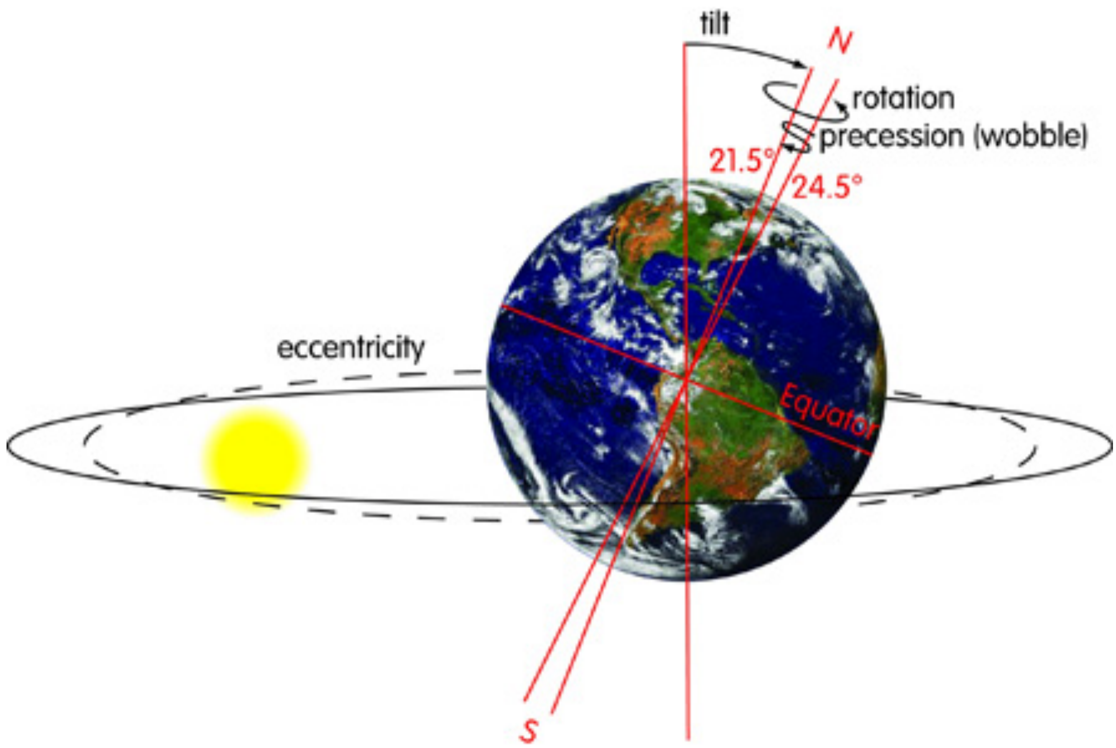
Milankovitch Cycles: Precession discovered and explained from Hipparchus to Newton



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January 2015

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1 INTRODUCTION

"From late antiquity to the 17th century, astronomy had two related goals: to show that the movements of the planets were not haphazard but regular and therefore predictable, and to predict them with accuracy. All else that concerned astronomers was peripheral."[1]

Heraclides of Pontos established already in the fourth century B.C. that the Earth rotates on its axis2. This movement is certainly not the only movement of the Earth and in this thesis I will discuss several of these movements. In particular the precession of the equinoxes, which is one of the Milankovitch cycles. Milankovitch cycles are astronomical units that create cyclic variations in the Earth's movement. These cycles influence the global climate. Due to time constraints I could not cover all cycles, so I decided to pick one representative example.

The goal of my thesis is to explain precession from Hipparchus (second century B.C.) until Newton (18^{th} century A.D.). Hipparchus was the first one to note the existence of precession and Newton was the first one to find its cause. In the next chapter I will describe some basic phenomena needed to read this thesis. The chapters that follow describe precession from Hipparchus till Newton.

[1] Hoskin, Cambridge Illustrated History of Astronomy, p. 22
[2] Ptolemy, p. 44n41

2 BASIC PHENOMENA

Before starting the story on the Milankovitch cycles, we need to familiarize ourselves with the various heavenly phenomena that lie at the basis of these cycles. In this thesis I will describe a system of the Sun, the Earth and its Moon, the five planets (Mars, Jupiter, Saturn, Pluto and Venus) and the sphere of the fixed stars. The astronomers of antiquity thought that the Earth stood still in the centre of the universe and that the heavenly bodies moved around it. Based on these ideas they were able to derive a system to describe the heavenly phenomena based on philosophical and mathematical ideas.

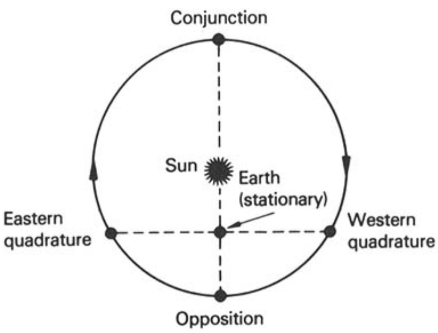

FIGURE 2-1

In antiquity astronomers made use of a gnomon, which is a primitive Sundial consisting of a stick placed vertically on a horizontal surface.[3] With this stick they could measure the length and direction of the Sun's shadow during the day. Of course the length will vary, but the direction of the shortest shadow will be the same every day. After observing these shadows long enough, astronomers noticed that these variations repeated itself and when combined with weather data this lead to the concept of seasons.

A heavenly body is in conjunction if it lies on a straight line from the Earth through the Sun and in opposition if it lies on the same line as the Sun with the Earth on the other side of the Sun (see figure 2-1). Collectively the conjunction and opposition are called syzygies.

Besides measuring the length of the days, astronomers were also interested in the nights when the stars were visible. The imaginary surface of the stars is called the celestial sphere. Where the point directly above the observer is called the zenith and the points perpendicular to the celestial equator at this celestial sphere are called the celestial poles. The circle midway between the celestial poles is called the celestial equator, the line connecting these poles is called the Earth's rotation axis. The celestial equator lies in the same plane as the Earth's equator and divides the Earth in a northern and southern hemisphere (see figure 2-2). The celestial sphere itself completes a revolution about the

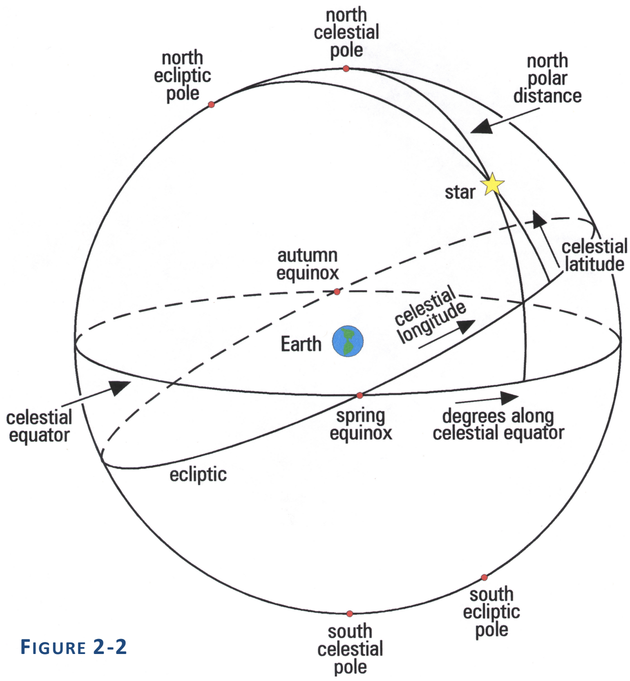

FIGURE 2-2

[3] Linton, p. 2

poles in 23 hours and 56 minutes, which is called a sidereal day. "A sidereal year is the time it takes for the Sun to return to the same position with respect to the stars."[4]

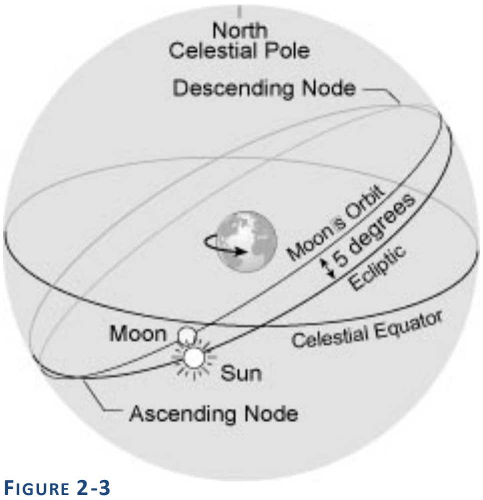

FIGURE 2-3

Another circle on the starry vault is the ecliptic on which the Sun always appears to be located during a year. The ecliptic cuts the celestial equator twice, at the equinoxes (see figure 2-2), and makes an angle of $\epsilon = 23.5°$ with the celestial equator at these points. This angle is called the obliquity of the ecliptic and is not a constant, as we will see later on. At the event of an equinox the lengths of the day and night are equal. The equinoxes are the two moments when the Sun crosses the celestial equator, a spring equinox in March and an autumn equinox in September.

