The *real* problem behind Russell's paradox

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Introduction:

At the beginning of the 20th century Russell found a contradiction in what is now known as naïve set-theory. This contradiction has become known as Russell's paradox and it has played a very important role in the development of logic. The essence of Russell's paradox is that in naïve set-theory one can define a set C as the set which contains all sets which are not members of themselves, i.e. $C = \{x \mid x \notin x\}$.[1] This is a paradoxical object since it can easily be shown that $C \in C \Leftrightarrow C \notin C$. The Russell-paradox was first made public by Russell in 1903 his book *The Principles of Mathematics* (Russell, 1903a), although he had informed several people of it in private.

Russell's paradox and its possible solutions have been extensively studied by logicians of world-class caliber like Ernst Zermelo, Kurt Gödel and Alonzo Church. In most axiomatizations of set-theory the paradox is solved by restricting the principle of comprehension, which essentially means that in these theories a set cannot simply be defined by stating a condition which determines whether an object is a member of that set or not. Russell himself believed that the paradox should be solved by some form of a *theory of types*, which essentially postulates that there are different types of objects in the universe. Simple type theory as we now know it basically solves the paradox because its syntax forbids that one can express that sets are elements of sets. It is therefore meaningless to ask whether sets can be elements of themselves. One can only express that sets are elements of sets of higher type, which are a different *type* of object. We can essentially already find simple type theory as we now know it in appendix B of the *Principles* (Russell, 1903a, p. 523).

There were, however, different reactions to the discovery of the Russell paradox. Georg Cantor, who had invented naïve set theory, postulated a distinction between the *transfinite* and the *absolutely infinite*, and claimed that the absolutely infinite, since it contained contradictory objects, was simply not accessible to human thought. Henry Poincaré, a French mathematician and defender of intuitionism, happily gloated that the attempts to use symbolic logic to analyze mathematics were now no longer sterile, but were begetting contradictions. Zermelo, who was the first to have rigorously axiomatised set theory, only discussed the technical solution of solving the paradox and explicitly put the philosophical issue's behind it on hold. Gottlob Frege was devastated by its discovery, believing the existence of the paradox to show that his life's work of trying to ground arithmetic on the basis of logic alone was mistaken. In the last years of his life he tried to ground arithmetic on the science of geometry which he took to be synthetic a priori. Furthermore, paradoxes like the *liar-paradox*, e.g. "this proposition is false", had been known for centuries and had been extensively studied by medieval logicians who called these kinds of paradoxes *Insolubilia*. But instead of taking these paradoxes as threatening to destroy their understanding of logic, most medieval authors seemed to have regarded them merely as argumentative nuisances, and their main concern was to come up with ways of dealing with them when they arose in disputation (Spade & Read, Winter 2009).

Russell, however, reacted in neither of these ways. Instead, Russell dedicated years of his life to try and solve the problems that were posed by the paradoxes. In his autobiographical *My Philosophical Development* Russell recounts that he felt that the paradoxes were almost

[1] Russell himself does not use the term "set", instead using the term "class". When I turn to an examination of Russell's work I will follow his terminology.

a personal challenge, and that he would have dedicated his life to solving it if necessary, even though the problem struck him as *trivial* (Russell, 1959, p. 79). All these facts suggest that there was something special with Russell. Why did he labor for almost 10 years to solve a paradox for which he basically seems to have had a technically acceptable solution in 1903 already?

In order to understand *all* the problems that the paradox posed to Russell, we need to thoroughly understand the philosophy behind his analysis of mathematics. Russell's paradox has mostly been studied by logicians, who have tended to focus on the *technical* aspects of the paradox. This thesis tries to correct this one-sided view of the paradox by showing the *philosophical* challenges that the paradox posed for Russell. Its central thesis can therefore be stated as:

> ***The main problem that Russell's paradox posed to Russell was not merely the technical problem of having an inconsistent deductive system. Instead, Russell's paradox destroyed Russell's metaphysical understanding of reality.***

Argumentation structure:

This thesis is intended to give a historical reconstruction of Russell's philosophical development which shows how Russell's philosophical views were related to the paradox. To facilitate the reader in grasping the overall structure of the argument presented in this thesis I will give a short summary of its main argument below.

To gain a thorough understanding of Russell's paradox we need to go back to the time that Russell was still an idealist. From an early age Russell had been fascinated by the demonstrative nature of mathematics, and in a large part of his early philosophical work Russell tries to understand the nature of mathematics and what mathematical concepts are.

During his "Hegelian" phase[2] Russell believed that mathematics was an abstraction of reality, and as such, was not completely true. This claim that mathematics was not completely true must be understood in the sense that we now know classical mechanics to be not completely true. We know that classical mechanics abstracts from relativity and quantum effects and as such its principles are not the true laws of nature. Nevertheless, it still gives a fairly accurate description of the motion of normal-sized objects which do not move at high speed. That mathematics was an abstraction of reality was visible because its concepts were not completely non-contradictory. Although he would later abandon idealism due to the arguments G.E. Moore was to give in his *The Nature of Judgment,* he would never abandon the crucial key insight which he had learned from F.H. Bradley and J.M.E. McTaggart that ultimate reality is such that it does not contradict itself.

At the end of the 19th-century Moore convinced Russell that Bradley's *idealistic logic* was wrong. Bradley believed that concepts could only exist in the mind. Since concepts were only ideal, this explained why concepts could be contradictory, even though reality could not. Moore, however, argued that concepts were real and had to exist independently of the mind. Furthermore, Moore believed that philosophy essentially consisted in the analysis of

[2] The relationship between *British Idealism* and Hegel's philosophy is a very complicated one. Bradley, for instance, explicitly claims that he is not a follower of Hegel (Bradley, 1883, p. iv). I therefore prefer not to call the *British Idealist* Hegelians, and will call them "Hegelians" instead.

concepts. Russell accepted Moore's arguments but lacked a method of analyzing mathematical concepts. This method he found in the work of Giuseppe Peano.

In 1900 Russell met Peano at the Second International Conference of Philosophy. Peano, who had been inspired by Leibniz, had developed a Universal Characteristic which was meant to analyze the content of mathematical concepts. Russell quickly accepted Peano's method, expanded it, and tried to use it to analyze all mathematical concepts in terms of primitive logical ideas and propositions only. This analysis pointed to the existence of a certain kind of special complex concepts which were able to somehow mean, or denote, other concepts. Russell called these denoting concepts.

However, there was a problem with these denoting concepts. Denoting concepts had to exist independently of the mind. As such, since reality was non-contradictory, these concepts had to be self-consistent. But they were not. In 1901 Russell discovered that certain denoting concepts were not consistent. The most famous example of this is the class of all classes which do not belong to themselves, but there were infinitely more. The paradox showed that there was something wrong with Russell's understanding of the nature of denoting concepts. But denoting concepts were crucial for his understanding of mathematics. The paradoxes therefore completely destroyed Russell's metaphysical understanding of the logical universe. This led him to the struggle which lasted for years in which he tried to show that, although denoting *phrases* are part of the Universal Characteristic, denoting *concepts* do not exist independently of the mind.

Sources:

A thorough historical understanding of something means to go back to the original sources. Russell's philosophical development did not occur in a vacuum. Instead Russell was part of the community of British philosophers, of which Bradley, McTaggart and Moore were the most important for him. Around the turn of the century he also became part of a community of philosophers of logic, of which Peano, Louis Couturat and Ernst Schröder were the most influential.

In this thesis I have tried to track down all of the original sources. A lot of the books and articles that are vital to understand the development of Russell's philosophy of mathematics were very difficult to find no less than ten years ago, and their contents have been mostly handed down via tradition. But due to modern technological innovations the work of the historian has become far more easy than it has ever been before. Because of the digitalization we can now go back to the original sources and see that their contents have not always been handed down correctly by tradition. In this thesis I have tried to make full use of this opportunity and I have seen all the sources I have cited in their original form. Some of these I have seen in hardcopy, but most of them I have only seen as digital copies. I have cited these works as if I have seen the hardcopies. However, if one really needs to see the digital editions that I have used, all of them were accessed via www.archive.org.

I generally do not discuss secondary literature on Russell, instead I believe that the primary sources must tell their own story. A thorough discussion of the secondary literature often has the downside of obscuring the real historical story. This does not mean that the secondary literature does not lie at the core of this thesis. For this thesis I have studied Bernard Linsky's *Russell's Metaphysical Logic* (Linsky, 1999), Gregory Landini's *Russell's Hidden Substitutional Theory* (Landini, 1998) and the relevant articles in *The Cambridge Companion* *to Russell* (Griffin, 2003b). They have provided me with valuable insights in Russell's philosophy and have often given me hints of where to look in the primary sources. However, I have tried to stay away from explicitly criticizing this literature. A proper historical understanding does not stem from polemics but from reading and returning to the original sources. There is however one book that I wish to especially mention. This thesis could not have been written without the monumental work *The search for Mathematical Roots 1870-1940* by historian Ivor Grattan-Guinness (Grattan-Guinness, 2000). Like myself, Grattan-Guinness is mainly interested in understanding the historical development of logic. His book is a valuable reserve of historical facts to which I have turned time and time again in order to understand the subtle historical details of the development of mathematical logic.

A word of gratitude:

Before I turn to the body of this thesis I wish to thank my supervisors Albert Visser and Paul Ziche. Not only did they provide extensive comments to earlier drafts of this thesis, despite the fact that they had to do so on a Sunday since I was late handing it in, but they have also guided me in my philosophical development during the years I was at Utrecht University.

Chapter 1: Reality is such that it is non-contradictory

In this chapter I discuss Russell's early search for knowledge of the nature of mathematics. This led him to Cambridge, where he became a member of the *British Idealist* movement. However, his idealist analysis of mathematics showed him that mathematics was an abstraction from experience, and as such, was not completely true. This could be seen in the ultimately contradictory nature of the concepts of mathematics. But what Russell most longed for was a reason to suppose that mathematics was true. These reasons were presented to him by Moore, who abandoned Bradley's view that concepts existed only in the mind. There was one idealist principle, however, that Russell would never abandon: Bradley's criterion that ultimate reality is such that it does not contradict itself.

Russell's earliest views

In 1883, when Russell was eleven years old, John Francis Stanley Russell (known as Frank) decided to teach his little brother Bertie Euclidean geometry. Together they worked through Stephen Thomas Hawtrey's *An introduction to the Elements of Euclid* (Hawtrey, 1874), which contains the first twelve propositions of the first book only. Little Bertie mastered the book within two months and made his brother proud (Monk, 1997, pp. 25-6). Russell himself would later believe that these were his first steps in his life-long search for demonstrative truth (Russell, 1956, p. 14). He was immediately enamored by the idea of mathematical demonstration, even though he would condemn Euclid's own proofs for their lack of mathematical rigor after he became familiar with Moritz Pasch's *Vorlesungen über Neuere Geometrie* (Pasch, 1882) and David Hilbert's *Grundlagen der Geometrie* (Hilbert, 1899) (Russell, 1917, pp. 94-5).

That geometrical propositions could be proved and need not be accepted simply on the basis of belief made a very great impression on Russell. Only one thing frustrated him: His brother could not give any further reasons why Bertie had to accept the axioms themselves, other than the pragmatic one: if he didn't, they simply couldn't go on (Monk, 1997, pp. 25-6). Russell accepted this for the moment, but even in his early years he never quite overcame his fundamental doubts as to the validity of mathematics (Russell, 1956, p. 15). This wish to understand the nature of mathematics would become a dominant theme of his life, and it would ultimately lead him to the logicist analysis of mathematics for which he is now most famous.

Cambridge

Having found much greater delight in the study of mathematics than in any other study, Russell applied to study mathematics at Trinity College and got accepted with a minor scholarship (Monk, 1997, p. 38).

