4}+\log\left(\frac{m^2}{\mu^2}\right)\right.\\
&\left.+\frac{1}{12}\left(\pi^2+6\gamma_E^2-12\gamma_E-12\right)\right]\\
&\times\underbrace{\int d\sigma\frac{1}{e^{\cosh\sigma}+1}}_{\simeq 0.333474}\equiv g_{\mu\nu}\Delta\Lambda^{2L,(2)}(k_BT)\,.
\end{aligned} \tag{47}$$

However, removing divergences first in the original theory and then integrating out torsion, both terms in Eq. (46) and in Eq. (47) should be disregarded as unphysical, and be replaced by 0. Here they are intended as counter terms which regularise the initial state, but do not evolve further. In contrast with this, Eq. (42) is a physical result from which we can deduce the running of the renormalised cosmological constant with the energy scale μ.

Part III

SEMI-CLASSICAL APPROXIMATION AND SELF CONSISTENT BACK-REACTION

5 SEMI-CLASSICAL APPROXIMATION AND SELF CONSISTENT BACK-REACTION

5.1 TWO LOOPS EFFECTIVE ACTION AND EQUATIONS OF MOTION

In this chapter we will turn our attention to the Dirac equation (37). However, we are going to derive it again, starting from the one loop effective action for the propagator. The reason for this procedural discordance stands in the complicated momentum dependence one finds in the interaction terms in Eq. (37). Such a complication arises from locality of torsion interactions: since the interaction happens locally at a single point, it spreads all through momentum space. This makes it harder to get the propagator equation from Eq. (37), so instead we are going to derive it from the principle of least action.

To this end we start from the action in Eq. (32) in terms of the conformally rescaled fields

$$\mathcal{S}_\chi = \int d^4x \Big[\frac{i}{2}\partial_a\big(\bar{\chi}(x)\gamma^a\chi(x) - \bar{\chi}(x)\gamma^a\chi(x)\big) \\ -\bar{\chi}(x)a(\eta)(m_R + i m_I\gamma^5)\chi(x) + \frac{3\pi G\xi^2}{2a(\eta)^2}(\bar{\chi}(x)\gamma^5\gamma^a\chi(x))^2\Big],$$

Our purpose is to make the interaction non local in x space, by rewriting

$$\mathcal{S}_\chi = \int d^4x d^4x'\delta^{(4)}(x-x')\Big[\frac{i}{2}\partial_a\big(\bar{\chi}(x')\gamma^a\chi(x) - \bar{\chi}(x)\gamma^a\chi(x')\big) \\ -\bar{\chi}(x')a(\eta)(m_R + i m_I\gamma^5)\chi(x) + \frac{3\pi G\xi^2}{2a(\eta)^2}(\bar{\chi}(x')\gamma^5\gamma^a\chi(x))^2\Big], \tag{48}$$

where $\partial_a = \frac{\partial}{\partial x^a}$ and we rewrote $\hat{M} = m_R + i\gamma^5 m_I$. Then we define the propagator

$$S^{+-}_{\alpha\beta}(x,x') = i\langle\bar{\chi}(x)\chi(x')\rangle\,.$$

Since in the propagator equation all one loop contribution appear at coincidence, we do not have to write equations for all the propagators in Eqs. (39a - 39d), because their expression at coincidence coincide. We therefore chose only one to work with, S^{+-}.

After partial integration of the kinetic term, and taking expectation value the action becomes:

$$\mathcal{S}_\chi = -i\int_{x,x'} \mathrm{Tr}\Big[\frac{i}{2}\big(\gamma^a S^{+-}(x,x') - \gamma^a S^{+-}(x',x)\big)\partial_a - a(\eta)\hat{M}\, S^{+-}(x,x') \\ + \frac{3\pi G\xi^2}{2a(\eta)^2}\langle\bar{\chi}(x)\gamma^5\gamma^a\chi(x')\bar{\chi}(x)\gamma^5\gamma_a\chi(x')\rangle\Big]\delta^{(4)}(x-x')\,.$$

Now, note that the interaction term can be rewritten, using Wick theorem

$$\begin{aligned}
&\int_{x,x'} \mathrm{Tr}\langle\bar{\chi}(x)\gamma^5\gamma^a\chi(x')\bar{\chi}(x)\gamma^5\gamma_a\chi(x')\rangle = \\
&= \int d^4x d^4x' (\gamma^5\gamma^a)_{\alpha\beta}(\gamma^5\gamma_a)_{\gamma\delta}\big(S^{+-}_{\alpha\beta}(x,x')S^{+-}_{\gamma\delta}(x,x') \\
&- S^{+-}_{\alpha\delta}(x,x')S^{+-}_{\gamma\beta}(x,x')\big)\delta^{(4)}(x-x')\,.
\end{aligned} \tag{49}$$

From the spinorial structure in Eq. (49), the diagrammatic interpretation of our approximation is clear: we are considering the Hartree and the Fock terms from Figure 8, and evaluating them at coincidence. This is analogous to glue the diagrams along the dashed line in Figure 8.

Having simplified the fermions four point function, we can rewrite the two loops effective action

$$\begin{aligned}
i\mathcal{S}_\chi = &\int_{x,x'} \mathrm{Tr}\Big[i\gamma^a S^{+-}(x,x')\partial_a - a(\eta)(m_R + i m_I\gamma^5)S^{+-}(x,x') \\
&+ \frac{3\pi G\xi^2}{2a(\eta)^2}\big(\mathrm{Tr}(\gamma^5\gamma^a S^{+-}(x,x'))\mathrm{Tr}(\gamma^5\gamma_a S^{+-}(x,x')) - \\
&- \mathrm{Tr}(\gamma^5\gamma^a S^{+-}(x,x')\gamma^5\gamma^a S^{+-}(x,x'))\big)\Big]\delta^{(4)}(x-x')\,.
\end{aligned} \tag{50}$$

Varying the action (50) with respect to the propagator, multiplying by $S^{+-}(x',x'')$ and integrating over the intermediate variable x' we get

$$\begin{aligned}
&\Big(i\gamma^a\partial_a - a(\eta)(m_R + i m_I\gamma^5)\Big)S^{+-}(x,x'') = \\
&= -\left(\frac{3\pi G\xi^2}{a(\eta)^2}\big(\mathrm{Tr}(\gamma^5\gamma^a S^{+-}(x,x''))\gamma^5\gamma_a - \gamma^5\gamma^a S^{+-}(x,x'')\gamma^5\gamma^a\right) \\
&\times S^{+-}(x,x'')\,.
\end{aligned} \tag{51}$$

We now wish to use Eq. (51) to evolve an initial thermal state during a collapsing phase, and prove that it does not lead to a singularity, but rather through a bounce. We therefore want to evolve the fermions interacting system through an evolving background and study the effect of the interaction terms in Eq. (51). In classical mechanics, we would use the Boltzmann equation

$$\left(\frac{\partial}{\partial t} + \frac{\vec{k}}{m}\cdot\vec{\nabla} + \vec{F}\cdot\frac{\partial}{\partial\vec{k}}\right) f(\vec{x},\vec{p},t) = \mathit{Coll}(f)\,, \tag{52}$$

where $\mathit{Coll}(f)$ denotes the collision term and $\vec{F}$ some external force field. Then the out of equilibrium dynamics will be described by the $\mathit{Coll}(f)$, and the phase space evolution of f by the left hand side of

Eq. (52). The Boltzmann equation contains derivatives with respect to both positions and momentum, and simultaneous evaluation of their eigenvalues. In quantum mechanics such a description is not possible, since the position and momentum operator do not commute. A clever trick around this problem was suggested by Wigner [36], who wrote the Wightman functions in a mixed representation (Wigner representation)

$$\begin{aligned} S^{+-}(x,k) \equiv & \int d^4 r e^{ik\cdot r} S^{+-}\left(x-\frac{r}{2}, x+\frac{r}{2}\right) = \\ =& i\int d^4 r e^{ik\cdot r} \langle \bar{\chi}\left(x-\frac{r}{2}\right)\chi\left(x+\frac{r}{2}\right)\rangle\,. \end{aligned} \tag{53}$$

In practice, the Wigner transform is a function of the “centre of mass”, x, of the system $\bar{\chi}\chi$, and of k, the momentum associated with the “distance” between $\bar{\chi}$ and χ. The utility of $S^{+-}(x,k)$ is that, when integrated over x, it gives the correct quantum probabilities to find the system in the state p and vice versa [36]. In formulas

$$\left|\int d^4 x \chi\left(x\right) e^{-ip\cdot x}\right|^2 = \mathrm{Tr}\left(-i\gamma^0 \int d^4 x S^{+-}(x,k)\right), \tag{54a}$$

$$\left|\chi\left(x\right)\right|^2 = \mathrm{Tr}\left(-i\gamma^0 \int d^4 k S^{+-}(x,k)\right). \tag{54b}$$

Now we can construct the Boltzmann kinetic equation, by rewriting the Dirac equation in this mixed representation. We would then find the equation of motion [30]

$$\left(\frac{i}{2}\not{\partial} + \not{k} - (m_R + i m_I) e^{-\frac{i}{2}\overleftarrow{\partial}\cdot\partial_k}\right) S^{+-}(x,k) = \mathit{Coll}\left(S^{+-}(x,k)\right)\,.$$

Proof. From Eq. (51) it is clear that, when taking the Wigner transform, we evaluate a general mass function of the coordinates as

$$\begin{aligned} m_{R,I}\left(x-\frac{r}{2}\right) \to & m_{R,I}\left(x - i\frac{\partial_k}{2}\right) = \sum_{p=0}^{\infty}\frac{m_{R,I}^{(p)}}{p!}\left(x - i\frac{\partial_k}{2}\right)^p = \\ &= \sum_{q=0}^{\infty}\sum_{p=q}^{\infty}\frac{m_{R,I}^{(p)}}{q!(p-q)!}x^{p-q}\cdot\left(-\frac{i}{2}\partial_k\right)^q = \\ &= \sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\frac{m_{R,I}^{(p)}}{q!p!}x^{p}\left(-\frac{i}{2}\partial_x\cdot\partial_k\right)^q\,. \end{aligned}$$

Which gives the last term in our Wigner representation of the Dirac equation. The other therms are trivial, and *Coll* $(S^{+-}(x,k))$ just denotes whatever comes from interactions.
□

We now managed to rewrite the momentum and position operator in a mixed representation needed to derive the quantum analogue of the Boltzmann kinetic equation.

Indeed, we can now generalise Eq. (41) to a system out of thermal equilibrium [7]

$$iS^{+-}(x,k) = -\left(\not{K} - \hat{M}\right)\tilde{f}(x,k)2\pi\delta(k^2 - m^2)\,, \tag{55}$$

where $\tilde{f}(x,k)$ is the non equilibrium, non homogeneous distribution function in the Wigner representation. The defining property of Boltzmann probability distribution, $f(t,\vec{x},\vec{p})$ in Eq. (52), is that the integral of f over the phase space gives the total number of particles. In a similar fashion, one can define the statistical particle number

$$N_{stat} = \int d^4x d^4k\, \tilde{f}(x,k)\,. \tag{56}$$

N_{stat} indeed gives the statistical particle number, because of Eqs. (54a - 54b). The situation is, however, more complicated when examined more closely: one can define several quantities[1] that reduce to the total particle number at thermal equilibrium, each of which provides different information. The reason for this is that in a system out of equilibrium, particles are constantly produced and annihilated by the microscopic processes. Because we study average functions, we cannot keep track of each single particle and tell exactly what the total number is.

We now wish to make the connection with this general framework and our cosmological field of fermions, in a space-time with torsion. Our starting point will be Eq. (51), evaluated at coincidence since we are primarily interested in the energy-momentum tensor: because of the symmetries of FLRW space-time, we can infer that

$$S^{+-}(\eta,\vec{x};\eta,\vec{x}'') = S^{+-}(\eta,\left|\vec{x}-\vec{x}''\right|)\,,$$

such that equal times analogue of S^{+-} in Wigner representation will only be a function of $k = \left|\vec{k}\right|$. This statement means that the propagator does not depend on the specific point it is evaluated, but only on the average spatial distance, $\vec{r}$, which we then transform into its Fourier counterpart $\vec{k}$. Also, because all propagators in the 2-loops energy momentum tensor (38) are evaluated at equal times, we do not need the information about the full propagator, but only require its projection on shell evaluated at coincidence.