Some other lines on Earth: the lines that connect the celestial poles are called lines of longitude, whereas the circles parallel to the celestial equator are called lines of latitude. There is also the longitude on the sky; this is the angle of an object measured eastwards along the ecliptic from the spring equinox points.[5]

Besides the Earth's and Sun's orbit, there is also the orbit of the Moon which inclines 5 degrees with the ecliptic. The points where the orbit of the Moon cuts the ecliptic are called the ascending and descending nodes. These points are situated on the line of nodes and have a retrograde motion of 18,6 years. In special cases, the Sun aligns with the line of nodes, which causes a solar or lunar eclipse, also referred to as a syzygy.

2.1 Systems

Now that I have introduced the necessary terminology I can explain the world systems of several astronomers from antiquity. Although they had different theories to explain the motions of the universe, they all had similar goals: to show that the movements of the planets were not haphazard but regular and therefore predictable, and to predict them with accuracy. All else that concerned astronomers was peripheral.[6]

However, this turned out to be more complicated than expected. Already in antiquity, astronomers observed that the planets did not move in a uniform circular motion. So the simple system of Aristotle, in which all heavenly bodies moved in concentric circles about the Earth, did not suffice. To explain these irregular observations they came up with several ideas. One of these ideas was an eccentric circle by Apollonius from Perga (ca. 262 – ca. 190 B.C.). This is still a perfect circle but without the central body being in the middle of the circle. The

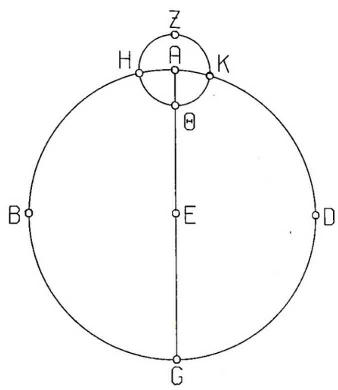

FIGURE 2-4

[4] http://astro.unl.edu/naap/motion3/sidereal_synodic.html
[5] Hoskin, Cambridge Illustrated History of Astronomy, p. 376
[6] Hoskin, Cambridge Illustrated History of Astronomy, p. 22

difference between the central body and the centre of the circle is called the eccentricity of the eccentric circle. If a Sun-centred system is assumed, in this orbit there is point of greatest distance from the Sun, the aphelion and the point of least distance, the perihelion. Collectively they are called the apsides. If an Earth-centred view is taken, these points are called apogee and perigee respectively (but of course the distance is then taken from the Earth instead of the Sun). Another idea by Apollonius was the epicyclic hypothesis (as in figure 2-4). In this case the circle $ABCD$ is concentric to E, but the body moves on the circle $ZH\Theta K$ about E. The main idea behind these hypothesises was to explain the observed irregular movements of the orbiting bodies by developing a model that had the bodies move uniform but only make them appear non-uniform.

2.2 Milankovitch Cycles

Milutin Milankovitch was a Serbian astronomer in the mid-twentieth century and he was the first one to calculate the impact of astronomical changes on the climate. He based this on three sorts of astronomical changes: eccentricity, obliquity and precession, as seen in the figure 2-5. These three astronomical changes together make up the Milankovitch cycles. In this section I will explain what these cycles are.

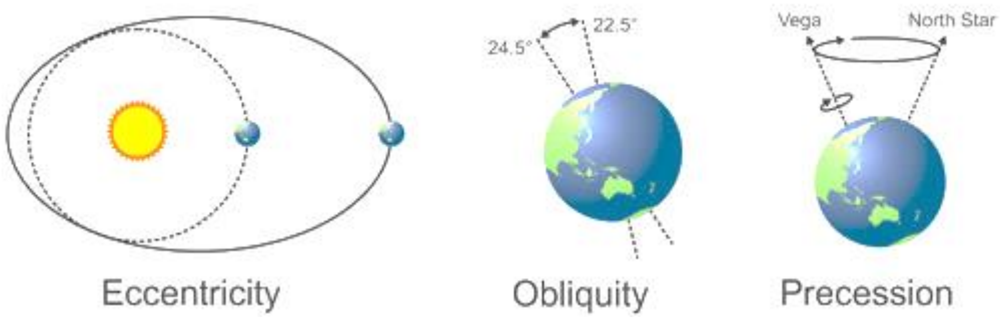

Figure 2-5

Orbital Eccentricity

Eccentricity was already discussed in section 2.1 as a parameter that defined the difference between the actual centre of the circle and the central body. If the circle is not a perfect circle, but for instance an ellipse, eccentricity is defined slightly differently.

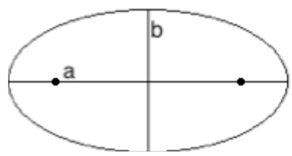

Figure 2-6

An ellipse is a curve with two focal points that lie on the ellipse's semi major axis, the longest axis in the ellipse. The semi minor axis is the longest axis perpendicular to the semi major axis, they are respectively *a* and *b* in the figure 2-6. In the case of planets orbiting each other, the Sun is at one of the two focal points, whereas the planet moves on the curve about the Sun.

Eccentricity is then the parameter that determines the amount by which its orbit about a body deviates from a perfect circle. In case of an ellipse this values lies strictly between 0 and 1.

The eccentricity is calculated by the following formula:

$$\epsilon = \sqrt{1 - \frac{b^2}{a^2}},$$

here a is the semi major axis and b the semi minor axis.

The eccentricity of the Earth has a period of 100.000 years and its value now is around 0.0167, but it varies between 0.0034 and 0.058. If this elliptical orbit would have been drawn to scale on this piece of paper it would look like a perfect circle, due to its very small eccentricity.

Obliquity

The angle of the Earth's obliquity is the angle between its rotational axis and its orbital axis (the axis perpendicular to the ecliptic). Obliquity has a period of 41.000 years and its current value is 23.4°, the value varies between 22° and 24.5°.

Precession

There are two types of precession, the axial precession and the precession of the ellipse, which make up the precession of the equinoxes. Axial precession (see figure 2-5) is where the Earth's axis makes a wobbling motion and has a period of about 26.000 years. Nowadays the northern hemisphere is pointed towards the Sun at perihelion in the summer. 13.000 years ago, the northern hemisphere was pointed towards the Sun in aphelion (see figure 2-7).

The period of the other type of precession is much slower, it has a period of about 134.000 years, and was not explained well until the 19th century, and so it is beyond the scope of this thesis. In the rest of the thesis I will use the term precession of the equinoxes for the axial precession, since this is what Hipparchus and Newton did as well.

