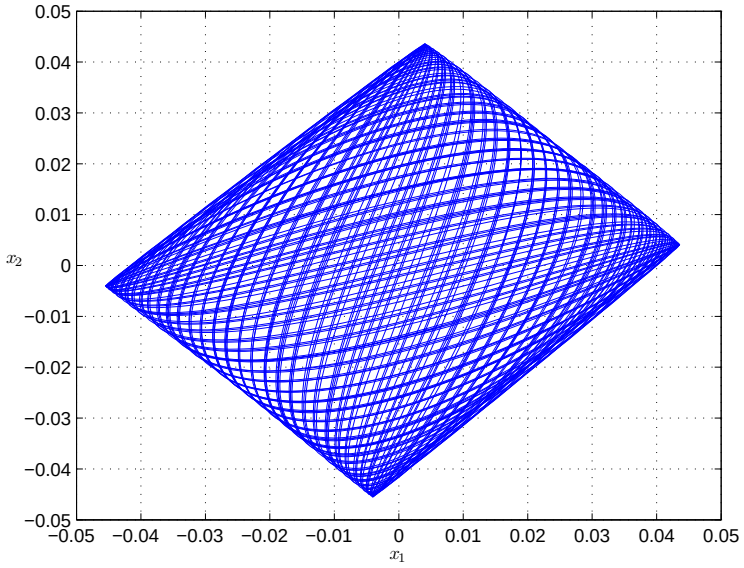
# ON A NORMALIZATION TECHNIQUE FOR CODIMENSION TWO BIFURCATIONS OF EQUILIBRIA OF DELAY DIFFERENTIAL EQUATIONS

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Master Thesis

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# Contents

# Chapter 1

# Introduction

> Nobody realizes that some people expend tremendous energy merely to be normal.
>
> Albert Camus (1913 - 1960)

Deterministic continuous-time models in the sciences often take the form of an ordinary differential equation. When the system under scrutiny is assumed to be free of external forcing, this equation will be autonomous and can be written as

$$\dot{x}(t) \equiv \frac{dx}{dt}(t) = f(x(t), \alpha) \tag{ODE}$$

with solutions $x$ depending on time $t$ and taking values in $\mathbf{R}^n$. Here $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is a smooth vector field depending on an $m$-dimensional parameter $\alpha = (\alpha_1, \ldots, \alpha_m)$.

A bifurcation analysis of (ODE) typically starts by locating its equilibria and then proceeds by analysing how their number and stability depend on the value of a certain one-dimensional control parameter, say $\alpha_1 \in \mathbf{R}^1$, the identification of which often requires good modelling insight. At certain values of $\alpha_1$ bifurcations may occur. For example, equilibria may collide in a fold-bifurcation or may spawn periodic solutions in a Hopf-bifurcation. These are the canonical examples of local codimension-*one* bifurcations: Their occurence depends on the fulfillment of a *one*-dimensional condition, the 'tuning' of a *one*-dimensional parameter.

The character of a codimension one bifurcation may depend on a secondary parameter. An investigation of this dependence requires a two-dimensional parameter space, $(\alpha_1, \alpha_2) \in \mathbf{R}^2$. For instance, at $\alpha_1 = \alpha_{1,c}$ an equilibrium may exhibit a Hopf bifurcation which, in turn, changes from supercritical to subcritical at $(\alpha_1, \alpha_2) = (\alpha_{1,c}, \alpha_{2,c})$. This is an example of a codimension-*two* bifurcation. (We will actually encounter this type of bifurcation in §4.2 in Chapter 4.)

Software such as `CONTENT` [20] and its successor `MATCONT` [4] is currently used to great advantage in the continuation of equilibria of ODE and their codimension-one bifurcations, as well as in normal form analysis at codimension-one/two critical points. We refer to Chapter 10 of [19] for an introduction to numerical continuation techniques for equilibria of ODE, to [17] for an up-to-date survey of numerical pathfollowing and its applications in various contexts, and to [19] for an introduction to applied finite-dimensional bifurcation analysis in general.

ODE models assume an instantaneous effect of the dependent variable $x$ on its rate of change $\dot{x}$. Depending on the modelling context this assumption may not be justified. The mathematical theory of structured biological populations has been one of the driving forces behind a systematic functional-analytic investigation of equations of the form

$$x(t) = f(x_t, \alpha) \tag{RE}$$

or

$$\dot{x}(t) = f(x_t, \alpha) \tag{DDE}$$

where $f$ is a parameter-dependent map from some infinite-dimensional function space $X$ to $\mathbf{R}^n$ that depends on the *history* $x_t \in X$ *at time* $t$ of the unknown $x$,

$$x_t : [-h, 0] \to \mathbf{R}^n, \quad x_t(\theta) \equiv x(t + \theta)$$

Here $h > 0$ is the delay parameter. We make a few remarks to fix our terminology. Note that (RE) is a purely functional equation, not involving differentiation. We will call it a **renewal equation**, also known as a Volterra functional equation. Its counterpart (DDE) shall be called a **delay differential equation**. Both classes form a subset of the larger class DE of **delay equations**. Finally, one often encounters mixed systems of the type (RE) + (DDE). It is natural to regard such systems as instances of DE as well. It has recently been shown in [5] that the functional analytic framework presented in [8] for the analysis of DDE is equally well-suited for dealing with RE and mixed systems. Indeed, on the abstract (semigroup) level, the perturbation theory of dual semigroups (so-called sun-star calculus) truely serves as a unifying device.

In this thesis we restrict our attention to *ordinary* DDE with *finite* delay parameter, i.e.

- we assume that $0 < h < \infty$,
- we work in $X = C([-h, 0], Y)$ with $Y = \mathbf{R}^n$.

Relaxing the first restriction leads to non-compactness of the history interval $[-h, 0]$ which slightly complicates the spectral analysis of linear equations. Admission of more general choices of the space $Y$ enabled the treatment partial delay differential equations or structured

population models with an infinite number of feedback variables or states-at-birth. We mention the paper [6] which discusses into some detail how the framework introduced in [8] and [5] can be adapted to apply to these situations.

Also, the restriction to DDE deserves an explaination.

- Although DDE and RE (as well as mixed systems) are very similar on an abstract semigroup level, we are interested in numerical algorithms and issues of implementation. We feel that this interest is served best by treating DDE and RE separately. We do plan a follow-up paper in which we discuss normalization for the case of RE and mixed RE-DDE systems.

- In contemporary applied mathematics DDE seem to be more prominent than RE and mixed systems. The recent book [11] by Erneux provides an up-to-date overview of the various application areas of time-invariant DDE. These range from biological to optical and mechanical systems in which feedback plays an important role.

- In line with this, software for the continuation of equilibria and periodic orbits of DDE is nowadays publicly available. We mention the `MATLAB` package `DDE-BIFTOOL` developed at the University of Leuven [10]. Also available is the `C++` software `Knut` (formerly `PDDE-CONT`) developed at the University of Bristol [25], [26], but this software is written specifically for the continuation of periodic orbits. At the moment `DDE-BITFOOL` is not capable of detecting local bifurcations and performing normal form analysis, not even for codimension-one singularities. It is here that we hope the methods presented in this manuscript may be of use.

This purpose of this thesis is to 'lift' the normalization method for local bifurcations of ODE presented in [18] and reprinted in Chapter 8 of [19] to the infinite-dimensional setting of DDE. We work out in detail how to compute the *critical* normal form coefficients for all five generic codimension-two bifurcations of equilibria,

- Cusp
- Bautin (Generalized Hopf)
- Bogdanov-Takens
- Fold-Hopf
- Double Hopf

and illustrate our results by means of examples. The formulas we derive are explicit and rather compact. They depend only on first and higher-order (Fréchet-)derivatives of the right-hand side of (DDE), as well as on eigenfunctions pertaining to the critical equilibrium. As we will see, these eigenfunctions can always be represented as finite-dimensional objects without requiring any intermediate discretization or truncation steps.

## 1.1 Structure of this thesis

Whenever we like to stress a particular phrase, we use *italic*. Definitions are printed in **boldface**. We have chosen to employ a 'theorem-proof' style of writing, but we have interspersed the text with (hopefully illuminating) comments of a less formal nature.

Chapter 2 is both introductory as well as preparatory. We collect and (where necessary) augment and adapt, in a fashion as concise and self-contained as reasonably possible, those elements of the theory of DDE that are required to understand the lifting procedure of the normalization technique mentioned in the introduction from the finite-dimensional ODE setting to the infinite-dimensional setting of DDE. Key elements in this regard are:

- A computational spectral theory, by which we mean an explicit procedure to obtain the eigenvalues *and* corresponding (generalized) eigenvectors associated with the linearization around an equilibrium of a nonlinear DDE. The task of finding such a procedure is more demanding than its ODE-analogue, but fortunately we shall require only a small part of the characteristic matrix formalism involved.
- An invariant (*center*) manifold theory for non-hyperbolic equilibria.

Chapters 3 and 4 form the core of the manuscript. In Chapter 3 we carry out the program of deriving explicit expressions for the critical normal form coefficients of the five codimension-two bifurcations mentioned in the introduction. At various points computational lemmas are presented and proven. The structure of this Chapter is similar in purposes to [18].

In Chapter 4 we illustrate our results by means of two examples that together cover all the bifurcations that were met in Chapter 3: A relatively simple, analytically tractable Van der Pol oscillator equation and a more elaborate system appearing in neurodynamics. In contrast to the first example, an analysis of the second example requires numerical aids. By providing an analytically as well as a numerically spirited example we hope to convince the reader that our expressions are equally well suited for paper-and-pencil computations and computer implementation. Together with a brief example on the cusp bifurcation in §3.3 the examples in Chapter 4 cover all five generically occuring local codimension-two bifurcations in DDE.

Finally, in Chapter 5 we briefly look ahead.

## 1.2 Existing literature

Two standard references for the theory of DDE are found most prominently in the literature. There is the book of Hale and Verduyn Lunel [15] (a revised edition of the 1977 original by Hale) and the book by Diekmann, Van Gils, Verduyn Lunel and Walther [8].

Whenever one wants to set up a dynamical theory for DDE, one inevitably encounters the functional analytic difficulty that the 'natural' phase space $C([-h, 0], \mathbf{R}^n)$ of continuous

functions is too 'small' for a successful linear or nonlinear (perturbation) theory. This is an important theoretical issue to which we devote §2.1 in Chapter 2. There are two ways to address the problem: In [15] it is essentially ignored by taking a 'formal adjoint' approach. This 'solution' lacks mathematical rigor and therefore it does not form a good basis for a method of normal form computation that we would like to extend (at a later stage) to RE and mixed DDE-RE systems.

It is primarily for this reason that we have chosen to adopt [8] as our main reference for the theory of DDE. In this work the 'state space problem' is tackled using sun-star calculus. As we noted above, this approach leads to a framework that is well suited for the analysis of RE and DDE-RE systems. Futhermore, expressions for critical normal form coefficients have an appearance that is strikingly similar to their finite-dimensional analogues.

A formula for the direction of bifurcation (the first Lyapunov coefficient) for Hopf singularities in the sun-star context was first derived and applied to DDE by Van Gils, see Chapter X of [8] and the literature comments in §X.4. We will encounter this formula again in §3.5.3 as a 'by-product' of our treatment of the Bautin (generalized Hopf) bifurcation. Although §IX.10 of [8] contains an example of a DDE exhibiting a Bogdanov-Takens bifurcation, the corresponding normal form calculation is performed by first computing the center manifold and then analysing the restricted system. This traditional two-stage approach is computationally involved and essentially obsolete. We do not know of other examples of systematic analysis of codimension-two points of DDE using the sun-star method.

For an introduction to the work of Faria and Magelhães on normal forms for DDE from the mid-1990s we refer to the review article [12] and the references therein. The work of these authors is based on the formal adjoint approach taken in [15]. Its purpose is to provide a method for the calculation of normal form coefficients (possibly depending on parameters) that avoids preliminary computation of the center manifold by employing a normalization that also linearizes the center manifold. This goal is shared by the approach proposed in the present manuscript. However, we find our method to be preferable for three reasons.

- From a theoretical viewpoint, the method in this manuscript is based on sun-star calculus. In contrast to an approach based on formal adjoint theory, it is therefore entirely rigorous and is expected to extend to RE and mixed DDE-RE systems with relatively little effort.

- It leads to explicit and ready-to-implement expressions for the critical normal form coefficients. These are compact, easy to evaluate and valid under weaker conditions than those imposed in the work of Faria and Magelhães. (For instance, in order to evaluate the formulas presented in this manuscript for a particular DDE, there is no need to solve functional equations or boundary value problems.) The example in §4.1 illustrates this point by comparing results using our method to results from the literature [16] that were derived by means of the Faria - Magelhães approach.

- It has been completely implemented, in the sense that formulas for the critical normal forms of all five codimension-two bifurcations of equilibria are available. (We do not limit ourselves to a discussion of the Hopf and Bogdanov-Takens bifurcations.) Moreover, as will be illustrated in §4.2, these formulas are also readily evaluated when only numerical (as opposed to symbolic) data is available.

The basic idea of the normalization approach used in this thesis goes back to the work of Coullet and Spiegel [2]. The introduction to Chapter 3 contains more references to applications of the method to ODEs and maps.

## 1.3 Retrospective and acknowledgements

The process of writing this thesis bore, at times, close similarities to some sort of quest. In fact, not so much the writing itself but rather the complications arising from a variety of factors introduced an unnecessarily large and regrettable delay [sic] in its completion.

The first version of the thesis was ready by September 2007 as the result of a coordinated effort between the Mathematical Institute of the University of Utrecht and the Department of Theoretical Physics of the Free University of Amsterdam. Unfortunately, what the mathematician may appreciate as useful formalism is sometimes mistaken by a physicist for unnecessary abstraction. By its very nature, normal form computation is a subtle matter that requires a bit of theoretical preparation, particularly when one deals with infinite-dimensional systems such as delay equations.

Ultimately, it was decided that the thesis would be completed independently of the Department of Theoretical Physics in Amsterdam. This warranted a rather thorough rewriting of parts of the material as well as the introduction of the second example in Chapter 4 that took the space of a foreseen but never completed example from laser physics.

However, it would be unjust to blame the classical mathematics - physics tension for the entire time gap from the Autumn of 2007 to the Autumn of 2010. In the years in between I have been plagued by problems of the mind that sometimes made it hard to work. This fact, combined with the start of my PhD track in Autumn 2007 that brought other tasks to the forefront, did not promote a swift completion of this manuscript. The quote at the top of the present chapter should be read in this light.

It is therefore a pleasure to report that, despite the above difficulties, every time I resumed the work on this thesis, I did so with pleasure. In particular, it was very stimulating to write §4.2, seeing the methods of Chapter 3 come to life. I would like to thank Stephan van Gils, Hil Meijer and Sid Visser from the Department of Mathematics of the University of Twente for suggesting their neural mass model [30] as a test case. I appreciate the opportunity I had in April 2010 to meet and speak with Dirk Roose from the Department of Computer Science of the University of Leuven during his visit to Utrecht. Furthermore, thanks are due to Odo

Diekmann for a careful reading of Chapter 2 and the first paragraph of Chapter 3. I am grateful to him and Stephan van Gils for giving me the time to come to peace with myself and calmly finish incomplete work. My thesis advisor Yuri Kuznetsov is to be thanked for his lasting patience and his stimulating work [19] without which this manuscript would not have been written.

On a personal level, I am much indebted to my parents, Simone and Jos, and my girlfriend Alina for their love and support. I would not be without the three of you.

Zeist, September 2007 - September 2010

# Chapter 2

# Stationary states of delay differential equations

This chapter introduces the theory of delay differential equations in so far as needed to understand the material in the subsequent chapter. Proofs that can be found in the literature are omitted. Instead, we provide detailed references.

For $h > 0$ let $C([-h, 0], \mathbf{R}^n)$ be the Banach-space of continuous functions $\phi : [-h, 0] \to \mathbf{R}^n$, endowed with the supremum-norm

$$\|\phi\| \equiv \sup\{|\phi(x)| \, : \, -h \leq x \leq 0\}$$

By $C([-h, 0], \mathbf{R}^n)^*$ we shall denote its dual space. A representation theorem by F. Riesz enables us to identify this dual space with the Banach space $\mathrm{NBV}([0, h], \mathbf{R}^n)$ of functions $\eta : [0, h] \to \mathbf{R}^n$ of bounded variation on $[0, h]$, normalized by requiring that $\eta(0) = 0$ and $\eta$ be continuous from the right on the open interval $(0, h)$.

Let $f : C([-h, 0], \mathbf{R}^n) \times \mathbf{R}^m \to \mathbf{R}^n$ be of class $C^k$, where $k \geq 1$ is assumed to be as large as necessary. We consider the parameter-dependent DDE

$$\dot{x}(t) = f(x_t, \alpha) \qquad (t \geq 0) \tag{DDE}$$

with initial condition $\phi \in C([-h, 0], \mathbf{R}^n)$ specified as

$$x_0 = \phi. \tag{IC}$$

Recall from the introduction that for every fixed $t \geq 0$ the history function $x_t : [-h, 0] \to \mathbf{R}^n$ is defined by

$$x_t(\theta) \equiv x(t + \theta) \qquad (-h \leq \theta \leq 0).$$

Sometimes we will want to regard the delay $h$ itself as a parameter. This situation is slightly

delicate from the point of view of smoothness, but in the case of a single discrete delay the subtleties can be circumvented, see the first example in Chapter 4.

By a solution of (DDE) with initial condition (IC) we shall mean a function $x(\cdot, \alpha, \phi) : [-h, t_+) \to \mathbf{R}^n$ which satisfies (IC), is differentiable on $(0, t_+)$ and satisfies (DDE) there. In this paper we shall always assume that solutions are in fact *global*, in the sense that we can take $t_+ = \infty$. For linear equations this holds generally, while for nonlinear equations it requires ad-hoc verification by deriving a-priori bounds on the solutions, just as in the case of ordinary differential equations, but often this step is omitted in applications.

In the case that $f$ is linear in the state variable, (DDE) may be written as

$$\dot{x}(t) = \int_0^h d\eta(\theta, \alpha) x_t(-\theta) \qquad (t \geq 0) \tag{LDDE}$$

where $\eta(\cdot, \alpha) \in \mathrm{NBV}([0, h], \mathbf{R}^n)$ is uniquely and explicitly determined by $f$ and the integral is a Riemann-Stieltjes integral, see Chapter I.1 of [8].

In this chapter we are interested in collecting a number of tools and results vital to an analysis of (DDE) near a constant solution.

In §2.1 we show how (LDDE) and its nonlinear perturbations generate a semiflow in the function space $C([-h, 0], \mathbf{R}^n)$. Although we will not get into technical questions of existence and uniqueness (these have been taken care of in [8]), in this paper we shall need parts of the formalism involved.

In §2.2 we explain how the asymptotics of the linearization near an equilibrium of the nonlinear semiflow corresponding to (DDE) may be analyzed by means of the so-called characteristic matrix.

In §2.3 we discuss parameter-dependence and loss of hyperbolicity. As the parameter $\alpha$ is varied, certain eigenvalues of an equilibrium of (DDE) may move across the imaginary axis in the complex plane. When these eigenvalues hit the imaginary axis, the equilibrium becomes non-hyperbolic and it may undergo local bifurcations. Near a non-hyperbolic equilibrium the semiflow dynamics are essentially finite-dimensional by virtue of the existence of a smooth center manifold. On this invariant manifold the semiflow enjoys particularly good differentiability properties that facilitate our approach to normal form calculations in Chapter 3.

## 2.1 Semiflows generated by delay differential equations

As it turns out, the space $C([-h, 0], \mathbf{R}^n)$ by itself is not readily suitable for a semigroup approach to (DDE) as the function $\eta$ encoding the particulars of the linear equation (LDDE) appears explicitly in the *domain* of the generator of the semigroup one would like to study. This complicates the development of a perturbation theory for dealing with nonlinear prob-

lems. (See the remarks on the problem on p. 39 of [8] and in the introductory section of [5].)

One way to resolve this difficulty is to make use of a general functional analytic perturbation framework known as *sun-star calculus* or *dual perturbation theory*. This approach allows us to treat DDE as bounded (in fact: finite rank) perturbations with values in a 'bigger' space, the so-called sun-star dual $X^{\odot\star}$ of the original state space $X$. As we will explain below, these perturbations enter additively in the *action* of a certain weak$^*$-generator $A^{\odot\star}$ on $X^{\odot\star}$, leaving its domain untouched. The price one has to pay for thus enlarging one's state space vocabulary is a loss of strong continuity of the corresponding sun-star semigroup $(T^{\odot\star}(t))_{t\geq 0}$, which we can however recover by taking a suitable restriction. We will now present the basic ideas and results in an abstract setting. For proofs of all the statements in this section we refer to Chapters II, III and VII and Appendix II.3 of [8]. For linear semigroup theory in general we refer to [9].

Let $X$ be a Banach space, let $\mathcal{L}(X)$ be the space of bounded linear operators on $X$ and let $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$ be a strongly continuous ($C_0$) one-parameter semigroup with generator $A$ having domain $D(A)$, which we denote by $(A, D(A))$. If $X$ is non-reflexive, for instance when $X = C([-h, 0], \mathbf{R}^n)$, the adjoint semigroup $(T^*(t))_{t\geq 0} \equiv (T(t)^*)_{t\geq 0}$ is in general only weak$^*$-continuous on $X^*$ and $(A^*, D(A^*))$ generates $(T^*(t))_{t\geq 0}$ merely in the weak$^*$-sense. However, the set

$$X^{\odot} \equiv \{x^* \in X^* \,:\, t \mapsto T^*(t)x^* \text{ is norm-continuous } \}$$

is a norm-closed $T^*(t)$-invariant subspace of $X^*$. In fact, one can prove that

$$X^{\odot} = \overline{D(A^*)} \tag{2.1}$$

where the closure in the right-hand side is with respect to the norm of $X^*$. By construction the restriction $(T^{\odot}(t))_{t\geq 0}$ of $(T^*(t))_{t\geq 0}$ to $X^{\odot}$ is strongly continuous. Moreover, one may prove that this restriction is exactly generated by the part of $A^*$ in $X^{\odot}$ which we denote by $A^{\odot}$, i.e.

$$D(A^{\odot}) \equiv \{x^{\odot} \in D(A^*) \,:\, A^*x^{\odot} \in X^{\odot}\}, \qquad A^{\odot}x^{\odot} \equiv A^*x^{\odot}.$$

At this stage we have a $C_0$ semigroup $(T^{\odot}(t))_{t\geq 0}$ on a Banach space $X^{\odot}$ which is norm-generated by $(A^{\odot}, D(A^{\odot}))$. Therefore, we may play the same game once more. We obtain an adjoint semigroup $(T^{\odot\star}(t))_{t\geq 0}$ on the dual space $X^{\odot\star}$ with weak$^*$-generator $(A^{\odot\star}, D(A^{\odot\star}))$. The set

$$X^{\odot\odot} \equiv \{x^{\odot\star} \in X^{\odot\star} \,:\, t \mapsto T^{\odot\star}(t)x^{\odot\star} \text{ is norm-continuous } \}$$

is a norm-closed $T^{\odot\star}(t)$-invariant subspace of $X^{\odot\star}$. One can prove that

$$X^{\odot\odot} = \overline{D(A^{\odot\star})}. \tag{2.2}$$

where the closure is with respect to the norm in $X^{\odot\star}$. The restricted semigroup $(T^{\odot\odot}(t))_{t\geq 0}$ is by construction strongly continuous and its generator is given by the part $A^{\odot\odot}$ of $A^{\odot\star}$ in $X^{\odot\odot}$, i.e.

$$D(A^{\odot\odot}) \equiv \{x^{\odot\odot} \in D(A^{\odot\star}) \,:\, A^{\odot\star}x^{\odot\odot} \in X^{\odot\odot}\}, \qquad A^{\odot\odot}x^{\odot\odot} \equiv A^{\odot\star}x^{\odot\odot}.$$

In this paper we will be concerned with the situation in which the spaces $X$ and $X^{\odot\odot}$ can be identified with each other via the canonical embedding $j : X \to X^{\odot\star}$ given by

$$\langle j(x), x^{\odot}\rangle \equiv \langle x^{\odot}, x\rangle \qquad \forall\, x \in X, \forall\, x^{\odot} \in X^{\odot}. \tag{2.3}$$

When such an identification is possible (i.e. when $j$ is onto $X^{\odot\odot}$) we shall say that $X$ is **sun-reflexive** with respect to the semigroup $(T(t))_{t\geq 0}$.

**Remark 2.1.** In this paper we will omit the embedding $j$ in our notation. For example, we shall write $X \subset X^{\odot\star}$ instead of $X^{\odot\odot} \subset X^{\odot\star}$ and $X = j^{-1}(X^{\odot\odot})$. The advantage of this choice is that our bifurcation formulas in Chapter 3 will look much cleaner, but the disadvantage is that the reader has to do his own bookkeeping-of-spaces. ◊

One can show that there exists a unique $C_0$ semigroup $(T(t))_{t\geq 0}$ on $X = C([-h, 0], \mathbf{R}^n)$ which is in one-to-one correspondence with the solutions of (LDDE). In fact, suppose that $\alpha$ is a fixed parameter value so that we can suppress it in our notation. Let us assume that $x(\cdot, \phi) : [-h, \infty) \to \mathbf{R}^n$ is a solution of (LDDE) with initial value $x_0 = \phi \in C([-h, 0], \mathbf{R}^n)$, then

$$T(t)\phi = x_t(\cdot, \phi) \qquad (t \geq 0) \tag{2.4}$$

Conversely, for any initial value $\phi \in C([-h, 0], \mathbf{R}^n)$ the function $x(\cdot, \phi) : [-h, \infty) \to \mathbf{R}^n$ defined by

$$x_0 \equiv \phi, \qquad x(t, \phi) \equiv (T(t)\phi)(0) \quad \forall\, t \geq 0 \tag{2.5}$$

is the unique solution of (DDE) with initial condition $\phi$. The sun-star construction outlined above behaves particularly well with respect to perturbations in $B(X, X^{\odot\star})$ of the weak$^*$-generator $A^{\odot\star}$ of the adjoint semigroup $(T^{\odot\star}(t))_{t\geq 0}$ on the 'big' space $X^{\odot\star}$. Indeed, the domains $D(A^*)$ and $D(A^{\odot\star})$ are the same for all linear equations (i.e. for all choices of $\eta$ in (LDDE) and for all parameter values $\alpha \in \mathbf{R}^m$) and by (2.1) and (2.2) the same then holds for the spaces $X^{\odot}$ and $X^{\odot\odot}$. In particular, $C([-h, 0], \mathbf{R}^n)$ is sun-reflexive with respect to every linear DDE. In Table 2.1 we list explicit representations for the spaces $X, X^*, X^{\odot}$ and $X^{\odot\star}$ as well as the dual pairings between them, for the case that $X = C([-h, 0], \mathbf{R}^n)$ and $(T(t))_{t\geq 0}$ is

the semigroup associated with (LDDE). We will frequently use these pairings in Chapter 3.

| space | representation | pairing |
|---|---|---|
| $X$<br>$X^*$ | $\phi \in C([-h,0],\mathbf{R}^n)$<br>$f \in \mathrm{NBV}([0,h],\underline{\mathbf{R}}^n)$ | $\langle f,\phi\rangle = \int_0^h df(\theta)\phi(-\theta)$ |
| $X^\odot$<br>$X^{\odot\star}$ | $(c,g) \in \underline{\mathbf{R}}^n \times L^1([0,h],\underline{\mathbf{R}}^n)$<br>$(\alpha,\phi) \in \mathbf{R}^n \times L^\infty([-h,0],\mathbf{R}^n)$ | $\langle(\alpha,\phi),(c,g)\rangle = c\alpha + \int_0^h g(\theta)\phi(-\theta)\,d\theta$ |
| $X$<br>$X^\odot$ | $\phi \in C([-h,0],\mathbf{R}^n)$<br>$(c,g) \in \underline{\mathbf{R}}^n \times L^1([0,h],\underline{\mathbf{R}}^n)$ | $\langle(c,g),\phi\rangle = c\phi(0) + \int_0^h g(\theta)\phi(-\theta)\,d\theta$ |

**Table 2.1:** Representations for the abstract spaces $X, X^*, X^\odot$ and $X^{\odot\star}$ for the case of the semigroup $(T(t))_{t\geq 0}$ associated with the linear equation (LDDE). The space $\underline{\mathbf{R}}^n$ is just $\mathbf{R}^n$, but in Chapter 3 it shall turn out to be convenient to regard its elements as *row* vectors instead of column vectors. Also indicated are the dual pairings that we will encounter in this manuscript.

For the remainder of this chapter, let the spaces $X, X^\odot$ etc. be as in Table 2.1. Using results from sun-star calculus one can also deal efficiently with perturbations of (LDDE) of the form

$$\dot{x}(t) = \int_0^h d\eta(\theta,\alpha)x_t(-\theta) + g(x_t,\alpha) \qquad (t \geq 0) \tag{2.6}$$

Such perturbations arise when studying stability of equilibria of (DDE) as in the next section. Here $g : X \times \mathbf{R}^m \to \mathbf{R}^n$ is assumed to be of class $C^k$ for sufficiently high $k$ and is supposed to satisfy

$$g(0,\alpha_0) = 0, \quad D_1 g(0,\alpha_0) = 0 \tag{2.7}$$

for some fixed $\alpha_0 \in \mathbf{R}^m$. Furthermore, for $j = 1,\ldots,n$ let us denote by $e_j$ the standard basis vectors of $\mathbf{R}^n$. Introduce the vectors $r_j^{\odot\star} \in X^{\odot\star}$ by putting

$$r_j^{\odot\star} \equiv (e_j, 0) \qquad (j = 1,\ldots,n)$$

Using this notation let us define a $C^k$-smooth mapping $R : X \times \mathbf{R}^m \to X^{\odot\star}$ by

$$R(\phi,\alpha) \equiv \sum_{j=1}^n \left\{ g(\phi,\alpha) + \int_0^h [d\eta(\theta,\alpha) - d\eta(\theta,\alpha_0)]\phi(-\theta) \right\}_j r_j^{\odot\star} \tag{2.8}$$

(Note that the finite-dimensional range of $R$ is contained in the linear span of the $\mathbf{R}^n$-component of $X^{\odot\star}$. This is a special feature of DDE and DE.) The subscript $j$ attached to the curley brackets denotes the $j$-th component of the quantity they enclose. Finally, also note that it follows from our assumptions (2.7) on $g$ that

$$R(0,\alpha_0) = 0, \qquad D_1 R(0,\alpha_0) = 0 \tag{2.9}$$

The kernel $\eta(\cdot,\alpha_0) \in \mathrm{NBV}([0,h],\mathbf{R}^n)$ defines a linear DDE. Let $(T(t))_{t\geq 0}$ be the corresponding semigroup of solution operators. Now consider the parameter-dependent nonlinear abstract

integral equation

$$u(t) = T(t)\phi + \int_0^t T^{\odot\star}(t-\tau)R(u(\tau),\alpha)\,d\tau \tag{AIE}$$

where $\phi \in X$ is given and the integral must be interpreted as a weak$^*$-integral (with values in $X$), see Lemma III.2.1 and Interlude 3.13 of Appendix II in [8]. Solutions of (AIE) are by definition continuous functions $u : [0, t_+) \to X$. We recall our running assumption that it is always possible to take $t_+ = \infty$. Analogous to the purely linear case, it can be shown that these solutions constitute a strongly continuous *nonlinear* semiflow on $X$ (see Chapter VII of [8]) and that there is a one-to-one correspondence between solutions of (2.6) and solutions of (AIE). Namely, if $x(\cdot,\alpha,\phi) : [-h,\infty) \to \mathbf{R}^n$ solves (2.6) with initial condition $x_0 = \phi$, then

$$u(t,\alpha,\phi) = x_t(\cdot,\alpha,\phi) \qquad (t \geq 0) \tag{2.10}$$

uniquely solves (AIE). Conversely, if $u(\cdot,\alpha,\phi)$ is a solution of (AIE) then the function $x(\cdot,\alpha,\phi) : [-h,\infty) \to \mathbf{R}^n$ defined by

$$x_0 \equiv \phi, \qquad x(t,\alpha,\phi) \equiv u(t,\alpha,\phi)(0) \quad \forall\, t \geq 0 \tag{2.11}$$

uniquely solves (2.6) with initial condition $\phi$.

