Lefschetz fibrations and symplectic structures

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Abstract

In this thesis, we study a relation between symplectic structures and Lefschetz fibrations to shed some light on 4-manifold theory. We introduce symplectic manifolds and state some results about them. We then introduce Lefschetz fibrations, which are a generalization of fiber bundles, and discuss them briefly to obtain an intuitive understanding. The main theorem of this thesis is a result obtained by Gompf [**4**]. It provides a way to construct a symplectic structure on a general Lefschetz fibration with homologeously nonzero fiber. We also discuss a generalization of this, achieved by Gompf [**6**], that generalizes this result to arbitrary even dimensions.

Contents

CHAPTER 1

Introduction

Symplectic geometry has been an important subject of research for the past decades. Two important questions are how to determine whether a certain manifold admits a symplectic structure, and whether symplectic structures occur often. We try to shed some light on both of these questions, especially in the 4-dimensional case.

After investigating the basic properties in Chapter 2, we introduce a gluing construction in Chapter 3. This is used to prove that any finitely presented group is the fundamental group of some symplectic 4-manifold [**4**]. On the one hand, this theorem is very convenient, because it provides a lot of examples of symplectic 4-manifolds. On the other hand, it shatters all hope of classifying symplectic 4-manifolds, because of the negative solution of the word problem for groups. However, although symplectic 4-manifolds can not be completely classified, it is nice to be able to describe them in more combinatorical terms. To achieve this, we introduce generalizations of fiber bundles: Lefschetz fibrations and Lefschetz pencils. In Chapter 4, we describe the behavior of these objects, and in Chapter 5, we introduce a construction of a symplectic structure on any Lefschetz fibration, which was first achieved by Gompf [**4**]. Since these fibrations can be described in a more combinatorical way, this gives an alternative way to describe symplectic structures. In light of the theorem of Donaldson [**1**], which states that any symplectic 4-manifold admits a Lefschetz pencil, this method fits to describe all symplectic 4-manifolds.

CHAPTER 2

Symplectic manifolds

1. Definition and basic properties

In this section, we define symplectic structures on manifolds and state some of their basic properties. These structures will be the main object of study in this thesis.

DEFINITION 2.1. A 2-form ω on a linear space V is called **nondegenerate** or **symplectic** if, for any nonzero vector $x \in \mathbb{R}^n$, we can find a vector $y \in V$ such that $\omega(x, y) \neq 0$.

The existence of a nondegenerate form implies that the dimension of V is even. This is because, using a basis, forms can be represented by matrices. Being nondegenerate and anti-symmetric imposes the conditions of invertibility and skew-symmetry, and this combination does not exist in odd dimensions.
On the vector space $\mathbb{R}^{2n}$, we have the form $x_1 \wedge y_1 + ... + x_n \wedge y_n$. This form is nondegenerate and is called the **standard symplectic form** on $\mathbb{R}^{2n}$. On $\mathbb{C}^n$, using the coordinates $(x_1+y_1i, ..., x_n+y_ni)$, we can define the form in the same way.
Although nondegenerate forms differ from inner products, it makes sense to talk about orthogonal complements, using the following definitions.

DEFINITION 2.2. Let V be a linear space and let ω be a nondegenerate form on V. Now a subspace $U \subset V$ is called a **symplectic subspace** if $\omega|_U$ is again a nondegenerate form.

DEFINITION 2.3. Let V be a linear space with a nondegenerate form ω and let $U \subset V$ be a symplectic subspace. The **(symplectic) orthogonal complement** of U with respect to ω is defined as $U^\perp = \{x \in V | \forall y \in U, \omega(x, y) = 0\}$. The symplectic orthogonal satisfies $\dim U + \dim U^\perp = \dim V$.

Using this, we can define different types of subspaces.

DEFINITION 2.4. Let V be a linear space and let ω be a nondegenerate form on V.
- A subspace $U \subset V$ is **isotropic** if $U \subset U^\perp$. This is equivalent with saying $\omega|_U = 0$.
- A subspace $U \subset V$ is **coisotropic** if $U^\perp \subset U$.
- A subspace that is both isotropic and coisotropic is **Lagrangian**.

REMARK 2.5. Since the dimensions of U and $U^\perp$ add up to the dimension of the whole space, isotropic spaces of V have at most half its dimension, and coisotropic at least half. This means that Lagrangian subspaces have exactly half the dimension of the whole space.

We now generalize the linear definition to manifolds.

DEFINITION 2.6. A smooth 2-form ω is called a **symplectic form** if it is pointwise nondegenerate and closed. The latter condition meaning that $d\omega = 0$, in which d is the de Rham differential.

REMARK 2.7. In the linear case, the words nondegenerate and symplectic are synonyms. In the smooth category they are not, since the term symplectic is usually reserved for closed forms.

A manifold M with a symplectic form $\omega \in \Omega^2(M)$ is called a **symplectic manifold** and is often denoted together with its symplectic form: (M, ω). The existence of a symplectic form implies that n is even.

Of course, a submanifold $N \subset M$ is called a **symplectic**, **isotropic**, **coisotropic** or **Lagrangian** submanifold if its T_xN is a symplectic, isotropic, coisotropic or Lagrangian subspace, respectively, of T_xM for any $x \in N$.

DEFINITION 2.8. Let (M, ω_M) and (N, ω_N) be symplectic manifolds. A diffeomorphism $f : M \to N$ is called a **symplectomorphism** if $f^*\omega_N = \omega_M$.

Symplectomorphisms are the isomorphisms in the category of symplectic manifolds.

When we restrict our attention to closed forms, we can locally write the symplectic form as the standard form, using the following well-known theorem of Darboux.

THEOREM 2.9 (Darboux). *Let M be a $2m$-dimensional manifold and let ω be a pointwise non-degenerate form. Closedness of ω is equivalent to the following condition.*
Around every $x \in M$, we can find a coordinate chart $(U, \phi) = (U, x_1, ..., x_m, y_1, ..., y_m)$ such that $\omega = dx_1 \wedge dy_1 + ... + dx_m \wedge dy_m$.

COROLLARY 2.10. *Let ω be a closed 2-form on a $2m$-dimensional manifold M. The wedge-product $\omega^m = \omega \wedge ... \wedge \omega$ (m times) is a volume form if and only if ω is symplectic.*

PROOF. If ω is symplectic, it can be written in standard coordinates. It follows immediately that ω^m is a volume form.
For the other direction, let ω be degenerate, so $i_X\omega = 0$ for some X. Now if we fill in this X in ω^m, we get $i_X(\omega^m) = m(i_X\omega)\omega^{m-1} = 0$, so this can not be a volume form. □

We now give some classes of examples, to motivate our definition.

EXAMPLE 2.11. Every orientable 2-manifold admits a symplectic structure. In the 2-dimensional case, being orientable is equivalent to admitting a volume form. A volume form μ on a 2-manifold is a symplectic form. Closedness is direct, since the de Rham differential d sends μ to a 3-form and there are no nonzero forms with degree higher than the manifolds dimension. It follows directly from Corollary 2.10 (the case $n = 1$) that μ is symplectic.

EXAMPLE 2.12. On the complex projective space $\mathbb{CP}^n$, which is even-dimensional as a real manifold, there is a well-known, canonically defined symplectic structure, called the Fubini-Study symplectic form. In complex coordinates $[z_0 : ... : z_n]$, this is given by $\omega = \frac{i}{2\pi}\overline{\partial}\partial \log \sum_{i=0}^{n} |z_i|^2$. Note that this form is well-defined on $\mathbb{CP}^n$. To see this, we can write the form in different coordinate patches.
For $z_0 \neq 0$, take linear coordinates $a_1 = \frac{z_1}{z_0}, ..., a_n = \frac{z_n}{z_0}$. The form becomes $\frac{i}{2\pi}\overline{\partial}\partial \log 1 + |a_1|^2 + ... + |a_n|^2$. Working on the coordinate patch with $z_1 \neq 0$, we can take $b_1 = \frac{z_0}{z_1}, b_2 = \frac{z_2}{z_1}, ..., b_n = \frac{z_n}{z_1}$. We can write the coordinates b_i in terms of the coordinates a_i: $b_1 = \frac{1}{a_1}, b_2 = \frac{a_2}{a_1}, ..., b_n = \frac{a_n}{a_1}$. This gives $\omega = \frac{i}{2\pi}\overline{\partial}\partial \log 1 + |b_1|^2 + ... + |b_n|^2 = \frac{i}{2\pi}\overline{\partial}\partial \log 1 + |\frac{1}{a_1}|^2 + |\frac{a_2}{a_1}|^2 + ... + |\frac{a_n}{a_1} = \frac{i}{2\pi}\overline{\partial}\partial \left(\log \frac{1}{a_1} + \log \frac{1}{\overline{a_1}} + \log 1 + a_1 + ... + a_n\right)$. Since the terms $\log \frac{1}{a_1}$ and $\log \frac{1}{\overline{a_1}}$ both only depend on one of the terms a_1, $\overline{a_1}$, both will vanish after applying both ∂ and $\overline{\partial}$, so the first two terms drop. We are left with the same formula, in the different coordinates, so we conclude that the descriptions agree on the overlap of the patches. Of course, this can be done for any two coordinates, so the form is well-defined on $\mathbb{CP}^n$.
This symplectic form is often denoted ω_{FS} or ω_{std}.

REMARK 2.13. Let X be a compact, $2m$-dimensional manifold and let ω be a symplectic form. Let η be an arbitrary closed 2-form. For small enough $t \in \mathbb{R}$, the form $\omega + t\eta$ is symplectic.

PROOF. Since ω is symplectic, ω^m is a volume form. At a certain point $x \in X$, take a basis $v_1, ..., v_{2m}$ of the tangent space. $\omega^m(v_1, ..., v_{2m}) = s \neq 0$.

Consider $(\omega+t\eta)^m(v_1,...,v_{2n})$. This can be written as $s+\mathcal{O}(t)$, in which $\mathcal{O}(t)$ denotes a polynomial in t. Since this is a polynomial expression, t can be shrank to make s dominate the term. This means that for small enough t, $(\omega+t\eta)^m$ is a volume form.
For each point $x \in X$, we can find a neighborhood in which $(\omega+t\eta)^m$ is a volume form, since it is a smooth section of $\bigwedge^n T^*X$. This means that around any point, we can find a neighborhood and a corresponding t, so we can use the compactness to find a finite subcover, take the smallest t and conclude there is an overall $t \in \mathbb{R}$ for which $(\omega+t\eta)^n$ is a volume form, hence $\omega+t\eta$ is symplectic. □

A corollary of this is that if a form is nondegenerate at a certain point, it is nondegenerate in a neighborhood of that point.

We end this section with a nice fact about the cohomology of symplectic manifolds. Consider a $2m$-dimensional manifold M and a symplectic form ω. We have that ω^m is a volume form (Corollary 2.10) and hence represents a nonzero cohomology class, we can conclude that any wedge-power of ω represents a nonzero cohomology class. It now follows that for any even k, $H^k_{dR}(M;\mathbb{R}) \neq \{0\}$. In words: M has nonzero cohomology in any even degree. For this reason, any sphere S^{2n}, for $n > 1$, can not support a symplectic structure.

2. Compatibility with complex structures

Some familiarity with complex structures is assumed, but we recall the definition here.

Definition 2.14. We first define the structure point-wise and then generalize to manifolds.

- Let V be a real vector space. A **complex structure** on V is a linear map $J : V \to V$ with the property that $J \circ J = -Id$.
- Let X be a manifold. A **almost-complex structure** on X is a smoothly varying collection of complex structures on every tangent space.
- Let X be a manifold with an almost-complex structure J. The structure J is called a **complex structure** on X if around any point $x \in X$, we can find a chart $\left(U, \frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n},\frac{\partial}{\partial y_1},...,\frac{\partial}{\partial y_n}\right)$ such that $J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$, for all $i \in \{1,...,n\}$. This property of a complex structure is called integrability.

Complex manifolds are often defined in a more direct way, very similar to the definition of real manifolds. Instead of taking charts to $\mathbb{R}^n$, one takes charts to $\mathbb{C}^n$. The transition functions between different charts are not only supposed to be smooth, but even holomorphic. Note that this definition also yields a complex structure in the above sense. If one has local complex coordinates, this defines a linear map J on the tangent space by multiplication by the complex unit i. Using the fact that the holomorphic transition functions of the maps yield complex linear transition functions between the tangent spaces, this map is easily proven to be well-defined on the entire manifold. The identity $J^2 = -Id$ is obvious, and the integrability condition is satisfied since we are already working in local coordinates: these coordinates $(z_1,...,z_n)$ are complex, but we can just use a transition $(z_1,...,z_n) \mapsto (x_1,...,x_n,y_1,...,y_n)$ to $\mathbb{R}^n$, so i acts on the coordinates as in the integrability condition. (In fact, this condition is imposed on complex structures for exactly this reason.)

In the rest of this section, we describe ways to match symplectic structures to (almost-)complex ones.

Definition 2.15. Let X be a manifold. A nondegenerate form ω on X is said to be **compatible** with the (almost-)complex structure J if for all $v,w \in TX$, we have $\omega(Jv,Jw) = \omega(v,w)$ and $\omega(v,Jv) > 0$.

EXAMPLE 2.16. Consider $\mathbb{CP}^n$. The Fubini-Study symplectic form ω_{std} is compatible with the complex structure obtained by multiplication with i.

The relation between almost-complex structures and nondegenerate 2-forms is made clear in the next theorem.