1 Other than N_{stat}, and with reference to Eq. (60a - 60d), we can have $N_{kin} = \sqrt{f_{1h}^2 + f_{2h}^2 + f_{3h}^2}$.

We then end up with the equation

$$\begin{aligned}&\left(i\gamma^0\partial_\eta - hk\gamma^0\gamma^5 - a(\eta)(m_R + im_I\gamma^5)\right)S_h^{+-}(\eta,k) = \\ &= -\frac{3\pi G\xi^2}{a(\eta)^2}\int\frac{d^3k'}{(2\pi)^3}\left(\sum_{h'}\mathrm{Tr}(\gamma^5\gamma^a S_{h'}^{+-}(\eta,k'))\gamma^5\gamma_a - \gamma^5\gamma^a S_h^{+-}(\eta,k')\gamma^5\gamma^a\right) \\ &\times S_h^{+-}(\eta,k)\,,\end{aligned} \tag{57}$$

where we have projected onto the helicity basis, in the same way as in section 4.2. The difference with section 4.2 is that now the momentum integral on the right hand side of Eq. (57) factorises, while in the analogous Eq. (37) it was not possible to obtain such a splitting.

To end this section, we remark that the Wigner representation is of particular interest on cosmological backgrounds, because of the symmetries of the system. Spatial isotropy and homogeneity imply that the $\vec{x}$ coordinate in Eq. (53) is superfluous. Furthermore, we only require the on shell projection of the propagator, since to write down the Friedmann equations this is all that we need, which implies also that we can neglect the k_0 dependence in Eq. (53).

5.2 FERMIONIC CURRENTS AND SEMI-CLASSICAL APPROXIMATION

Let us consider again the propagator equation (57): we want to construct a ansatz for $S_h(\eta,k)$, which exploit the helicity conservation properties derived in section 4.2. To this end, we note that

$$\begin{aligned}S_{hh'}(\eta,k) =& i\langle\bar{\chi}_{h,\alpha}(\eta,k)\chi_{h',\beta}(\eta,k)\rangle = i\delta_{hh'}\langle\bar{\chi}_{h,\alpha}\chi_{h,\beta}\rangle\otimes\xi^\dagger_{h,\alpha}\xi_{h,\beta} \\ \equiv& \delta_{hh'}S_h(\eta,k)\,,\end{aligned} \tag{58}$$

as we decomposed the field χ in the helicity basis. In Eq. (58) we have dropped the superscript $+-$, since in the time coincidence limit it is not required to distinguish between different polarities. Note that the propagator is diagonal in helicity representation of the fields, which is what we needed to construct our ansatz.

The helicity eigenvector ξ_h are given by [19]

$$\xi_\pm = \frac{1}{\sqrt{2(1\mp\hat{k}_z)}}\begin{pmatrix}\pm(\hat{k}_x - i\hat{k}_y)\\ 1\mp\hat{k}_z\end{pmatrix},$$

such that

$$\xi_{h,\alpha}\xi_{h,\beta} = \frac{1}{2}\begin{pmatrix}1+h\hat{k}_z & h(\hat{k}_x + i\hat{k}_y)\\ h(\hat{k}_x - i\hat{k}_y) & 1+h\hat{k}_z\end{pmatrix} = \frac{\mathbb{1}+h\hat{k}\cdot\vec{\sigma}}{2}\,.$$

While for the remaining degrees of freedom, we choose to split in the chirality basis, in such a way that

$$\langle \chi^\dagger_{h,\alpha}\chi_{h,\beta}\rangle = \begin{pmatrix} \langle \chi^\dagger_{h,L}\chi_{h,L}\rangle & \langle \chi^\dagger_{h,L}\chi_{h,R}\rangle \\ \langle \chi_{h,R}\chi^\dagger_{h,L}\rangle & \langle \chi^\dagger_{h,R}\chi_{h,R}\rangle \end{pmatrix},$$

since χ_h is a two component spinor. Here the subscript L, R refer to left-handed and right-handed chiralities. The chirality projectors are given by $P_{L,R} = \frac{\mathbb{1}\pm\gamma^5}{2}$.

We are now ready to write the ansatz for the propagator [9]

$$-i\gamma^0 S_h(\eta,k) \equiv \frac{1}{4}\left(f_{ah}\rho^a\right)\otimes\left(\mathbb{1}+h\hat{k}\cdot\vec{\sigma}\right). \tag{59}$$

Where we defined the f_{ah} as the expectation value of the fermionic currents

$$\begin{aligned} f_{0h}(\eta,k) &= -i\mathrm{Tr}\left(\gamma^0 S_h(\eta,k)\right) &&= \mathrm{Tr}\langle\chi^\dagger_h\chi_h\rangle\,, && \text{(60a)}\\ f_{1h}(\eta,k) &= i\mathrm{Tr}\left(S_h(\eta,k)\right) &&= \mathrm{Tr}\langle\chi^\dagger_h\gamma^0\chi_h\rangle\,, && \text{(60b)}\\ f_{2h}(\eta,k) &= \mathrm{Tr}\left(\gamma^5 S_h(\eta,k)\right) &&= -i\mathrm{Tr}\langle\chi^\dagger_h\gamma^0\gamma^5\chi_h\rangle\,, && \text{(60c)}\\ f_{3h}(\eta,k) &= i\mathrm{Tr}\left(\gamma^5\gamma^0 S_h(\eta,k)\right) &&= \mathrm{Tr}\langle\chi^\dagger_h\gamma^5\chi_h\rangle\,. && \text{(60d)} \end{aligned}$$

And ρ^a are Pauli matrices, with $\rho^0 = \mathbb{1}$. We write the γ matrices in Weyl basis

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \rho^1\otimes\mathbb{1}\,,\\ \gamma^i &= \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = -i\rho^2\otimes\sigma^i\,,\\ \gamma^5 &= \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = -\rho^3\otimes\mathbb{1}\,. \end{aligned} \tag{61}$$

Such that the propagator written in Eq. (58) coincides with the one written in the right hand side of Eq. (59).

We can now consider the Dirac equation Eq. (57), and take the combinations given in Eqs. (60a - 60d), to get the following set of equations

$$\partial_\eta f_{0h}(k) = 0\,, \tag{62a}$$

$$\partial_\eta f_{1h}(k) + 2hkf_{2h}(k) - 2am_I f_{3h}(k) = \tag{62b}$$

$$= \frac{6\pi G\xi^2}{a^2}\int d\vec{p}\left(\sum_{h'} f_{3h'}(p)f_{2h}(k) - \frac{1}{4}\left(f_{3h}(p)f_{2h}(k) + f_{3h}(k)f_{2h}(p)\right)\right),$$

$$\partial_\eta f_{2h}(k) - 2hkf_{1h}(k) + 2am_R f_{3h}(k) = \tag{62c}$$

$$= -\frac{6\pi G\xi^2}{a^2}\int d\vec{p}\left(\sum_{h'} f_{3h'}(p)f_{1h}(k) - \frac{1}{4}\left(f_{3h}(p)f_{1h}(k) + f_{3h}(k)f_{1h}(p)\right)\right),$$

$$\partial_\eta f_{3h} - 2am_R f_{2h} + 2am_I f_{1h} = 0\,, \tag{62d}$$

where we mean $f(k) = f(\eta, k)$.

We have therefore arrived to the semi-classical approximation we were looking for: we have switched from a full quantum mechanical formalism, to a set of four equations, for the variables f_{ah}. The attribute semi-classical here is intended in the following sense: the quantities f_{ah} are not quantum mechanical wave functions, but expectation values of them. One can therefore regard the set of Eqs. (62a - 62d) as a classical system, since f_{ah} are real commuting variables. However the underlying quantum structure is still captured by the set of Eqs. (62a - 62d), since we derived them starting from the microscopic Dirac theory and the number of degrees of freedom stays the same. We have lost information about the phase dependence of the fields χ and the off shell propagator.

Finally, a comment on Eqs. (62a - 62d): as derived in Appendix B, the interactions terms induced by torsion generate pseudo-vector fields, given in terms of the fermionic fields themselves. Such interactions produce an effective shift on the momentum and mass terms in Eqs. (62b - 62c), as we can see by comparing the momentum and mass term on the left hand side with the interactions on the right hand side[2]. Since this terms leads to a time dependent change in the mass, they can lead to particle production when the temperature reaches the shifted mass scale.

5.3 SOLUTION IN THE MASSLESS REGIME

The structure of Eqs. (62a - 62d) invites to formulate a system of equations for integral quantities of the f's. This way the momentum integrals that come from the interaction terms will disappear, and we would have an ordinary system of first order differential equations. We define the momenta

$$n_{ah}^{(m)} = \int \frac{d^3k}{(2\pi)^3} k^m f_{ah}(k)\,, \tag{63}$$

and define

$$\begin{aligned} f_h &= f_{1h} + i f_{2h}\,, \\ n_h^{(m)} &= n_{1h}^{(m)} + i n_{2h}^{(m)}\,, \\ m &= m_R + i m_I\,, \\ \alpha_5 &= 3\pi G \xi^2/2\,. \end{aligned}$$

2 The reason why such effect does not appear in Eq. (62d) is explained in Appendix B.

Then Eqs. (62a - 62d) will transform to

$$\partial_\eta n_h^{(m)} - 2hi\, n_h^{(m+1)} + \frac{2i\alpha_5}{a^2}\left(\sum_{h'} n_{3h'}^{(0)} - \frac{n_{3h}^{(0)}}{4}\right) n_h^{(m)} \qquad (64a)$$

$$+i\left(am - \frac{\alpha_5 n_h^{(0)}}{2a^2}\right) n_{3h}^{(m)} = 0\,,$$

$$\partial_\eta n_{3h}^{(m)} + iam\, n_h^{*(m)} - iam^* n_h^{(m)} = 0\,. \qquad (64b)$$

Plus the complex conjugate of Eq. (64a), which we do not write because it is not an independent equation: the real degrees of freedom, f_{1h}, f_{2h} are here replaced by a complex one.

We notice that in the system (64a - 64b) all equations are coupled to each others, by the terms $n^{(m+1)}$. In other words, the system can be rewritten as

$$\left[\left(\partial_\eta + \frac{2i\alpha_5}{a^2}\left(\sum_{h'} n_{3h'}^{(0)} - \frac{n_{3h}^{(0)}}{4}\right)\right)\mathbb{1} - hk\mathcal{P}^{(1)}\right]\cdot \vec{n}_h \equiv \Lambda \cdot \vec{n}_h$$

$$= -i\left(am - \frac{\alpha_5 n_h^{(0)}}{2a^2}\right)\mathbb{1}\,\vec{n}_{3h}\,,$$

$$\partial_\eta\, \vec{n}_{3h} = -i(am\, \vec{n}_h^* - am^*\, \vec{n}_h)\,.$$

Where $\mathcal{P}^{(1)}$ is a permutation of the identity matrix. This means that[3]

$$\Lambda = \begin{pmatrix} (\partial_\eta + i\Gamma(\eta)) & -2hi & 0 & 0 & \cdots \\ 0 & (\partial_\eta + i\Gamma(\eta)) & -2hi & 0 & \\ 0 & 0 & (\partial_\eta + i\Gamma(\eta)) & -2hi & \\ \vdots & & & & \ddots \end{pmatrix}, \qquad (65)$$

is a matrix where each line is obtained from the previous line by a constant permutation. This type of matrices can always be diagonalised, even when they are infinitely dimensional, if they satisfy convergence properties. In Section C.1, we explicitly show how to construct such a diagonalisation in general, and how to switch to the continuous limit. Because the matrix for the basis change is unitary and constant, this procedure is exact.