## 2.2 Linearization and analysis near an equilibrium

Let $\overline{\phi} \in X$ be a *constant* function. Suppose that for the parameter value $\alpha_0 \in \mathbf{R}^m$ the function $\overline{x} : [-h,\infty) \to \mathbf{R}^n$ defined by

$$\overline{x}(t,\alpha_0) \equiv \overline{\phi}(0) \qquad \forall\, t \geq 0$$

is a stationary solution of (DDE), i.e. $f(\overline{\phi},\alpha_0) = 0$. By a change of coordinates it can always be arranged that $\overline{\phi} = 0$. We can then write (DDE) in the form (2.6) satisfying conditions (2.7). Namely,

$$\begin{aligned} \dot{x}(t) &= D_1 f(0,\alpha)x_t + (f(x_t,\alpha) - D_1 f(0,\alpha)x_t) \\ &= \int_0^h d\eta(\theta,\alpha)x_t(-\theta) + (f(x_t,\alpha) - D_1 f(0,\alpha)x_t) \end{aligned} \tag{2.12}$$

where $D_1 f(0,\alpha) \in \mathcal{L}(X,\mathbf{R}^n)$ denotes the partial (Fréchet) derivative of $F$ with respect to its first argument evaluated at the point $(0,\alpha) \in X \times \mathbf{R}^m$ and $\eta(\cdot,\alpha)$ denotes its $\mathrm{NBV}([0,h],\mathbf{R}^n)$ representation.

We would like to have an instrument to decide about stability of the origin as an equilibrium of the nonlinear semiflow associated with (2.12, $\alpha = \alpha_0$). Recall from the previous section that this semiflow corresponds to the solution of (AIE, $\alpha = \alpha_0$) with $(T(t))_{t\geq 0}$ the strongly continuous solution of the linear DDE defined by $\eta(\cdot,\alpha_0)$. Note that for $\alpha = \alpha_0$ the

nonlinearity $R$ from (2.8) simply becomes

$$R(\phi, \alpha_0) = \sum_{j=1}^{n} \{f(\phi, \alpha_0) - D_1 f(0, \alpha_0)\phi\}_j \, r_j^{\odot\star} \tag{2.13}$$

Let $(A, D(A))$ be the generator of the semigroup $(T(t))_{t\geq 0}$. If its spectrum $\sigma(A)$ does not contain any purely imaginary points, then the question of stability of the equilibrium at the origin at the parameter value $\alpha_0$ is answered by the location of $\sigma(A)$ in the complex plane. This is formalized in the Principle of Linearized Stability for DDE, see Theorem VII.6.8 of [8].

**Remark 2.2.** As soon as spectral theory is applied to the analysis of a real-valued problem, one should complexify all spaces involved, as well as the operators acting on them. For the sun-star framework introduced in the previous section this is not a trivial task. It has however been carried out in detail in Section III.7 of [8] and that is why we refrain from discussing the issue here. We do not expect this omission to cause the reader major difficulties. ◊

In order to find $\sigma(A)$ it is not necessary to work with the generator $(A, D(A))$ directly. Indeed, there exists a holomorphic matrix-valued function, called the **characteristic matrix**, from which all spectral information can be obtained. The following theorem summarizes what we will need. The proofs of the statements given can be found in Chapter IV of [8]. One can exploit the fact that, although $A$ itself is unbounded, for *finite* delays $h > 0$ its resolvent $(zI - A)^{-1}$ is a compact operator on $X$, for which the non-zero spectrum consists of isolated eigenvalues only. For an introduction to the spectral theory of closed linear operators see Chapter V of [29] and Chapter IV.1 of [9].

A point $\lambda \in \sigma(A)$ is called an **eigenvalue of finite type** if it is an isolated point of $\sigma(A)$ and its algebraic multiplicity is finite, see the definitions on p. 96 of [8].

**Theorem 2.3.** *Let $(A, D(A))$ be the generator of the semigroup $(T(t))_{t\geq 0}$ corresponding to the linear part of (2.12, $\alpha = \alpha_0$).*

*i* $\sigma(A) = \sigma(A^*) = \sigma(A^{\odot}) = \sigma(A^{\odot\star})$. *These spectra consist solely of eigenvalues of finite type.*

*ii* *The matrix-valued function $\Delta : \mathbf{C} \to \mathbf{C}^{n\times n}$ defined by*

$$\Delta(z) \equiv zI - \int_0^h e^{-z\theta} d\eta(\theta) \tag{2.14}$$

*is holomorphic, and $\lambda \in \sigma(A)$ if and only if $\det \Delta(\lambda) = 0$. In that case the order of $\lambda$ as a root of $\det \Delta$ equals the algebraic multiplicity of $\lambda$ as an eigenvalue and the dimension of the nullspace $N[\Delta(\lambda)]$ is equal to the geometric multiplicity of $\lambda$ as an eigenvalue.*

*Finally, the (generalized) eigenspaces corresponding to $\lambda$ are given by the nullspaces*

$$N[(\lambda I - A)^{k_\lambda}] = N[(\lambda I - A^{\odot\star})^{k_\lambda}] \text{ and } N[(\lambda I - A^*)^{k_\lambda}] = N[(\lambda I - A^{\odot})^{k_\lambda}]$$

*where $k_\lambda$ is the order of $\lambda$ as a pole of $z \mapsto \Delta(z)^{-1}$.*

The transcendental equation $\det \Delta(z) = 0$ is known as the **characteristic equation**. In all but the simplest cases, finding eigenvalues by locating its roots requires numerical analysis.

In addition to knowledge about the eigenvalues we will also require the corresponding eigenvectors. These, too, can be obtained from the characteristic matrix. We do not need a systematic result, but merely expressions for special cases when $\lambda \in \sigma(A)$ is simple (i.e. of algebraic multiplicity one, for cusp, Bautin, fold-Hopf and double Hopf points) or when $\lambda$ is a double eigenvalue (for Bogdanov-Takens points).

First we consider the 'simple' case.

**Lemma 2.4.** *Let $\lambda$ be a simple eigenvalue of $A$. If the non-zero column vector $q$ is a right null vector of $\Delta(\lambda)$ (i.e. $\Delta(\lambda)q = 0$) then*

$$\phi = (\theta \mapsto e^{\lambda\theta} q) \tag{2.15}$$

*is an eigenvector of $A$ corresponding to $\lambda$. Furthermore, if the non-zero row vector $p$ is a left null vector of $\Delta(\lambda)$ (i.e. $p\Delta(\lambda) = 0$) then*

$$\phi^{\odot} = (0,\, \theta \mapsto p\left(I + \int_0^\theta \int_\sigma^h e^{\lambda(\sigma - s)}\, d\eta(s)\, d\sigma\right)) \tag{2.16}$$

*is an eigenvector of $A^*$ corresponding to $\lambda$. Finally,*

$$\langle \phi^{\odot}, \phi \rangle = p\Delta'(\lambda)q \neq 0 \tag{2.17}$$

*where $\Delta'(\lambda)$ denotes the derivative of $z \mapsto \Delta(z)$ at $z = \lambda$.*

*Proof.* The statements are identical to or follow directly from Theorems IV.5.5 and IV.5.9 (eigenvector for $A$ and $A^*$, respectively) and Corollary 5.12 (the identity (2.17) for their pairing) in [8]. □

The expression (2.17) will be used frequently in Chapter 3 to achieve a mutual normalization of (adjoint) eigenvectors. Note that for $\phi^{\odot} \in X^{\odot}$ we employed the representation for elements in $X^{\odot}$ given in Table 2.1.

When bifurcations involve different simple eigenvalues (such is the case for e.g. the fold-Hopf and the double Hopf bifurcation), we are required to calculate pairings of the sort (2.17) with $\phi^{\odot}$ and $\phi$ pertaining to different eigenvalues. The following lemma shows by means of an explicit (and hopefully illuminating) computation that such pairings always vanish.

**Lemma 2.5.** *Let $\lambda$ and $\mu$ be simple eigenvalues of $A$, with $\lambda \neq \mu$. Let $\phi_\lambda$ be an eigenvector corresponding to $\lambda$ and let $\phi_\mu^\odot$ be an eigenvector of $A^*$ corresponding to $\mu$. Then $\langle \phi_\mu^\odot, \phi_\lambda \rangle = 0$.*

*Proof.* Since $\lambda$ and $\mu$ are simple, by Lemma 2.4 $\phi_\lambda$ is as in (2.15) and $\phi_\mu^\odot$ is as in (2.16) with $\lambda$ replaced by $\mu$, for some non-zero column vector $q$ satisfying $\Delta(\lambda)q = 0$ and some non-zero row vector $p$ satisfying $p\Delta(\mu) = 0$. We calculate

$$\begin{aligned}
\langle \phi_\mu^\odot, \phi_\lambda \rangle &= pq + p \int_0^h \left( \int_\theta^h e^{\mu(\theta - s)}\, d\eta(s) \right) e^{-\lambda\theta}\, d\theta q \\
&= pq + p \int_0^h \int_\theta^h e^{-\mu s}\, d\eta(s) e^{(\mu-\lambda)\theta}\, d\theta q \\
&= pq + p \int_0^h \int_0^s e^{(\mu-\lambda)\theta}\, d\theta e^{-\mu s}\, d\eta(s) q \\
&= pq + p \int_0^h \frac{e^{-\mu s} - e^{-\lambda s}}{\lambda - \mu}\, d\eta(s) q \\
&= \left\{ \frac{1}{2} pq - \frac{p}{\mu - \lambda} \int_0^h e^{-\mu s}\, d\eta(s) q \right\} + \left\{ \frac{1}{2} pq - \frac{p}{\lambda - \mu} \int_0^h e^{-\lambda s}\, d\eta(s) q \right\} \qquad (2.18)
\end{aligned}$$

Now, the first term in braces can be written as

$$\begin{aligned}
\frac{1}{2} pq - \frac{p}{\mu - \lambda} \int_0^h e^{-\mu s}\, d\eta(s) q &= \frac{p}{\mu - \lambda} \left\{ \frac{\mu - \lambda}{2} I - \int_0^h e^{-\mu s}\, d\eta(s) \right\} q \\
&= \frac{p}{\mu - \lambda} \left\{ \mu I - \int_0^h e^{-\mu s}\, d\eta(s) \right\} q - \frac{1}{2} \frac{\mu + \lambda}{\mu - \lambda} pq \\
&= \frac{p}{\mu - \lambda} \Delta(\mu) q - \frac{1}{2} \frac{\mu + \lambda}{\mu - \lambda} pq
\end{aligned}$$

The second term in (2.18) is equal to the first term with $\mu$ and $\lambda$ permuted. It follows that

$$\begin{aligned}
\langle \phi_\mu^\odot, \phi_\lambda \rangle &= \frac{p}{\mu - \lambda} \Delta(\mu) q - \frac{1}{2} \frac{\mu + \lambda}{\mu - \lambda} pq - \frac{p}{\mu - \lambda} \Delta(\lambda) q + \frac{1}{2} \frac{\mu + \lambda}{\mu - \lambda} pq \\
&= \frac{1}{\mu - \lambda} p[\Delta(\mu) - \Delta(\lambda)] q = 0
\end{aligned}$$

since $p\Delta(\mu) = 0$ and $\Delta(\lambda)q = 0$. □

Next, we turn to the case of a double eigenvalue.

**Definition 2.6.** A sequence of column vectors $q_0, q_1, \ldots, q_{k-1}$ in $\mathbf{R}^n$ is called a **right Jordan chain** for $\Delta(z)$ at $z = \lambda$ if $q_0 \neq 0$ and

$$\Delta(z)(q_0 + (z - \lambda)q_1 + \ldots + (z - \lambda)^{k-1} q_{k-1}) = O((z - \lambda)^k) \text{ as } z \to \lambda$$

The number $k$ is called the **rank** of the chain. Similarly, a sequence of row vectors $p_0, \ldots, p_{k-1}$

in $\mathbf{R}^n$ is called a **left Jordan chain** for $\Delta(z)$ at $z = \lambda$ if $p_0 \neq 0$ and

$$(p_{k-1} + (z-\lambda)p_{k-2} + \ldots + (z-\lambda)^{k-1}p_0)\Delta(z) = O((z-\lambda)^k) \text{ as } z \to \lambda$$

Althought the above definition is quite usable for hand calculations, we remark that Exercise IV.5.11 in [8] presents a way to calculate Jordan chains from the zero-eigenvectors of a matrix whose entries involve powers of $\Delta(z)$ at $z = \lambda$. This method may be preferable for computer implementations.

Suppose that $\lambda$ is a double eigenvalue of $A$ of geometric multiplicity one. It is straightforward to verify that there exists an eigenvector $\phi_0 \in D(A)$ and a generalized eigenvector $\phi_1 \in D(A)$ such that

$$A\phi_0 = \lambda\phi_0, \qquad A\phi_1 = \lambda\phi_1 + \phi_0$$

Also, there exists an eigenvector $\phi_1^\odot \in D(A^*)$ and a generalized eigenvector $\phi_0^\odot \in D(A^*)$ such that

$$A^*\phi_1^\odot = \lambda\phi_1^\odot, \qquad A^*\phi_0^\odot = \lambda\phi_0^\odot + \phi_1^\odot.$$

**Lemma 2.7.** *Let $\lambda$ be an eigenvalue of $A$ with geometric multiplicity one and algebraic multiplicity two. Let*

$$\{q_0, q_1\} \in \mathbf{R}^n, \qquad \{p_1, p_0\} \in \underline{\mathbf{R}}^n$$

*be rank-two right and left Jordan chains of $\Delta(z)$ at $z = \lambda$. Then the column vector valued functions*

$$\phi_0 = (\theta \mapsto e^{\lambda\theta}q_0), \quad \phi_1 = (\theta \mapsto e^{\lambda\theta}(\theta q_0 + q_1))$$

*are an eigenvector and a generalized eigenvector for $A$ corresponding to $\lambda$ and the row vector valued functions*

$$\phi_1^\odot = (0,\, p_1\left(I + \int_0^\theta \int_\sigma^h e^{\lambda(\sigma-s)}\, d\eta(s)\, d\sigma\right))$$
$$\phi_0^\odot = (0,\, p_0\left(I + \int_0^\theta \int_\sigma^h e^{\lambda(\sigma-s)}\, d\eta(s)\, d\sigma\right) + p_1\left(\int_0^\theta \int_\sigma^h e^{\lambda(\sigma-s)}(\sigma - s)\, d\eta(s)\, d\sigma\right))$$

*are an eigenvector and a generalized eigenvector for $A^*$ corresponding to $\lambda$. Moreover, the following identities hold:*

$$\langle \phi_0^\odot, \phi_0 \rangle = p_0\Delta'(\lambda)q_0 + \frac{1}{2!}p_1\Delta''(\lambda)q_0 \tag{2.19a}$$
$$\langle \phi_1^\odot, \phi_1 \rangle = p_1\Delta'(\lambda)q_1 + \frac{1}{2!}p_1\Delta''(\lambda)q_0 \tag{2.19b}$$
$$\langle \phi_1^\odot, \phi_0 \rangle = p_1\Delta'(\lambda)q_0 \tag{2.19c}$$
$$\langle \phi_0^\odot, \phi_1 \rangle = p_0\Delta'(\lambda)q_1 + \frac{1}{2!}p_0\Delta''(\lambda)q_0 + \frac{1}{2!}p_1\Delta''(\lambda)q_1 + \frac{1}{3!}p_1\Delta'''(\lambda)q_0. \tag{2.19d}$$

*Here $\Delta'(\lambda), \Delta''(\lambda)$ and $\Delta'''(\lambda)$ are derivatives of orders one to three of $\Delta(z)$ at $z = \lambda$.*

*Proof.* The formulas for the (generalized) eigenvectors can be found from Theorems IV.5.5 and IV.5.9 in [8]. The expressions for the pairings in (2.19) cannot be found there (except for (2.19c) which is just the same as (2.17)), but they are derived in a fashion completely analogous to the derivation of (2.17) and the proof of Lemma 2.5. Namely, one compares the pairing between $X^\odot$ and $X$ given in Table 2.1 with the definition (2.14) of $\Delta(z)$. We omit the details. □

By induction one verifies that identities such as those in (2.19) hold more generally, i.e. pairings may be expressed as appropriately truncated series involving derivates of $z \mapsto \Delta(z)$ as well as right and left Jordan chains for $\Delta(z)$ at $z = \lambda$. Since we do not have any need for such results in this paper, we refrain from stating them explicitly.

We conclude this section by noting that Theorem 2.3 provides a simple criterion for calculating algebraic and geometric multiplities of a given eigenvalue. Therefore, it is easy to check which of the two foregoing lemmas should be applied in a concrete case.

## 2.3 The center manifold

We suppose that for $\alpha_0$ the zero-function is a stationary solution of (DDE) with right-hand side depending on state and parameter in a $C^k$-fashion, and we return to the splitting (2.12, $\alpha = \alpha_0$),

$$\dot{x}(t) = \int_0^h d\eta(\theta, \alpha_0) x_t(-\theta) + (f(x_t, \alpha_0) - D_1 f(0, \alpha_0) x_t) \tag{2.20}$$

In this section we will be concerned with the case of a non-hyperbolic equilibrium, i.e.

$$\sigma(A) \cap i\mathbf{R} \neq \emptyset. \tag{2.21}$$

Here $i\mathbf{R}$ denotes the imaginary axis in the complex plane and $A$ is the generator of the semigroup $(T(t))_{t\geq 0}$ solving the linear DDE defined by $\eta(\cdot, \alpha_0)$. Until considering parameter-dependence at the end of this section, we shall once more suppress dependence on $\alpha_0$ in our notation, for it will stay fixed.

The proof of existence of a *smooth* center manifold for DDE is more subtle than the corresponding proof for the ODE case, but the essential ideas are the same. Let the **center subspace** $X_0 \subset X$ be defined as the direct sum

$$X_0 = \bigoplus \{\mathcal{M}_\lambda \,:\, \lambda \in \sigma(A) \cap i\mathbf{R}\}$$

Here $\mathcal{M}_\lambda$ is the generalized eigenspace corresponding to $\lambda$. By the first statement of Theorem 2.3 the sum contains only a finite number of finite-dimensional terms. Consequently $X_0$ is

finite-dimensional. Let $P_0 \in \mathcal{L}(X)$ be the spectral projection of $X$ onto $X_0$ and denote its extension to $X^{\odot\star}$ with range $X_0$ by $P_0^{\odot\star} \in \mathcal{L}(X^{\odot\star}, X)$.

**Remark 2.8.** The subspace $X_0$ is spanned by a basis consisting of (generalized) eigenvectors of $A$ corresponding to eigenvalues on the imaginary axis. It is this basis, available explicitly for the cases of our interest from Lemmas 2.4 and 2.7 above, that we shall use in Chapter 3 as a coordinate system with respect to which we express the dynamics of $y$ generated by (2.24) below. ◊

Now let $\xi : \mathbf{R}_+ \to \mathbf{R}$ be a $C^\infty$-smooth *cut-off* function with the properties

$$\xi(s) \in \begin{cases} \{1\} & (0 \le s \le 1) \\ [0,1] & (1 \le s \le 2) \\ \{0\} & (2 \le s) \end{cases} \tag{2.22}$$

Define, for any $\delta > 0$, the *modified nonlinearity* $R_\delta : X \to X^{\odot\star}$ by

$$R_\delta(\phi) \equiv R(\phi, \alpha_0)\, \xi\left(\frac{\|P_0\phi\|}{\delta}\right) \xi\left(\frac{\|(I-P_0)\phi\|}{\delta}\right) \tag{2.23}$$

where $R(\phi, \alpha_0)$ is given by (2.13). For any $\phi \in X$ denote by $u_\delta(\cdot, \phi)$ the solution of (AIE, $R = R_\delta$, $\alpha = \alpha_0$). (The modification of the nonlinearity is necessary to overcome certain technical difficulties related to non-invariance of spaces of continuous functions with limited exponential growth under the so-called substitution (or: Nemytskii) operator associated with $R$. For a detailed explaination we refer to Chapter IX of [8].)

**Theorem 2.9.** *For $\delta > 0$ sufficiently small there exists a $C^k$-smooth injection $\mathcal{C}_\delta : X_0 \to X$ such that its image $\mathcal{C}_\delta(X_0)$, called the* ***global center manifold*** *or* ***center manifold*** *for short and denoted by $\mathcal{W}_\delta^c$, has the following properties:*

*i $\mathcal{W}_\delta^c$ is conditionally locally forward-invariant, in the following sense. If $\phi \in X_0$ and $\sup\{\|u_\delta(t, \mathcal{C}_\delta(\phi))\| : t \in [0,T]\} \le \delta$ then $u_\delta(t, \mathcal{C}_\delta(\phi)) = \mathcal{C}_\delta(P_0 u_\delta(t, \mathcal{C}_\delta(\phi)))$ for all $t \in [0,T]$.*

*ii $\mathcal{W}_\delta^c$ contains all solutions of (AIE, $R = R_\delta$, $\alpha = \alpha_0$) that are defined for all time and satisfy $\sup\{\|u_\delta(t,\psi)\| : t \in \mathbf{R}\} \le \delta$.*

*iii $\mathcal{W}_\delta^c$ contains the origin since $\mathcal{C}_\delta(0) = 0$ and it is tangent to $X_0$ there, i.e. $D\mathcal{C}_\delta(0)\phi = \phi$ for all $\phi \in X_0$.*

*iv If $\psi \in \mathcal{W}_\delta^c$ and $u_\delta(\cdot, \psi)$ exists for all time, then $y(t) \equiv P_0 u_\delta(t, \psi) \in X_0$ satisfies the ordinary differential equation*

$$\dot{y}_\delta(t) = A y_\delta(t) + P_0^{\odot\star} R_\delta(\mathcal{C}_\delta(y_\delta(t))) \qquad (t \in \mathbf{R}) \tag{2.24}$$

*Proof.* The construction of the center manifold is carried out by the Perron-Frobenius method, exploiting the variation-of-constants formula (AIE, $R = R_\delta$, $\alpha = \alpha_0$) to define an appropriate fixed-point operator. Statement one to three are Theorem IX.5.3 and (regarding the tangency) Corollary IX.7.10 in [8]. Establishing $C^k$-smoothness of $\mathcal{C}$ requires work: See Section IX.7. As pointed out in [8], one may alternatively exploit the fact that for DDE the nonlinearity takes values in a finite-dimensional subspace of $X^{\odot\star}$ to arrive at a smooth (as opposed to merely Lipschitz) modified nonlinearity. As far as the last statement is concerned, projection of (AIE, $R = R_\delta$, $\alpha = \alpha_0$) onto $X_0$ readily leads to (2.24), see Section IX.8. □

**Remark 2.10.** The following two remarks will be of relevance in Chapter 3.

(i) It is important to realize that $\mathcal{W}^c_\delta$ is a *global* center manifold for the solution of (AIE, $R = R_\delta$, $\alpha = \alpha_0$) which involves the *modified* nonlinearity. When calculating *local* normal forms in Chapter 3, we will be interested in the dynamics of this solution in a small neighbourhood of zero in $X$. From item (i) of the previous theorem we see that solutions that start on the center manifold will remain on it in forward time as long as they stay in the ball $B_\delta(0)$ of radius $\delta$ centered at the origin. Item (ii) tells us that the center manifold captures all solutions that exist and remain in $B_\delta(0)$ for all time, such as small periodic and homo- or heteroclinic orbits. As long as a solution stays in $B_\delta(0)$ one sees from (2.22) and (2.23) that the modification of the nonlinearity becomes immaterial. Furthermore, for such solutions (2.24) reduces to

$$\dot{y}(t) = Ay(t) + P_0^{\odot\star} R(\mathcal{C}_\delta(y(t))) \qquad (t \in \mathbf{R}) \tag{2.25}$$

with a right-hand side that is $C^k$-smooth in $y$. When it is clear from the context that we are only interested in the local dynamics of solutions (i.e. the dynamics in $B_\delta(0)$), we will write $\mathcal{W}^c$ and $\mathcal{C}$ instead of $\mathcal{W}^c_\delta$ and $\mathcal{C}_\delta$.

(ii) If $\sigma(A)$ does not contain points in the open right half-plane, then $\mathcal{W}^c_\delta$ is conditionally locally exponentially stable. This implies that if a solution of (2.25) that lies in $B_\delta(0)$ for all time is locally exponentially stable within $\mathcal{W}^c$, then it is locally exponentially stable in $X$.

It are these properties that make the center manifold a very useful tool for the analysis of equilibria of DDE. ◊

We need the following counterpart to (2.24) on the center manifold.

**Proposition 2.11.** *Let $\psi \in \mathcal{W}^c$ and suppose that the solution $t \mapsto u(t, \psi)$ of (AIE, $\alpha = \alpha_0$) exists as a map from $\mathbf{R}$ to $X$ and lies in $B_\delta(0)$ for all $t \in \mathbf{R}$. Then $u(t, \psi) \in \mathcal{W}^c$ for all $t \in \mathbf{R}$ and $u(t, \psi)$ is differentiable with respect to $t$ and satisfies*

$$\frac{du(t,\psi)}{dt} = A^{\odot\star} u(t, \psi) + R(u(t, \psi)) \qquad \forall\, t \in \mathbf{R} \tag{2.26}$$

*Proof.* Note that in the formulation of the above proposition we employ the convention put forward in the last sentence of item (i) of Remark 2.10. The first part repeats item (ii) of Theorem 2.9. The differential equation (2.26) on the center manifold follows directly from the differential equation (2.24) on the center subspace and the fact that the center manifold consists of smooth functions, i.e. functions that are at least continuously differentiable. □

Note that we cannot replace $A^{\odot\star}$ by $A$ in (2.26), since although we know that the left-hand side must be in $X$, the same cannot be said of $R(u(t,\psi))$.

In conclusion of this section we comment on parameter-dependence. Although these results will not be used explicitly, they are relevant when one would like to extend the methods of this thesis to computation of normal forms depending on parameters. Until now we have been concerned with the existence and properties of a center manifold for a non-hyperbolic equilibrium at $\alpha = \alpha_0$. Without loss of generality we may assume that $\alpha_0 = 0$. As we allow $\alpha$ to vary in a small ball around the origin in $\mathbf{R}^m$, the equilibrium need no longer be non-hyperbolic or may even cease to exist. However, if we augment (2.12) with the trivial equation $\dot{\alpha} = 0$ to obtain the extended system

$$\begin{cases} \dot{x}(t) = \displaystyle\int_0^h d\eta(\theta,\alpha)x_t(-\theta) + (f(x_t,\alpha) - D_1 f(0,\alpha)x_t) \\ \dot{\alpha}(t) = 0 \end{cases} \tag{2.27}$$

then (2.27) constitutes a DDE-ODE system on the extended phase space $X \times \mathbf{R}^m$ to which the center manifold theorem applies, see Section IX.9 of [8] and also Section 5.2 of [19] for the analogous finite-dimensional case. Because the second equation of (2.27) always contributes a zero eigenvalue of algebraic multiplicity $m$, one has

$$\dim \overline{\mathcal{W}}^c_\delta = \dim \mathcal{W}^c_\delta + m$$

where $\overline{\mathcal{W}}^c_\delta$ is a center manifold of the non-hyperbolic equilibrium $(0,0) \in X \times \mathbf{R}^m$ of (2.27) and $\mathcal{W}^c_\delta$ is the usual center manifold of the equilibrium $0 \in X$ of (2.20). In fact, the center subspace of the extended system is given by $X_0 \times \mathbf{R}^m$ and $\overline{\mathcal{W}}^c_\delta$ equals the range of a smooth mapping $\overline{\mathcal{C}}_\delta : X_0 \times \mathbf{R}^m \to X \times \mathbf{R}^m$ of the form $\overline{\mathcal{C}}_\delta(\phi,\alpha) = (\mathcal{C}_{1,\delta}(\phi,\alpha),\alpha)$, satisfying statements analogous to those in Theorem 2.9. Moreover, since $\dot{\alpha} = 0$ the sections

$$\mathcal{W}^c_\delta(\alpha) \equiv \overline{\mathcal{W}}^c_\delta \cap \{(x,\alpha) \,:\, x \in X\}$$

are invariant under the flow generated by the modified nonlinearity corresponding to (2.27), of dimension $\dim \mathcal{W}^c_\delta$ and together they foliate $\overline{\mathcal{W}}^c_\delta$. We shall call $\mathcal{W}^c_\delta(\alpha)$ a **global center manifold for** $0 \in X$ **at parameter** $\alpha$.

We conclude with parameter-dependent versions of (2.24) and Proposition 2.11. The

former becomes

$$\dot{y}_\delta(t) = Ay_\delta(t) + P_0^{\odot\star} R_\delta(\mathcal{C}_{1,\delta}(y_\delta(t), \alpha), \alpha) \qquad (t \in \mathbf{R}) \tag{2.28}$$

while the latter reads

$$\frac{du(t,\psi,\alpha)}{dt} = A^{\odot\star} u(t,\psi,\alpha) + R(u(t,\psi,\alpha),\alpha) \qquad \forall\, t \in \mathbf{R} \tag{2.29}$$

# Chapter 3

# Codimension-two critical normal forms

The primary advantage of the existence of a center manifold for a non-hyperbolic equilibrium of a DDE is that it enables 'lifting' of local bifurcation theory from the finite-dimensional ODE-setting to the infinite-dimensional setting of DDE. Indeed, solutions of a DDE that remain in the vicinity of such an equilibrium for all (positive and negative) time satisfy a finite-dimensional differential equation, as we saw in §2.3. Examples of application of this principle are the proof of the Hopf bifurcation theorem for DDE in Chapter X of [8] and its analogue for DE in Theorem 2.21 of [5].

In this chapter we discuss an approach which goes back to Coullet and Spiegel [2] and was applied by Kuznetsov in [18] to obtain explicit expressions for the critical normal form coefficients for all generically occuring codimension-one and codimension-two bifurcations of equilibria in ODE, also see §8.7 of [19].

The difference between this and other approaches is essentially twofold. Firstly, the method does not require a preliminary reduction to the center manifold but rather solves for the critical normal form and center manifold coefficients simultaneously. Secondly, as we shall see in this chapter, the expressions obtained for the normal form coefficients involve numerically accessible data, rendering them suitable for symbolic as well as numerical evaluation. Indeed, their ODE counterparts are implemented in the packages `CONTENT` [20] and `MATCONT` [4] for continuation and bifurcation analysis of ODE. As part of his thesis [22] Meijer applied the method to iterated maps and implemented the results in the package `CL_MATCONT` for maps.

In §3.1 we start with a worked-out example to illustrate the method in the context of DDE. We shall discuss normalization for the Cusp bifurcation, which is the simplest of all codimension-two cases. However, it includes the essential steps involved in the general method. During the discussion it will become clear which mathematical ingredients are needed in order to proceed with our derivations and arrive at our goal: An expression for the cubic

normal form coefficient at criticality.

In §3.2 we provide these ingredients by formulating a few results on solvability of linear operator equations and the bordered operator matrix approach to dealing with under-determined systems. The basic concepts are well-known and not difficult, but when specific DDE-related results are presented, we give detailed proofs.

In §3.3 we briefly return to our initial example. Our discussion that started in §3.1 can now be concluded with the help of the results derived in §3.2.

In §3.4 we list expressions for the relevant critical normal form coefficients for the Bogdanov-Takens, Bautin, fold-Hopf and double-Hopf bifurcations.

In the remainder of this chapter all the spaces $X, X^\odot$ etc. are as in Table 2.1.

## 3.1 The method by example: Cusp bifurcation

In this section we shall assume that our parameter space is two-dimensional: $m = 2$.