But his hopes of finding the same delight as during his own studies of mathematics were quickly smashed against the rocks of the tedious training for the Mathematical Tripos in Cambridge, which emphasized a series of useful techniques to facilitate the practical application of mathematics instead of formal proofs (Monk, 1997, p. 45). Russell felt that the proofs that they offered were full of logical fallacies (Russell, 1956, p. 15) and he became so disgusted with the mathematical education at Trinity, which he believed to be an insult to the logical intelligence (Russell, 1959, p. 38), that he sold his books immediately after the Tripos and vowed never to look in a mathematical book again (Monk, 1997, p. 51).

But what he still most desired was to find some reason to suppose that mathematics was true (Russell, 2010, p. 57). What were the grounds of the axioms which his brother had told him to assume? So when he started to study for his Moral Sciences Tripos he focused on the philosophy of mathematics and logic in the hope of finding answers to the questions that had plagued him for so long.

He had already read Mill's *Logic* before coming to Cambridge, who's empiricist views concerning mathematics he believed to be inadequate (Russell, 2010, p. 57). Because of a chance encounter with Harold Joachim, who was the neighbor and son-in-law of Russell's uncle Rollo and fellow at Merton College Oxford, he had also read Bradley's *Principles of Logic* (Bradley, The Principles of Logic, 1883), which Joachim had said was good but hard, and Bernard Bosanquet's *Logic, or the Morphology of Knowledge* (Bosanquet, 1888), which Joachim said was better but harder (Russell, 1959, p. 37).

In the meantime Russell had converted himself to what he called "Hegelianism". Russell believed to have found certainty in the dialectical method of Hegel which he had learned from his friend and mentor John McTaggart Ellis McTaggart. McTaggart, whose maternal great-uncle was so rich that he was named after him twice (Geach, 1995, p. 567), had claimed that he could prove by logic that the world was good and the soul immortal, although the proof was long and difficult (Russell, 1959, p. 38). This idea that the dialectical method could prove theses connected with philosophy and religion greatly attracted Russell, who was struggling with the loss of his own faith. Russell had met McTaggart early on in his studies on the instigation of Alfred North Whitehead and both were members of the Cambridge secret society known as The Apostles.

McTaggart was famous for his study of Hegel and he had written his fellowship dissertation on Hegel's dialectical method in 1891. An extended version of this dissertation was published as *Studies in the Hegelian Dialectics* in 1896 (McTaggart, 1896). In the *Studies* McTaggart depicts Hegelian dialectics as a method of demonstrating and systematizing the pure, i.e. non-empirical, concepts of the understanding, which are better known as *categories* (McTaggart, 1896, p. 1). The dialectics proceeds from the more abstract categories towards the more concrete ones by way of contradiction, until the absolute category is reached which understands reality as it is (McTaggart, 1896, pp. 3-4).

McTaggart depicts this as a reconstruction in thought of what is given in experience. Thought tries to understand what is given in experience, but can only do so by making use of concepts. The contradictions which drive the dialectic are caused by the imperfect nature of these concepts, that is, if thought tries to think that which is immediately given in experience in an incomplete manner, then it does so contradictorily. Contradictions are resolved by understanding that the category used is incomplete, it is one-sided in that it only captures a moment of the given, not its totality, and a more encompassing concept must be derived which reconciles the contradictory concept with what is immediately given in experience (McTaggart, 1896, pp. 8-10). Only when one thinks reality as it really is does one think it non-contradictory.

Roughly the same idea can be found in Russell's second main "Hegelian" influence, F.H. Bradley. I will discuss Bradley's view in more detail below, but for now it is sufficient to say that Bradley believed that, since all concepts are universals and reality itself is concrete, all conceptions of reality are ultimately abstractions, although they can be more or less so, and therefore ultimately false (Bradley, 1893, pp. 162-183). This process of abstraction leads to the ultimate contradictory nature of al conceptions of reality. Only ultimate reality was such that it did not contradict itself (Bradley, 1893, p. 135).

The way in which Russell understood the idealist claim that only reality was such that it did not contradict itself, was that any science was a (conscious) reconstruction in thought of the reality that we experience. In this reconstruction, since it is an abstraction, we are faced with contradictions which force us towards a more concrete understanding of reality, i.e. a higher science, until we think reality as it really is (Russell, 1959, pp. 52-53). Russell therefore took the dialectical development of the sciences to proceed from the more abstract sciences to the more concrete, i.e. from arithmetic towards geometry, physics, psychology, etc.

An Essay on the Foundations of Geometry

Continuing his search for reasons to believe the validity of mathematics, Russell wrote an essay five months after the Mathematical Tripos in which he argued the Kant-like thesis that the Euclidean axioms were necessarily true for the way in which humans intuited objects in space (Monk, 1997, p. 64) and, after finishing his Moral Tripos with a 'starred first' distinction, Russell decided to write his fellowship dissertation on the same subject under the supervision of Ward and Whitehead (Monk, 1997, pp. 79-80). Unfortunately the dissertation itself is now lost, but Russell published a later reworking of it in 1897 as *An Essay on the Foundations of Geometry* (Russell, 1897), which he dedicated to McTaggart.

In the *Essay* Russell defends the Kant-like view that there are things about space which can be known *a priori*. In the *Kritik der Reinen Vernunft* (Kant, 1787) Kant had defended the thesis that the axioms of mathematics could be known to be certain, because they were conditions of experience. But because of the development of non-Euclidean geometries during the 19th-century, this claim had come under attack. At least one of the axioms of geometry, the parallel-postulate, could be denied without contradiction. In the *Essay* Russell claimed that, although Kant's claim was too strong, we can actually know three things with absolute certainty about space: Space has to be homogenous, space has to have a finite number of dimensions and every two points have to determine a line which is their distance (Russell, 1897, p. 148). This can be known, according to Russell, because these are conditions of any form of externality, and as such are axioms shared by any possible science of space, i.e. any geometry (Russell, 1897, p. 176). Any other property of space is empirical. This meant in particular that the question whether space is flat or curved could only be decided by measurement (Russell, 1897, p. 175). Russell ends the *Essay* in an Hegelian vein and tries to show that geometry itself contains three fundamental contradictions, which arise from the fact that geometry is an abstraction of concrete reality (Russell, 1897, p. 188). These contradictions, Russell continues, can only be solved form the "higher", i.e. more concrete, standpoint of physics by understanding the contradictory notion of the geometrical point in terms of the concept of *matter* (Russell, 1897, p. 198).

The *Essay* can be seen as a first attempt to answer Russell's question about the nature of mathematics. His answer was basically twofold: First of all, Russell, as an Hegelian, believed that mathematics could not be completely true, since mathematics was an abstraction of reality. Secondly, mathematical concepts could not be completely analyzed because they had inherent contradictions in them since these concepts were abstractions of real things. But Russell would not be satisfied with this answer. After he read Hegel's *Logic* himself in the spring of 1897, which he previously did not think necessary trusting McTaggart's judgment, Russell became disgusted by Hegel's own dialectical arguments on the nature of mathematical concepts, believing Hegel's own views to be scarcely better than puns (Monk, 1997, p. 114). This had mainly to do with Hegel's treatment of continuity. Russell, who had recently learned about the German developments in mathematics which had made this concept rigorous, precise and self-consistent, did not find this concepts treated by Hegel in the same way. Instead, Hegel had emphasized its contradictory nature.

Conclusion

Dissatisfied with his Hegelian analysis of mathematics Russell kept searching for a way in which he could understand mathematics to be certain and true. In the next chapter we will see that Moore would convince Russell that concepts had to exist independently of the mind and were therefore real. Mathematical concepts could therefore be real and non-contradictory. Russell himself seems to have believed that he abandoned all elements of idealist doctrine when he revolted into realism. But he did not. One main principle of idealism would form the core of Russell's struggle with the paradox. Ultimate reality had to be such that it did not contradict itself.

Chapter 2: Logical Atomism as the basis of a philosophy of mathematics

In his autobiographical *My Philosophical Development* Russell recounts that there was only one major revolution in his philosophy, which took place in the years 1899-1900. The rest of the changes in his philosophical view could in general better be seen as an evolution of his thought (Russell, 1959, p. 11). This revolution in his thinking was brought about by two major events in the final years of the 19th-century:[3] His acceptance of the philosophy of logical atomism due to Moore and his coming to know Giuseppe Peano's symbolic method.

In 1899 Moore published *The Nature of Judgment* (Moore G. , 1899) in *Mind*, which has been considered to be the birth-certificate of Analytic Philosophy because in it Moore claims that the true method of philosophy is the analysis of concepts. Originally read at the Aristotelian Society in 1898 Moore's argument would lead Russell to completely disavow the logical metaphysics of *British Idealism* and accept the doctrine of *Logical Atomism*. In this chapter I will discuss why Moore, and Russell in his wake, abandoned the idealist doctrine that concepts were ideal, and what consequences this had for Russell's metaphysical view on the nature of mathematics.

Bradley's "Hegelian" logic

As far as we know, the term "*Logical Atomism*" was first used by Russell during a meeting of the French Society of Philosophy (La Société française de philosophie) held on the 23rd of March in 1911, and his contribution was published in their proceedings as *Le Réalisme Analytique* (Russell, 1911). Although Russell referred to his own position as *Analytic Realism* in the lecture, he claimed during the discussion that "On verra que cette philosophie est un atomisme logique".[4]

But the term is better known because of the series of eight lectures Russell gave in the winter of 1917-18, which were called *The Philosophy of Logical Atomism* and were published in *The Monist* in the following years. In the first lecture of *The Philosophy of Logical Atomism* Russell distinguishes *Logical Atomism* from "the monistic logic of the people who more or less follow Hegel." (Russell, 1918, p. 496) The monistic logic which Russell refers to here is the logic that he had studied during the time he was still a follower of *British Idealism*. In the preface of the *Essay on the Foundations of Geometry* Russell had explicitly credited Bradley as the main source for his own understanding of logic, although he also mentioned Bosanquet and Christoph von Sigwart, a German Logician. And it was Bradley's logical doctrines, and his view of judgment in particular, that were the target of Moore's *The nature of Judgment*.

In Britain, as elsewhere, logic had long been seen as the science of inference, i.e. the mental operation which proceeds by combining two premises, which consisted of judgments, so as to form a consequent conclusion, which was also a judgment itself.[5] It is therefore no surprise that the first book of Bradley's *Principles of Logic* is about judgment, while the remaining two are about inference.

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[3] Since Russell was a mathematician he believed the 20th century started on the 1st of January 1901.
[4] My translation: "*We will see that such a philosophy is a logical atomism.*"
[5] For a very interesting summary article on how logic was viewed just before the modern reinterpretation due to the development of mathematical logic, see the entry on Logic in the Encyclopaedia Britannica of 1911, which was written by the "old-school" Oxford logician Thomas Case.

The first thing Bradley concerns himself with in his logic is the question when a judgment is true. It was generally thought that a judgment consisted of the connection of two ideas, the subject and the predicate, which could be seen to be a true connection when the predicate was contained in the subject, when the subject and the predicate were connected immediately,[6] something which could be perceived by the senses, or when the subject and the predicate were connected via one or more mediating terms, which could be understood by inference. But Bradley did not follow this tradition and instead claimed that the truth of a judgment depended on the relationship of ideas with reality (Bradley, 1883, p. 2). Bradley quickly clarified that he did not mean that it is the idea itself which has to be compared to reality. The idea itself is only a singular event in the mind of a thinker and one cannot predicate a singular idea, in all its particularity, of anything else. Only when the idea is used to stand for something universal can it be used to predicate something of reality.

It is therefore not the ideas themselves, but their meanings, i.e. the universals that they stand for or symbolize, which are true or false of reality (Bradley, 1883, p. 3). Bradley continues by explaining that the meaning of an idea "consists of a part of the content (original or acquired), cut off, fixed by the mind, and considered apart from the existence of the sign." (Bradley, 1883, p. 4). By abstracting and cutting off a part of its content the idea no longer is a full particular but becomes a universal. Bradley himself uses the example of the idea of a horse (Bradley, 1883, p. 6). Suppose that I want to think of horses. Now, any ideas I have of horses are of particular horses because I have only seen particular horses. There are no universals roaming the world. Any memories I have of horses are therefore memories of particular ones. If I want to use any of my ideas of a horse to think of horses in general, then I have to abstract away the particularity of the idea I want to use, for instance I abstract away the color of the particular horse that I have seen and its exact height.