We now use such a framework to simplify our system. First, let us notice that Eq. (100) gives, inserting Eq. (63)[4]

$$\nu_{ah}(\theta) = \sum_{n=0}^{\infty}\int \frac{d^3k}{(2\pi)^3}\,(\beta k)^n\, e^{-in\theta} f_{ah}(\eta, k)\,, \qquad (66)$$

3 Meaning $\Gamma(\eta) = \frac{2\alpha_5}{a^2}\left(\sum_{h'} n_{3h'}^{(0)} - \frac{n_{3h}^{(0)}}{4}\right)$.

4 Note that we inserted a factor of $\beta = \frac{1}{k_B T_{in}}$ for dimensional reasons. T_{in} here denotes the temperature of the initial thermal state we want to evolve.

which does not converge in general, and most notably it does not converge for the thermal distributions $f_{ah,th}$(see Section C.2 for more details on this). Thus we need to modify our definitions, and the diagonalisation procedure defined in Section C.1. We apply Borel summation and redefine (100) as

$$n_{ah}^{(n)} = \frac{n!}{\beta^n}\int\frac{d\theta}{2\pi}e^{in\theta}\nu_{ah}(\theta)\,, \tag{67a}$$

$$\nu_{ah}(\theta) = \sum_{n=0}^{\infty}\frac{e^{-in\theta}}{n!}\beta^n n_{ah}^{(n)}\,. \tag{67b}$$

Which is well defined as long as $f_{ah}(\eta, k)$ goes to zero faster than any power of k. This, however, modifies the diagonal form of Λ, in Eq. (65). To see how, consider, in Eqs. (64a - 64b) the term

$$\begin{aligned} n_{ah}^{(n+1)} &= \frac{n!}{\beta^{n+1}}\int\frac{d\theta}{2\pi}(n+1)e^{i(n+1)\theta}\nu_{ah}(\theta) \overset{\text{P. I.}}{=} \\ &= \frac{n!}{\beta^n}\int\frac{d\theta}{2\pi}e^{in\theta}\left(\frac{ie^{i\theta}}{\beta}\partial_\theta\nu_{ah}(\theta)\right)\,. \end{aligned}$$

Now it is clear that Eqs. (64a - 64b) will transform, under (67a - 67b) as

$$\partial_\eta\nu_h + \frac{2he^{i\theta}}{\beta}\partial_\theta\nu_h + 2iam\nu_{3h} = \tag{68a}$$

$$= -\frac{2i\alpha_5}{a^2}\int\frac{d\theta'}{2\pi i}\left(\sum_{h'}\nu_{3h'}(\theta') - \frac{\nu_{3h}(\theta')}{4}\nu_h - \int\frac{d\theta'}{2\pi i}\nu_h(\theta')\frac{\nu_{3h}}{4}\right)\,,$$

$$\partial_\eta\nu_{3h} - 2am_R\nu_{2h} + 2am_I\nu_{1h} = 0\,. \tag{68b}$$

We have not done much: we switched from an integral equation in the variables (η, k) into an integral equation in the variables (η, θ). However, the system (68a - 68b) can be further simplified by defining the new coordinate

$$\begin{aligned} \rho &= e^{-i\theta}\,, \\ e^{i\theta}\partial_\theta &= -i\partial_\rho\,, \\ \int\frac{d\theta}{2\pi} &= \oint\frac{d\rho}{2\pi i}\frac{1}{\rho}\,. \end{aligned}$$

Then we find

$$\partial_\eta\nu_h - \frac{2hi}{\beta}\partial_\rho\nu_h + 2ia(\eta)m\nu_{3h} = \tag{69a}$$

$$= -\frac{2i\alpha_5}{a(\eta)^2}\left(\sum_{h'}\nu_{3h'}(\rho)\Big|_{\rho=0}\nu_h - \frac{1}{4}\nu_{3h}(\rho)\Big|_{\rho=0}\nu_h - \frac{1}{4}\nu_h(\rho)\Big|_{\rho=0}\nu_{3h}\right)\,,$$

$$\partial_\eta\nu_{3h} - 2a(\eta)m_R\nu_{2h} + 2a(\eta)m_I\nu_{1h} = 0\,. \tag{69b}$$

We now can appreciate the advantage of this technique: the interaction terms, originally integrals of the distributions f_{ah}, now are expressed by boundary terms in the ρ space. The rest of the information can be reconstructed by analytically continuation, if one allows ρ to be a complex variable. There is of course a price to pay: our system is now differential in two variables. However, such a additional dependence is not too difficult to deal with, because it comes linearly and at first order.

It is now time to find a solution to Eqs. (69a - 69b). We claim that, in the limit $a(\eta)m_{R,I} \to 0$, describing the late phase of the collapse, the solution is given by

$$\begin{aligned} \nu_h(\rho,\eta) =& \exp\left(\frac{i\alpha_5 \nu_{3h}(0)}{2}\int_{\eta_0}^{\eta} d\eta' a(\eta')^{-2}\right) \\ \times&\left[F\left(\eta+\frac{\beta\rho}{2hi}\right)-\frac{i\alpha_5}{2}\int_0^{\frac{\beta\rho}{2hi}} ds' a\left(-s'+\eta+\frac{\beta\rho}{2hi}\right)^{-2} F\left(-s'+\eta+\frac{\beta\rho}{2hi}\right)\nu_{3h}\left(\frac{2hi}{\beta}s'\right)\right], \\ \nu_{3h}(\eta,\rho) =& H(\rho)\,, \end{aligned} \tag{70}$$

where $F(\rho)$ and $H(\rho)$ are analytical functions determined by the initial conditions. This massless limit is useful in particular in the situation of a collapsing universe, or the bulk of a star collapsing into a singularity. In these situations, the conformal factor $a(\eta) \to 0$ in the late phase of the collapse.

Proof. Let us start by rewriting Eqs. (69a - 69b), in the limit $m_{R,I} \to 0$

$$\begin{aligned} \left[\left(\partial_\eta + \sum_{h'} \frac{2i\alpha_5}{a(\eta)^2}\nu_{3h'}(0,\eta) - \frac{i\alpha_5}{2a(\eta)^2}\nu_{3h}(0,\eta)\right) - \frac{2hi}{\beta}\partial_\rho\right]\nu_h(\rho,\eta) &= \\ = \frac{i\alpha_5}{2a(\eta)^2}\nu_h(0,\eta)\nu_{3h}(\rho,\eta)\,, \\ \partial_\eta \nu_{3h}(\rho,\eta) = 0\,. \end{aligned}$$

At thermal equilibrium, the term $\sum_{h'} \nu_{3h'}(0,\eta)$ simplifies(Section C.2), so we can make the ansatz[5]

$$\nu_h(\rho,\eta) = \mathrm{Exp}\left(\frac{i\alpha_5 \nu_{3h}(0)}{2}\int_{\eta_0}^{\eta} d\eta' \frac{1}{a^2(\eta')}\right) V_h(\rho,\eta)\,.$$

We thus get the following equation to solve

$$\left[\partial_\eta - \frac{2hi}{\beta}\partial_\rho\right] V_h(\rho,\eta) = \frac{i\alpha_5}{2a(\eta)^2} V_h(0,\eta)\nu_{3h}(\rho)\,. \tag{71}$$

5 Note that this simplification is not necessary if we want to apply this method in a more general situation, but it simplifies the notation.

To solve this equation, we are going to make the ansatz

$$V_h(\rho,\eta) = F\left(\eta + \frac{\beta\rho}{2hi}\right) + G(\rho,\eta) \text{ such that } G(\rho=0,\eta) = 0 \tag{72}$$

Then the equation we have to solve is

$$\left[\partial_\eta - \frac{2hi}{\beta}\partial_\rho\right] G(\rho,\eta) = \frac{i\alpha_5}{2a(\eta)^2} F(\eta)\nu_{3h}(\rho)\,.$$

This equation can be solved with the method of characteristic curves [8]. This means looking for solutions of the system of ordinary equations

$$\frac{d\eta(s,r)}{ds} = 1\,, \tag{73a}$$

$$\frac{d\rho(s,r)}{ds} = -\frac{2hi}{\beta}\,, \tag{73b}$$

$$\frac{dG(s,r)}{ds} = \frac{i\alpha_5}{2a(\eta(s,r))^2} F(\eta(s,r))\nu_{3h}(\rho(s,r))\,, \tag{73c}$$

$$G(0,r) = 0\,. \tag{73d}$$

A solution with the boundary condition $G(\rho=0,\eta)=0$ is

$$\begin{aligned} \eta &= s + r\,,\\ \rho &= -\frac{2hi}{\beta}s\,,\\ G(s,r) &= \int_0^s ds'(s'+r)^2 F(s'+r)\nu_{3h}(s')\,. \end{aligned} \tag{74}$$

Note that (74) is always a solution of the set of equations (73a - 73d). However, we are eventually interested in the inverse curves $r(\eta,\rho)$, $s(\eta,\rho)$. These can only be found if the determinant of the Jacobian of the transformation (74) is non zero. Luckily this is our case, and we find:

$$\begin{aligned} r &= \eta + \frac{\beta\rho}{2hi}\,,\\ s &= -\frac{\beta\rho}{2hi}\,,\\ G(\eta,\rho) &= \frac{i\alpha_5}{2}\int_0^{-\frac{\beta\rho}{2hi}} ds' a\left(s'+\eta+\frac{\beta\rho}{2hi}\right)^{-2} F\left(s'+\eta+\frac{\beta\rho}{2hi}\right)\nu_{3h}\left(-\frac{2hi}{\beta}s'\right)\,. \end{aligned} \tag{75}$$

Eq. (75) leads to the solution (70). Note that we have not specified F, which must therefore be deduced from initial conditions, nor $\nu_{3h}(\rho,\eta) = H(\rho)$.

□

At this point we should impose initial conditions, our initial thermal state of fermions, and derive a relation which will allow us to calculate F, and specify the solution. With reference to Section C.2, we need to impose

$$\begin{aligned} F\left(\eta_0+\frac{\beta\rho}{2hi}\right)-\frac{i\alpha_5}{2}\int_0^{\frac{\beta\rho}{2hi}} ds'\frac{F\left(-s'+\eta_0+\frac{\beta\rho}{2hi}\right)}{a\left(-s'+\eta_0+\frac{\beta\rho}{2hi}\right)^2}\nu_{3h}\left(\frac{2hi}{\beta}s'\right) &= \\ =\frac{m(k_BT)^2}{2\pi^2}\left(\psi'(1-\rho)-\frac{1}{2}\psi'(1-\rho/2)\right), \\ \nu_{3h}(\rho) =\frac{(k_BT)^3}{2\pi^2}\left(\psi''(1-\rho)-\frac{1}{2}\psi''(1-\rho/2)\right). \end{aligned} \tag{76}$$

Which can be rewritten purely in terms of the time coordinate η, by identifying $\eta\equiv\eta_0+\frac{\beta\rho}{2hi}$. Then Eq. (76) can be rewritten as

$$\begin{aligned} F(\eta)-\frac{i\alpha_5}{2}\int_0^{\eta-\eta_0} ds\,\frac{F(\eta-s)}{a(\eta-s)^2}\nu_{3h}\left(\frac{2hi}{\beta}s\right) &= \\ =\frac{m(k_BT)^2}{2\pi^2}\left(\psi'(1-\frac{2hi}{\beta}(\eta-\eta_0))-\frac{1}{2}\psi'(1-\frac{hi}{\beta}(\eta-\eta_0))\right). \end{aligned} \tag{77}$$

Eq. (77) started as an equation for the auxiliary variable ρ, but eventually became an equation for the physical time η. What the solution to Eq. (77) physically represent is

$$F_h(\eta)=\exp\left(-\frac{i\alpha_5\nu_{3h}(0)}{2}\int_{\eta_0}^{\eta} d\eta' a(\eta')^{-2}\right)\int d\vec{k}\,(f_{1h}(\eta,k)+if_{2h}(\eta,k)) .$$

Where f_{1h} and f_{2h} are the distribution functions of the fermionic fields, given in Eqs. (60b - 60c). In deriving the former equation, we have to use the relation $n^{(m)}_{ah}=\frac{1}{\beta^m}\frac{\partial^{(m)}}{\partial\rho^{(m)}}\nu_{ah}(\rho)\big|_{\rho=0}$.

Note that the method we have used does not require any assumptions on the scale factor $a(\eta)$, as long as it can be analytically extended to the whole complex plane[6]. Because of this feature, we can actually study the space-time back reaction, since the gravitational field equations, the Friedmann equations, will be given purely in terms of $a(\eta)$. Naturally the analytical character of our solution will stop here, and we will use numerical tools to solve the Friedmann equations.