The minimal and maximal values of the Milankovitch cycles were obtained by numerical integration methods. The methods of the French astronomer Laskar could obtain these values for a time period of 250 Myr.[7]

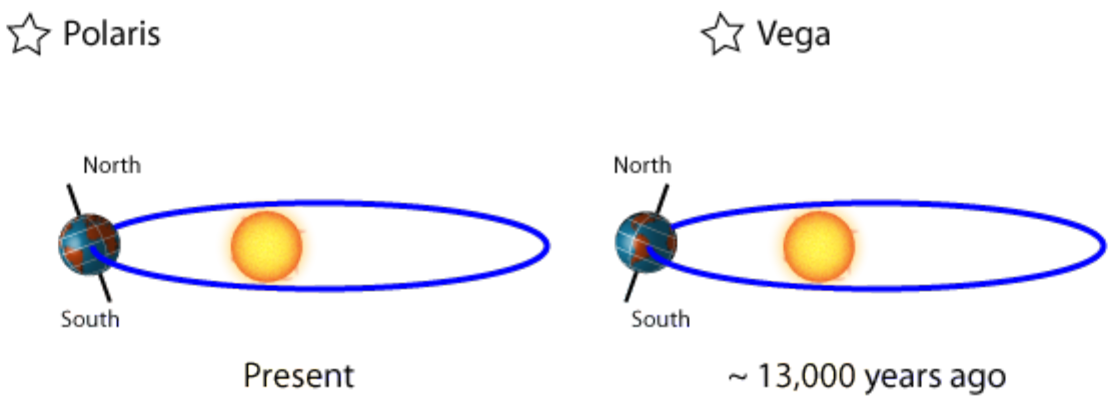

FIGURE 2-7

[7] Laskar et al. 2011

2.2.1 *Effect on climate*

Figure 2-7 shows the present situation and the situation 13.000 years ago. During the northern hemisphere summer, the Earth stands in aphelion and the northern hemisphere is tilted towards the Sun, while in the winter the Earth stands in perihelion and the northern hemisphere is tilted away from the Sun. This means that Earth gains more solar radiation during the northern hemisphere winter, because it stands closer to the Sun than during the northern hemisphere summer. However, during northern hemisphere summer, the northern hemisphere is tilted towards the Sun. This means that the angle of the incoming radiation is much smaller, so it receives more of the incoming radiation. During northern hemisphere winter, this effect is opposite. For the southern hemisphere, timing of summer and winter are exactly opposite, which has the consequence that Earth is closest to the Sun during southern hemisphere summer while it is furthest from the Sun during southern hemisphere winter. The result of this situation is that climate on the southern hemisphere is more extreme than on the northern hemisphere. 13.000 year ago, this situation was exactly opposite.[8]

An example: ice ages

Contrary to the expectation, the growth of ice caps is determined by the summer temperature instead of the winter temperatures. When there are hot summers, ice caps will melt faster than during mild summers and cold winters will not be able to restore the loss of ice. Ice caps will melt slower during mild summers, while mild winters are still cold enough for the ice to grow back. As a result of this, ice caps keep shrinking during these years of hot summers, which result in minima in ice volumes. The years of mild summers and mild winter, could result in "ice ages", especially when eccentricity is in a minimum.

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[8] Ruddiman. p. 157

3 Classic Antiquity – Scientific Revolution

3.1 Classic Antiquity: Hipparchus

In the second century B.C. an astronomical observatory was built by one of the greatest astronomers of antiquity, Hipparchus of Nicea (ca. 190 – ca. 120 B.C.), on the island of Rhodes. From all the observations and retrieved data astronomical problems arose and from these, lots of calculations. Mostly, they involved triangles and it was for this reason that the subject of trigonometry was developed. Hipparchus is nowadays still considered to be the founder of trigonometry. Without us knowing how he did it, he constructed a table of chords, which is equivalent to a table of sines. Like we still do, he subdivided the circle into 360 degrees, an idea that came from the Babylonians and was brought to him by the *On Ascensions* of Hypsicles of Alexandria (ca. 180 B.C.):[9]

Although, sometimes it is difficult to distinguish between the discoveries of Hipparchus and Ptolemy, since most of the works of Hipparchus were lost and everything we know about him now was brought to us by Ptolemy's *Almagest*. In *On the Length of the Year*[10], Hipparchus calculated very precisely the length of the tropical year. The tropical year is the time between identical equinoxes, which was already set to 365.25 days by Aristotle. Hipparchus corrected this value to 365.247 days or equivalently 365 days 5 hours 55 minutes and0 12 seconds. This exceeds the modern value by only 6 and a half minutes. After having determined this exact value, he could calculate the parameters needed for the eccentric circle (or epicycle-deferent system) theory by Apollonius of Perga (ca. 240 – ca. 190 B.C.) as explained in chapter 2.

About 150 years earlier, two other astronomers lived, Timocharis and Aristillos. Their observations of the longitude of certain stars were read and compared by Hipparchus, who used this to discover the precession of the equinoxes. He discovered this by comparing the length of the tropical year (slightly less than 365.25 days) to the length of the sidereal year (which is slightly greater than 365.25 days). Hence, Hipparchus came to the idea that the sphere of the fixed stars too has a very slow motion. This is just like that of the planets, towards the rear with respect to the revolution producing the first daily motion, which is that of a great circle drawn through the poles of both equator and ecliptic.[11] The value of this rearward motion was set to 1° per century by both Hipparchus and Ptolemy (the modern value is 1° per 72 years).

Ptolemy (ca. 90 – 160 A.D.) was a great successor of Hipparchus. He wrote the famous *Almagest*, which was originally called *Mathematical Synthesis* or *Mathematical Collection and the work*. It was written around 150 A.D. after Ptolemy had made a lot of observations in the years before. However, as already said before, it is really hard and sometimes even impossible to distinguish between Hipparchus' and Ptolemy's discoveries, because all of Hipparchus' his work was lost due to the huge success of the *Almagest*. The *Almagest* (and

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[9] Gow, p. 275
[10] Linton, p. 55
[11] Ptolemy, III.1

its geocentric system) would dominate mathematical astronomy for almost 1500 years, until *De Revolutionibus* from Copernicus in 1543.

In the *Almagest* Ptolemy also discusses the obliquity of the ecliptic and he contributes this discovery to Eratosthenes of Alexandria (276 B.C. – 196 B.C.). His value for the obliquity was 22/83 of a right angle:

$$\frac{22}{83}\times 90^\circ = 23^\circ 51'.$$

Which only was wrong by less than 1 percent of the true value at the time. Ptolemy did not explain how he found this value.

3.2 Post-Ptolemaic ideas

After Ptolemy the Greek made few advances in the field of astronomy. In Europe, the Western Roman Empire fell (around 476 A.D.) which marked the beginning of the Middle Ages and astronomical science was continued in the Islamic world (around 622 A.D.). They had a slow start, because they began with reading and translating the significant astronomical Greek and Indian works into the Arabic language. After they had studied this material they tried to establish their own, new theories of astronomy. The most significant theory was the one on trepidation, which says that the precession of the equinoxes is a variable that changes periodically. Consequently the obliquity of the ecliptic is also a periodic variable. This theory held until the invention of the telescope and then Giovanni Magini refuted the theory.[12]

In the 14th century there was a revival of science in Western Europe, the beginning of the renaissance. Copernicus changed the geocentric model to a heliocentric one with the Sun at the centre of the universe. He explained his theory in *De Revolutionibus* [13] (1543), which is a revision of Ptolemy's Almagest. He kept Ptolemy's system, but he put the Sun in the centre and made the Earth move about it.