Abstraction is an activity of the mind, and Bradley says that "an idea, if we use idea of the meaning, is neither given nor presented but is taken." (Bradley, 1883, p. 8). Universal ideas cannot exist, according to Bradley, apart from the particular ideas from which they are abstracted and as such cannot exist independently of any mind. This is what makes Bradley's logic *idealistic*. But in the *Nature of Judgment* Moore will argue that these ideas actually do exist independently of any mind, and Russell will follow him.

That Bradley's logic is also *Monistic* can be seen from Bradley's insistence that a judgment does not consist of the connection of two ideas, nor is it the case that it ascribes an ideal content, the predicate, to the subject of a proposition, i.e. the judgment "this rose is red" does not express the connection of my idea of this rose with my idea of red, nor does it ascribe redness to *this* rose. Instead Bradley claimed that a judgment was "the act which refers an ideal content (recognized as such) to a reality beyond the act." (Bradley, 1883, p. 10). That is, there is a unified ideal content, which is, rightly or falsely, attributed to reality, depending on whether reality indeed is as it is thought to be.

It is sometimes argued that Russell misrepresents Bradley's view when he argues that Bradley believed that all judgments are of subject-predicate form, because Bradley explicitly denies that a judgment is the connection of two ideas, the subject and the predicate. But Bradley did believe that all judgments were of subject-predicate form, but with the

[6] That is, without middle term.

qualification that there was only one subject, namely reality and we can explicitly find him claim so in *The Principles of Logic*:

> *I will anticipate no further except to remark, that in every judgment there is a subject of which the ideal content is asserted. But this subject of course can not belong to the content or fall within it, for, in that case, it would be the idea attributed to itself. We shall see that the subject is, in the end, no idea but always reality; (Bradley, 1883, p. 14).*

What is also important to note is that the ideal content which is predicated of reality, however complex, was still a single idea according to Bradley (Bradley, 1883, p. 12). Russell differed from him on this account already in his idealist phase (Griffin, 2003a, p. 87) but this will grow out to their famous dispute about whether all relations are internal. A consequence of Bradley's view is that any relations which are thought to hold within this complex idea are not real. Since there is only one subject, i.e. reality, any relations which are thought to exist only exist within the ideal content. Russell will later call this the doctrine of internal relations and contrasts it with his own doctrine of external relations in which relations are real, i.e. in which relations exist independently of any mind.

Bradley's claim that there are no relations in reality is famously argued for by Bradley in the third chapter of *Appearance and Reality* (Bradley, 1893). There Bradley argues that relations cannot be thought of as real, because between every relation and the terms it relates there must exist a relating relation, which in turn is related to the term and the relation it relates, ad infinitum. Bradley believed that this regress showed that real relations are absurd. But in the *Principles* Russell will accept this argument, and then claim that the regress is not vicious because the relational proposition itself only contains the relation and the terms it relates, not the infinity of relations holding between the relation and the terms it relates, which are only implied by that proposition (Russell, 1903a, pp. 99-100).

Moore's *The nature of Judgment* and the birth of analytic philosophy

In *The Nature of Judgment* Moore attacks this theory of judgment by Bradley, and Moore's main argument is aimed at Bradley's conception of a universal idea, or, as Moore comes to call them, concepts. According to Moore it is wrong to see concepts as abstractions, and he explicitly claims that his main object in *The Nature of Judgment* is to "protest against this description of a concept as an abstraction from ideas." (Moore G. , 1899, p. 177). Instead, Moore will argue that the concepts themselves must be thought to exist independently of any mind. Moore does so by arguing that Bradley's *idealism* cannot show how it is possible for an idea to mean anything at all, because Bradley's theory cannot explain how we are able to take any content from an idea.

Moore argues as follows (Moore G. , 1899, pp. 177-178): In order for me to abstract a part of an idea, i.e. a concept, from that idea I must already be able to identify the ideal content within the idea from which I wish to abstract the concept. But this presupposes that I already know the conceptual content of the idea from which I want to abstract, at least in part, namely, that part which I want to abstract. However, this content is itself conceptual, and as such, ideal according to Bradley. Since this ideal content is a universal, I cannot have been given this content, but, like any other conceptual content, it has to be taken from an idea. But this again presupposes that I know the ideal content from which the content of the complete idea could be taken. But that again is an ideal content, which must have been taken, ad infinitum. This leads to an infinite regress.

According to Moore this infinite regress shows that Bradley's doctrine cannot be correct. Somewhere along the way we must presuppose that the ideal content already exists independent of any abstraction. Moore therefore takes concepts to be primitive. They exist independently of any mind, although a mind is capable of thinking a concept. This is Moore's famous revolt into realism. This meant that Moore became a realist concerning propositions as well, since he believed that propositions were complexes made up from concepts which were supposed to be connected (Moore G. , 1899, p. 179). And because of this, Moore emphasized *conceptual analysis* as the most important task of philosophy: "A thing becomes intelligible first when it is analyzed into its constituent concepts" (Moore G. , 1899, p. 182).

Russell was quick to accept Moore's argument against Bradley, and followed Moore in becoming a realist. This is the basis of Russell's philosophy of *Logical atomism* which he set against the monistic logic of those who followed Hegel. It is *atomistic* because instead of a single unified reality, we now have a whole domain of simple concepts which exist independently of any mind, and which stand in complex relations towards other simple concepts and as such form complexes. While Moore seems to have emphasized the rejection of *idealism,* Russell himself emphasized a different consequence of this revolt in *My Philosophical Development* (Russell, 1959, p. 54).

Russell had come to realize that relations were crucial for an understanding of mathematics, in particular, asymmetrical relations were needed to understand the notions of Number, Quality, Order, Space, Time and Motion (Russell, 1903a, p. 226). But relations could not be understood under the assumption that all propositions were of subject-predicate form, and asymmetric relations were especially problematic on this view (Russell, 1903a, pp. 218-226). This meant that the view that all propositions are of subject-predicate form must be false. But it was only Moore's *pluralistic* realism which allowed Russell to understand how asymmetric relations could exist independently of any mind and he explicitly acknowledges that it is this conceptual pluralism that destroys the theory of *Monism* (Russell, 1903a, p. 44).

After Russell abandoned *British Idealism* he came to disown all his previous philosophical work as worthless, and his estimation of the *Essay on the Foundations of Geometry* in particular was harsh. Even though it had drawn the attention of first class reviewers like Couturat (Couturat, 1898), who could only read it with an English dictionary at hand, and Henri Poincaré (Poincaré, 1899), he refused to let it be reprinted later in life. The main problem with the *Essay* was, according to *My Philosophical Development,* that its argument contradicts Einstein's general theory of relativity (Russell, 1959, p. 38). Russell had claimed that empirical space must have constant curvature, since this was a pre-condition for something to be a form of externality at all, but according to the general theory of relativity the curvature of space-time is related to the matter and radiation present within that space-time and need not be constant. Being in contradiction with scientific discovery was more than enough to condemn the *Essay* to the dustbin.[7]

[7] But for Russell scholars there is still an interesting question here. Einstein only started to develop his general theory of relativity in 1907, publishing the field-equations themselves in 1915 (Einstein, 1915). This is after Russell published *The Principles of Mathematics* (Russell, 1903) and during the time Whitehead and Russell were writing *Prinicpia Mathematica* (Whitehead & Russell, 1910, 1912, 1913).

Conclusion:

Moore's argument that concepts could not be purely *ideal* lead Russell to his pluralistic conception of concepts, which had to exist independently of any mind. However, in the last chapter I argued that there was one *idealistic* doctrine that remained as an important part of Russell's metaphysical understanding of the universe, i.e. ultimate reality was such that it did not contradict itself. The doctrine that reality had to be non-contradictory coupled to Russell's new doctrine that concepts were real meant that Russell could no longer accept any contradictions in the concepts of mathematics as he had done in the *Essay*.

As far as I know it is still an open question what Russell's attitude was concerning the argument in the *Essay* after he abandoned Kantian-Hegelian philosophy but before he understood the implications of the general theory of relativity.

What is clear from Russell's discussion of geometry in the *Principles* is that Russell came to believe that geometry was a branch of pure mathematics (Russell, 1937, pp. 372-374). Just like any other purely mathematical propositions he thus came to consider all geometrical theorems to be of the form that if certain axioms were true, certain theorems would follow from necessity. This meant that, according to Russell in the *Principles*, geometry is not about empirical space.

Chapter 3: Peano's Characteristica Universalis

Now that Russell had a new metaphysical understanding of what concepts were and how they related to each other he immediately started working on an analysis of mathematical thought, which he planned to call *The Fundamental Ideas and Axioms of Mathematics* (Monk, 1997, p. 123). But surely mathematical concepts are complex. Moore's revolt however had not shown Russell how to determine the conceptual content of mathematical concepts such as the concept of *number*. Russell was therefore in need of a method of analysis. This he found in Peano.

Russell's meeting with Peano

In 1899 Russell was invited by Couturat to present a paper at the First International Congress of Philosophy,[8] which would be held in the summer of 1900 in Paris (Monk, 1997, p. 124). It was here that Russell first met Giuseppe Peano. Russell was immediately impressed by Peano and considered Peano to exhibit a level of precision and logical rigor unsurpassed by any of the other participants (Russell, 1959, p. 65).

In 1900 Russell had already known about symbolic logic, due to Whitehead's *A Treatise on Universal Algebra* (Whitehead A. N., 1898) and he briefly mentions it in his book on Leibniz when he discusses Leibniz's vision of a *Characteristica Universalis* (Russell, 1900, p. 169), which Russell there conflates with the *Calculus Ratiocinator*. He was also already familiar with Peano's symbolic logic from Couturat's article *La Logique Mathématique de M. Peano* (Couturat, 1899), but what is certain is that he had never actually read any of his works. In 1900 Russell still judged that symbolic methods were of no use to philosophy, because, although they provided a theory of deduction which was fruitful for mathematics, they did not constitute an analysis of the simple concepts involved nor did they help with finding the primitive axioms (Russell, 1900, p. 70). However, seeing how precise and rigorous Peano was in dealing with his subject, Russell approached him after his presentation and asked for all his work. He started to read it all immediately and adopted his notation (Russell, 1959, p. 65).

What must have attracted Russell in Peano against the algebraic logic as he found it in Whitehead's *Universal Algebra* was that the main aim of Peano's project of the *Formulaire de Mathématiques*[9] was to state mathematical theorems and their proofs very precisely with the help of mathematical symbols. But in contrast to Whitehead, who's main aim was to compare different algebraic structures, each symbol of Peano's symbolism stood for a primitive concept, and as such, this reduction of a theory to symbols consisted in a precise analysis of the ideas involved in a certain mathematical theory (Peano, 1895, pp. III-IV). Russell, who had answered Moore's call that philosophy essentially was analysis of concepts, was precisely in need of such a method which could analyze the content of mathematical ideas. In the preface of the third edition of the *Formulaire* Peano links his project of analysis explicitly to Leibniz's dream of a *Characteristica Universalis* (Peano, 1901).

[8] Not to be confused with the Second International Congress of Mathematicians, which was also held in Paris that year, where David Hilbert presented 10 of his famous 23 unsolved mathematical problems.

[9] Peano published five editions of the *Formulaire*. The last two were written in his uninflected Latin and called *Formulairio Mathematico*.

Leibniz's dream of a Universal Characteristic

It is well known that the ideal of a *mathematical logic* was in some way anticipated by Leibniz, something which has frequently been remarked by logicians. In books on (the history of) logic we often find references to Leibniz's dream that when reasoning has been turned into symbolic manipulation two philosophers might solve a dispute by simply calculating through the argument (Leibniz, 1890, p. 200).

