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Parabolic Initial-Boundary Value Problems with Inhomogeneous Data A Maximal Weighted L^q-L^p-Regularity Approach

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Abstract

We develop a new function space theoretic weighted L^q-L^p-maximal regularity approach for linear vector-valued parabolic initial-boundary value problems with inhomogeneous boundary conditions of static type. The weights we consider are power weights in time and in space, and yield flexibility in the optimal regularity of the initial-boundary data. The novelty of our approach is the use of weighted anisotropic mixed-norm Banach space-valued function spaces of Sobolev, Bessel potential, Triebel-Lizorkin, and Besov type. The main tools are maximal functions and Fourier multipliers, of which we also give a detailed treatment.

Preface

This thesis was written as part of the master's program Mathematical Sciences at Utrecht University, under the supervision of Mark Veraar (Analysis Group, Delft University of Technology).

Prerequisites: We assume the reader to be familiar with the basics of measure theory, functional analysis, and Fourier analysis and distribution theory, as can be found in e.g. [87, 19, 46]; also see the appendix.

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Notations and Conventions

- $\mathbb{N} = \{0, 1, 2, \ldots\}$;
- $\mathbb{R}_+ =]0, \infty[$ and $\mathbb{R}^d_+ = \mathbb{R}_+ \times \mathbb{R}^{d-1}$;
- We write $X \hookrightarrow Y$ if a topological space X embeds continuously into another topological space Y in the sense that X is canonically included in Y with a continuous inclusion mapping;
- Let X and Y be two Banach spaces which are continuously included in a common Hausdorff topological vector space Z. Then $X \cap Y$ is a Banach space for the norm $\|z\|_{X \cap Y} = \|z\|_X + \|z\|_Y$;
- We write $a \lesssim_{p_1,\ldots,p_n} b$ if $a \leq Cb$ holds with a constant C only depending on $p_1, \ldots, p_n$. We write $a \eqsim_{p_1,\ldots,p_n} b$ when both $a \lesssim_{p_1,\ldots,p_n} b$ and $b \lesssim_{p_1,\ldots,p_n} a$ hold;
- Throughout this thesis we fix a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. However, except for Sections 3.1-3.3 of Chapter 3 and for Appendices A and B, we only consider the case $\mathbb{K} = \mathbb{C}$ (as is customary in Fourier analysis).
- As is customary in Fourier analysis, for a multi-index $\alpha \in \mathbb{N}^d$ we write $D^\alpha = \frac{1}{i^{|\alpha|}}\partial^\alpha$;
- $(\epsilon_n)_{n \in \mathbb{N}}$ denotes a (fixed) Rademacher sequence on some probablility space $(\Omega, \mathcal{F}, \mathbb{P})$, see Appendix E.1.

Contents

Chapter 1

Introduction

1.1 Introduction to the Subject

During the last 20 years, the theory of maximal L^q-L^p-regularity turned out to be an important tool in the theory of nonlinear parabolic partial differential equations (PDE's). Maximal regularity means that there is an isomorphism between the data and the solution of the linear problem in suitable function spaces. Having established such sharp estimates for the linearized problem (in fact the best possible), the nonlinear problem can be treated with quite simple tools as the contraction principle and the implicit function theorem. Let us mention [5, 17, 81] in the direction of abstract parabolic problems, and [36] in the direction of parabolic PDE's with inhomogeneous boundary conditions. The full range $q, p \in]1, \infty[$ enables one to treat more nonlinearities, where q is the integrability in time and p the integrability in space. For instance, one often requires large q and p due to better Sobolev embeddings, and $q \neq p$ due to scaling invariance of PDE's (see e.g. [43]).

An abstract Cauchy problem

$$\dot{u}(t) + Au(t) = f(t) \ \ (t \in J), \quad u(0) = 0, \tag{1.1}$$

in a Banach space E on a time interval $J = [0, T]$ with $T \in]0, \infty]$, where A is a densely defined closed linear operator on E with domain $D(A)$, is said to have the property of *maximal L^q-regularity*, $q \in]1, \infty[$, if for each function $f \in L^q(J; E)$ there exists a unique solution $u \in W^1_q(J; E) \cap L^q(J; D(A))$ of (1.1). Having maximal L^q-regularity for (1.1), the corresponding version

$$\dot{u}(t) + Au(t) = f(t) \ \ (t \in J), \quad u(0) = u_0, \tag{1.2}$$

with a non-zero initial value can be easily treated via an application of related trace theorems. As a consequence of the closed graph theorem, an equivalent formulation of maximal L^q-regularity for (1.1) is that the map

$$\frac{d}{dt} + A : {}_0W^1_q(J; E) \cap L^q(J; D(A)) \longrightarrow L^q(J; E)$$

is an isomorphism of Banach spaces, where ${}_0W^1_q(J; E)$ denotes the closed subspace of $W^1_q(J; E)$ consisting of all functions which have a vanishing time trace at $t = 0$. It was already observed in [91] that (1.1) has maximal L^q-regularity for some $q \in]1, \infty[$ if and only if it has maximal L^q-regularity for every $q \in]1, \infty[$. As an application of its operator-valued Fourier multiplier theorem, Weis [102] characterized maximal L^q-regularity in terms of $\mathcal{R}$-sectoriality in the setting of Banach spaces E which are of class UMD[1]. A second approach to the maximal L^q-regularity problem is via the operator sum method, as initiated by Da Prato & Grisvard [20] and extended by Dore & Venni [31] and Kalton & Weis [63]. For more details on these approaches and for more information on (the history of) the maximal L^q-regularity problem in general, we refer to [64].

Many concrete linear parabolic PDE's can be formulated as an abstract Cauchy problem (1.1) (or (1.2)). For this thesis an important class of examples are the autonomous vector-valued parabolic initial-boundary value problems with boundary conditions of static type subject to homogeneous initial-boundary data, i.e. problems of the form

$$\begin{aligned} \partial_t u(x,t) + \mathcal{A}(x,D)u(x,t) &= f(x,t), && x \in \Omega, && t \in J, \\ \mathcal{B}_j(x,D)u(x,t) &= 0, && x \in \partial\Omega, && t \in J, \quad j = 1,\dots,n, \\ u(x,0) &= 0, && x \in \Omega, \end{aligned} \tag{1.3}$$

where $J = [0,T]$ for some $T \in]0,\infty]$, Ω is a domain in $\mathbb{R}^d$ with a compact smooth boundary $\partial\Omega$, $\mathcal{A}(x,D)$ is a partial differential operator of order $2n$ having $\mathcal{B}(X)$-valued variable coefficients, and the $\mathcal{B}_j(x,D)$ are partial boundary differential operators of order $n_j < 2n$ having $\mathcal{B}(X)$-valued variable coefficients, with X a fixed Banach space. For these problems an abstract formulation of the form (1.1) is possible in the L^p-setting, $p \in]1,\infty[$: just take A to be the L^p-realization of the corresponding differential boundary value problem, i.e., consider the Banach space $E = L^p(\Omega;X)$ and the operator A on E given by

$$\begin{aligned} D(A) &= \{v \in W^{2n}_p(\Omega;X) \,:\, \mathcal{B}_j v = 0 \text{ (on } \partial\Omega), j = 1,\dots,n\}, \\ Av &= \mathcal{A}v. \end{aligned}$$

Then the associated abstract Cauchy problem (1.1) has maximal L^q-regularity if and only if for each $f \in L^q(J;L^p(\Omega;X))$ there exists a unique solution $u \in W^1_q(J;L^p(\Omega;X)) \cap L^q(J;W^{2n}_p(\Omega;X))$ of (1.3), in which case we say that (1.3) enjoys the property of *maximal L^q-L^p-regularity*. With normal ellipticity and conditions of Lopatinskii-Shapiro type as the basic structural assumptions, Denk, Hieber & Prüss [25] proved maximal L^q-L^p-regularity for a large class of problems of the form (1.3) in the setting of UMD spaces; in fact, also non-autonomous versions were treated in which the top order coefficients of the operators are assumed to be bounded and uniformly continuous (allowing for perturbation arguments). Earlier works in this direction include [33, 32, 83, 34, 50, 51, 24], all concerning scalar-valued 2nd order problems having special boundary conditions (mainly Dirichlet).

The linear parabolic initial-boundary value problems (1.3) include linearizations of reaction-diffusion systems and of phase field models with Dirichlet, Neumann and Robin conditions. However, if one wants to use linearization techniques to treat such problems with non-linear boundary conditions, then one needs to study a versions (1.3) with boundary inhomogeneities. It is in fact crucial to have a sharp theory for the fully inhomogeneous version of the linear problem (1.3): The problem

$$\begin{aligned} \partial_t u(x,t) + \mathcal{A}(x,D)u(x,t) &= f(x,t), && x \in \Omega, && t \in J, \\ \mathcal{B}_j(x,D)u(x,t) &= g_j(x,t), && x \in \partial\Omega, && t \in J, \quad j = 1,\dots,n, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned} \tag{1.4}$$

[1] The class of UMD Banach spaces is defined in Appendix E.5, where UMD stands for the unconditionality of martingale differences.

is said to enjoy the property of *maximal L^q-L^p-regularity* if there exists a (necessarily unique) space of initial-boundary data $\mathscr{D}_{i.b.} \subset L^q(J; L^p(\partial\Omega; X))^n \times L^p(\Omega; X)$ such that for every $f \in L^q(J; L^p(\Omega; X))$ it holds that (1.4) has a unique solution $u \in W^1_q(J; L^p(\Omega; X)) \cap L^q(J; W^{2n}_p(\Omega; X))$ if and only if $(g = (g_1, \ldots, g_n), u_0) \in \mathscr{D}_{i.b.}$. In this situation there exists a Banach norm on $\mathscr{D}_{i.b.}$, unique up to equivalence, with

$$\mathscr{D}_{i.b.} \hookrightarrow L^q(J; L^p(\partial\Omega; X))^n \oplus L^p(\Omega; X)$$

which makes the associated solution operator a topological linear isomorphism between the data space $L^q(J; L^p(\Omega; X)) \oplus \mathscr{D}_{i.b.}$ and the solution space $W^1_q(J; L^p(\Omega; X)) \cap L^q(J; W^{2n}_p(\Omega; X))$. The *maximal L^q-L^p-regularity problem* for (1.4) consists of establishing maximal L^q-L^p-regularity for (1.4) and explicitly determining the space $\mathscr{D}_{i.b.}$ together with a Banach norm as above.

Combining operator sum methods with tools from vector-valued harmonic analysis, Denk, Hieber & Prüss [26] solved the maximal L^q-L^p-regularity problem for (1.4); as in [25], also non-autonomous versions were considered in which the top order coefficient of the operators are assumed to be bounded and uniformly continuous. Earlier works on this problem are [66] ($q = p$) and [101] ($p \leq q$) for scalar-valued 2nd order problems with Dirichlet and Neumann boundary conditions. Later, Denk, Prüss & Zacher [28] solved the maximal L^q-L^p-regularity problem, in case $q = p$, for a large class of linear vector-valued parabolic initial-boundary problems with inhomogeneous boundary conditions of relaxation type, which include dynamic boundary conditions as well as problems arising as linearizations of free boundary value problems that are transformed to a fixed domain.

The above mentioned results of [26, 28] have been extended by Meyries & Schnaubelt [76] to the setting of temporal power weights $v_\mu(t) = t^\mu$, $\mu \in [0, q-1[$, for the case that $q = p$; also see [73]. The weighted framework allows to reduce the initial regularity and to avoid compatibility conditions at the boundary, and it provides an inherent smoothing effect of the solutions. Here the main tools are interpolation and trace theory for anisotropic fractional Sobolev spaces (of intersection type) with temporal weights, operator-valued functional calculus, as well as localization and perturbation arguments, of which the required interpolation and trace theory was already studied systematically in an earlier paper [75]. In [73, 74], this weighted maximal regularity approach was used to establish convergence to equilibria and the existence of global attractors in high norms.

Preceding the weighted maximal regularity approach in [76], Prüss & Simonett [82] initiated a weighted maximal L^q-regularity approach for abstract Cauchy problems (1.1)/(1.2). Here it is proposed to work in the weighted Lebesgue-Bochner spaces[2]

$$L^q(\mathbb{R}_+, v_\mu; E) = \left\{ u \in L^0(\mathbb{R}_+; E) \, : \, \int_{\mathbb{R}_+} \|u(t)\|_E^q \, v_\mu(t) dt < \infty \right\},$$

equipped with the natural norm, for the power weights $v_\mu(t) = t^\mu$, $\mu \in [0, q-1[$.[3] The abstract Cauchy problem (1.1) (for $J = \mathbb{R}_+$) then is said to enjoy the property of *maximal L^q_μ-regularity* if for each function $f \in L^q(\mathbb{R}_+, v_\mu; E)$ there exists a unique solution $u \in W^1_q(\mathbb{R}_+, v_\mu; E) \cap L^q(\mathbb{R}_+, v_\mu; D(A))$ of (1.1), where $W^1_q(\mathbb{R}_+, v_\mu; E)$ stands for the 1st order weighted Sobolev space

[2] E-valued Lebesgue-Bochner spaces on $\mathbb{R}_+$ are subspaces of $L^0(\mathbb{R}_+; E)$ determined by certain integrability conditions, where $L^0(\mathbb{R}_+; E)$ stands for the space of equivalence classes of strongly measurable functions $\mathbb{R}_+ \longrightarrow E$; see Appendix A.1.

[3] The authors actually use a different parametrization of the weights.

on $\mathbb{R}_+$ associated with $L^q(\mathbb{R}_+, v_\mu; E)$. Having maximal L^q_μ-regularity for (1.1), the problem (1.2) can be solved for initial values u_0 belonging to the real interpolation space $(E, D(A))_{1-\frac{1}{q}(1+\mu),q}$.[4] The space $(E, D(A))_{1-\frac{1}{q}(1+\mu),q}$ gets closer to the space E when μ gets closer to $q-1$, giving a reduction in the required initial regularity.

It is the main purpose of the present thesis to extend the results of [26, 76], concerning the maximal L^q-L^p-regularity problem for (1.4), to the setting of power weights in time and in space for the full range $q, p \in]1, \infty[$, using a different approach on the function space theoretic part of the problem. Here we do not only aim at giving a systematic treatment of the maximal weighted L^q-L^p-regularity problem for (1.4) itself, but also of the required function space theory (in which maximal functions play a crucial role) and Fourier multiplier theory. The weights we consider are the power weights

$$v_\mu(t) = t^\mu, \mu \in]-1, q-1[, \quad \text{and} \quad w_\gamma(x) = \mathrm{dist}(x, \partial\Omega)^\gamma, \gamma \in]-1, p-1[.$$

The main feature of this weighted approach is the flexibility for regularity of the initial-boundary data as μ and γ vary in $]-1, q-1[$ and $]-1, p-1[$, respectively. For simplicity (and for reasons of time and size of this thesis), we in fact restrict ourselves to model problems with homogeneous constant coefficient operators on the half space $\Omega = \mathbb{R}^d_+ = \mathbb{R}_+ \times \mathbb{R}^{d-1}$, which are very important because the general case can be reduced to them using standard PDE-techniques (which may be quite technically involved) as freezing the coefficients, localization and perturbation; such a reduction is worked out in great detail in Sections 2.3 and 2.4 of [73].

1.2 Our Function Space Theoretic Approach

The aim of this section and the next section is to give an overview of this thesis. In this section we start with giving a description of the main idea behind our approach to the weighted maximal L^q-L^p-regularity problem for the model problems for (1.4) on the half-space, for which we also need to rigorously formulate this maximal regularity problem and to briefly comment on the existing literature in this direction. Having this description, we give an outline of the contents of this thesis, together with a description of the structure and the organization, in the next section.

In order to settle ideas, we just consider the following 'simple' vector-valued parabolic initial-boundary value problem on the half-space $\mathbb{R}^d_+ = \mathbb{R}_+ \times \mathbb{R}^{d-1}$:

$$\begin{aligned} \partial_t u(y, x', t) + (1-\Delta)u(y, x', t) &= f(y, x', t), \quad (y, x') \in \mathbb{R}^d_+, \quad t \geq 0 \\ u(0, x', t) &= g(x', t), \quad x' \in \mathbb{R}^{d-1}, \quad t \geq 0, \\ u(y, x', 0) &= u_0(y, x'), \quad (y, x') \in \mathbb{R}^d_+. \end{aligned} \tag{1.5}$$

Let X be a Banach space, $q, p \in]1, \infty[$, $\mu \in]-1, q-1[$, $\gamma \in]-1, p-1[$, and consider the weights

$$v_\mu(t) := |t|^\mu \qquad (t \in \mathbb{R}), \qquad w_\gamma(y, x') := |y|^\gamma \qquad ((y, x') \in \mathbb{R} \times \mathbb{R}^{d-1} = \mathbb{R}^d). \tag{1.6}$$

In order to give a rigorous description of the maximal L^q_μ-L^p_γ-regularity problem for (1.5), we need the concept of trace operator. Given a Banach space E, we have $W^1_q(\mathbb{R}_+, v_\mu; E) \hookrightarrow C([0, \infty[; E)$ and we can define the trace in 0 as the continuous linear operator

$$\mathrm{tr}_{t=0} : W^1_q(\mathbb{R}_+, v_\mu; E) \longrightarrow E, \; u \mapsto u(0). \tag{1.7}$$

[4] An elementary introduction to interpolation theory can be found in [69].

Furthermore, it can be shown that there exists a (necessarily unique) continuous linear operator

$$\mathrm{tr}_{y=0} : W^1_p(\mathbb{R}^d_+, w_\gamma; E) \longrightarrow L^p(\mathbb{R}^{d-1}; E) \tag{1.8}$$

which maps continuous functions on $\overline{\mathbb{R}^d_+} = [0,\infty[\times\mathbb{R}^{d-1}$ to their restriction with respect to the boundary $\{0\}\times\mathbb{R}^{d-1}$ of $\mathbb{R}^d_+$ (which we identify with $\mathbb{R}^{d-1}$). Taking $E = L^p(\mathbb{R}^{d-1}; X)$ and $E = X$, respectively, these two trace operators induce (in the natural way) operators

$$\mathrm{tr}_{t=0} \in \mathcal{B}\Big(W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+, v_\mu; W^2_p(\mathbb{R}^d_+, w_\gamma; X)), L^p(\mathbb{R}^d_+, w_\gamma; X)\Big) \tag{1.9}$$

and

$$\mathrm{tr}_{y=0} \in \mathcal{B}\Big(W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+, v_\mu; W^2_p(\mathbb{R}^d_+, w_\gamma; X)), L^q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^{d-1}; X))\Big). \tag{1.10}$$

In the maximal L^q_μ-L^p_γ-regularity approach to (1.5) we want to uniquely solve the problem

$$\begin{aligned} \partial_t u + (1-\Delta)u &= f, \\ \mathrm{tr}_{y=0} u &= g, \\ \mathrm{tr}_{t=0} u &= u_0. \end{aligned} \tag{1.11}$$

in the solution space $W^1_q(\mathbb{R}_+; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+; W^2_p(\mathbb{R}^d_+, w_\gamma; X))$; to be more precise, we want to find[5] a Banach space of initial-boundary data

$$\mathscr{D}_{i.b.} \hookrightarrow L^q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^{d-1}; X)) \oplus L^p(\mathbb{R}^d; X),$$

which is necessarily unique up to an equivalence of norms, such that the problem (1.5) admits, for each $f \in L^q(J; L^p(\Omega; X))$, a unique solution $u \in W^1_q(\mathbb{R}_+; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+; W^2_p(\mathbb{R}^d_+, w_\gamma; X))$ if and only if the data g, u_0 satisfy $(g, u_0) \in \mathscr{D}_{i.b.}$.

Having available a rich theory of maximal regularity for abstract Cauchy problems (for which we need X to be a UMD space), the main difficulty in (1.11) is the boundary inhomogeneity g. A very important step in treating treating this boundary inhomogeneity is to determine the trace space of

$$\mathbb{E} := W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+, v_\mu; W^2_p(\mathbb{R}^d_+, w_\gamma; X))$$

for the trace operator $\mathrm{tr}_{y=0}$, that is, to determine a Banach space

$$\mathbb{F} \hookrightarrow L^q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^{d-1}; X)),$$

which is necessarily unique up to an equivalence of norms, such that $\mathrm{tr}_{y=0}$ is a continuous surjection $\mathbb{E} \longrightarrow \mathbb{F}$ having a continuous right-inverse.

In [26] it was established that, as a byproduct of one of the main results (concerning the solution to the maximal L^q-L^p-regularity problem [26, Theorem 2.3]),

$$\mathbb{F} = F^{1-\frac{1}{2p}(1+\gamma)}_{q,p}(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}_+, v_\mu; B^{2-\frac{1}{p}(1+\gamma)}_{p,p}(\mathbb{R}^{d-1}; X)) \tag{1.12}$$

for the unweighted case $\mu = \gamma = 0$ under the restriction that X is a UMD space, where $F^{1-\frac{1}{2p}(1+\gamma)}_{q,p}$ denotes a Triebel-Lizorkin space and $B^{2-\frac{1}{p}(1+\gamma)}_{p,p}$ denotes a Besov space; for the case $p \leq q$,

[5] Establish its existence and determine it explicitly.

$\mu = \gamma = 0$, $X = \mathbb{C}$, this was already known in [101]. Whereas Denk, Hieber & Prüss [26] solved the maximal L^q-L^p-regularity problem for the general case $q, p \in]1, \infty[$ [26, Theorem 2.3] by using very clever ad hoc arguments (in an operator theoretic way), they solved the special case $q = p$ [26, Theorem 2.1] in a more systematic way making use of (1.12) with a specific choice of right-inverse. The latter was later extended and worked out in more detail, in a more systematic way, for the case $q = p$, $\mu \in [0, q-1[$, $\gamma = 0$ in [73, 75, 76]. Here the approach to the trace problem under consideration is operator theoretic, making use of operators with a bounded $\mathcal{H}^\infty$-calculus, interpolation theory, and operator sums. Whereas this approach heavily relies on the equality $q = p$, we will follow a different approach, based on distribution theory and harmonic analysis, which works for the general case $q, p \in]1, \infty[$, $\mu \in]-1, q-1[$, $\gamma \in]-1, p-1[$. Our approach roughly consists of viewing intersection spaces like $\mathbb{E}$ and $\mathbb{F}$ not only as intersection spaces but also as 'concrete' spaces of X-valued distributions on $\mathbb{R}^d_+ \times \mathbb{R}_+$ and $\mathbb{R}^{d-1} \times \mathbb{R}_+$, respectively, and study these spaces via their versions on the full Euclidean spaces $\mathbb{R}^d \times \mathbb{R}$ and $\mathbb{R}^{d-1} \times \mathbb{R}$, respectively. The main advantage of this approach is the availability of tools from Euclidean harmonic analysis. In this approach, $\mathbb{E}$ and $\mathbb{F}$ are identified with so called *weighted anisotropic mixed-norm function spaces*, where *anisotropic* has to be interpreted as different 'smoothness' in the different coordinate directions, and where *mixed-norm* comes from the fact that these spaces are defined in terms of (weighted) mixed-norm Lebesgue-Bochner spaces which consist of (equivalence classes of) Lebesgue-strongly measurable functions having different integrability in the different coordinate directions (in a certain order of integration).

1.3 Outline

Let us now describe the organization of this thesis. Besides this introductory chapter, this thesis consists of five chapters and an appendix. Each of these five chapters ends with a section called "Notes", in which we provide some historical background on the subject under consideration (or at least give references doing so) and in which we give a description of the literature used. In the appendix we present some material from measure theory, Banach function spaces, distribution theory, harmonic analysis, and Banach space theory, which is used throughout the thesis. Some part of this material forms a prerequisite for some of the chapters of the thesis and some part of this material is just for convenience of reference in the main text of the thesis; this is described in the appendix itself. Before we comment on the relations between the next five chapters of the thesis, let us first describe the contents of these chapters.

The title of Chapter 2 is "Preliminaries". Besides presenting some material which is needed for the rest of the thesis, the main aim of this chapter is to give an efficient introduction to some concepts needed to understand the maximal L^q_μ-L^p_γ-regularity problem discussed in Section 1.2 (and in Section 6.1.1) and (in combination with the reference given to Section 5.1 and Section 5.2.1.a of Chapter 5) to get an idea of our approach based on weighted anisotropic mixed-norm function spaces.

In Chapter 3 we prove a generalization of the boundedness of the Hardy-Littlewood maximal function operator on the A_p-weighted Lebesgue space $L^p(\mathbb{R}^d, w)$, $p \in]1, \infty[$, (see Appendix D) to the UMD Banach function space valued setting, which is of independent interest. Here the main tools are martingale theory and the theory of mixed-norm spaces (see Appendix B.2). As a consequence of this general result we obtain several more concrete maximal and weighted norm inequalities, which are very important for Chapter 5.

In Chapter 4 we prove several anisotropic Mikhlin Fourier multiplier theorems on the weighted mixed-norm Lebesgue-Bochner spaces for operator-valued symbols. We proceed via the abstract theory of unconditional Schauder decompositions and via extrapolation theory for Caldéron-Zygmund operators.

Chapter 5 is concerned with the theory of weighted anisotropic mixed-norm Banach space-valued function spaces, with as one of the main interests trace theory. We study function spaces of Sobolev, Bessel potential, Triebel-Lizorkin, and Besov type, first on the full Euclidean space and then on the domains $\mathbb{R}^d_+ \times \mathbb{R}_+$, $\mathbb{R}^{d-1} \times \mathbb{R}_+$ and $\mathbb{R}^d_+$ via a restriction procedure. The main tools are the anisotropic Mikhlin theorem from Chapter 4 and the maximal and weighted norm inequalities from Chapter 4, where the first is used to treat Bessel potential spaces and where the second is used to treat Triebel-Lizorkin and Besov spaces.

In the final chapter, Chapter 6, we use the developed function space theory from Chapter 5 in combination with isotropic non-mixed-norm versions of the Mikhlin Fourier multiplier theorems from Chapter 4 in order to solve the maximal L^q_μ-L^p_γ-regularity problem for (1.5).

Having described the structure of this thesis and the contents of each chapter separately, for convenience of the reader we finally would like to give a brief description for each of the Chapters 2 - 5 which material will be needed in later chapters:

- *Chapter 2:* Except for a large part of Chapter 4 (see below), this chapter forms an important basis for the rest of the thesis. Here we need to remark that Section 2.1 is not needed for Chapters 3 and 4.

- *Chapter 3:* Besides Theorem 3.1.4 and Section 3.4, which are very important for Chapter 5, this chapter is independent of the rest of the thesis.

- *Chapter 4:* For applications in Chapters 5 and 6, the material in this chapter up to (and including) Proposition 4.2.4 is sufficient. In Chapter 6 we in fact even only directly use the isotropic non-mixed-norm case.

- *Chapter 5:* The results needed for direct application in Chapter 6 are all contained in Section 5.3, which is about function spaces on domains. However, Section 5.3 heavily relies on Section 5.2, which is about function spaces on the full Euclidean space; not only for the proofs of essentially all the stated results but also for the definition via restriction of Bessel potential, Triebel-Lizorkin, and Besov spaces.

Chapter 2

Preliminaries

2.1 Weighted Sobolev Spaces

2.1.1 Definitions and Basic Properties

Let X be a Banach space, let $U \subset \mathbb{R}^d$ be an open subset, let $p \in]1, \infty[$, and let $w \in A_p(\mathbb{R}^d)$. Recall that $A_p(\mathbb{R}^d)$ stands for the class of Muckenhaupt A_p-weights on $\mathbb{R}^d$, see Definition D.2.1. We define the *weighted Lebesgue-Bochner space*

$$L^p(U, w; X) := \left\{ f \in L^0(U; X) \;:\; \int_U \|f(x)\|_X^p \, w(x)\, dx < \infty \right\},$$

which becomes a Banach space when equipped with the norm

$$\|f\|_{L^p(U,w;X)} := \left(\int_U \|f\|_X^p \, w \, d\lambda_U \right)^{1/p}.$$

Note that

$$L^p(U, w; X) \hookrightarrow L^1_{loc}(\overline{U}; X) \hookrightarrow \mathcal{D}'(U; X), \tag{2.1}$$

which can be seen in the same way as (D.4). For $k \in \mathbb{N}$ we define the corresponding *weighted Sobolev space*

$$W_p^k(U, w; X) := \{ f \in \mathcal{D}'(U; X) \;:\; D^\alpha f \in L^p(U, x; X), |\alpha| \le k \},$$

which becomes a Banach space when equipped with the norm

$$\|f\|_{W_p^k(U,w;X)} := \sum_{|\alpha| \le k} \|D^\alpha f\|_{L^p(U,w;X}.$$

The following characterization of weighted Sobolev spaces on intervals will be the basis for the definition of the trace operators below (in Section 2.1.3).

Lemma 2.1.1. *Let X be a Banach space, $J =]a, b[$ with $-\infty < a < b < \infty$, $p \in]1, \infty[$, and $w \in A_p(\mathbb{R})$. Then we have*

$$W_p^1(J, w; X) \hookrightarrow C(\overline{J}; X).$$

Moreover,

$$W_p^1(J, w; X) = \left\{ f \in C(\overline{J}; X) \;:\; f(x) = f(a) + \int_a^x g(t) dt, g \in L^p(J, w; X) \right\}.$$

The proof of this lemma is analogous to the proof of the unweighted case, see for instance [46, Theorem 4.13].

A linear map $\mathscr{E} : L^1_{loc}(\overline{U}; X) \longrightarrow L^1_{loc}(\mathbb{R}^d; X)$ is called an *extension operator* from U to $\mathbb{R}^d$ if it satisfies

$$(\mathscr{E} f)_{|U} = f, \qquad f \in L^1_{loc}(\overline{U}; X).$$

An extension operator from U to $\mathbb{R}^d$ which restricts to a bounded linear operator from $W^n_p(U, w; X)$ to $W^n_p(\mathbb{R}^d, w; X)$ is very useful to derive many properties of $W^n_p(U, w; X)$ from that of $W^n_p(\mathbb{R}^d, w; X)$, the latter having the advantage of the availability of many tools from Euclidean harmonic analysis. For the half space $U = \mathbb{R}^d_+$ we have:

Lemma 2.1.2. *Let E be a Banach space and let $k \in \mathbb{N}$. Then there exists an extension operator $\mathscr{E}_{E,k} : L^1_{loc}(\overline{\mathbb{R}^d_+}; E) \longrightarrow L^1_{loc}(\mathbb{R}^d; E)$ which restricts to a bounded linear operator from $W^n_p(\mathbb{R}^d_+, w; E)$ to $W^n_p(\mathbb{R}^d, w; E)$ for each $n \in \{0, \ldots, k\}$, $p \in]1, \infty[$, and $w \in A_p(\mathbb{R}^d)$ which is symmetric with respect to reflection in $\{0\} \times \mathbb{R}^{d-1}$ (i.e. satisfying $w(x, y) = w(-x, y)$ for almost all $(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$) and for which, in case $k \geq 1$, there exist $C \in]0, \infty[$ and $\lambda \in]0, \infty[\setminus \{1\}$ such that $w(\lambda x, y) \leq C w(x, y)$ for almost all $(x, y) \in \mathbb{R} \times \mathbb{R}^{d-1}$. Moreover, if F is a Banach space with $F \hookrightarrow E$, then $\mathscr{E}_{F,k}$ is the restriction of $\mathscr{E}_{E,k}$ to $L^1_{loc}(\overline{\mathbb{R}^d_+}; F)$.*

Comments on the proof: For the proof of this lemma we can simply follow the construction in [1, Theorem 5.19], which is based on successive reflections in $\{0\} \times \mathbb{R}^{d-1}$; such an approach is standard and is also followed in [46, Theorem 4.12] and [98, Theorem 4.5.2]. We would like to remark that the existence of an extension operator $\mathscr{E}_{E,K}$ (as in the statement of the lemma) for some $K \in \mathbb{Z}_{\geq 1}$ implies the existence of $\mathscr{E}_{E,k}$ for all $k \in \{0, \ldots, K\}$; indeed, we could simply take $\mathscr{E}_{E,k} := \mathscr{E}_{E,K}$ for $k \in \{0, \ldots, K-1\}$. However, the construction of $\mathscr{E}_{E,k}$ in [1, Theorem 5.19] follows for each $k \in \mathbb{N}$ the same procedure, yielding an extension operator $\mathscr{E}_{E,k}$ which does not restrict to a bounded linear operator from $W^n_p(\mathbb{R}^d_+, w; E)$ to $W^n_p(\mathbb{R}^d, w; E)$ when $n \geq k + 1$.

Lemma 2.1.3. *Let X be a Banach space, $n \in \mathbb{N}$, $p \in]1, \infty[$, and $w \in A_p(\mathbb{R}^d)$.*

(i) $C^\infty_c(\mathbb{R}^d; X)$ is dense in $W^n_p(\mathbb{R}^d, w; X)$.

(ii) If w is symmetric with respect to reflection in $\{0\} \times \mathbb{R}^{d-1}$ (as in Lemma 2.1.2), then $C^\infty_{(c)}(\overline{\mathbb{R}^d_+}; X) = \{f_{|\mathbb{R}^d_+} : f \in C^\infty_c(\mathbb{R}^d; X)\}$ is dense in $W^n_p(\mathbb{R}^d_+, w; X)$.

Proof. Since (ii) follows directly from (i) thanks to Lemma 2.1.2, we only need to prove (i). Let $f \in W^n_p(\mathbb{R}^d, w; X)$ be given. Pick a $\phi \in C^\infty_c(\mathbb{R}^d)$ such that $\phi \geq 0$ and $\int_{\mathbb{R}^d} \phi(x) dx = 1$. In accordance with (D.2), for each $n \in \mathbb{N}$ we write $\phi_n = n^d \phi(n \cdot)$. By Proposition D.2.5 and the basic properties of the convolution product (see Proposition C.5.2), we have $(\phi_n * f)_{n \in \mathbb{N}} \subset C^\infty(\mathbb{R}^d; X) \cap W^n_p(\mathbb{R}^d, w; X)$ and $f = \lim_{n \to \infty} \phi_n * f$ in $W^n_p(\mathbb{R}^d, w; X)$. This shows that $C^\infty(\mathbb{R}^d; X) \cap W^n_p(\mathbb{R}^d, w; X)$ is dense in $W^n_p(\mathbb{R}^d, w; X)$. With a standard truncation argument, this can be improved to the denseness of $C^\infty_c(\mathbb{R}^d; X)$ in $W^n_p(\mathbb{R}^d, w; X)$; see for instance [46, Theorem 4.10]. □

2.1.2 Anisotropic Sobolev Spaces

Let us now turn to the weighted anisotropic Sobolev space of the intersection type

$$W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+, v_\mu; W^2_p(\mathbb{R}^d_+, w_\gamma; X)) \tag{2.2}$$

from Section 1.2. Viewed as a subspace of $\mathcal{D}'(\mathbb{R}_+;\mathcal{D}'(\mathbb{R}^d_+;X))$, this is just the space of all $u \in \mathcal{D}'(\mathbb{R}_+;\mathcal{D}'(\mathbb{R}^d_+;X))$ with

$$u, \partial_t u, D^\alpha_x \in L^q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \subset \mathcal{D}'(\mathbb{R}_+;\mathcal{D}'(\mathbb{R}^d_+;X)), \qquad |\alpha| \leq 2,$$

having

$$u \mapsto \|u\|_{L^q(\mathbb{R}_+,v_\mu;L^p(\mathbb{R}^d_+,w_\gamma;X))} + \|\partial_t u\|_{L^q(\mathbb{R}_+,v_\mu;L^p(\mathbb{R}^d_+,w_\gamma;X))} + \sum_{|\alpha|\leq 2} \left\|D^\alpha_x\right\|_{L^q(\mathbb{R}_+,v_\mu;L^p(\mathbb{R}^d_+,w_\gamma;X))}$$

as an equivalent norm. Under the canonical identification $\mathcal{D}'(\mathbb{R}_+;\mathcal{D}'(\mathbb{R}^d_+;X)) = \mathcal{D}'(\mathbb{R}^d_+ \times \mathbb{R}_+;X)$ from Appendix C.7, $L^q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X))$ corresponds to the *weighted mixed-norm Lebesgue-Bochner space* $L^{(p,q),(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X)$ consisting of all $f \in L^0(\mathbb{R}^d_+ \times \mathbb{R}_+; X)$ with

$$\|f\|_{L^{(p,q),(d,1)}(\mathbb{R}^d_+\times\mathbb{R}_+,(w_\gamma,v_\mu);X)} := \left(\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}^d_+} \|f(x,t)\|^p_X \, dx\right)^{q/p} dt\right)^{1/q} < \infty.$$

As a consequence we have that, under the canonical identification $\mathcal{D}'(\mathbb{R}_+;\mathcal{D}'(\mathbb{R}^d_+;X)) = \mathcal{D}'(\mathbb{R}^d_+\times \mathbb{R}_+;X)$, the intersection space (2.2) corresponds with the *weighted anisotropic mixed-norm Sobolev space* $W^{(2,1)}_{(p,q),(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X)$ consisting of all $u \in \mathcal{D}'(\mathbb{R}^d_+ \times \mathbb{R}_+; X)$ with $D^{(\alpha,\beta)}u \in L^{(p,q),(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X)$ for every $(\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}$ satisfying $\alpha = 0 \,\&\, |\beta| \leq 1$ or $|\alpha| \leq 2 \,\&\, \beta = 0$; here we equip $W^{(2,1)}_{(p,q),(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X)$ with its natural norm, which turns it into a Banach space. The same can of course be done with $\mathbb{R}_+$ replaced by $\mathbb{R}$ and/or with $\mathbb{R}^d_+$ replaced by $\mathbb{R}^d$.

Lemma 2.1.4. *The restriction operator*

$$r_{\mathbb{R}^d\times\mathbb{R},\mathbb{R}^d_+\times\mathbb{R}_+;X} \in \mathcal{L}\left(\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}; X), \mathcal{D}'(\mathbb{R}^d_+ \times \mathbb{R}_+; X)\right), u \mapsto u_{|\mathbb{R}^d_+\times\mathbb{R}_+}$$

restricts to a continuous surjection

$$W^{(2,1)}_{(p,q),(d,1)}(\mathbb{R}^d \times \mathbb{R}, (w_\gamma, v_\mu); X) \longrightarrow W^{(2,1)}_{(p,q),(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X)$$

with a continuous right-inverse $\mathscr{E}$.

Proof. It is enough to prove this statement for restriction from $\mathbb{R}^d \times \mathbb{R}$ to $\mathbb{R}^d \times \mathbb{R}_+$ and for restriction from $\mathbb{R}^d \times \mathbb{R}_+$ to $\mathbb{R}^d_+ \times \mathbb{R}_+$. Let us only do the first: Let $\mathscr{E}_{E,1}$ and $\mathscr{E}_{F,1}$ be the extension operators from Lemma 2.1.2 for the Banach spaces $E = L^p(\mathbb{R}^d, w_\gamma; X)$ and $F = W^2_p(\mathbb{R}^d, w_\gamma; X)$. Since the weight $v_\mu \in A_q(\mathbb{R})$ (1.6) is symmetric with respect to reflections in the origin, we in particular have $\mathscr{E}_{E,1} \in \mathcal{B}(W^1_q(\mathbb{R}_+, v_\mu; E), W^1_q(\mathbb{R}, v_\mu; E))$ and $\mathscr{E}_{F,1} \in \mathcal{B}(L^q(\mathbb{R}_+, v_\mu; F), L^q(\mathbb{R}, v_\mu; F))$ (after restriction). Furthermore, $\mathscr{E}_{E,1}$ extends $\mathscr{E}_{F,1}$ as $F \hookrightarrow E$. As a consequence,

$$\mathscr{E}_{E,1} \in \mathcal{B}\left(W^1_q(\mathbb{R}_+, v_\mu; E), W^1_q(\mathbb{R}, v_\mu; E)\right) \cap \mathcal{B}\left(L^q(\mathbb{R}_+, v_\mu; F), L^q(\mathbb{R}, v_\mu; F)\right),$$

from which it follows that $\mathscr{E}_{E,1}$ restricts to a bounded linear operator

$$\mathscr{E}_t : W^1_q(\mathbb{R}_+, v_\mu; E) \cap L^q(\mathbb{R}_+, v_\mu; F) \longrightarrow W^1_q(\mathbb{R}, v_\mu; E) \cap L^q(\mathbb{R}, v_\mu; F)$$

satisfying $(\mathscr{E}_t u)_{|\mathbb{R}_+} = u$. Identifying these intersection spaces with the corresponding weighted anisotropic mixed-norm Sobolev spaces, we obtain the desired result. □

2.1.3 Trace Operators

Let X be a Banach space, $p \in]1,\infty[$, and $w \in A_p(\mathbb{R})$. Recall that

$$W_p^1(\mathbb{R}_+, w; X) \hookrightarrow C([0,\infty[; X) \tag{2.3}$$

by Lemma 2.1.1. So the boundary value $f_{|\partial\mathbb{R}_+} = f_{|\{0\}}$ can just be defined in the classical sense of evaluation of continuous functions when $f \in W_p^1(\mathbb{R}_+, w; X)$, giving rise to the continuous linear operator

$$\mathrm{tr}_{\{0\}} : W_p^1(\mathbb{R}_+, w; X) \longrightarrow X,\ f \mapsto f(0).$$

This operator is called the *trace* in 0.

We would also like to give a precise meaning to the boundary value $f_{|\partial\mathbb{R}^d_+} = f_{|\{0\}\times\mathbb{R}^{d-1}}$ when $f \in W_p^1(\mathbb{R}^d_+, w; X)$ for general d, at least for the weight $w = w_\gamma$ from (1.6). But it does not hold that $W_p^1(\mathbb{R}^d_+, w; X) \subset C(\overline{\mathbb{R}^d_+}; X)$ when $d > 1$. However, the concept of boundary value can be introduced in the sense of traces:

Lemma 2.1.5. *Let X be a Banach space, $p \in]1,\infty[$, $w_+ \in A_p(\mathbb{R})$, and $w' \in A_p(\mathbb{R}^{d-1})$. Let $w := w_+ \otimes w' \in A_p(\mathbb{R}^d)$, i.e. $w(x) := w_+(x_1)w'(x_2,\dots,x_d)$ for $x = (x_1,\dots,x_d) \in \mathbb{R}^d$. Then we have*

$$W_p^1(\mathbb{R}^d_+, w; X) \hookrightarrow W^1(\mathbb{R}^d_+, w_+; L^p(\mathbb{R}^{d-1}, w'; X)) \hookrightarrow C([0,\infty[; L^p(\mathbb{R}^{d-1}, w'; X)) \tag{2.4}$$

under the canonical identification $\mathcal{D}'(\mathbb{R}^d_+; X) = \mathcal{D}'(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}^{d-1}; X))$ from Appendix C.7. Moreover, the induced continuous linear operator

$$\mathrm{tr}_{\{0\}\times\mathbb{R}^{d-1}} : W_p^1(\mathbb{R}^d_+, w; X) \hookrightarrow C([0,\infty[; L^p(\mathbb{R}^{d-1}, w'; X)) \longrightarrow L^p(\mathbb{R}^{d-1}, w'; X),\ f \mapsto f(0), \tag{2.5}$$

is on the dense subspace $C(\overline{\mathbb{R}^d_+}; X) \cap W_p^1(\mathbb{R}^d_+, w; X)$ of $W_p^1(\mathbb{R}^d_+, w; X)$ just given by restriction with respect to the hyperplane $\{0\}\times\mathbb{R}^{d-1}$ (which we identify with $\mathbb{R}^{d-1}$). This operator is called the trace operator *on $W_p^1(\mathbb{R}^d_+, w; X)$ with respect to $\{0\}\times\mathbb{R}^{d-1}$.*

Proof. We only need to show the last assertion, the first inclusion in (2.4) being trivial and the second inclusion in (2.4) being a special case of Lemma 2.1.1. Viewing $L^p(\mathbb{R}^{d-1}, w'; X)$ as a linear subspace of $\mathcal{D}'(\mathbb{R}^{d-1}; X)$ and accordingly viewing $W_p^1(\mathbb{R}^d_+, w; X)$ as a linear subspace of $C([0,\infty[; \mathcal{D}'(\mathbb{R}^{d-1}; X))$, the operator $\mathrm{tr} = \mathrm{tr}_{\{0\}\times\mathbb{R}^{d-1}}$ is obtained by restricting the evaluation in 0 map

$$\mathrm{ev}_0 : C([0,\infty[; \mathcal{D}'(\mathbb{R}^{d-1}; X)) \longrightarrow \mathcal{D}'(\mathbb{R}^{d-1}; X), f \mapsto f(0) \tag{2.6}$$

to the subspace $W_p^1(\mathbb{R}^d_+, w; X)$. On the other hand, viewing $C(\mathbb{R}^{d-1}; X)$ as a linear subspace of $\mathcal{D}'(\mathbb{R}^{d-1}; X)$ and accordingly viewing $C(\overline{\mathbb{R}^d_+}; X)$ as the linear subspace $C([0,\infty[; C(\mathbb{R}^{d-1}; X))$ of $C([0,\infty[; \mathcal{D}'(\mathbb{R}^{d-1}; X))$, the restriction map

$$C(\overline{\mathbb{R}^d_+}; X) \longrightarrow C(\mathbb{R}^{d-1}; X),\ f \mapsto f_{|\{0\}\times\mathbb{R}^{d-1}},$$

is obtained by restricting the evaluation in 0 map ev_0 (2.6) to the subspace $C(\overline{\mathbb{R}^d_+}; X)$. Hence we obtain the desired result. □

In the notation from Section 1.2, the above applies to the weights $v_\mu \in A_q(\mathbb{R})$ and $w_\gamma \in A_p(\mathbb{R}^d)$ (1.6); see Example D.2.12. Given a Banach space E, we can thus define the trace operators $\mathrm{tr}_{t=0}$ (1.7) and $\mathrm{tr}_{y=0}$ (1.8) from Section 1.2. Taking $E = L^p(\mathbb{R}^{d-1};X)$ and $E = X$, respectively, these two trace operators induce (in the natural way) the operators $\mathrm{tr}_{t=0}$ (1.9) and $\mathrm{tr}_{y=0}$ (1.10), which are needed in the formulation of the maximal L^q_μ-L^p_γ-regularity problem for (1.5).

Just as in the proof of Lemma 2.1.5 concerning the trace operator (2.5), the just defined trace operators $\mathrm{tr}_{t=0}$ (1.9) and $\mathrm{tr}_{y=0}$ (1.10) can also naturally be viewed as restrictions of 'distributional' trace operators. Indeed, concerning trace operator $\mathrm{tr}_{t=0}$ (1.9), we have

$$\begin{aligned} W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+, v_\mu; W^2_p(\mathbb{R}^d_+, w_\gamma; X)) &\hookrightarrow W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \\ \hookrightarrow C([0,\infty[; L^p(\mathbb{R}^d_+, w_\gamma; X)) &\hookrightarrow C([0,\infty[; \mathcal{D}'(\mathbb{R}^d_+; X)), \end{aligned}$$

$$L^p(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow \mathcal{D}'(\mathbb{R}^d_+; X),$$

and $\mathrm{tr}_{t=0}$ (1.9) is accordingly the restriction of the evaluation in 0 map

$$C([0,\infty[; \mathcal{D}'(\mathbb{R}^d_+; X)) \longrightarrow \mathcal{D}'(\mathbb{R}^d_+; X),\ f \mapsto f(0), \tag{2.7}$$

where the intersection space (2.3) and $C([0,\infty[; \mathcal{D}'(\mathbb{R}^d_+; X))$ may also be viewed as subspaces of $\mathcal{D}'(\mathbb{R}^d_+ \times \mathbb{R}_+; X)$ via the canonical identification $\mathcal{D}'(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}^d_+; X)) = \mathcal{D}'(\mathbb{R}^d_+ \times \mathbb{R}_+; X)$; also see Section 2.1.2. Furthermore, concerning the trace operator $\mathrm{tr}_{y=0}$ (1.10), we have

$$\begin{aligned} W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+, v_\mu; W^2_p(\mathbb{R}^d_+, w_\gamma; X)) &\hookrightarrow L^q(\mathbb{R}_+, v_\mu; W^2_p(\mathbb{R}_+ \times \mathbb{R}^{d-1}, w_\gamma; X)) \\ \hookrightarrow L^q(\mathbb{R}, v_\mu; W^1_p(\mathbb{R}, |\cdot|^\gamma; L^p(\mathbb{R}^{d-1}; X))) &\hookrightarrow \mathcal{D}'(\mathbb{R}_+; C([0,\infty[; \mathcal{D}'(\mathbb{R}^{d-1}; X))) \\ &\overset{!}{=} C([0,\infty[; \mathcal{D}'(\mathbb{R}^{d-1} \times \mathbb{R}_+; X)), \end{aligned}$$

$$L^q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^{d-1}; X)) \hookrightarrow \mathcal{D}'(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}^{d-1}; X)) = \mathcal{D}'(\mathbb{R}^{d-1} \times \mathbb{R}_+; X),$$

and $\mathrm{tr}_{y=0}$ is accordingly the restriction of the evaluation in 0 map

$$C([0,\infty[; \mathcal{D}'(\mathbb{R}^{d-1} \times \mathbb{R}_+; X)) \longrightarrow \mathcal{D}'(\mathbb{R}^{d-1} \times \mathbb{R}_+; X),\ f \mapsto f(0); \tag{2.8}$$

here we again did the usual identifications as in Appendix C.7, where the identification in '$\overset{!}{=}$' is based on a combination of Lemma C.7.4 and the canonical isomorphism. This motivates us to define the distributional trace operators

$$\mathrm{tr}_{t=0} : D(\mathrm{tr}_{t=0}) \subset \mathcal{D}'(\mathbb{R}^d_+ \times \mathbb{R}_+) \longrightarrow \mathcal{D}'(\mathbb{R}^d_+; X)$$

and

$$\mathrm{tr}_{t=0} : D(\mathrm{tr}_{y=0}) \subset \mathcal{D}'(\mathbb{R}^d_+ \times \mathbb{R}_+) \longrightarrow \mathcal{D}'(\mathbb{R}^{d-1} \times \mathbb{R}_+; X)$$

simply as the mappings (2.7) and (2.8), respectively.

In Chapter 5 we will determine the trace spaces of $\mathrm{tr}_{t=0}$ (1.9) and $\mathrm{tr}_{y=0}$ (1.10) by making use of weighted anisotropic Triebel-Lizorkin spaces; see Section 5.1.

2.2 Weighted Mixed-Norm Lebesgue-Bochner Spaces

Convention 2.2.1 (*$\mathpzc{d}$-decomposition* of $\mathbb{R}^d$.)**.** Let $d = |\mathpzc{d}|_1 = \mathpzc{d}_1 + \ldots + \mathpzc{d}_l$ with $\mathpzc{d} = (\mathpzc{d}_1, \ldots, \mathpzc{d}_l) \in (\mathbb{Z}_{\geq 1})^l$. The decomposition

$$\mathbb{R}^d = \mathbb{R}^{\mathpzc{d}_1} \times \ldots \times \mathbb{R}^{\mathpzc{d}_l}.$$

is called the *$\mathpzc{d}$-decomposition* of $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ we accordingly write $x = (x_1, \ldots, x_l)$ and $x_j = (x_{j,1}, \ldots, x_{j,\mathpzc{d}_j})$, where $x_j \in \mathbb{R}^{\mathpzc{d}_j}$ and $x_{j,i} \in \mathbb{R}$ ($j = 1, \ldots, l; i = 1, \ldots, \mathpzc{d}_j$). We also say that we view $\mathbb{R}^d$ as being *$\mathpzc{d}$-decomposed*. Furthermore, for each $k \in \{1, \ldots, l\}$ we define the inclusion map

$$\iota_k = \iota_{[\mathpzc{d};k]} : \mathbb{R}^{\mathpzc{d}_k} \longrightarrow \mathbb{R}^d,\ x_k \mapsto (0, \ldots, 0, x_k, 0, \ldots, 0), \tag{2.9}$$

and the projection map

$$\pi_k = \pi_{[\mathpzc{d};k]} : \mathbb{R}^d \longrightarrow \mathbb{R}^{\mathpzc{d}_k},\ x = (x_1, \ldots, x_l) \mapsto x_k. \tag{2.10}$$

Definition 2.2.2. Suppose that $\mathbb{R}^d$ is $\mathpzc{d}$-decomposed as above. Let X be a Banach space, $U_j \subset \mathbb{R}^{\mathpzc{d}_j}$, $j = 1, \ldots, l$, open subsets, $p = (p_1, \ldots, p_l) \in [1, \infty[^l$, and $w = (w_1, \ldots, w_l) \in \prod_{j=1}^l W(\mathbb{R}^{\mathpzc{d}_j})$. We define the *weighted mixed-norm Lebesgue-Bochner space* $L^{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w; X)$ as the space of all $f \in L^0(U_1 \times \ldots \times U_l; X)$ satisfying

$$\|f\|_{L^{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w; X)} := \left(\int_{U_l} \ldots \left(\int_{U_2} \left(\int_{U_1} \|f(x)\|_X^{p_1}\, w_1(x_1) dx_1\right)^{p_2/p_1} w_2(x_2) dx_2\right)^{p_3/p_2} \ldots w_l(x_l) dx_l\right)^{1/p_l} < \infty.$$

We equip $L^{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w; X)$ with the norm $\|\cdot\|_{L^{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w; X)}$, which turns it into a Banach space. For $X = \mathbb{K}$ we simply write $L^{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w) = L^{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w; \mathbb{K})$.

We would like to remark that $L^{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w; X)$ is just the Köthe-Bochner space $E(X)$ associated with the mixed-norm Banach function space $E = L^{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w)$ on $U_1 \times \ldots \times U_l$ and the Banach space X; see Appendix B for the notions of Banach function space and Köthe-Bochner space. Furthermore, there is a canonical isometric isomorphism

$$L^{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w; X) \cong L^{p_l}(U_l, w_l; \ldots L^{p_1}(U_1, w_1; X) \ldots),$$

where $L^{p_l}(U_l, w_l; \ldots L^{p_1}(U_1, w_1; X) \ldots)$ stands for the iterated weighted Lebesgue-Bochner space.

Lemma 2.2.3. *Suppose that $\mathbb{R}^d$ is $\mathpzc{d}$-decomposed as above. Let X be a Banach space, $p \in [1, \infty[^l$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{\mathpzc{d}_j})$. Then*

$$\mathcal{S}(\mathbb{R}^d; X) \overset{d}{\hookrightarrow} L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X).$$

Moreover, there in fact exists an $L \in \mathbb{N}$ and a constant $C > 0$ such that

$$\|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)} \leq C \left\| \prod_{j=1}^l (1 + |\pi_{[\mathpzc{d};j]}(\,\cdot\,)|^2)^L f \right\|_{L^\infty(\mathbb{R}^d; X)} \tag{2.11}$$

for all strongly measurable functions $f : \mathbb{R}^d \longrightarrow X$.

Proof. The inequality (2.11), thereby the continuous inclusion part of this lemma, follows easily from l applications of Corollary D.2.8. Finally, the density follows from the denseness of $L^{p_l}(\mathbb{R}^{d_l}, w_l) \otimes \ldots \otimes L^{p_1,w_1}(\mathbb{R}^{d_1}) \otimes X$ in $L^{p,d}(\mathbb{R}^d, w; X) \cong L^{p_l}(\mathbb{R}^{d_l}, w_l; \ldots L^{p_1}(\mathbb{R}^{d_1}, w_1; X) \ldots)$ and Lemma D.2.6. □

Lemma 2.2.4. *Suppose that $\mathbb{R}^d$ is $\mathpzc{d}$-decomposed as above. Let X be a Banach space, $p \in]1, \infty[^l$, and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$. Then it holds that*

$$\mathcal{S}(\mathbb{R}^d; X) \overset{d}{\hookrightarrow} L^{p,d}(\mathbb{R}^d, w; X) \hookrightarrow L^1_{loc}(\mathbb{R}^d; X), \mathcal{S}'(\mathbb{R}^d, w; X).$$

Proof. By Lemma 2.2.3, we only need to show the second inclusion. Denoting by $p' = (p'_1, \ldots, p'_l) \in]1, \infty[^l$ the vector of Hölder conjugates and by $w' = (w_1^{-\frac{1}{p_1-1}}, \ldots, w_l^{-\frac{1}{p_l-1}}) \in \prod_{j=1}^{l} A_{p'_j}(\mathbb{R}^{d_j})$ the vector of associated dual weights, we have $\mathcal{S}(\mathbb{R}^d) \hookrightarrow L^{p',d}(\mathbb{R}^d, w')$ by Lemma 2.2.3. Since

$$L^{p',d}(\mathbb{R}^d, w') \times L^{p,d}(\mathbb{R}^d, w; X) \longrightarrow X, \, (\phi, f) \mapsto \int_{\mathbb{R}^d} \phi f \, dx$$

is a well-defined bounded bilinear mapping, we thus obtain the continuous inclusions

$$L^{p,d}(\mathbb{R}^d, w; X) \hookrightarrow \mathcal{B}(L^{p',d}(\mathbb{R}^d, w'), X) = \mathcal{L}(L^{p',d}(\mathbb{R}^d, w'), X) \hookrightarrow \mathcal{L}(\mathcal{S}(\mathbb{R}^d), X) = \mathcal{S}'(\mathbb{R}^d; X).$$

Finally, the denseness of $\mathcal{S}(\mathbb{R}^d; X)$ in $L^{p,d}(\mathbb{R}^d, w; X)$ follows from Lemma 2.2.3. □

2.3 Anisotropic Distance Functions

Throughout this section we suppose that $\mathbb{R}^d$ is $\mathpzc{d}$-decomposed as in Convention 2.2.1. In our study of anisotropic function spaces in Chapter 5 we will need the notions of anisotropic dilation and anisotropic distance function (with respect to this $\mathpzc{d}$-decomposition); see for example Section 5.2.1.a for anisotropic Bessel potential spaces.

Given $a \in]0, \infty[^l$, we define the $(\mathpzc{d}, a)$-anisotropic dilation $\delta_\lambda^{[\mathpzc{d},a]}$ on $\mathbb{R}^d$ by $\lambda > 0$ to be the mapping $\delta_\lambda^{[\mathpzc{d},a]}$ on $\mathbb{R}^d$ given by the formula

$$\delta_\lambda^{[\mathpzc{d},a]} x := (\lambda^{a_1} x_1, \ldots, \lambda^{a_l} x_l), \qquad x \in \mathbb{R}^d. \tag{2.12}$$

We shall furthermore frequently use the notation

$$b \cdot_{\mathpzc{d}} y := \sum_{j=1}^{l} \sum_{i=1}^{d_j} b_j y_{j,i}, \qquad b \in \mathbb{R}^l, y \in \mathbb{R}^d; \tag{2.13}$$

mostly with $b = a \in]0, \infty[^l$.

Definition 2.3.1. A $(\mathpzc{d}, a)$-anisotropic distance function on $\mathbb{R}^d$ is a function $u : \mathbb{R}^d \longrightarrow [0, \infty[$ satisfying

(i) $u(x) = 0$ if and only if $x = 0$.

(ii) $u(\delta_\lambda^{[\mathpzc{d},a]} x) = \lambda u(x)$ for all $x \in \mathbb{R}^d$ and $\lambda > 0$.

(iii) There exists a $c > 0$ such that $u(x+y) \leq c(u(x)+u(y))$ for all $x, y \in \mathbb{R}^d$.

In this thesis we will mainly use two $(\vec{d}, a)$-anisotropic distance functions on $\mathbb{R}^d$, namely $\rho_{\vec{d},a}$ and $|\cdot|_{\vec{d},a}$, to be defined below. The advantage of $\rho_{\vec{d},a}$ is that it allows us to define $(\vec{d}, a)$-anisotropic polar coordinates and the advantage of $|\cdot|_{\vec{d},a}$ is that it is explicitly given by a formula, so that it is (in most situations) more suitable for doing computations and estimations.

The $(\vec{d}, a)$-anisotropic distance function $|\cdot|_{\vec{d},a} : \mathbb{R}^d \longrightarrow [0, \infty[$ is given by the formula

$$|x|_{\vec{d},a} := \left(\sum_{j=1}^{l} |x_j|^{2/a_j} \right)^{1/2} \qquad (x \in \mathbb{R}^d), \tag{2.14}$$

and the $(\vec{d}, a)$-anisotropic distance function $\rho_{\vec{d},a} : \mathbb{R}^d \longrightarrow [0, \infty[$ is defined as follows: For $x \in \mathbb{R}^d \setminus \{0\}$ we define $\rho_{\vec{d},a}(x)$ to be the unique number $\rho_{\vec{d},a}(x) = \lambda > 0$ for which we have $\delta^{[\vec{d},a]}_{\lambda^{-1}} x \in S^{d-1}$, and we put $\rho_{\vec{d},a}(0) := 0$. Observe that $\rho_{\vec{d},a}(x) = 1$ if and only if $x \in S^{d-1}$.

Via the $(\vec{d}, a)$-anisotropic distance function $\rho_{\vec{d},a}$ we can define $(\vec{d}, a)$-anisotropic polar coordinates on $\mathbb{R}^d \setminus \{0\}$: For every $x \in \mathbb{R}^d \setminus \{0\}$ there is a unique $(u, \lambda) \in S^{d-1} \times \mathbb{R}_{>0}$ so that $x = \delta^{[\vec{d},a]}_{\lambda} u$; just take $\lambda = \rho_{\vec{d},a}(x)$ and $u = \delta^{[\vec{d},a]}_{\rho_{\vec{d},a}(x)^{-1}} x$.

For the coordinate transformation $\mathbb{R}^d \setminus \{0\} \ni x \mapsto (u, \lambda)$ (the change to $(\vec{d}, a)$-anisotropic polar coordinates), the following associated change-of-variables formula for integration can be obtained via a standard calculus computation (which we omit):

$$dx = \lambda^{\vec{d} \cdot a - 1} \sum_{j=1}^{l} a_j |u_j|^2 d\sigma(u) d\lambda, \tag{2.15}$$

where $d\sigma$ is the surface measure on S^{d-1}.

It is not difficult to show that the two $(\vec{d}, a)$-anisotropic distance functions $\rho_{\vec{d},a}$ and $|\cdot|_{\vec{d},a}$ are equivalent. Similar to the fact that all norms on $\mathbb{R}^d$ are equivalent, it can in fact be shown that:

Lemma 2.3.2. *All $(\vec{d}, a)$-anisotropic distance functions on $\mathbb{R}^d$ are equivalent: Given two $(\vec{d}, a)$-anisotropic distance functions u and v on $\mathbb{R}^d$, there exist constants $m, M > 0$ such that $mu(x) \leq v(x) \leq Mu(x)$ for all $x \in \mathbb{R}^d$*

The statement of the above lemma is of course equivalent with the statement that every $(\vec{d}, a)$-anisotropic distance function on $\mathbb{R}^d$ is equivalent with $|\cdot|_{\vec{d},a}$ (which is the way to proceed in the proof of this lemma). Using an argument based on compactness, it is not difficult to see that a function $u : \mathbb{R}^d \longrightarrow [0, \infty[$, which satisfies (i) and (ii) of Definition 2.3.1, is continuous if and only if it is equivalent with $|\cdot|_{\vec{d},a}$. As a consequence, the $(\vec{d}, a)$-anisotropic distance functions on $\mathbb{R}^d$ are precisely the continuous functions $u : \mathbb{R}^d \longrightarrow [0, \infty[$ satisfying (i) and (ii) of Definition 2.3.1; also see [29] and [104].

Chapter 3

Maximal and Weighted Norm Inequalities

In this chapter we prove the boundedness (and well-definedness) of the Hardy-Littlewood maximal function operator taking values in a UMD Banach function space, which is defined by taking the supremumum in the order of the Banach function space, in the corresponding A_p-weighted L^p-Bochner spaces ($p \in]1,\infty[$). Reformulating this result in terms of mixed-norm spaces, it is a small step to obtain the boundedness of partial Hardy-Littlewood maximal function operators, which are defined by taking the Hardy-Littlewood maximal operator in the separate variables, in the weighted mixed-norm Lebesgue spaces $L^{p,d}(\mathbb{R}^d, w)$ plus corresponding Fefferman-Stein inequalities. In Section 3.1 we state the main results and in Section 3.4 we collect several important consequences.

3.1 Introduction

Recall from Appendix D.2 that the Hardy-Littlewood maximal function operator M (D.1) is bounded on the weighted space $L^p(\mathbb{R}^n, w)$, where $w \in \mathcal{W}(\mathbb{R}^n)$ and $p \in]1,\infty[$, if and only if $w \in A_p(\mathbb{R}^n)$. Here the reverse implication, i.e. the sufficiency of the A_p-condition for the boundedness of M on $L^p(\mathbb{R}^n, w)$, can be interpreted as follows: *Let $p \in]1,\infty[$ and $w \in A_p(\mathbb{R}^n)$ be given. For each $f \in L^p(\mathbb{R}^n, w; \mathbb{K})$ we have:*

(i) for almost all $x \in \mathbb{R}^n$, the supremum

$$M(f)(x) = \sup_{\delta>0} \fint_{B(x,\delta)} |f(y)|dy = \sup_{\delta>0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)|dy$$

exists in $\mathbb{K}$ (or in $\mathbb{K}^+$);

(ii) the function $\mathbb{R}^n \ni x \mapsto M(f)(x) \in \mathbb{K}$, which is almost everywhere well-defined, defines an element of $L^p(\mathbb{R}^n, w; \mathbb{K})$.

Moreover, the resulting sublinear operator

$$M : L^p(\mathbb{R}^n, w; \mathbb{K}) \longrightarrow L^p(\mathbb{R}^n, w; \mathbb{K})\, f \mapsto M(f),$$

is bounded.

It is a natural question whether the above remains valid when we replace the Banach lattice $\mathbb{K}$ by a general Banach lattice F; note here that $\left\{\fint_{B(x,\delta)} |f(y)|dy \,:\, \delta > 0\right\} \subset F^+$ because $|f| \in$

$L^p(\mathbb{R}^n, w; F)^+ \subset L^1_{loc}(\mathbb{R}^n; F)^+$ (whenever $f \in L^p(\mathbb{R}^n, w; F)$). In this chapter we will give a positive answer in the case that F is a UMD Banach function space on a σ-finite measure space (from which the case of a general Banach lattice can be derived, see Remark 3.1.2.(v)). This is one of the main results of this chapter and is stated as the next theorem:

Theorem 3.1.1. *Let F be a UMD Banach function space on a σ-finite measure space, let $p \in]1, \infty[$, and let $w \in A_p(\mathbb{R}^n)$. Then the formula*

$$M(f)(x) := \sup_{\delta>0} \fint_{B(x,\delta)} |f(y)|\, dy, \qquad x \in \mathbb{R}^n,$$

*gives rise to a well-defined bounded sublinear operator M on $L^p(\mathbb{R}^n, w; F)$, where the well-definedness of M on $L^p(\mathbb{R}^n, w; F)$ means that, for each $f \in L^p(\mathbb{R}^n, w; F)$, (i) and (ii) above hold with $\mathbb{K}$ replaced by F. Moreover, the implicit constant in $\|M(f)\|_{L^p(\mathbb{R}^n,w;F)} \lesssim \|f\|_{L^p(\mathbb{R}^n,w;F)}$ only depends on n, p, w and the UMD-constant $\beta_F = \beta_{2,F}$ of F in an A_p-*consistent *way, i.e. there exists an increasing function $C_{n,p,\beta_F} : [1, \infty[\longrightarrow [1, \infty[$ (only depending on n, p and β_F) such that $\|M(f)\|_{L^p(\mathbb{R}^n,w;F)} \leq C_{p,\beta_F}([w]_{A_p}) \|f\|_{L^p(\mathbb{R}^n,w;F)}$ for all $f \in L^p(\mathbb{R}^n, w; F)$.*

The statement of this theorem also makes sense for general Banach lattices. Although Banach function spaces are enough for our purposes, for the interested reader we give several reformulations (depending on certain properties of the Banach lattice under consideration) of this statement, allowing us to derive the case of general UMD Banach lattices from the case of UMD Banach function spaces:

Remark 3.1.2. Let E be a Banach lattice.

(i) Let $f \in L^1_{loc}(\mathbb{R}^n; E)$ and $x \in \mathbb{R}^n$ be given. Then the two sets

$$\left\{ \fint_{B(x,\delta)} |f(y)| dy \,:\, \delta \in \mathbb{Q}_+ \right\}, \left\{ \fint_{B(x,\delta)} |f(y)| dy \,:\, \delta > 0 \right\} \subset E^+ \tag{3.1}$$

have the same upper bounds in the Banach lattice E (as can be seen similarly to the last part of the proof of Lemma 3.2.1). Therefore, the first set has a supremum in E if and only if the second set has a supremum in E, in which case the two suprema coincide.

(ii) Let $f \in L^1_{loc}(\mathbb{R}^n; E)$ be such that, for almost every $x \in \mathbb{R}^n$, the first set (or equivalently the second set) in (3.1) has a supremum $M(f)(x)$ in E. Furthermore, suppose that the function $\mathbb{R}^n \ni x \mapsto M(f)(x) \in E$, which is almost everywhere well-defined, defines an element of $L^0(\mathbb{R}^n; E)$. Then $M(f)$ is the supremum of both of the sets

$$\left\{ x \mapsto \fint_{B(x,\delta)} |f(y)| dy \,:\, \delta \in \mathbb{Q}_+ \right\}, \left\{ x \mapsto \fint_{B(x,\delta)} |f(y)| dy \,:\, \delta > 0 \right\} \subset L^0(\mathbb{R}^n; E)^+ \tag{3.2}$$

in $L^0(\mathbb{R}^n; E)$.

(iii) Suppose that E has an order continuous norm, or equivalently that E is σ-Dedekind complete and has a σ-order continuous norm. Given $f \in L^1_{loc}(\mathbb{R}^n; E)$, the following are equivalent:

(a) For almost every $x \in \mathbb{R}^n$, the first set (or equivalently the second set) in (3.1) has a supremum $M(f)(x)$ in E.

(b) The first set in (3.2) a supremum $M(f)$ in $L^0(\mathbb{R}^n; E)$.

(c) The second set in (3.2) a supremum $M(f)$ in $L^0(\mathbb{R}^n; E)$.

Moreover, $x \mapsto M(f)(x)$ is the same mapping in (a)/(b)/(c).

For the implications (a)⇒(b)/(c) it enough to show that the function $\mathbb{R}^n \ni x \mapsto M(f)(x) \in E^+$, which is almost everywhere well-defined, defines an element of $L^0(\mathbb{R}^n; E)$ (see (ii)), which can be done using the σ-order continuity of E[1], whereas the reverse implications (b)/(c)⇒(a) follow from the σ-Dedekind completes of E.

(iv) Suppose that E is a Kantorovich-Banach space (a KB-space); so E in particularly satisfies the hypotheses of (iii). Let $p \in]1,\infty[$, $w \in A_p(\mathbb{R}^n)$, and $f \in L^p(\mathbb{R}^n, w; E)$. Then $L^p(\mathbb{R}^n, w; E)$ is a KB-space, so that (a)/(b)/(c) from (iii) hold with $M(f) \in L^p(\mathbb{R}^n, w; E)$, if and only if,

$$\left\{ M_J(f) := \sup_{\delta \in J} \left[x \mapsto \fint_{B(x,\delta)} |f(y)| dy \right] \,:\, J \subset \mathbb{R}_+ \text{ finite} \right\} \subset L^0(\mathbb{R}^n; E)^+$$

is a norm bounded set in $L^p(\mathbb{R}^n, w; E)$, if and only if,

$$\left\{ M_J(f) = \sup_{\delta \in J} \left[x \mapsto \fint_{B(x,\delta)} |f(y)| dy \right] \,:\, J \subset \mathbb{Q}_+ \text{ finite} \right\} \subset L^0(\mathbb{R}^n; E)^+ \tag{3.3}$$

is a norm bounded set in $L^p(\mathbb{R}^n, w; E)$, in which case $M(f)$ is the supremum of each of these two sets in $L^p(\mathbb{R}^n, w; E)$.

(v) Suppose that E is a UMD Banach lattice (so E is in particularly a KB-space). Let $p \in]1,\infty[$ and $w \in A_p(\mathbb{R}^n)$. Then there exists a constant $C \geq 0$, depending on E, n, p and w in the same way as in Theorem 3.1.1, such that, for every $f \in L^p(\mathbb{R}^n, w; F)$, the equivalent conditions of (iv) are satisfied with the norm estimate $\|M(f)\|_{L^p(\mathbb{R}^n,w;E)} \leq C \|f\|_{L^p(\mathbb{R}^n,w;E}$, which can be seen as follows: Given $f \in L^p(\mathbb{R}^n, w; E)$, it can be shown that there exists a separable closed Riesz subspace E_0 of E in which f takes its values almost everywhere (so we may view f as an element of $L^p(\mathbb{R}^n, w; E_0)$). By [67, Theorem 1.b.14] there exists a Banach function space F on some probability space which is isometrically isomorphic with E_0 as Banach lattices. But then F is a UMD Banach function space on a probability space with UMD-constant $\beta_{2,F} = \beta_{2,E_0} \leq \beta_{2,E}$, so that the desired result now easily follows from Theorem 3.1.1 (reformulated as in (iii)/(iv)).

We reduce Theorem 3.1.1 to a martingale theoretic problem (which is in fact equivalent to the original problem), which we solve by using the UMD property of F in combination with the theory of mixed-norm spaces from Appendix B.2 (mainly via the canonical isomorphism $L^p(\mathbb{R}^n, w; F) \simeq L^p(\mathbb{R}^n, w)[F]$).

Corollary 3.1.3. *Let F be a Banach function space on a σ-finite measure space, $p \in [1,\infty[$, and $w \in A_\infty(\mathbb{R}^n)$. Suppose that $r \in]0,\infty[$ is such that F^r, as defined in* (B.1)*, is a Banach function space*[2] *with the UMD property and such that $w \in A_{p/r}(\mathbb{R}^n)$. Then the formula*

$$M_r(f)(x) := \sup_{\delta>0} \left(\fint_{B(x,\delta)} |f(y)|^r \, dy \right)^{1/r}, \qquad x \in \mathbb{R}^n,$$

gives rise to a well-defined bounded sublinear operator M_r on $L^p(\mathbb{R}^n, w; F)$.

[1] Instead of the σ-order continuity of E, here we could also assume that E is a separable σ-normal Banach lattice (e.g. a separable Banach function space on a σ-finite measure space); we just have to use the Pettis measurability theorem. We call a Riesz space σ-normal if its σ-order continuous dual is point separating.

[2] F^r is always a Banach function space for $r \in]0,1]$.

Reformulating this corollary in terms of mixed-norm spaces, it is a small step to obtain the next theorem, which (together with its consequences in Section 3.4) will be one of the crucial ingredients in Chapter 5.

Theorem 3.1.4. *Suppose that* $\mathbb{R}^d$ *is* $\mathpzc{d}$*-decomposed as in Convention 2.2.1 and let* $a \in]0,\infty[^l$ *and* $w \in \prod_{j=1}^{l} \mathcal{W}(\mathbb{R}^{d_j})$*. Let* $j_0 \in \{1,\dots,l\}$ *and* $r_{j_0} \in]0, \min\{p_{j_0},\dots,p_l\}[$ *be such that* $w_{j_0} \in A_{p_{j_0}/r_{j_0}}(\mathbb{R}^{d_{j_0}})$*. Then the formula*

$$M_{[\mathpzc{d};j_0],r_{j_0}}(f)(x) := \sup_{\delta>0}\left(\fint_{B(x_{j_0},\delta)} |f(x_1,\dots,x_{j_0-1},y,x_{j_0+1},\dots,x_l)|^{r_{j_0}}\,dy\right)^{1/r_{j_0}}, \qquad x \in \mathbb{R}^d,$$

gives rise to a well-defined bounded sublinear operator $M_{[\mathpzc{d};j_0],r_{j_0}}$ *on* $L^{p,\mathpzc{d}}(\mathbb{R}^d,w)$*. Moreover, there holds a Fefferman-Stein inequality for* $M_{[\mathpzc{d};j_0],r_{j_0}}$*: for every* $q \in]\max\{1,r\},\infty]$ *there exists a constant* $C \in]0,\infty[$ *such that, for all sequences* $(f_{i\in I})_{i\in\mathbb{Z}} \subset L^{p,\mathpzc{d}}(\mathbb{R}^d,w)$*,*

$$\left\|\left\|(M_{[\mathpzc{d};j_0],r_{j_0}}(f_i))_{i\in\mathbb{Z}}\right\|_{\ell^q(I)}\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)} \leq C\left\|\|(f_i)_{i\in\mathbb{Z}}\|_{\ell^q(\mathbb{Z})}\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)}.$$

We shall write $M_{[\mathpzc{d};j_0]} := M_{[\mathpzc{d};j_0],1}$.

Corollary 3.1.5. *Suppose that* $\mathbb{R}^d$ *is* $\mathpzc{d}$*-decomposed as in Convention 2.2.1, and let* $p \in]1,\infty[^l$ *and* $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$*. Then*

$$M^{[\mathpzc{d},a]}(f)(x) := \sup_{\delta>0}\fint_{|y-x|_{\mathpzc{d},a}<\delta} |f(y)|\,dy, \qquad x \in \mathbb{R}^d,$$

defines a bounded sublinear operator $M^{[\mathpzc{d},a]}$ *on* $L^{p,\mathpzc{d}}(\mathbb{R}^d,w)$*.*

3.2 Maximal Funtions on Mixed-Norm Spaces

Let us first focus on Theorem 3.1.1. To this end, let F be a UMD Banach function space on the σ-finite measure space $(T,\mathscr{B},\nu)$ and let $w \in A_p(\mathbb{R}^n)$, $p \in]1,\infty[$. Then, being a UMD space, F is reflexive and thus has a σ-order continuous norm (see Propositions E.5.5 and B.1.8). So, by Theorem B.2.7, we have a canonical isometric isomorphism $L^p(\mathbb{R}^d,w;F) \cong L^p(\mathbb{R}^n,w)[F]$ of Banach lattices. Moreover, if $f \in L^p(\mathbb{R}^n,w;F)$ corresponds to $\tilde{f} \in L^p(\mathbb{R}^d,w)[F]$ under this isomorphism, then we have, for ν-a.a. $t \in T$,

$$\left(\fint_{B(x,\delta)} |f|\,d\lambda\right)(t) = \fint_{B(x,\delta)} |\tilde{f}(y,t)|\,dy, \qquad \delta > 0; \tag{3.4}$$

see Corollary B.2.3. This suggests to reformulate Theorem 3.1.1 in terms of the mixed-norm space $L^p(\mathbb{R}^d,w)[F]$, having the advantage that we can work with (equivalence classes) of $\mathbb{C}$-valued measurable functions on the product measure space $(\mathbb{R}^n \times T, \mathcal{B}(\mathbb{R}^n)\otimes\mathscr{B}, \lambda\otimes\nu)$ and that we can always take suprema in the extended positive real numbers $[0,\infty]$.