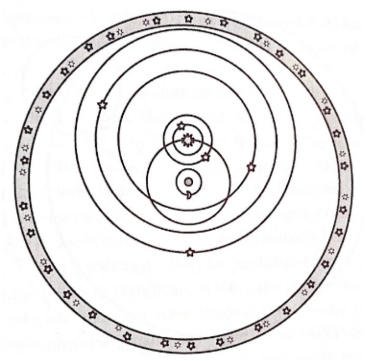

Figure 3-1 - Tychonic system

After Copernicus came Tycho Brahe who built an observatory, the *Uraniborg*, at the island Hven in OreSund, Denmark in 1576. He did this, because Brahe had studied astronomical tables and had discovered that these data were far from exact. So he made it his life goal to collect very precise data. At this observatory he collected very much valuable data on the motions of the heavenly bodies. He even came up with his own model, the Tychonic system; the Earth with its orbiting Moon was set at the centre of the system and all the other planets orbited the Sun, which then orbited the Earth (see figure 3-1). He tried to make this system

[12] Linton, p. 208. *I also read somewhere that Girolamo Fracastoro already in the Renaissance refuted this theory of trepidation, however I could not find back the source.*

[13] De Revolutionibus Orbium Coelestium (On the Revolutions of the Heavenly Spheres)

work by using his collected data, but he could not do this on his own and required help from the very intelligent Johannes Kepler (1571 – 1630). So, from 1600 until his death in 1601 Tycho Brahe was assisted by Kepler. Brahe made Kepler promise to work out the Tychonic system, but Kepler could not make this work and was more a “Copernicus guy”. Also, he knew that the data was very exact and that he needed to find a working system instead of trying to make a system work with the data. By using all this data and his excellent skills, Kepler tried to work with ellipses instead of circular orbits and he succeeded.

Kepler was not satisfied yet, because he did not only want to find a model that fitted the seemingly irregular movements, but he also wanted to find a cause for it. He attempted this by using magnetism and a force coming from the Sun, but this did not work out. In this attempt he was motivated by God, everything had to be driven by the same force.

3.3 Johannes Kepler

Kepler was maybe one of last the astronomers who practiced the science of ancient astronomy[14], but he was also the first one to confront planetary motions as a physical problem. The most important contributions of Kepler are his laws of planetary motion, which were the foundations of Newton’s *Principia*. The first two of his laws were published in the *Astronomia nova* (1609) and the third one in *Harmonices Mundi* (1619).

Laws

1. The planets move in elliptical orbits, with the Sun at one of the foci.
2. “The line connecting the planet and the Sun sweeps out equal areas in equal intervals of time.”[15]
3. The ratio of the period of revolution of a planet squared over the semi major axis cubed is the same for all planets. Also called: the law of harmonies.

Motivations for the laws

Law 1 - Kepler thought the Sun’s centrality to be essential, because he wanted one driving force for the planetary motions. As already mentioned, Kepler was motivated by God. Kepler concluded from Brahe’s data that distant planets were slower in absolute speeds and so he searched for a connection between the period and the distances of the planets. He tried to show this by using simple systems, “because often a single cause will produce many effects”[16] (also inspired by God). An ellipse was simple and it showed the desired results.

Law 2 - A result of this law is that a planet close to the Sun moves faster than if its less close to the Sun. Hence, a planet moves quickest around perihelion and slowest around aphelion, when it’s close to its driving force.

Law 3 - The third law becomes clearer if you look at it in the form of a formula:

[14] Voelkel, p. 1
[15] Voelkel, p. 65
[16] Hoskin, The General History of Astronomy, part 2A, p. 57

$$\frac{P_1^2}{P_2^2} = \frac{R_1^3}{R_2^3},$$

here P_1, P_2 are the revolution times of two planets and R_1, R_2 the lengths of their semi major axis. In the table below are the data from Brahe that Kepler used for his calculations. The periods are given relatively to the Earth's period (in days), and the distances in astronomical units. Astronomical units are given in Earth-Sun distance.

	Mercury	**Venus**	**Earth**	**Mars**	**Jupiter**	**Saturn**
Period	0,241	0,616	1,000	1,882	11,871	29,477
Distance	0,388	0,724	1,000	1,524	5,200	9,510
$\frac{\text{Period}^2}{\text{Distance}^3}$	0,99	1,00	1,00	1,00	1,00	1,01

3.4 The Scientific Revolution

The scientific revolution is referred to as the time span of the lives of two great scientists; Galileo Galilei (1564 – 1642)[17] and Sir Isaac Newton (1642 – 1727)[18]. Astronomy has always been divided, there was a mathematical side and a philosophical side and the latter one dominated. The scientific revolution changed this division.

Galileo, sometimes referred to as the founder of modern astronomy[19], devoted much of his life to the movement of bodies in a mathematical way. From 1609 on he used a telescope for his research. Galileo was not the discoverer of the telescope, but probably the first one to use this apparatus to change the worldview. For instance, he observed that the Moon was not as smooth as people used to think and that Jupiter was surrounded by small "satellites". The observations Galileo made by using the telescope denied about all of Aristotle's ideas. Also the Ptolemaic worldview was finally set aside and the Copernican theory took over. During his research, Galileo was in contact with Kepler, who was very pleased with the telescopic observations since they confirmed his theories.

Next to Galileo, Descartes and Huygens were also of important in the scientific revolution. Descartes (1596 – 1650) tried to explain the movements of the planets by the vortex theory: "The fundamental idea of his theory was that the space between the planets was filled with fluid matter containing a number of rotating vortices that carried the planets around in their orbits."[20] Huygens (1629 – 1695) tried to work on gravity and explained the movements of the planets by circular motion. Huygens had a good run, but Newton saw how circular motion actually worked and he finished Huygens' work.

3.5 Summary: Ancient Greek – Scientific Revolution

The ancient Greek were able to discover the precession of the equinoxes (Hipparchus) and the obliquity of the ecliptic (Eratosthenes). They even did a very good job on calculating their values and Johannes Kepler in the Renaissance discovered that the Earth did not move

[17] In England the Julian calendar was still in use at this time.
[18] In Italy they were already using the Gregorian calendar, so actually Galileo did not die in the same year as Newton.
[19] Linton, p. 201
[20] Linton, p. 241

in a circular orbit but in an elliptical one. Hipparchus already worked with eccentric circles, but he did not use elliptical orbits, Kepler was the first to do so. Though, what these astronomers did not know were the variations in the values of the precession, obliquity and eccentricity. Of course it was very difficult and maybe even impossible for them to know, because the rate of change is so slow. Nowadays, we calculate this by using numerical methods. Though, scientists could have known that there was variation in these values by knowing exactly why the Earth moves that way it does.

4 NEWTON

At this point Sir Isaac Newton comes into the picture, the man behind the Universal Gravitation Theory. He wrote the *Philosophiae Naturalis Principia Mathematica* or in English *Mathematical Principles of Natural Philosophy*[21]. The first edition was published in 1687 and it was revised twice in 1713 and 1726. Hereafter I will refer to this work as the *Principia*. The *Principia* is split into three books; the first two on the Motion of Bodies and the third one on the System of the World. The books on the Motion of Bodies develop a lot of theory. The book on the System of the World applies this theoretical knowledge to the actual universe.

Newton's *Principia* had two main goals:[22]

1. To prove Kepler's laws of planetary motion
2. To apply these laws in the observed universe and to take the perturbations of the planets and their Moons into account.

At first, the *Principia* would be just an expanded version of *De Motu Corporum in Gyrum* ("On the motion of bodies in an orbit"). *De Motu* was a manuscript of Newton, sent to Edmond Halley in November 1684. Halley was a clerk to the Royal Society when Newton, a member of the Royal Society then, was working on his *Principia.* Although he certainly was not paid well, Halley did pay for the *Principia* to be printed.

With help of the *Principia*, I will explain why the Earth moves in the Universe the way it does. Newton was not able to explain every detail, simply because he did not possess all the mathematics necessary.

4.1 PREPARATIONS

Before consulting book three to explain Precession, we need some definitions, propositions and laws that Newton uses regularly in his proofs.

Centripetal force

At first, the definition of centripetal force is needed. Centripetal force was not entirely discovered by Newton, Christiaan Huygens (around 1659) preceded him in this field, but Newton finished it up in a nice definition, which will be cited next:

> "Centripetal force is the force by which bodies are drawn from all sides, are impelled, or in any way tend, toward some point as to a centre."[23]

Newton explains here that there are three kinds of centripetal force, namely gravity, magnetic force and the force that makes the planets move in circular orbits instead of rectilinear ones. Nowadays we wouldn't make a difference between the first and last force. There is no need to discuss these forces separately, however it might be helpful to give the modern notation for a better understanding of Newton's definition of centripetal force:

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[21] In this thesis I will be using the English translation by I. Bernard Cohen and Anne Whitman of the third edition.

