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# **A method for investigating off-axis Gaussian beams**

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## **Abstract**

In order to achieve a way to propagate a Gaussian beam through a set of optical elements, several aspects of optics need to be combined. We will show that by using Gaussian optics in combination with ray tracing, we can find a method to include several complexities that are not possible with the common rules of Gaussian optics. We will check our theory with experiments where we shoot Gaussian beam at an offset with respect to the optical axis of the lens-system. For small deviations of 1.0 cm from the optical axis on a 2.54 cm wide lens, the theory is in good agreement with the measurements.

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# 1 Introduction

The field of nanophotonics considers the interaction between light and matter on the nanoscale. This research branch has different aspects to it.

In general practice lenses are designed for light to pass through the center in order to get the desired effect. Different types of theories have been developed to describe the light trajectory through lenses. But most of them consider either an infinitely thin lens, or light that only hits the center of a thick lens. This does not really take the spherical form of the lens into account. And because of that, it is not that easy to calculate what happens to a beam that hits the lens off-center.

In our experiment group we try to bring an ultra-cold atom cloud next to a nanostructure. We do this with an optical trap which requires that we hit one of the lenses off-center. And in order to understand what happens with a beam that is going off-axis we aim to derive a theory that can describe the off-axis beam after propagating through a lens.

This thesis will derive a set of equations to calculate the beam parameters when dealing with spherical aberration.

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## 2 Theory

There exist ways of propagating light rays that give us a good indication of the properties of an optical setup. Information such as focal distance, refraction and magnification can be calculated and visualized with simple rules. However, problems arise when looking at more complex situations where you might want to know the intensity profile of your beam. In the first section of the theory, the ray-picture of light propagation will be briefly explained to get a basic understanding of propagating light. After that we shall use ray optics in combination with Gaussian optics to derive a set of operations where we can trace beams through a set of lenses and calculate the intensity field of a beam that is affected by spherical aberration.

### 2.1 Ray optics

We will need to find a way to describe the path that light follows. A simple form of light has always been the wave-picture. You can follow these waves by looking at their wavefronts, and these wavefronts of light can be tracked by drawing a line perpendicular to these wavefronts. [1] This line represents the direction in which the energy of the light is flowing towards.

One way to describe this ray path is by defining two variables, namely the distance and the angle which the ray makes with respect to the optical axis.<sup>1</sup>

An important component in order to propagate the ray along its path is refraction. For instance, when the ray encounters a lens it will cross over into a different medium with a different refractive index. In order to know how the ray continues in the new medium, the direction of the transmitted ray can be determined by simply using Snell's law.

Every interface between two media can be viewed as a two-dimensional situation, the incoming wave-vector can therefore be cast into a parallel and a perpendicular component with respect to the surface normal-vector. The wave-vector  $\vec{k} = n_i \vec{k}_0$  then splits into  $\vec{k}_{\parallel} + \vec{k}_{\perp}$  where  $|\vec{k}_0| = 2\pi/\lambda_0$  and  $n_i$  is the refractive index of the material. The  $k$ -component parallel to the surface must remain unchanged during the transmission into the lens. From this we can derive Snell's law immediately.

$$k_{\parallel, in} = k_{\parallel, out} \quad (2.1)$$

$$n_{in} k_0 \sin \alpha_{in} = n_{out} k_0 \sin \alpha_{out} \quad (2.2)$$

$$n_{in} \sin \alpha_{in} = n_{out} \sin \alpha_{out} \quad (2.3)$$

In this equation  $\alpha$  represents the angle between the surface normal and the incoming and outgoing ray.

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<sup>1</sup>Of course this only defined in 2D, but it can be easily expanded to 3D

Already when calculating the focal point of a single lens setup, things are going to get difficult, since the rays will not have a single focal point anymore as can be seen in Fig. 2.1.

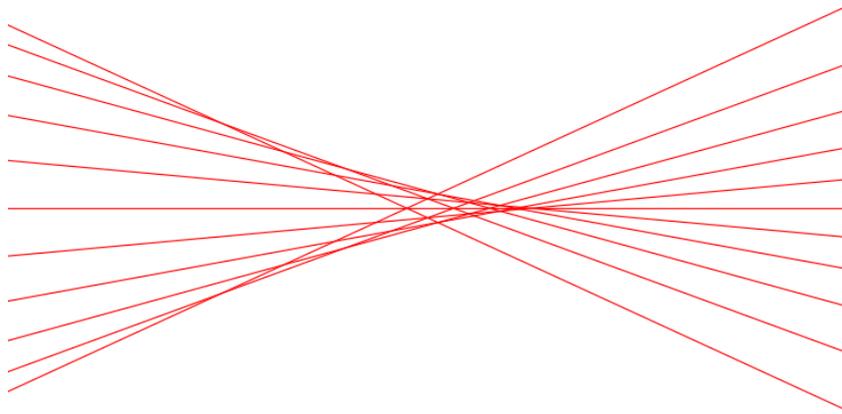


Figure 2.1: The effects of spherical aberration on the focus

Lenses and mirrors are probably the most used calculations in a setup. And now that we have a way to calculate refraction, we also need a way to deal with reflection. Reflection on a mirror can simply be considered as "folding out" the space as pictured in figure 2.2. The ray can be propagated along a path where the mirror is not installed and the rest of the setup is rotated so that they are behind the mirror. This way, the ray can continue alongside one axis during the whole setup.

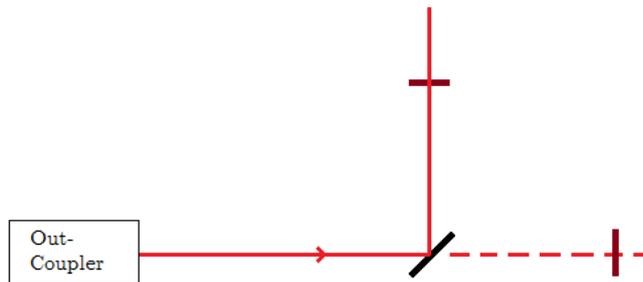


Figure 2.2: A representation of folded out space

### 2.1.1 Ray transfer matrix analysis

Instead of calculating the refraction and reflection at each interface, we could also make better use of the two variables,  $x$  and  $\theta$ , we defined earlier by making an approximation. This approximation, known as the "paraxial approximation", requires  $\theta$  to be small in order to make use of the fact that  $\sin \theta \approx \theta$ . This severely simplifies Snell's law, because all that has to be done now is just basic multiplication and addition. These approximations can be used to describe a set of lenses, mirrors, slabs or other objects in terms of a matrix multiplication. Finally the newly found ray-vector is defined with the refractive index  $n$  of the current region as  $\begin{pmatrix} x \\ n\theta \end{pmatrix}$

For each optical element there exists a matrix of the form  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  where  $AD - BC = 1$ . If that matrix is multiplied with the ray-vector it returns a new transformed ray-vector that describes the propagated ray through that optical element.

The only thing we have to find out now are the ABCD coefficients for each region. For a spherically curved material, with radius of curvature  $R$ , it is possible to derive the following two equations after assuming the paraxial approximation

$$x_2 = x_1 \quad (2.4)$$

$$\theta_2 = \arcsin \left[ \frac{n_1}{n_2} \sin \left( \theta_1 + \arcsin \frac{x_1}{R} \right) \right] - \arcsin \frac{x_1}{R} \quad (2.5)$$

$$n_2 \theta_2 \approx n_1 \theta_1 + \frac{n_1 - n_2}{R} x_1 \quad (2.6)$$

In matrix notation this becomes  $\begin{pmatrix} x_2 \\ n_2 \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{R} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ n_1 \theta_1 \end{pmatrix}$

If the ABCD values for all of the elements in the loop are given, then the whole optical setup can be chained together into one matrix. To visualize this, let's say there is a setup with two curved surfaces with radii of curvature  $R_1$  and  $R_2$  enclosing a piece of material with a thickness  $d$  and refractive index  $n_2$ , then you can rewrite it to the following matrix-multiplication

$$\begin{pmatrix} x' \\ \theta' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{n_2 - n_1}{R} & 1 \end{pmatrix}}_{\text{surface 2}} \begin{pmatrix} 1 & d/n_2 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{n_1 - n_2}{R} & 1 \end{pmatrix}}_{\text{surface 1}} \begin{pmatrix} x \\ \theta \end{pmatrix} \quad (2.7)$$

This shows the ray-parameters in the final plane right after the last lens.<sup>2</sup> What this equation describes is in fact the matrix for a thick lens, contrived of two spherical surfaces and a finite thickness of the medium in between the surfaces.

## 2.2 Gaussian Beam

Now that we have a way to describe ray optics in a simpler way, we can move on to Gaussian optics. We need Gaussian optics, because ray optics on its own does not really tell you anything about the shape of your beam after passing through optical elements.