Recall the definitions of $\mathcal{M}(S)$, $\overline{\mathcal{M}}_+(S)$, $L^0(S)$ and $\overline{L^0}_+(S)$ from Appendix A.1.

Lemma 3.2.1. *Let $(T, \mathscr{B}, \nu)$ be a σ-finite measure space. For each function $f \in \mathcal{M}(\mathbb{R}^n \times T)$,*

$$Mf(x,t) := \sup_{\delta>0} \fint_{B(x,\delta)} |f(y,t)|dy = \sup_{\delta\in\mathbb{Q}_+} \fint_{B(x,\delta)} |f(y,t)|dy, \qquad (x,t) \in \mathbb{R}^n \times T, \tag{3.5}$$

defines an element of $\overline{\mathcal{M}}_+(\mathbb{R}^n \times T)$. Moreover, we obtain a mapping

$$M : L^0(\mathbb{R}^n \times T) \longrightarrow \overline{L^0}_+(\mathbb{R}^n \times T).$$

Proof. Let $f \in \mathcal{M}(\mathbb{R}^n \times T)$. For each $\delta > 0$ we have, by Tonelli's theorem (cf. Theorem A.1.1) and the fact that $\mathbb{R}^n$ admits an exhaustion by compacts, that $(x,t) \mapsto \fint_{B(x,r)} |f(y,t)|dy$ defines a measurable function $\mathbb{R}^n \times T \longrightarrow [0,\infty]$ whose equivalence class does not depend on the equivalence class of the given f. By Tonelli's theorem we furthermore have

$$B := \{t \in T : [y \mapsto f(y,t)] \in L^1_{loc}(\mathbb{R}^n)\} \in \mathscr{B}.$$

Therefore, as both suprema in (3.5) are equal to ∞ for $(x,t) \in \mathbb{R}^n \times [T \setminus B] \in \mathcal{B}(\mathbb{R}^n) \otimes \mathscr{B}$, it remains to establish equality of the two suprema in (3.5) for $(x,t) \in \mathbb{R}^n \times B$.

Let $(x,t) \in \mathbb{R}^n \times B$. The inequality '$\geq$' in (3.5) holds trivially. For the reverse inequality '$\leq$', let $\epsilon > 0$ and $\delta > 0$ be arbitrary. Then we can pick a $\delta' \in \mathbb{Q}_{>0}$ such that $\delta' < \delta$ and

$$\frac{1}{|B(x,\delta)|} \int_{B(x,\delta)\setminus B(x,\delta')} |f(y,t)|dy \leq \epsilon;$$

here we use $[y \mapsto f(y,t)] \in L^1_{loc}(\mathbb{R}^d)$ in combination with the Lebesgue dominated convergence theorem. So

$$\begin{aligned}
\fint_{B(x,\delta)} |f(y,t)|dy &= \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)\setminus B(x,\delta')} |f(y,t)|dy + \frac{1}{|B(x,\delta)|} \int_{B(x,\delta')} |f(y,t)|dy \\
&\leq \epsilon + \frac{1}{|B(x,\delta')|} \int_{B(x,\delta')} |f(y,t)|dy \\
&\leq \epsilon + \sup_{\delta''\in\mathbb{Q}_{>0}} \fint_{B(x,\delta'')} |f(y,t)|dy.
\end{aligned}$$

First taking the supremum over $\delta > 0$ and then letting $\epsilon \to 0$, we obtain the desired inequality '$\leq$'. □

In view of Remark 3.1.2.(iii)/(iv) (and the discussion preceding this lemma), Theorem 3.1.1 can now be reformulated in terms of the mixed-norm space $L^p(\mathbb{R}^d, w)[F]$, as follows.

Theorem 3.2.2. *Let F be a UMD Banach function space on the σ-finite measure space $(T, \mathscr{B}, \nu)$, $p \in]1,\infty[$, and $w \in A_p(\mathbb{R}^n)$. Then M, defined by (3.5), restricts to a bounded sublinear operator on $L^p(\mathbb{R}^n, w)[F]$ with norm bound only depending on n, p, w and the UMD-constant of F in an A_p-consistent way (see Theorem 3.1.1).*

Remark 3.2.3. Let F be a Banach function space on the σ-finite measure space $(T, \mathscr{B}, \nu)$, $p \in]1,\infty[$, and $w \in A_p(\mathbb{R}^n)$. Suppose that F has a σ-Levi norm (which is certainly the case when F is a KB space). By Theorem B.2.7, we can view $L^p(\mathbb{R}^n, w; F)$ as a closed Riesz subspace of $L^p(\mathbb{R}^n, w)[F]$. Moreover, if $f \in L^p(\mathbb{R}^n, w; F)$ gets identified with $\tilde{f} \in L^p(\mathbb{R}^n, w)[F]$, then for

ν-a.a. $t \in T$ we have that (3.4) holds true. Since $L^p(\mathbb{R}^n, w)[F]$ is easily seen to have a σ-Levi norm as well, it follows that the set in (3.3) is norm bounded in $L^p(\mathbb{R}^n, w; F)$ if and only if $M(f) \in L^p(\mathbb{R}^n, w)[F]$, in which case $\|M(f)\|_{L^p(\mathbb{R}^n,w)[F]}$ coincides with the smallest norm bound of this set.

Every UMD Banach function space being a reflexive Banach lattice and thus a KB space, a combination of the above and Remark 3.1.2.(iii)/(iv) shows that Theorem 3.2.2 is indeed a reformulation of Theorem 3.1.1.

We will prove Theorem 3.2.2 in the next section. Let us first look at some consequences.

Corollary 3.2.4. *Let F a Banach function space on the σ-finite measure space $(T, \mathscr{B}, \nu)$, $p \in [1, \infty[$, $w \in A_\infty(\mathbb{R}^n)$, and $\emptyset \neq I \subset \mathbb{Z}$. Suppose that $r \in]0, p[$ is such that $w \in A_{p/r}$ and such that F^r (see* (B.1)*) is a Banach function space*[3] *with the UMD property. Then*

$$M_r(f_i)_{i\in I} := ([M|f_i|^r]^{1/r})_{i\in I}$$

defines a bounded sublinear operator on $[L^p(\mathbb{R}^n, w)[F]][\ell^\infty(I)]$.

Proof. We first consider the case $I = \{0\}$; so we just have $\ell^\infty(I) = \mathbb{C}$, $[L^p(\mathbb{R}^n, w)[F]][\ell^\infty(I)] = L^p(\mathbb{R}^n, w)[F]$, and $M_r f = [M|f|^r]^{1/r}$ for $f \in L^p(\mathbb{R}^n, w)[F]$. Since

$$\begin{aligned} \|M_r f\|_{L^p(\mathbb{R}^n,w)[F]} &= \left\|[M|f|^r]^{1/r}\right\|_{L^p(\mathbb{R}^n,w)[F]} = \|M|f|^r\|^r_{(L^p(\mathbb{R}^n,w)[F])^r} \\ &= \|M|f|^r\|^r_{L^{p/r}(\mathbb{R}^n,w)[F^r]}, \end{aligned}$$

it follows from Theorem 3.2.2 that

$$\|M_r f\|_{L^p(\mathbb{R}^n,w)[F]} \leq C^r \, \| \, |f|^r \, \|^r_{L^{p/r}(\mathbb{R}^n,w)[F^r]} = C^r \, \|f\|_{L^p(\mathbb{R}^n,w)[F]} \, .$$

Finally, the general case now follows from the observation that

$$\|M_r(f_i)_{i\in I}\|_{\ell^\infty(I)} \leq M_r \, \|(f_i)_{i\in I}\|_{\ell^\infty(I)}$$

for all $f \in [L^p(\mathbb{R}^n, w)[F]][\ell^\infty(I)]$. □

Let $(S, \mathscr{A}, \mu)$ and $(T, \mathscr{B}, \nu)$ be two σ-finite measure spaces. Then the formula

$$M(f)(s, x, t) := \sup_{\delta>0} ⨍_{B(x,\delta)} |f(s, y, t)| \, dy, \qquad (s, x, t) \in S \times \mathbb{R}^n \times T,$$

gives rise to a well defined mapping

$$M : \overline{L^0}(S \times \mathbb{R}^n \times T) \longrightarrow \overline{L^0}_+(S \times \mathbb{R}^n \times T);$$

just apply Lemma 3.2.1 to the σ-finite measure space $(S \times T, \mathscr{A} \otimes \mathscr{B}, \mu \otimes \nu)$.

Corollary 3.2.5. *Let E and F be Banach function spaces on the σ-finite measures space $(S, \mathscr{A}, \mu)$ and $(T, \mathscr{B}, \nu)$, respectively, $p \in [1, \infty[$, $q \in [1, \infty]$, $w \in A_\infty(\mathbb{R}^n)$, and $\emptyset \neq I \subset \mathbb{Z}$. Suppose that $r \in]0, \min\{p, q\}[$ is such that $w \in A_{p/r}$ and such that F^r (see* (B.1)*) is a Banach function space*[3] *with the UMD property. Then*

$$M_r(f_i)_{i\in I} := ([M|f_i|^r]^{1/r})_{i\in I}$$

defines a bounded sublinear operator on $E[L^p(\mathbb{R}^n, w)[F]][\ell^q(I)]$

[3] F^r is always a Banach function space for $r \in]0, 1]$.

Proof. This follows easily from Corollary 3.2.4. Note here that, in case $q < \infty$, $F[\ell^q(I)]$ is a Banach function space on the σ-finite measure space $T \times I$, for which we have $F[\ell^q(I)]^r = F^r[\ell^{q/r}(I)]$; so, if F^r is a Banach function space with the UMD property, then so is $F[\ell^q(I)]^r = F^r[\ell^{q/r}(I)] \simeq F^r(\ell^{q/r}(I))$ by Proposition E.5.6. □

Corollary 3.1.3 and Theorem 3.1.4 are easy consequences of this corollary. For Corollary 3.1.3 we just have to reformulate this corollary in terms of the Lebesgue-Bochner space for the special case that $E = \mathbb{K}$ and $I = \{0\}$.

Let us finally give the proof of Corollary 3.1.5:

Proof of Corollary 3.1.5. That $M^{[d,a]}$ is well defined as a sublinear operator $L^1_{loc}(\mathbb{R}^d) \longrightarrow L^0(\mathbb{R}^d)$ can be shown as in Lemma 3.2.1 (but easier). So it remains to be shown that $M^{[d,a]}$ is bounded on $L^{p,d}(\mathbb{R}^d, w)$. From the equivalence $|x|'_{d,a} := \max\{|x_1|^{1/a_1}, \ldots, |x_l|^{1/a_l}\} \eqsim |x|_{d,a}$ $(x \in \mathbb{R}^d)$ it follows that, for every $f \in L^1_{loc}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $\delta > 0$,

$$\begin{aligned} \fint_{|y-x|_{d,a}<\delta} |f(y)|dy &\eqsim \fint_{|y-x|'_{d,a}<\delta} |f(y)|dy = \fint_{|y_1-x_1|<\delta^{a_1}} \ldots \fint_{|y_l-x_l|<\delta^{a_l}} |f(y)|\, dy_l \ldots dy_1 \\ &\leq M_{[d;1]} \ldots M_{[d;l]} f. \end{aligned}$$

By Theorem 3.1.4 we thus obtain that, for every $f \in L^{p,d}(\mathbb{R}^d, w)$,

$$0 \leq M^{[d,a]}(f) \leq M_{[d;1]} \ldots M_{[d;l]} f \in L^{p,d}(\mathbb{R}^d, w) \quad \text{and} \quad \left\| M_{[d;1]} \ldots M_{[d;l]} f \right\|_{L^{p,d}(\mathbb{R}^d,w)} \lesssim \|f\|_{L^{p,d}(\mathbb{R}^d,w)},$$

yielding the boundedness of $M^{[d,a]}$ on $L^{p,d}(\mathbb{R}^d, w)$. □

3.3 Proof of Theorem 3.2.2

In this Section we prove Theorem 3.2.2. The strategy is to reduce this theorem to a problem in martingale theory (Theorem 3.3.5), which gives an explanation for the assumption that the Banach function space under consideration is of class UMD. This reduction we carry out in Section 3.3.1. In Section 3.3.2 we subsequently prove the reduced problem, for which we need two extra martingale theoretic results (Proposition 3.3.6 and Lemma 3.3.9), whose proofs we give in Section 3.3.3.

3.3.1 Reduction to a Martingale Theoretic Problem

Let the notations be as in Theorem 3.2.2. As a first step in the proof of Theorem 3.2.2, we reduce the boundedness of M to a problem in martingale theory.

Below we often make use of the identifications between Köthe-Bochner spaces and mixed-norm space from Appendix B.2.

Denote by $\mathscr{Q}$ the collection of all cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes. Then it is elementary to see that there exist constants $c, c' \geq 1$ such that:

(i) for every $x \in \mathbb{R}^d$ and $\delta > 0$ there exists a cube $Q \in \mathscr{Q}$ containing x such that

$$|Q| \leq c|B(x, \delta)| \quad \text{and} \quad B(x, \delta) \subset Q;$$

(ii) for every $x \in \mathbb{R}^n$ and every cube $Q \in \mathcal{Q}$ containing x there exists a $\delta > 0$ such that

$$|B(x,\delta)| \leq c'|Q| \quad \text{and} \quad Q \subset B(x,\delta).$$

This motivates to define, similarly to Lemma 3.2.1, the maximal function operator

$$M' : L^0(\mathbb{R}^n \times T) \longrightarrow \overline{L^0}_+(\mathbb{R}^n \times T).$$

by

$$(M'f)(x,t) := \sup_{\mathcal{Q} \ni Q \ni x} ⨍_Q |f(y,t)| dy, \qquad (x,t) \in \mathbb{R}^n \times T.$$

By (i) and (ii) above we then have $Mf \leq cM'f$ and $M'f \leq c'Mf$ for all $f \in L^0(\mathbb{R}^n \times T)$. Accordingly, the boundedness of M on $L^p(\mathbb{R}^n, w)[F]$ is equivalent to the boundedness of M' on $L^p(\mathbb{R}^d, w)[F]$.

As next step we relate the maximal function operator M' to maximal function operators associated with certain dyadic systems, for which we have a natural martingale interpretation. We call $\mathcal{D}$ a dyadic system (of cubes) in $\mathbb{R}^n$ if $\mathcal{D} = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k$ is a collection of cubes, where each where each $\mathcal{D}_k$ is a partition of $\mathbb{R}^n$ consisting of cubes of the form $x + [0, 2^{-k}[^n$ (for some $x \in \mathbb{R}^n$), and each cube $D \in \mathcal{D}_k$ is a union of 2^n cubes from $\mathcal{D}_{k+1}$. The most easiest example of a dyadic system is *standard dyadic system* $\mathcal{D}^0$, which is defined as

$$\mathcal{D}^0 = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k^0, \qquad \mathcal{D}_k^0 := \left\{ 2^{-k}([0,1[^n + m) \,:\, m \in \mathbb{Z} \right\};$$

also see Example A.3.17. It is not difficult to see that each dyadic system $\mathcal{D}$ has to be of the form $\mathcal{D} = \bigcup_{k \in \mathbb{Z}} [\mathcal{D}_k^0 + b_k]$ for some sequence $(b^{[k]})_k \subset \mathbb{R}^n$ satisfying $b^{[k]} - b^{[k+1]} \in 2^{-k}\mathbb{Z}^n$ for all k, and reversely that each such system defines a dyadic system. Here it is of course enough to consider $(b^{[k]})_k \subset \mathbb{R}^n$ with $b^{[k]} \in [0, 2^{-k}[^n$. Via binary expansions we thus see that

$$\mathcal{D}^\omega = \bigcup_{k \in \mathbb{Z}} \mathcal{D}_k^\omega, \qquad \mathcal{D}_k^\omega := \mathcal{D}_k^0 + \sum_{i > k} 2^{-i} \omega_i, \qquad \omega \in (\{0,1\}^n)^{\mathbb{Z}},$$

constitute all the dyadic systems.

Lemma 3.3.1 (Covering lemma)**.** *Define* $\omega^{odd} = (\omega_i^{odd})_{i \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ *and* $\omega^{even} = (\omega_i^{even})_{i \in \mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ *by*

$$\omega_i^{odd} := \begin{cases} 0 & \text{, if } i \text{ is even} \\ 1 & \text{, if } i \text{ is odd} \end{cases} \qquad \text{and} \qquad \omega_i^{even} := \begin{cases} 1 & \text{, if } i \text{ is even} \\ 0 & \text{, if } i \text{ is odd.} \end{cases}$$

For every cube $Q \in \mathcal{Q}$ *there exists an* $\omega \in \{0, \omega^{odd}, \omega^{even}\}^n$ *and a dyadic cube* $D \in \mathcal{D}^\omega$ *such that*

$$5^n|Q| \leq |D| \leq 10^n|Q| \quad \text{and} \quad Q \subset D.$$

Proof. This is not very difficult and can be proved analogously to the covering lemma from [57]. □

For $\omega \in \{0, \omega^{odd}, \omega^{even}\}^d$, we define the *shifted dyadic maximal function operator* (or the maximal function operator with respect to the dyadic system $\mathcal{D}^\omega$)

$$M'_\omega : L^0(\mathbb{R}^n \times T) \longrightarrow \overline{L^0}_+(\mathbb{R}^n \times T)$$

by

$$(M'_\omega f)(x,t) := \sup_{\mathscr{D}^\omega \ni D \ni x} \fint_D |f(y,t)|dy, \qquad (x,t) \in \mathbb{R}^n \times T.$$

It is an easy consequence of the covering lemma that

$$M'f \leq 10^d \sup_{\omega \in \{0,\omega^{odd},\omega^{even}\}^n} M'_\omega f \leq 10^n \sum_{\omega \in \{0,\omega^{odd},\omega^{even}\}^n} M'_\omega f \leq 10^n \sum_{\omega \in \{0,\omega^{odd},\omega^{even}\}^n} M'f = 30^n M'f$$

for all f. Therefore, M is bounded on $L^p(\mathbb{R}^n, w)[F]$ if and only if each M'_ω is bounded on $L^p(\mathbb{R}^n, w)[F]$.

For $\omega \in \{0,\omega^{odd},\omega^{even}\}^n$, let $(\mathscr{F}^\omega_k)_{k \in Z} := (\sigma(\mathscr{D}^\omega_k))_{k\in\mathbb{Z}}$ be the filtration generated by the the dyadic system $(\mathscr{D}^\omega_k)_{k\in\mathbb{Z}}$; then $\mathscr{F}^{\omega,atom}_k = \mathscr{D}^\omega_k$. To each $f \in L^1_{loc}(\mathbb{R}^n; F)$ we associate the martingale $F^{|f|,\omega} = (F^{|f|,\omega}_k)_{k\in\mathbb{Z}} \subset L^1_{loc}(\mathbb{R}^n; F)$ (with respect to this filtration) given by

$$F^{|f|,\omega}_k := \mathbb{E}(|f| \mid \mathscr{F}^\omega_k) = \sum_{D \in \mathscr{D}^\omega_k} 1_D \fint_D |f(y)|dy, \qquad k \in \mathbb{Z};$$

see Examples A.3.16 and A.3.4.(v). Then, under the identifications from Corollary B.2.3, we have

$$F^{|f|,\omega}_k(x,t) = \sum_{D \in \mathscr{D}^\omega_k} 1_D(x) \fint_D |f(y,t)|dy, \qquad k \in \mathbb{Z},$$

and thus

$$M'_\omega f = \sup_{k\in\mathbb{Z}} F^{|f|,\omega}_k \quad \text{a.e. on} \quad \mathbb{R}^n \times T.$$

Since $L^p(\mathbb{R}^d, w)[F] \simeq L^p(\mathbb{R}^n, w; F)$ as Banach lattices (with as ordering in $L^p(\mathbb{R}^d, w)[F]$ the induced one from $L^0(\mathbb{R}^n \times T)$, which is the pointwise a.e. ordering), it follows that the boundedness of M on $L^p(\mathbb{R}^n, w)[F]$ is equivalent to: for each $\omega \in \{0,\omega^{odd},\omega^{even}\}^n$, $(F^{|f|,\omega}_k)_{k\in\mathbb{Z}}$ belongs to $L^p(\mathbb{R}^n, w; F)$ and has a supremum $\sup_{k\in\mathbb{Z}} F^{|f|,\omega}_k$ in $L^p(\mathbb{R}^n, w; F)$ which is of norm $\leq C\,\|f\|_{L^p(\mathbb{R}^n,w;F)}$ for some constant independent of f. We will in fact consider this martingale theoretic problem in a more general setting.

Let $\omega \in \{0,\omega^{odd},\omega^{even}\}^n$ be arbitrary. Observe that the σ-finite measure space $(\Sigma, \mathscr{F}, \mu) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda)$ and the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}} = (\mathscr{F}^\omega_k)_{k\in\mathbb{Z}}$ in particular satisfy the following properties:

- $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ generates $\mathscr{F}$: $\mathscr{F} = \mathscr{F}_\infty \overset{def}{=} \sigma\left(\bigcup_{k\in\mathbb{Z}} \mathscr{F}_k\right)$;
- $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ it is regular with respect to μ: each $\mathscr{F}_k$ is countably-atomic with respect to μ (in the sense of Definition A.1.2) and there exists a $\theta \geq 1$ such that

$$\mu(A) \leq \theta\mu(B), \qquad k \in \mathbb{Z}, A \in \mathscr{F}^{atom}_{k-1}, B \in \mathscr{F}^{atom}_k, B \subset A. \tag{3.6}$$

It is not difficult to see that the latter condition is equivalent to

$$\mathbb{E}(f \mid \mathscr{F}_k) \leq \theta \mathbb{E}(f \mid \mathscr{F}_{k-1}), \qquad k \in \mathbb{Z}, f \in \mathcal{M}^+(\Sigma, \mathscr{F}). \tag{3.7}$$

Furthermore, as $w \in A_p(\mathbb{R}^n)$, the weight $W = w \in \mathcal{W}(\Sigma, \mathscr{F}, \mu)$ in particular satisfies the A_p-condition (D.3) with the supremum just taken over all dyadic cubes $D \in \mathscr{D}^\omega = \bigcup_{k\in\mathbb{Z}} \mathscr{F}_k^{atom}$. Since

$$1_D \mathbb{E}[W \mid \mathscr{F}_k] \left(\mathbb{E}[W^{-1/(p-1)} \mid \mathscr{F}_k\right)^{p-1} = 1_D \left(\fint_D W\, d\mu\right)\left(\fint_D W^{-1/(p-1)}\, d\mu\right)^{p-1}$$

for all atoms D in $\mathscr{F}_k$, the A_p condition over these dyadic cubes is equivalent to the existence of a constant $C \in [1, \infty[$ for which we have

$$\sup_{k\in\mathbb{Z}} \mathbb{E}[W \mid \mathscr{F}_k] \left(\mathbb{E}[W^{-1/(p-1)} \mid \mathscr{F}_k\right)^{p-1} \leq C \quad \text{a.e..} \tag{3.8}$$

This motivates to define the class $A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$ as the set of all weights $W \in \mathcal{W}(\Sigma, \mathscr{F}, \mu)$ with $W, W^{-1/(p-1)} \in L^1_\sigma((\mathscr{F}_k)_{k\in\mathbb{Z}})$ satisfying (3.8) for some $C \in [1, \infty[$; we denote by $[W]_{A_p}$ the least possible constant $C \in [1, \infty[$.

Given a weight $W \in \mathcal{W}(\Sigma, \mathscr{F}, \mu)$, we will use the following notation: We write $L^p(W; X) := L^p((\Sigma, \mathscr{F}, W\mu); X)$, which we view as subspace of $L^0(\Sigma; X)$. Furthermore, given a measurable set $F \in \mathscr{F}$, we write $W(F) := W\mu(F)$. Finally, we write $\mathbb{E}_W(\,\cdot \mid \mathscr{F}_k)$ for the conditional operator with respect to the weighted measure $W\mu$. Then we have $f \in L^1_\sigma(W\mu, (\mathscr{F}_k)_{k\in\mathbb{Z}}; X)$ if and only if $fW \in L^1_\sigma(\mu, \mathscr{F}_k; X)$, in which case we have the following relation between the conditional expectations:

$$\mathbb{E}_W(f \mid \mathscr{F}_k) = \frac{1}{\mathbb{E}(W \mid \mathscr{F}_k)} \mathbb{E}(fW \mid \mathscr{F}_k), \qquad k \in \mathbb{Z}. \tag{3.9}$$

This identity also holds true for arbitrary $f \in \mathcal{M}^+(\Sigma, \mathscr{F})$ in the sense of extended conditional expectation. Since this is a very important identity which will be used frequently, let us prove it: We just fix an atom D of $\mathscr{F}_k$ and compute

$$\begin{aligned} 1_D \mathbb{E}(fW \mid \mathscr{F}_k) &= 1_D \frac{1}{\mu(D)} \int_D fW\, d\mu = 1_D \frac{\int_D W\, d\mu}{\mu(D)} \frac{1}{W(D)} \int f\, W d\mu \\ &= 1_D \mathbb{E}(W \mid \mathscr{F}) \mathbb{E}_W(f \mid \mathscr{F}_k). \end{aligned}$$

Let X be a Banach space. To each $f \in L^1_\sigma((\mathscr{F}_k)_{k\in\mathbb{Z}}; X) = L^1_\sigma((\Sigma, \mathscr{F}, \mu)(\mathscr{F}_k)_{k\in\mathbb{Z}}; X)$ we associate the X-valued martingale $F^f = (F^f_k)_{k\in\mathbb{Z}}$ (with respect to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$) given by

$$F^f_k := \mathbb{E}(f \mid \mathscr{F}_k) = \sum_{D \in \mathscr{F}_k^{atom}} 1_D \fint_D f\, d\mu. \tag{3.10}$$

Given a weight $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$, in the next proposition we in particularly show that $L^p(W; X) \subset L^1_\sigma((\mathscr{F}_k)_{k\in\mathbb{Z}}; X)$ and that $(F^f_k)_{k\in\mathbb{Z}}$ is a bounded sequence in $L^p(W; X)$ whenever $f \in L^p(W; X)$.

Proposition 3.3.2. *Suppose $(\Sigma, \mathscr{F}, \mu)$ is a σ-finite measure space equipped with a regular filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ generating $\mathscr{F}$; let $\theta \geq 1$ be the constant in (3.6). Let X be a Banach space, $W \in \mathcal{W}(\Sigma, \mathscr{F}, \mu)$, and $p \in]1, \infty[$.*

(i) $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$ if and only if $W^{-\frac{1}{p-1}} \in A_{p'}((\mathscr{F}_k)_{k\in\mathbb{Z}})$ ($\frac{1}{p} + \frac{1}{p'} = 1$).

(ii) $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$ if and only if there exists a constant $C \in [1, \infty[$ such that, for every $f \in \mathcal{M}^+(\Sigma, \mathscr{F})$,

$$\mathbb{E}(f \mid \mathscr{F}_k) \leq C^{1/p} \left(\mathbb{E}_W[f^p \mid \mathscr{F}_k]\right)^{1/p}, \qquad k \in \mathbb{Z}. \tag{3.11}$$

Moreover, in this situation the smallest such constant C equals $[W]_{A_p}$.

(iii) *Let $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$. Then $L^p(W;X) \subset L^1_\sigma((\mathscr{F}_k)_{k\in\mathbb{Z}};X)$ and $\mathbb{E}(\cdot \mid \mathscr{F}_k)$ is a contraction on $L^p(W;X)$ for each $k \in \mathbb{Z}$.*

(iv) *For every $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$ there exists a $q \in]1,p[$ such that $W \in A_q((\mathscr{F}_k)_{k\in\mathbb{Z}})$. If $[W]_{A_p} \le C \in [1,\infty[$, then q and $[W]_{A_q}$ only depend on C, p and θ.*

(v) *Let $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$ with $[W]_{A_p} \le \tilde{C} \in [1,\infty[$. Given an $f \in L^p(W;X)$ with associated X-valued martingale $F^f = (F^f_k)_{k\in\mathbb{Z}}$ (as in (3.10)), let $(F^f)^* := \sup_{k\in\mathbb{Z}} \left\|F^f_k\right\|_X$ be the corresponding maximal function. Then there exists a constant $C \in [0,\infty[$, only depending on $\tilde{C}$, p and θ, such that, for all $f \in L^p(W;X)$,*

$$\left\|(F^f)^*\right\|_{L^p(W)} \le C \left\|f\right\|_{L^p(W;X)}.$$

(vi) *Let $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$. For every $f \in L^p(W;X)$ with generated martingale $F^f = (F^f_k)_{k\in\mathbb{Z}} \subset L^p(W;X)$ we have the convergence*

$$\lim_{k\to\infty} F^f_k = f \quad \text{in } L^p(W;X).$$

Proof.

(i) This follows easily from the definition.

(ii) This can be shown completely analogously to [45, Proposition 9.1.5.(8)] (which is Proposition D.2.2.(iv)). Let us just treat the direct implication. Given $f \in \mathcal{M}^+(\Sigma,\mathscr{F})$, we use Hölder (with $\frac{1}{p}+\frac{1}{p'}=1$) and the definition of $A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$, to estimate

$$\begin{aligned}
(\mathbb{E}[f \mid \mathscr{F}_k])^p &= \left(\mathbb{E}(fW^{\frac{1}{p}}W^{-\frac{1}{p}} \mid \mathscr{F}_k)\right)^p \\
&\le \mathbb{E}[f^pW \mid \mathscr{F}_k]\left(\mathbb{E}[W^{-\frac{p'}{p}} \mid \mathscr{F}_k]\right)^{\frac{p}{p'}} \\
&\overset{(3.9)}{=} \mathbb{E}_W[f^p \mid \mathscr{F}_k]\,\mathbb{E}[W \mid \mathscr{F}_k]\left(\mathbb{E}[W^{-\frac{1}{p-1}} \mid \mathscr{F}_k]\right)^{p-1} \\
&\le \mathbb{E}_W[f^p \mid \mathscr{F}_k][W]_{A_p}.
\end{aligned}$$

(iii) This follows from Proposition A.3.11 as $W^{-\frac{1}{p-1}} \in L^1_\sigma((\mathscr{F}_k)_{k\in\mathbb{Z}})$.

(iv) This can be shown as in [45, Corollary 9.2.6] (which is about the usual A_p weights on $\mathbb{R}^n$).

(v) By (iii) we can pick a $q \in]1,p[$ such that $W \in A_q((\mathscr{F}_k)_{k\in\mathbb{Z}})$. Then we can estimate

$$\left\|F^f_k\right\|_X^q \le \|\mathbb{E}[f \mid \mathscr{F}_k]\|_X^q \le \mathbb{E}[\|f\|_X \mid \mathscr{F}_k]^q \overset{(ii)}{\le} [W]_{A_q}\mathbb{E}_W[\|f\|_X^q \mid \mathscr{F}_k].$$

Applying Corollary A.3.19 to $\|f\|_X^q \in L^{p/q}(W)$, we finally get

$$\begin{aligned}
\left\|(F^f)^*\right\|_{L^p(W)} &= \left\|\sup_{k\in\mathbb{Z}} \left\|F_k^f\right\|_X^q\right\|_{L^{p/q}(W)}^{1/q} \\
&\leq [W]_{A_q}^{1/q} \left\|\sup_{k\in\mathbb{Z}} \mathbb{E}_W(\|f\|_X^q \mid \mathscr{F}_k)\right\|_{L^{p/q}(W)}^{1/q} \\
&\leq [W]_{A_q}^{1/q} \left([p/q]' \left\|\|f\|_X^q\right\|_{L^{p/q}(W)}\right)^{1/q} \\
&= [W]_{A_q}^{1/q} \left(1-\frac{q}{p}\right)^{-1/q} \|f\|_{L^p(W;X)}\,.
\end{aligned}$$

(vi) Let $\epsilon > 0$. Then, in view of the hypothesis $\mathscr{F} = \mathscr{F}_\infty$ and Lemma A.3.26 (applied to the weighted measure $W\mu$), we can find $K \in \mathbb{Z}$ and $\tilde{f} \in L^p(W;\mathscr{F}_K;X)$ such that $\left\|f-\tilde{f}\right\|_{L^p(W;X)} < \epsilon/2$; Observing that $\tilde{f} = \mathbb{E}(\tilde{f} \mid \mathscr{F}_k)$ for all $k \geq K$, we obtain

$$\left\|f - g_k^f\right\|_{L^p(W;X)} = \left\|f - \tilde{f} + \mathbb{E}(\tilde{f} - f \mid \mathscr{F}_k)\right\|_{L^p(W;X)} \overset{(iv)}{\leq} 2\left\|f-\tilde{f}\right\|_{L^p(W;X)} < \epsilon$$

for all $k \geq K$.

□

Let the notations be as in the proposition. We denote by $\mathscr{M}(X) = \mathscr{M}((\mathscr{F}_k)_{k\in\mathbb{Z}};X)$ the vector space of all X-valued martingales on Σ with respect to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$. We write ${}_{00}\mathscr{M}(X)$ for the linear subspace of $\mathscr{M}(X)$ consisting of all $g = (g_k)_{k\in\mathbb{Z}}$ with the property that $g_k = 0$ for all $k \leq K$ for some $K \in \mathbb{Z}$. Given $\mathcal{W}(\Sigma,\mathscr{F},\mu)$, we define

$$\mathscr{M}_{L^p(W)}(X) := \mathscr{M}(X) \cap \ell^\infty(\mathbb{Z};L^p(W;X)) \quad \text{and} \quad {}_{00}\mathscr{M}_{L^p(W)}(X) := {}_{00}\mathscr{M}(X) \cap \ell^\infty(\mathbb{Z};L^p(W;X)).$$

In case $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$ the following holds true:

Lemma 3.3.3. *Let the notations be as in Proposition 3.3.2 and suppose that $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$. Then $\mathscr{M}_{L^p(W)}(X)$ is a Banach space when equipped with the norm*

$$\|g\|_{\mathscr{M}_{L^p(W)}(X)} := \sup_{k\in\mathbb{Z}} \|g_k\|_{L^p(W;X)} = \lim_{k\to\infty} \|g_k\|_{L^p(W;X)}\,, \qquad (g = (g_k)_{k\in\mathbb{Z}} \in \mathscr{M}_{L^p(W)}(X)), \tag{3.12}$$

for which we have the isometric embedding

$$L^p(W;X) \longrightarrow \mathscr{M}_{L^p(W)}(X),\ f \mapsto F^f = (F_k^f)_{k\in\mathbb{Z}} = (\mathbb{E}[f \mid \mathscr{F}_k])_{k\in\mathbb{Z}}. \tag{3.13}$$

Moreover, $g \mapsto \left\|\sup_{k\in\mathbb{Z}} \|g_k\|_X\right\|_{L^p(W)}$ defines an equivalent norm on $\mathscr{M}_{L^p(W)}(X)$; in fact, there exists an increasing function $C_{p,\theta} : [1,\infty[\longrightarrow [1,\infty[$ (only depending on p and θ) such that

$$\|g\|_{\mathscr{M}_{L^p(W)}(X)} \leq \left\|\sup_{k\in\mathbb{Z}} \|g_k\|_X\right\|_{L^p(W)} \leq C_{p,\theta}([W]_{A_p})\, \|g\|_{\mathscr{M}_{L^p(W)}(X)}\,, \qquad g \in \mathscr{M}_{L^p(W)}(X).$$

Remark 3.3.4. When X has the Radon-Nikodým property (RNP) it can be shown (at least in the unweighted case [57]) that every $L^p(W)$-bounded martingale g with respect to $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ is generated by some $f \in L^p(W)$, i.e. $g = (\mathbb{E}[f \mid \mathscr{F}_k])_{k\in\mathbb{Z}}$. The latter just means that the isometric embedding (3.13) is surjective, or equivalently, that (3.13) is an isometric isomorphism. Reflexive Banach spaces (and thus in particular UMD spaces) and separable dual spaces are examples of Banach spaces having the RNP property; see [57].

Proof. First observe that, given $g \in \mathscr{M}_{L^p(W)}(X)$, $(\|g_k\|_{L^p(W;X)})_{k\in\mathbb{Z}}$ is an increasing sequence. Indeed, from the martingale property $g_k = \mathbb{E}(g_{k+1} \mid \mathscr{F}_k)$ and the contractivity of $\mathbb{E}(\,\cdot \mid \mathscr{F}_k)$ on $L^p(W;X)$ (see Lemma 3.3.2.(ii)) it follows that $\|g_k\|_{L^p(W;X)} = \|\mathbb{E}\|\,(g_{k+1} \mid \mathscr{F}_k)_{L^p(W;X)} \leq \|g_{k+1}\|_{L^p(W;X)}$. Therefore, we have $\lim_{k\to\infty}\|g_k\|_{L^p(W;X)} = \sup_{k\in\mathbb{Z}}\|g_k\|_{L^p(W;X)}$.

To show that $\mathscr{M}_{L^p(W)}(X)$ is a Banach space we must show that $\mathscr{M}_{L^p(W)}(X)$ is a closed subspace of $\ell^\infty(\mathbb{Z}; L^p(W;X))$. To this end, let $(g^{[n]})_{n\in\mathbb{N}}$ be a sequence in $\mathscr{M}_{L^p(W)}(X)$ converging to some $g = (g_k)_{k\in\mathbb{Z}}$ in $\ell^\infty(\mathbb{Z}; L^p(W;X))$. Then we have $g_k = \lim_{n\to\infty} g_k^{[n]}$ in $L^p(W;X)$ for all $k \in \mathbb{Z}$. From the martingale property of each $g^{[n]}$ and the fact that $\mathbb{E}(\,\cdot \mid \mathscr{F}_k) \in \mathcal{B}(L^p(W;X))$ (see Lemma 3.3.2.(ii)) it follows that

$$g_k = \lim_{n\to\infty} g_k^{[n]} = \lim_{n\to\infty} \mathbb{E}(g_{k+1}^{[n]} \mid \mathscr{F}_k) = \mathbb{E}(\lim_{n\to\infty} g_{k+1}^{[n]} \mid \mathscr{F}_k) = \mathbb{E}(g_{k+1} \mid \mathscr{F}_k), \quad \forall k \in \mathbb{Z},$$

showing that $g \in \mathscr{M}(X)$ and thus that $g \in \mathscr{M}_{L^p(W)}(X)$.

The isometric embedding part in the last part of the lemma is immediate from (vi) of Proposition 3.3.2. So it remains to be shown that $g \mapsto \left\|\sup_{k\in\mathbb{Z}}\|g_k\|_X\right\|_{L^p(W)}$ defines an equivalent norm on $\mathscr{M}_{L^p(W)}(X)$. For this we just need to show that $\left\|\sup_{k\in\mathbb{Z}}\|g_k\|_X\right\|_{L^p(W)} \lesssim \|g\|_{\mathscr{M}_{L^p(W)}(X)}$ for all $g \in \mathscr{M}_{L^p(W)}(X)$; then it is easily seen that $g \mapsto \left\|\sup_{k\in\mathbb{Z}}\|g_k\|_X\right\|_{L^p(W)}$ defines a norm on $\mathscr{M}_{L^p(W)}(X)$ which of course also satisfies the reverse inequality. So let $g \in \mathscr{M}_{L^p(W)}(X)$ be given. Choose an arbitrary $K \in \mathbb{Z}$ and consider the stopped martingale $g^K = (g_{K\wedge k})_{k\in\mathbb{Z}} = F^{g_K}$. By Proposition 3.3.2.(v) we have

$$\left\|\sup_{k\leq K}\|g_k\|_X\right\|_{L^p(W)} = \left\|\sup_{k\in\mathbb{Z}}\left\|F_k^{g_K}\right\|_X\right\|_{L^p(W)} \leq C\,\|g_K\|_{L^p(W)} \leq C\,\|g\|_{\ell^\infty(\mathbb{Z};L^p(W;X))} = C\,\|g\|_{\mathscr{M}_{L^p(W)}(X)}\,.$$

Letting $K \to \infty$ we get the desired inequality. □

From the lemma it follows that

$$L^p(W;X) \longrightarrow \mathscr{M}_{L^p(W)}(X),\; f \mapsto F^{|f|} = (F_k^{|f|})_{k\in\mathbb{Z}}$$

defines an isometric sublinear operator. For the proof of Theorem 3.2.2 is thus suffices to prove the following result, which is an immediate consequence of the lemma in case $F = \mathbb{C}$.

Theorem 3.3.5. *Suppose $(\Sigma, \mathscr{F}, \mu)$ is a σ-finite measure space equipped with a regular filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ generating $\mathscr{F}$; let $\theta \geq 1$ be the constant in (3.6). Let F be a UMD Banach function space over the σ-finite measure space $(T, \mathscr{B}, \nu)$ and let $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$, $p \in]1,\infty[$. For all $g \in \mathscr{M}_{L^p(W)}(F)$ it holds that $\sup_{k\in\mathbb{Z}}|g_k|$ exists in $L^p(W;F)$. Moreover, the induced operator*

$$\mathcal{M} : \mathscr{M}_{L^p(W)}(F) \longrightarrow L^p(W;F),\; g \mapsto \sup_{k\in\mathbb{Z}}|g_k|$$

is bounded, with norm bound only depending on p, W, θ and the UMD constant of F in an A_p-consistent way (which is defined as in Theorem 3.1.1)

Proof. We will give the proof of this theorem in Section 3.3.2. □

Explicitly writing out the boundedness of $\mathcal{M}$ gives the visually attractive inequality

$$\left\| \sup_{k \in \mathbb{Z}} |g_k| \right\|_{L^p(W;F)} \lesssim \sup_{k \in \mathbb{Z}} \|g_k\|_{L^p(W;F)}, \qquad g \in \mathscr{M}_{L^p(W)}(F),$$

which says that we can take the supremum outside at the cost of an inequality (for some constant independent of the martingale g under consideration).

3.3.2 Proof of Theorem 3.3.5

Throughout this subsection we assume that $(\Sigma, \mathscr{F}, \mu)$ is a σ-finite measure space equipped with a regular filtration $(\mathscr{F}_k)_{k \in \mathbb{Z}}$ generating $\mathscr{F}$; we let $\theta \geq 1$ be the constant in (3.6).

Before we can start with the proof of Theorem 3.3.5, we need to do some preparations.

Let F be a Banach function space on the σ-finite measure space $(T, \mathscr{B}, \nu)$. For each F-valued martingale $g = (g_k)_{k \in \mathbb{Z}}$ on Σ with respect to the filtration $(\mathscr{F}_k)_{k \in \mathbb{Z}}$ (in symbols $g \in \mathscr{M}(F)$) we define the partial square function $S_K(g) \in L^0(\Sigma; F)$ by

$$S_K(g) := \left(\sum_{k=-K}^{K} |dg_k|^2 \right)^{1/2} \qquad (K \in \mathbb{N});$$

here $dg = (dg_k)_{k \in \mathbb{Z}}$ is the difference sequence corresponding to g ($dg_k = g_k - g_{k-1}$). We write $S(f) := S(F^f)$ when $f \in L^1_\sigma((\mathscr{F}_k)_{k \in \mathbb{Z}}; F)$; here F^f is the martingale associated with f as in (3.10). In the scalar case $F = \mathbb{C}$ we can define, for each $g \in \mathscr{M}(\mathbb{C})$, the *square function* $S(g) \in \overline{L^0}_+(\Sigma)$ by

$$S(g) := \sup_{K \in \mathbb{N}} S_K g = \left(\sum_{k \in \mathbb{Z}} |d^g_k|^2 \right)^{1/2}. \tag{3.14}$$

If $p \in]1, \infty[$ and $W \in A_p((\mathscr{F}_k)_{k \in \mathbb{Z}})$, then we have, for each $g \in \mathscr{M}_{L^p(W)}(F)$, $S_K(g) \in L^p(W; F)$. In case that F has the UMD-property we can take the supremum over $K \in \mathbb{Z}$ in $L^p(W; F)$:

Proposition 3.3.6. *Suppose that F is a UMD Banach function space on the σ-finite measure space $(T, \mathscr{B}, \nu)$. Let $p \in]1, \infty[$ and $W \in A_p((\mathscr{F}_k)_{k \in \mathbb{Z}})$. Then $\sup_{K \in \mathbb{N}} S_K(g)$ exists in $L^p(W; F)$ for all $g \in \mathscr{M}_{L^p(W)}(F)$, and the induced sublinear operator*

$$S : \mathscr{M}_{L^p(W)}(F) \longrightarrow L^p(W; F),\ g \mapsto \sup_{K \in \mathbb{N}} S_K(g)$$

is bounded. Moreover, we have

$$\|S(g)\|_{L^p(W;F)} \eqsim \|g\|_{\mathscr{M}_{L^p(W)}(F)}, \qquad g \in {}_{00}\mathscr{M}_{L^p(W)}(F). \tag{3.15}$$

Here all the implicit constants only depend on p, W, θ and the UMD constant of F in an A_p-consistent way (which is defined as in Theorem 3.1.1).

Proof. We will give the proof in Section 3.3.3. □

Remark 3.3.7. Let the notations be as in the proposition. For each Banach space X we define

$$_0\mathscr{M}_{L^p(W)}(X) := \left\{ g \in \mathscr{M}_{L^p(W)}(X) \,:\, \lim_{k\to-\infty} g_k = 0 \text{ in } L^p(W;X) \right\}.$$

(i) The the equivalence (3.15) can be improved to hold for all martingales $g \in {}_0\mathscr{M}_{L^p(W)}(F)$ (with the same implicit constants). Indeed, given $g \in {}_0\mathscr{M}_{L^p(W)}(F)$, it is not difficult to see that

$$\|g\|_{\mathscr{M}_{L^p(W)}(F)} = \lim_{l\to-\infty} \left\| {}^l g \right\|_{\mathscr{M}_{L^p(W)}(F)} \eqsim \lim_{l\to-\infty} \left\| S({}^l g) \right\|_{L^p(W;F)} = \|S(g)\|_{L^p(W;F)};$$

where, for each $l \in \mathbb{Z}$, ${}^l g = (g_k - g_{k\wedge l})_{k\in\mathbb{Z}} \in {}_{00}\mathscr{M}_{L^p(W)}(F)$ denotes the started martingale.

(ii) As a consequence of the representation of L^p-functions in terms of their martingale differences from [57], we have the topological direct sum

$$\mathscr{M}_{L^p(W)}(X) = L^p(W, \mathscr{F}_{-\infty}; X) \oplus {}_0\mathscr{M}_{L^p(W)}(X)$$

for any Banach space X (at least in the unweighted case $W = 1$), where $\mathscr{F}_{-\infty} = \bigcap_{k\in\mathbb{Z}} \mathscr{F}_k$. Accordingly, given a $g \in \mathscr{M}_{L^p(W)}(X)$, we denote by $g_{-\infty}$ the corresponding projection of g onto $L^p(W, \mathscr{F}_{-\infty}; X)$. Note that $L^p(W, \mathscr{F}_{-\infty}; X) = \{0\}$ when μ is purely infinite on $\mathscr{F}_{-\infty}$ (i.e. $\mu(F) \in \{0, \infty\}$ for all $F \in \mathscr{F}_{-\infty}$). Defining the square function operator $\tilde{S}$ on $\mathscr{M}_{L^p(W)}(F)$ by

$$\tilde{S}(g) := \left(|g_{-\infty}|^2 + |S(g)|^2 \right)^{1/2} = \left(|g_{-\infty}|^2 + |S(g - g_{-\infty})|^2 \right)^{1/2}, \qquad g \in \mathscr{M}_{L^p(W)}(F),$$

we have the following equivalence of norms on $\mathscr{M}_{L^p(W)}(F)$:

$$\|g\|_{\mathscr{M}_{L^p(W)}(F)} \overset{(i)}{\eqsim} \left(\|g_{-\infty}\|^p_{L^p(W,\mathscr{F}_{-\infty};F)} + \|S(g - g_{-\infty})\|^p_{L^p(W;F)} \right)^{1/p} \eqsim \left\| \tilde{S}(g) \right\|_{L^p(W;F)}.$$

Remark 3.3.8. In the unweighted case $W = 1$ this proposition is a consequence of Khintchine-Maurey (Theorem E.2.2/Proposition E.1.4), the UMD-property of F, and some basic martingale theory.

Proof. Let $g \in \mathscr{M}_{L^p}(F)$. Since the Banach function space $L^p(\Sigma; F) \simeq L^p(\Sigma)[F]$ is of class UMD and thus has finite cotype (see Propositions E.5.6 and E.5.5), it follows from Khintchine-Maurey (cf. Theorem E.2.2) that

$$\|S_K(g)\|_{L^p(\Sigma;F)} = \left\| \left(\sum_{k=-K}^{K} |dg_k|^2 \right)^{1/2} \right\|_{L^p(\Sigma;F)} \eqsim \mathbb{E} \left\| \sum_{k=-K}^{K} \varepsilon_k dg_k \right\|_{L^p(\Sigma;F)}.$$

Invoking the UMD-property of F (in the form of Lemma E.5.2), we thus get

$$\|S_K(g)\|_{L^p(\Sigma;F)} \eqsim \left\| \sum_{k=-K}^{K} dg_k \right\|_{L^p(\Sigma;F)} = \|g_K - g_{-K-1}\|_{L^p(\Sigma;F)}. \tag{3.16}$$

From (3.16) and the L^p-contractivity of conditional expectations it follows that there is some constant $C > 0$ such that, for all $g \in \mathscr{M}_{L^p}(F)$, the increasing sequence of partial square functions $(S_K(g))_{K\in\mathbb{N}}$ is norm bounded by $C\|g\|_{\mathscr{M}_{L^p}(F)}$. Being a UMD-space and thus a reflexive space, $L^p(\Sigma; F)$ is a KB-space (see Propositions E.5.6 and E.5.5). Hence $S(g) := \sup_{K\in\mathbb{N}} S_K(g)$ exists in $L^p(\Sigma; F)$, is of norm $\|S(g)\|_{L^p(\Sigma;F)} \le C\|g\|_{\mathscr{M}_{L^p(F)}}$, and is given as the limit $S(g) = \lim_{K\to\infty} S_K(g)$ in $L^p(\Sigma; F)$.

It remains to establish the estimate from below for $g \in {}_{00}\mathscr{M}_{L^p(W)}(F)$. Pick $K_0 \in \mathbb{N}$ such that $g_k = 0$ for all $k \le -K_0$. Then then LHS of (3.16) equals $\|g_K\|_{L^p(\Sigma;F)}$ for all $K \ge K_0$. Letting the $K \to \infty$ on both sides of (3.16) we get the desired inequality. □

Lemma 3.3.9. *Let $\mathscr{G}$ be a collection of positive functions $Z \in L^1_\sigma((\mathscr{F}_k)_{k\in\mathbb{Z}})^+$ such that $Z > 0$ and $S(Z) \le cZ$ a.e. for all $Z \in \mathscr{G}$ and some constant $c > 0$. Then there exists a constant $C > 0$, only depending on c, such that*

$$\int_\Sigma g^* Z\, d\mu \le C \int_\Sigma S(g) Z\, d\mu$$

for every $Z \in \mathscr{G}$ and each $g \in {}_{00}\mathscr{M}(\mathbb{C})$. Here $g^* := \sup_{k\in\mathbb{Z}} |g_k|$.

Proof. We will prove this in Section 3.3.3. □

Remark 3.3.10. The proof of this lemma will be a combination of Lemma 3.3.20, Lemma 3.3.12 and inequality (3.43) from Theorem 3.3.19 (with $F = \mathbb{C}$ and $r = 1$): from the two lemmas it follows that $\mathscr{G}$ belongs uniformly to some A_p, so that we can use inequality (3.43) (with $F = \mathbb{C}$ and $r = 1$) uniformly in $W = Z \in \mathscr{G}$.

The inequality (3.43) from Theorem 3.3.19 (with $F = \mathbb{C}$ and $r = 1$) will also be used in the proof of Lemma 3.3.12. The motivation for having both inequalities in Theorem 3.3.19 in its full generality is that it will be the main ingredient for the proof of Proposition 3.3.6 in the weighted case (in which we have $r = p$). However, for the reader which is only interested in the unweighted case $W = 1$ in Theorem 3.3.5, it suffices to have inequality (3.43) from Theorem 3.3.19 in the special case $F = \mathbb{C}$ and $r = 1$. This inequality can be obtained by modifying a classical result of Gundy and Wheeden [47, Theorem 2] concerning the case of $\Sigma = [0, 1]$ with the dyadic filtration. The latter result is in the unweighted case also known as Davis' inequality [21].

We are now ready to give the proof of Theorem 3.3.5.

Proof of Theorem 3.3.5. In this proof we identify $L^p(W; F)$ with the mixed-norm space $L^p(W)[F]$; here we use Theorem B.2.7 and the fact that F has a σ-order continuous norm (being a UMD space and thus a reflexive space). Then note that $L^p(W)[F] \simeq L^p(W; F)$ is a KB-space (being a UMD and thus a reflexive Banach lattice, see Propositions E.5.6, E.5.5 and B.1.8) Furthermore, we without loss of generality assume that $\mathrm{supp}(F) = T$ and we denote by $W' := W^{-\frac{1}{p-1}} \in A_{p'}((\mathscr{F}_k)_{k\in\mathbb{Z}})$ the p-dual weight of $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$. Then we have $L^p(W)[F]^* \simeq L^p(W)^\times[F^\times] = L^{p'}(W')[F^\times]$ by Proposition B.2.8; in particular, $L^{p'}(W')[F^\times]$ is norming for $L^p(W)[F]$ via the natural pairing

$$\langle f, \phi\rangle = \int_{\Sigma\times T} f\phi\, d(\mu\otimes\nu), \qquad f \in L^p(W)[F], \phi \in L^{p'}(W')[F^\times].$$

From this it easily follows that

$$\|f\|_{L^p(W)[F]} = \sup\left\{\langle f, \phi\rangle \,:\, \phi \in L^{p'}(W')[F^\times]^+, \|\phi\|_{L^{p'}(W')[F^\times]} \le 1\right\}, \qquad f \in L^p(W)[F]^+. \quad (3.17)$$

Let $g \in \mathscr{M}_{L^p(W)}(F)$ be given. It is enough to find a constant C, only depending on p, W, θ and the UMD constant of F in an A_p-consistent way, such that $\left\|\sup_{k=-K,\dots,K}|g|_k\right\|_{L^p(W)[F]} \leq C\,\|g\|_{\mathscr{M}_{L^p(W)}(F)}$ for all $K \in \mathbb{N}$. Indeed, since $L^p(W)[F]$ is a KB-space, it then follows that $\mathcal{M}(g) := \sup_{k\in\mathbb{Z}}|g_k| = \sup_{K\in\mathbb{N}}\sup_{k=-K,\dots,K}|g|_k$ exists in $L^p(W)[F]$ and satisfies the norm estimate $\|\mathcal{M}(g)\|_{L^p(W)[F]} \leq C\,\|g\|_{\mathscr{M}_{L^p(W)}(F)}$. In view of (3.17), it actually suffices to find a constant C (with dependence as above) such that

$$\langle \sup_{k=-K,\dots,K}|g_k|\,,\phi\rangle \leq C\,\|g\|_{\mathscr{M}_{L^p(W)}(F)}\,\|\phi\|_{L^{p'}(W')[F^\times]}\,, \qquad K\in\mathbb{N}, \phi\in L^{p'}(W')[F^\times]^+. \tag{3.18}$$

We first show that it is enough to establish (3.18) for $g \in {}_{00}\mathscr{M}_{L^p(W)}(F)$ (with a different constant C): Let $K\in\mathbb{N}$ and denote by ${}^{-K}g = ({}^{-K}g_k)_{k\in\mathbb{Z}} \in {}_{00}\mathscr{M}_{L^p(W)}(F)$ the started martingale corresponding to the constant stopping time $-K$, i.e. ${}^{-K}g_k = g_k - g_{-K\wedge k}$. For each $K\in\mathbb{N}$ and every $\phi \in L^{p'}(W')[F^\times]^+$, we then have

$$\begin{aligned}
\langle \sup_{k=-K,\dots,K}|g_k|\,,\phi\rangle &= \langle \sup_{k=-K,\dots,K}|g_k - g_{-K} + g_{-K}|\,,\phi\rangle\\
&\leq \langle \sup_{k=-K,\dots,K}|g_k - g_{-K}|\,,\phi\rangle + \langle|g_{-K}|,\phi\rangle\\
&\leq \langle \sup_{k=-K,\dots,K}|{}^{-K}g_k|\,,\phi\rangle + \|g_{-K}\|_{L^p(W)[F]}\,\|\phi\|_{L^{p'}(W')[F^\times]}\\
&\leq \langle \sup_{k=-K,\dots,K}|{}^{-K}g_k|\,,\phi\rangle + \|g\|_{\mathscr{M}_{L^p(W)}(F)}\,\|\phi\|_{L^{p'}(W')[F^\times]}\,.
\end{aligned}$$

and

$$\left\|{}^{-K}g\right\|_{\mathscr{M}_{L^p(W)}(F)} \leq \|g\|_{\mathscr{M}_{L^p(W)}(F)} + \|g_{-K}\|_{L^p(W)[F]} \leq 2\,\|g\|_{\mathscr{M}_{L^p(W)}(F)}\,.$$

Therefore, it is indeed enough to consider the case $g \in {}_{00}\mathscr{M}_{L^p(W)}(F)$.

In order to establish (3.18) for $g \in {}_{00}\mathscr{M}_{L^p(W)}(F)$, to each $\phi \in L^{p'}(W')[F^\times]^+$ we will associate a function $\Phi_\phi \in L^{p'}(W')[F^\times]^+$ such that

(i) $\phi \leq \Phi_\phi$;

(ii) $\left\|\Phi_\phi\right\|_{L^{p'}(W')[F^\times]} \leq 4\,\|\phi\|_{L^{p'}(W')[F^\times]}$;

(iii) $S(\Phi_\phi) \leq c\Phi_\phi$ in $L^{p'}(W')[F^\times]$;

(iv) $\Phi_\phi(\varsigma,t) > 0$ for $\mu\otimes\nu$-a.e. $(\varsigma,t)\in\Sigma\times T$,

for some constant $c > 0$ only depending on p, W, θ and the UMD constant of F in an A_p-consistent way. Before we describe the construction of Φ_ϕ, let us first continue with (3.18) (using the function Φ_ϕ). The idea is to apply, for ν-a.a. $t\in T$, Lemma 3.3.9 to the family $\mathscr{G}_t := \{\Phi_\phi(\,\cdot\,,t) \,:\, \phi\in L^{p'}(W')[F^\times]^+\}$: Let $g \in {}_{00}\mathscr{M}_{L^p(W)}(F)$, $\phi\in L^{p'}(W')[F^\times]^+$, and $K\in\mathbb{N}$ be given. Then we have, by Corollary B.2.3, for ν-a.a. $t\in T$, $\tilde{g}(\,\cdot\,,t) := (g_k(\,\cdot\,,t))_{k\in\mathbb{Z}} \in {}_{00}\mathscr{M}(\mathbb{C})$, and we can compute

$$\begin{aligned}
\langle \sup_{k=-K,\dots,K}|g_k|\,,\phi\rangle &= \int_{\Sigma\times T}\left(\sup_{k=-K,\dots,K}|g_k|\right)\phi\,d(\mu\otimes\nu)\\
&\overset{(i)}{\leq} \int_{\Sigma\times T}[\tilde{g}(\,\cdot\,,t)]^*(\varsigma)\,\Phi_\phi(\varsigma,t)\,d(\mu\otimes\nu)(\varsigma,t)\\
&= \int_T\int_\Sigma[\tilde{g}(\,\cdot\,,t)]^*(\varsigma)\,\Phi_\phi(\varsigma,t)\,d\mu(\varsigma)d\nu(t).
\end{aligned}$$

Furthermore, for ν-a.a. $t \in T$ we have $S(g)(t) = S(\tilde{g}(\,\cdot\,,t))$ as a consequence of Corollary B.2.3, where S is both the square function operator on $\mathcal{M}_{L^p(W)}(F)$ from Proposition 3.3.6 as the square function operator on $\mathcal{M}(\mathbb{C})$ from (3.14). By (iii) and (iv), for ν-a.a. $t \in T$ we may thus apply Lemma 3.3.9 to the martingale $\tilde{g}(\,\cdot\,,t) \in {}_{00}\mathcal{M}(\mathbb{C})$ and the family $\mathscr{G}_t$, to obtain

$$\begin{aligned}\langle \sup_{k=-K,\dots,K} |g_k|\,,\phi\rangle &\lesssim \int_T \int_\Sigma S(\tilde{g}(\,\cdot\,,t))(\varsigma)\,\Phi_\phi(\varsigma,t)\,d\mu(\varsigma)d\nu(t)\\ &= \int_{\Sigma\times T} S(g)(\varsigma,t)\,\Phi_\phi(\varsigma,t)\,d(\mu\otimes\nu)(\varsigma,t)\end{aligned}$$

Using Proposition 3.3.6 and (ii), we find

$$\langle \sup_{k=-K,\dots,K} |g_k|\,,\phi\rangle \lesssim \|S(g)\|_{L^p(W)[F]}\left\|\Phi_\phi\right\|_{L^{p'}(W')[F^\times]} \lesssim \|g\|_{\mathcal{M}_{L^p(W)}(F)}\,\|\phi\|_{L^{p'}(W')[F^\times]}\,.$$

Finally, to finish the proof, we provide the construction of Φ_ϕ: From the identification $F^\times \simeq F^*$ (see Theorem B.1.12) and the fact that duals of UMD spaces are again UMD (see Proposition E.5.5), it follows that $F^\times$ is a UMD Banach function space. Accordingly, we denote by $B = B_{p',F^\times}$ the operator norm of the square function operator S on $L^{p'}(W')[F^\times] \cong L^{p'}(W';F^\times)$; here we use Proposition 3.3.6 and identify $L^{p'}(W')[F^\times] \cong L^{p'}(W';F^\times)$ with a closed subspace of $\mathcal{M}_{L^{p'}(W')}(F^\times)$ in the natural way (see Lemma 3.3.3). By Lemma B.1.11 there exists a $u \in L^{p'}(W')[F^\times]^+$ of norm $\|u\| = 1$ such that $u(\varsigma,t) > 0$ for $\mu\otimes\nu$-a.e. $(\varsigma,t) \in \Sigma\times T$. So, to each $\phi \in L^{p'}(W')[F^\times]^+$ we can associate the function $\Phi_\phi \in L^{p'}(W')[F^\times]^+$ defined by

$$\Phi_\phi := \sum_{n=0}^{\infty}(2B)^{-n}S^n(\phi + \|\phi\|\,u),$$

where $S^n = S\circ\ldots\circ S$ (n times) for each $n \in \mathbb{N}$. Then Φ_ϕ clearly satisfies (i)-(iv). □

3.3.3 Proofs of Proposition 3.3.6 and Lemma 3.3.9

In this subsection we prove Proposition 3.3.6 and Lemma 3.3.9.

3.3.3.a Outline

Let $(\Sigma,\mathscr{F},\mu)$ and $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ be as Sections 3.3.1 and 3.3.2 (also see Theorem 3.3.5). For Proposition 3.3.6 we could try to get a generalization with weights of the UMD inequality and then simply proceed as in the unweighted case $W = 1$ (which is in Remark 3.3.8). However, we follow a different approach, which simultaneously gives a part of Lemma 3.3.9.

Let F be a UMD Banach function space. Given $r \in [1,\infty[$ and

$$W \in A_\infty((\mathscr{F}_k)_{k\in\mathbb{Z}}) := \bigcup_{p\in]1,\infty[} A_p((\mathscr{F}_k)_{k\in\mathbb{Z}}),$$

we show that

$$\left\|\sup_{K\in\mathbb{N}}\|S_K(g)\|_F\right\|_{L^r(W)} \lesssim \|g^*\|_{L^r(W)}\,, \qquad \forall g \in \mathcal{M}(F), \tag{3.19}$$

and

$$\|g^*\|_{L^r(W)} \lesssim \left\|\sup_{K\in\mathbb{N}}\|S_K(g)\|_F\right\|_{L^r(W)}, \qquad \forall g \in {}_{00}\mathcal{M}(F); \tag{3.20}$$

here $g^* = \sup_{k\in\mathbb{Z}} \|g_k\|_F$. Moreover, we show that the implicit constants in these inequalities only depend on $[W]_{A_p}$ (with $W \in A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$, $p \in]1,\infty[$,), $r \in [1,\infty[$, the UMD constant of F, and the constant $\theta \geq 1$ from (3.6); see Theorem 3.3.19. Proposition 3.3.6 then follows by taking $r = p$ and applying Lemma 3.3.3. And, by taking $r = 1$ and $G = \mathbb{C}$, for Lemma 3.3.9 it then "just" remains to be shown that $\mathscr{G}$ belongs uniformly to some $A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$, $p \in]1,\infty[$ (Lemma 3.3.20 and Lemma 3.3.12).

The idea behind the proof of (3.19) and (3.20) (Theorem 3.3.19) is to first derive the two inequalities for $r = 2$ and $W = 1$ from Remark 3.3.8 and Lemma 3.3.3, and next to extrapolate these two inequalities for $g \in {}_{00}\mathscr{M}(F)$ to all r and W (via an application of Corollary 3.3.16); for the first inequality (3.19), the case $g \in \mathscr{M}(F)$ is then easily derived from the case $g \in {}_{00}\mathscr{M}(F)$.

3.3.3.b Some $A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$-Theory

Recall that, for $p \in]1,\infty[$, the class $A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$ is defined as the set of all weights $W \in \mathcal{W}(\Sigma,\mathscr{F},\mu)$ with $W, W^{-1/(p-1)} \in L^1_\sigma((\mathscr{F}_k)_{k\in\mathbb{Z}})$ for which there exists a constant $C > 0$ such that

$$\sup_{k\in\mathbb{Z}} \mathbb{E}(W \mid \mathscr{F}_k)\left(\mathbb{E}(W^{-1/(p-1)} \mid \mathscr{F}_k)\right)^{p-1} \leq C \quad \text{a.e.},$$

and that $[W]_{A_p}$ denotes the least possible constant $C > 0$. Moreover, we write

$$A_\infty((\mathscr{F}_k)_{k\in\mathbb{Z}}) = \bigcup_{p\in]1,\infty[} A_p((\mathscr{F}_k)_{k\in\mathbb{Z}}).$$

For convenience of notation we from now on write $A_p = A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$ ($p \in]1,\infty]$).

An alternative definition of the class $A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$, $p \in]1,\infty[$, is given in Proposition 3.3.2.(ii). Interchanging the roles of $\mathbb{E}[\,\cdot \mid \mathscr{F}_k]$ and $\mathbb{E}[\,\cdot \mid \mathscr{F}_k]$ in (3.11) gives the inequality

$$\mathbb{E}_W[f \mid \mathscr{F}_k] \leq C^{1/p}\mathbb{E}[f^p \mid \mathscr{F}_k]^{1/p}, \qquad f \in \mathcal{M}^+(\Sigma,\mathscr{F}), k \in \mathbb{Z}. \tag{3.21}$$

If we would like to modify the definition of A_p in order to be able to follow the proof of Proposition 3.3.2.(ii) with the inequality (3.11) replaced by the inequality (3.21), we naturally come to the class $\widehat{A}_p$, which is defined as follows: Given $p \in]1,\infty[$, we define $\widehat{A}_p$ as the class of all weights $W \in \mathcal{W}(\Sigma,\mathscr{F},\mu)$ with $W \in L^1_\sigma((\mathscr{F}_k)_{k\in\mathbb{Z}})$ and $W^{1/(p-1)} \in L^1_\sigma(W,(\mathscr{F}_k)_{k\in\mathbb{Z}})$ for which there exists a constant $C > 0$ such that

$$\sup_{k\in\mathbb{Z}} \mathbb{E}(W \mid \mathscr{F}_k)^{-1}\left(\mathbb{E}_W[W^{1/(p-1)} \mid \mathscr{F}_k\right)^{p-1} \leq C \quad \text{a.e.}; \tag{3.22}$$

we denote by $[W]_{\widehat{A}_p}$ the least possible constant $C > 0$. Moreover, we write

$$\widehat{A}_\infty := \bigcup_{p\in]1,\infty[} \widehat{A}_p.$$

Lemma 3.3.11. *Let $p \in]1,\infty[$ and $W \in \widehat{A}_p$. Then* (3.21) *holds true with $C \leq [W]_{\widehat{A}_p}$.*

Proof. As in the proof of Proposition 3.3.2.(ii), we use Hölder's inequality (with $\frac{1}{p} + \frac{1}{p'} = 1$) and the definition of $\widehat{A_p}$, to estimate

$$\begin{aligned} \mathbb{E}_W[f \mid \mathscr{F}_\tau]^p &= \mathbb{E}_W[f W^{-\frac{1}{p}} \cdot W^{\frac{1}{p}} \mid \mathscr{F}_k]^p \\ &\leq \mathbb{E}_W[f^p W^{-1} \mid \mathscr{F}_k]\, \mathbb{E}_W[W^{\frac{1}{p-1}} \mid \mathscr{F}_k]^{p-1} \\ &\overset{(3.9)}{=} \mathbb{E}[f^p \mid \mathscr{F}_k] \frac{1}{\mathbb{E}[W \mid \mathscr{F}_k]}\, \mathbb{E}_W[W^{\frac{1}{p-1}} \mid \mathscr{F}_k]^{p-1} \\ &\leq \mathbb{E}[f^p \mid \mathscr{F}_k]\, [W]_{\widehat{A_p}}. \end{aligned}$$

□

The following characterization of the class A_∞ constitutes an important part of the proof of Lemma 3.3.9.

Lemma 3.3.12. *The following are equivalent:*

(i) $W \in A_\infty$.

(ii) There exist $p \in]1, \infty[$ *and* $C \in]0, \infty[$ *such that* $W \in A_p$ *with* $[W]_{A_p} \leq C$.

(iii) $W \in \mathcal{W}(\Sigma, \mathscr{F}, \mu)$ *and there exist* $\alpha, \beta \in]0, 1[$ *such that*

$$\mathbb{E}(1_A \mid \mathscr{F}_k) \leq \alpha \Longrightarrow \mathbb{E}_W(1_A \mid \mathscr{F}_k) \leq \beta$$

for all $A \in \mathscr{F}$, $k \in \mathbb{Z}$.

Moreover, the constants in (ii) (resp. (iii)) only depend on the constants in (iii) (resp. (ii)) and θ.

Proof. Because of our definition of A_∞, we only need to show that (ii) and (iii) are equivalent (plus dependence of the involved constants). For this we observe that (iii) can be reformulated as:

(iii)' $W \in \mathcal{W}(\Sigma, \mathscr{F}, \mu)$ and there exist $\alpha, \beta \in]0, 1[$ such that for all $k \in \mathbb{Z}$ and all atoms D of $\mathscr{F}_k$ it holds that

$$\forall E \subset D, E \in \mathscr{F} : \ \mu(E) \leq \alpha\mu(D) \Longrightarrow W(E) \leq \beta W(D)$$

Indeed, for every $A \in \mathscr{F}$ and $k \in \mathbb{Z}$ it holds that

$$\mathbb{E}(1_A \mid \mathscr{F}_k) = \sum_{D \text{ atom of } \mathscr{F}_k} 1_D \fint_D 1_A \, d\mu = \sum_{D \text{ atom of } \mathscr{F}_k} 1_D \frac{\mu(A \cap D)}{\mu(D)}$$

and, similarly,

$$\mathbb{E}_W(1_A \mid \mathscr{F}_k) = \sum_{D \text{ atom of } \mathscr{F}_k} 1_D \frac{W(A \cap D)}{W(D)}.$$

The equivalence between (ii) and (iii)' (plus dependence of the involved constants) can now be derived as in [45, Theorem 9.3.3] (which is about the usual A_p-weights on $\mathbb{R}^n$). □

Lemma 3.3.13. *Let $W \in A_\infty$, say $W \in A_p$ with $[W]_{A_p} \le C$, $p \in]1,\infty[$, $C \in [1,\infty[$. Then $W \in \widehat{A_\infty}$; in fact, $W \in \widehat{A_q}$ for some $q \in]1,\infty[$, with q and $[W]_{\widehat{A_q}}$ only depending on p, C and θ. Furthermore, for $l := \theta^p C \ge 1$ we have*

$$W(A) \le lW(B), \qquad A \in \mathscr{F}_{k-1}^{atom}, B \in \mathscr{F}_k^{atom}, B \subset A, k \in \mathbb{Z}. \tag{3.23}$$

As a consequence, for each $F \in \mathscr{F}_{k-1}$ there exists a $G \in \mathscr{F}_k$ with $F \subset G$ and $W(G) \le lW(F)$.

Proof. It can be shown that the reverse Hölder inequality holds true for $A_p((\mathscr{F}_k)_{k\in\mathbb{Z}})$ weights, just as for the usual A_p weights on $\mathbb{R}^n$ [45, Theorem 9.2.2]. This reverse Hölder inequality says that there exist $\gamma, \tilde{C} \in]0,\infty[$, only depending on C, p and θ, such that

$$\mathbb{E}(W \mid \mathscr{F}_k)^{-1}\mathbb{E}(W^{1+\gamma} \mid \mathscr{F}_k)^{\frac{1}{1+\gamma}} \le \tilde{C} \quad \text{a.e.}$$

Now it is not difficult to see that, for $q := 1 + \frac{1}{\gamma}$, we have $W \in \widehat{A_q}$ with $[W]_{\widehat{A_q}} \le \tilde{C}$.

Next we establish the inequality (3.23). Applying Proposition 3.3.2.(ii) to $f = 1_B$, we find

$$\mathbb{E}[1_B \mid \mathscr{F}_{k-1}]^p \le [W]_{A_p}\mathbb{E}_W[1_B \mid \mathscr{F}_{k-1}].$$

This implies that

$$1_A\left(\frac{\mu(B)}{\mu(A)}\right)^p = 1_A\mathbb{E}[1_B \mid \mathscr{F}_{k-1}]^p \le [W]_{A_p}1_A\mathbb{E}_W[1_B \mid \mathscr{F}_{k-1}] = [W]_{A_p}1_A\frac{W(B)}{W(A)}.$$

Since $\mu(A) \le \theta\mu(B)$ by the hypothesis (3.6), it follows that $w(A) \le \theta^p[W]_{A_p} \le \theta^p C$.

Finally, let us treat the last statement. In case $F \in \mathscr{F}_k$ is contained in a single atom A of $\mathscr{F}_{k-1}$, we can simply take $G = A$, for F certainly contains an atom B of $\mathscr{F}_k$. Each element of $\mathscr{F}_k$ being a countable union of such F, the general case follows. □

Lemma 3.3.14. *Let $q \in]1,\infty[$, $c > 0$, and $W \in \hat{A}_q$ with $[W]_{\hat{A}_q} \le c$. Then, for any stopping time $\tau : \Sigma \longrightarrow \mathbb{Z} \cup \{\infty\}$ and $F \in \mathscr{F}$, $F \subset \{\tau < \infty\}$, we have*

$$\mathbb{E}_W(1_F \mid \mathscr{F}_\tau) \le c^{1/q}\left(\mathbb{E}(1_F \mid \mathscr{F}_\tau)\right)^{1/q} \quad a.e.$$

Proof. As $F \in \mathscr{F}$, $F \subset \{\tau < \infty\}$, it suffices to show that

$$1_{\{\tau=k\}}\left(\mathbb{E}_W(1_F \mid \mathscr{F}_\tau)\right)^q \le 1_{\{\tau=k\}}c\mathbb{E}(1_F \mid \mathscr{F}_\tau) \quad a.e.$$

for all $k \in \mathbb{Z}$. Since $1_{\{\tau=k\}}\left(\mathbb{E}_W(1_F \mid \mathscr{F}_\tau)\right)^q = 1_{\{\tau=k\}}\left(\mathbb{E}_W(1_F \mid \mathscr{F}_k)\right)^q$ and $1_{\{\tau=k\}}c\mathbb{E}(1_F \mid \mathscr{F}_\tau) = 1_{\{\tau=k\}}c\mathbb{E}(1_F \mid \mathscr{F}_k)$ (see Lemma A.3.22), this follows from Lemma 3.3.11. □

3.3.3.c Weighted Inequalities Between Maximal Operators

We write $\mathscr{P}^+ = \mathscr{P}^+((\mathscr{F}_k)_{k\in\mathbb{Z}})$ for the set of all positive processes $V = (V_k)_{k\in\mathbb{Z}}$ on Σ which are adapted to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$. We define ${}_{00}\mathscr{P}^+$ to be the set of all $V = (V_k)_{k\in\mathbb{Z}} \in \mathscr{P}^+$ with the property that there exists a $K \in \mathbb{Z}$ for which $V_k = 0$ for all $k \le K$. For a process $V = (V_k)_{k\in\mathbb{Z}} \in \mathscr{P}^+$ and a stopping times $\tau, \sigma : \Sigma \longrightarrow \mathbb{Z} \cup \{\infty\}$ we write

$$V^* := \sup_{k\in\mathbb{Z}} V_k, \quad V^*_\tau = \sup_{k\le\tau} V_k, \quad {}^\sigma V^*_\tau = V^*_\tau - V^*_{\tau\cap\sigma}.$$

Theorem 3.3.15. *Let X be a Banach space, $r \in [1, \infty[$, and $W \in A_\infty$. Let U and V be mappings ${}_{00}\mathscr{M}(X) \longrightarrow {}_{00}\mathscr{P}^+$ with the property that, for some constant $c > 0$,*

$$U^*_{\tau\wedge\sigma}(g) = U^*_\tau(g^\sigma) \quad \textit{and} \quad V^*_{\tau\wedge\sigma}(g) = V^*_\tau(g^\sigma) \tag{3.24}$$

and

$$\mathbb{E}([{}^\tau U^*_k(g)]^2 \mid \mathscr{F}_\tau) \leq c\mathbb{E}([V^*_k(g)]^2 \mid \mathscr{F}_\tau) \tag{3.25}$$

for all $g \in {}_{00}\mathscr{M}(X)$, all stopping times $\tau, \sigma : \Sigma \longrightarrow \mathbb{Z} \cup \infty$, and all $k \in \mathbb{Z}$. Furthermore, for each $g \in {}_{00}\mathscr{M}(X)$ let $L_g \in \mathbb{Z}$ be such that $g_k = 0$ for all $k \leq L_g$ and such that

$$1_A U^*_k(g) = U^*_k(1_A g), \qquad k \in \mathbb{Z}, A \in \mathscr{F}_{L_g}, \tag{3.26}$$

where $1_A g = (1_A g_k)_{k\in\mathbb{Z}} = (\mathbb{E}[1_A g_k \mid \mathscr{F}_k])_{k\in\mathbb{Z}} \in {}_{00}\mathscr{M}(X)$. Then there exists a constant $C > 0$ such that

$$\|U^*(g)\|_{L^r(W)} \leq C \|V^*(g)\|_{L^r(W)}$$

for all $g \in {}_{00}\mathscr{M}(X)$. Moreover, if $p, q \in]1, \infty[$ are such that $W \in A_p \cap \hat{A}_q$ (see Lemma 3.3.13), then the constant C can be chosen in such a way that $C \lesssim_{c,r} \theta^{p(q/2+1/r)}[W]^{1/2}_{\hat{A}_q}[W]^{q/2+1/r}_{A_p}$.