6 Required to define $a\left(-s'+\eta_0+\frac{\beta\rho}{2hi}\right)$.

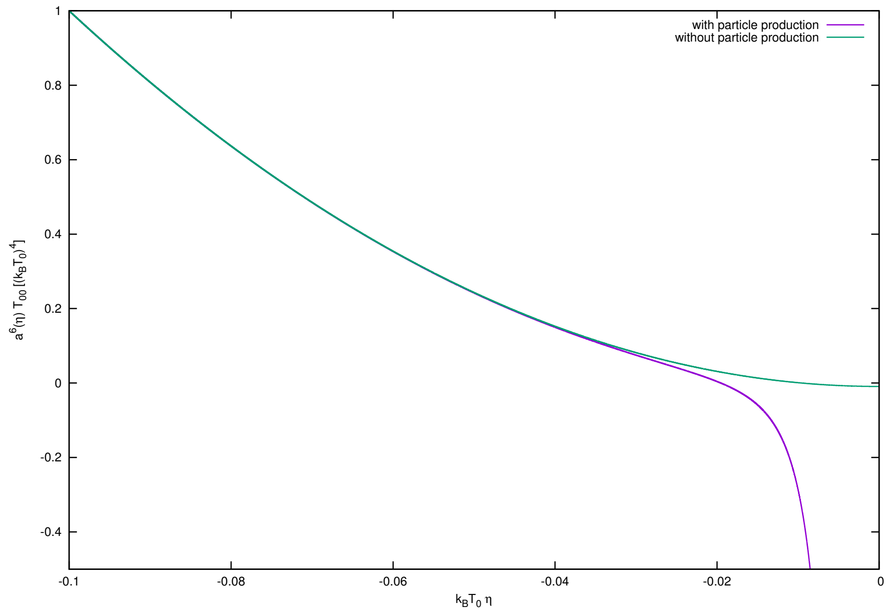


Figure 9: The energy density of the conformally rescaled fields. The epsilon parameter, $\epsilon = \left(\frac{1}{H}\right)'$, has been set to 2, its value during radiation domination. The parameters are $\xi \simeq 10^3$, $\frac{|m|}{k_B T_0} \simeq 10^{-2}$ and $k_B T_0 \simeq 0.01 M_p$. Note that the tail diverges towards $-\infty$, when $a(\eta)$ becomes small, only if particle production is included.

5.4 APPROXIMATIONS AND SELF CONSISTENT BACK REACTION

Eq. (77) always has a solution that can be written as the series [10, 37]

$$
\begin{aligned}
F(\eta) &= \sum_{n=0}^{\infty} \left(\frac{i\alpha_5}{2}\right)^n F_n(\eta)\,, \\
F_0(\eta) &= \frac{m(k_B T)^2}{2\pi^2} \left(\psi'\left(1 - \frac{2hi}{\beta}(\eta - \eta_0)\right) - \frac{1}{2}\psi'\left(1 - \frac{hi}{\beta}(\eta - \eta_0)\right)\right), \\
F_n(\eta) &= \frac{h(k_B T)^3}{2\pi^2} \int_0^{\eta - \eta_0} ds\, \frac{F_{n-1}(\eta - s)}{a(\eta - s)^2} \left(\psi''\left(1 - \frac{2hi}{\beta}s\right) - \frac{1}{2}\psi''\left(1 - \frac{hi}{\beta}s\right)\right),
\end{aligned}
\tag{78}
$$

which converges in any finite interval of the real line, as long as all the involved functions are continuous there[7]. Eq. (78) therefore represents a solution for F, which can be used to construct the energy density evolution in a fixed background. In Figure 9 we report the evolution, in a fixed background, of the rescaled energy density [8],

$$a^6(\eta) T^{\psi}_{00}.$$

7 The poles of the ψ functions are avoided here, because they occur at imaginary values of η(see Figure 12).

8 We choose to rescale the zeroth component of the energy-momentum tensor with the highest power of the scale factor found in it, the torsion contributions scaling $\propto a^{-6}(\eta)$.

Note that the contribution of torsion makes T^{ψ}_{00} negative, and particle production acts to enhance this effect at late time. Since this evolution is obtained by keeping ϵ constant, it does not reproduce a physical behaviour: in reality, T^{ψ}_{00} appears on the right hand side of the Friedmann equations, in such a way that the sum of the fermionic contribution and the other fluids composing the universe remains positive. In the realistic situation, the total energy density will never become negative, but ϵ will adjust itself accordingly.

We construct our approximation by looking at the structure of (77), and acknowledging that α_5 is a rather small parameter, $\mathcal{O}(10^{-39}\,GeV^{-2})$. Then it is clear that torsion contributions are only going to matter at late stages of the gravitational collapse. This means that $F(\eta)$ is going to evolve approximately free until $a(\eta)$ becomes small. The asymptotic expansion we are talking about, $\eta \to 0$, describes the late phase of the collapse. In this regime (77) can be expanded as(after changing integration variable to $s' = \eta - s$)

$$F(\eta) - \frac{i\alpha_5}{2}\int_{\eta_0}^{\eta} ds\, \frac{F(s)}{a(s)^2}\nu_{3h}\left(-\frac{2hi}{\beta}s\right) \simeq \\ \simeq \frac{m(k_B T_0)^2}{2\pi^2}\left(\psi'\left(1+\frac{2hi}{\beta}\eta_0\right) - \frac{1}{2}\psi'\left(1+\frac{hi}{\beta}\eta_0\right)\right), \tag{79}$$

which can be solved exactly with

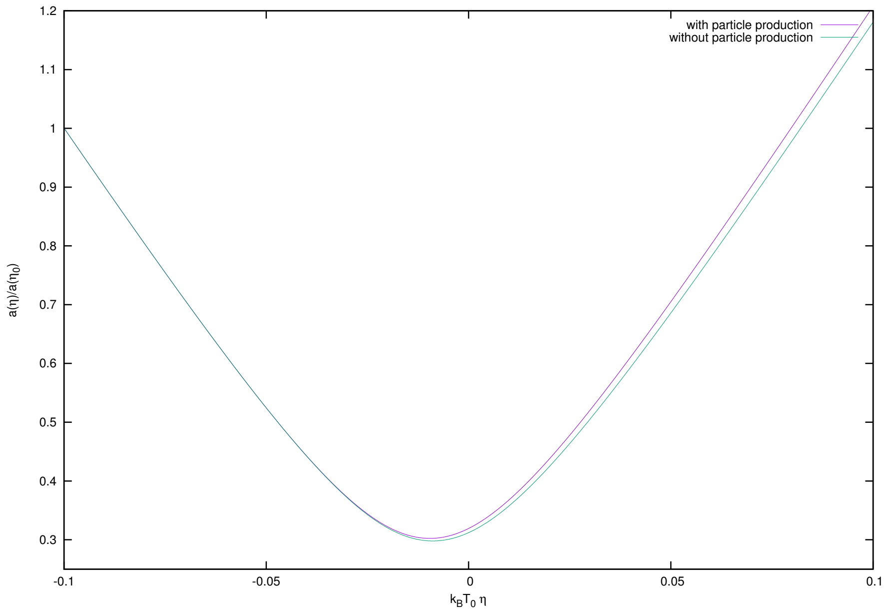
$$F(\eta) = F_0 \exp\left(\frac{i\alpha_5}{2}\int_{\eta_0}^{\eta} ds\,(a(s))^{-2}\,\nu_{3h}\left(-\frac{2hi}{\beta}s\right)\right), \tag{80}$$

$$F_0 = \frac{m(k_B T_0)^2}{2\pi^2}\left(\psi'\left(1+\frac{2hi}{\beta}\eta_0\right) - \frac{1}{2}\psi'\left(1+\frac{hi}{\beta}\eta_0\right)\right),$$

as it can be easily confirmed by differentiating (79). We can express the energy-momentum tensor in terms of F starting from the energy-momentum tensor (38). Some algebra, using the Dirac equation to get rid of the time derivatives, and our ansatz for the propagator (59) lead to

$$T_{00} = \sum_h \int \frac{d\vec{p}}{(2\pi)^3}\left(\frac{1}{a^4}h|\vec{p}|f_{3h} + \frac{1}{a^3}(m_R f_{1h} + m_I f_{2h})\right) - \\ - \frac{\alpha_5}{a^6}\int \frac{d\vec{p}}{(2\pi)^3}\frac{d\vec{p}'}{(2\pi)^3}\left(\sum_{hh'} f_{3h}f_{3h'} + \sum_h 2(f_{3h}^2 + f_{0h}^2) + \frac{5}{2}(f_{1h}^2 + f_{2h}^2)\right), \tag{81}$$

$$T_{ij} = \delta_{ij}\Bigg\{\sum_h \int \frac{d\vec{p}}{(2\pi)^3}\frac{1}{3}\left(\frac{1}{a^4}h|\vec{p}|f_{3h}\right) - \\ - \frac{\alpha_5}{a^6}\sum_{hh'}\int \frac{d\vec{p}}{(2\pi)^3}\frac{d\vec{p}'}{(2\pi)^3}\left(\sum_{hh'} f_{3h}f_{3h'} + \sum_h 2(f_{3h}^2 + f_{0h}^2) + \frac{5}{2}(f_{1h}^2 + f_{2h}^2)\right)\Bigg\}, \tag{82}$$



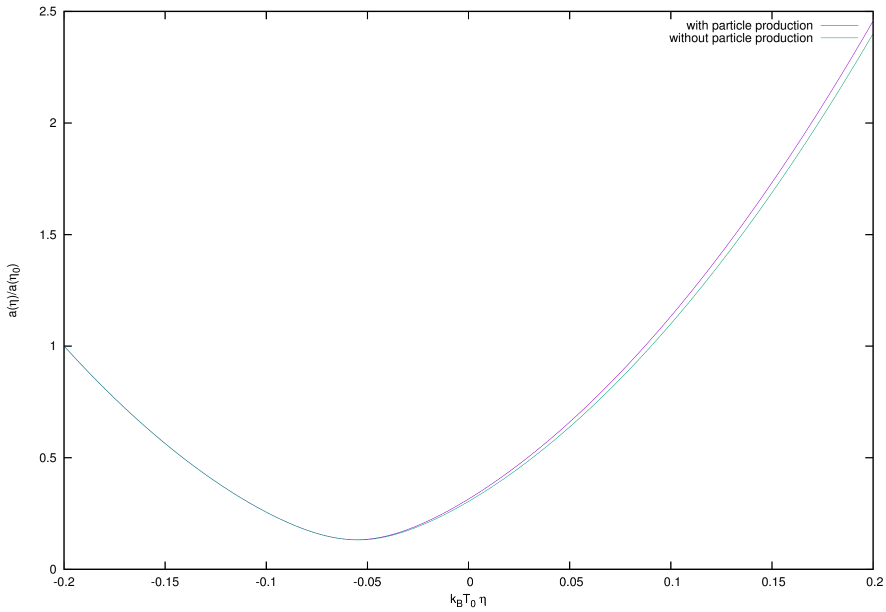


Figure 10: Scale factor evolution as a function of conformal time η, for $\xi \simeq 10^3$, $\frac{|m|}{k_B T_0} \simeq 10^{-2}$ and $k_B T_0 \simeq 0.01 M_p$. The evolution starts in a Radiation dominated background, i.e. $a(\eta) = \left(\frac{\eta}{\eta_0}\right)$, in the upper figure, and in matter domination, i.e. $a(\eta) = \frac{1}{2}\left(\frac{\eta}{\eta_0}\right)^2$, in the lower. We notice that the scale factor does not become singular, but it reaches a minimum value a_{min}. The effect of particle production is to enhance the bounce, making a_{min} slightly bigger. Moreover, introduction of particle production makes the hubble parameter after the bounce bigger: the universe collapses and starts expanding slightly faster than it was collapsing.