Russell himself also often spoke of mathematical logic in relation to Leibniz's dream of a *Calculus Ratiocinator*, the first of which can be found in print, as far as I know, in his article *Recent work on the Principles of Mathematics*[10] which was published in 1901 in the American journal *The International Monthly*. Russell there explicitly claims that Peano's work is the perfection of Leibniz's dream (Russell, 1993a, p. 369).

The *Characteristica Universalis* and the *Calculus Ratiocinator* were extensively discussed by Couturat in his book on Leibniz's logic: *La Logique de Leibniz* (Couturat, 1901). Because of its heavy focus on the idea of *logical analysis* this book is indispensable for anyone with an interest in the history of Analytic Philosophy, especially because of the influence it has had on Russell, who wrote a very favorably review of it in *Mind* (Russell, 1903b). Unfortunately it does not seem to be widely read, perhaps because it was written in French and has not been fully translated yet.[11]

Leibniz's dream of a *Characteristica Universalis*, or Universal Characteristic, can already be found in his dissertation called *Dissertatio de Arte Combinatoria* (Leibniz, 1880), written in 1665 when Leibniz was 17 and published in 1666. The main idea in the *Dissertatio* was to construct an *alphabet of human thought*, i.e. find the most basic ideas from which all complex ideas are made up. Leibniz claims that he was inspired by Ramon Llull's *Ars Magna*, published in 1305. In the *Ars Magna* Llull had distinguished 6 categories, *absolute attributes*, *relations*, *questions*, *subjects*, *virtues* and *vices*, containing 9 primitive terms each. The idea was that within each of the categories the simple terms could be combined with one another to form more complex ideas of that category. Simple and complex terms from each of the categories could then be combined to form propositions. The young Leibniz correctly calculated that if this categorization was correct, 17,804,320,388,674,561 different propositions could be formed (Couturat, 1901, p. 37).

But what is wrong with Llull's *Ars Magna*, according to Leibniz (Couturat, 1901, p. 38), is that the *Ars Magna* does not help with analyzing what the simple categories and terms are. Leibniz charges Llull that he had arbitrarily set the terms within each category and the number of categories to 9 and 6 respectively. But Llull's method can only show the true number of possible propositions after the simples are given. Therefore, an analysis of the simple terms and categories had to be carried out. Leibniz believed that in order to find the simples one must start with complexes and work back towards the simples, reducing complex ideas to simpler ones until one reaches the most simple ideas. Only then can one build up all possible complexes from these simples (Couturat, 1901, p. 39).

[10] This paper was reprinted in *Mysticism and Logic* as *Mathematics and the Metaphysicians* (Russell, 1917, p. 79). It can also be found in the third volume of the Collected Papers (Russell, 1993a).
[11] There is a partial translation of it by Donald Rutherford and R. Timothy Monroe on the web (see http://philosophyfaculty.ucsd.edu/faculty/rutherford/leibniz/contents.htm). Unfortunately it seems that it will stay unfinished since it was last updated 10 years ago.

The Universal Characteristic is formed by assigning a symbol to each of the simple ideas, that is, after one has correctly identified them. All complex ideas can then be stated by combining the symbols which stand for simple ideas. A definition of a complex idea would then state nothing but all the simple ideas from which the complex idea is made up. By making these symbols something which can be universally shared one creates a universal language that could be read by anyone, regardless of the languages he or she commanded.

In this Leibniz was inspired by the many attempts to create an international or universal language by Renaissance thinkers. Leibniz mentions three contemporary attempts in the *Dissertatio*, an anonymous one, one by Johann Joachim Becher and one by Athanasius Kircher. Each of them tried to make a correspondence between numbers and the words in different languages which meant the same. Strings of those numbers could then serve as a universal language, since their meaning could be looked up by anyone (Couturat, 1901, p. 54). Of course, although these language might work in practice, the shortcomings of these projects were obvious: they would have been difficult to remember, words in different languages are not completely synonymous, languages having different syntax, etc. The true Universal Characteristic would not have these defects. Leibniz dreamed of a language where the simple signs did not merely conventionally signify the simple ideas that they stood for but one where the signs did so intrinsically, thinking of the way in which he believed that Egyptian hieroglyph's and Chinese characters directly depict what they stand for (Couturat, 1901, pp. 60-61).[12]

Leibniz believed that the analysis of the simple ideas should be done by analyzing language. But, instead of studying any language directly, he first analyzed and regimented Latin in order to see how an ideal language functioned in expressing thought, also hoping to create a universal scientific language (Couturat, 1901, p. 60). In the beginning of the 20th-century Peano, who had studied Couturat's book, resurrected the program of regimenting Latin for use as a scientific language and created *Latin without inflection* (*Latin sine flexione*) in 1903 (Peano, 1903). Before having used Scholastic Latin and French, he used this remarkably easy to read language for his scientific publications, among which the 4th and 5th editions of the *Formulaire*.

But in parallel to the Universal Characteristic, Leibniz also developed his idea of a calculus of thought, the *Calculus Ratiocinator*, although according to Couturat he kept the ideas strictly separated (Couturat, 1901, pp. 78-79). The idea of a Calculus Ratiocinator was inspired by Thomas Hobbes, who had said in his *De Corpore* that all reasoning consisted in the addition and subtraction of ideas (Hobbes, 1839). Although not very deeply developed, the idea seems to have been that a proposition is a sum of two terms, while a syllogism is a sum of two proposition, i.e. a sum of three different terms, since the two propositions shared a middle term (Hobbes, 1839, p. 42).

If Hobbes was correct in this, then all reasoning with ideas could in principle be reduced to the manipulation of the signs for these ideas, i.e. a calculus of thought, and the Universal Characteristic could, in principle, be turned into a calculus of thought by assigning numbers

[12] We now know that this view of hieroglyphs and Chinese characters is wrong. Both of these scripts mainly make use of phono-semantic symbols, which stand for sounds, not for ideas.

to the primitive ideas and then calculate with these numbers (Couturat, 1901, pp. 96-103). It is here that the famous anecdote of Leibniz vision belongs:[13]

> *If we had it* [i.e. the Calculus Ratiocinator]*, we should be able to reason in metaphysics and morals in much the same way as in geometry and analysis...*
>
> *If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other (with a friend as witness, if they liked): let us calculate.* (Russell, 1900, p. 170)

The ideal of a *Calculus Ratiocinator* was first realized by George Boole in 1847 in his *The Mathematical Analysis of Logic* (Boole, 1848), although Boole's algebraic calculus of thought is better known from the more mature exposition he gives of it in *An Investigation of the Laws of Thought* (Boole, 1854). Boole did not seem to have known about Hobbes and Leibniz's dream (Peckhaus, 2009). The logicians who followed Boole, those now known as the logicians of the algebraic tradition, found a way to symbolize term-logic and used this symbolization to calculate arguments with. This turned syllogistic reasoning into something a machine could do, a machine which was quickly built by William Stanley Jevons who tells of its successful construction in *The Substitution of Similars* (Jevons, 1869, pp. 55-60).

Peano's symbolism

Peano, however, put less emphasis on the *Calculus Ratiocinator*, and instead tried to develop a *Characteristica Universalis*. According to Peano, ordinary language was full of ambiguities. He therefore invented an artificial language which did not suffer from these defects and which could, like mathematics, be read by anyone who knew the notation regardless of the languages he or she commanded. And it was clarity and precision that was the main purpose of his most famous work, *Arithmeticis Principia,* in which Peano set forth the axioms now known as the *Dedekind-Peano axioms* (Peano, 1889, pp. III-V).

His symbolic script also formed the basis of his project of the *Formulaire de Mathématique*, which Peano started in 1895. The *Formulaire* was meant as an encyclopedia of mathematical theorems and their proofs, in the unambiguous symbolic language he had invented (Peano, 1895, p. IV). The main purpose of symbolization was to give a precise analysis of the concepts involved in these theorems, thereby adding precision and rigor to the exposition of the theory. Peano believed that symbolization was impossible when ideas were still obscure (Peano, 1895, p. IV).

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[13] What follows is Russell's translation which seems widely used. Russell's anecdote actually consists of two unrelated parts which were merged by him. Although it is clear in Russell's original text that these are two distinct quotes, the quote as a whole seems to have started a life of its own.

The first part of the quote is from a letter Leibniz sent to Jean Gallois in 1677 and is written in French. The second part is from an unpublished essay by Leibniz written in Latin. The originals read:

> *Car si nous l'avions telle que je la conçois, nous pourrions raisonner en metaphysique et en morale à peu pres comme en Geometrie et en Analyse,* […] (Leibniz, 1890, p. 21)
>
> *Quo facto, quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistat. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo (accito si placet amico) dicere: calculemus.* (Leibniz, 1890, p. 200)

Although Peano aimed to create a philosophical language, he realized that the meaning of his symbols were still in a sense conventional (Peano, 1895, p. IV).[14] Simple symbols were assigned their meaning either via stipulation or via definition. The primitive simple symbols stood for simple ideas, and got their meaning assigned to them via ordinary language. A definition of a non-primitive symbol consisted in the substitution of a simple symbol for a complex idea symbolized by a complex of symbols whose meaning was already fully determined. Peano insisted that defined ideas did not exist independently of any mind as an entity, i.e. they were not real, and Peano called that which could still be analyzed a pseudo-entity. As such, definitions were not truly a part of mathematics but existed either as historical facts or because of the author's will (Peano, 1901, pp. 6-7).

Peano knew from Couturat that Leibniz had also tried to reduce logical operations to a calculus and realized that this was different from Leibniz' project of a Characteristica Universalis. But Peano always emphasized that his symbolic logic should be seen as a precise and rigorous language, not as a calculus. Peano's symbolism even technically fails as a logical calculus because Peano had confounded implication with deduction. In particular, Peano's logic does not contain any rules of inference, instead, his proofs merely consisted of symbolic script written one under the other. Of course, Peano takes it that the reader will accept that if $p \supset q$ and p are given, q follows from necessity, but he never explicitly states this rule of deduction and as such his deductive system is susceptible to the argument that Lewis Carroll gives in his *What the Tortoise said to Achilles* (Carroll, 1895), as Russell realized in the *Principles* (Russell, 1903a, p. 35).

Conclusion

It is perhaps correct to say that the algebraic tradition to which Boole, Charles Saunders Peirce and Schröder belong put more emphasis on the idea of a *calculus* of thought, and less on the notion of a universal characteristic, which was the main emphasis of Peano's symbolism, although the matter is more complicated for Schröder. In response to Schröder's remark in the mostly favorable review of the *Begriffsschrift* that Frege's *Begriffsschrift* was more of a *Calculus Ratiocinator* then a *Universal Characteristic* (Schröder, 1880, p. 82) Frege attacks the algebraic tradition, personified by Boole, in his *Ueber den Zweck der Begriffsschrift* (Frege, 1882/1883) for being just that, a pure *Calculus Ratiocinator*. He then claims that Schröder has misunderstood his purpose, and that the *Begriffsschrift* instead was designed to express content in a more precise and clear manner then was possible in ordinary language and therefore was a Universal Characteristic. But in the 1890's Schröder's view of algebraic logic was clearly linked to the project of finding a philosophical language which depicted the true structure of the world, as is clear from his contribution to the First International Congress of Mathematicians which was held in Zürich in 1897 (Schröder, 1898). Schröder there even likened the different kinds of logical symbols to categories in the Kantian and Aristotelian sense.

After Russell met Peano he immediately adopted Peano's symbolism, together with the crucial idea that a correct symbolic representation of the concepts of mathematics at the same time consisted in an analysis of its concepts. Although it was Moore who convinced Russell that all philosophy was analysis, it was Peano who showed Russell how to actually analyze the concepts of mathematics. Finally, Russell could understand what mathematics was about and what mathematical concepts were. In the last months of the 19th-century he

[14] Note that Peano's symbolic language is a fully interpreted language.

quickly finished the draft of a book which analyzed all of mathematics using only the primitive ideas which were suggested to him by Peano's symbolic analysis, writing an impressive 200,000 word draft in three months. He now called his book *the Principles of Mathematics*.