We will really explicitly use the exponent 2 from condition (3.25) in some computation in the proof of this theorem. However, in the next corollary the exponent 2 will be irrelevant in the sense that the argumentation remains valid for other exponents for which the above theorem holds as well.

Before we can state the next corollary, we first need to observe the following: Given $g \in {}_{00}\mathscr{M}(X)$, a stopping time $\tau : \Sigma \longrightarrow \mathbb{Z} \cup \{\infty\}$, and $A \in \mathscr{F}_\tau$, we have $(1_A {}^\tau g_n)_{n\in\mathbb{Z}}$. Indeed, since

$$1_A {}^\tau g_n - 1_A {}^\tau g_{n-1} = 1_{A\cap\{\tau\leq n-1\}} d^\tau g_n \quad \text{and} \quad A \cap \{\tau \leq n-1\} \in \mathscr{F}_{n-1}, \qquad n \in \mathbb{Z},$$

and since ${}^\tau g \in {}_{00}\mathscr{M}(X)$, this is a consequence of Proposition A.3.12 (and Remark A.3.15.(i)).

Corollary 3.3.16. *Let X be a Banach space, $r \in [1, \infty[$, and $W \in A_\infty$. Let U and V be mappings ${}_{00}\mathscr{M}(X) \longrightarrow {}_{00}\mathscr{P}^+$ satisfying* (3.24). *Furthermore, suppose that there exist constants $c_1, c_2, c_3 > 0$ such that*

$$|{}^\tau U^*_k(g)| \leq c_1 U^*_k({}^\tau g), \tag{3.27}$$

$$V^*_k({}^\tau g) \leq c_2 V^*_k(g), \tag{3.28}$$

$$U^*_k((1_A {}^\tau g_n)_{n\in\mathbb{Z}}) = 1_A U^*_k({}^\tau g) \quad \textit{and} \quad V^*_k((1_A {}^\tau g_n)_{n\in\mathbb{Z}}) = 1_A V^*_k({}^\tau g), \tag{3.29}$$

and

$$\|U^*(g)\|_{L^2} \leq c_3 \|V^*(g)\|_{L^2} \tag{3.30}$$

for all $g \in {}_{00}\mathscr{M}(X)$, all stopping times $\tau : \Sigma \longrightarrow \mathbb{Z} \cup \{\infty\}$, $A \in \mathscr{F}_\tau$, and all $k \in \mathbb{Z}$. Then there exists a constant $C > 0$ such that

$$\|U^*(g)\|_{L^r(W)} \leq C \|V^*(g)\|_{L^r(W)}, \qquad g \in {}_{00}\mathscr{M}(X).$$

Moreover, if $p, q \in]1, \infty[$ are such that $W \in A_p \cap \hat{A}_q$ (see Lemma 3.3.13), then the constant C can be chosen in such a way that $C \lesssim_{c_1c_2c_3,r} \theta^{p(q/2+1/r)}[W]^{1/2}_{\hat{A}_q}[W]^{q/2+1/r}_{A_p}$.

Proof. We must check that U and V satisfy (3.25), (3.26) being an immediate consequence of (3.29). So let be given: $g \in {}_{00}\mathscr{M}(X)$, $\tau : \Sigma \longrightarrow \mathbb{Z} \cup \{\infty\}$ a stopping time, and $k \in \mathbb{Z}$. Choose an arbitrary $A \in \mathscr{F}_\tau$ and consider the martingale $f = (f_n)_{n\in\mathbb{Z}} := (1_A{}^\tau g_n)_{n\in\mathbb{Z}} \in {}_{00}\mathscr{M}(X)$. Applying (3.30) to the stopped martingale f^k, we obtain

$$\left\|U_k^*(f)\right\|_{L^2} \stackrel{(3.24)}{=} \left\|U^*(f^k)\right\|_{L^2} \leq c_3 \left\|V^*(f^k)\right\|_{L^2} \stackrel{(3.24)}{=} c_3 \left\|V_k^*(f)\right\|_{L^2}.$$

Recalling the definition of f and invoking the identities in (3.29), we just have the inequality

$$\left\|1_A U_k^*({}^\tau g)\right\|_{L^2} \leq c_3 \left\|1_A V_k^*({}^\tau g)\right\|_{L^2}.$$

Via the pointwise inequalities (3.27) and (3.28), we can now estimate

$$\left\|1_A{}^\tau U_k^*(g)\right\|_{L^2} \leq c_1 c_2 c_3 \left\|1_A V_k^*(g)\right\|_{L^2}.$$

As $A \in \mathscr{F}_\tau$ was arbitrarily given, we obtain the desired (3.25) with $c := c_1c_2c_3$. □

In the proof of the inequality (3.19) we will apply this extrapolation result to the mappings $U, V : {}_{00}\mathscr{M}(F) \longrightarrow {}_{00}\mathscr{P}^+$ given by

$$U_k(g) = \left\|\left(\sum_{n=-\infty}^{k} |dg_n|^2\right)^{1/2}\right\|_F \quad \text{and} \quad V_k(g) = \sup_{n\leq k} \|g_n\|_F ,$$

and for the inequality (3.20) we will apply this corollary to the same mappings with the roles interchanged; see Theorem 3.3.19.

For the proof of Theorem 3.3.15 we need two lemmata. The first lemma is a so called good λ inequality.

Lemma 3.3.17. *Let $u, v \in \mathcal{M}^+(\Sigma)$ be two positive measurable functions on Σ such that $W(u > \lambda) < \infty$ for every $\lambda > 0$. Suppose that there exist constants $\alpha > 1$ and $\beta, \epsilon, \delta > 0$ such that*

$$W(u > \alpha\lambda) \leq \epsilon W(u > \lambda) + \delta W(v > \beta\lambda), \qquad \lambda > 0. \tag{3.31}$$

Let $r \in [1, \infty[$ and write $\gamma := \alpha^r$ and $\eta := \beta^{-r}$. If $\gamma\epsilon < 1$, then we have

$$\|u\|_{L^r(W)}^r \leq \frac{\gamma\eta\delta}{1-\gamma\epsilon} \|v\|_{L^r(W)}^r. \tag{3.32}$$

Proof. It suffices to show that, for each $n \in \mathbb{N}$, (3.32) holds true with u replaced by $u \wedge n = \min\{u, n\}$. As (3.31) is also satisfied with u replaced by $u \wedge n$, we may as well without loss of generality assume that u is bounded. For each $n \in \mathbb{Z}_{>0}$ we then have $u1_{\{u\geq\frac{1}{n}\}} \in L^r(W)$ because

$W(\{u \geq \frac{1}{n}\}) < \infty$ (in view of (3.31)), for which we compute

$$\begin{aligned}
\left\|u1_{\{u\geq\frac{1}{n}\}}\right\|^r_{L^r(W)} &\overset{(A.3)}{=} \alpha^r \int_0^\infty r\lambda^{r-1} W\left(u1_{\{u\geq\frac{1}{n}\}} > \alpha\lambda\right) d\lambda \\
&= \alpha^r \int_{\frac{\alpha}{n}}^\infty r\lambda^{r-1} W(u > \alpha\lambda) d\lambda \\
&\overset{(3.31)}{\leq} \alpha^r \left(\epsilon \int_{\frac{\alpha}{n}}^\infty r\lambda^{r-1} W(u > \lambda) d\lambda + \delta \int_0^\infty r\lambda^{r-1} W(v > \beta\lambda) d\lambda\right) \\
&= \alpha^r \left(\epsilon \int_0^\infty r\lambda^{r-1} W\left(u1_{\{u\geq\frac{\alpha}{n}\}} > \lambda\right) d\lambda + \delta\beta^r\beta^{-r} \int_0^\infty r\lambda^{r-1} W(v > \beta\lambda) d\lambda\right) \\
&\overset{(A.3)}{=} \gamma\epsilon \left\|u1_{\{u\geq\frac{\alpha}{n}\}}\right\|^r_{L^r(W)} + \gamma\delta\eta \|v\|^r_{L^r(W)} \\
&\leq \gamma\epsilon \left\|u1_{\{u\geq\frac{1}{n}\}}\right\|^r_{L^r(W)} + \gamma\delta\eta \|v\|^r_{L^r(W)}
\end{aligned}$$

Since $\gamma\epsilon < 1$, it follows that (3.32) is satisfies with u replaced by $u1_{\{u\geq\frac{1}{n}\}}$. The desired result now follows by letting $n \to \infty$. □

The second lemma produces a stopping time satisfying certain useful properties.

Lemma 3.3.18. *Let* $W \in A_\infty$ *and let* $l \geq 1$ *be as in* (3.23). *For every* $V = (V_k)_{k\in\mathbb{Z}} \in {}_{00}\mathscr{P}^+$ *and* $\lambda > 0$ *there exists a stopping time* $\tau : \Sigma \longrightarrow \mathbb{Z} \cup \{\infty\}$ *such that*

$$W(\{\tau < \infty\}) \leq lW(\{V^* > \lambda\})$$

and

$$V^*_\tau \leq \lambda \quad a.e., \ \{V^* > \lambda\} \subset \{\tau < \infty\}.$$

Proof. We define the stopping time $\sigma : \Sigma \longrightarrow \mathbb{Z} \cup \{\infty\}$ by

$$\sigma := \inf\{k \mid V_k > \lambda\} = \inf\{k \mid V_k \in]\lambda, \infty[\},$$

that is, σ is the first hitting time of $]\lambda, \infty[$ associated with the adapted process $V \in {}_{00}\mathscr{P}^+$ (see Example A.3.21). By the last part of Lemma 3.3.13, as $\{\sigma = k\} \in \mathscr{F}_k$, there exists a $G_k \in \mathscr{F}_{k-1}$ such that $\{\sigma = k\} \subset G_k$ and $W(G_k) \leq lW(\{\sigma = k\})$; indeed, Now we can define the stopping time $\tau : \Sigma \longrightarrow \mathbb{Z} \cup \{\infty\}$ by

$$\tau := \inf\{k \mid 1_{G_{k+1}} = 1\},$$

that is, τ is the first hitting time of $\{1\}$ corresponding to the adapted process $(1_{G_{k+1}})_{k\in\mathbb{Z}}$. To finish the proof we show that τ is as desired.

Firstly, since it clearly holds that $\{\tau = k\} \subset G_{k+1}$, we have

$$W(\{\tau < \infty\}) = \sum_{k\in\mathbb{Z}} W(\{\tau = k\}) \leq \sum_{k\in\mathbb{Z}} W(G_{k+1}) \leq l \sum_{k\in\mathbb{Z}} W(\{\sigma = k\}) = lW(\{\sigma < \infty\}) = lW(\{V^* > \lambda\}).$$

Secondly, since

$$\{k \leq \tau\} \subset \bigcap_{j\leq k} \Sigma \setminus G_j \subset \bigcap_{j\leq k} \Sigma \setminus \{\sigma = j\} = \{\sigma > k\} \subset \bigcap_{j\leq k} \{V_j \leq \lambda\},$$

we have

$$V_\tau^* = \sup_{k\le\tau} V_k \le \lambda.$$

Finally, if $V^*(\varsigma) > \lambda$, then there exists a smallest $k \in \mathbb{Z}$ such that $V_k(\varsigma) > \lambda$, implying that $\varsigma \in \{\sigma = k\} \subset G_k$ and thus that $\tau(\varsigma) \le k-1 < \infty$. □

We are now ready to prove Theorem 3.3.15.

Proof of Theorem 3.3.15. It suffices to find a constant $C > 0$ such that

$$\left\|U_k^*(g)\right\|_{L^r(W)}^r \le C^r \left\|V_k^*(g)\right\|_{L^r(W)}^r, \qquad k \in \mathbb{Z}, g \in {}_{00}\mathscr{M}(X),$$

or equivalently, such that

$$\left\|1_D U_k^*(g)\right\|_{L^r(W)}^r \le C^r \left\|V_k^*(g)\right\|_{L^r(W)}^r, \qquad k \in \mathbb{Z}, g \in {}_{00}\mathscr{M}(X), D \in \mathscr{F}_{L_g}^{atom}. \tag{3.33}$$

For the positive measurable functions $u = 1_D U_k^*(g) \overset{(3.26)}{=} U^*(1_D g)$ and $v = V^*(g)$ on Σ we have $W(u > \lambda) \le W(D) < \infty$ for every $\lambda > 0$. By Lemma 3.3.17 it thus suffices to find constants $\alpha > 1$ and $\beta, \epsilon, \delta > 0$ such that $\alpha^r\epsilon < 1$ and such that (3.31) is satisfied for this choice of u and v; then, in the notation of this lemma, (3.33) is satisfied for the constant $C = [\gamma\eta\delta/(1-\gamma\epsilon)]^{1/r}$.[4] Since $1_D g \in {}_{00}\mathscr{M}(X)$ for all $g \in {}_{00}\mathscr{M}(X)$ and $D \in \mathscr{F}_{L_g}^{atom}$, for this it is certainly enough to find constants $\alpha > 1$ and $\beta, \epsilon, \delta > 0$ such that $\alpha^r\epsilon < 1$ and such that

$$W(\{U_k^*(g) > \alpha\lambda\}) \le \epsilon W(\{U_k^*(g) > \lambda\}) + \delta W(\{V_k^*(g) > \beta\lambda\}), \qquad k \in \mathbb{Z}, g \in {}_{00}\mathscr{M}(X). \tag{3.34}$$

Let $p \in]1,\infty[$ be such that $W \in A_p$ and let $q \in]1,\infty[$ be such that $W \in \hat{A}_q$ (see Lemma 3.3.13), put $C_{q,W} := [W]_{\hat{A}_q}^{1/q}$, and let $l = \theta^p[W]_{A_p} \ge 1$ be as in Lemma 3.3.13.[5] Fix $\alpha > 1$ and write $\delta := l$. Put $\gamma := \alpha^r > 1$, choose $\beta > 0$ so small that, for the constant $\epsilon := C_{q,W}l(c\beta^2/(1-\alpha)^2)^{1/q} > 0$, we have $\gamma\epsilon < 1$, and put $\eta := \beta^{-r}$. We will show that (3.34) is satisfied for these choices of constants. Here we have to check that

$$C = \left(\frac{\gamma\eta\delta}{1-\gamma\epsilon}\right)^{1/r} = \beta^{-1}l^{1/r}\alpha(1-\gamma\epsilon)^{-1/r} > 0$$

can be chosen as in the last statement of the theorem: we just note that we can take

$$\beta := \left(2^{-q/2}\alpha^{-rq}\frac{C_{q,W}^q l^q c}{1-\alpha^2}\right)^{-1/2} \eqsim_{\alpha,c} (2\alpha^r)^{q/2} C_{q,W}^{-q/2} l^{-q/2} = (2\alpha^r)^{q/2}[W]_{\hat{A}_q}^{-1/2} l^{-q/2},$$

for which we have $\gamma\epsilon = \frac{1}{2} < 1$, so that

$$C \eqsim_{\alpha,c,r} (2\alpha^r)^{-q/2}[W]_{\hat{A}_q}^{1/2} l^{q/2+r} \le (2\alpha^r)^{-1/2}[W]_{\hat{A}_q}^{1/2}(\theta^p[W]_{A_p})^{q/2+1/r} \lesssim_{\alpha,r} \theta^{p(q/2+1/r)}[W]_{\hat{A}_q}^{1/2}[W]_{A_p}^{q/2+1/r}.$$

Fix $g \in {}_{00}\mathscr{M}(X)$ and $k \in \mathbb{Z}$. Let $\lambda > 0$. First, applying Lemma 3.3.18 to the stopped process $V^k(g) = (V_{k\wedge n}(g))_{n\in\mathbb{N}} \in {}_{00}\mathscr{P}^+$ and the constant $\beta\lambda > 0$, we get a stopping time $\sigma : \Sigma \longrightarrow \mathbb{Z}\cup\{\infty\}$ such that

$$W(\{\sigma < \infty\}) \le lW(\{V_k^*(g) > \beta\lambda\}) \tag{3.35}$$

[4] Here we also have to pay attention on the dependence of the obtained constant $C = [\gamma\eta\delta/(1-\gamma\epsilon)]^{1/r}$ as in the statement of the theorem.

[5] Here the definition of the constant $C_{q,W}$ is motivated by Lemma 3.3.14.

and

$$V^*_{k\wedge\sigma}(g) \le \beta\lambda \quad a.e., \; \{V^*_k(g) > \beta\lambda\} \subset \{\sigma < \infty\}. \tag{3.36}$$

Next, applying Lemma 3.3.18 to the stopped process $U^{k\wedge\sigma}(g) = (U_{k\wedge\sigma\wedge n}(g))_{n\in\mathbb{N}} \in {}_{00}\mathscr{P}^+$ and the constant $\lambda > 0$ we get a stopping time $\tau : \Sigma \longrightarrow \mathbb{Z} \cup \{\infty\}$ such that

$$W(\{\tau < \infty\}) \le lW(\{U^*_{k\wedge\sigma}(g) > \lambda\}) \tag{3.37}$$

and

$$U^*_{k\wedge\sigma\wedge\tau}(g) \le \lambda \quad a.e., \; \{U^*_{k\wedge\sigma}(g) > \lambda\} \subset \{\tau < \infty\}. \tag{3.38}$$

For later, let us observe that

$$\{U^*_k(g) > \alpha\lambda\} \subset \{U^*_{k\wedge\sigma}(g) > \alpha\lambda\} \cup \{\sigma < \infty\}; \tag{3.39}$$

for this we just have to note that the complement of the RHS is contained in the complement of the LHS.

Since $U^*_{k\wedge\sigma\wedge\tau}(g) \le \lambda$ almost everywhere by (3.38), it follows that

$$\{U^*_{k\wedge\sigma}(g) > \alpha\lambda\} \subset \left\{ [U^*_{k\wedge\sigma}(g) - U^*_{k\wedge\sigma\wedge\tau}(g)]^2 > \lambda^2(\alpha-1)^2 \right\} \quad a.e..$$

This implies that, for all $A \in \mathscr{F}_\tau$,

$$\int_A 1_{\{U^*_{k\wedge\sigma}(g)>\alpha\lambda\}}\, d\mu \le \frac{1}{\lambda^2(\alpha-1)^2} \int_A [U^*_{k\wedge\sigma}(g) - U^*_{k\wedge\sigma\wedge\tau}(g)]^2\, d\mu,$$

or equivalently, that

$$\begin{aligned} \mathbb{E}(1_{\{U^*_{k\wedge\sigma}(g)>\alpha\lambda\}} \mid \mathscr{F}_\tau) \quad &\le \quad \frac{1}{\lambda^2(\alpha-1)^2}\mathbb{E}([U^*_{k\wedge\sigma}(g) - U^*_{k\wedge\sigma\wedge\tau}(g)]^2 \mid \mathscr{F}_\tau) \\ &\overset{(3.24)}{=} \quad \frac{1}{\lambda^2(\alpha-1)^2}\mathbb{E}([{}^\tau U^*_k(g^\sigma)]^2 \mid \mathscr{F}_\tau). \end{aligned}$$

Via the inequality (3.25), the identity $V^*_k(g^\sigma) = V^*_{k\wedge\sigma}(g)$ (see (3.24)), and the first inequality in (3.36), this can be further estimated as

$$\mathbb{E}(1_{\{U^*_{k\wedge\sigma}(g)>\alpha\lambda\}} \mid \mathscr{F}_\tau) \le \frac{c\beta^2}{(\alpha-1)^2}. \tag{3.40}$$

From

$$\{U^*_{k\wedge\sigma}(g) > \alpha\lambda\} \subset \{U^*_{k\wedge\sigma}(g) > \lambda\} \overset{(3.38)}{\subset} \{\tau < \infty\} \in \mathscr{F}_\tau \tag{3.41}$$

it follows that

$$\begin{aligned} W(\{U^*_{k\wedge\sigma}(g) > \alpha\lambda\}) \quad &= \quad \int_{\{\tau<\infty\}} 1_{\{U^*_{k\wedge\sigma}(g)>\alpha\lambda\}}\, W d\mu \\ &= \quad \int_{\{\tau<\infty\}} \mathbb{E}_W(1_{\{U^*_{k\wedge\sigma}(g)>\alpha\lambda\}} \mid \mathscr{F}_\tau)\, W d\mu. \end{aligned}$$

Using Lemma 3.3.14 (for which we need to recall $W \in \hat{A}_q$ with $[A]_{\hat{A}_q} \leq C_{q,W}$ and (3.41)), (3.40), and (3.37), we can estimate this as

$$\begin{aligned} W(\{U^*_{k\wedge\sigma}(g) > \alpha\lambda\}) &\leq C_{q,W} \int_{\{\tau<\infty\}} \left(\mathbb{E}(1_{\{U^*_{k\wedge\sigma}(g)>\alpha\lambda\}} \mid \mathscr{F}_\tau)\right)^{1/q} W d\mu \\ &\leq C_{q,W} \left(\frac{c\beta^2}{(\alpha-1)^2}\right)^{1/q} W(\{\tau < \infty\}) \\ &\leq C_{q,W} l \left(\frac{c\beta^2}{(\alpha-1)^2}\right)^{1/q} W(\{U^*_k(g) > \lambda\}). \end{aligned} \tag{3.42}$$

Combining (3.39), (3.42), and (3.35), we find

$$\begin{aligned} W(\{U^*_k(g) > \alpha\lambda\}) &\leq W(\{U^*_{k\wedge\sigma} > \alpha\lambda\}) + W(\{\sigma < \infty\}) \\ &\leq C_{q,W} l \left(\frac{c\beta^2}{(\alpha-1)^2}\right)^{1/q} W(\{U^*_k(g) > \lambda\}) + lW(\{V^*_k(g) > \beta\lambda\}) \\ &= \epsilon W(\{U^*_k(g) > \lambda\}) + \delta W(\{V^*_k(g) > \beta\lambda\}) \end{aligned}$$

for every $\lambda > 0$. This proves the desired inequality (3.34). □

3.3.3.d Weighted Inequalities for the Square Function Operator

Theorem 3.3.19. *Let F be a UMD Banach function space, $W \in A_\infty$, and $r \in [1, \infty[$. Then*

$$\left\| \sup_{K\in\mathbb{N}} \|S_K(g)\|_F \right\|_{L^r(W)} \lesssim \|g^*\|_{L^r(W)}, \qquad \forall g \in \mathscr{M}(F), \tag{3.43}$$

and

$$\|g^*\|_{L^r(W)} \lesssim \left\| \sup_{K\in\mathbb{N}} \|S_K(g)\|_F \right\|_{L^r(W)}, \qquad \forall g \in {}_{00}\mathscr{M}(F). \tag{3.44}$$

Moreover, if $W \in A_p$, $p \in]1, \infty[$, with $[W]_{A_p} \leq C \in [1, \infty[$, then the implicit constants only depend on C, p, $r \in [1, \infty[$, the UMD(2) constant of F, and the constant $\theta \geq 1$ from (3.6).

Proof. We first show the two inequalities (3.43),(3.44) for $g \in {}_{00}\mathscr{M}(F)$. Note that for such g it holds that

$$S_K(g) = \left(\sum_{k=-\infty}^{K} |dg_k|^2\right)^{1/2}$$

for large enough $K \in \mathbb{N}$. So, defining the mappings $\bar{S}, M : {}_{00}\mathscr{M}(F) \longrightarrow {}_{00}\mathscr{P}^+$ by

$$\bar{S}_k(g) := \left\| \left(\sum_{n=-\infty}^{k} |dg_n|^2\right)^{1/2} \right\|_F \quad \text{and} \quad M_k(g) := \sup_{n\leq k} \|g_n\|_F,$$

the two inequalities (3.43),(3.44) for $g \in {}_{00}\mathscr{M}(F)$ can be reformulated as

$$\left\|\bar{S}^*(g)\right\|_{L^r(W)} \eqsim \|M^*(g)\|_{L^r(W)}, \qquad \forall g \in {}_{00}\mathscr{M}(F). \tag{3.45}$$

To prove these inequalities, we check the conditions of Corollary 3.3.16 for $(U,V) \in \{(\bar{S},M),(M,\bar{S})\}$; note that the dependence of the implicit constants in (3.45) as in the statement of the theorem then follow from the last statement of Corollary 3.3.16 in combination with Lemma 3.3.13.

First we check the L^2-inequalities. For $g \in {}_{00}\mathscr{M}_{L^2}(F)$ we have, using Remark 3.3.8 and Lemma 3.3.3,

$$\begin{aligned}\left\|\bar{S}^*(g)\right\|_{L^2} &= \left\|\sup_{K\in\mathbb{N}}\left\|\left(\sum_{k=-\infty}^{K}|dg_k|^2\right)^{1/2}\right\|_F\right\|_{L^2} = \left\|\sup_{K\in\mathbb{N}}\|S_K(g)\|_F\right\|_{L^2} \\ &= \|S(g)\|_{L^2(\Sigma;F)} \eqsim \|g\|_{\mathscr{M}_{L^2}(F)} \eqsim \left\|\sup_{k\in\mathbb{Z}}\|g_k\|_F\right\|_{L^2} \\ &= \|M^*(g)\|_{L^2}.\end{aligned}$$

To extend this to all $g \in {}_{00}\mathscr{M}(F)$, it suffices to show that $\left\|\bar{S}^*(g)\right\|_{L^2} = \infty$ implies that $\|M^*(g)\|_{L^2} = \infty$ whenever $g \in {}_{00}\mathscr{M}(F)$. For this we may assume that $(g_k)_{k\in\mathbb{Z}} \in L^2(\Sigma;F)$; otherwise it certainly holds that $\|M^*(g)\|_{L^2} = \infty$. Then, for each $N \in \mathbb{N}$, we have $g^N = (g_{k\wedge N})_{k\in\mathbb{Z}} = g^{g_N} \in {}_{00}\mathscr{M}_{L^2}(F)$ for the stopped sequence. Hence,

$$\left\|\bar{S}^*_N(g)\right\|_{L^2} = \left\|\bar{S}^*(g^N)\right\|_{L^2} \lesssim \left\|M^*(g^N)\right\| = \left\|M^*_N(g)\right\|_{L^2}.$$

Letting $N \to \infty$ we obtain $\infty = \left\|\bar{S}^*(g)\right\| \lesssim \|M^*(g)\|_{L^2}$, as desired. Therefore, $(U,V) \in \{(\bar{S},M),(M,\bar{S})\}$ satisfy the L^2-estimate (3.30). We next check the other conditions of Corollary 3.3.16 in order to extrapolate these L^2-estimates to the weighted inequalities (3.45).

First we look at $\bar{S}$: From $S_k(g) \geq S_{k\wedge\tau}(g) \geq 0$ it follows that

$$0 \leq S_k(g) - S_{k\wedge\tau}(g) \leq \left(S_k(g)^2 - S_{k\wedge\tau}(g)^2\right)^{1/2} = S_k({}^\tau g)$$

in the Banach function space F. Since $\bar{S}_j(g) = \bar{S}^*_j(g)$ for all $j \in \mathbb{Z}$, it follows that

$$|{}^\tau\bar{S}^*_k(g)| = \left|\|S_k(g)\|_F - \|S_{k\wedge\tau}(g)\|_F\right| \leq \|S_k(g) - S_{k\wedge\tau}(g)\|_F \leq \|S_k({}^\tau g)\|_F = \bar{S}_k({}^\tau g),$$

that is, (3.27) is satisfied for $U = \bar{S}$. That $U = V = \bar{S}$ satisfies (3.24), (3.29) and that $V = \bar{S}$ satisfies (3.28) are both trivial.

Next we look at M: That $U = M$ and $V = M$ satisfy (3.27) and (3.28), respectively, are easy consequence of the triangle inequality. That $U = V = M$ satisfies (3.24), (3.29) and that $V = M$ satisfies (3.28) are both trivial.

We may thus apply Corollary 3.3.16 (with $(U,V) \in \{(\bar{S},M),(M,\bar{S})\}$) to obtain (3.45), which in turn implies the two inequalities (3.43),(3.44) for $g \in {}_{00}\mathscr{M}(F)$.

To finish we show that (3.43) is also valid for $g \in \mathscr{M}(F)$. For each $K \in \mathbb{N}$ we consider the started martingale ${}^{-(K+1)}g = (g_k - g_{k\wedge-(K+1)})_{k\in\mathbb{Z}} \in {}_{00}\mathscr{M}(F)$, for which we already know that (3.43) is valid. Then

$$\begin{aligned}\left\|\|S_K(g)\|_F\right\|_{L^r(W)} &= \left\|\left\|S_K({}^{-(K+1)}g)\right\|_F\right\|_{L^r(W)} \leq \left\|\sup_{N\in\mathbb{N}}\left\|S_N({}^{-(K+1)}g)\right\|_F\right\|_{L^r(W)} \\ &\lesssim \left\|({}^{-(K+1)}g)^*\right\|_{L^r(W)} \leq \|g^*\|_{L^r(W)} + \left\|\left\|g_{-(K+1)}\right\|_F\right\|_{L^r(W)} \leq 2\|g^*\|_{L^r(W)}.\end{aligned}$$

As $(\|S_K(g)\|_F)_{K\in\mathbb{N}}$ is an increasing sequence in $\overline{L^0}_+(\Sigma)$, taking the supremum over $K \in \mathbb{N}$ gives the desired inequality. □

3.3.3.e A Lemma due to Bourgain

For the proof of Lemma 3.3.9 we need the following lemma.

Lemma 3.3.20. *Suppose $Z \in L^1_\sigma((\mathscr{F}_k)_{k\in\mathbb{Z}})^+$ satisfies $Z > 0$ and $S(Z) \le cZ$ a.e. for some constant $c > 0$. Then there exist $\alpha, \beta \in]0, 1[$, only depending on c and the constant $\theta \ge 1$ from (3.6), such that $Z \in A_\infty$ with α, β as in Lemma 3.3.12.(iii).*

Proof. I) In the proof of Lemma 3.3.12 we saw that it is equivalent to show the following: There exist $\alpha, \beta \in]0, 1[$ such that for all $k \in \mathbb{Z}$ and all atoms D of $\mathscr{F}_k$ it holds that

$$\forall E \subset D, E \in \mathscr{F} : \ \mu(E) \le \alpha\mu(D) \Longrightarrow \int_E Z\, d\mu \le \beta \int_D Z\, d\mu.$$

To this end, we fix a $k \in \mathbb{Z}$ and an atom D of $\mathscr{F}_k$. We define $\tilde{Z} := Z_{|D}$, $\widetilde{\mathscr{F}} := \mathscr{F} \cap D$, $(\widetilde{\mathscr{F}_n})_{n\in\mathbb{Z}} := (\mathscr{F}_n \cap D)_{n\in\mathbb{Z}}$, and we let $\widetilde{\mu}$ be the normalized restricted measure on $\widetilde{\mathscr{F}}$, that is,

$$\tilde{\mu}(A) := \frac{1}{\mu(D)}\mu(A) \qquad (A \in \tilde{\mathscr{F}});$$

then note that $\tilde{Z} \in L^1(D, \widetilde{\mathscr{F}}, \tilde{\mu})^+$ and $\widetilde{\mathscr{F}_n} = \{\emptyset, D\}$ for $n \le k$. In this notation we must establish the existence of $\alpha, \beta \in]0, 1[$, independent of k and D, such that

$$\forall E \in \widetilde{\mathscr{F}} : \ \tilde{\mu}(E) \le \alpha \Longrightarrow \int_E \tilde{Z}\, d\tilde{\mu} \le \beta \int_E \tilde{Z}\, d\tilde{\mu}. \tag{3.46}$$

II) Let $\Phi, \Psi : [0, \infty[\longrightarrow [0, \infty[$ be the complementary Young's functions from Appendix A.2, Example A.2.3:

$$\Phi(t) = t \log(1 + t) \quad \text{and} \quad \Psi(t) = \exp(t) - 1.$$

By Hölder's inequality for Orlicz spaces (cf. Lemma A.2.2),

$$\int_E \tilde{Z}\, d\tilde{\mu} = \left|\int_D \tilde{Z} 1_E\, d\tilde{\mu}\right| \le 2 \left\|\tilde{Z}\right\|_{\Phi(\tilde{\mu})} \|1_E\|_{\Psi(\tilde{\mu})}, \qquad \forall E \in \widetilde{\mathscr{F}}.$$

Given $\alpha \in]0, 1]$ and $E \in \widetilde{\mathscr{F}}$ with $\tilde{\mu}(E) \le \alpha$, for $\lambda = \log(1/\alpha)^{-1} > 0$ we have

$$\int_D \Psi(1_E/\lambda)\, d\tilde{\mu} = \int_D 1_E \left(\frac{1}{\alpha} - 1\right) d\tilde{\mu} = \tilde{\mu}(E)\left(\frac{1}{\alpha} - 1\right) \le 1 - \alpha \le 1,$$

and thus $\|1_E\|_{\Psi(\tilde{\mu})} \le \log(1/\alpha)^{-1}$. Therefore,

$$\forall \alpha \in]0, 1], \forall E \in \widetilde{\mathscr{F}} : \ \tilde{\mu}(E) \le \alpha \Longrightarrow \int_E \tilde{Z}\, d\tilde{\mu} \le 2 \log(1/\alpha)^{-1} \left\|\tilde{Z}\right\|_{\Phi(\tilde{\mu})}. \tag{3.47}$$

III) We claim that, for every $f \in L^1(\tilde{\mu})^+$ and $\gamma \ge \int f\, d\tilde{\mu}$,

$$\frac{1}{\gamma} \int_{\{f>\gamma\}} f\, d\tilde{\mu} \le \theta\tilde{\mu}(\{f^* > \gamma\}), \tag{3.48}$$

where $f^* := \sup_{n\in\mathbb{Z}} \mathbb{E}(f \mid \widetilde{\mathscr{F}_n}) = \sup_{n\geq k} \mathbb{E}(f \mid \widetilde{\mathscr{F}_n})$ and where $\theta \geq 1$ is the constant from (3.6).

To see this, let $\tau : D \longrightarrow \mathbb{Z}_{\geq k+1} \cup \{\infty\}$ be the first hitting time of $]\gamma, \infty[$ corresponding to the martingale $(\mathbb{E}(f \mid \widetilde{\mathscr{F}_n}))_{n\in\mathbb{Z}}$, that is,

$$\tau := \inf\{n \mid \mathbb{E}(f \mid \widetilde{\mathscr{F}_n}) > \gamma\};$$

note here that $\mathbb{E}(f \mid \widetilde{\mathscr{F}_n}) = \int f \, d\tilde{\mu} \leq \gamma$ for $n \leq k$. On the one hand, as $f = \lim_{n\to\infty} \mathbb{E}(f \mid \widetilde{\mathscr{F}_n})$ pointwise a.e. by Theorem A.3.25, we have

$$1_{\{\tau=\infty\}} f \leq 1_{\{\tau=\infty\}} \gamma$$

and

$$1_{\{\tau<\infty\}} f^* \geq \sum_{n=k+1}^{\infty} 1_{\{\tau=n\}} \mathbb{E}(f \mid \widetilde{\mathscr{F}_n}) > \sum_{n=k+1}^{\infty} 1_{\{\tau=n\}} \gamma = 1_{\{\tau<\infty\}} \gamma,$$

implying that

$$\{f > \gamma\} \subset \{\tau < \infty\} \subset \{f^* > \gamma\}.$$

On the other hand,

$$1_{\{\tau<\infty\}} \mathbb{E}(f \mid \widetilde{\mathscr{F}_\tau}) = \sum_{n=k+1}^{\infty} 1_{\{\tau=n\}} \mathbb{E}(f \mid \widetilde{\mathscr{F}_n}) \overset{(3.7)}{\leq} \theta \sum_{n=k+1}^{\infty} 1_{\{\tau=n\}} \mathbb{E}(f \mid \widetilde{\mathscr{F}_{n-1}}) \leq \theta \sum_{n=k+1}^{\infty} 1_{\{\tau=n\}} \gamma \leq \theta\gamma 1_{\{\tau<\infty\}}.$$

Therefore,

$$\frac{1}{\gamma} \int_{\{f>\gamma\}} f \, d\tilde{\mu} \leq \frac{1}{\gamma} \int_{\{\tau<\infty\}} f \, d\tilde{\mu} = \frac{1}{\gamma} \int_{\{\tau<\infty\}} \mathbb{E}(f \mid \widetilde{\mathscr{F}_\tau}) \, d\tilde{\mu} \leq \theta\tilde{\mu}(\{\tau < \infty\}) \leq \theta\tilde{\mu}(\{f^* > \gamma\}),$$

proving the claim.

IV) We show that

$$\|g\|_{\Phi(\tilde{\mu})} \lesssim_\theta \int_D g^* \, d\tilde{\mu}, \qquad \forall g \in L^1(\tilde{\mu})^+. \tag{3.49}$$

Let $g \in L^1(\tilde{\mu})^+ \setminus \{0\}$. For every $C > 0$ we define

$$\lambda_C := C \int_D g^* \, d\tilde{\mu} > 0$$

and $f = f_C := g/\lambda_C \in L^1(\tilde{\mu})^+$. Then note that $0 \leq \int_D f \, d\tilde{\mu} \leq 1/C$ in view of $f \leq f^*$ (which follows from Theorem A.3.25). Using the identity (A.3) for the function $\phi(t) = \log(1 + t)$ and the measure space $(D, \widehat{\mathscr{F}}, f\tilde{\mu})$, we obtain

$$\begin{aligned}
\int_D \Psi(g/\lambda_C) \, d\tilde{\mu} &= \int_D \log(1 + f) \, f d\tilde{\mu} \\
&= \int_0^\infty \frac{1}{1+t} \left(\int_{\{f>t\}} f \, d\tilde{\mu} \right) dt \\
&= \int_0^{\int_D f \, d\tilde{\mu}} \frac{1}{1+t} \left(\int_{\{f>t\}} f \, d\tilde{\mu} \right) dt + \int_{\int_D f \, d\tilde{\mu}}^\infty \frac{1}{1+t} \left(\int_{\{f>t\}} f \, d\tilde{\mu} \right) dt
\end{aligned}$$

The first term on the RHS of the last inequality can be estimated by $(\int_D f\,d\tilde{\mu})^2 \leq 1/C^2$, whereas the second can be estimated as

$$\begin{aligned}
\int_{\int_D f\,d\tilde{\mu}}^{\infty} \frac{1}{1+t}\left(\int_{\{f>t\}} f\,d\tilde{\mu}\right)dt &\overset{(3.48)}{\leq} \theta \int_{\int_D f\,d\tilde{\mu}}^{\infty} \frac{t}{1+t}\tilde{\mu}(\{f^* > \gamma\})dt \\
&\leq \theta \int_0^{\infty} \tilde{\mu}(\{f^* > \gamma\})dt \\
&= \theta \int_D f^*\,d\tilde{\mu} \\
&= \theta \frac{1}{\lambda_C} \int_D g^*\,d\tilde{\mu} \\
&= \frac{\theta}{C}.
\end{aligned}$$

It thus follows that

$$\int_D \Psi(g/\lambda_C)\,d\tilde{\mu} \leq \frac{1}{C^2} + \frac{\theta}{C}.$$

Choosing C so large that the RHS becomes ≤ 1, we obtain (3.49).

V) Combining (3.47) and (3.49) yields

$$\forall \alpha \in]0,1], \forall E \in \widetilde{\mathscr{F}}: \ \tilde{\mu}(E) \leq \alpha \Longrightarrow \int_E \tilde{Z}\,d\tilde{\mu} \lesssim \log(1/\alpha)^{-1} \int_D \tilde{Z}^*\,d\tilde{\mu}$$

Note that $(D, \widetilde{\mathscr{F}}, \tilde{\mu})$ and $(\widetilde{\mathscr{F}_n})_{n\in\mathbb{Z}}$ satisfy the same hypotheses as $(\Sigma, \mathscr{F}, \mu)$ and $(\mathscr{F}_n)_{n\in\mathbb{Z}}$. So, letting $m := \int_D \tilde{Z}\,d\tilde{\mu}$, we may apply Theorem 3.3.19 to $(D, \widetilde{\mathscr{F}}, \tilde{\mu})$, $(\widetilde{\mathscr{F}_n})_{n\in\mathbb{Z}}$, $W = 1$, $r = 1$, and $(\mathbb{E}(Z - m \mid \widetilde{\mathscr{F}_n}))_{n\in\mathbb{Z}} \in {}_{00}\mathscr{M}((\widetilde{\mathscr{F}_n})_{n\in\mathbb{Z}})$ (note $\mathbb{E}(Z \mid \widetilde{\mathscr{F}_n}) = m$ for $n \leq k$), to obtain that

$$\int_D \tilde{Z}^*\,d\tilde{\mu} \leq m + \int_D (\tilde{Z} - m)^*\,d\tilde{\mu} \lesssim_\theta m + \int_D S(\tilde{Z} - m)\,d\tilde{\mu}.$$

Using the hypothesis that $S(Z) \leq cZ$, we can dominate the integrand of the second term on the RHS as follows:

$$\begin{aligned}
S(\tilde{Z} - m) &= \left(\sum_{n=k+1}^{\infty} |\mathbb{E}(\tilde{Z} \mid \widehat{\mathscr{F}_n}) - \mathbb{E}(\tilde{Z} \mid \widehat{\mathscr{F}_{n-1}})|^2\right)^{1/2} \\
&= 1_D \left(\sum_{n=k+1}^{\infty} |\mathbb{E}(Z \mid \mathscr{F}_n) - \mathbb{E}(Z \mid \mathscr{F}_{n-1})|^2\right)^{1/2} \\
&\leq 1_D S(Z) \\
&\leq 1_D cZ.
\end{aligned}$$

We thus find that

$$\forall \alpha \in]0,1], \forall E \in \widetilde{\mathscr{F}}: \ \tilde{\mu}(E) \leq \alpha \Longrightarrow \int_E \tilde{Z}\,d\tilde{\mu} \lesssim_{\theta,c} \log(1/\alpha)^{-1} \int_D \tilde{Z}\,d\tilde{\mu}.$$

Choosing $\alpha \in]0,1[$ sufficiently small we obtain (3.46), as desired. □

3.3.3.f The Proofs of Proposition 3.3.6 and Lemma 3.3.9.

Proof of Proposition 3.3.6. This is immediate from Theorem 3.3.19 (taking $r = p$) and the fact (from Lemma 3.3.3) that $g \mapsto \|g^*\|_{L^p(W)} = \left\|\sup_{k\in\mathbb{Z}} \|g_k\|_F\right\|_{L^p(W)}$ defines an equivalent norm on $\mathscr{M}_{L^p(W)}(F)$ when $W \in A_p$ (with implicit constants only depending on $[W]_{A_p}$, p and θ as in Lemma 3.3.3). Here we of course use that $(S_K)_{K\in\mathbb{N}}$ is an increasing sequence in the KB-space $L^p(W;F)$. □

Proof of Lemma 3.3.9. By a combination of Lemma 3.3.20 and Lemma 3.3.12, $\mathscr{G}$ belongs uniformly to some A_p in the sense that there exist $p \in]1,\infty[$ and $C_1 > 0$ such that $Z \in A_p$ with $[Z]_{A_p} \leq C_1$ for each $Z \in \mathscr{G}$. Invoking Theorem 3.3.19 (with $r = 1$) we obtain the desired result. □

3.4 Several Maximal and Weighted Norm Inequalities

Suppose that $\mathbb{R}^d$ is $\mathpzc{d}$-decomposed as in Convention 2.2.1. In this section we collect several important consequences (mainly inequalities) of the main results of this chapter (which are stated in Section 3.1).

Lemma 3.4.1. *Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. Suppose $\phi \in L^1(\mathbb{R}^d)$ is such that*

$$\psi^{[\mathpzc{d},a]}(x) := \sup\{\, |\phi(y)| \,:\, |y|_{\mathpzc{d},a} \geq |x|_{\mathpzc{d},a} \,\}$$

defines a function $\psi^{[\mathpzc{d},a]} \in L^1(\mathbb{R}^d)$. Then there exists a constant $C_{p,w,\mathpzc{d},a} > 0$ (only depending on p, w, $\mathpzc{d}$, and a) such that, for all $f \in L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$,

$$\left\| \sup_{t>0} \left\|\phi_t^{[\mathpzc{d},a]} * f\right\|_X \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)} \leq C_{p,w,\mathpzc{d},a}\, \|\psi\|_{L^1(\mathbb{R}^d)}\, \|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)}\,.$$

Here $\phi_t^{[\mathpzc{d},a]} \in L^1(\mathbb{R}^d)$ is defined by $\phi_t^{[\mathpzc{d},a]} := t^{a\cdot\mathpzc{d}}\phi(\delta_t^{[\mathpzc{d},a]}x)$.

Proof. A straightforward modification of Lemma D.1.2 yields the pointwise domination

$$\left\|\phi_t^{[\mathpzc{d},a]} * f\,(x)\right\|_X \leq \left\|\psi^{[\mathpzc{d},a]}\right\|_{L^1(\mathbb{R}^d)} M^{[\mathpzc{d},a]}\, \|f\|_X\,(x), \qquad x \in \mathbb{R}^d.$$

The desired result now follows from Corollary 3.1.5. □

Lemma 3.4.2. *Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. Suppose that $\phi \in C_c(\mathbb{R}^d)$ is such that $\phi \geq 0$ and $\int_{\mathbb{R}^d}\phi(x)dx = 1$. For $f \in L^{p,d}(\mathbb{R}^d, w; X)$ we have $\phi_t * f \overset{t\to\infty}{\longrightarrow} f$ both in $L^{p,d}(\mathbb{R}^d, w; X)$ and pointwise almost everywhere, where ϕ_t is given by $\phi_t = t^d\phi(t\,\cdot)$.*

Proof. The pointwise almost everywhere convergence is a consequence of Lemma D.1.4 and Theorem D.1.5; recall here that $L^{p,d}(\mathbb{R}^d, w; X) \hookrightarrow L^1_{loc}(\mathbb{R}^d; X)$ (see Lemma 2.2.4). For the convergence in $L^{p,d}(\mathbb{R}^d, w; X)$ we pick $(t_k)_{k\in\mathbb{N}} \subset]0,\infty[$ such that $t_k \nearrow \infty$ (as $k \to \infty$). Since we have the domination $\left\|\phi_{t_k} * f\right\|_X \lesssim M(\|f\|_X)$ by Lemma D.1.2, and since $M(\|f\|_X) \in L^{p,d}(\mathbb{R}^d, w; X)$ by Corollary 3.1.5, the Lebesgue dominated convergence theorem tells us that $\phi_{t_k} * f \overset{k\to\infty}{\longrightarrow} f$ in $L^{p,d}(\mathbb{R}^d, w; X)$. □

Lemma 3.4.3. *Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$. Then*

$$\mathcal{S}_0(\mathbb{R}^d;X) := \{f \in \mathcal{S}(\mathbb{R}^d;X) \,:\, 0 \notin \operatorname{supp}\hat{f} \text{ compact}\}$$

is dense in $L^{p,d}(\mathbb{R}^d,w;X)$.

Proof. In view of $\mathscr{F}^{-1}C_c^\infty(\mathbb{R}^d;X) \stackrel{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^d;X)$ (see Appendix C.3) and $\mathcal{S}(\mathbb{R}^d;X) \stackrel{d}{\hookrightarrow} L^{p,d}(\mathbb{R}^d,w;X)$ (see Lemma 2.2.4), it suffices to approximate an $f \in \mathscr{F}^{-1}C_c^\infty(\mathbb{R}^d;X)$ with a sequence from $\mathcal{S}_0(\mathbb{R}^d;X)$ in the $L^{p,d}(\mathbb{R}^d,w;X)$-norm. So fix such an f. Let $\phi \in \mathscr{F}^{-1}C_c^\infty(\mathbb{R}^d)$ be such that $\hat{\phi}$ is 1 on a neighborhood of 0. Then we have

$$f - \phi_t * f = \mathscr{F}^{-1}[(1-\widehat{\phi_t})\hat{f}] = \mathscr{F}^{-1}\left[(1-\hat{\phi}(t^{-1}\,\cdot\,))\hat{f}\right] \in \mathcal{S}_0(\mathbb{R}^d;X), \qquad t > 0,$$

where $\phi_t = t^d\phi(t\,\cdot\,)$. So it suffices to find a sequence $(t_k)_{k\in\mathbb{N}} \subset]0,\infty[$ such that $\lim_{k\to\infty}\phi_{t_k} * f = 0$ in $L^{p,d}(\mathbb{R}^d,w;X)$. By the Lebesgue dominated convergence theorem, since $\|\phi_t * f\|_X \lesssim M(\|f\|_X)$ by Lemma D.1.2, and since $M(\|f\|_X) \in L^{p,d}(\mathbb{R}^d,w;X)$ by Corollary 3.1.5, for this it is in turn enough to find a sequence $(t_k)_{k\in\mathbb{N}} \subset]0,\infty[$ such that $\lim_{k\to\infty}\phi_{t_k} * f = 0$ pointwise almost everywhere. To this end, let $q \in]1,\infty[$. Then, by Young's inequality (cf. Theorem A.1.4),

$$\limsup_{t\to 0}\|\phi_t * f\|_{L^q(\mathbb{R}^d;X)} \leq \limsup_{t\to 0}\|\phi_t\|_{L^q(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^d;X)} = \limsup_{t\to 0} t^{d(1-\frac{1}{q})}\|\phi\|_{L^q(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^d;X)} = 0.$$

Hence, $\lim_{t\to 0}\phi_t * f = 0$ in $L^q(\mathbb{R}^d;X)$. In particular, there exists a sequence $(t_k)_{k\in\mathbb{N}} \subset]0,\infty[$ with $t_k \stackrel{k\to\infty}{\longrightarrow} 0$ such that $\phi_t * f \stackrel{k\to\infty}{\longrightarrow} 0$ pointwise almost everywhere. □

Corollary 3.4.4. *Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$. Then $\mathcal{S}_0(\mathbb{R}^{d_1}) \otimes \ldots \otimes \mathcal{S}_0(\mathbb{R}^{d_l}) \otimes X$ is dense in $L^{p,d}(\mathbb{R}^d,w;X)$. As a consequence,*

$$\mathcal{S}_{0,d}(\mathbb{R}^d;X) := \left\{ f \in \mathcal{S}(\mathbb{R}^d;X) \,:\, \operatorname{supp}\hat{f} \text{ compact}, \operatorname{supp}\hat{f} \cap \prod_{j=1}^{l}[\mathbb{R}^{d_j}\setminus\{0\}] = \emptyset \right\}.$$

is dense in $L^{p,d}(\mathbb{R}^d,w;X)$.

We next present some inequalities (which are very important for Chapter 5). For this we first need to introduce the following notation:

Notation 3.4.5. Let X be a Banach space, $p \in [1,\infty[^l$ and $w \in \prod_{j=1}^{l}\mathcal{W}(\mathbb{R}^{d_j})$. For a sequence $(f_k)_{k\in\mathbb{N}}$ of (equivalence classes of) strongly measurable functions $\mathbb{R}^d \longrightarrow X$ we use the notations:

$$\|(f_k)_{k\in\mathbb{N}}\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} := \left\|\left(\big\|\|f_k\|_X\big\|_{L^{p,d}(\mathbb{R}^d,w)}\right)_{k\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})}$$

$$\|(f_k)_{k\in\mathbb{N}}\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)} := \left\|\big\|(\|f_k\|_X)_{k\in\mathbb{N}}\big\|_{\ell^q(\mathbb{N})}\right\|_{L^{p,d}(\mathbb{R}^d,w)}.$$

Remark 3.4.6. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence of (equivalence classes of) strongly measurable functions $\mathbb{R}^d \longrightarrow X$ with $\|(f_k)_{k\in\mathbb{N}}\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} < \infty$. If $q < \infty$, then $(f_k)_{k\in\mathbb{N}}$ can be identified with an element F of $L^{p,d}(\mathbb{R}^d,w;\ell^q(\mathbb{N};X))$ in the natural way. This can be shown in an elementary way, but can also be seen as a special case of Theorem B.2.7.

Given a function $f : \mathbb{R}^d \longrightarrow X$, $r \in]0,\infty[^l$ and $b \in]0,\infty[^l$, we define the *maximal function of Peetre-Fefferman-Stein type* $f^*(r,b,\mathpzc{d};\,\cdot\,)$ by

$$f^*(r,b,\mathpzc{d};x) := \sup_{z\in\mathbb{R}^d} \frac{\|f(x-z)\|_X}{(1+|b_1z_1|^{d_1/r_1})\dots(1+|b_lz_l|^{d_l/r_l})}, \qquad x \in \mathbb{R}^d. \tag{3.50}$$

Proposition 3.4.7. *Let X be a Banach space, $p \in [1,\infty[^l$, $q \in [1,\infty]$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Let $r \in]0,1[^l$ be such that $w_j \in A_{p_j/r_j}(\mathbb{R}^{d_j})$ for $j = 1,\dots,l$. Then there exists a constant $C > 0$ such that, for all $(f_n)_{n\in\mathbb{N}} \in L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N};X))$ and $(b^{[n]})_{n\in\mathbb{N}} \subset]0,\infty[^l$ with $f_n \in \mathcal{S}'(\mathbb{R}^d;X)$ and* $\operatorname{supp}\hat{f} \subset Q_{\mathpzc{d},b^{[n]}}$ *for all $n \in \mathbb{N}$, we have the inequalities*

$$\left\|(f_n^*(r,b^{[n]},\mathpzc{d};\,\cdot\,))_{n\geq 0}\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N}))} \leq C\,\|(f_n)_n\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N};X))}$$

and

$$\left\|(f_n^*(r,b^{[n]},\mathpzc{d};\,\cdot\,))_{n\geq 0}\right\|_{\ell^q(\mathbb{N};L^{p,\mathpzc{d}}(\mathbb{R}^d,w))} \leq C\,\|(f_n)_n\|_{\ell^q(\mathbb{N};L^{p,\mathpzc{d}}(\mathbb{R}^d,w))}\,.$$

Proof. We only treat the first inequality, the second one being similar (and easier). As in the proof of [62, Proposition 3.12], it can be shown that

$$f_n^*(r,b,\mathpzc{d};x) \leq c\left[M_{[\mathpzc{d};l],r_l}(\dots M_{[\mathpzc{d};1],r_1}(\|f_n\|_X)\dots)\right](x), \qquad n \in \mathbb{N}, x \in \mathbb{R}^d$$

for some constant $c > 0$ only depending on r. The desired result now follows from Theorem 3.1.4. □

Proposition 3.4.8. *Let X and Y be Banach spaces, $p \in [1,\infty[^l$, $q \in [1,\infty]$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Let $r \in]0,\infty[^l$ be such that $r_j < \min\{p_1,\dots,p_j,q\}$ and $w_j \in A_{p_j/r_j}(\mathbb{R}^{d_j})$ for $j = 1,\dots,l$. Then, for each $c > 0$, there exists a constant $C > 0$ such that, for all $(M_n)_{n\in\mathbb{N}} \subset \mathscr{F}L^1(\mathbb{R}^d;\mathcal{L}(X,Y))$ and all $(f_n)_{n\in\mathbb{N}} \in L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N};X))$ and $(b^{[n]})_{n\in\mathbb{N}} \subset]0,\infty[^l$ with $f_n \in \mathcal{S}'(\mathbb{R}^d;X)$ and* $\operatorname{supp}\hat{f} \in Q_{\mathpzc{d},cb^{[n]}}$ *for all $n \in \mathbb{N}$, it holds that*

$$\begin{aligned}&\left\|(\mathscr{F}^{-1}(M_n\mathscr{F}f_n))_{n\in\mathbb{N}}\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N};Y))}\\ &\quad\leq C\sup_{k\geq 0}\left\|\prod_{j=1}^l(1+|\,\cdot\,|^{d_j/r_j}))\mathscr{F}^{-1}M_k(b_1^{[k]}\,\cdot\,,\dots,b_l^{[k]}\,\cdot\,)\right\|_{L^1(\mathbb{R}^d;\mathcal{L}(X,Y))}\|(f_n)_{n\in\mathbb{N}}\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N};X))}\end{aligned}$$

and

$$\begin{aligned}&\left\|(\mathscr{F}^{-1}(M_n\mathscr{F}f_n))_{n\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N};L^{p,\mathpzc{d}}(\mathbb{R}^d,w))}\\ &\quad\leq C\sup_{k\geq 0}\left\|\prod_{j=1}^l(1+|\,\cdot\,|^{d_j/r_j}))\mathscr{F}^{-1}M_k(b_1^{[k]}\,\cdot\,,\dots,b_l^{[k]}\,\cdot\,)\right\|_{L^1(\mathbb{R}^d;\mathcal{L}(X,Y))}\|(f_n)_{n\in\mathbb{N}}\|_{\ell^q(\mathbb{N};L^{p,\mathpzc{d}}(\mathbb{R}^d,w))}\,.\end{aligned}$$

Proof. First we observe that, for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$,

$$\left\|\mathscr{F}^{-1}(M_n\mathscr{F}f_n)(x)\right\|_Y \leq \sup_{z\in\mathbb{R}^d}\frac{\left\|\mathscr{F}^{-1}(M_n\mathscr{F}f_n)(x-z)\right\|_Y}{(1+|cb_1^{[n]}z_1|^{d_1/r_1})\dots(1+|cb_m^{[n]}z_m|^{d_m/r_m})}. \tag{3.51}$$

Next we estimate

$$
\begin{aligned}
\left\|\mathscr{F}^{-1}(M_n\mathscr{F}f_n)(x-z)\right\|_Y &\leq \int_{\mathbb{R}^d}\left\|\mathscr{F}^{-1}M_n(x-z-y)\right\|_{\mathcal{L}(X,Y)}\|f_n(y)\|_X\,dy \\
&\leq \int_{\mathbb{R}^d}\left\|\mathscr{F}^{-1}M_n(x-z-y)\right\|_{\mathcal{L}(X,Y)}\prod_{j=1}^m(1+|cb_j^{[n]}(x_j-y_j)|^{d_j/r_j})dy \\
&\qquad\cdot\sup_{u\in\mathbb{R}^d}\frac{\|f_n(u)\|_X}{\prod_{j=1}^m(1+|cb_j^{[n]}(x_j-u_j)|^{d_j/r_j})} \\
&= \int_{\mathbb{R}^d}\left\|\mathscr{F}^{-1}M_n(x-z-y)\right\|_{\mathcal{L}(X,Y)}\prod_{j=1}^m(1+|cb_j^{[n]}(x_j-y_j)|^{d_j/r_j})dy \\
&\qquad\cdot f_n^*(r,cb^{[n]},\mathpzc{d};x),
\end{aligned}
$$

and note that

$$
\prod_{j=1}^m(1+|cb_j^{[n]}(x_j-y_j)|^{d_j/r_j})\leq C_1\prod_{j=1}^m\left[(1+|b_j^{[n]}(x_j-z_j-y_j)|^{d_j/r_j})(1+|cb_j^{[n]}z_j)|^{d_j/r_j})\right]
$$

for some constant $C_1>0$ independent of n,x,y,z. Combining this with (3.51) we obtain

$$
\begin{aligned}
\left\|\mathscr{F}^{-1}(M_n\mathscr{F}f_n)(x)\right\|_Y &\leq C_1\sup_{z\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left\|\mathscr{F}^{-1}M_n(x-z-y)\right\|_{\mathcal{L}(X,Y)}\prod_{j=1}^m(1+|b_j^{[n]}(x_j-z_j-y_j)|^{d_j/r_j})dy \\
&\qquad\cdot f_n^*(r,b^{[n]},\mathpzc{d};x) \\
&= C_1\int_{\mathbb{R}^d}\left\|\mathscr{F}^{-1}M_n(y)\right\|_{\mathcal{L}(X,Y)}\prod_{j=1}^m(1+|b_j^{[n]}y_j|^{d_j/r_j})dy\,f_n^*(r,cb^{[n]},\mathpzc{d};x).
\end{aligned}
$$

The proof is now completed by observing that

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\left\|\mathscr{F}^{-1}M_n(y)\right\|_{\mathcal{L}(X,Y)}\prod_{j=1}^m(1+|b_j^{[n]}(y_j)|^{d_j/r_j})dy \\
&= \int_{\mathbb{R}^d}\prod_{j=1}^m(1+|y_j|^{d_j/r_j})\left\|(\mathscr{F}^{-1}[M_n(b_1^{[n]}\cdot,\ldots,b_m^{[n]}\cdot)])(y)\right\|_{\mathcal{L}(X,Y)}dy \\
&\leq \sup_{k\geq 0}\left\|\prod_{j=1}^m(1+|\cdot|^{d_j/r_j}))\mathscr{F}^{-1}[M_k(b_1^{[k]}\cdot,\ldots,b_m^{[k]}\cdot)]\right\|_{L^1(\mathbb{R}^d;\mathcal{L}(X,Y))}
\end{aligned}
$$

and applying Proposition 3.4.7 to $(f_n)_{n\in\mathbb{N}}$. □

Proposition 3.4.9. *Let X be a Banach space, $p\in[1,\infty[^l$, $q\in[1,\infty]$, and $w\in\prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Suppose $r\in]0,1[^l$ is such that $w_j\in A_{p_j/r_j}(\mathbb{R}^{d_j})$ for $j=1,\ldots,l$. Let $\psi\in\mathcal{S}(\mathbb{R}^d)$ be such that $\operatorname{supp}\hat{\psi}\subset\{\xi\in\mathbb{R}^d\mid|\xi|_{\mathpzc{d},a}\leq 2\}$, and set $\psi_n:=\psi(\delta_{2^n}^{[\mathpzc{d},a]}\cdot)$ for each $n\in\mathbb{N}$. Then there exists a constant $C>0$ such that, for all $(f_n)_{n\in\mathbb{N}}\subset\mathcal{S}'(\mathbb{R}^d;X)$ with $\operatorname{supp}\hat{f}_n\subset\prod_{j=1}^l[-R2^{na_j},R2^{na_j}]^{d_j}$ for some $R\geq 1$, the following inequality holds true:*

$$
\|(\psi_n*f_n)_{n\geq 0}\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N};X))}\leq CR^{\sum_{j=1}^l a_jd_j(\frac{1}{r_j}-1)}\|(f_n)_{n\geq 0}\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N};X))}\,.
$$

Proof. As in the proof of [62, Proposition 3.14], it can be shown that

$$\|(\psi_n * f_n)(x)\|_X \leq cR^{\sum_{j=1}^{m} a_j \mathpzc{d}_j(\frac{1}{r_j}-1)} \left[M_{[\mathpzc{d};l],r_l}(\ldots M_{[\mathpzc{d};1],r_1}(\|f_n\|_X)\ldots)\right](x), \qquad n \in \mathbb{N}, x \in \mathbb{R}^d,$$

for some constant $c > 0$ independent of n. The desired result now follows from Theorem 3.1.4. □

Lemma 3.4.10. *Fix $m \in \mathbb{N}$ and $r \in]0,1]$. Let X be a Banach space, E a UMD Banach function space, $p \in]1,\infty[$, and $w \in A_p(\mathbb{R}^d)$. For each $c > 0$ there exists a constant $C > 0$ such that, for all $f \in L^p(\mathbb{R}^d, w; E(X)) \subset \mathcal{S}'(\mathbb{R}^d; E(X))$ with $\operatorname{supp} \hat{f} \subset B(0,R)$ for some $R > 0$, the following inequality holds for all $x, h \in \mathbb{R}^d$:*

$$\left\|\Delta_h^m f(x)\right\|_X \leq C\,(M_r\,\|f\|_X)\,(x) \begin{cases} (R|h|)^m, & \text{if } |h| \leq R^{-1}; \\ (R|h_{j_0}|)^{d/r}, & \text{if } |h| > R^{-1}. \end{cases}$$

Here we write

$$\Delta_h^m f(x) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(x + (m-k)h), \qquad x, h \in \mathbb{R}^d.$$

Proof. This can be shown similarly to [88, Lemma 4]; here we just have to do some of the computations pointwise in the Banach function space E. □

3.5 Notes

3.5.1 General Notes

The unweighted version of Theorem 3.1.1 is basically due to Bourgain [11]. Actually, Bourgain considered a UMD Banach space X with a normalized unconditional basis and defines the Hardy-Littlewood maximal function operator coordinatewise (which can be interpreted as a generalization of the classical Feffermann-Stein inequality for $X = \ell^q(\mathbb{N})$, $q \in]1,\infty[$). However, such a Banach space can be naturally viewed as a Banach function space on the σ-finite measure space $(\mathbb{N}, \mathcal{P}, \#)$, and in this way the coordinatewise defined Hardy-Littlewood maximal function coincides with the one from Theorem 3.1.1. This was extended by Rubio de Francia [84] to general UMD Banach function spaces (on σ-finite measure spaces) for the case $n = 1$ by basically following the argumentation of Bourgain pointwise in the Banach function space instead of coordinatewise in the Banach space X (which of course can be viewed as pointwise in $\mathbb{N}$). Here one simplification is made which explains the restriction $n = 1$, but without this simplification the argumentation of Bourgain can be extended to general n; also see below (Section 3.5.2). Motivated by [11, 84], Garcia-Cuerva, Macias & Torrea [41] introduced the Hardy-Littlewood (H.L.) property for Banach lattices: a Banach lattice E is said to have the H.L. property if there exists a constant $C \geq 0$ and a $p \in]1,\infty[$ for which the sets (3.3) (with f ranging over $L^p(\mathbb{R}^n; E)$) are norm bounded by $C\,\|f\|_{L^p(\mathbb{R}^d;E)}$ in $L^p(\mathbb{R}^n; E)$; here it is shown that the H.L. property does not depend on the dimension n (so that it is a well-defined notion). We would like to remark that in [41] it is only mentioned that every UMD Banach function space on a σ-finite measure space has the H.L. property but that nothing is said about general UMD Banach lattices (as in Remark3.1.2). The main idea in [41] is to pass, for each J, to a smooth version $M_{\phi,J}$ of M_J (which is equivalent to M_J in the sense of domination), to which a Caldéron-Zygmun operator can be associated in a natural way, which allows the authors to apply the theory of vector-valued singular integrals to obtain several characterizations of the H.L. property (amongst which the irrelevance of the exponent $p \in]1,\infty[$ in the definition). As a direct consequence of one of these characterizations, we find that the H.L. property implies its A_p-weighted version H.L.$_w$ (which is defined by replacing $L^p(\mathbb{R}^n;E)$ by $L^p(\mathbb{R}^n,w;E)$ in the definition of H.L. plus stating the dependence of the constant C on the A_p-weight w). Therefore, as every UMD Banach function space has the H.L. property [84, 41] (also see Remark 3.1.2), we obtain Theorem 3.1.1 as a consequence of Remark 3.1.2 (at least without the precise dependence of the constant). Combining the ideas from [41] with [49, Corollary 2.10] (see Theorem 4.4.3), it can in fact be shown that the constant $C \geq 0$ in Theorem 3.1.1 can be chosen in such a way that $C \leq [w]_{A_p}^{\max\{1,1/(p-1)\}}C'$ for some constant C' only depending on p and F (via the UMD constant of F). For a more elementary approach to Theorem 3.1.1 in the special case that F is a mixed-norm L^q-space, we would like to mention [40]

Further works in this direction include [95],[39], [42], and [71].

Finally, we would like to mention that unweighted version of Theorem 3.1.4 (in the case $r_{j_0} = 1$) is a classical result due to Bagby [7].

3.5.2 Comparison to the Literature

- *Section 3.1*: As already mentioned above, Theorem 3.1.1 can be obtained by a combination of [84], [41] and Remark 3.1.2. However, we have decided to follow a different strategy (see the discussions of Sections 3.2 and 3.3 below): we prove a more general maximal inequality for martingales with values in a UMD Banach function space (Theorem 3.3.5) from which Theorem 3.1.1 can be deduced. An important tool is the theory of mixed-norm spaces from Appendix B.2. In this chapter we aim at giving a reasonably self-contained systematic treatment. For example, we have (explicitly) included Remark 3.1.2 for more transparency in the relations between the several possible definitions of M.

- *Section 3.2*: In [84, 41], (equivalence classes of) functions $f \in L^p(\mathbb{R}^n,w;F)$, or more generally $f \in L^1_{loc}(\mathbb{R}^n;F)$, are also viewed as (equivalence classes of) measurable functions on the product $\mathbb{R}^n \times T$ (without explicit reference to the mixed norm space $L^p(\mathbb{R}^n,w)[F]$ and the canonical identification $L^p(\mathbb{R}^n,w;F) \simeq L^p(\mathbb{R}^n,w)[F]$, for which we need restrictions on F). For example, in [41] it was already observed that, a Banach function space F on a σ-Finite measure space having the σ-Fatou property satisfies the H.L. property (see Section 3.5.1) if and only if M (defined by the RHS of (3.5)) is bounded on $L^p(\mathbb{R}^n,w)[F]$; a detailed description of this equivalence can be found in Remark 3.2.3. The maximal inequality in UMD Banach function spaces due to Rubio de Francia [84] basically corresponds to (the formulation of) Theorem 3.2.2; in [84], M is actually defined on $L^p(\mathbb{R}) \otimes F$, and $L^p(\mathbb{R};F)$ is identified with a Banach function space (without explicit mentioning $L^p(\mathbb{R})[F]$). For convenience of the reader, in this section (and also in Section 3.3) we have tried to be more detailed on and explicit in these identifications (for which the required theory is documented in Appendix B.2).

  Theorem 3.2.2 is for us not just a convenient reformulation of Theorem 3.1.1 (in terms of abstract mixed-norm spaces), but is actually (more or less) the abstraction of the main motivation for this chapter, Theorem 3.1.4. As already mentioned before, this theorem is an extension to the weighted setting of a classical result of Bagby [7], which was obtained in a completely different and more elementary way (before the start of UMD-theory).

Finally, we would like to mention that the argumentation used in the proof of Lemma 3.2.1 is a straightforward extension (of a slight modification) of the argumentation used in the proof of [87, Lemma 19.16] (about the ordinary Hardy-Littlewood maximal function).

- *Section 3.3*: The most important references for this section are [57],[12, 84], [94, 95], and [44, 45].
  - *Section 3.3.1*: The reduction of Theorem 3.2.2 to the boundedness of the shifted dyadic maximal function operators M'_{ω}, $\omega \in \{0, \omega^{odd}, \omega^{even}\}^n$, (which is in fact an equivalent problem) is based on [57], where such a reduction is performed for (basically) the ordinary Hardy-Littlewood maximal function operator (in the unweighted setting). Here we have made a minor modification in the covering lemma, Lemma 3.3.1, in order that the corresponding filtrations $(\mathscr{F}^{\omega}_k)_{k\in\mathbb{Z}}$ get the right structure (the property of being regular).

    The abstract setting (concerning the measure space $(\Sigma, \mathscr{F}, \mu)$, the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$, and the weight W) can be seen as an extension of the weighted setting considered by Tozoni [94, 95] (where the measure space is a probability space and the filtration is indexed by the natural numbers $\mathbb{N}$). Proposition 3.3.2.(v) corresponds to [94, Lemma 4.1] and we follow the proof of the reference given there, which is [59, Theorem 2]. The proof of Proposition 3.3.2.(vi) is taken from [57] (which is about the unweighted case). Surprisingly, Proposition 3.3.2.(iii), or in fact Proposition A.3.11, is new; it is also not treated in the probabilistic setting of Tozoni [94, 95] (and predecessors). The main advantage of having this $L^p(W;X)$-contractivity of the conditional expectation is that it allows us to define the Banach space $\mathscr{M}_{L^p(W)}(X)$ of all $L^p(W)$-bounded X-valued martingales with its natural norm $\|\cdot\|_{\mathscr{M}_{L^p(W)}(X)}$ (3.12), in which $L^p(W;X)$ is isometrically contained in the natural way (3.13).[6] This space is (at least implicitly) also in the work of Burkholder in the unweighted probabilistic setting; see the survey article [22]. Finally, Theorem 3.3.5 is new and should be interpreted as a natural (abstract martingale theoretic) generalization of the well-definedness and boundedness of the (shifted) dyadic Banach lattice Hardy-Littlewood maximal function operators on $L^p(\mathbb{R}^n, w; F)$; see the discussions about Sections 3.3.2 and 3.3.3 below for more information on the proof of this theorem. For a similar observation we refer to [71, Remark 6], which is concerned with the probilistic Hardy-Littlewood property of Banach lattices.
  - *Section 3.3.2*: The idea of the proof of Theorem 3.3.5 given in this subsection is basically due to Bourgain [12] (who restricts itself to the dyadic filtration on $[0,1]$ for simplicity). Actually, our proof is an extension and an adaption of a slight modification of the proof of [84, Theorem 3.i)⇒iii)], in which the argumentation of Bourgain was already extended to Banach function spaces; also see the beginning of Section 3.5.1 above and [95, Theorem 3.1]. We would like to mention that one modification of Bourgain's proof made by Rubio de Francia [84] is the use of the Hilbert

[6] Without this contractivity we could take $\|g\|_{\mathscr{M}_{L^p(W)}(X)} := \limsup_{k\to\infty} \|g_k\|_{L^p(W;X)}$ in order to have the isometric embedding (3.13), which is less elegant.

transform H followed by the modulus in place of the square function operator S in the construction of Φ_ϕ; for clarity, here the dyadic Hardy-Littlewood maximal function operator M_d on the 1-dimensional torus $\mathbb{T}$ is considered (as the main part of the proof). The main advantage of this modification is that Lemma 3.3.9 can be replaced by a much simpler variant for the Hilbert transform. However, this modification can only be made in the 1-dimensional case (which was enough for Rubio de Francia's purposes anyway) and certainly cannot be done in our abstract setting.

In the improved form of Remark 3.3.7.(ii), Proposition 3.3.6 can be seen as a natural generalization of the weighted-L^p square function estimates of Tozoni [94, 95] (see especially the formulation given in [95, Theorem III]) concerning the probabilistic setting with one-sided filtrations (i.e. filtrations indexed by $\mathbb{N}$). The unweighted version of Proposition 3.3.6 is due to Bourgain [12]; we present a slightly different more modern proof of this unweighted version (as Remark 3.3.8) using the notion of finite cotype together with the Khintchine-Maurey theorem (cf. Theorem E.2.2).

For more information on (the weighted version of) Proposition 3.3.6 and on Lemma 3.3.9, we refer to the discussion about Section 3.3.3 below; also see Remark 3.3.10.

– *Section 3.3.3*: The main point of this subsection is to prove Proposition 3.3.19 and Lemma 3.3.20, the desired Proposition 3.3.6 and Lemma 3.3.9 being easy consequences of these two results.

  Proposition 3.3.19 is (together with the required preparations) mainly based on the work of Tozoni [94, 95], which in turn was to a large extend based on [10, 14, 68, 59]. In fact, besides some technical modifications (mostly required for our setting) and a slightly different presentation, Sections 3.3.3.b-3.3.3.d are completely based on [94] (and on some of the references given therein).

  As should be clear from the title of Subsubsection 3.3.3.e, Lemma 3.3.9 is due to Bourgain [12]. Actually, Bourgain only proved this result for the dyadic filtration on $[0, 1]$, but the same proof can be used to establish (3.46) (which corresponds to the reduced situation obtained in Step I of our proof); also see [95, Theorem I] and the comments after it. This is worked out in a detailed self-contained way in Steps II-V, where we replaced the application of the $L \log L$ result in Bourgain's proof by Steps III&IV. The argumentation used in Steps III&IV is inspired by the proofs of [45, Lemma 7.5.4] and [44, Corollary 2.1.21].

• *Section 3.4*: Proposition 3.4.8 and Lemma 3.4.3 are extensions of [77, Proposition 2.4] and [105, Lemma 2.3], respectively. For the remaining literature used in this section, see the references given in the main text.

Chapter 4

Fourier Multipliers

In this chapter we prove several $(\mathpzc{d}, a)$-anisotropic Mikhlin Fourier multiplier theorems on the weighted mixed-norm Lebesgue-Bochner spaces $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ for UMD spaces X.

4.1 Introduction

Unless stated otherwise or unless clear from the context, throughout this chapter we view $\mathbb{R}^d$ as being $\mathpzc{d}$-decomposed as in Convention 2.2.1.

Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$. A function $m \in L^\infty(\mathbb{R}^d; \mathcal{B}(X))$ is called a *Fourier multiplier* on $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ if the Fourier multiplier operator

$$T_m : \mathcal{S}(\mathbb{R}^d; X) \longrightarrow C_0^\infty(\mathbb{R}^d; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X),\ f \mapsto \mathscr{F}^{-1}[m\hat{f}]$$

(takes its values in $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ and) extends to a bounded linear operator T_m on $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$.

The classical Mikhlin theorem says that for $m \in L^\infty(\mathbb{R}^d)$ to be a Fourier multiplier on $L^p(\mathbb{R}^d)$ ($p \in]1,\infty[$), it is sufficient that m belongs to $C^N(\mathbb{R}^d \setminus \{0\})$, where $N = N_d := [d/2] + 1 \in \mathbb{N}$, and satisfies the Mikhlin condition

$$C_m := \sup\{|\xi|^{|\theta|}\,|D^\theta m(\xi)| \,:\, \xi \in \mathbb{R}^d \setminus \{0\}, |\theta| \leq N\} < \infty, \tag{4.1}$$

in which case we have $\|T_m\|_{\mathcal{B}(L^p(\mathbb{R}^d))} \lesssim_{p,d} C_m$.

The classical Mikhlin theorem can, for example, be used to prove that

$$W_p^k(\mathbb{R}^d) = H_p^k(\mathbb{R}^d), \qquad p \in]1,\infty[, k \in \mathbb{N}, \tag{4.2}$$

with an equivalence of norms. Here $H_p^s(\mathbb{R}^d)$ stands for the Bessel potential space of order $s \in \mathbb{R}$, which is defined as follows: Since $\xi \mapsto (1 + |\xi|^2)^{s/2}$ belongs to $\mathscr{O}_M(\mathbb{R}^d)$, we may define the Bessel potential operator $\mathcal{J}_s \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^d))$ of order s by

$$\mathcal{J}_s f := \mathscr{F}^{-1}[(1 + |\cdot|^2)^{s/2}\hat{f}], \qquad f \in \mathcal{S}'(\mathbb{R}^d).$$

Having this Bessel potential operator, we define

$$H_p^s(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) \,:\, \mathcal{J}_s f \in L^p(\mathbb{R}^d)\}, \qquad \|f\|_{H_p^s(\mathbb{R}^d)} := \|\mathcal{J}_s f\|_{L^p(\mathbb{R}^d)}. \tag{4.3}$$

For Chapter 6 it will be important to generalize the identity (4.2) to the weighted anisotropic vector-valued setting. This requires a $(\mathpzc{d}, a)$-anisotropic version of the Mikhlin theorem on the weighted mixed-norm Lebesgue-Bochner space $L^{p,d}(\mathbb{R}^d, w; X)$ for scalar-valued symbols. In Chapter 6 we will furthermore need a version of the Mikhlin theorem on the weighted Lebesgue-Bochner space $L^p(\mathbb{R}^d, w; X)$ for operator-valued symbols. This motivates to investigate a $(\mathpzc{d}, a)$-anisotropic version of the Mikhlin theorem on the weighted mixed-norm Lebesgue-Bochner space $L^{p,d}(\mathbb{R}^d, w; X)$ for operator-valued symbols, containing both situations as special cases.