### 2.2.1 Gaussian optics

A Gaussian beam has, as the name suggests, a Gaussian field distribution in the beam's waist that looks like

$$u(x, y, z) = \frac{U_0}{q(z)} \exp \left[ -\frac{x^2 + y^2}{w(z)} \right] \quad (2.8)$$

Where the waist at any given point  $z$  along the beam with waist at  $z_w$  can be worked out with

$$w(z) = w_0 \sqrt{1 + \left( \frac{z - z_w}{z_R} \right)^2} \quad (2.9)$$

$$z_R = \frac{\pi w_0^2}{\lambda} \quad (2.10)$$

Where  $z_R$  is the Rayleigh range of the beam. The Gaussian field equation can also be written in terms of the complex beam parameter  $q$

$$q = z - z_w - iz_R \quad (2.11)$$

<sup>2</sup>Note that you have to read the setup from right to left. The first lens is the first operation on the ray-vector

This parameter gives helpful information, namely the distance from the waist and the Rayleigh range of the beam. The Rayleigh range of the beam gives us the distance the beam can travel before the area of the cross-section of the beam is doubled. Thus giving information about how fast the beam diverges. If the parameters  $z - z_w$  and  $z_R$  are combined, then the waist can be calculated at an arbitrary position  $z$  along the beam. Writing the parameters of the beam in this way is very useful for certain calculations as shall be seen later on.

### 2.2.2 Propagating an on-axis Gaussian beam

The propagation of rays through optical elements can be described by a simple matrix multiplication. However to propagate a general scalar field from one plane (entrance plane) to another plane parallel to it (exit plane), one has to solve the scalar Kirchhoff integration [3]. Mathematically this amounts to solving an integral of the following form

$$u_{\text{out}}(x, y, z) = \int \int K(x, y, z; \xi, \eta, z') u_{\text{in}}(\xi, \eta, z') d\xi d\eta, \quad (2.12)$$

where  $K(x, y, z; \xi, \eta, z')$  is the kernel that relates the field in the entrance plane to the field in the exit plane.  $u_{\text{in}}$  and  $u_{\text{out}}$  denote the field in the entrance and exit plane, respectively. In other words, the kernel describes the propagation of a Huygens' secondary wave from a point  $\{\xi, \eta, z'\}$  in the entrance plane to a point  $\{x, y, z\}$ .

Within the paraxial approximation, it can be shown that this kernel is given by the so-called Baues-Collins-integral kernel [4] [5],

$$K(x, y, z; \xi, \eta, 0) = \frac{k_0}{2\pi i} \sqrt{\frac{n}{n'}} \frac{e^{ik_0 L_0}}{B} \exp \left[ ik_0 \frac{A(\xi^2 + \eta^2) - 2(\xi x + \eta y) + D(x^2 + y^2)}{2B} \right], \quad (2.13)$$

which makes use of the same A, B, C and D coefficients as we derived through ray optics. The pre-factor in this kernel make sure, amongst other things, that energy is conserved.

The coordinates at the entrance plane  $\{\xi, \eta, z'\}$  are a set of coordinates that can be chosen by us, so in order to make it easier the  $z'$  is chosen to be 0 in the entrance plane.

The expression for the incoming field that will be used has its focus at  $z' = 0$  and can be re-expressed from Eq.2.14 to the following form which contains the  $q$ -parameter

$$u_{\text{in}}(\xi, \eta, 0) = \frac{u_0}{\sqrt{q_{x0} q_{y0}}} \exp \left[ \frac{ik_0}{2} \left( \frac{x^2}{q_{x0}} + \frac{y^2}{q_{y0}} \right) \right] \quad (2.14)$$

Where  $q_{x0}$  and  $q_{y0}$  are just the  $q$ -parameters at the focus. Plugging these values in the integral, gives us the following equation

$$\begin{aligned} u_{\text{out}}(x, y, z) &= \frac{k_0}{2\pi i} \frac{e^{ik_0 L_0}}{B} \sqrt{\frac{n_{\text{in}}}{n_{\text{out}}}} \exp \left[ \frac{ik_0}{2B} D (x^2 + y^2) \right] \\ &\times \int_{-\infty}^{\infty} \exp \left[ \frac{ik_0}{2B} \left( \xi^2 \left( \frac{B}{q_{x0}} + A \right) - 2\xi x \right) \right] d\xi \\ &\times \int_{-\infty}^{\infty} \exp \left[ \frac{ik_0}{2B} \left( \eta^2 \left( \frac{B}{q_{y0}} + A \right) - 2\eta y \right) \right] d\eta \end{aligned} \quad (2.15)$$

The integrals of the form  $\int_{-\infty}^{\infty} e^{-ax^2-2bx} dx$  are given by this simple identity  $\sqrt{\frac{\pi}{a}} e^{b^2/a}$  if  $\text{Re}(a) > 0$ . So solving it for the expression of  $u_{\text{out}}$  gives us:

$$\begin{aligned} u_{\text{out}}(x, y, z) &= \frac{u_0}{\sqrt{q_{x0}q_{y0}}} \frac{k_0}{2\pi i} \frac{e^{ik_0L_0}}{B} \sqrt{\frac{n_{\text{in}}}{n_{\text{out}}}} \sqrt{\frac{2\pi i q_{x0} B}{k_0(Aq_{x0} + B)}} \times \sqrt{\frac{2\pi i q_{y0} B}{k_0(Aq_{y0} + B)}} \\ &\quad \times \exp \left[ \frac{ik_0}{2} \left( \frac{D}{B} - \frac{q_{x0}}{B(Aq_{x0} + B)} \right) x^2 \right] \times \exp \left[ \frac{ik_0}{2} \left( \frac{D}{B} - \frac{q_{y0}}{B(Aq_{y0} + B)} \right) y^2 \right] \\ &= \frac{u_0 e^{ik_0L_0}}{\sqrt{Aq_{x0} + B} \sqrt{Aq_{y0} + B}} \exp \left[ \frac{ik_0}{2} \left\{ \left( \frac{D}{B} - \frac{q_{x0}}{B(Aq_{x0} + B)} \right) x^2 + \left( \frac{D}{B} - \frac{q_{y0}}{B(Aq_{y0} + B)} \right) y^2 \right\} \right] \end{aligned} \quad (2.16)$$

We can see that the amplitude has changed, and the pre-factor before  $x^2$  and  $y^2$  can be seen as the new  $1/q'_x$  and  $1/q'_y$ . When we work this out we get the relation between the old and the new  $q$ -parameters.

$$\frac{1}{q'_x} = \frac{D}{B} - \frac{q_{x0}}{B(Aq_{x0} + B)} = \frac{D(Aq_{x0} + B)}{B(Aq_{x0} + B)} - \frac{q_{x0}}{B(Aq_{x0} + B)} \quad (2.17)$$

We also know that  $AD - BC = 1$ , so if we replace the factor  $DA$  with  $(1 + BC)$  then we can see a solution forming:

$$\frac{1}{q'_x} = \frac{(1 + BC)q_{x0} + DB}{B(Aq_{x0} + B)} - \frac{q_{x0}}{B(Aq_{x0} + B)} = \frac{BCq_{x0} + DB}{B(Aq_{x0} + B)} = \frac{Cq_{x0} + D}{Aq_{x0} + B} \quad (2.18)$$

The same steps can be taken for the  $y$  components of the function, but the end result comes down to

$$u_{\text{out}}(x, y, z) = \frac{u_0}{\sqrt{Aq_{x0} + B} \sqrt{Aq_{y0} + B}} \exp \left[ \frac{ik_0}{2} \left( \frac{x^2}{q'_x} + \frac{y^2}{q'_y} \right) \right] \quad (2.19)$$

And this very much resembles the starting equation, so the only thing that changes during the propagation through the optical setup is the amplitude and the  $q$ -parameter.

### 2.2.3 Gaussian ABCD

Now that the Gaussian  $q$ -parameter allows one to make use of the ABCD matrices once again to describe, in the paraxial approximation, what happens with a Gaussian beam. The  $q$ -parameter can travel through the system and the beam parameters in the final plane can be simply calculated from the initial  $q$ -parameter with [2]

$$q_f = \frac{Aq_0 + B}{Cq_0 + D} \quad (2.20)$$

Where the A, B, C and D coefficients are the same elements of the ray transfer matrices used in ray optics. This  $q_f$  is then again the  $q$ -parameter in the final plane where the real part is the  $z$ -distance from the waist and the imaginary part gives back the Rayleigh range.

This form of propagation through ABCD law with Gaussian optics is only of use when you look at very simple systems where you relate two parallel planes with an on-axis Gaussian beam. In this method you define optical elements in one location on the optical axis, this means that spherical aberration can also not be entered this way. The beam has to go through the middle of the lens or object in order for Eq. 2.20 to hold. And since we want to know how the profile of the beam changes due to spherical aberration, or what happens off-axis, we can no longer rely on the fact that our beam remains Gaussian.

# 3 Methods

As mentioned in Section 2.2 the ABCD law is just not enough to discuss the beam profile for an off-axis beam. And the ray tracing is not enough in order to achieve a way to find the profile of the beam. So in order to propagate a Gaussian beam through a set of optical elements, several aspects of these methods need to be combined. We will show that by using Gaussian optics in combination with ray tracing, we can find a method to include several complexities that are not possible with the common rules of Gaussian optics.

## 3.1 Off-axis beams

For off-axis beams we cannot simply use the normal ABCD transfer matrix that is defined for the on-axis beam. The main issue is that it relates an incoming and outgoing field with the help of two planes. These two planes are defined such that they are perpendicular to the beam direction, therefore a different approach is required since the two planes are not perpendicular to the optical axis.