Chapter 4: The logicist analysis of mathematics

Driven by Moore to accept that all philosophical progress was due to analysis, a view which Russell would never again abandon (Russell, 1959, pp. 14-15), and armed with Peano's logical symbolism, Russell set out to give a new analysis of mathematics. The main aim was to see what primitive concepts and propositions were needed to be able to deduce the whole of mathematics. He would describe this wonderful time in his *My Philosophical Development* as an intellectual honeymoon:

> *Every day I found myself understanding something that I had not understood on the previous day. I thought all difficulties were solved and all problems were at an end.* (Russell, 1959, p. 73)

Russell would argue that all mathematical concepts and propositions could be understood to be complex concepts and propositions of logic respectively. As such, mathematics was about the most general structure of the universe. This thesis is now known as logicism. But before Russell could fully analyze all mathematical concepts he first had to upgrade Peano's sail ship with the steam engine of Peirce and Schröder's logic of relations.

Adding the Theory of Relations:

It is a mystery why Peano does not have a theory of relations, even though he sometimes makes use of them, since he was familiar with the logic of relations as it had been developed by Peirce and Schröder (Grattan-Guinness, 2000, p. 267). Schröder had already argued against Peano that his algebraic notation was superior because Peano could not handle relations and he characterized using Peano's symbolic logic as using a sailboat when steamships were already invented (Schröder, 1898, p. 161). But Russell would give Peano's sailboat an upgrade.

Russell had already understood the importance of relations in his book on Leibniz, stating that we should reject the doctrine that all propositions are of subject-predicate form (Russell, 1900, pp. 12-15). Russell would always see it as Leibniz's greatest defect that he did not dare challenge the Aristotelian doctrine that all propositions were of subject-predicate form even though he had good reasons to believe this, and he claimed that Leibniz would have invented symbolic logic 200 years before Boole had he dared to do so (Russell, 1993b, p. 369).

In the *Principles of Mathematics* Russell explains why this is such a defect. According to Russell, the view that all propositions about relations can be analyzed in terms of subject-predicate propositions either leads to *Monism*, in which all predicates are about a single subject and relations between objects are analyzed as properties of the whole of reality, or to a pluralistic *Monadism*, where, although there are many subjects, each and every subject must contain all the relations it has to other monads within itself as a property. However, both of these theories were false according to Russell because neither one of them could account for asymmetric relations, which are crucial for the understanding of mathematics (Russell, 1903a, pp. 218-226). Russell needed these asymmetric relations to be real, i.e. exist independently, in order to be able to analyze the crucial notion of *order*. Hence his claim that, in contrast to Moore, he was more interested in the rejection of monism than in the rejection of idealism (Russell, 1959, p. 54). In the lecture *Analytic Realism* Russell even explicitly claims that this logical pluralism and the independent existence of relations is the most fundamental part of his new doctrine, the one we now know as *Logical Atomism* (Russell, 1911).

In 1901 Russell recasts Peirce and Schröder's theory of relations in terms of Peano's symbolism in a paper written in French called *Sur la logique des relations avec des applications à la théorie des séries* (Russell, 1901),[15] which was published in Peano's journal *Revue de Mathématiques*. The theory of relations had been developed by Peirce in the second half of the 19th-century (Grattan-Guinness, 2000, p. 149), and had been taken over by Schröder and discussed in detail in the third part of his *Vorlesungen über die Algebra der Logik* (Schröder, 1895). These lectures were published at Schröders own expense and were never actually given since Schröder did not have a teaching position at a university but worked as a mathematics teacher at the Polytechnic school in Karlsruhe (Grattan-Guinness, 2000, p. 161).

Schröder had dealt with relations in terms of what we would now call ordered pairs, and a relation was expressed in terms of a sum of pairs, e.g. the relation a was expressed as $\sum_{ij} a_{ij}\,(i : j)$. But what is wrong with the symbolism of Schröder, according to Russell, is that his notation is cumbersome and he found it practically unfeasible, even if it was philosophically correct (Russell, 1901, p. 115). Russell remained agnostic on whether Peirce and Schröder had a philosophical acceptable account of relations, although he believed that they did not accept the real existence of relations because they still tended to think of relations as classes of couples (Russell, 1903a, p. 24). He found Peano's notation, on the other hand, to be elegant even though it lacked a correct rendering of relations. He therefore quickly added the calculus of relations to Peano's system. But the revolt into pluralism due to Moore had given Russell reason to believe that relations were primitive ideas and this allowed him to assign relations their own symbol, e.g. $x\boldsymbol{R}y$, instead of dealing with them as classes of couples. This greatly simplified the technical handling of relations.

The analysandum; Peano's axioms

Russell's logicist analysis of mathematics heavily relies on the work that had been done in the 19th-centrury by German mathematicians. Russell describes these developments in his article *Recent work on the principles of mathematics* (Russell, 1993a). One crucial part of this analysis is now known as the arithmetization of analysis. During this period the concept of *limit* was given a precise meaning in order to get rid of the unintelligible concept of the *infinitesimal*.

Before the arithmetization, analysis had been mainly grounded on geometry, using the vague concept of the infinitesimal, which was a quantity that was infinitely small but not zero. Bishop Berkeley would attack this concept vehemently in *The Analyst* as completely unintelligible, characterizing infinitesimals as "Ghosts of departed quantities" (Berkeley, 1734, p. 59). One reason for this being that in calculations of derivatives using infinitesimals, infinitesimal terms like "dx" start out as non-zero but departed as zero at the end, as we can see, for instance, in the following "old-school" calculation of the derivative of x^2:

$$\frac{d}{dx}(x^2) = \frac{(x+dx)^2-x^2}{dx} = \frac{(2x+dx)(dx)}{dx} = 2x + dx,$$

[15] A translation can be found in the third volume of the Collected Papers as *The Logic of Relations with Some Applications to the Theory of Series* (Russell, 1993b).

which, since dx is infinitely small, is equal to $2x$. In this calculation dx, although infinitely small, cannot be 0 during the calculation, since one is not allowed to divide by zero. But immediately after the calculation it is treated as if it is zero.

These problems with the concept of the *infinitesimal* disappeared after Karl Weierstrass rigorously defined the concept of a *limit* which does not need to make recourse to such a problematic entity as the infinitesimal. Russell highly valued Weierstrass's $\varepsilon - \delta$ definition of the limit, which was anticipated by Bernard Bolzano and Augustin Louis Cauchy.

The concept of *limit* can be fully stated using concepts of logic and arithmetic alone:

$$\lim_{x \to c} f(x) = L \qquad =_{df} \qquad \forall \varepsilon > 0\ \exists \delta : \ \forall x (0 < |x - c| < \ \delta \ \supset |f(x) - L| < \varepsilon).$$

The derivative of the function $y = x^2$ could now be calculated using this concept to be:

$$\frac{d}{dx}(x^2) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{(2x+h)(h)}{h} = \lim_{h \to 0} (2x + h) = 2x$$

This derivation makes no recourse to any infinitesimal quantity dx.

Furthermore, Richard Dedekind and George Cantor had analyzed two other problematic concepts in terms of set theory, the concepts of *continuity* and *infinity* (Russell, 1993a, p. 370).

A third important part of the analysis was the reduction of the different kinds of numbers to the natural numbers. Dedekind had defined the real numbers in terms of the rational numbers by defining a real number as the set of rational numbers which was determined by a cut in the rationals. The rational numbers could in turn be defined in terms of relations between integers, which in turn could be defined in terms of natural numbers only. This led Leopold Kronecker to his famous claim that God had made the natural numbers while everything else was the work of man. This meant that at the end of the 19th-century a very large part of mathematics was analyzed in terms of arithmetic, natural numbers and set-theory.

At the end of the 19th-century Peano successfully axiomatized arithmetic in his famous little booklet *Arithmetices Principia* (Peano, 1889) and managed to state the theory of arithmetic in an early form of his Characteristica Universalis. But Peano was not a logicist. Instead, Peano believed that mathematics had its own primitive concepts, and therefore had primitive mathematical symbols as well as logical ones.

Peano postulated four primitive concepts of arithmetic, the concept of *number* (N), *one* (1), *successor* $(a + 1)$ and *identity* $(=)$. The *Arithmetices* even explicitly begins with two tables in which the two different kinds of signs are distinguished. The logical part of the *Arithmetices* consisted of a theory of propositions and a theory of classes similar to Cantor's set theory, to which Peano added a notion of *such that*, which Peano symbolized as $[x \in]$. Peano used these simple concepts to express nine primitive propositions which were sufficient to demonstrate the whole theory of arithmetic. Here I will give them in Peano's original notation (we would now symbolize "Ɔ" as "→" and "−=" as "≠"):[16]

1. $1 \in N.$
2. $a \in N . Ɔ . a = a.$
3. $a, b, c \ \in N . Ɔ : a = b . = . b = a.$
4. $a, b \ \in N . Ɔ \therefore a = b . b = c : Ɔ . a = c.$
5. $a = b . b \in N : Ɔ . a \in N.$
6. $a \in N . Ɔ . a + 1 \in N.$
7. $a, b \ \in N . Ɔ : a = b . = . a + 1 = b + 1.$
8. $a \in N . Ɔ . a + 1 -= a$
9. $k \in K \therefore 1 \in k \therefore x \in N . x \in k : Ɔ_x . x + 1 \in k :: Ɔ . N \, Ɔ \, k.$

Dedekind had already formulated similar axioms in his *Was sind und was sollen die Zahlen* (Dedekind, 1888, p. 20), as Peano acknowledges in the preface of the *Arithmetices*, and the axioms have become known as the Dedekind-Peano axioms. However, as Grattan-Guiness mentions, what most commentators have failed to realize is that, although Peano was aware in 1889 that his axioms were similar to those that had been given by Dedekind, he had discovered them independently (Grattan-Guinness, 2000, p. 228).

The proof of Russell's logicist thesis now basically rested on showing that Peano's mathematical concepts could further be analyzed in terms of the simple concepts of logic and that the Peano-axioms could be demonstrated using only propositions of logic.

Analyzing the primitives: number, zero and successor

Although Peano's axioms made recourse to the primitive notion *one*, it is customary to start with *zero*, and this makes no essential difference to the Peano-axioms. Russell therefore had to give an analysis of the concepts of *number, zero, successor* and *identity*. But unlike Peano, Russell did not make a distinction between the logical and mathematical concept of identity and simply analyzed the concept of identity in terms of Leibniz' law:[17]

$$a = b \, . =_{df} . \forall \varphi (\varphi a \supset \varphi b . \varphi b \supset \varphi a).$$

As was mentioned, Russell mainly took over Peano's theories of propositions and of classes, which contains the primitive concepts of Cantor's set-theory, although surprisingly, Russell does not take classes to be primitive in the *Principles*, nor does he believe that the null-class exists as a primitive. Instead, Russell analyzes the theory of classes in terms of propositional functions by taking the notion *such that* (∋) and *is an element of* (∈) to be primitive, although in the principles he doubts the philosophical correctness of this (Russell, 1903a, p. 19). This essentially meant that Russell's theory of classes depended on Cantor's comprehension principle, which basically states that any thinkable propositional function defines a class of objects which satisfies this function (Cantor, 1895, p. 481).

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[16] From a modern perspective we can clearly see the development that symbolic logic has made with respect to the use of certain symbols. Note that Peano uses both the logical "=" and the primitive mathematical "=" together in the axioms below, which might be confusing. Furthermore, "Ɔ" is used both to symbolize implication and superset. Lastly, it might be confusing that Peano uses a small k to stand for a class variable, which we would now normally depict using a capital letter.
[17] Russell replaced Peano's "Ɔ" with the horseshoe.