The following theorem is roughly a (non sharp) collection of all the operator-valued Mikhlin theorems from this chapter. Here the appropriate version of the classical Mikhlin condition (4.1) is an $\mathcal{R}$-boundedness $(\mathpzc{d}, a)$-anisotropic Mikhlin condition. For the notion of $\mathcal{R}$-boundedness we refer to Appendix E.3 and for more background on operator-valued Fourier multiplier theorems we refer to the notes of this chapter (and mainly the references given therein). For property (α) we refer to Appendix E.4; here we only need that property (α) is sufficient condition for Proposition E.4.4.

Theorem 4.1.1. *Suppose that $\mathbb{R}^d$ is $\mathpzc{d}$-decomposed as in Convention 2.2.1. Let X be a UMD space, $a \in]0,\infty[^l$, $p \in]1,\infty[^l$, and*

$$w \in \begin{cases} \prod_{j=1}^{l} A_{p_j}^{rec}(\mathbb{R}^{d_j}), & l > 1; \\ \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j}), & l = 1 \text{ or } X \text{ has property } (\alpha). \end{cases} \tag{4.4}$$

Then there exists an $N \in \mathbb{N}$ such that, for every $m \in L^\infty(\mathbb{R}^d; \mathcal{B}(X)) \cap C^N(\mathbb{R}^d \setminus \{0\}; \mathcal{B}(X))$ satisfying the anisotropic $\mathcal{R}$-boundedness Mikhlin condition

$$\kappa_m := \mathcal{R}\{\, |\xi|_{\mathpzc{d},a}^{a \cdot_{\mathpzc{d}} \theta} D^\theta m(\xi) \;:\; \xi \in \mathbb{R}^d \setminus \{0\}, |\theta| \leq N \,\} < \infty, \tag{4.5}$$

we have that the linear operator

$$T_m : \mathcal{S}(\mathbb{R}^d; X) \longrightarrow L^\infty(\mathbb{R}^d; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X),\; f \mapsto \mathscr{F}^{-1}[m\hat{f}],$$

takes its values in $L^{p,d}(\mathbb{R}^d, w; X)$ and extends to a (necessarily unique) bounded linear operator $T_m \in \mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X))$ of norm $\|T_m\|_{\mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))} \lesssim_{X,\mathpzc{d},a,p,w} \kappa_m$. Moreover, if in addition X has property (α), then for every collection of symbols $\mathscr{M} \subset L^\infty(\mathbb{R}^d; \mathcal{B}(X)) \cap C^N(\mathbb{R}^d \setminus \{0\}; \mathcal{B}(X))$ satisfying

$$\kappa_{\mathscr{M}} := \mathcal{R}\{\, |\xi|_{\mathpzc{d},a}^{a \cdot_{\mathpzc{d}} \theta} D^\theta m(\xi) \;:\; \xi \in \mathbb{R}^d \setminus \{0\}, |\theta| \leq N, m \in \mathscr{M} \,\} < \infty, \tag{4.6}$$

we have

$$\mathcal{R}\{T_m \;:\; m \in \mathscr{M}\} \lesssim_{X,\mathpzc{d},a,p,w} C_{\mathscr{M}} \quad \text{in} \quad \mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X)).$$

For scalar-valued symbols m the $\mathcal{R}$-bound in (4.5) coincides with the uniform bound of the set under consideration (see Example 4.5). As a consequence, we obtain the following anisotropic Mikhlin theorem for scalar-valued symbols:

Corollary 4.1.2. *Let X be a UMD space, $a \in]0,\infty[^l$, $p \in]1,\infty[^l$, and*

$$w \in \begin{cases} \prod_{j=1}^{l} A_{p_j}^{rec}(\mathbb{R}^{d_j}), & l > 1; \\ \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j}), & l = 1 \text{ or } X \text{ has property } (\alpha). \end{cases}$$

Then there exists an $N \in \mathbb{N}$ such that, for every $m \in L^\infty(\mathbb{R}^d) \cap C^N(\mathbb{R}^d \setminus \{0\})$ satisfying the anisotropic Mikhlin condition

$$C_m := \sup\{\, |\xi|_{\mathpzc{d},a}^{a \cdot_{\mathpzc{d}} \theta} |D^\theta m(\xi)| \;:\; \xi \in \mathbb{R}^d \setminus \{0\}, |\theta| \leq N \,\} < \infty, \tag{4.7}$$

we have that the linear operator

$$T_m : \mathcal{S}(\mathbb{R}^d; X) \longrightarrow L^\infty(\mathbb{R}^d; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X),\ f \mapsto \mathscr{F}^{-1}[m\hat{f}],$$

takes its values in $L^{p,d}(\mathbb{R}^d, w; X)$ *and extends to a (necessarily unique) bounded linear operator* $T_m \in \mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X))$ *of norm* $\|T_m\|_{\mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))} \lesssim_{X,d,a,p,w} C_m$*. Moreover, if in addition* X *has property* (α)*, then for every collection of symbols* $\mathscr{M} \subset L^\infty(\mathbb{R}^d) \cap C^N(\mathbb{R}^d \setminus \{0\})$ *satisfying*

$$C_{\mathscr{M}} := \sup\{ |\xi|_{d,a}^{a \cdot_d \theta} |D^\theta m(\xi)| \ : \ \xi \in \mathbb{R}^d \setminus \{0\}, |\theta| \leq N, m \in \mathscr{M} \} < \infty, \tag{4.8}$$

we have

$$\mathcal{R}\{T_m \ : \ m \in \mathscr{M}\} \lesssim_{X,d,a,p,w} C_{\mathscr{M}} \quad \text{in} \quad \mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X)).$$

Remark 4.1.3. We expect that Theorem 4.1.1 (and therefore also its corollary) remains valid for general weight-vectors $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$. The reason for the restriction (4.4) is as follows: First of all, we will see that it is only possible to follow a proof via unconditional Schauder decompositions (as in the unweighted case) for weight-vectors $w \in \prod_{j=1}^{l} A_{p_j}^{rec}(\mathbb{R}^{d_j})$ (Theorem 4.5.16). Second of all, having in particularly the unweighted case, the case of general weight-vectors $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$ could be obtained via extrapolation. However, we will only be able to do an extrapolation argument in the isotropic non-mixed norm case $l = 1$, $a = 1$ (Theorem 4.5.20). The only problem for the general case is that we do not have a (d, a)-anisotropic version of the extrapolation result for Caldéron-Zygmund operators, Theorem 4.4.2; the rest of the computations (Lemma 4.4.7) and arguments still work. Finally, in case that X has property (α), we can bootstrap this weighted isotropic non-mixed norm case, yielding a sufficient condition which is weaker than the condition from Theorem 4.1.1 (Theorem 4.5.21)

4.2 Definitions and Basic Properties

In this chapter our interest is Fourier multipliers on the mixed-norm weighted Lebesgue-Bochner spaces $L^{p,d}(\mathbb{R}^d, w; X)$ with $p \in]1, \infty[^l$ and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$. In order to give a meaningful definition of a Fourier multiplier m on $L^{p,d}(\mathbb{R}^d, w; X)$, i.e. which symbols m to allow and on which dense space to initially define the associated Fourier multiplier operator T_m, let us first look at the possibility of locally integrable symbols. Since

$$L^1_{loc}(\mathbb{R}^d; \mathcal{B}(X)) \times C_c^\infty(\mathbb{R}^d; X) \longrightarrow L^1(\mathbb{R}^d; X),\ (m, g) \mapsto mg$$

is a continuous bilinear map and since the inverse Fourier transform $\mathscr{F}^{-1}$ is continuous from $L^1(\mathbb{R}^d; X)$ to $C_0(\mathbb{R}^d; X) \subset L^\infty(\mathbb{R}^d; X)$ (by the Riemann-Lebesgue Theorem), it follows that

$$L^1_{loc}(\mathbb{R}^d; \mathcal{B}(X)) \times \mathscr{F}^{-1}C_c^\infty(\mathbb{R}^d; X) \longrightarrow C_0(\mathbb{R}^d; X),\ (m, f) \mapsto \mathscr{F}^{-1}[m\hat{f}]$$

is a continuous bilinear map when we equip $\mathscr{F}^{-1}C_c^\infty(\mathbb{R}^d; X)$ with the locally convex topology which makes $\mathscr{F}$ a topological linear isomorphism from $\mathscr{F}^{-1}C_c^\infty(\mathbb{R}^d; X)$ onto $C_c^\infty(\mathbb{R}^d; X) = \mathcal{D}(\mathbb{R}^d; X)$. In particular, given a function $m \in L^1_{loc}(\mathbb{R}^d; \mathcal{B}(X))$ we can define the continuous linear operator

$$T_m : \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^d; X)) \longrightarrow C_0(\mathbb{R}^d; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X),\ f \mapsto \mathscr{F}^{-1}[m\hat{f}]. \tag{4.9}$$

The linear space $\mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^d;X))$ being a dense subspace of $L^{p,d}(\mathbb{R}^d,w;X)$, it makes sense to call m a *Fourier multiplier* on $L^{p,d}(\mathbb{R}^d,w;X)$ provided that T_m (takes its values in $L^{p,d}(\mathbb{R}^d,w;X)$ and) satisfies the norm estimate $\|T_mf\|_{L^{p,d}(\mathbb{R}^d,w;X)} \leq C_m \|f\|_{L^{p,d}(\mathbb{R}^d,w;X)}$ for some constant $C_m > 0$ and all $f \in \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^d;X))$, so that T_m extends to a bounded linear operator T_m on $L^{p,d}(\mathbb{R}^d,w;X)$, called the associated *Fourier multiplier operator*. The following lemma says that for this to be the case it is necessary that $m \in L^\infty(\mathbb{R}^d;X)$:

Lemma 4.2.1. *Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. If $m \in L^1_{loc}(\mathbb{R}^d;\mathcal{B}(X))$ is a Fourier multiplier on $L^{p,d}(\mathbb{R}^d,w;X)$ (in the sense discussed above), then $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X))$ and $\|m\|_{L^\infty(\mathbb{R}^d;\mathcal{B}(X))} \lesssim \|T_m\|_{\mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))}$.*

Proof. Pick two Schwartz functions $\phi,\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}\hat{\phi}(\xi)\hat{\psi}(\xi)d\xi = 1$ and $\hat{\phi},\hat{\psi} \in C_c^\infty(\mathbb{R}^d;X)$. Let $\xi_0 \in \mathbb{R}^d$ be a Lebesgue point of m (see Definition D.1.3) and let $x \in B_X$. Then, by Proposition D.1.4 and the basic properties of the Fourier transform, we have

$$\begin{aligned} m(\xi_0)x &= \lim_{\epsilon\searrow 0}\frac{1}{\epsilon^d}\int_{\mathbb{R}^d} m(\xi)\hat{\phi}\left(\frac{\xi-\xi_0}{\epsilon}\right)x\,\hat{\psi}\left(\frac{\xi-\xi_0}{\epsilon}\right)d\xi \\ &= \lim_{\epsilon\searrow 0}\epsilon^d\int_{\mathbb{R}^d} m(\xi)\mathscr{F}[e_{\xi_0}\phi(\epsilon\,\cdot\,)](\xi)x\,\mathscr{F}[e_{-\xi_0}\psi(\epsilon\,\cdot\,)]d\xi \\ &= \lim_{\epsilon\searrow 0}\epsilon^d\int_{\mathbb{R}^d} T_m[e_{\xi_0}\phi(\epsilon\,\cdot\,)x](y)\,e_{-\xi_0}(y)\psi(\epsilon y)dy. \end{aligned}$$

Denoting be $w' = (w'_1,\ldots,w'_l) \in \prod_j^l A_{p'_j}$ the vector of p-dual weights, we can estimate

$$\begin{aligned} \|m(\xi_0)x\|_X &\leq \liminf_{\epsilon\searrow 0}\epsilon^d\int_{\mathbb{R}^d}\left\|T_m[e_{\xi_0}\phi(\epsilon\,\cdot\,)x](y)\right\|_X|\psi(\epsilon y)|dy \\ &\leq \liminf_{\epsilon\searrow 0}\epsilon^d\left\|T_m[e_{\xi_0}\phi(\epsilon\,\cdot\,)x]\right\|_{L^{p,d}(\mathbb{R}^d,w;X)}\|\psi(\epsilon\,\cdot\,)\|_{L^{p',d}(\mathbb{R}^d,w')} \\ &\leq \liminf_{\epsilon\searrow 0}\epsilon^d\|T_m\|_{\mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))}\left\|e_{\xi_0}\phi(\epsilon\,\cdot\,)x\right\|_{L^{p,d}(\mathbb{R}^d,w;X)}\|\psi(\epsilon\,\cdot\,)\|_{L^{p',d}(\mathbb{R}^d,w')} \\ &\leq \|T_m\|_{\mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))}\liminf_{\epsilon\searrow 0}\epsilon^d\|\phi(\epsilon\,\cdot\,)\|_{L^{p,d}(\mathbb{R}^d,w)}\|\psi(\epsilon\,\cdot\,)\|_{L^{p',d}(\mathbb{R}^d,w')} \end{aligned}$$

Since

$$\epsilon^d\|\phi(\epsilon\,\cdot\,)\|_{L^{p,d}(\mathbb{R}^d,w)}\|\psi(\epsilon\,\cdot\,)\|_{L^{p',d}(\mathbb{R}^d,w')} = \|\phi\|_{L^{p,d}(\mathbb{R}^d,w;X)}\|\psi\|_{L^{p',d}(\mathbb{R}^d,w')}$$

by a change of variables, it follows that

$$\|m(\xi_0)x\|_X \leq \|\phi\|_{L^{p,d}(\mathbb{R}^d,w;X)}\|\psi\|_{L^{p',d}(\mathbb{R}^d,w')}.$$

Almost every point ξ_0 in $\mathbb{R}^d$ being a Lebesgue point of m (see Theorem D.1.5), this shows that $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X))$ with $\|m\|_{L^\infty(\mathbb{R}^d;\mathcal{B}(X))} \leq \|\phi\|_{L^{p,d}(\mathbb{R}^d,w;X)}\|\psi\|_{L^{p',d}(\mathbb{R}^d,w')}\|T_m\|_{\mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))}$. □

In view of the lemma, for Fourier multipliers on $L^{p,d}(\mathbb{R}^d,w;X)$ we only need to focus on symbols m from $L^\infty(\mathbb{R}^d;\mathcal{B}(X))$. To this end we note that, similarly to (4.9), given $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X))$,

$$T_m : \mathcal{S}(\mathbb{R}^d;X) \longrightarrow C_0(\mathbb{R}^d;X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d;X),\; f \mapsto \mathscr{F}^{-1}[m\hat{f}] \tag{4.10}$$

defines a continuous linear operator. The linear space $\mathcal{S}(\mathbb{R}^d;X)$ being a dense subspace of $L^{p,d}(\mathbb{R}^d,w;X)$, we may define the notion of Fourier multiplier on $L^{p,d}(\mathbb{R}^d,w;X)$ as follows:

Definition 4.2.2. Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. A function $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X))$ is called a *Fourier multiplier* on $L^{p,d}(\mathbb{R}^d,w;X)$ if the linear operator from (4.10) (takes its values in $L^{p,d}(\mathbb{R}^d,w;X)$ and) extends to a bounded linear operator T_m on $L^{p,d}(\mathbb{R}^d,w;X)$, which is then called the *Fourier multiplier operator* associated with m. We denote by $\mathcal{M}_{p,d,w}(X)$ the space of all Fourier multipliers on $L^{p,d}(\mathbb{R}^d,w;X)$ equipped with the norm $\|m\|_{\mathcal{M}_{p,d,w}(X)} := \|T_m\|_{\mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))}$.

Proposition 4.2.3. *Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$.*

(i) Suppose $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X)) \subset L^1_{loc}(\mathbb{R}^d;\mathcal{B}(X))$. Then m is a Fourier multiplier on $L^{p,d}(\mathbb{R}^d,w;X)$ in the sense of Definition 4.2.2 if and only if it is a Fourier multiplier on $L^{p,d}(\mathbb{R}^d,w;X)$ in the sense discussed before Lemma 4.2.1, in which case the bounded linear extensions of (4.9) *and* (4.10) *of course yield the same associated Fourier multiplier operator $T_m \in \mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))$.*

(ii) Suppose $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X)) \cap \mathcal{O}_M(\mathbb{R}^d;\mathcal{B}(X))$. Then m is a Fourier multiplier on $L^{p,d}(\mathbb{R}^d,w;X)$ if and only if the continuous linear operator

$$\tilde{T}_m : \mathcal{S}'(\mathbb{R}^d;X) \longrightarrow \mathcal{S}'(\mathbb{R}^d;X),\ f \mapsto \mathscr{F}^{-1}[m\hat{f}] \tag{4.11}$$

restricts to a bounded linear operator on $L^{p,d}(\mathbb{R}^d,w;X)$, in which case $\tilde{T}_m$ extends the associated Fourier multiplier operator $T_m \in \mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))$.

(iii) Suppose $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X)) \cap \mathcal{O}_M(\mathbb{R}^d;\mathcal{B}(X))$ and let D be a dense subspace of $L^{p,d}(\mathbb{R}^d,w;X)$. Then the operator $\tilde{T}_m$ from (4.11) *restricts to a bounded linear operator on $L^{p,d}(\mathbb{R}^d,w;X)$ if and only if $\tilde{T}_m D \subset L^{p,d}(\mathbb{R}^d,w;X)$ and $\|\tilde{T}_m f\|_{L^{p,d}(\mathbb{R}^d,w;X)} \lesssim \|f\|_{L^{p,d}(\mathbb{R}^d,w;X)}$ for all $f \in D$.*

Proof. (i) We only need to show that, if (4.9) extends to a bounded linear operator on $L^{p,d}(\mathbb{R}^d,w;X)$, then so does (4.10). To this end we denote, for obvious reasons of notation, by $\tilde{T}_m$ both the operator from (4.9) as its bounded extension to an operator on $L^{p,d}(\mathbb{R}^d,w;X)$. Then it suffices to show that $\tilde{T}_m$ coincides on $\mathcal{S}(\mathbb{R}^d;X)$ with the operator T_m from (4.10). As $\mathcal{S}(\mathbb{R}^d;X) \hookrightarrow L^{p,d}(\mathbb{R}^d,w;X)$ and $L^{p,d}(\mathbb{R}^d,w;X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d;X)$, we may view $\tilde{T}_m$ as a continuous linear operator

$$\tilde{T}_m : \mathcal{S}(\mathbb{R}^d;X) \longrightarrow \mathcal{S}'(\mathbb{R}^d;X). \tag{4.12}$$

Since the operator T_m from (4.10) may also be viewed as a continuous linear operator

$$T_m : \mathcal{S}(\mathbb{R}^d;X) \longrightarrow \mathcal{S}'(\mathbb{R}^d;X) \tag{4.13}$$

and since $\tilde{T}_m$ and T_m coincide on the dense subspace $\mathscr{F}^{-1}C_c^\infty(\mathbb{R}^d;X)$ of $\mathcal{S}(\mathbb{R}^d;X)$, it follows that (4.12) and (4.13) are the same operators.

(ii) Since $\tilde{T}_m$ extends the operator T_m from (4.10), we only need to show that, if m is a Fourier multiplier operator on $L^{p,d}(\mathbb{R}^d,w;X)$, then $\tilde{T}_m$ restricts to a bounded linear operator on $L^{p,d}(\mathbb{R}^d,w;X)$. As in (i), we may view the associated Fourier multiplier operator $T_m \in \mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))$ as a continuous linear operator

$$T_m : L^{p,d}(\mathbb{R}^d,w;X) \longrightarrow \mathcal{S}'(\mathbb{R}^d;X) \tag{4.14}$$

and we may view $\tilde{T}_m$ as a continuous linear operator

$$\tilde{T}_m : L^{p,d}(\mathbb{R}^d, w; X) \longrightarrow \mathcal{S}'(\mathbb{R}^d; X). \tag{4.15}$$

Now note that it is enough to show that the operators (4.14) and (4.15) coincide, for which it is already enough that they coincide on the dense subspace $\mathcal{S}(\mathbb{R}^d; X)$ of $L^{p,d}(\mathbb{R}^d, w; X)$. But this is immediate from the original definitions (4.10) and (4.11).

(iii) We only need to establish the reverse implication. So suppose $\tilde{T}_m D \subset L^{p,d}(\mathbb{R}^d, w; X)$ and $\|\tilde{T}_m f\|_{L^{p,d}(\mathbb{R}^d,w;X)} \lesssim \|f\|_{L^{p,d}(\mathbb{R}^d,w;X)}$ for all $f \in D$. Then, by denseness of D in $L^{p,d}(\mathbb{R}^d, w; X)$, the restriction of $\tilde{T}_m$ to D extends to a bounded linear operator T on $L^{p,d}(\mathbb{R}^d, w; X)$. To finish, we show that $Tf = \tilde{T}_m$ for all $f \in L^{p,d}(\mathbb{R}^d, w; X)$. As in (i), we may view $\tilde{T}_m$ and T as continuous linear operators $L^{p,d}(\mathbb{R}^d, w; X) \longrightarrow \mathcal{S}'(\mathbb{R}^d; X)$. Since these operators coincide on the dense space D of $L^{p,d}(\mathbb{R}^d, w; X)$, it follows that $Tf = \tilde{T}_m$ for all $f \in L^{p,d}(\mathbb{R}^d, w; X)$, as desired.

□

The following proposition contains the basic properties of the space of Fourier multipliers $\mathcal{M}_{p,d,w}(X)$.

Proposition 4.2.4. *Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$.*

(i) $\mathcal{M}_{p,d,w}(X)$ is a Banach algebra (w.r.t. the pointwise a.e. operations). Moreover,

$$\mathcal{M}_{p,d,w}(X) \hookrightarrow \mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X)),\ m \mapsto T_m \tag{4.16}$$

is an isometric algebra homomorphism and

$$\mathcal{M}_{p,d,w}(X) \hookrightarrow L^\infty(\mathbb{R}^d; \mathcal{B}(X)),\ m \mapsto m \tag{4.17}$$

is a homomorphism of Banach algebras

(ii) If $m \in \mathcal{M}_{p,d,w}(X)$ and $b \in \mathbb{R}^d$, then $m_b(\xi) := m(\xi - b)$ defines a Fourier multiplier $m_b \in \mathcal{M}_{p,d,w}(X)$ with $\|m\|_{\mathcal{M}_{p,d,w}(X)} = \|m_b\|\, \mathcal{M}_{p,d,w}(X)$, which is given by $T_{m_b} = M_{e_{\imath b}} T_m M_{e_{-\imath b}}$. Recall here that, given a $g \in L^\infty(\mathbb{R}^d)$, we denote by M_g the associated multiplication operator.

(iii) Suppose that $m \in \mathcal{M}_{p,d,w}(X)$. Let $p' = (p'_1, \ldots, p'_l) \in]1,\infty[^l$ be the vector of Hölder conjugates of $p = (p_1, \ldots, p_l)$ and denote by $w' = (w'_1, \ldots, w'_l) \in \prod_{j=1}^l A_{p'_j}(\mathbb{R}^{d_j})$ the p-dual weight vector of $w = (w_1, \ldots, w_l)$. Then $\tilde{m}^*(\xi) := m(-\xi)^*$ defines a Fourier multiplier $\tilde{m}^* \in \mathcal{M}_{p',d,w'}(X^*)$ which is the restriction of the adjoint operator T_m^* to $L^{p',d}(\mathbb{R}^d, w'; X^*) \subset (L^{p,d}(\mathbb{R}^d, w; X))^*$.

(iv) Let $m \in L^\infty(\mathbb{R}^d; \mathcal{B}(X))$ and $\{m_k\}_{k\in\mathbb{N}} \subset \mathcal{M}_{p,d,w}(X)$ be such that, for almost all $\xi \in \mathbb{R}^d$ and every $x \in X$,

$$m(\xi)x = \lim_{k\to\infty} m_k(\xi)x, \quad \sup_{k\in\mathbb{N}} \|m_k\|_{\mathcal{M}_{p,d,w}(X)} < \infty.$$

Then we have $m \in \mathcal{M}_{p,d,w}(X)$ with $\|m\|_{\mathcal{M}_{p,d,w}(X)} \leq \sup_{k\in\mathbb{N}} \|m_k\|_{\mathcal{M}_{p,d,w}(X)}$.

(v) *Let $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X))$ and let $\{m_k\}_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{M}_{p,d,w}(X)$. Suppose that, for almost all $\xi \in \mathbb{R}^d$ and every $x \in X$,*

$$m(\xi)x = \lim_{k\to\infty} m_k(\xi)x.$$

Then we have $m = \lim_{k\to\infty} m_k$ in $\mathcal{M}_{p,d,w}(X)$.

Proof. (ii) This follows easily from Lemma C.6.2.

(iii) This is for instance proved in [57] in the setting of unweighed Lebesgue-Bochner spaces, using arguments which can be extended to our setting.

(iv) Let $f \in \mathcal{S}(\mathbb{R}^d;X)$. Observe that, in view of $\sup_{k\in\mathbb{N}} \|m_k\|_{\mathcal{M}_{p,d,w}(X)} < \infty$ and the continuous inclusion $\mathcal{M}_{p,d,w}(X) \hookrightarrow L^\infty(\mathbb{R}^d;\mathcal{B}(X))$ (see Lemma 4.2.1 and Remark 4.2.3.(i)),

$$\sup_{k\in\mathbb{N}} \|m_k\|_{L^\infty(\mathbb{R}^d;\mathcal{B}(X))} < \infty.$$

By Lebesgue's dominated convergence theorem we thus obtain that $m\hat{f} = \lim_{k\to\infty} m_k\hat{f}$ in $L^1(\mathbb{R}^d;X)$, from which it follows that $T_m f = \lim_{k\to\infty} T_{m_k} f$ in $C_0(\mathbb{R}^d;X) \subset L^\infty(\mathbb{R}^d;X)$ because the inverse Fourier transform $\mathscr{F}^{-1}$ is continuous from $L^1(\mathbb{R}^d;X)$ to $C_0(\mathbb{R}^d;X)$. By Fatou's lemma (applied l times) we thus get

$$\|T_m f\|_{L^{p,d}(\mathbb{R}^d,w;X)} \leq \liminf_{k\to\infty} \left\|T_{m_k} f\right\|_{L^{p,d}(\mathbb{R}^d,w;X)} \leq C\,\|f\|_{L^{p,d}(\mathbb{R}^d,w;X)}$$

for the constant $C := \sup_{k\in\mathbb{N}} \|m_k\|_{\mathcal{M}_{p,d,w}(X)} \in [0,\infty[$.

(v) From (iv) and the fact the Cauchy sequence are bounded it follows that $m \in \mathcal{M}_{p,d,w}(X)$. To show that $m = \lim_{k\to\infty} m_k$ in $\mathcal{M}_{p,d,w}(X)$, let $\epsilon > 0$ be arbitrary and pick an $N \in \mathbb{N}$ such that $\|m_k - m_N\|_{\mathcal{M}_{p,d,w}(X)} \leq \epsilon$ for all $k \geq N$. Then, applying (iii) to the function $m - m_N \in L^\infty(\mathbb{R}^d;\mathcal{B}(X))$ and the sequence $\{m_k - m_N\}_{k\geq N} \subset \mathcal{M}_{p,d,w}(X)$, we find $\|m - m_N\|_{\mathcal{M}_{p,d,w}(X)} \leq \epsilon$.

(i) We first show that $\mathcal{M}_{p,d,w}(X)$ is complete. For this suppose we are given a Cauchy sequence $(m_k)_{k\in\mathbb{N}}$ in $\mathcal{M}_{p,d,w}(X)$. Then $(m_k)_{k\in\mathbb{N}}$ also is a Cauchy sequence in the Banach space $L^\infty(\mathbb{R}^d;\mathcal{B}(X))$ because of the continuous inclusion $\mathcal{M}_{p,d,w}(X) \hookrightarrow L^\infty(\mathbb{R}^d;\mathcal{B}(X))$. Denote by m the limit of $(m_k)_{k\in\mathbb{N}}$ in $L^\infty(\mathbb{R}^d;\mathcal{B}(X))$. Via (v) we then obtain that also $m = \lim_{k\to\infty} m_k$ in $\mathcal{M}_{p,d,w}(X)$.

A proof of the remaining assertions can be found in [57].

□

For convenience of later reference, we state two observations about multiplier symbols which only depend on a single variable as the following simple lemma, of which we do not give a proof.

Lemma 4.2.5.

(i) *Let $j \in \{1,\dots,l\}$ and $m \in L^\infty(\mathbb{R}^{d_j};\mathcal{B}(X))$. Define $M \in L^\infty(\mathbb{R}^d;\mathcal{B}(X))$ by $M(\xi) := m(\xi_j)$, that is, $M = m \circ \pi_{[d;j]}$. Then the following are equivalent:*

(a) *$M \in \mathcal{M}_{p,d,w}(X)$.*

(b) $m \in \mathcal{M}_{p_j,d_j,w_j}(Y_{j,w})$.

(c) $m \in \mathcal{M}_{p_j,d_j,w_j}(Y_j)$.

Moreover, in this situation we have $\|M\|_{\mathcal{M}_{p,d,w}(X)} = \|m\|_{\mathcal{M}_{p_j,d_j,w_j}(Y_{j,w})} = \|m\|_{\mathcal{M}_{p_j,d_j,w_j}(Y_j)}$.

(ii) *Suppose that* $l = 1$. *Let* $m \in L^\infty(\mathbb{R}; \mathcal{B}(X))$ *be a symbol for which there exists an increasing function* $C : [0,\infty[\longrightarrow [0,\infty[$ *such that, for all weights* $v \in A_p(\mathbb{R})$, *it holds that* $m \in \mathcal{M}_{p,1,v}(X)$ *with norm* $\|m\|_{\mathcal{M}_{p,1,v}(X)} \leq C([v]_{A_p})$. *Then for all* $i \in \{1,\dots,d\}$ *and all* $w \in A_p^{rec}(\mathbb{R}^d)$, *we have that* $M_i := [\xi \mapsto m(\xi_i)] \in L^\infty(\mathbb{R}^d)$ *belongs to* $\mathcal{M}_{p,d,w}(X)$ *with norm* $\|M_i\|_{\mathcal{M}_{p,d,w}(X)} \leq C([w]_{A_p^{rec}})$.

We will see that a very wide class of symbols can be built from the single symbol $1_{]0,\infty[^d}$. So, in order to prove Fourier multiplier theorems, it is natural to pay special attention to those Banach spaces X for which $1_{]0,\infty[^d} \in \mathcal{M}_{p,d,w}(X)$ holds true, i.e., for which the Fourier multiplier operator

$$R : \mathcal{S}(\mathbb{R}^d; X) \longrightarrow C_0(\mathbb{R}^d; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X),\ f \mapsto \mathscr{F}^{-1}[1_{]0,\infty[^d}\hat{f}]$$

takes its values in $L^{p,d}(\mathbb{R}^d, w; X)$ and extends to a bounded linear operator R on $L^{p,d}(\mathbb{R}^d, w; X)$. As R is known under the name *Riesz projection*, this motivates the following definition (and the chosen terminology therein).

Definition 4.2.6. Let $p \in]1,\infty[^l$ and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. We say that a Banach space X is of class $\mathcal{RP}_{p,d,w}$ if $1_{]0,\infty[^d} \in \mathcal{M}_{p,d,w}(X)$, in which case we write $\alpha_{p,d,w,X} := \left\|1_{]0,\infty[^d}\right\|_{\mathcal{M}_{p,d,w}(X)} = \|R\|_{\mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))}$. In the unweighted case $w = 1$ we just write $\mathcal{RP}_{p,d} = \mathcal{RP}_{p,d,w}$.

We denote by $\mathscr{J}^d$ the collection of all rectangles in $\mathbb{R}^d$ with sides parallel to the coordinate axes, i.e., rectangles in $\mathbb{R}^d$ of the form $I_1 \times \ldots \times I_d$ for some intervals $I_1,\dots,I_d \subset \mathbb{R}$.

Lemma 4.2.7. *Let* X *be a Banach space,* $p \in]1,\infty[^l$, *and* $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$.

(i) *The following are equivalent:*

(a) X *is of class* $\mathcal{RP}_{p,d,w}$.

(b) $1_{\mathbb{R}^{d-1}\times]0,\infty[}, 1_{\mathbb{R}^{d-2}\times]0,\infty[\times\mathbb{R}}, \dots, 1_{]0,\infty[\times\mathbb{R}^{d-1}} \in \mathcal{M}_{p,d,w}(X)$.

(c) $\{1_J : J \in \mathscr{J}^d\} \subset \mathcal{M}_{p,d,w}(X)$

Moreover, if (a)/(b)/(c) hold, then we have $\|1_J\|_{\mathcal{M}_{p,d,w}(X)} \leq (2\alpha_{p,d,w,X})^d$ *for all* $J \in \mathscr{J}^d$. *In this situation we write, for each* $J \in \mathscr{J}^d$, $\Delta(J) := T_{1_J} \in \mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X))$ *for the Fourier multiplier operator associated with* 1_J.

(ii) *Suppose* $l = 1$. *Then* $X = \mathbb{C}$ *is of class* $\mathcal{RP}_{p,d,w}$ *if and only if* $w \in A_p^{rec}(\mathbb{R}^d)$.

(iii) *If* $X \neq \{0\}$ *is of class* $\mathcal{RP}_{p,d,w}$, *then* $w \in \prod_{j=1}^l A_{p_j}^{rec}(\mathbb{R}^{d_j})$.

(iv) X *is of class* $\mathcal{RP}_{p,d}$ *if and only if* X *is a UMD space.*

(v) *Let* $p' = (p'_1,\dots,p'_l) \in]1,\infty[^l$ *be the vector of Hölder conjugates of* $p = (p_1,\dots,p_l)$ *and denote by* $w' = (w'_1,\dots,w'_l) \in \prod_{j=1}^l A_{p'_j}(\mathbb{R}^{d_j})$ *the p-dual weight vector of* $w = (w_1,\dots,w_l)$. *Then* X *is of class* $\mathcal{RP}_{p,d,w}$ *if and only if* X^* *is of class* $\mathcal{RP}_{p',d,w'}$.

Proof. (i) '(a)⇒(b)': By Proposition 4.2.4.(ii), $1_{]-n,\infty[\times]0,\infty[^{d-1}} \in \mathcal{M}_{p,d,w}(X)$ of norm $\alpha_{p,d,w,X}$ for all $n \in \mathbb{N}$. Since $1_{\mathbb{R}\times]0,\infty[^{d-1}} = \lim_{n\to\infty} 1_{]-n,\infty[\times]0,\infty[^{d-1}}$ pointwise, it follows from Proposition 4.2.4.(iv) that $1_{\mathbb{R}\times]0,\infty[^{d-1}} \in \mathcal{M}_{p,d,w}(X)$ of norm $\leq \alpha_{p,d,w,X}$. Repeating this argument $d-2$ times, we find $1_{\mathbb{R}^{d-1}\times]0,\infty[} \in \mathcal{M}_{p,d,w}(X)$ of norm $\leq \alpha_{p,d,w,X}$. The remaining symbols $1_{\mathbb{R}\times]0,\infty[\times\mathbb{R}^{d-1}}, \ldots, 1_{]0,\infty[\times\mathbb{R}^{d-1}}$ can be treated similarly.

'(b)⇒(c)': Let $J = I_1 \times \ldots I_d$ be a product of bounded non-empty intervals. If $I_1 =]b,c]$ (up to measure zero) with $b < c$, then we have $1_{I_1\times\mathbb{R}^{d-1}} = 1_{]b,\infty[\times\mathbb{R}^{d-1}} - 1_{]c,\infty[\times\mathbb{R}^{d-1}} \in \mathcal{M}_{p,d,w}(X)$ of norm $\leq 2\left\|1_{]0,\infty[\times\mathbb{R}^{d-1}}\right\|_{\mathcal{M}_{p,d,w}(X)}$ by (i) and (ii) of Proposition 4.2.4. Similarly we have $1_{\mathbb{R}^{i-1}\times I_i\times\mathbb{R}^{d-i}} \in \mathcal{M}_{p,d,w}(X)$ of norm $\leq 2\left\|1_{\mathbb{R}^{i-1}\times]0,\infty[\times\mathbb{R}^{d-i}}\right\|_{\mathcal{M}_{p,d,w}(X)}$ for $i = 1,\ldots,d$. From this it follows that $1_J = \prod_{i=1}^d 1_{\mathbb{R}^{i-1}\times I_i\times\mathbb{R}^{d-i}} \in \mathcal{M}_{p,d,w}(X)$ with

$$\|1_J\|_{\mathcal{M}_{p,d,w}(X)} \leq \prod_{i=1}^{d} 2\left\|1_{\mathbb{R}^{i-1}\times]0,\infty[\times\mathbb{R}^{d-i}}\right\|_{\mathcal{M}_{p,d,w}(X)}. \tag{4.18}$$

Since for a general rectangle $J \in \mathscr{J}^d$ it holds that the indicator function 1_J can be written as the pointwise limit of indicator functions of bounded rectangles from $\mathscr{J}^d$, it follows from Proposition 4.2.4.(iv) that $1_J \in \mathcal{M}_{p,d,w}(X)$ with norm estimate (4.18) for each $J \in \mathscr{J}^d$.

(ii) This follows immediately from Proposition D.2.14.

(iii) Suppose $X \neq \{0\}$ is of class $\mathcal{RP}_{p,d,w}$. Then we have $1_{\mathbb{R}^{d_1+\ldots+d_{j-1}}\times]0,\infty[^{d_j}\times\mathbb{R}^{d_{j+1}+\ldots+d_l}} \in \mathcal{M}_{p,d,w}(X)$ for each $j \in \{1,\ldots,l\}$ by (i). From Lemma 4.2.5.(i) it follows that $1_{]0,\infty[^{d_j}} \in \mathcal{M}_{p_j,d_j,w_j}(Y_j)$ for each $j \in \{1,\ldots,l\}$, where $Y_j := L^{(p_{j+1},\ldots,p_l),(d_{j+1},\ldots,d_l)}(\mathbb{R}^{d_{j+1}+\ldots+d_l},(w_{j+1},\ldots,w_l);X)$. Since $Y_j \neq \{0\}$ (as $X \neq \{0\}$), it follows that $1_{]0,\infty[^{d_j}} \in \mathcal{M}_{p_j,d_j,w_j}(\mathbb{C})$, that is, $\mathbb{C}$ is of class $\mathcal{RP}_{p_j,d_j,w_j}$ for each $j \in \{1,\ldots,l\}$. By (ii) we conclude that $w \in \prod_{j=1}^l A_{p_j}^{rec}(\mathbb{R}^{d_j})$.

(iv) Writing

$$q := (\underbrace{p_1,\ldots,p_1}_{d_1 \text{ times}}, \underbrace{p_2,\ldots,p_2}_{d_2 \text{ times}}, \ldots, \underbrace{p_l,\ldots,p_l}_{d_l \text{ times}}) \in]1,\infty[^d$$

and $\tilde{d} := \mathbf{1} \in \mathbb{N}^d$, we have $L^{p,d}(\mathbb{R}^d;X) = L^{q,\tilde{d}}(\mathbb{R}^d;X)$. Therefore, X is of class $\mathcal{RP}_{p,d}$ if and only if it is of class $\mathcal{RP}_{q,\tilde{d}}$. By (i), this is equivalent with

$$1_{\mathbb{R}^{d-1}\times]0,\infty[}, 1_{\mathbb{R}\times]0,\infty[\times\mathbb{R}^{d-1}}, \ldots, 1_{]0,\infty[\times\mathbb{R}^{d-1}} \in \mathcal{M}_{p,d}(X).$$

Writing, for each $i \in \{1,\ldots,d\}$,

$$Y_i := L^{(q_{i+1},\ldots,q_d),(1,\ldots,1)}(\mathbb{R}^{d-i};X),$$

the latter is equivalent with Y_i being of class $\mathcal{RP}_{q_i,1}$ for each $i \in \{1,\ldots,d\}$; see Lemma 4.2.5. By Theorem E.5.7, this is in turn is equivalent with Y_i being a UMD space for each $i \in \{1,\ldots,d\}$, for which it is necessary and sufficient that X is a UMD space (see E.5.6).

(v) This follows from a combination of Proposition 4.2.4.(iii) and (the proof of) (i).

□

Lemma 4.2.8. *Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$ such that X is of class $\mathcal{RP}_{p,d,w}$. Then we have*

$$\mathcal{R}\{\Delta([y,\bar{\infty}[) \,:\, y \in \mathbb{R}^d\} \leq \alpha_{p,d,w,X} \quad \text{and} \quad \mathcal{R}\{\Delta(J) \,:\, J \in \mathscr{J}^d\} \leq (2\alpha_{p,d,w,X})^d \tag{4.19}$$

in $\mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))$. Here we write $[y,\bar{\infty}[= y + [0,\infty[^d$.

Proof. The first $\mathcal{R}$-bound is immediate from a combination of Proposition 4.2.4.(ii) and Lemma E.3.8. For the second $\mathcal{R}$-bound, we write $\mathscr{S} := \{\Delta([y,\bar{\infty}[) \,:\, y \in \mathbb{R}^d\} \subset \mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))$. Then, using the denseness of $\mathscr{F}^{-1}C_c^\infty(\mathbb{R}^d;X)$ in $L^{p,d}(\mathbb{R}^d,w;X)$ and the uniform boundedness of $\{\Delta(J) \,:\, J \in \mathscr{J}^d\}$ in $\mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))$, it is not difficult to see that

$$\{\Delta(J) \,:\, J \in \mathscr{J}^d\} = \mathrm{cl}^{\mathrm{SOT}}\left(\left[\mathrm{cl}^{\mathrm{SOT}}(\mathscr{S}) - \mathrm{cl}^{\mathrm{SOT}}(\mathscr{S})\right]^d\right);$$

the steps are basically the same as in Lemma 4.2.7.(i) (where we have to replace pointwise limits of symbols with limits of the associated Fourier multiplier operators in the SOT topology). □

Besides indicator functions of rectangles from $\mathscr{J}^d$ and linear combinations thereof, we now turn to a more wide class of symbols which can be built out of $1_{]0,\infty[^d}$.

Lemma 4.2.9. *Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}^{rec}(\mathbb{R}^{d_j})$, such that X is of class $\mathcal{RP}_{p,d,w}$. Suppose that $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X))$ has the following representation: there exist a complex Borel measure μ on $\mathbb{R}^d$ and a bounded WOT-measurable function $\tau : \mathbb{R}^d \longrightarrow \mathcal{B}(X)$ such that, for all $x \in X$ and $x^* \in X^*$,*

$$\langle m(\xi)x, x^*\rangle = \int_{]-\bar{\infty},\xi]} \langle \tau(y)x, x^*\rangle d\mu(y). \tag{4.20}$$

Then we have $m \in \mathcal{M}_{p,d,w}(X)$ with $\|m\|_{\mathcal{M}_{p,d,w}(X)} \leq \|\tau\|_\infty \, \alpha_{p,d,w,X} \, \|\mu\|$. In fact, for every $f \in L^{p,d}(\mathbb{R}^d,w;X)$ and $g \in L^{p',d}(\mathbb{R}^d,w';X^*)$, it holds that $\mathbb{R}^d \ni y \mapsto \langle g, \tau(y)\Delta([y,\bar{\infty}[)f\rangle \in \mathbb{C}$ is a Borel measurable function which is uniformly bounded by $\|\tau\|_\infty \, \alpha_{p,d,w,X} \, \|f\| \, \|g\|$, and the Fourier multiplier operator $T_m \in \mathcal{B}(L^{p,d}(\mathbb{R}^d,w;X))$ satisfies

$$\langle T_m f, g\rangle = \int_{\mathbb{R}^d} \langle \tau(y)\Delta([y,\bar{\infty}[)f, g\rangle d\mu(y). \tag{4.21}$$

Proof. Let's first prove the measurabilty of $\mathbb{R}^d \ni y \mapsto \langle g, \tau(y)\Delta([y,\bar{\infty}[)f\rangle \in \mathbb{C}$ for every $f \in L^{p,d}(\mathbb{R}^d,w;X)$ and $g \in L^{p',d}(\mathbb{R}^d,w';X^*)$. For each fixed $y \in \mathbb{R}^d$ it holds that

$$L^{p,d}(\mathbb{R}^d,w;X) \times L^{p',d}(\mathbb{R}^d,w';X^*) \longrightarrow \mathbb{C},\ (f,g) \mapsto \langle \tau(y)\Delta([y,\bar{\infty}[)f, g\rangle$$

is a continuous bilinear map. Since $L^{p,d}(\mathbb{R}^d,w)\otimes X$ and $L^{p',d}(\mathbb{R}^d,w')\otimes X^*$ are dense in $L^{p,d}(\mathbb{R}^d,w;X)$ and $L^{p',d}(\mathbb{R}^d,w';X^*)$, respectively, it thus is enough to consider $f = \phi \otimes x$ and $g = \psi \otimes x^*$ with $\phi \in L^{p,d}(\mathbb{R}^d,w)$, $\psi \in L^{p',d}(\mathbb{R}^d,w')$, $x \in X$, and $x^* \in X^*$. Then $y \mapsto \langle \tau(y), \Delta([y,\infty[)f\rangle = \langle \Delta([y,\infty[)\phi, \psi\rangle\langle \tau(y)x, x^*\rangle$ is measurable, being the product of two measurable functions.

Since the measurable function $y \mapsto \langle \tau(y)\Delta([y,\bar{\infty}[)f, g\rangle$ is uniformly bounded by

$$\|\tau\|_\infty \, \|f\| \, \|g\|,$$

it follows that $y \mapsto \langle \tau(y)\Delta([y,\bar{\infty}[)f, g\rangle$ is integrable with respect to the positive finite Borel measure μ. Hence, the expression on the right hand-side of (4.21) is well defined and, in fact, gives rise to a bounded bilinear form

$$B_{\tau,\mu} : L^{p,d}(\mathbb{R}^d, w; X) \times L^{p',d}(\mathbb{R}^d, w'; X^*) \longrightarrow \mathbb{C},\ (f,g) \mapsto \int_{\mathbb{R}^d} \langle \tau(y), \Delta([y,\bar{\infty}[)f\rangle d\mu(y) \qquad (4.22)$$

of norm $\leq \|\tau\|_\infty\, \alpha_{p,d,w,X}\, \|\mu\|$. Since $\mathcal{S}(\mathbb{R}^d; X)$ is dense in $L^{p,d}(\mathbb{R}^d, w; X)$, since $\mathcal{S}(\mathbb{R}^d; X^*)$ is dense in $L^{p',d}(\mathbb{R}^d, w'; X^*)$, and since $L^{p',d}(\mathbb{R}^d, w'; X^*)$ is norming for $L^{p,d}(\mathbb{R}^d, w; X)$, it is thus enough to show that (4.21) holds for all $f \in \mathcal{S}(\mathbb{R}^d; X)$ and $g \in \mathcal{S}(\mathbb{R}^d; X^*)$; here T_m is at this moment of course still the operator from (4.10). So let $f \in \mathcal{S}(\mathbb{R}^d; X)$ and $g \in \mathcal{S}(\mathbb{R}^d; X^*)$. Then

$$\begin{aligned}
\langle T_m f, g\rangle &= \langle m\hat{f}, \check{g}\rangle = \int_{\mathbb{R}^d} \int_{]-\bar{\infty},\xi]} \langle \tau(y)\hat{f}(\xi), \check{g}(\xi)\rangle d\mu(y) d\xi \\
&\overset{Fubini}{=} \int_{\mathbb{R}^d} \int_{[y,\bar{\infty}[} \langle \tau(y)\hat{f}(\xi), \check{g}(\xi)\rangle d\xi d\mu(y) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \langle \tau(y) 1_{[y,\bar{\infty}[} \hat{f}(\xi), \check{g}(\xi)\rangle d\xi d\mu(y) \\
&= \int_{\mathbb{R}^d} \langle \tau(y)\Delta([y,\bar{\infty}[)f, g\rangle d\mu(y).
\end{aligned}$$

□

We finally come to a concrete example of a symbol admitting a representation as in the lemma. This example will be an important ingredient in the proof of the operator-valued Mikhlin theorem for the case that $d = 1$, where we apply it to $m1_I$ for appropriate compact intervals I not containing 0; see Theorem 4.5.5 and Theorem 4.5.13.

Example 4.2.10. Let X be a Banach space and let $0 < a < b < \infty$. Suppose that we have a function $m : \mathbb{R} \longrightarrow \mathcal{B}(X)$ which is C^1 on the interval $[a, b]$. Then $m1_{[a,b[}$ has the representation (4.20) from the above lemma for the positive finite Borel measure $\mu := \delta_a + 1_{[a,b]}\frac{dy}{y} + \delta_b$ and the integrand $\tau(y) := m(a)1_a(y)+ym'(y)1_{]a,b[}(y)-m(b)1_b(y)$. Note $\|\mu\| = \mu(\mathbb{R}^d) = \mu(\mathbb{R}^d) \leq 2+\log\left(\frac{b}{a}\right)$ and $\|\tau\|_\infty \leq C_m := \sup\{|m(\xi)|, |\xi m'(\xi)| \mid \xi \in [a, b]\}$. So, in case that X is of class $\mathcal{RP}_{p,w}$ for some $p \in]1, \infty[$ and $w \in A_p(\mathbb{R})$, we find $m1_{[a,b[} \in \mathcal{M}_{p,w}(X)$ with $\|m\|_{\mathcal{M}_{p,w}(X)} \leq \alpha_{p,w,X} C_m \left[2 + \log\left(\frac{b}{a}\right)\right]$

Remark 4.2.11. Let $-\infty < a < b < 0$. For every function $m : \mathbb{R} \longrightarrow \mathcal{B}(X)$ which is C^1 on the interval $[a, b]$ we have a similar representation for $m1_{]a,b]}$; just do a reflection argument.

Proof. We check that $m1_{[a,b[}$ has indeed the claimed representation (4.20): For $\xi \in \mathbb{R} \setminus [a, b[$ we have $(m1_{[a,b[})(\xi) = 0 = \int_{]-\infty,\xi]} \tau(y)d\mu(y)$, and for $\xi \in [a, b[$ we have

$$\begin{aligned}
(m1_{[a,b[})(\xi) &= m(a) + \int_a^\xi m'(y)dy \\
&= \int_{]-\infty,\xi]} m(a)d\delta_a(y) + \int_{]-\infty,\xi]} ym'(y)1_{]a,b[}(y)\frac{dy}{y} - \int_{]-\infty,\xi]} m(b)d\delta_b(y) \\
&= \int_{]-\infty,\xi]} \tau(y)d\mu(y).
\end{aligned}$$

□

4.3 Unconditional Schauder Decompositions

In this section we discuss the notion of (unconditional) Schauder decomposition of a Banach space, with as goal to get an abstract framework to prove Fourier multiplier theorems. The notion of (unconditional) Schauder decomposition is a generalization of the notion of (unconditional) Schauder basis (of which orthonormal bases in Hilbert space are a special case), and the idea is to decompose vectors x of a Banach space X into convergent sums $x = \sum_k x_k$, where $x_k \in X_k$ and X_k are certain distinct subspaces of X. The connection with Fourier multiplier operators and multiplier operators with respect to unconditional Schauder decompositions, leading naturally to the notion of $\mathcal{R}$-boundedness of a set of operators on a Banach space (as defined in Appendix E.3).

4.3.1 Introduction

The first approach to Fourier multiplier theorems on $L^{p,\mathscr{d}}(\mathbb{R}^d, w; X)$ taken in this chapter consists of decomposing $\mathbb{R}^d$, possibly op to a set of measure zero, into a collection of disjoint rectangles $(E_n)_{n\in\mathbb{N}}$, proving estimates for the operators $T_{m1_{E_n}}$ associated with the compactly supported pieces $m1_{E_n}$ of m, and assembling the pieces together in a way which gives the desired norm estimate for T_m.

In order to give a more precise description of this approach, which can be more easily translated into an abstract functional analytic problem, let X be Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}^{rec}(\mathbb{R}^{d_j})$, such that X is of class $\mathcal{RP}_{p,\mathscr{d},w}$. Suppose that we have a decomposition, possibly up to a set of measure zero, of $\mathbb{R}^d$ into a collection of disjoint intervals $(E_n)_{n\in\mathbb{N}}$ with $1_{E_n} \in \mathcal{M}_{p,\mathscr{d},w}$ for each $n \in \mathbb{N}$, in such a way that the projections $D_n := \Delta(E_n) = T_{1_{E_n}} \in \mathcal{B}(L^{p,\mathscr{d}}(\mathbb{R}^d, w; X))$ (see Lemma 4.2.7.(i)) give a decomposition of $L^{p,\mathscr{d}}(\mathbb{R}^d, w; X)$ in the sense that $f = \sum_{n=0}^{\infty} D_n f$ for all $f \in L^{p,\mathscr{d}}(\mathbb{R}^d, w; X)$. The idea is to represent T_m as a 'multiplier operator' with respect to this decomposition $(D_n)_{n\in\mathbb{N}}$. Note that

$$E_0 := \bigcup_{N=0}^{\infty} \mathrm{Ran}\left(\sum_{n=0}^{N} D_n\right) \tag{4.23}$$

is a dense subspace of $L^{p,\mathscr{d}}(\mathbb{R}^d, w; X)$. So, by Remark 4.2.3.(ii)/(iii), in order to show that a symbol $m \in L^\infty(\mathbb{R}^d; \mathcal{B}(X)) \cap \mathscr{O}_M(\mathbb{R}^d; \mathcal{B}(X))$ belongs to $\mathcal{M}_{p,\mathscr{d},w}(X)$, it is necessary and sufficient that the operator from (4.11) maps E_0 into $L^{p,\mathscr{d}}(\mathbb{R}^d, w; X)$ and satisfies the norm estimate $\left\|\tilde{T}_m f\right\|_{L^{p,\mathscr{d}}(\mathbb{R}^d,w;X)} \lesssim \|f\|_{L^{p,\mathscr{d}}(\mathbb{R}^d,w;X)}$ for all $f \in E_0$. Now suppose that we are able to show that

$$m_n := m1_{E_n} \in \mathcal{M}_{p,\mathscr{d},w}(X), \qquad \forall n \in \mathbb{N}, \tag{4.24}$$

by using for instance Lemma 4.2.9, something which is necessary for $m \in \mathcal{M}_{p,\mathscr{d},w}(X)$ to hold true in light of Proposition 4.2.4.(i) (as $1_{E_n} \in \mathcal{M}_{p,\mathscr{d},w}(X)$). Writing $T_n := T_{m_n} \in \mathcal{B}(L^{p,\mathscr{d}}(\mathbb{R}^d, w; X))$ for every $n \in \mathbb{N}$, we have $T_n D_n = D_n T_n D_n$ by Proposition 4.2.4.(i), i.e. the operator T_n leaves the subspace $\mathrm{Ran}(D_n)$ invariant. Since $\tilde{T}_m D_n = T_{m_n}$ for each $n \in \mathbb{N}$, which can be easily seen on the dense space of Schwartz functions, it follows that $\tilde{T}_m$ restricts to the 'multiplier operator' $T : E_0 \longrightarrow E_0$ (with respect to the decomposition $(D_n)_{n\in\mathbb{N}}$) given by

$$Tf := \sum_{n=0}^{N} T_n D_n f, \qquad \forall f \in E_0, f \in \mathrm{Ran}\left(\sum_{n=0}^{N} D_n\right). \tag{4.25}$$

Therefore, having (4.24), for $m \in \mathcal{M}_{p,d,w}(X)$ it is necessary and sufficient that T is bounded with respect to the $L^{p,d}(\mathbb{R}^d, w; X)$-norm, in which case this operator has T_m as its bounded linear extension to $L^{p,d}(\mathbb{R}^d, w; X)$. For a general symbol $m \in L^\infty(\mathbb{R}^d; \mathcal{B}(X))$ satisfying (4.24), the last statement remains valid: If $m \in \mathcal{M}_{p,d,w}(X)$, then it is easy to see that T_m extends the operator $T : E_0 \longrightarrow E_0$ (use Proposition 4.2.4.(i)); in particular, T is bounded (with respect to the $L^{p,d}(\mathbb{R}^d, w; X)$-norm). Conversely, assume that T is bounded. Now observe that the boundedness of T is equivalent with $\{\sum_{n=0}^N T_n D_n \mid N \in \mathbb{N}\}$ being uniformly bounded, which (by Proposition 4.2.4.(i)) in turn is equivalent with $(\sum_{n=0}^N m 1_{E_n})_{N\in\mathbb{N}}$ being a bounded sequence in $\mathcal{M}_{p,d,w}(X)$. So, in view of Proposition 4.2.4.(iv) and the pointwise a.e. convergence $m = \sum_{n=0}^\infty m 1_{E_n}$, for $m \in \mathcal{M}_{p,d,w}(X)$ it is indeed sufficient that T is bounded.

This approach can be translated into the following abstract functional analytic problem: Let E be a Banach space with a *Schauder decomposition* $D = (D_n)_{n\in\mathbb{N}}$, that is, $D = (D_n)_{n\in\mathbb{N}}$ is a sequence of bounded linear projections in E satisfying (i) $D_k D_n = 0$ whenever $k \neq n$, and (ii) $x = \sum_{n=0}^\infty D_n x$ for all $x \in E$. Suppose we are given a collection of bounded linear operators $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(E)$ with the property that $D_n T_n D_n = T_n D_n$ for every $n \in \mathbb{N}$. Then the problem is to determine whether the linear operator $T : E_0 \longrightarrow E_0$ given in (4.25), where E_0 is the dense subspace of E defined in (4.23), extends to a bounded linear operator on E.

Let's first consider the case that $E = H$ is a Hilbert space and the D_n are orthogonal projections. Then we have $\mathrm{Ran}(D_k) \perp \mathrm{Ran}(D_n)$ whenever $k \neq n$ because of the assumption $D_k D_n = 0$ whenever $k \neq n$. Since T_n leaves $\mathrm{Ran}(D_n)$ invariant for every $n \in \mathbb{N}$, it follows that

$$\left\| \sum_{n=0}^N T_n D_n x \right\|_H = \left(\sum_{n=0}^N \|T_n D_n x\|_H^2 \right)^{1/2} \leq \sup_{n=0,\dots,N} \|T_n\| \left(\sum_{n=0}^N \|D_n x\|_H^2 \right)^{1/2} = \sup_{n=0,\dots,N} \|T_n\| \left\| \sum_{n=0}^N D_n x \right\|_H \tag{4.26}$$

for all $N \in \mathbb{N}$ and $x \in H$. Hence, the operator $T : E_0 \longrightarrow E_0$ from (4.25) satisfies the norm estimate $\|Tx\|_H \leq \sup_{n\in\mathbb{N}} \|T_n\| \, \|x\|$ for all x in the dense space E_0 of H. So, if we assume that $M := \sup_{n\in\mathbb{N}} \|T_n\| < \infty$, then we obtain that T extends to a bounded linear operator T on H of norm $\leq M$.

In order to find an appropriate substitute for this orthogonality for a general Banach space E, first recall that

$$\left(\sum_{n=0}^N \|x_n\|_H \right)^{1/2} \overset{(E.3)}{=} \left\| \sum_{n=0}^N \epsilon_n x_n \right\|_{L^2(\Omega;H)}$$

for all $N \in \mathbb{N}$ and $x_0, \dots, x_N \in H$, where $(\epsilon_n)_{n\in\mathbb{N}}$ denotes a Rademacher sequence on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. So the orthogonality of the subspaces $(\mathrm{Ran}(D_n))_{n\in\mathbb{N}}$ is equivalent with

$$\left\| \sum_{n=0}^N x_n \right\|_H = \left\| \sum_{n=0}^N \epsilon_n x_n \right\|_{L^2(\Omega;H)}$$

for all $N \in \mathbb{N}$ and $x_0, \dots, x_N \in H$ with $x_n \in \mathrm{Ran}(D_n)$ for each $n \in \{0, \dots, N\}$, which is of course equivalent with

$$\left\| \sum_{n=0}^N D_n x \right\|_H = \left\| \sum_{n=0}^N \epsilon_n D_n x \right\|_{L^2(\Omega;H)} \tag{4.27}$$

for all $N \in N$ and $x \in H$. This suggests to study Schauder decompositions D of general Banach spaces E which satisfy (4.27) with equality '=' replaced by equivalence '$\eqsim$'. In order to do a computation similar to (4.26) for such Schauder decompositions, we have to replace the uniform bounded of the family $\{T_n \;:, n \in \mathbb{N}\} \subset \mathcal{B}(E)$ with its $\mathcal{R}$-bound (see Definition E.3.1). This notion of $\mathcal{R}$-boundedness then will provide a sufficient condition on the $(T_n)_{n\in\mathbb{N}}$ for the operator $T : E_0 \longrightarrow E_0$ given in (4.25) to extend to a bounded linear operator on E; see Theorem 4.3.14. We will see that there are several characterizations of Schauder decompositions D satisfying (4.27) with equality '=' replaced by equivalence '$\eqsim$', of which unconditional convergence of $x = \sum_{n=0}^{\infty} D_n x$ for every $x \in X$ will be our starting point.

4.3.2 Unconditional Schauder Decompositions

Throughout this subsection we let E be a Banach space. We furthermore shall use the following notation: Let $D = (D_n)_{n\in\mathbb{N}}$ be a of bounded linear projections in E with the property that $D_n D_k = 0$ whenever $n \neq k$. Then we define the sequence of *partial sum projections* $(P_N)_{N\in\mathbb{N}}$ by

$$P_N := \sum_{n=0}^{N} D_n,$$

and we define the linear subspace $\mathrm{Ran}(D)$ of E by

$$\mathrm{Ran}(D) := \bigcup_{N\in\mathbb{N}} \mathrm{Ran}(P_N).$$

4.3.2.a Definitions and Basic Properties

Definition 4.3.1. A sequence $D = (D_n)_{n\in\mathbb{N}}$ of bounded linear projections in E is called a *Schauder decomposition* of E if

(i) $D_k D_n = 0$ whenever $k \neq n$,

(ii) $x = \sum_{n=0}^{\infty} D_n x$ for all $x \in E$.

From the strong convergence $P_N \overset{N\to\infty}{\longrightarrow} I$ and the Principle of Uniform Boundedness it follows that the sequence of partial sum projections $(P_N)_{N\in\mathbb{N}}$ is uniformly bounded. Since $D_n = P_n - P_{n-1}$ for all $n \geq 1$, it follows that the sequence $(D_n)_{n\in\mathbb{N}}$ is uniformly bounded as well.

Let $D = (D_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear projections in E satisfying (i). Then note that D is a Schauder decomposition of E if and only if $\bigcup_{N\in\mathbb{N}} \mathrm{Ran}(\sum_{n=0}^{N} D_n)$ is dense in E and $\{\sum_{n=0}^{N} D_n \mid N \in \mathbb{N}\}$ is uniformly bounded.

Before we define the notion of unconditional Schauder decomposition, we first state a standard fact about unconditional convergence of series as the following lemma:

Lemma 4.3.2. *Let E be a Banach space and let $(x_n)_{n\in\mathbb{N}} \subset E$. The following statements are equivalent:*

(i) For every $(\lambda_n)_{n\in\mathbb{N}} \in \ell^{\infty}(\mathbb{N})$, the series $\sum_{n=0}^{\infty} \lambda_n x_n$ converges.

(ii) The series $\sum_{k=0}^{\infty} x_k$ is unconditionally convergent*, i.e., for any permutation σ of $\mathbb{N}$ the series $\sum_{n=0}^{\infty} x_{\sigma(n)}$ is convergent.*

(iii) The series $\sum_{k=0}^{\infty} x_k$ is summable*, i.e., there exits an $x \in E$ such that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all finite subsets $F \subset \{N, N+1, \ldots\}$ we have $\|\sum_{n\in F} x_n\| < \epsilon$.*

(iv) For every sequence of unimodular scalars $(\varepsilon_n)_{n\in\mathbb{N}}$, the series $\sum_{n=0}^{\infty} \varepsilon_n x_n$ converges.

(v) For every sequence of signs $(\varepsilon_n)_{n\in\mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$, the series $\sum_{n=0}^{\infty} \varepsilon_n x_n$ converges.

(vi) For every sequence $(\alpha_n)_{n\in\mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$, the series $\sum_{n=0}^{\infty} \alpha_n x_n$ converges.

Furthermore, if the series is unconditionally convergent, then the sum is independent of the order of summation.

Proof. See for example [103, Lemma 1.2.5] or [54, Lemma 2.2]. □

Example 4.3.3.

(i) Every absolutely convergent series $\sum_{n\in\mathbb{N}} x_n$ (in the Banach space E) converges unconditionally.

(ii) Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal system in a Hilbert space H. Then the series $\sum_{n\in\mathbb{N}} \frac{1}{n+1} e_n$ converges unconditionally but not absolutely.

(iii) It is a classical result of Dvoretzky & Rogers [35] that the Banach spaces E in which the unconditionally convergent series coincide with the absolutely convergent series are precisely the finite dimensional Banach spaces E.

Given a countable collection $(x_\gamma)_{\gamma\in\Gamma} \subset E$ it makes sense to call the series unconditionally convergent with limit x if we have $x = \sum_{n=0}^{\infty} x_{\gamma_n}$ for any enumeration $\mathbb{N} \longrightarrow \Gamma, n \mapsto \gamma_n$, or equivalently, if we have $x = \sum_{n=0}^{\infty} x_{\gamma_n}$ unconditionally for some/any enumeration $\mathbb{N} \longrightarrow \Gamma, n \mapsto \gamma_n$. In this situation we write $x = \sum_{\gamma\in\Gamma} x_\gamma$. Using this terminology, we define:

Definition 4.3.4. A countable collection $D = (D_\gamma)_{\gamma\in\Gamma}$ of bounded linear projections in E is called an *unconditional Schauder decomposition* of E if

(i) $D_\gamma D_{\gamma'} = 0$ whenever $\gamma \neq \gamma'$,

(ii) $x = \sum_\gamma D_\gamma x$ unconditionally for all $x \in E$.

A Schauder decomposition $D = (D_n)_{n\in\mathbb{N}}$ is said to be *unconditional* if $D = (D_n)_{n\in\mathbb{N}}$ is an unconditional Schauder decomposition.

We next come to several characterizations of unconditionallity for a Schauder decomposition $D = (D_n)_{n\in\mathbb{N}}$ of E. In order to state some of these characterizations in a notionally compact way, we define, for each $\lambda \in \mathbb{C}^{\mathbb{N}}$, the linear operator $T_\lambda : \mathrm{Ran}(D) \longrightarrow \mathrm{Ran}(D)$ by

$$T_\lambda x := \sum_{n=0}^{N} T_n D_n x, \qquad \forall x \in \mathrm{Ran}(D), x \in \mathrm{Ran}\left(\sum_{n=0}^{N} D_n\right).$$

Lemma 4.3.5. *Let $D = (D_n)_{n\in\mathbb{N}}$ be a Schauder decomposition of E. The following statements are equivalent:*

(i) D is unconditional.

(ii) There exists a constant $C_1 > 0$ such that $\|T_\varepsilon\|_{\mathcal{B}(\mathrm{Ran}(D))} \leq C_1$ for all $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}} \in \{-1, 1\}^{\mathbb{N}}$.

(iii) There exists a constant $C_2 > 0$ such that $\|T_\alpha\|_{\mathcal{B}(\mathrm{Ran}(D))} \leq C_2$ for all all $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$.

(iv) There exists a constant $C_3 > 0$ such that $\|T_\varepsilon\|_{\mathcal{B}(\mathrm{Ran}(D))} \leq C_3$ for all sequences of unimodular scalars $\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}}$.

(v) There exists a constant $C_4 > 0$ such that $\|T_\lambda\|_{\mathcal{B}(\mathrm{Ran}(D))} \leq C_4 \|\lambda\|_\infty$ for all $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$.

The smallest constant such that (iv) holds is called the unconditional constant *of the decomposition D and will be denoted by C_D.*[1]

Proof. "(i)⇒(v)": Using the characterization (i) of Lemma 4.3.2 of unconditional convergence, we see that, for each $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in \ell^\infty$, the sequence $(\sum_{n=0}^N \lambda_n D_n)_{N\in\mathbb{N}} \subset \mathcal{B}(E)$ converges pointwise, i.e., the limit $\lim_{N\to\infty} \sum_{n=0}^N \lambda_n D_n x$ exists for all $x \in E$. By the uniform boundedness principle, these poitwise limits define a bounded linear operator $\tilde{T}_\lambda \in \mathcal{B}(X)$, which clearly extends $T_\lambda : \mathrm{Ran}(D) \longrightarrow \mathrm{Ran}(D)$. To get (iv), it thus suffices to show that

$$\ell^\infty(\mathbb{N}) \longrightarrow \mathcal{B}(E),\ \lambda \mapsto \tilde{T}_\lambda$$

defines a bounded linear operator. By the closded graph theorem, for this we may show that $\lambda \mapsto \tilde{T}_\lambda$ is a closed operator: Suppose that $\lambda^{[i]} \overset{i\to\infty}{\longrightarrow} 0$ in $\ell^\infty(\mathbb{N})$ and $T_{\lambda^{[i]}} \overset{i\to\infty}{\longrightarrow} T$ in $\mathcal{B}(E)$. In order to show that $T = 0$, it is enough to show that $D_k T D_n = 0$ for all $k, n \in \mathbb{N}$, which can be seen as follows:

$$D_k T D_n = \lim_{i\to\infty} D_k T_{\lambda^{[i]}} D_n = \lim_{i\to\infty} D_k \lambda_n^{[i]} D_n = D_k 0 D_n = 0.$$

"(v)⇒(iv)⇒(ii)" & "(v)⇒(iii)": These implications are trivial.

"(iii)⇒(ii)": Given $(\varepsilon_n)_{n\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$ there exists a unique $(\alpha_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $\varepsilon_n = 2\alpha_n - 1$ for each $n \in \mathbb{N}$. From this it easily follows that we can take $C_1 = 2C_2 + 1$.

"(ii)⇒(i)": We just have to check the characterization (iii) of Lemma 4.3.2 of unconditional convergence, which is not difficult (and can be found in [103, Lemma 1.2.5]). □

Based on the characterization (iii) of unconditionallity from this lemma, we can take infinite sums (in the strong operator topology) of (distinct) projections from an unconditional Schauder decomposition:

Corollary 4.3.6. *Let $D = (D_\gamma)_{\gamma\in\Gamma}$ be an unconditional Schauder decomposition. For each $G \subset \Gamma$, $\sum_{\gamma\in G} D_\gamma$ is unconditionally convergent/summable with respect to the SOT; we denote by D_G the associated limit. In this way we obtain a collection $\{D_G : G \subset \Gamma\}$ of bounded linear projections in E with the property that $D_G D_{G'} = 0$ whenever $G \cap G' \neq \emptyset$. Moreover, given any partition $(G_k)_{k\in\mathbb{N}}$ of Γ, $\Delta := (D_{G_k})_{k\in\mathbb{N}}$ defines an unconditional Schauder decomposition of E with unconditional constant $C_\Delta \leq C_D$.*

Proof. Let's fix an enumeration $\mathbb{N} \longrightarrow \Gamma, n \mapsto \gamma_n$ of Γ. For each $G \subset \Gamma$, we define $\alpha^G = (\alpha^G_n)_{n\in\mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$ by $\alpha^G_n := 1$ if $\gamma_n \in G$ and $\alpha^G_n := 0$ if $\gamma_n \notin G$. Then $T_{\alpha^G} : \mathrm{Ran}(D) \longrightarrow \mathrm{Ran}(D)$ extends to a bounded linear projection Δ_G on E, which coincides with the SOT-limit $\sum_{n=0}^\infty D_{\gamma_n}$.

[1] In the literature the unconditional constant is usually defined as the smallest constant such that (ii) holds. The reason to choose for unimodular scalars instead of just signs is that it is more convenient when working with $\mathbb{K}$-Rademacher sequences $(\epsilon_n)_{n\in\mathbb{N}}$ instead of real-Rademacher sequences $(r_n)_{n\in\mathbb{N}}$.

Since T_{α^G} is independent of the chosen enumeration, it follows that $\sum_{\gamma \in G} D_\gamma$ is unconditionally convergent/summable with limit Δ_G. From this construction it is clear that $D_G D_{G'} = 0$ whenever $G \cap G' \neq \emptyset$. We skip the proof of the last statement since we will not need it later on. □

Note that (iv) of Lemma 4.3.5 is equivalent with the existence of a constant $C > 0$ such that

$$\left\| \sum_{n=0}^{N} \varepsilon_n D_n x \right\|_E \leq C \left\| \sum_{n=0}^{N} D_n x \right\|_E \tag{4.28}$$

holds for all $(\varepsilon_n)_{n=0}^N \in \mathbb{T}^{N+1}$, $N \in \mathbb{N}$, and $x \in E$, in which case the smallest such constant coincides with the unconditional constant C_D of D. Recall that we denote by $(\epsilon_n)_{n\in\mathbb{N}}$ a fixed Rademacher sequence on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$; see Appendix E.1. Just as the UMD property of Banach spaces has a randomized characterization in terms of $(\epsilon_n)_{n\in\mathbb{N}}$ (see Lemma E.5.2), we have:

Lemma 4.3.7. *Let $D = (D_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear projections in E with the property that $D_n D_k = 0$ for all $n \neq k$. Then the following statements are equivalent:*

(i) There exists a constant $C > 0$ such that (4.28) holds for all $(\varepsilon_n)_{n=0}^N \in \mathbb{T}^{N+1}$, $N \in \mathbb{N}$, and $x \in E$.

(ii) There exists a $p \in [1, \infty[$ and constants $C_p^+, C_p^- > 0$ such that

$$\frac{1}{C_p^-} \left\| \sum_{n=0}^{N} D_n x \right\|_E \leq \left\| \sum_{n=0}^{N} \epsilon_n D_n x \right\|_{L^p(\Omega;E)} \leq C_p \left\| \sum_{n=0}^{N} D_n x \right\|_E \tag{4.29}$$

holds for all $x \in E$ and $N \in \mathbb{N}$.

(iii) For every $p \in [1, \infty[$ there exist constants $C_p^+, C_p^- > 0$ such that

$$\frac{1}{C_p^-} \left\| \sum_{n=0}^{N} D_n x \right\|_E \leq \left\| \sum_{n=0}^{N} \epsilon_n D_n x \right\|_{L^p(\Omega;E)} \leq C_p \left\| \sum_{n=0}^{N} D_n x \right\|_E \tag{4.30}$$

holds for all $x \in E$ and $N \in \mathbb{N}$.

Moreover, in this situation we can take $C_p^-, C_p^+ \leq C \leq C_p^+ C_p^-$.

As a consequence, a Schauder decomposition $D = (D_n)_{n\in\mathbb{N}}$ of E is unconditional if and only if it satisfies one of the equivalent conditions (i),(ii),(iii), in which case the smallest constant $C \geq 1$ for which (i) holds coincides with the unconditional constant C_D of D; in particular, we can take $C_p^-, C_p^+ \leq C_D \leq C_p^+ C_p^-$.

Proof. First note that (4.28) holds true if and only if the reverse inequality holds true (with a different constant of course); just replace x by $\sum_{n=0}^N \varepsilon_n D_n x$ in (4.28). With this in mind, the statement follows simply from the fact that, for any $(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$, $(\alpha_n \epsilon_n)_{n\in\mathbb{N}}$ is identically distributed with $(\epsilon_n)_{n\in\mathbb{N}}$. For more details we refer to [103, Lemma 1.3.6]. □

The following definition is motivated by the above lemma.

Definition 4.3.8. A sequence $D = (D_n)_{n\in\mathbb{N}} \subset \mathcal{B}(E)$ is called U^+ if there exists a constant $C^+ > 0$ such that

$$\left\| \sum_{n=0}^{N} \epsilon_n D_n x \right\|_{L^2(\Omega;E)} \leq C^+ \left\| \sum_{n=0}^{N} D_n x \right\|_E$$

for all $N \in \mathbb{N}$ and $x \in E$, and is called called U^- if there exists a constant $C^- > 0$ such that

$$\left\| \sum_{n=0}^{N} D_n x \right\|_E \leq C^- \left\| \sum_{n=0}^{N} \epsilon_n D_n x \right\|_{L^2(\Omega;E)}$$

for all $N \in \mathbb{N}$ and $x \in E$. We denote the smallest such constants $C^+ > 0$ and $C^- > 0$ by $C_D^+ > 0$ and $C_D^- > 0$, respectively.

Note that, by Lemma 4.3.7, a Schauder decomposition $D = (D_n)_{n\in\mathbb{N}}$ of E is unconditional if and only if it is both U^+ and U^-.

Example 4.3.9. Let X be a Banach space, $(S, \mathscr{A}, \mu)$ a measure space with a σ-finite filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$, and $p \in]1, \infty[$. By Theorem A.3.25, $D = (D_n)_{n\in\mathbb{N}}$ defined by

$$D_n := \begin{cases} \mathbb{E}(\,\cdot \mid \mathscr{F}_0) & n = 0; \\ \mathbb{E}(\,\cdot \mid \mathscr{F}_n) - \mathbb{E}(\,\cdot \mid \mathscr{F}_{n-1}) & n \geq 1, \end{cases}$$

defines a Schauder decomposition of $L^p(S; X)$. Then X has the UMD_p property with respect to $(S, \mathscr{A}, (\mathscr{F}_n)_{n\in\mathbb{N}}, \mu)$, if and only if D is unconditional, if and only if D is U^+ and U^-.

Let $D = (D_\gamma)_{\gamma\in\Gamma}$ be a countable collection of bounded linear operators in E with the property that $D_\gamma D_{\gamma'} = 0$ whenever $\gamma \neq \gamma'$. Then note that there exists a constant $C^+ > 0$ such that

$$\left\| \sum_{\gamma\in F} \epsilon_\gamma D_\gamma x \right\|_{L^2(\Omega;E)} \leq C^+ \left\| \sum_{\gamma\in\Gamma} D_\gamma x \right\|_E$$

holds for all finite subsets $F \subset \Gamma$ and all $x \in E$, if and only if, for any enumeration $\mathbb{N} \longrightarrow \Gamma, n \mapsto \gamma_n$, $\Delta = (D_{\gamma_n})_{n\in\mathbb{N}}$ is U^+, in which case the smallest such constant C^+ coincides with C_Δ^+. This allows us say that $D = (D_\gamma)_{\gamma\in\Gamma}$ is U^+ provided that there exists such a constant $C^+ > 0$, in which case we denote by C_D^+ the smallest such constant. We do a similar thing for the notion of U^-.