### 3.1.1 Chief Ray

We consider a small Gaussian beam that is off-axis. First the optical path of the beam needs to be found, which can be calculated by initially treating the beam as a ray. Ray calculations can be done without approximations by using Snell's law and trivial geometry. From now on the ray that describes the path of the beam will be known as the "chief-ray".

$$\left. \begin{aligned} \vec{r}_{\text{chief}} &= \vec{o}_{\text{chief}} + d\hat{p} \\ \|\vec{r}_{\text{sphere}} - \vec{c}\|^2 &= R^2 \end{aligned} \right\} d = -\hat{p} \cdot (\vec{o} - \vec{c}) \pm \sqrt{\hat{p} \cdot (\vec{o} - \vec{c})^2 - \|\vec{o} - \vec{c}\|^2 + R^2} \quad (3.1)$$

This gives a solution for the path length  $d$  that the chief-ray, with starting point  $\vec{o}$  and direction  $\hat{p}$ , has to travel before it hits the spherical lens surface with radius  $R$  with its center located at point  $\vec{c}$ . With the length now known, the point of intersection can be easily found by plugging  $d$  into the first equation for the chief-ray.

### 3.1.2 Waist rays and local coordinate system

We pick our initial coordinate system to be a Cartesian coordinate system where our beam is Gaussian. This coordinate system is chosen as the fixed frame. We always express the direction and location of the beam in terms of this fixed frame coordinate system. However the field of the beam is only known along the propagation direction of the beam, therefore it is necessary to define a local coordinate system along the beam's trajectory.

When the beam intersects with an optical element its direction might change and since the Gaussian beam profile depends on the distance it has traveled, a change of local coordinate system needs to take

place. Not only the direction of the beam, but also the direction of the  $x$ - and  $y$ -axes are calculated after passing every interface, this is done because the  $x$ - and  $y$ -axes are not necessarily under an angle of  $90^\circ$  with respect to each other. To track the changing of these axes we introduce a set of "waist-rays".

These waist-rays are rays that run parallel to the chief-ray, and when the chief-ray intersects with a surface, the waist-rays are intersected with the plane perpendicular to the beam's direction, its own  $z$ -axis. The difference between the chief-intersection and the waist-ray intersection will give a vector that lies in the  $xy$ -plane. For the  $x$  waist-ray vector we should get the new  $x$ -axis direction. If this is calculated at every interface, then this provide us with a way to keep track of the local  $x$ - and  $y$ -axes. These local coordinate systems can later be used to calculate the beam profile and translate it to the final plane.

## 3.2 New ABCD for off-axis beams

### 3.2.1 Modifying ABCD with the delta shift

With the direction of the beam known between every interface from the chief-ray, we can derive a set of parallel planes where we can relate incoming beam and outgoing beam again using a set of modified ABCD coefficients. The outgoing plane is defined by the direction of the beam towards the exit plane as seen in Fig. 3.1(a). The exit plane has a local coordinate system that we need to transfer back through the path of the beam. We drag that plane back until it hits the first part of the lens surface at point  $Q$ . The red line denotes the beam path up until the exit plane  $P_{\text{out}}$ . This is necessary because in order to work with ABCD matrices we need to relate two parallel planes to each other.

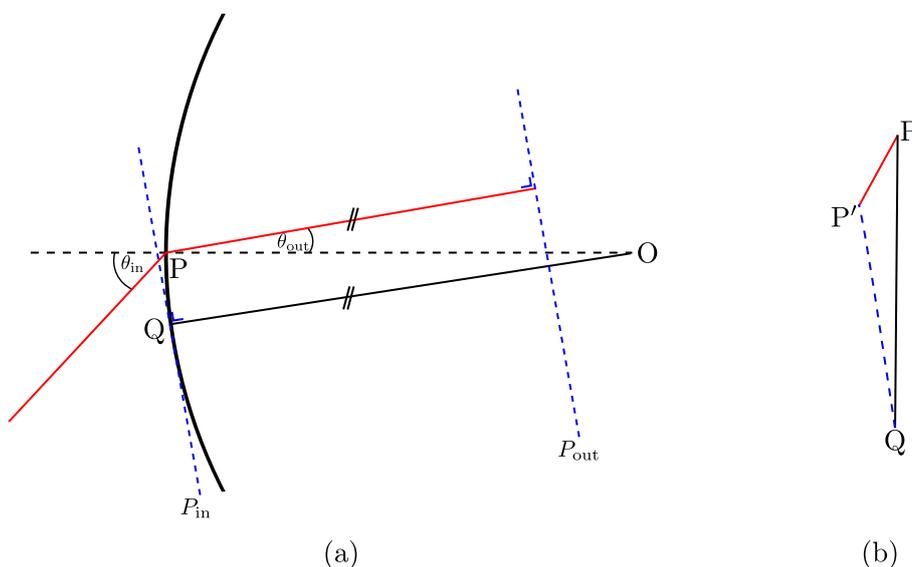


Figure 3.1: Zooming in at the intersection on (a) gives us (b) where the  $\Delta$ -shift can be seen as the distance from  $P'$  to  $Q$

We can then find the distance between these two planes and use those to define our new ray transfer matrix ABCD coefficients to propagate the off-axis Gaussian beam to the next interface. As one can see, line segment  $\overline{QO}$  now functions as the new optical axis, and we have a beam that is shifted upwards from that axis. So we need to define a shift in the local coordinate system of the beam with

that amount. We call this shift the  $\Delta$ -shift. This shift is necessary to better relate the plane of income  $P_{\text{in}}$  to the new plane  $P_{\text{out}}$ . We can take aberration into account because we defined the new ABCD coefficients for every beam with a different offset. Resulting in a different traveled distance, and therefore also a different ABCD matrix operation on each beam.

If we now look closely where the beam enters the lens at point  $P$ , we can see that this beam crosses the incoming plane at point  $P'$  as shown in Fig. 3.1(b).

$$\vec{\Delta} = \vec{Q} - \vec{P}' \quad (3.2)$$

$$\vec{Q} = \vec{o}_{\text{out}} - R\hat{z}_{\text{out}} \quad (3.3)$$

$$\vec{P}' = \vec{o}_{\text{in}} + d_{P'}\hat{z}_{\text{in}} \quad (3.4)$$

$$d_{P'} = \frac{(\vec{o}_{\text{in}} - \vec{Q}) \cdot \hat{z}_{\text{out}}}{\hat{z}_{\text{in}} \cdot \hat{z}_{\text{out}}} \quad (3.5)$$

Where  $R$  is the radius of curvature of the lens,  $\vec{o}$  is the origin of the beam and  $\hat{z}$  is the direction it is going in.

Now that we have defined our  $\Delta$ , we still need to split it into two parts in the perpendicular plane to the beam direction  $\vec{\Delta} = \Delta_x\hat{x} + \Delta_y\hat{y}$ , where  $\hat{x}$  and  $\hat{y}$  are the local  $x$ - and the  $y$ -components. We will talk more about the next section.

### 3.2.2 Expressing the incoming field in the new local coordinates

The incoming field distribution at the waist was defined as Eq. 2.14. The field equation for any position along the propagation direction can be given by

$$u_{\text{in}}(x, y, z) = \frac{U_0}{q(z)} \exp \left[ \frac{-ik}{2} \left( \frac{x^2}{q_x(z)} + \frac{y^2}{q_y(z)} \right) \right] \quad (3.6)$$

When the beam hits the next interface in the system, it changes direction, and with that it also changes its local coordinate system. In order to calculate the beam at the entrance plane of our optical element, we need to do a projection from  $\{x, y, z\} \rightarrow \{\xi, \eta, z'\}$ , the coordinate system after passing the interface. This way, the local  $z$ -axis is always pointing in the beam's direction. We can achieve this with the following change of variables

$$x = \xi d_{\xi x} + \eta d_{\eta x} + z' d_{z' x} \quad (3.7)$$

$$y = \xi d_{\xi y} + \eta d_{\eta y} + z' d_{z' y} \quad (3.8)$$

$$z = \xi d_{\xi z} + \eta d_{\eta z} + z' d_{z' z} \quad (3.9)$$

In perfect Cartesian systems, the  $d_{\xi x}$  can be defined as the dot product between the new  $x$ -axis direction  $\hat{\xi}$  and the old direction  $\hat{x}$ . In other words:  $d_{\alpha\beta} = \hat{\alpha}' \cdot \hat{\beta}$ . But this is not entirely correct due to the fact that the local coordinate axes  $\hat{x}$  and  $\hat{y}$  are not always perpendicular. These new axes-directions are found from the waist-rays as explained in section 3.1.2. The coordinate transformation always goes to a new coordinate system with the  $z$ -direction in the direction of the outgoing beam. This direction is known from ray-tracing the chief-ray. So when we want to know a certain beam profile at the entrance plane  $\{\xi, \eta, z'\}$  then we can get the values that we need to plug in for  $x$ ,  $y$  and  $z$  in the fixed frame. Those values in the function can then return the complex field in the new coordinate system.

$$u_{\text{in}}(\xi, \eta, 0) = U_0 \exp \left[ ik_0 \left\{ \frac{([\xi + \Delta_\xi] d_{\xi x} + [\eta + \Delta_\eta] d_{\eta x})^2}{2q_{x0}} + \frac{([\xi + \Delta_\xi] d_{\xi y} + [\eta + \Delta_\eta] d_{\eta y})^2}{2q_{y0}} \right\} \right] \quad (3.10)$$