[22] Newton, p. 20

[23] Newton, p. 405

$$F_c = \frac{m \cdot v^2}{r}.$$

here F_c is the centripetal force, m is the mass, v the velocity of the object and r the distance to the fixed centre of force. So this force depends not only on the body's distance from the centre, but also on its mass and velocity.

Besides the law of centripetal force, Newton's law of universal gravitation is stated in the *Principia*. Though, it surprises me that this very important law of him was not stated until book III. Since in my opinion, universal gravitation is Newton's most important discovery. With this law it could finally be explained why bodies move about each other and do not fall out of their orbits.

This law was preceded by the Moon test in the system of the world. It says that the Moon gravitates toward the Earth and is being kept in his orbit by the force of gravity, otherwise it would continue a rectilinear motion. A simple idea, but it has some important consequence, namely that gravity does not only work on the Earth but also to the Moon. Now, the law can be extended to something universal.

Law of Universal Gravitation

This law states that: "Gravity exists in all bodies universally and is proportional to the quantity of matter in each."[24] A corollary to this law is "the gravitation toward each of the individual equal particles of a body is inversely as the square of the distance of places from those particles."[25]

The proof of universal gravitation is mostly physical. Consider two planets, A and B. The main idea of this proof is that all the parts of planet A gravitate toward all the parts of planet B. And because to every action there is an equal reaction (by Law 3 in the next section) it follows that also the reverse holds.

The universal law of gravitation contrasts sharply with Aristotle's worldview. Aristotle thought that all objects were to be at rest, in his theory the planets moved independently of each other in concentric circles. In the universal law of gravitation the planets gravitate towards each other, so they are most certainly not at rest.

Newton's Laws of motion

The basics for Newton´s dynamics were stated in the laws of motion and their corollaries. We will now look at these three laws:

Law 1

"Every body perseveres in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by forces impressed."[26]

---

[24] Newton, p. 810
[25] Newton, p. 810
[26] Newton, p. 416

It means that: an object that is at rest will stay at rest and an object that is in motion will continue doing so in the same direction and with the same velocity unless it is being disturbed by an external force.

Law 2

"A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed."[27]

A more modern interpretation would be that force (*F*) implies a change in motion and motion equals mass (*m*) times velocity, and change in velocity is acceleration (*a*):

$$F = m \times a.$$

This formula only holds for objects with a constant mass.

Law 3

"To any action there is always an opposite and equal reaction; in other words, the actions of two bodies upon each other are always equal and always opposite in direction."[28]

Nowadays we would write this like:

$$F_{action} = -F_{reaction}.$$

Newton thought of this more philosophically, namely that if you press a stone with your finger, the finger is also pressed by the stone. He then clarifies that this equal reaction is in motion and not in velocity.

A corollary to the laws is then formed. If a body is acted on by two forces that are acting jointly on it and these forces make the body describe a diagonal of parallelogram (see figure 4-1), in the same time the body could describe its two sides if the forces were acting separately on the body.

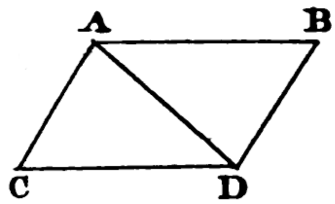

FIGURE 4-1

Proof. Let these two forces be F_m and F_n acting on the body in A, then if F_m and F_n would act separately, F_m m will send the body from A to B and F_n will send the body from A to C in a uniform motion. The force F_m could also work on the body in point C and will then send it to D, and likewise F_n could send the body from B to D. When the forces would act jointly on the body in A, they will send A also to D, but then in a straight line with uniform motion by Law 1 and this will be done in the same time as the forces acting separately.■

Given these definitions, propositions and laws we are now ready to study book III of the *Principia* on the System of the World to explain precession.

[27] Newton, p. 416
[28] Newton, p. 417

4.2 PRINCIPIA – BOOK III

In book III, the System of the World, of the *Principia*, Isaac Newton explains how the Earth and all the other bodies are driven by universal gravitation. Books I and II are on the motion of bodies and they provide the necessary theorems to accomplish this. Newton begins the propositions in this book by explaining the centripetal force, a key ingredient for universal gravitation and therefore essential to explain precession. In the second proposition he states the operation of this force and its magnitude:

***Prop III.2* – "The forces by which the primary planets are continually drawn away from rectilinear motions and are maintained in their respective orbits are directed to the Sun and are inversely as the squares of their distances from its centre."**

To prove this proposition, I will start with the first two propositions of the *Principia* and then I will separate the above proposition into two parts, the first part being the direction of the force and the second part being that this force is inversely as the squares of the distances.

Section two of book I is called: to find the centripetal forces, in this section he proves Kepler's area law. This law is proved in two steps.

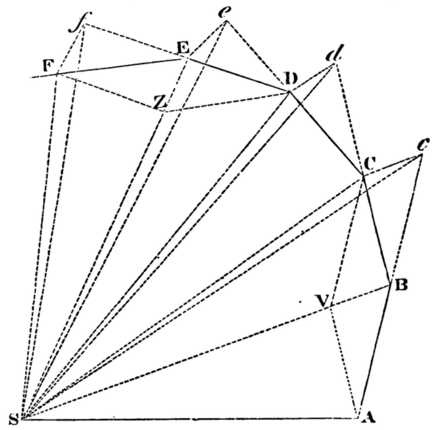

FIGURE 4-2

First step

Newton says that: "The areas which bodies made to move in orbits describe by radii drawn to an unmoving centre of forces lie in unmoving planes and are proportional to the times." [29]

Proof. Let ABCDEF be a polygonal (see figure 4-2) divided into equal intervals of times (AB = BC = … and AB = Bc = …) and let Cc // SB, Dd // SC etc. Also let CV // Bc and AV // BC.

Let a body at A move in a rectilinear motion to B in the first part of time by its inherent force – so the force it possesses to move in a constant velocity. When no other forces would interact with the body, in the second part of time it would continue to move to c in a rectilinear motion by Law 1. However, when the body arrives at B the centripetal force interacts with the body and gives a single but great impulse to the body which makes the body deflect from its rectilinear motion to c and instead moves to C. By a corollary to the laws, this point C is reached in the same timeframe.

Since SB // Cc, it follows that,

$$\text{area } \Delta SBC = \text{area } \Delta SBc = \text{area } \Delta SAB$$

One could continue this for the rest of the polygonal. Therefore, in equal times, equal areas are described in this unmoving plane.■

A corollary to this proposition:

[29] Newton, p. 444

If arcs AB and BC are successively described by the same body in equal time frame and they lie in the parallelogram ABCV with the diagonal BV, the diagonal will pass through the centre of forces, S.

Second step

Newton says that: "Every body that moves in some curved line described in a plane and, by a radius drawn to a point, either unmoving or moving uniformly forward with a rectilinear motion, describes areas around that point proportional to the times, is urged by a centripetal force tending toward that same point."[30]

Proof. This proposition states the inverse of proposition 1. Together the two propositions say that a force is central if and only if the area law holds. ∎

Proof of proposition III.2

The first part (that the primary planets are directed to the Sun) is proven by the two steps above and the second part (about the distances) is proven by the following proposition:

***Prop I.4* – "The centripetal forces of bodies that describe different circles with uniform motion tend toward the centres of those circles and are to one another as the squares of the arcs described in the same time divided by the radii of the circles."[31]**

Proof. Let *B* and *b* be bodies revolving around an unmoving centre *S* (see figure 4-3) with centripetal forces *CD* and *cd* by the propositions in the steps above. Let *BD* and *bd* be their arcs and let figure *tKb* be similar to figure *DCB*. Let BD and bt be travelled in the same amount of time.