THEOREM 2.17. *For any almost-complex structure J on a manifold X, there is a compatible nondegenerate form.*

PROOF. Let g be a Riemannian metric on X. Take $\omega(u,v) = \frac{1}{2}(g(Ju,v) + g(-u,Jv))$. This is a nondegenerate 2-form on X that is compatible with J. Bilinearity follows directly from the (bi)linearity of g and J.
Anti-symmetry: $\omega(u,v) = \frac{1}{2}(g(Ju,v) + g(-u,Jv)) = -\frac{1}{2}(g(Jv,u) + g(-v,Ju)) = -\omega(u,v)$.
For nondegeneracy, let $u \in TX$ be arbitrary. Take $\omega(u,Ju) = \frac{1}{2}(g(Ju,Ju) + g(-u,-u)) > 0$. Compatibility is satisfied since we also have $\omega(Ju,Jv) = \frac{1}{2}(g(-u,Jv) + g(-Ju,-v)) + \frac{1}{2}(g(Ju,v) + g(-u,Jv)) = \omega(u,v)$. □

DEFINITION 2.18. A manifold is called **Kähler** if it has both a complex structure and a symplectic structure, such that these two structures are compatible.

Kähler manifolds also have a Riemannian structure, so we then have a compatible triple J, ω, g, any of which can be expressed in terms of the other two. In order to do this, note that since ω and g are nondegenerate, both of them induce a bundle isomorphism $\phi : TM \to T^*M$ by taking $\phi_\omega(u) = i_u\omega = \omega(u,\cdot)$ and $\phi_g(u) = i_u g = g(u,\cdot)$. Using this, we can take the following.

$$g(u,v) = \omega(u,Jv)$$
$$\omega(u,v) = g(Ju,v)$$
$$J(u) = \phi_g^{-1}(\phi_\omega(u))$$

If we have any two of these structures that are compatible to one another, the third one is automatically defined in a unique way. This means that any complex submanifold N of a Kähler M manifold is itself Kähler, since the restriction of the Riemannian metric to N yields a second structure on N (we also have the complex structure), and we get back the restriction of the symplectic form as a symplectic structure. So N is also a symplectic submanifold. This yields a lot of examples: since the projective space $\mathbb{CP}^n$ is Kähler using the Fubini-Study form, any projective variety has a Kähler structure.

When we have almost-complex and nondegenerate structures defined on different manifolds, they can also be compatible, relative to a map between the manifolds.

DEFINITION 2.19. Let $f : X \to Y$ be a smooth map between manifolds. Let ω be a 2-form on Y and J be an almost complex structure on X.

- J is called (ω, f)**-tame** (or that J tames ω) if $df^*\omega(v,Jv) = \omega(df(v), df(Jv)) > 0$ holds for all $v \in TX \setminus \ker(df)$.
- J is called (ω, f)**-compatible** if it is tame and $df^*\omega$ is also J-invariant. This last condition means that $df^*\omega(Jv,Jw) = df^*\omega(v,w)$ for all $v,w \in TY$.
- If $X = Y$ and f is the identity funtion, the complex structure will just be called ω-tame or ω-compatible. Note that this definition of compatibility matches the one previously given.

REMARK 2.20.
- If a form ω tames some J, it is nondegenerate, since $\omega(v,Jv) > 0$ holds for any v. So a closed, taming 2-form is symplectic.
- Being ω-tame is an open condition. It is easy to see this, since, for fixed ω, $\omega(v,Jv)$ is a continuous map from $TX \times \mathcal{J}$ to $\mathbb{R}$ (with $\mathcal{J}$ the set of complex structures) and the set of ω-tame complex structures is the inverse image of an open set.
- If J is (ω, f)-tame, then $\ker(df)$ pointwise is a complex subspace of TX

3. Blow-ups

In this section, we introduce a way to deal with singularities. When two lines intersect transversely in a point, we cannot distinguish them in that point, since they only differ in direction. A way to deal with this is, intuitively speaking, by replacing the point by the set of all directions through that point, obtaining a slightly different manifold. This is called blowing up the point. Of course, two lines intersection in that point will not intersect in the new manifold, since their direction in the point is different.

Consider the manifold $\tau = \{(\ell, p) \in \mathbb{CP}^{n-1} \times \mathbb{C}^n | p \in \ell\}$. Let $\pi_2 : \tau \to \mathbb{C}^n$ denote projection onto the second coordinate. For any nonzero $x \in \mathbb{C}^n$, $\pi_2^{-1}(x)$ is a one-point space, while for $x = 0$, the inverse image consists of an entire copy of $\mathbb{CP}^{n-1}$. We conclude that the map $\pi_2 : \tau \setminus \pi_2^{-1}(0) \to \mathbb{C}^n \setminus \{0\}$ is a biholomorphic function, in other words, a holomorphic bijection with a holomorphic inverse.
Now assume we have two (complex) lines L_1, L_2 in $\mathbb{C}^n$ intersecting at the origin. Their preimages $\pi_2^{-1}(L_1)$ and $\pi_2^{-1}(L_2)$ are clearly not manifolds, since they consist of a line and a copy of $\mathbb{CP}^{n-1}$. However, when we take out the origin and consider the closure of what is left, $\tilde{L_i} = \overline{\pi_2^{-1}(L_i \setminus \{0\})}$, we do get a manifold. Note that $\tilde{L_1}$ and $\tilde{L_2}$ do not intersect, since the directions of the lines in the origin are taken into account here. This is the blow-up of the complex space. Now if we have a complex manifold X, we can blow up at a point p by taking a neighborhood U of p that is biholomorphic to an open subset V of $\mathbb{C}^n$ by a map that sends p to 0. When we remove U and replace it by $\pi_2^{-1}(V) \subset \tau$, we get a new complex manifold, which is the blow-up of X at p. Denote this blow-up by X'. We can define a map $\pi : X' \to X$, in the obvious way, that is a biholomorphism anywhere except at $\pi^{-1}(p)$. The inverse image $\pi^{-1}(p)$ is diffeomorphic to $\mathbb{CP}^{n-1}$ and is called the **exceptional divisor** or (**exceptional sphere** if $n = 2$). The blow-up of a manifold X is obtained by taking out a point and gluing back a copy of $\mathbb{CP}^{n-1}$. In fact, it is diffeomorphic to $X \# \overline{\mathbb{CP}^n}$, in which the bar denotes changing the orientation.

Blow-ups can not only be defined in the complex category, but also in the symplectic category [**2**]. To do this, we will denote the standard ball of radius δ in $\mathbb{C}^n$ by $B(\delta)$. Let $L(\delta)$ denote the inverse image of this ball under the second projection: $L(\delta) = \pi_2^{-1}(B(\delta)$. On this, we have a symplectic form $\rho(\lambda) = \pi_2^* \omega_0 + \lambda^2 \pi_1 \omega_{std}$, with ω_0 and ω_{std} the standard forms on $\mathbb{C}^n$ and $\mathbb{CP}^{n-1}$ respectively. Let L_0 denote the zero-section of $(L, \rho(\lambda))$

Lemma 2.21. *For any λ and any $\delta > 0$, the set $(L(\delta) \setminus L_0, \rho(\lambda))$ is symplectomorphic to the spherical shell $(B(\sqrt{\lambda^2 + \delta^2}) \setminus B(\lambda), \omega_0)$ in $\mathbb{C}^n$.*

Proof. Consider the form $\omega(\lambda)$ on $\mathbb{C}^n$ given by

$$\omega(\lambda) = \frac{i}{2}(\partial\overline{\partial}(|z|^2 + \lambda^2 \log |z|^2))$$

This form is pulled back by π_2 to $\rho(\lambda)$ on $L \setminus L_0$.
The map $F : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus B(\lambda)$ given by $F(z) = \sqrt{|z|^2 + \lambda^2} \frac{z}{|z|}$ satisfies $F^* \omega(\lambda)$. □

Using this, we can define the symplectic blow-up of a manifold (M, ω) of weight λ as follows. Take a symplectic embedding $\psi : B(\lambda) \to M$ and extend it to a symplectic embedding $\psi_0 : B(\lambda + \epsilon) \to M$. Replace the image $\psi_0(B(\sqrt{\lambda^2 + \delta^2})$ by the standard neighborhood $L(\delta)$, for some δ such that $\sqrt{\lambda^2 + \delta^2} - \lambda < \epsilon$. The blow-up of M can be defined as

$$\tilde{M} = \Big(M \setminus \psi_0(\text{int } B(\sqrt{\lambda^2 + \delta^2}))\Big) \cup L(\delta)$$

The new form equals ω on $M \setminus \phi_0(\text{int } B(\lambda+\delta))$ and equals $\rho(\lambda)$ on $L(\delta)$. The symplectomorphism class of the resulting manifold does not depend on the choice of δ, ψ or ψ_0.

We conclude that we can blow a symplectic manifold up into another one. For Kähler manifolds, this can be done in such a way that on the resulting manifold, the symplectic and complex structure are again compatible, so Kähler manifolds can also be blown up.

4. $E(1)$

In this section we introduce the construction of a certain 4-manifold as a singular fibration. In the next section, there will be more on fibrations and a generalization of this structure, the Lefschetz fibration, will be introduced and discussed in detail. First of all, we take a look at a particular 4-manifold.

Consider $\mathbb{CP}^2$. Take two generic homogeneous polynomials, p_0 and p_1, of degree 3. Their zero-sets intersect transversely in 9 points $I(p_0, p_1) = \{P_1, ..., P_9\}$. Away from these points, we can define a fibration to $\mathbb{CP}^1$ as follows.

For any point Q away from the set $\{P_i\}$, define a map π by $Q \mapsto [p_0(Q) : p_1(Q)]$. Since p_0 and p_1 are homogeneous, this is a map from $\mathbb{CP}^2 \setminus \{P1, ..., P_9\}$ to $\mathbb{CP}^1$. We can not extend π to the points P_i, because choosing Q to be one of the intersection points would send it to $[0 : 0]$. However, when we blow up the points, we can extend the map. Since we picked the cubics to be generic, their zero-sets intersect transversely in the points P_i. This means that, for a certain P_i, $z \mapsto (p_0(z), p_1(z))$ is a local chart around P_i. Using this chart, the manifold τ looks like $\{([p_0(z) : p_1(z)], (p_0(z), p_1(z))) \in \mathbb{CP}^1 \times \mathbb{C}^2\}$. When we use this to blow up, it determines π in the copy of $\mathbb{CP}^1$. Any point in this closure can be approached by a curve $\{(p_0(\lambda z), p_1(\lambda z)) \in \mathbb{C}^2 | \lambda \in \mathbb{R}\}$ by letting λ go to zero. The map π is defined anywhere except when $\lambda = 0$, and it is constant where it is defined, so it can be extended in its closure, hence to the blow-up.

This gives a map π from the nine point blow-up $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ to $\mathbb{CP}^1$. The space $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ is called $E(1)$.

The map π behaves much like a fiber bundle. The regular fibers are generic cubic curves, hence diffeomorphic to the torus T^2. However, torus bundles over the sphere have Euler characteristic 0, while the connected sum $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ has Euler characteristic 12. We conclude that there are also fibers that are not diffeomorphic to the torus. These will be called singular fibers. Since $\mathbb{CP}^2$ is simply connected, $E(1)$ is as well. Moreover, when we take out (a regular neighborhood of) a regular fiber, which is diffeomorphic to the torus, the resulting space is still simply connected, since the only possible nontrivial loop arising from this would be one around the fiber taken out, but this curve can be contracted over a section $\mathbb{CP}^1$ (any exceptional sphere forms a section), which is still simply connected when a point is taken out.

The manifold $E(1)$ can be given a symplectic structure. There is a Kähler structure on $\mathbb{CP}^2$ and the blow-ups can be performed in a way that preserves this Kähler structure. This means that we can get a 2-form that is not only symplectic, but also has generic (toral) fibers as symplectic submanifolds, since they are complex submanifolds of a Kähler manifold.

CHAPTER 3

The symplectic normal sum

1. Construction and proof

Two symplectic $2n$-manifolds can be connected by the construction of symplectic sum. This is a version of a connected sum, which preserves the symplectic structures, to give a symplectic structure to the connected sum of the manifolds. This construction is specified in the following theorem.

THEOREM 3.1 (Gompf [**5**]). *Let M_1 and M_2 be $2n$-dimensional manifolds which both have the same symplectic codimension 2 manifold V embedded as a symplectic submanifold. Assume that the normal bundles $N_{M_1}V$ and $N_{M_2}V$ have opposite Euler numbers. For any choice of an orientation-reversing isomorphism between these bundles, there is a canonical (up to isotopy) symplectic structure on the connected sum $M_1 \#_V M_2$.*

PROOF. We begin the proof with a lemma, which is a special case of Weinstein's neighborhood theorem..