which in our approximation ($m \to 0$) and written in terms of the generating functions ν_{ah} becomes

$$T_{00} = \sum_h \left(\frac{1}{a^4} h \frac{1}{\beta} \frac{\partial \nu_{3h}}{\partial \rho} \right) \Big|_{\rho=0} - \frac{\alpha_5}{a^6} \left(\sum_h 2(\nu_{3h}^2 + \frac{5}{2} \nu_h^* \nu_h) \right) \Big|_{\rho=0}, \tag{83}$$

$$T_{ij} = \delta_{ij} \left\{ \sum_h \frac{1}{3} \left(\frac{1}{a^4} h \frac{1}{\beta} \frac{\partial \nu_{3h}}{\partial \rho} \right) \Big|_{\rho=0} - \frac{\alpha_5}{a^6} \left(\sum_h 2(\nu_{3h}^2 + \frac{5}{2} \nu_h^* \nu_h) \right) \Big|_{\rho=0} \right\}. \tag{84}$$

Equations (83 - 84) can be substituted in the second Friedmann equation, which in conformal time reads

$$\frac{a''}{a^3} = \frac{4\pi G}{3} (\rho - 3p),$$

which yields to

$$\frac{a''}{a^3} = \frac{4\pi G}{3} (\rho_b - 3p_b) + \frac{4\pi G}{3} \frac{2\alpha_5}{a^6} \sum_h \left(2\nu_{3h}^2|_{\rho=0} + \frac{5}{2} |F_h(\eta)|^2 \right). \tag{85}$$

where $F_h(\eta)$ is given by (80) and ρ_b, p_b are the energy density and pressure of the fluid driving the collapse. Once the background evolution is given (i.e. $w_b(\eta) = p_b/\rho_b$ is known), Eq. (85) becomes the equation for the scale factor $a(\eta)$ alone and can be solved self consistently. Given initial conditions a_0, H_0 and $k_B T_0$ we can evolve in the background until the second term in (85) becomes significant and then write a numerical code that construct the remainder of the solution. Again, since the torsion contributions are going to be important at late times, we can use the expansion (80) around $k_B T_0 \eta \simeq 0$.

The results of this numerical analysis are shown in Figure 10 and Figure 11. The effect of including particle production, is that the bounce gets enhanced and happens sooner. The magnitude of this effect depends on the initial conditions, but the rest of our conclusions are general. These examples illustrate the importance of taking the backreaction self-consistenly into account for the comparison of a classical evolution with a quantum evolution where perturbative loop effects are self-consistently accounted for. Before proceeding to the conclusion, let us verify that the energy scale reached in the bounce scenario is enough away from the Planck scale. From Figure 10, it is clear that the classical solution is enough for this purpose. The torsion contribution to the Friedmann equation is

$$H^2 = \frac{1}{m_P^2} \left(\rho - \frac{\xi^2}{m_P^2} n^2 \right) \overset{bounce}{=} 0,$$

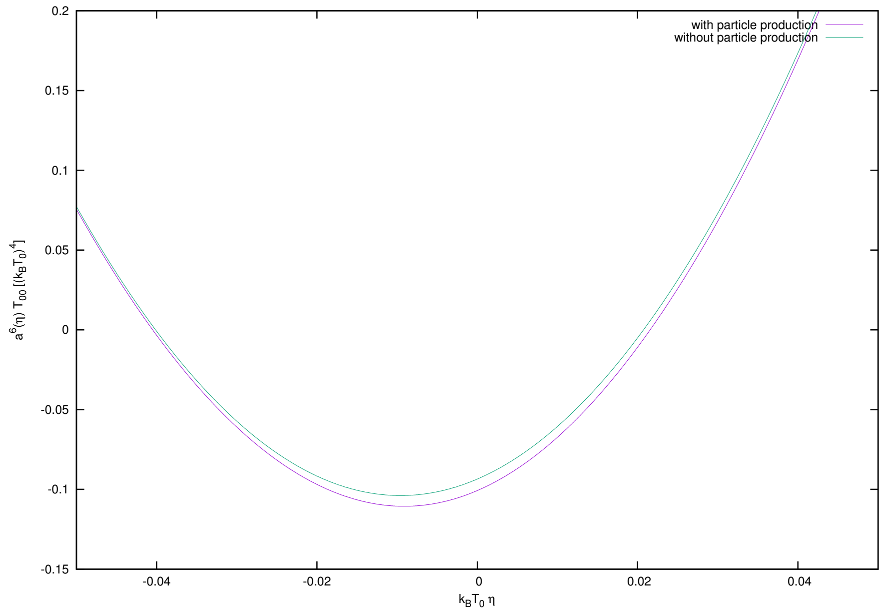


Figure 11: Rescaled energy density evolution through the bounce(for radiation domination). Compared to what happens in Figure 9, the energy density does not diverge here. The difference is that in the plot in Figure 9 the back reaction of the fields with gravity is turned off. In this case, the fermionic fields gain more and more energy, without a bound. When the back reaction is turned on, however, the fermions energy stops the collapse once it becomes comparable with the background energy density, therefore preventing the singularities in both T_{00} and $a(\eta)$.

where $\rho \simeq \rho_0 \frac{(k_B T)^4}{(k_B T_0)^4}$ is the (relativistic) density driving the collapse and $n \simeq n_0 \frac{(k_B T)^3}{(k_B T_0)^3}$ is the fermions number density. Solving the Friedmann equation yields to

$$k_B T_{bounce} = \sqrt{\frac{m_P \sqrt{\rho_0}}{|\xi| n_0}} k_B T_0 \,, \tag{86}$$

$$R_{bounce} = \frac{m_P \rho_0^{3/4}}{|\xi| n_0} \,. \tag{87}$$

From these relations we can deduce that

- The temperature at the bounce remains in general lower than the Planck scale. It can be small even in the minimally coupled case ($\xi = 1$) if the initial number density $n_0 \gg \rho_0$, a situation analogous to that found in neutron stars. The same applies to the curvature scale.

- The coefficient that multiplies $k_B T_0$, in (86), can be made smaller by a large $\xi \gg 1$. In case of a single fermionic fluid which couple to torsion and initially drives the collapse, we find that the only way for the curvature scale to be physically acceptable is to have a large ξ. This, however, does not apply to the temperature, which in this situation can still be small, since $k_b T_{bounce} = \sqrt{\frac{m_P k_B T_0}{\xi}}$.

From this we can conclude that the Planck scale is not reached if $n_0 \gg \rho_0$, or $\xi \gg 1$.

6 CONCLUSIONS

In this thesis we have analysed the effect of torsion of Einstein-Cartan theory on the evolution of the Universe. In particular, we have studied the torsion contribution on a matter and radiation dominated collapsing universe and find that – instead of ending in a big crunch singularity – the universe undergoes a bounce. We have evidence that this behaviour is generic, and is not affected by the nature of the collapse (see Appendix D). In contrast to older works [28, 29, 35], we did not assume a classical form of the spin fluid sourcing torsion, but instead we derived our description from a full microscopic treatment of fermions. We did both: (a) a classical treatment (in which the fermionic fluid is described by an initial thermal state and particle production due to Universe's contraction is switched off) and (b) perturbative quantum treatment (in which particle production is accounted for at the one-loop level in the fermionic dynamical equations and at the two-loops level in the Friedmann equation). Our analysis shows that the bounce is not largely affected by the quantum particle production. We found that, when fermion production is taken account of, then the bounce occurs somewhat earlier, indicating that fermion production induces a negative backreaction on the Universe's evolution.

The reason why we chose a matter dominated collapse is that the resulting bounce might present a viable alternative to inflation. Namely, it is well known that the (Bunch-Davies) vacuum state in matter era yields a flat spectrum of perturbations. [1] Therefore, it would be of particular interest to derive the power spectrum of cosmological perturbations and investigate whether it can be used to seed the large scale structure and fluctuations in cosmic microwave background that match the data.

A second situation in our study can be of use is that of a collapsing star turning into a black hole. In this case the interior of the star can be modelled by a FLRW metric [23], as long as the collapse respect spherical symmetry. Therefore, at least in the bulk of the star, the analysis of this paper applies, and can be used to infer black holes formation. As it happens with the singularity at the beginning of our

1 This is so because the equation of motion for a conformally rescaled massless scalar, $\phi_c = a\phi$) in momentum space and in conformal time, $(\partial_\eta^2 + k^2 - a''/a)(a\phi(\eta, k)) = 0$, is identical to that in de Sitter space. This follows immediately from the form of the scale factors, which is in de Sitter space, $a \propto -1/\eta$, and in matter era, $a \propto \eta^2$, such that in both cases $a''/a = 2/\eta^2$. Since the spectrum of a massless scalar in a Bunch-Davies vacuum in de Sitter space is scale invariant, so must be the corresponding spectrum of a massless scalar in matter era.

universe, it is probable that also formation of black holes singularities is prevented by the introduction of torsion.

Part IV

APPENDIX

A

APPENDIX A

In this appendix we discuss Lorentz invariance of the matter lagrangian and see what it implies. The transformation law for the tetraed and the spin connection are as follows [13, 26]

$$
\begin{aligned}
e'^{\mu}_{a} &= \Lambda^{b}_{a} e^{\mu}_{b} \,, \\
\omega'^{ab}{}_{\mu} &= \omega^{cd}{}_{\mu} \Lambda^{a}_{c} \Lambda^{b}_{d} - \Lambda^{db} \partial_{\mu} \Lambda^{a}_{d} \,.
\end{aligned}
\tag{88}
$$

We consider an infinitesimal transformation, i.e. $\Lambda^{b}_{a} = \delta^{b}_{a} + \Omega^{b}_{a}$, where $\Omega_{ab} = -\Omega_{ba}$ is the infinitesimal parametrization. We require the matter lagrangian to be invariant under lorentz transformations, which means

$$
\delta\mathcal{L}_m = \frac{\delta\mathcal{L}_m}{\delta\omega^{ab}{}_{\lambda}} \delta\omega^{ab}{}_{\lambda} + \frac{\delta\mathcal{L}_m}{\delta e^{\mu}_{a}} \delta e^{\mu}_{a} \equiv \frac{1}{2} \left(e\Sigma^{\lambda}{}_{ab} \delta\omega^{ab}{}_{\lambda} + eT^{a}_{\mu} \delta e^{\mu}_{a} \right) \,,
$$

By using the infinitesimal form of Λ as we wrote it above, we find

$$
\begin{aligned}
\delta\omega^{ab}{}_{\lambda} &= \omega^{ad}{}_{\lambda} \Omega^{b}_{d} + \omega^{cb}{}_{\lambda} \Omega^{a}_{c} - \partial_{\lambda} \Omega^{ab} \,, \\
\delta e^{\mu}_{a} &= e^{\mu}_{b} \Omega^{b}_{a} \,.
\end{aligned}
$$

Which will give us a conservation equation as a consequence of imposing $\delta\mathcal{L}_m = 0$. We find

$$
\begin{aligned}
2\delta\mathcal{L}_m &= e\Sigma^{\lambda}{}_{ab} \left(\omega^{ad}{}_{\lambda} \Omega^{b}_{d} + \omega^{cb}{}_{\lambda} \Omega^{a}_{c} - \partial_{\lambda} \Omega^{ab} \right) + T_{[ab]} \Omega^{ab} \\
&= e \left(\Sigma^{\lambda}{}_{cb} \omega^{c}{}_{a\lambda} + \Sigma^{\lambda}{}_{ac} \omega^{c}{}_{b\lambda} + \frac{1}{e} \partial_{\lambda} \left(e\Sigma^{\lambda}{}_{ba} \right) - T_{[ab]} \right) \Omega^{ba} ,
\end{aligned}
\tag{89}
$$

where we renamed indices, integrated one term by parts and used the fact that Ω^{ab} is antisymmetric. Using the well known identity

$$
\frac{1}{\sqrt{-g}} \partial_{\lambda} \sqrt{-g} = \frac{1}{e} \partial_{\lambda} e = \{^{\sigma}{}_{\lambda\sigma}\} ,
$$

we can rewrite the conservation law as

$$
\nabla_{\lambda} \Sigma^{\lambda}{}_{[ab]} - S_{\lambda} \Sigma^{\lambda}{}_{[ab]} - T_{[ab]} = 0 \,.
$$

To make the connection with the formulation in terms of the torsion tensor note

$$
\Sigma^{\lambda}{}_{[\mu\nu]} = \frac{2}{e} \frac{\delta\mathcal{L}_m}{\delta\omega^{ab}{}_{\lambda}} = \frac{2}{e} \frac{\delta\mathcal{L}_m}{\delta S^{\alpha\beta\gamma}} \frac{\delta S^{\alpha\beta\gamma}}{\delta\omega^{ab}{}_{\lambda}} = \Pi_{[\mu\nu]}{}^{\lambda} \,,
$$

and use the solution (21) to finally get

$$
\nabla_{\lambda} \Pi_{[\mu\nu]}{}^{\lambda} - 2\pi G \Pi_{\lambda} \Pi_{[\mu\nu]}{}^{\lambda} - T_{[\mu\nu]} = 0 \,.
\tag{90}
$$

Which is precisely the anti symmetric part of Eq. (25), as one can calculate it starting from Eq. (23).