Let $D = (D_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear projections in E with the property that $D_n D_k = 0$ whenever $n \neq k$. By Proposition E.1.1 (or by the Kahane contraction principle, Proposition E.1.2), if D is U^+, then we have

$$\left\| \sum_{n=0}^{N} \epsilon_n D_n x \right\|_{L^2(\Omega;E)} \leq C \left\| \sum_{n=0}^{M} D_n x \right\|_E$$

for all $N, M \in \mathbb{N}, N \leq M$ and $x \in E$ for the constant $C = C_D^+$, which is of course equivalent with

$$\left\| \sum_{n=0}^{N} \epsilon_n D_n x \right\|_{L^2(\Omega;E)} \leq C \, \|x\|_E \tag{4.31}$$

for all $N \in \mathbb{N}$ and $x \in \mathrm{Ran}(D)$. Conversely, if (4.31) holds for some constant $C > 0$, then D is U^+ with $C_D^+ \leq C$ (just replace x by $\sum_{n=0}^{N} D_n x$).

Lemma 4.3.10. *Let $D = (D_n)_{n\in\mathbb{N}}$ be an unconditional Schauder decomposition of E. Then the sequence of adjoint operators $D^* = (D_n^*)_{n\in\mathbb{N}}$ satisfies the inequality*

$$\left\|\sum_{n=0}^{N} \varepsilon_n D_n^* x^*\right\| \leq C_D \|x^*\|$$

for all $(\varepsilon_n)_{n=0}^N \in \mathbb{T}^{N+1}$, $N \in \mathbb{N}$, and $x^* \in E^*$. As consequence, D^* is U^+ with $C_{D^*}^+ \leq C_D$.

Proof. Let $(\varepsilon_n)_{n=0}^N \in \mathbb{T}^{N+1}$, $N \in \mathbb{N}$, and $x^* \in E^*$. Then we have, for every x in the dense subspace $\mathrm{Ran}(D)$ of E (and thus norming for E^*),

$$\left|\langle \sum_{n=0}^{N} \varepsilon_n D_n^* x^*, x\rangle\right| = \left|\langle x^*, \sum_{n=0}^{N} \varepsilon_n D_n x\rangle\right| \leq \|x^*\| \left\|\sum_{n=0}^{N} \varepsilon_n D_n x\right\| \leq C_D \|x^*\| \|x\|.$$

This gives the desired norm inequality. □

Lemma 4.3.11. *Let $D = (D_n)_{n\in\mathbb{N}}$ be a sequence of bounded linear projections in E with the property that $D_k D_n = 0$ whenever $k \neq n$. Denote by D^* the sequence of adjoints $D^* = (D_n^*)_{n\in\mathbb{N}}$. If there exists a constant $C > 0$ such that*

$$\left\|\sum_{n=0}^{N} \epsilon_n D_n^* x^*\right\|_{L^2(\Omega;E^*)} \leq \|x^*\|$$

for all $N \in \mathbb{N}$ and $x^* \in Z$ for some linear subspace Z of E^* which is norming for $\mathrm{Ran}(D)$*, then D is U^- with $C_D^- \leq C$. In particular, if D^* is U^+ and* $\mathrm{Ran}(D^*)$ *is norming for* $\mathrm{Ran}(D)$*, then D is U^- with $C_D^- \leq C_{D^*}^+$; see* (4.31)*. Moreover, the same statement holds true with the roles of D and D^* interchanged.*

Proof. We only prove the first part, the second part (with the roles of D and D^* interchanged) being completely similar. Let $x \in E$ and $x^* \in Z$. Then we have

$$\begin{aligned}
\left|\langle \sum_{n=0}^{N} D_n x, x^*\rangle\right| &= \int_\Omega \left|\langle \sum_{n=0}^{N} D_n x, x^*\rangle\right| d\omega = \int_\Omega \left|\langle \sum_{n=0}^{N} |\epsilon_n(\omega)|^2 D_n^2 x, x^*\rangle\right| d\omega \\
&= \int_\Omega \left|\langle \left(\sum_{k=0}^{N} \epsilon_k(\omega) D_k\right)\left(\sum_{n=0}^{N} \overline{\epsilon_n}(\omega) D_n\right) x, x^*\rangle\right| d\omega \\
&= \int_\Omega \left|\langle \sum_{n=0}^{N} \epsilon_n(\omega) D_n x, \sum_{k=0}^{N} \overline{\epsilon_k}(\omega) D_k^* x^*\rangle\right| d\omega \\
&\leq \left\|\sum_{n=0}^{N} \epsilon_n D_n x\right\|_{L^2(\Omega;E)} \left\|\sum_{k=0}^{N} \overline{\epsilon_k} D_k^* x^*\right\|_{L^2(\Omega;E^*)} \\
&\leq \left\|\sum_{n=0}^{N} \epsilon_n D_n x\right\|_{L^2(\Omega;E)} \left\|\sum_{k=0}^{N} \epsilon_k D_k^* x^*\right\|_{L^2(\Omega;E^*)} \\
&\leq C \left\|\sum_{n=0}^{N} \epsilon_n D_n x\right\|_{L^2(\Omega;E)} \|x^*\|.
\end{aligned}$$

Since Z is norming for $\mathrm{Ran}(D)$, it follows that D is U^- with $C_D^- \leq C$. □

Corollary 4.3.12. *Let $D = (D_n)_{n\in\mathbb{N}}$ be a Schauder decomposition of E. Then D is unconditional if and only if both D and D^* are U^+. In this situation we have $C^-_D \le C^+_{D^*}$.*

Proof. We only need to show that $\mathrm{Ran}(D^*)$ is norming for $\mathrm{Ran}(D)$; then the statement follows immediately from Lemma's 4.3.10 and 4.3.11. To this end, let $x \in \mathrm{Ran}(D) \setminus \{0\}$, say $x \in \mathrm{Ran}(P_N)$. By Hahn-Banach, there exists a functional $x^* \in S_{E^*}$ such that $\langle x, x^* \rangle = \|x\|$ and which vanishes on the closed subspace $\mathrm{Ran}(1 - P_N)$. Then, for all $y \in E$,

$$\langle y, x^* \rangle = \langle y, x^* \rangle - \langle (1 - P_N)y, x^* \rangle = \langle P_N y, x^* \rangle = \langle y, (P_N)^* x^* \rangle,$$

whence $x^* = (P_N)^* x^*$; in particular, $x^* \in \mathrm{Ran}(D^*)$. □

Corollary 4.3.13. *Let $D = (D_\gamma)_{\gamma\in\Gamma}$ be countable collection of bounded linear projections in E such that $D_\gamma D_{\gamma'} = 0$ whenever $\gamma \neq \gamma'$. Then D is an unconditional Schauder decomposition provided that $\mathrm{Ran}(D)$ is dense in E and that there exist constants $C, C^* > 0$ such that*

$$\left\| \sum_{\gamma\in F} \epsilon_\gamma D_\gamma x \right\|_{L^2(\Omega;E)} \le C \|x\|_E \tag{4.32}$$

holds for all finite subsets $F \subset \Gamma$ and all $x \in E$, and

$$\left\| \sum_{\gamma\in F} \epsilon_\gamma D_\gamma x^* \right\|_{L^2(\Omega;E)} \le C^* \|x^*\|_E \tag{4.33}$$

holds for all finite subsets $F \subset \Gamma$ and all $x^* \in E^*$. Moreover, in this situation we have $C^+_D \le C$ and $C^-_D \le C^+_{D^*} \le C^*$.

Proof. Let $\mathbb{N} \longrightarrow \Gamma, n \mapsto \gamma_n$ be any enumeration of Γ and write $\Delta := (D_{\gamma_n})_{n\in\mathbb{N}}$. From (4.33) and Lemma 4.3.11 it follows that Δ is U^- with $C^-_\Delta \le C^*$. Combining this with (4.32), we in particular obtain that the sequence of partial sum projections associated with Δ is uniformly bounded. Since $\mathrm{Ran}(\Delta) = \mathrm{Ran}(D)$ is dense in E by assumption, it follows that Δ is a Schauder decomposition of E. By (4.32) and (4.33), both Δ and Δ^* are U^+ with $C^+_D \le C$ and $C^+_{D^*} \le C^*$. The result now follows from Corollary 4.3.12. □

4.3.2.b A Multiplier Theorem

We now turn back to the multiplier problem considered in Section 4.3.1. In view of the randomized characterization of unconditional Schauder decomposition given in Lemma 4.3.7, a solution to this problem would be to restrict to unconditional Schauder decompositions D and require the sequence of operators $(T_n)_{n\in\mathbb{N}}$ to act boundedly on Rademacher sums. The latter is precisely the defining property of $\mathcal{R}$-boundedness, see Appendix E.3.

Theorem 4.3.14. *Let $D = (D_n)_{n\in\mathbb{N}}$ be an unconditional Schauder decomposition of E, with unconditional constant C_D. Suppose that $(T_n)_{n\in\mathbb{N}} \subset \mathcal{B}(E)$ is $\mathcal{R}$-bounded, with $\mathcal{R}$-bound $M = \mathcal{R}((T_n)_{n\in\mathbb{N}})$, and has the property that $T_n D_n = D_n T_n D_n$ for every $n \in \mathbb{N}$. Then*

$$S x := \sum_{n=0}^{\infty} T_n D_n x, \qquad x \in E,$$

gives rise to a well-defined bounded linear operator $S \in \mathcal{B}(E)$ of norm $\|S\| \le C_D^2 M$.

Proof. We just have to combine the characterization of unconditional Schauder decomposition given in Lemma 4.3.7 with the definition of $\mathcal{R}$-boundedness, Definition E.3.1. For a detailed proof we refer to [103, Theorem 2.2.4]. □

4.3.2.c Unconditional Blockings of Product Decompositions

In our application to Fourier multipliers we want to use a Schauder decomposition $D = (D_\gamma)_{\gamma\in\Gamma}$ of $L^{p,d}(\mathbb{R}^d, w; X)$ corresponding to an appropriate decomposition of $\mathbb{R}^d$, possibly up to a set of measure zero, by a countable collection of disjoint rectangles $(E_\gamma)_{\gamma\in\Gamma}$; here X is a UMD Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}^{rec}(\mathbb{R}^{d_j})$. In order to find such an unconditional Schauder decomposition, we would like to use the case $d = 1$. To settle ideas, suppose that $d = 2$ and $\mathpzc{d} = (1,1)$. Then we have

$$L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X) = L^{(p_1,p_2),(1,1)}(\mathbb{R}^2, (w_1, w_2); X) \cong L^{p_2}(\mathbb{R}, w_2; L^{p_1}(\mathbb{R}, w_2; X)). \tag{4.34}$$

Now suppose that we have two decompositions of $\mathbb{R} \setminus \{0\}$, possibly up to a set of measure zero, by countable collections of disjoint intervals $(I_i)_{i\in I}$ and $(J_j)_{j\in J}$, respectively, in such a way that, for any $q \in]1,\infty[$ and $v \in A_q(\mathbb{R})$, it holds that the collections of Fourier multiplier operators on $L^q(\mathbb{R}, v; Y)$ associated with $(1_{I_i})_{i\in I}$ and $(1_{J_j})_{j\in J}$ both define unconditional Schauder decompositions of $L^q(\mathbb{R}, v; Y)$. Then, using (4.34), it is not difficult to see that $(1_{I_i\times\mathbb{R}})_{i\in I}, (1_{\mathbb{R}\times J_j})_{j\in J} \subset \mathcal{M}_{p,\mathpzc{d},w}(X)$, with the sequences of associated Fourier multiplier operators $D^1 = (D^1_i)_{i\in I} := (T_{1_{I_i\times\mathbb{R}}})_{i\in I}$ and $D^2 = (D^2_j)_{j\in J} := (T_{1_{\mathbb{R}\times J_j}})_{j\in J}$ defining two commuting unconditional Schauder decompositions of $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$. Furthermore, we have $(D^1_i D^2_j)_{(i,j)\in I\times J} = (T_{1_{I_i\times J_j}})_{(i,j)\in I\times J}$, the collection of Fourier multiplier operators on $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ corresponding to the decomposition of $\mathbb{R}^2$, possibly up to a set of measure zero, by the rectangles $(I_i \times J_j)_{(i,j)\in I\times J}$. This raises the question whether $(D^1_i D^2_j)_{(i,j)\in I\times J}$ is an unconditional Schauder decomposition of $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$, so that we can take $E_\gamma := I_i \times J_j$ for $\gamma = (i, j) \in \Gamma := I \times J$. However, in the literature it is well known that the answer to this is not 'yes' unless we impose further conditions, in addition to UMD, on the Banach space X; see Remark 4.5.19. Since we want to prove Fourier multiplier theorems in the generality of all UMD spaces, we will construct a different partition $(E_\gamma)_{\gamma\in\Gamma}$ out of $(I_i \times J_j)_{(i,j)\in I\times J}$.

In Lemma 4.5.18, Section 4.5.3, we will construct a decomposition $(E_\gamma)_{\gamma\in\Gamma}$ of $(\mathbb{R} \setminus \{0\})^2$, consisting of countably many rectangles, using the collection of rectangles $(I_i \times J_j)_{(i,j)\in I\times J}$ in which $(I_i)_{i\in I}$ and $(J_j)_{j\in J}$ are dyadic partitions of $\mathbb{R} \setminus \{0\}$. A key ingredient will be the following abstract result about unconditional blockings of product decompositions:

Theorem 4.3.15. *Let $D^1 = (D^1_i)_{i\in\mathbb{Z}}$ and $D^2 = (D^2_i)_{i\in\mathbb{Z}}$ be two commuting unconditional Schauder decompositions of E. Suppose that the following $\mathcal{R}$-boundedness condition holds true:*

$$\kappa_k := \mathcal{R}\left\{\sum_{i=M}^{N} D^k_i \;:\; M,N \in \mathbb{Z}\right\} < \infty, \quad \kappa^*_k := \mathcal{R}\left\{\sum_{i=M}^{N} (D^k_i)^* \;:\; M,N \in \mathbb{Z}\right\} < \infty, \quad k = 1,2. \tag{4.35}$$

Define the partition $(I_n)_{n\in\mathbb{Z}}$ of the index set $\mathbb{Z}^2$ by

$$I_n := \begin{cases} \{r+1\} \times \{\ldots, r-1, r\}, & n = 2r+1, r \in \mathbb{Z} \\ \{\ldots, r, r+1\} \times \{r+1\}, & n = 2r+2, r \in \mathbb{Z}. \end{cases}$$

For each $n \in \mathbb{Z}$ we define the bounded linear projection

$$\Delta_n := \text{SOT} - \sum_{(i_1,i_2)\in I_n} D^1_{i_1} D^2_{i_2} = \begin{cases} D^1_{r+1} D^2_{\{\dots,r-1,r\}} & n = 2r+1, r \in \mathbb{Z} \\ D^1_{\{\dots,r,r+1\}} D^2_{r+1} & n = 2r+2, r \in \mathbb{Z} \end{cases}$$

in E, where $D^2_{\{\dots,r-1,r\}}$ and $D^1_{\{\dots,r,r+1\}}$ are as in Corollary 4.3.6. Then $\Delta = (\Delta_n)_{n\in\mathbb{Z}}$ is an unconditional Schauder decomposition of E for which we have

$$C^+_\Delta \leq C^+_{D^1}\kappa_2 + C^+_{D^2}\kappa_1, \qquad C^-_\Delta \leq C^+_{\Delta^*} \leq C^+_{[D^1]^*}\kappa^*_2 + C^+_{[D^2]^*}\kappa^*_1 \tag{4.36}$$

and

$$\mathcal{R}\left\{\sum_{n=M}^{N} \Delta_n \;:\; N, M \in \mathbb{Z}\right\} \leq 2\kappa_1\kappa_2, \qquad \mathcal{R}\left\{\sum_{n=M}^{N} \Delta^*_n \;:\; N, M \in \mathbb{Z}\right\} \leq 2\kappa^*_1\kappa^*_2. \tag{4.37}$$

Proof. We start with observing the following: Defining for each $k \in \{1, 2\}$ and $n \in \mathbb{Z}$ the bounded linear projection $P^k_n := D^k_{\{\dots,n-1,n\}} \in \mathcal{B}(E)$, where $D^k_{\{\dots,n-1,n\}}$ is as in Corollary 4.3.6, we have

$$\mathcal{R}\{P^k_n \;:\; n \in \mathbb{Z}\} \leq \kappa_k \qquad \text{and} \quad \mathcal{R}\{(P^k_n)^* \;:\; n \in \mathbb{Z}\} \leq \kappa^*_k, \qquad k = 1, 2. \tag{4.38}$$

Indeed, it holds that

$$P^k_n = \text{SOT} - \lim_{M\to-\infty} \sum_{i_k=M}^{n} D^k_n \quad \text{and} \quad (P^k_n)^* = \text{W}^*\text{OT} - \lim_{M\to-\infty} \sum_{i_k=M}^{n} (D^k_n)^*,$$

whence

$$\{P^k_n \;:\; n \in \mathbb{Z}\} \subset \text{SOT} - \text{cl}\left\{\sum_{i=M}^{N} D^1_n \;:\; M, N \in \mathbb{Z}\right\}$$

and

$$\{(P^k_n)^* \;:\; n \in \mathbb{Z}\} \subset \text{W}^*\text{OT} - \text{cl}\left\{\sum_{i=M}^{N} (D^1_n)^* \;:\; M, N \in \mathbb{Z}\right\}.$$

By the $\mathcal{R}$-boundedness assumption (4.35) and the basic properties of $\mathcal{R}$-boundedness (see Proposition E.3.5), we obtain (4.38).

Now, let's show that $(\Delta_{\gamma_n})_{n\in\mathbb{N}}$ is a Schauder decomposition, where $\mathbb{N} \longrightarrow \mathbb{Z}, n \mapsto \gamma_n$ is any enumeration of $\mathbb{Z}$ with the property that, for every $h \in \mathbb{N}$, $\{\gamma_n \;:\; n = 0, \dots, h\} = \{M_h, \dots, N_h\}$ for some $M_h, N_h \in \mathbb{Z}$: Since $D^1 = (D^1_i)_{i\in\mathbb{Z}}$ and $D^2 = (D^2_i)_{i\in\mathbb{Z}}$ are both unconditional Schauder decompositions of E, it is easy to see that

$$\text{Ran}(\Delta) = \bigcup\left\{\text{Ran}\left(\sum_{i_1\in F_1}\sum_{i_2\in F_2} D^1_{i_1} D^2_{i_2}\right) \;:\; F_1, F_2 \subset \mathbb{Z} \text{ finite}\right\}$$

is dense in E. In order to show that partial sum projections associated with $(\Delta_{\gamma_n})_{n\in\mathbb{N}}$ form a uniformly bounded family, we may show the stronger statement (4.37). To this end, we define $(\Pi_n)_{n\in\mathbb{Z}}$ by $\Pi_n := \Delta_{\{\dots,n-1,n\}}$. Then we have $\sum_{n=M}^{N} \Delta_n = \Pi_N - \Pi_{M-1}$ for $N \geq M$ and $\sum_{n=M}^{N} \Delta_n = 0$ otherwise, so that by the basic properties of $\mathcal{R}$-bounds (see Proposition E.3.5), it is enough to show that

$$\mathcal{R}\{\Pi_n \;:\; n \in \mathbb{Z}\} \leq \kappa_1\kappa_2, \qquad \mathcal{R}\{\Pi^*_n \;:\; n \in \mathbb{Z}\} \leq \kappa^*_1\kappa^*_2. \tag{4.39}$$

For this we note that

$$\Pi_n = \begin{cases} P^1_{r+1}P^2_r & n = 2r+1, r \in \mathbb{Z}, \\ P^1_{r+1}P^2_{r+1} & n = 2r+2, r \in \mathbb{Z}. \end{cases}$$

In particular, $(\Pi_n)_{n\in\mathbb{Z}} \subset \{P^1_k : n \in \mathbb{Z}\} \cdot \{P^2_n : n \in \mathbb{Z}\}$ and thus $(\Pi^*_n)_{n\in\mathbb{Z}} \subset \{(P^2_k)^* : n \in \mathbb{Z}\} \cdot \{(P^1_n)^* : n \in \mathbb{Z}\}$. From (4.38) and the basic properties of $\mathcal{R}$-boundeds (see Proposition E.3.5) we obtain (4.39), as desired.

By Corollary 4.3.12, to finish the proof it remains to be shown that Δ and Δ^* are both U^+, $C^+_\Delta \le C^+_{D^1}\kappa_2 + C^+_{D^2}\kappa_1$, $C^+_{\Delta^*} \le C^+_{[D^1]^*}\kappa^*_2 + C^+_{[D^2]^*}\kappa^*_1$. We only consider Δ, the case Δ^* being completely similar. To this end, let $x \in \mathrm{Ran}(\Delta)$ and a finite subset F of $\mathbb{Z}$ be given. Writing $F = F_1 \cup F_2$ with $F_k := F \cap \{2r+k \mid r \in \mathbb{Z}\}$ for $k \in \{1, 2\}$, it suffices to show that

$$\left\| \sum_{n\in F_1} \epsilon_n \Delta_n x \right\|_{L^2(\Omega;E)} \le C^+_{D^1}\kappa_2, \qquad \left\| \sum_{n\in F_2} \epsilon_n \Delta_n x \right\|_{L^2(\Omega;E)} \le C^+_{D^2}\kappa_1.$$

We only treat the random sum over F_1, the sum over F_2 being similar. Using that

$$\sum_{n\in F_1} \Delta_n = \sum_{r\in\tilde{F}_1} D^1_{r+1}P^2_{r+1} = \sum_{r\in\tilde{F}_1} P^2_{r+1}D^1_{r+1},$$

where $\tilde{F}_1 := \{r \in \mathbb{Z} : 2r+1 \in F_1\}$, and that $x \in \mathrm{Ran}(\Delta) \subset \overline{\mathrm{Ran}(D^1)}$, we find

$$\begin{aligned} \left\| \sum_{n\in F_1} \epsilon_n \Delta_n x \right\|_{L^2(\Omega;E)} &= \left\| \sum_{r\in\tilde{F}_1} \epsilon_{2r+1} P^2_{r+1} D^1_{r+1} x \right\|_{L^2(\Omega;E)} \le \kappa_2 \left\| \sum_{r\in\tilde{F}_1} \epsilon_{2r+1} D^1_{r+1} x \right\|_{L^2(\Omega;E)} \\ &\le \kappa_2 C^+_{D^1} \|x\|_E, \end{aligned}$$

as desired. □

Remark 4.3.16.

(i) The $\mathcal{R}$-boundedness of the first two collections in (4.35) is automatic when the space E has the so-called property *weak-*(α); see [103, Definition 2.4.1], [103, Corollary 2.4.3].[2] Having the $\mathcal{R}$-boundedness of the first two collections, the $\mathcal{R}$-boundedness of the last two collections then is a consequence when the space E has non-trivial type; see Theorem E.3.7. In particular, as the UMD property implies property weak-(α) and non-trivial type, the $\mathcal{R}$-boundedness of (4.35) is automatic when E is a UMD space; see [103], [57].

In the next subsection we will apply the above theorem to a concrete situation (in the setting of Fourier multipliers) in which the $\mathcal{R}$-boundedness of (4.35) can be checked directly, with explicit bounds, so that we do not have to rely on the just mentioned abstract results.

(ii) If we assume $\{\sum_{n\in G} D^1_n \mid G \subset \mathbb{Z} \text{ finite}\}$, $\{\sum_{n\in G} D^2_n \mid G \subset \mathbb{Z} \text{ finite}\}$, $\{\sum_{n\in G}(D^1_n)^* \mid G \subset \mathbb{Z} \text{ finite}\}$, and $\{\sum_{n\in G}(D^2_n)^* \mid G \subset \mathbb{Z} \text{ finite}\}$ to be $\mathcal{R}$-bounded, then we can follow (a slightly modified version of) the above proof to get the result that $(D^1_{i_1}D^2_{i_2})_{(i_1,i_2)\in\mathbb{N}^2}$ is an unconditional Schauder decomposition. A sufficient condition for the $\mathcal{R}$-boundedness of the first two collections is that E has property (α), and, having the $\mathcal{R}$-boundedness of the first

[2] Property weak-(α) is in the literature also known under the name *triangular contraction property*; see [57].

two, a sufficient condition for the $\mathcal{R}$-boundedness of the last two collections then is that E is has non-trivial type; see [103, Corollary 2.3.5] and Theorem E.3.7. In particular, the $\mathcal{R}$-boundedness of these collections is automatic when E is UMD space with (α).

In the context of Fourier multipliers it even occurs that property (α) is not only sufficient for the above, but also necessary; see Remark 4.5.19.

Corollary 4.3.17. *Let $D^1 = (D^1_i)_{i\in\mathbb{N}}$ and $D^2 = (D^2_i)_{i\in\mathbb{N}}$ be two commuting unconditional Schauder decompositions of E, with corresponding collections of partial sum projections $\{P^1_N\}_{N\in\mathbb{N}}$ and $\{P^2_N\}_{N\in\mathbb{N}}$, respectively. Suppose that*

$$\kappa_k := \mathcal{R}\left\{P^k_N \,:\, N\in\mathbb{N}\right\} < \infty, \quad \kappa^*_k := \mathcal{R}\left\{(P^k_N)^* \,:\, N\in\mathbb{N}\right\} < \infty, \quad k = 1,2. \tag{4.40}$$

Define the partition $(I_n)_{n\in\mathbb{N}}$ of the index set $\mathbb{N}^2$ by

$$I_n := \begin{cases} \{(0,0)\}, & n = 0 \\ \{r+1\}\times\{0,\ldots,r\}, & n = 2r+1, r\in\mathbb{N} \\ \{0,\ldots,r+1\}\times\{r+1\}, & n = 2r+2, r\in\mathbb{N} \end{cases}.$$

For each $n\in\mathbb{N}$ we define the bounded linear projection

$$\Delta_n := \sum_{(i_1,i_2)\in I_n} D^1_{i_1} D^2_{i_2}$$

in E. Let $(\Pi_N)_{N\in\mathbb{N}}$ be the associated family of partial sum projections. Then $\Delta = (\Delta_n)_{n\in\mathbb{N}}$ is an unconditional Schauder decomposition of E for which we have

$$C^+_\Delta \leq C^+_{D^1}\kappa_2 + C^+_{D^2}\kappa_1, \qquad C^-_\Delta \leq C^+_{\Delta^*} \leq C^+_{[D^1]^*}\kappa^*_2 + C^+_{[D^2]^*}\kappa^*_1 \tag{4.41}$$

and

$$\mathcal{R}\{\Pi_N \,:\, N\in\mathbb{N}\} \leq 2\kappa_1\kappa_2, \qquad \mathcal{R}\{(\Pi_N)^* \,:\, N\in\mathbb{N}\} \leq 2\kappa^*_1\kappa^*_2. \tag{4.42}$$

Proof. Just define $D^k_n := 0$ for every $n\in\mathbb{Z}_{<0}$ and $k\in\{1,2\}$, and apply Theorem 4.3.15 to the commuting unconditional Schauder decompositions $(D^1_n)_{n\in\mathbb{Z}}$ and $(D^2_n)_{n\in\mathbb{Z}}$. □

4.4 Calderón-Zygmund Operators

In this section we state the A_p theorem for vector-valued Calderón-Zygmund operators due to Hänninen and Hytönen [49], which says that a bounded linear operator on the unweighted Lebesgue-Bochner space $L^p(\mathbb{R}^d;X)$ corresponding to a singular kernel K also acts boundedly on the weighted Lebesgue-Bochner spaces $L^p(\mathbb{R}^d,w;X)$ for weights w from the class A_p. We furthermore give sufficient conditions on a symbol m in order that the associated Fourier multiplier operator T_m is a Calderón-Zygmund operator with corresponding singular kernel estimates depending on concrete estimates for m.

4.4.1 Introduction

Our first approach to Fourier multipliers on $L^{p,d}(\mathbb{R}^d, w; X)$ taken in this chapter will be based on the abstract theory of unconditional Schauder decompositions discussed in the previous section and requires the Banach space X to be of class $\mathcal{RP}_{p,w}$, for which it is necessary that $w \in \prod_{j=1}^{l} A_{p_j}^{rec}(\mathbb{R}^{d_j})$ by Lemma 4.2.7.(iii) . In order to also get Fourier multiplier results for general weight-vectors $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$, our second approach to Fourier multipliers consists of extrapolation from the unweighted case (for sufficiently regular symbols and then obtain more general symbols via approximation). The idea is to view Fourier multiplier operators (associated with sufficiently regular symbols) as integral operators of convolution type, so that we can apply the theory of (singular) integral operators.

In order to describe the connection between Fourier multiplier operators and integral operators of convolution type for symbols from the Schwartz class, let X be a Banach space, $p \in]1, \infty[^l$, and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$, and suppose that we are given a symbol $m \in \mathcal{S}(\mathbb{R}^d; \mathcal{B}(X))$. Then, recall that $m \in \mathcal{M}_{p,d,w}(X)$ with $\|m\|_{\mathcal{M}_{p,d,w}(X)} \leq C$ if and only if the operator

$$T_m : \mathcal{S}'(\mathbb{R}^d; X) \longrightarrow \mathcal{S}'(\mathbb{R}^d; X),\ f \mapsto \mathscr{F}^{-1}[m\hat{f}] = \check{m} * f$$

maps some dense subspace D of $L^{p,d}(\mathbb{R}^d, w; X)$ into $L^{p,d}(\mathbb{R}^d, w; X)$ with the norm estimate $\|T_m f\|_{L^{p,d}(\mathbb{R}^d,w;X)} \leq C\,\|f\|_{L^{p,d}(\mathbb{R}^d,w;X)}$ for all $f \in D$; see Remark 4.2.3.(iii). By Young's inequality, since $\check{m} \in \mathcal{S}(\mathbb{R}^d; \mathcal{B}(X)) \subset L^1(\mathbb{R}^d; \mathcal{B}(X))$, we have that T_m restricts to a bounded linear operator on $L^{p,d}(\mathbb{R}^d; X)$ of norm $\|\check{m}\|_{L^1(\mathbb{R}^d;\mathcal{B}(X))}$, which is given by convolution with $\check{m}$ in the classical sense (of measure theory):

$$T_m f(x) = \int_{\mathbb{R}^d} \check{m}(x-y)f(y)\,dy, \qquad f \in L^{p,d}(\mathbb{R}^d; X), x \in \mathbb{R}^d. \tag{4.43}$$

Now the idea is to obtain the boundedness of T_m on $L^{p,d}(\mathbb{R}^d, w; X)$ by proving a norm estimate on the dense space $D := L_c^\infty(\mathbb{R}^d; X)$ of $L^{p,d}(\mathbb{R}^d, w; X)$ via an extrapolation theorem for bounded linear operators on $L^{p,d}(\mathbb{R}^d, w; X)$ corresponding to a singular kernel; see Definition 4.4.1, Definition 4.4.2, and Theorem 4.4.3. This norm estimate will depend on $[w_1]_{A_{p_1}}, \ldots, [w_l]_{A_{p_l}}$, $\|m\|_{\mathcal{M}_{p,d}(X)}$, and the implicit constants in the definition of a singular kernel. In Lemma 4.4.7 we will give explicit bounds for these implicit constants in terms of estimates on the symbol m, and in the next section we will prove Fourier multiplier theorems for $L^{p,d}(\mathbb{R}^d; X)$, also giving bounds for $\|m\|_{\mathcal{M}_{p,d}(X)}$ in terms of a certain $\mathcal{R}$-boundedness condition for the symbol m. We will see that these estimates/bounds on m are well suited to treat more general symbols, which also satisfy such estimates/bounds, using approximation arguments; see (the proofs) of Proposition 4.5.8 and Theorem 4.5.20.

4.4.2 An Extrapolation Theorem

We start with (d, a)-anisotropic generalizations of the definitions of singular kernel and Calderón-Zygmund operator given in [49].

Definition 4.4.1. Let $a \in]0, \infty[^l$ and let $c \geq 1$ be the smallest constant for which the c-relaxed triangle inequality

$$|x+y|_{d,a} \leq c(|x|_{d,a} + |y|_{d,a}), \qquad x, y \in \mathbb{R}^d, \tag{4.44}$$

is fulfilled. Then a Lebesgue strongly measurable function $K : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) \mid x \in \mathbb{R}^d\} \longrightarrow \mathcal{B}(X)$ is called a (d, a)-*anisotropic singular kernel* if

(i) K satisfies the decay estimate

$$\|K(x,y)\| \lesssim \frac{1}{|x-y|_{d,a}^{a\cdot_d \mathbf{1}}} \quad \text{for} \quad x \neq y.$$

(ii) K satisfies the Hölder-type estimates

$$\|K(x,y)-K(x',y)\| \lesssim \left(\frac{|x-x'|_{d,a}}{|x-y|_{d,a}}\right)^{\epsilon} \frac{1}{|x-y|_{d,a}^{a\cdot_d \mathbf{1}}}, \qquad 0 < |x-x'|_{d,a} < \frac{1}{2c}|y-y'|_{d,a}$$

and

$$\|K(x,y)-K(x,y')\| \lesssim \left(\frac{|y-y'|_{d,a}}{|x-y|_{d,a}}\right)^{\epsilon} \frac{1}{|x-y|_{d,a}^{a\cdot_d \mathbf{1}}}, \qquad 0 < |x-x'|_{d,a} < \frac{1}{2c}|y-y'|_{d,a}$$

for some Hölder exponent $\epsilon \in]0,1]$.

Definition 4.4.2. Let $p \in]1,\infty[^l$. A linear operator $T : L^{p,d}(\mathbb{R}^d;X) \longrightarrow L^{p,d}(\mathbb{R}^d;X)$ is called a (d,a)-*anisotropic Calderón-Zygmund operator* with Hölder exponent $\epsilon \in]0,1]$ if

(i) $T \in \mathcal{B}(L^{p,d}(\mathbb{R}^d;X))$.

(ii) There exists a (d,a)-anisotropic singular kernel $K : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) \mid x \in \mathbb{R}^d\} \longrightarrow \mathcal{B}(X)$ with Hölder exponent $\epsilon \in]0,1]$ such that

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy$$

for every $f \in L^\infty_c(\mathbb{R}^d;X)$ and $x \notin \operatorname{supp}(f)$.

In case $l=1$ and $a=1$ we just speak of *singular kernel* and *Calderón-Zygmund operator*. Using this terminology, we now state the vector-valued A_p theorem.

Theorem 4.4.3 (A_p theorem for vector-valued Calderón-Zygmund operators)**.** *Let X be a Banach space. Suppose that T is a Calderón-Zygmund operator on $L^p(\mathbb{R}^d;X)$. Then we have*

$$\|Tf\|_{L^p(\mathbb{R}^d,w;X)} \lesssim [w]_{A_p}^{\max\{1,1/(p-1)\}} \|f\|_{L^p(\mathbb{R}^d,w;X)}$$

for all $f \in L^\infty_c(\mathbb{R}^d;X)$ and $w \in A_p(\mathbb{R}^d)$.

Proof. This result is [49, Corollary 2.10]; also see [53]. □

Example 4.4.4. Let X be a UMD space, $l=1$, $d=1$, $a=1$, and $p \in]1,\infty[$. Then the Hilbert transform $H = H_X$ is a Calderón-Zygmund operator with scalar-valued kernel $K(x,y) = \frac{1}{\pi}\frac{1}{x-y}$. This operator coincides with the Fourier multiplier operator with symbol $m(\xi) = -\imath\operatorname{sign}(\xi)$, where sign is the signum function.

Due to restrictions of time and size of this thesis we have decided not to investigate a (d,a)-anisotropic version of the theorem.

Recall that it is our aim of applying this extrapolation result in the context of Fourier multipliers. So let us take a look at kernels of convolution type, i.e., kernels $\bar{K}$ of the form $\bar{K}(x,y) := K(x-y)$ form some $K : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathcal{B}(X)$.

Example 4.4.5. Let $K : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathcal{B}(X)$ be a strongly measurable function and let $c \geq 1$ be as in Definition 4.4.1. Then we have that $\bar{K}(x,y) := K(x-y)$ defines a $(\mathpzc{d}, a)$-anisotropic singular kernel with Hölder exponent $\epsilon \in]0,1]$ if and only if

(i') K satisfies the decay estimate

$$\|K(x)\| \lesssim \frac{1}{|x|_{\mathpzc{d},a}^{a\cdot_{\mathpzc{d}}\mathbf{1}}} \quad \text{for} \quad x \neq 0.$$

(ii') K satisfies the Hölder-type estimate

$$\|K(x-y) - K(x)\| \lesssim \left(\frac{|y|_{\mathpzc{d},a}}{|x|_{\mathpzc{d},a}}\right)^{\epsilon} \frac{1}{|x|_{\mathpzc{d},a}^{a\cdot_{\mathpzc{d}}\mathbf{1}}}, \qquad 0 < |y|_{\mathpzc{d},a} < \frac{1}{2c}|x|_{\mathpzc{d},a}$$

for the Hölder exponent $\epsilon \in]0,1]$.

Moreover, if (i') and (ii') hold we can take the for implicit constants in (i) and (ii) the implicit constants from (i') and (ii'), respectively. In this situation we call K a $(\mathpzc{d}, a)$-anisotropic singular kernel of convolution type with Hölder exponent $\epsilon \in]0,1]$.

Lemma 4.4.6. *Let $K \in C^1(\mathbb{R}^d \setminus \{0\}; \mathcal{B}(X))$ be a function satisfying the estimates*

$$\|\partial^{\alpha} K(x)\| \leq C|x|_{\mathpzc{d},a}^{-a\cdot_{\mathpzc{d}}[\mathbf{1}+\alpha]}, \qquad |\alpha| \leq 1, x \in \mathbb{R}^d \setminus \{0\}$$

for some constant $C > 0$. Then K is a $(\mathpzc{d}, a)$-anisotropic singular kernel of convolution type with Hölder exponent $\epsilon = a_{min} \in]0,1]$. Moreover, the implicit constant in the estimate (ii') of Example 4.4.5 can be chosen $\lesssim_{\mathpzc{d},a} C$.

Proof. Let $x, y \in \mathbb{R}^d$ satisfy $0 < |y|_{\mathpzc{d},a} < \frac{1}{2c}|x|_{\mathpzc{d},a}$. Then we have

$$\begin{aligned} \|K(x-y) - K(x)\| &= \left\| \int_0^1 \frac{\partial}{\partial t} K(x-ty) dt \right\| \\ &\leq \sum_{j=1}^{d} \sum_{i=1}^{d_j} \int_0^1 \left\| \partial_{x_{j,i}} K(x-ty) y_j \right\| dt \\ &\leq C \sum_{j=1}^{d} \sum_{i=1}^{d_j} \int_0^1 \frac{1}{|x-ty|_{\mathpzc{d},a}^{a\cdot_{\mathpzc{d}}\mathbf{1}+a_j}} |y_{j,i}| dt. \end{aligned}$$

For every $t \in [0,1]$ it holds that

$$|x-ty|_{\mathpzc{d},a} \geq \frac{1}{c}|x|_{\mathpzc{d},a} - |ty|_{\mathpzc{d},a} \geq \frac{1}{c}|x|_{\mathpzc{d},a} - |y|_{\mathpzc{d},a} > \frac{1}{2c}|x|_{\mathpzc{d},a}.$$

Therefore,

$$
\begin{aligned}
\|K(x-y)-K(x)\| &\lesssim_{\mathpzc{d},a} C\sum_{j=1}^{d}\sum_{i=1}^{d_j}\int_0^1 \frac{1}{|x|_{\mathpzc{d},a}^{a\cdot_{\mathpzc{d}}\mathbf{1}+a_j}}|y_{j,i}|dt \\
&\leq C\sum_{j=1}^{d}\sum_{i=1}^{d_j}\frac{1}{|x|_{\mathpzc{d},a}^{a\cdot_{\mathpzc{d}}\mathbf{1}}}\cdot\frac{|y|_{\mathpzc{d},a}^{a_j}}{|x|_{\mathpzc{d},a}^{a_j}} \\
&\lesssim_{\mathpzc{d}} C\sum_{j=1}^{d}\frac{1}{|x|_{\mathpzc{d},a}^{a\cdot_{\mathpzc{d}}\mathbf{1}}}\left(\frac{|y|_{\mathpzc{d},a}}{|x|_{\mathpzc{d},a}}\right)^{a_j} \\
&\overset{\frac{|y|_{\mathpzc{d},a}}{|x|_{\mathpzc{d},a}}<\frac{1}{2c}<1}{\lesssim_{\mathpzc{d}}} C\frac{1}{|x|_{\mathpzc{d},a}^{a\cdot_{\mathpzc{d}}\mathbf{1}}}\left(\frac{|y|_{\mathpzc{d},a}}{|x|_{\mathpzc{d},a}}\right)^{a_{min}}.
\end{aligned}
$$

□

4.4.3 From Multiplier Symbol to Singular Kernel

Lemma 4.4.7. *Let Y be a Banach space and suppose that $m \in L^\infty(\mathbb{R}^d;Y)$ is such that, for all $\theta \in \mathbb{N}^d$ with $a\cdot_{\mathpzc{d}}\theta \leq a\cdot_{\mathpzc{d}}\mathbf{1}+a_{max}+1$, $D^\theta m \in \mathcal{D}'(\mathbb{R}^d;Y)$ coincides on $\mathbb{R}^d\setminus\{0\}$ with a function $m_\theta \in C(\mathbb{R}^d\setminus\{0\};Y)$ satisfying $\|m_\theta(\xi)\|_Y \leq C_m|\xi|_{\mathpzc{d},a}^{-a\cdot_{\mathpzc{d}}\theta}$ for some constant $C_m>0$. Then $\mathscr{F}^{-1}m|_{\mathbb{R}^d\setminus\{0\}} \in \mathcal{D}'(\mathbb{R}^d\setminus\{0\};Y)$ can be represented by a function $K \in C^1(\mathbb{R}^d\setminus\{0\};Y)$ satisfying the estimates*

$$\|\partial^\alpha K(x)\| \lesssim_{\mathpzc{d},a} C_m|x|_{\mathpzc{d},a}^{-a\cdot_{\mathpzc{d}}[\mathbf{1}+\alpha]}, \qquad |\alpha|\leq 1, x\in\mathbb{R}^d\setminus\{0\}. \tag{4.45}$$

Proof. We pick a $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that

$$0\leq\hat{\psi}\leq 1, \quad \hat{\psi}(\xi)=1 \text{ if } |\xi|_{\mathpzc{d},a}\leq 1, \quad \hat{\psi}(\xi)=0 \text{ if } |\xi|_{\mathpzc{d},a}\geq\frac{3}{2},$$

and we define $(\psi_k)_{k\in\mathbb{Z}} \subset \mathcal{S}(\mathbb{R}^d)$ via the relation

$$\hat{\psi}_k(\xi)=\hat{\psi}(\delta_{2^{-k}}^{[\mathpzc{d},a]}\xi)-\hat{\psi}(\delta_{2^{-k+1}}^{[\mathpzc{d},a]}\xi), \qquad \xi\in\mathbb{R}^d.$$

Then it holds that

$$\sum_{k\in\mathbb{Z}}\hat{\psi}_k(\xi)=1 \text{ with } \left|\sum_{k\in I}\hat{\psi}_k(\xi)\right|\leq 1, \qquad \xi\in\mathbb{R}^d, I\subset\mathbb{Z}, \tag{4.46}$$

and

$$\operatorname{supp}\hat{\psi}_k \subset \left\{\xi\in\mathbb{R}^d \mid 2^{k-1}\leq|\xi|_{\mathpzc{d},a}\leq\frac{3}{2}2^k\right\}, \qquad k\in\mathbb{Z}. \tag{4.47}$$

For each $k\in\mathbb{Z}$ we define

$$K_k := \psi_k * \mathscr{F}^{-1}m = \mathscr{F}[\hat{\psi}_k m] \in \mathcal{S}'(\mathbb{R}^d)\cap C^\infty(\mathbb{R}^d;Y).$$

Since $\sum_{k\in\mathbb{Z}}\hat{\psi}_k m = m$ in $\mathcal{S}'(\mathbb{R}^d;Y)$ as a consequence of (4.46), $m\in L^\infty(\mathbb{R}^d;Y)$ and the Lebesgue dominated convergence theorem, it follows that

$$\mathscr{F}^{-1}m=\sum_{k\in\mathbb{Z}}K_k \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d;Y).$$

Hence,

$$K := \mathscr{F}^{-1}m|_{\mathbb{R}^d\setminus\{0\}} = \sum_{k\in\mathbb{Z}} K_k|_{\mathbb{R}^d\setminus\{0\}} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d \setminus \{0\}; Y). \tag{4.48}$$

In order to show that K belongs to $C^1(\mathbb{R}^d \setminus \{0\}; Y)$ and satisfies the estimates (4.45), we claim that each $K_k \in C^\infty(\mathbb{R}^d; Y)$ satisfies, for every $M \in \mathbb{N}$ with $M \leq a \cdot_d \mathbf{1} + a_{max} + 1$, the estimates

$$|x|_{d,a}^M \|\partial^\alpha K_k(x)\| \lesssim_{d,a} C 2^{k(a\cdot_d[\mathbf{1}+\alpha]-M)}, \qquad |\alpha| \leq 1, x \neq 0. \tag{4.49}$$

Then (4.45) can be derived as follows: Fix an arbitrary $|\alpha| \leq 1$. Let $M \in \mathbb{N}$ be the smallest natural number such that $M > a \cdot_d [\mathbf{1} + \alpha]$; so $M \leq a \cdot_d [\mathbf{1} + \alpha] + 1 \leq a \cdot_d [\mathbf{1}] + a_{\max} + 1$. Then we have, for $x \neq 0$,

$$\begin{aligned}
|x|_{d,a}^M \sum_{k:\, 2^k > |x|_{d,a}^{-1}} |\partial^\alpha K_k(x)| &\overset{(4.49)}{\lesssim}_{d,a} C \sum_{k:\, 2^k > |x|_{d,a}^{-1}} 2^{k(a\cdot_d[\mathbf{1}+\alpha]-M)} \\
&= C|x|_{d,a}^{-a\cdot_d[\mathbf{1}+\alpha]+M} \sum_{k > \log_2\left(|x|_{d,a}^{-1}\right)} 2^{\left(k-\log_2\left(|x|_{d,a}^{-1}\right)\right)(a\cdot_d[\mathbf{1}+\alpha]-M)} \\
&\leq C|x|_{d,a}^{-a\cdot_d[\mathbf{1}+\alpha]+M} \sum_{n\geq 0} 2^{n(a\cdot_d[\mathbf{1}+\alpha]-M)} \\
&\lesssim_{d,a} C|x|_{d,a}^{-a\cdot_d[\mathbf{1}+\alpha]+M}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k:\, 2^k \leq |x|_{d,a}^{-1}} |\partial^\alpha K_k(x)| &\overset{(4.49)}{\lesssim}_{d,a} C \sum_{k:\, 2^k \leq |x|_{d,a}^{-1}} 2^{k(a\cdot_d[\mathbf{1}+\alpha])} \\
&= C|x|_{d,a}^{-a\cdot_d[\mathbf{1}+\alpha]} \sum_{k \leq \log_2\left(|x|_{d,a}^{-1}\right)} 2^{\left(k-\log_2\left(|x|_{d,a}^{-1}\right)\right)(a\cdot_d[\mathbf{1}+\alpha])} \\
&\leq C|x|_{d,a}^{-a\cdot_d[\mathbf{1}+\alpha]} \sum_{n\leq 0} 2^{n(a\cdot_d[\mathbf{1}+\alpha])} \\
&\lesssim_{d,a} C|x|_{d,a}^{-a\cdot_d[\mathbf{1}+\alpha]},
\end{aligned}$$

implying that

$$\sum_{k\in\mathbb{Z}} |\partial^\alpha K_k(x)| \lesssim_{d,a} C|x|_{d,a}^{-a\cdot_d[\mathbf{1}+\alpha]}.$$

This implies that $\sum_{k\in\mathbb{Z}} \partial^\alpha K_k|_{\mathbb{R}^d\setminus\{0\}}$ converges in $C(\mathbb{R}^d\setminus\{0\}; Y) \hookrightarrow \mathcal{D}'(\mathbb{R}^d\setminus\{0\}; Y)$ to some function $\tilde{K}_\alpha$ satisfying

$$|\tilde{K}_\alpha(x)| \lesssim_{d,a} C|x|_{d,a}^{-a\cdot_d[\mathbf{1}+\alpha]}, \qquad x \in \mathbb{R}^d \setminus \{0\}.$$

In view of (4.48) $\partial^\alpha K = \tilde{K}_\alpha$ we must then have $\partial^\alpha K = \tilde{K}_\alpha$. This proves the claim.

To finish the proof it remains to establish the claim, that is, given $M \in \mathbb{N}$ with $M \leq a \cdot_d \mathbf{1} + a_{max} + 1$, it remains to establish the estimates (4.49). To this end, fix $|\alpha| \leq 1$ and $x \neq 0$.

We first estimate

$$
\begin{aligned}
|x|_{d,a}^{M}\,\|\partial^{\alpha}K_k(x)\| &\lesssim_{d,a} \sum_{\beta\in\mathbb{N}^d:\, a\cdot_d\beta=M}\left\|x^{\beta}\partial^{\alpha}K_k(x)\right\| \\
&= \sum_{\beta\in\mathbb{N}^d:\, a\cdot_d\beta=M}\left\|[\mathscr{F}^{-1}\partial_\xi^{\beta}\xi^{\alpha}(\hat{\psi}_k m)](x)\right\| \\
&= (2\pi)^{-d}\sum_{\beta\in\mathbb{N}^d:\, a\cdot_d\beta=M}\left\|\int_{\mathbb{R}^d} e^{\imath x\cdot\xi}\left[\partial_\xi^{\beta}M_{\xi^\alpha}\hat{\psi}_k m\right](\xi)d\xi\right\| \\
&\lesssim_d \sum_{\beta\in\mathbb{N}^d:\, a\cdot_d\beta=M}\int_{\mathbb{R}^d}\left\|\partial_\xi^{\beta}M_{\xi^\alpha}\hat{\psi}_k m(\xi)\right\|d\xi \\
&\overset{(4.47)}{=} \sum_{\beta\in\mathbb{N}^d:\, a\cdot_d\beta=M}\int_{\{\xi\in\mathbb{R}^d:2^{k-1}\le|\xi|_{d,a}\le\frac{3}{2}2^k\}}\left\|\partial_\xi^{\beta}M_{\xi^\alpha}\hat{\psi}_k m(\xi)\right\|d\xi \\
&\lesssim_{d,a} \mathrm{Vol}\left(\prod_{j=1}^{l}[-2^{(k+1)a_j},2^{(k+1)a_j}]^{d_j}\right)\sup_{\beta\in\mathbb{N}^d:\, a\cdot_d\beta=M}\ \sup_{2^{k-1}\le|\xi|_{d,a}\le 2^{k+1}}\left\|\partial_\xi^{\beta}M_{\xi^\alpha}\hat{\psi}_k m(\xi)\right\| \\
&\le 2^{(k+1)a\cdot_d\mathbf{1}}\sup_{\beta\in\mathbb{N}^d:\, a\cdot_d\beta=M}\ \sup_{2^{k-1}\le|\xi|_{d,a}\le 2^{k+1}}\left\|\partial_\xi^{\beta}M_{\xi^\alpha}\hat{\psi}_k m(\xi)\right\|. \qquad (4.50)
\end{aligned}
$$

Since

$$
\partial_\xi^{\beta}M_{\xi^\alpha}=\sum_{\gamma\le\beta}\binom{\beta}{\gamma}M_{\partial_\xi^{\gamma}\xi^\alpha}\partial_\xi^{\beta-\gamma}=\sum_{\gamma\le\alpha,\beta}c_{\gamma,\beta}M_{\xi^{\alpha-\gamma}}\partial_\xi^{\beta-\gamma} \qquad (4.51)
$$

by the Leibiz' rule, the estimate (4.50) motivates to estimate, for $\tilde{\alpha},\tilde{\beta}\in\mathbb{N}^d$ with $|\tilde{\alpha}|\le 1$ and $\tilde{\beta}\le a\cdot_d\mathbf{1}+a_{max}+1$ and $\xi\in\mathbb{R}^d$ with $2^{k-1}\le|\xi|_{d,a}\le 2^{k+1}$,

$$
\begin{aligned}
\left\|\xi^{\tilde{\alpha}}\partial_\xi^{\tilde{\beta}}\hat{\psi}_k m(\xi)\right\| &= \left\|\xi^{\alpha}\sum_{\tilde{\gamma}\le\tilde{\beta}}\binom{\tilde{\beta}}{\tilde{\gamma}}2^{-k(a\cdot_d\tilde{\gamma})}(\partial_\xi^{\tilde{\gamma}}\hat{\psi})(\delta_{2^{-k}}^{[d,a]}\xi)\partial_\xi^{\tilde{\beta}-\tilde{\gamma}}m(\xi)\right\| \\
&\lesssim_{d,a} \sum_{\tilde{\gamma}\le\tilde{\beta}}2^{-k(a\cdot_d\tilde{\gamma})}|\xi|_{d,a}^{a\cdot_d\tilde{\alpha}-a\cdot_d[\tilde{\beta}-\tilde{\gamma}]}|\xi|_{d,a}^{a\cdot_d[\tilde{\beta}-\tilde{\gamma}]}\left\|m_{\tilde{\beta}-\tilde{\gamma}}(\xi)\right\| \\
&\overset{\tilde{\beta}-\tilde{\gamma}\le a\cdot_d\mathbf{1}+a_{max}+1}{\le} C\sum_{\tilde{\gamma}\le\tilde{\beta}}2^{-k(a\cdot_d\tilde{\gamma})}|\xi|_{d,a}^{a\cdot_d\tilde{\alpha}-a\cdot_d[\tilde{\beta}-\tilde{\gamma}]} \\
&\approx C\sum_{\tilde{\gamma}\le\tilde{\beta}}2^{-k(a\cdot_d\tilde{\gamma})}2^{k(a\cdot_d\tilde{\alpha}-a\cdot_d[\tilde{\beta}-\tilde{\gamma}])} \\
&= C\sum_{\tilde{\gamma}\le\tilde{\beta}}2^{k(a\cdot_d\tilde{\alpha}-a\cdot_d\tilde{\beta})} \\
&\lesssim_{d,a} C2^{k(a\cdot_d\tilde{\alpha}-a\cdot_d\tilde{\beta})}. \qquad (4.52)
\end{aligned}
$$

Combining (4.50), (4.51), and (4.52), we arrive at (4.49), as desired. □

4.5 Fourier Multiplier Theorems

4.5.1 A Generic Fourier Multiplier Theorem

Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^{l} A_{p_j}^{rec}(\mathbb{R}^{d_j})$, such that X is of class $\mathcal{RP}_{p,d,w}$. Suppose that we have a decomposition of $\mathbb{R}^d$, possibly up to a set of measure zero, into rectangles $(E_n)_{n\in\mathbb{N}} \subset \mathscr{J}^d$, in such a way that $\Delta = (\Delta(E_n))_{n\in\mathbb{N}}$ forms an unconditional Schauder decomposition of $L^{p,d}(\mathbb{R}^d, w; X)$; see Lemma 4.5.4. for the dyadic Schauder decomposition that we will be using in practice. Consider a symbol $m \in L^\infty(\mathbb{R}^d; \mathcal{B}(X))$ for which we have, for each $n \in \mathbb{N}$, a Borel measure μ_n on $\mathbb{R}^d$ and a bounded WOT-measurable function $\tau_n : \mathbb{R}^d \longrightarrow \mathcal{B}(X)$ such that $m_n := 1_{E_n} m \in L^\infty(\mathbb{R}^d; \mathcal{B}(X))$ has the representation (4.20). Then we have (4.24) by Lemma 4.2.9. Therefore, in order that $m \in \mathcal{M}_{p,d,w}(X)$, it is necessary and sufficient that the linear operator $T : E_0 \longrightarrow E_0$ given in (4.25) is bounded with respect to the $L^{p,d}(\mathbb{R}^d, w; X)$-norm; see the discussion from the beginning of Section 4.3.1. In view of Theorem 4.3.14, for this it suffices that $(T_n)_{n\in\mathbb{N}} = (T_{m_n})_{n\in\mathbb{N}}$ is an $\mathcal{R}$-bounded collection in $\mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X))$. We will see that the notion of uniformly $\mathcal{R}$-bounded variation, to be introduced in the following definition, provides a sufficient condition (on the μ_n and τ_n) for this $\mathcal{R}$-boundedness.

Definition 4.5.1. Let X be a Banach space. We say that a set of functions $\mathscr{M} \subset L^\infty(\mathbb{R}^d; \mathcal{B}(X))$ is of *uniformly $\mathcal{R}$-bounded variation* if there exist a constant $C > 0$, an $\mathcal{R}$-bounded set $\mathscr{T}$, and for each $m \in \mathscr{M}$ a Borel measure μ_m on $\mathbb{R}^d$ and a bounded WOT-measurable function $\tau_m : \mathbb{R}^d \longrightarrow \mathcal{B}(X)$, with $\|\mu_m\| \leq C$ and $\tau_m(\mathbb{R}^d) \subset \mathscr{T}$, such that, for all $\xi \in \mathbb{R}^d$, $x \in X$, and $x \in X^*$,

$$\langle m(\xi)x, x^* \rangle = \int_{]-\overline{\infty},\xi]} \langle \tau(y)x, x^* \rangle d\mu(y).$$

For each $q \in [1,\infty[$ we define

$$\mathrm{var}_{\mathcal{R}_q}(\mathscr{M}) := \inf\{ C\mathcal{R}_q(\mathscr{T}) \mid C > 0, \mathscr{T} \subset \mathcal{B}(X) \text{ as above}\}$$

Proposition 4.5.2. *Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^{l} A_{p_j}^{rec}(\mathbb{R}^{d_j})$, such that X has property $R_{p,d,w}$. Suppose that $\mathscr{M} \subset L^\infty(\mathbb{R}^d; \mathcal{B}(X))$ is of uniformly $\mathcal{R}$-bounded variation. Then we have $m \in \mathcal{M}_{p,d,w}(X)$ for all $m \in \mathscr{M}$, and moreover,*

$$\mathcal{R}_q\{T_m \mid m \in \mathscr{M}\} \leq \alpha_{p,d,w,X}\, \mathrm{var}_{\mathcal{R}_q}(\mathscr{M}) < \infty \quad \text{in} \quad \mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X)),$$

for every $q \in [1,\infty[$.

Proof. Let $C > 0$ and $\mathscr{T}$ as in the definition of uniformly $\mathcal{R}$-bounded variation for $\mathscr{M}$, and define $\mathscr{S} := \{\Delta([y,\overline{\infty}[) \mid y \in \mathbb{R}^d\} \subset \mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X))$. Each $m \in \mathscr{M}$ in particular satisfies the hypotheses of Lemma 4.2.9. Hence, for each $m \in \mathscr{M}$ we have $m \in \mathcal{M}_{p,d,w}(X)$, with the associated Fourier multiplier operator $T_m \in \mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X))$ having the representation (4.21). This representation yields that

$$T_m \in C\,\overline{\mathrm{abs\,conv}}(\mathscr{T}\mathscr{S});$$

this is similar to Proposition A.1.3. From the basic properties of $\mathcal{R}_q$-bounds (see Proposition E.3.5) it thus follows that

$$\mathcal{R}_q\{T_m \mid m \in \mathscr{M}\} \leq C\mathcal{R}_q(\mathscr{T})\mathcal{R}_q(\mathscr{S}) \overset{(4.19)}{\leq} \alpha_{p,d,w,X} C\mathcal{R}_q(\mathscr{T}) < \infty.$$

Taking the infimum over all admissible $C > 0$ and $\mathscr{T}$, we obtain the desired result. □

Combining the discussion from the beginning of this subsection with the above proposition, we arrive at the following generic Fourier multiplier theorem:

Theorem 4.5.3. *Let X be a Banach space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^{l} A^{rec}_{p_j}(\mathbb{R}^{d_j})$, such that X is of class $\mathcal{RP}_{p,d,w}$. Suppose that we have a decomposition of $\mathbb{R}^d$, possibly up to a set of measure zero, into rectangles $(E_n)_{n\in\mathbb{N}} \subset \mathscr{J}^d$, in such a way that $\Delta = (\Delta(E_n))_{n\in\mathbb{N}}$ forms an unconditional Schauder decomposition of $L^{p,d}(\mathbb{R}^d, w; X)$. Let $m \in L^\infty(\mathbb{R}^d; \mathcal{B}(X))$ be a symbol with the property that $(m1_{E_n})_{n\in\mathbb{N}}$ is of uniformly $\mathcal{R}$-bounded variation. Then we have $m \in \mathcal{M}_{p,d,w}(X)$ with $\|m\|_{\mathcal{M}_{p,d,w}(X)} \leq \alpha_{p,d,w,X} C_\Delta^2 \mathrm{var}_{\mathcal{R}_q}(\mathscr{M})$, where C_Δ is the unconditional constant of Δ.*

4.5.2 Fourier Multipliers in One Variable

In this section we consider the case $d = 1$. We will obtain operator-valued Mikhlin Fourier multiplier theorems for the weighted spaces $L^p(\mathbb{R}^d, w; X)$, $p \in]1,\infty[$, $w \in A_p(\mathbb{R})$, as a consequence of the generic Fourier multiplier theorem of Theorem 4.5.3. We will start with the unweighted case, which we will subsequently use to treat the weighted case by using the theory from Section 4.4.

Let us consider the dyadic partition $(I_{(k,\varepsilon)})_{(k,\varepsilon)\in\mathbb{Z}\times\{-1,1\}}$ of $\mathbb{R} \setminus \{0\}$, which is defined by

$$I_{(k,\epsilon)} := \epsilon[2^k, 2^{k+1}[, \qquad k \in \mathbb{Z}, \varepsilon \in \{-1, 1\}. \tag{4.53}$$

Since $\frac{2^{k+1}}{2^k} = 2$ and $\frac{-2^k}{-2^{k+1}} = \frac{1}{2}$ for all $k \in \mathbb{Z}$, we have control of $\left\|\mu_{k,\epsilon}\right\|$ for the measures $\mu_{k,\varepsilon}$ from Example 4.2.10/Remark 4.2.11 corresponding to the compact intervals $\overline{I_{(k,\varepsilon)}}$. Furthermore, in case X is a UMD space, this partition gives rise to an unconditional Schauder decomposition of the unweighted Lebesgue-Bochner space $L^p(R; X)$:

Lemma 4.5.4. *Let X be a UMD Banach space and $p \in]1,\infty[$. Then we have that $\Delta = (\Delta(I_{(k,\epsilon)}))_{(k,\varepsilon)\in\mathbb{Z}\times\{-1,1\}}$ defines an unconditional Schauder decomposition of $L^p(\mathbb{R}; X)$, with unconditional constant C_Δ depending only on p and X.*

Comments on the proof: This result is due to Bourgain [12]; also see [57]. The proof is martingale theoretic and exploits the direct definition of UMD (Definition E.5.3). For a similar result in case of the one-dimensional torus $\mathbb{T}$ we refer to [103, Theorem 3.3.3]. We furthermore refer to [64, 3.14] for a proof for the special case of X being a closed subspace of an $L^q(S; \mathbb{C})$-space, with $(S, \mathcal{B}, \nu)$ a σ-finite measure space and $q \in]1,\infty[$. These spaces are easier to treat, because here it is possible to use many tools from classical harmonic analysis instead of martingale transforms.

Having available this unconditional Schauder decomposition, the following operator-valued Mikhlin theorem can now be obtained as a consequence of the generic Fourier multiplier theorem of Theorem 4.5.3:

Theorem 4.5.5. *Let X be a UMD Banach space and $p \in]1,\infty[$. Suppose that the symbol $m \in L^\infty(\mathbb{R}; \mathcal{B}(X)) \cap C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(X))$ satisfies the following $\mathcal{R}$-boundedness version of the classical Mikhlin condition:*

$$\kappa_m := \mathcal{R}\{\, \xi^j m^{[j]}(\xi) \ : \ j \in \{0, 1\}, \xi \in \mathbb{R} \setminus \{0\} \,\} = \mathcal{R}\{\, m(\xi), \xi m'(\xi) \ : \ \xi \in \mathbb{R} \setminus \{0\} \,\} < \infty. \tag{4.54}$$

Then we have $m \in \mathcal{M}_{p,1}(X)$ with $\|m\|_{\mathcal{M}_{p,1}(X)} \lesssim_{p,X} \kappa_m$.

The $\mathcal{R}$-boundedness of the range of the multiplier symbol is in fact a necessary condition:

Remark 4.5.6. Let $m \in \mathcal{M}_{p,d,w}(X)$ with $p \in]1,\infty[^l$ and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$. Pick a representative m of $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X)) \subset L^1_{loc}(\mathbb{R}^d;\mathcal{B}(X))$. Then, similarly to Lemma 4.2.1, it can be shown that

$$\mathcal{R}\{m(\xi) \,:\, \xi \text{ is a Lebesgue point of } m\} \lesssim_{p,w} \|m\|_{\mathcal{M}_{p,d,w}(X)}\,;$$

see [57].

Proof. By Theorem 4.5.3 and Lemma 4.5.4, it suffices to show that

$$\mathscr{M} := \{\, m1_{I_{k,\varepsilon}} \,:\, (k,\varepsilon) \in \mathbb{Z} \times \{-1,1\}\,\} \subset L^\infty(\mathbb{R}^d;\mathcal{B}(X))$$

is of uniformly $\mathcal{R}$-bounded variation with $\operatorname{var}_{\mathcal{R}_2}(\mathscr{M}) \lesssim \kappa_m$. Using Example 4.2.10 we find, for each $k \in \mathbb{Z}$ and $\varepsilon \in \{-1,1\}$, a Borel measures $\mu_{k,\varepsilon}$ on $\mathbb{R}$ and a bounded measurable function $\tau_{k,\varepsilon} : \mathbb{R} \longrightarrow \mathcal{B}(X)$ such that $m1_{I_{k,\varepsilon}}$ has the representation (4.20). Moreover, we have $\sup_{k,\varepsilon} \left\|\mu_{k,\varepsilon}\right\| \leq 2 + \log(2) \leq 3$ and

$$\mathscr{T} := \{\, \tau_{k,\varepsilon} \,:\, (k,\varepsilon) \in \mathbb{Z} \times \{-1,1\}\,\} \subset \operatorname{abs\,conv}\{m(\xi), \xi m'(\xi) \,:\, \xi \in \mathbb{R} \setminus \{0\}\}.$$

Therefore, $\mathscr{M}$ is indeed of $\mathcal{R}$-bounded variation with

$$\operatorname{var}_{\mathcal{R}_2}(\mathscr{M}) \leq \sup_{k,\varepsilon} \left\|\mu_{k,\varepsilon}\right\| \mathcal{R}(\mathscr{T}) \leq 3 \cdot 2\kappa_m;$$

see Proposition E.3.5. □

By Proposition E.3.4, for scalar valued symbols m the $\mathcal{R}$-boundedness of the set in (4.62) is equivalent to the uniform boundedness of this set. As a consequence, we obtain the following Mikhlin theorem for scalar-valued symbols:

Corollary 4.5.7. *Let X be a UMD Banach space and $p \in]1,\infty[$. Suppose that the symbol $m \in L^\infty(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ satisfies the Mikhlin condition:*

$$C_m := \sup\{\, |m(\xi)|, |\xi m'(\xi)| \,:\, \xi \in \mathbb{R} \setminus \{0\}\,\} < \infty.$$

Then we have $m \in \mathcal{M}_{p,1}(X)$ with $\|m\|_{\mathcal{M}_{p,1}(X)} \lesssim_{p,X} C_m$.

We now turn to the weighted setting. As a first step to get a generalization of the above theorem with A_p-weights, we first prove the following version in which we in addition assume an extra Mikhlin condition on the symbol in order to be able to apply the extrapolation result of Theorem 4.4.3:

Proposition 4.5.8. *Let X be a UMD space, $p \in]1,\infty[$, and $w \in A_p(\mathbb{R})$. Suppose that $m \in L^\infty(\mathbb{R};\mathcal{B}(X)) \cap C^3(\mathbb{R} \setminus \{0\};\mathcal{B}(X))$ satisfies*

$$\kappa_m := \mathcal{R}\{\, m(\xi), \xi m'(\xi) \,:\, \xi \in \mathbb{R} \setminus \{0\}\,\} < \infty.$$

and

$$C_m := \sup\{\, |\xi^k| \left\|m^{(k)}(\xi)\right\| \,:\, k \in \{0,1,2,3\}, \xi \in \mathbb{R} \setminus \{0\}\,\} < \infty. \tag{4.55}$$

Then we have $m \in \mathcal{M}_{p,1}(X)$ with $\|m\|_{\mathcal{M}_{p,1}(X)} \lesssim_{p,X} [w]_{A_p}^{\max\{1,1/(p-1)\}} \max\{\kappa_m, C_m\}$.

Recall that $\mathcal{R}$-bounded sets are also uniformly bounded with uniform bound less than or equal the $\mathcal{R}$-bound; see Remark E.3.2. In particular, we have

$$\max\{\kappa_m, C_m\} = \max\{\kappa_m, C'_m\},$$

where $C'_m := \sup\{\, |\xi^k m^{(k)}(\xi)| \,:\, k \in \{2,3\}, \xi \in \mathbb{R} \setminus \{0\}\,\}$.

Proof. **Step I:** $m \in \mathcal{S}(\mathbb{R}; \mathcal{B}(X))$.

We may without loss of generality assume that $\max\{\kappa_m, C_m\} \leq 1$. Then, in order to prove that $m \in \mathcal{M}_{p,1}(X)$ with $\|m\|_{\mathcal{M}_{p,1}(X)} \lesssim_{p,X} [w]_{A_p}^{\max\{1,1/(p-1)\}}$, we may show that

$$T_m : \mathcal{S}'(\mathbb{R}^d; X) \longrightarrow \mathcal{S}'(\mathbb{R}^d; X),\ f \mapsto \mathscr{F}^{-1}[mf] = \check{m} * f$$

maps the dense space $D := L^\infty_c(\mathbb{R}; X)$ of $L^p(\mathbb{R}, w; X)$ into $L^p(\mathbb{R}, w; X)$ and satisfies a norm estimate

$$\|T_m f\|_{L^p(\mathbb{R},w;X)} \lesssim_{p,X} [w]_{A_p}^{\max\{1,1/(p-1)\}} \|f\|_{L^p(\mathbb{R},w;X)}, \qquad f \in D; \tag{4.56}$$

see Remark 4.2.3.(iii). By the Mikhlin theorem of Theorem 4.5.5 and Remark 4.2.3, T_m restricts to a bounded linear operator on the unweighted space $L^p(\mathbb{R}; X)$ of norm $\lesssim_{p,X} \kappa_m \leq 1$, which is in fact given by convolution with $\check{m} \in \mathcal{S}(\mathbb{R}; \mathcal{B})$ in the classical sense (4.43). Furthermore, by Lemma 4.4.7 and Lemma 4.4.6, $K := \check{m}|_{\mathbb{R}\setminus\{0\}} \in C^\infty(\mathbb{R} \setminus \{0\}; \mathcal{B}(X))$ is a singular kernel of convolution type with implicit constants $\lesssim C_m \leq 1$ and with Hölder exponent $\epsilon = 1$. In particular, T_m is a Calderón-Zygmund operator on $L^p(\mathbb{R}; X)$ of norm $\|T_m\|_{\mathcal{B}(L^p(\mathbb{R};X))} \lesssim_{p,X} 1$, with K as singular kernel of convolution type having implicit constants $\lesssim 1$ and Hölder exponent $\epsilon = 1$. So we may apply the extrapolation result of Theorem 4.4.3 to obtain the desired estimate (4.56).

Step II: $m \in C^\infty(\mathbb{R}; \mathcal{B}(X))$.

Using Proposition 4.2.4.(iv), this can derived from Step I as in Step 2 in the proof of [93, Theorem 4.4].

Step III: m arbitrary as in the assumption.

Using Proposition 4.2.4.(iv), this can derived from Step II as in Step 3 in the proof of [93, Theorem 4.4]. □

Corollary 4.5.9. *Let $p \in]1,\infty[$ and $w \in A_p(\mathbb{R})$. Then every UMD space X is of class $\mathcal{R}_{p,1,w}$ with $\alpha_{p,1,w,X} \lesssim_{p,X} [w]_{A_p}^{\max\{1,1/(p-1)\}}$.*

Proof. The symbol $m := 1_{]0,\infty[}$ certainly satisfies the conditions of the above proposition with $\kappa_m = C_m = 1$, whence $1_{]0,\infty[} \in \mathcal{M}_{p,1,w}(X)$ with $\left\|1_{]0,\infty[}\right\|_{\mathcal{M}_{p,1,w}(X)} \lesssim_{X,p} [w]_{A_p}^{\max\{1,1/(p-1)\}}$.

We can also establish $1_{]0,\infty[} \in \mathcal{M}_{p,1,w}(X)$ in a more direct way from Theorem 4.4.3, as follows: Since $-\imath\mathrm{sign} \in \mathcal{M}_{p,1,w}(X)$ with norm $\left\|-\imath\mathrm{sign}\right\|_{\mathcal{M}_{p,1,w}(X)} \lesssim_{p,X} [w]_{A_p}^{\max\{1,1/(p-1)\}}$ by Example 4.4.4 (which is an application of Theorem 4.4.3), it follows that

$$1_{]0,\infty[} = \frac{1}{2}(\imath \cdot -\imath\mathrm{sign} + 1) \in \mathcal{M}_{p,1,w}(X)$$

with norm $\left\|1_{]0,\infty[}\right\|_{\mathcal{M}_{p,1,w}(X} \lesssim_{p,X} \frac{1}{2}([w]_{A_p}^{\max\{1,1/(p-1)\}} + 1)$. □

Using the Mikhlin theorem from Proposition 4.5.8 (in combination with its corollary), we can now prove a weighted version of Lemma 4.5.4.

Lemma 4.5.10. *Let X be a UMD Banach space, $p \in]1,\infty[$, and $w \in A_p(\mathbb{R})$. Then we have that $\Delta = (\Delta(I_{(k,\epsilon)}))_{(k,\epsilon)\in\mathbb{Z}\times\{-1,1\}}$ defines an unconditional Schauder decomposition of $L^p(\mathbb{R};X)$, with $C^+_{\Delta,p,w,X} \lesssim_{p,X} [w]_{A_p}^{2\max\{1,1/(p-1)\}}$ and $C^-_{\Delta,p,w,X} \leq C^+_{\Delta^*,p',w',X^*} \lesssim_{p,X} [w]_{A_p}^{2\max\{1,1/(p'-1)\}}$.*

We will use the following denseness result in order to prove that $\mathrm{Ran}(\Delta)$ is dense in $L^p(\mathbb{R},w;X)$:

Lemma 4.5.11. $L^p_0(\mathbb{R},w) := \{f \in L^p(\mathbb{R},w) \mid 0 \notin \mathrm{supp}(\hat{f})$ *compact*$\}$ *is dense in* $L^p(\mathbb{R},w)$.

Remark 4.5.12. This lemma is an immediate consequence of special case of Lemma 3.4.3. However, we give a different proof which straightforwardly extends to a proof of the more general denseness result: *Given* $p \in]1,l[$ *and* $w \in \prod_{j=1}^{l} A^{rec}_{p_j}(\mathbb{R}^{d_j})$,

$$\left\{ f \in L^{p,d}(\mathbb{R}^d,w) \;:\; \mathrm{supp}(\hat{f}) \textit{ compact}, \mathrm{supp}(\hat{f}) \cap [\mathbb{R}^d \setminus \{0\}]^d = \emptyset \right\}$$

is dense in $L^{p,d}(\mathbb{R}^d,w)$. This will be needed in Step III of the proof of Lemma 4.5.17.