LEMMA 3.2. *Let (M, ω_M) and (V, ω_V) be symplectic manifolds and let $N \subset V$ be a compact, codimension 2, symplectic submanifold. Let $f : V \to M$ be an orientation preserving embedding, such that $f|_N$ is symplectic. There exists a compact supported isotopy relative to N from f to an embedding $\tilde{f}$ that is symplectic in a neighborhood of N.*

PROOF. $N \subset V$ and $f(N) \subset M$ are symplectic submanifolds, so their normal bundles can be specified using orthogonal complements with respect to their forms ω_V and ω_M. We can use an isotopy of f to get the fibers of these bundles to correspond under f_*, since all tubular neighborhoods of diffeomorphic submanifolds are diffeomorphic.
Define $\eta = f^*\omega_M - \omega_V$ and $\omega_t = \omega_V + t\eta = tf^*\omega_M + (1-t)\omega_v$, with $t \in [0, 1]$. Let $j : N \to V$ denote the inclusion. Then $j^*\eta = 0$, because $f|_N$ is symplectic by assumption, so all ω_t will agree on the tangent space of N. The normal spaces (induced by symplectic orthogonality) to N in V under the forms ω_V and $f^*\omega_M$ agree by construction and these two forms are area forms on these spaces. So, the forms ω_t are all convex combinations of compatibly oriented area forms, hence nonzero, hence nondegenerate on the tangent bundle $TV|_N$. Since nondegeneracy is an open condition, we can find a neighborhood $U \subset V$ of N on which all forms ω_t are nondegenerate. Since they are combinations of closed forms, they are symplectic on U.
We now define an operator, using integration. For this, we assume U is identified with an open normal disk bundle of N. If this is not the case, U can be shrank to a tubular neighborhood, since all neighborhoods contain tubular neighborhoods. For $s \in [0, 1]$, let $m_s : U \to U$ be multiplication (in the disk) with s. Let $X_s = \frac{d}{ds} m_s$ to be the corresponding vector field. We now define the integral operator $I : \Omega^p(U) \to \Omega^{p-1}(U)$ on p-forms by $I(\rho) = \int_0^1 m_s^*(X_s \rfloor \rho) ds$, where the $\rfloor$ denotes contraction. For contraction, consider the following standard differentiation formula.

$$\frac{d}{ds}(m_s^*\rho) = m_s^*(X_s \rfloor \rho) + d(m_s^*(X_s \rfloor \rho)) \tag{1}$$

Using this, we obtain an important property of the operator I, namely that if $d\rho = 0$ and $j^*\rho = 0$, then $dI(\rho) = \int_0^1 dm_s^*(X_s \rfloor \rho) ds = \int_0^1 \frac{d}{ds}(m_s^*\rho) ds = m_1^*\rho - m_0^*\rho = \rho$.

Now define the isotopy from f to $\tilde{f}$ in the following way. Let $\phi = I(\eta) \in \Omega^1(U)$. Note that $d\phi = 0$ on U, since $d\eta = 0$ and $j^*\eta = 0$. Define the vector field Y_t on U by $Y_t^{\lrcorner}\omega_t = -\phi$. The vector field Y_t is uniquely determined since all forms ω_t are nondegenerate on U. Since ϕ vanishes $TU|_N$, so does Y_t. Integrate Y_t for $t \in [0,1]$ to obtain a flow g_t on a neighborhood of N. This flow g_t is compactly supported and g_0 is the identity on V. We now show that $g_t^*\omega_t$ does not depend on t. Differentiation yields:

$$\begin{aligned}
&\frac{d}{dt}(g_t^*\omega_t) && \\
=&dg_t^*(Y_t^{\lrcorner}\omega_t) + g_t^*(\frac{d}{dt}\omega_t) && \text{Using (1) again} \\
=&-g_t^*d\phi + g_t^*\eta && \text{Since } \phi = -Y_t^{\lrcorner}\omega_t \text{ and } \frac{d}{dt}\omega_t = \eta \\
=&0 && \text{Since } d\phi = \eta \text{ on } U
\end{aligned}$$

So $g_1^*\omega_1 = g_0 * \omega_0 = \omega_0$. In other words, $g_1^* f^*\omega_M = \omega_V$ near N. This gives the required embedding $\tilde{f} = f \circ g_1$, which is symplectic near N and isotopic (relative to N) to f. □

We now construct models for tubular neighborhoods of the embedded submanifolds in M. Let $j_i : N \to M$ be a symplectic submanifold of a symplectic manifold (M, ω_M) and let $e = e(\nu_i)$ be the Euler number of its normal bundle. Let $\pi : E \to N$ be vector bundle bundle with Euler class e, on which we have a fiberwise $SO(2)$-structure, and let $E^0 \subset E$ be the disk bundle with disks of radius $\frac{1}{\sqrt{\pi}}$. We now obtain a sphere bundle by gluing the disk fibers of E to the disk fibers of $\overline{E}$, the bundle with opposite orientation. For the gluing, we use the fiberwise map given by

$$x \mapsto \sqrt{\frac{1}{\pi||x||^2 - 1}}x$$

This map turns the punctured open disk inside out and preserves the standard area form, up to its sign, which gets reversed. The map is therefore symplectic. We define $i_0, i_\infty : N \to S$ to be the zero-sections of E and $\overline{E}$, respectively, with images N_0 and N_∞. The SO(2)-action now induces and action r on S.

Lemma 3.3. *There exists a closed, $SO(2)$-invariant 2-form η on S with $i_0^*\eta = 0$ and η restricting to a symplectic form of area 1 on any fiber.*
For such η, there is a $t_1 > 0$ such that $\omega_{t_1} = \pi^*\omega_N + t\eta$ is an $SO(2)$-invariant symplectic form on S.

Proof. We begin by obtaining a closed 2-form η' on S, that restricts to the standard form ω_{S^2} on each fiber: pick a representative of the Poincaré dual of $[N_0] \in H_2(S;\mathbb{R})$. Notice that $\int_F \beta = 1$ for each fiber F. Choose local trivializations $\phi_i : \pi^{-1}(U_i) \to U_i \times S^2$ for S, with a partition of unity $\{\rho_i\}$ subordinate to the cover $\{U_i\}$ of N. Now $\phi_i^*\pi_{S^2}^*\omega_{S^2} - \beta$ is the difference of two forms in the same cohomology class, since the integrals over any fiber coincide. Thus, there must be some 1-form α_i such that it is equal to $d\alpha_i$. Take $\eta' = \beta + d\sum_i(\rho_i \circ \pi) \cdot \alpha_i$. This is the desired closed form on S. Now consider $\eta'' = \eta - \pi^* i_0^*\eta'$. This is also closed and restricts to ω_{S^2} on each fiber. Moreover, it satisfies $i_0^*\eta'' = 0$.
We now define

$$\eta = \frac{1}{2\pi}\int_{\mathrm{SO}(2)} r(\theta)^*\eta'' d\theta$$

by averaging the form η'' over the previously defined SO(2)-action r. This is the form we are looking for, since this integral is obviously SO(2)-invariant and it keeps the properties of closedness and $i_0^*\eta = 0$. It is now an observation by Thurston that for any such η, the form $\pi^*\omega_N + t\eta$ is symplectic for small enough t. This comes down to the observation that $\pi^*\omega_N$ is nondegenerate along a horizontal subspace H, and $t\eta$ is nondegenerate along a vertical one V. This would be enough if $\pi^*\omega_N|_V$ and $t\eta|_H$ would both vanish. The former does, and the latter can be shrank (by shrinking t) to make $\pi^*\omega$ dominate on H. Since nondegeneracy is an open property (Remark 2.13), this is enough. Since N is compact, we can pick an overall t. This proves the lemma. $\square$

We are now ready to prove the theorem, so let us return to the situation described in it. Let $j_1(N) \subset M_1$, $j_2(N) \subset M_2$ be symplectic embeddings of N and let $\psi : \nu_1 \to \nu_2$ be an orientation reversing isomorphism between their normal bundles. We construct E, S, η and ω_{t_1} over N as in the lemmas. The bundles E and ν_1 are isomorphic, so we have an orientation preserving embedding $f : E^0 \to M$ (since E^0 is a subbundle of E) with $f \circ i_0 = j_1$. This embedding is symplectic when restricted to N, so now we can shrink t_1 even more to obtain a symplectic embedding $\hat{f} : (E^0, \omega_{t_1}) \to (M, \omega_M)$, with $\hat{f} \circ_0 = j_1$, which is isotopic (relative to N) to f.
The next thing to do is to identify (a neighborhood of) N_∞ to (a neighborhood of) $j_2(N)$. By construction, we have $S \setminus N_0 = \overline{E^0}$, so the normal bundles ν_0 and ν_∞ of N_0 and N_∞ are identified by an orientation reversing map h. Consider $\psi'' = \psi \circ f^* \circ h : \nu_\infty \to \nu_2$. This map preserves orientation, since ψ (by assumption) and h reverse it and f^* does not.
Now we have a smooth embedding $g : S \setminus N_0 \to M_1$ with $g \circ i_\infty = j_2$ and $g_* = \psi''$ on ν_∞. We want to use $f^{-1} \circ g$ to obtain smoothness on the connected sum. The problem is that g is not necessarily symplectic. To fix this, let $\mu : S \to S$ be a map that rescales the fibers of E^0 in such a way that a neighborhood of N_∞ stays intact and a neighborhood of N_0 gets collapsed to N_0. This can be done in a smooth way. We now compose g^{-1} with μ and conclude that g^{-1} extends to a map λ from a neighborhood of $\overline{g(S \setminus N_0)}$ into S, which sends the points in the complement of $g(S \setminus N_0)$ into N_0. Let $\zeta = \lambda^*\eta$. This is a closed 2-form, vanishing outside $g(S \setminus N_0)$, so it can be extended over M_2 by just taking the zero 2-form everywhere else. We have $j_2^*\zeta = i_\infty^*\eta$. We now replace ω_{M_2} by $\tilde{\omega}_{M_2} = \omega_{M_2} + t\zeta$. The form $\tilde{\omega}_{M_2}$ is symplectic on both M_2 and $j_2(N)$ for small enough t, because of the openness of the nondegeneracy condition. Also, $g|_\infty : N_\infty \to M$ is a symplectic embedding (sending the form ω_t to $\omega_{\tilde{M_2}}$). From Lemma 3.2, we get an isotopy (relative to N_∞) from g to $\tilde{g} : s \setminus N_0 \to M$, with $\tilde{g}$ a map that is symplectic on a neighborhood U_∞ of N_∞.
It is now time to glue the manifolds together. Let $W = \tilde{g}(U_\infty \setminus N_\infty)$. The map $\tilde{g}^{-1} : W \to E$ is a symplectomorphism. Since E^0 is symplectically embedded in M_1 by $\hat{f}$, it can be taken out, and be replaced by W, to obtain $M_1 \#_\psi M_2$. We now have a symplectic structure on $M_1 \#_\psi M_2$. $\square$

The construction can be used to glue symplectic manifolds together, and thus yields a lot of examples of symplectic manifolds.

2. Application: examples of symplectic manifolds

Although the definition of a symplectic structure seems quite obscure, symplectic structures are very common and there are a lot of different symplectic manifolds. This is illustrated by the following theorem.

Theorem 3.4 (Gompf [**4**]). *Let G be a finitely generated group. Now there is a closed symplectic 4-manifold which has G as its fundamental group.*

Proof. Let $G = \langle g_1, ..., g_k \mid r_1, ..., r_l \rangle$, we will construct a corresponding 4-manifold.
Let F be a surface with genus k, the number of generators. Fix a collection of circles $\alpha_i, \beta_j \subset F$ that represent a basis of $H_1(F; \mathbb{Z})$, such that the intersections $\alpha_i \cap \beta_j$ consists of one point precisely when $i = j$ and are otherwise empty, and the intersections $\alpha_i \cap \alpha_j$ and $\beta_i \cap \beta_j$ are empty.
Now take l immersed curves γ_j in F representing the relators r_j in the free group $\pi_1(F)/\langle \beta_1, ..., \beta_k \rangle$, and take $\gamma_{l+i} = \beta_i$ $(i = 1, ..., k)$. We now have a set of $k + l$ curves γ_i on F.
Take a torus T^2 with generating circles α and β and consider the 4-manifold $F \times T^2$. In this, we have a collection of $k + l$ tori $T_i = \gamma_i \times \alpha$ and the torus $T_0 = \{\text{pt.}\} \times T^2$. These tori can be perturbed to be disjoint. To do this, note that α can be moved a little bit for every i, since α has

self-intersection number zero. The torus T_0 can be chosen disjointly by taking a point in F that is not in any of the curves. Such a point exists, since the curves are embedded submanifolds of a lower dimension, so finitely many of them can never cover the entire manifold. Since F and T are both orientable surfaces, they allow symplectic structures ω_F and ω_T. On their product, we have the symplectic structure $\omega_F + \omega_T$. The torus T_0 is a symplectic submanifold, but the other tori T_i are Lagrangian (the symplectic form vanishes on them). To fix this, we alter the form on $F \times T$ a little bit. Consider a specific torus T_i, with $i \geq 1$. Denote (a representation of) its Poincaré dual by α_i. Take $\alpha = \sum_i \alpha_i$. Around the tori T_i, we define normal neighborhoods νT_i and a set of larger normal neighborhoods $\nu' T_i$, such that for each i, $\overline{\nu T_i}$ is contained in $\nu' T_i$ and all neighborhoods $\nu' T_i$ are disjoint. We can pick such a set of neighborhoods since the tori T_i are closed, disjoint subsets. Take smooth bump functions ρ_i that are 1 on νT_i and 0 outside $\nu' T_i$. Take volume forms μ_i on each torus, by the inherited orientation of $F \times T$. We can assume $[\mu_i] = [\alpha|_{\nu' T_i}] \in H^2_{dR}(\nu' T_i)$ by rescaling the forms μ_i, so we can find local 1-forms θ_i such that $d\theta = \mu_i - \alpha$. Using this, consider:

$$\eta = \alpha + d \sum_i \rho_i \theta_i$$

This form is closed, and it restricts to a volume (hence symplectic) form on each of the tori T_i. Now we can use Corollary 2.13 to find a positive number t such that $\omega_F + \omega_T + t\eta$ is symplectic on the manifold $F \times T$. Moreover, since the first two terms are equal to the zero form on the tori, the third term is the only thing that is left and it is symplectic. We now have a symplectic manifold in which all tori T_i are symplectic submanifolds. This situation is the one we need to apply the symplectic summation. We have $k + l + 1$ disjoint symplectic submanifolds of $F \times T^2$.

We now use the 4-manifold $E(1)$, which is defined above. This is a manifold with symplectic fibers. Since $E(1)$ is the connected sum of simply connected manifolds, it is simply connected. When we cut out a regular neighborhood νT of such a fiber, the result $E(1) \setminus \nu T$ is still simply connected.

Take the symplectic sum of $F \times T^2$ with $k + l + 1$ copies of $E(1)$, glued along the tori T_i' and a regular fiber of every $E(1)$.