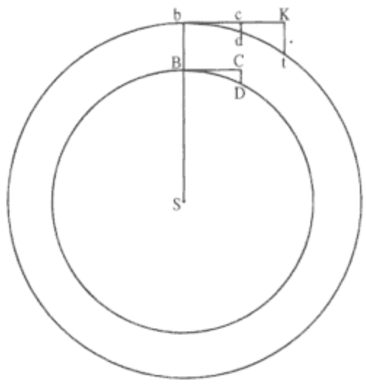

FIGURE 4-3

Newton refers to lemma I.5[32] to state the following,

$$CD : Kt = BD : bt$$

The lemma says that BCD and bKt are similar figures and in similar figures the corresponding sides are proportional to each other, this also holds for curvilinear sides.

He also refers to another lemma, I.11[33], to state that,

$$Kt : cd = bt^2 : bd^2$$

From the combination of ratios, it follows that,

---

[30] Newton, p. 446
[31] Newton, p. 803
[32] "All the mutually corresponding sides – curvilinear as well as rectilinear – of similar figures are proportional, and the areas of such figures are as the squares of their sides." Principia Book I Lemma 5
[33] "In all curves having a finite curvature at the point of contact, the vanishing sub tense of the angle of contact is ultimately in the squared ratio of the sub tense of the conterminous arc." Principia Book I Lemma 11. I do not quite understand this lemma, but Newton uses this one to prove his proposition, as do I in this thesis.

$$CD : cd = BD \times bt : bd^2$$

Or equivalently,

$$CD : cd = \frac{BD \times bt}{bS} : \frac{bd^2}{bS}$$

From the equality of ratios bt/bS and BD/BS it follows that,

$$CD : cd = \frac{BD^2}{BS} : \frac{bd^2}{bS}$$

So, the centripetal forces *CD* and *cd* are to one another as the squares of the arcs *BD* and *bd* described in the same time divided by the radii *BS* and *bS* of the circles.■

By combining all the steps above, prop III.2 has been proven.

4.3 Precession explained by Newton

Next to his explanations of eccentricity and obliquity (which are not in this thesis due to a lack of time), Newton explains the precession of the equinoxes. Precession would not occur if there was no obliquity or if the Earth would be a perfect spheroid. It is then explained in three lemmas followed by a proposition in the system of the world.

In book I, Newton treated the three body problem and added 22 corollaries to this problem. Several of these corollaries are used by him to explain the precession of the equinoxes.

The conclusion of corollary 11 of the three-body problem

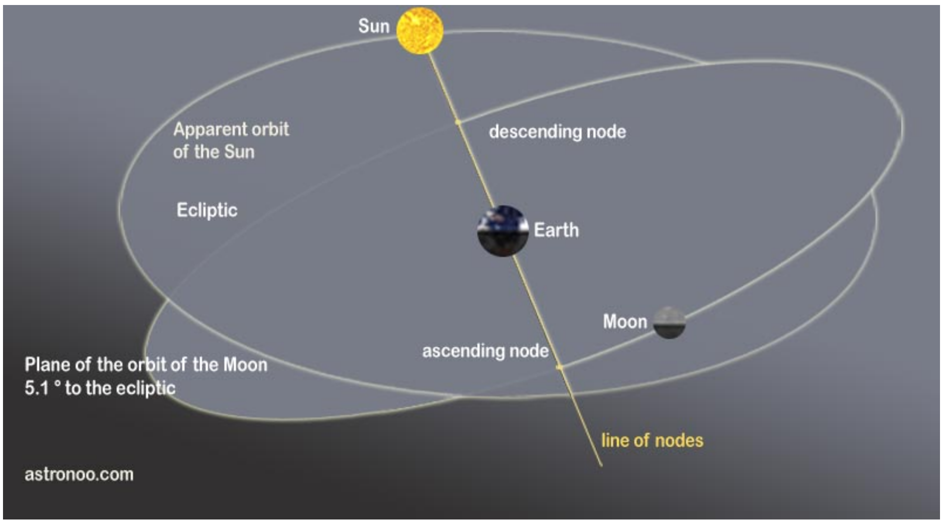

Figure 4-4

In figure 4-4 can be seen that the plane of the orbit of the Moon is not equal to the ecliptic but inclines with it, with angle of about 5 degrees, as already mentioned before.

If the Moon is positioned between the Earth and the Sun the Moon is new, if the Moon is positioned on the other side of the Earth as seen from the Sun, the Moon is full. Notice in these two cases that the Moon will be slightly above or under the Sun-Earth line. If the

Moon is positioned perpendicular to the line through the Sun and the Earth it is in one of its quadratures. In figure 4-4 there is a special case visible, namely a syzygy. A syzygy occurs when the Sun, Moon and Earth are aligned on a straight line. However, keep in mind that the Sun will not always align with the line of nodes, this figure represents a very special case.

Since the ecliptic makes an angle with the orbit plane of the Moon, the Sun wants to correct for this angle. This correction takes place because of the gravity and pulls at the Moon. The effect of this pull is concluded by Newton as follows: "and therefore, since the nodes always either have a retrograde motion or are stationary, they are carried backward in each revolution."[34] This effect is a kind of wobbling motion, so the result of the Sun's gravitational perturbation is a sort of precession and makes the nodes regress.

The use of the last corollaries of the three-body problem

In the last corollaries Newton assumes there are many little, fluid Moons orbiting the Earth at equal distances. Then he imagines the little, fluid Moons to form a fluid ring, the motion of this ring obeys by the same laws of motion as the actual Moon orbiting the Earth, namely that its nodes will regress (by corollary 11). The next step is that the Earth grows and grows until it touches the ring, this ring then solidifies and makes a so called equatorial bulge to the Earth (see the shades area in figure 4-5). Hence, the Earth has to obey by the same motions as the equatorial bulge and this results in precession.

After Newton explained the cause of the precession of the equinoxes, he calculates its period. Before I begin with this calculation, some preparations are needed.

[34] Newton, p. 579

4.3.1 PREPARATIONS TO CALCULATE THE PRECESSION

Figures

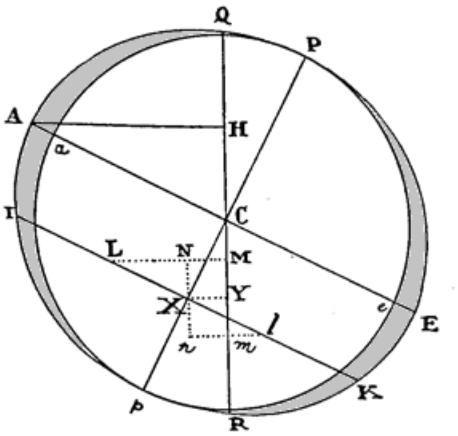

FIGURE 4-5

Let *APEp* represent the Earth (see figure 11-2) with uniform density with centre *C*, poles *P* and *p* and equator *AE*. The sphere *Pape* is inscribed in the Earth, also with centre *C* and a radius *CP*. Let *QR* be the plane that stands perpendicular on the Earth-Sun line. Let *PApaPepE* be the equatorial bulge to the sphere.

Some values

Newton does not clarify how he obtained the distances in the table below. In the guide to Newton's Principia, 19.658.600 feet and 19.573.000 feet[35] are taken for the Earth's radii and Newton remarks that 5000 feet are approximately a mile.[36] However, he does use the values in the table below for his calculations.

	Orbital period[37]	Radius of the globe	Radius bulge
Earth	1.436 minutes	52.44[38]	4.590[39]
Moon	39.343 minutes	NA	NA

Lemmas

"In Lemmas I and II, Newton introduces the concepts of the moment of momentum and of the moments of inertia."[40] Moment of momentum is a measure of the amount of rotation an object has, whereas "the moment of inertia measures a body's response to efforts to rotate it."[41] These concepts are modern interpretations of what Newton described in the two lemmas that follow.