We recall the setting of Section 2.3. Suppose that for the critical parameter value $\alpha_0 = 0$ the zero-function is a stationary solution of (DDE) and write this equation as (2.20),

$$\dot{x}(t) = \int_0^h d\eta(\theta, 0)x_t(-\theta) + g(x_t, 0) \tag{3.1}$$

where $g : X \times \mathbf{R}^2 \to \mathbf{R}^n$ is a $C^k$-smooth function defined by $g(x_t, \alpha) \equiv f(x_t, \alpha) - D_1 f(0, \alpha)x_t$ and satisfying

$$g(0, 0) = 0, \qquad D_1 g(0, 0) = 0. \tag{3.2}$$

In the remainder of this section we shall suppress the dependence of $g$ and $\eta$ on the parameter, since it will stay fixed at its critical value. Hence we write, with abuse of notation,

$$\eta(\theta) \equiv \eta(\theta, 0) \quad \forall\, \theta \in [0, h] \quad \text{and} \quad g(\phi) \equiv g(\phi, 0) \quad \forall\, \phi \in X. \tag{3.3}$$

Let $(A, D(A))$ be the generator of the semigroup $(T(t))_{t\geq 0}$ solving the linear DDE associated with $\eta$ and suppose that $\lambda = 0$ is a simple eigenvalue of $A$, giving rise to a center manifold $\mathcal{W}^c_\delta$ of the origin, linearly approximated by the center eigenspace $X_0$. We assume that there are no other eigenvalues on the imaginary axis. Let $\phi$ and $\phi^\odot$ be eigenvectors of $A$ and $A^*$ corresponding to $\lambda$. It is always possible to scale these vectors such that the normalization

$$\langle \phi^\odot, \phi \rangle = 1 \tag{3.4}$$

holds. Since $X_0$ is the linear span of $\phi$ it follows that any point $y$ in $X_0$ can be expressed as a multiple of $\phi$. Indeed,

$$y = \langle \phi^\odot, y \rangle \phi$$

see Theorem IV.2.5.vi of [8].

Let $\psi \in \mathcal{W}^c$. Recalling the discussion in Remark 2.10, in this section we shall consider solutions $u = u(\cdot, \psi)$ of (AIE, $\alpha = \alpha_0 = 0$) that exist for all time and lie in the ball $B_\delta(0)$ of radius $\delta$ centered at zero. Such solutions lie on $\mathcal{W}^c$ and satisfy

$$\dot{y}(t) = Ay(t) + G(y(t)) \quad \forall\, t \in \mathbf{R}$$

where $y(t)$ is the projection of $u(t)$ onto $X_0$ and $G : X_0 \to X_0$ is a $C^k$-smooth function defined by $G(\phi) \equiv P_0^{\odot\star} R(\mathcal{C}(\phi))$. If we let $z(t) \equiv \langle \phi^\odot, y(t) \rangle$ then, by the chain rule and the fact that $\phi^\odot$ is a zero-eigenvector,

$$\dot{z} = \langle \phi^\odot, G(z\phi) \rangle$$

The right-hand side of this ODE is $C^k$-smooth in $y$. Therefore, by (3.2) it has a Taylor expansion starting with quadratic terms, say

$$\dot{z} = bz^2 + cz^3 + O(|z|^4) \tag{3.5}$$

We shall regard the above equation with $b = 0$ and without the $O(|z|^4)$ term as a normal form at criticality of the cusp bifurcation, discussed in detail in Chapter 8.2 of [19]. It is the goal of this example to explain a way to find an expression for the yet unknown cubic coefficent $c$ in terms of $\phi$, $\phi^\odot$ and derivatives of the function $f$ appearing in the right-hand side of (DDE). The cusp bifurcation is *degenerate*, in the sense that it is a fold bifurcation with vanishing quadratic coefficient: $b = 0$. Therefore, calculation of the third-order coefficient $c$ becomes necessary, but inclusion of still higher order terms in (3.5) is not required. Generically (i.e. in systems without special symmetries) one needs to tune two parameters to enforce such a degeneracy: One parameter to hit a fold point and a second parameter to kill the quadratic coefficient in the fold normal form. Therefore, the cusp is an example of a local codimension-two bifurcation.

On the center manifold itself $u$ satisfies the differential equation (2.26) which we succinctly write as

$$\frac{du(t)}{dt} = A^{\odot\star} u(t) + R(u(t)) \qquad \forall\, t \in \mathbf{R}. \tag{3.6}$$

We recall from (2.13) with $\alpha_0 = 0$ that the map $R : X \to X^{\odot\star}$ is given by

$$R(\phi) = \sum_{j=1}^{n} g_j(\phi) r_j^{\odot\star} \quad \text{with} \quad r_j^{\odot\star} = (e_j, 0) \in X^{\odot\star} \tag{3.7}$$

where $g_j$ is the $j$-th component of $g$ and $e_j$ is the $j$-th vector in the standard basis of $\mathbf{R}^n$. By the smoothness of $g$ and (3.2) we may expand $R$ in a power series around the origin in $X$

starting with quadratic terms,

$$R(u) = \frac{1}{2}B(u,u) + \frac{1}{6}C(u,u,u) + O(\|u\|^4) \tag{3.8}$$

The terms $B$ and $C$ are symmetric bounded multilinear forms from $X$ to $X^{\odot\star}$. These represent the derivatives of order two and three of $R$ at the origin. The $k$-th derivative of $R$ at the origin henceforth is a mapping from $X^k$ to $X^{\odot\star}$. (See e.g. Chapter 1 of [1] for differential calculus in abstract spaces.) In the case of DDE it follows from (3.7) and the definition of the function $g$ that for arbitrary vectors $\xi_1$ and $\xi_2$ in $X$,

$$\begin{aligned} B(\xi_1,\xi_2) &= \sum_{j=1}^{n} [D^2 g(0)(\xi_1,\xi_2)]_j r_j^{\odot\star} \\ &= \sum_{j=1}^{n} [D^2 f(0)(\xi_1,\xi_2)]_j r_j^{\odot\star} \\ &= D^2 f(0)(\xi_1,\xi_2) r^{\odot\star} \end{aligned}$$

Here $f$ is the function appearing in the right-hand side of (DDE). The last line is an 'inner-like' product of $D^2 f(0)(\xi_1,\xi_2)$ in $\mathbf{R}^n$ with $r^{\odot\star} \equiv (r_1^{\odot\star},\ldots,r_n^{\odot\star})$, used for notational convenience. Analogously, $C(\xi_1,\xi_2,\xi_3) = D^3 f(0)(\xi_1,\xi_2,\xi_3) r^{\odot\star}$ and so forth.

In addition to the expansions in (3.5) and (3.8) we also expand the $C^k$-smooth center manifold mapping $\mathcal{C} : U \subset X_0 \to X$ introduced in Theorem 2.9, as follows. We recall that $U$ is some open ball around the origin and that $\mathcal{W}^c_{loc}$ is tangent to $X_0$ there. Now let $\xi$ be a point in $X_0$ with coordinate $z = \langle \phi^{\odot}, \xi\rangle$ in $\mathbf{R}^1$. Since the coordinate mapping $\xi \mapsto z(\xi)$ is a $C^k$-smooth injection onto some neighbourhood $V$ of the origin in $\mathbf{R}^1$, we may introduce a coordinate-version of $\mathcal{C}$, defined by

$$\mathcal{H} : V \subset \mathbf{R}^1 \mapsto X, \qquad \mathcal{H}(z) \equiv \mathcal{C}(\xi(z))$$

and expand it as

$$\mathcal{H}(z) = z\phi + \frac{1}{2}h_2 z^2 + \frac{1}{6}h_3 z^3 + O(|z|^4). \tag{3.9}$$

with unknown coefficients $h_\nu$ in $X$.

**Remark 3.1.** We have already used the letter $h$ to indicate the delay, following a literature convention. Hence there is the risk of confusion when using the same character for the coefficients in the expansion of $\mathcal{H}$. We expect that it will be clear from the context which of the two denotations is meant. ◊

Now we are ready to state the so-called **homological equation** (3.10) below.[1] The key to it is the invariance of the center manifold. More precisely, if $y(t)$ is the projection of the

[1] Quoting Sanders in Section 2 of [23]: 'Why this is called a homological equation is seldomly explained and

small solution $u(t)$ onto $X_0$ and $z(t)$ is its coordinate with respect to $\phi$, then

$$u(t) = \mathcal{H}(z(t)) \qquad \forall\, t \in \mathbf{R}$$

Differentiating both sides of this relation with respect to time and using (3.6) we obtain

$$A^{\odot\star}\mathcal{H}(z) + R(\mathcal{H}(z)) = D\mathcal{H}(z)\dot{z} \tag{3.10}$$

Using (3.8) and (3.5) we can substitute for $R$ and $\dot{z}$ their power series and order terms in powers of $z$. After some straightforward calculations one arrives at

$$\frac{1}{2}z^2[A^{\odot\star}h_2 + D^2f(0)(\phi,\phi)r^{\odot\star}] + \frac{1}{6}z^3[A^{\odot\star}h_3 + 3D^2f(0)(\phi,h_2)r^{\odot\star} + D^3f(0)(\phi,\phi,\phi)r^{\odot\star}] = bz^2\phi + (c\phi + bh_2)z^3 + O(|z|^4) \tag{3.11}$$

Calculating the normal form coefficients $b$ and $c$ now simply amounts to recursively solving the above equation by equating coefficients of like powers and solving the corresponding linear systems. In order to do this we need some results which are explained in the next section. We shall return to the calculation of the normal form coefficients in Section 3.3.

## 3.2 Solvability and bordered operators

We consider the operator $A^{\odot\star}$ appearing in (3.11). When solving the homological equation we shall encounter operator equations of the form

$$(\lambda I - A^{\odot\star})(v_0, v) = (w_0, w) \tag{3.12}$$

Here $\lambda$ is a real or complex number, $(w_o, w)$ is a given vector in $X^{\odot\star}$ and $(v_0, v)$ in $D(A^{\odot\star})$ is unknown. Recall that we shall use the representations in Table 2.1 for the spaces $X, X^*, X^{\odot}$ and $X^{\odot\star}$. For example, $(w_0, w)$ and $(v_0, v)$ are elements in $\mathbf{R}^n \times L^\infty([-h, 0], \mathbf{R}^n)$.

There are two possible cases: If $\lambda$ is *not* an eigenvalue of $A$ we call $\lambda$ a regular point. Then by Theorem 2.3 the closed operator

$$(\lambda I - A^{\odot\star}) : D(A^{\odot\star}) \subset X^{\odot\star} \to X^{\odot\star} \tag{3.13}$$

has a bounded inverse (the resolvent at $\lambda$) and

$$(v_0, v) = (\lambda I - A^{\odot\star})^{-1}(w_0, w) \tag{3.14}$$

this paper is written to provide an explanation of this terminology and to define the so-called unique normal form in terms of spectral sequences.'

is the unique solution in $D(A^{\odot\star})$ of (3.12). However, if $\lambda$ *is* an eigenvalue of $A$, then (3.12) need not have a solution. Moreover, any solution that does exist is not unique, since we may add to it an arbitrary linear combination of eigenvectors of $A^{\odot\star}$ corresponding to $\lambda$.

We shall use the following lemma as a solvability condition on the right-hand side of (3.12). (In fact, since decompositions analogous to (3.15) below hold for $\lambda I - A, \lambda I - A^*$ and $\lambda I - A^\odot$, it follows from the lemma that all these operators are of Fredholm type for any choice of $\lambda$, but the adjoints among them are in general not norm-densely defined.)

**Lemma 3.2** (**Fredholm Alternative**). *Let $\lambda$ be arbitrary. Then (3.12) has a solution $(v_0, v) \in D(A^{\odot\star})$ if and only if $(w_0, w)$ annihilates $N(\lambda I - A^*)$.*

*Proof.* Recall from item (ii) of Theorem 2.3 that $N(\lambda I - A^*) = N(\lambda I - A^\odot)$. Hence by the Closed Range Theorem (see e.g. Section IV.10 of [29]) the assertion of the lemma is equivalent to (3.13) having closed range, so we only need to consider the case that $\lambda$ is an eigenvalue. Suppose that the order of $\lambda$ as a pole of $z \mapsto \Delta(z)^{-1}$ is equal to $k_\lambda \in \mathbf{N}$. Then $X^{\odot\star}$ has the direct sum decomposition

$$X^{\odot\star} = N[(\lambda I - A^{\odot\star})^{k_\lambda}] \oplus R[(\lambda I - A^{\odot\star})^{k_\lambda}] \tag{3.15}$$

where the first component is finite-dimensional and the second component is closed, see Theorem IV.2.5 of [8]. Because $R(\lambda I - A^{\odot\star})$ contains $R[(\lambda I - A^{\odot\star})^{k_\lambda}]$ it too is closed by Lemma 5.6 of [24]. □

Let us now assume that at least one solution of (3.12) exists, for which we would like to obtain a representation formula. From Corollary III.2.12 of [8] we know that the adjoint generator $A^{\odot\star}$ on $X^{\odot\star}$ is given by

$$D(A^{\odot\star}) = \{(\alpha, \psi) \in X^{\odot\star} \,:\, \psi \in \mathrm{Lip}(\alpha)\}, \qquad A^{\odot\star}(\alpha, \psi) = \begin{bmatrix} \int_0^h d\eta(\theta)\psi(-\theta) \\ \dot{\psi} \end{bmatrix}$$

Here $\mathrm{Lip}(\alpha)$ is the subspace of all (classes of) functions in $L^\infty([-h, 0], \mathbf{R}^n)$ that have a Lipschitz-continuous representative with value $\alpha$ in zero. Therefore a vector $(v_0, v)$ is a solution of (3.12) if and only if $v$ is Lipschitz-continuous with $v(0) = v_0$ and satisfies

$$\begin{cases} \lambda v - \dot{v} = w \\ \lambda v_0 - \displaystyle\int_0^h d\eta(\theta) v(-\theta) = w_0 \end{cases} \tag{3.16}$$

If $\lambda$ is not an eigenvalue, then this problem has a unique solution. It can be found by variation-of-constants:

**Lemma 3.3.** *Suppose $\lambda$ is not an eigenvalue. Then the unique solution $(v_0, v) \in D(A^{\odot\star})$ of (3.12) is given by*

$$v(\theta) = e^{\lambda\theta}v_0 + \int_\theta^0 e^{\lambda(\theta-\sigma)}w(\sigma)\,d\sigma \qquad (\theta \in [-h, 0]) \tag{3.17}$$

*with*

$$v_0 = \Delta(\lambda)^{-1}\left\{w_0 + \int_0^h d\eta(\tau)\int_0^\tau e^{-\lambda\sigma}w(\sigma-\tau)\,d\sigma\right\} \tag{3.18}$$

*where $\Delta(\lambda)$ is the characteristic matrix from (2.14).*

**Corollary 3.4.** *The following two special cases will appear frequently in our calculations.*

- *Let $(w_0, w) = (w_0, 0)$. Then the solution $(v_0, v)$ of (3.12) is given by*

$$(v_0, v) = \begin{pmatrix} \Delta(\lambda)^{-1}w_0 \\ \theta \mapsto e^{\lambda\theta}\Delta(\lambda)^{-1}w_0 \end{pmatrix}$$

- *Let $(w_0, w) = (0,\ \theta \mapsto e^{\lambda\theta}\Delta(\lambda)^{-1}\zeta)$ for some fixed vector $\zeta$ in $\mathbf{R}^n$. Then*

$$(v_0, v) = \begin{pmatrix} \Delta(\lambda)^{-1}[\Delta'(\lambda) - I]\Delta(\lambda)^{-1}\zeta \\ \theta \mapsto \Delta(\lambda)^{-1}[\Delta'(\lambda) - I - \theta\Delta(\lambda)]w(\theta) \end{pmatrix}$$

*Proof.* The first representation follows immediately by substitution into (3.17) and (3.18). For the second representation, we first calculate $v_0$ from (3.18) as

$$\begin{aligned} v_0 &= \Delta(\lambda)^{-1}\int_0^h d\eta(\tau)\int_0^\tau e^{-\lambda\sigma}e^{\lambda(\sigma-\tau)}\,d\sigma\Delta(\lambda)^{-1}\zeta \\ &= \Delta(\lambda)^{-1}\int_0^h\int_0^\tau e^{-\lambda\tau}\,d\sigma\,d\eta(\tau)\Delta(\lambda)^{-1}\zeta \\ &= \Delta(\lambda)^{-1}\int_0^h \tau e^{-\lambda\tau}\,d\eta(\tau)\Delta(\lambda)^{-1}\zeta \\ &= \Delta(\lambda)^{-1}\Delta'(\lambda)\Delta(\lambda)^{-1}\zeta - \Delta(\lambda)^{-2}\zeta \\ &= \Delta(\lambda)^{-1}[\Delta'(\lambda) - I]\Delta(\lambda)^{-1}\zeta \end{aligned}$$

Substituting this into (3.17) we obtain, for $\theta \in [-h, 0]$,

$$\begin{aligned} v(\theta) &= e^{\lambda\theta}v_0 + \int_\theta^0 e^{\lambda(\theta-\sigma)}e^{\lambda\sigma}\, d\sigma\Delta(\lambda)^{-1}\zeta \\ &= e^{\lambda\theta}v_0 - \theta e^{\lambda\theta}\Delta(\lambda)^{-1}\zeta \\ &= e^{\lambda\theta}\Delta(\lambda)^{-1}[\Delta'(\lambda) - I]\Delta(\lambda)^{-1}\zeta - \theta e^{\lambda\theta}\Delta(\lambda)^{-1}\zeta \\ &= \Delta(\lambda)^{-1}[\Delta'(\lambda) - I - \theta\Delta(\lambda)]e^{\lambda\theta}\Delta(\lambda)^{-1}\zeta \\ &= \Delta(\lambda)^{-1}[\Delta'(\lambda) - I - \theta\Delta(\lambda)]w(\theta) \end{aligned}$$

□

If $\lambda$ *is* an eigenvalue, a solution of (3.12) is not unique, if it exists. In a sense the following simple lemma makes an arbitrary but (as we shall see) convenient choice among all solutions available: It singles out the solution 'orthogonal' to the eigenspace of $\lambda$. (Quotes are used because we work in Banach spaces that are not inner product spaces.)

**Lemma 3.5.** *Let $L : D(L) \subset E \to E$ be a closed, densely defined operator on a Banach space $E$. Suppose that zero is a simple eigenvalue of $L$ and $L^*$ with corresponding eigenvectors $\psi$ and $\psi^*$. Let $P$ be the spectral projection operator of $E$ onto the zero-eigenspace. Assume that for given $y^* \in E^*$ there exists a particular solution $x_0^*$ in $D(L^*)$ of the equation*

$$L^*x^* = y^* \tag{3.19}$$

*Then the augmented system*

$$\begin{cases} L^*x^* + s\psi^* = y^* \\ \langle x^*, \psi\rangle = 0 \end{cases} \tag{3.20}$$

*has a unique solution $x^* = (I - P^*)x_0^*$ and $s = 0$, and $x^*$ is the unique solution of (3.19) that annihilates $\psi$.*

Hence, if $\lambda$ is a simple eigenvalue of $A$ and (3.12) is consistent, then we can apply Lemma 3.5 to the (closed and densely defined) operator $L = \lambda I - A^\odot$ on $X^\odot$ with domain $D(A^\odot)$ to obtain the unique solution of (3.12) that vanishes on the eigenspace corresponding to $\lambda$.

*Proof of Lemma 3.5.* The adjoint $P^*$ is exactly the spectral projection of $E^*$ onto the kernel of $L^*$. It can be represented as

$$P^* = \frac{\langle \cdot, \psi\rangle}{\langle \psi^*, \psi\rangle}\psi^*$$

This shows that $x^*$ is in $D(L^*)$ and satisfies (3.20). Suppose $(x_1^*, s_1)$ is another solution. Then

$$L^*(x^* - x_1^*) + (s - s_1)\psi^* = 0 \tag{3.21}$$

and by pairing with $\psi$,

$$\langle x^* - x_1^*, L\psi\rangle + (s - s_1)\langle \psi^*, \psi\rangle = 0$$

The first term vanishes because $\psi$ is an eigenvector. Since $\langle \psi^*, \psi\rangle \neq 0$ it follows that $s_1 = s$ and from (3.21) we see that $x^* - x_1^* = \gamma\psi^*$ for some scalar $\gamma$. Pairing with $\psi$ leads to $\gamma = 0$ and thus $x_1^* = x^*$. □

One may note that (3.20) at least formally looks like the operator matrix equation

$$\begin{bmatrix} L^* & \psi^* \\ \psi & 0 \end{bmatrix} \begin{bmatrix} x^* \\ s \end{bmatrix} = \begin{bmatrix} y^* \\ 0 \end{bmatrix}$$

Such systems containing auxiliary equations and unknowns are called **bordered systems**. The unique solution $x^*$ mentioned in the lemma shall be denoted by $x^* = [L^*]^{INV} y^*$. Those familiar with [18] may notice that the above lemma is nothing more than an adaptation of equation 4.6 in this reference to the present operator setting of delay equations. For a discussion of bordered systems in $\mathbf{R}^n$ and their numerical analysis, see Chapter 3 of [13] and the references therein.

The only thing still lacking is a representation for $(\lambda I - A^{\odot\star})^{INV}$ when $\lambda$ is a simple eigenvalue, analogous to Lemma 3.3 which applies to the non-singular case.

**Proposition 3.6.** *Suppose $\lambda$ is a simple eigenvalue of $A$ and assume that (3.12) is consistent for a given $(w_0, w) \in X^{\odot\star}$ with bordered inverse $(v_0, v) = (\lambda I - A^{\odot\star})^{INV}(w_0, w)$ in $X^{\odot\star}$. Let $q \in \mathbf{R}^n, \phi \in X, p \in \underline{\mathbf{R}}^n$ and $\phi^{\odot} \in X^{\odot}$ be as in Lemma 2.4, normalized to $\langle \phi^{\odot}, \phi\rangle = 1$. Then*

$$v(\theta) = e^{\lambda\theta} v_0 + \int_\theta^0 e^{\lambda(\theta-\sigma)} w(\sigma)\, d\sigma \qquad (\theta \in [-h, 0]) \tag{3.22}$$

*with*

$$v_0 = \xi + \gamma q, \qquad \xi \equiv \Delta(\lambda)^{INV} \left[ w_0 + \int_0^h d\eta(\tau) \int_0^\tau e^{-\lambda\sigma} w(\sigma - \tau)\, d\sigma \right] \tag{3.23}$$

*The constant $\gamma$ is given by*

$$\gamma = -p\Delta'(\lambda)\xi - p \int_0^h \int_\tau^h e^{-\lambda s}\, d\eta(s) \int_{-\tau}^0 e^{-\lambda\sigma} w(\sigma)\, d\sigma\, d\tau \tag{3.24}$$

*Proof.* By assumption we know that the system (3.16) has a solution $(v_0, v)$ in $X^{\odot\star}$. By variation-of-constants every such solution is of the form (3.22) for some constant $v_0$ in $\mathbf{R}^n$. The second condition in (3.16) can then be rewritten as

$$\Delta(\lambda) v_0 = w_0 + \int_0^h d\eta(\tau) \int_0^\tau e^{-\lambda\sigma} w(\sigma - \tau)\, d\sigma$$

Since system (3.16) is consistent, there must exist at least one constant $v_0$ in $\mathbf{R}^n$ for which the

above equality holds. Consequently the bordered inverse $\Delta(\lambda)^{INV}$ appearing in the definition of $\xi$ in (3.23) is well-defined. Since the nullspace of $\Delta(\lambda)$ is spanned by $q$, the expression for $v_0$ follows for some scalar $\gamma$.

Next, we derive the value for $\gamma$ given in (3.24). In order to single out the bordered inverse $(\lambda I - A^{\odot\star})^{INV}(w_0, w)$ we see that $\gamma$ must be chosen in such a way that $\langle (v_0, v), (\phi^{\odot}(0+), \phi^{\odot})\rangle = 0$. Recall from (2.3) in Section 2.1 that

$$\langle j(x), x^{\odot}\rangle = \langle x^{\odot}, x\rangle \quad \forall\, x \in X,\ x^{\odot} \in X^{\odot}.$$

Since $v \in X$ and $(v_0, v) = j(v)$ we may evaluate

$$\begin{aligned}\langle (v_0, v), (\phi^{\odot}(0+), \phi^{\odot})\rangle &= \langle \phi^{\odot}, v\rangle \\ &= \langle \phi^{\odot}, e^{\lambda\theta}\xi + \gamma e^{\lambda\theta} q + \int_{\theta}^{0} e^{\lambda(\theta-\sigma)} w(\sigma)\, d\sigma\rangle \\ &= \gamma + \langle \phi^{\odot}, e^{\lambda\theta}\xi + \int_{\theta}^{0} e^{\lambda(\theta-\sigma)} w(\sigma)\, d\sigma\rangle\end{aligned}$$

Hence we fix $\gamma$ at

$$\gamma = -\langle \phi^{\odot}, e^{\lambda\theta}\xi + \int_{\theta}^{0} e^{\lambda(\theta-\sigma)} w(\sigma)\, d\sigma\rangle \tag{3.25}$$

Using the short-hand notation

$$\psi(\theta) \equiv e^{\lambda\theta}\xi + \int_{\theta}^{0} e^{\lambda(\theta-\sigma)} w(\sigma)\, d\sigma \qquad (\theta \in [-h, 0])$$

we evaluate the dual pairing appearing in the right-hand side of (3.25) as

$$\begin{aligned}\langle \phi^{\odot}, \psi\rangle &= p\xi + \int_{0}^{h} [\phi^{\odot}]'(\tau)\psi(-\tau)\, d\tau \\ &= p\xi + p\int_{0}^{h}\int_{\tau}^{h} e^{\lambda(\tau-s)}\, d\eta(s)\psi(-\tau)\, d\tau \\ &= p\xi + p\int_{0}^{h}\int_{\tau}^{h} e^{\lambda(\tau-s)}\, d\eta(s) e^{-\lambda\tau}\xi\, d\tau + p\int_{0}^{h}\int_{\tau}^{h} e^{\lambda(\tau-s)}\, d\eta(s)\int_{-\tau}^{0} e^{-\lambda(\tau+\sigma)} w(\sigma)\, d\sigma\, d\tau \\ &= p\xi + p\int_{0}^{h}\int_{0}^{s} e^{-\lambda s}\, d\tau\, d\eta(s)\xi + p\int_{0}^{h}\int_{\tau}^{h} e^{-\lambda s}\, d\eta(s)\int_{-\tau}^{0} e^{-\lambda\sigma} w(\sigma)\, d\sigma\, d\tau \\ &= p\Delta'(\lambda)\xi + p\int_{0}^{h}\int_{\tau}^{h} e^{-\lambda s}\, d\eta(s)\int_{-\tau}^{0} e^{-\lambda\sigma} w(\sigma)\, d\sigma\, d\tau\end{aligned}$$

In the first line we used the fact that the NBV-function $\phi^{\odot}$ is continuously differentiable on $(0, h]$ but has a jump equal to $\phi^{\odot}(0+) = p$ at zero. In the fourth line we used Fubini's theorem to interchange the order of integration in the first iterated integral. Substitution of this result into (3.25) yields (3.24). □

We shall frequently encounter the following special case.

**Corollary 3.7.** *Suppose in addition that* $(w_0, w) = (\zeta, 0) + \kappa(q, \phi)$ *where* $\zeta$ *in* $\mathbf{R}^n$ *is an arbitrary vector and* $\kappa$ *is a scalar. Then*

$$v_0 = \xi + \gamma q, \qquad v(\theta) = e^{\lambda\theta}(\xi + \gamma q - \kappa\theta q) \qquad (\theta \in [-h, 0]) \tag{3.26}$$

*with*

$$\xi = \Delta(\lambda)^{INV}(\zeta + \kappa\Delta'(\lambda)q) \quad \textit{and} \quad \gamma = -p\Delta'(\lambda)\xi + \frac{1}{2}\kappa p\Delta''(\lambda)q \tag{3.27}$$

In this case we shall employ the notation $v = B_\lambda^{INV}(\zeta, \kappa)$ to succinctly denote the bordering inverse.

## 3.3 The cusp bifurcation revisited

We now continue the discussion that we left in Section 3.1 after the derivation of (3.11). From this point on all the computations are completely analogous to those carried out for the cusp bifurcation in ODE in Section 5 of [18]. Indeed, the resulting formulas will formally (that is, in appearance) be almost identical. This is the virtue of the center manifold reduction and the relatively simple spectral theory of delay equations.

Equating coefficients of $z^2$ in the left and right-hand sides of (3.11) leads to the linear system

$$A^{\odot\star}h_2 = -D^2f(0)(\phi, \phi)r^{\odot\star} + 2b\phi \tag{3.28}$$

This equation is singular, since $\lambda = 0$ is assumed to be a simple eigenvalue of $A$. By the Fredholm Alternative (Lemma 3.2) and the chosen normalization (3.4) it has a solution if and only if

$$-\langle D^2f(0)(\phi, \phi)r^{\odot\star}, \phi^{\odot}\rangle + 2b = 0$$

which happens if and only if

$$b = \frac{1}{2}\langle D^2f(0)(\phi, \phi)r^{\odot\star}, \phi^{\odot}\rangle \tag{3.29}$$

We thus found the quadratic coefficient in the critical normal form (3.5) and recall that at the cusp bifurcation $b = 0$. Had we been interested in the fold bifurcation for which it is part of the genericity requirements that $b \neq 0$, then we could have stopped here. We note that (3.28) may now be assumed consistent, and using Lemma 3.5 we can write

$$h_2 = -[A^{\odot\star}]^{INV}D^2f(0)(\phi, \phi)r^{\odot\star} \tag{3.30}$$

for the unique solution $h_2$ in $X^{\odot\star}$ satisfying $\langle h_2, \phi^{\odot}\rangle = 0$.

Proceeding with the coefficients of $z^3$ terms we find the linear equation

$$A^{\odot\star}h_3 = c\phi - \frac{1}{6}[3D^2f(0)(\phi,h_2)r^{\odot\star} + D^3f(0)(\phi,\phi,\phi)r^{\odot\star}]$$

which is singular as well. The Fredholm Alternative implies that

$$c = \frac{1}{6}\langle 3D^2f(0)(\phi,h_2)r^{\odot\star} + D^3f(0)(\phi,\phi,\phi)r^{\odot\star}, \phi^{\odot}\rangle \tag{3.31}$$

This equation gives an expression for the cubic coefficient in the critical normal form, but we are not done yet: We still need to evaluate the 'abstract' pairings in (3.29) and (3.31).

Firstly, using Table 2.1 and Lemma 2.4 we observe that (3.29) becomes

$$b = \frac{1}{2}p \cdot D^2f(0)(\phi,\phi) \tag{3.32}$$

which cannot be made any more concrete since $f$ clearly depends on the specific system under investigation. Note, however, that the right-hand side of (3.32) is now an ordinary inner product in $\mathbf{R}^n$ which can be evaluated straightforwardly, e.g. on a computer. Analogously, (3.31) becomes

$$c = \frac{1}{6}p \cdot [3D^2f(0)(\phi,h_2) + D^3f(0)(\phi,\phi,\phi)] \tag{3.33}$$

where $h_2$ is calculated from (3.30) using Corollary 3.7 with $\zeta = -D^2f(0)(\phi,\phi)$ and $\kappa = 0$. Plugging this in and using that $\lambda = 0$ yields

$$h_2 = -\Delta(0)^{INV}D^2f(0)(\phi,\phi) + [p\Delta'(0)\Delta(0)^{INV}D^2f(0)(\phi,\phi)]q \tag{3.34}$$

which we interpret as a constant function in $X = C([-h,0],\mathbf{R}^n)$.

We conclude our discussion of the cusp bifurcation with a rather trivial example that serves to illustrate the application of the formulas just derived in the simplest possible setting. For more elaborate examples the reader is referred to Chapter 4.