The Seifert-van Kampen theorem directly implies that the fundamental group of this manifold is indeed G. The symplectic sum gives a symplectic structure on our manifold. $\square$

CHAPTER 4

Lefschetz fibrations

1. A generalization of fiber bundles

Recall the manifold $E(1)$ introduced in Section 4. This manifold has a structure that resembles the structure of a fiber bundle, but it has a finite number of points in which the fiber differs. In this section, we give a generalization of this behavior: the Lefschetz fibration. Lefschetz fibrations are generalizations of the well-known concept of fiber bundles. We first give a definition of the latter.

Definition 4.1. A **fiber bundle** consists of two manifolds E, B and a surjective map $\pi : E \to B$ called the projection with the following local triviality condition. For every point $x \in B$ there is a neighborhood $U \subset B$ such that π restricted to $\pi^{-1}(U)$ is the projection map of a topological product $U \times F$. The topological space F is called the fiber and does not depend on x or U.
B is called the base space

Of course, products of spaces $E = B \times F$ are examples of fiber bundles. These bundles are called trivial bundles. An easy nontrivial example of a fiber bundle is the Möbius band, which is a fiber bundle with base space $B = S^1$ and fiber $F = I$.
The definition of Lefschetz fibration generalizes this concept, by allowing a finite number of singular points. In these singular points, the projection map will not be a product, but we do restrict the behaviour of π.

Definition 4.2. Let X be a compact, connected, oriented, smooth 4-manifold and Σ a compact, connected, oriented, smooth 2-manifold. A **Lefschetz fibration** is a map $\pi : X \to \Sigma$ with the following properties.

- $\pi^{-1}(\partial\Sigma) = \partial X$
- Each critical point of π lies in the interior (complement of the boundary) of X.
- For each critical point of π, we can find a pair of orientation preserving complex coordinate charts, one on X, centered at the critical point, and one on Σ, such that $\pi(z_1, z_2) = z_1^2 + z_2^2$ on these charts.

A priori, this definition does not look like the definition of a fiber bundle, but they actually are quite alike: when we restrict the map to fibers of regular values, a Lefschetz fibration is a fiber bundle.

Proposition 4.3. *Let $\pi : X \to \Sigma$ be a Lefschetz fibration and let $C \subset \Sigma$ be the collection of critical values of π. The map $\tilde{\pi} = \pi|_{X \setminus \pi^{-1}(C)}$ is a fiber bundle.*

Proof. Recall the theorem of Ehresmann [**3**], which states that any proper, surjective submersion is a fiber bundle. $\tilde{\pi}$ is a submersion by definition of critical points. To prove that it is surjective, note that there can only be finitely many critical points, since X is compact and the critical points all have non-intersecting neighborhoods. This means that $\Sigma \setminus C$ is still connected. The map $\tilde{\pi}$ is open, since it is a submersion, and its image is closed since X is compact. We conclude that it is surjective. The map π is proper since it is a continuous map between compact sets. The properness of $\tilde{\pi}$ follows directly, since every compact subset of $\Sigma \setminus C$ has to be a closed subset of Σ which does not contain a critical value, hence its inverse images under $\tilde{\pi}$ and π coincide. ∎

We conclude that away from a finite number of singular fibers, π is a fiber bundle, so this definition can really be considered a generalization of fiber bundles.

2. Basic properties and topology of Lefschetz fibrations

To give some intuition of what a Lefschetz fibration looks like, we take a closer look at a critical point and try to construct a regular neighborhood of this. Around the critical point, π is given by $(z_1, z_2) \mapsto z_1^2 + z_2^2$ in complex coordinates. This means a nearby regular fiber (the inverse image of a non-singular point) is given by the formula $z_1^2 + z_2^2 = t$, with $t \neq 0$. We assume t to be a positive real number, because we can multiply by a complex unit to make it one. The fiber is locally given by the formula $x_1^2 + x_2^2 - y_1^2 - y_2^2 = t$, which gives a submanifold of $\mathbb{C}^2$. This submanifold can be identified with the cotangent bundle of S^1 by viewing the latter as a submanifold of $\mathbb{C}^2$ and using the formula $(x, y) \mapsto (|x|^{-1}x, |x|y)$. When we intersect this fiber with $\mathbb{R}^2$ (and hence only view the coordinates x_i), we get a circle S^1 given by the formula $x_1^2 + x_2^2 = t$, which bounds a disk D_t. As t approaches 0, this disk shrinks to a point. Its boundary ∂D_t is called the vanishing cycle of this critical point. We explicitly see the singular fiber being obtained from the regular fiber by collapsing this vanishing cycle to a point. Its neighborhood cylinder does not vanish.
Regular fibers are oriented 2-manifolds, which are classified: all connected oriented 2-manifolds are spaces with a genus n, in which $n = 0$ represents the sphere S^2. Now it is important to classify different vanishing cycles.
For the sphere, there is only one cycle, and it separates the sphere into two pieces. For the torus

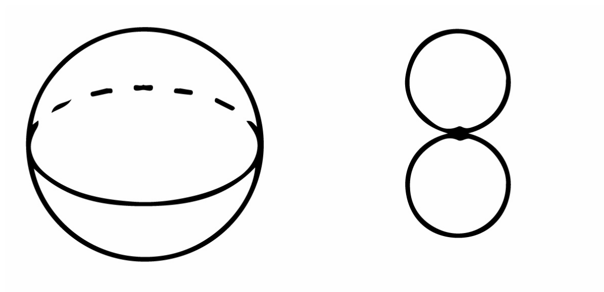

Figure 1. A sphere with its equator as vanishing cycle. When the vanishing cycle is collapsed to a point, an almost disconnected space arises.

T^2, there are two different cycles, one of which separates the torus. For surfaces with higher genus, it is possible to have a separating cycle, as well as a non-separating cycle. Having a separating vanishing cycle will result in a singular fiber that is almost disconnected, but for one (singular) point connecting the two components.

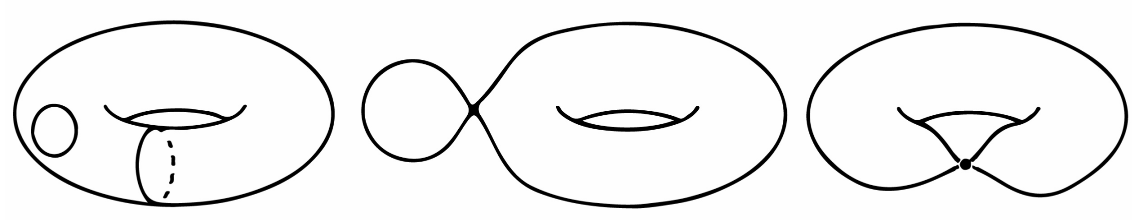

Figure 2. On the left is a torus with two possible vanishing cycles. Collapsing them gives two different pinched manifolds, which are exibited on the right.

3. The exact sequence of a Lefschetz fibration

The fibers of Lefschetz fibrations are not necessarily connected, but it would be nice to focus on cases where they are. In order to do that, we prove the following.

THEOREM 4.4. *Let $\pi : X \to \Sigma$ be a Lefschetz fibration with fiber F. The maps $F \hookrightarrow X \to \Sigma$ induce an exact sequence*

$$\pi_1(F) \to \pi_1(X) \to \pi_1(\Sigma) \to \pi_0(F) \to 0$$

REMARK 4.5. The function $\pi_1(\Sigma) \to \pi_0(F)$ is defined as a function between sets, but since $\pi_0(F)$ is a pointed set, it still makes sense to talk of exactness here.

PROOF. This exact sequence is much like the standard exact sequence of a fiber bundle, for which the above statement holds. Just like in the exact sequence of fiber bundles, we define the last map, let us call it τ, with the homotopy lifting property. This states that for every path $\gamma : [0,1] \to \Sigma$ and point x in the fiber of $\gamma(0)$, there is a unique (up to homotopy) path $\tilde{\gamma} : [0,1] \to X$ such that $\pi \circ \tilde{\gamma}(t) = \gamma(t)$ for every $t \in [0,1]$ and $\tilde{\gamma}(0) = x$. Elements γ of $\pi_1(\sigma)$ are represented by loops. We apply the above procedure to γ, taking x in the base component of the fiber. Now define $\tau(\gamma)$ to be the component of the endpoint $\tilde{\gamma}(1)$.
The problem with translating this to Lefschetz fibrations, is that it only works in the regular part. In the singular points, X is not a fiber bundle. If the path avoids singular values, we are done. If it does not, we need to use a local homotopy to make the path go around the singular value. The only risk here is that a path in the base space jumps to another connected component of the fiber when passing through a singular point. However, in the previous section we saw what the singular fibers look like: relative to a regular fiber, an embedded copy of S^1 is collapsed. Since S^1 itself is connected, it can only be embedded into one connected components. Collapsing this vanishing cycle will never connect two components that were previously disconnected. We conclude that there is no risk of the lifted path ending up in a different connected component.
The endpoint of the path does not depend on the homotopy or the lift, so τ is well-defined.
The exactness at $\pi_1(\Sigma)$ is easy to see: a loop in Σ lifts to a loop in X precisely when its lift begins and ends in the same component of the fiber. For exactness at $\pi_1(X)$, notice that a path in Σ is a constant path precisely when its lift is a path in one fiber. □

COROLLARY 4.6. *Without loss of generality, we can assume the fibers of a Lefschetz fibration to be connected.*

PROOF. If the fibers of a Lefschetz Fibration $\pi : X \to \Sigma$ are not connected, then $\pi_1(X)$ maps to a finite-index subgroup of $\pi_1(\Sigma)$, which corresponds to a finite covering $\tilde{\Sigma}$ of Σ, so we can lift π to this finite covering space and view the new Lefschetz fibration $\tilde{\pi} : X \to \tilde{\Sigma}$, which has connected fibers $\tilde{F}$.

$$\begin{tikzcd} & \tilde{\Sigma} \arrow[d] \\ X \arrow[ur, dotted, "\tilde{\pi}"] \arrow[r, "\pi"] & \Sigma \end{tikzcd}$$

This new Lefschetz fibration looks locally the same, but it sends different connected components of the fiber to different points in $\tilde{\Sigma}$. To see this, we can take the exact sequence of this new Lefschetz fibration:

$$\pi_1(\tilde{F}) \to \pi_1(X) \to \pi_1(\tilde{\Sigma}) \to \pi_0(\tilde{F}) \to 0$$

Since it is exact at $\pi_1(\tilde{\Sigma})$ and the map $\pi_1(X) \to \pi_1(\tilde{\Sigma})$ is surjective by the construction of the covering space, the fact that $\pi_0(\tilde{F})$ is a one-point space follows directly. □

CHAPTER 5

The main theorem in 4 dimensions

1. Main theorem: Lefschetz fibrations and symplectic structures

We are now ready to state and prove the following important theorem.

THEOREM 5.1 (Gompf [**4**]). *Let X be a closed 4-manifold and let $\pi : X \to \Sigma$ be a Lefschetz fibration. Let $[F]$ denote the homology class of the fiber. Now X admits a symplectic structure with symplectic fibers if and only if $[F] \neq 0$ in $H_2(X;\mathbb{R})$.*

PROOF. One of the implications is easy to prove: if X has a symplectic structure ω that restricts to a symplectic structure on the fibers, we would have a symplectic 2-form on F, which is a volume form. This means we have $\langle[\omega],[F]\rangle = \int_F \omega \neq 0$, so $[F]$ can not be equal to the zero class.
The other direction is more complicated. It requires a quite explicit construction of a symplectic structure on X.
By Corollary 4.6, we can assume that the fibers are connected. In light of the classification of 2-manifolds, this means the fibers are surfaces of a certain genus. We now perturb π such that any fiber contains at most one critical point.
Before we proceed, we need the following lemma, which allows us to define a useful closed form. This form will help us to glue closed forms together later, without losing their closedness.

LEMMA 5.2. *For X as above, there is a closed 2-form ζ on X such that for any closed surface F_0 contained in a single fiber, we have $\int_{F_0} \zeta > 0$. Of course, we assume F_0 to have the orientation induced by the fiber it is contained in.*

PROOF. The important thing to notice here, is that there are not many closed surfaces in the fibers. If the vanishing cycle of a critical point separates the fiber (meaning its complement is not connected), the singular fiber can we written as the union of two closed surfaces, $F = F_0 \cup F_1$, and these are the only nonempty closed surfaces, apart from F itself. If the vanishing cycle does not separate the fiber, the only closed surface in F is F itself. It is important to note that there is only one vanishing cycle per critical fiber, since we assumed π to be injective on the set of critical points.
Since we assumed F to be in a nonzero cohomology class, we can find an element $a \in H^2(X;\mathbb{R})$ such that $\langle a,[F]\rangle > 0$. Looking at a specific singular fiber F, assume it can be written as $F = F_0 \cup F_1$, the union of two closed surfaces. If this is not the case, we are already done.
We know $s := \langle a,[F]\rangle > 0$. Because of the local model of the singular point, described in Section 2 of the previous chapter, we know the two surfaces F_0 and F_1 intersect transversely in one point, so we have $\langle a,[F]\rangle = \langle a,[F_0]\rangle + \langle a,[F_1]\rangle$. A problem will arise if one of these terms is negative. We can assume it is the first one.
If $r := \langle a,[F_0]\rangle \leq 0$, we take the cohomology class $a' = a + (\lambda s - r)PD[F_1]$, with PD the Poincaré dual and $\lambda \in (0,1)$. This class evaluates positively on any surface in F. Note that for any surface S, $\langle PD[F_1], S\rangle$ is given by the intersection number of S and F_1, and the intersection number $[F_0]\cdot[F_1]$ is equal to one. Furthermore, note that $[F]^2 = 0$, since the fiber F can be moved around freely in a fibration without creating a self-intersection. This gives $0 = [F]^2 = [F_0 + F_1]^2 = [F_0]^2 + [F_1]^2 + 2[F_0]\cdot[F_1] = [F_0]^2 + [F_1]^2 + 2$.
Although F_0 and F_1 might be very different, $[F_0]^2$ and $[F_1]^2$ have to be equal, since any nonzero

intersection must come from a neighborhood of the singular point, in which F_0 and F_1 look symmetric. We can now conclude that $\langle PD[F_1], [F_1]\rangle = -1$. Keeping this in mind, evaluating a' on the surfaces gives

$$\langle a', [F_0]\rangle = \langle a, [F_0]\rangle + (\lambda s - r)\langle PD[F_1], F_0\rangle = r + \lambda s - r = \lambda s > 0$$

$$\langle a', [F_1]\rangle = \langle a, [F_1]\rangle + (\lambda s - r)\langle PD[F_1], [F_1]\rangle = s - r - (\lambda s - r) = (1 - \lambda)s > 0$$

$$\langle a', [F]\rangle = \langle a, [F]\rangle + (\lambda s - r)\langle PD[F_1], [F]\rangle = \langle a, [F]\rangle + (\lambda s - r)(\langle PD[F_1], [F_0]\rangle + [F_1]^2 = s + (\lambda s - r)(1-1) = s > 0$$

This modification $a \mapsto a'$ only alters our cohomology class on the specific fibers, not on any other fiber. Since there are only finitely many singular fibers, we only have to modify the cohomology class a finite number of times. After these modification, represent the class by a 2-form ζ. This provides the solution. □

Note that we not only proved the statement, but kept some freedom of choice in any separated singular fiber; the procedure works for any λ strictly between 0 and 1. This means that, for a separated fiber, we can choose how the integral is divided over the two surfaces.