To express the Gaussian field from  $\{x, y, z\} \rightarrow \{\xi, \eta, z'\}$  we need to do a coordinate system transformation. Both frames are not necessarily orthogonal systems, but  $z \perp \{x, y\}$  and  $z' \perp \{\xi, \eta\}$ . In order to project on our new  $x$ -axis  $\hat{\xi}$ , the vector needs to be split up in components  $\{\hat{x}, \hat{y}, \hat{z}\}$  from the old frame. The  $\hat{z}$ -term is first taken away, so that we only get the  $\hat{x}$  and  $\hat{y}$  component of the direction. Now only the parallel component of  $\hat{\xi}$  is left. That can then be split again into old frame  $\hat{x}$  and  $\hat{y}$ .

$$\left. \begin{aligned} \hat{\xi}_{\parallel} \cdot \hat{x} &= \xi_x + \xi_y \cos \theta \\ \hat{\xi}_{\parallel} \cdot \hat{y} &= \xi_y + \xi_x \cos \theta \end{aligned} \right\} \rightarrow \begin{cases} \xi_x = \frac{\vec{\xi}_{\parallel} \cdot (\hat{x} - \hat{y} \cos \theta)}{\sin^2 \theta} \\ \xi_y = \frac{\vec{\xi}_{\parallel} \cdot (\hat{y} - \hat{x} \cos \theta)}{\sin^2 \theta} \end{cases} \quad (3.11)$$

Where  $\vec{\xi}_{\parallel}$  is obtained from  $\vec{\xi} - (\vec{\xi} \cdot \hat{e}_z) \hat{e}_z$ . So now to recalculate the transformation matrix for the new  $x$ -axis  $\vec{\xi}$

$$d_{\xi x} = \frac{\hat{\xi}_{\parallel} \cdot (\hat{x} - \hat{y} \cos \theta)}{\sin^2 \theta} \quad (3.12)$$

$$d_{\xi y} = \frac{\hat{\xi}_{\parallel} \cdot (\hat{y} - \hat{x} \cos \theta)}{\sin^2 \theta} \quad (3.13)$$

$$d_{\xi z} = \hat{\xi} \cdot \hat{z} \quad (3.14)$$

After doing the same for  $\eta$  and  $z'$ , the  $3 \times 3$  transformation coefficient matrix can be created in order to do the transformation. And it is done while taking into account that  $\hat{x}$  and  $\hat{y}$  are not necessarily perpendicular.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_{\xi x} & d_{\eta x} & d_{z'x} \\ d_{\xi y} & d_{\eta y} & d_{z'y} \\ d_{\xi z} & d_{\eta z} & d_{z'z} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ z' \end{pmatrix} \quad (3.15)$$

### 3.2.3 Outgoing field

To get the outgoing field, all we need to know now is what the expression for our new kernel is. Because that is part of the equation that relates the field right before the interface with the field after through A, B, C and D. Again we can choose our own set of  $\{\xi, \eta, z'\}$  coordinates, so  $z'$  is chosen to be 0 in the entrance plane. This also makes the projection a bit easier since  $z'$  terms can be dropped.

In order to speed up calculations, the  $z$  projection will not be done at the interfaces. The reason why can be shown by looking at a 2D case where the translation from  $x, z \rightarrow x', z'$  is given by

$$x = x' \cos \theta - z' \sin \theta \quad (3.16)$$

$$z = z' \cos \theta + x' \sin \theta \quad (3.17)$$

The  $q$ -parameter, which is the only  $z$ -dependent part of the equation looks like

$$\begin{aligned} q(z) &= z - z_w - iz_R \\ &= \underbrace{z' \cos \theta}_{=0} + x' \sin \theta - z_w - iz_R \end{aligned} \quad (3.18)$$

We want to be able to neglect this  $z$ -dependence for calculating the change of coordinates. And since we always evaluate  $x$  in the order of magnitude of the beam waist,  $x' \sim w_0$ , this can be neglected if

either  $z_w$  or  $z_R \gg w_0$ . It is easier to see that  $z_R$  will always be larger than the beam waist at the focus if we write it out

$$\begin{aligned} \frac{\pi w_0^2}{\lambda} &\gg w_0 \\ w_0 &\gg \lambda \end{aligned} \quad (3.19)$$

This can be easily satisfied if our beam waist at the focus is about the same order of magnitude as the lens diameter. But we do have to be careful when calculating the  $z$ -dependence of the final projection, since this is where we need the real and imaginary parts of  $q$ .

In our case it is necessary to remodel the kernel due to the fact that the  $x$ - and  $y$ -axes are not always perpendicular. Our final kernel will have the form<sup>1</sup>

$$K(x, y, z; \xi, \eta, 0) = |A_{\text{kernel}}| \exp \left[ \frac{ik_0}{2B} \left\{ \begin{aligned} &A(\xi^2 + \eta^2 + 2\xi\eta \cos \theta) - \\ &2(\xi x + \eta y + (\eta x + \xi y) \cos \theta) + \\ &+ D(x^2 + y^2 + 2xy \cos \theta) \end{aligned} \right\} \right] \quad (3.20)$$

Everything that was needed to evaluate equation 2.12 has been gathered. After doing the integration, the outgoing field is given by<sup>2</sup>

$$\begin{aligned} u_{\text{out}}(x, y, z) = U_0 \frac{k_0}{2\pi i} \sqrt{\frac{n_{\text{in}}}{n_{\text{out}}}} \frac{e^{ik_0 L_0}}{B} \sqrt{\frac{\pi}{a_1}} \sqrt{\frac{\pi}{a_2}} \exp \left[ \frac{ik_0}{2B} \left( D + \frac{ik_0}{2B} \left( \frac{1}{a_1} + \frac{\cos^2 \theta}{a_2} \right) \right) x^2 \right] \\ \exp \left[ \frac{ik_0}{2B} \left( D + \frac{ik_0}{2B} \left( \frac{\cos^2 \theta}{a_1} + \frac{1}{a_2} \right) \right) y^2 \right] \end{aligned} \quad (3.21)$$

where  $a_1$  and  $a_2$  are defined as follows

$$a_1 = -ik_0 \left( \frac{d_{\xi x}^2}{2q_{x0}} + \frac{d_{\xi y}^2}{2q_{y0}} + \frac{A}{2B} \right) \quad (3.22)$$

$$a_2 = -ik_0 \left( \frac{d_{\eta x}^2}{2q_{x0}} + \frac{d_{\eta y}^2}{2q_{y0}} + \frac{A}{2B} \right) \quad (3.23)$$

The new propagated beam profile function has retained the form where  $x^2$  and  $y^2$  are multiplied by a new constant, these values can be seen as the  $1/q_f$  after propagation. Now just as with Eq 2.20 a rule between the old  $q_{x0}, q_{y0}$  and the new  $q_{xf}, q_{yf}$  can be defined using the A, B, C and D coefficients. These rules can be applied after every element up to the interface of the next element. A lens system from the first interface of that lens would be calculated in steps:

- 1 The ABCD coefficients of the diffraction are found and used to find the new angle of the chief-ray and the beam.
- 2 The chief-ray is translated along a certain distance  $d$  from the left to the right surface of the lens with the next ABCD matrix.
- 3 The Gaussian beam, known up until the first lens surface is then calculated with the new  $q_f$ -rule using the combined ABCD coefficients from step 1 and 2.

<sup>1</sup>See appendix B

<sup>2</sup>See appendix A

### 3.2.4 The final plane

The beam profile still needs to be projected one last time after it has gone through all the optical elements in order to evaluate it in the final plane. This final projection can be done to the fixed coordinate frame by using the transformation matrix with the fixed frame as the end coordinate system. This returns the profile of the beamlet in the final plane which can then be summed together with the other beamlets to produce the end result.

## 3.3 Splitting the beam into beamlets

In the end we would like to evaluate this equation for a big beam, but then we cannot use all the approximations that we need. So in order to make this valid for a big beam, we can split the beam into a superposition of little beamlets. And for this to work, we are looking for a way to split the incoming single Gaussian function into a sum of little beamlets. It is important that these little beamlets have a Gaussian shape themselves and have a waist at the focus that is at least one or two orders of magnitude smaller than the waist of the original beam. It is also important that the waist of the beamlets are bigger at their focus than the wavelength of the beam, as otherwise we would not be able to use the d-coefficients for the projection derived in the previous section.

### 3.3.1 Sampling function

To sample the original Gaussian function, it makes sense to choose smaller Gaussian beamlets. They shall then each contribute the exact wanted amount that when they are superposed they will return the shape of the original function.

$$u_s(x) = \sum_{n=-\infty}^{\infty} u(n\Delta)g(x - n\Delta) \quad (3.24)$$

This function basically says that you can create your sampled function  $u_s(x)$  out of a sum of multiple little functions  $g(x)$ . These beamlets  $g(x)$  will be displaced from the origin with respective amplitude of the original function  $f(x)$  at that displacement. In our case we want our function  $g(x)$  to have the form of a Gaussian so that we can propagate them with the derived rules. So we have to pick the waist of  $g(x)$  in the right way, that the beamlets all add up to our original Gaussian profile.