**Example 3.8.** Consider the scalar DDE

$$\begin{aligned}\dot{x}(t) &= \alpha_1 x(t) + g(x(t-1)) \\ &\equiv f(x_t)\end{aligned} \tag{3.35}$$

with $g : \mathbf{R} \to \mathbf{R}$ a function of class $C^3$ satisfying $g(0) = 0$ and $\alpha_1$ a real scalar parameter. Expansion around the zero-equilibrium yields

$$\dot{x}(t) = \alpha_1 x(t) + g'(0)x(t-1) + \frac{1}{2}g''(0)[x(t-1)]^2 + \frac{1}{6}g'''(0)[x(t-1)]^3 + O([x(t-1)]^4) \tag{3.36}$$

Theorem 2.3 justifies substitution of $e^{\lambda t}$ into the linearized equation to obtain the character-

istic 'matrix'

$$\begin{aligned}\Delta(\lambda) &= \alpha_1 - \lambda + g'(0)e^{-\lambda} \\ &= \alpha_1 + g'(0) - (1 + g'(0))\lambda + \frac{1}{2}g'(0)\lambda^2 + O(\lambda^3)\end{aligned}$$

We see that $\lambda = 0$ is a simple eigenvalue provided $\alpha_1 = -g'(0) \neq 1$. This indicates the occurence of a fold bifurcation. When $\alpha_1 = -g'(0) = 1$ we observe that $\lambda$ has multiplicity two, in which case generically a Bogdanov-Takens bifurcation takes place. The latter was analyzed in Section IX.10 of [8] using the standard two-step approach of center manifold reduction and subsequent normalization. Here we shall concentrate on the former case and henceforth we assume that

$$\alpha_1 = -g'(0) \neq 1. \tag{3.37}$$

We start by regarding $\alpha_1$ as our parameter, leaving all other quantities fixed. We choose

$$q = 1, \qquad p = \frac{1}{g'(0) - 1}$$

By Lemma 2.4 for this choice the corresponding eigenvectors $\phi$ and $\phi^\odot$ are properly normalized to satisfy $\langle \phi^\odot, \phi \rangle = 1$. The second derivative of $f$ equals

$$D^2 f(0)(\xi_1, \xi_2) = g''(0)\xi_1(t-1)\xi_2(t-1) \qquad (\xi_1, \xi_2 \in X)$$

Note that this is indeed a symmetric bilinear $\mathbf{R}$-valued form on $X$. Substitution into (3.32) then yields the quadratic critical normal form coefficient

$$b = \frac{1}{2}\frac{g''(0)}{g'(0) - 1}$$

Therefore we may conclude that if (3.37) holds and the parameter $\alpha_1$ enters the system (3.35) generically, then the trivial equilibrium of (3.35) is a fold point, *provided* $g''(0) \neq 0$. This is not very surprising, since $g''(0)$ is proportional to the quadratic term in the expansion (3.36).

Next we free $\alpha_2 \equiv g''(0)$ as a second parameter. Suppose that $\alpha_2 = 0$, leading to a vanishing quadratic coefficient $b$. (Such may also occur in the one-pameter situation when (3.35) has a $\mathbf{Z}_2$-symmetry, i.e. is invariant under the substitution $x \to -x$. This illustrates the general phenomenon that symmetries lower the codimension of a bifurcation, i.e. the number of defining bifurcation conditions.) A simple calculation shows that all bordered inverses in (3.34) vanish, leaving us with $h_2 = 0$ identically. The third derivative of $f$ equals

$$D^3 f(0)(\xi_1, \xi_2, \xi_3) = g'''(0)\xi_1(t-1)\xi_2(t-1)\xi_3(t-1) \qquad (\xi_1, \xi_2, \xi_3 \in X)$$

Consequently we see from (3.33) that the cubic critical normal form coefficient is given by

$$c = \frac{1}{6}\frac{g'''(0)}{g'(0) - 1}$$

Hence if (3.37) holds and there is generic dependence on the two-dimensional parameter $\alpha = (\alpha_1, \alpha_2)$, then the zero-equilibrium of (3.35) exhibits a non-degenerate cusp singularity, *provided* $g'''(0) \neq 0$. At such a point two fold branches meet tangentially in the $(\alpha_1, \alpha_2)$ parameter plane. This causes a hysteresis effect. (See Section 8.2 of [19].) ◊

## 3.4 Computation of critical normal form coefficients

In the three foregoing sections we explained in detail how the normalization method works in the case of the simplest local codimension-two bifurcation, the cusp. To this end some auxiliary techniques were introduced in Section 3.2. In this section we first summarize the method in generality for the computation of critical normal form coefficients. Next, it is applied to derive expressions for the critical normal form coefficients of the remaining four out of five generically possible codimension-two bifurcations of equilibria in DDE.

### 3.4.1 The method

Suppose once more that at the parameter value $\alpha = \alpha_0 = 0$ the zero-function is a stationary solution of (DDE),

$$f(0, 0) = 0$$

Let $(A, D(A))$ be the generator of the semigroup $(T(t))_{t\geq 0}$ solving the linear DDE associated with the linearized equation

$$\dot{x}(t) = D_1 f(0, \alpha_0) x_t \tag{3.38}$$

and suppose that one of the bifurcation conditions in Table 3.1 is satisfied and $A$ has no other eigenvalues on the imaginary axis.

This implies the existence of a non-trivial center subspace $X_0$ of finite dimension $n_c$ and spanned by some basis $\Phi$ consisting of eigenvectors and, in the case of Bogdanov-Takens bifurcation, generalized eigenvectors corresponding to the eigenvalues of $A$ that lie on the imaginary axis. Tangent to $X_0$ there exists a local center manifold $\mathcal{W}^c_{loc}$. We consider solutions $u$ of (AIE, $\alpha_0$) that are defined and lie on $\mathcal{W}^c_{loc}$ for all (positive and negative) time. Let $y(t)$ be the projection of $u(t)$ onto $X_0$. Then $y(t)$ can be expressed uniquely relatively to $\Phi$. The corresponding coordinate vector $z(t)$ of $y(t)$ satisfies some ODE admitting an expansion of the form

$$\dot{z}(t) = \sum_{|\nu|=1}^{N} \frac{1}{\nu!} g_\nu z^\nu(t) + O(\|z(t)\|^{N+1}) \qquad \forall t \in \mathbf{R} \tag{3.39}$$

| **Name** | **Bifurcation Condition** |
|---|---|
| Cusp | $\lambda_1 = 0, b = 0$ |
| Bogdanov-Takens | $\lambda_1 = \lambda_2 = 0$ |
| Bautin (generalized Hopf) | $\lambda_{1,2} = \pm i\omega_0$, $\omega_0 > 0$, $l_1(0) = 0$ |
| Fold-Hopf | $\lambda_1 = 0$, $\lambda_{2,3} = \pm i\omega_0$, $\omega_0 > 0$ |
| Double Hopf | $\lambda_{1,4} = \pm i\omega_1$, $\lambda_{2,3} = \pm i\omega_2$, $\omega_{1,2} > 0$ |

**Table 3.1:** All generically possible two-parameter bifurcations of equilibria in DDE. The bifurcation condition lists the eigenvalues of the generator $(A, D(A))$ of the semigroup $(T(t))_{t\geq 0}$ corresponding to the solution of (3.38). The coefficient $b$ is the quadratic coefficient in the fold normal form, which vanishes in the cusp case. The *first Lyapunov coefficient* $l_1(0)$ is the cubic coefficient in the complex Hopf normal form, evaluated at criticality. It vanishes in the case of a Bautin bifurcation.

with unknown critical normal form coefficients $g_\nu \in \mathbf{R}^{n_c}$. Here $\nu$ denotes a multi-index of length $n_c$ and the series is supposed to be truncated after some sufficiently high order $N$. Clearly we shall always tacitly assume that $k \geq N$. (One may compare this expansion with (3.5). In the cusp case $n_c = 1$ and the series was truncated after third order, hence $N = 3$ there.)

On $\mathcal{W}^c_{loc}$ itself $u$ satisfies the differential equation

$$\dot{u}(t) = A^{\odot\star}u(t) + R(u(t)) \qquad \forall t \in \mathbf{R}$$

The nonlinearity $R : X \to X^{\odot\star}$ is $C^k$-smooth and can be expanded as

$$R(u) = \sum_{j>1}^{N} \frac{1}{j!} D^j f(0)(\overbrace{u, \ldots, u}^{j \text{ times}}) r^{\odot\star} + O(\|u\|^{N+1}) \tag{3.40}$$

where $D^j f(0)(\cdot)r^{\odot\star}$ is a continuous $j$-linear form from $X$ to $X^{\odot\star}$ representing the $j$th derivative of $f$ at the origin in $X$.

Let $\mathcal{H} : V \subset \mathbf{R}^{n_c} \to X$ be a $C^k$-smooth mapping, defined on a neighbourhood $V$ of the origin in the coordinate space $\mathbf{R}^{n_c}$ with image $\mathcal{H}(V) = \mathcal{W}^c_{loc}$. Then $\mathcal{H}$ admits an expansion

$$\mathcal{H}(z) = \sum_{|\nu|=1}^{N} \frac{1}{\nu!} h_\nu z^\nu + O(\|z\|^{N+1}) \tag{3.41}$$

where $\nu$ is a multi-index of length $n_c$ and $h_\nu \in X$ is an unknown coefficient.

Substituting the expansions (3.39), (3.40) and (3.41) into the homological equation

$$A^{\odot\star}\mathcal{H}(z) + R(\mathcal{H}(z)) = D\mathcal{H}(z)\dot{z} \tag{3.42}$$

and equating coefficients of like powers of $z$, one may solve recursively for the unknown

coefficients $g_\nu$ and $h_\nu$ by applying the Fredholm alternative and taking bordered inverses as discussed in Section 3.2.

### 3.4.2 List of codimension-two normal forms

We list the critical as well as parameter-dependent normal forms that we will use in this paper for the five bifurcations in Table 3.1. For a detailed derivation and bifurcation analysis of the normal forms presented here, we refer to Chapter 8 of [19].

**1. Cusp ($\lambda_1 = 0$, $b = 0$)**

The cusp bifurcation has already been treated in Sections 3.1 and 3.3. Suppose that the real coefficient $c$ appearing in (3.5) and (3.33) does not vanish. Then the projection onto the center subspace of the restriction to the local center manifold of the flow corresponding to (AIE, $\alpha$) is locally topologically equivalent to the normal form

$$\dot{z} = \beta_1(\alpha) + \beta_2(\alpha)z + cz^3 \qquad (z \in \mathbf{R}) \tag{3.43}$$

where center subspace, local center manifold and flow now all depend on the parameter $\alpha$. Here $\beta_1$ and $\beta_2$ are smooth real functions of $\alpha$ that satisfy $\beta_1(0) = 0 = \beta_2(0)$. Note that the above normal form is *exact* in the sense that higher-order terms have been dropped with impunity.

**Remark 3.9.**

(i) We deliberately treat $\beta_1$ and $\beta_2$ as functions of the original parameter $\alpha = (\alpha_1, \alpha_2)$. This allows us to make the statement leading to (3.43) without the need to mention any transversality conditions.

(ii) However, if such transversality conditions happen to be met, then the map $(\alpha_1, \alpha_2) \mapsto (\beta_1, \beta_2)$ is in fact a local diffeomorphism of parameter spaces and consequently $\beta_1$ and $\beta_2$ may be regarded as new parameters. It is in terms of these new parameters that the bifurcation diagram of (3.43) is analysed in §8.3.2 of [19]. In this case we will say that the system under investigation **depends generically on its parameters** at the bifurcation point.

(iii) The same terminology will be used for the other codimension-two bifurcations that follow. The precise transversality conditions for these bifurcations can be found in the corresponding sections of Chapter 8 of [19]. In the present work we do not address the problem of their verification. ◊

As was already noted at the end of Example 3.8, if there is generic dependence on the parameter $\alpha$, then the system (3.43) predicts a hysteresis phenomenon.

### 2. Bogdanov-Takens ($\lambda_1 = \lambda_2 = 0$)

Suppose that $a_2$ and $b_2$ are real coefficients such that $a_2 b_2 \neq 0$. Then the projection onto the center subspace of the restriction to the local center manifold of the flow corresponding to (AIE, $\alpha$) is locally topologically equivalent to the normal form

$$\begin{cases} \dot{z}_0 = z_1 \\ \dot{z}_1 = \beta_1(\alpha) + \beta_2(\alpha) z_0 + a_2 z_0^2 + b_2 z_0 z_1 \end{cases} \qquad (z \in \mathbf{R}^2) \tag{3.44}$$

where $\beta_{1,2}$ are smooth real functions of $\alpha$ satisfying $\beta_1(0) = 0 = \beta_2$. Again, the above normal form does not contain any higher-order terms. If the system depends generically on the parameters, then (3.44) predicts curves of generic fold and Hopf bifurcations emanating from the origin in the $(\alpha_1, \alpha_2)$ parameter plane, as well as a unique curve of saddle-homoclinic bifurcations. At this latter curve the unique periodic solution born in the Hopf bifurcation becomes a homoclinic orbit as its period tends to infinity.

The Bogdanov-Takens bifurcation has become a popular object of analysis in concrete systems. By computing normal form expansions only to quadratic order, one is already rewarded with the existence of a curve of global bifurcations.

In fact, in addition to quadratic formulas we shall also provide expressions for critical coefficients $a_3$ and $b_3$ of order *three* in (3.44). This is done for the purpose of discussing degeneracies due to symmetry in our first example in Chapter 4. We refrain from stating possible unfoldings, also see Remark 4.2.

### 3. Bautin (generalized Hopf) ($\lambda_{1,2} = \pm i\omega_0$, $l_1(0) = 0$)

Normal forms of Hopf-related bifurcations are conveniently cast in complex coordinates, due to the periodic nature of the underlying bifurcation. Let $l_2(0) \neq 0$ be a real constant called the *second Lyapunov coefficient*. Then the projection onto the center subspace of the restriction to the local center manifold of the flow corresponding to (AIE, $\alpha$) is locally topologically equivalent to the normal form

$$\dot{z} = (\beta_1(\alpha) + i\omega_0) z + \beta_2(\alpha) z|z|^2 + l_2(0) z|z|^4 \qquad (z \in \mathbf{C}) \tag{3.45}$$

where $\beta_{1,2}$ are smooth real functions that satisfy $\beta_1(0) = 0 = \beta_2$. Again, higher order terms need not be incorporated. At a Bautin point a Hopf bifurcation changes its criticality. The first Lyapunov coefficient $l_1(0)$ switches sign from negative (supercritical Hopf) to positive (subcritical Hopf). If there is generic dependence on parameters, then (3.45) predicts the presence of a curve in the $(\alpha_1, \alpha_2)$ parameter plane emanating from the origin and corresponding to a generic fold bifurcation of a stable and an unstable periodic orbit, born in the respective super- and subcritical Hopf bifurcations.

One needs to take care of a small subtlety. The normal form (3.45) is derived with the help of a time-reparametrization and therefore cannot be used directly in conjunction with the homological equation, since in the latter equation all time-derivatives are supposed to be with respect to the same unit of time. Therefore, instead one uses the truncated *Poincaré smooth normal form* for the Bautin bifurcation, at criticality ($\alpha = 0$) given by

$$\dot{z} = i\omega_0 z + c_1(0)z^2\overline{z} + c_2(0)z^3\overline{z}^2 \tag{3.46}$$

where $c_1(0)$ and $c_2(0)$ are complex constants. The derivation of (3.46) does not involve any reparametrization of time, see Lemmas 8.3 and 8.4 of [19]. Once $c_1(0)$ and $c_2(0)$ are computed, the two Lyapunov coefficients can be recovered from the relations

$$l_1(0) = \frac{1}{\omega_0}\mathrm{Re}\, c_1(0), \qquad l_2(0) = \frac{1}{\omega_0}\mathrm{Re}\, c_2(0)$$

**4. Fold-Hopf ($\lambda_1 = 0$, $\lambda_{2,3} = \pm i\omega_0$)**

Suppose that $b(0)$ and $c(0)$ are real constants and $b(0)c(0) \neq 0$. Then the projection onto the center subspace of the restriction to the local center manifold of the flow corresponding to (AIE, $\alpha$) is locally *smoothly* orbitally equivalent to

$$\begin{cases} \dot{z}_0 = \delta(\alpha) + b(\alpha)z_0^2 + c(\alpha)|z_1|^2 + O(\|(z_0, z_1, \overline{z}_1\|^4) \\ \dot{z}_1 = \sigma(\alpha)z_1 + d(\alpha)z_0z_1 + e(\alpha)z_0^2z_1 + O(\|(z_0, z_1, \overline{z}_1\|^4) \end{cases} \qquad (z_0 \in \mathbf{R}, z_1 \in \mathbf{C}) \tag{3.47}$$

Here $\delta, b, c$ and $e$ are smooth real-valued functions of $\alpha$, while $\sigma$ and $d$ are smooth complex-valued functions of $\alpha$, and

$$\delta(0) = 0, \qquad \sigma(0) = i\omega_0 \tag{3.48}$$

The system (3.47) is known a the *Gavrilov normal form* for the fold-Hopf bifurcation, see Lemma 8.10 of [19]. This is the first time we encounter a normal form that includes higher-order terms that cannot be truncated, since they may influence the qualitative dynamics one may find in (3.47). Therefore, a complete analysis of a fold-Hopf point in a concrete system, if at all possible, is much more complicated than in the cusp, Bogdanov-Takens or Bautin case. Assuming generic dependence on parameters, depending on the values of the critical coefficients one may encounter invariant tori, chaotic dynamics, Neimark-Sacker bifurcations of cycles and Shilnikov homoclinic bifurcations.

For the same reason as in the Bautin-case, (3.47) cannot be used directly for our computations. Rather, we need to use the *Poincaré smooth normal form* for the fold-Hopf bifurcation (see Lemma 8.9 of [19]) which at criticality reads

$$\begin{cases} \dot{z}_0 = g_{200}z_0^2 + g_{011}|z_1|^2 + g_{300}z_0^3 + g_{111}z_0|z_1|^2 + O(\|(z_0, z_1, \overline{z}_1)\|^4) \\ \dot{z}_1 = i\omega_0 z_1 + g_{110}z_0z_1 + g_{210}z_0^2z_1 + g_{021}z_1|z_1|^2 + O(\|(z_0, z_1, \overline{z}_1)\|^4) \end{cases} \tag{3.49}$$

where the coefficients $g_{jkl}$ are real in the first and complex in the second equation. At criticality and under the condition that $g_{200}g_{011} \neq 0$ the coefficients in the Gavrilov normal form can be recovered from those in the Poincaré normal form through the relations (3.48) and

$$b(0) = g_{200}, \quad c(0) = g_{011} \tag{3.50}$$

and

$$e(0) = \mathrm{Re}\left[g_{210} + g_{110}\left(\frac{\mathrm{Re}\, g_{021}}{g_{011}} - \frac{3g_{300}}{2g_{200}} + \frac{g_{111}}{2g_{011}}\right) - \frac{g_{021}g_{200}}{g_{011}}\right] \tag{3.51}$$

for the real coefficients and

$$d(0) = g_{110} - i\omega_0\frac{g_{300}}{g_{200}} \tag{3.52}$$

for the complex coefficients.

### 5. Double Hopf ($\lambda_{1,4} = \pm i\omega_1$, $\lambda_{2,3} = \pm i\omega_2$)

In addition to the bifurcation conditions, we assume the non-resonance conditions

$$k\omega_1 \neq l\omega_2 \quad \text{for all } k, l \in \mathbf{N} \text{ with } k + l \leq 5 \tag{3.53}$$

and we assume that $p_{ij}(0)$ are complex constants for $1 \leq i, j \leq 2$ such that

$$(\mathrm{Re}\, p_{11}(0))(\mathrm{Re}\, p_{12}(0))(\mathrm{Re}\, p_{21}(0))(\mathrm{Re}\, p_{22}(0)) \neq 0 \tag{3.54}$$

Then the projection onto the center subspace of the restriction to the local center manifold of the flow corresponding to (AIE, $\alpha$) is locally *smoothly* orbitally equivalent to

$$\begin{cases} \dot{z}_1 = \lambda_1(\alpha)z_1 + p_{11}(\alpha)z_1|z_1|^2 + p_{12}(\alpha)z_1|z_2|^2 + ir_1(\alpha)z_1|z_1|^4 + s_1(\alpha)z_1|z_2|^4 \\ \qquad + O(\|(z_1, \overline{z}_1, z_2, \overline{z}_2)\|^6) \\ \dot{z}_2 = \lambda_2(\alpha)z_2 + p_{21}(\alpha)z_2|z_1|^2 + p_{22}(\alpha)z_2|z_2|^2 + s_2(\alpha)z_2|z_1|^4 + ir_2(\alpha)z_2|z_2|^4 \\ \qquad + O(\|(z_1, \overline{z}_1, z_2, \overline{z}_2)\|^6) \end{cases} \tag{3.55}$$

Here $\lambda_{1,2}$, $p_{ij}$ and $s_i$ are smooth complex functions of $\alpha$ such that $\lambda_1(0) = i\omega_1$ and $\lambda_2(0) = i\omega_2$ and $r_i$ is a smooth real function of $\alpha$, for $1 \leq i, j \leq 2$. Again, (3.55) is not suitable for our computations, because it is derived using a time rescaling. So instead we use the *Poincaré smooth normal form* for the double Hopf bifurcation, at criticality given by

$$\begin{cases} \dot{z}_1 = i\omega_1 z_1 + g_{2100}z_1|z_1|^2 + g_{1011}z_1|z_2|^2 + g_{3200}z_1|z_1|^4 + g_{2111}z_1|z_1|^2|z_2|^2 \\ \qquad + g_{1022}z_1|z_2|^4 + O(\|(z_1, \overline{z_1}, z_2, \overline{z_2}\|^6) \\ \dot{z}_2 = i\omega_2 z_2 + g_{1110}z_2|z_1|^2 + g_{0021}z_2|z_2|^2 + g_{2210}z_2|z_1|^4 + g_{1121}z_2|z_1|^2|z_2|^2 \\ \qquad + g_{0032}z_2|z_2|^4 + O(\|(z_1, \overline{z_1}, z_2, \overline{z_2}\|^6) \end{cases} \tag{3.56}$$

where the constants $g_{jklm}$ are all complex. The critical coefficients in (3.55) can be recovered from those in (3.56) through the relations

$$\operatorname{Re}\begin{bmatrix} p_{11}(0) & p_{12}(0) \\ p_{21}(0) & p_{22}(0) \end{bmatrix} = \operatorname{Re}\begin{bmatrix} g_{2100} & g_{1011} \\ g_{1110} & g_{0021} \end{bmatrix} \tag{3.57}$$

The real parts of $s_i(0)$ are given by

$$\operatorname{Re} s_1(0) = \operatorname{Re} g_{1022} + \operatorname{Re} g_{1011} \times \left[ \frac{\operatorname{Re} g_{1121}}{\operatorname{Re} g_{1110}} - 2\frac{\operatorname{Re} g_{0032}}{\operatorname{Re} g_{0021}} - \frac{(\operatorname{Re} g_{3200})(\operatorname{Re} g_{0021})}{(\operatorname{Re} g_{2100})(\operatorname{Re} g_{1110})} \right]$$

and

$$\operatorname{Re} s_2(0) = \operatorname{Re} g_{2210} + \operatorname{Re} g_{1110} \times \left[ \frac{\operatorname{Re} g_{2111}}{\operatorname{Re} g_{1011}} - 2\frac{\operatorname{Re} g_{3200}}{\operatorname{Re} g_{2100}} - \frac{(\operatorname{Re} g_{2100})(\operatorname{Re} g_{0032})}{(\operatorname{Re} g_{1011})(\operatorname{Re} g_{0021})} \right]$$

The real constants $r_i(0)$ are of secondary importance in the bifurcation analysis of (3.55) and so we omit expressions for these. They can be extracted from the proof of Lemma 8.14 in [19].

The double Hopf bifurcation is the most complicated bifurcation, both from a computational as well as a conceptual viewpoint. System (3.55) is best analyzed by rewriting it in polar coordinates. As in the fold-Hopf case, the sixth-order terms may not be truncated, since they may affect the qualitative dynamics. Depending on the sign of

$$(\operatorname{Re} p_{11}(0))(\operatorname{Re} p_{22}(0)) = (\operatorname{Re} g_{2100})(\operatorname{Re} g_{0021}) \tag{3.58}$$

this bifurcation exhibits either 'simple' or 'difficult' dynamics, see §8.6.2 of [19]. Assuming generic dependence on parameters, one may encounter invariant tori, chaotic dynamics, Neimark-Sacker bifurcations of cycles and Shilnikov homoclinic orbits. Note that, although computations up to and including fifth order are required to determine *all* critical coefficients, computations up to and including third order suffice to distinguish between 'simple' and 'difficult' cases.

### 3.4.3 A remark on generality and presentation

Before we present our formulas for the various critical normal form coefficients, we pause to comment on our presentation of the various bifurcation formulas that will follow. For those readers familiar with finite-dimensional applications of the normalization method, e.g. to ODEs in Chapter 8.7 of [19] or to maps in Chapter 3 of [22], it will by now have become quite clear that all the formulas found in these references carry over, *formally*, to the case of DDE. Put simplistically, one merely need add a $\odot$ here and a $*$ there and one is done. This is entirely due to the existence of a local center manifold for DDE. Clearly then, it is to

be expected that *any* infinite-dimensional dynamical system admitting reduction to a finite-dimensional center manifold near an equilibrium is amenable to the normalization method discussed here, *provided* the generators of its linearizations are sufficiently well-behaved to allow for application of the Fredholm Alternative, i.e. they should have closed range, see the proof of Lemma 3.2.

With these considerations in mind, one could present the bifurcation formulas below in a rather general form, leaving expressions involving dual pairings as well as bordered inverse *unevaluated*. By proceeding in this way, one achieves results that are applicable to any evolution equation fitting in the general sun-star framework of Section 2.1, and not only to DDE. For instance, the formulas found would be equally well applicable to Volterra renewal equations (RE) and hybrid DDE-RE systems, see [5]. Moreover, these results would be formally almost identical to those found in the finite-dimensional case.

However, as I already expressed in the introductory Chapter 1, it is my opinion that one should not only strive for generality but also for what one might call call *evaluability*. In order to actually use the formulas found below in applications, one should be able to evaluate the dual pairings and (bordered) inverses appearing in them, ideally as ordinary inner products and matrix-vector multiplications. At this lower level of abstraction the characteristic matrix plays a prominent role, see e.g. Lemma 3.3, Proposition 3.6 and their respective corollaries. In essence it allows us to replace an operator $(A, A^*, \ldots)$ on an infinite-dimensional space by a matrix *without any additional discretization or limit procedure*. I want to stress the far-reaching consequences of this result by exploiting it to make the bifurcation formulas as explicit as possible, and this is what has been done below. I hope that the reader will see the general applicability of these formulas even though I have chosen not to present them in their most general form.

## 3.5 Critical normal form coefficients

Here we shall derive the critical coefficients for the remaining four bifurcations mentioned in Table 3.1, but we will be slightly more brief than in the cusp case.

### 3.5.1 Cusp ($\lambda_1 = 0$, $b = 0$)

The cubic normal form coefficient $c$ appearing in the cusp normal form has been derived in Section 3.3, see (3.33).

### 3.5.2 Bogdanov-Takens ($\lambda_1 = \lambda_2 = 0$)

At this bifurcation $\sigma(A)$ contains a zero-eigenvalue of geometric (algebraic) multiplicity one (two) and there are no other eigenvalues on the imaginary axis. Therefore, there exist eigen-

vectors $\phi_0$ and $\phi_1^{\odot}$ and generalized eigenvectors $\phi_1$ and $\phi_0^{\odot}$ of $A$ and $A^*$,

$$A\phi_0 = 0, \quad A\phi_1 = \phi_0, \quad A^*\phi_1^{\odot} = 0, \quad A^*\phi_0^{\odot} = \phi_1^{\odot}$$

and these span the respective generalized eigenspaces. Let $q_0, q_1$ be the column vectors and $p_1, p_0$ be the row vectors mentioned in Lemma 2.7. By an application of the Fredholm Alternative (Lemma 3.2) to the decomposition of $X$ into the direct sum of the closed range and finite-dimensional kernel of the spectral projection associated with the zero-eigenvalue, it is easy to see that it is always possible to achieve a scaling such that the following 'bi-orthogonality' relation is satisfied:

$$\langle \phi_i^{\odot}, \phi_j \rangle = \delta_{ij} \qquad (i, j = 0, 1) \tag{3.59}$$

In practise this scaling can be achieved by an application of formulas (2.19). If we let $z \in \mathbf{R}^2$ represent a coordinate vector with respect to $\{\phi_0, \phi_1\}$, then the homological equation (3.42) becomes

$$A^{\odot\star}\mathcal{H}(z) + R(\mathcal{H}(z)) = D_{z_0}\mathcal{H}(z)\dot{z}_0 + D_{z_1}\mathcal{H}(z)\dot{z}_1 \tag{3.60}$$

with $\mathcal{H}$ in (3.41) taking the form

$$\mathcal{H}(z) = z_0\phi_0 + z_1\phi_1 + \frac{1}{2}h_{20}z_0^2 + h_{11}z_0z_1 + \frac{1}{2}h_{02}z_1^2 + \frac{1}{6}h_{30}z_0^3 + \frac{1}{2}h_{21}z_0^2z_1 + \frac{1}{2}h_{12}z_0z_1^2 + \frac{1}{6}h_{03}z_1^3 + O(\|z\|^4)$$

and $\dot{z}$ given by (3.44). Note that $N = 3$ because both quadratic and cubic coefficients are sought. Collecting the $z_0^2$-terms and the $z_0z_1$-terms in (3.60) yields two singular linear systems:

$$\begin{aligned} A^{\odot\star}h_{20} &= 2a_2\phi_1 - D^2f(0)(\phi_0, \phi_0)r^{\odot\star} \\ A^{\odot\star}h_{11} &= h_{20} + b_2\phi_1 - D^2f(0)(\phi_0, \phi_1)r^{\odot\star} \end{aligned} \tag{3.61}$$

By the Fredholm Alternative the first of these has a solution if and only if

$$a_2 = \frac{1}{2}p_1 \cdot D^2f(0)(\phi_0, \phi_0) \tag{3.62}$$

which determines the first quadratic coefficient. Now that we know there exists a solution $h_{20}$ in $D(A^{\odot\star})$ wee see by virtue of (3.59) that

$$\langle \phi_1^{\odot}, h_{20} \rangle = \langle A^{\odot}\phi_0^{\odot}, h_{20} \rangle = \langle A^{\odot\star}h_{20}, \phi_0^{\odot} \rangle = -p_0 \cdot D^2f(0)(\phi_0, \phi_0)$$

Demanding solvability of the second equation in (3.61) then yields

$$b_2 = p_0 \cdot D_1^2f(0)(\phi_0, \phi_0) + p_1 \cdot D_1^2f(0)(\phi_0, \phi_1) \tag{3.63}$$

We observe that for the quadratic coefficients $a_2$ and $b_2$ no (bordered) inverses need to be evaluated.

We give expressions for the cubic coefficients $a_3$ and $b_3$ without proof. For hints in an ODE-context, see [21], where issues of simplification and computation for coefficients of orders up to and including four are discussed. We find:

$$a_3 = \frac{1}{2}p_1 \cdot D^2 f(0)(h_{20}, \phi_0) - \frac{1}{2}a_2 p_1 \cdot D^2 f(0)(\phi_1, \phi_1) + \frac{1}{6}p_1 \cdot D^3 f(0)(\phi_0, \phi_0, \phi_0) \tag{3.64}$$

and

$$\begin{aligned} b_3 = {} & \frac{1}{2}p_1 \cdot \left\{ D^2 f(0)(h_{20}, \phi_1) + 2D^2 f(0)(h_{11}, \phi_0) + D_1^3 f(0)(\phi_0, \phi_0, \phi_1) \right\} \\ & + \frac{1}{2}p_0 \cdot \left\{ 3D^2 f(0)(h_{20}, \phi_0) + D^3 f(0)(\phi_0, \phi_0, \phi_0) \right\} \\ & + a_2 p_0 \cdot D^2 f(0)(\phi_1, \phi_1) - 5a_2 \langle \phi_0^{\odot}, h_{11} \rangle - \frac{1}{2}b_2 p_1 \cdot D^2 f(0)(\phi_1, \phi_1) \end{aligned} \tag{3.65}$$

where $h_{20}$ and $h_{11}$ are found by applying the bordered inverse $\Delta(0)^{INV}$ to the respective right-hand sides of (3.61). Explicit expressions for the solutions require a non-simple counterpart to Proposition 3.6, but since our examples in Chapter 4 do not require these, we refrain from stating such a result.