With this in mind, we will start constructing a symplectic structure on X. We do this in four steps:

(1) we define a symplectic form on every fiber,
(2) we extend it to regular neighborhoods of the fibers,
(3) we glue the forms together,
(4) we take a sum with a symplectic form on Σ, to get nondegeneracy in every direction, hence on the entire X.

Step (1): Let F_y denote the fiber of $y \in \Sigma$. Since X is a Lefschetz fibration, we can choose open balls V_j in X around the critical points of X, such that in these balls, π is given as $\pi(z_1, z_2) = z_1^2 + z_2^2$. Shrink these balls to be disjoint. Now take the standard symplectic form $\omega_{V_j} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. This is symplectic on $V_j \cap F_y$, since F_y is a holomorphic curve. To extend it to all of F_y, pick an open ball U_j with closure in V_j, which still contains the critical value. Since $F_y \setminus U_j$ is an oriented surface, there is a symplectic form $\omega_{F\setminus U_j}$ on it (Example 2.11), which possibly does not agree with ω_{V_j}. These two forms can be glued together using a partition of unity subordinate to the cover $\{V_j \cap F_y, F_y \setminus U_j\}$. In this way, we get a symplectic form ω_y on every fiber F_y, for $y \in \bigcup_j U_j$. This form is indeed symplectic, since by the definition of a Lefschetz fibration, the chart on U_j has an orientation compatible with the one of the fiber (which is enherited from the orientation of X). For the points y outside U_j, define ω_y such that it is symplectic along F_y. This can be done since all oriented 2-manifolds are symplectic (Example 2.11 again). We now have a symplectic form on every fiber.

This is where Lemma 5.2 comes in again; we can assume $[\omega_y] = [\zeta|_{F_y}]$ in $H^2_{dR}(F_y)$. The cohomology class of a 2-form is completely determined by its integral on all closed surfaces contained in one fiber. First of all, we can just rescale ω_y to obtain the same integral on the fiber F_y as ζ. At the critical point, we have to be careful to make sure that the integral is separated in the right way, but remember, we still have some freedom regarding the λ we choose, so we can completely determine how ζ splits the integral. Just make sure it does it in the same way as ω_y.

Step (2): The problem is that these forms are only symplectic along the 2-dimensional fibers, while we want to have a symplectic form defined on the 4-dimensional X. The next step in accomplishing this is extending the forms to regular neighborhoods of the fibers. For regular $y \in \Sigma$, define a neighborhood $W_y \subset \Sigma$ containing no critical values. Define $\tilde{W}_y = \pi^{-1}(W_y)$ and let $r_y : \tilde{W}_y \to F_y$ be a retraction. Now define η_y as the pullback, $\eta_y = r_y^*\omega_y = \omega_y \circ r_y$.

For $y \in \Sigma$ a critical value, we proceed a bit differently. Define W_y to contain only y as critical value and again let $\tilde{W}_y = \pi^{-1}(W_y)$. Assume $\tilde{W}_y$ contains $\overline{U_j}$, the closure of the ball chosen around y before. This can be obtained by shrinking U_j. This time, define a retraction $r_y : \tilde{W}_y \to F_y \cup \overline{U_j}$

and let $\eta_y = r_y^*(\omega_y \text{ or } \omega_{U_j})$.

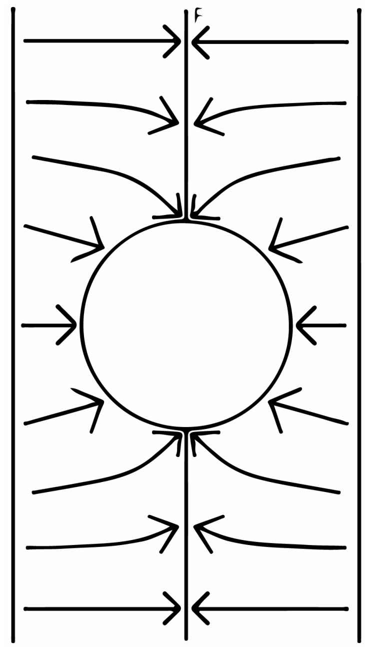

Figure 1. The retraction r for y a singular value.

Step (3): Glue the forms together. Note that the regular neighborhoods of F_y define an open cover of a compact manifold, so we can take a finite subcover $\{W_y | y \in I\}$ of Σ, for I some finite subset of Σ. Let $\{\rho_y | y \in I\}$ be a partition of unity subordinate to this cover. We have $[\eta_y] = [\zeta|_{\tilde{W}_y}]$ in $H^2_{dR}(F_y)$, and $H^2_{dR}(\tilde{W}_y) = H^2_{dR}(F_y)$ (because of the retraction), so $[\eta_y - \zeta|_{\tilde{W}_y}] = 0 \in H^2_{dR}(\tilde{W}_y)$. This means there exists a 1-form θ_y on $\tilde{W}_y$ such that $d\theta_y = \eta_y - \zeta|_{\tilde{W}_y}$. Take $\eta = \zeta + d(\sum_{y\in I}(\rho_y \circ \pi)\theta_y)$ on X.

This is a closed form, and it is symplectic along the fibers of π. Closedness is obvious, since $d\eta = d\zeta = 0$. For symplecticness along the fibers, note

$$\eta|_{F_x} = \zeta|_{F_x} + \sum_{y\in I} \rho_y(x) d\eta_y|_{F_x} = \zeta|_{F_x} + \sum_{y\in I} \rho_y(x)(\eta_y|_{F_x} - \zeta|_{F_x}) = \sum_{y\in I} \rho_y(x)\eta_y|_{F_x}$$

The last expression is a convex sum of volume forms, hence it is symplectic.

Step (4): It is now time to construct the final form. On the surface Σ, define a symplectic form ω_Σ that is compatible with the complex structure on $\bigcup_j \pi(U_j)$. At the charts $\pi(U_j)$ (or better: at their neighborhoods $\pi(V_j)$, as used above), we have such a form by Theorem 2.17, and it can be extended to the entire manifold by defining a volume form compatible to the orientation on the remaining part $\Sigma \setminus \bigcup_j \pi(U_j)$ and gluing using a subordinate partition of unity.

Now, for $t > 0$ a real number, define

$$\omega_t = t\eta + \pi^*\omega_\Sigma$$

For sufficiently small t, ω_t is the symplectic form we are looking for.

There are three things we have to check: closedness, nondegeneracy along the fibers and overall nondegeneracy. The first two are pretty straightforward. The form ω_t is closed since η and ω_Σ are both closed. Restricted to a fiber, ω_t is equal to $t\eta$ since π is constant on a fiber, hence $\pi^* = 0$.

The form η was already proven to be symplectic along the fibers, and so is any nonzero multiple of it.

For overall nondegeneracy, we view points within and outside U_j separately. First, pick $x \in F_y$ a regular point outside U_j and consider its tangent space T_xX. We take a look at the orthogonal complement of T_xF_y. This complement is the same with respect to η as it is with respect to ω_t: let $v \in T_xF_y$ and $u \in (T_xF_y)^{\perp_\eta}$, the orthogonal complement with respect to η. By definition, we have $\eta(u,v) = 0$, so $\omega_t(v,u) = t\eta(v,u) + \omega_\Sigma(\pi_*(v), \pi_*(u)) = 0 + \omega_\Sigma(0, \pi_*(u)) = 0$, so $(T_xF_y)^{\perp_\eta} \subset (T_xF_y)^{\perp_{\omega_t}}$. This is one inclusion. Because they are vector spaces of the same dimension, the other one follows. The subspaces T_xF_y and $(T_xF_y)^\perp$ are complementary: because η is nondegenerate on T_xF_y, they can not have an intersection, but their dimensions add up to the total space.
So, since $\pi^*\omega_\Sigma$ is nondegenerate on $(T_xF_y)^{\perp_{\omega_t}}$, it is nondegenerate on $(T_xF_y)^{\perp_\eta}$. Now we can pick t small enough to have the restriction of ω_t nondegenerate on $(T_xF_y)^\perp$, since nondegeneracy is an open property (Remark 2.13). We also know that η is symplectic on T_xF_y, and for any vectors $v \in T_xF_y$ and $w \in (T_xF_y)^\perp$, we have $\eta(v,w) = 0$. Moreover, $\pi^*\omega_\Sigma$ will give zero if one of its entries is in T_xF_y, regardless of the other one.
For certain $u \in T_xX$, write $u = v + w$, with $v \in T_xF_y$ and $w \in (T_xF_y)^\perp$. Using the symplecticness of η and $\omega_t|_{(T_xF_y)^\perp}$, we obtain $v' \in T_xF_y$ and $w' \in (T_xF_y)^\perp$ such that $\eta(v,v') > 0$ and $\omega_t(w,w') > 0$. Now take $u' = v' + w'$, we have:

$$\omega_t(u,u') = t\eta(v+w, v'+w') + \pi^*\omega_\Sigma(v+w, v'+w')$$

$$= t\eta(v,v') + t\eta(v,w') + t\eta(w,v') + t\eta(w,w') + \pi^*\omega_\Sigma(v,v') + \pi^*\omega_\Sigma(v,w') + \pi^*\omega_\Sigma(w,v') + \pi^*\omega_\Sigma(w,w')$$

$$= t\eta(v,v') + t\eta(w,w') + \pi^*\omega_\Sigma(w,w') = t\eta(v,v') + \omega_t|_{(T_xF_y)^\perp}(w,w') > 0$$

We conclude that ω_t is nondegenerate at x. Since ω_t varies smoothly on the manifold and $X \setminus \bigcup_j U_j$ is compact, we can choose an overall $t > 0$ such that ω_t is symplectic on the entire $X \setminus \bigcup_j U_j$.
We now only need to worry about the open balls U_j. In here, $\eta = \omega_{U_j}$ is given by the standard symplectic form $x_1 \wedge y_1 + x_2 \wedge y_2$, since the retraction r^* is equal to the identity here. For a nonzero tangent vector v, we have

$$\omega_t(v, iv) = t\eta(v, iv) + \omega_\Sigma(\pi_*v, \pi_*iv)$$

We view the terms separately. First of all, within U_j, η is given explicitly by $\eta = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. It is now a pretty standard result that applying this to (v, iv) gives $||v||^2$, but since we have not introduced a lot of compatibility results here, it is better to do the explicit calculation. Take $v = \lambda_1 \frac{\partial}{\partial x_1} + \lambda_2 \frac{\partial}{\partial y_1} + \lambda_3 \frac{\partial}{\partial x_2} + \lambda_4 \frac{\partial}{\partial y_2}$. We get:

$$\eta(v, iv) = (dx_1 \wedge dy_1 + dx_2 \wedge dy_2)\left(\lambda_1 \frac{\partial}{\partial x_1} + \lambda_2 \frac{\partial}{\partial y_1} + \lambda_3 \frac{\partial}{\partial x_2} + \lambda_4 \frac{\partial}{\partial y_2}, \lambda_2 \frac{\partial}{\partial x_1} - \lambda_1 \frac{\partial}{\partial y_1} + \lambda_4 \frac{\partial}{\partial x_2} - \lambda_3 \frac{\partial}{\partial y_2}\right)$$

$$= \lambda_1^2 - (-\lambda_2)^2 + \lambda_3^2 - (-\lambda_4^2) > 0$$

Second, since π is given by $(z_1, z_2) \mapsto z_1^2 + z_2^2$ and is hence holomorphic, π_* is a complex linear map, so $\pi_* iv = i\pi_* v$ for any v. Using this, we get

$$\omega_t(v, iv) = t||v||^2 + \omega_\Sigma(\pi_*v, i\pi_*v) > 0$$

The positivity of the first term is obvious and the second term is nonnegative because of the compatibility of ω_Σ with the complex structure. We conclude that ω_t is everywhere nondegenerate, hence the symplectic form we are looking for. □

2. Lefschetz pencils

In this section, a slightly different structure is introduced: the Lefschetz pencil. Like a Lefschetz fibration, a Lefschetz pencil is like a fiber bundle, but with a bit more freedom. Intuitively speaking, Lefschetz pencils are fibrations over the 2-sphere that are allowed to have singular points, and its fibers are allowed to intersect in a finite number of points. Here is the definition.