The class of Number, i.e. N, was analyzed as a class of classes. Peano himself had already analyzed the concept of number himself in a later work as the property which all classes which were similar had in common (Peano, 1901, p. 70). Two classes are similar if there is a one-one correspondence between their terms.[18] This relation of similarity is reflexive, symmetric and transitive and as such defines an equivalence class. Like Peano, Russell believed that numbers were properties of classes (Russell, 1903a, p. 113), but Russell criticized Peano's analysis of *Number* because he believed that it did not define a definite object, instead there are an infinite number of properties which could be the concept of *Number* (Russell, 1903a, p. 114). Instead, Russell had a definite class in mind which defined the concept of *Number*. He defined the number of a class as *the class of classes which are similar to that class* (Russell, 1903a, p. 115). This definition is now known as the Frege-Russell definition of number. Since Russell had not understood any Frege when he developed this definition it is generally believed that Russell came up with this definition independently. However, it seems probable that Russell might have gotten this idea via Peano, who had actually read Frege. Peano explicitly rejected the Frege-Russell definition of *Number* in the *Formulaire* because he claimed that numbers and classes of classes obviously have different properties (Peano, 1901, p. 70). Russell in turn explicitly dismissed Peano's rejection because he could not see what this obvious difference was, which indicates that he knew of Peano's critique on Frege (Russell, 1903a, p. 115).

The concept of *zero* could now easily be defined. It is the class of classes which are similar to the null-class (Russell, 1903a, p. 128). However, Russell did not believe, unlike Zermelo, that the null-class existed as a primitive concept. Instead, Russell needed a propositional function in order to define the null-class. In the *Principles* Russell says that any function which is false for all values of x will do (Russell, 1903a, p. 23). In *Principia Mathematica* Whitehead and Russell will use the propositional function $x \neq x$ for this, thereby defining *zero* as the class of classes which are similar to the x such that $x \neq x$, following Frege.

All that was left now was to analyze the concept of *successor,* i.e. $a + 1$. Russell defined this concept as the class which was similar to the union of any member of a with an x which is not a member of the class of a which was chosen (Russell, 1903a, p. 128).

This analysis of the concepts of arithmetic in terms of logical concepts showed Russell that there were two very important ideas which were crucial for the analysis of mathematics. Both can be expressed by the expression "*the* x*…".* In his *Introduction to Mathematical Philosophy*, Russell calls these the concepts of *the* in the *singular* and in the *plural* (Russell, 1919, p. 167) and he considers them to be the key-concepts within the philosophy of mathematics. As we have seen, these two ideas were necessary since *the* in the plural defines classes (in symbols $x \ni \varphi x$), while *the* in the singular (in symbols $\imath x \varphi x$) defines a definite object which is needed for the theory of functions. Without these concepts the logicist analysis would fail. However, as we will see, Russell realized that these complex concepts were special. Propositions containing these complex concepts were not *about* these concepts. Instead, these concepts somehow meant, or denoted, other concepts. In the principle Russell calls these concepts *denoting concepts*.

[18] Peano does not actually give this relation a name, the name "similarity" is probably due to Russell. Instead Peano says that a and b are the same number if there exists "une correspondance réciproque entre a et b" (Peano, 1901, p. 70).

Demonstrating the primitive propositions

Russell did not need to give an analysis of the 2nd, 3rd, 4th and 5th axiom, which govern the behavior of the concept of *identity*, since they were already a part of his logical theory. Nowadays these axioms are generally omitted when the Peano-axioms are discussed. That *zero* was a number was easy to demonstrate, because *zero* was defined to be a number. That the successor of a number was a number followed from the fact that the successor of a number determined an equivalency class under the similarity relation. Since *zero* is the class of classes which are similar to the null-class, it is easy to prove that *zero* is not the successor of any number because if *zero* would stand in the relation of *successor* to any other class of classes this would imply that one is able to remove a term from a class which is a member of *zero*, which is impossible.

That the successor of a number is always different from that number could only be proved if there were infinitely many individuals, something which Russell believed that he could easily prove in the *Principles* because Cantor had shown that there were infinitely many different classes and classes were individuals. I will discuss this proof in the next chapter. However, the theory of types will make this proof unavailable, because it will demand that classes are not individuals, and in *Principia* Whitehead and Russell would need the *axiom of infinity* in order to demonstrate this proposition. The final axiom, the axiom of induction, was actually not a proposition which needed to be demonstrated, according to Russell, but a definition, defining the concept of hereditary classes. These are classes that can be ordered using an asymmetric transitive relation.

Apart from the two crucial denoting concepts we see here that Russell's logicist analyses of mathematics requires the real existence of relations. On the one hand one-one relations were crucial to analyze the concept of *similarity.* On the other hand Russell required asymmetrical transitive relations such as the *ancestor* relation in order to define the concept of mathematical induction. This he took as the crucial reason why *Monism* and *Monadism* were mistaken. These asymmetric relations could not be expressed by propositions of subject-predicate form. *Monism* could only treat these relations as properties which either hold or do not hold of the whole, thereby making it impossible to treat the relation as asymmetric, while *Monadism* reduces the relation to two unrelated properties of two different objects and thereby cannot show the crucially asymmetric *connection* between the relata.

The Conceptual Realism behind the logicist analysis

Russell recounts that after his revolt into pluralism he started out as a complete Platonist (Russell, 1959, p. 62). However, by the time he wrote the *Principles* his method of analysis had shown him that a lot of the entities he had previously taken to be primitive, such as numbers, actually only existed as relations between concepts. These complex concepts Russell came to call *complexes*.

Although the picture will become more complicated after Russell developed the *Theory of Descriptions* and the *Multiple Relation Theory of Judgment*, a *complex* is basically a complex concept in the sense that we have seen in Moore's *The Nature of Judgment* and Russell explicitly refers to Moore's notion of the concept in the *Principles* (Russell, 1903a, p. 44). But Russell would make the simple picture painted by Moore slightly more complicated. He called whatever was symbolized, whether simple or complex, a *term* (Russell, 1903a, p. 43) and he distinguished between two types of terms, *things* and *concepts* (Russell, 1903a, p.

44). *Things* are concepts which only exist in complexes as logical subject and are denoted by proper names, all other *terms* are concepts.[19]

Combining Moore's view of *concepts* with Peano's idea of a Universal Characteristic, a *complex* is a *term* that is expressed by a complex symbol, and each simple symbol of this complex symbol stands for a primitive concept, e.g. "φa" symbolizes the complex concept consisting of the object a and the concept $\hat{x}(\varphi x)$, and "aRb" symbolizes a complex concept made up from a, the relation R (as relating) and b.[20] And, due to his realism, all these complexes existed independently of any mind.

Propositions were complex concepts which had a special property. Russell did not take the property of *being a proposition* to be a primitive property of a complex, instead Russell defined propositions to be all those complexes which had the *implication relation*, which was a primitive, to themselves (Russell, 1903a, p. 15). However, following Moore, Russell did take the *truth* or *falsehood* of propositions to be an indefinable property of propositions. As such, all propositions, both true and false ones, were *terms* and as such existed independently of any mind. Perhaps surprisingly, it is this doctrine that all propositions have *Being* which is one of the main differences between Russell's theory of *complexes* and Meinong's *Gegenstandstheorie*, as was emphasized by Russell himself in his article *Meinong's theory of Complexes and Assumptions* (Russell, 1904, p. 58). In contrast to Russell, Meinong had denied that false propositions have *Being*.

Because of his conceptual realism Russell believed that it is always false to deny *Being* of any term. And *any* term could be spoken about, because *all terms* formed complexes with *all* concepts. This meant that Russell believed that *any* term could be the argument of a propositional function. Given that the propositions of logic and mathematics were about the most general structure of the universe, the variables that were employed in its propositions had to be completely unrestricted, i.e. Russell needed to have quantification over the entire universe. According to Russell a variable is nothing but an ambiguous denotation. As such, φx denotes a class of propositions, namely the class of all propositions such that $\varphi a, \varphi b, \ldots$ for *all* terms $a, b, \ldots$ (Russell, 1903a, p. 93). Russell did realize that variables could not be pure ambiguity, since variables clearly have some form of individuality since xRy expresses a different relation from xRx. Russell therefore says that propositional functions need to be attained in order although the order in which this is done is unimportant, e.g. aRb could be formed into xRy by first substituting x for a and then y for b, or vice versa. Russell realized that his account of the variable was ultimately unsatisfactory but he had nothing better to offer (Russell, 1903a, p. 94).

However, since Russell was a conceptual realist, this meant that *every* term had to be able to be the value of the variable. This meant complete quantification not only over *things* (i.e. *individuals)*, but also over *propositional functions, relations*, *classes* and *propositions*. As we now know, this doctrine is the main cause of the logical paradoxes.

[19] In *Principia* things are called "individuals".
[20] Wittgenstein will criticize this view of relational complexes in the *Tracatatus Logico-Philosophicus* (Wittgenstein, 1922, 3.1432). Wittgenstein changed Russell's metaphysical picture by denying the existence of relations and removing the distinction between *things* and *concepts*, subsuming them both under the notion of *object*. In contrast to Russell's view in the *Principles*, not all Wittgensteinian objects combine with one another, since this depends on their form. This is why "$\varphi(\varphi)$" and "RRR" do not symbolize *complexes*, or *sachverhalten*, in the Tractatus. They do, however, in Russell's theory of the *Principles*.

Russell also quickly understood that concepts did not all function in complexes in the same way. Not all propositions were *about* their grammatical subject. “I met a man” is not about the concept *a man*, which, although it has Being, does not have a wife or a bank account. Nor does “any number is odd or even” mean that the concept *any number* is odd or even (Russell, 1903a, p. 47). Instead, these propositions were about other concepts[21] which were *denoted* by these concepts, i.e. their *logical* subject. Russell calls these kinds of concepts *denoting* concepts.

The existence of these special kinds of concepts was one of the two fundamental things he claimed to have learned from adopting Peano’s symbolism, and the existence of denoting concepts is particularly clear in quantification theory (Russell, 1959, pp. 66-67).[22] Independent of Frege, who has priority but who’s theory was unknown to Russell when he developed these ideas, Peirce had developed a theory of quantification and this theory had been taken over by Schröder (Grattan-Guinness, 2000, p. 181). But Peirce and Schröder’s theory of quantification works by first restricting the domain of discourse, and then asserting that all, or some, of the entities in this domain have a certain property.

According to this view quantification is basically of subject-predicate form: a certain predicate is asserted of a *class* of objects. “All men are mortal” is asserted by first forming the class *all men* and then predicating mortality of it. Peano, however, always quantified over the entire universe. He therefore analyzed all statements to be of the form *if x is a φ, then x is a ψ* (in Peano's symbolism: $\varphi x \supset_x \psi x$). Russell believed Peano’s theory to be superior to that of Peirce and Schröder because he believed that the primitive propositions of logic, and those of pure mathematics as well, were of this form (Russell, 1903a, pp. 3-8). Since propositions of logic and pure mathematics were the most *general* propositions, they had to be about the entire universe. Furthermore, Peirce and Schröder’s could not express true propositions about empty classes, e.g. they could not express the proposition that all winged horses are horses, which is obviously an analytical truth (Russell, 1959, pp. 66-67).

These denoting concepts, such as *all* x and *the* x, somehow involved other concepts with the proposition although these concepts were not part of it, as could be directly seen from their symbolization. Russell believed this to be a special relation between the *denoting concept* and the other concept involved which he called the *denoting relation.* Russell explicitly says that denoting concepts themselves have meaning, not just the symbols which stand for them, which Russell considered uninteresting logically. It is not *we*, nor the *symbols* which denote, it is the *denoting concept* itself (Russell, 1903a, p. 53). As we have seen in the last section, two *kinds* of denoting complexes are especially important for Russell, since his analysis of mathematics crucially depends on it: both can be expressed by “the x …”.