We start by defining the basic form of our little beamlet

$$g(x) = \exp\left[-\frac{x^2}{w_s^2}\right] \quad (3.25)$$

Where  $w_s$  is the beam-waist of the Gaussian sampling function  $g(x)$ , this waist needs to be larger than the wavelength  $\lambda = 780$  nm but smaller than the waist of the original Gaussian profile. This can also be generalized to a 2D array of little Gaussian beamlets where the all the parameters and constants are split into multiplications of  $x$  and  $y$ . We need a summation of beamlets, but of course a summation from  $-\infty$  to  $\infty$  is not a very fast operation, therefore we must pick a region that just spans the beam. So a sum of  $N$ -beamlets will need to cover at least twice the beam-waist in  $x$  and  $y$  direction.

$$\sum_{n=-N}^N u(n\Delta)g(x - n\Delta) \quad (3.26)$$

Where  $N\Delta \geq 2w_0$ . In the smallest possible valid summation, we shall only sum over a region that spans twice the waist-size of the original beam. This should give us a good portion of the beam to make our sampled function from.

It is also wise to pick  $\Delta$  in the same order of magnitude as the waist of the beamlet  $w_s$ . By increasing the value of  $\Delta$  with respect to  $w_s$ , you might encounter so-called aliasing where you can individually distinguish the little Gaussian beamlets from each other.

The amplitude of the beamlets cannot exactly be the same as the value of the incoming beam. The beamlets will have some residuals that overlap and those would cause higher values than the original function. To counter this form of over-counting, a factor of  $\frac{\Delta}{w_s\sqrt{\pi}}$  must be added to the amplitude of the small Gaussian beamlets. [6]

After some qualitative experimenting with the parameters and ratio between the spacing and the width of the beamlets, it can be seen in Fig. 3.2 that the sampled function becomes better when the width of the beamlets approach the same value as the spacing between the beamlets.

As can be seen in the top row, the shape of the beam is not quite good enough yet. The sampled beam looks a bit pointy with respect to the original Gaussian. This is also reflected in the difference plot where an amplitude error of 20-30% in the middle can be seen. The next row is already much better, where the difference of the two plots is now only 4% of the original amplitude. In the final row where we chose  $\Delta = w_s$ , the error is almost not visible anymore.

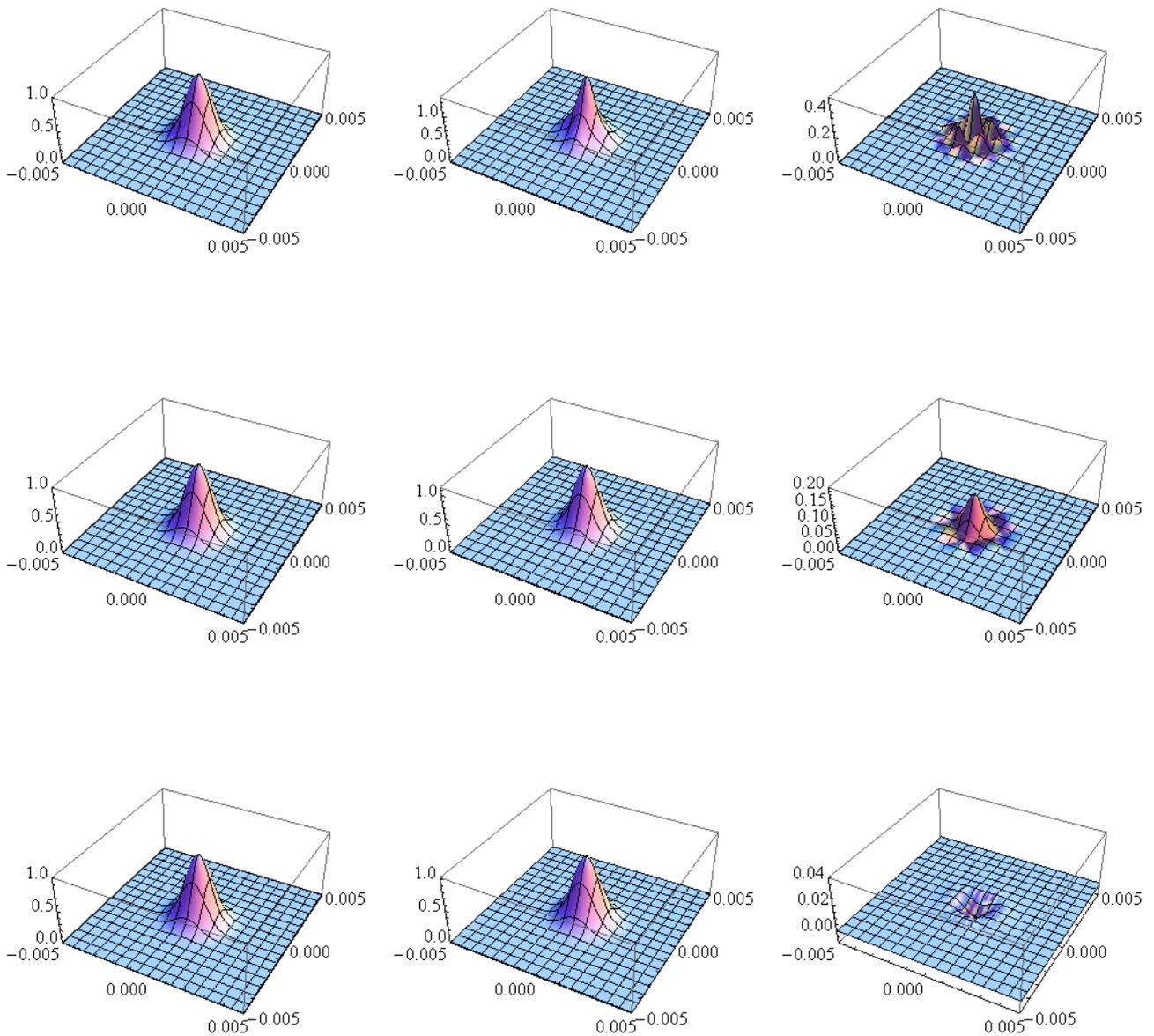


Figure 3.2: From left to right one can see the original beam, the sampled beam and the difference of the two plotted in the  $xy$ -plane. The beamlet's width to spacing ratio is increased as you go down.

# 4 Results and Discussion

## 4.1 Experiment

In order to test if the theory works for off-axis beams we have written a code. This code provides us with the final beam parameters for a given beam at the entrance plane. In our case we will test the code by running an experiment where the beam is given an offset with respect to the optical axis. The beam then travels through the defined optical elements and we can collect the data again at the final plane. There we can see how the beam shape changes due to the effect of spherical aberration.

### 4.1.1 Position of the focus with offset

In order to test our calculations, we have also thought of a measurement that allows us to check the code. The setup that is shown in Fig. 4.1 shows a fiber collimator which couples out light with a wavelength of 780 nm. For this experiment we chose an achromatic lens with a diameter of 2.54 cm and a focal length of  $f = 35$  mm. Achromatic lenses are also used in the ultra-cold atom setup. An achromatic lens is specifically made to cancel shifts of the focus due to chromatic aberration, where beams with different wavelengths are refracted differently. So in order to understand what happens in the setup we will try to make use of the same concepts.

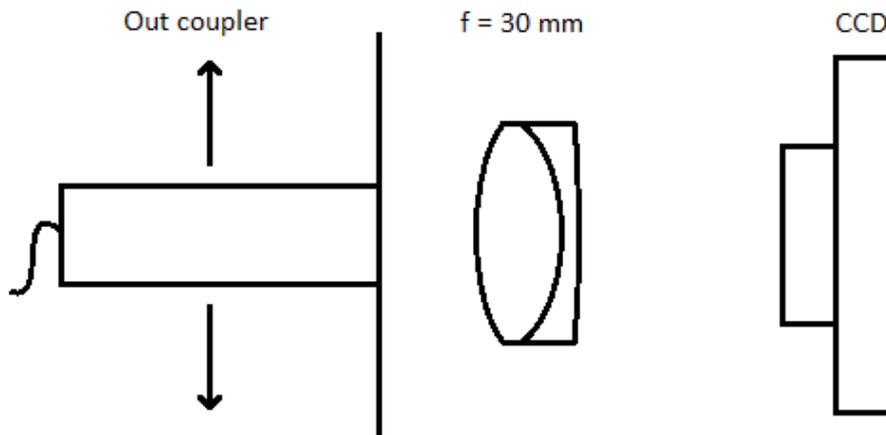


Figure 4.1: The fiber collimator is put on a translation stage so that we can vary the offset on the lens

The beam coming out of the fiber collimator has a certain Rayleigh length and a displacement from the waist. So in order to be able to calculate the beam parameters on the camera, we need to do a measurement to calculate the initial  $q$ -parameters. This is done by measuring the waist of the beam with the CCD camera at several distances from the out-coupler. We can fit the measured values to Eq. 2.9 which calculates the beam waist for a given distance from  $w_0$ . In Fig. 4.2 and 4.3 the measurements are shown with the result of the fits to Eq.2.9.