Definition 5.3. Let X be a closed, connected, oriented, smooth 4-manifold. A **Lefschetz pencil** on X consists of a nonempty finite subset B and a smooth map $\pi : X \setminus B \to \mathbb{CP}^1$, such that the following properties hold.

- Around every point $b \in B$ we can define an orientation preserving coordinate chart such that π is given by projectivization $\mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$, $(z_1, z_2) \mapsto [z_1, z_2]$.
- For each critical point of π, we can find a pair of orientation preserving complex coordinate charts, one on X, centered at the critical point, and one on $\mathbb{CP}^1$, such that $\pi(z_1, z_2) = z_1^2+z_2^2$ on these charts.

Lefschetz pencils can be viewed as loosened versions of Lefschetz fibrations, but they are not technically a generalization, since B is not allowed to be empty, and only the surface $\mathbb{CP}^1$ can be the base space. There is, however, a way to obtain a Lefschetz fibration (over $\mathbb{CP}^1$) from a Lefschetz pencil: by blowing up. The points in B are in the closure of any fiber and can be viewed as the intersections of the fibers. The fibers are usually defined by $F_y = \overline{\pi^{-1}(y)} = \pi^{-1}(y) \cup B$. Through a point $b \in B$, there are fibers in any (complex) direction. These directions are separated by blowing up b, and this creates a copy of $\mathbb{CP}^1$, which is a section of π since it intersects every fiber exactly once. By exploiding this procedure, we can get symplectic structures on Lefschetz pencils. A Lefschetz pencil can be blown up to get a Lefschetz fibration, and on this there is a symplectic structure because of Theorem 5.1. This Lefschetz fibration can then be blown down again, preserving the symplectic structure, hence inducing a structure on the pencil. For the details, see [**4**]. There is

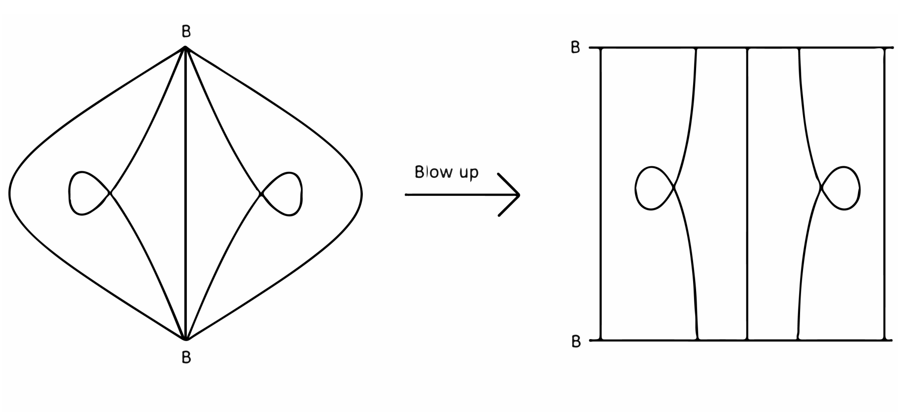

Figure 2. A lower dimensional model of a blow up of a Lefschetz Pencil. The pencil has two singular fibers. When the base locus B is blown up, it becomes a section.

also an implication in the other direction. In this thesis, we will state this theorem, but not prove it.

Theorem 5.4. [**1**] *Any symplectic* 4*-manifold admits a Lefschetz pencil.*

These theorems together imply that a 4-manifold X admits a symplectic structure precisely if it admits a Lefschetz pencil. This is a useful tool in classifying symplectic 4-manifolds.

CHAPTER 6

Generalization

1. Hyperpencils

Lefschetz fibrations and Lefschetz pencils are defined as a 4-dimensional concept, but a higher-dimensional equivalent can be defined. In the remaining part of the thesis, we will take a look at hyperpencils, a generalization of Lefschetz pencils to arbitrary even dimensions. Like Lefschetz pencils, hyperpencils are like fibrations with 2-dimensional fibers, but some critical points and intersections of the fibers are allowed, although their behavior is restricted. Using this, we can use the well-understood case of 2-manifolds to study higher dimensional manifolds and their symplectic structures.

Before we can proceed to the definition of hyperpencils, we first need a technical condition for the restriction of the critical points.

Definition 6.1. Let $f : X \to Y$ be a smooth map between manifolds and let $K \subset X$ be the set of critical points. Let $P = \overline{\bigcup_{x\in X\setminus K} \ker f_*|_x} \subset TX$. Let $P_x = P \cap T_xX$. A point $x \in X$ is **wrapped** if span P_x has (real) codimension at most 2 in $\ker f_*|_x$.

Note that regular points are always wrapped, so this definition is only relevant for critical points. Intuitively speaking, a wrapped critical point is "less critical" than an unwrapped one.
The definition of wrapped critical points is quite general. The critical points of Lefschetz fibrations are wrapped, but in four dimensions, there is way more freedom. In fact, any critical point of a map f from a 4-manifold to a 2-manifold is wrapped. This is because with these dimensions, the dimension of $\ker f_*$ is 2 in any regular point, while in singular points, the dimension of the kernel can not exceed 4. This means that the codimension of span P_x is at most 2.
In higher dimensions, this definition actually is distinctive. An example of a critical point that is not wrapped is the point $(0, 0, 0)$ of the map $f(z_1, z_2, z_3) = (z_1^2, z_2^2)$. In this point, P_x has (real) codimension 4.

We are now ready to give the definition of a hyperpencil.

Definition 6.2. A **hyperpencil** on a smooth, closed, oriented $2n$-manifold X consists of a finite subset $B \subset X$ (called the base locus) and a map $\pi : X \to \mathbb{CP}^{n-1}$ with the following properties.

- Around every point $b \in B$, we can find an orientation preserving complex chart on which π becomes projectivization $\mathbb{C}^n \to \mathbb{CP}^{n-1}$, $(z_1, ..., z_n) \mapsto [z_1 : ... : z_n]$.
- Each critical point of π is wrapped and has a neighborhood with a continuous (ω_{std}, f)-compatible almost-complex structure, in which ω_{std} denotes the standard Fubini-Study metric on $\mathbb{CP}^{n-1}$.
- Each fiber $F_y = \overline{\pi^{-1}(y)} \subset X$ ($y \in \mathbb{CP}^1$) contains only finitely many critical points of π and each connected component of $F \setminus \{\text{critical points}\}$ intersects B.

The remaining part of this thesis will consists of the construction of complex and symplectic structures on hyperpencils. A more precise formulation of the statement will be given in Theorem 6.3.

2. The main theorem in 2n dimensions

With the generalizations of the definitions, we are ready to state and prove a generalization of Theorem 5.1 to arbitrary even dimensions. The generalization uses pencils instead of fibrations, since we want to keep it as general as possible.

THEOREM 6.3 (Gompf [**6**]). *Let $f : X \setminus B \to \mathbb{CP}^{n-1}$ be a hyperpencil and let ω_{std} denote the standard Fubini-Study symplectic form on $\mathbb{CP}^{n-1}$.*

(1) *There exists an almost-complex structure J on X that is (ω_{std}, f)-compatible on $X \setminus B$, and agrees near B with the complex structure given there by the definition of hyperpencils.*
(2) *For any such J, there is a symplectic structure ω on X that tames J.*

PROOF. Since the proof of this theorem is quite long and has a complicated structure, we first spend some lines on a simple outline, before going into the details.
For part (1), the most important step is proving a lemma that allows us to glue together almost-complex structures on the total space of the hyperpencil. For this lemma, we first need some linear algebra and then a partition of unity, to glue existing local almost-complex structures into a global one. This almost-complex structure can be chosen to be compatible with the already defined symplectic form on $\mathbb{CP}^{n-1}$.
Once we have an almost-complex structure on the hyperpencil, we can proceed with part (2) of the theorem, that has a proof that resembles the proof of Theorem 5.1. We generalize the proof to arbitrary dimensions, using a lemma. Once we have done this, we translate this into the situation at hand and construct a symplectic structure on the hyperpencil, thereby proving the theorem.

Part (1) of the proof

For the first part, we will first introduce a lemma and apply it to the hyperpencil:

LEMMA 6.4. *Let $f : X \to Y$ be a smooth map between manifolds of dimension $2n$ and $2n-2$ respectively. We view df as a fiberwise map between the two bundles $TX \to X$ and $f^*TY \to X$. Let $C \subset D \subset X$ be closed subsets. Assume that the regular points of $f|_{X \setminus C}$ form a dense set in $X \setminus C$. Let ω_Y be a nondegenerate 2-form on f^*TY. Let J_C be an (ω_Y, f)-compatible almost-complex structure on a neighborhood U of C.*
Suppose that each $x \in X \setminus U$ has a neighborhood W_x with an (ω_Y, f)-compatible almost-complex structure on its tangent bundle, and that for some neighborhood V of D, for all $x \in V \setminus U$, the induced structures on f^*TY agree with each other and with f_*J_C wherever the domains overlap. Let J_D denote the resulting almost-complex structure on the fibers of f^*TY.

(1) *Assume all critical points of f in $X \setminus D$ are wrapped. Then $J_C|_C$ extends to an $\omega_Y, f)$-compatible almost-complex structure J on X, with $f_*J|_D = J_D$.*
(2) *Assume $D = X$ and ω_X is a 2-form on X. If the local almost-complex structures on X giben above (including J_C) can be chosen to be ω_X-tame, then J can be assumed to be ω_X-tame.*

Once we have proven this lemma, we will apply it to the hyperpencil, using the same f and replacing X of the lemma by $X \setminus B$. Let $C = D \subset X \setminus B$ be a finite collection of closed balls, one around each $b \in B$, contained in the local charts given by the definition of hyperpencils. Let $U = V$ be a collection of slightly bigger open balls, still contained in the charts. The structure J_C of the lemma will of course be the complex structure on C obtained from the charts. For any x, we should have a complex structure in a neighborhood W_x. For critical points x, these are available by the definition of hyperpencils. For regular points, they can be constructed easily. The remaining assumption of the lemma is the denseness of regular points of f, which is immediate, since an open ball of singular points would intersect a fiber F in more than a finite number of points, which contradicts the definition of a hyperpencil. To prove this lemma, we need another (sub)lemma, about complex linear algebra, to define taking powers of matrices properly. This will later be used to define a useful retraction of linear maps to almost-complex structures.

SUBLEMMA 6.5. *Let $\mathcal{A} \subset GL_n(\mathbb{C})$ denote the open subset of matrices with no real, nonpositive eigenvalues. For $r \in \mathbb{R}$, let $\rho_r : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ denote the branch of the holomorphic map $z \mapsto z^r$ with $\rho_r(1) = 1$. There is a unique holomorphic map $\rho : \mathcal{A} \times \mathbb{R} \to GL_n(\mathbb{C})$, which will be denote by $\rho(A, r) = A^r$, with the following properties.*

(1) *Each λ-eigenvector of Z is a $\rho_r(\lambda)$-eigenvector of A^r.*
(2) *For $m \in \mathbb{Z}$, A^n is the usual exponential of matrices, with A^{-1} the inverse and A^0 the identity matrix.*
(3) *If $|r| < 1$ or $s \in \mathbb{Z}$, $(A^r)^s = A^{rs}$.*
(4) *If $T : \mathbb{C}^n \to \mathbb{C}^k$ is a linear transformation with $TA = BT$, then $TA^r = B^r T$ whenever both sides are defined.*
(5) *If A is real, than so is A^r for any r.*
(6) *Let m be a nonzero integer. $A^{\frac{1}{m}}$ is the unique solution to the equation $X^m = A$ for which all eigenvalues of X lie in the image of ρ_m.*

PROOF. Let $A \in \mathcal{A}$ and $r \in \mathbb{R}$. Consider the Jordan form of A. This splits $\mathbb{C}^n$ as the sum of generalizes eigenspaces $V_\lambda = \ker(A - \lambda I)^n$. On such a space V_λ, we write $A = \lambda(N_A + I)$ for $N_A = \frac{1}{\lambda}A - I$. Note that N_A is a nilpotent transformation on this eigenspace, since it is a multiple of the nilpotent $A - \lambda I$. Define p_r to be the power series expansion of the function $\rho_r(1+z)$ around 0. Notice that, since N_A is nilpotent, $p_r(N_A)$ is a polynomial of transformations.
Set $\rho(A, r) = \rho_r(\lambda)p_r(N_A)$.
First of all, we want to check that ρ is indeed holomorphic. Consider the polynomial equation $\det A - \lambda I = 0$. The solutions of this are the eigenvalues of matrices $A \in \mathcal{A}$, and because the equation is polynomial, this set forms an algebraic variety in $\mathcal{A} \times \mathbb{C}$. The subset $\mathcal{S} \subset \mathcal{A}$ of matrices that have less than n distinct eigenvalues is also an algebraic variety, because we can express this is polynomial equation as well. Over $\mathcal{A} \setminus \mathcal{S}$, the set of matrices with n distinct eigenvalues, the eigenvalues vary holomorphically. Locally, we can now construct holomorphically varying bases of eigenvectors over $\mathcal{A} \setminus \mathcal{S}$ and in these bases, ρ is holomorphic.

We now check the properties listed in the lemma.