As has now become clear, this theory did not necessarily commit Russell to the existence of non-existing objects like the King of France. There was no problem with denoting concepts which did not denote anything. Interestingly, Russell’s early theory seems to have been very

[21] And yes, strange as it may sound, Russell did indeed believe that a particular man was a complex concept with whom we could become acquainted immediately via sense perception. For Russell the perception of a particular man is the direct apprehension of the sense datum $\exists x(Man(x).\varphi_1(x).\varphi_2(x)\ldots)$, which is a complex existential concept (Russell, 1904, pp. 213-218). Here the string of φ_ns stands for whatever properties (and relations) that man is *perceived* to have. Only the man himself is directly acquainted with the value of x.

[22] The other thing being that a singleton, i.e. a class with only one member, is not identical to its only member.

similar to the one given by Peter Strawson fifty years later in his paper *On Referring* (Strawson, 1950). Like Strawson, Russell insisted that propositions containing denoting phrases presuppose that that phrase denotes and that if this presupposition is not fulfilled the proposition is meaningless (Russell, 1994, p. 286). But this theory, unfortunately, does not help against the paradox. Even if a denoting phrase does not denote, the denoting concept *itself* has to exist. Since classes are fully determined by their defining conditions the only reason to suppose that the class of classes which are not members of themselves does not exist seems to be fully *ad hoc* to avoid the paradox.

Conclusion:

At the end of the 19th century all looked well, and Russell believed to have successfully analyzed mathematics in terms of logical concepts. From Bradley Russell had learned that reality was such that it did not contradict itself. Due to Moore, Russell had come to accept that mathematical concepts were real and Peano had taught him how to analyze the concepts of mathematics by making use of his Universal Characteristic, which showed the true structure of complex concepts. Russell could now understand what he had been seeking for years. Mathematical concepts were nothing but complex logical concepts, and mathematical propositions were true propositions about the most general structure of the universe.

Unfortunately, the intellectual honeymoon he had experienced due to his analysis of mathematics could not last. All the ingredients were now in place to form the logical contradiction which would baffle him for years. When he found his paradox, intellectual sorrow descended upon him in full measure (Russell, 1959, p. 73).

Chapter 5: The *real* problem of the paradox

In the first year of the new century Russell discovered the paradox now known as Russell's paradox. After making a thorough study of Cantor's proof that there was no greatest cardinal Russell found an inconsistent denoting complex: the class of classes that are not members of themselves. This meant that his analysis of mathematics implied that there were real inconsistent concepts. But reality was such that it did not contradict itself. Therefore the paradox showed Russell that his analysis of the most general structure of the universe was wrong.

It would take Russell years to figure out how to solve the paradox. Every solution which naturally presented itself seemed to be in contradiction with one of the fundamental principles he held. But an insight by Frege proved to hold the key. Perhaps not all concepts could form complex concepts together. This led Russell to his program of trying to turn all problematic concepts into incomplete symbols, starting with denoting complexes in *On Denoting* (Russell, 1905).

Russell's paradox

The story of how Russell discovered the Russell paradox has been well documented (see (Coffa, 1979) and (Moore G. , 1988)) but there is some doubt about when Russell found the paradox exactly, because Russell himself keeps recalling different moments. In his *Autobiography*, Russell says that he discovered the paradox in May 1901 (Russell, 2010, p. 138) and in *My Philosophical Development* he says that he discovered the paradox in the spring of 1901 (Russell, 1959, p. 75). But he also mentions June 1901 twice in print and in a letter to Jourdain we find that he says that he discovered the paradox in January 1901 (Moore G. , 1988, p. 52). However, it is certain that he had discovered the paradox at least by the 15th of May 1901 (Moore G. , 1988, p. 52). At first he did not realize that he had discovered a paradox (Moore G. , 1988, p. 52), and later he thought it to be trivial (Russell, 2010, p. 138). The problem of dating the discovery of the paradox is probably due to the fact that Russell did not seem to have realized the importance of his discovery until he received a letter back from Frege in 1902, who was devastated by it. In any case, it took him more than a year to tell anyone at all of his discovery (Moore G. , 1988, p. 53).

But what is certain is that Russell found the paradox when he was examining Cantor's proof that there is no largest set, a claim which he would repeat again and again (see for instance (Russell, 1903a, p. 101) and (Russell, 1959, p. 75)). This theorem is known as *Cantor's Theorem*. Russell first came to know of Cantor's Theorem from Arthur Hannequin's book *Essai critique sur l'hypothèse des atomes dans la science contemporaine* (Hannequin, 1895). But Russell did not accept Cantor's Theorem before he found the paradox. Instead, Russell believed that Cantor's argument had to contain a subtle fallacy somewhere (Russell, 1993a, p. 375), because surely philosophical reflection showed that the class of everything, i.e. the class of all terms, had the highest cardinal number.

The thesis that any set is strictly smaller than its power set is known as *Cantor's Theorem*. It was essentially first stated by Cantor in 1891, when he held a lecture at the opening meeting of the *Deutsche Mathematiker-Vereinigung* in Bremen titled *Über eine elementare Frage der Mannigfaltigkeitslehre*. This society had been founded to combat the mathematical hegemonies of Göttingen and Berlin (Grattan-Guinness, 2000, p. 110). Although it is often cited as "(Cantor, 1891)", it was actually published in 1892 in the first proceedings of this society (Cantor, 1892).

In his paper of 1892 Cantor does not state the theorem as we know it today, instead he phrased the argument that there is no largest cardinal in terms of characteristic functions. A characteristic function is a function on a set that assigns either the value 1 or 0 to each of its elements, and as such indicates a subset (Cantor, 1892, p. 77). The manner in which we now know *Cantor's Theorem* is due to Zermelo, as is its name. Zermelo published it as part of his axiomatization of set theory as "32. Satz von Cantor" (Zermelo, 1908, p. 276).

In this theorem Zermelo showed that the power set of a set is always strictly greater than that set. The power set of a set is the set which contains all sub-sets of that set. For instance $\wp(\{a,b\}) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$. Zermelo's proof roughly goes as follows:

> *Cantor's Theorem*
>
> The cardinality of S is smaller or equal to the cardinality of $\wp(S)$, because there is an injective function from S to $\wp(S)$ which sends each element of S to its singleton in $\wp(S)$.
>
> Suppose that the cardinality of $\wp(S)$ is smaller or equal to the cardinality of S. Then there is an injective function $f: \wp(S) \longrightarrow S$. We now define a subset of S which we call C such that x is an element of C if and only if x is in the range of f and x is not an element of $f^{-1}(x)$. By definition C is a subset of S, so C is an element of $\wp(S)$. Since f is an injective function from $\wp(S)$ to S there must be an element x of S such that $f^{-1}(x) = C$. Either this x is an element of C or it is not. Suppose that x is an element of $C.$ Then x is an element of $f^{-1}(x)$ and therefore x is not an element of C. But if x is not an element of C then x is not an element of $f^{-1}(x)$, and therefore by definition of C x is an element of C. Both cases are impossible. Therefore there is no injective function from $\wp(S)$ to S, which means that the cardinality of $\wp(S)$ is not smaller or equal to the cardinality of S.
>
> Since S does not have bigger or equal cardinality as $\wp$(S), S has strictly smaller cardinality than $\wp(S)$.

Since any set has a power set this means that *Cantor's Theorem* proves that there is no largest cardinal number.

In *Recent work in the philosophy of mathematics* Russell states that he believed that Cantor's argument had to contain a subtle fallacy (Russell, 1993a, p. 375). In a letter Russell wrote to Couturat he explains that he has found a counter-example to Cantor's argument, the class of classes. Let us call the class of classes CLS. According to Russell it is clear that since the power set of the class of classes contains only classes, $\wp(CLS)$ is a subset of CLS, thereby invalidating Cantor's argument.

However, in a later reprint of this article as *Metaphysics and the mathematicians*, which was published in *Mysticism and Logic*, Russell admits that Cantor's argument was not fallacious. Instead, the mistake was his own (Russell, 1993a, p. 375).

Cantor's theorem shows that CLS is strictly smaller than $\wp(CLS)$. But how is this possible? Since all elements of $\wp(CLS)$ are classes, every element of $\wp(CLS)$ is an element of CLS. We should be able to define a one-one function from $\wp(CLS)$ to CLS by simply corresponding all classes of $\wp(CLS)$ to themselves in CLS. We therefore take $f(x)$ to have the value x for all values of $f(x)$ and plug it into *Cantor's Theorem*. Why can't C be correlated to C? Substituting CLS for S, $\wp(CLS)$ for $\wp(S)$ and x for $f(x)$ in the proof of *Cantor's Theorem* we get:

> *Russell's paradox*
>
> We define a subset of CLS which we call C such that x is an element of C if and only if x is not an element of x. We ask whether C is an element of itself. There are two possibilities. C is an element of C or it is not. Suppose that C is an element of itself. Then C is an element of C and therefore, by definition of C, C is not an element of C. But if C is not an element of C then by definition of C, C is an element of C.
>
> Therefore C is an element of C if and only if C is not an element of C.

This was the paradox that would baffle Russell for years.

In the *Principles* Russell had not taken classes to exist as primitives concepts, instead they were defined by denoting phrases which used a propositional function to state a condition of membership. He therefore looked at the propositional function he used in forming C. This led him to a second way of formulating a contradictory concept, the propositional function "*x is not predicable of x"*. Let us call this predicate, which is a function of x, $\hat{x}(Rx)$. Now, remember that Russell believed that the variable was unrestricted. Since $\hat{x}(Rx)$ is a primitive term and therefore has *Being*, $\hat{x}(Rx)$ is a possible value of $\hat{x}(Rx)$. Now, is $R(\hat{x}(Rx))$ true or false? Clearly $R(\hat{x}(Rx))$ is false, because otherwise it would be predicable of itself. But that would mean that it is not predicable of itself and $R(\hat{x}(Rx))$ is therefore true. A second contradiction emerged.

Russell was the first to publish the paradox now known as Russell's Paradox. He did so in 1903 in *The Principles of Mathematics* (Russell, 1903a, pp. 101-106). But Russell actually does not have priority finding this paradox. Russell's paradox was independently discovered by Zermelo in 1899. After Frege told David Hilbert of Russell's paradox, Hilbert replied that Zermelo had found it independently and had priority by three or four years (Grattan-Guinness, 2000, pp. 216-217). He seems to have found the paradox quickly after Hilbert convinced him of studying set-theory, but he never published it and would always refer to it as "*Russell'*s paradox". Zermelo only claims independent discovery of Russell's Paradox in print once, in his *Neuer Beweis für die Möglichkeit einer Wohlordnung* (Zermelo, 1908, pp. 118-119). Zermelo's independent discovery was not fully realized until the 1970s, when a note by Husserl was found in Husserl's own copy of his review of Schröder's *Vorlesungen* where Husserl recalls that Zermelo told him in April 1902, a year before the *Principles* was published, that: "the set of all sets which do not contain themselves as elements […] does not contain itself as element" (Grattan-Guinness, 2000, pp. 216-217).

After Russell published his paradox a whole industry of finding paradoxes emerged. The paradoxes of the greatest cardinal and the greatest ordinal, which are intimately connected to Russell's Paradox, had already been found in *Mengenlehre* by Cantor himself and Cesare Burali-Forti, who was a student of Peano, respectively. Although these paradoxes seem to have worried Cantor, Cantor reacts to this discovery by making a distinction between the transfinite and the absolutely infinite (Cantor, 1967). Cantor believed that absolutely infinite sets, like the set of all ordinal numbers, could not be understood by the finite human mind and were therefore not a part of his *Mengenlehre*. Russell understood this solution, but would never accept it. He believed this solution to be fully *ad hoc,* since according to Cantor, the absolutely infinite sets were precisely those that were contradictory.