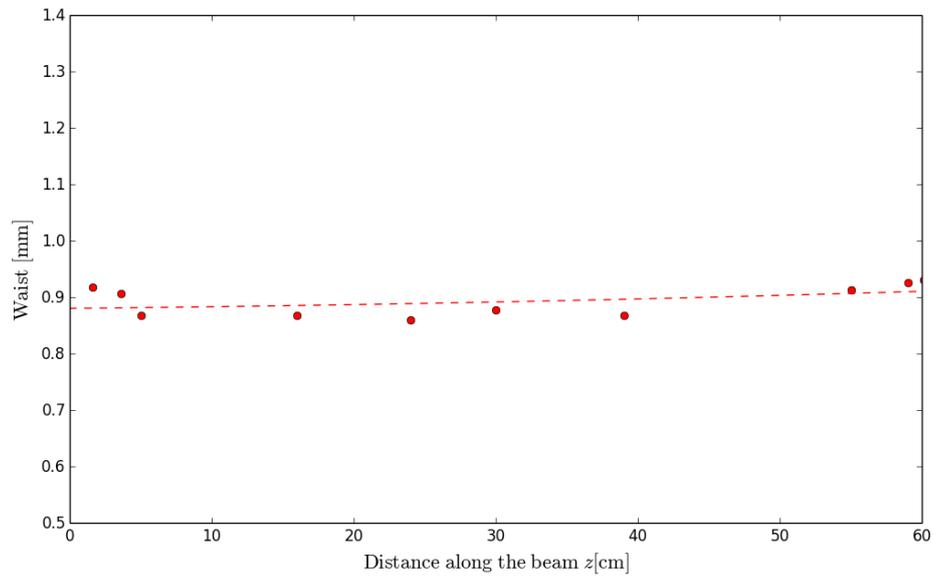


Figure 4.2: The  $x$ -waist function fitted through the measured data points

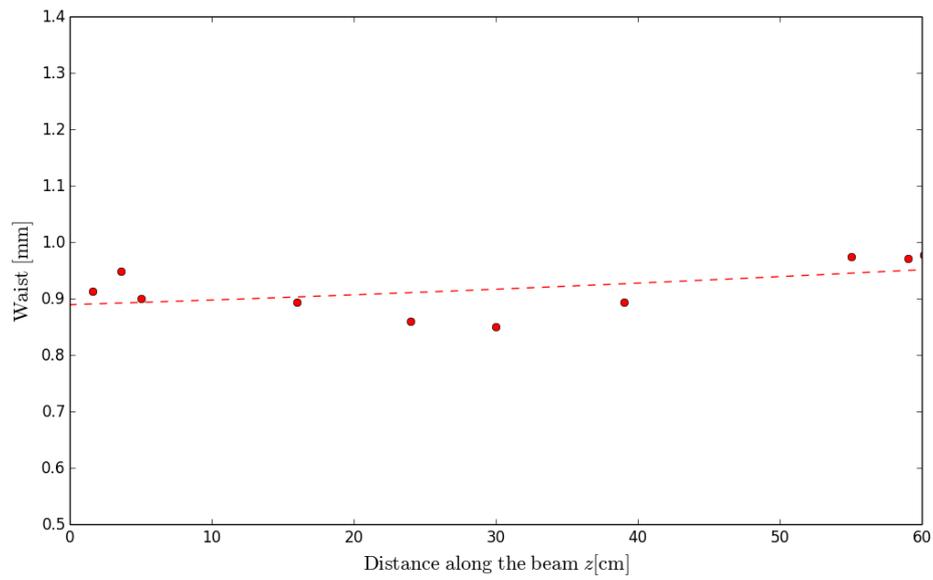


Figure 4.3: The  $y$ -waist function fitted through the measured data points

After the measurement of the initial beam parameters we moved the lens into place. We then set the distance between the camera and the lens by moving the camera until we had the smallest possible waist on the camera. With the camera in focus we began the measurement of the waists when giving the incoming beam an offset with respect to the optical axis. To achieve this offset the out-coupler is shifted back and forth with a distance of 7.5 mm from the optical axis, a total range of 15 mm in the  $x$ -direction. In Fig. 4.4 and Fig. 4.5 the effect of this shift on the  $x$ -waist and the  $y$ -waist is made visible by the red dots.

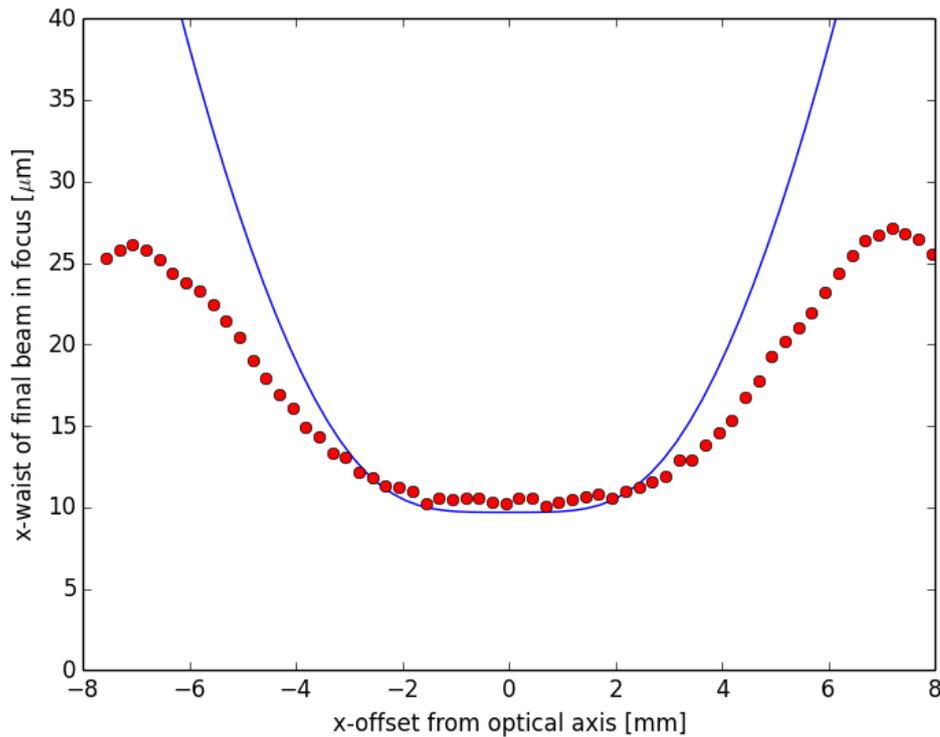


Figure 4.4: Here the  $x$ -waists of the beam in the final planes are shown as function of the initial  $x$ -offset

One can see that although the calculated model does not exactly correspond with the measurement, in the middle it is in quite good agreement with the measurements. As the beam moves further away from the center, the paraxial approximation will not hold, and therefore the results start to diverge. It is also quite a small lens, so there is not much distance to scan. If the measurement were to be improved, they would need to be done on a lens with a larger focal length where the paraxial approximation would hold over a longer region.

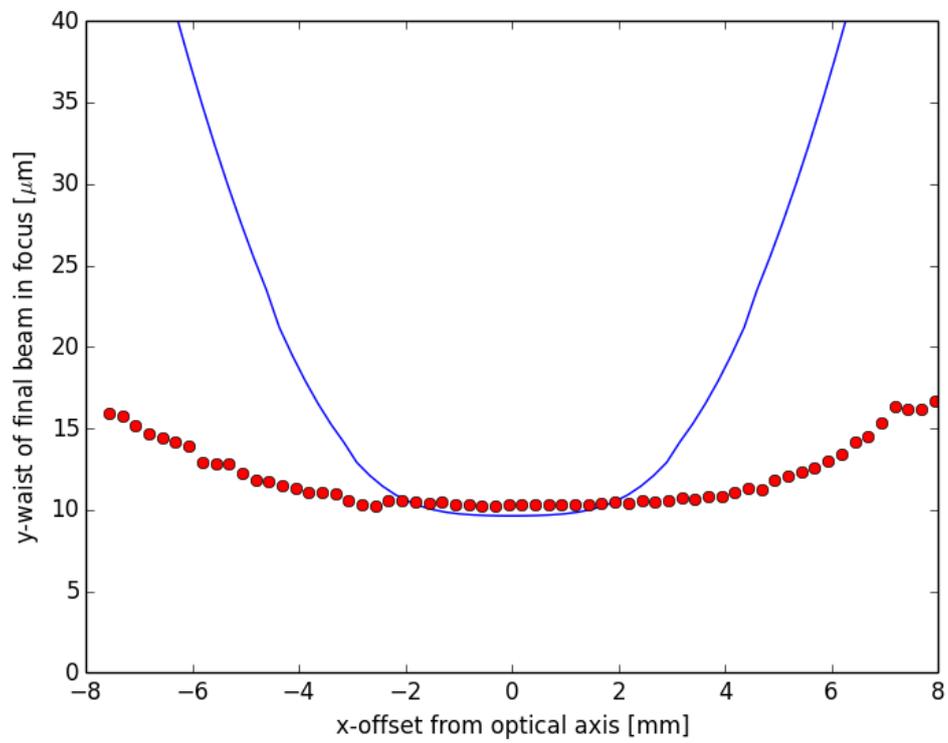


Figure 4.5: Here the  $y$ -waists of the beam in the final planes are shown as function of the initial  $x$ -offset

## 5 Outlook

In order to achieve better results, it would be beneficial to study the same focus changes for off-axis beams with a different set of lenses. In our case we used the AC254-035-B, with 35 mm focal length and 1 inch diameter, but for lenses with larger focal lengths, the range over which you can measure increases which allows for more accurate measurements.