APPENDIX B

In this appendix we discuss a different way of looking at Eqs. (62a - 62d). Namely, we will derive that torsion interactions are analogous to interactions between torsion and pseudo-vector fields. Indeed, let us consider the Lagrangian

$$\mathcal{L} = \frac{i}{2}(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\gamma^{\mu}\psi) - \bar{\psi}\hat{M}\psi - eZ_{\mu}\bar{\psi}\gamma^{5}\gamma^{\mu}\psi - \frac{i\eta}{2}eB_{\mu\nu}\bar{\psi}\gamma^{[\mu}\gamma^{\nu]}\psi\,, \tag{91}$$

where η, e are dimensionless coupling constants, one gets the equations of motion, for spinorial the currents in Eqs. (60a - 60d)

$$\begin{aligned}
f'_{0h} &= 0\,, && (92a)\\
f'_{1h} + 2hkf_{2h} - 2m_I f_{3h} &= && (92b)\\
&= (2eZ_0)f_{2h} + 2\eta h(B_{0i}\hat{k}^i)f_{3h}\,, \\
f'_{2h} - 2hkf_{1h} + 2m_R f_{3h} &= && (92c)\\
&= -(2eZ_0)f_{1h} + 2\eta h(B_{ij}\epsilon^{ijk}\hat{k}_k)f_{3h}\,, \\
f'_{3h} - 2m_R f_{2h} + 2m_I f_{1h} &= && (92d)\\
&= -2\eta h\left(B_{0i}\hat{k}^i f_{1h} + B_{ij}\epsilon^{ijk}\hat{k}_k f_{2h}\right)\,.
\end{aligned}$$

While the equations of motion one gets from integrating out torsion, namely Eqs. (62a - 62d) are:

$$\begin{aligned}
& f'_{0h} = 0\,, \\
& f'_{1h} + 2hkf_{2h} - 2m_I f_{3h} = \\
= \;& 2\alpha_5\left(\int d^3p\sum_{h'} f_{3h'}(p) - \frac{f_{3h}(p)}{4}\right) f_{2h} - 2\alpha_5\left(\int d^3p f_{2h}(p)\right) f_{3h}\,, \\
& f'_{2h} - 2hkf_{1h} + 2m_R f_{3h} = \\
= \;& -2\alpha_5\left(\int d^3p\sum_{h'} f_{3h'}(p) - \frac{f_{3h}(p)}{4}\right) f_{1h} + 2\alpha_5\left(\int d^3p f_{1h}(p)\right) f_{3h}\,, \\
& f'_{3h} - 2m_R f_{2h} + 2m_I f_{1h} = 0\,.
\end{aligned}$$

Meaning $f_{ah} = f_{ah}(k) = f_{ah}(\eta, k)$. Now, we see that, under the identifications:

$$(2eZ_0) = 2\alpha_5\left(\int d^3p\sum_{h'} f_{3h'}(p) - \frac{f_{3h}(p)}{4}\right)\,, \tag{93a}$$

$$2\eta h(B_{0i}\hat{k}^i) = -2\alpha_5\left(\int d^3p f_{2h}(p)\right)\,, \tag{93b}$$

$$2\eta h(B_{ij}\epsilon^{ijk}\hat{k}_k) = 2\alpha_5\left(\int d^3p f_{1h}(p)\right)\,. \tag{93c}$$

Equations (92a - 92c) become identical to (62a - 62d).

Proof. The difference in the two sets of equations comes into play in equation (92d), which would become:

$$f'_{3h} - 2m_R f_{2h} + 2m_I f_{1h} = -2\eta h \left(B_{0i}\hat{k}^i f_{1h} + B_{ij}\epsilon^{ijk}\hat{k}_k f_{2h}\right) = \\ = -2\alpha_5 \left(\int d^3p f_{2h}(p) f_{1h}(k) - \int d^3p f_{1h}(p) f_{2h}(k)\right), \tag{94}$$

The caveat here is that $B_{\mu\nu}$ actually depends on $\bar{\psi}\psi$, so we have to include its variation in the equations. We get the corrected equations

$$\begin{aligned} (i\gamma^\mu\partial_\mu - m)\psi &= e\left(Z_\mu\gamma^\mu\psi + \bar{\psi}\gamma^\mu\psi\frac{\delta Z_\mu}{\delta\bar{\psi}}\right) && (95a)\\ &\quad + \frac{i\eta}{2}\left(B_{\mu\nu}\sigma^{\mu\nu}\psi + \bar{\psi}\sigma^{\mu\nu}\psi\frac{\delta B_{\mu\nu}}{\delta\bar{\psi}}\right), \\ \bar{\psi}\left(i\overleftarrow{\partial}_\mu\gamma^\mu + m\right) &= -e\left(\bar{\psi}\gamma^\mu Z_\mu + \bar{\psi}\gamma^\mu\psi\frac{\delta Z_\mu}{\delta\psi}\right) && (95b)\\ &\quad - \frac{i\eta}{2}\left(\bar{\psi}\sigma^{\mu\nu}B_{\mu\nu} + \bar{\psi}\sigma^{\mu\nu}\psi\frac{\delta B_{\mu\nu}}{\delta\psi}\right), \end{aligned}$$

where $\sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$. To get the equation of motion for f_{3h} (in position space), we will consider the linear combination $\psi^\dagger\gamma^5\gamma^0\times$ (95a) + (95b) $\times\gamma^5\psi$, but worry only about the right hand side of the equation, since we know the left hand side will combine the correct way. We get:

$$\begin{aligned} & \frac{i\eta}{2}\left(B_{\mu\nu}\psi^\dagger\gamma^5\gamma^0\sigma^{\mu\nu}\psi + \bar{\psi}\sigma^{\mu\nu}\psi\psi^\dagger\gamma^5\gamma^0\frac{\delta B_{\mu\nu}}{\delta\bar{\psi}}\right) \\ - & \frac{i\eta}{2}\left(B_{\mu\nu}\psi^\dagger\gamma^0\sigma^{\mu\nu}\gamma^5\psi + \bar{\psi}\sigma^{\mu\nu}\psi\frac{\delta B_{\mu\nu}}{\delta\psi}\gamma^5\psi\right) = \\ = & \ 2\eta h(B_{0i}\hat{k}^i f_{1h} + B_{ij}\epsilon^{ijk}\hat{k}_k f_{2h}) \\ + & \ i\eta h\hat{k}^i f_{2h}\left(-\frac{\delta B_{0i}}{\delta\psi}\gamma^5\psi + \psi^\dagger\gamma^5\gamma^0\frac{\delta B_{0i}}{\delta\bar{\psi}}\right) \\ + & \ i\eta h\hat{k}_k\epsilon^{ijk} f_{1h}\left(-\frac{\delta B_{ij}}{\delta\psi}\gamma^5\psi + \psi^\dagger\gamma^5\gamma^0\frac{\delta B_{ij}}{\delta\bar{\psi}}\right). \end{aligned} \tag{96}$$

Now, consider the definitions (93b - 93c), and plug it in (96):

$$\begin{aligned} & -2\alpha_5\int_{\vec{p}} f_{2h}(p)f_{1h}(k) + 2\alpha_5\int_{\vec{p}} f_{1h}(p)f_{2h}(k) \\ & -2\alpha_5\int_{\vec{p}} f_{1h}(p)f_{2h}(k) + 2\alpha_5\int_{\vec{p}} f_{2h}(p)f_{1h}(k) \\ & = 0. \end{aligned} \tag{97}$$

□

APPENDIX C

C.1 DIAGONALIZATION OF CIRCULANT MATRICES

Let us consider a matrix that can be written as

$$C = \begin{pmatrix} c_0 & c_1 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & \cdots & c_{N-2} \\ \vdots & & \ddots & \\ c_1 & \cdots & & c_0 \end{pmatrix} . \tag{98}$$

That is, each line is obtained in the previous by permuting each element one place to the right. Clearly, such a matrix is of a special kind, because it can be expressed entirely in terms of the vector

$$\vec{c} = (c_0, c_1, \cdots, c_{N-1}) \, ,$$

and as a matter of fact this will allow us to diagonalise any matrix in the form (98). We find the eigenvectors

$$\vec{v}_k = \frac{1}{\sqrt{N}} (1, \omega_k, \omega_k^2, \cdots, \omega_k^{N-1}), \; k = 0, 1, \cdots, N-1 \, ,$$

where

$$\omega_k = e^{2\pi i \frac{k}{N}}, \; k = 0, 1, \cdots, N-1 \, .$$

Proof. We get, with the notational understanding that $c_{-1} = c_{N-1}, c_{-2} = c_{N-2}, \cdots, c_{-N+1} = c_1$

$$\begin{aligned} C \cdot \vec{v}_k = & \frac{1}{\sqrt{N}} \left(\sum_{n=0}^{N-1} c_n \omega_k^n, \sum_{n=0}^{N-1} c_{n-1} \omega_k^n, \cdots, \sum_{n=0}^{N-1} c_{n-N+1} \omega_k^n \right) = \\ & = \sum_{n=0}^{N-1} c_n \omega_k^n \frac{1}{\sqrt{N}} \left(1, \, \omega_k, \, \cdots, \, \omega_k^{N-1}\right) = \left(\sum_{n=0}^{N-1} c_n \omega_k^n \right) \vec{v}_k \, . \end{aligned}$$

So the system is diagonal in $\vec{v}_k$ basis, as long as

$$\sum_{n=0}^{N-1} c_n \left(e^{2\pi i \frac{k}{N}} \right)^n < \infty \, ,$$

which later, when we switch to the continuous limit, will be the convergence condition.

□

The next thing to notice is that the eigenvector are orthonormal:

$$\vec{v}_k^\dagger \cdot \vec{v}_l = \frac{1}{N} \sum_{a=0}^{N-1} e^{2\pi i\, a(l-k)/N} = \delta_{l,k} \, .$$

Since the basis vector are orthonormal, the basis change matrix is unitary.