Proof. In view of denseness of $\mathscr{F}^{-1}C^\infty_c(\mathbb{R})$ in $L^p(\mathbb{R},w)$, it suffices to show that $\mathscr{F}^{-1}C^\infty_c(\mathbb{R})$ is contained in the closure of $L^p_0(\mathbb{R},w)$ in $Lp(\mathbb{R},w)$. So fix an $f \in \mathscr{F}^{-1}C^\infty_c(\mathbb{R})$ and denote by $R \in \mathcal{B}(L^p(\mathbb{R},w))$ the Riesz projection in $L^p(\mathbb{R},w)$. Then we have $f = f^+ + f^-$, where $f^+ := Rf \in L^p(\mathbb{R},w)$ and $f^- := (1-R)f \in L^p(\mathbb{R},w)$ have compact Fourier support contained in $[0,\infty[$ and $]-\infty,0]$, respectively. Then $(f^+_n)_{n\in\mathbb{N}} := (e_{\frac{1}{1+n}}f_+)_{n\in\mathbb{N}}$ and $(f^-_n)_{n\in\mathbb{N}} := (e_{-\frac{1}{1+n}}f_+)_{n\in\mathbb{N}}$ are sequences in $L^p_0(\mathbb{R},w)$ which converge to f^+ and f^- in $L^p(\mathbb{R},w)$ as $n \to \infty$, respectively, from which the desired result follows. □

Proof of Lemma 4.5.10. Let us check the conditions of Corollary 4.3.13. We start with denseness of $\mathrm{Ran}(\Delta)$ in $L^p(\mathbb{R},w;X)$. Since $L^p_0(\mathbb{R},w)\otimes X$ is contained in $\mathrm{Ran}(\Delta)$, and since $L^p(\mathbb{R},w)\otimes X$ is dense in $L^p(\mathbb{R},w;X)$, this follows from Lemma 4.5.11. So it remains to be shown that (4.32) and (4.33) are satisfied with $C \lesssim_{p,X} [w]_{A_p}^{2\max\{1,1/(p-1)\}}$ and $C^* \lesssim_{p,X} [w]_{A_p}^{2\max\{1,1/(p'-1)\}}$. We shall only treat Δ, the collection of dual operators $\Delta^* \subset \mathcal{B}(L^{p'}(\mathbb{R},w';X^*))$ being of a similar form thanks to Proposition 4.2.4.(iii) while $[w]_{A_p} = [w']_{A_{p'}}$; note here that X^* has the RNP because X is reflexive (being a UMD space). To this end, we pick a $\rho \in C^\infty_c(\mathbb{R})$ such that $\rho \equiv 1$ on $I_{0,1} = [1,2[$ and $\mathrm{supp}(\rho) \subset I_{-1,1} \cup I_{0,1} \cup I_{1,1} = [\frac{1}{2},4[$, and subsequently define $(\rho_{(k,\varepsilon)})_{(k,\varepsilon)\in\mathbb{Z}\times\{-1,1\}} \subset C^\infty_c(\mathbb{R})$ by $\rho_{(k,\varepsilon)}(\xi) := \rho(\varepsilon 2^{-k}\xi)$; then note that

$$\rho_{(k,\varepsilon)} \equiv 1 \text{ on } I_{k,\varepsilon}, \qquad \mathrm{supp}(\rho_{(k,\varepsilon)}) \subset I_{(k-1,\varepsilon)} \cup I_{(k,\varepsilon)} \cup I_{(k+1,\varepsilon)}. \tag{4.57}$$

Given $\eta = (\eta_{(k,\varepsilon)})_{(k,\varepsilon)\in\mathbb{Z}\times\{-1,1\}} \in \{-1,1\}^{\mathbb{Z}\times\{-1,1\}}$ and a finite subset $F \subset \mathbb{Z}\times\{-1,1\}$, we define the symbol $m_{\eta,F} \in C^\infty(\mathbb{R})$ by

$$m_{\eta,F} := \sum_{(k,\varepsilon)\in F} \eta_{(k,\varepsilon)}\rho_{(k,\varepsilon)}.$$

Since each $\xi \in \mathbb{R}\setminus\{0\}$ is contained in an open neighborhood on which at most of three of the $\rho_{(k,\varepsilon)}$ do not vanish (see (4.57)), and since the Mikhlin condition is invariant under dilations, it follows from Proposition 4.5.8 that $m_{\eta,F} \in \mathcal{M}_{p,1,w}(X)$ with $\left\| m_{\eta,F} \right\|_{\mathcal{M}_{p,1,w}(X)} \lesssim_{p,X} [w]_{A_p}^{\max\{1,1/(p-1)\}} M$ for some constant $M \in]0,\infty[$ independent of η and F. Therefore, we have $(\rho_{(k,\varepsilon)})_{(k,\varepsilon)\in\mathbb{Z}\times\{-1,1\}} \subset \mathcal{M}_{p,1,w}(X)$ with the property that

$$\left\| \sum_{(k,\varepsilon)\in F} \eta_{(k,\varepsilon)} T_{\rho_{(k,\varepsilon)}} \right\|_{\mathcal{B}(L^p(\mathbb{R},w;X))} \lesssim_{p,X} M[w]_{A_p}^{\max\{1,1/(p-1)\}}$$

holds true for all $\eta = (\eta_{(k,\varepsilon)})_{(k,\varepsilon)\in\mathbb{Z}\times\{-1,1\}} \in \{-1,1\}^{\mathbb{Z}\times\{-1,1\}}$ and all finite subsets $F \subset \mathbb{Z}\times\{-1,1\}$. As a consequence,

$$\left\| \sum_{(k,\varepsilon)\in F} \epsilon_{(k,\varepsilon)} T_{\rho_{(k,\varepsilon)}} f \right\|_{L^2(\Omega;L^p(\mathbb{R},w;X))} \lesssim_{p,X} M[w]_{A_p}^{\max\{1,1/(p-1)\}} \|f\|_{L^p(\mathbb{R},w;X))}$$

holds true for all finite subsets $F \subset \mathbb{Z}\times\{-1,1\}$ and $f \in L^p(\mathbb{R},w;X)$. Since

$$\mathcal{R}\{\Delta_{(k,\varepsilon)} : (k,\varepsilon) \in \mathbb{Z}\times\{-1,1\}\} \leq 2\alpha_{p,1,w,X} \lesssim_{p,X} [w]_{A_p}^{\max\{1,1/(p-1)\}} \quad \text{in } \mathcal{B}(L^p(\mathbb{R},w;X)) \tag{4.58}$$

by Lemma 4.2.8 and Corollary 4.5.9, and since $\Delta_{(k,\varepsilon)} = \Delta_{(k,\varepsilon)} T_{\rho_{(k,\varepsilon)}}$ in view of (4.57), it follows that (4.32) is satisfied with $C \lesssim_{p,X} [w]_{A_p}^{2\max\{1,1/(p-1)\}}$, as desired. □

Having this unconditional Schauder decomposition, the following weighted version of Theorem 4.5.5 now also is a consequence of the generic Fourier multiplier theorem of Theorem 4.5.3.

Theorem 4.5.13. *Let X be a UMD Banach space, $p \in]1,\infty[$, and $w \in A_p(\mathbb{R})$. Suppose that the symbol $m \in L^\infty(\mathbb{R};\mathcal{B}(X)) \cap C^1(\mathbb{R}\setminus\{0\};\mathcal{B}(X))$ satisfies*

$$\kappa_m := \mathcal{R}\{\, m(\xi), \xi m'(\xi) \;:\; \xi \in \mathbb{R}\setminus\{0\} \,\} < \infty. \tag{4.59}$$

Then we have $m \in \mathcal{M}_{p,1}(X)$ with $\|m\|_{\mathcal{M}_{p,1}(X)} \lesssim_{p,X} [w]_{A_p}^{2(1+\max\{1/(p-1),1/(p'-1)\})}\kappa_m$.

Proof. The proof is completely analogous to the proof of Theorem 4.5.5. Of course, now we have to use Lemma 4.5.10 instead of Lemma 4.5.4, and we furthermore have to note that

$$C^+_{\Delta,p,w,X} C^-_{\Delta,p,w,X} \lesssim_{p,X} [w]_{A_p}^{2\max\{1,1/(p-1)\}} [w]_{A_p}^{2\max\{1,1/(p'-1)\}} = [w]_{A_p}^{2(1+\max\{1/(p-1),1/(p'-1)\})}.$$

□

In case our space X possesses property (α), we can bootstrap the $\mathcal{R}$-boundedness to obtain the following result:

Corollary 4.5.14. *Let X be an (α)-UMD Banach space, $p \in]1,\infty[$, and $w \in A_p(\mathbb{R})$. Suppose that $\mathscr{M} \subset L^\infty(\mathbb{R};\mathcal{B}(X)) \cap C^1(\mathbb{R}\setminus\{0\};\mathcal{B}(X))$ is a collection of symbols with the property that*

$$\kappa_{\mathscr{M}} := \mathcal{R}\{\, m(\xi), \xi m'(\xi) \;:\; \xi \in \mathbb{R}\setminus\{0\}, m \in \mathscr{M} \,\} < \infty. \tag{4.60}$$

Then we have $\mathscr{M} \subset \mathcal{M}_{p,1,w}(X)$ with

$$\mathcal{R}\{T_m : m \in \mathscr{M}\} \lesssim_{p,X} [w]_{A_p}^{2(1+\max\{1/(p-1),1/(p'-1)\})}\kappa_{\mathscr{M}} \quad \textit{in } \mathcal{B}(L^p(\mathbb{R},w;X)).$$

Proof. By Theorem 4.5.13 we have $\mathscr{M} \subset \mathcal{M}_{p,1,w}(X)$. Let us accordingly write $\mathcal{T} := \{T_m : m \in \mathscr{M}\} \subset \mathcal{B}(L^p(\mathbb{R},w;X))$. By Lemma E.3.6, in order to show the required $\mathcal{R}$-bound, we must show that

$$\sup_{\tilde{T}\in\tilde{\mathcal{T}}} \|\tilde{T}\|_{\mathcal{B}(\mathrm{Rad}(L^p(\mathbb{R},w;X)))} \lesssim_{p,X} [w]_{A_p}^{2(1+\max\{1/(p-1),1/(p'-1)\})}\kappa_{\mathscr{M}}.$$

This can done as in [64, 5.2] or [57]. The idea is basically that, under the canonical isometric isomorphism $\mathrm{Rad}_p(L^p(\mathbb{R}, w; X)) \simeq L^p(\mathbb{R}, w; \mathrm{Rad}_p(X))$ (E.2), each $\tilde{T} = (T_{m_j})_{n\in\mathbb{N}} \in \tilde{\mathcal{T}}$ corresponds to the Fourier multiplier operator $T_{\mathbf{m}}$ associated with the symbol $\mathbf{m} \in L^\infty(\mathbb{R}; \mathcal{B}(\mathrm{Rad}_p(X))) \cap C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(\mathrm{Rad}_p(X)))$ given by

$$[\mathbf{m}(\xi)]\,((x_n)_{n\in\mathbb{N}}) := (m_n(\xi)x_n)_{n\in\mathbb{N}}, \qquad \xi \in \mathbb{R},$$

for which it can be shown, by checking the conditions of Theorem 4.5.13 with the help of Proposition E.4.4 (which requires property (α)), that

$$\mathbf{m} \in \mathcal{M}_{p,1,w}(\mathrm{Rad}_p(X)) \quad \text{with} \quad \|\mathbf{m}\|_{\mathcal{M}_{p,1,w}(\mathrm{Rad}_p(X))} \lesssim_{p,X} [w]_{A_p}^{2(1+\max\{1/(p-1),1/(p'-1)\})}\kappa_{\mathscr{M}}.$$

□

View $\mathbb{R}^d$ as being $\mathpzc{d}$-decomposed as in Convention 2.2.1. We write

$$\mathbb{R}^d_{*^{\mathpzc{d}}} := [\mathbb{R}^{\mathpzc{d}_1} \setminus \{0\}] \times \ldots \times [\mathbb{R}^{\mathpzc{d}_l} \setminus \{0\}].$$

In case $\mathpzc{d} = \mathbf{1} = (1, \ldots, 1)$, we just write $\mathbb{R}^d_* = \mathbb{R}^d_{*^{\mathpzc{d}}}$. Given a Banach space Y and $n \in \mathbb{N}^l$, we denote by $C^{n,\mathpzc{d}}_*(Y)$ the space of all functions $f : \mathbb{R}^d_{*^{\mathpzc{d}}} \longrightarrow Y$ whose partial derivatives $D^\theta f$ exist and are continuous for all multi-indices $\alpha \in \mathbb{N}^d$ with $|\alpha_j| \leq n_j$, $j = 1, \ldots, l$. In this notation we have:

Corollary 4.5.15. *Let X be an (α)-UMD Banach space, $\mathpzc{d} = \mathbf{1}$, $p \in]1, \infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R})$. Suppose that $\mathscr{M} \subset L^\infty(\mathbb{R}^d; \mathcal{B}(X)) \cap C^{\mathbf{1},\mathbf{1}}_*(\mathcal{B}(X))$ is a collection of symbols with the property that*

$$\kappa_{\mathscr{M}} := \mathcal{R}\{\, \xi^\theta D^\theta m(\xi) \;:\; \xi \in \mathbb{R}^d_*, \theta \leq \mathbf{1}, m \in \mathscr{M} \,\} < \infty. \tag{4.61}$$

Then we have $\mathscr{M} \subset \mathcal{M}_{p,\mathbf{1},w}(X)$ with

$$\mathcal{R}\{T_m : m \in \mathscr{M}\} \lesssim_{p,X} \left(\prod_{j=1}^l [w_j]_{A_{p_j}}^{2(1+\max\{1/(p_j-1),1/(p_j'-1)\})} \right) \kappa_{\mathscr{M}} \quad \text{in } \mathcal{B}(L^{p,\mathbf{1}}(\mathbb{R}^d, w; X)).$$

Proof. This can be shown via an induction on $l \geq 1$, for which we refer to [57]. Let us indicate the main idea for the case $l = 2$: First note that it is enough to consider the case that $\mathscr{M}$ consists of a single symbol m; the desired result can then be derived via a bootstrap argument as in Corollary 4.5.14. So, given a symbol $m \in C^{\mathbf{1},\mathbf{1}}(\mathcal{B}(X))$ satisfying

$$\kappa_m := \mathcal{R}\{\, \xi^\theta D^\theta m(\xi) \;:\; \xi \in \mathbb{R}^2_*, \theta \leq \mathbf{1} \,\} < \infty,$$

we must show that $m \in \mathcal{M}_{p,\mathbf{1},w}(X)$ with $\|m\|_{\mathcal{M}_{p,\mathbf{1},w}(X)} \lesssim_{p,X} \left(\prod_{j=1}^2 [w_j]_{A_{p_j}}^{2(1+\max\{1/(p_j-1),1/(p_j'-1)\})} \right) \kappa_m$. Clearly, the collection of symbols $\mathscr{M} := \{m(\,\cdot\,, \xi_2), \xi_2\partial_2 m(\,\cdot\,, \xi_2) : \xi_2 \in \mathbb{R} \setminus \{0\}\}$ satisfies the conditions of Corollary 4.5.14 with $\kappa_{\mathscr{M}} \leq \kappa_m$, so $\mathscr{M} \subset \mathcal{M}_{p_1,1,w_1}$ with

$$\mathcal{R}\left\{ T_{m(\,\cdot\,,\xi_2)}, \xi_2 T_{\partial_2 m(\,\cdot\,,\xi_2)} \;:\; \xi_2 \in \mathbb{R} \setminus \{0\} \right\} \lesssim_{p,X} [w_j]_{A_{p_j}}^{2(1+\max\{1/(p_j-1),1/(p_j'-1)\})} \kappa_m \quad \text{in } \mathcal{B}(L^{p_1}(\mathbb{R}, w_1; X)).$$

Now the idea is that

$$\mathbb{R} \setminus \{0\} \ni \xi_2 \mapsto T_{m(\,\cdot\,,\xi_2)} \in \mathcal{B}(L^{p_1}(\mathbb{R}, w_1; X))$$

defines a C^1-function with $M'(\xi_2) = T_{\partial_2 m(\,\cdot\,,\xi_2)}$, so that M satisfies the $\mathcal{R}$-boundedness Mikhlin condition from Theorem 4.5.13 with $\kappa_M \lesssim_{p,X} [w_j]_{A_{p_j}}^{2(1+\max\{1/(p_j-1),1/(p_j'-1)\})} \kappa_m$, whose associated Fourier multiplier operator coincides with T_m on $\mathcal{S}(\mathbb{R}^2; X)$ (under the usual identifications). Here we have to note that $L^{p_1}(\mathbb{R}, w_1; X)$ is a UMD space (see Proposition E.5.6) which is isometrically isomorphic with $L^{p_1}(\mathbb{R}; X)$. □

4.5.3 Fourier Multipliers in Several Variables

In this subsection we prove several Mikhlin Fourier multiplier theorems in several variables. We start with an extension of Theorem 4.5.13. Recall that this theorem was obtained as a consequence of the generic Fourier multiplier theorem, Theorem 4.5.3. For this generic Fourier multiplier theorem it is required that X is of class $\mathcal{RP}_{p,d,w}$, for which it is in turn necessary that $w \in \prod_{j=1}^{l} A_{p_j}^{rec}(\mathbb{R}^{d_j})$ (see Lemma 4.2.7).

Theorem 4.5.16. *Let X be a UMD space, $a \in]0,\infty[^l$, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^{l} A_{p_j}^{rec}(\mathbb{R}^{d_j})$. Suppose that the symbol $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X)) \cap C^1(\mathbb{R}\setminus\{0\};\mathcal{B}(X))$ satisfies the anisotropic $\mathcal{R}$-boundedness Mikhlin condition*

$$\kappa_m := \mathcal{R}\{|\xi|_{d,a}^{a \cdot_d \theta} D^\theta m(\xi) \,:\, \xi \in \mathbb{R}\setminus\{0\}, \theta \leq \mathbf{1}\,\} < \infty. \tag{4.62}$$

Then we have $m \in \mathcal{M}_{p,d,w}(X)$ with $\|m\|_{\mathcal{M}_{p,d,w}(X)} \lesssim_{d,p,X,w} \kappa_m$.

In order to derive this theorem from the generic Fourier multiplier theorem, Theorem 4.5.3, we must show that X is of class $\mathcal{RP}_{p,d,w}$ and we must find a suitable unconditional Schauder decomposition corresponding to a decomposition of $\mathbb{R}^d$, possibly up to measure zero, by a countable collection of rectangles. The idea it to exploit the one variable case from the previous subsection by using the fact that $A_{p_j}^{rec}(\mathbb{R}^{d_j})$ consists precisely of those weights on $\mathbb{R}^{d_j}$ which are uniformly A_{p_j} in each of their variables. As a first step we have the following lemma.

Lemma 4.5.17. *Let X be a UMD space, $b \in]1,\infty[^l$, $I_k^{b_j} := [-b_j^{k+1}, -b_j^k[\cup]b_j^k, b_j^{k+1}]$ $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^{l} A_{p_j}^{rec}(\mathbb{R}^{d_j})$. Then X is of class $\mathcal{RP}_{p,d,w}$ with $\alpha_{p,d,w,X} \lesssim_{p,X} \prod_{j=1}^{l} [w_j]_{A_{p_j}^{rec}}^{d_j \max\{1,1/(p_j-1)\}}$. For each $j \in \{1,\dots,l\}$ and $i \in \{1,\dots,d_j\}$, we have that*

$$\Delta^{(j,i)} = (\Delta_k^{(j,i)})_{k\in\mathbb{Z}} := \left(\Delta\left[\mathbb{R}^{d_1+\ldots+d_{j-1}+(i-1)} \times I_k^{b_j} \times \mathbb{R}^{(d_j-i)+d_{j+1}+\ldots+d_l}\right]\right)_{k\in\mathbb{Z}}$$

defines an unconditional Schauder decomposition of $L^{p,d}(\mathbb{R}^d, w; X)$ with $C^+_{\Delta^{(j,i)}} \lesssim_{p,X} [w_j]_{A_{p_j}^{rec}}^{2\max\{1,1/(p_j-1)\}}$ and $C^-_{\Delta^{(j,i)}} \leq C^+_{[\Delta^{(j,i)}]^*} \lesssim_{p,X} [w_j]_{A_{p_j}^{rec}}^{2\max\{1,1/(p'_j-1)\}}$. Furthermore, we have

$$\mathcal{R}\left\{\sum_{k=M}^{N} \Delta_k^{(j,i)} \,:\, M,N\in\mathbb{Z}\right\} \lesssim_{p,X} [w_j]_{A_{p_j}^{rec}}^{\max\{1,1/(p_j-1)\}}, \quad \mathcal{R}\left\{\sum_{k=M}^{N} [\Delta_k^{(j,i)}]^* \,:\, M,N\in\mathbb{Z}\right\} \lesssim_{p,X} [w_j]_{A_{p_j}^{rec}}^{\max\{1,1/(p'_j-1)\}}$$

Proof. **Step I:** *Let $j \in \{1,\dots,l\}$ and $i \in \{1,\dots,d_j\}$. Suppose that $m \in L^\infty(\mathbb{R}) \cap C^3(\mathbb{R}\setminus\{0\})$ satisfies the Mikhlin condition* (4.55) *and define $M \in L^\infty(\mathbb{R}^d)$ by $M(\xi) := m(\xi_{j,i})$.* Then we have $M \in \mathcal{M}_{p,w,d}(X)$ with $\|M\|_{\mathcal{M}_{p,w,d}(X)} \lesssim_{p,X} [w_j]_{A_{p_k}^{rec}}^{\max\{1,1/(p_{j-1})\}}$.

This follows from a combination of Lemma 4.2.5 and Proposition 4.5.8.

Step II: *X is of class $\mathcal{RP}_{p,d,w}$ with $\alpha_{p,d,w,X} \lesssim_{p,X} \prod_{j=1}^{l} [w_j]_{A_{p_j}^{rec}}^{d_j \max\{1,1/(p_j-1)\}}$.*

From Step I it follows that, for all $j \in \{1,\dots,l\}$ and $i \in \{1,\dots,d_j\}$,

$$1_{\mathbb{R}^{d_1+\ldots+d_{j-1}+(i-1)} \times [0,\infty[\times \mathbb{R}^{(d_j-i)+d_{j+1}+\ldots+d_l}} \in \mathcal{M}_{p,d,w}(X) \text{ of norm } \lesssim_{p,X} [w_j]_{A_{p_j}^{rec}}^{\max\{1,1/(p_j-1)\}}.$$

The desired result now follows with the help of Proposition 4.2.4.(i).

Step III: *For all $j \in \{1, \ldots, l\}$ and $i \in \{1, \ldots, \mathpzc{d}_j\}$, $\Delta^{(j,i)}$ defines an unconditional Schauder decomposition of $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ with $C^+_{\Delta^{(j,i)}} \lesssim_{p,X} [w_j]^{2\max\{1,1/(p_j-1)\}}_{A^{rec}_{p_j}}$ and $C^-_{\Delta^{(j,i)}} \leq C^+_{[\Delta^{(j,i)}]^*} \lesssim_{p,X} [w_j]^{2\max\{1,1/(p'_j-1)\}}_{A^{rec}_{p_j}}$.*

This can be derived from Step I by using the argumentation from the proof of Lemma 4.5.10; see Remark 4.5.12.

Step IV: We finally prove the $\mathcal{R}$-bounds in the last statement. The first $\mathcal{R}$-bound can be shown as in Lemma 4.2.8 (and Proposition E.3.5), and the second $\mathcal{R}$-bound can be derived in the same way since the collection of adjoint operators is of the same form in view of Proposition 4.2.4.(iii). □

Using the abstract result about unconditional blockings of product decompositions from Theorem 4.3.15, we can now construct an unconditional Schauder decomposition of $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ out of the unconditional Schauder decompositions from the above lemma. For simplicity of notation we restrict ourselves to the case $\mathpzc{d} = (1,1)$, the general case being completely similar.

Lemma 4.5.18. *Let X be a UMD space, $\mathpzc{d} = (1,1)$, $b \in]1,\infty[^2$, $p \in]1,\infty[^2$, and $w \in \prod_{j=1}^2 A^{rec}_{p_j}(\mathbb{R})$. Define $(E_n)_{n\in\mathbb{Z}}$ by $E_n := \left([-b_1^{r+1}, -b_1^r[\cup]b_1^r, b_1^{r+1}]\right) \times [-b_2^r, b_2^r]$ for $n = 2r+1, r \in \mathbb{Z}$ and $E_n := [-b_1^{r+1}, b_1^{r+1}] \times \left([-b_2^{r+1}, -b_2^r[\cup]b_1^2, b_2^{r+1}]\right)$ for $n = 2r+2, r \in \mathbb{Z}$. Then $\Delta := (\Delta[E_n])_{n\in\mathbb{Z}}$ is an unconditional Schauder decomposition of $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ with C^+_Δ and $C^+_{\Delta^*}$ only depending on $\mathpzc{d}$, p, X and w.*

Proof. This can be shown by combining Lemma 4.5.17 with Theorem 4.3.15: Let $D^1 := \Delta^{(1,1)}$ and $D^2 := \Delta^{(2,1)}$ be the unconditional Schauder decompositions of $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ from Lemma 4.5.17. Then D^1 and D^2 clearly commute. Furthermore, it is not difficult to see that $\Delta = (\Delta[E_n])_{n\in\mathbb{Z}}$ coincides with the Δ defined in Theorem 4.3.15. By Theorem 4.3.15, as D^1 and D^2 satisfy the $\mathcal{R}$-boundedness contition (4.35) (see the last statement of Lemma 4.5.17), we may thus conclude that Δ is an unconditional Schauder decomposition of $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$. Moreover,

$$\begin{aligned} C^+_\Delta &\lesssim_{p,X} [w_1]^{2\max\{1,1/(p_1-1)\}}_{A^{rec}_{p_1}}[w_2]^{\max\{1,1/(p_2-1)\}}_{A^{rec}_{p_2}} + [w_1]^{\max\{1,1/(p_1-1)\}}_{A^{rec}_{p_1}}[w_2]^{2\max\{1,1/(p_2-1)\}}_{A^{rec}_{p_2}} \\ &= \sum_{k=1}^{l} \mathpzc{d}_k \prod_{j=1}^{l} [w_j]^{(\mathpzc{d}_j+\delta_{k,j})\max\{1,1/(p_j-1)\}}_{A^{rec}_{p_j}} \end{aligned}$$

and

$$\begin{aligned} C^-_\Delta \leq C^+_{\Delta^*} &\lesssim_{p,X} [w_1]^{2\max\{1,1/(p'_1-1)\}}_{A^{rec}_{p_1}}[w_2]^{\max\{1,1/(p'_2-1)\}}_{A^{rec}_{p_2}} + [w_1]^{\max\{1,1/(p'_1-1)\}}_{A^{rec}_{p_1}}[w_2]^{2\max\{1,1/(p'_2-1)\}}_{A^{rec}_{p_2}} \\ &= \sum_{k=1}^{l} \mathpzc{d}_k \prod_{j=1}^{l} [w_j]^{(\mathpzc{d}_j+\delta_{k,j})\max\{1,1/(p'_j-1)\}}_{A^{rec}_{p_j}}. \end{aligned}$$

□

Remark 4.5.19. In case X is a UMD space with property (α), we could just take the product decomposition $\Delta = (\Delta_k^{(1,1)}\Delta_k^{(2,1)})_{k\in\mathbb{Z}}$ above; see Remark 4.3.16.(ii). In [58] is was shown that, in the unweighted non-mixed-norm case, for the product decomposition to be an unconditional Schauder decomposition it is even necessary that X is an (α)-UMD space.

Having available this unconditional Schauder decomposition, the Theorem 4.5.16 now also is a consequence of the generic Fourier multiplier theorem of Theorem 4.5.3:

Proof of Theorem 4.5.16. The proof is essentially the same as the proof of Theorem 4.5.5. For detailed computations (in the isotropic case) we refer to [57]. □

We next come to general A_p-weights. As already observed several times, in this situation we cannot proceed via an unconditional Schauder decomposition corresponding to a decomposition of $\mathbb{R}^d$, possibly up to measure zero, by a countable collection of rectangles. Instead, we proceed via an extrapolation argument from the unweighted case.

Theorem 4.5.20. *Let X be a UMD space, $p \in]1,\infty[$, and $w \in A_p(\mathbb{R}^d)$. Suppose that the symbol $m \in L^\infty(\mathbb{R}^d;\mathcal{B}(X)) \cap C^{d+2}(\mathbb{R}^d \setminus \{0\};\mathcal{B}(X))$ satisfies*

$$\kappa_m := \mathcal{R}\{|\xi|^{|\theta|} D^\theta m(\xi) \,:\, \xi \in \mathbb{R} \setminus \{0\}, \theta \leq \mathbf{1}\} < \infty.$$

and

$$C_m := \sup\{|\xi|^{|\theta|} \left\|D^\theta m(\xi)\right\| \,:\, \xi \in \mathbb{R} \setminus \{0\}, |\theta| \leq d+2\} < \infty.$$

Then we have $m \in \mathcal{M}_{p,d,w}(X)$ with $\|m\|_{\mathcal{M}_{p,d,w}(X)} \lesssim_{d,p,X} [w]_{A_p}^{\max\{1,1/(p-1)\}} \max\{\kappa_m, C_m\}$.

Proof. This can be shown completely similar to the proof of Proposition 4.5.8. Of course, now we have to use (the isotropic case of) Theorem 4.5.16 instead of Theorem 4.5.5. □

Since

$$\max\{\kappa_m, C_m\} \leq \mathcal{R}\{|\xi|^{|\theta|} D^\theta m(\xi) \,:\, \xi \in \mathbb{R} \setminus \{0\}, |\theta| \leq d+2\},$$

we can use the same bootstrapping procedure as in Corollary 4.5.15. In the notation introduced before Corollary 4.5.15, we accordingly obtain:

Corollary 4.5.21. *Let X be an (α)-UMD space, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_p(\mathbb{R}^{d_j})$. Suppose that $\mathscr{M} \subset L^\infty(\mathbb{R}^d;\mathcal{B}(X)) \cap C_*^{d+2}(\mathcal{B}(X))$ is a collection of symbols satisfying*

$$\kappa_{\mathscr{M}} := \mathcal{R}\left\{|\xi_1|^{|\theta_1|} \ldots |\xi_l|^{|\theta_l|} D^\theta m(\xi) \,:\, \xi \in \mathbb{R}^d_{*^d}, \theta \in \mathbb{N}^d, \theta_j \leq d_j + 2, j = 1,\ldots,l, m \in \mathscr{M}\right\} < \infty.$$

Then we have $\mathscr{M} \subset \mathcal{M}_{p,d,w}(X)$ with

$$\mathcal{R}\{T_m : m \in \mathscr{M}\} \lesssim_{d,p,X} \left(\prod_{j=1}^l [w_j]_{A_{p_j}}^{\max\{1,1/(p_j-1)\}}\right) \kappa_{\mathscr{M}} \quad \text{in} \quad \mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X)).$$

4.6 Notes

4.6.1 General Notes

For a nice survey of Banach-valued Fourier multipliers (in the unweighted setting) we refer to [56]; also see the notes of [64, 25, 57]. Let us mention that the unweighted version of Theorem 4.5.16 is due to Hytonen [55] (as a consequence of a slightly stronger anisotropic Fourier multiplier theorem); also see [57]. Concerning the weighted setting, the isotropic non-mixed-norm scalar-valued ($l = 1, a = 1, X = \mathbb{C}$) version of this theorem is due to Kurtz [65]. For Banach-valued Fourier multiplier theorems in the A_p-weighted setting we refer to [73, 76, 78]; also see below (Section 4.6.2).

4.6.2 Comparison to the Literature

- *Section 4.2*: Except for Proposition 4.2.3, Lemma 4.2.5 and Lemma 4.2.7, which are very basic, this section is completely based on a combination of [64] and [57] (which are about the unweighted case). Here the definition of the Banach spaces of class $\mathcal{RP}_{p,d,w}$ (cf. Definition 4.2.6) is new and is a natural replacement of the UMD property in the unweighted case (see Lemma 4.2.7.(iv)).

- *Section 4.3*: This section is based on the PhD thesis [103] (mostly some slight modifications); also see the articles [23, 16]. We would like to comment on Theorem 4.3.15 (and Corollary 4.3.6, which its needed for the formulation of this theorem). First of all, Theorem 4.3.15 can be seen as a version of its corollary, Corollary 4.3.17, for Schauder decompositions indexed by $\mathbb{Z}$ instead of $\mathbb{N}$. This corollary in turn corresponds to [103, Theorem 2.5.1], where the UMD property is taken as a hypothesis in place of the $\mathcal{R}$-boundedness condition (4.40) (see Remark 4.3.16). Whereas the inspiration for the construction in [103, Theorem 2.5.1] comes from the work in [105] on multi-dimensional Fourier multipliers on the torus $\mathbb{T}^d$, Theorem 4.3.15 was inspired by the case $\mathbb{R}^d$ in [105] and was in fact already implicitly used in [55] (with reference to [103]). Finally, the simple Corollary 4.3.6 is motivated by the construction in Theorem 4.3.15.

- *Section 4.4*: The inspiration for this section comes from [78] (also see [73, 76]), where [49, Corollary 2.10] was already used to derive an isotropic Mikhlin Fourier multiplier theorem on $L^p(\mathbb{R}^d, w; X)$ for X a UMD space, $p \in]1, \infty[$, and $w \in A_p(\mathbb{R}^d)$ (also see the discussion about Proposition 4.5.8 and Theorem 4.5.20 below). The inspiration for the (computations in the proof of) Lemma 4.4.7 comes from (the computations in the proofs of) [92, V.I.4.Lemma & V.I.4.Proposition.2], which (via Section 4.4.1) forms the motivation for Section 4.4.2, where Definition 4.4.1 is new.

- *Section 4.5*: This section is mainly based on [57] (which is about the unweighted non-mixed-norm case). Here Section 4.5.1 is directly based on [57] (and [56] regarding the terminology introduced in Definition 4.5.1). In Section 4.5.2 and 4.5.3, the proofs of Theorem 4.5.5, Corollary 4.5.14, and Corollary 4.5.15/Corollary 4.5.21, are taken from [57]. The idea to use [49, Corollary 2.10] (cf. Theorem 4.4.3) in order to derive Proposition 4.5.8 and Theorem 4.5.20 from the unweighted case is taken from [78] (also see [73, 76]). The argumentation in the proof of Lemma 4.5.10 is directly based on [105], where it is used to derive Lemma 4.5.18 in the unweighted non-mixed-norm case (for a general dimension $d \geq 1$) from the multi-dimensional Mikhlin theorem on $L^p(\mathbb{R}^d; X)$ for scalar-valued symbols, which was in turn obtained from the case $\mathbb{T}^d$ via a so-called transference argument. The unweighted version of Theorem 4.5.16 can be found in [56], where it actually is a consequence of a slightly stronger anisotropic Fourier multiplier theorem (whose proof also works in the weighted setting). The most crucial ingredient for Theorem 4.5.16 is the unconditional Schauder decomposition from Lemma 4.5.18, which is also used in [56] (and which is actually just an anisotropic version of the unconditional Schauder decomposition of $L^p(\mathbb{R}^d; X)$ used in [105]). Here it was our main aim to give a detailed and self-contained proof of Lemma 4.5.18 by a doing a blocking argument, similar to the approach in [103, Remark 3.5.2] to the unconditionality of the Schauder decomposition of $L^p(\mathbb{T}^d; X)$ from [105]. As a first step we have Lemma 4.5.17. In order to derive Lemma 4.5.18 from it, we naturally come to the abstract Theorem 4.3.15

about unconditional blockings of product decompositions; also see the discussion about Section 4.3 above.

Chapter 5

Anisotropic Function Spaces

In this chapter we study weighted anisotropic mixed-norm Banach space-valued function spaces, which naturally occur in the theory of maximal weighted L^q-L^p-regularity for parabolic initial-boundary value problems. We start with spaces on the full Euclidean space $\mathbb{R}^d$, where we follow a Fourier analytic approach with as main tools maximal functions and Fourier multipliers, and subsequently treat spaces on domains (open subsets of $\mathbb{R}^d$) via their versions on $\mathbb{R}^d$.

5.1 Introduction

As already mentioned in Section 1.2, a very important step in the solution of the maximal L^q_μ-L^p_γ problem for the parabolic initial-boundary value problem (1.5) is to determine the spatial trace space of the weighted anisotropic Sobolev space of intersection type

$$W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+, v_\mu; W^2_p(\mathbb{R}^d_+, w_\gamma; X)),$$

where the weights v_μ and w_γ are as in (1.6). In Section 2.1 we saw that this intersection space can be naturally identified with the weighted anisotropic Sobolev space of distribution type

$$W^{(2,1)}_{(p,q),(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X) \hookrightarrow \mathcal{D}'(\mathbb{R}^d_+ \times \mathbb{R}_+; X).$$

In view of Lemma 2.1.4, in order to determine the trace space of this anisotropic Sobolev space we may proceed via its version on $\mathbb{R}^d \times \mathbb{R}$, $W^{(2,1)}_{(p,q),(d,1)}(\mathbb{R}^d \times \mathbb{R}, (w_\gamma, v_\mu); X)$. The main advantage of this strategy is that it opens the door to Euclidean harmonic analysis.

In [86] Scharf, Schmeißer & Sickel characterized the trace space of vector-valued Sobolev spaces $W^n_p(\mathbb{R}^d; X)$, where $p \in]1, \infty[$ and X is an arbitrary Banach space. Here the strategy was to use a sandwich argument, based on the embeddings

$$F^n_{p,1}(\mathbb{R}^d; X) \hookrightarrow W^n_p(\mathbb{R}^d; X) \hookrightarrow F^n_{p,\infty}(\mathbb{R}^d; X) \tag{5.1}$$

and the independence of the trace space of the vector-valued Triebel-Lizorkin space $F^n_{p,q}(\mathbb{R}^d; X)$ on the microscopic parameter $q \in [1, \infty]$. In this chapter we will extend this to the weighted anisotropic mixed-norm Banach space-valued setting, to a large extend based on the work of Johnsen & Sickel [62] concerning the trace problem for unweighted anisotropic scalar-valued Triebel-Lizorkin spaces; also see the notes of this chapter.

5.2 Function Spaces on $\mathbb{R}^d$

Throughout this section we view $\mathbb{R}^d$ as being $\mathpzc{d}$-decomposed as in Convention 2.2.1 unless otherwise stated, where $\mathpzc{d} = (d_1, \dots, d_l) \in (\mathbb{Z}_{>0})^l$.

5.2.1 Definitions and Basic Properties

5.2.1.a Definition Anisotropic Sobolev and Bessel-Potential Spaces

Definition 5.2.1. Let X be a Banach space, $n \in \mathbb{N}^l$, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$. Put

$$J_{n,\mathpzc{d}} := \left\{ \alpha \in \bigcup_{j=1}^{l} \iota_{[\mathpzc{d};j]}\mathbb{N}^{d_j} \;:\; |\alpha_j| \leq n_j \right\}; \tag{5.2}$$

note here that $\iota_{[\mathpzc{d};j]}\mathbb{N}^{d_j} = \{n \in \mathbb{N}^d \;:\; n = (0,\dots,0,n_j,0,\dots,0)\}$, see Convention 2.2.1. We define the *weighted anisotropic mixed-norm Sobolev space* $W^n_{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ as the space of all $f \in \mathcal{S}'(\mathbb{R}^d; X)$ for which $D^\alpha f \in L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ for all $\alpha \in J_{n,\mathpzc{d}}$. We equip this space with the norm

$$\|f\|_{W^n_{p,\mathpzc{d}}(\mathbb{R}^d,w;X)} := \sum_{\alpha \in J_{n,\mathpzc{d}}} \|D^\alpha f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)}.$$

It will be convenient to have a Fourier analytic description of the weighted anisotropic mixed-norm Sobolev space $W^n_{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$. In Chapter 4 we already mentioned the identity $W^n_p(\mathbb{R}^d) = H^n_p(\mathbb{R}^d)$, $p \in]1,\infty[$, (4.2) in the scalar-valued unweighted isotropic setting, where $H^n_p(\mathbb{R}^d)$ (4.3) stands for the Bessel potential space of order $n \in \mathbb{N}$. This identity also has a vector-valued analogue, but under a restriction on the Banach space under consideration. In fact, it is well known that the identity $W^n_p(\mathbb{R}^d; X) = H^n_p(\mathbb{R}^d; X)$, $p \in]1,\infty[$, $n\mathbb{Z}_{\geq 1}$, holds true if and only if X is a UMD space; see for instance [3]. Below we will define a weighted anisotropic mixed-norm version $H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ (see Definition 5.2.2), for which we will prove the identity $W^n_{p,\mathpzc{d}}(\mathbb{R}^d, w; X) = H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ in Proposition 5.2.45. Before we do this, we first give a motivation for its definition in a simple situation: Suppose that $X = H$ is a Hilbert space, $n \in (\mathbb{Z}_{>0})^l$, $p = (2,\dots,2)$, and $w = (1,\dots,1)$. Furthermore, pick $s > 0$ and $a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$ such that $a_j = \frac{s}{n_j}$ (so $s = a_j n_j$) for each $j \in \{1,\dots,l\}$. By the equivalence

$$\sum_{\alpha \in J_{n,\mathpzc{d}}} |\xi^\alpha| \eqsim (1 + |\xi|^2_{\mathpzc{d},a})^{s/2}, \qquad \xi \in \mathbb{R}^d,$$

the Plancherel theorem (see Theorem C.6.3), and the fact that $(1 + |\cdot|^2_{\mathpzc{d},a})^{s/2} \in \mathscr{O}_M(\mathbb{R}^d)$, for every $f \in \mathcal{S}'(\mathbb{R}^d; X)$ we then have:

$$\begin{aligned}
f \in W^n_{p,\mathpzc{d}}(\mathbb{R}^d, w; X) \quad &\Leftrightarrow \quad D^\alpha f \in L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X),\ \ \alpha \in J_{n,\mathpzc{d}} \\
&\Leftrightarrow \quad \mathscr{F}^{-1}[\xi^\alpha \hat{f}] \in L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X) = L^2(\mathbb{R}^d; H),\ \ \alpha \in J_{n,\mathpzc{d}} \\
&\Leftrightarrow \quad \xi^\alpha \hat{f} \in L^2(\mathbb{R}^d; H),\ \ \alpha \in J_{n,\mathpzc{d}} \\
&\Leftrightarrow \quad \sum_{\alpha \in J_{n,\mathpzc{d}}} |\xi^\alpha| \hat{f} \in L^2(\mathbb{R}^d; H) \\
&\Leftrightarrow \quad (1 + |\cdot|^2_{\mathpzc{d},a})^{s/2} \hat{f} \in L^2(\mathbb{R}^d; H) \\
&\Leftrightarrow \quad \mathscr{F}^{-1}[(1 + |\cdot|^2_{\mathpzc{d},a})^{s/2} \hat{f}] \in L^2(\mathbb{R}^d; H) = L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X),
\end{aligned}$$

in which case

$$\|f\|_{W^n_{p,d}(\mathbb{R}^d,w;X)} \eqsim \left\| \mathscr{F}^{-1}[(1+|\,\cdot\,|^2_{d,a})^{s/2}\hat{f}] \right\|_{L^{p,d}(\mathbb{R}^d,w;X)}.$$

For an arbitrary anisotropy $a \in (\frac{1}{\mathbb{Z}_{\geq 1}})^l$ it holds that $(1+|\,\cdot\,|^2_{d,a})^{\sigma/2}$ belongs to $\mathscr{O}_M(\mathbb{R}^d)$ for all $\sigma \in \mathbb{R}$. We may thus define the Fourier multiplier operator $\mathcal{J}^{d,a}_\sigma \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^d;X))$ by

$$\mathcal{J}^{d,a}_\sigma f := \mathscr{F}^{-1}[(1+|\,\cdot\,|^2_{d,a})^{\sigma/2}\hat{f}] \qquad (f \in \mathcal{S}'(\mathbb{R}^d;X)). \tag{5.3}$$

The operator $\mathcal{J}^{d,a}_\sigma$ and is called the (d,a)*-anisotropic Bessel potential operator of order* σ.

Definition 5.2.2. Let X be a Banach space, $a \in (\frac{1}{\mathbb{Z}_{\geq 1}})^l$, $p \in]1,\infty[^l$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. We define the *weighted anisotropic mixed-norm Bessel-potential space* $H^{s,a}_{p,d}(\mathbb{R}^d,w;X)$ as the space of all $f \in \mathcal{S}'(\mathbb{R}^d;X)$ for which $\mathcal{J}^{d,a}_s f \in L^{p,d}(\mathbb{R}^d,w;X)$. We equip this space with the norm

$$\|f\|_{H^{s,a}_{p,d}(\mathbb{R}^d,w;X)} := \left\| \mathcal{J}^{d,a}_s f \right\|_{L^{p,d}(\mathbb{R}^d,w;X)}.$$

Remark 5.2.3. For $a \in]0,\infty[^l \setminus (\frac{1}{\mathbb{Z}_{\geq 1}})^l$, $|\,\cdot\,|^2_{d,a}$ is not smooth in the origin, so that we cannot define the Bessel potential operator $\mathcal{J}^{d,a}_\sigma$ via (5.3). In order to overcome this problem, we could also define the Bessel potential operator by

$$\mathcal{J}^{d,a}_\sigma : f \mapsto \mathscr{F}^{-1}\left[\sum_{j=1}^{l} (1+|\pi_{[d;j]}(\,\cdot\,)|^2)^{\sigma/2a_k}\hat{f} \right];$$

this is the definition taken in [37]. Since we will only need the case $a \in (\frac{1}{\mathbb{Z}_{\geq 1}})^l$, we decide to just stay with (5.3).

5.2.1.b Motivation for Triebel-Lizorkin Spaces

Since the definitions of Triebel-Lizorkin spaces and Besov spaces may look somewhat complicated at first glance, we now first give some motivation for the Triebel-Lizorkin case before we give the formal definitions; also see [97, Section 2.2.4].

For doing estimates of functions it is often convenient to split the function into pieces, estimate the several pieces separately, and assemble these together. In Chapter 4 we have seen that in the context of Fourier multipliers on $L^{p,d}(\mathbb{R}^d,w;X)$ (with $p \in]1,\infty[^l$, $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$ and X UMD) this can be succesfully done in the framework of unconditional Schauder decompositions. In the one-dimensional case $d=1$, the operator-valued Mikhlin theorem on $L^p(\mathbb{R},w;X)$ was proved by using the unconditional Schauder decomposition from Lemmas 4.5.4 and 4.5.10. Consider the blocking $(\Delta_k)_{k\in\mathbb{N}}$ of this unconditional Schauder decomposition given by

$$\Delta_k f := \mathscr{F}^{-1}[1_{J_k}\hat{f}] \quad \text{with } J_k := \begin{cases}]-1,1[, & k=0 \\ ]-2^k,-2^{k-1}] \cup [2^{k-1},2^k[, & k \geq 1. \end{cases}$$

Recall that $(\epsilon_k)_{k\in\mathbb{N}}$ denotes a fixed Rademacher sequence on some probability space $(\Omega,\mathcal{F},\mathbb{P})$;

see Appendix E.1. For every $f \in L^p(\mathbb{R}, w; X)$ we have $f = \sum_{k=0}^{\infty} \Delta_k f$ in $L^p(\mathbb{R}, w; X)$ and

$$\begin{aligned} \|f\|_{L^p(\mathbb{R},w;X)} &= \lim_{K\to\infty} \left\| \sum_{k=0}^{K} \Delta_k f \right\| \\ &\overset{(4.30)}{\eqsim} \lim_{K\to\infty} \left\| \sum_{k=0}^{K} \epsilon_k \Delta_k f \right\|_{L^p(\Omega;L^p(\mathbb{R},w;X))} \\ &\overset{Prop. E.1.1}{=} \sup_{K\in\mathbb{N}} \left\| \sum_{k=0}^{K} \epsilon_k \Delta_k f \right\|_{L^p(\Omega;L^p(\mathbb{R},w;X))}. \end{aligned} \tag{5.4}$$

Now let us look what the equivalence of norms (5.4) means for the Bessel potential space $H^s_p(\mathbb{R}^d, w; X)$, $s \in \mathbb{R}$. Given a g in the dense space $\mathscr{F}^{-1}C^\infty_c(\mathbb{R}; X)$ of $H^s_p(\mathbb{R}, w; X)$[1] we can take $f = \mathcal{J}_k g \in L^p(\mathbb{R}, w; X)$ in (5.4), to obtain

$$\|g\|_{H^s_p(\mathbb{R}^d,w;X)} = \|\mathcal{J}_s g\|_{L^p(\mathbb{R},w;X)} \eqsim \sup_{K\in\mathbb{N}} \left\| \sum_{k=0}^{K} \epsilon_k \Delta_k \mathcal{J}_k g \right\|_{L^p(\Omega;L^p(\mathbb{R},w;X))}.$$

As $(1 + |\xi|^2)^{s/2} 1_{I_k}(\xi) \eqsim 2^{ks} 1_{I_k}(\xi)$ for all $k \in \mathbb{N}$ and $\xi \in \mathbb{R}$, this suggests that (perhaps)

$$\|g\|_{H^s_p(\mathbb{R}^d,w;X)} \eqsim \sup_{K\in\mathbb{N}} \left\| \sum_{k=0}^{K} \epsilon_k 2^{ks} \Delta_k g \right\|_{L^p(\Omega;L^p(\mathbb{R},w;X))}. \tag{5.5}$$

Note the RHS of (5.5) defines a norm on $\mathscr{F}^{-1}C^\infty_c(\mathbb{R}; X)$. So we could try to define an extended norm $|||\cdot|||_{s,p,w} : \mathcal{S}'(\mathbb{R}^d; X) \longrightarrow [0, \infty]$ (in some reasonable way) which on $\mathscr{F}^{-1}C^\infty_c(\mathbb{R}; X)$ is given by the expression on the RHS of (5.5), and define a space consisting of all $f \in \mathcal{S}'(\mathbb{R}^d; X)$ with $|||f|||_{s,p,w} < \infty$. However, we will not do this. We instead replace the multiplier symbols $(1_{I_k})_{k\in\mathbb{N}}$ corresponding to $(\Delta_k)_{k\in\mathbb{N}} = (\Delta[I_k])_{k\in\mathbb{N}}$ by smooth symbols $(\psi_k)_{k\in\mathbb{N}} \subset C^\infty_c(\mathbb{R})$. The advantage of this is that it allows us to define $S_k \in \mathcal{L}(\mathcal{S}'(\mathbb{R}; X))$ by

$$S_k f := \mathscr{F}^{-1}[\psi_k \hat{f}] = \check{\psi}_k * f, \qquad f \in \mathcal{S}'(\mathbb{R}; X).$$

Moreover, by the Paley-Wiener-Schwartz theorem (cf. Theorem C.6.4), each $S_k f$ is an analytic function; in particular, we have $S_k f \in \mathcal{S}'(\mathbb{R}; X) \cap C^\infty(\mathbb{R}; X)$ for every $k \in \mathbb{N}$. So we may define an extended norm $|||\cdot|||_{s,p,w} : \mathcal{S}'(\mathbb{R}^d; X) \longrightarrow [0, \infty]$ by

$$|||g|||_{s,p,w} := \sup_{K\in\mathbb{N}} \left\| \sum_{k=0}^{K} \epsilon_k 2^{ks} S_k g \right\|_{L^p(\Omega;L^p(\mathbb{R},w;X))}, \qquad g \in \mathcal{S}'(\mathbb{R}; X). \tag{5.6}$$

In order to relate $|||\cdot|||_{s,p,w}$ to the Bessel potential space $H^s_p(\mathbb{R}, w; X)$ in the same heuristic way as for the RHS of (5.5), we have to pick $(\psi_k)_{k\in\mathbb{N}}$ in such a way that $\|\cdot\|_{L^p(\mathbb{R},w;X)} \eqsim |||\cdot|||_{0,p,w}$ on $L^p(\mathbb{R}, w; X)$ and that $(1 + |\xi|^2)^{s/2}\psi_k(\xi) \eqsim 2^{ks}\psi_k(\xi)$ for all $k \in \mathbb{N}$ and $\xi \in \mathbb{R}$. In fact, in order to try to get the characterization

$$H^s_p(\mathbb{R}, w; X) = \{f \in \mathcal{S}'(\mathbb{R}; X) : |||f|||_{s,p,w} < \infty\} \text{ with } \|f\|_{H^s_p(\mathbb{R},w;X)} \eqsim |||f|||_{s,p,w}, \ \forall f \in H^s_p(\mathbb{R}, w; X), \tag{5.7}$$

we would like to take a sequence $(\psi_k)_{k\in\mathbb{N}} \subset C^\infty_c(\mathbb{R})$ having the properties that

[1]This denseness will be proven in Proposition 5.2.14.

(i) $L^p(\mathbb{R}, w; X) = \{f \in \mathcal{S}'(\mathbb{R}; X) : |||f|||_{0,p,w} < \infty\}$ plus $\|f\|_{L^p(\mathbb{R},w;X)} \eqsim |||f|||_{0,p,w}$ for all $f \in L^p(\mathbb{R}, w; X)$;

(ii) $\operatorname{supp}\psi_0 \subset \{\xi \in \mathbb{R} : |\xi| \leq 2\}$, $\operatorname{supp}\psi_k \subset \{\xi \in \mathbb{R} : 2^{k-1} \leq |\xi| \leq 2^{k+1}\}$ for $k \geq 1$;

(iii) $f = \sum_{k=0}^{\infty} S_k f$ in $\mathcal{S}'(\mathbb{R}; X)$ for all $f \in \mathcal{S}'(\mathbb{R}; X)$.

For (iii) it is sufficient that

$$\sum_{k=0}^{\infty} \psi_k = 1 \quad \text{in} \quad \mathscr{O}_M(\mathbb{R}) \tag{5.8}$$

because of the facts the pointwise multiplication map $\mathscr{O}_M(\mathbb{R}) \times \mathcal{S}'(\mathbb{R}; X) \longrightarrow \mathcal{S}'(\mathbb{R}; X)$ is continuous in its first variable (see Appendix C.4) and that the Fourier transform $\mathscr{F}$ is a topological linear isomorphism of $\mathcal{S}'(\mathbb{R}^d; X)$ (see Appendix C.6). For the inequality '$\lesssim$' in (i) we could try to modify the proof of Lemma 4.5.10. Since this proof is based on the dilation invariance of the Mikhlin condition, the support condition (ii) suggests to require

$$\psi_k(\xi) = \psi_1(2^{(-k+1)}\xi), \qquad \xi \in \mathbb{R}, k \geq 1. \tag{5.9}$$

For the reverse inequality '$\gtrsim$' we could then try to proceed via a duality argument (similar to Lemma 4.3.11).

Now suppose we have successfully established the characterization (5.7). Then the Rademacher sequence $(\epsilon_k)_{k\in\mathbb{N}}$ in (5.6) may not be very convenient for doing estimates. If X is a Hilbert space, then we have the non-random description

$$\begin{aligned} |||g|||_{s,p,w} &\overset{Fubini}{=} \sup_{K\in\mathbb{N}} \left\| \sum_{k=0}^{K} \epsilon_k 2^{ks} S_k g \right\|_{L^p(\mathbb{R},w;L^p(\Omega;X))} \overset{(E.3)}{=} \sup_{K\in\mathbb{N}} \left\| \left(\sum_{k=0}^{K} \left\| 2^{ks} S_k g \right\|_X^2 \right)^{1/2} \right\|_{L^p(\mathbb{R},w)} \\ &= \left\| \left(\sum_{k=0}^{\infty} \left\| 2^{ks} S_k g \right\|_X^2 \right)^{1/2} \right\|_{L^p(\mathbb{R},w)}, \end{aligned}$$

that is,

$$H^s_p(\mathbb{R}, w; X) \cong F^s_{p,2}(\mathbb{R}, w; X),$$

where

$$F^s_{p,q}(\mathbb{R}, w; X) := \left\{ g \in \mathcal{S}'(\mathbb{R}; X) : \|g\|_{F^s_{p,q}(\mathbb{R},w;X)} := \left\| \left(\sum_{k=0}^{\infty} \left\| 2^{ks} S_k g \right\|_X^q \right)^{1/q} \right\|_{L^p(\mathbb{R},w)} < \infty \right\}, \quad q \in [1, \infty],$$

is a Triebel-Lizorkin space (to be defined rigorously below). For a general Banach space X, say with type $\tau \in [1, 2]$ and cotype $q \in [2, \infty]$ (see Appendix E.2), replacing the identity (E.3) by the type and cotype inequalities, we only have the embeddings

$$F^s_{p,\tau}(\mathbb{R}, w; X) \hookrightarrow H^s_p(\mathbb{R}, w; X) \hookrightarrow F^s_{p,q}(\mathbb{R}, w; X). \tag{5.10}$$

However, we will see that a lot of properties of $F^s_{p,q}(\mathbb{R}, w; X)$ are independent of the microscopic parameter $q \in [1, \infty]$.

In Section 5.2.1.c we will define Littlewood-Paley sequences $\varphi = (\varphi_k)_{k\in\mathbb{N}}$, for which the Fourier transformed sequence $(\hat{\varphi}_k)_{k\in\mathbb{N}}$ will play the role of $(\psi_k)_{k\in\mathbb{N}}$ in the definition of the (weighted anisotropic mixed-norm) Triebel-Lizorkin space, for which we do not require (i) from above. In Proposition 5.2.31 we will see that, for arbitrary Banach spaces X, the embedding (5.10) holds true with $\tau = 1$ and $q = \infty$, and similarly for Sobolev spaces ($s \in \mathbb{N}$).

Before we give the definition of Littlewood-Paley sequence in Section 5.2.1.c, let us first have a look at an arbitrary sequence $(\psi_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ satisfying (ii), (5.8), and (5.9). First note that (5.8) implies the pointwise convergence $\sum_{k=0}^{\infty} \psi_k(\xi) = 1$ for $\xi \in \mathbb{R}^d$. In view of the support condition (ii), we must thus in particular have

$$\sum_{k=0}^{K} \psi_k \equiv 1 \text{ on } \{\xi \in \mathbb{R} : |\xi| \leq 2^K\}, \qquad K \in \mathbb{N}.$$

From this it follows that

$$\begin{aligned} \sum_{k=0}^{2} \psi_k &= 1_{\{|\xi|\leq 2^2\}} + \psi_2 1_{\{|\xi|>2^2\}} \overset{(5.9)}{=} 1_{\{|\xi|\leq 2\}}(2^{-1}\cdot) + \psi_1(2^{-1}\cdot)1_{\{|\xi|>2\}}(2^{-1}\cdot) \\ &= \left(\sum_{k=0}^{1} \psi_k 1_{\{|\xi|\leq 2\}}\right)(2^{-1}\cdot) + \psi_1(2^{-1}\cdot)1_{\{|\xi|>2\}}(2^{-1}\cdot) = \sum_{k=0}^{1} \psi_k(2^{-1}\cdot), \end{aligned}$$

implying that

$$\psi_1 = \sum_{k=0}^{2} \psi_k - \psi_0 - \psi_2 = \psi_0(2^{-1}\cdot) + \psi_1(2^{-1}\cdot) - \psi_0 - \psi_2 \overset{(5.9)}{=} \psi_0(2^{-1}\cdot) - \psi_0.$$

In conclusion, the given sequence $(\psi_k)_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ must in particular satisfy $\psi_0 \equiv 1$ on $\{\xi \in \mathbb{R} : |\xi| \leq 2\}$ and $\psi_1 = \psi_0(2^{-1}\cdot) - \psi_0$.

5.2.1.c Anisotropic Littlewood-Paley Sequences

Let $a \in]0,\infty[^l$ be fixed. Recall the definition of the associated anisotropic dilation $\delta_\lambda^{[d,a]}$ by $\lambda > 0$ (2.12) and the associated anisotropic distance function $|\cdot|_{d,a}$ (2.14).

Definition 5.2.4. For $0 < A < B < \infty$ we define $\Phi_{A,B}^{d,a}(\mathbb{R}^d)$ as the set of all sequences $\varphi = (\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ which are constructed in the following way: given a $\varphi_0 \in \mathcal{S}(\mathbb{R}^d)$ satisfying

$$0 \leq \hat{\varphi}_0 \leq 1,\ \hat{\varphi}_0(\xi) = 1 \text{ if } |\xi|_{d,a} \leq A,\ \hat{\varphi}_0(\xi) = 0 \text{ if } |\xi|_{d,a} \geq B, \tag{5.11}$$

$(\varphi_n)_{n\geq 1} \subset \mathcal{S}(\mathbb{R}^d)$ is defined via the relations

$$\hat{\varphi}_n(\xi) = \hat{\varphi}_1(\delta_{2^{-n+1}}^{[d,a]}\xi) = \hat{\varphi}_0(\delta_{2^{-n}}^{[d,a]}\xi) - \hat{\varphi}_0(\delta_{2^{-n+1}}^{[d,a]}\xi), \quad \xi \in \mathbb{R}^d, n \geq 1. \tag{5.12}$$

We put $\Phi^{d,a}(\mathbb{R}^d) := \bigcup_{0<A<B<\infty} \Phi_{A,B}^{d,a}(\mathbb{R}^d)$ and we call a sequence $\varphi \in \Phi^{d,a}(\mathbb{R}^d)$ a (d,a)-*anisotropic Littlewood-Paley sequence*.

In case $l = 1$ we write $\Phi^a(\mathbb{R}^d) = \Phi^{d,a}(\mathbb{R}^d)$, $\Phi(\mathbb{R}^d) = \Phi^1(\mathbb{R}^d)$, $\Phi_{A,B}^a(\mathbb{R}^d) = \Phi_{A,B}^{d,a}(\mathbb{R}^d)$, and $\Phi_{A,B}(\mathbb{R}^d) = \Phi_{A,B}^1(\mathbb{R}^d)$.

The flexibility in $0 < A < B < \infty$ can in some situations be very useful. However, in most situations it will be enough to consider the specific choice $A = 1$ and $B = \frac{3}{2}$, for which the structure of a Littlewood-Paley sequence becomes a little bit simpler:

Example 5.2.5. Let $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in \Phi^{d,a}_{1,\frac{3}{2}}(\mathbb{R}^d)$ be given and put $\varphi_{-1} := 0$. Then we have

$$\operatorname{supp}\hat{\varphi}_0 \subset \left\{\xi \mid |\xi|_{d,a} \leq \frac{3}{2}\right\} \quad \text{and} \quad \operatorname{supp}\hat{\varphi}_n \subset \left\{\xi \mid 2^{n-1} \leq |\xi|_{d,a} \leq 2^n\frac{3}{2}\right\}, \quad n \geq 1;$$

in particular, $\operatorname{supp}\hat{\varphi}_n \cap \operatorname{supp}\hat{\varphi}_n \neq \emptyset$ if and only if $|n-m| \leq 1$. Furthermore,

$$\hat{\varphi}_{n-1} + \hat{\varphi}_n + \hat{\varphi}_{n+1} \equiv 1 \quad \text{on} \quad \left\{\xi \mid 2^{n-2}\frac{3}{2} \leq |\xi|_{d,a} \leq 2^{n+1}\right\} \supset \operatorname{supp}\hat{\varphi}_n, \qquad n \in \mathbb{N}.$$

Let $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ be a general (d,a)-anisotropic Littlewood-Paley sequence. Observe that $\sum_{n=0}^{\infty}\hat{\varphi}_n(\xi) = 1$ for all $\xi \in \mathbb{R}^d$ and that

$$\operatorname{supp}\hat{\varphi}_0 \subset \{\xi \mid |\xi|_{d,a} \leq B\} \quad \text{and} \quad \operatorname{supp}\hat{\varphi}_n \subset \{\xi \mid 2^{n-1}A \leq |\xi|_{d,a} \leq 2^n B\}, \quad n \geq 1; \tag{5.13}$$

in particular, there exists an $h \in \mathbb{N}$ such that $\operatorname{supp}\hat{\varphi_n} \cap \operatorname{supp}\hat{\varphi_m} = \emptyset$ whenever $|n-m| > h$. To φ we associate the family of convolution operators $(S_n)_{n\in\mathbb{N}} = (S_n^{\varphi})_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{S}'(\mathbb{R}^d;X))$ given by

$$S_n f = S_n^{\varphi} f := \varphi_n * f = \mathscr{F}^{-1}[\hat{\varphi}_n\hat{f}] \qquad (f \in \mathcal{S}'(\mathbb{R}^d;X)). \tag{5.14}$$

By the Paley-Wiener-Schwartz Theorem (cf. Theorem C.6.4), $S_n f$ belongs to $C^{\infty}(\mathbb{R}^d;X) \cap \mathcal{S}'(\mathbb{R}^d;X)$ and has an extension to an entire analytic function on $\mathbb{C}^d$ for all $f \in \mathcal{S}'(\mathbb{R}^d;X)$. Moreover, we have $f = \sum_{n=0}^{\infty} S_n f$ in $\mathcal{S}'(\mathbb{R}^d;X)$ respectively in $\mathcal{S}(\mathbb{R}^d;X)$ whenever $f \in \mathcal{S}'(\mathbb{R}^d;X)$ respectively $f \in \mathcal{S}(\mathbb{R}^d;X)$. Recalling that the pointwise multiplication maps

$$\mathscr{O}_M(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d;X) \longrightarrow \mathcal{S}(\mathbb{R}^d;X) \quad \text{and} \quad \mathscr{O}_M(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d;X) \longrightarrow \mathcal{S}'(\mathbb{R}^d;X)$$

are continuous in their first variable and that the Fourier transform $\mathscr{F}$ is a topological linear isomorphism of both $\mathcal{S}(\mathbb{R}^d;X)$ and $\mathcal{S}'(\mathbb{R}^d;X)$, this is immediate from the following lemma.

Lemma 5.2.6. $\sum_{n=0}^{\infty}\hat{\varphi}_n = 1$ *with convergence in* $\mathscr{O}_M(\mathbb{R}^d)$.

Proof. In light of the pointwise convergence $\sum_{n=0}^{\infty}\hat{\varphi}_n = 1$ on $\mathbb{R}^d$, we must show that $\sum_{n=N}^{\infty}\hat{\varphi}_n \stackrel{N\to\infty}{\longrightarrow} 0$ in $\mathscr{O}_M(\mathbb{R}^d)$; here $\sum_{n=N}^{\infty}\hat{\varphi}_n(\xi)$ is defined in the pointwise sense and is in fact given by

$$\sum_{n=N}^{\infty}\hat{\varphi}_n(\xi) = \begin{cases} 1 & \text{if } |x|_{d,a} \geq 2^N A; \\ \hat{\varphi}_N(\xi) & \text{if } |x|_{d,a} \leq 2^N B. \end{cases}$$

To establish this convergence, let $\psi \in \mathcal{S}(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$ be arbitrary. Then

$$\left\|\psi D^{\alpha}\left[\sum_{n=N}^{\infty}\hat{\varphi}_n\right]\right\|_{\infty} \leq \|\psi D^{\alpha}\hat{\varphi}_N\|_{\infty} \stackrel{N\to\infty}{\longrightarrow} 0,$$

where the limit follows from $\psi \in \mathcal{S}(\mathbb{R}^d)$, $\operatorname{supp} D^{\alpha}\hat{\varphi}_N \subset \{\xi \mid 2^{N-1}A \leq |\xi|_{d,a} \leq 2^N B\}$ and $\|D^{\alpha}\hat{\varphi}_N\|_{\infty} = 2^{(-N+1)a\cdot_d\alpha}\left\|(D^{\alpha}\hat{\varphi}_1)(\delta^{[d,a]}_{2^{-N+1}}\,\cdot)\right\|_{\infty} \leq C$ (for some $C \in]0,\infty[$) for $N \geq 1$. □

The following lemma provides a sufficient condition for the convergence of series in $\mathcal{S}'(\mathbb{R}^d;X)$ and covers the convergence of $\sum_{n\in\mathbb{Z}} S_n f$ with some extra information on the speed of convergence.

Lemma 5.2.7. *(i) Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^d;X)$ be such that*

$$\operatorname{supp}\hat{f}_0 \subset \{\xi \mid |\xi|_{d,a} \le c\} \quad \text{and} \quad \operatorname{supp}\hat{f}_n \subset \{\xi \mid c^{-1}2^n \le |\xi|_{d,a} \le c2^n\}\ \forall n \ge 1 \tag{5.15}$$

and

$$\|f_n(x)\|_X \le \tilde{c}b^{nN}(1+|x|)^N \qquad (x \in \mathbb{R}^d, n \in \mathbb{N}) \tag{5.16}$$

for some $c, \tilde{c}, b > 1$ and $N \in \mathbb{N}$. Then $\sum_{n\in\mathbb{N}} f_n$ is a convergent series in $\mathcal{S}'(\mathbb{R}^d;X)$; in fact, for every $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $K \in \mathbb{Z}_{>0}$ there is a constant $C_{\phi,K} > 0$ such that

$$\|f_n(\phi)\|_X \le C_{\phi,K}[b^N 2^{-(N+K)}]^n \qquad (n \in \mathbb{N}). \tag{5.17}$$

(ii) For every $f \in \mathcal{S}'(\mathbb{R}^d;X)$, the sequence $(f_n)_{n\in\mathbb{N}} = (S_n f)_{n\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^d;X)$ satisfies the hypotheses of (i).

Proof. (i) It is enough to establish convergence of the series $\sum_{n\in\mathbb{N}} f_n$ in the topology of pointwise convergence on $\mathcal{S}'(\mathbb{R}^d;X)$; see Proposition C.2.1. Pick a $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\hat{\psi}(\xi) \equiv 1$ for $c^{-1} \le |\xi|_{d,a} \le c$ and $\operatorname{supp}\hat{\psi} \subset \{\xi \in \mathbb{R}^d : |\xi|_{d,a} \ge \frac{1}{2}c^{-1}\}$, and define $\psi_n \in \mathcal{S}(\mathbb{R}^d)$ via $\hat{\psi}_n := \hat{\psi}(\delta^{[d,a]}_{2^n}\,\cdot)$. Now let $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then we have

$$f_n(\phi) = f_n(\psi_n * \phi) = \int_{\mathbb{R}^d} f_n(x)(\psi_n * \phi)(x)dx,$$

so that

$$\begin{aligned}
\|\langle f_n, \phi\rangle\|_X &\le \int_{\mathbb{R}^d} \|f_n(x)\|_X\, |(\psi_n * \phi)(x)|dx \\
&\le \left\|(1+|\cdot|)^{-(N+d)}\|f_n\|_X\right\|_2 \left\|(1+|\cdot|)^{N+d}\psi_n * \phi\right\|_2 \\
&\overset{(5.16)}{\le} \tilde{c}b^{nN}\left\|(1+|\cdot|)^{-d}\right\|_2 \left\|(1+|\cdot|)^{N+d}\psi_n * \phi\right\|_2 \\
&\le C_1 b^{nN}\left\|(1+|\cdot|)^{N+d}\mathscr{F}^{-1}[\hat{\psi}_n\hat{\phi}]\right\|_2.
\end{aligned} \tag{5.18}$$

Using that

$$(1+|x|)^{N+d} \le (1+|x|_1)^{N+d} = \sum_{|\alpha|\le N+d} c_\alpha |x^\alpha| \qquad (x \in \mathbb{R}^d)$$

for some $\{c_\alpha : \alpha \in \mathbb{N}^d, |\alpha| \le N+d\} \subset \mathbb{N}$, $x^\alpha \circ \mathscr{F}^{-1} = (-1)^{|\alpha|}\mathscr{F}^{-1} \circ D^\alpha$, the Plancherel theorem, the Leibniz rule, $\operatorname{supp}\hat{\psi}_n \subset \{\xi : |\xi|_{d,a} \ge c^{-1}2^{n-1}\}$, and $\hat{\phi} \in \mathcal{S}(\mathbb{R}^d)$, we can estimate, for every $k \in \mathbb{Z}_{>0}$,

$$\begin{aligned}
&\left\|(1+|\cdot|)^{N+d}\mathscr{F}^{-1}[\hat{\psi}_n\hat{\phi}]\right\|_2 \\
&\quad\le \sum_{|\alpha|\le N+d} c_\alpha \left\|x^\alpha \mathscr{F}^{-1}[\hat{\psi}_n\hat{\phi}]\right\|_2 \\
&\quad= \sum_{|\alpha|\le N+d} c_\alpha \left\|D^\alpha[\hat{\psi}_n\hat{\phi}]\right\|_2 \\
&\quad\le C_2 \sum_{|\beta|+|\gamma|\le N+d} \left\|(D^\beta\hat{\psi}_n)(D^\gamma\hat{\phi})\right\|_2 \\
&\quad\le C_2 \sum_{|\beta|+|\gamma|\le N+d} \left\|D^\beta\hat{\psi}_n\right\|_\infty \left\||\cdot|_{d,a}^{k+\frac{1}{2}a\cdot d} D^\gamma\hat{\phi}\right\|_\infty \left\|1_{\{\xi \mid |\xi|_{d,a}\ge c^{-1}2^{n-1}\}}|\cdot|_{d,a}^{-(k+\frac{1}{2}a\cdot d)}\right\|_2 \\
&\quad= C_2 \sum_{|\beta|+|\gamma|\le N+d} 2^{-n(a\cdot_d\beta)} \left\|D^\beta\hat{\psi}\right\|_\infty \left\||\cdot|_{d,a}^{k+\frac{1}{2}a\cdot d} D^\gamma\hat{\phi}\right\|_\infty \left\|1_{\{\xi \mid |\xi|_{d,a}\ge c^{-1}2^{n-1}\}}|\cdot|_{d,a}^{-(k+\frac{1}{2}a\cdot d)}\right\|_2 \\
&\quad\le \tilde{C}_{\phi,k}\left\|1_{\{\xi \mid |\xi|_{d,a}\ge c^{-1}2^{n-1}\}}|\cdot|_{d,a}^{-(k+\frac{1}{2}a\cdot d)}\right\|_2.
\end{aligned}$$

Via the inequality $\rho_{\mathscr{d},a} \lesssim |\cdot|_{\mathscr{d},a}$ (see Lemma 2.3.2), say $\mu\rho_{\mathscr{d},a} \leq |\cdot|_{\mathscr{d},a}$, and a computation in $(\mathscr{d}, a)$-anisotropic polar coordinates (see Section 2.3, formula (2.15)), this can be further estimated as

$$\begin{aligned} \left\|(1+|\cdot|)^{N+d}\mathscr{F}^{-1}[\hat{\psi}_n\hat{\phi}]\right\|_2 &\leq \tilde{C}_{\phi,k}\mu^{-(k+\frac{1}{2}a\cdot\mathscr{d})}\left\|1_{\{\xi \,|\, \rho_{\mathscr{d},a}(\xi)\geq(\mu c)^{-1}2^{n-1}\}}\,\rho_{\mathscr{d},a}^{-(k+\frac{1}{2}a\cdot\mathscr{d})}\right\|_2 \\ &\leq \tilde{C}_{\phi,k}\mu^{-(k+\frac{1}{2}a\cdot\mathscr{d})}C_3\frac{(\mu c)^k}{\sqrt{2k}}2^{-(n-1)k}. \end{aligned} \tag{5.19}$$

Finally, combining the estimates (5.18) and (5.19), we get that for every $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $K \in \mathbb{Z}_{>0}$ ($K = k - N$, $N > k$) there is a constant $C_{\phi,K} > 0$ for which (5.17) holds. Choosing K such that $b^N 2^{-(N+K)} < 1$, the desired convergence now follows from the sequential completeness of $\mathcal{S}'(\mathbb{R}^d; X)$ with respect to the topology of pointwise convergence.

(ii) From (5.13) it is immediate that $(f_n)_n$ fulfills (5.15). For the growth condition (5.16), we first observe that

$$f_n(x) \overset{\text{(C.11)}}{=} (2\pi)^{-d}\hat{f}_n(e_{\imath x}) = (2\pi)^{-d}[\hat{\varphi}_n\hat{f}](e_{\imath x}) = (2\pi)^{-d}\hat{f}(\hat{\varphi}_n e_{\imath x}), \qquad x \in \mathbb{R}^d.$$

For $\hat{f} \in \mathcal{S}'(\mathbb{R}^d; X)$ there exist a constant $c_N > 0$ and a seminorm p_N on $\mathcal{S}(\mathbb{R}^d)$ from the generating family (C.1) as in (C.3): $\left\|\langle\hat{f},\phi\rangle\right\|_X \leq c_N p_N(\phi)$ for all $\phi \in \mathcal{S}(\mathbb{R}^d)$. We can thus estimate

$$\begin{aligned} \|f_n(x)\|_X &\leq (2\pi)^{-d}c_N p_N(\hat{\varphi}_n e_{\imath x}) \\ &= (2\pi)^{-d}c_N \sup_{|\alpha|\leq N, |\beta|\leq N, \xi\in\mathbb{R}^d} |\xi^\beta D^\alpha(\hat{\varphi}_n e_{\imath x})(\xi)|. \end{aligned}$$

Using $\hat{\varphi}_n = \hat{\varphi}_1(\delta^{[\mathscr{d},a]}_{2^{-n+1}}\,\cdot)$ (for $n \geq 1$), the Leibniz rule, and the chain rule, for $n \geq 1$ we can estimate this as

$$\begin{aligned} \|f_n(x)\|_X &\leq (2\pi)^{-d}c_N \sup_{|\alpha|\leq N, |\beta|\leq N, \xi\in\mathbb{R}^d} |\xi^\beta| \sum_{\gamma\leq\alpha}\binom{\alpha}{\gamma}2^{(-n+1)a\cdot_{\mathscr{d}}\gamma}|(D^\gamma\hat{\varphi}_1)(\delta^{[\mathscr{d},a]}_{2^{-n+1}}\xi)|\,|(\imath x)^{|\alpha-\gamma|}e_{\imath x}(\xi)| \\ &\leq \tilde{c}_1 \sup_{|\gamma|\leq N, |\beta|\leq N, \xi\in\mathbb{R}^d} |\xi^\beta|\,|(D^\gamma\hat{\varphi}_1)(\delta^{[\mathscr{d},a]}_{2^{-n+1}}\xi)|(1+|x|)^N \\ &\leq \tilde{c}_1 \sup_{|\gamma|\leq N, |\beta|\leq N, \zeta\in\mathbb{R}^d} 2^{(n-1)a\cdot_{\mathscr{d}}\beta}|\zeta^\beta|\,|D^\gamma\hat{\varphi}_1(\zeta)|(1+|x|)^N \\ &\leq \tilde{c}_2(2^{|a|_\infty})^{nN}(1+|x|)^N. \end{aligned}$$

For $n = 0$ this estimate can be obtained similarly. □

5.2.1.d Definition Anisotropic Besov and Triebel-Lizorkin Spaces

We first define the weighted anisotropic mixed-norm Besov and Triebel-Lizorkin spaces with respect to a $\varphi \in \Phi^{\mathscr{d},a}(\mathbb{R}^d)$, and then show that they actually do not depend on this chosen φ. For these definitions we need to recall the notation from Notation 3.4.5.

Definition 5.2.8. Let X be a Banach space, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Given $\varphi \in \Phi^{\mathscr{d},a}(\mathbb{R}^d)$ with associated $(S_n^\varphi)_{n\in\mathbb{N}}$ given by (5.14), we define the *weighted anisotropic mixed-norm Besov space* $B^{s,a}_{p,q,\mathscr{d},\varphi}(\mathbb{R}^d, w; X)$ with respect to φ as the space of all $f \in \mathcal{S}'(\mathbb{R}^d; X)$ for which

$$\|f\|_{B^{s,a}_{p,q,\mathscr{d},\varphi}(\mathbb{R}^d,w;X)} := \left\|(2^{ns}S_n^\varphi f)_{n\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})[L^{p,\mathscr{d}}(\mathbb{R}^d,w)](X)} < \infty.$$

Definition 5.2.9. Let X be a Banach space, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Given $\varphi \in \Phi^{d,a}(\mathbb{R}^d)$ with associated $(S_n^\varphi)_{n\in\mathbb{N}}$ given by (5.14), we define the *anisotropic mixed-norm Triebel-Lizorkin space* $F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)$ with respect to φ as the space of all $f \in \mathcal{S}'(\mathbb{R}^d;X)$ for which

$$\|f\|_{F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)} := \left\|(2^{ns}S_n^\varphi f)_{n\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)} < \infty.$$

It is easy to see that $(B^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X), \|\cdot\|_{B^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)})$ and $(F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X), \|\cdot\|_{F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)})$ define normed linear spaces. To show that $B^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)$ (resp. $F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)$) does not depend, as topological vector spaces, on the chosen φ, or equivalently, $B^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)$ (resp. $F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)$) defines the same linear subspaces of $\mathcal{S}'(\mathbb{R}^d;X)$ with equivalent norms, we need the following lemma.