> <u>Note: efficacy</u>
>
> *In the following lemma, Newton uses the term efficacy multiple times. In figure 4-6 the efficacy of the particle F (to turn the Earth around its centre) equals the force FG multiplied by the distance CG. Based on this given definition, one could describe the word efficacy better with the term moment or torque. The formula for torque is:*

[35] Newton, p. 234
[36] Newton, p. 234n§50.
[37] Newton, p. 820
[38] Newton, p. 886
[39] Newton, p. 886
[40] Chandrasekhar, p. 466
[41] Marsden and Tromba, p. 400

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F},$$

here $\boldsymbol{\tau}$ is the torque vector, $\boldsymbol{r}$ is the displacement vector and $\boldsymbol{F}$ is the force vector.

Lemma 1

Let the equatorial bulge exist of many individual particles, uniformly scattered. Then only consider the particles that form a ring around the equator AE. Newton then states the following:

Force and efficacy of N particles around AE : Force and efficacy of N particles at A = 1 : 2,

where the force is the force to rotate the Earth around its centre and N is a huge number of particles. This force will be performed around the common section of the equator and the plane QR.

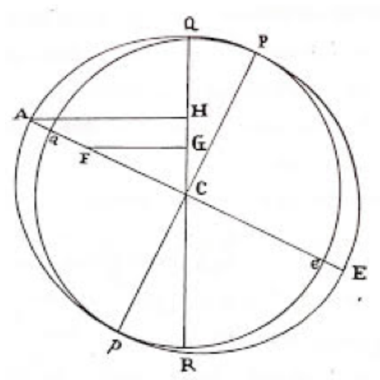

FIGURE 4-6

Proof. Now, let these particles be spread evenly over the perimeter of the circle AE. From all these particles let perpendiculars FG drop to the plane QR. So F is not a fixed point, but are all the particles from *a* till C. Do the same for AH to the plane QR.

The force by which the particle at F recedes from the plane QR is equal to the perpendicular FG. If FG is multiplied by CG, you will get the efficacy of that particle to turn the Earth around its centre. For the particle at A, this efficacy is AH times HC. So,

$$\text{efficacy of } F : \text{efficacy } A = FG \times GC : AH \times HC.$$

Here F and A are not fixed points, but particles in the places F and A.

Since ΔAHC and ΔFGC are similar,

$$FG : AH = FC : AC$$

and

$$GC : HC = FC : AC,$$

it follows that

$$FG \times GC : AH \times HC = FC^2 : AC^2.$$

So,

$$\text{efficacy of } F : \text{efficacy } A = FC^2 : AC^2.$$

At the beginning of this lemma was stated:

Force and efficacy of N particles around AE : Force and efficacy of N particles at A = 1 : 2.

By combining these last two equations, Newton concluded the following:

$$\text{efficacy of } F : \text{efficacy } A = 1 : 2.$$

Hence, the total force of the particles around the equator are to the total force of the particles around *A* as 1 is to 2. ■

Lemma 2

Under the same conditions as the previous lemma, Newton now states that:

Force of N particles outside globe : force of N particles uniformly at equator = 2 : 5,

here these forces around the equator are disposed uniformly in a ring around the equator.

This proof is even less clear than the previous one. So here I will just give the result: The total force of the particles around the Earth is to the total force of the particles around the equator as 2 is to 5.

Lemma 3

This lemma, I think, is very unclear. The motion of a cylinder revolving around its axis is compared to the motion of a very thin ring surrounding the sphere.

"Under the same conditions, I say, thirdly, that the motion of the whole Earth around the axis described above, a motion that is composed of the motions of all the particles, will be to the motion of the above-mentioned ring around the same axis in a ratio that is compounded of the ratio of the matter in the Earth to the matter in the ring and the ratio of three times the square of the quadrantal arc of any circle to two times the square of the diameter – that is, in the ratio of the matter to the matter and of the number 925.275 to the number 1.000.000."[42]

The noteworthy thing about the proof that he gives here is that he does not use any numbers at all. It is more of a reasoning that he gives. When you try to interpret this lemma the modern way, circulation comes to mind. Mathematically, circulation is defined as the line integral of the velocity along a closed curve of a velocity field.

4.3.2 To find the precession of the equinoxes

Newton takes the annual rate of regression of the nodes of the lunar orbit to be $20°11'46''$. This follows from:

$$\text{mean hourly motion of the nodes in an orbit} \times 24^h \times \text{length sidereal year}$$[43]

$$8''17'''38^{iv}18^{v} \times 24 \times 365{,}256360417 = 20°11'46''$$

[42] Newton, p. 884
[43] See Principia, p. 852 (end of corol. 2 to prop. 30)

As already established, we are dealing with a ring of Moons (attached to the Earth) instead of just one Moon. From corol. 16 to the three body problem (prop. I.66) he uses that "the motion of the nodes of each would be as the periodic times"[44].

Hence, if the ring of Moons revolves near the Earth, in the space of a sidereal day, the annual motion of the nodes of the ring is to the annual rate of regression of the nodes of the lunar orbit as a sidereal day is to the orbital period of the Moon.

The following equation is obtained:

$$\text{annual motion of the nodes} : \; -20°11'46'' \propto \text{sidereal day : orbital period of the Moon}$$

($\propto$ means that the left side is proportional to the right side, but not equal, there will be some extra factors to equal the equation)

This can also be written as:

$$\text{annual motion of the nodes} \propto \; -20°11'46'' \times \frac{1436}{39.343}$$

It does not matter for the above theory whether those Moons touch one another and are solid or if they become liquid bodies and form a continuous ring.

To obtain the final equation some factors are needed, one of them being the correct shape of the Earth. This ring around the Earth can be seen as two bulges at each side of the Earth (see figure 9-2). Let this bulge *PapAPepE* (the shaded area) lie outside of the globe *Pape*.

Newton then calculates the ratio bulge : Earth the following way:[45]

> He assumes that the globe is to the bulges as the radius squared of the globe is to the absolute distance of the semi major axis squared minus the radius squared.
>
> So, this gives the following ratio:
>
> $$\frac{Pape}{PapAPepE} = \frac{aC^2}{AC^2 - aC^2} = \frac{PC^2}{AC^2 - PC^2} = \frac{52.441^2}{459^2}$$
>
> Hence, if this ring adheres to the Earth along the equator and they both revolve about the diameter of the ring, the motion of the ring would be to the motion of Pape (by lemma 3) as $AC - PC = Aa$ to PC and 1.000.000 to 925.275 jointly, that is;
>
> $$motion\ ring : Pape \; = \frac{459}{52441} \times \frac{1.000.000}{925.275} = \frac{4.590}{485.223}$$

However, personally I think this is very unclear. Another way to obtain this factor – the ratio bulge : Earth – is by using the ratio from lemma 3 and the oblateness of the Earth. The oblateness of the Earth is $\frac{PC}{Aa} = \frac{459}{52.441} = \frac{2}{230}$ (from prop. III.19). Hence, this results in the following ratio:

---

[44] Newton, p. 885

[45] The calculations are directly from the Principia, p. 885 - 7

$$\frac{PapaPepE}{Pape} = \frac{1.000.000}{925.275} \times \frac{2}{230} \approx \frac{4.590}{485.223}$$

Slightly different, but gaining the same result as Newton and with the same factors.