We can find the basis change matrix easily: let $\vec{e}_l$ denotes the canonical basis[1] in $\mathbb{C}^N$, then

$$U_{kl} = (\vec{v}_k \cdot \vec{e}_l) = \frac{1}{\sqrt{N}} e^{2\pi i (k\, l)/N} \, ,$$

which implies that

$$U_{kl}^\dagger U_{lk'} = \frac{1}{N} \sum_{l=0}^{N-1} e^{2\pi i (l\,(k-k'))/N} = \delta_{k,k'} \, .$$

We can then write the vector components

$$n^{(n)} \vec{e}_n = \frac{1}{N} \sum_{k=0}^{N-1} \left(e^{2\pi i\, n\, k/N} \right) v^{(k)} \vec{v}_k \, . \tag{99}$$

Now, since our original problem is infinite dimensional, we should carefully study the limit in which $N \to \infty$. Clearly, in this limit, the roots of identity (ω_k) are going to be parametrised by a continuous angle. In the sense that:

$$\omega_k \to \omega(\theta) = e^{i\theta}, \; \theta \in [0, 2\pi]$$

In this limit, we can make replacements

$$\begin{aligned} \frac{1}{N} \sum_{k=0}^{N-1} &\to \int \frac{d\theta}{2\pi} \, , \\ \lambda_k &\to \lambda(\theta) \, , \\ v^{(k)} &\to v(\theta) \, . \end{aligned}$$

Then the vectorial component become the Fourier transformation of one another

$$\begin{aligned} n^{(n)} &= \int \frac{d\theta}{2\pi} e^{in\theta} v(\theta) \, , \\ v(\theta) &= \sum_{n=0}^{\infty} e^{-in\theta} n^{(n)} \, . \end{aligned} \tag{100}$$

1 i. e.the vectors $(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots$

While the eigenvalues will be given by

$$\lambda(\theta) = \left(\sum_{n=0}^{\infty} c_n e^{in\theta} \right) . \tag{101}$$

C.2 CALCULATION OF THE GENERATING FUNCTION ν_h FOR A THERMAL DISTRIBUTION

In the following we are going to derive the expressions for ν_h and ν_{3h} at thermal equilibrium. Going back to Eq. (41), and removing the divergent part $\theta(k_0)$, we find the initial conditions

$$\begin{aligned} f_{0h} &= 0\,, &\text{(102a)}\\ f_{1h} &= \frac{2m_R}{\sqrt{k^2+m^2}} \frac{1}{e^{\beta\sqrt{k^2+m^2}}+1}\,, &\text{(102b)}\\ f_{2h} &= \frac{2m_I}{\sqrt{k^2+m^2}} \frac{1}{e^{\beta\sqrt{k^2+m^2}}+1}\,, &\text{(102c)}\\ f_{3h} &= \frac{2hk}{\sqrt{k^2+m^2}} \frac{1}{e^{\beta\sqrt{k^2+m^2}}+1}\,, &\text{(102d)} \end{aligned}$$

as it can be easily verified by plugging the propagator (41) into the definitions (60a - 60d). Removing the divergent part from this initial state as shown in Section 4.3, assures us that Eqs. (102a - 102b) are the physical part of the thermal fluid we are considering.

We now want to find an expression for the generating functions $\nu_h = \nu_{1h} + i\nu_{2h}$ and ν_{3h} at thermal equilibrium. To be able to do this calculation analytically, we are going to expand in powers of m. First, consider the definitions of the generating functionals, from Eq. (67b) we get

$$\nu_{ah}(\rho) = \sum_{n=0}^{\infty} \frac{n_{ah}^{(n)}}{n!} (\beta\rho)^n = \int d\vec{k} e^{\beta k \rho} f_{ah}(\eta, k)\,. \tag{103}$$

Note that Eq. (103) can be always evaluated numerically, which will give the functions $\nu_{ah}(\rho)$. Now, however, we exploit the ultra relativistic limit, to get

$$\begin{aligned} n_h^{(n)} &= \frac{m}{2\pi^2} \int dk \frac{k^{n+1}}{e^{\beta k}+1} = \frac{m}{2\pi^2}\left(1-\frac{1}{2^{n+1}}\right)\frac{(n+1)!\zeta(n+2)}{\beta^{n+2}} &\text{(104a)}\\ &+\mathcal{O}(m^3)\\ n_{3h}^{(n)} &= \frac{h}{2\pi^2} \int dk \frac{k^{n+2}}{e^{\beta k}+1} = \frac{h}{2\pi^2}\left(1-\frac{1}{2^{n+2}}\right)\frac{(n+2)!\zeta(n+3)}{\beta^{n+3}} &\text{(104b)}\\ &+\mathcal{O}(m^2)\,. \end{aligned}$$

Plugging (102b - 102c) into the definition (67b) we find:

$$\begin{aligned}\nu_h(\rho) =& \frac{m(k_B T)^2}{2\pi^2}\Bigg(\sum_{n=0}^{\infty}(n+1)\rho^n\zeta(n+2)\\ &-\frac{1}{2}\sum_{n=0}^{\infty}(n+1)\rho^n\zeta(n+2)/2^n\Bigg) =\\ =&\frac{m(k_B T)^2}{2\pi^2\rho^2}\Bigg(\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(n+1)\left(\frac{\rho}{k}\right)^{n+2}\\ &-2\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(n+1)\left(\frac{\rho}{2k}\right)^{n+2}\Bigg).\end{aligned}$$

We wrote everything in the variable ρ, rather than the original θ, because for $\rho = e^{-i\theta}$ the sums in Eq. (105) do not converge. However, we can calculate what Eq. (105) yields to where the sum is convergent(i. e. $|\rho| < 1$), and then analytically extend the result to the whole complex plane, where it is non singular. This procedure does not affect the solution we found in Section 5.3, because all we need is the form of $\nu_{ah}(\rho)$ around $\rho = 0$. This is clear from Eqs. (69a) and (77): to reconstruct the data about the physical functions f_{ah}, we need to evaluate ν_{ah} and their derivatives at $\rho = 0$. So it suffice that (105) converges in a set including a neighbourhood of $\rho = 0$.

With this caveat in our mind, we can use the result

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}(n+1)\left(\frac{x}{k}\right)^{n+2} =& \frac{\partial}{\partial x}\sum_{k=1}^{\infty}\frac{x^2}{k^2}\sum_{n=0}^{\infty}\left(\frac{x}{k}\right)^n =\\ =&\frac{\partial}{\partial x}\sum_{k=1}^{\infty}\frac{x^2}{k(k-x)} = x^2\psi'(1-x),\end{aligned}$$

where $\psi'(x) = \frac{d^2}{dx^2}\log\Gamma(x)$ is the polygamma function, and $\Gamma(x)$ the gamma function that generalises the factorial. Then we find

$$\nu_h(\rho) = \frac{m(k_B T)^2}{2\pi^2}\left(\psi'(1-\rho) - \tfrac{1}{2}\psi'(1-\tfrac{\rho}{2})\right), \tag{105a}$$

$$\nu_{3h}(\rho) = -\frac{h(k_B T)^3}{2\pi^2}\left(\psi''(1-\rho) - \tfrac{1}{2}\psi''(1-\tfrac{\rho}{2})\right). \tag{105b}$$

From Eqs. (105a - 105b) we can tell that ν_{ah} are regular functions near the origin. In Figure 12 we can see the pole structure of ν_{ah}: all poles come at integers values of ρ, and are double poles. We can also infer that the integral in Eq. (77) does not encounter any poles, since it is evaluated on a vertical trajectory in the complex plane, i. e. $\frac{2hi}{\beta}\eta$.

Now, to clarify how the inverse transformation works, namely how to switch back from $\nu_{ah}(\rho)$ to $n_{ah}^{(m)} = \int_{\vec{k}} k^m f_{ah}$.

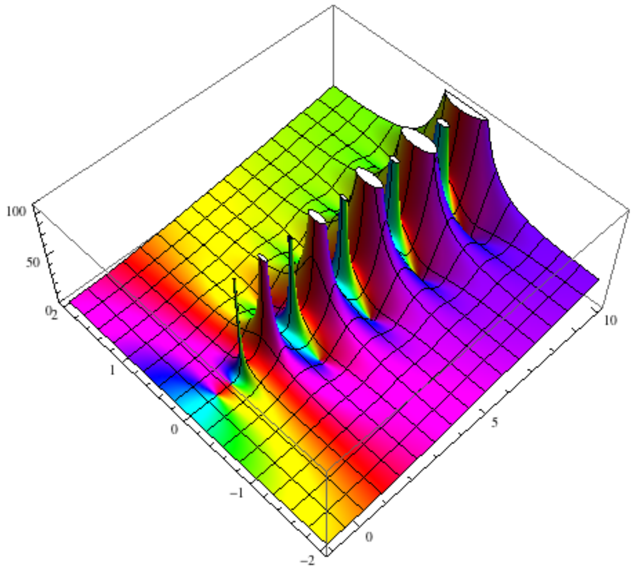


Figure 12: Poles structure for the function $\psi'(1-x) - \frac{1}{2}\psi'(1-x/2)$. In the plot the vertical axe is the absolute value of the function, the horizontal plane is the complex plane and the colour coding contains information about the phase of the function. All the poles are double poles and they occur for $\rho = 1, 2, \cdots$.

Consider Eq. (67a), plugging in the variable $\rho = e^{-i\theta}$

$$\begin{aligned} n_h^{(n)} &= n!\frac{m(k_B T)^{2+n}}{2\pi^2}\int \frac{d\theta}{2\pi} e^{in\theta}\left(\psi'(1-e^{-i\theta}) - \frac{1}{2}\psi'(1-\frac{1}{2}e^{-i\theta})\right) = \\ &= n!\frac{m(k_B T)^{2+n}}{2\pi^2}\oint \frac{d\rho}{2\pi i}\rho^{-(n+1)}\left(\psi'(1-\rho) - \frac{1}{2}\psi'(1-\rho/2)\right), \end{aligned}$$

where the integration is on a counterclockwise circuit surrounding the origin, and not enclosing any of the integer ρ poles. Using the complex analysis residue theorem we can easily evaluate this integral, by rewriting

$$\begin{aligned} &\oint_\gamma \frac{d\rho}{2\pi i}\rho^{-n-1}\left(\psi'(1-\rho) - \frac{1}{2}\psi'(1-\rho/2)\right) = \\ &= \left(1 - \frac{1}{2^{n+1}}\right)\mathrm{Res}\left(\rho^{-n-1}\psi'(1-\rho)|_{\rho=0}\right) = \\ &= \left(1 - \frac{1}{2^{n+1}}\right)\frac{1}{n!}\lim_{\rho\to 0}\frac{d^{(n)}}{d\rho^{(n)}}\psi'(1-\rho) = \\ &= \left(1 - \frac{1}{2^{n+1}}\right)\frac{(-1)^n}{n!}\psi^{(n+1)}(1) = \\ &= \left(1 - \frac{1}{2^{n+1}}\right)(n+1)\zeta(n+2) \end{aligned}$$

Where we used that $\psi^{(n)}(1) = (-1)^{n+1} n!\zeta(n+1)$.

APPENDIX D

In this appendix, we use the results from Section 5.4 to derive the general behaviour of the fields energy density in the energy momentum tensor. We are going to calculate the asymptotic behaviour of the function $F(\eta)$ given by Eq. (80).

In a general background where the parameter $\epsilon = \left(\frac{1}{H}\right)' = -\frac{H'}{H^2}$ is constant, we can write, by making use of Eq. (80)

$$a(\tau) = |H_0(\epsilon-1)\eta|^{1/(\epsilon-1)} \equiv |\tau|^{1/(\epsilon-1)}\,;\ \epsilon > 1\,, \tau \in [-1,0]\,,$$

$$\begin{aligned} F(\tau) =& F_0 \mathrm{Exp}\left[\frac{i\alpha_5}{2H_0(\epsilon-1)}\frac{h(k_B T)^3}{2\pi^2}\left(\int_{-1}^{\tau} ds \frac{Z\left(-2hi\frac{s}{\beta H_0(\epsilon-1)}\right)}{|s|^\delta}\right)\right]\,; \\ \delta =& \frac{2}{\epsilon-1}\,, \end{aligned} \tag{106}$$

where

$$Z(\rho) = \psi''(1-\rho) - \frac{1}{2}\psi''(1-\rho/2)\,,$$

is the ρ dependent part of $\nu_{3h}(\rho)$, as seen in Eq. (105b).

- For $\epsilon > 3$ we can write

$$\begin{aligned} |F(\tau)| =& \left| F_0 \sum_{n=0}^{\infty} \left[\frac{i\alpha_5}{2H_0(\epsilon-1)}\frac{h(k_B T)^3}{2\pi^2}\left(\int_{-1}^{\tau} ds \frac{Z\left(-2hi\frac{s}{\beta H_0(\epsilon-1)}\right)}{|s|^\delta}\right)\right]^n /n! \right| \\ \leqslant& |F_0| \sum_{n=0}^{\infty} \left[\frac{\alpha_5}{2H_0(\epsilon-1)}\frac{(k_B T)^3}{2\pi^2}|Z(0)|\int_{-1}^{0} ds \frac{1}{|s|^\delta}\right]^n /n! = \\ =& |F_0| \sum_{n=0}^{\infty} \left[\frac{\alpha_5}{2H_0(\epsilon-1)}\frac{(k_B T)^3}{2\pi^2}|Z(0)|/\delta\right]^n /n! = \\ =& |F_0| \mathrm{Exp}\left(\frac{\alpha_5}{2H_0(\epsilon-1)}\frac{(k_B T)^3}{2\pi^2}\frac{|Z(0)|}{\delta}\right)\,. \end{aligned}$$

Such that $|F(\tau)| \leqslant C$, for a constant $C = |F_0|\,\mathrm{Exp}\left(\frac{\alpha_5}{2H_0(\epsilon-1)}\frac{(k_B T)^3}{2\pi^2}\frac{|Z(0)|}{\delta}\right)$. Furthermore, if $\epsilon > 3$, the collapse is driven by a matter whose density scales as

$$\rho(\tau) = \frac{\rho_0}{a^{3(1+w)}} = \frac{\rho_0}{a^{2\epsilon}}\,,$$

which will be always dominant with respect to torsion corrections in the energy density, which scales $\propto \frac{1}{a^6}$ as $a \to 0$. This implies that there is no bounce for $\epsilon > 3$.