### 3.5.3 Bautin (generalized Hopf) ($\lambda_{1,2} = \pm i\omega_0$, $l_1(0) = 0$)

In this case $\sigma(A)$ contains a simple purely imaginary pair $\lambda_{1,2} = \pm i\omega_0$ with $\omega_0 > 0$ and no other purely imaginary eigenvalues. Let $\phi$ and $\phi^{\odot}$ be complex eigenvectors of $A$ and $A^*$ corresponding to $\lambda_1 = +i\omega_0$ and let $q$ and $p$ be as in Lemma 2.4. It is always possible to achieve the normalization

$$\langle \phi^{\odot}, \phi \rangle = 1 \tag{3.66}$$

Any point $y$ in the *real* two-dimensional center subspace $X_0$ corresponding to $\lambda_{1,2}$ may be uniquely expressed with respect to the set $\{\phi, \overline{\phi}\}$ by means of the smooth complex coordinate mapping

$$y \mapsto (z, \overline{z}), \quad z \equiv \langle \phi^{\odot}, y \rangle$$

The homological equation presently becomes

$$A^{\odot\star}\mathcal{H}(z, \overline{z}) + R(\mathcal{H}(z, \overline{z})) = D_z\mathcal{H}(z, \overline{z})\dot{z} + D_{\overline{z}}\mathcal{H}(z, \overline{z})\dot{\overline{z}}.$$

with center manifold expansion

$$\mathcal{H}(z, \overline{z}) = z\phi + \overline{z}\overline{\phi} + \sum_{2 \le j+k \le 5} \frac{1}{j!k!} h_{jk} z^j \overline{z}^k + O(|z|^6)$$

Note that since the image of $\mathcal{H}$ lies in the real space $X$, it follows that its coefficients satisfy $h_{kj} = \overline{h}_{jk}$. The derivates $\dot{z}$ and $\dot{\overline{z}}$ are given by (3.46) and its complex conjugate. Since fifth-order terms will be computed, we set $N = 5$ in the expansion (3.40) of the non-linearity $R$.

Comparing coefficients of the quadratic terms $z^2$ and $z\overline{z}$ leads to two non-singular linear systems. Using Corollary 3.4 their solutions can be found explicitly:

$$\begin{aligned} h_{20} &= e^{2i\omega_0\theta}\Delta(2i\omega_0)^{-1}D^2f(0)(\phi,\phi) \\ h_{11} &= \Delta(0)^{-1}D^2f(0)(\phi,\overline{\phi}) \end{aligned} \tag{3.67}$$

There are two systems corresponding to the cubic terms $z^3$ and $z^2\overline{z}$, the first of which is non-singular and may be solved by ordinary inversion to yield

$$h_{30} = e^{3i\omega_0\theta}\Delta(3i\omega_0)^{-1}[3D^2f(0)(\phi,h_{20}) + D^3f(0)(\phi,\phi,\phi)] \tag{3.68}$$

while the second system equals

$$(i\omega_0 I - A^{\odot\star})h_{21} = \left[D^3f(0)(\phi,\phi,\overline{\phi}) + D^2f(0)(\overline{\phi},h_{20}) + 2D^2f(0)(\phi,h_{11})\right] r^{\odot\star} - 2c_1(0)\phi \tag{3.69}$$

By application of the Fredholm Alternative one obtains

$$\begin{aligned} c_1(0) = \frac{1}{2}p\cdot\Big[&D^2f(0)(\overline{\phi},e^{2i\omega_0\theta}\Delta(2i\omega_0)^{-1}D^2f(0)(\phi,\phi)) \\ &+ 2D^2f(0)(\phi,\Delta(0)^{-1}D^2f(0)(\phi,\overline{\phi})) + D^3f(0)(\phi,\phi,\overline{\phi})\Big] \end{aligned} \tag{3.70}$$

This expression should be compared to the expression found for the constant $c$ in Theorem X.3.9 of [8] on the direction of Hopf bifurcation for DDE. Note that $h_{21}$ is obtained by applying the bordered inverse $(i\omega_0 I - A^{\odot\star}h_{21})^{INV}$ to the right-hand side of (3.69) and that, consequently, $\langle\phi^{\odot},h_{21}\rangle = 0$. One gets:

$$h_{21} = B^{INV}_{i\omega_0}[D^3f(0)(\phi,\phi,\overline{\phi}) + D^2f(0)(\overline{\phi},h_{20}) + 2D^2f(0)(\phi,h_{11}),-2c_1(0)] \tag{3.71}$$

where we employed the notation for the bordered inverse introduced after Corollary 3.7. Also, from now on we assume that

$$l_1(0) = \frac{1}{2\omega_0}(c_1(0) + \overline{c}_1(0)) = \frac{1}{\omega_0}\mathrm{Re}\,c_1(0) = 0 \tag{3.72}$$

since otherwise the Hopf bifurcation would not be degenerate and there would be no reason to proceed with our calculations. This simplifies certain bifurcation formulas below.

Comparing coefficients of the fourth-order terms $z^4, z^3\overline{z}$ and $z^2\overline{z}^2$ leads to three non-singular linear systems. Only two of them will appear in the expression for $c_2(0)$ below.

Using the linearity of $(2i\omega_0 I - A^{\odot\star})^{-1}$ and both parts of Corollary 3.4 we find their solutions to be:

$$\begin{aligned}
h_{31} &= e^{2i\omega_0\theta}\Delta(2i\omega_0)^{-1}[D^2f(0)(\overline{\phi},h_{30}) + 3D^2f(0)(h_{20},h_{11}) + 3D^2f(0)(\phi,h_{21}) \\
&\quad + 3D^3f(0)(\phi,\overline{\phi},h_{20}) + 3D^3f(0)(\phi,\phi,h_{11}) + D^4f(0)(\phi,\phi,\phi,\overline{\phi})] \\
&\quad - 6c_1(0)\Delta(2i\omega_0)^{-1}[\Delta'(2i\omega_0) - I - \theta\Delta(2i\omega_0)]h_{20} \\
h_{22} &= \Delta(0)^{-1}[2D^2f(0)(\overline{\phi},h_{21}) + 2D^2f(0)(h_{11},h_{11}) + 2D^2f(0)(\phi,\overline{h}_{21}) \\
&\quad + D^2f(0)(h_{20},\overline{h}_{20}) + D^3f(0)(\overline{\phi},\overline{\phi},h_{20}) + D^3f(0)(\phi,\phi,\overline{h}_{20}) \\
&\quad + 4D^3f(0)(\phi,\overline{\phi},h_{11}) + D^4f(0)(\phi,\phi,\overline{\phi},\overline{\phi})]
\end{aligned} \tag{3.73}$$

As far as the fifth-order terms are concerned, only the system corresponding to the coefficient of the $z^3\overline{z}^2$-term involves $c_2(0)$. Because this system is singular, we apply the Fredholm Alternative. This leads to

$$\begin{aligned}
c_2 = \frac{1}{12}p\cdot\Big[&6D^2f(0)(h_{11},h_{21}) + 3D^2f(0)(\overline{h}_{21},h_{20}) + D^2f(0)(\overline{h}_{20},h_{30}) \\
&+ 3D^2f(0)(\phi,h_{22}) + 2D^2f(0)(\overline{\phi},h_{31}) + 6D^3f(0)(\overline{\phi},h_{20},h_{11}) \\
&+ 6D^3f(0)(\phi,h_{11},h_{11}) + 3D^3f(0)(\phi,h_{20},\overline{h}_{20}) + 6D^3f(0)(\phi,\overline{\phi},h_{21}) \\
&+ 3D^3f(0)(\phi,\phi,\overline{h}_{21}) + D^3f(0)(\overline{\phi},\overline{\phi},h_{30}) + 6D^4f(0)(\phi,\phi,\overline{\phi},h_{11}) \\
&+ 3D^4f(0)(\phi,\overline{\phi},\overline{\phi},h_{20}) + D^4f(0)(\phi,\phi,\phi,\overline{h}_{20}) + D^5f(0)(\phi,\phi,\phi,\overline{\phi},\overline{\phi})\Big]
\end{aligned} \tag{3.74}$$

with all the appearing coefficients $h_{jk}$ derived above. From $c_2(0)$ we may calculate the second Lyapunov coefficient as

$$l_2(0) = \frac{1}{\omega_0}\mathrm{Re}\, c_2(0) \tag{3.75}$$

### 3.5.4 Fold-Hopf ($\lambda_1 = 0$, $\lambda_{2,3} = \pm i\omega_0$)

At this bifurcation the spectrum $\sigma(A)$ contains a simple zero-eigenvalue $\lambda_1$, a simple purely imaginary pair $\lambda_{2,3} = \pm i\omega_0$ with $\omega_0 > 0$, and there are no other purely imaginary eigenvalues. Let $\phi_0, \phi_1, \phi_0^{\odot}$ and $\phi_1^{\odot}$ be such that

$$A\phi_0 = 0, \qquad A\phi_1 = i\omega_0\phi_1, \qquad A^*\phi_0^{\odot} = 0, \qquad A^*\phi_1^{\odot} = i\omega_0\phi_1^{\odot}$$

Akin to (3.59) it is always possible to choose these vectors such that the 'bi-orthogonality' relation

$$\langle \phi_i^{\odot}, \phi_j\rangle = \delta_{ij} \qquad (i,j = 0,1) \tag{3.76}$$

is satisfied. Furthermore, let $q_0, q_1, p_0$ and $p_1$ be corresponding column- and row vectors, as in Lemma 2.4. Any point $y$ in the *real* three-dimensional center subspace $X_0$ corresponding to $\lambda_{1,2,3}$ can be uniquely expressed with respect to the set $\{\phi_0, \phi_1, \overline{\phi}_1\}$ by means of the smooth

real-complex coordinate mapping[2]

$$y \mapsto (z_0, z_1, \overline{z}_1), \qquad z_0 = \langle \phi_0^{\odot}, y \rangle \text{ and } z_1 = \langle \phi_1^{\odot}, y \rangle \tag{3.77}$$

with $z = (z_0, z_1) \in \mathbf{R} \times \mathbf{C}$. The homological equation then becomes

$$A^{\odot\star}\mathcal{H}(z,\overline{z}) + R(\mathcal{H}(z,\overline{z})) = D_{z_0}\mathcal{H}(z,\overline{z})\dot{z}_0 + D_{z_1}\mathcal{H}(z,\overline{z})\dot{z}_1 + D_{\overline{z}_1}\mathcal{H}(z,\overline{z})\dot{\overline{z}}_1$$

with center manifold expansion given by

$$\mathcal{H}(z_0, z_1, \overline{z}_1) = z_0\phi_0 + z_1\phi_1 + \overline{z}_1\overline{\phi}_1 + \sum_{2 \le j+k+l \le 3} \frac{1}{j!k!l!} h_{jkl} z_0^j z_1^k \overline{z}_1^l + O(\|(z_0, z_1, \overline{z}_1)\|^4)$$

and with $\dot{z}$ according to (3.49). In the expansion (3.40) of the non-linearity $R$ we set $N = 3$ since we will be calculating up to and including third-order coefficients. Note that, for the same reason as in the Bautin-case, one has $h_{jlk} = \overline{h}_{jkl}$.

There are seven critical coefficients to be determined. The first three of them are found by looking at terms $z_0^j z_1^k \overline{z}_1^l$ with $j + k + l = 2$ in the homological equation. This leads to four relevant systems, two of which are singular and correspond to resonant terms in the normal form. Their solutions are:

$$\begin{aligned}
h_{200} &= B_0^{INV}[D^2 f(0)(\phi_0, \phi_0) - [p_0 \cdot D^2 f(0)(\phi_0, \phi_0)]\phi_0] \\
h_{020} &= e^{2i\omega_0}\Delta(2i\omega_0)^{-1} D^2 f(0)(\phi_1, \phi_1) \\
h_{110} &= B_{i\omega_0}^{INV}[D^2 f(0)(\phi_0, \phi_1) - [p_1 \cdot D^2 f(0)(\phi_0, \phi_1)]\phi_1] \\
h_{011} &= B_0^{INV}[D^2 f(0)(\phi_1, \overline{\phi}_1) - [p_0 \cdot D^2 f(0)(\phi_1, \overline{\phi}_1)]\phi_0]
\end{aligned} \tag{3.78}$$

By the Fredholm Alternative the appearing bordered inverses exist, provided

$$\begin{aligned}
g_{200} &= \frac{1}{2} p_0 \cdot D^2 f(0)(\phi_0, \phi_0) \quad g_{110} = p_1 \cdot D^2 f(0)(\phi_0, \phi_1) \\
g_{011} &= p_0 \cdot D^2 f(0)(\phi_1, \overline{\phi}_1)
\end{aligned} \tag{3.79}$$

which fixes the quadratic normal form coefficients.

For the four remaining coefficients we once more apply the Fredholm Alternative to the

[2]Clearly, our notation is a bit sloppy, in the sense that the second pairing in the r.h.s. of (3.77) really is the complexified counterpart of the first pairing. See Remark 2.2.

resonant $z_0^j z_1^k \overline{z}_1^l$ terms with $j+k+l=3$. This leads to the expressions

$$\begin{aligned}
g_{300} &= \frac{1}{6} p_0 \cdot [3D^2 f(0)(\phi_0, h_{200}) + D^3 f(0)(\phi_0, \phi_0, \phi_0)] \\
g_{111} &= p_0 \cdot [D^2 f(0)(\phi_0, h_{011}) + D^2 f(0)(\overline{\phi}_1, h_{110}) + D^2 f(0)(\phi_1, \overline{h}_{110}) \\
&\quad + D^3 f(0)(\phi_0, \phi_1, \overline{\phi}_1)] \\
g_{210} &= \frac{1}{2} p_1 \cdot [D^2 f(0)(\phi_1, h_{200}) + 2D^2 f(0)(\phi_0, h_{110}) + D^3 f(0)(\phi_0, \phi_0, \phi_1)] \\
g_{021} &= \frac{1}{2} p_1 \cdot [D^2 f(0)(\overline{\phi}_1, h_{020}) + 2D^2 f(0)(\phi_1, h_{011}) + D^3 f(0)(\phi_1, \phi_1, \overline{\phi}_1)]
\end{aligned} \tag{3.80}$$

for the cubic normal form coefficients, with the appearing $h_{jkl}$ derived above.

### 3.5.5 Double Hopf ($\lambda_{1,4} = \pm i\omega_1$, $\lambda_{2,3} = \pm i\omega_2$)

At this bifurcation the spectrum $\sigma(A)$ contains two pairs $\lambda_{1,4} = \pm i\omega_1$ and $\lambda_{2,3} = \pm i\omega_2$ of purely imaginary eigenvalues. We assume that $\omega_1 > \omega_2 > 0$ and there are no other eigenvalues on the imaginary axis. Additionally, we suppose that the non-resonance condition (3.53) is satisfied. Let $\phi_{1,2}$ and $\phi_{1,2}^{\odot}$ be eigenvectors of $A$ and $A^*$,

$$A\phi_1 = +i\omega_1\phi_1, \quad A\phi_2 = +i\omega_2\phi_2, \quad A^*\phi_1^{\odot} = +i\omega_1\phi_1^{\odot}, \quad A^*\phi_2^{\odot} = +i\omega_2\phi_2^{\odot}$$

and let $q_{1,2}$ and $p_{1,2}$ be corresponding column- and row vectors, as in Lemma 2.4. It is always possible to scale these vectors such that the 'bi-orthogonality' relation

$$\langle \phi_i^{\odot}, \phi_j \rangle = \delta_{ij} \qquad (1 \le i, j \le 2) \tag{3.81}$$

is satisfied. Moreover, any point $y$ in the *real* four-dimensional center subspace $X_0$ can be expressed uniquely with respect to the set $\{\phi_1, \overline{\phi}_1, \phi_2, \overline{\phi}_2\}$ by means of the smooth complex coordinate mapping

$$y \mapsto (z_1, z_2, \overline{z}_1, \overline{z}_2), \qquad z_1 = \langle \phi_1^{\odot}, y \rangle \text{ and } z_2 = \langle \phi_2^{\odot}, y \rangle$$

where $z = (z_1, z_2)$ is in $\mathbf{C}^2$. The homological equation presently reads

$$A^{\odot\star}\mathcal{H}(z,\overline{z}) + R(\mathcal{H}(z,\overline{z})) = D_{z_1}\mathcal{H}(z,\overline{z})\dot{z}_1 + D_{\overline{z}_1}\mathcal{H}(z,\overline{z})\dot{\overline{z}}_1 + D_{z_2}\mathcal{H}(z,\overline{z})\dot{z}_2 + D_{\overline{z}_2}\mathcal{H}(z,\overline{z})\dot{\overline{z}}_2$$

with $\dot{z}$ given by (3.56) and a center manifold expansion of the form

$$\begin{aligned}
\mathcal{H}(z_1, \overline{z}_1, z_2, \overline{z}_2) = z_1\phi_1 + \overline{z}_1\overline{\phi}_1 + z_2\phi_2 + \overline{z}_2\overline{\phi}_2 \\
+ \sum_{2 \le j+k+l+m \le 5} \frac{1}{j!k!l!m!} h_{jklm} z_1^j \overline{z}_1^k z_2^l \overline{z}_2^m + O(\|(z_1, \overline{z}_1, z_2, \overline{z}_2)\|^6)
\end{aligned}$$

satisying $h_{kjml} = \overline{h}_{jklm}$, and with $N = 5$ in expansion (3.40) of the non-linearity $R$.

A total of ten critical coefficients needs to be determined. We start by collecting $z_1^j \overline{z}_1^k z_2^l \overline{z}_2^m$-terms with $j+k+l+m = 2$ in the homological equation. All relevant systems are non-singular and their solutions are:

$$\begin{aligned}
h_{1100} &= \Delta(0)^{-1} D^2 f(0)(\phi_1, \overline{\phi}_1) \\
h_{2000} &= e^{2i\omega_1} \Delta(2i\omega_1)^{-1} D^2 f(0)(\phi_1, \phi_1) \\
h_{1010} &= e^{i(\omega_1+\omega_2)} \Delta(i(\omega_1+\omega_2))^{-1} D^2 f(0)(\phi_1, \phi_2) \\
h_{1001} &= e^{i(\omega_1-\omega_2)} \Delta(i(\omega_1-\omega_2))^{-1} D^2 f(0)(\phi_1, \overline{\phi}_2) \\
h_{0020} &= e^{2i\omega_2} \Delta(2i\omega_2)^{-1} D^2 f(0)(\phi_2, \phi_2) \\
h_{0011} &= \Delta(0)^{-1} D^2 f(0)(\phi_2, \overline{\phi}_2)
\end{aligned} \tag{3.82}$$

Note how the non-resonance condition is used to guarantee invertibility in the ordinary sense.

Proceeding with cubic coefficients, we encounter a total of ten relevant systems. Six of these can be solved by ordinary inversion:

$$\begin{aligned}
h_{3000} &= e^{3i\omega_1} \Delta(3i\omega_1)^{-1} [3D^2 f(0)(h_{2000}, \phi_1) + D^3 f(0)(\phi_1, \phi_1, \phi_1)] \\
h_{2010} &= e^{i(2\omega_1+\omega_2)} \Delta(i(2\omega_1+\omega_2))^{-1} [2D^2 f(0)(h_{1010}, \phi_1) + D^2 f(0)(h_{2000}, \phi_2) + D^3 f(0)(\phi_1, \phi_1, \phi_2)] \\
h_{2001} &= e^{i(2\omega_1-\omega_2)} \Delta(i(2\omega_1-\omega_2))^{-1} [2D^2 f(0)(h_{1001}, \phi_1) + D^2 f(0)(h_{2000}, \overline{\phi}_2) + D^3 f(0)(\phi_1, \phi_1, \overline{\phi}_2)] \\
h_{1020} &= e^{i(\omega_1+2\omega_2)} \Delta(i(\omega_1+2\omega_2))^{-1} [2D^2 f(0)(h_{1010}, \phi_2) + D^2 f(0)(h_{0020}, \phi_1) + D^3 f(0)(\phi_1, \phi_2, \phi_2)] \\
h_{1002} &= e^{i(\omega_1-2\omega_2)} \Delta(i(\omega_1-2\omega_2))^{-1} [2D^2 f(0)(h_{1001}, \overline{\phi}_2) + D^2 f(0)(\overline{h}_{0020}, \phi_1) + D^3 f(0)(\phi_1, \overline{\phi}_2, \overline{\phi}_2)] \\
h_{0030} &= e^{3i\omega_2} \Delta(3i\omega_2)^{-1} [3D^2 f(0)(h_{0020}, \phi_2) + D^3 f(0)(\phi_2, \phi_2, \phi_2)]
\end{aligned}$$

The remaining four systems correspond to resonant terms in the normal form. Using the Fredholm Alternative to ensure their solvability leads to the following expressions for the four cubic critical normal form coefficients:

$$\begin{aligned}
g_{2100} &= \frac{1}{2} p_1 \cdot [2D^2 f(0)(h_{1100}, \phi_1) + D^2 f(0)(h_{2000}, \overline{\phi}_1) + D^3 f(0)(\phi_1, \phi_1, \overline{\phi}_1)] \\
g_{1011} &= p_1 \cdot [D^2 f(0)(h_{0011}, \phi_1) + D^2 f(0)(h_{1001}, \phi_2) + D^2 f(0)(h_{1010}, \overline{\phi}_2) \\
&\quad + D^3 f(0)(\phi_1, \phi_2, \overline{\phi}_2)] \\
g_{1110} &= p_2 \cdot [D^2 f(0)(\overline{h}_{1001}, \phi_1) + D^2 f(0)(h_{1010}, \overline{\phi}_1) + D^2 f(0)(h_{1100}, \phi_2) \\
&\quad + D^3 f(0)(\phi_1, \overline{\phi}_1, \phi_2)] \\
g_{0021} &= \frac{1}{2} p_2 \cdot [2D^2 f(0)(h_{0011}, \phi_2) + D^2 f(0)(h_{0020}, \overline{\phi}_2) + D^3 f(0)(\phi_2, \phi_2, \overline{\phi}_2)]
\end{aligned} \tag{3.83}$$

We refrain from listing the fourth and fifth order coefficients here, since the expressions are rather lengthy and, as remarked in §3.4.2, the cubic coefficients suffice to distinguish between 'simple' and 'difficult' double Hopf points. In case higher order coefficients are desired, we

are confident that the reader will by now be able to 'translate' the coefficients given for the ODE case in [18] to the present setting.

# Chapter 4

# Examples

In this chapter we work out two examples that illustrate the application of the formulas derived in §3.5. By contrasting a relatively simple and analytically accessible DDE with a more involved and numerically challenging example, we hope to convince the reader that our results can be applied to normalization problems of a varying degree of computational complexity. Together with the simple Example 3.8 on the cusp bifurcation, the examples in this chapter exhaust the list of all but one of the codimension-two bifurcations treated in this manuscript.

In §4.1 we obtain symbolic critical normal form coefficients for a Bogdanov-Takens bifurcation in a Van der Pol equation subject to delayed feedback. We explain how a $\mathbf{Z}_2$-symmetry in the system forces us to calculate not only quadratic but also cubic coefficients and compare our findings with literature results.

In §4.2 we investigate codimension-two points on the stability boundary of the rest state of a neural mass model. We encounter double Hopf, fold-Hopf and Bautin (generalized Hopf) points and compute their critical normal forms. Our findings are presented largely in the form of a commented `Maple` worksheet. We hope that this example will serve as a sort of computational prototype for the numerical normal form analysis of other systems. The computations in this section are new.

Both examples feature a DDE of so-called *point type*, i.e. at a certain fixed time $t \in \mathbf{R}_+$ the history $x_t$ is evaluated at a *finite* number of points in the interval $[-h, 0]$, see §4.2.3 below for a precise definition. This subclass of DDE appears most often in applications and it is the only subclass that can be analysed by software tools such as `DDE-BIFTOOL` or `Knut`. (Note that, perhaps contrary to suggestion, discrete DDE are still infinite-dimensional systems.) On the other hand, DDE that are *not* of point type (in this case one sometimes speaks of *distributed delay*) do occur naturally in e.g. the theory of physiologically structured populations. Such warrants the development of new computational tools to deal with this broader class of DDE. Indeed, population theorists have recently taken up this challenge, see [3].

## 4.1 Bogdanov-Takens bifurcation in a Van der Pol oscillator

Our first example is of a mechanical nature. Consider the equation

$$\ddot{x}(t) + \varepsilon(x^2(t) - 1)\dot{x}(t) + x(t) = \varepsilon k(x(t-\tau)) \tag{4.1}$$

where $\varepsilon > 0$ is a parameter, $\tau > 0$ is a delay and $k$ is a smooth function on $\mathbf{R}$ with $k(0) = 0$. In the most direct interpretation, this equation models a unit point mass subject to nonlinear damping and delayed feedback.

- If $k \equiv 0$ then system (4.1) is the well-known Van der Pol equation (without forcing), see for example Section 2.1 of [14]. The Van der Pol equation is representative for a broader class of ordinary differential equations, known as Liénard systems, exhibiting nonlinear oscillations.

- There seems to be a recent interest in the delayed case, i.e. $k$ in (4.1) does not vanish identically. A one-parameter study of (4.1) with linear delayed feedback $k(x(t-\tau)) \propto x(t-\tau)$ was done in [31]. The authors identified a sequence of delays $\{\tau_j\}$ for which their system exhibits Hopf-bifurcations and calculated their directions. In the subsequent article [16] a normal form analysis of a Bogdanov-Takens (BT) bifurcation in (4.1) with *nonlinear* delayed feedback was performed. This analysis allowed for the occurence of degeneracies for certain choices of $k$ in (4.1).

In this first example we test the methods from Chapter 3 by performing a normal form coefficient calculation of a BT-bifurcation occuring in (4.1) for general (smooth) $k$. We are interested in comparing our results with those found in [16], both in the non-degenerate and the degenerate case. We have refrained from including `Maple` worksheets in this example, since all calculations are relatively simple and can be carried out by hand. However, for those interested the author's worksheets are available upon request by email.

We start by rewriting (4.1) as

$$\begin{cases} \dot{x}_1(t) = \tau x_2(t) \\ \dot{x}_2(t) = \tau\left\{-x_1(t) - \varepsilon(x_1^2(t) - 1)x_2(t) + \varepsilon k(x_1(t-1))\right\} \end{cases} \tag{4.2}$$

where we have rescaled time as $t \to \frac{t}{\tau}$. The advantage of this is that the delay $\tau$ can now be treated as an ordinary parameter and the history is convenienty defined on the unit interval, i.e. the phase space for (4.2) is $C([-1, 0], \mathbf{R}^2)$.

### 4.1.1 Linearization around the trivial equilibrium

Clearly, the origin is a trivial equilibrium of (4.2). Linearizing around it yields

$$\begin{cases} \dot{x}_1(t) = \tau x_2(t) \\ \dot{x}_2(t) = -\tau x_1(t) + \varepsilon\tau x_2(t) + \varepsilon\tau\alpha x_1(t-1) \end{cases}$$

where $\alpha \equiv k'(0)$. With $x \equiv (x_0, x_1)$ this can be written as

$$\dot{x}(t) = \tau \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix} x(t) + \varepsilon\tau\alpha \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-1) \tag{4.3}$$

We regard $\alpha$ and $\tau$ as control parameters and assume that $\varepsilon$ is fixed. Corresponding to (4.3) there exists a unique kernel $\eta_{\alpha,\tau} \in \mathrm{NBV}([0,1], \mathbf{R}^2)$ such that

$$\dot{x}(t) = \int_0^1 d\eta_{\alpha,\tau}(\theta) x_t(-\theta)$$

From (4.3) we infer that it is given by

$$\eta_{\alpha,\tau}(\theta) = \tau \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix} \mathbb{1}_{(0,\infty)}(\theta) + \varepsilon\tau\alpha \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbb{1}_{[1,\infty)}(\theta) \qquad (\theta \in \mathbf{R}) \tag{4.4}$$

where $\mathbb{1}_D : \mathbf{R} \to \mathbf{R}$ denotes the indicator function on $D \subseteq \mathbf{R}$,

$$\mathbb{1}_D(\theta) = \begin{cases} 1 & (\theta \in D) \\ 0 & (\theta \notin D) \end{cases}$$

Using (4.4) we calculate the characteristic equation. From (2.14) we observe that the characteristic matrix is

$$\Delta_{\alpha,\tau}(\lambda) = \lambda I - \int_0^1 e^{-\lambda\theta}\, d\eta_{\alpha,\tau}(\theta) = \begin{bmatrix} \lambda & -\tau \\ \tau(1 - \varepsilon\alpha e^{-\lambda}) & \lambda - \tau\varepsilon \end{bmatrix}$$

and thus we see that

$$\det \Delta_{\alpha,\tau}(\lambda) = \lambda^2 - \varepsilon\tau\lambda + \tau^2 - \varepsilon\tau^2\alpha e^{-\lambda} \tag{4.5}$$

The roots of this equation determine the local stability and possible bifurcation of the trivial equilibrium $x = (0,0)$ under variation of $\alpha$ and $\tau$. Since we assume that the delay $\tau$ is positive, we can write (4.5) set equal to zero as

$$\left(\frac{\lambda}{\tau}\right)^2 - \varepsilon\left(\frac{\lambda}{\tau}\right) + 1 - \varepsilon\alpha e^{-\frac{\lambda}{\tau}\tau} = 0$$

and in terms of $z \equiv \frac{\lambda}{\tau}$ this becomes

$$z^2 - \varepsilon z + 1 - \varepsilon\alpha e^{-z\tau} = 0 \tag{4.6}$$

There is a one-to-one correspondence between the roots of (4.5) and the solutions of (4.6). Indeed, let $\alpha$ and $\tau$ be given. Then $z$ is a solution of (4.6) in the left half plane (right half plane, imaginary axis) if and only if $\lambda = \tau z$ is a root of (4.5) in the left half plane (right half plane, imaginary axis). The multiplicities of $z$ and $\lambda$ are clearly the same.

**Remark 4.1.** The characteristic equation (4.6) would be the one we had found if we had not done the rescaling $t \to \frac{t}{\tau}$ which is necessary if one regards the delay as a parameter. So, although we do perform this rescaling, as far as the characteristic equation is concerned it is more convenient to work with the equivalent (4.6) instead of (4.5) because the parameter $\tau$ appears only in the exponent. One just has to be careful to correct nonzero eigenvalues with a factor $\tau$ or $\tau^{-1}$ when passing from one equation to the other. ◊

One checks that for $\alpha = \varepsilon^{-1}$ equation (4.6) has a zero root for all values of $\tau > 0$ and no other roots on the imaginary axis. Expand (4.6) in a Taylor series around $z = 0$ as

$$z^2 - \varepsilon z + 1 - \varepsilon\alpha e^{-z\tau} = 1 - \varepsilon\alpha + \varepsilon(\alpha\tau - 1)z + \frac{1}{2}(2 - \varepsilon\alpha\tau^2)z^2 + \frac{1}{6}\varepsilon\alpha\tau^3 z^3 + O(z^4)$$

Upon inspection of the coefficients of the powers in this expansion, we infer that zero is

- a simple eigenvalue if $\alpha = \varepsilon^{-1}$ and $\tau \neq \varepsilon$,
- a double eigenvalue if $\alpha = \varepsilon^{-1}$ and $\tau = \varepsilon$ with $\varepsilon \neq \sqrt{2}$,
- a triple eigenvalue if $\alpha = \varepsilon^{-1}$ and $\tau = \varepsilon$ with $\varepsilon = \sqrt{2}$.