The fact that these contradictions could be formed in the deductive system he had sketched in the *Principles* was of course a technical problem for Russell. One of the propositions of logic that was deducible from his primitive propositions was the principle of explosion, i.e. $(p. {\sim} p) \supset q$, which is also known as *ex falso sequitur quodlibet*. This meant that any proposition could now be proved to be true, even propositions such as $1 + 1 = 3$, thereby trivializing the deductive system. This was clearly unacceptable. But Russell does not even mention this problem that the paradoxes posed for his logicist analysis of mathematics.

Instead, Russell emphasizes another problem that the contradiction poses for him. According to Russell the paradox seems to show that not all class concepts define classes, i.e. that Cantor's principle of comprehension is false. But it gives absolutely no reason to suppose *why* this is so (Russell, 1903a, p. 102). Zermelo would solve the technical problem that the paradox posed by accepting that Cantor's comprehension principle was false, and he gave a new comprehension principle, although he explicitly states that for the moment he will not discuss the philosophical question concerning the origin and validity of his new principle, which he considered to be deep problems (Zermelo, 1908, p. 262). But Russell could never accept such a temporary *ad hoc* solution. What Russell wanted was to understand what mathematical concepts *were*, and *why* mathematical propositions were true. This is why the discovery of the paradox shook Russell to the core. It showed him that his logicist analysis of mathematics was wrong.

In *My Philosophical Development* Russell recounts that any solution to the paradox had to abide by three conditions. It had to remove all contractions, it had to save as much of mathematics as possible, although Russell mentions that this is not logically compulsive, and it had to make logical sense. Once the solution was found it should seem that this is what one ought to have been expecting all along (Russell, 1959, p. 79). Russell would therefore never accept any solution to the paradox which he deemed to be *ad hoc*.

Russell's method of analyzing mathematics had fully depended on the idea that his symbolism was a Universal Characteristic and therefore showed the true conceptual content of a complex concept. However, C, i.e. $x \ni (x \notin x)$ uses only three primitive concepts: the unrestricted variable, the concept of *such that* and the concept of *is an*, all of which were crucial to his logicist analysis of mathematics. Russell could not really abandon any of them. Furthermore, the form of Russell's paradox is so simple that it readily seemed to suggest its own possible solutions. Russell believed that the only presuppositions of Russell's Paradox were the comprehension principle and the claim that CLS can be the value of the variable in $x \notin x$ (Russell, 1903a, p. 103). However, both presuppositions seemed to Russell to be absolutely true. There was no reason to suppose that comprehension was false, other than the contradiction, and the fact that logic and mathematics were about the most general structure of the universe suggested that the variable had to be absolutely unrestricted. Although the paradox looked trivial at first, there seemed to be absolutely no philosophically acceptable solution to it. There simply was no reason to suppose that C was not a concept, other than the fact that if it was, it was a contradictory one.

A glimmer of hope: incomplete symbols

Having been baffled by the contradiction for more than a year, Russell decided to contact Frege. He sent a letter to Frege in the hope that Frege would immediately point out Russell's mistake. This letter is now famous. But instead of pointing out the mistake, Frege was devastated by Russell's discovery.

Frege quickly wrote a letter back in which he confesses that the paradox that Russell had found had shaken the basis on which he had intended to build arithmetic, because it had shown that his *Grundgesetze V,* which was his axiom that guaranteed that all propositional functions had an extension, was not a principle of logic. Frege would try to amend his *Grundgesetze* by an appendix to the second volume of his *Grundgesetze* in which he tried to restrict *Grundgesetze V* (Frege, 1903), but this emendation ultimately proved to be problematic as well, since the system is still inconsistent on the assumption that there are two objects (Quine, 1955). After realizing that he could not salvage *Grundgesetze V,* Frege gave up.

However, this letter by Frege contained the key insight which would ultimately be the philosophical basis of the *theory of types*. Frege mentions that his theory in the *Grundgesetze* is immune to the predicate version of Russell's paradox. The reason for this is that Frege had made a distinction between two different types of entities, *objects* and *concepts*, which is Frege's term for a Russellian propositional function. In Frege's theory, propositional functions are incomplete entities, and they can only form a judgment, which for our purposes can be equated with a Russellian complex, if they are completed by an object (Frege, 1969). Incomplete objects did not combine with one another into complexes.

This had at first suggested to Russell that the solution to the paradox should be sought in postulating a hierarchy of objects, each of which formed complexes with certain concepts and not with others. He took his universe to contain a myriad of different type of concepts, such as individuals, classes, classes of classes, classes of classes of classes etc. This would restrict what could be a member of what, thereby restricting the *is an* relation and as such gave a philosophical motivated reason for restricting the variable. It is this theory which Russell tries to work out in the appendix of the *Principles*, and which has become known as the *simple theory of types* (Russell, 1903a, p. 523).

But Russell quickly realized that a universe which contained hierarchies of different types of objects could not be the philosophically correct solution. First of all, the structure of the typified ontology could not be stated in the Characteristica Universalis, which is always fully interpreted, without making recourse to an unrestricted variable. Furthermore, in the appendix of the *Principles* Russell repeats the argument that restricting the domain over which is quantified cannot be the solution to the paradox, because the propositions of logic and mathematics have to be the most general propositions about the structure of the universe. Lastly, typified variables do not solve the semantic paradoxes like the liar unless a hierarchy of propositions is postulated as well. But Russell believed that there was absolutely no non *ad hoc* reason to suppose that a hierarchy of propositions existed (Russell, 1903a, p. 528). Postulating a hierarchy of objects was therefore not the answer.

Russell would therefore first dedicate years trying to find a natural restriction of the comprehension principle. These are now known as his Zig-Zag theories. Only after having worked fruitlessly for years did he find the key insight that he needed. It was his renewed understanding of *denoting phrases* in his paper *On Denoting* (Russell, 1905) which made the possible solution clear to him. The key insight that Frege could offer Russell was not the idea of a stratified ontology, but the idea that some types of concepts were *incomplete* and could only form complex concepts with concepts which were complete. This made him realize that not every combination of symbols of his Characteristica Universalis had to stand for a complex concept, only a fully completed symbol, such as a proposition, needed to. Denoting *phrases* could therefore be understood to be *incomplete symbols.* They did not need to stand for a denoting *complex*.

After *On Denoting* Russell started a program of reducing all the problematic denoting complexes which his Characteristica Universalis had made him believe in into incomplete symbols. The no-class theory, of which the substitutional theory was a version, reduced classes and relations to incomplete symbols and the multiple relation theory of judgment gets rid of the *Being* of unasserted propositions. And this reduction solved all of the logical paradoxes, because all the problematic complex concepts no longer had *Being* and as such could no longer be the value of a variable. No artificial restriction on the variable was therefore needed. After this reduction, Russell was no longer confronted with contradictory denoting concepts and could therefore safely again believe that the concepts of mathematics existed independently of any mind. However, dark clouds were still present. Only asserted propositions lay behind the semantic paradoxes and these could not be reduced to incomplete symbols. Asserted propositions had to be complete. To solve the semantic paradoxes Russell would ultimately need to capitulate and postulate a hierarchy of propositional function. This hierarchy of functions he would call *orders*.

Chapter 6: Conclusion

Summary of the main argument

In this thesis I have tried to defend the claim that the main problem that Russell's paradox posed was that it destroyed his metaphysical understanding of reality and as such Russell could either solve it or return to the idealist metaphysics he had so gladly revolted against.

From an early age Russell had wanted to understand what mathematical concepts *were* and *why* mathematical propositions were true. After he had accepted Moore's revolt against idealism, thereby becoming a conceptual realist, and Peano's method of analysis using mathematical logic, Russell believed that he was finally able to understand that mathematical concepts were nothing but complex logical concepts and mathematical propositions were true because they were propositions about the most general structure of the universe. This is the essence of his logicist analysis of mathematics. However, this analysis convinced him that there had to exist a special kind of concept which was capable of denoting another concept. Russell called these concepts *denoting concepts*.

But the paradox showed him that his logicist analysis of mathematics could not be correct. Some of these denoting concepts, such as the class of classes which are not members of themselves, turned out to be contradictory objects and proved to be in conflict with the fundamental principle of metaphysics which Russell had kept from his idealist past: ultimate reality was such that it did not contradict itself. He had been so close to a real understanding of the nature of mathematics, which he had been longing for since he first studied Euclid, and now it threatened to all slide away. But he would refuse to return to his old idealist view that mathematical concepts were self-contradictory and that mathematics was only a false abstraction of reality. He therefore considered the problems that the paradox posed as a personal challenge and would have dedicated his life to solving it.

Concluding remarks for further study

In this thesis I have tried to show how important it is to study all of the influences on Russell's philosophical development if one truly wants to understand why Russell's philosophy developed itself as it does. Russell's idealist phase is now generally considered to be an important part of Russell's philosophical baggage, and studies have been made in order to understand Russell's relationship with the British Idealists such as Nicholas Griffin's *Russell's Idealist Apprenticeship* (Griffin, 1991) and Peter Hylton's *Russell, Idealism and the Emergence of Analytic Philosophy* (Hylton, 1990). However, understanding Russell's idealist past is not enough for a complete understanding of Russell's philosophy. If a deeper understanding of Russell's development is to be sought, and with it a better understanding of the roots of Analytic Philosophy, then the 19th-century project of developing a Leibnizian Characteristica Universalis must be studied as well.

In general it seems to be the case that contributions in languages different from English are poorly read. If there is ever to be a complete history of Analytic Philosophy, this must change. A proper understanding of the roots of Analytic Philosophy cannot bypass the 19th-century movement of the development of a Characteristica Universalis. A good place to start this program would be by translating all the relevant works into English and creating an anthology of texts on the 19th-century project of developing a Characteristica Universalis. Neither Couturat's book on Leibniz, Schröder's *Vorlesungen* nor Peano's *Formulaire* have been fully translated into English and I believe this is one of the reasons why they are generally not studied. It is even the case that an English translation of Leibniz *Dissertatio de Arte Combinatoria* is difficult to come by. Nor have other important texts been translated such as Adolf Trendelenburg's seminal lecture *Über Leibnizens Entwurf einer allgemeinen Charakteristik* (Trendelenburg, 1856). In this lecture Trendelenburg coins the term "*Begriffsschrift*" as the German alternative for the Latin "*Characteristica Universalis"* (Trendelenburg, 1856, p. 39) and it is certain that Trendelenburg inspired Frege, Schröder and Peano, who we can consider to be the founding fathers of modern symbolic logic (Peckhaus, 2009).

Secondly, although it is well known that the *theory of descriptions* is somehow linked to the paradox, it has remained unclear to many what the exact relation is between the *theory of descriptions* and the *theory of types*. The same goes for Russell's other theories developed during the 1900s, which is particularly true for his *multiple relation theory of judgment*. The analysis given in this thesis suggests that all of these theories were clearly meant to try and deal with the problems that had been raised by the paradox since they all aim for the same thing, reducing something which seems to be a problematic entity into an incomplete symbol.

This might help vindicate Russell's *multiple relation theory of judgment* which is generally considered to be rubbish due to the apparently obvious problem of the unity of the proposition. I do not share this harsh view of Russell's multiple relation theory. The multiple relation theory of judgment is clearly part of Russell's attempted solution to the paradox and it seems to cohere strongly with Russell's other metaphysical views. As such, it deserves a lot more credit than it gets (See also (Griffin, 1985)).

Lastly, this thesis contributes to a better understanding of Russell's ramified theory of types. I believe that a proper understanding of the paradox shows that the theory of types that we find in the *Principia* cannot basically be the simple theory of types we find in the *Principles* to which a theory of orders is added because the theory in the *Principles* postulates a hierarchy of classes and relations, while the *theory of types* in the *Principia* does not have any classes, relations nor unasserted propositions in its ontology instead claiming that these objects only exist as a *façon de parler*. The key to the *theory of types* in the *Principia* therefore has to be found in a complete understanding of the notion of the *incomplete symbol*. This insight might not only prove of worth for a better historical understanding of the *Principia*, but could provide valuable insights into the nature of type theory itself as well.

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