(1) For a λ-eigenvector v, $N_A v = 0$, so A^r acts as $\rho_r(\lambda)I$.
(2) Since the functions ρ_r and p_r have the same property, this follows directly.
(3) ρ_r and p_r also have this property.
(4) Take a transformation T such that $TA = BT$. This implies that T sends every generalized λ-eigenspace to itself. Now note that we also have $TN_A = N_B T$. Together with the fact that, on every λ-eigenspace, $\rho_r(\lambda)$ is a constant, we now have $TA^r = T\rho_r(\lambda)p_r(N_A) = \rho_r(\lambda)p_r(N_B)T = B^r T$.
(5) Take A to be real. Since the generalized eigenvectors span $\mathbb{C}^n$, $\mathbb{R}^n$ is spanned by the vectors $v + \overline{v}$, $v \in V_\lambda$, in which λ ranges over all eigenvalues of A. Since $\overline{v} \in V_{\overline{\lambda}}$, $A^r(v + \overline{v}) = A^r(v) + \overline{A^r(v)}$ follows immediately, so A^r gives real values on every vector in $\mathbb{R}^n$.
(6) Write $X = \lambda(I + N_X)$ on each of its generalized eigenspaces. Now $A = X^m = \lambda^m(I + N_X)^m$ on each V_λ. The last term has the form $I + N'$ with N' a nilpotent matrix. Hence $A^{\frac{1}{m}} = \rho(X^n, \frac{1}{n} = X$, since the only nth root of λ^m that is in the image of $\rho_{\frac{1}{m}}$ is λ itself.

What remains is uniqueness, which follows directly from the holomorphicity of ρ. Since ρ is unique on the dense subset of diagonalizable matrices, and holomorphicity implies continuity, ρ extends in a unique way to all of $\mathcal{A}$. $\square$

Using this, we can prove Lemma 6.4 to obtain suitable almost-complex structures.

Proof. *(of Lemma 6.4)* To define a global almost-complex structure on X, first assume that $D = C$ (the other case will be studied later). Recall that we have an open set W_x around every point $x \in X$ with an almost-complex structure on it. Since all manifolds are paracompact, we can take a locally finite subcover $\{W_\alpha\}$ of X. Now define a partition of unity ρ_α subordinate to the cover $\{W_\alpha\}$ and define the global map $A = \sum_\alpha \rho_\alpha J_\alpha$. This is a linear map, but a convex combination of (almost-)complex structures is not in general an (almost-)complex structure again. To obtain a continuously varying almost-complex structure, we define a retraction, using the power-operator described in the previous lemma.
Let V be a real, finite-dimensional vector space and let $\mathcal{B}$ be the set of linear operators with no real eigenvalues. Let $\mathcal{J}$ be the set of complex structures on V. Now the function $r : \mathcal{B} \to \mathcal{J}$, defined $B \mapsto B(-B^2)^{-\frac{1}{2}}$ is a real-analytic retraction. To see that this is well-defined, notice that $-B^2$ indeed has no real, nonpositive eigenvalues, for if $-\lambda^2$ would be an eigenvalue of $-B^2$, we would have $0 = \det(-B^2 - (-\lambda^2)I) = \det(B + \lambda I)\det(B - \lambda I)$, implying that $\pm\lambda$ would be an eigenvalue of B. So, we can indeed take powers of $-B^2$. To see that r is a retraction, notice that for arbitrary $B \in \mathcal{B}$, we have $r(B)^2 = B(-B^2)^{-\frac{1}{2}}(-B^2)^{-\frac{1}{2}}B = B(-B^2)^{-1}B = -I$, using the fact that powers of commuting operators commute, by point 4 of the previous lemma. Furthermore, if B is a complex structure, $r(B) = B$ follows immediately, so we indeed have a retraction.
We want to take $r\sum_\alpha \rho_\alpha J_\alpha = r(A)$ as the required almost-complex structure. For that, we first need to make sure A has no real eigenvalues.

Sublemma 6.6. *A has no real eigenvalues on any tangent space.*

Proof. First note that the map $B = \sum_\alpha \rho_\alpha f_* J_\alpha$ has no real eigenvalues on any tangent space, since each term $f_* J_\alpha$ is ω_Y-tame. This would induce a contradiction, since ω is anti-symmetric. Since $df A = B df$, each λ-eigenvector of A is mapped to either an eigenvector of B or to 0, and since the former do not exist for real λ, each real eigenvector of A is in the kernel of df.
Now we must make sure there are no real eigenvectors in ker df. Recall the definition of the set $P \subset \ker df \subset TX$ of the wrapped points definition, which is the limit (taken over all regular values x) of the kernel of df_x. For $x \in X$, a tangent vector $v \in T_xX$ can be written as the limit of a sequence $\{v_i\}$ with each v_i in the kernel of df_{x_i} for some sequence $\{x_i\}$ of regular points converging to x.
The sequence of 2-planes ker df_{x_i} might not converge, but we can take a subsequence that does. Its limit will be a 2-plane $\Pi \subset P_x$, containing v. For large i, the plane ker df_{x_i} will be a J_α-complex subspace for any α, since every J_α is (ω_Y, f)-compatible.
Of course, now we can find a $v' \in T_xX/\Pi$ and repeat this procedure. In the end, we find a decomposition $\mathrm{span}_{\mathbb{R}} P_x = \bigoplus \Pi_j$, in which each plane Π_j is a J_α-complex subspace of T_xX for each J_α defined at x. The quotient $Q_x = T_xX/\mathrm{span}_{\mathbb{R}} \bigoplus \Pi_j$ now inherits an almost-complex structure $\tilde{J}_\alpha$ from each J_α. The (complex) dimension of Q_x is at most 1 by assumption, since all critical points are wrapped. On a (complex) 1-dimensional vector space, we can always find a form ω that tames the almost-complex structure, so it can not have real eigenvectors here as well. The only thing left to exclude are real eigenvectors in $\mathrm{span}_{\mathbb{R}} \bigoplus \Pi_j$. However, a direct sum of taming forms on the subspaces P_x tames each J_α, so here, too, are no real eigenvectors. $\square$

Since A has no real eigenvalues in any point, the retraction r is well-defined and there is an almost-complex structure satisfying the requirements.
To finish the proof of part 1, we must consider the general case, in which D and C do not necessarily coincide. By intersecting each W_α with the neighborhood V of D, adding a new $W_\alpha = X \setminus D$ (on which we have $J_\alpha = r(A)|_{X \setminus D}$) to the set of opens and extending J_D to $r(B)$, we reduce this general case to the case where $D = X$.
To prove this case, consider a vector $v \in T_xX$. This lies in a 2-plane $\Pi = \lim \ker T_{x_i}X$ and Π is a J_α-complex line for any J_α (including J_C) defined at T_xX. Each J_α induces an almost-complex structure on T_xX, namely the limit of the structures induced on the sequence of spaces

$T_{x_i}X/\ker df_{x_i}$. Since these structures are also determined by J_D, using the isomorphism between im df_{x_i} and $T_{x_i}X/\ker df_{x_i}$, they do not depend on α.
Let $Z \subset X$ denote the subset of X for which $\dim_{\mathbb{C}} \operatorname{span}_{\mathbb{R}} P_x$ is at least 2. (Note that for n at least 3, Z is empty, as consequence of all critical points being wrapped, so this is an extension only needed in a very particular case.) For $x \in Z$, T_xX contains at least two linearly independent planes Π_1 and Π_2, and the J_α-complex map $T_xX \to T_xX/\Pi_1 \oplus T_xX/\Pi_2$ determines an almost-complex structure on T_xX, since it is injective. Thus, over $C' = C \cup \overline{Z}$, the structures J_α match and fit together into a continuous, (ω_Y, f)-compatible structure $J_{C'}$, that depends only on J_C and J_D.
We can now construct a Riemannian metric.

P is an oriented subbundle of the tangent space $T(X \setminus C')$. We can now define J on $X \setminus C'$ as a counterclockwise rotation over an angle of $\frac{\pi}{2}$. For each $x \in X \setminus C'$, df_x factors through T_xX/P_x, on which J agrees with each J_α, so $J|_{X\setminus C'}$ is (ω_Y, f)-compatible.
We have defined structures on X and on $X \setminus C'$. Now all we have to do is prove they can be glued together along the boundary of C'. To do this, consider a point $x \in C'$ that is a limit point of $X \setminus C'$. On this, we have the almost-complex structure J_x. If it is not continuously approached by $J|_{X\setminus C'}$, there is a sequence $\{x_i\}$ in $X \setminus C'$ such that J_{x_i} does not approach J_x. By passing to a subsequence, we can assume that the 2-planes P_{x_i} converge to some Π, since the 2-subplanes of a tangent space form a compact Grassmann-manifold. We now otain $J_{x_i} \to J_x$: take a J_α that is defined in a neighborhood of x. This coincides (by definition of $J_{C'}$) with $J_{C'}$. Because of the continuity of g, J_{x_i} converges to this. This contradiction implies that the structures fit together at the boundary, so we obtained an overall structure.

For part 2 of the lemma, consider a 2-form ω_X on X that tames all J_α and J_C. Recall the construction of the 2-bundle P, which is a subbundle of the tangent space TX. On $X \setminus C'$, define Q as its symplectic orthogonal complement with respect to ω_X, so $Q = \{v \in T(X\setminus C') | \omega_X(v,p) = 0 \forall p \in P\}$. (We have not established the symplecticness of ω_X, but we can use this definition anyway.) $Q \oplus P$ is a ω_X-orthogonal sum splitting of the tangent space, since ω_X is nondegenerate on P.
Note that any subbundle of TX that is complementary to P can be written as the graph of a continuous section ψ of $\operatorname{Hom}(Q,P)$. For such a ψ, we define an almost-complex structure J_ψ on $X \setminus C'$, using the structure J as defined above. On the subbundle P, take J_ψ to be just J. There is a canonical vector space isomorphism between the graph of ψ and Q, since this graph is a section of the bundle $\operatorname{Hom}(Q,P)$. We can use this isomorphism to obtain an almost-complex structure on graph ψ as well, so we now have an almost-complex structure J_ψ on $T(X \setminus C')$.
Any J_ψ can be expressed in terms of J_0 (defined in the same way, with $\psi = 0$ the 0-section). The structure J_0 agrees with J_ψ on TxX/Q_x, since the graph at x, graph(ψ_x), equals Q_x if $\psi = 0$. Now take a point $(q, \psi(q)) \in \operatorname{graph}(\psi) \subset Q \oplus P$. We have $J_\psi(q,\psi(q)) = (J_0(q), \psi(J_0(q)))$. Now take any $(q,p) \in Q \oplus P$. We have

$$J_\psi(q,p) = J\psi(q,\psi(q)) + J_\psi(p - \psi(q)) = (J_0(q), \psi(J_0(q)) - J_\psi(q) + J(p))$$

Since J and J_ψ coincide on P. We now claim that J_ψ tames ω_X for certain ψ. First of all, using the equation above, we have

$$\omega_X((q,p), J_\psi(q,p))\omega_X(q, J_0(q)) + \omega_X(p, J_0(q)) + \omega_X(q, (\psi J_0 - J\psi)(q) + J(p)) + \omega_X(p, (\psi J_0 - J\psi)(q) + J(p))$$

$$= \omega_X(q, J_0(q)) + \omega_X(p, J(p)) + \omega_X(p, (\psi J_0 - J\psi)q) \tag{2}$$

For the last equality, we use the fact that P and Q are J-complex and J_ψ-complex subspaces, and ω_X-orthogonal to each other, hence terms of the form $\omega_X(p, J(q))$ vanish.
We view the three terms on the right hand side separately. All three of them are positive if $(p,q) \neq (0,0)$. We prove them in different order, since the positivity of the first term is the most difficult and its proof relies on the other two.

- The first term is immediate, since J is ω_X-tame on P_x for every $x \in X \setminus C'$, since all fibers are correctly oriented J-complex lines.
- For the third term, we can choose ψ such that its graph is a J-complex subbundle of $T(X \setminus C)$. This is easy, since any complex subbundle is the graph of some ψ. Now $J_\psi = J|_{X \setminus C'}$. Since J agrees with some J_α at every point, it is ω_X-tame.
- The positivity of the first term follows from the fact that $J_0|_{Q_x}$ is ω_X-tame for any $x \in X \setminus C'$. To prove this tameness, define symmetric 2-form g by $g(v,w) = \frac{1}{2}(\omega_X(v, J_0(w)) + \omega_X(w, j_0(v)))$. Since $g(q,q) = \omega(q, J_0(q))$, we can rephrase the statement and prove that g is positive definite. For this, define a J_0-invariant inner product on Q_x. Let Q_- be the span of vectors $v \in Q$ that satisfy $g(v,v) \leq 0$. We prove that this is the zero space. Since both g and the defined inner product are J_0-invariant, Q_- is a J_0-complex subspace of Q_x: if $v \in Q_-$, which means $g(v,v) \leq 0$, so $g(J(v), J(v)) \leq 0$ as well, so $J(v) \in Q_-$. If we assume ψ to be a linear transformation such that graph ψ is a J_α-complex subbundle of T_xX, as we did above, then $J_\psi|_{\text{graph } \psi} = J_\alpha|_{\text{graph } \psi}$ is ω_X-tame. So Equation 2 is positive for any pair $(q,p) = (q, \psi(q))$. But this condition cancels two of the terms, so we are left with the equation $\omega_X(q, J_0(q)) + \omega_X(p, \psi(J_0(q)))$. The first term is precisely $g(q,q)$, which is nonpositive on Q_-. This means that the other term can not vanish here. Hence $p = \psi(q)$ can not vanish on Q_- unless q does, so ψ has a trivial kernel. Since this means we have an injection of Q_x into P_x, we can conclude that $\dim_{\mathbb{C}} Q_- \leq \dim_{\mathbb{C}} P_x = 1$, so we only have to rule out the case $\dim_{\mathbb{C}} Q_- = 1$. If this is the case, and Q_x also has dimension 1, then $J_0|_{Q_x}$ is immediately ω_X-tame, so we are done. Assume that Q_x has dimension higher than 1. Consider the function $q \mapsto g(q,q)$, which takes both positive and nonpositive values on $Q_x \setminus \{0\}$, so there is a nonzero vector q in Q_x on which the function vanishes because of the connectedness of $Q_x \setminus \{0\}$. Now the function vanishes on the entire J_0-complex line Q_0 containing q, so it can not be nondegenerate on $Q_0 \oplus P_x$, which is the direct sum of ω_X-orthogonal subspaces. However, this direct sum is a J_α-complex subspace, since both its terms are J_α-complex subspaces. This contradicts the assumption that J_α is ω_X-tame, so we have a contradiction.