Lemma 5.2.10. *Let X be a Banach space, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. For every $\varphi \in \Phi^{d,a}(\mathbb{R}^d)$ and $c > 1$ there exists a constant $C > 0$ such that, for all $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^d;X)$ with*

$$\operatorname{supp}\hat{f}_0 \subset \{\xi \in \mathbb{R}^d \mid |\xi|_{d,a} \leq c\} \quad \textit{and} \quad \operatorname{supp}\hat{f}_k \subset \{\xi \in \mathbb{R}^d \mid c^{-1}2^k \leq |\xi|_{d,a} \leq c2^k\}\ (k \geq 1) \quad (5.20)$$

and for which we have convergence of $f := \sum_{k\in\mathbb{N}} f_k$ in $\mathcal{S}'(\mathbb{R}^d;X)$, it holds that

$$\begin{aligned} \|f\|_{B^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)} &\leq C\left\|(2^{sk}f_k)_{k\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)}, \\ \|f\|_{F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)} &\leq C\left\|(2^{sk}f_k)_{k\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)}. \end{aligned}$$

Proof. We only treat the Triebel-Lizorkin case, the Besov case being completely similar. Let $(S_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{S}'(\mathbb{R}^d;X))$ be the family of convolution operators associated with φ and put $f_k := 0$ for $k \in \mathbb{Z}_{<0}$. In view of the Fourier supports of the φ_n (5.13) and the Fourier support assumption (5.20), there exists a fixed $h \in \mathbb{N}$ such that $S_n f_k = 0$ for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}\setminus\{n-h,\ldots,n+h\}$. From the convergence $f = \sum_{k\in\mathbb{Z}} f_k$ in $\mathcal{S}'(\mathbb{R}^d;X)$ and $S_n \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^d;X))$ it follows that

$$S_n f = \sum_{k\in\mathbb{Z}} S_n f_k = \sum_{k=n-h}^{n+h} S_n f_k = \sum_{m=-h}^{h} S_n f_{m+n}, \quad \forall n \in \mathbb{N}. \tag{5.21}$$

We pick an $r \in]0,1[^l$ such that $w_j \in A_{p_j/r_j}(\mathbb{R}^{d_j})$ for each $j = 1,\ldots,l$. Then, for each $m \in \{-h,\ldots,h\}$, an application of Proposition 3.4.8 to $(2^{ns}f_{m+n})_{n\in\mathbb{N}}$ yields the existence of a constant $C_1 > 0$ independent of m and $(f_k)_k$ such that

$$\begin{aligned} \|(2^{ns}S_n f_{m+n})_{n\in\mathbb{N}}\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)} &= \left\|(\mathscr{F}^{-1}\hat{\varphi}_n\mathscr{F}[2^{ns}f_{m+n}])_{n\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)} \\ &\leq C_1 \sup_{n\geq 0}\left\|\prod_{j=1}^{l}(1+|\pi_j(\,\cdot\,)|^{d_j/r_j})\mathscr{F}^{-1}[\hat{\varphi}_n(\delta^{[d,a]}_{2^{m+n}}\,\cdot\,)]\right\|_{L^1(\mathbb{R}^d)} \\ &\quad \cdot \left\|(2^{ks}f_{m+k})_{k\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)}. \end{aligned} \tag{5.22}$$

For each $m \in \{-h,\ldots,h\}$ it holds that

$$\mathscr{F}^{-1}[\hat{\varphi}_n(\delta^{[d,a]}_{2^{m+n}}\,\cdot\,)] = \begin{cases} \mathscr{F}^{-1}[\hat{\varphi}_1(\delta^{[d,a]}_{2^{m-1}}\,\cdot\,)] & =: \phi_m \in \mathcal{S}(\mathbb{R}^d) \quad \text{if } n \geq 1; \\ \mathscr{F}^{-1}[\hat{\varphi}_0(\delta^{[d,a]}_{2^{m}}\,\cdot\,)] & =: \psi_m \in \mathcal{S}(\mathbb{R}^d) \quad \text{if } n = 0. \end{cases}$$

Hence

$$\sup_{m\in\{-h,\ldots,h\}}\sup_{n\geq 0}\left\|\prod_{j=1}^{l}(1+|\pi_j(\,\cdot\,)|^{d_j/r_j})\mathscr{F}^{-1}[\hat{\varphi}_n(\delta^{[d,a]}_{2^{m+n}}\,\cdot\,)]\right\|_{L^1(\mathbb{R}^d)}$$
$$=\max_{g\in\{\phi_m,\psi_m\,:\,m\in\{-h,\ldots,h\}\}}\left\|\prod_{j=1}^{l}(1+|\pi_j(\,\cdot\,)|^{d_j/r_j})g\right\|_{L^1(\mathbb{R}^d)} =: C_2\in]0,\infty[.$$

Combining (5.21), (5.22) and (5.23), we obtain

$$\begin{aligned}\|f\|_{F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)} &\leq \sum_{m=-h}^{h}\|(2^{ns}S_nf_{m+n})_{n\in\mathbb{N}}\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)}\\ &\leq C_3\sum_{m=-h}^{h}\left\|(2^{ks}f_{m+k})_{k\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)}\\ &= C_3\sum_{m=-h}^{h}2^{-ms}\left\|(2^{(m+k)s}f_{m+k})_{k\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)}\\ &\leq C_4\left\|(2^{sk}f_k)_{k\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)}.\end{aligned}$$

□

From this lemma (and the Fourier supports of the φ_n (5.13)) the independence of φ is immediate:

Proposition 5.2.11 (Independence of φ)**.** *Let X be a Banach space, $a\in]0,\infty[^l$, $p\in[1,\infty[^l$, $q\in[1,\infty]$, $s\in\mathbb{R}$, and $w\in\prod_{j=1}^{l}A_\infty(\mathbb{R}^{d_j})$.*

(i) $B^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)$ does not depend on the choice of $\varphi\in\Phi^{d,a}(\mathbb{R}^d)$, up to an equivalence of norms.

(ii) $F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)$ does not depend on the choice of $\varphi\in\Phi^{d,a}(\mathbb{R}^d)$, up to an equivalence of norms.

Now we may define the anisotropic mixed-norm weighted Besov and Triebel-Lizorkin spaces:

Definition/Convention 5.2.12. Let X be a Banach space, $a\in]0,\infty[^l$, $p\in[1,\infty[^l$, $q\in[1,\infty]$, $s\in\mathbb{R}$, and $w\in\prod_{j=1}^{l}A_\infty(\mathbb{R}^{d_j})$.

(i) We define the *anisotropic mixed-norm Besov space* $B^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)$ as the locally convex space $B^{s,a}_{p,q,d}(\mathbb{R}^d,w;X) := B^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)$, where $\varphi\in\Phi^{d,a}(\mathbb{R}^d)$ is arbitrary, with the topology generated by the norm $\|\cdot\|_{B^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)}$.

(ii) We define the *anisotropic mixed-norm Triebel-Lizorkin space* $F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)$ as the locally convex space $F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X) := F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)$, where $\varphi\in\Phi^{d,a}(\mathbb{R}^d)$ is arbitrary, with the topology generated by the norm $\|\cdot\|_{F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)}$.

We will always assume that there is a fixed choice of $\varphi \in \Phi^{d,a}(\mathbb{R}^d)$ (which may be specified in specific situations) and view $B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ respectively $F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ as a normed linear space with the norm $\|\cdot\|_{B^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} = \|\cdot\|_{B^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)}$ respectively $\|\cdot\|_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} = \|\cdot\|_{F^{s,a}_{p,q,d,\varphi}(\mathbb{R}^d,w;X)}$.

In case $l = 1$ we drop the d from the notation, and in case $l = 1$ and $a = 1$ we both drop d and a from the notation. In the latter case we have the usual weighted isotropic Besov and Triebel-Lizorkin spaces, $B^s_{p,q}(\mathbb{R}^d, w; X)$ and $F^s_{p,q}(\mathbb{R}^d, w; X)$, respectively.

Remark 5.2.13. Let $p \in [1,\infty[^l$ and $a \in]0,\infty[^l$ be such that $p_{j_0} = p_{j_0+1}$ and $a_{j_0} = a_{j_0+1}$ for some $j_0 \in \{1,\dots,l-1\}$. Furthermore, let $\mathcal{A} \in \{B, F\}$, $s \in \mathbb{R}$, $q \in [1,\infty]$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Define

$$\begin{aligned}
\tilde{d} &:= (d_1,\dots,d_{j_0-1}, d_{j_0} + d_{j_0+1}, d_{j_0+2},\dots,d_l) \in (\mathbb{Z}_{>0})^{l-1},\\
\tilde{a} &:= (a_1,\dots,a_{j_0}, a_{j_0+2},\dots,a_l) \in]0,\infty[^{l-1},\\
\tilde{p} &:= (p_1,\dots,p_{j_0}, p_{j_0+2},\dots,p_l) \in [1,\infty[^{l-1}\\
\tilde{w} &:= (w_1,\dots,w_{j_0-1}, w_{j_0} \otimes w_{j_0+1}, w_{j_0+2},\dots,w_l) \in \textstyle\prod_{j=1}^{l-1} A_\infty(\mathbb{R}^{\tilde{d}_j}).
\end{aligned}$$

Then we have

$$\mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) = \mathcal{A}^{s,\tilde{a}}_{\tilde{p},q,\tilde{d}}(\mathbb{R}^d, \tilde{w}; X).$$

5.2.1.e Basic Properties

Sobolev and Bessel Potential Spaces Recall Definition 5.2.1 of the weighted anisotropic Sobolev space $W^n_{p,d}(\mathbb{R}^d, w; X)$ and Definition 5.2.2 of the weighted anisotropic Bessel potential space $H^{s,a}_{p,d}(\mathbb{R}^d, w; X)$.

Proposition 5.2.14. *(i) Let X be a Banach space, $n \in \mathbb{N}^l$, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. Then $W^n_{p,d}(\mathbb{R}^d, w; X)$ is a Banach space with*

$$\mathcal{S}(\mathbb{R}^d; X) \overset{d}{\hookrightarrow} W^n_{p,d}(\mathbb{R}^d, w; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X). \tag{5.23}$$

As a consequence, it also holds that

$$C^\infty(\mathbb{R}^d; X), \mathscr{F}^{-1}C^\infty(\mathbb{R}^d; X) \overset{d}{\hookrightarrow} W^n_{p,d}(\mathbb{R}^d, w; X).$$

(ii) Let X be a Banach space, $a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. Then $H^{s,a}_{p,d}(\mathbb{R}^d, w; X)$ is a Banach space with

$$\mathcal{S}(\mathbb{R}^d; X) \overset{d}{\hookrightarrow} H^{s,a}_{p,d}(\mathbb{R}^d, w; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X).$$

As a consequence, it also holds that

$$C^\infty(\mathbb{R}^d; X), \mathscr{F}^{-1}C^\infty(\mathbb{R}^d; X) \overset{d}{\hookrightarrow} H^{s,a}_{p,d}(\mathbb{R}^d, w; X).$$

Proof. (i) Completeness of $W^n_{p,d}(\mathbb{R}^d, w; X)$ and the continuous inclusions in (5.23) (without the denseness) can be easily derived from the corresponding assertions for $L^{p,d}(\mathbb{R}^d, w; X)$ (which are stated in Lemma 2.2.4). In the same way as in Lemma 2.1.3, it can be shown that $C_c^\infty(\mathbb{R}^d; X)$ (and thus $\mathcal{S}(\mathbb{R}^d; X)$) is dense in $W^n_{p,d}(\mathbb{R}^d, w; X)$; of course, now we have to use Lemma 3.4.2 instead of Proposition D.2.5.

(ii) From the definition of $H^{s,a}_{p,d}(\mathbb{R}^d, w; X)$ it is clear that the Bessel potential operator $\mathcal{J}^{d,a}_s$ restricts to a topological linear isomorphism from $H^{s,a}_{p,d}(\mathbb{R}^d, w; X)$ to $L^{p,d}(\mathbb{R}^d, w; X)$. Since the Bessel potential operator $\mathcal{J}^{d,a}_s$ also is a topological linear isomorphism on $\mathcal{S}(\mathbb{R}^d; X)$ and $\mathcal{S}'(\mathbb{R}^d; X)$, the desired result follows from the corresponding assertions for $L^{p,d}(\mathbb{R}^d, w; X)$ (which are stated in Lemma 2.2.4).

□

Proposition 5.2.15. *Let X be a UMD space, $a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$, $p \in]1, \infty[^l$, $s \in \mathbb{R}$, and*

$$w \in \begin{cases} \prod_{j=1}^l A^{rec}_{p_j}(\mathbb{R}^{d_j}), & l > 1; \\ \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j}), & l = 1 \text{ or } X \text{ has } (\alpha). \end{cases}$$

Then, for each multi-index $\alpha \in \mathbb{N}^d$, the partial derivative operator $D^\alpha \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^d; X))$ restricts to a bounded linear operator

$$D^\alpha : H^{s,a}_{p,d}(\mathbb{R}^d, w; X) \longrightarrow H^{s-a\cdot_d\alpha,a}_{p,d}(\mathbb{R}^d, w; X).$$

Proof. We just need to check that $\xi \mapsto \xi^\alpha \mathcal{J}^{d,a}_{-k} = \xi^\alpha(1+|\xi|^2_{d,a})^{-k/2}$ is a Mikhlin Fourier multiplier as in Corollary 4.1.2, which is an easy but tedious computation which we omit. □

Remark 5.2.16. We expect that (i) of the above proposition remains true under the assumption $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$, also see Remark 4.1.3.

Triebel-Lizorkin and Besov Spaces Recall the definition of anisotropic weighted Triebel-Lizorkin and Besov spaces, $F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ and $B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$, respectively, in our conventions concerning these spaces; see Definition/Convention 5.2.12.

Proposition 5.2.17. *Let X be a Banach space, $a \in]0, \infty[^l$, $p \in [1, \infty[^l$, $q \in [1, \infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Let $\mathcal{A} \in \{B, F\}$. Then $\mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ is a Banach space with*

$$\mathcal{S}(\mathbb{R}^d; X) \hookrightarrow \mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X). \tag{5.24}$$

Moreover, if $q < \infty$, then the following spaces are dense in $\mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$: $C^\infty_c(\mathbb{R}^d; X)$, $\mathcal{S}(\mathbb{R}^d; X)$, $\mathscr{F}^{-1}C^\infty_c(\mathbb{R}^d; X)$, $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d; X) \cap \mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$.

We will prove the completeness part in the proposition via the Fatou property, which in case $q = \infty$ also serves a useful substitute for the denseness of the Schwartz space $\mathcal{S}(\mathbb{R}^d; X)$. A normed space $E \subset \mathcal{D}'(\mathbb{R}^d; X)$ is said to have the *Fatou property* if for all $(f_n)_{n\in\mathbb{N}} \subset E$ the following implication holds:

$$\lim_{n\to\infty} f_n = f \text{ in } \mathcal{D}'(\mathbb{R}^d; X),\ \liminf_{n\to\infty} \|f_n\|_E < \infty \implies f \in E,\ \|f\|_E \leq \liminf_{n\to\infty} \|f_n\|_E\,.$$

Proposition 5.2.18. *Let X be a Banach space, $a \in]0, \infty[^l$, $p \in [1, \infty[^l$, $q \in [1, \infty]$, $s \in \mathbb{R}$ and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Then $B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ and $F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ have the Fatou property.*

Proof. This can be shown using Fatou's lemma (from measure theory). For a detailed proof of this in the unweighted isotropic case, which can be extended to our situation, we refer to [88, Proposition 4]. □

Lemma 5.2.19. *Let X be a Banach space. Every normed space $E \hookrightarrow \mathcal{D}'(\mathbb{R}^d;X)$ with the Fatou property is a Banach space.*

Proof. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E. Then, on the one hand, $\liminf_{n\to\infty}\|f_n\|_E \leq \sup_n \|f_n\|_E < \infty$. On the other hand, $(f_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in the sequentially complete space $\mathcal{D}'(\mathbb{R}^d;X)$ because of $E \hookrightarrow \mathcal{D}'(\mathbb{R}^d;X)$, whence converges to some f in $\mathcal{D}'(\mathbb{R}^d;X)$. By the Fatou property of E, $f \in E$. To finish the proof we show that we also have convergence $f_n \overset{n\to\infty}{\longrightarrow} f$ in the norm topology of E. To this end, let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\|f_l - f_k\| \leq \epsilon$ for all $l, k \geq N$. Then, for all $k \geq N$, it holds that $f_l - f_k \in E$, $\liminf_{l\to\infty}\|f_l - f_k\|_E \leq \epsilon$ and $f_l - f_k \overset{l\to\infty}{\longrightarrow} f - f_k$ in $\mathcal{D}'(\mathbb{R}^d;X)$. So applying, for each $k \geq N$, the Fatou property of E to the sequence of differences $(f_l - f_k)_{l\in\mathbb{N}}$ we obtain that $\|f - f_k\|_E \leq \epsilon$ for all $k \geq N$. □

Proof of Proposition 5.2.17. We start with the first inclusion in (5.24). In view of the elementary embedding $B^{s+1,a}_{p,\infty,d}(\mathbb{R}^d,w;X) \hookrightarrow B^{s,a}_{p,1,d}(\mathbb{R}^d,w;X) \hookrightarrow \mathscr{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)$ (see Proposition 5.2.30.(ii)&(iii), which is proven directly from the definition of Besov and Triebel-Lizorkin spaces; so no circularities occur in the argumentation), we may without loss of generality assume that $\mathscr{A} = B$ and $q = \infty$. Let $f \in \mathcal{S}(\mathbb{R}^d;X)$. Then

$$\begin{aligned}\|f\|_{B^{s,a}_{p,\infty,d}(\mathbb{R}^d,w;X)} &= \sup_{k\in\mathbb{N}} 2^{ks}\|S_k f\|_{L^{p,d}(\mathbb{R}^d,w;X)} \\ &\overset{(2.11)}{\leq} C_1 \sup_{k\in\mathbb{N}} 2^{ks}\sum_{|\alpha|\leq L}\|x \mapsto x^\alpha S_k f(x)\|_{L^\infty(\mathbb{R}^d;X)}\end{aligned}$$

for some $L \in \mathbb{N}$ and $C_1 > 0$ independent of f; see Lemma 2.2.3. Using $x^\alpha \circ \mathscr{F}^{-1} = (-1)^{|\alpha|}\mathscr{F}^{-1}\circ D^\alpha$ and the boundedness of $\mathscr{F}^{-1}$ from $L^1(\mathbb{R}^d;X)$ to $L^\infty(\mathbb{R}^d;X)$, we can estimate this as

$$\begin{aligned}\|f\|_{B^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} &\leq C_1 \sup_{k\in\mathbb{N}} 2^{ks}\sum_{|\alpha|\leq L}\left\|\mathscr{F}^{-1}(D^\alpha\mathscr{F}[S_k f])\right\|_{L^\infty(\mathbb{R}^d;X)} \\ &\leq C_2 \sup_{k\in\mathbb{N}} 2^{ks}\sum_{|\alpha|\leq L}\left\|D^\alpha[\hat{\varphi}_k\hat{f}]\right\|_{L^1(\mathbb{R}^d;X)}. \end{aligned} \tag{5.25}$$

Using $\hat{\varphi}_k = \hat{\varphi}_1(\delta^{[d,a]}_{2^{-k+1}}\,\cdot\,)$ (for $k \geq 1$), the Leibniz rule, the chain rule, and $\varphi_1 \in \mathcal{S}(\mathbb{R}^d)$, we can estimate, for all $k \geq 1$ and $|\alpha| \leq L$,

$$\left\|D^\alpha[\hat{\varphi}_k\hat{f}]\right\|_X \leq C_3\sum_{\beta\leq\alpha} 1_{\operatorname{supp}\hat{\varphi}_k}\left\|D^{\alpha-\beta}\hat{f}\right\|_X \leq C_4\sup_{|\beta|\leq L} 1_{\{\xi\,|\,2^{k-1}A\leq|\xi|_{d,a}\leq 2^k B\}}\left\|D^\beta\hat{f}\right\|_X.$$

Similarly we have, for $k = 0$ and $|\alpha| \leq L$,

$$\left\|D^\alpha[\hat{\varphi}_k\hat{f}]\right\|_X \leq C_5\sup_{|\beta|\leq L}\left\|D^\beta\hat{f}\right\|_X.$$

Combining these two estimates with (5.25), we obtain

$$\begin{aligned}\|f\|_{B^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} \leq\ & C_6\sup_{|\beta|\leq L}\left\|D^\beta\hat{f}\right\|_{L^1(\mathbb{R}^d;X)} \\ & + \sup_{k\geq 1} 2^{ks}\left\|(1+|\cdot|_{d,a})^{-M}1_{\{\xi\,|\,2^{k-1}A\leq|\xi|_{d,a}\leq 2^k B\}}\right\|_{L^\infty(\mathbb{R}^d)}\sup_{|\beta|\leq L}\left\|(1+|\cdot|_{d,a})^M D^\beta\hat{f}\right\|_{L^1(\mathbb{R}^d;X)}\end{aligned}$$

for all $M \in \mathbb{N}$. Since $\mathcal{S}(\mathbb{R}^d;X) \hookrightarrow L^1(\mathbb{R}^d;X)$, since $(1+|\cdot|_{d,a})^M$ is of polynomial growth, and since $\mathscr{F}$ is a continuous linear operator on $\mathcal{S}(\mathbb{R}^d;X)$, this can in turn be further estimated as

$$\|f\|_{B^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} \leq c_M p_{N_M}(f)\left(1+\sup_{k\geq 1} 2^{ks}(1+2^{-k+1}A)^{-M}\right)$$

for some $c_M > 0$ and $N_M \in \mathbb{N}$ depending on M; here p_{N_M} is a seminorm on $\mathcal{S}(\mathbb{R}^d;X)$ from the generating family (C.1). Choosing $M \in \mathbb{N}$ big enough now gives the desired estimate.

We now continue with the second inclusion in (5.24). We only need to establish the continuity of this inclusion with respect to the topology of pointwise convergence on $\mathcal{S}'(\mathbb{R}^d;X)$; see Proposition C.2.1. For simplicity we only treat the case $p \in]1,\infty[^l$ and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$[2]; for general $p \in [1,\infty[^l$ and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$ a proof can be found in [52] for the case that $l=1$, $a=1$, and $X=\mathbb{C}$, which we expect to extend to our situation. So suppose that $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. We may again assume without loss of generality that $q=\infty$ and $\mathcal{A}=B$. Let's pick an $h \in \mathbb{N}$ such that $S_{k+n}S_k = 0$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{Z}$ with $|n| > h$; here we write $S_k := 0$ for $k<0$. Now let $f \in B^{s,a}_{p,\infty,d}(\mathbb{R}^d,w;X)$ and $\phi \in \mathcal{S}(\mathbb{R}^d)$ be given. From the convergence $f=\sum_{k=0}^\infty S_kf$ in $\mathcal{S}'(\mathbb{R}^d;X)$ it then follows that

$$f(\phi) = \sum_{k=0}^\infty S_kf(\phi) = \sum_{k=0}^\infty [\sum_{n=-h}^h S_{k+n}S_kf](\phi) = \sum_{n=-h}^h\sum_{k=0}^\infty S_kf(S_{k+n}\phi).$$

Denoting by $p'=(p_1',\ldots,p_l') \in]1,\infty[^l$ the vector of Hölder conjugates and by $w'=(w_1^{-\frac{1}{p_1-1}},\ldots,w_l^{-\frac{1}{p_l-1}}) \in \prod_{j=1}^l A_{p_j'}(\mathbb{R}^{d_j})$ the p-dual weight vector, we obtain

$$\begin{aligned}
\|f(\phi)\|_X &\overset{(C.10)}{\leq} \sum_{n=-h}^h\sum_{k=0}^\infty \int_{\mathbb{R}^d} \|S_kf(x)\|_X\,|\phi(x)|\,dx \\
&\leq \sum_{n=-h}^h\sum_{k=0}^\infty \|S_kf\|_{L^{p,d}(\mathbb{R}^d,w;X)}\,\|S_{k+n}\phi\|_{L^{p',d}(\mathbb{R}^d,w';X)} \\
&\leq \sum_{n=-h}^h 2^{ns}\left\|(2^{ks}\|S_kf\|_{L^{p,d}(\mathbb{R}^d,w;X)})_{k\in\mathbb{N}}\right\|_{\ell^\infty(\mathbb{N})}\left\|(2^{-(k+n)s}\|S_{k+n}\phi\|_{L^{p',d}(\mathbb{R}^d,w';X)})_{k\in\mathbb{N}}\right\|_{\ell^1(\mathbb{N})} \\
&\leq \sum_{n=-h}^h 2^{ns}\,\|f\|_{B^{s,a}_{p,\infty,d}(\mathbb{R}^d,w;X)}\,\|\phi\|_{B^{-s,a}_{p',1,d}(\mathbb{R}^d,w')} \\
&\leq C_\phi\,\|f\|_{B^{s,a}_{p,\infty,d}(\mathbb{R}^d,w;X)};
\end{aligned}$$

here we used $\mathcal{S}(\mathbb{R}^d) \hookrightarrow B^{-s,a}_{p',1,d}(\mathbb{R}^d,w')$ in the last step.

That $\mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)$ is a Banach space now follows from a combination of the second inclusion in (5.24), Proposition 5.2.18, and Lemma 5.2.19.

Finally, we must establish the denseness of the listed spaces in the case $q<\infty$. Let us first treat the subspace $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d;X) \cap \mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)$. For this we fix an $f \in \mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)$. From Lemma 5.2.10 (and the hypothesis $q<\infty$ in combination with the Lebesgue dominated

[2]This is the case needed for applications in Chapter 6 anyway.

convergence theorem) it then follows that $(\sum_{k=0}^{K} S_K f)$ is a Cauchy sequence in the Banach space $\mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X)$. Since $f = \sum_{k=0}^{\infty} S_k f$ in $\mathcal{S}'(\mathbb{R}^d; X)$, it thus follows that

$$f = \lim_{K \to \infty} \sum_{k=0}^{K} S_k f \quad \text{in} \quad \mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X),$$

which proves the desired denseness. In view of

$$C_c^\infty(\mathbb{R}^d; X) = \mathcal{D}(\mathbb{R}^d; X), \mathscr{F}^{-1}C_c^\infty(\mathbb{R}^d; X) = \mathscr{F}^{-1}\mathcal{D}(\mathbb{R}^d; X) \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^d; X),$$

to finish it suffices to approximate an $f \in \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d; X) \cap \mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ by a sequence of Schwartz functions (in the norm of $\mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$). To this end we fix an $f \in \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d; X) \cap \mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$, say $f = \sum_{k=0}^{K} S_k f$, $K \in \mathbb{N}$. Picking $\eta \in \mathscr{F}^{-1}C_c^\infty(\mathbb{R}^d)$ with $\eta(0) = 1$ and with small enough support and using Lemma 5.2.10, we can find a constant $C > 0$ independent of $r \in]0, 1]$ such that

$$\left\| f - \eta(\delta_r^{[d,a]}(\,\cdot\,) f \right\|_{\mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} \leq C \left\| f - \eta(\delta_r^{[d,a]}(\,\cdot\,) f \right\|_{L^{p,d}(\mathbb{R}^d,w;X)}, \qquad r \in]0, 1];$$

for more details we refer to the proof of [77, Lemma 3.8] and the reference given therein. Since the RHS tends to 0 as $r \to 0$, and since $\eta(\delta_r^{[d,a]}(\,\cdot\,) f \in \mathcal{S}(\mathbb{R}^d; X)$ as a consequence of the Paley-Wiener-Schwartz theorem (cf. TheoremC.6.4), the desired denseness result follows. □

Proposition 5.2.20. *Let X be a Banach space, $a \in]0, \infty[^l$, $p \in]1, \infty[^l$, $q \in [1, \infty[$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$. Suppose that $\phi \in C_c^\infty(\mathbb{R}^d)$ is such that $\phi \geq 0$ and $\int_{\mathbb{R}^d} \phi(x)dx = 1$. Let $\mathcal{A} \in \{B, F\}$. For $f \in \mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ we have $\phi_t * f \overset{t\to\infty}{\longrightarrow} f$ in $\mathcal{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$, where ϕ_t is as in* (D.2)

Proof. We first consider the Besov case $\mathcal{A} = B$. Let's fix an $f \in B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ and choose a sequence $(t_n)_{n\in\mathbb{N}} \subset]0, \infty[$ satisfying $t_n \nearrow \infty$ (as $n \to \infty$). We need to show that

$$(\phi_t * 2^{sk} S_k f)_{k\in\mathbb{N}} = (2^{sk} S_k(\phi_t * f))_{k\in\mathbb{N}} \overset{t\to\infty}{\longrightarrow} (2^{sk} S_k f)_{k\in\mathbb{N}} \qquad \text{in} \quad \ell^q(\mathbb{N}; L^{p,d}(\mathbb{R}^d, w; X)). \quad (5.26)$$

From Lemma 3.4.2 we know that $\phi_t * 2^{sk} S_k f \overset{t\to\infty}{\longrightarrow} 2^{sk} S_k f$ in $L^{p,d}(\mathbb{R}^d, w; X)$ for every $k \in \mathbb{N}$. Since $\left\|\phi_t * 2^{sk} S_k f\right\|_{L^{p,d}(\mathbb{R}^d,w;X)} \lesssim \left\|2^{sk} S_k f\right\|_{L^{p,d}(\mathbb{R}^d,w;X)}$ for every $k \in \mathbb{N}$ by Lemma 3.4.1, while of course $(2^{sk} S_k f)_{k\in\mathbb{N}} \in \ell^q(\mathbb{N}; L^{p,d}(\mathbb{R}^d, w; X))$, the required convergence (5.26) follows from the Lebesgue dominated convergence theorem.

Next we consider the Triebel-Lizorkin case $\mathcal{A} = F$. Given $f \in F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$, we need to show that

$$(\phi_t * 2^{sk} S_k f)_{k\in\mathbb{N}} = (2^{sk} S_k(\phi_t * f))_{k\in\mathbb{N}} \overset{t\to\infty}{\longrightarrow} (2^{sk} S_k f)_{k\in\mathbb{N}} \qquad \text{in} \quad L^{p,d}(\mathbb{R}^d, w)[\ell^q(\mathbb{N})](X). \quad (5.27)$$

To this end we view $(2^{sk} S_k f)_{k\in\mathbb{N}} \in L^{p,d}(\mathbb{R}^d, w)[\ell^q(\mathbb{N})](X)$ as a strongly measurable function $F : \mathbb{R}^d \longrightarrow \ell^q(\mathbb{N}; X)$ belonging to $L^{p,d}(\mathbb{R}^d, w; \ell^q(\mathbb{N}; X))$; see Remark 3.4.6.[3] By Lemma 3.4.2, $\phi_t * F \overset{t\to\infty}{\longrightarrow} F$ in $L^{p,d}(\mathbb{R}^d, w; \ell^q(\mathbb{N}; X))$. The convolution $\phi_t * F$ corresponding to $(\phi_t * 2^{sk} S_k f)_{k\in\mathbb{N}}$ under the identification $L^{p,d}(\mathbb{R}^d, w; \ell^q(\mathbb{N}; X)) = L^{p,d}(\mathbb{R}^d, w)[\ell^q(\mathbb{N})](X)$, this proves the required convergence (5.27). □

[3]Here we really use $q \in [1, \infty[$.

Proposition 5.2.21. *Let X be a Banach space, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$ and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Let $\mathscr{A} \in \{B,F\}$. Then, for each multi-index $\alpha \in \mathbb{N}^d$, the partial derivative operator $D^\alpha \in \mathcal{S}'(\mathbb{R}^d;X)$ restricts to a bounded linear operator*

$$D^\alpha : \mathscr{A}^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X) \longrightarrow \mathscr{A}^{s-a\cdot_{\mathpzc{d}}\alpha,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)$$

Proof. We only treat the Triebel-Lizorkin case $\mathscr{A} = F$, the Besov case being completely similar. Writing $\varphi_{-1} := 0$, we define $(M_k)_k \in \mathcal{S}(\mathbb{R}^d)$ by $M_k := \hat{\varphi}_{k-1} + \hat{\varphi}_k + \hat{\varphi}_{k+1}$. Then we have $\hat{\varphi}_k = M_k\hat{\varphi}_k$ by construction. Hence

$$D^\alpha f = D^\alpha \sum_{k=0}^\infty S_k f = \sum_{k=0}^\infty D^\alpha \mathscr{F}^{-1}\mathscr{F}[S_k f] = \sum_{k=0}^\infty \mathscr{F}^{-1}\xi^\alpha M_k \mathscr{F}[S_k f] \quad \text{in } \mathcal{S}'(\mathbb{R}^d;X).$$

Since $\xi^\alpha M_k\mathscr{F}[S_k f]$ has support contained in $\operatorname{supp}\mathscr{F}[S_k f] \subset \operatorname{supp}\hat{\varphi}_k$, and since the φ_k satisfy the Fourier support condition (5.13), it follows from Lemma 5.2.10 that

$$\begin{aligned} \|D^\alpha f\|_{F^{s-a\cdot_{\mathpzc{d}}\alpha,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)} &\lesssim \left\| (2^{(s-a\cdot_{\mathpzc{d}}\alpha)k}\mathscr{F}^{-1}\xi^\alpha M_k\mathscr{F}[S_k f])_{k\in\mathbb{N}} \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)} \\ &= \left\| (\mathscr{F}^{-1}(2^{-(a\cdot_{\mathpzc{d}}\alpha)k}\xi^\alpha M_k)\mathscr{F}[2^{sk}S_k f])_{k\in\mathbb{N}} \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)}. \end{aligned} \tag{5.28}$$

Since

$$[2^{-(a\cdot_{\mathpzc{d}}\alpha)k}\xi^\alpha M_k](\delta^{[\mathpzc{d},a]}_{2^k}\,\cdot\,) = \phi, \qquad k \geq 3,$$

for some $\phi \in \mathcal{S}(\mathbb{R}^d)$, it follows from Proposition 3.4.8 that

$$\left\| (\mathscr{F}^{-1}(2^{-(a\cdot_{\mathpzc{d}}\alpha)k}\xi^\alpha M_k)\mathscr{F}[2^{sk}S_k f])_{k\in\mathbb{N}} \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)} \lesssim \left\| (2^{sk}S_k f)_{k\in\mathbb{N}} \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)}. \tag{5.29}$$

Combining (5.28) and (5.29), we obtain

$$\|D^\alpha f\|_{F^{s-a\cdot_{\mathpzc{d}}\alpha,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)} \lesssim \|f\|_{F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)}.$$

□

5.2.1.f Convergence Criteria for Series

We now present some technical lemmas giving sufficient conditions for convergence of series in $\mathcal{S}'(\mathbb{R}^d;X)$ plus norm estimations in weighted anisotropic Triebel-Lizorkin and Besov spaces. In each of these results we impose some kind of Fourier support condition on the given series.

Lemma 5.2.22. *Let X be a Banach space, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s > 0$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Suppose that there exists an $r \in]0,1[^l$ such that $s > \sum_{j=1}^l a_j d_j(\frac{1}{r_j} - 1)$ and $w \in \prod_{j=1}^l A_{p_j/r_j}(\mathbb{R}^{d_j})$. Then, for every $c > 0$, there exists a constant $C > 0$ such that, for all $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^d;X)$ satisfying $\operatorname{supp}\hat{f}_k \subset \prod_{j=1}^l[-c2^{ka_j}, c2^{ka_j}]^{d_j}$ and*

$$\mathfrak{B} := \left\| (2^{ks}f_k)_{k\geq 0} \right\|_{\ell^q(\mathbb{N})[L^{p,\mathpzc{d}}(\mathbb{R}^d,w)](X)} < \infty,$$

it holds that $\sum_{k\in\mathbb{N}} f_k$ defines a convergent series in $\mathcal{S}'(\mathbb{R}^d;X)$ with limit $f \in B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)$ of norm $\leq C\mathfrak{B}$.

Proof. We write $T_n := 2^{na\cdot d}\varphi_0(\delta^{[d,a]}_{2^n}\,\cdot)*$ for each $n \in \mathbb{N}$; then $S_0 = T_0$ and $S_n = T_n - T_{n+1}$ for $n \geq 1$. Furthermore, we put $f_k := 0$ for each $k \in \mathbb{Z}_{<0}$. In view of the support condition $\operatorname{supp}\hat{f}_k \subset \prod_{j=1}^{l}[-c2^{ka_j}, -c2^{ka_j}]^{d_j}$ for $k \geq 0$, there exists a fixed $h \in \mathbb{N}$ such that $S_n f_k = 0$ for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}_{<n-h}$. As a consequence,

$$S_n \sum_{k=0}^{K} f_k = \sum_{k=n-h}^{K} S_n f_k = \sum_{l=-h}^{K-n} S_n f_{l+n},$$

implying that

$$\begin{aligned}\left\| \sum_{k=0}^{K} f_k \right\|_{B^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} &\leq \sum_{l=-h}^{K} \|(2^{ns}S_n f_{l+n})_{n\geq 0}\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} \\ &\leq \sum_{l=-h}^{K} \left(\|(2^{ns}T_n f_{l+n})_{n\geq 0}\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} + \|(2^{ns}T_{n+1} f_{l+n})_{n\geq 1}\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} \right).\end{aligned}$$

Applying Proposition 3.4.9 with $R = 2^{l_+}$, we can estimate

$$\begin{aligned}\left\| \sum_{k=0}^{K} f_k \right\|_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} &\leq C_1 \sum_{l=-h}^{K} 2^{l_+ \sum_{j=1}^{m} a_j d_j (\frac{1}{r_j}-1)} \|(2^{ns} f_{l+n})_{n\geq 0}\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} \\ &\leq C_2 \sum_{l=-h}^{K} 2^{l_+(-s+\sum_{j=1}^{m} a_j d_j (\frac{1}{r_j}-1))} \left\|(2^{(l+n)s} f_{l+n})_{n\geq 0}\right\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} \\ &\leq C_3 \mathfrak{B};\end{aligned}$$

here we used $s > \sum_{j=1}^{m} a_j d_j(\frac{1}{r_j} - 1)$ for the last estimate. This shows that $(\sum_{k=0}^{K} f_k)_{K\in\mathbb{N}}$ is a sequence in $B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ which is bounded by $C_3\mathfrak{B}$. In light of the Fatou property of $B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ (cf. Proposition 5.2.18), it remains to be shown that $\sum_{k\in\mathbb{N}} f_k$ is a convergent series in $\mathcal{S}'(\mathbb{R}^d; X)$.

To finish, we establish the convergence of $\sum_{k\in\mathbb{N}} f_k$ in $\mathcal{S}'(\mathbb{R}^d; X)$. To this end we fix an $\tilde{s} \in]\sum_{j=1}^{l} a_j d_j(\frac{1}{r_j} - 1), s[$. By Proposition 5.2.17, it is enough that $(\sum_{k=0}^{K} f_k)_{K\in\mathbb{N}}$ is a Cauchy sequence in $B^{\tilde{s},a}_{p,1,d}(\mathbb{R}^d, w; X)$. So let $\epsilon > 0$. We observe that

$$\left\|(2^{k\tilde{s}} f_k)_{k\geq 0}\right\|_{\ell^1(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} \leq C_4 \mathfrak{B} < \infty, \tag{5.30}$$

which can be shown as in the proof of Proposition 5.2.30(ii). For each $\kappa \in \mathbb{N}$ we define the sequence $(f^\kappa_k)_{k\in\mathbb{Z}}$ by $f^\kappa_k := f_k$ for $k \geq \kappa$ and $f^\kappa_k := 0$ for $k < \kappa$. In this notation we can derive, as in the first part of the proof, that

$$\begin{aligned}\left\| \sum_{k=\kappa}^{K} f_k \right\|_{B^{\tilde{s},a}_{p,1,d}(\mathbb{R}^d,w;X)} &= \left\| \sum_{k=0}^{K} f^\kappa_k \right\|_{B^{\tilde{s},a}_{p,1,d}(\mathbb{R}^d,w;X)} \\ &\leq C_5 \sum_{l=-h}^{K} 2^{\ell_+(-\tilde{s}+\sum_{j=1}^{l} a_j d_j (\frac{1}{r_j}-1))} \left\|2^{(l+n)\tilde{s}} f^\kappa_{l+n})_{n\geq 0}\right\|_{\ell^1(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)}.\end{aligned} \tag{5.31}$$

Now we first choose $L \in \mathbb{N}$ such that

$$C_5 \sum_{l=L+1}^{\infty} 2^{\ell_+(-\tilde{s}+\sum_{j=1}^{l} a_j d_j(\frac{1}{r_j}-1))} < \frac{\epsilon}{2(C_4\mathfrak{B}+1)}, \tag{5.32}$$

and next pick $\kappa_0 \in \mathbb{N}$ such that, for all $\kappa \geq \kappa_0$,

$$C_5 \sum_{l=-h}^{L} 2^{\ell_+(-\tilde{s}+\sum_{j=1}^{l} a_j d_j(\frac{1}{r_j}-1))} \left\| 2^{(l+n)\tilde{s}} f^{\kappa}_{l+n})_{n\geq 0} \right\|_{\ell^1(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} < \frac{\epsilon}{2}; \tag{5.33}$$

note that the existence of L and κ_0 are assured by $\tilde{s} > \sum_{j=1}^{l} a_j d_j(\frac{1}{r_j} - 1)$ and (5.30), respectively. Finally, combining (5.31), (5.33), (5.32), and (5.30), we obtain, for all $K \geq \kappa \geq \kappa_0$,

$$\left\| \sum_{k=\kappa}^{K} f_k \right\|_{B^{\tilde{s},a}_{p,1,d}(\mathbb{R}^d,w;X)} < \frac{\epsilon}{2} + \frac{\epsilon}{2(C_4\mathfrak{B}+1)} C_4\mathfrak{B} < \epsilon.$$

□

Lemma 5.2.23. *Let X be a Banach space, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s > 0$, and $w \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{d_j})$. Suppose that there exists an $r \in]0,1[^l$ such that $s > \sum_{j=1}^{l} a_j d_j(\frac{1}{r_j} - 1)$ and $w \in \prod_{j=1}^{l} A_{p_j/r_j}(\mathbb{R}^{d_j})$. Then, for every $c > 0$, there exists a constant $C > 0$ such that, for all $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^d;X)$ satisfying* $\operatorname{supp} \hat{f}_k \subset \prod_{j=1}^{l}[-c2^{ka_j}, -c2^{ka_j}]^{d_j}$ *and*

$$F := \left\| (2^{ks} f_k)_{k\geq 0} \right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)} < \infty,$$

it holds that $\sum_{k\in\mathbb{N}} f_k$ defines a convergent series in $\mathcal{S}'(\mathbb{R}^d;X)$ with limit $f \in F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)$ of norm $\leq CF$.

Proof. The proof goes completely analogous to the proof of Lemma 5.2.22. □

Lemma 5.2.24. *Let X be a Banach space, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{d_j})$. For every $\lambda > 0$ and $c > 1$ there exists a constant $C > 0$ such that, for all $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^d;X)$ satisfying*

$$\operatorname{supp} \hat{f}_0 \subset \{\xi \in \mathbb{R}^d \mid |\xi|_{d,a} \leq c\}, \qquad \operatorname{supp} \hat{f}_k \subset \{\xi \in \mathbb{R}^d \mid c^{-1}2^{\lambda k} \leq |\xi|_{d,a} \leq c2^{\lambda k}\} \;\; (k \geq 1), \tag{5.34}$$

and

$$\mathfrak{B} := \left\| (2^{\lambda sk} f_k)_{k\geq 0} \right\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} < \infty,$$

it holds that $\sum_{k\in\mathbb{N}} f_k$ defines a convergent series in $\mathcal{S}'(\mathbb{R}^d;X)$ with limit $f \in B^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)$ of norm $\leq C\mathfrak{B}$.

Proof. We claim that there exists a constant $C > 0$ independent of $(f_k)_{k\in\mathbb{N}}$ such that

$$\left\| \sum_{k=0}^{K} f_k \right\|_{B^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} \leq C \left\| (2^{\lambda sk} f_k)_{k\geq 0} \right\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)}, \qquad K \in \mathbb{N}. \tag{5.35}$$

By the Fatou property of $B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ (cf. Proposition 5.2.18), it then remains to be shown that $\sum_{k\in\mathbb{N}} f_k$ defines a convergent series in $\mathcal{S}'(\mathbb{R}^d; X)$. This can be shown via the space $B^{\tilde{s},a}_{p,1,d}(\mathbb{R}^d, w; X)$, $\tilde{s} < s$, in the same spirit as in the end of the proof of Lemma 5.2.22 (but a lot easier).

To finish the proof we must establish the claim (5.35): For convenience of notation we put $f_k := 0$ for $k \in \mathbb{Z}_{<0}$. Then, in view of the Fourier support condition (5.34) and the Fourier supports of the φ_n (5.13), there exists a $h \in \mathbb{N}$ such that $S_n f = 0$ for all $n \in \mathbb{N}$ and $k \in \mathbb{Z} \setminus \left\{\left\lfloor \frac{n}{\lambda} \right\rfloor - h, \ldots, \left\lfloor \frac{n}{\lambda} \right\rfloor + h\right\}$; here $\left\lfloor \frac{n}{\lambda} \right\rfloor \in \mathbb{Z}$ denotes the least integer part of $\frac{n}{\lambda}$; we shall write $\frac{n}{\lambda} = \left\lfloor \frac{n}{\lambda} \right\rfloor + \nu_n$ with $\nu_n \in [0, 1[$. As a consequence, we have

$$S_n \sum_{k=0}^{K} f_k = \sum_{m=-h}^{h} S_n f_{m+\lfloor \frac{n}{\lambda} \rfloor}, \qquad n, K \in \mathbb{N}. \tag{5.36}$$

We pick $r \in]0, 1[^l$ such that $w_j \in A_{p_j/r_j}(\mathbb{R}^{d_j})$ for $j = 1, \ldots, l$. Then, for each $m \in \{-h, \ldots, h\}$, an application of Proposition 3.4.8 to $(2^{ns} f_{m+\lfloor \frac{n}{\lambda} \rfloor})_{n\in\mathbb{N}}$ yields the existence of a constant $C_1 > 0$ independent of m and $(f_k)_k$ such that

$$\begin{aligned} \left\|(2^{ns} S_n f_{m+\lfloor \frac{n}{\lambda} \rfloor})_{n\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} &= \left\|(\mathscr{F}^{-1}\hat{\varphi}_n \mathscr{F}[2^{ns} f_{m+\lfloor \frac{n}{\lambda} \rfloor}])_{n\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)} \\ &\leq C_1 \sup_{n\geq 0} \left\| \prod_{j=1}^{l} (1 + |\pi_j(\,\cdot\,)|^{d_j/r_j}) \mathscr{F}^{-1}[\hat{\varphi}_n(\delta^{[d,a]}_{2^{\lambda(m+\lfloor n/\lambda \rfloor)}} \cdot)] \right\|_{L^1(\mathbb{R}^d)} \\ &\quad \cdot \left\|(2^{ks} f_{m+\lfloor \frac{k}{\lambda} \rfloor})_{k\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)}. \end{aligned} \tag{5.37}$$

In order to estimate the term $\left\|(2^{sk} f_{m+\lfloor \frac{k}{\lambda} \rfloor})_{k\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})[L^{p,d}(\mathbb{R}^d,w)](X)}$ in (5.37), we first have a look at the mapping $i_\lambda : \mathbb{Z} \longrightarrow \mathbb{Z}, k \mapsto \lfloor k/\lambda \rfloor$. We observe that

- if $\lambda \in]0, 1]$, then i_λ is an injection;
- if $\lambda \in [1, \infty[$, then i_λ is a surjection with $\# i_\lambda^{-1}(n) \in \{\lfloor \lambda \rfloor, \lfloor \lambda \rfloor + 1\}$ for all $n \in \mathbb{Z}$;

in particular, for all $n \in \mathbb{Z}$ it holds that $\# i_\lambda^{-1}(n) \leq \lfloor \lambda \rfloor + 1$. Since it furthermore clearly holds that $2^{ks} \lesssim 2^{\lambda s \lfloor \frac{k}{\lambda} \rfloor}$ for $k \in \mathbb{Z}$, it follows that

$$\left\|(2^{sk} f_{m+\lfloor \frac{k}{\lambda} \rfloor})_{k\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w;\ell^q(\mathbb{N};X))} \leq C_2 \left\|(2^{\lambda s n} f_{m+n})_{n\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w;\ell^q(\mathbb{N};X))} \tag{5.38}$$

For each $m \in \{-h, \ldots, h\}$ we have

$$\mathscr{F}^{-1}[\hat{\varphi}_n(\delta^{[d,a]}_{2^{\lambda(m+\lfloor n/\lambda \rfloor)}} \cdot)] = \begin{cases} \mathscr{F}^{-1}[\hat{\varphi}_1(\delta^{[d,a]}_{2^{\lambda(m-\nu_n)-1}} \cdot)] & \text{if } n \geq 1; \\ \mathscr{F}^{-1}[\hat{\varphi}_0(\delta^{[d,a]}_{2^{\lambda m}} \cdot)] & \text{if } n = 0. \end{cases}$$

Defining $\phi_m := \mathscr{F}^{-1}[\hat{\varphi}_1(\delta^{[d,a]}_{2^{\lambda m-1}} \cdot)] \in \mathcal{S}(\mathbb{R}^d)$ and $\psi_m := \mathscr{F}^{-1}[\hat{\varphi}_0(\delta^{[d,a]}_{2^{\lambda m}} \cdot)]$, this can be rewritten as

$$\mathscr{F}^{-1}[\hat{\varphi}_n(\delta^{[d,a]}_{2^{\lambda(m+\lfloor n/\lambda \rfloor)}} \cdot)] = \begin{cases} 2^{(a\cdot_d \mathbf{1})\lambda\nu_n} \phi_m(\delta^{[d,a]}_{2^{\lambda\nu_n}} \cdot) & \text{if } n \geq 1; \\ \psi_m & \text{if } n = 0. \end{cases}$$

Therefore, as $\nu_n \in [0,1[$ for all n, we get

$$
\begin{aligned}
\sup_{m\in\{-h,\dots,h\}} \sup_{n\geq 0} &\Big\| \prod_{j=1}^{l}(1+|\pi_j(\,\cdot\,)|^{d_j/r_j})\mathscr{F}^{-1}[\hat{\varphi}_n(\delta^{[d,a]}_{2^{m+n}}\,\cdot\,)] \Big\|_{L^1(\mathbb{R}^d)} \\
&\leq \max_{g\in\{\phi_m,\psi_m\,:\,m\in\{-h,\dots,h\}\}} \sup_{b\in[1,2^\lambda[} \Big\| \prod_{j=1}^{l}(1+|\pi_j(\,\cdot\,)|^{d_j/r_j})b^{a\cdot_d \mathbf{1}}g(\delta^{[d,a]}_{b}\,\cdot\,) \Big\|_{L^1(\mathbb{R}^d)} \\
&= \max_{g\in\{\phi_m,\psi_m\,:\,m\in\{-h,\dots,h\}\}} \sup_{b\in[1,2^\lambda[} \Big\| \prod_{j=1}^{l}(1+b^{-a_j}|\pi_j(\,\cdot\,)|^{d_j/r_j})g \Big\|_{L^1(\mathbb{R}^d)} \\
&= \max_{g\in\{\phi_m,\psi_m\,:\,m\in\{-h,\dots,h\}\}} \Big\| \prod_{j=1}^{l}(1+|\pi_j(\,\cdot\,)|^{d_j/r_j})g \Big\|_{L^1(\mathbb{R}^d)} =: C_3 \in]0,\infty[\qquad (5.39)
\end{aligned}
$$

Finally, as in the end of the proof of Lemma 5.2.10, a combination of (5.36), (5.38), (5.37) and (5.39) gives (5.35), as desired. □

Lemma 5.2.25. *Let X be a Banach space, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{d_j})$. For every $\lambda > 0$ and $c > 1$ there exists a constant $C > 0$ such that, for all $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^d;X)$ satisfying*

$$
\operatorname{supp}\hat{f}_0 \subset \{\xi\in\mathbb{R}^d \mid |\xi|_{d,a} \leq c\}, \qquad \operatorname{supp}\hat{f}_k \subset \{\xi\in\mathbb{R}^d \mid c^{-1}2^{\lambda k} \leq |\xi|_{d,a} \leq c2^{\lambda k}\} \ \ (k\geq 1), \quad (5.40)
$$

and

$$
F := \big\|(2^{\lambda s k}f_k)_{k\geq 0}\big\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)} < \infty,
$$

it holds that $\sum_{k\in\mathbb{N}} f_k$ defines a convergent series in $\mathcal{S}'(\mathbb{R}^d;X)$ with limit $f \in F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)$ of norm $\leq CF$.

Proof. The proof goes completely analogous to the proof of Lemma 5.2.24. □

5.2.1.g A Fourier Multiplier Theorem

Recall that each m in $\mathscr{O}_M(\mathbb{R}^d;\mathcal{L}(X,Y))$, the space of slowly increasing $\mathcal{L}(X,Y)$-valued functions, defines a multiplication operator $f \mapsto mf$ which maps $\mathcal{S}'(\mathbb{R}^d;X)$ continuously into $\mathcal{S}'(\mathbb{R}^d;Y)$; see Appendix C.4. As a consequence, we may associate to such an m the Fourier multiplier operator $T_m \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^d;X),\mathcal{S}'(\mathbb{R}^d;Y))$ given by $T_m f := \mathscr{F}^{-1}[m\hat{f}]$. The next proposition says that for m it is sufficient to satisfy some kind of anisotropic Mikhlin condition in order that this Fourier multiplier operator T_m restricts to a bounded linear operator on anisotropic mixed-norm weighted Besov and Triebel-Lizorkin spaces.

Proposition 5.2.26. *Let X and Y be Banach spaces, $a \in]0,\infty[^l$, $p \in]1,\infty[^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{d_j})$. Let $r \in]0,\infty[^l$ be such that $r_j < \min\{p_1,\dots,p_j,q\}$ and $w_j \in A_{p_j/r_j}(\mathbb{R}^{d_j})$ for $j = 1,\dots,l$. Let $\mathscr{A} \in \{B,F\}$. Then there exists a constant $C > 0$ such that, for all $m \in \mathscr{O}_M(\mathbb{R}^d;\mathcal{L}(X,Y))$ satisfying*

$$
\sup_{\alpha\in\mathbb{N}^d:|\alpha_j|<d_j+\frac{d_j}{r_j}+1} \ \sup_{\xi\in\mathbb{R}^d} \big\|(1+|\xi|_{d,a})^{a\cdot_d\alpha}D^\alpha m(\xi)\big\|_{\mathcal{L}(X,Y)} =: \kappa_m < \infty, \qquad (5.41)
$$

the associated Fourier multiplier operator $T_m \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^d;X),\mathcal{S}'(\mathbb{R}^d;Y))$ restricts to a bounded linear operator from $\mathscr{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)$ to $\mathscr{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;Y)$ of norm $\leq C\kappa_m$.

Proof. We only consider the case $\mathscr{A} = F$, the case $\mathscr{A} = B$ being completely similar. Put $f_k := \mathscr{F}^{-1}\hat{\varphi}_k\hat{f}$ for $k \in \mathbb{Z}_{\geq 0}$ and $f_k := 0$ for $k \in \mathbb{Z}_{<0}$. For each $n \in \mathbb{N}$ we have $S_nT_mf = \sum_{k\in\mathbb{Z}} S_nT_mf_k$ in $\mathcal{S}'(\mathbb{R}^d;Y)$ because $f = \sum_{k\in\mathbb{Z}} f_k$ in $\mathcal{S}'(\mathbb{R}^d;Y)$, $T_m \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^d;X),\mathcal{S}'(\mathbb{R}^d;Y))$ and $S_n \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^d;Y))$. But, in view of the Fourier supports (5.13) of the φ_n, there is some fixed $h \in \mathbb{N}$ independent of m and f such that $S_nT_mS_kf = 0$ for all $n \in \mathbb{N}$ and $k \notin \mathbb{Z} \setminus \{-h,\ldots,h\}$, implying that $S_nT_mf = \sum_{\kappa=-h}^{h} S_nT_mf_{\kappa+n}$. We thus obtain

$$\|T_mf\|_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;Y)} \leq \sum_{\kappa=-h}^{h} \left\|(\mathscr{F}^{-1}[(\hat{\varphi}_n m\mathscr{F}(2^{ns}f_{n+\kappa})])_{n\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)}. \tag{5.42}$$

For each $\kappa \in \{-h,\ldots,h\}$ we invoke Proposition 3.4.8 to obtain a constant $C_1 > 0$ (independent of m, κ, and $(f_k)_k$) for which have the following estimate:

$$\begin{aligned}
&\left\|(\mathscr{F}^{-1}[\hat{\varphi}_n m]\mathscr{F}2^{sn}f_{n+\kappa})_{n\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)} \\
&\quad\leq C_1 \sup_{n\geq 0}\left\|\prod_{j=1}^{l}(1+|\,\cdot\,|^{d_j/r_j}))\mathscr{F}^{-1}[(\hat{\varphi}_n m)(\delta^{[d,a]}_{2^{\kappa+n}}\,\cdot\,)]\right\|_{L^1(\mathbb{R}^d;\mathcal{L}(X,Y))} \\
&\qquad\cdot\left\|(2^{ks}f_{\kappa+k})_{k\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)}.
\end{aligned} \tag{5.43}$$

Since $\hat{\varphi}_n = \varphi_1(\delta^{[d,a]}_{2^n}\,\cdot\,)$ for $n \geq 1$, for each $\kappa \in \{-h,\ldots,h\}$ we have

$$\hat{\varphi}_n(\delta^{[d,a]}_{2^{\kappa+n}}\,\cdot\,) = \begin{cases} \hat{\varphi}_1(\delta^{[d,a]}_{2^{\kappa+n}}\,\cdot\,) & =: \phi_\kappa \in C_c^\infty(\mathbb{R}^d) \quad \text{if } n \geq 1; \\ \hat{\varphi}_0(\delta^{[d,a]}_{2^{\kappa}}\,\cdot\,) & =: \psi_\kappa \in C_c^\infty(\mathbb{R}^d) \quad \text{if } n = 0. \end{cases}$$

Now we define $i \in (\mathbb{N})^l$ by setting each i_j to be the smallest natural number satisfying $i_j > d_j + \frac{d_j}{r_j}$. Then we can estimate

$$\begin{aligned}
&\sup_{n\geq 0}\left\|\prod_{j=1}^{l}(1+|\,\cdot\,|^{d_j/r_j})\mathscr{F}^{-1}[(\hat{\varphi}_n m)(\delta^{[d,a]}_{2^{\kappa+n}}\,\cdot\,)]\right\|_{L^1(\mathbb{R}^d;\mathcal{L}(X,Y))} \\
&\quad\leq \sup_{(n,g)\,\in\,\mathbb{Z}_{>0}\times\{\phi_\kappa\}\cup\{0\}\times\{\psi_\kappa\}}\left\|\prod_{j=1}^{l}(1+|\,\cdot\,|^{d_j/r_j})\mathscr{F}^{-1}[g\,m(\delta^{[d,a]}_{2^{\kappa+n}}\,\cdot\,)]\right\|_{L^1(\mathbb{R}^d;\mathcal{L}(X,Y))} \\
&\quad\leq \sup_{(n,g)\,\in\,\mathbb{Z}_{>0}\times\{\phi_\kappa\}\cup\{0\}\times\{\psi_\kappa\}}\left\|\prod_{j=1}^{l}(1+|\,\cdot\,|^{i_j})\mathscr{F}^{-1}[g\,m(\delta^{[d,a]}_{2^{\kappa+n}}\,\cdot\,)]\right\|_{L^\infty(\mathbb{R}^d;\mathcal{L}(X,Y))} \\
&\qquad\cdot\left\|\prod_{j=1}^{l}(1+|\,\cdot\,|^{-i_j+d_j/r_j})\right\|_{L^1(\mathbb{R}^d)} \\
&\quad\leq C_2 \sup_{(n,g)\,\in\,\mathbb{Z}_{>0}\times\{\phi_\kappa\}\cup\{0\}\times\{\psi_\kappa\}}\left\|\prod_{j=1}^{l}(1+|\,\cdot\,|^{i_j})\mathscr{F}^{-1}[g\,m(\delta^{[d,a]}_{2^{\kappa+n}}\,\cdot\,)]\right\|_{L^\infty(\mathbb{R}^d;\mathcal{L}(X,Y))}
\end{aligned}$$

Since

$$\prod_{j=1}^{l}(1+|x_j|^{i_j}) \leq \prod_{j=1}^{l}(1+|x_j|_1^{i_j}) = \sum_{\alpha\in J} c_\alpha|x^\alpha|$$

for some $J \subset \mathbb{N}^d$ and $(c_\alpha)_{\alpha\in J} \subset \mathbb{N}$ with the property that $|\alpha_j| \leq i_j$ $(j = 1, \ldots, l)$ for all $\alpha \in J$, and since the inverse Fourier transform $\mathscr{F}^{-1}$ is bounded from $L^1(\mathbb{R}^d; \mathcal{L}(X, Y))$ to $L^\infty(\mathbb{R}^d; \mathcal{L}(X, Y))$, this can be further estimated as

$$\begin{aligned}
&\sup_{n\geq 0}\left\|\prod_{j=1}^{l}(1+|\cdot|^{d_j/r_j})\mathscr{F}^{-1}[(\hat{\varphi}_n m)(\delta^{[d,a]}_{2^{\kappa+n}}\cdot)]\right\|_{L^1(\mathbb{R}^d;\mathcal{L}(X,Y))} \\
&\quad\leq C_2\sum_{\alpha\in J}c_\alpha \sup_{(n,g)\,\in\,\mathbb{Z}_{>0}\times\{\phi_\kappa\}\cup\{0\}\times\{\psi_\kappa\}}\left\|\mathscr{F}^{-1}\left(D^\alpha[g\,m(\delta^{[d,a]}_{2^{\kappa+n}}\cdot)]\right)\right\|_{L^\infty(\mathbb{R}^d;\mathcal{L}(X,Y))} \\
&\quad\leq C_3\sum_{\alpha\in J}\sup_{(n,g)\,\in\,\mathbb{Z}_{>0}\times\{\phi_\kappa\}\cup\{0\}\times\{\psi_\kappa\}}\left\|D^\alpha[g\,m(\delta^{[d,a]}_{2^{\kappa+n}}\cdot)]\right\|_{L^1(\mathbb{R}^d;\mathcal{L}(X,Y))}. \qquad (5.44)
\end{aligned}$$

By a combination of the Leibniz rule and the chain rule,

$$D^\alpha[g\,m(\delta^{[d,a]}_{2^{\kappa+n}}\cdot)] = \sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(D^{\alpha-\beta}g)\,2^{(\kappa+n)a\cdot_d\beta}(D^\beta m)(\delta^{[d,a]}_{2^{\kappa+n}}\cdot).$$

Since $\phi_\kappa, \psi_\kappa \in C_c^\infty(\mathbb{R}^d)$ and since $\operatorname{supp}\phi_\kappa \subset \{\xi \mid |\xi|_{d,a} \geq R\}$ for some $R > 0$, we can thus estimate, for all $\kappa \in \{-h, \ldots, h\}$ and $\alpha \in J$,

$$\begin{aligned}
&\sup_{(n,g)\,\in\,\mathbb{Z}_{>0}\times\{\phi_\kappa\}\cup\{0\}\times\{\psi_\kappa\}}\left\|D^\alpha[g\,m(\delta^{[d,a]}_{2^{\kappa+n}}\cdot)]\right\|_{L^1(\mathbb{R}^d;\mathcal{L}(X,Y))} \\
&\quad\leq C_4\sup_{n\in\mathbb{N}}\sum_{\beta\leq\alpha}\sup_{\xi\in\mathbb{R}^d}\left\|(1+2^{\kappa+n}|\xi|_{d,a})^{a\cdot_d\beta}(D^\beta m)(\delta^{[d,a]}_{2^{\kappa+n}}\xi)\right\|_{\mathcal{L}(X,Y)} \\
&\quad= C_4\sum_{\beta\leq\alpha}\sup_{\zeta\in\mathbb{R}^d}\left\|(1+|\zeta|_{d,a})^{a\cdot_d\beta}(D^\beta m)(\zeta)\right\|_{\mathcal{L}(X,Y)}. \qquad (5.45)
\end{aligned}$$

Finally, combining (5.42), (5.43), (5.44), (5.45) and (5.41), we obtain

$$\begin{aligned}
\|T_m f\|_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;Y)} &\leq \sum_{\kappa=-h}^{h}C_5\sum_{\alpha\in J}\sum_{\beta\leq\alpha}\kappa_m\left\|(2^{ks}f_{\kappa+k})_{k\in\mathbb{N}}\right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{N})](X)} \\
&\leq C_6\kappa_\mu\,\|f\|_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)}.
\end{aligned}$$

□

Example 5.2.27. Let $m \in C^\infty(\mathbb{R}^d; \mathcal{L}(X, Y))$ be such that $m(\delta^{[d,a]}_\lambda\xi) = m(\xi)$ for all $\xi \in \mathbb{R}^d$ with $|\xi|_{d,a} \geq R$ for some fixed $R > 0$ and $\lambda > 1$. Then m satisfies the condition of the theorem.

Proof. Inductively we see that $m(\delta^{[d,a]}_{\lambda^k}\xi) = m(\xi)$ for all $k \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$ with $|\xi|_{d,a} \geq R$. Differentiation yields

$$\lambda^{ka\cdot_d\alpha}D^\alpha m(\delta^{[d,a]}_{\lambda^k}\xi) = D^\alpha m(\xi) \qquad (|\xi|_{d,a} \geq R+1)$$

for all $\alpha \in \mathbb{N}^d$. Since each $\zeta \in \mathbb{R}^d$ with $|\zeta|_{d,a} \geq R + 1$ can be written as $\zeta = \delta^{[d,a]}_{\lambda^k}\xi$ for some

$k \in \mathbb{N}$ and $\xi \in \mathbb{R}^d$ with $R+1 \leq |\xi|_{\mathpzc{d},a} \leq \lambda(R+1)$, it follows that

$$
\begin{aligned}
\sup_{\zeta \in \mathbb{R}^d} \left\| (1+|\zeta|_{\mathpzc{d},a})^{a \cdot_{\mathpzc{d}} \alpha} D^\alpha m(\zeta) \right\|_{\mathcal{L}(X,Y)} &\leq \sup_{|\zeta|_{\mathpzc{d},a} \leq R+1} \left\| (1+|\zeta|_{\mathpzc{d},a})^{a \cdot_{\mathpzc{d}} \alpha} D^\alpha m(\zeta) \right\|_{\mathcal{L}(X,Y)} \\
&\quad + \sup_{|\zeta|_{\mathpzc{d},a} \geq R+1} \left\| (1+|\zeta|_{\mathpzc{d},a})^{a \cdot_{\mathpzc{d}} \alpha} D^\alpha m(\zeta) \right\|_{\mathcal{L}(X,Y)} \\
&\leq C_{1,\alpha} + 2^{a \cdot_{\mathpzc{d}} \alpha} \sup_{|\zeta|_{\mathpzc{d},a} \geq R+1} \left\| |\zeta|_{\mathpzc{d},a}^{a \cdot_{\mathpzc{d}} \alpha} D^\alpha m(\zeta) \right\|_{\mathcal{L}(X,Y)} \\
&= C_{1,\alpha} + 2^{a \cdot_{\mathpzc{d}} \alpha} \sup_{R+1 \leq |\xi|_{\mathpzc{d},a} \leq \lambda(R+1)} \left\| |\xi|_{\mathpzc{d},a}^{a \cdot_{\mathpzc{d}} \alpha} D^\alpha m(\delta^{[\mathpzc{d},a]}_{\lambda^k} \xi) \right\|_{\mathcal{L}(X,Y)} \\
&\leq C_{2,\alpha}.
\end{aligned}
$$

This shows that m belongs to $\mathscr{O}_M(\mathbb{R}^d; \mathcal{L}(X,Y))$ and satisfies condition (5.41) □

5.2.2 Isomorphisms and Embeddings

5.2.2.a Lifting Property and Equivalent Norms involving Derivatives

The isomorphisms in the next two propositions will be crucial to relate the Bessel potential and Sobolev spaces with the Besov and Triebel-Lizorkin spaces (see Proposition 5.2.31).

The first proposition basically says that $F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)$ can be defined out of $F^{0,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)$ via the Bessel potential operator $\mathcal{J}^{d,a}_a$ in the same way as $H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ is defined out of $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$.

Proposition 5.2.28. *Let X be a Banach space, $a \in \left(\frac{1}{\mathbb{Z}_{\geq 1}}\right)^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Then for all $\sigma \in \mathbb{R}$, the $(\mathpzc{d}, a)$-anisotropic Bessel-potential operator $\mathcal{J}^{d,a}_\sigma \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^d; X))$ restricts to isomorphisms of Banach spaces*

$$
\begin{aligned}
\mathcal{J}^{d,a}_\sigma : B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X) &\xrightarrow{\simeq} B^{s-\sigma,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X) \\
\mathcal{J}^{d,a}_\sigma : F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X) &\xrightarrow{\simeq} F^{s-\sigma,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X).
\end{aligned}
$$

Proof. We only treat the Triebel-Lizorkin case, the Besov case being completely similar. It suffices to show that $\mathcal{J}^{d,a}_\sigma$ restricts to a bounded linear operator from $F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)$ to $F^{s-\sigma,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)$ (for arbitrary s and σ), the inverse of $\mathcal{J}^{d,a}_\sigma$ being $\mathcal{J}^{d,a}_{-\sigma}$ (as operators on $\mathcal{S}'(\mathbb{R}^d; X)$). We assume that $\varphi \in \Phi^{d,a}(\mathbb{R}^d)$ is such that $\operatorname{supp} \hat{\varphi}_k \cap \neq \emptyset \operatorname{supp} \hat{\varphi}_n$ implies $|n-k| \leq 1$; this can for example be achieved by taking $A = 1$ and $B = \frac{3}{2}$ in Definition 5.2.4, see Example 5.2.5. Now let us put $\varphi_{-1} := 0$ and define $\psi_k := \varphi_{k-1} + \varphi_k + \varphi_{k+1}$ for each $k \in \mathbb{N}$. Then note that

$$
\hat{\varphi}_k = \hat{\psi}_k \hat{\varphi}_k \quad \forall k \geq 0 \quad \text{and} \quad \hat{\psi}_k = \phi(\delta^{[\mathpzc{d},a]}_{2^{-k}} \cdot) \quad \forall k \geq 3
$$

for some $\phi \in \mathcal{S}(\mathbb{R}^d)$ with $\phi \equiv 0$ in a neighborhood of 0. For each $k \in \mathbb{N}$ we furthermore define $\mu_k \in C^\infty_c(\mathbb{R}^d)$ by

$$
m_k(\xi) := 2^{-\sigma k}(1 + |\xi|^2_{\mathpzc{d},a})^{\sigma/2} \hat{\psi}_k(\xi) \qquad (\xi \in \mathbb{R}^d).
$$

Then, for all $f \in F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)$,

$$
\left\| \mathcal{J}^{d,a}_\sigma f \right\|_{F^{s-\sigma,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)} = \left\| (\mathscr{F}^{-1} m_k \mathscr{F} 2^{sk} S_k f)_{k \geq 0} \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d, w; \ell^q(\mathbb{N}; X))},
$$

which by Proposition 3.4.8 can be estimated as

$$\left\|\mathcal{J}^{d,a}_{\sigma} f\right\|_{F^{s-\sigma,a}_{p,q,d}(\mathbb{R}^d,w;X)} \leq C_1 \sup_{k\geq 0} \left\| \prod_{j=1}^{l} (1 + |\pi_{[d;j]}(\,\cdot\,)|^{d_j/r_j}) \mathscr{F}^{-1}[m_k(\delta^{[d,a]}_{2^k}\,\cdot\,)] \right\|_{L^1(\mathbb{R}^d)} \|f\|_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)},$$

where $r \in]0,\infty[^l$ is as in this proposition. Choosing $i \in (2\mathbb{N})^l$ such that $i_j > d_j + \frac{d_j}{r_j}$ ($j = 1,\dots,m$), this can be further estimated as

$$\begin{aligned}
&\left\|\mathcal{J}^{d,a}_{\sigma} f\right\|_{F^{s-\sigma,a}_{p,q,d}(\mathbb{R}^d,w;X)} \\
&\quad \leq C_1 \sup_{k\geq 0} \left\| \mathscr{F}^{-1} \prod_{j=1}^{l} (1 - (-\Delta_{[d;j]})^{i_j/2})[m_k(\delta^{[d,a]}_{2^k}\,\cdot\,)] \right\|_{L^\infty(\mathbb{R}^d)} \|f\|_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} \\
&\qquad \cdot \sup_{k\geq 0} \left\| \prod_{j=1}^{l} (1 + |\pi_{[d;j]}(\,\cdot\,)|^{-i_j + d_j/r_j}) \right\|_{L^1(\mathbb{R}^d)} \\
&\quad \leq C_2 \sup_{k\geq 0} \left\| \mathscr{F}^{-1} \left(\prod_{j=1}^{l} (1 + (-\Delta_{[d;j]})^{i_j/2})[m_k(\delta^{[d,a]}_{2^k}\,\cdot\,)] \right) \right\|_{L^\infty(\mathbb{R}^d)} \|f\|_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} \\
&\quad \leq C_3 \sup_{k\geq 0} \left\| \prod_{j=1}^{l} (1 + (-\Delta_{[d;j]})^{i_j/2})[m_k(\delta^{[d,a]}_{2^k}\,\cdot\,)] \right\|_{L^1(\mathbb{R}^d)} \|f\|_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)}, \qquad (5.46)
\end{aligned}$$

where we used the boundedness of $\mathscr{F}^{-1}$ from L^1 to L^∞ in the last inequality. Since $m_0, m_1, m_2 \in C^\infty_c(\mathbb{R}^d)$ and since

$$\begin{aligned}
m_k(\delta^{[d,a]}_{2^k}\xi) &= 2^{-\sigma k}(1 + |\delta^{[d,a]}_{2^k}\xi|^2_{d,a})^{\sigma/2}\hat{\psi}_k(\delta^{[d,a]}_{2^k}\xi) \\
&= (2^{-2k} + |\xi|^2_{d,a})^{\sigma/2}\phi(\xi) \qquad \forall k \geq 3
\end{aligned}$$

with $\phi \in C^\infty_c(\mathbb{R}^d)$ satisfying $\phi \equiv 0$ in a neighborhood of 0, there exists a constant $C_4 > 0$ (independent of k) such that

$$\left\| \prod_{j=1}^{l} (1 + (-\Delta_{[d;j]})^{i_j/2})[m_k(\delta^{[d,a]}_{2^k}\,\cdot\,)] \right\|_{L^1(\mathbb{R}^d)} \leq C_4.$$

Combining this with (5.46), we obtain

$$\left\|\mathcal{J}^{d,a}_{\sigma} f\right\|_{F^{s-\sigma,a}_{p,q,d}(\mathbb{R}^d,w;X)} \leq C_5 \|f\|_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)}.$$

□

Let $n \in (\mathbb{Z}_{>0})^l$, $a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$, and $s \in \mathbb{R}$, such that $n_j = \frac{s}{a_j}$ for $j = 1,\dots,l$. Then the next proposition basically says that $F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ can be defined in terms of derivatives and $F^{0,a}_{p,q,d}(\mathbb{R}^d, w; X)$ in the same way as $W^n_{p,d}(\mathbb{R}^d, w; X)$ is defined in terms of derivatives and $L^{p,d}(\mathbb{R}^d, w; X)$. In fact, here we have the freedom to replace $J_{d,n}$ (5.2) by suitable other sets of multi-indices $J \subset \mathbb{N}^d$. In order to give an easy description of which J are allowed, we define

$$\underline{J}_{n,d} := \{0\} \cup \left\{ \alpha \in \bigcup_{j=1}^{l} \iota_{[d;j]}\mathbb{N}^{d_j} \,:\, |\alpha_j| \leq n_j,\, j = 1,\dots,l \right\}$$

and

$$\overline{J}_{n,d} := \left\{\alpha \in \mathbb{N}^d \,:\, \frac{1}{n} \cdot_d \alpha \leq 1\right\}.$$

Proposition 5.2.29. *Let X be a Banach space, $a \in \left(\frac{1}{\mathbb{Z}_{\geq 1}}\right)^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$, $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$, and $k \in \mathbb{N}$. Suppose that $J \subset \mathbb{N}^d$ is such that $\underline{J}_{\frac{k}{a},d} \subset J \subset \overline{J}_{\frac{k}{a},d}$, where $\frac{k}{a} = (\frac{k}{a_1},\ldots,\frac{k}{a_l}) \in \mathbb{N}^l$. Let $\mathscr{A} \in \{B,F\}$. Then $f \mapsto \sum_{\alpha\in J} \|D^\alpha f\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)}$ defines an equivalent norm on $\mathscr{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)$.*

Proof. Observe that $f \mapsto \sum_{\alpha\in J} \|D^\alpha f\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)}$ defines a norm on the subspace of $\mathcal{S}'(\mathbb{R}^d;X)$ on which it is finite. We first show that $\sum_{\alpha\in J}\|D^\alpha \cdot\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)} \lesssim \|\cdot\|_{\mathscr{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)}$. For each $\alpha \in J$ it is an easy but tedious computation to check that $\xi \mapsto \xi^\alpha(1+|\xi|^2_{d,a})^{-k/2}$ is a Fourier multiplier on $\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)$ as in Proposition 5.2.26, say of norm c_α. With the lifting property from Proposition 5.2.28 it thus follows that

$$\begin{aligned}
\sum_{\alpha\in J}\|D^\alpha f\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)} &= \sum_{\alpha\in J}\left\|\mathscr{F}^{-1}[\xi^\alpha \mathscr{F} f]\right\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)} \\
&= \sum_{\alpha\in J}\left\|\mathscr{F}^{-1}[\xi^\alpha(1+|\xi|^2_{d,a})^{-k/2}\mathscr{F}\mathcal{J}^{d,a}_k f]\right\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)} \\
&\leq \sum_{\alpha\in J} c_\alpha \left\|\mathcal{J}^{d,a}_k f\right\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)} \\
&\leq C\,\|f\|_{\mathscr{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)}\,.
\end{aligned}$$

Next, we show the reverse inequality. For this it suffices to take the minimal J in the statement,that is , to take $J = \underline{J}_{\frac{k}{a},d}$. We claim that there exist Fourier multipliers $\rho_{j,i}$ ($j = 1,\ldots,l$, $i = 1,\ldots,d_j$) on $\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)$ as in Proposition 5.2.26 and a constant $c > 0$ such that

$$1 + \sum_{j=1}^{l}\sum_{i=1}^{d_j}\rho_{j,i}(\xi)\xi_{j,i}^{k/a_j} \geq c(1+|\xi|^2_{d,a})^{k/2} \qquad (\xi \in \mathbb{R}^d). \tag{5.47}$$

Assuming this is true, it is again an easy but tedious computation to check that

$$m(\xi) := (1+|\xi|^2_{d,a})^{k/2}\left[1 + \sum_{j=1}^{l}\sum_{i=1}^{d_j}\rho_{j,i}(\xi)\xi_{j,i}^{k/a_j}\right]^{-1}$$

is a Fourier multiplier on $\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)$ as in Proposition 5.2.26. With the lifting property from Proposition 5.2.28 it thus follows that

$$\begin{aligned}
\|f\|_{\mathscr{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} &= \left\| \mathcal{J}^{d,a}_{-k} T_\mu \mathscr{F}^{-1}\left[1+\sum_{j=1}^{m}\sum_{i=1}^{d_j}\rho_{j,i}\xi_{j,i}^{k/a_j}\right]\mathscr{F} f\right\|_{\mathscr{A}^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} \\
&\leq C_1\|f\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)} + C_1\sum_{j=1}^{m}\sum_{i=1}^{d_j}\left\|\mathscr{F}^{-1}\rho_{j,i}\xi_{j,i}^{k/a_j}\mathscr{F} f\right\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)} \\
&\leq C_2\|f\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)} + C_2\sum_{j=1}^{m}\sum_{i=1}^{d_j}\left\|D_{j,i}^{k/a_j} f\right\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)} \\
&= C_2\sum_{\alpha\in J}\|D^\alpha f\|_{\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)}.
\end{aligned}$$

To finish, it remains to establish the claim. For each (j,i), with $j\in\{1,\ldots,l\}$ and $i\in\{1,\ldots,d_j\}$, we construct $\rho_{j,i}\in C^\infty(\mathbb{R}^d)$ as follows: Let $\psi\in C^\infty(\mathbb{R}^d)$ be a smooth function taking values in $\mathbb{R}_{\geq 0}$ such that $\psi\equiv 1$ on $\{\xi : |\xi|_{d,a}\geq 1\}$ and $0\notin \operatorname{supp}\psi$. Since

$$K^{\pm}_{j,i} := \{u\in S^{d-1} \mid \pm u_{i,j}>0 \ \text{ and } \ |u_{\tilde{j},\tilde{i}}|^{1/a_{\tilde{j}}}\leq |u_{i,j}|^{1/a_j} \ \ \forall(\tilde{j},\tilde{i})\neq(j,i)\}$$

is a compact in the unit sphere $S^{d-1}\subset\mathbb{R}^d$ which is contained in the open $O^{\pm}_{j,i} := \{u\in S^{d-1} \mid \pm u_{i,j}>0\}\subset S^{d-1}$ (in the topology of S^{d-1}), there exists a $\vartheta^{\pm}_{i,j}\in C^\infty(S^{d-1})$ taking values in $[0,1]$ with $\vartheta^{\pm}_{i,j}\equiv 1$ on $K^{\pm}_{j,i}$ and $\operatorname{supp}\vartheta^{\pm}_{i,j}\subset O^{\pm}_{j,i}$. Now we let $\phi^{\pm}_{i,j}\in C^\infty(\mathbb{R}^d\setminus\{0\})$ be the function which in (d,a)-anisotropic polar coordinates is given by $\phi^{\pm}_{i,j} : (\lambda,u)\mapsto\vartheta^{\pm}_{i,j}(u)$, and define

$$\rho_{j,i} := \psi(\vartheta^{+}_{i,j} + (-1)^{k/a_j}\vartheta^{-}_{i,j}).$$

To see that (5.47) is fulfilled by the constructed functions $(\rho_{j,i})_{j,i}$, let $\xi\in\mathbb{R}^d$. Then we have that $|\xi_{j,i}|^{1/a_j}\leq|\xi_{i_0,j_0}|^{1/a_j}$ for all $(i,j)\neq(i_0,j_0)$ and some (i_0,j_0). By construction of the $\rho_{j,i}$, we have

$$1+\sum_{j=1}^{l}\sum_{i=1}^{d_j}\rho_{j,i}(\xi)\xi_{j,i}^{k/a_j} \geq 1+\psi(\xi)|\xi_{i_0,j_0}|^{k/a_{j_0}} \geq 1+\frac{\psi(\xi)}{d}\sum_{j=1}^{l}\sum_{i=1}^{d_j}|\xi_{j,i}|^{k/a_j}.$$

In particular, this is ≥ 1 because we have chosen $\psi\geq 0$. Therefore, it suffices to consider the case $|\xi|_{d,a}\geq 1$. Then $\psi(\xi)=1$, so that

$$1+\sum_{j=1}^{l}\sum_{i=1}^{d_j}\rho_{j,i}(\xi)\xi_{j,i}^{k/a_j} \geq 1+\frac{1}{d}\sum_{j=1}^{l}\sum_{i=1}^{d_j}|\xi_{j,i}|^{k/a_j} \geq \frac{1}{d}|(1,|\xi_{1,1}|^{2/a_1},\ldots,|\xi_{1,d_1}|^{2/a_1},|\xi_{2,1}|^{2/a_2},\ldots,|\xi_{l,d_l}|^{2/a_l})|_{k/2}^{k/2},$$

where $|\cdot|_k$ is the k-norm on $\mathbb{R}^{1+d}$. Since $|\cdot|_k\eqsim|\cdot|_1$ on $\mathbb{R}^{1+d}$ and since $\left(\sum_{j=1}^{l}\sum_{i=1}^{d_j}|\zeta_{j,i}|^{2/a_j}\right)^{1/2}\eqsim|\zeta|_{d,a}$ for $\zeta\in\mathbb{R}^d$ (equivalence of norms and (d,a)-anisotropic distance functions, respectively), the desired estimate (5.47) follows.