Now that a calculation can be done to propagate an off-axis Gaussian beam, the next thing would be to simulate a situation where you split one big Gaussian beam into an array of little Gaussian beams that can each propagate through the system on their own. In the final plane they would then be added together to possibly form into complex patterns due to the accumulated phase-differences.

# Appendix

## A - Outcoming field

We write  $\vec{\Delta} = \Delta_\xi \hat{\xi} + \Delta_\eta \hat{\eta}$

$$u_{\text{in}}(\xi, \eta, 0) = U_0 \exp \left[ ik_0 \left\{ \frac{([\xi + \Delta_\xi] d_{\xi x} + [\eta + \Delta_\eta] d_{\eta x})^2}{2q_{x0}} + \frac{([\xi + \Delta_\xi] d_{\xi y} + [\eta + \Delta_\eta] d_{\eta y})^2}{2q_{y0}} \right\} \right]$$

$$K(x, y, z; \xi, \eta, 0) = A_{\text{Kernel}} \exp \left[ \frac{ik_0}{2B} \left\{ \begin{aligned} &A(\xi^2 + \eta^2 + 2\xi\eta \cos \theta) - \\ &2(\xi x + \eta y + [\eta x + \xi y] \cos \theta) + \\ &D(x^2 + y^2 + 2xy \cos \theta) \end{aligned} \right\} \right]$$

$$u_{\text{out}}(x, y, z) = \int \int K(x, y, z; \xi, \eta, z') u_{\text{in}}(\xi, \eta, z') d\xi d\eta$$

First comes the integration over  $\xi$  by collecting all  $\xi$  terms

$$f_1(\xi) = \exp \left[ ik_0 \left\{ \begin{aligned} &\xi^2 \left( \frac{d_{\xi x}^2}{2q_{x0}} + \frac{d_{\xi y}^2}{2q_{y0}} + \frac{A}{2B} \right) + \\ &\xi \left( \frac{d_{\xi x} d_{\eta x} (\eta + \Delta_\eta) + d_{\xi x}^2 \Delta_\xi}{q_{x0}} + \frac{d_{\xi y} d_{\eta y} (\eta + \Delta_\eta) + d_{\xi y}^2 \Delta_\xi}{q_{y0}} + \right. \\ &\left. \frac{A}{B} \eta \cos \theta - \frac{x + y \cos \theta}{B} \right) \end{aligned} \right\} \right]$$

$$\int_{-\infty}^{\infty} e^{-ax^2 - 2bx} dx = \sqrt{\frac{\pi}{a}} e^{b^2/a}, \quad \text{Re}(a) > 0$$

$$a_1 = -ik_0 \left( \frac{d_{\xi x}^2}{2q_{x0}} + \frac{d_{\xi y}^2}{2q_{y0}} + \frac{A}{2B} \right)$$

$$b_1 = -\frac{ik_0}{2} \left( \underbrace{\eta \left( \frac{d_{\xi x} d_{\eta x}}{q_{x0}} + \frac{d_{\xi y} d_{\eta y}}{q_{y0}} + \frac{A \cos \theta}{B} \right)}_{\gamma_1} + \Delta_\eta \underbrace{\left( \frac{d_{\xi x} d_{\eta x}}{q_{x0}} + \frac{d_{\xi y} d_{\eta y}}{q_{y0}} \right)}_{\gamma_1 - \frac{A \cos \theta}{B}} - \frac{x + y \cos \theta}{B} + \Delta_\xi \left( \frac{d_{\xi x}^2}{q_{x0}} + \frac{d_{\xi y}^2}{q_{y0}} \right) \right)$$

The outcome of the integral is then

$$\begin{aligned} \int_{-\infty}^{\infty} f_1(\xi) d\xi &= \sqrt{\frac{\pi}{a_1}} \exp \left[ -\frac{k_0^2}{4a_1} \left\{ \frac{(x+y \cos \theta)^2}{B^2} - \frac{2(x+y \cos \theta)}{B} \left( \Delta_\eta \left( \gamma - \frac{A \cos \theta}{B} \right) + \Delta_\xi ( \quad ) \right) \right\} \right] \\ &\times \exp \left[ -\frac{k_0^2}{4a_1} \left\{ \eta^2 \gamma_1^2 + 2\eta \gamma_1 \left[ \Delta_\eta \left( \gamma_1 - \frac{A \cos \theta}{B} \right) + \Delta_\xi \left( \frac{d_{\xi x}^2}{q_{x0} + \frac{d_{\xi y}^2}{q_{y0}}} \right) - \frac{x+y \cos \theta}{B} \right] \right\} \right] \\ &\underbrace{\hspace{10em}}_{f'_2(\eta)} \\ &\times \exp \left[ -\frac{k_0^2}{4a_1} \left[ \Delta_\eta \left( \gamma_1 - \frac{A \cos \theta}{B} \right) + \Delta_\xi \left( \frac{d_{\xi x}^2}{q_{x0}} + \frac{d_{\xi y}^2}{q_{y0}} \right) \right]^2 \right] \end{aligned}$$

There are some remaining  $\eta$ -dependent terms from the first equation we group as  $f''_2(\eta)$ . Now combining the result of the previous integral with the  $\eta$ -dependent terms of the first equation, we can form the new integrand  $f_2(\eta)$

$$\begin{aligned} \int_{-\infty}^{\infty} f_2(\eta) d\eta &= \int_{-\infty}^{\infty} f'_2(\eta) f''_2(\eta) d\eta \\ &= \int_{-\infty}^{\infty} \exp \left[ -\frac{k_0^2}{4a_1} \left\{ \eta^2 \gamma_1^2 + 2\eta \gamma_1 \left[ \Delta_\eta \left( \gamma_1 - \frac{A \cos \theta}{B} \right) + \Delta_\xi \left( \frac{d_{\xi x}^2}{q_{x0} + \frac{d_{\xi y}^2}{q_{y0}}} \right) - \frac{x+y \cos \theta}{B} \right] \right\} \right] \\ &\times \exp \left[ ik_0 \left\{ \eta^2 \left( \frac{d_{\eta x}}{2q_{x0}} + \frac{d_{\eta y}}{2q_{y0}} + \frac{A}{2B} \right) \right\} \right] \\ &\times \exp \left[ ik_0 \left\{ \eta \left( \Delta_\xi \left( \gamma_1 - \frac{A \cos \theta}{B} \right) + \Delta_\eta \left( \frac{d_{\eta x}^2}{q_{x0} + \frac{d_{\eta y}^2}{q_{y0}}} \right) - \frac{x \cos \theta + y}{B} \right) \right\} \right] d\eta \end{aligned}$$

We can now again find our  $a$  and  $b$ -terms to evaluate this integral. Even before writing out  $b_2^2$ , you can see that it will bring about some  $xy$ -crossterms and since we want a relation between two Gaussian functions on the entrance and exit plane, we say that these crossterms with prefactor  $(-\frac{k_0^2}{4a_1})(\frac{ik_0}{2})$  must vanish. This immediately implies that  $\gamma_1 = 0$ .

$$a_2 = -ik_0 \left( \frac{d_{\eta x}^2}{2q_{x0}} + \frac{d_{\eta y}^2}{2q_{y0}} + \frac{A}{2B} \right)$$

$$b_2 = -ik_0 \left( \Delta_\xi \left( \gamma_1 - \frac{A \cos \theta}{B} \right) + \Delta_\eta \left( \frac{d_{\eta x}^2}{q_{x0} + \frac{d_{\eta y}^2}{q_{y0}}} \right) - \frac{x \cos \theta + y}{B} \right)$$

The final result of the integration can now be written out by filling in the expression

$$\begin{aligned} \int_{-\infty}^{\infty} f_2(\eta) d\eta &= \sqrt{\frac{\pi}{a_2}} \times \exp \left[ -\frac{k_0^2}{4a_2} \left\{ \left( \frac{(x \cos \theta + y)^2}{B^2} \right) - \frac{2(x \cos \theta + y)}{B} \left( \Delta_\eta \left( \frac{d_{\eta x}^2}{q_{x0}} + \frac{d_{\eta y}^2}{q_{y0}} \right) - \frac{\Delta_\xi A \cos \theta}{B} \right) \right\} \right] \\ &\times \exp \left[ -\frac{k_0^2}{4a_1} \left( \Delta_\eta \left( \frac{d_{\eta x}^2}{q_{x0}} + \frac{d_{\eta y}^2}{q_{y0}} \right) - \frac{\Delta_\xi A \cos \theta}{B} \right)^2 \right] \end{aligned}$$

Then the remaining terms that did not depend on  $\varepsilon$  or  $\eta$  are added to the final result to collect the terms from the outgoing field equation.