In this example we are interested in the second case, so let us assume that

$$0 < \varepsilon < \sqrt{2} \tag{4.7}$$

It is known from [31] that under this condition all roots of (4.5) and (4.6) except the zero root are strictly in the left half-plane. Therefore the center manifold is locally attracting. We introduce new parameters $\mu = (\mu_1, \mu_2)$ defined by

$$\alpha = \frac{1}{\varepsilon} + \mu_1, \quad \tau = \varepsilon + \mu_2. \tag{4.8}$$

For $\mu = (0, 0)$ the origin undergoes a Bogdanov-Takens (BT) bifurcation with critical normal form given by (3.44) including only quadratic terms,

$$\begin{aligned} \dot{z}_0 &= z_1 \\ \dot{z}_1 &= a_2 z_0^2 + b_2 z_0 z_1 \end{aligned} \tag{4.9}$$

*provided* that $a_2 \neq 0$ and $b_2 \neq 0$. If, for example, both $a_2 = 0$ and $b_2 = 0$ due to $\mathbf{Z}_2$ symmetry, then the BT-bifurcation is degenerate and an unfolding (4.9) must be augmented with cubic terms,

$$\begin{aligned} \dot{z_0} &= z_1 \\ \dot{z_1} &= \beta_1 y_0 + \beta_2 z_1 + a_3 z_0^3 + b_3 z_0^2 z_1 \end{aligned} \tag{4.10}$$

where $(\beta_1, \beta_2)$ are new parameters. So, in order to investigate a possible degeneracy we need to calculate $a_2$ and $b_2$.

**Remark 4.2.** The third-order unfolding (4.10), required in case of degeneracy, was first studied in [27], available in reprint as [28] and re-presented in [14]. We also refer to the discussion of equation 8 in [21]. ◊

### 4.1.2 Calculation of Jordan chains

At criticality (i.e. $\mu = 0$) we suppress parameter dependence in our notation and write

$$\eta_{\mu=0} \equiv \eta, \quad \Delta_{\mu=0}(\lambda) \equiv \Delta(\lambda)$$

From (3.62) and (3.63) we see that we need (generalized) eigenvectors $\phi_{0,1}$ of the generator $A$ corresponding to $\eta$, but we only require a left Jordan chain

$$\{p_1, p_0\}$$

of $\Delta(\lambda)$ at $\lambda = 0$ and not the full (generalized) eigenvectors $\phi_{2,1}^{\odot}$ of Lemma 2.7.

**Lemma 4.3.** *Let*

$$q_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad q_1 = \begin{bmatrix} 0 \\ \varepsilon^{-1} \end{bmatrix}$$

*be column-vectors in* $\mathbf{R}^2$ *and let*

$$p_1 = \begin{bmatrix} -\varepsilon & 1 \end{bmatrix}, \quad p_0 = \begin{bmatrix} \varepsilon^{-1} & 0 \end{bmatrix}$$

*be row-vectors in* $\underline{\mathbf{R}}^2$. *Then* $\{q_0, q_1\}$ *constitutes a* right *Jordan chain of rank two for* $\Delta(0)$. *Similarly,* $\{p_1, p_0\}$ *is a* left *Jordan chain of rank two for* $\Delta(0)$.

*Proof.* According to Definition 2.6 we have to choose $c, d \in \mathbf{R}$ such that

$$\Delta(\lambda)(q_0 + \lambda q_1) = \begin{bmatrix} (1 - d\varepsilon)\lambda + c\lambda^2 \\ \varepsilon(1 - \frac{\lambda}{2})\lambda + \varepsilon c\lambda^2 - d(\varepsilon^2 - \lambda)\lambda \end{bmatrix} = O(\lambda^2) \text{ as } \lambda \to 0$$

Taking into account (4.7) we see that $c = 0$ and $d = \varepsilon^{-1}$ does the job.

Likewise, observe that $p_1^T = (-\varepsilon, 1)$ spans the nullspace of $\Delta(0)^T$. Writing $p_0^T = (c, d)$ for scalars $c, d$ to be determined, we see that

$$(p_1 + \lambda p_0)\Delta(\lambda) = \begin{bmatrix} (c - \frac{1}{2}\varepsilon + \varepsilon d)\lambda^2 & (1 - \varepsilon c - \varepsilon^2 d)\lambda + d\lambda^2 \end{bmatrix}$$

up to and including $O(\lambda^2)$. Picking $d = 0$ and $c = \varepsilon^{-1}$ assures that this expression is $O(\lambda^2)$. □

From Lemma 2.7 it now follows that

$$\phi_0(\theta) \equiv q_0, \quad \phi_1(\theta) \equiv \theta q_0 + q_1 \qquad (-1 \leq \theta \leq 0) \tag{4.11}$$

are (generalized) eigenvectors of $A$ at $\lambda = 0$ satisfying $A\phi_0 = 0$ and $A\phi_1 = \phi_0$.

Before we can compute the normal form coefficients it remains to make sure that the normalization condition (3.59) is satisfied. For this we can use the identities (2.19). Note that we have a freedom in $q_1$: If $q_1$ is a right Jordan vector such that $\phi_1(\theta) = \theta q_0 + q_1$ is a generalized eigenvector of $A$ corresponding to $\lambda = 0$ then the same is true for $q_1 + \delta q_0$ where $\delta \in \mathbf{R}$ is any constant. So we substitute $q_0$ and $p_{1,0}$ and a re-labeled

$$q_1 = \begin{bmatrix} 0 \\ \varepsilon^{-1} \end{bmatrix} + \delta q_0$$

into (2.19) and choose $\delta$ appropriately. Namely, the following normalized vectors are such that (3.59) holds:

$$q_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad q_1 = \frac{1}{3}\begin{bmatrix} \varepsilon^2(2 + \varepsilon^2)^{-1} \\ 3\varepsilon^{-1} \end{bmatrix} \tag{4.12}$$

and

$$p_1 = \frac{2\varepsilon}{2 + \varepsilon^2}\begin{bmatrix} -\varepsilon & 1 \end{bmatrix}, \qquad p_0 = \frac{2}{2 + \varepsilon^2}\begin{bmatrix} 1 & 0 \end{bmatrix} \tag{4.13}$$

### 4.1.3 Quadratic critical normal form coefficients

Now it remains to substitute the previously computed ingredients into (3.62) and (3.63). We write (4.2) at criticality as

$$\dot{x}(t) = f(x_t)$$

where the right-hand side $f : C([-1, 0], \mathbf{R}^2) \to \mathbf{R}^2$ is defined by

$$f(\varphi) \equiv \varepsilon^2 \begin{bmatrix} \varepsilon^{-1}\varphi_2(0) \\ -\varepsilon^{-1}\varphi_1(0) - (\varphi_1^2(0) - 1)\varphi_2(0) + k(\varphi_1(-1)) \end{bmatrix}$$

Here we denote by $\varphi = (\varphi_1, \varphi_2)$ the two-component function $\varphi \in C([-1, 0], \mathbf{R}^2)$.

**Remark 4.4.** Note that we use the symbol $\varphi$ as opposed to the symbol $\phi$ employed for the eigenvectors in e.g. (4.11). We hope that this will help the reader to make the distinction between the two different uses of the same Greek character. ◊

We compute the second derivative of $f$ at zero and recall that this is a continuous bilinear form from $C([-1,0],\mathbf{R}^2) \times C([-1,0],\mathbf{R}^2)$ to $\mathbf{R}^2$. For $\varphi = (\varphi_1,\varphi_2)$ and $\psi = (\psi_1,\psi_2)$ in $C([-1,0],\mathbf{R}^2)$ we find

$$D^2 f(0)(\varphi,\psi) = \varepsilon^2 \begin{bmatrix} 0 \\ k''(0)\varphi_1(-1)\psi_1(-1) \end{bmatrix} \tag{4.14}$$

**Remark 4.5.** In §4.2.3 below we discuss a method to compute such derivatives systematically. We have deliberately postponed this discussion to the second example in order not to obscure the present discussion of an easy case in which hand calculations are feasible. ◊

Using (4.14) together with (4.11), (4.12) and (4.13) we evaluate $a_2$ and $b_2$ from (3.62) and (3.63) to obtain

$$a_2 = \frac{\varepsilon^3 k''(0)}{2-\varepsilon^2}, \quad b_2 = -\frac{4\varepsilon^3(3-\varepsilon^2)k''(0)}{3(2-\varepsilon^2)^2} \tag{4.15}$$

and we conclude that these expressions are identical to those found in [16] using a more elaborate method.

There are some things worth noting:

1. Suppose that $k$ is an odd function: $k(-y) = -k(y)$. Then system (4.2) is $\mathbf{Z}_2$-equivariant: It is invariant under the substitution $x \to -x$. Because a smooth odd function has a vanishing second derivative at the origin it follows that $k''(0) = 0$ and $a_2$ and $b_2$ in (4.15) vanish and the BT-bifurcation is degenerate. The unfolding (4.9) is no longer valid, but instead we must calculate the third-order coefficients $a_3$ and $b_3$ and use (4.10). This will be done below.

2. If $\varepsilon = \sqrt{2}$ condition (4.7) is violated and $a_2$ and $b_2$ either diverge (if $k''(0) \neq 0$) or are indeterminate (if $k''(0) = 0$). Recall that for the parameter values

$$\varepsilon = \sqrt{2}, \quad \alpha = \frac{1}{\sqrt{2}}, \quad \tau = \sqrt{2}$$

$\lambda = 0$ is a triple root of the characteristic equation. This is another kind of degeneracy due to the extra zero eigenvalue. We will not deal with it here, although there are no fundamental reasons for this: The normalization method described in Chapter 3 still works, but the amount of computation increases considerably. (This is mainly due to the fact that our Jordan chains will have length three instead of two because the generalized eigenspace corresponding to $\lambda = 0$ will be three-dimensional. We henceforth also require an extended version of Lemma 2.7.)

3. If $\varepsilon = \sqrt{3}$ we see that $b_2$ vanishes while $a_2$ may or may not vanish depending on whether or not $k''(0)$ vanishes. As in Item 1 this forces us to compute the cubic coefficients $a_3$ and $b_3$ and use the higher-order unfolding (4.10).

### 4.1.4 Degeneracy due to symmetry. Cubic critical normal form coefficients

Let us assume that (4.7) holds and that

$$k''(0) = 0. \tag{4.16}$$

Then, as we saw above, the BT bifurcation is degenerate,

$$a_2 = b_2 = 0 \tag{4.17}$$

and calculation of $a_3$ and $b_3$ from formulas (3.64) and (3.65) is required. These identities are formulated in terms of $a_2$ and $b_2$ and they simplify considerably under (4.16) and (4.17). Namely, using (4.14) together with (4.16) we observe that

$$a_3 = \frac{1}{6} p_1 \cdot D^3 f(0)(\phi_0, \phi_0, \phi_0). \tag{4.18}$$

The third-order derivative appearing here is computed to be

$$\begin{aligned} [D^3 f(0)(\varphi, \psi, \chi)]_1 &= 0, \\ [D^3 f(0)(\varphi, \psi, \chi)]_2 &= -2\varepsilon^2 \{\varphi_2(0)\psi_1(0)\chi_1(0) - \varphi_1(0)\psi_2(0)\chi_1(0) \\ &\quad - \varphi_1(0)\psi_1(0)\chi_2(0) - \frac{1}{2} k^{(3)}(0)\varphi_1(-1)\psi_1(-1)\chi_1(-1)\} \end{aligned} \tag{4.19}$$

for $\varphi, \psi, \chi \in C([-1, 0], \mathbf{R}^2)$. Substituting (4.11) with (4.12) and (4.13) into (4.19) we evaluate (4.18) to obtain

$$a_3 = \frac{\varepsilon^3 k^{(3)}(0)}{3(2 - \varepsilon^2)} \tag{4.20}$$

By (4.16) and (4.17) formula (3.65) for $b_3$ reduces to

$$b_3 = \frac{1}{2} p_1 \cdot D^3 f(0)(\phi_0, \phi_0, \phi_1) + \frac{1}{2} p_0 \cdot D^3 f(0)(\phi_0, \phi_0, \phi_0)$$

which is readily evaluated using (4.19) to

$$b_3 = -\frac{2\varepsilon^2}{2 - \varepsilon^2} \left( 1 + \frac{\varepsilon(3 - \varepsilon^2) k^{(3)}(0)}{3(2 - \varepsilon^2)} \right) \tag{4.21}$$

Summarizing, under conditions (4.7) and (4.16) the BT-bifurcation of the trivial equilibrium at the critical parameter values $\alpha = \varepsilon^{-1}, \tau = \varepsilon$ unfolds on the locally attracting (parameter-

dependent) center manifold according to (4.10) with $a_3$ and $b_3$ as in (4.20) and (4.21). (For this we also have to assume that the system (4.2) depends generically on its parameters. In other words, we require that certain transversality conditions are met. Verification of this assumption is not difficult, but outside of the scope of this work.) The expressions for $a_3$ and $b_3$ are identical to those found in [16] using a more elaborate technique.

## 4.2 Codimension-two Hopf bifurcations in a neural mass model

The calculations in the previous section were of an symbolic nature and could be carried out by hand without too much difficulty. In contrast, in this section we will discuss an example that requires a numerical approach. All calculations were performed in `Maple 13` using standard double precision. Self-contained parts of the `Maple` worksheet have been reproduced below. Since the code is quite elementary, we hope that it is clear how to adapt it to other platforms. The worksheet used to perform all computations in the following subsections can be obtained by email from the author.

### 4.2.1 Model introduction

In [30] the following non-dimensionalized model of two interacting layers of neurons was considered:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) - ag(bx_1(t-\tau_1)) + cg(dx_2(t-\tau_2)) \\ \dot{x}_2(t) = -x_2(t) - ag(bx_2(t-\tau_1)) + cg(dx_1(t-\tau_2)) \end{cases} \tag{4.22}$$

We will not address modelling questions here, but only give a brief summary. The variables $x_1(t)$ and $x_2(t)$ represent the population-averaged neural activity at time $t$ in layers one and two, respectively. The parameter $a > 0$ is a measure of the strength of inhibitory feedback, while $c > 0$ measures the strength of the excitatory effect of one layer on the other. The parameters $b > 0$ and $d > 0$ are saturation rates and the delays $\tau_{1,2} > 0$ represent time lags in the inhibitory feedback loop and excitatory inter-layer connection. Finally, the function $g : \mathbf{R} \to \mathbf{R}$ is of the sigmoidal form

$$g(z) \equiv [\tanh(z-1) + \tanh(1)]\cosh^2(1) \qquad (z \in \mathbf{R}) \tag{4.23}$$

In fact, the detailed form of $g$ is not relevant to the subsequent calculations. Only the values of $g$ and its first five derivatives at zero enter the calculations. In accordance with [30] we fix the numerical values

$$b = 2.0, \quad d = 1.2, \quad \tau_1 = 12.7, \quad \tau_2 = 20.2 \tag{4.24}$$

We consider the feedback strengths $a$ and $c$ as free control parameters.

Analysis of (4.22) by the authors of [30] is still in progress. I am happy that they nonetheless allowed me to use their model as a test case for the normalization techniques decribed in this manuscript.

### 4.2.2 Linearization and the characteristic equation

It is apparent from (4.22) and (4.23) that the origin $(x_1, x_2) = (0, 0)$ is an equilibrium for all possible parameter values. Since the system is symmetric with respect to the interchange of the labels of layers one and two, equilibria are always of the form $(x_1, x_2) = (x^*, x^*)$ for some $x^* \in \mathbf{R}$. Clearly, only the case $x^* \geq 0$ is physically relevant. In total, the number of simultaneously present equilibria lies between one and three and they are found as solutions of the transcendental equation

$$x^* + ag(bx^*) - cg(dx^*) = 0 \tag{4.25}$$

Note that this equation (and henceforth the location of equilibria) does not depend on the values of the delays $\tau_1$ and $\tau_2$. In Figure **??** a specific one-parameter bifurcation diagram is presented. Obviously this diagram is not exhaustive, but it serves to give an impression of the behaviour of solutions of (4.25).

Linearizing (4.22) around an equilibrium $(x^*, x^*)$ yields the system

$$\begin{cases} \dot{x}_1(t) = -x_1(t) - k_1 x_1(t - \tau_1) + k_2 x_2(t - \tau_2) \\ \dot{x}_2(t) = -x_2(t) - k_1 x_2(t - \tau_1) + k_2 x_1(t - \tau_2) \end{cases} \tag{4.26}$$

where

$$k_1 \equiv abg'(bx^*), \qquad k_2 \equiv cdg'(dx^*) \tag{4.27}$$

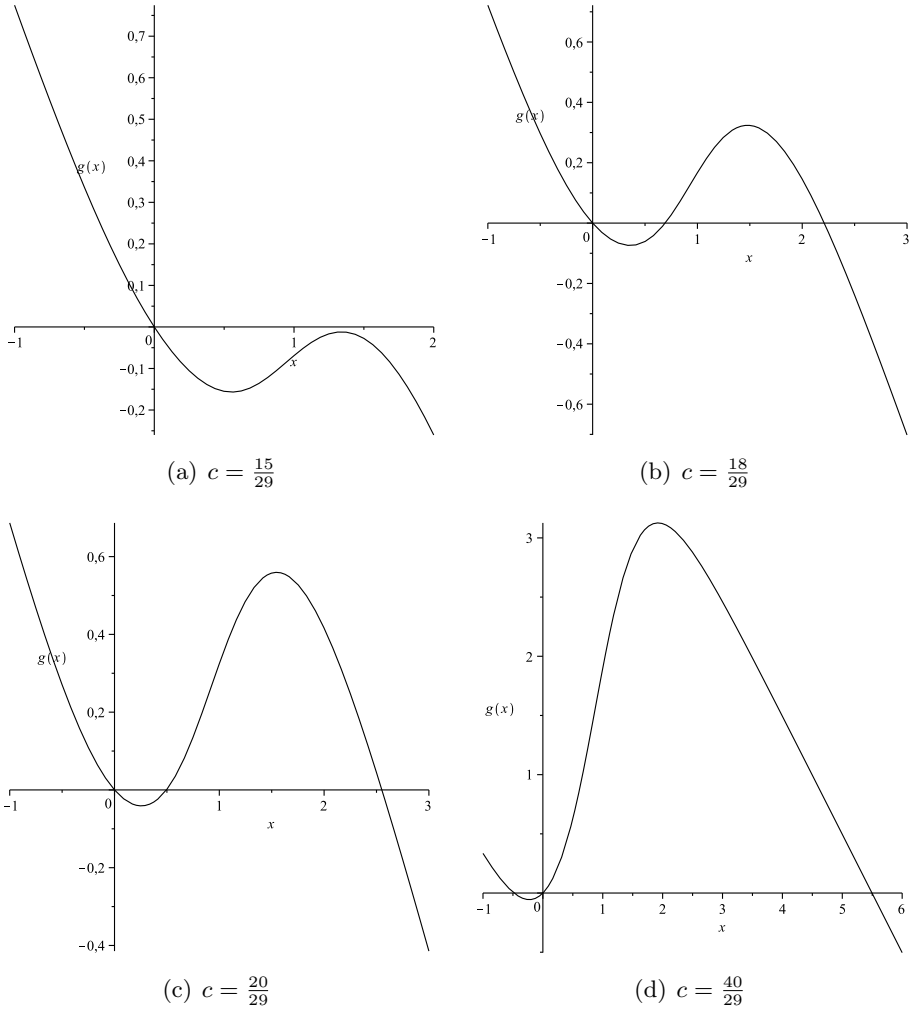
Introducing $x \equiv (x_0, x_1)$ we can write (4.26) in the form

$$\dot{x}(t) = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) - k_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t - \tau_1) + k_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t - \tau_2) \tag{4.28}$$

From here on we will focus exclusively on the trivial equilibrium $(x^*, x^*) = (0, 0)$. This equilibrium corresponds to a quiescent state in which both neural layers are at rest. Following [30] we will analyse the linearisation in terms of $k_1$ and $k_2$, noting from (4.27) that

$$a = \frac{k_1}{bg'(0)}, \qquad c = \frac{k_2}{dg'(0)} \tag{4.29}$$

so there is a one-to-one correspondence between critical values of the control parameters $a$ and $c$ on the one hand and $k_{1,2}$ on the other hand. Since we have no desire to treat one of the delays $\tau_{1,2}$ as a bifurcation parameter, there is no need to perform a scaling by the delay




(a) $c = \frac{15}{29}$ (b) $c = \frac{18}{29}$

(c) $c = \frac{20}{29}$ (d) $c = \frac{40}{29}$

**Figure 4.1:** The graph of the left-hand side of (4.25) as a function of $x^*$ for $a = \frac{2}{29}$, $b$ and $d$ as in (4.24) and values of the parameter $c$ as indicated. The origin is always an equilibrium. In (a) it is the only equilibrium present, but in (b) it coexists with two non-zero equilibria that were born in a saddle-node bifurcation occuring for some $c \in (\frac{15}{29}, \frac{18}{29})$. In (c) this situation persists qualitatively, but in (d) a transcritical bifurcation has lead to an exchange of stability between the trivial equilibrium and the left-most member of the saddle-node pair.

time as in §4.1. Setting $h \equiv \max(\tau_1, \tau_2) > 0$ we observe that corresponding to (4.28) there exists a unique kernel $\eta_{k_1,k_2} \in \mathrm{NBV}([0,h], \mathbf{R}^2)$ such that

$$\dot{x}(t) = \int_0^h d\eta_{k_1,k_2}(\theta) x_t(-\theta)$$

From (4.28) we see that $\eta_{k_1,k_2}$ is given by

$$\eta_{k_1,k_2}(\theta) = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbb{1}_{(0,\infty)}(\theta) - k_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbb{1}_{[\tau_1,\infty)}(\theta) + k_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbb{1}_{[\tau_2,\infty)}(\theta) \tag{4.30}$$

for $\theta \in \mathbf{R}$. Hence the characteristic matrix is

$$\Delta_{k_1,k_2}(\lambda) = \begin{bmatrix} \lambda + 1 + k_1 e^{-\lambda\tau_1} & -k_2 e^{-\lambda\tau_2} \\ -k_2 e^{-\lambda\tau_2} & \lambda + 1 + k_1 e^{-\lambda\tau_1} \end{bmatrix} \tag{4.31}$$

the determinant of which leads to the characteristic equation

$$\Delta^+_{k_1,k_2}(\lambda)\Delta^-_{k_1,k_2}(\lambda) = 0 \tag{4.32}$$

with

$$\Delta^\pm_{k_1,k_2} \equiv 1 + \lambda + k_1 e^{-\lambda\tau_1} \pm k_2 e^{-\lambda\tau 2}$$

As was shown in [30] and can easily be checked by the reader, fold bifurcations occur on the curves in the $(k_1, k_2)$-plane defined by the equations

$$1 + k_1 + k_2 = 0, \qquad 1 + k_1 - k_2 = 0 \tag{4.33}$$

while Hopf bifurcations from an eigenvalue $i\omega$ occur on the curves parametrized by $\omega$ as

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{1}{\sin(\omega(\tau_2 - \tau_1))} \begin{bmatrix} \sin\omega\tau_2 & -\cos\omega\tau_2 \\ -\sin\omega\tau_1 & \cos\omega\tau_1 \end{bmatrix} \begin{bmatrix} -1 \\ \omega \end{bmatrix} \qquad (\omega > 0) \tag{4.34}$$

and

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \frac{1}{\sin(\omega(\tau_2 - \tau_1))} \begin{bmatrix} \sin\omega\tau_2 & -\cos\omega\tau_2 \\ \sin\omega\tau_1 & -\cos\omega\tau_1 \end{bmatrix} \begin{bmatrix} -1 \\ \omega \end{bmatrix} \qquad (\omega > 0) \tag{4.35}$$

with singularities for

$$\omega = \frac{m\pi}{\tau_2 - \tau_1} \equiv \omega_s(m) \qquad (m \in \mathbf{N}) \tag{4.36}$$

It will come as no surprise that analytical results are difficult to obtain for the characteristic equation (4.32). Some observations can however be made. In [30] it was shown that the origin is a locally stable equilibrium of (4.22) for $(k_1, k_2)$ strictly contained in the circle of radius $\frac{1}{2}\sqrt{2}$ centered at the origin in the $(k_1, k_2)$ parameter plane.

In Figure 4.2 we depict and discuss the structure of the fold and Hopf bifurcation curves in the $(k_1, k_2)$-plane near the origin. It is seen that the quiescent (zero) equilibrium typically looses its stability in an Andronov-Hopf bifurcation, but additionally various codimension-two points exist. In §4.2.5 we will compute the direction of Hopf bifurcation and give a precise account of all the relevant codimension-two points that exist in Figure 4.2.

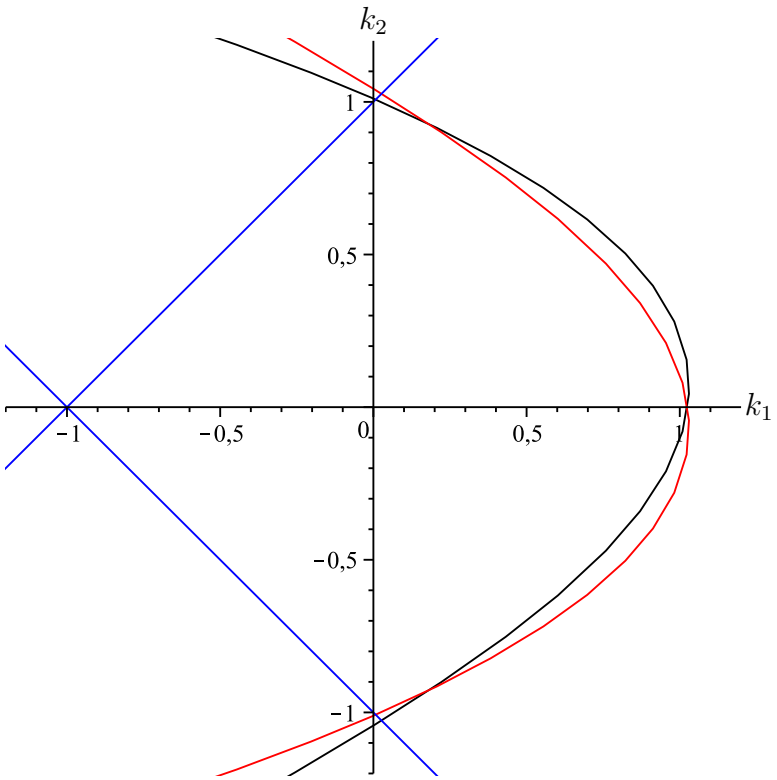



**Figure 4.2:** The $(k_1, k_2)$-parameter plane near the origin for parameter values as in (4.24). Shown in black and red are curves of Hopf bifurcation, parametrized by (4.34) and (4.35), respectively. The straight blue lines correspond to the fold curves (4.33). The curve parameter $\omega$ was varied in the interval $[0, \omega_s(1)]$ with $\omega_s(1)$ as in (4.36). For values $\omega > \omega_s(1)$ the plane becomes gradually filled with Hopf curves, but the stable region surrounding the origin remains unaltered. As can be seen from the curve parametrizations, the fold curves do not depend on any fixed system parameters (excluding the free parameters $k_1$ and $k_2$) while the Hopf curves depend only on $\tau_1$ and $\tau_2$. We note a symmetry between the upper and lower half-plane and remark that only the positive quadrant is physically relevant, although in general bifurcation points sufficiently close to the quadrant's boundary may still be 'felt' in the quadrant's interior. However, such points do not exist in the above figure. For example, the Bogdanov-Takens point at the crossing of the two blue lines lies too far in the negative half-plane to warrant analysis.

In order to do this, we need some preparations. In the following subsection we take a break and divert from the main course of argument by explaining how the bookkeeping of higher-order derivatives of the system (4.22) may be done in a clear, efficient manner.

### 4.2.3 Intermezzo. Symbolic calculation of higher-order derivatives

Consider a DDE of the form

$$\dot{x}(t) = f(x_t) \tag{4.37}$$

where $f : C([-h, 0], \mathbf{R}^n) \to \mathbf{R}^n$ is at least five times continuously differentiable. Suppose that the delay in (4.37) is of point-type, i.e. there is $r \in \mathbf{N}$, there are

$$0 = \tau_0 < \tau_1 < \tau_2 < \ldots < \tau_r = h \tag{4.38}$$

and there exists a function $G \in C^5(\mathbf{R}^{n\times(r+1)}, \mathbf{R}^n)$ such that

$$f(\varphi) = G(\Phi) \tag{4.39}$$

where $\Phi = \left\{\varphi_j^k\right\}_{\substack{j=1\\k=0}}^{n,r}$ is the $n \times (r+1)$ matrix

$$\left\{\varphi_j^k\right\}_{\substack{j=1\\k=0}}^{n,r} \equiv \begin{bmatrix} \varphi_1(-\tau_0) & \ldots & \varphi_1(-\tau_r) \\ \varphi_2(-\tau_0) & \ldots & \varphi_2(-\tau_r) \\ & \vdots & \\ \varphi_n(-\tau_0) & \ldots & \varphi_n(-\tau_r) \end{bmatrix} \qquad (\varphi \in C([-h, 0], \mathbf{R}^n)) \tag{4.40}$$

Here we again use $\varphi$ instead of $\phi$ to avoid confusion with our notation for eigenvectors, see Remark 4.4.

In this subsection we show how to compute derivatives of $f$ in terms of derivatives of $G$. (All derivatives are implicitly understood to be Fréchet-derivatives, as usual.) This is not difficult, but one has to be a bit careful with notation. We recall that for $\ell \in \mathbf{N}$ the $\ell$th-order derivative $D^\ell f(0)$ of $f$ at zero is a bounded $\ell$-linear form from $C([-h, 0], \mathbf{R}^n)$ to $\mathbf{R}^n$. We denote the derivative at zero of the $i$th component of $G$ with respect to its $(j, k)$th variable as $D_j^k G_i(0) \in \mathbf{R}$. Likewise, the second derivative at zero of the $i$th component of $G$ with respect to its $(j_1, k_1)$th and $(j_2, k_2)$th variables is denoted by $D_{j_1 j_2}^{k_1 k_2} G_i(0) \in \mathbf{R}$ and so on for higher-order derivatives of $G$. When $\ell = 1$ we recover the ordinary first-order derivative $Df(0) = D^1 f(0)$, the $i$th component of which is given by

$$D^1 f_i(0)\varphi = \sum_{j_1=1}^{n} \sum_{k_1=0}^{r} D_{j_1}^{k_1} G_i(0) \varphi_{j_1}^{k_1} \qquad (\varphi \in C([-h, 0]) \tag{4.41}$$

The second derivative $D^2 f(0)$ takes two functions in $C([-h, 0], \mathbf{R}^n)$ as input and produces

from these a vector in $\mathbf{R}^n$. Thus the $i$th component is given by

$$\begin{aligned} D^2 f_i(0)(\varphi,\psi) &= \sum_{j_1,j_2=1}^{n} \sum_{k_1,k_2=0}^{r} D^{k_1 k_2}_{j_1 j_2} G_i(0) \varphi^{k_1}_{j_1} \psi^{k_2}_{j_2} \\ &\equiv D^{k_1 k_2}_{j_1 j_2} G_i(0) \varphi^{k_1}_{j_1} \psi^{k_2}_{j_2} \end{aligned} \qquad (\varphi,\psi \in C([-h,0],\mathbf{R}^n) \qquad (4.42)$$

where in the second line we used the convention that repeated lower (or upper) indices imply summation. This is convenient for typographical reasons when higher order derivatives are involved, but we will not use it except in (4.42) and (4.43) below. By now it is probably clear how to compute these higher order derivatives. We have, up to order five (since this is what is required for the normal forms in §3.5),

$$\begin{aligned} D^3 f_i(0)(\varphi,\psi,\chi) &= D^{k_1 k_2 k_3}_{j_1 j_2 j_3} G_i(0) \varphi^{k_1}_{j_1} \psi^{k_2}_{j_2} \chi^{k_3}_{j_3} \\ D^4 f_i(0)(\varphi,\psi,\chi,\zeta) &= D^{k_1 k_2 k_3 k_4}_{j_1 j_2 j_3 j_4} G_i(0) \varphi^{k_1}_{j_1} \psi^{k_2}_{j_2} \chi^{k_3}_{j_3} \zeta^{k_4}_{j_4} \\ D^5 f_i(0)(\varphi,\psi,\chi,\zeta,\eta) &= D^{k_1 k_2 k_3 k_4 k_5}_{j_1 j_2 j_3 j_4 j_5} G_i(0) \varphi^{k_1}_{j_1} \psi^{k_2}_{j_2} \chi^{k_3}_{j_3} \zeta^{k_4}_{j_4} \eta^{k_5}_{j_5} \end{aligned} \qquad (4.43)$$

for $\varphi,\psi,\chi,\zeta,\eta \in C([-h,0],\mathbf{R}^n)$ and $i = 1,\ldots,n$.