- The case $\epsilon = 3$ is of interest, and in this case we can write, after countably many partial integrations

$$\int_{-1}^{\tau} ds \frac{Z\left(-2hi\frac{s}{\beta H_0(\epsilon-1)}\right)}{s} =$$
$$= \log(|\tau|) Z\left(-2hi\frac{\tau}{\beta H_0(\epsilon-1)}\right) + \sum_{n=1}^{\infty} \frac{\left(-2hi\frac{\tau}{\beta H_0(\epsilon-1)}\right)^n (\log(\tau) - H_n)}{n!} \times$$
$$\times Z^{(n)}\left(-2hi\frac{\tau}{\beta H_0(\epsilon-1)}\right) \overset{\tau\to 0}{\sim} \log(|\tau|) Z\left(-2hi\frac{\tau}{\beta H_0(\epsilon-1)}\right), \tag{107}$$

where H_n are the harmonic numbers (i. e.$H_n = \sum_{k=0}^{n} \frac{1}{k}$). The part that we have neglected in (107) is an analytical function that goes to zero as $\tau \to 0$. Thus the asymptotic behaviour for $\epsilon = 3$ is

$$F(\tau) = F_0 |\tau|^{\frac{i\alpha_5}{2H_0(\epsilon-1)} \frac{h(k_B T)^3}{2\pi^2} Z\left(-2hi\frac{\tau}{\beta H_0(\epsilon-1)}\right)} \to \tilde{F},\ \tau \to 0$$

Where $\tilde{F}$ would some constant which can in principle be determined by calculating all the terms in (107).

- The case $2 < \epsilon < 3$ is also interesting: also in this case we can partial integrate to get

$$\int_{-1}^{\tau} ds \frac{Z\left(-2hi\frac{s}{\beta H_0(\epsilon-1)}\right)}{|s|^{\delta}} = \frac{Z\left(-2hi\frac{\tau}{\beta H_0(\epsilon-1)}\right)}{|\tau|^{\delta-1}(1-\delta)} +$$
$$+ \sum_{n=1}^{\infty} \frac{\left(\frac{2hi}{\beta H_0(\epsilon-1)}\right)^n |\tau|^{n-\delta}}{(n-\delta)(n-\delta-1)\cdots(1-\delta)} Z^{(n)}\left(-2hi\frac{\tau}{\beta H_0(\epsilon-1)}\right) - \tag{108}$$
$$-\left[\tau \to -1\right] \overset{\tau\to 0}{\simeq} \frac{Z\left(-2hi\frac{\tau}{\beta H_0(\epsilon-1)}\right)}{|\tau|^{\delta-1}(1-\delta)} + const\,,$$

which gives, for F

$$F(\tau) = F_0 \mathrm{Exp}\left(\frac{i\alpha_5}{2H_0(\epsilon-1)} \frac{h(k_B T)^3}{2\pi^2} \frac{Z\left(-2hi\frac{\tau}{\beta H_0(\epsilon-1)}\right)}{\tau^{\delta-1}(1-\delta)}\right)$$
$$\Longrightarrow |F(\tau)| \to \tilde{F},\ \tau \to 0\,,$$

which oscillates infinitely many times, but remain finite in absolute value. It is noteworthy that, since $|F|$ does not diverge at $\tau = 0$ for $\epsilon > 2$, it means that the particle production in this regime remains finite: only a finite number of fermions are produced during the collapse before a singularity is formed. Still, since the torsion part of the energy density becomes dominant at some stage during the collapse, a bounce will always happen in this regime, even if the particle density only increases to a finite value.

- The case $\epsilon = 2$ is when the above behaviour in the particle production changes. Proceeding in the way we derived Eqs. (107 - 108) we would find

$$\begin{aligned} |F(\tau)| &= \frac{|F_0|}{\tau^{\gamma_1}}, \\ \gamma_1 &= \frac{9\alpha_5(k_B T)^3\zeta(4)}{4H_0(\epsilon-1)\pi^2} = \frac{\alpha_5(k_B T)^3}{40H_0(\epsilon-1)}, \end{aligned} \tag{109}$$

where we expanded $Z\left(-2hi\frac{\tau}{\beta H_0(\epsilon-1)}\right)$ up to second order in τ.

- The case $\epsilon < 2$ is the regime when a similar behaviour as during radiation happens. In this case the particle number diverges in a exponential way,

$$|F(\tau)| \sim \mathrm{Exp}(A/\tau^{\Gamma});\ A > 0,\ \Gamma > 0$$

In this case we have a bounce, plus the particle production is significantly more intense that what it was during a radiation domination collapse. Lowering ϵ but keeping it bigger than 1 has therefore the only effect of making the particle number diverge faster and faster.

To recap:

- $\epsilon > 3$: there is no bounce, the particle number is finite and $F(\tau)$ is analytical in $\tau = 0$.
- $\epsilon = 3$: there is a bounce only if the initial densities are fine tuned in a particular way, since $F(\tau)$ will grow but to a finite amount during the collapse. $F(\tau)$ will not be analytical in $\tau = 0$, but its absolute value will.
- $2 < \epsilon < 3$: there is a bounce, the particle number remains finite all the way through the collapse. $F(\tau)$ will not be analytical in $\tau = 0$, but its absolute value will.
- $\epsilon \leqslant 2$: there is a bounce, and the particle number diverges before a singularity is formed. For $\epsilon < 2$ the particle number goes to infinity exponentially.

The case $\epsilon \leqslant 1$ is harder to solve, because in this case the singularity happens at $\tau = \infty$. Therefore there is no easy way to derive the asymptotic behaviour as we did in this section. However, since we found that, when $\epsilon \leqslant 2$, the particle production diverges, an educated guess is that it does so even when $\epsilon < 1$.

From the results of this Appendix, we conclude that the particle production induced by torsion can be quite intense at very late time, when $\epsilon < 2$. Studying the back reaction of the fermionic fields on the space-time is therefore necessary, to inquire whether such behaviour is indeed realised or not.

APPENDIX E

Here we will derive the equations of motion for the fermionic currents and the energy momentum tensor in the case $\xi = 0$ and $\xi' \neq 0$. The Dirac equation, in this case reads

$$\left(i\gamma^\mu D_\mu - m_R - i m_I \gamma^5\right)\psi = -(3\pi G \xi'^2)(\bar{\psi}\gamma^\sigma\psi)\gamma_\sigma\psi\,. \tag{110}$$

Which in semi-classical approximation, for the currents f_{ah} becomes

$$\partial_\eta f_{0h}(\vec{k}) = 0, \tag{111a}$$

$$\partial_\eta f_{1h}(\vec{k}) + 2h|\vec{k}|f_{2h}(\vec{k}) - 2a m_I f_{3h}(\vec{k}) = \tag{111b}$$

$$= -\frac{18\pi G\xi'^2}{a^2}\int\frac{d\vec{p}}{(2\pi)^3}\left(\left(f_{3h}(\vec{p})f_{2h}(\vec{k}) + f_{3h}(\vec{k})f_{2h}(\vec{p})\right)\right),$$

$$\partial_\eta f_{2h}(\vec{k}) - 2h|\vec{k}|f_{1h}(\vec{k}) + 2a m_R f_{3h}(\vec{k}) = \tag{111c}$$

$$= \frac{18\pi G\xi'^2}{a^2}\int\frac{d\vec{p}}{(2\pi)^3}\left(\left(f_{3h}(\vec{p})f_{1h}(\vec{k}) + f_{3h}(\vec{k})f_{1h}(\vec{p})\right)\right),$$

$$\partial_\eta f_{3h} - 2a m_R f_{2h} + 2a m_I f_{1h} = 0. \tag{111d}$$

Note the absence of terms containing $\sum_h$, which in the first part of this paper was mainly due to the presence of the γ^5 matrix in the interaction term, and the consequent violation of parity symmetry. Aside from this and a factor of 3 difference in the torsional coupling constant, the structure of these equations is precisely the same of Eq. (62a - 62d). In both cases the torsion interactions induce a shift in the mass and in the momenta of the fermionic fields. This effect in case $\xi = 0$ is due to the Fock contraction, while in case $\xi' = 0$ to both Hartree and Fock. This suggests that both interaction terms can be treated in the same way. Some interest could come from the mixed situation, in which $\xi, \xi' \neq 0$, in which case we would find Eq. (62a - 62d) again, but now the Hartree and Fock terms will have a different coupling strength (respectively, ξ^2 and $\xi^2 + 3\xi'^2$).

The energy momentum tensor for $\xi = 0$ is

$$\begin{aligned} T_{00} &= \sum_h \int \frac{d\vec{p}}{(2\pi)^3}\left(\frac{1}{a^4}h|\vec{p}|f_{3h} + \frac{1}{a^3}(m_R f_{1h} + m_I f_{2h})\right) - \\ &- \frac{\alpha_5}{a^6}\int\frac{d\vec{p}}{(2\pi)^3}\frac{d\vec{p}'}{(2\pi)^3}\left(\sum_{hh'} f_{0h}f_{0h'} + \sum_h (f_{1h}^2 + f_{2h}^2) - \frac{1}{2}(f_{3h}^2 + f_{0h}^2)\right), \end{aligned} \tag{112}$$

$$\begin{aligned} T_{ij} &= \delta_{ij}\left\{\sum_h \int \frac{d\vec{p}}{(2\pi)^3}\frac{1}{3}\left(\frac{1}{a^4}h|\vec{p}|f_{3h}\right) - \right. \\ &\left. - \frac{\alpha_5}{a^6}\sum_{hh'}\int\frac{d\vec{p}}{(2\pi)^3}\frac{d\vec{p}'}{(2\pi)^3}\left(\sum_{hh'} f_{0h}f_{0h'} + \sum_h (f_{1h}^2 + f_{2h}^2) - \frac{1}{2}(f_{3h}^2 + f_{0h}^2)\right)\right\}, \end{aligned} \tag{113}$$

where now $\alpha_5 = \frac{9\pi G \xi'^2}{2}$. Again there is a structural difference between the vector interaction and the pseudo-vector interaction. Most notably, in the vector case, the term $\propto \sum_h f_{3h}^2$ comes with the opposite sign, when compared to the pseudo-vector case. This will have the effect of delaying the bounce, or eliminating it completely. In this case the classical solution will only lead to a bounce if $|m| > k_B T$. However particle production might prevent the singularity formation in the case $|m| < k_B T$ too, since it causes f_{1h} and f_{2h} to scale faster than f_{3h}, which remains constant in the massless regime.

To answer this question, one should calculate the scaling behaviour of f_{ah} in a fixed background with $\epsilon = 3$. What we found for the function F in Appendix D is:

$$\begin{aligned} F\left(\frac{\eta}{\beta}\right) =& F_0 \exp\left(\frac{i\alpha_5}{2}\frac{h(k_B T)^2}{2\pi^2}\log\left|\frac{\eta}{\beta}\right| Z\left(-2hi\frac{\eta}{\beta}\right)\right) \\ & \to \tilde{F}, \ \frac{\eta}{\beta} \to 0\,. \end{aligned} \tag{114}$$

Where $\tilde{F}$ is a constant. Whether or not this case leads to a bounce depends on the initial conditions, and on the energy density dominating the phase before the time $\epsilon \to 3$. Quantum corrections and particle production, therefore, are expected to lead to a bounce for $|m| \lesssim k_B T$, and $|m| > k_B T$ but not when the masses are negligible with respect to the temperature.

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