At this point, Newton generalizes the result:

"Hence, if the ring adheres to the globe and communicates to the globe its own motion with which its nodes or equinoctial points regress, the motion that will remain in the ring will be to its former motion as 4.590 to 485.223 ($485.223 - 4.590 = 489.813$), and therefore the motion of the equinoctial points will be diminished in the same ratio."[46]

Upon inclusion of this factor, the equation becomes:

$$\text{annual motion of the nodes} \propto -20°11'46'' \times \frac{1436}{39.343} \times \frac{4.590}{489.813}$$

Or,

$$\text{annual motion of the nodes} \propto -20°11'46'' \times \frac{100}{292.369}$$

However PapAPepE is not just a ring around the equator, but its matter is scattered over the whole surface of Pape. So, by lemma 2, the forces of these particles act in the ratio 2 : 5.

$$\text{annual motion of the nodes} \propto -20°11'46'' \times \frac{100}{292.369} \times \frac{2}{5}$$

Rewrite by using that $20°11'46'' = 72.706''$:

$$\text{annual motion of the nodes} \propto -72.706'' \times \frac{100}{292.369} \times \frac{2}{5}$$

$$\text{annual motion of the nodes} \propto -9''56'''50^{iv}$$

This equation would hold if the plane of the equator was equal to the plane of the ecliptic, but there is an inclination of 23.5°. Therefore, to complete the equation, it has to be multiplied by $\cos 23.5$.

$$\text{annual motion of the nodes from the Sun} = -9''56'''50^{iv} \times \cos 23.5 = 9''7'''20^{iv},$$

which is the annual precession of the equinoxes based on the perturbations of the Sun. Then there are also the perturbations of the Moon. Newton calculated the force of the Moon as 4,4815 times as large as the force of the Sun, by studying the tides at Bristol.

$$\text{annual motion of the nodes from the Moon} = 9''7'''20^{iv} \times 4.4815 = 40''52'''52^{iv}$$

Adding the perturbations of the Sun and the Moon results in:

[46] Newton, p. 886

$$9''7'''20^{iv} + 40''52'''52^{iv} = 50''00'''12^{iv}$$

This value, about 50 arc seconds, agrees with the accepted value in Newton's time.

4.3.3 *Discussion*

The obtained value, about 50 arc seconds agreed with the accepted value in his time, but there are some flaws in his calculations. For instance, he used 4,815 as the force of the Moon on the Earth in comparison with the Sun by studying the tides of Bristol. Firstly, I think it is strange that he believed that studying tides at a bay would be representative for the force of the Moon on the (open) seas. Secondly, this value should be about 2.18, so it would mess up his values. Perturbations by the Moon then would be:

$$2.18 \times 9''7'''20^{iv} = 19''53'''11^{iv}$$

Instead of $40''52'''52^{iv}$.

Also, when he made this error but ended up with the correct end value, he should have made at least one other error to compensate for he first one. Some other errors I could think of that he made are:

The oblateness of the Earth: Newton takes $\frac{2}{230}$ as the oblateness of the Earth, but it should haven been about $\frac{1}{298}$.

Inclination angles: Newton uses the angle of $23.5°$ $(\cos(23.5) \approx 0.917)$ for the calculation for the force of the Sun, but he also uses this value for the Moon which is inclined at $5°$ $(\cos(5) \approx 0.996)$ with the ecliptic and so should have a different value than the inclination of the ecliptic with the equator. However, this difference is very small.

If we adapt the formulation with the errors made by Newton, we get the following results:

$$\text{annual motion of the nodes from the Sun} = 72.706 \times \frac{1.436}{39.343} \times \frac{1}{298} \times \frac{2}{5} \times \cos 5° = 3''32'''55^{iv}$$

$$\text{annual motion of the nodes form the Moon} = 3''32'''55^{iv} \times 2.18 \approx 7''44'''8^{iv}$$

Sums up to $11''17''3^{iv}$ instead of about 50 arc seconds. So, I think it can be concluded that his theory was incorrect.

Newton cannot really be blamed for his miscalculations, the gravitational theory that he used was completely new. Also he lacked mathematical techniques, differential equations and vectors were not available to Newton in his time. Vectors would have made his calculations on forces much easier and differential equations would have made the values more precise. So, given the amount of ingredients he had to work with, I think he did quite well.

5 Conclusion

Hipparchus discovered precession by noticing that the lengths of the tropical year and the sidereal year differed slightly and concluded that the sphere of the fixed stars should have a movement as well as the Sun. Ptolemy recalculated this discovery and published it in his *Almagest*.

Johannes Kepler was very lucky to have the valuable data of Tycho Brahe at his hands, he knew that the data was very precise and so his model should honour this data. Kepler's three laws of planetary motion lay the foundations for Newton's *Principia*. Kepler attempted to explain his laws of planetary motion by a central force, the Sun, but he did not succeed. Several others tried this as well, but it wasn't until Newton and his theory of Universal Gravitation that planetary motions were explained.

Newton was even able to explain the motion of precession of the equinoxes. He attempted this by using the three-body problem applied to the Sun-Earth-Moon system. He imagined many little, fluid Moons orbiting the Earth around the equator and forming a concentric ring. The Earth then grew and grew, until it attached to the ring, which then created a bulge to the Earth. This bulge then had to obey by the same laws of motion as the Moon did. Also the Sun affects the motion of the Earth, by pulling on the bulges of the Earth to correct them into the right orbit. Although he was able to explain precession, calculating it was a different story. He made quite a few mistakes, and when correct values were used, he came nowhere near the correct value for precession. In further research it would be interesting to find out where Newton went wrong and why.

More research is required to explain all of the Milankovitch cycles by Newton, also it would be very interesting to learn more about how Johannes Kepler obtained his three laws of planetary motion.

6 FIGURES

Cover picture	http://sparkcharts.sparknotes.com/
Figure 2-1	http://history.nasa.gov/
Figure 2-2	http://elfindingpolaris.files.wordpress.com/
Figure 2-3	http://ase.tufts.edu/cosmos/
Figure 2-4	*Almagest*, p.145
Figure 2-5	http://www.skepticalscience.com/
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Figure 3-1	*Eudoxus to Einstein*, p. 161
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7 BIBLIOGRAPGY

Primary works

Newton, the Principia. Translated by I. Bernard Cohen and Anne Whitman. University of California Press, 1999.

Ptolemy's Almagest. Translated and annotated by G.J. Toomer. London: Duckworth, 1984.

Secondary works

Brackenridge, J. Bruce. The Key to Newton's Dynamics The Kepler Problem and the Principia. Berkeley: University of California Press, 1996.

Chandrasekhar, S. Newton's Principia for the Common Reader, Oxford: Clarendon Press, 1996.

Crowe, Michael J. Theories of the World from Antiquity to the Copernican Revolution. New York: Dover Publications, INC., 1990.

Gow, James M.A. A Short History of Greek Mathematics. New York: Chelsea Publishing Company, 1968 (revised reprint).

Guicciardini, Niccolo. Reading the Principia. Cambridge: University Press, 1999.

Helden, Albert van. Measuring the Universe Cosmic Dimension from Aristarchus to Halley. Chicago: The University of Chicago Press, 1986 (Paperback edition).

Hoskin, Michael Anthony. The Cambridge Illustrated History of Astronomy, Cambridge: Cambridge University Press, 1997.

Hoskin, Michael Anthony. The General History of Astronomy, Cambridge: Cambridge University Press, 1984. Part 2A.

Kepler, Johannes. Mysterium Cosmographicum (The Secret of the Universe). Translated by A.M. Duncan. New York: Abaris Books, 1981.

Laskar, J., Fienga, A., Gastineau, M. and Manche, H. La2010: a new orbital solution for the long-term motion of the Earth. Astronomy and Physics, volume 532, article 89, 2011.

Linton, C.M. From Eudoxus to Einstein A History of Mathematical Astronomy. Cambridge: University Press, 2004.

Marsden and Tromba. Vector Calculus, fifth edition. New York: W.H. Freeman and Company. 2003.

Ruddiman, William F. Earth's Climate *Past and Future*, New York: W.H. Freemand and Company, 2008.

Stephenson, Charles Bruce. Kepler's Physical Astronomy. New York: Springer, 1987.

Struik, Prof. DR. D.J. Geschiedenis van de Wiskunde. Utrecht: Aula-boeken Antwerpen, 1965.

Vermij, Rienk. Kleine Geschiedenis van de Wetenschap. Amsterdam: Uitgeverij Nieuwezijds, 2007 (2nd edition).

Voelkel, James R. Johannes Kepler and the New Astronomy, Oxford University Press 1999, P. 65

Internet

http://astro.unl.edu/naap/motion3/sidereal_synodic.html