The following simple `Maple` code computes the second and third order derivatives $D^2 f(0)$ and $D^3 f(0)$ pertaining to the system (4.22) but is set up to be easily adaptable to other systems. The first derivative is computed as well, so it can be compared to the linearization (4.26). We start by defining the function $G$ from (4.39) and (4.37). Using $y \equiv \{y\}^{n,r}_{j=1,k=0}$ as a placeholder variable, we enter

```
> restart;
> n:=2; r:=2;
> G:=<-y[1,0]-a*g(b*y[1,1])+c*g(d*y[2,2]),-y[2,0]-a*g(b*y[2,1])+c*g(d*y[1,2])>;
```

We set up three, five and seven dimensional arrays `A`, `B` and `C` to store the first, second and third order derivatives of $G$:

```
> A:=Array(1..n,1..n,0..r);
> B:=Array(1..n,1..n,0..r,1..n,0..r);
> C:=Array(1..n,1..n,0..r,1..n,0..r,1..n,0..r);
```

The first index runs over the components of $G$, the second and third indicate differentiation with respect to the $(j_1,k_1)$th variable, the fourth and fifth correspond to differentiation with respect to the $(j_2,k_2)$th variable, and so on. Hence, the arrays are filled as follows.

```
> for i from 1 to n do
>    for j from 1 to n do
>       for k from 0 to r do
>          A[i,j,k]:=eval(diff(G[i],y[j,k]),y=0);
```

```
>       end do:
>    end do:
> end do:
```

for `A`,

```
> for i from 1 to n do
>    for j1 from 1 to n do
>       for k1 from 0 to r do
>          for j2 from 1 to n do
>             for k2 from 0 to r do
>                B[i,j1,k1,j2,k2]:=eval(diff(G[i],y[j1,k1],y[j2,k2]),y=0);
>             end do:
>          end do:
>       end do:
>    end do:
> end do:
```

for `B`, and

```
> for i from 1 to n do
>    for j1 from 1 to n do
>       for k1 from 0 to r do
>          for j2 from 1 to n do
>             for k2 from 0 to r do
>                for j3 from 1 to n do
>                   for k3 from 0 to r do
>                      C[i,j1,k1,j2,k2,j3,k3]:=
>                      eval(diff(G[i],y[j1,k1],y[j2,k2],y[j3,k3]),y=0);
>                   end do:
>                end do:
>             end do:
>          end do:
>       end do:
>   end do:
> end do:
```

for `C`. Next, we define three $n$-vectors to store the derivatives of $f$,

```
> DF:=Array(1..n);
> D2F:=Array(1..n);
> D3F:=Array(1..n);
```

and we fill these with products of elements of `A`, `B` and `C` and arbitrary $n \times (r+1)$ arrays `PHI`, `PSI` and `CHI`, as in (4.41), (4.42) and (4.43).

```
> for i from 1 to n do
>    DF[i]:=add(add(A[i,j,k]*PHI[j,k],j=1..n),k=0..r);
> end do:

> for i from 1 to n do
>    D2F[i]:=add(add(add(add(
>            B[i,j1,k1,j2,k2]*PHI[j1,k1]*PSI[j2,k2],
>            j1=1..n),k1=0..r),j2=1..n),k2=0..r);
> end do:
```

and

```
> for i from 1 to n do
>    D3F[i]:=add(add(add(add(add(add(
>            C[i,j1,k1,j2,k2,j3,k3]*PHI[j1,k1]*PSI[j2,k2]*CHI[j3,k3],
>            j1=1..n),k1=0..r),j2=1..n),k2=0..r),j3=1..n),k3=0..r);
> end do:
```

which yields

```
> DF[1];
> DF[2];
```

$$\begin{aligned} &-\Phi_{1,0} - abg'(0)\Phi_{1,1} + cdg'(0)\Phi_{2,2} \\ &-\Phi_{2,0} - abg'(0)\Phi_{2,1} + cdg'(0)\Phi_{1,2} \end{aligned} \tag{4.44}$$

for the first derivative $Df(0)\varphi$. We note that the resulting expression agrees with the linearisation obtained earlier in (4.26). Likewise, we find

```
> D2F[1];
> D2F[2];
```

$$\begin{aligned} &-ab^2g^{(2)}(0)\Phi_{1,1}\Psi_{1,1} + cd^2g^{(2)}(0)\Phi_{2,2}\Psi_{2,2} \\ &-ab^2g^{(2)}(0)\Phi_{2,1}\Psi_{2,1} + cd^2g^{(2)}(0)\Phi_{1,2}\Psi_{1,2} \end{aligned} \tag{4.45}$$

and

```
> D3F[1];
> D3F[2];
```

$$\begin{aligned} &-ab^3g^{(3)}(0)\Phi_{1,1}\Psi_{1,1}X_{1,1} + cd^3g^{(3)}(0)\Phi_{2,2}\Psi_{2,2}X_{2,2} \\ &-ab^3g^{(3)}(0)\Phi_{2,1}\Psi_{2,1}X_{2,1} + cd^3g^{(3)}(0)\Phi_{1,2}\Psi_{1,2}X_{1,2} \end{aligned} \tag{4.46}$$

So, if one wants to evaluate $Df(0)$, $D^2f(0)$ or $D^3f(0)$ at concrete choices for $\varphi$, $\psi$ and $\chi$, one forms the corresponding arrays `PHI`, `PSI` and `CHI` and evaluates the above expressions `DF`, `D2F` or `D3F`.

A drawback of expressions such as (4.44), (4.45) and (4.46) is that the 'user' is required to convert each function (such as $\varphi$) to an array (such as $\Phi$) before the derivatives may be evaluated. In §4.2.4 we will therefore replace the expressions `DF`, `D2F` and `D3F` by `Maple` procedures. These procedures will handle the function $\to$ array conversion themselves.

### 4.2.4 System specification in Maple

The computations that follow are convenienty performed in a computer algebra system such as `Maple`, capable of both symbolic as well as numerical calculations. To present our results, we proceed as follows. In the present subsection we propose an 'initialization' of the system (4.22), its fixed parameters and derivatives. The reader who is interested in checking our computations should enter this code into an empty `Maple` worksheet. In the later subsections we will add bits of code to this worksheet to perform various normal form computations.

From here on, we adopt the convention that variables writting in `verbatim` font correspond to `Maple` variables. So, if we write `q` or `lambda` we refer to `Maple` variables and these correspond to the 'ordinary' variables $q$ and $\lambda$ in the main text. We start with

```
> restart;
> with(LinearAlgebra):
> Digits:=15;
> interface(showassumed=0);
> assume(theta,'real');
```

Then we define system parameters. The parameters `n` and `r` are the dimension of the system (4.37) and the number of delays. The other parameters are as in (4.24). Note that `tau[0]` is the 'zero-delay' appearing in (4.38).

```
> n:=2; r:=2;                                                    (*)
> b:=2; d:=1.2; tau[0]:=0.0; tau[1]:=12.7; tau[2]:=20.2;
```

These parameters are global in the worksheet. They are read (but not altered) by the various procedures that follow. Next, we code the function $g$ from (4.23) and its derivatives (up to and including order three for double Hopf and fold-Hopf points and up to and including order five for Bautin points) at zero.

```
> g:=z->(tanh(z-1.0)+tanh(1.0))*cosh(1.0)^2;
> dg:=D(g)(0);
> d2g:=(D@@2)(g)(0);
> d3g:=(D@@3)(g)(0);
```

```
> d4g:=(D@@4)(g)(0);
> d5g:=(D@@5)(g)(0);
```

We also provide the characteristic matrix as a function of $\lambda$, as well as its first two derivatives. (The second derivative is not needed in the present subsection, but it will be required in §4.2.7 below.)

```
> Delta:=lambda->
>        Matrix([[lambda+1+k1*exp(-lambda*tau[1]),-k2*exp(-lambda*tau[2])],
>                [-k2*exp(-lambda*tau[2]),lambda+1+k1*exp(-lambda*tau[1])]]);
> DDelta:=lambda->eval(map(diff,Delta(z),z),z=lambda);
> D2Delta:=lambda->eval(map(diff,DDelta(z),z),z=lambda);
```

Moreover, we provide procedures `D2F3` and `D3F` that take as input two or three expressions (*not* functions) in $\mathbf{R}^n$ depending on a variable `theta` and output the second or third Fréchet-derivative of the right-hand side of (4.22) at zero, evaluated at the input expression(s). For these procedures we use the results (4.45) and (4.46) found in the previous subsection. The conversion of input expressions into arrays (see the remark at the end of §4.2.3) is done internally. For `D2F` we have

```
> D2F:=proc(phi,psi)
> local PHI,PSI,i,j;
> global a,b,c,d,n,r,g,tau;
>
> PHI:=Array(1..n,0..r);
> PSI:=Array(1..n,0..r);
>
> for i from 1 to n do
>    for j from 0 to r do
>       PHI[i,j]:=eval(phi[i],theta=-tau[j]);
>       PSI[i,j]:=eval(psi[i],theta=-tau[j]);
>    end do;
> end do;
>
> return(<-a*d2g*b^2*PHI[1,1]*PSI[1,1]+c*d2g*d^2*PHI[2,2]*PSI[2,2],
>         -a*d2g*b^2*PHI[2,1]*PSI[2,1]+c*d2g*d^2*PHI[1,2]*PSI[1,2]>)
>
> end:
```

while for `D3F` we have

```
> D3F:=proc(phi,psi,chi)
```

```
> local PHI,PSI,CHI,i,j;
> global a,b,c,d,n,r,g,tau;
>
> PHI:=Array(1..n,0..r);
> PSI:=Array(1..n,0..r);
> CHI:=Array(1..n,0..r);
>
> for i from 1 to n do
>    for j from 0 to r do
>       PHI[i,j]:=eval(phi[i],theta=-tau[j]);
>       PSI[i,j]:=eval(psi[i],theta=-tau[j]);
>       CHI[i,j]:=eval(chi[i],theta=-tau[j]);
>    end do;
> end do;
>
> return(<-a*d3g*b^3*PHI[1,1]*PSI[1,1]*CHI[1,1]
>         +c*d3g*d^3*PHI[2,2]*PSI[2,2]*CHI[2,2],
>         -a*d3g*b^3*PHI[2,1]*PSI[2,1]*CHI[2,1]
>         +c*d3g*d^3*PHI[1,2]*PSI[1,2]*CHI[1,2]>)
>
> end:
```

This suffices for double Hopf and fold-Hopf points. In §4.2.8 we will compute the second Lyapunov coefficient (a fifth order coefficient) for a Bautin point that we shall encounter in §4.2.5. For this we also need to provide `Maple` procedures for fourth and fifth order Fréchet-derivatives of the right-hand side of (4.22). These were not computed in §4.2.3 to avoid tedious repetitions, but we believe that their form is easily deducibe from the pattern provided by (4.45) and (4.46). For `D4F` we define

```
> D4F:=proc(phi,psi,chi,zeta)
> local PHI,PSI,CHI,ZETA,i,j;
> global a,b,c,d,n,r,g,tau;
>
> PHI:=Array(1..n,0..r);
> PSI:=Array(1..n,0..r);
> CHI:=Array(1..n,0..r);
> ZETA:=Array(1..n,0..r);
>
> for i from 1 to n do
>     for j from 0 to r do
```

```
>         PHI[i,j]:=eval(phi[i],theta=-tau[j]);
>         PSI[i,j]:=eval(psi[i],theta=-tau[j]);
>         CHI[i,j]:=eval(chi[i],theta=-tau[j]);
>         ZETA[i,j]:=eval(zeta[i],theta=-tau[j]);
>     end do;
> end do;
>
> return(<-a*d4g*b^4*PHI[1,1]*PSI[1,1]*CHI[1,1]*ZETA[1,1]
>         +c*d4g*d^4*PHI[2,2]*PSI[2,2]*CHI[2,2]*ZETA[2,2],
>         -a*d4g*b^4*PHI[2,1]*PSI[2,1]*CHI[2,1]*ZETA[2,1]
>         +c*d4g*d^4*PHI[1,2]*PSI[1,2]*CHI[1,2]*ZETA[1,2]>)
>
> end:
```

and, finally, for `D5F` we have

```
> D5F:=proc(phi,psi,chi,zeta,eta)
> local PHI,PSI,CHI,ZETA,ETA,i,j;
> global a,b,c,d,n,r,g,tau;
>
> PHI:=Array(1..n,0..r);
> PSI:=Array(1..n,0..r);
> CHI:=Array(1..n,0..r);
> ZETA:=Array(1..n,0..r);
> ETA:=Array(1..n,0..r);
>
> for i from 1 to n do
>     for j from 0 to r do
>         PHI[i,j]:=eval(phi[i],theta=-tau[j]);
>         PSI[i,j]:=eval(psi[i],theta=-tau[j]);
>         CHI[i,j]:=eval(chi[i],theta=-tau[j]);
>         ZETA[i,j]:=eval(zeta[i],theta=-tau[j]);
>         ETA[i,j]:=eval(eta[i],theta=-tau[j]);
>     end do;
> end do;
>
> return(<-a*d5g*b^5*PHI[1,1]*PSI[1,1]*CHI[1,1]*ZETA[1,1]*ETA[1,1]
>         +c*d5g*d^5*PHI[2,2]*PSI[2,2]*CHI[2,2]*ZETA[2,2]*ETA[2,2],
>         -a*d5g*b^5*PHI[2,1]*PSI[2,1]*CHI[2,1]*ZETA[2,1]*ETA[2,1]
>         +c*d5g*d^5*PHI[1,2]*PSI[1,2]*CHI[1,2]*ZETA[1,2]*ETA[1,2]>)
```

```
>
> end:
```

This concludes the initialization part of the worksheet. In all our subsequent `Maple` computations we will assume that these definitions have been executed in the active `Maple` session.

### 4.2.5 Identification of codimension-two points

We now return to Figure 4.2. Various codimension-two points reveal their presence as points of intersection of the black, red and blue lines. Furthermore, the direction of bifurcation along the Hopf curves may change at a Bautin bifurcation.

It is natural to start our computations by calculating this direction of bifurcation, i.e. calculating the first Lyapunov coefficient along the black and red curves in the positive quadrant of the $(k_1, k_2)$-plane. Using the `Maple` definitions provided in §4.2.4 this is not very difficult. Suppose that `L` is a vector that parametrize the black curve in Figure 4.2. Let `omega0` be an admissible frequency. We are interested in computing the direction of bifurcation at the point

```
> k1:=eval(L[1],omega=omega0);
> k2:=eval(L[2],omega=omega0);
```

in the $(k_1, k_2)$-plane. Using (4.29) we set the system control parameters and the critical eigenvalue accordingly:

```
> a:=k1/(b*dg); c:=k2/(d*dg);
> lambda0:=I*omega0;
```

It is easily checked that the vectors $q \equiv (1, -1)$ and $p \equiv (1, -1)^T$ satisfy $\Delta(\lambda_0)q = 0$ and $p\Delta(\lambda_0) = 0$, independently of `omega0`. Using Lemma 2.4 to satisfy the 'bi-orthogonality' condition (3.66) we normalize these vectors as

```
> q:=<1,-1>; p:=Transpose(<1,-1>);
> alpha:=1/sqrt(p.DDelta(lambda0).q);
> q:=alpha*q; p:=alpha*p;
```

Next, we define the eigenfunction $\phi$ corresponding to the eigenvector $q$, as well as its complex conjugate.

```
> phi:=exp(lambda0*theta)*q;
> phibar:=map(conjugate,phi);
```

Everything is ready to compute the quantity in (3.70) and, from that and (3.72), the first Lyapunov coefficient $l_1$.

```
> h:=Array(0..2,0..2);
> h[2,0]:=exp(2*lambda*theta)*MatrixInverse(Delta(2*lambda)).D2F(phi,phi);
> h[1,1]:=MatrixInverse(Delta(0)).D2F(phi,phibar);
> c1:=(1/2)*p.(D2F(phibar,h[2,0])+2*D2F(phi,h[1,1])+D3F(phi,phi,phibar));
> l1:=(1/omega0)*Re(c1);
```

This is all there is to calculating the direction of Hopf bifurcation along the black curve in Figure 4.2. A similar procedure can be followed to compute the direction along the red curve. We have wrapped the above code in a `Maple` procedure (not reproduced here, but available by email) to graph the plot in Figure 4.3.

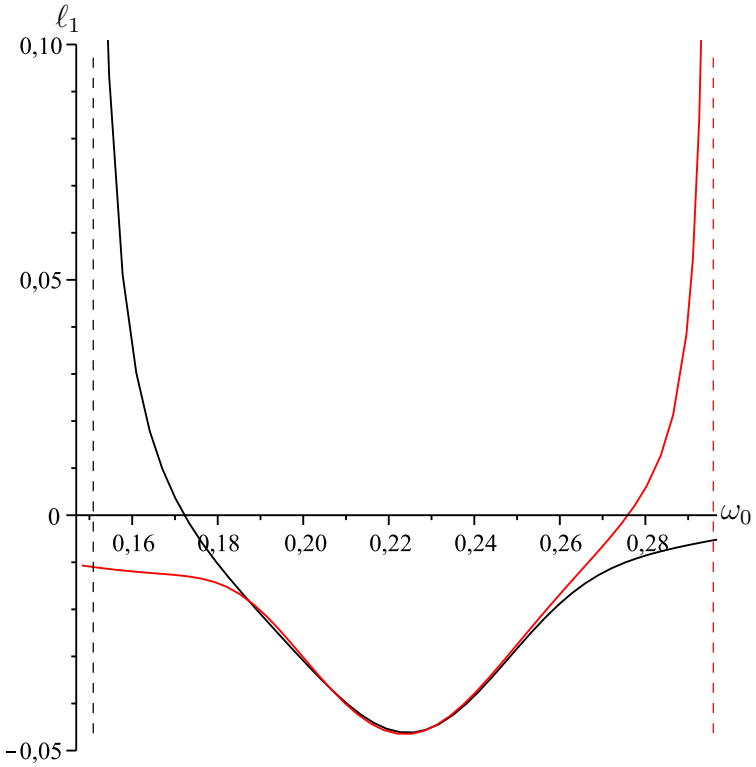
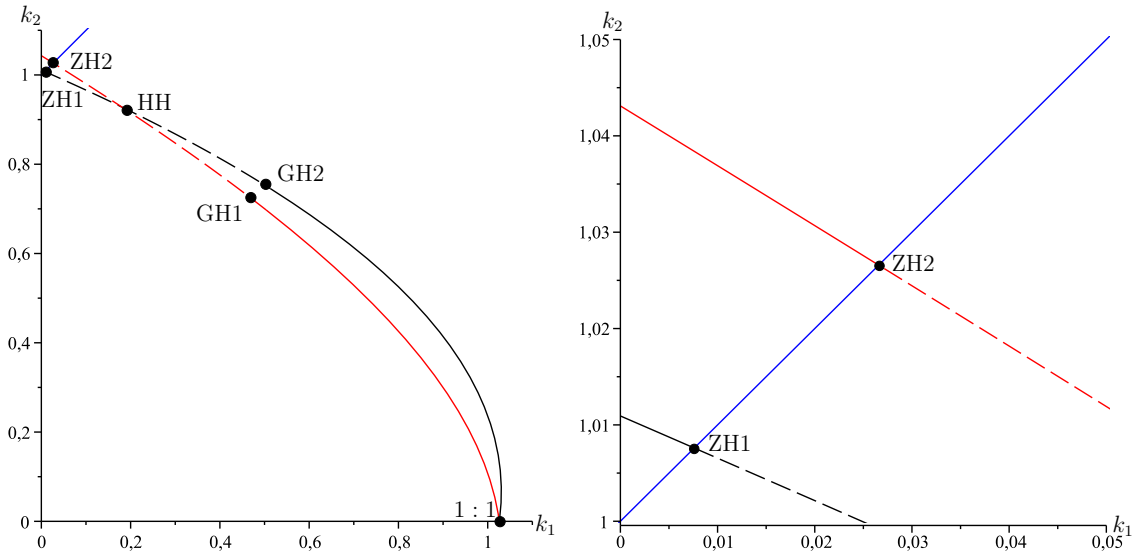



**Figure 4.3:** The first Lyapunov coefficient computed along the portions of the black and red curves that lie in right-half plane in Figure 4.2. Fixed parameter values are as in (4.24). Both graphs cross the horizontal axis, each once. Thus there exist two Bautin-points, one on each curve. In fact, the Lyapunov coefficient along the black curve diverges near the start of the $\omega_0$-interval and becomes negative left of the singularity. (This is not visible in the plot.) Likewise, $l_1$ diverges along the red curve near the end of this interval and becomes negative right of the singularity. The points of divergence coincide with points of fold-Hopf bifurcation, as explained in the main text.

We are now in a position to identify the codimension-two points in the positive quadrant in Figure 4.2, also see Figure 4.4. Solving numerically for the two Bautin (generalized Hopf)




(a) (b)

**Figure 4.4:** In 4.4(a) we see the positive quadrant of Figure 4.2 to which we added the data computed in §4.2.5. Fixed parameter values are as in (4.24). A dashed line corresponds to a subcritical Andronov-Hopf bifurcation, while a solid line indicates a supercritical direction. The labels for the codimension-two points are chosen for consistency with standard `MATCONT` notation. **ZH** = zero-Hopf (fold-Hopf) **HH** = Hopf-Hopf (double Hopf) **GH** = generalized Hopf. 4.4(b) is a magnification of the upper left portion of 4.4(a). Note that at the fold-Hopf points ZH1 and ZH2 the direction of Hopf bifurcation changes. At these points the first Lyapunov coefficient does not vanish, but rather it diverges, as was already noted in the caption of Figure 4.3.

points yields

$$\begin{aligned}(k_1^c, k_2^c) &= (0.503730243249497, 0.697442362012240)\\ \omega_0 &= 0.275909434388554\end{aligned} \tag{GH1}$$

for the point on the red curve, and

$$\begin{aligned}(k_1^c, k_2^c) &= (0.513230584432908, 0.745286219214126)\\ \omega_0 &= 0.172217243841191\end{aligned} \tag{GH2}$$

for the point on the black curve.

The $1:1$-resonance occurs at

$$\begin{aligned}(k_1^c, k_2^c) &= (1.02601931196937, 0.0)\\ (\omega_1, \omega_2) &= (0.229598842623607, 0.229598842623607)\end{aligned} \tag{1:1}$$

and the charateristic matrix $\Delta_{k_1,k_2}(\lambda)$ from (4.31) reduces to the zero-matrix at $\lambda = i\omega_1 = i\omega_2$. Therefore the resonance is *semisimple* in the sense that the characteristic matrix possesses two independent null-vectors. This situation requires special treatment for which §3.5 lacks

results. (Compare this to the remark made at the bottom of p.424 in [19] in conjunction with a $1:1$ resonance for maps.)

Instead, we choose to focus on the more 'standard' fold-Hopf and double Hopf points in Figure 4.2. The double Hopf point occurs at

$$\begin{aligned}(k_1^c, k_2^c) &= (0.180751807497717, 0.927849704599635)\\ (\omega_1, \omega_2) &= (0.289979003927627, 0.156040086681052)\end{aligned} \tag{HH}$$

The non-resonance condition (3.53) is satisfied,

$$n_1\omega_1 \neq n_2\omega_2 \quad \text{for all } n_1, n_2 \in \mathbf{N} \text{ with } n_1 + n_2 \leq 5 \tag{4.47}$$

For $(n_1, n_2) = (1, 2)$ we have

$$n_1\omega_1 - n_2\omega_2 = 0.022101169434477$$

and this $(n_1, n_2)$-pair minimizes the absolute value of the difference of the left-hand and right-hand sides of (4.47) over all relevant combinations of $n_1$ and $n_2$.

The third codimension-two point present in the positive quadrant is the fold-Hopf point. Its coordinates are

$$\begin{aligned}(k_1^c, k_2^c) &= (0.00760034373723105, 1.00760034373723)\\ \omega_0 &= 0.148557497656540\end{aligned} \tag{ZH1}$$

In this example we will compute the critical normal forms of the points DH, FH1 and GH1, since these are the codimension-two points that lie on the stability boundary of the origin in the positive quadrant of the $(k_1, k_2)$-plane. The points labeled FH2 and GH2 do no lie on this boundary and therefore we will not compute their critical normal forms. (We are however confident that the interested reader can do this himself after learning about the computations below.)

### 4.2.6 The double Hopf point

In this subsection we will show that for the critical parameter values

$$a_c = \frac{k_1^c}{bg'(0)} = 0.0903759037488591, \qquad c_c = \frac{k_2^c}{dg'(0)} = 0.773208087166367 \tag{4.48}$$

with $(k_1^c, k_2^c)$ as in (HH) and all other parameters as in (4.24), the equilibrium $(0, 0)$ of (4.22) exhibits a 'simple' non-degenerate double Hopf bifurcation. (Here 'simple' refers to the positive sign of the product appearing in (3.58). We will return to this point below.) Using the definitions and procedures from §4.2.4 we process data specific to the double Hopf point to

calculate the third-order critical normal form coefficients.

First we set the critical parameters and eigenvalues.

```
> k1:=0.180751807497717; k2:=0.927849704599635;                (**)
> a:=k1/(b*dg); c:=k2/(d*dg);
> omega1:=0.289979003927627; omega2:=0.156040086681052;
> lambda1:=I*omega1; lambda2:=I*omega2;
```

We proceed by computing the critical eigenvectors. Let $\Delta(\lambda_{1,2})$ be the characteristic matrices from (4.31) evaluated at the critical eigenvalues $\lambda_{1,2} = i\omega_{1,2}$. The column- and row vectors

$$q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad p_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

are such that

$$\Delta(\lambda_1)q_1 = 0, \quad p_1\Delta(\lambda_1) = 0, \quad \Delta(\lambda_2)q_2 = 0, \quad p_2\Delta(\lambda_2) = 0$$

(Compare this with the introduction in §3.5.5.) We have to scale $q_1$, $p_1$ and $q_2, p_2$ with constants $\alpha_1, \beta_1 \in \mathbf{C}$ and $\alpha_2, \beta_2 \in \mathbf{C}$ in such a way that the 'bi-orthogonality' relationship (3.81) is satisfied. For this we use (2.17) from Lemma 2.4 and Lemma 2.5.

```
> q1:=<1,1>;> p1:=Transpose(<1,1>);
> q2:=<1,-1>; p2:=Transpose(<1,-1>);
> alpha1:=1/sqrt(p1.DDelta(lambda1).q1); beta1:=alpha1;
> alpha2:=1/sqrt(p2.DDelta(lambda2).q2); beta2:=alpha2;
> q1:=alpha1*q1; p1:=beta1*p1;
> q2:=alpha2*q2; p2:=beta2*p2;
```

(We have chosen the scaling constants in such a way that, in addition to satisfying (3.81), the vectors are of comparable numerical magnitude.) Using Lemma 2.4 we define the corresponding critical eigenfunctions and their complex conjugates. (Here we use the assumption that `theta` is real.) Note that these are expressions, rather than `Maple` functions.

```
> phi1:=exp(lambda1*theta)*q1;
> phi1bar:=map(conjugate,phi1);
> phi2:=exp(lambda2*theta)*q2;
> phi2bar:=map(conjugate,phi2);
```

Now it is easy to compute the quadratic critical center manifold coefficients from (3.82).

```
> h:=Array(0..2,0..2,0..2,0..2);
> h[1,1,0,0]:=MatrixInverse(Delta(0)).D2F(phi1,phi1bar);
```

```
> h[2,0,0,0]:=exp(2*lambda1)*MatrixInverse(Delta(2*lambda1)).D2F(phi1,phi1);
> h[1,0,1,0]:=exp(lambda1+lambda2)*
>             MatrixInverse(Delta(lambda1+lambda2)).D2F(phi1,phi2);
> h[1,0,0,1]:=exp(lambda1-lambda2)*
>             MatrixInverse(Delta(lambda1-lambda2)).D2F(phi1,phi2bar);
> h[0,0,2,0]:=exp(2*lambda2)*
>             MatrixInverse(Delta(2*lambda2)).D2F(phi2,phi2);
> h[0,0,1,1]:=MatrixInverse(Delta(0)).D2F(phi2,phi2bar);
```

We also require their complex conjugates.

```
> hbar[1,1,0,0]:=map(conjugate,h[1,1,0,0]);
> hbar[2,0,0,0]:=map(conjugate,h[2,0,0,0]);
> hbar[1,0,1,0]:=map(conjugate,h[1,0,1,0]);
> hbar[1,0,0,1]:=map(conjugate,h[1,0,0,1]);
> hbar[0,0,2,0]:=map(conjugate,h[0,0,2,0]);
> hbar[0,0,1,1]:=map(conjugate,h[0,0,1,1]);
```

At last we are able to compute the cubic critical normal form coefficients as in (3.83).

```
> gg[2,1,0,0]:=(1/2)*p1.(2*D2F(h[1,1,0,0],phi1)+D2F(h[2,0,0,0],phi1bar)
>              +D3F(phi1,phi1,phi1bar));
> gg[1,0,1,1]:=p1.(D2F(h[0,0,1,1],phi1)+D2F(h[1,0,0,1],phi2)
>              +D2F(h[1,0,1,0],phi2bar)+D3F(phi1,phi2,phi2bar));
> gg[1,1,1,0]:=p2.(D2F(hbar[1,0,0,1],phi1)+D2F(h[1,0,1,0],phi1bar)
>              +D2F(h[1,1,0,0],phi2)+D3F(phi1,phi1bar,phi2));
> gg[0,0,2,1]:=(1/2)*p2.(2*D2F(h[0,0,1,1],phi2)+D2F(h[0,0,2,0],phi2bar)
>              +D3F(phi2,phi2,phi2bar));
```

(We use `gg` instead of `g` since the latter symbol has already been defined.) They evaluate to

$$
\begin{aligned}
g_{2100} &= 0.0113599727138386 + 0.0025880313644929\,i \\
g_{1011} &= 0.0065770995240054 - 0.0112835232180977\,i \\
g_{1110} &= 0.0072326241332179 + 0.0129547559869050\,i \\
g_{0021} &= 0.0099439410875781 - 0.0028471712578469\,i
\end{aligned}
$$

These are the critical normal form coefficients that we are looking for. Under the additional hypothesis that the eigenvalues $\lambda_{1,2}$ cross the imaginary axis transversally as the control parameters $a$ and $c$ are varied (this can be verified numerically), we conclude from our discussion of the double Hopf normal form in §3.4.2 (and in particular the condition (3.54)) that a non-degenerate double Hopf bifurcation occurs at the critical parameter values (HH) with all

other parameters set at their values in (4.24). Moreover, this bifurcation is of 'simple' type, since

$$(\operatorname{Re} g_{2100})(\operatorname{Re} g_{0021}) = 0.000112962899422905 > 0$$

For an analysis of the bifurcation diagram of the corresponding truncated normal form, we refer to the discussion of the 'simple' case on p. 359 of [19]. (Note that since $\operatorname{Re} g_{2100} > 0$ and $\operatorname{Re} g_{0021} > 0$ it is necessary to reverse time!) In fact, since the quantities

$$\theta(0) \equiv \frac{\operatorname{Re} g_{1011}}{\operatorname{Re} g_{0021}} = 0.661417788589018, \qquad \delta(0) \equiv \frac{\operatorname{Re} g_{1110}}{\operatorname{Re} g_{2100}} = 0.636676188879151$$

are such that $0 < \delta(0) \leq \theta(0)$ and $\theta(0)\delta(0) < 1$, it follows from p. 360 of [19] that we are in subcase II of the 'simple' double Hopf bifurcation. Hence the bifurcation diagram displayed in subfigure II of Figure 8.25 in [19] applies, with time reversal. It predicts the presence of two-dimensional invariant tori, but these are repelling and cannot be visualized by direct integration of the DDE (4.22). We will return to this point in §4.2.9. Moreover, this bifurcation diagram predicts two curves of subcritical Hopf bifurcation emanating from the codimension-two point in the parameter plane. Such is consistent with Figure 4.4(a) in which we see that the double Hopf point lies at the intersection of two *subcritical* Hopf branches.