$x^2$ -term:

$$D_{\text{term}}^x = \frac{ik_0}{2B} x^2 \left\{ D + \frac{ik_0}{2B} \left( \frac{1}{a_1} + \frac{\cos^2 \theta}{a_2} \right) \right\}$$

$y^2$ -term:

$$D_{\text{term}}^y = \frac{ik_0}{2B} y^2 \left\{ D + \frac{ik_0}{2B} \left( \frac{\cos^2 \theta}{a_1} + \frac{1}{a_2} \right) \right\}$$

$x$ -term:

$$\Delta_{\text{term}}^x = \frac{ik_0}{B} x \left\{ \frac{ik_0}{2a_1} \left[ \frac{\Delta_\eta A \cos \theta}{B} - \Delta_\xi \left( \frac{d_{\xi x}^2}{q_{x0}} + \frac{d_{\xi y}^2}{q_{y0}} \right) \right] + \frac{ik_0 \cos \theta}{2a_2} \left[ \frac{\Delta_\xi A \cos \theta}{B} - \Delta_\eta \left( \frac{d_{\eta x}^2}{q_{x0}} + \frac{d_{\eta y}^2}{q_{y0}} \right) \right] \right\}$$

$y$ -term:

$$\Delta_{\text{term}}^y = \frac{ik_0}{B} y \left[ \frac{ik_0}{2a_2} \left[ \frac{\Delta_\xi A \cos \theta}{B} - \Delta_\eta \left( \frac{d_{\eta x}^2}{q_{x0}} + \frac{d_{\eta y}^2}{q_{y0}} \right) \right] + \frac{ik_0 \cos \theta}{2a_1} \left[ \frac{\Delta_\eta A \cos \theta}{B} - \Delta_\xi \left( \frac{d_{\xi x}^2}{q_{x0}} + \frac{d_{\xi y}^2}{q_{y0}} \right) \right] \right]$$

$$\rightarrow \frac{ik_0}{2B} D_{\text{term}}^x \left\{ x^2 - \frac{2\Delta_{\text{term}}^x}{D_{\text{term}}^x} x \right\} \quad ; \quad \frac{\Delta_{\text{term}}^x}{D_{\text{term}}^x} = \Delta_\xi$$

$$= \frac{ik_0}{2B} D_{\text{term}}^x (x + \Delta_\xi)^2 - \frac{ik_0}{2B} D_{\text{term}}^x \Delta_\xi^2$$

$$\rightarrow \frac{ik_0}{2B} D_{\text{term}}^y \left\{ y^2 - \frac{2\Delta_{\text{term}}^y}{D_{\text{term}}^y} y \right\} \quad ; \quad \frac{\Delta_{\text{term}}^y}{D_{\text{term}}^y} = \Delta_\eta$$

$$= \frac{ik_0}{2B} D_{\text{term}}^y (y + \Delta_\eta)^2 - \frac{ik_0}{2B} D_{\text{term}}^y \Delta_\eta^2$$

In the end the final amplitude of the outgoing field equation can be rewritten as

$$\begin{aligned} A_{\text{out}} &= A_{\text{Kernel}} U_0 \sqrt{\frac{\pi}{a_1}} \sqrt{\frac{\pi}{a_2}} \exp \left[ \frac{ik_0}{2} \left\{ \frac{(\Delta_\xi d_{\xi x} + \Delta_\eta d_{\eta x})^2}{q_{x0}} + \frac{(\Delta_\xi d_{\xi y} + \Delta_\eta d_{\eta y})^2}{q_{y0}} - \frac{1}{B} (D_{\text{term}}^x \Delta_\xi^2 + D_{\text{term}}^y \Delta_\eta^2) \right\} \right] \\ &\quad \times \exp \left[ \frac{ik_0}{2a_1} \left[ \Delta_\xi \left( \frac{d_{\xi x}^2}{q_{x0}} + \frac{d_{\xi y}^2}{q_{y0}} \right) - \frac{\Delta_\eta A \cos \theta}{B} \right]^2 \right] \times \exp \left[ \frac{ik_0}{2a_2} \left[ \Delta_\eta \left( \frac{d_{\eta x}^2}{q_{x0}} + \frac{d_{\eta y}^2}{q_{y0}} \right) - \frac{\Delta_\xi A \cos \theta}{B} \right]^2 \right] \\ &= A_{\text{Kernel}} U_0 \frac{\pi}{\sqrt{a_1 a_2}} e^{ik_0 \Delta L_0} \end{aligned}$$

Where  $A_{\text{Kernel}}$  is the same as derived in Appendix B. Everything above here is the new amplitude or phase term. Now the total expression looks like

$$u_{\text{out}}(x, y, z) = A_{\text{out}} \times \exp \left[ \frac{ik_0}{2B} D_{\text{term}}^x (x + \Delta_\xi)^2 - \frac{ik_0}{2B} D_{\text{term}}^x \Delta_\xi^2 \right] \times \exp \left[ \frac{ik_0}{2B} D_{\text{term}}^y (y + \Delta_\eta)^2 - \frac{ik_0}{2B} D_{\text{term}}^y \Delta_\eta^2 \right]$$

## B - Kernel modification

The kernel for a normal on axis beam is given by

$$K(x, y, z; \xi, \eta, 0) = \frac{k_0}{2\pi i} \sqrt{\frac{n_{\text{in}}}{n_{\text{out}}}} \frac{e^{ik_0 L_0}}{B} \exp \left[ ik_0 \frac{A(\xi^2 + \eta^2) - 2(\xi x + \eta y) + D(x^2 + y^2)}{2B} \right]$$

The fact that the pre-factors  $\frac{k}{2\pi i} \sqrt{\frac{n_{\text{in}}}{n_{\text{out}}}} \frac{e^{ik_0 L_0}}{B}$  are in here, is determined by the conservation of energy. This does not work when our  $x$ - and  $y$ -axes are not perpendicular, so we need to modify it to the following for to allow for non-orthogonal  $\hat{x}$  and  $\hat{y}$ .

$$K(x, y, z; \xi, \eta, 0) = A_{\text{Kernel}} \exp \left[ \frac{ik_0}{2B} \left\{ \begin{aligned} &A(\xi^2 + \eta^2 + 2\xi\eta \cos \theta) - \\ &2(\xi x + \eta y + (\eta x + \xi y) \cos \theta) + \\ &+ D(x^2 + y^2 + 2xy \cos \theta) \end{aligned} \right\} \right]$$

To calculate this, we would simply need to propagate a beam with a kernel of amplitude  $A_{\text{Kernel}}$ . Then we calculate the energy before and after propagation with the Poynting-vector.

$$\vec{S} = \frac{1}{2} (\vec{E} \times \vec{B}) = \left( \frac{1}{2} n |\vec{E}|^2 \right) \hat{s}$$

Then we integrate the poynting vector over the entire surface of the plane, with normalized unitvector  $\hat{s}$  where the beam is known. This no longer gives us an energy per unit area, but the intensity of the beam. This is of course related to the energy in the beam and by dividing the incoming and outgoing intensity we can find the prefactor  $A_{\text{Kernel}}$

$$\frac{1}{2} n_{\text{out}} \iint |u_{\text{out}}|^2 dx dy = \frac{1}{2} n_{\text{out}} \frac{\pi}{\sqrt{c_x c_y}} \frac{\pi^2}{a_1 a_2} |U_0|^2 |A_{\text{Kernel}}|^2 \underbrace{\exp [ik_0 \Delta L_0] \exp [-ik_0 \Delta L_0^*]}_{\exp [-2k_0 \text{Im}(\Delta L_0)]}$$

Where  $c_x$  and  $c_y$  are the resulting terms that come from the integral over  $|u_{\text{out}}|^2$  and  $a_1$  and  $a_2$  are as derived in Appendix A.

$$c_x = -k_0 \text{im} \left( \frac{1}{q_x} \right)$$

$$c_y = -k_0 \text{im} \left( \frac{1}{q_y} \right)$$

Doing the same for the incoming complex beam amplitude, we get

$$\begin{aligned} \frac{1}{2} n_{\text{in}} \iint |u_{\text{in}}|^2 dx dy &= \frac{1}{2} n_{\text{in}} |U_0|^2 \int_{-\infty}^{\infty} \exp \left[ -k_0 x^2 \text{Im} \left( \frac{1}{q_{x0}} \right) \right] \int_{-\infty}^{\infty} \exp \left[ -k_0 y^2 \text{Im} \left( \frac{1}{q_{y0}} \right) \right] \\ &= \frac{1}{2} n_{\text{in}} |U_0|^2 \sqrt{\frac{\pi}{k_0 \text{Im} \left( \frac{1}{q_{x0}} \right)}} \sqrt{\frac{\pi}{k_0 \text{Im} \left( \frac{1}{q_{y0}} \right)}} \end{aligned}$$

Since we know the intensity of the beam must be equal before and after propagation, we can now relate the two equations and solve for  $A_{\text{Kernel}}$

$$\begin{aligned} |A_{\text{Kernel}}|^2 &= \frac{n_{\text{in}}}{n_{\text{out}}} \frac{a_1 a_2 \sqrt{c_x c_y}}{\pi^3} \sqrt{\frac{\pi}{k_0 \text{Im} \left( \frac{1}{q_{x0}} \right)}} \sqrt{\frac{\pi}{k_0 \text{Im} \left( \frac{1}{q_{y0}} \right)}} \exp [-2k_0 \text{Im}(\Delta L_0)] \\ &= \frac{n_{\text{in}}}{n_{\text{out}}} \frac{k_0 a_1 a_2}{k_0 \pi^2} \sqrt{\frac{\text{Im} \left( \frac{1}{q_x} \right) \text{Im} \left( \frac{1}{q_y} \right)}{\text{Im} \left( \frac{1}{q_{x0}} \right) \text{Im} \left( \frac{1}{q_{y0}} \right)}} \end{aligned}$$

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