On $X \setminus C'$, we now have the required almost-complex structure. All that is left is extending it over all of X. Pick a point $x \in C'$. At this point, J agrees with some J_α, which is ω_X-tame on the closed set C'. Since taming is an open condition, J is ω_X-tame on some open neighborhood U' of C'. We use this neighborhood to glue the structures together. Define a smooth function $\rho : X \to [0,1]$ that is 1 at C' and 0 outside U'. Now $J_{\rho\psi}$ is an ω_X-tame almost-complex structure on $X \setminus C'$ that extends as J over C'. We have the almost-complex structure we wanted. □

We can now immediately apply Lemma 6.4. By the first part of the lemma, which we can use since all critical points are wrapped by definition, we obtain an almost-complex structure J on $X \setminus B$. Since B is a finite set, J is defined on a dense set of X, and hence extends immediately over X.

Part (2) of the proof

Part 2 will follow from yet another lemma, that resembles Theorem 5.1. This lemma is used to define the symplectic structure.

LEMMA 6.7. *Let $f : X \to Y$ be a map between manifolds. Let $C \subset X$ be closed, and such that $X \setminus intC$ is compact. Let W_C be a neighborhood of C. Let ω_Y be a symplectic form on Y and let J be an (ω_Y, f)-tame almost-complex structure on X.*
Fix a cohomology class $c \in H^2_{dR}(X)$. Suppose that for each $y \in Y$, the inverse image $f^{-1}(y)$ has a neighborhood $W_y \subset X$ that contains W_C, such that the restriction $H^1_{dR}(W_y) \to H^1_{dR}(W_C)$ is surjective. Let η_y be a closed 2-form on W_y such that $[\eta_y] = c|_{W_y} \in H^2_{dR}(W_y)$ and such that η_y tames J on each of the complex subspaces ker df_x for $x \in W_y$. Suppose that all these forms η_y agree

on W_C, and that the resulting form η_C on W_C tames J on $TX|_C$.
Now there is a closed 2-form η on X that agrees with η_C in a neighborhood of C, with $[\eta] = c \in H^2_{dR}(X)$, and such that for $t > 0$ small enough, the form $\omega_t = t\eta + f^*\omega_Y$ on X tames J.

PROOF. First, we fix a representative ζ of c, so $[\zeta] = c$. For each $y \in Y$, $[\eta_y] = c|_{W_y}$, so on W_y, we can write $\eta_y = \zeta + d\alpha_y$, in which α_y is a 1-form. Fix a point $y_0 \in Y$ and set $\alpha_C = \alpha_{y_0}|_{W_C}$. For each y, we have $d(\alpha_C - \alpha_y) = (\eta_{y_0} - \zeta) - (\eta_{y_0} - \zeta) = 0$ on W_C, so $[\alpha_C - \alpha_y] \in H^1_{dR}(W_C)$. We can now use the assumed surjectiveness of the restriction to find an extension of this form to W_y.
Choose a function $g : W_y \to \mathbb{R}$ with $dg = \alpha_C - \alpha_y$ near C. Define $\alpha'_y = \alpha_y + dg$, to obtain $\alpha'_y = \alpha_C$ near C, for every $y \in Y$.
Cover Y by sets U_i, such that each $f^{-1}(U_i)$ is contained in some W_y, and let $\{\rho_i\}$ be a partition of unity subordinate to this cover.
Let $\eta = \zeta + d\sum_i(\rho_i \circ f)\alpha_{y_i}$, in which α_{y_i} denotes the 1-forms corresponding to a point $y_i \in U_i$. It is immediate that η is a closed form in the same cohomology class as ζ, which is c, since they differ by a differential.
To finish the proof, we have to show that ω_t indeed tames J for small $t > 0$. We perform the differentiation to obtain

$$\eta = \zeta + d\sum_i(\rho_i \circ f)\alpha_{y_i} = \zeta + \sum_i(\rho \circ f)d\alpha_{y_i} + \sum_i(d\rho_i \circ df) \wedge \alpha_{y_i}$$

and look what this does on the subspaces ker df_x. The last term obviously vanishes when applied to a pair of vectors in the kernel, so on each individual ker df_x, we have $\eta = \zeta + d\sum_i(\rho_i \circ f)\eta_{y_i} = \sum_i(\rho_i \circ f)\eta_{y_i}$. Since we assumed all the forms η_y to tame J, this is a convex combination of taming forms, so J is η-tame when J is restricted to a space ker df_x.

$$\omega_t(v, Jv) = t\eta(v, Jv) + f^*\omega_Y(v, Jv)$$

We assumed J to be (ω_Y, f)-tame, so the last term is positive as long as v is not in the kernel of df and zero otherwise. On ker df, $\eta(v, Jv)$ is positive, and since $v \mapsto \eta(v, Jv)$ is a continuous function, it takes positive values on a neighborhood U of ker df.
We take a look at the unit ball Σ of the tangent space. For this, we can use any convenient metric. If we just consider C, we have $\eta(v, Jv) = \eta_C(v, Jv) > 0$, so $\omega_t(v, Jv)$ is just the sum of positive terms. Outside C, look at the compact set $\Sigma|_{X\setminus \text{int } C\setminus U}$. Since this is a compact set and $v \mapsto \eta(v, Jv)$ is continuous, it is bounded on this set. Since this set has empty intersection with ker df, $f^*\omega_Y(v, Jv)$ is strictly positive, and hence bounded by a positive constant from below, since $\Sigma|_{X\setminus \text{int } C\setminus U}$ is constant. We conclude that for small enough t, $\omega_t(v, Jv)$ only takes positive values on the unit sphere, hence on the entire tangent space. The form ω_t tames J on X. □

This lemma can be applied to the hyperpencil in the following way. Let C and U be as above, collections of closed balls. Let W_C be a neighborhood of C with closure contained in U. Let $y \in \mathbb{CP}^{n-1}$ and let F_y denote its fiber. Let $K \subset F_y$ denote the set of critical points lying in the fiber. Recall that this is a finite set. Let $\Delta \subset X \setminus U$ be a collection of disjoint open balls, one around each point of K. On $\Delta \cup U$, we define a closed 2-form σ in the following way. Since K is a finite set, we can choose σ to tame J at K. Since taming is an open condition, we can assume that σ tames J on all of Δ, which can be achieved by shrinking Δ. On U, we have complex coordinates, so we just take σ to be the standard symplectic form on $\mathbb{C}^n$. We scale this such that the integral of σ is always smaller than $\frac{1}{2}$ on each complex line through 0 intersected with U. The structure J is now σ-tame on $\Delta \cup U$.
J is also (ω_{std}, f)-tame on $X \setminus B$, so $F_y \setminus K$ is a smooth J-holomorphic curve in $X \setminus K$ with complex orientation agreeing with its pre-image orientation, and each components intersects B nontrivially.
We now make a small detour to investigate the possibility of the fiber F_y being knotted at K. We can assume $\partial\Delta$ and F_y are transverse, so they intersect in a finite collection of circles. Every such circle can be connected to a point $b \in B$ by a path in $F_y \setminus K$, since each circle lies in a connected

component, and any component intersects B. Let Δ' be a collection of smaller balls, contained in the interior of Δ, that still contains K, disjoint from these paths. Make sure Δ' and F_y are again transverse. Each component F_i of the compact surface $F_y \setminus \Delta'$ either is contained in Δ or intersects B. Let W_y be the union of int $\Delta' \cup W_C$ with a tubular neighborhood of $F_y \setminus \text{int } \Delta' \subset X \setminus \text{int } \Delta'$. Extend each F_i to a closed, oriented surface $\hat{F}_i$ by attaching a surface in Δ'. Now the classes $[\hat{F}_i]$ form a basis of $H_2(W_y, \mathbb{Z})$.

It is now time to start constructing our required form, after which we will apply Theorem 6.7. First notice that $\sigma|F_i \cap (\Delta \sup W_C)$ is a positive area form, since F_y is J-holomorphic and J is σ-tame on $\Delta \cup U$. We rescale σ on Δ such that $\int_{\hat{F}_i \cap \Delta} \sigma < \frac{1}{2}$ for each i. Extend σ over each F_i intersecting B as a positive area form, such that $\int_{\hat{F}_i \sigma}$ equals the number of points in which B and F_i intersect. Define $\pi : W_y \to W_y$ by smoothly gluing the identity on $W_y \cap (\Delta \cup U)$ to the normal bundle projection on $W_y \setminus \Delta$. Now $\Im \pi \subset F_y \cap \Delta \cap U$ and π restricted to $F_y \cup \Delta_0 \cup W_C$ is the identity (since this is a subset of $W_y \cap (\Delta \cup U)$).

$\eta_y = \pi^* \sigma$ is a well-defined closed 2-form on W_y. On $W_y \cap (\text{int } \Delta \cup W_C)$, η_y equals σ, so it tames J. More specific, it tames each $J|_{\ker\, df_x}$, as is required in order to apply Theorem 6.4. The form $\eta_y|_{F_y \setminus K}$ tames J on each $T_x F_y = \ker\, df_x$. Using the openness of taming again, we can shrink W_y and assume η_y tames $J|_{\ker\, df_x}$ for all $x \in W_y$.

For the application of Theorem 6.4, we will take the cohomology class c to be c_f, the Poincaré dual of the fiber, so that we have $\langle \eta_y, \hat{F}_i \rangle = \langle \sigma, \hat{F}_i \rangle = \# F_i \cap B = \langle c, \hat{F}_i \rangle$ for each $\hat{F}_i$ intersecting B. For $\hat{F}_i$ not intersecting B, we have $\langle c, \hat{F}_i \rangle = 0 = \langle \eta_y, \hat{F}_i \rangle$, since $\eta_y = \sigma$ is exact on $\hat{F}_i$. Since c and η_y agree on a basis of $H_2(W_y; \mathbb{Z})$, we can conclude that $[\eta_y] = c|_{W_y} \in H^2_{dR}(W_y)$. We can now apply Theorem 6.7 to $X \setminus B$, using $\omega_y = \omega_{std}$ on $Y = \mathbb{CP}^{n-1}$. Note that $H^1_{dR}(W_C) = 0$, since W_C consists of punctured $2n$-balls (they are punctured because the points of B are taken out). The 2-form η on $X \setminus B$ is now canonical, and the standard form, relative to the charts, on C. This extends to X. Now a $t > 0$ can be chosen such that $\omega_t = t\eta + f^* \omega_{std}$.

This form is well-defined and J-tame on $X \setminus B$, but it is singular at B. Next, we describe the singularities and find a way to eliminate them. Since J and η are standard in the local coordinates and f is projectivization, we can write η in spherical coordinates: $\eta = r^2 f^* \omega_{std} + \frac{1}{2\pi} d(r^2) \wedge \beta$ [**2**]. In this formula, r is the radial coordinate on $\mathbb{C}^n$ and β is the pull-back to $\mathbb{C}^n \setminus \{0\} = S^{2n-1} \times \mathbb{R}$ of the connection 1-form on S^{2n-1}. To verify this formula, note that H on $\mathbb{C}^n \setminus \{0\}$ is orthogonal to each complex line L through 0 under both η and the standard symplectic form. On each L, $d(r^2) \wedge \beta = 2r dr \wedge d\theta$ is standard (up to a scale factor that does not depend on L). The two terms are scaled compatibly since $d\eta = 0$.

$\omega_t = (1 + tr^2) f^* \omega_{std} + \frac{t}{2\pi} d(r^2) \wedge \beta$ in these coordinates. This is clearly singular at 0. We now perform a substitution of variables $R = \frac{1+tr^2}{1+t}$ (t is constant here) and obtain $\eta(R) = \frac{1}{1+t} \omega_t(r)$. So, we have a radial, symplectic embedding $\phi : (\mathbb{C}^n \setminus \{0\}, \frac{1}{1+t}\omega_t) \to (\mathbb{C}^n, \eta)$. For $V \subset \mathbb{C}^n$ the image of the given coordinate chart at $b \in B$, define a radially symmetric diffeomorphism onto an open ball, that agrees with ϕ outside of a closed ball around 0 in V. In any such open ball, let ω be $\phi_0^* \eta$ near b and $\frac{1}{1+t}$ elsewhere. These pieces fit together since ϕ is a symplectic embedding, so this construction defines a symplectic form on X.

The only thing left to do now is verifying whether ω indeed has the desired properties. Away from B, we have $\omega = \frac{1}{1+t}\omega_t$, and any multiple of ω_t tames J. Let $b \in B$. Near b, we have local coordinates such that J is the standard complex structure and $\omega = \phi_0^* \eta$, wich η standard up to a constant factor. By the radial symmetry of ϕ_0, H is preserved on $\mathbb{C}^n \setminus \{0\}$, and so is $\eta|_H$. The form ϕ_0 also preserves each complex line through 0, and $\phi_0^* \eta$ is a positive area form on each of these complex lines. Since these complex lines and H are η-orthogonal and J-holomorphic, J is also ω-tame near B. We conclude that ω tames J on all of X.// The cohomology class $[\omega]$ is easily computed, since we can work outside of C. We have

$$[\omega] = \frac{1}{1+t}[\omega_t] = \frac{1}{1+t}(tc + [f^*\omega_{std}]) = \frac{1}{1+t}(c+tc) = c$$

So, the constructed form has the desired properties. $\square$

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