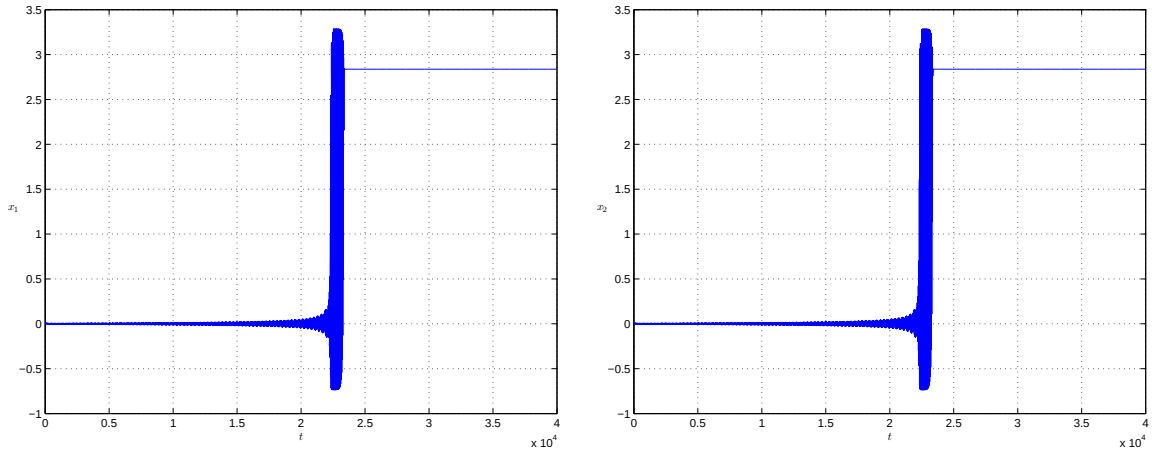
Questions about the effect of higher-order terms and persistence of 'truncated' dynamics are addressed in §8.6.3 of [19]. Since the determinant of the matrix (3.57),

$$\begin{vmatrix} \operatorname{Re} g_{2100} & \operatorname{Re} g_{1011} \\ \operatorname{Re} g_{1110} & \operatorname{Re} g_{0021} \end{vmatrix} = 0.0000653932106790078$$

is non-zero, Lemma 8.16 of [19] implies the presence of invariant two-dimensional tori near the bifurcation point. (However, by the remark made earlier, such tori are unstable.) These come into existence via Neimark-Sacker bifurcations of cycles that were in turn born in subcritical Hopf bifurcations. Curves of Neimark-Sacker and subcritical Hopf bifurcation emanate from the critical point in the parameter plane. In summary, the features of bifurcation diagram 8.25.II of [19] (with time reversed) persist under the addition of higher-order terms, but the motion *on* the unstable torus may no longer be quasi-periodic due to phase locking.

It is interesting to note that there are no attractors other than the origin present in bifurcation diagram 8.25.II of [19] (with time reversed). The origin is an attractor for certain parameter values that correspond to points in the interior of the stability region in Figure 4.4(a). On the other hand, for 'unstable' parameter values we expect that small initial conditions will 'fly off' to a remote attractor that is not present in the local bifurcation diagram. This is illustrated in Figure 4.5.

We refrain from computing the higher-order critical normal form coefficients, since their added value is small: We do not require them to guarantee non-degeneracy of the double Hopf bifurcation and, as we have just seen, dynamical features of the bifurcation are determined



(a) (b)

**Figure 4.5:** Time evolution of the variables $x_1$ and $x_2$ according to (4.22) starting from a constant history function $(x_1(\theta), x_2(\theta)) \equiv (0.01, 0.0175)$ for $\theta \in [-h, 0]$ with $h \equiv \max(\tau_1, \tau_2)$. Note that this history function may be considered 'small' in the usual supremum norm. Fixed parameter values are as in (4.24), while the control variables were set to $a = 0.0905, c = 0.774$, i.e. just above the critical values (4.48). After a long transcient, during which the system spends time close to the stable manifolds of the saddles predicted by the bifurcation's normal form, the dynamics seem to 'fly off' and converge to a stable steady state far away from the origin. This behaviour is not due to numerical instability, as can be checked by *starting* the integration from the distant steady state.

predominantly by the third order coefficients.

### 4.2.7 The fold-Hopf point

In this subsection we show that for the critical parameter values

$$a_c = \frac{k_1^c}{bg'(0)} = 0.00380017186861556, \qquad c_c = \frac{k_2^c}{dg'(0)} = 0.839666953114366$$

with $(k_1^c, k_2^c)$ as in (ZH1) and all other parameter values as in (4.24), the equilibrium $(0, 0)$ of (4.22) exhibits a non-degenerate fold-Hopf bifurcation. As in §4.2.6 we will present the computations in the form of a commented `Maple` worksheet, thus enabling easy verification of our results.

We assume that all commands in the initialization part of the worksheet as presented in §4.2.6 have been executed in the `Maple` shell and we continue the worksheet from there by providing point-specific data. First we enter the coordinates of the point, as well as the critical eigenvalues.

```
> k1 := 0.00760034373723105; k2 := 1.00760034373723;
> a:=k1/(b*dg); c:=k2/(d*dg);
> omega0:=0.148557497656540;
```

```
> lambda0:=0; lambda1:=I*omega0;
```

Next, we compute the critical eigenvectors. Let $\Delta(\lambda_0)$ and $\Delta(\lambda_1)$ be the characteristic matrices corresponding to the eigenvalues $\lambda_0 = 0$ and $\lambda_1 = i\omega_0$. As can easily be verified by direct substitution, the vectors

$$q_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad p_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad q_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad p_1 = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

are such that

$$\Delta(\lambda_0)q_0 = 0, \qquad p_0\Delta(\lambda_0) = 0, \qquad \Delta(\lambda_1)q_1 = 0, \qquad p_1\Delta(\lambda_1) = 0$$

Again, we need to scale $q_0$, $p_0$ and $q_1, p_1$ with constants $\alpha_0, \beta_0 \in \mathbf{R}$ and $\alpha_1, \beta_1 \in \mathbf{C}$ in such a way that the 'bi-orthogonality' relationship (3.76) is satisfied. For this we use once more (2.17) from Lemma 2.4 and Lemma 2.5.

```
> q0:=<1,1>; p0:=Transpose(<1,1>);
> q1:=<1,-1>; p1:=Tranpose(<1,-1>);
> alpha0:=1/sqrt(p0.DDelta(lambda0).q0); beta0:=alpha0;
> alpha1:=1/sqrt(p1.DDelta(lambda1).q1); beta1:=alpha1;
> q0:=alpha0*q0; p0:=beta0*p0
> q1:=alpha1*q1; p1:=beta1*p1;
```

Using Lemma 2.4 we define the corresponding critical eigenfunctions and their complex conjugates.

```
> phi0:=exp(lambda0*theta)*q0;
> phi0bar:=map(conjugate,phi0);
> phi1:=exp(lambda1*theta)*q1;
> phi1bar:=map(conjugate,phi1);
```

The quadratic normal form coefficients are now easily computed from (3.79).

```
> gg:=Array(0..3,0..3,0..3);
> gg[2,0,0]:=(1/2)*p0.D2F(phi0,phi0);
> gg[1,1,0]:=p1.D2F(phi0,phi1);
> gg[0,1,1]:=p0.D2F(phi1,phi1bar);
```

This yields the following output.

$$\begin{aligned} g_{200} &= 0.00656040565870122470 \\ g_{110} &= 0.0132803410870657 + 0.000182334796002904\, i \\ g_{011} &= 0.0130415057497976 \end{aligned}$$

In order the calculate the cubic coefficients, we need to solve for the quadratic center manifold coefficients using (3.78). Since these expressions involve bordered inverses of the type addressed in Corollary 3.7, we first write a small procedure that implements (3.26) and (3.27).

```
> BINV:=proc(lambda,zeta,kappa)
> global q0,p0,q1,p1,lambda0,lambda1;
> local q,p,xi,gam,A,B,X;
>
> if lambda=lambda0 then
>     q:=q0; p:=p0;
> elif lambda=lambda1 then
>     q:=q1: p:=p1;
> else
>     error "input is not an eigenvalue";
> end if;
>
> A:=Matrix([[Delta(lambda),q],[p,0]]);
> B:=<zeta+kappa*DDelta(lambda).q,0>;
> X:=LinearSolve(A,B);
> xi:=<X[1,1],X[2,1]>;
> gam:=-p.DDelta(lambda).xi+(1/2)*kappa*p.D2Delta(lambda).q;
>
> return(exp(lambda*theta)*(xi+gam*q-kappa*theta*q));
> end:
```

Now it is easy to evaluate the formulas in (3.78).

```
> h:=Array(0..2,0..2,0..2);
> h[2,0,0]:=BINV(lambda0,D2F(phi0,phi0),-p0.D2F(phi0,phi0));
> h[0,2,0]:=exp(2*lambda1)*MatrixInverse(Delta(2*lambda1)).D2F(phi1,phi1);
> h[1,1,0]:=BINV(lambda1,D2F(phi0,phi1),-p1.D2F(phi0,phi1));
> h[0,1,1]:=BINV(lambda0,D2F(phi1,phi1bar),-p0.D2F(phi1,phi1bar));
```

As usual, we also require the complex conjugates.

```
> hbar:=Array(0..2,0..2,0..2);
> hbar[2,0,0]:=map(conjugate,h[2,0,0]);
> hbar[0,2,0]:=map(conjugate,h[0,2,0]);
> hbar[1,1,0]:=map(conjugate,h[1,1,0]);
> hbar[0,1,1]:=map(conjugate,h[0,1,1]);
```

At last we are able to evaluate the formulas (3.80) for the cubic critical normal form coefficients.

```
> gg[3,0,0]:=(1/6)*p0.(3*D2F(phi0,h[2,0,0])+D3F(phi0,phi0,phi0));
> gg[1,1,1]:=p0.(D2F(phi0,h[0,1,1])+D2F(phi1bar,h[1,1,0])
>            +D2F(phi1,hbar[1,1,0])+D3F(phi0,phi1,phi1bar));
> gg[2,1,0]:=(1/2)*p1.(D2F(phi1,h[2,0,0])+2*D2F(phi0,h[1,1,0])
>            +D3F(phi0,phi0,phi1));
> gg[0,2,1]:=(1/2)*p1.(D2F(phi1bar,h[0,2,0])+2*D2F(phi1,h[0,1,1])
>            +D3F(phi1,phi1,phi1bar));
```

This yields

$$\begin{aligned} g_{300} &= -0.000529267105230375018 \\ g_{111} &= -0.00320689049366882 \\ g_{210} &= -0.00158236276273148 - 0.0000075492163283651 7\, i \\ g_{021} &= 0.0501035490853393 - 0.0226017769063655\, i \end{aligned}$$

All critical normal form coefficients have been computed. Using (3.50), (3.51) and (3.52) and the values for $g_{jkl}$ found above, we can compute the critical coefficients in the Gavrilov normal form (3.47). These are:

$$\begin{aligned} b(0) &= 0.00656040565870122470 \\ c(0) &= 0.0130415057497976 \\ e(0) &= 0.0242089386247185 \\ \sigma(0) &= 0.148557497656540\, i \\ d(0) &= 0.0132803410870657 + 0.0121673553626291\, i \end{aligned}$$

Thus it follows by Theorem 8.6 of [19] that the fold-Hopf bifurcation is non-degenerate. (Of course, this statement is true provided we also verify that the critical eigenvalues cross the imaginary axis transversally at the bifurcation point.)

We can extract more information from the critical coefficients by calculating the quantities

$$s \equiv \operatorname{sign}[b(0)c(0)] = +1, \qquad \theta \equiv \frac{\operatorname{Re} g_{110}}{g_{200}} = 2.02431705872512 > 0$$

Indeed, since $s = +1$ and $\theta$ is positive, we may apply Theorem 8.7 of [19] to conclude that the fold-Hopf bifurcation is of the 'simple' type: It can locally be described by a *quadratic* normal form and no higher order terms of any sort need to be incorporated. (Hence the cubic coefficients were calculated in vain, but we have included their computation anyway, for the purpose of illustrating the implementation of the bordered inverse.) Only fold and Hopf curves emanate from the codimension-two point. No global bifurcation curves or invariant tori are present. For proofs of these statements we refer the reader to the detailed analysis

performed in §§8.5.2 and 8.5.3 of [19].

### 4.2.8 The Bautin (generalized Hopf) point

The third and last codimension-two bifurcation point for which we calculate the critical normal form is the Bautin point, given by the values (GH1) in the $(k_1, k_2)$-plane. These correspond to the values

$$a_c = \frac{k_1^c}{bg'(0)} = 0.251865121624750, \qquad c_c = \frac{k_2^c}{dg'(0)} = 0.581201968343537$$

of the control parameters, with all other parameters fixed at their values given in (4.24). The method is probably clear by now. Again, we assume that the worksheet presented in §4.2.4 has been executed in the active `Maple` session. We fix the critical parameter values and the critical eigenvalues.

```
> k1:=0.503730243249497; k2:=0.697442362012240;
> a:=k1/(b*dg); c:=k2/(d*dg);
> omega0:=0.275909434388554; lambda0:=I*omega0;
```

Next, we compute the critical eigenvectors. Let $\Delta(\lambda_0)$ be the characteristic matrix corresponding to the critical eigenvalue $\lambda_0 = i\omega_0$. It is easily verified that

$$q = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad p = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

are such that

$$\Delta(\lambda_0)q = 0, \qquad p\Delta(\lambda_0) = 0$$

The following code uses (2.17) from Lemma 2.4 to scale the vectors $q$ and $p$ in such a way that the 'bi-orthogonality' relation (3.66) is satisfied.

```
> alpha:=1/sqrt(p.DDelta(lambda0).q); beta:=alpha;
> q:=alpha*q; p:=beta*p;
> p.DDelta(lambda0).q;
```

Next, we use Lemma 2.4 to define the corresponding eigenfunction.

```
> phi:=exp(lambda0*theta)*q;
> phibar:=map(conjugate,phi);
```

Everything is ready to evaluate the formulas from §3.5.3. We start with (3.67) and (3.68).

```
> h:=Array(0..3,0..3);
> h[2,0]:=exp(2*lambda0*theta)*MatrixInverse(Delta(2*lambda0)).D2F(phi,phi);
```

```
> h[1,1]:=MatrixInverse(Delta(0)).D2F(phi,phibar);
> h[3,0]:=exp(3*lambda0*theta)*
>         MatrixInverse(Delta(3*lambda0)).(3*D2F(phi,h[2,0])+D3F(phi,phi,phi));
```

The reader can check that the real part of (3.70) vanishes, in accordance with the fact that we find ourselves at a point where the first Lyapunov coefficient is zero.

```
> c1:=(1/2)*p.(D2F(phibar,h[2,0])+2*D2F(phi,h[1,1])+D3F(phi,phi,phibar));
```

For (3.71) we need to evaluate a bordered inverse. We could write a procedure for this (as in §4.2.7), but in fact a simple calculation shows that $\zeta + \kappa\Delta'(\lambda_0)q = 0$, in the notation of Corollary 3.7. It follows that

```
> h[2,1]:=exp(lambda0*theta)*kappa*((1/2)*p.D2Delta(lambda0).q-theta)*q;
```

We also make the complex conjugates of the center manifold coefficients available.

```
> hbar:=Array(0..3,0..3);
> hbar[2,0]:=map(conjugate,h[2,0]);
> hbar[3,0]:=map(conjugate,h[3,0]);
> hbar[2,1]:=map(conjugate,h[2,1]);
```

We continue with (3.73) and obtain

```
> h[3,1]:=exp(2*lambda0*theta)*
>         MatrixInverse(Delta(2*lambda0)).(D2F(phibar,h[3,0])+3*D2F(h[2,0],h[1,1])
>         +3*D2F(phi,h[2,1])+3*D3F(phi,phibar,h[2,0])+3*D3F(phi,phi,h[1,1])
>         +D4F(phi,phi,phi,phibar))-6*c1*MatrixInverse(Delta(2*lambda0)).
>         (DDelta(2*lambda0)-IdentityMatrix(2)-theta*Delta(2*lambda0)).h[2,0];
```

and

```
> h[2,2]:=MatrixInverse(Delta(0)).(2*D2F(phibar,h[2,1])+2*D2F(h[1,1],h[1,1])
>         +2*D2F(phi,hbar[2,1])+D2F(h[2,0],hbar[2,0])+D3F(phibar,phibar,h[2,0])
>         +D3F(phi,phi,hbar[2,0])+4*D3F(phi,phibar,h[1,1])
>         +D4F(phi,phi,phibar,phibar));
```

All cards are on the table to compute the second Lyapunov coefficient from (3.74) and (3.75).

```
> c2:=(1/12)*p.(6*D2F(h[1,1],h[2,1])+3*D2F(hbar[2,1],h[2,0])
>     +D2F(hbar[2,0],h[3,0])+3*D2F(phi,h[2,2])+2*D2F(phibar,h[3,1])
>     +6*D3F(phibar,h[2,0],h[1,1])+6*D3F(phi,h[1,1],h[1,1])
>     +3*D3F(phi,h[2,0],hbar[2,0])+6*D3F(phi,phibar,h[2,1])
>     +3*D3F(phi,phi,hbar[2,1])+D3F(phibar,phibar,h[3,0])
>     +6*D4F(phi,phi,phibar,h[1,1])+3*D4F(phi,phibar,phibar,h[2,0])
>     +D4F(phi,phi,phi,hbar[2,0])+D5F(phi,phi,phi,phibar,phibar));
```

Taking the real part we find

```
l2:=(1/omega0)*Re(c2);
```

with output

$$l_2(0) = 0.00110327860627586$$

Let us in addition we assume that the map $(a, c) \mapsto \ell_1(a, c)$ is regular at $(a, c) = (a_c, c_c)$, where $\ell_1(a, c)$ is the first Lyapunov coefficient. Then the zero equilibrium of (4.22) exhibits a non-degenerate Bautin bifurcation at the critical point (GH1). Since $l_2(0) > 0$ we find ourselves in the reverse of the situation discussed in §8.3.2 of [19]. In particular, by calculating $l_2(0)$ we have proved the existence of a curve of fold bifurcations of limit cycles emanating from the Bautin point in the $(k_1, k_2)$-plane. At this curve two cycles annihilate and we are left with an *unstable* equilibrium. (This is why the case $l_2(0) > 0$ is sometimes considered 'hard' or 'dangerous', in the spirit of the terminology used to describe a subcritical (non-degenerate) Hopf bifurcation.)

### 4.2.9 In pursuit of a stable invariant torus

In conclusion of this example we return to the double Hopf point analysed in §4.2.6. There we concluded that this point is of 'simple' type. Moreover, we saw that the values of the critical normal form coefficients predicted the existence of two-dimensional invariant tori near the bifurcation point. However, these tori were repelling and therefore they could not be expected to appear in simulations.

In this section we re-run the computations performed in §4.2.6 but we set our 'fixed' parameters at the values

$$b = 2.0, \quad d = 1.2, \quad \tau_1 = 12.99, \quad \tau_2 = 20.15 \tag{4.49}$$

instead of the values given in (4.24). We note that $b$ and $d$ remain unaltered while the delays $\tau_1$ and $\tau_2$ are changed slightly, since only these latter parameters affect the position of the codimension-one curves in the $(k_1, k_2)$-plane. The effect of this change is that the double Hopf point labeled **HH** in Figure 4.4(a) now occurs at the intersection of two *supercritical* Hopf branches. More specifically, its coordinates are

$$\begin{aligned}(k_1^{c'}, k_2^{c'}) &= (0.559667089973705, 0.688875991374739)\\ (\omega_1', \omega_2') &= (0.272554827172345, 0.174659443775867)\end{aligned} \tag{HH'}$$

In order to compute the critical normal form coefficients for these new fixed parameter values, we are required to change the scripts presented in §§4.2.4 and 4.2.6 at two places. Firstly, the `Maple` code block labeled `(*)` in §4.2.4 must be updated to use the values (4.49). Secondly, the `Maple` code block labeled `(**)` in §4.2.6 should be replaced by

```
> k1:=0.559667089973705; k2:=0.688875991374739;
> a:=k1/(b*dg); c:=k2/(d*dg);
> omega1:=0.272554827172345; omega2:=0.174659443775867;
> lambda1:=I*omega1; lambda2:=I*omega2;
```

reflecting the new coordinates of the double Hopf point in the $(k_1, k_2)$-plane and the new values for the corresponding frequencies $\omega_{1,2}$ given by (HH'). The non-resonance condition (4.47) since for $(n_1, n_2) = (2, 3)$ we have

$$n_1\omega 1 - n_2\omega 2 = 0.021131323017089$$

and this value of $(n_1, n_2)$ minimizes the absolute value of the difference of the left- and right hand sides of (4.47) over all admissible pairs $(n_1, n_2)$. The remainder of the script in §4.2.6 remains unaltered. Upon execution we now find the following values for the critical normal form coefficients:

$$\begin{aligned} g_{2100} &= -0.00158423502629251 + 0.00128155174197111\, i \\ g_{1011} &= -0.00076572821118787 - 0.00382491890256949\, i \\ g_{1110} &= -0.00044023276677625 + 0.00371800958543468\, i \\ g_{0021} &= -0.00176942031197673 - 0.00141734227810451\, i \end{aligned}$$

Again, as in §4.2.6, the bifurcation is of 'simple' type, since

$$(\operatorname{Re} g_{2100})(\operatorname{Re} g_{0021}) = 0.00000280317763446696 > 0$$

We also note that

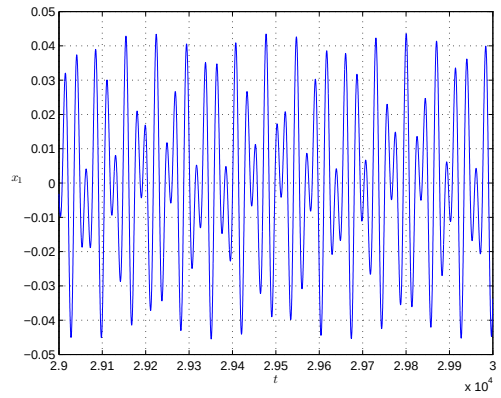
$$\begin{vmatrix} \operatorname{Re} g_{2100} & \operatorname{Re} g_{1011} \\ \operatorname{Re} g_{1110} & \operatorname{Re} g_{0021} \end{vmatrix} = 0.00000246607898545709$$

is non-zero. We compute the quantities

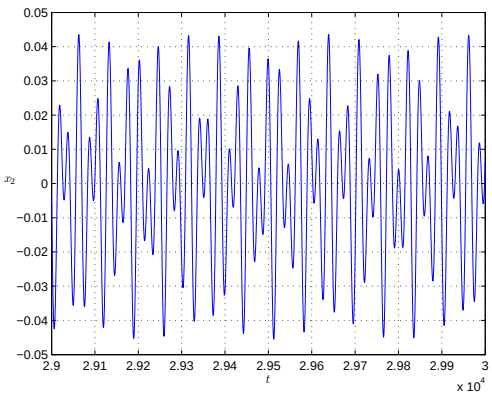
$$\theta(0) \equiv \frac{\operatorname{Re} g_{1011}}{\operatorname{Re} g_{0021}} = 0.432756539531544, \qquad \delta(0) \equiv \frac{\operatorname{Re} g_{1110}}{\operatorname{Re} g_{2100}} = 0.277883495485201$$

and observe that $0 < \delta(0) \leq \theta(0)$ and $\theta(0)\delta(0) < 1$. It follows from p. 360 of [19] that we find ourselves again in subcase II of the 'simple' double Hopf bifurcation. However, in contrast to the situation of §4.2.6 we presently have $\operatorname{Re} g_{2100} < 0$ and $\operatorname{Re} g_{0021} < 0$, so bifurcation diagram 8.25.II of [19] applies now *without time reversal*. In particular, we expect the existence of two Neimark-Sacker curves emanating from the codimension-two point in the $(k_1, k_2)$-plane. At these curves a *stable* two-dimensional invariant torus is born and this torus should persist for parameter values sufficiently close to the Neimark-Sacker curves. The dynamics on the torus

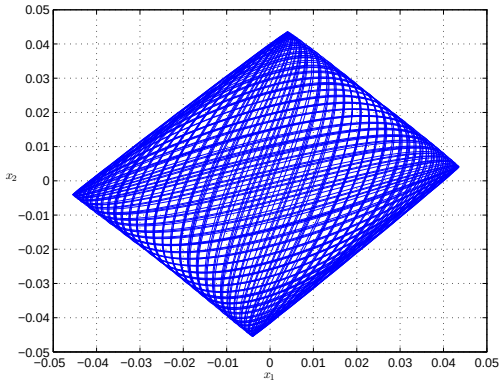
are generically not quasi-periodic but rather phase-locked due to the effect of higher-order terms. This is illustrated in Figure 4.6. Note that the increased accuracy used in the time integration is required because we restrict ourselves to a very small neighbourhood of the bifurcation point. Such a restriction is necessary due to the nearby presence of the Bautin point and the breakdown of the invariant torus only slightly away from the Neimark-Sacker curves.



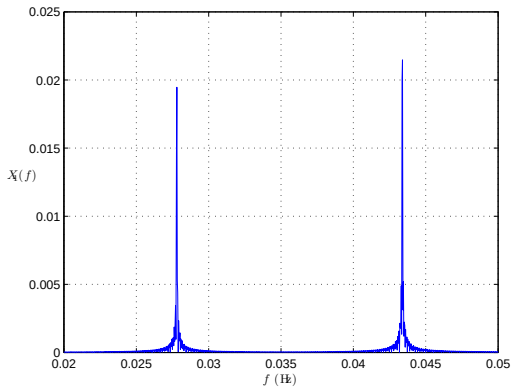

(a)



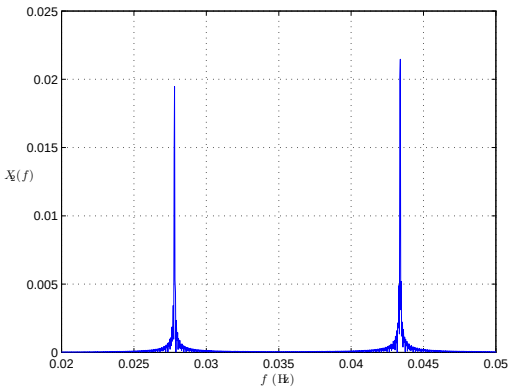

(b)




(c)




(d)




(e)

**Figure 4.6:** The result of an integration over 30.000 time units of the system (4.22) for $(k_1, k_2) = (0.5598, 0.6890)$ close to the critical values $(k_1^{c'}, k_2^{c'})$ given by (HH'). The other parameters are fixed at their values (4.49). As an initial condition we chose the constant history function $(x_1(t), x_2(t)) \equiv (0.050, 0.075)$. To produce the time series 4.6(a) and 4.6(b) the `MATLAB` routine `dde23` was used with an absolute tolerance of $10^{-8}$ and a relative tolerance of $10^{-6}$. Only the last 1000 time steps were reproduced for clarity of presentation. In 4.6(c) we represent the dynamics during the last 2500 time steps in the 'physical' phase space consisting of the $(x_1, x_2)$-plane. The torus structure is clearly visible. Finally, 4.6(d) and 4.6(e) display the outcome of a fast Fourier analysis of the last 20.000 time steps. As expected, two frequency peaks reveal themselves very closely to the values $\frac{\omega_{1,2}}{2\pi}$ with $\omega_{1,2}$ as in (HH').

# Chapter 5

# Final remarks and future work

In this thesis we have generalized (or rather: *lifted*) the techniques used in [18] and [22] for the calculation of critical normal form coefficients for finite dimensional ODEs and mappings to the infinite-dimensional setting of delay differential equations. We argued that these techniques fit naturally in the abstract functional-analytic framework used in [8] for the treatment of DDE using semigroup methods. For an overview of the contents of this thesis and its goals, we refer the reader to the introductory Chapter 1. In the remainder of this short chapter we like to point out some directions for future work.

From the abstract viewpoint of dual perturbation theory (sun-star calculus) there is no principal difference between delay differential equations such as (DDE) and renewal equations such as (RE). Differences are however predominant at a lower, more computational level. Therefore, we have chosen to restrict ourselves to a discussion of critical normalization for DDE. This immediately suggests one natural direction for future work: An extension of the normalization formulas to allow for DDE-RE systems, thereby making them potentially useful for the local analysis of structured population models such as [7], [3]. We plan to take this task at hand in the near future.

Another obvious direction of future activity is in the realm of software development. The continuation package `DDE-BIFTOOL` mentioned in the introductory Chapter 1 is currently incapable of bifurcation detection or normal form computations at critical points. The results from Chapter 3 of this thesis may readily be implemented to take care of the latter part of this task (i.e. the calculation of critical normal forms). In §4.2 `Maple` examples were provided to stimulate work in this direction. Clearly, the code presented these was not optimal. For instance, explicit matrix inversion was used liberally instead of e.g. LU decomposition to solve linear systems because the appearance of the resulting `Maple` code is closer to the appearance of the formulas derived in §3.5. For large problems such an approach is clearly not advisable.

Elaborating on the previous point, I would like to add that I do believe strongly that numerical methods can only flourish when they are available in a usable form. The local analysis of real-world DDE is, apart from a small class of very simple systems, beyond the reach of

pencil-and-paper methods. Since the characteristic equation is intrinsically transcendental (due to the infinite-dimensionality of DDE), there is little hope for symbolic formulas for the location of the critical points of a system. Ideally, I envision a software package for DDE capable of the same sort of analysis as `CONTENT` or `MatCont`.

This brings me to my final point. Normal form theorems for local bifurcations always hold under the assumption of certain *genericity conditions*. These typically divide into so-called *nondegeneracy conditions* and *transversality conditions*. If both are fulfilled, then a parameter-dependent normal form provides detailed insight into the qualitative *unfolding* of the dynamics near a critical point, see e.g. Chapter 8 of [19]. Calculating *critical* normal form coefficients amounts to verifying nondegeneracy: If certain critical coefficients vanish, then a normal form description requires computations at (even) higher order. However, in this thesis we almost completely neglected the issue of transversality, which is intimately related to the existence of a smooth and smoothly invertible function $K$,

$$(\beta_1, \beta_2) = K(\alpha_1, \alpha_2)$$

relating the original model parameters $(\alpha_1, \alpha_2)$ to the unfolding parameters $(\beta_1, \beta_2)$ that appear in the parameter-dependent normal form of the bifurcation under scrutiny. As demonstrated in §3.3 of [22] in the context of maps, the normalization method discussed in this thesis is easily adaptable to apply to parameter-dependent normalization, in which case *branch switching* capabilities for continuation software for DDE are within reach. It is merely necessary to replace the homological equation intoduced in Chapter 3 by a parameter-dependent counterpart.

Critical normalization tells us *what* to expect, parameter-dependent normalization shows *where* to expect it. Both are necessary for a fruitful bifurcation analysis of dynamical systems. There is interesting work ahead.

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