Finally, we show that each $\rho_{j,i}$ is a Fourier multiplier on $\mathscr{A}^{s-k,a}_{p,q,d}(\mathbb{R}^d,w;X)$ as in Proposition 5.2.26. But $\rho_{j,i}\in C^\infty(\mathbb{R}^d)$ is constructed in such a way that $\rho_{j,i}(\delta^{[d,a]}_\lambda\xi)=\rho_{j,i}(\xi)$ for all $\lambda>1$ and $|\xi|_{d,a}\geq 1$, whence Example 5.2.27 applies to get the desired conclusion. □

5.2.2.b Elementary Embeddings

Proposition 5.2.30. *Let X be a Banach space, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^{l} \in A_\infty(\mathbb{R}^{d_j})$.*

(i) For all $1 \leq q_0 \leq q_1 \leq \infty$,

$$\begin{aligned} B^{s,a}_{p,q_0,d}(\mathbb{R}^d, w; X) &\hookrightarrow B^{s,a}_{p,q_1,d}(\mathbb{R}^d, w; X), \\ F^{s,a}_{p,q_0,d}(\mathbb{R}^d, w; X) &\hookrightarrow F^{s,a}_{p,q_1,d}(\mathbb{R}^d, w; X). \end{aligned}$$

(ii) For all $q_0, q_1 \in [1,\infty]$ and $\epsilon > 0$,

$$\begin{aligned} B^{s+\epsilon,a}_{p,q_0,d}(\mathbb{R}^d, w; X) &\hookrightarrow B^{s,a}_{p,q_1,d}(\mathbb{R}^d, w; X), \\ F^{s+\epsilon,a}_{p,q_0,d}(\mathbb{R}^d, w; X) &\hookrightarrow F^{s,a}_{p,q_1,d}(\mathbb{R}^d, w; X). \end{aligned}$$

(iii) For $q \in [1,\infty]$,

$$B^{s,a}_{p,\min\{p_1,\ldots,p_m,q\},d}(\mathbb{R}^d, w; X) \hookrightarrow F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) \hookrightarrow B^{s,a}_{p,\max\{p_1,\ldots,p_m,q\},d}(\mathbb{R}^d, w; X)$$

(iv) For all $q \in [1,\infty]$ and Banach spaces $X \hookrightarrow Y$,

$$F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) \hookrightarrow F^{s,a}_{p,q,d}(\mathbb{R}^d, w; Y) \quad \text{and} \quad B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) \hookrightarrow B^{s,a}_{p,q,d}(\mathbb{R}^d, w; Y)$$

Proof. (i) is immediate from the monotonicity of ℓ^q-spaces ($\ell^{q_0} \hookrightarrow \ell^{q_1}$ for $1 \leq q_0 \leq q_1 \leq \infty$), (ii) follows from a combination of (i) and the estimate

$$\left\|(2^{sk}b_k)_{k\in\mathbb{N}}\right\|_{\ell^{q_1}(\mathbb{N})} \leq \left\|(2^{-\epsilon k})_k\right\|_{\ell^{q_1}(\mathbb{N})} \left\|(2^{(s+\epsilon)k}b_k)_{k\in\mathbb{N}}\right\|_{\ell^{q_\infty}(\mathbb{N})} \leq C \left\|(2^{(s+\epsilon)k}b_k)_{k\in\mathbb{N}}\right\|_{\ell^{q_\infty}(\mathbb{N})}, \quad (b_k)_k \in \mathbb{C}^{\mathbb{N}},$$

and (iv) is completely trivial. Finally, (iii) is also not very difficult and can be proven as [97, Section 2.3.2, Proposition 2.(iii)]. □

The embeddings between Sobolev and Triebel-Lizorkin spaces in the next proposition will form the basis for determining the trace space of anisotropic Sobolev spaces in Section 5.2.3; also see the introductory section of this chapter, Section 5.1. Similarly this will be the case for anisotropic Bessel potential spaces. Here we do not require any restrictions on the Banach space X; also see the discussion in Section 5.2.1.b.

Proposition 5.2.31. *Let X be a Banach space, $a \in \left(\frac{1}{\mathbb{Z}_{\geq 1}}\right)^l$, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$.*

(i) For all $s \in \mathbb{R}$,

$$\begin{aligned} B^{s,a}_{p,1,d}(\mathbb{R}^d, w; X) &\hookrightarrow H^{s,a}_{p,d}(\mathbb{R}^d, w; X) \hookrightarrow B^{s,a}_{p,\infty,d}(\mathbb{R}^d, w; X), \\ F^{s,a}_{p,1,d}(\mathbb{R}^d, w; X) &\hookrightarrow H^{s,a}_{p,d}(\mathbb{R}^d, w; X) \hookrightarrow F^{s,a}_{p,\infty,d}(\mathbb{R}^d, w; X). \end{aligned} \tag{5.48}$$

(ii) For all $s \in \mathbb{R}$ and $n \in \mathbb{N}^l$ such that $n_j = \frac{s}{a_j}$ ($j = 1, \ldots, l$),

$$\begin{aligned} B^{s,a}_{p,1,d}(\mathbb{R}^d, w; X) &\hookrightarrow W^{n}_{p,d}(\mathbb{R}^d, w; X) \hookrightarrow B^{s,a}_{p,\infty,d}(\mathbb{R}^d, w; X), \\ F^{s,a}_{p,1,d}(\mathbb{R}^d, w; X) &\hookrightarrow W^{n}_{p,d}(\mathbb{R}^d, w; X) \hookrightarrow F^{s,a}_{p,\infty,d}(\mathbb{R}^d, w; X). \end{aligned} \tag{5.49}$$

Proof. In view of Proposition 5.2.30, it suffices to prove (5.48) and (5.49). By Propositions 5.2.28 and 5.2.29, for this it is in turn enough to establish the embeddings

$$F^{0,a}_{p,1,\mathpzc{d}}(\mathbb{R}^d, w; X) \hookrightarrow L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X) \hookrightarrow F^{0,a}_{p,\infty,\mathpzc{d}}(\mathbb{R}^d, w; X). \tag{5.50}$$

For the first inclusion in (5.50), let f be in the dense subspace $\mathcal{S}(\mathbb{R}^d; X)$ of $F^{0,a}_{p,1,\mathpzc{d}}(\mathbb{R}^d, w; X)$. Then $f = \sum_{n=0}^{\infty} S_n f$ in $\mathcal{S}(\mathbb{R}^d; X)$; in particular, $f = \sum_{n=0}^{\infty} S_n f$ pointwise. It follows that

$$\|f\|_X \leq \sum_{n=0}^{\infty} \|S_n f\|_X = \|(S_n f)_{n\geq 0}\|_{\ell^1(\mathbb{N};X)},$$

and taking $L^{p,\mathpzc{d}}(\mathbb{R}^d, w)$-norms we obtain

$$\|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)} \leq \|f\|_{F^{0,a}_{p,1,\mathpzc{d}}(\mathbb{R}^d,w;X)}.$$

For the second inclusion in (5.50), let $f \in L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$. By Lemma 3.4.1,

$$\|S_0 f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)} \leq C_1 \|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)}.$$

For $n \geq 1$ we have

$$\begin{aligned} \|S_n f(x)\|_X &\leq \int_{\mathbb{R}^d} |\varphi_n(y)| \, \|f(x-y)\|_X \, dy \\ &= \int_{\mathbb{R}^d} |2^{(n-1)a\cdot\mathpzc{d}} \varphi_1(\delta^{[\mathpzc{d},a]}_{2^{n-1}} y)| \, \|f(x-y)\|_X \, dy \\ &\leq \sup_{k\in\mathbb{N}} \int_{\mathbb{R}^d} |2^{ka\cdot\mathpzc{d}} \varphi_1(\delta^{[\mathpzc{d},a]}_{2^k} y)| \, \|f(x-y)\|_X \, dy, \end{aligned}$$

so that, by Lemma 3.4.1,

$$\|(S_n f)_{n\geq 1}\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)[\ell^\infty(\mathbb{Z}_{\geq 1})](X)} \leq C_2 \|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)}.$$

Therefore,

$$\|f\|_{F^{0,a}_{p,\infty,\mathpzc{d}}(\mathbb{R}^d,w;X)} \leq \|S_0 f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)} + \|(S_n f)_{n\geq 1}\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)[\ell^\infty(\mathbb{Z}_{\geq 1})](X)} \leq C_3 \|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)}.$$

□

5.2.2.c Sobolev Embeddings

The following inequality is an inequality of Plancherel-Pólya-Nikol'skii type. It allows us to easily prove the Sobolev embedding result Corollary 5.2.33 for weighted anisotropic mixed-norm Besov spaces. We will furthermore use this inequality in the trace problem for Besov spaces in Section 5.2.3 to the case $\gamma \geq p-1$, where γ the parameter in the power weight $w_\gamma(t) = |t|^\gamma$ on $\mathbb{R}$; see the end of the proof of Lemma 5.2.53 for this application. We would like to remark that for applications in Chapter 6 we will only need traces for $\gamma \in]-1, p-1[$ (corresponding to $w_\gamma \in A_p$; see Example D.2.12).

Proposition 5.2.32. *Let X be a Banach space, $p, \tilde{p} \in]1,\infty[^l$, and $w, \tilde{w} \in \prod_{j=1}^{l} \mathcal{W}(\mathbb{R}^{d_j})$. Suppose that $J \subset \{1,\ldots,l\}$ is such that*

- $p_j = \tilde{p}_j$ *and* $w_j = \tilde{w}_j$ *for* $j \notin J$;

- $w_j(x_j) = |x_j|^{\gamma_j}$ *and* $\tilde{w}_j(x_j) = |x_j|^{\tilde{\gamma}_j}$ *for* $j \in J$ *for some* $\gamma_j, \tilde{\gamma}_j > -d_j$ *satisfying*

$$\frac{\tilde{\gamma}_j}{\tilde{p}_j} \leq \frac{\gamma_j}{p_j} \quad \text{and} \quad \frac{d_j + \tilde{\gamma}_j}{\tilde{p}_j} < \frac{d_j + \gamma_j}{p_j}.$$

Then there exists a constant $C > 0$ *such that, for all* $f \in \mathcal{S}'(\mathbb{R}^d; X)$ *with* $\operatorname{supp}(\hat{f}) \subset \prod_{j=1}^{l}[-R_1, R_1]^{d_j}$ *for some* $R_1, \ldots, R_l > 0$, *we have the inequality*

$$\|f\|_{L^{\tilde{p},d}(\mathbb{R}^d,\tilde{w};X)} \leq C \left(\prod_{j \in J} R_j^{\delta_j} \right) \|f\|_{L^{p,d}(\mathbb{R}^d,w;X)},$$

where $\delta_j := (d_j + \gamma_j)/p_j - (d_j + \tilde{\gamma}_j)/\tilde{p}_j > 0$ *for each* $j \in J$.

Proof. **Step I.** *The case* $l = 1$:

We refer to [77, Proposition 4.1].

Step II. *The case* $J = \{l\}$:

By Lemma C.7.2, we may view f as an element of $\mathcal{S}'(\mathbb{R}^{d_l}; C(\mathbb{R}^{d_1+\ldots+d_{l-1}}; X))$ having compact Fourier support contained in $[-R_l, R_l]^{d_l}$. Given a compact subset $K \subset \mathbb{R}^{d_1+\ldots+d_{l-1}}$ we have the continuous linear operator

$$m_{1_K} : C(\mathbb{R}^{d'}; X) \longrightarrow L^{\infty}_K(\mathbb{R}^{d'}; X) \hookrightarrow L^{p',d'}(\mathbb{R}^{d'}, w'; X),\ g \mapsto 1_K g,$$

where $d' := d_1 + \ldots + d_{l-1}$, $d' = (d_1, \ldots, d_{l-1})$, $p' := (p_1, \ldots, p_{l-1})$, and $w' = (w_1, \ldots, w_{l-1})$.[4] Accordingly, for each compact $K \subset \mathbb{R}^{d'}$ we have $1_K f = m_{1_K} f \in \mathcal{S}'(\mathbb{R}^{d_l}; L^{p',d'}(\mathbb{R}^{d'}, w'; X))$ with compact Fourier support contained in $[-R_l, R_l]^{d_l}$, so that we may apply Step I to obtain that

$$\|1_K f\|_{L^{\tilde{p}_l}(\mathbb{R}^{d_l},\tilde{w}_l;L^{p',d'}(\mathbb{R}^{d'},w';X))} \leq C R_l^{\delta_l} \|1_K f\|_{L^{p_l}(\mathbb{R}^{d_l},w_l;L^{p',d'}(\mathbb{R}^{d'},w';X))}$$

for some constant $C > 0$ independent of f and K. Since $L^{\tilde{p},d}(\mathbb{R}^d, \tilde{w}; X) = L^{\tilde{p}_l}(\mathbb{R}^{d_l}, \tilde{w}_l; L^{p',d'}(\mathbb{R}^{d'}, w'; X))$ and $L^{p,d}(\mathbb{R}^d, w; X) = L^{p_l}(\mathbb{R}^{d_l}, w_l; L^{p',d'}(\mathbb{R}^{d'}, w'; X))$, the desired result follows by taking $K = K_n = [-n, n]^{d_l}$ and letting $n \to \infty$.

Step III. *The case* $\#J = 1$:

Let's say that $J = \{j_0\}$. Then, by Corollary C.6.5, for each fixed $x'' = (x_{j_0+1}, \ldots, x_l) \in \mathbb{R}^{d_{j_0+1}+\ldots+d_l}$ we have that $f(\cdot, x'')$ defines an X-valued tempered distribution having compact Fourier support contained in $\prod_{j=1}^{j_0}[-R_j, R_j]^{d_j}$. The desired inequality follows by applying Step II to $f(\cdot, x'')$ for each x'' and subsequently taking $L^{(p_{j_0+1},\ldots,p_l),(d_{j_0+1},\ldots,d_l)}(\mathbb{R}^{d_{j_0+1}+\ldots+d_l}, (w_{j_0+1}, \ldots, w_l); X)$-norms with respect to x''.

Step IV. *The general case:*

Just apply Step III repeatedly ($\#J$ times). □

Corollary 5.2.33. *Let X be a Banach space,* $p, \tilde{p} \in]1, \infty[^l$, $q, \tilde{q} \in [1, \infty]$, $s, \tilde{s} \in \mathbb{R}$, *and* $w, \tilde{w} \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{d_j})$. *Suppose that* $J \subset \{1, \ldots, l\}$ *is such that*

- $p_j = \tilde{p}_j$ *and* $w_j = \tilde{w}_j$ *for* $j \notin J$;

[4] The notations here do not mean Hölder conjugates and p-dual weights.

- $w_j(x_j) = |x_j|^{\gamma_j}$ *and* $\tilde{w}_j(x_j) = |x_j|^{\tilde{\gamma}_j}$ *for* $j \in J$ *for some* $\gamma_j, \tilde{\gamma}_j > -d_j$ *satisfying*

$$\frac{\tilde{\gamma}_j}{\tilde{p}_j} \leq \frac{\gamma_j}{p_j} \quad \text{and} \quad \frac{d_j + \tilde{\gamma}_j}{\tilde{p}_j} < \frac{d_j + \gamma_j}{p_j}.$$

Furthermore, assume that $q \leq \tilde{q}$ *and that* $s - \frac{d_j+\gamma_j}{p_j} = \tilde{s} - \frac{d_j+\tilde{\gamma}_j}{\tilde{p}_j}$ *for each* $j \in J$. *Then*

$$B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) \hookrightarrow B^{\tilde{s},a}_{\tilde{p},\tilde{q},d}(\mathbb{R}^d, \tilde{w}; X).$$

Proof. This is an easy consequence of the above proposition and Proposition 5.2.30.(i); also see [77, Theorem 1.1,(2)⇒(1)] for the isotropic case. □

5.2.2.d Invariance under rescaling in Smoothness-Anisotropy

The next proposition says that the anisotropic Besov and Triebel-Lizorkin spaces, $B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ and $F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$, are invariant under the reparametrization $(s, a) \mapsto (\lambda s, \lambda a)$, $\lambda > 0$.

Proposition 5.2.34. *Let* X *be a Banach space,* $a \in]0, \infty[^l$, $p \in [1, \infty[^l$, $q \in [1, \infty]$, $s \in \mathbb{R}$ *and* $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. *Let* $\mathscr{A} \in \{B, F\}$. *Then we have, for each* $\lambda > 0$,

$$\mathscr{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) = \mathscr{A}^{\lambda s,\lambda a}_{p,q,d}(\mathbb{R}^d, w; X)$$

up to an equivalence of norms.

Proof. It is enough to show that

$$\mathscr{A}^{\lambda s,\lambda a}_{p,q,d}(\mathbb{R}^d, w; X) \hookrightarrow \mathscr{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X), \tag{5.51}$$

the opposite inclusion being of the same kind. To this end, let $\varphi^\lambda \in \Phi^{d,\lambda a}(\mathbb{R}^d)$ be fixed the Littlewood-Paley sequence in the definition of $\mathscr{A}^{\lambda s,\lambda a}_{p,q,d}(\mathbb{R}^d, w; X)$. For each $n \in \mathbb{N}$ we define $f_n := \varphi^\lambda_n * f$. Then $(f_n)_{n\in\mathbb{N}}$ satisfies the Fourier support condition

$$\operatorname{supp} \hat{f}_0 \subset \{\xi \in \mathbb{R}^d \mid |\xi|_{d,\lambda a} \leq \tilde{c}\}, \qquad \operatorname{supp} \hat{f}_n \subset \{\xi \in \mathbb{R}^d \mid \tilde{c}^{-1}2^n \leq |\xi|_{d,\lambda a} \leq \tilde{c}2^n\} \quad (n \geq 1)$$

for some constant $\tilde{c} > 1$. Since $|\cdot|_{d,\lambda a} \eqsim |\cdot|^\lambda_{d,a}$, it follows that $(f_n)_{n\in\mathbb{N}}$ satisfies the Fourier support condition (5.34) for some constant $c > 1$. With a modification of the proof of Lemma 5.2.10 (as in the proof of (5.35), or just with Lemma 5.2.24 and Lemma 5.2.25), we obtain (5.51). □

5.2.2.e Representations by Intersections and Difference Norms for Triebel-Lizorkin Spaces

In Corollary 5.2.58/Remark 5.2.59 we will prove that, in the notation of Sections 1.2 and 2.1, the spatial trace operator $\operatorname{tr}_{y=0}$ is a continuous surjection

$$\operatorname{tr}_{y=0} : W^{(2,1)}_{(p,q),(d,1)}(\mathbb{R}^d \times \mathbb{R}, (w_\gamma, v_\mu); X) \longrightarrow F^{1-\frac{1}{p}(1+\gamma),(\frac{1}{2},1)}_{p,q,(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}, (1, v_\mu); X)$$

with a continuous right-inverse. We would like to represent the anistropic Triebel-Lizorkin space on the right as a space of intersection type like (1.12). This is achieved in the following theorem in a more general setting.

Theorem 5.2.35. *Let X be a Banach space, $l=2$, $a \in]0,\infty[^2$, $p,q \in]1,\infty[$, $\tilde{p} := (p,q)$, $s>0$, and $w \in A_p(\mathbb{R}^{d_1}) \times A_q(\mathbb{R}^{d_2})$. Then*

$$F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X) = F^{s/a_2}_{q,p}(\mathbb{R}^{d_2}, w_2; L^p(\mathbb{R}^{d_1}, w_1; X)) \cap L^q(\mathbb{R}^{d_2}, w_2; F^{s/a_1}_{p,p}(\mathbb{R}^{d_1}, w_1; X)) \tag{5.52}$$

with equivalence of norms.

To prove this theorem, we introduce partial Triebel-Lizorkin spaces on $\mathbb{R}^d$ and prove that the general anisotropic Triebel-Lizorkin space $F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ (with $s>0$ and $q>1$) can be represented as an intersection of these spaces. This representation result will be proved by choosing suitable equivalent norms on $F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ and on these partial Triebel-Lizorkin spaces in terms of differences. Recall here that $F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) \hookrightarrow L^{p,d}(\mathbb{R}^d, w; X)$ for every $s>0$, $p \in]1,\infty[^l$, $q \in [1,\infty]$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$; see Propositions 5.2.30 and 5.2.31

We start by defining the partial Triebel-Lizorkin spaces. Let X be a Banach space, $a \in]0,\infty[^l$, $p \in]1,\infty[^l$, $q \in [1,\infty]$, $s>0$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. Let $j_0 \in \{1,\ldots,l\}$. To a $\varphi^{[d;j_0]} = (\varphi^{[d;j_0]}_n)_{n\in\mathbb{N}} \in \Phi^{a_{j_0}}(\mathbb{R}^{d_{j_0}})$ we associate the family of partial convolution operators $(S^{[d;j_0]}_n)_{n\in\mathbb{N}} \subset \mathcal{B}(L^{p,d}(\mathbb{R}^d, w; X))$ given by

$$S^{[d;j_0]}_n f := \varphi^{[d;l]}_n *_{[d;j_0]} f, \qquad f \in L^{p,d}(\mathbb{R}^d, w; X). \tag{5.53}$$

Here the partial convolution product $\phi *_{[d;j_0]} f$ for a $\phi \in \mathcal{S}(\mathbb{R}^{d_{j_0}})$ and an $f \in L^{p,d}(\mathbb{R}^d, w; X)$ is defined by

$$(\phi *_{[d;j_0]} f)(x) := \int_{\mathbb{R}^{d_{j_0}}} f(x - \iota_{[d;j_0]} y_{j_0}) \phi(y_{j_0})\, dy_{j_0}, \qquad x \in \mathbb{R}^d, \tag{5.54}$$

where $\iota_{[d;j_0]} : \mathbb{R}^{d_{j_0}} \longrightarrow \mathbb{R}^d$ is the inclusion map from Convention 2.2.1. Under the identification

$$L^{p,d}(\mathbb{R}^d, w; X) = L^{p'^{j_0}, d'^{j_0}}\left(\mathbb{R}^{|d'^{j_0}|_1}, w'^{j_0}; L^{p_{j_0}}(\mathbb{R}^{d_{j_0}}, w'^{j_0}; L^{p''^{j_0}, d''^{j_0}}(\mathbb{R}^{|d''^{j_0}|_1}, w''^{j_0}; X))\right),$$

where $d'^{j_0} = (d_1, \ldots, d_{j_0-1})$ and $d''^{j_0} = (d_{j_0+1}, \ldots, d_l)$, and similarly for p and w, the partial convolution $\phi *_{[d;j_0]}$ with $\phi \in \mathcal{S}(\mathbb{R}^{d_j})$ just corresponds to the operator which is pointwise induced by the convolution operator on the weighted $L^{p''^{j_0}, d''^{j_0}}(\mathbb{R}^{|d''^{j_0}|_1}, w''^{j_0}; X)$-valued Lebesgue-Bochner space$L^{p_{j_0}}(\mathbb{R}^{d_{j_0}}, w'^{j_0}; L^{p''^{j_0}, d''^{j_0}}(\mathbb{R}^{|d''^{j_0}|_1}, w''^{j_0}; X))$; note here that the latter yields a well-defined continuous linear operator because of the assumption $w_{j_0} \in A_{p_{j_0}}$ (see Lemma 3.4.1).

Given a $\phi \in \mathcal{S}(\mathbb{R}^{d_{j_0}})$, we can define the partial convolution operator $\phi *_{[d;j_0]} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^{d_j}; X))$ via the formula which used in (5.54). In the usual way, we can then define the operator $\phi *_{[d;j_0]} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^{d_j}; X))$ by

$$(\phi *_{[d;j_0]} f)(\psi) := f(\phi *_{[d;j_0]} \psi), \qquad f \in \mathcal{S}'(\mathbb{R}^d; X), \psi \in \mathcal{S}(\mathbb{R}^d).$$

It is standard to see that this operator coincides with the previous definitions for $f \in L^{p,d}(\mathbb{R}^d, w; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X)$ and $f \in \mathcal{S}(\mathbb{R}^{d_j}); X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X)$.

We now come to the definition of the partial Triebel-Lizorkin spaces.

Definition 5.2.36. Let X be a Banach space, $j_0 \in \{1,\ldots,l\}$, $a_{j_0} \in]0,\infty[$, $p \in]1,\infty[^l$, $q \in [1,\infty]$, $s>0$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. Let $\varphi^{[d;j_0]} \in \Phi^{a_{j_0}}(\mathbb{R}^{d_{j_0}})$ with associated family $(S^{[d;j_0]}_n)_{n\in\mathbb{N}}$ as in (5.53). We define the *partial Triebel-Lizorkin space* $F^{s,j_0,a_{j_0}}_{p,q,d,\varphi^{[d;j_0]}}(\mathbb{R}^d, w; X)$ with respect to $\varphi^{[d;j_0]}$ as the space of all $f \in L^{p,d}(\mathbb{R}^d, w; X) \subset \mathcal{S}'(\mathbb{R}^d; X)$ for which

$$\|f\|_{F^{s,j_0,a_{j_0}}_{p,q,d}(\mathbb{R}^d, w; X)} := \left\|(2^{ns} S^{[d;j_0]}_n f)_{n\geq 0}\right\|_{L^{p,d}(\mathbb{R}^d, w)[\ell^q(\mathbb{N})](X)} < \infty.$$

Remark 5.2.37.

(i) In case $l = 1$ this is the ordinary (anisotropic) Triebel-Lizorkin space $F^{s,a}_{p,q}(\mathbb{R}^d, w; X)$, which coincides with the usual (isotropic) Triebel-Lizorkin space $F^{s/a}_{p,q}(\mathbb{R}^d, w; X)$ by Proposition 5.2.34. Here we use that
$$F^{s,a}_{p,q}(\mathbb{R}^d, w; X) = F^{s/a}_{p,q}(\mathbb{R}^d, w; X) \subset L^{p,d}(\mathbb{R}^d, w; X)$$
(in view of $s > 0$) and that the (partial) convolution operator on $L^{p,d}(\mathbb{R}^d, w; X)$ is obtained by restriction of the (partial) convolution operator $\mathcal{S}'(\mathbb{R}^d; X)$.

(ii) In case $l = 2$, $p, q \in]1, \infty[$, $\tilde{p} := (p, q)$ we have, by Fubini,
$$F^{s,1,a_1}_{\tilde{p},p,d,\varphi^{[d;1]}}(\mathbb{R}^d, w; X) = F^{s,a_1}_{q,p}(\mathbb{R}^{d_2}, w_2; L^p(\mathbb{R}^{d_1}, w_1; X))$$
and
$$F^{s,2,a_2}_{\tilde{p},p,d,\varphi^{[d;2]}}(\mathbb{R}^d, w; X) = L^q(\mathbb{R}^{d_2}, w_2; F^{s,a_1}_{p,p}(\mathbb{R}^{d_1}, w_1; X)).$$
By Proposition 5.2.34, the above spaces are precisely the two spaces occurring in the intersection (5.52).

(iii) It is a natural question whether the partial Triebel-Lizorkin spaces $F^{s,j_0,a_{j_0}}_{p,q,d,\varphi^{[d;j_0]}}(\mathbb{R}^d, w; X)$ depend on the chosen $\varphi^{[d;j_0]} \in \Phi^{a_{j_0}}(\mathbb{R}^{d_{j_0}})$. We expect not, but we will not care about this. The reason is that, thanks to (ii), this will not matter for our proof of Theorem 5.2.35.

We can now state the representation result of the general anisotropic Triebel-Lizorkin space $F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ (with $s > 0$ and $q > 1$) as an intersection of partial Triebel-Lizorkin spaces:

Proposition 5.2.38. *Let X be a Banach space, $a \in]0, \infty[^l$, $p \in]1, \infty[^l$, $q \in]1, \infty]$, $s > 0$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. Then there exist $\varphi^{[d;j_0]} \in \Phi^{a_{j_0}}(\mathbb{R}^{d_{j_0}})$, $j_0 = 1, \dots, l$, such that*
$$F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) = \bigcap_{j_0=1}^{l} F^{s,j_0,a_{j_0}}_{p,q,d,\varphi^{[d;j_0]}}(\mathbb{R}^d, w; X)$$
with an equivalence of norms.

Note that Theorem 5.2.35 is an immediate consequence of this proposition and Remark 5.2.37.(ii). So it remains to prove this proposition.

As already announced above, we will prove this proposition via a characterization by differences. For $f \in L^{p,d}(\mathbb{R}^d, w; X)$, $m \in \mathbb{N}$ and $j_0 \in \{1, \dots, l\}$, the m-th order difference operator is defined as
$$\Delta^m_h f(x) := \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} f(x + (m-\nu)h), \quad x, h \in \mathbb{R}^d,$$
and the m-th order $[d; j_0]$-partial difference operator is defined as
$$\Delta^m_{[d;j_0],h_{j_0}} f(x) := \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu} f(x + (m-\nu)\iota_{[d;j_0]} h_{j_0}), \quad x \in \mathbb{R}^d, h_{j_0} \in \mathbb{R}^{d_{j_0}}.$$
With these difference operators we define:
$$[f]^{[m],cont}_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} := \left\| \left(\int_0^\infty t^{-sq} \left(t^{-d\cdot a} \int_{|h|_{d,a} \leq t} \left\| \Delta^m_h f \right\|_X dh \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{L^{p,d}(\mathbb{R}^d,w)}$$

$$[f]^{[m],discr}_{F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)} := \left\| \left(2^{sk} \int_{|h|_{\mathpzc{d},a}\leq 1} \left\| \Delta^m_{\delta^{[\mathpzc{d},a]}_{2^{-k}}h} f \right\|_X dh \right)_{k\in\mathbb{Z}} \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)[\ell^q(\mathbb{Z})]}$$

$$|||f|||^{[m],cont/discr}_{F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)} := \|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)} + [f]^{[m],cont/discr}_{F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)}$$

$$[f]^{[m],cont}_{F^{s,j_0,a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)} := \left\| \left(\int_0^\infty t^{-sq} \left(t^{-d_{j_0}a_{j_0}} \int_{|h_{j_0}|\leq t^{a_{j_0}}} \left\| \Delta^m_{[\mathpzc{d};j_0],h_{j_0}} f \right\|_X dh_{j_0} \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)}$$

$$[f]^{[m],discr}_{F^{s,j_0,a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)} := \left\| \left(2^{sk} \int_{|h_{j_0}|\leq 1} \left\| \Delta^m_{[\mathpzc{d};j_0],2^{-ka_{j_0}}h_{j_0}} f \right\|_X dh_{j_0} \right)_{k\in\mathbb{Z}} \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)[\ell^q(\mathbb{Z})]}$$

$$|||f|||^{[m],cont/discr}_{F^{s,j_0,a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)} := \|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)} + [f]^{[m],cont/discr}_{F^{s,j_0,a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)}.$$

Having this notation, we start the proof of Proposition 5.2.38 with the following proposition, which gives a characterization of the partial Triebel-Lizorkin spaces in terms of the partial difference operators.

Proposition 5.2.39. *Let X be a Banach space, $j_0 \in \{1,\dots,l\}$, $a_{j_0} \in]0,\infty[$, $p \in]1,\infty[^l$, $q \in [1,\infty]$, $s > 0$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. Let $m \in \mathbb{N}$ be such that $m > \frac{s}{a_{j_0}}$. Then there exists a $\varphi^{[\mathpzc{d};j_0]} \in \Phi^{a_{j_0}}(\mathbb{R}^{d_{j_0}})$ such that $|||\cdot|||^{[m],cont/discr}_{F^{s,j_0,a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)}$ and $\|\cdot\|_{F^{s,j_0,a_{j_0}}_{p,q,\mathpzc{d},\varphi^{[\mathpzc{d};j_0]}}(\mathbb{R}^d,w;X)}$ define equivalent extended norms on $L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)$.*

By Remark 5.2.37.(i) we obtain the following difference norm characterization of $F^{s,a}_{p,q}(\mathbb{R}^d,w;X)$, which for $a = 1$ includes the ordinary weighted isotropic Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^d,w;X)$.

Corollary 5.2.40. *Let X be a Banach space, $l = 1$, $a \in]0,\infty[$, $p \in]1,\infty[$, $q \in [1,\infty]$, $s > 0$, and $w \in A_p(\mathbb{R}^d)$. Choose $m \in \mathbb{N}$ such that $m > s/a$. Then $|||\cdot|||^{[m],cont/discr}_{F^{s,1,a}_{p,q,d}(\mathbb{R}^d,w;X)}$ and $\|\cdot\|_{F^{s,a}_{p,q}(\mathbb{R}^d,w;X)}$ define equivalent extended norms on $L^p(\mathbb{R}^d,w;X)$.*

In the proof of Proposition 5.2.39 we need the following lemma.

Lemma 5.2.41. *Let $m \in \mathbb{N}$, $0 < r \leq 1$, and $\varphi^{[\mathpzc{d};j_0]} \in \Phi^{a_{j_0}}(\mathbb{R}^{d_{j_0}})$. Then there exists a constant $C > 0$ such that for $f_n = S^{[\mathpzc{d};j_0]}_n f$ with $f \in L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)$ it holds that*

$$\left\| \Delta^m_{[\mathpzc{d};j_0],h_{j_0}} f_n(x) \right\|_X \leq C \left(M_{[\mathpzc{d};j_0],r} \|f_n\|_X \right)(x) \begin{cases} (2^{na_{j_0}}|h_{j_0}|)^m, & \text{if } |h_{j_0}| \leq 2^{-na_{j_0}}; \\ (2^{na_{j_0}}|h_{j_0}|)^{d_{j_0}/r}, & \text{if } |h_{j_0}| > 2^{-na_{j_0}}. \end{cases}$$

for allmost all $x \in \mathbb{R}^d$ and all $h_{j_0} \in \mathbb{R}^{d_{j_0}}$.

Proof. Write $\mathpzc{d} = (\mathpzc{d}', d_{j_0}, \mathpzc{d}'')$, $p = (p', p_{j_0}, p'')$, $w = (w', w_{j_0}, w'')$ and put $d' := |\mathpzc{d}'|_1$, $d'' := |\mathpzc{d}''|_1$. For almost all $x'' \in \mathbb{R}^{d''}$ we have that $g_{x''}(x_{j_0}) := f(\cdot, x_{j_0}, x'')$ and $g_{n,x''}(x_{j_0}) := f_n(\cdot, x_{j_0}, x'')$

define elements $g_{n,x''}, g_{x''} \in L^{p_{j_0}}(\mathbb{R}^{d_{j_0}}, w_{j_0}; L^{p',d'}(\mathbb{R}^{d'}, w'; X)) \subset \mathcal{S}'(\mathbb{R}^{d_{j_0}}; L^{p',d'}(\mathbb{R}^{d'}, w'; X))$ related by $g_{n,x''} = \varphi_n^{[d;j_0]} * g_{x''}$. So each $g_{n,x''}$ is a $L^{p',d'}(\mathbb{R}^{d'}, w'; X)$-valued tempered distribution belonging to $L^{p_{j_0}}(\mathbb{R}^{d_{j_0}}, w_{j_0}; L^{p',d'}(\mathbb{R}^{d'}, w'; X))$ and having Fourier support contained in $B(0, c2^{na_{j_0}})$ for some $c > 0$ independent of f, n, x''. With Lemma 3.4.10 we thus obtain that

$$\left\|\Delta^m_{h_{j_0}} g_{n,x''}(x_{j_0})\right\|_X \leq C\left(M_r \left\|g_{n,x''}\right\|_X\right)(x_{j_0}) \begin{cases} (2^{na_{j_0}}|h_{j_0}|)^m, & \text{if } |h_{j_0}| \leq 2^{-na_{j_0}}; \\ (2^{na_{j_0}}|h_{j_0}|)^{d_{j_0}/r}, & \text{if } |h_{j_0}| > 2^{-na_{j_0}}, \end{cases}$$

for some constant $C > 0$ independent of f, n, x''. The desired result now follows by definition of $g_{n,x''}$ and $M_{[d;j_0],r}$. □

With a slight modification of the above proof we can get the following lemma, which we will need later on in the proof of Proposition 5.2.44.

Lemma 5.2.42. *Let $m \in \mathbb{N}$, $0 < r \leq 1$, and $j_0 \in \{1, \ldots, l\}$. Let $(S_n)_{n\in\mathbb{N}}$ be the family of convolution operators associated with a $\varphi \in \Phi^{d,a}(\mathbb{R}^d)$. Then there exists a constant $C > 0$ such that for $f_n = S_n f$ with $f \in L^{p,d}(\mathbb{R}^d, w; X)$ it holds that*

$$\left\|\Delta^m, \iota_{[d;j_0]} h_{j_0} f_n(x)\right\|_X \leq C\left(M_{[d;j_0],r}\|f_n\|_X\right)(x) \begin{cases} (2^{na_{j_0}}|h_{j_0}|)^m, & \text{if } |h_{j_0}| \leq 2^{-na_{j_0}}; \\ (2^{na_{j_0}}|h_{j_0}|)^{d_{j_0}/r}, & \text{if } |h_{j_0}| > 2^{-na_{j_0}}. \end{cases}$$

for allmost all $x \in \mathbb{R}^d$ and all $h_{j_0} \in \mathbb{R}^{d_{j_0}}$.

We are now ready to give the proof of Proposition 5.2.39:

Proof of Proposition 5.2.39. Since $|||\cdot|||^{[m],cont}_{F^{s,l,a_l}_{p,q,d}(\mathbb{R}^d,w;X)}$ and $|||\cdot|||^{[m],discr}_{F^{s,l,a_l}_{p,q,d}(\mathbb{R}^d,w;X)}$ both clearly define norms on the subspaces of $L^{p,d}(\mathbb{R}^d, w; X)$ on which they are finite while $F^{s,l,a_l}_{p,q,d}(\mathbb{R}^d, w; X) \hookrightarrow L^{p,d}(\mathbb{R}^d, w; X)$, it suffices to establish the following:

(I) $[\,\cdot\,]^{[m],cont}_{F^{s,l,a_l}_{p,q,d}(\mathbb{R}^d,w;X)} \sim [\,\cdot\,]^{[m],discr}_{F^{s,l,a_l}_{p,q,d}(\mathbb{R}^d,w;X)}$;

(II) $[\,\cdot\,]^{[m],discr}_{F^{s,l,a_l}_{p,q,d}(\mathbb{R}^d,w;X)} \lesssim \|\cdot\|_{F^{s,l,a_l}_{p,q,d}(\mathbb{R}^d,w;X)}$;

(III) $\|\cdot\|_{F^{s,l,a_l}_{p,q,d}(\mathbb{R}^d,w;X)} \lesssim |||\cdot|||^{[m],discr}_{F^{s,l,a_l}_{p,q,d}(\mathbb{R}^d,w;X)}$.

(I): We only treat the inequality $[\,\cdot\,]^{[m],cont}_{F^{s,j_0,a_{j_0}}_{p,q,d}(\mathbb{R}^d,w;X)} \lesssim [\,\cdot\,]^{[m],discr}_{F^{s,j_0,a_{j_0}}_{p,q,d}(\mathbb{R}^d,w;X)}$, the reverse being similar. Via a discretization and monotonicity argument we obtain

$$\begin{aligned}
&\left(\int_0^\infty t^{-sq}\left(t^{-d_{j_0}a_{j_0}}\int_{|h_{j_0}|\leq t^{a_{j_0}}}\left\|\Delta^m_{[d;j_0],h_{j_0}}f(x)\right\|_X dh_{j_0}\right)^q \frac{dt}{t}\right)^{1/q} \\
&\quad\leq \left(\sum_{k=-\infty}^{\infty}\int_{[2^{-k},2^{-k+1}[} t^{-sq}\left(t^{-d_{j_0}a_{j_0}}\int_{|h_{j_0}|\leq t^{a_{j_0}}}\left\|\Delta^m_{[d;j_0],h_{j_0}}f(x)\right\|_X dh_{j_0}\right)^q \frac{dt}{t}\right)^{1/q} \\
&\quad\leq \left(\sum_{k=-\infty}^{\infty}\int_{[2^{-k},2^{-k+1}[} \frac{dt}{t} 2^{ksq}\left(2^{kd_{j_0}a_{j_0}}\int_{|h_{j_0}|\leq 2^{-ka_{j_0}}}\left\|\Delta^m_{[d;j_0],h_{j_0}}f(x)\right\|_X dh_{j_0}\right)^q\right)^{1/q} \\
&\quad= \log(2)^{1/q}\left(\sum_{k=-\infty}^{\infty} 2^{ksq}\left(\int_{|y_{j_0}|\leq 1}\left\|\Delta^m_{[d;j_0],2^{-ka_{j_0}}y_{j_0}}f(x)\right\|_X dy_{j_0}\right)^q\right)^{1/q},
\end{aligned}$$

and the desired inequality follows by taking $L^{p,d}(\mathbb{R}^d, w)$-norms.

(II): Let $f \in F^{s,j_0,a_{j_0}}_{p,q,d}(\mathbb{R}^d, w; X)$, put $f_n := S^{(d,j_0)}_n$ for $n \in \mathbb{Z}_{\geq 0}$ and $f_n := 0$ for $n \in \mathbb{Z}_{<0}$, and pick $r \in]0,1[$ and $\lambda \in]0,1[$ such that $s > a_{j_0}\frac{d_{j_0}}{r}(1-\lambda)$. For $g \in L^{p,d}(\mathbb{R}^d, w; X)$ we write

$$I^{\mu,a_{j_0}}_{[d;j_0]}(g, 2^{-k}))(x) := \int_{|h_{j_0}|\leq 1} \left\| \Delta^m_{[d;j_0],2^{-ka_{j_0}}h_{j_0}} g(x) \right\|_X dh_{j_0}.$$

In this notation, using that $f = \sum_{n\in\mathbb{Z}} f_{n+k}$ in $L^{p,d}(\mathbb{R}^d, w; X)$ for each $k \in \mathbb{Z}$, we can estimate

$$[f]^{[m],discr}_{F^{s,j_0,a_{j_0}}_{p,q,d}(\mathbb{R}^d,w;X)} \leq \sum_{n\in\mathbb{Z}} \left\| (2^{sk} I^{m,a_{j_0}}_{(d,j_0)}(f_{n+k}, 2^{-k}))_{k\in\mathbb{Z}} \right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{Z})]}. \tag{5.55}$$

We first estimate the sum over $n \in \mathbb{Z}_{\leq 0}$ in (5.55). For this we first observe that in light of Lemma 5.2.41, for $n \in \mathbb{Z}_{\leq 0}$ and $k \in \mathbb{Z}$,

$$\begin{aligned} I^{m,a_{j_0}}_{(d,j_0)}(f_{n+k}, 2^{-k})(x) &= \int_{|h_{j_0}|\leq 1} \left\| \Delta^m_{[d;j_0],2^{-ka_{j_0}}h_{j_0}} f(x) \right\|_X dh_{j_0} \\ &\leq C_1 \int_{|h_{j_0}|\leq 1} \left(M_{[d;j_0],r} \|f_{n+k}\|_X \right)(x) (2^{na_{j_0}})^m dh_{j_0} \\ &\leq C_2 2^{na_{j_0}m} \left(M_{[d;j_0],r} \|f_{n+k}\|_X \right)(x). \end{aligned}$$

Multiplying this with 2^{sk} and taking $\ell^q(\mathbb{Z})$-norms yields

$$\begin{aligned} \left\| \left(2^{sk} I^{m,a_{j_0}}_{(d,j_0)}(f_{n+k}, 2^{-k})(x)\right)_{k\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} &\leq C_2 \left(\sum_{k\in\mathbb{Z}} 2^{skq} 2^{na_{j_0}mq} \left[\left(M_{[d;j_0],r} \|f_{n+k}\|_X \right)(x) \right]^q \right)^{1/q} \\ &= C_2 2^{n(ma_{j_0}-s)} \left(\sum_{k\in\mathbb{Z}} 2^{s(k+n)q} \left[\left(M_{[d;j_0],r} \|f_{n+k}\|_X \right)(x) \right]^q \right)^{1/q} \\ &= C_2 2^{n(ma_{j_0}-s)} \left(\sum_{i\in\mathbb{N}} 2^{siq} \left[\left(M_{[d;j_0],r} \|f_i\|_X \right)(x) \right]^q \right)^{1/q}. \end{aligned}$$

Since $ma_{j_0} - s > 0$ and $M_{[d;j_0],r}$ is bounded on $L^{p,d}(\mathbb{R}^d, w)[\ell^q(\mathbb{N})]$ (see Theorem 3.1.4), it follows that

$$\begin{aligned} \sum_{n\geq 0} \left\| (2^{sk} I^{m,a_{j_0}}_{(d,j_0)}(f_{n+k}, 2^{-k}))_{k\in\mathbb{Z}} \right\|_{L^{p,d}(\mathbb{R}^d,w)[\ell^q(\mathbb{Z})]} &\leq C_3 \left\| \left(\sum_{i\in\mathbb{N}} 2^{siq} M_{[d;j_0],r} \|f_i\|_X \right) \right\|_{L^d(\mathbb{R}^d,w)} \\ &\leq C_4 \|f\|_{F^{s,j_0,a_{j_0}}_{p,q,d}(\mathbb{R}^d,w;X)}. \end{aligned}$$

Next we estimate the sum over $n \in \mathbb{Z}_{>0}$ in (5.55). For $n \geq 1$ and $k \in \mathbb{Z}$, we have

$$I^{m,a_{j_0}}_{(d,j_0)}(f_{n+k}, 2^{-k})(x) \leq \int_{|h_{j_0}|\leq 1} \left\| \Delta^m_{[d;j_0],2^{-ka_{j_0}}h_{j_0}} f(x) \right\|^\lambda_X dh_{j_0} \sup_{|h_{j_0}|\leq 1} \left\| \Delta^m_{[d;j_0],2^{-ka_{j_0}}h_{j_0}} f(x) \right\|^{1-\lambda}_X.$$

Observing that for the first term in this product we have, for almost all $x \in \mathbb{R}^d$, the estimate

$$\begin{aligned} \int_{|h_{j_0}|\leq 1} \left\| \Delta^m_{[d;j_0],2^{-ka_{j_0}}h_{j_0}} f(x) \right\|^\lambda_X dh_{j_0} &\leq C_5 \left(\|f_{n+k}(x)\|^\lambda + \sum_{\nu=1}^m (\nu^{-1} 2^{a_{j_0}k})^{d_{j_0}} \int_{|y_{j_0}|\leq \nu 2^{-a_{j_0}k}} \left\| f_{n+k}(x + \iota_{[d;j_0]} y_{j_0}) \right\|^\lambda_X dy_{j_0} \right) \\ &\leq C_6 M_{[d;j_0]} \|f_{n+k}\|^\lambda_X (x), \end{aligned}$$

and applying Lemma 5.2.41 to the second term in this product, we obtain

$$I^{m,a_{j_0}}_{(\mathpzc{d},j_0)}(f_{n+k},2^{-k})(x) \leq C_7 M_{[\mathpzc{d};j_0]}\|f_{n+k}\|_X^{\lambda}(x) 2^{\mathpzc{d}_{j_0}a_{j_0}n(1-\lambda)/r}(M_{[\mathpzc{d};j_0]}\|f_{n+k}\|_X^{r}(x))^{(1-\lambda)/r}.$$

Multiplying this with 2^{sk}, taking $\ell^q(\mathbb{Z})$ norms, summing over $n \in \mathbb{Z}_{\geq 1}$, using that $s > \mathpzc{d}_{j_0}a_{j_0}(1-\lambda)/r$, and applying Hölder's inequality, we get

$$\begin{aligned}
&\sum_{n\geq 1}\left\|(2^{sk}I^{m,a_{j_0}}_{(\mathpzc{d},j_0)}(f_{n+k},2^{-k})(x))_{k\in\mathbb{Z}}\right\|_{\ell^q(\mathbb{Z})} \\
&\quad\leq C_7\sum_{n\geq 1}2^{-sn}2^{\mathpzc{d}_{j_0}a_{j_0}n(1-\lambda)/r}\left(\sum_{k\in\mathbb{Z}}2^{s(n+k)q}(M_{[\mathpzc{d};j_0]}\|f_{n+k}\|_X^{\lambda}(x))^q(M_{[\mathpzc{d};j_0]}\|f_{n+k}\|_X^{r})^{q(1-\lambda)/r}(x)\right)^{1/q} \\
&\quad\leq C_8\left(\sum_{i\geq 0}2^{siq}(M_{[\mathpzc{d};j_0]}\|f_i\|_X^{\lambda}(x))^q(M_{[\mathpzc{d};j_0]}\|f_i\|_X^{r}(x))^{q(1-\lambda)/r}\right)^{1/q} \\
&\quad\leq C_8\left(\sum_{i\geq 0}2^{siq}(M_{[\mathpzc{d};j_0]}\|f_i\|_X^{\lambda}(x))^{q/\lambda}\right)^{\lambda/q}\left(\sum_{i\geq 0}2^{siq}(M_{[\mathpzc{d};j_0]}\|f_i\|_X^{r}(x))^{q/r}\right)^{(1-\lambda)/q} \\
&\quad= C_8\left\|\left(2^{si}M_{[\mathpzc{d};j_0],\lambda}(\|f_i\|_X)(x)\right)_{i\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})}^{\lambda}\left\|\left(2^{si}M_{[\mathpzc{d};j_0],r}(\|f_i\|_X)(x)\right)_{i\in\mathbb{N}}\right\|_{\ell^q(\mathbb{N})}^{1-\lambda}.
\end{aligned}$$

Finally, taking $L^{p,\mathpzc{d}}(\mathbb{R}^d,w)$-norms, applying Hölder's inequality l times, and using Theorem 3.1.4, we obtain

$$\begin{aligned}
&\sum_{n\geq 1}\left\|(2^{sk}I^{m,a_{j_0}}_{(\mathpzc{d},j_0)}(f_{n+k},2^{-k}))_{k\in\mathbb{Z}}\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w)[\ell^q(\mathbb{Z})]} \\
&\quad\leq C_8\left\|\left(2^{si}M_{[\mathpzc{d};j_0],\lambda}(\|f_i\|_X)\right)_{i\geq 0}\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N}))}^{\lambda}\left\|\left(2^{si}M_{[\mathpzc{d},j_0],r}(\|f_i\|_X)\right)_{i\geq 0}\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N}))}^{1-\lambda} \\
&\quad\leq C_9\left\|(2^{is}\|f_i\|_X)_{i\geq 0}\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N}))}^{\lambda}\left\|(2^{is}\|f_i\|_X)_{i\geq 0}\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N}))}^{1-\lambda} \\
&\quad= C_9\|f\|_{F^{s,j_0,a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)}.
\end{aligned}$$

(III): Let $f \in L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)$. Choose $\psi \in \mathcal{S}(\mathbb{R}^{\mathpzc{d}_l})$ such that $\hat{\psi}(\xi_{j_0}) = 1$ for $|\xi_{j_0}|^{1/a_{j_0}} \leq 1$ and $\hat{\psi}(\xi) = 0$ if $|\xi|^{1/a_{j_0}} \geq 3/2$. Now let $(\varphi_n)_{n\in\mathbb{N}} \in \Phi^{a_{j_0}}(\mathbb{R}^{\mathpzc{d}_{j_0}})$ be determined by

$$\hat{\varphi}_0(\xi_{j_0}) := (-1)^{m+1}\sum_{\nu=0}^{m-1}(-1)^{\nu}\binom{\mu}{\nu}\hat{\psi}(-(m-\nu)\xi_{j_0});$$

note that $\hat{\varphi}_0(\xi_{j_0})$ for $|\xi_{j_0}|^{1/a_{j_0}} \leq 1/\mu^{1/a_{j_0}}$ and $\hat{\varphi}_0(\xi_{j_0}) = 0$ for $|\xi_{j_0}|^{1/a_{j_0}} \geq 3/2$. Let $(S_n^{[\mathpzc{d};j_0]})_{n\in\mathbb{N}}$ be the corresponding partial convolution operators and let $(T_n^{[\mathpzc{d};j_0]})_{n\in\mathbb{N}}$ be the partial convolution operators corresponding to the functions $(2^{na_{j_0}\mathpzc{d}_{j_0}}\varphi_0(2^{na_{j_0}}\,\cdot\,))_{n\in\mathbb{N}}$. Then $S_n^{[\mathpzc{d};j_0]}f = T_{n+1}^{[\mathpzc{d};j_0]}f - T_n^{[\mathpzc{d};j_0]}f$ for all $n \geq 1$, so that

$$\begin{aligned}
\|f\|_{F^{s,j_0,a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)} &= \left\|\left(2^{sn}\left\|S_n^{[\mathpzc{d};j_0]}f\right\|_X\right)_{n\geq 0}\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N}))} \\
&\leq \left\|S_0^{[\mathpzc{d};j_0]}f\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)} + C_1\left\|\left(2^{sn}\left\|f - T_n^{[\mathpzc{d};j_0]}f\right\|_X\right)_{n\geq 1}\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;\ell^q(\mathbb{N}_{\geq 1}))}.
\end{aligned}$$

As $w_{j_0} \in A_{p_{j_0}}(\mathbb{R}^{\mathpzc{d}_{j_0}})$ we have

$$\left\|S_0^{[\mathpzc{d};j_0]}f\right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)} \leq C_2\|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)}.$$

So it remains to estimate the second term on the RHS.

First we compute

$$\begin{aligned} f(x) - T_n^{[\mathscr{d};j_0]} f(x) &= f(x) - 2^{na_{j_0}\mathscr{d}_{j_0}} \int_{\mathbb{R}^{\mathscr{d}_{j_0}}} f(x - \iota_{[\mathscr{d};j_0]} z_l)\varphi_0(2^{na_{j_0}} z_{j_0}) dz_{j_0} \\ &= f(x) - (-1)^{m+1} \sum_{\nu=0}^{m-1} \binom{m}{\nu} (-1)^\nu (m-\nu)^{-\mathscr{d}_{j_0}} 2^{na_{j_0}\mathscr{d}_{j_0}} \int_{\mathbb{R}^{\mathscr{d}_{j_0}}} f(x - \iota_{[\mathscr{d};j_0]} z_{j_0})\psi(-(m-\nu)^{-1} 2^{na_{j_0}} z_{j_0}) \\ &= f(x) + (-1)^m \sum_{\nu=0}^{m-1} \binom{m}{\nu} (-1)^\nu \int_{\mathbb{R}^{\mathscr{d}_{j_0}}} f(x + (m-\nu)2^{-na_{j_0}} \iota_{[\mathscr{d};j_0]} y_{j_0})\psi(y_{j_0}) dy_{j_0} \\ &= (-1)^m \sum_{\nu=0}^{m} \binom{m}{\nu} (-1)^\nu \int_{\mathbb{R}^{\mathscr{d}_{j_0}}} f(x + (m-\nu)2^{-na_{j_0}} \iota_{[\mathscr{d};j_0]} y_{j_0})\psi(y_{j_0}) dy_{j_0} \\ &= (-1)^m \int_{\mathbb{R}^{\mathscr{d}_{j_0}}} \Delta^m_{[\mathscr{d};j_0], 2^{-na_{j_0}} y_{j_0}} f(x)\psi(y_{j_0}) dy_{j_0}. \end{aligned}$$

Next, picking $r > s + a_{j_0}\mathscr{d}_{j_0}$ and choosing a corresponding constant $C_\psi > 0$ for $\psi \in \mathcal{S}(\mathbb{R}^{\mathscr{d}_l})$ such such that $|(1 + |x_{j_0}|^{r/a_{j_0}})\psi(x_{j_0})| \leq C_\psi$ for all $x_{j_0} \in \mathbb{R}^{\mathscr{d}_{j_0}}$, we can estimate

$$\begin{aligned} \left\| f(x) - T_n^{[\mathscr{d};j_0]} f(x) \right\| &\leq \int_{\mathbb{R}^{\mathscr{d}_{j_0}}} \left\| \Delta^m_{[\mathscr{d};j_0], 2^{-na_{j_0}} y_{j_0}} f(x)\psi(y_{j_0}) \right\|_X dy_{j_0} \\ &\leq C_\psi \int_{|y_{j_0}|^{1/a_{j_0}} \leq 1} \left\| \Delta^m_{[\mathscr{d};j_0], 2^{-na_{j_0}} y_{j_0}} f(x) \right\|_X dy_{j_0} \\ &\quad + C_\psi \sum_{k\geq 0} 2^{-kr} \int_{2^k \leq |y_{j_0}|^{1/a_{j_0}} \leq 2^{k+1}} \left\| \Delta^\mu_{[\mathscr{d};j_0], 2^{-na_{j_0}} y_{j_0}} f(x) \right\|_X dy_{j_0} \\ &= C_\psi 2^{na_{j_0}\mathscr{d}_{j_0}} \left(\int_{|h_{j_0}|^{1/a_{j_0}} \leq 2^{-n}} \left\| \Delta^m_{[\mathscr{d};j_0], h_{j_0}} f(x) \right\|_X dh_{j_0} \right. \\ &\quad \left. + \sum_{k\geq 0} 2^{-kr} \int_{2^{k-n} \leq |h_{j_0}|^{1/a_{j_0}} \leq 2^{k+1-n}} \left\| \Delta^m_{[\mathscr{d};j_0], h_{j_0}} f(x) \right\|_X dh_{j_0} \right) \\ &\leq C_3 \sum_{k\geq -1} 2^{-kr} 2^{(k+1)a_{j_0}\mathscr{d}_{j_0}} 2^{-(k+1-n)a_{j_0}\mathscr{d}_{j_0}} \int_{|h_{j_0}|^{1/a_{j_0}} \leq 2^{k+1-n}} \left\| \Delta^\mu_{[\mathscr{d};j_0], h_{j_0}} f(x) \right\|_X dh_{j_0}. \end{aligned}$$

Finally, multiplying with 2^{sn}, taking $L^{p,\mathscr{d}}(\mathbb{R}^d, w; \ell^q(\mathbb{N}_{\geq 1}))$-norms, using a monotonicity argument similar to the proof of (I), and recalling that $r > s + a_{j_0}\mathscr{d}_{j_0}$, we obtain

$$\begin{aligned} &\left\| \left(2^{sn} \left\| f - T_n^{[\mathscr{d};j_0]} f \right\|_X \right)_{n\geq 1} \right\|_{L^{p,\mathscr{d}}(\mathbb{R}^d, w; \ell^q(\mathbb{N}_{\geq 1}))} \\ &\leq C_3 \sum_{k\geq -1} 2^{-kr} 2^{(k+1)a_{j_0}\mathscr{d}_{j_0}} 2^{s(k+1)} \left\| \left(\sum_{n\geq 1} 2^{-sq(k+1-n)} 2^{-(k+1-n)a_{j_0}\mathscr{d}_{j_0} q} \left(\int_{|h_{j_0}|^{1/a_{j_0}} \leq 2^{k+1-n}} \left\| \Delta^\mu_{[\mathscr{d};j_0], h_{j_0}} f(x) \right\|_X dh_{j_0} \right)^q \right)^{1/q} \right\|_{L^{p,\mathscr{d}}} \\ &\leq C_4 \sum_{k\geq -1} 2^{(s + a_{j_0}\mathscr{d}_{j_0} - r)k} \left\| \left(\int_0^\infty t^{-sq} \left(t^{-\mathscr{d}_{j_0} a_{j_0}} \int_{|h_{j_0}| \leq t^{a_{j_0}}} \left\| \Delta^m_{[\mathscr{d};j_0], h_{j_0}} f(x) \right\|_X dh_{j_0} \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{L^{p,\mathscr{d}}(\mathbb{R}^d, w)} \\ &\leq C_5 \, |||f|||^{[m],cont}_{F^{s,j_0,a_{j_0}}_{p,q,\mathscr{d}}(\mathbb{R}^d, w; X)}. \end{aligned}$$

□

With an adaption of the proof of (II) above we can show that, in case $q > 1$, the following "shifted" version of (II) holds as well, which we will need for the proof of Proposition 5.2.38.

Lemma 5.2.43. *Let X be a Banach space, $j_0 \in \{1, \ldots, l\}$, $a_{j_0} \in]0, \infty[$, $p \in]1, \infty[^l$, $q \in]1, \infty]$, $s > 0$, and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$. Let $m \in \mathbb{N}$ be such that $m > \frac{s}{a_{j_0}}$ and let $c \in \mathbb{R}$. Then there is a constant $C > 0$ such that*

$$\left\| \left(2^{sk} \int_{|h_{j_0}| \leq 1} \left\| \Delta^m_{(\mathpzc{d}, j_0), 2^{-ka_{j_0}} h_{j_0}} f(x + c2^{-ka_{j_0}} \iota_{[\mathpzc{d}; j_0]} h_{j_0}) \right\|_X dh_{j_0} \right)_{k \in \mathbb{Z}} \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d, w; \ell^q(\mathbb{Z}))} \leq C \|f\|_{F^{s, j_0, a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)}$$

for all $f \in F^{s, j_0, a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)$.

Proof. Let $f \in F^{s, j_0, a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)$ and put $f_n := S^{[\mathpzc{d}; j_0]}_n f$ for $n \in \mathbb{Z}_{\geq 0}$ and $f_n := 0$ for $n \in \mathbb{Z}_{<0}$. For $g \in L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ we write

$$I^{m, a_{j_0}}_{[\mathpzc{d}; j_0], c}(g, 2^{-k}))(x) := \int_{|h_{j_0}| \leq 1} \left\| \Delta^m_{[\mathpzc{d}; j_0], 2^{-ka_{j_0}} h_{j_0}} g(x + c2^{-ka_{j_0}} \iota_{[\mathpzc{d}; j_0]} h_{j_0})) \right\|_X dh_{j_0}.$$

In this notation, using that $f = \sum_{n \in \mathbb{Z}} f_{n+k}$ in $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ for each $k \in \mathbb{Z}$, we again have an estimate

$$\left\| (2^{sk} I^{m, a_{j_0}}_{[\mathpzc{d}; j_0], c}(f_{n+k}, 2^{-k})) \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d, w; \ell^q(\mathbb{Z}))} \leq \sum_{n \in \mathbb{Z}} \left\| (2^{sk} I^{m, a_{j_0}}_{[\mathpzc{d}; j_0], c}(f_{n+k}, 2^{-k}))_{k \in \mathbb{Z}} \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d, w; \ell^q(\mathbb{Z}))}. \tag{5.56}$$

The sum over $n \in \mathbb{Z}_{\geq 1}$ in (5.56) can be estimated in the same way as in (II) of the proof of Proposition 5.2.39; note that it could also similarly be estimated more directly without making use of λ and r and using the Boundedness of $M_{[\mathpzc{d}; j_0]}$ on $L^{p,\mathpzc{d}}(\mathbb{R}^d, w)[\ell^q(\mathbb{N})]$.

To finish, we estimate the sum over $n \in \mathbb{Z}_{\leq 0}$ in (5.55). Of course, we only need to consider the case $c \neq 0$. Again using Lemma 5.2.41, for $n \in \mathbb{Z}_{\leq 0}$ and $k \in \mathbb{Z}$, we now have

$$\begin{aligned} I^{m, a_{j_0}}_{[\mathpzc{d}; j_0], c}(f_{n+k}, 2^{-k})(x) &= \int_{|h_{j_0}| \leq 1} \left\| \Delta^m_{[\mathpzc{d}; j_0], 2^{-ka_{j_0}} h_{j_0}} f(x + c2^{-ka_{j_0}} \iota_{[\mathpzc{d}; j_0]} h_{j_0})) \right\|_X dh_{j_0} \\ &\leq C_1 \int_{|h_{j_0}| \leq 1} M_{[\mathpzc{d}; j_0]}(\|f_{n+k}\|_X)(x + c2^{-ka_{j_0}} \iota_{[\mathpzc{d}; j_0]} h_{j_0}))(2^{na_{j_0}})^{\mu} dh_{j_0} \\ &\leq C_2 2^{na_{j_0} m} M^2_{[\mathpzc{d}; j_0]}(\|f_{n+k}\|_X)(x) \end{aligned}$$

Now the proof can be completed in the same way as in (II) of the proof of Proposition 5.2.39, now using the boundedness of $M_{[\mathpzc{d}; j_0]}$ on $L^{p,\mathpzc{d}}(\mathbb{R}^d, w)[\ell^q(\mathbb{N})]$. □

Finally, we prove the following proposition, which contains Proposition 5.2.38 in it.

Proposition 5.2.44. *Let X be a Banach space, $a \in]0, \infty[^l$, $p \in]1, \infty[^l$, $q \in]1, \infty]$, $s > 0$, and $w \in \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{d_j})$. Let $m \in \mathbb{N}$ be such that $m > \frac{s}{a_j}$ for all $j \in \{1, \ldots, l\}$. Then the following define equivalent extended norms on $L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$:*

(i) $\|\cdot\|_{F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)}$;

(ii) $|||\cdot|||^{[lm], cont}_{F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)} = \|\cdot\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d, w; X)} + [\,\cdot\,]^{[lm], cont}_{F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, w; X)}$;

(iii) $|||\cdot|||^{[lm],discr}_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} = \|\cdot\|_{L^{p,d}(\mathbb{R}^d,w;X)} + [\,\cdot\,]^{[lm],discr}_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)}$;

(iv) $\sum_{j=1}^{l} \|\cdot\|_{F^{s,j,a_j}_{p,q,d}(\mathbb{R}^d,w;X)}$;

(v) $\|\cdot\|_{L^{p,d}(\mathbb{R}^d,w;X)} + \sum_{j=1}^{l} [\,\cdot\,]^{[m],cont}_{F^{s,j,a_j}_{p,q,d}(\mathbb{R}^d,w;X)}$;

(vi) $\|\cdot\|_{L^{p,d}(\mathbb{R}^d,w;X)} + \sum_{j=1}^{l} [\,\cdot\,]^{[m],discr}_{F^{s,j,a_j}_{p,q,d}(\mathbb{R}^d,w;X)}$.

Proof. First note that they indeed define norms on the subspaces of $L^{p,d}(\mathbb{R}^d, w; X)$ on which they are finite. From Proposition 5.2.39 it follows that the norms in (iv),(v),(vi) are equivalent. Analogously to (I) in the proof Proposition 5.2.39 it can be shown that the norms in (ii) and (iii) are equivalent. So it is enough to show that (ii) is stronger than (i), (iv) is stronger than (iii), and (i) is stronger than (vi).

$\|\cdot\|_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} \lesssim |||\cdot|||^{[lm],cont}_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)}$: This can be done completely analogously to (III) in the proof of Proposition 5.2.39.

$|||\cdot|||^{[lm],discr}_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} \lesssim \sum_{j=1}^{l} \|\cdot\|_{F^{s,j,a_j}_{p,q,d}(\mathbb{R}^d,w;X)}$: Let $f \in L^{p,d}(\mathbb{R}^d, w; X)$. Since $F^{s,j,a_j}_{p,q,d}(\mathbb{R}^d, w; X) \hookrightarrow L^{p,d}(\mathbb{R}^d, w; X)$, we only need to estimate $[f]^{[lm],discr}_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)}$. To this end, we first note that

$$\left\|\Delta^{lm}_{\delta^{[d,a]}_{2^{-k}}h} f(x)\right\|_X \leq C_1 \sum_{n=1}^{N} \sum_{j=1}^{l} \left\|\Delta^m_{[d;j],2^{-ka_j}h_j} f(x + \sum_{i=1}^{m} c_i^{[n]} 2^{-ka_i}\iota_{[d;i]}h_i)\right\|_X \qquad (x, h \in \mathbb{R}^d, k \in \mathbb{Z})$$

for certain $C_1 > 0$, $N \in \mathbb{N}$ and $\{c_j^{[n]}\}_{j=1,\dots,m;n=1,\dots,N} \subset \mathbb{R}$; this can, for instance, be seen by writing out $\Delta^{lm}_y f = \mathscr{F}^{-1}[(e_{\iota y} - 1)^{lm}\hat{f}]$. So we can estimate

$$\begin{aligned}
&\int_{|h|_{d,a}\leq 1} \left\|\Delta^{lm}_{(2^{-k})^{d,a}} f(x)\right\|_X dh \\
&\quad\leq C_1 \sum_{n=1}^{N} \sum_{j=1}^{l} \int_{|h_1|,\dots,|h_l|\leq 1} \left\|\Delta^m_{[d;j],2^{-ka_j}h_j} f(x + \sum_{i=1}^{l} c_j^{[n]} 2^{-ka_i}\iota_{[d;i]}h_i)\right\|_X d(h_1, \dots, h_l) \\
&\quad\leq C_2 \sum_{n=1}^{N} \sum_{j=1}^{l} M_{[d;1]} \dots M_{[d;l]} \left(\int_{|h_j|\leq 1} \left\|\Delta^m_{[d;j],2^{-ka_j}h_j} f(\,\cdot\, + c_j^{[n]} 2^{-ka_j}\iota_{[d;j]}h_j)\right\|_X dh_j\right)(x).
\end{aligned}$$

Multiplying with 2^{sk}, taking $L^{p,d}(\mathbb{R}^d, w)[\ell^q(\mathbb{Z})]$-norms, using the boundedness of $M_{[d;1]} \dots M_{[d;l]}$ on $L^{p,d}(\mathbb{R}^d, w)[\ell^q(\mathbb{Z})]$, and invoking Lemma 5.2.43, we obtain

$$\begin{aligned}
&[f]^{[lm],discr}_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} \\
&\quad\leq \sum_{n=1}^{N} \sum_{j=1}^{l} \left\|\left(2^{sk} M_{[d;1]} \dots M_{[d;l]} \int_{|h_j|\leq 1} \left\|\Delta^m_{[d;j],2^{-ka_j}h_j} f(\,\cdot\, + c_j^{[n]} 2^{-ka_j}\iota_{[d;j]}h_j)\right\|_X dh_j\right)_{k\in\mathbb{Z}}\right\|_{L^{p,d}(\mathbb{R}^d,w;\ell^q(\mathbb{Z}))} \\
&\quad\leq C_3 \sum_{n=1}^{N} \sum_{j=1}^{l} \left\|\left(2^{sk} \int_{|h_j|\leq 1} \left\|\Delta^m_{[d;j],2^{-ka_j}h_j} f(\,\cdot\, + c_j^{[n]} 2^{-ka_j}\iota_{[d;j]}h_j)\right\|_X dh_j\right)_{k\in\mathbb{Z}}\right\|_{L^{p,d}(\mathbb{R}^d,w;\ell^q(\mathbb{Z}))} \\
&\quad\leq C_4 N \sum_{j=1}^{l} \|f\|_{F^{s,j,a_j}_{p,q,d}(\mathbb{R}^d,w;X)}.
\end{aligned}$$

$\|\cdot\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)}+\sum_{j_0=1}^{l}[\,\cdot\,]^{[m],discr}_{F^{s,j_0,a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)} \lesssim \|\cdot\|_{F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)}$: Since $F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X) \hookrightarrow L^{p,\mathpzc{d}}(\mathbb{R}^d,w;X)$ ($s>0$), we just need to show that $[\,\cdot\,]^{[m],discr}_{F^{s,j_0,a_{j_0}}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)} \lesssim \|\cdot\|_{F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,w;X)}$ for each j_0. This can be done completely analogously to (II) in the proof of Proposition 5.2.39, replacing $S^{[\mathpzc{d};l]}_n$ by S_n and using Lemma 5.2.42 instead of Lemma 5.2.41. □

5.2.2.f Embeddings and Isomorphisms under Restrictions on the Banach Space

As already announced, the identity $W^n_{p,\mathpzc{d}}(\mathbb{R}^d,w;X) = H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d,w;X)$ holds under some restrictions on the Banach space X and/or the weight-vector w:

Proposition 5.2.45. *Let X be a UMD space, $p\in]1,\infty[^l$, $n\in\mathbb{N}^l$, $s\in\mathbb{R}$, $a\in\left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$ with $n_j=\frac{s}{a_j}$ for each $j\in\{1,\dots,l\}$, and*

$$w \in \begin{cases} \prod_{j=1}^{l} A^{rec}_{p_j}(\mathbb{R}^{\mathpzc{d}_j}), & l>1; \\ \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{\mathpzc{d}_j}), & l=1 \text{ or } X \text{ has } (\alpha). \end{cases}$$

Then we have

$$W^n_{p,\mathpzc{d}}(\mathbb{R}^d,w;X) = H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d,w;X)$$

with an equivalence of norms.

Proof. This can be shown in exactly the same way as Proposition 5.2.29, now using Theorem 4.1.1 instead of Proposition 5.2.26. □

Next we come the an intersection representation for anisotropic Bessel potential spaces.

Proposition 5.2.46. *Let X be a UMD space, $a\in\left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$, $p\in]1,\infty[^l$, $s>0$, and*

$$w \in \begin{cases} \prod_{j=1}^{l} A^{rec}_{p_j}(\mathbb{R}^{\mathpzc{d}_j}), & l>1; \\ \prod_{j=1}^{l} A_{p_j}(\mathbb{R}^{\mathpzc{d}_j}), & l=1 \text{ or } X \text{ has } (\alpha). \end{cases}$$

For each $k\in\{1,\dots,l\}$ we write $\mathpzc{d}'^k=(\mathpzc{d}_1,\dots,\mathpzc{d}_{k-1})$ and $\mathpzc{d}''^k=(\mathpzc{d}_{k+1},\dots,\mathpzc{d}_l)$, and similarly for p and w. Then we have

$$H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d,w;X) = \bigcap_{k=1}^{l} L^{p'^k,\mathpzc{d}'^k}\left(\mathbb{R}^{|\mathpzc{d}'^k|_1}, w'^k; H^{s/a_k}_{p_k}(\mathbb{R}^{\mathpzc{d}_k}, w_k; L^{p''^k,\mathpzc{d}''^k}(\mathbb{R}^{|\mathpzc{d}''^k|_1}, w''^k; X))\right)$$

with an equivalence of norms.

Proof. Using Corollary 4.1.2, this can be shown similarly to [6, Theorem 3.7.2]. For more comments we refer to the notes of this chapter. □

The next two results are specially designed for applications in Chapter 6, for which it is more convenient to replace X by E in the notation.

Lemma 5.2.47. *Let E be a UMD Banach space, let $p \in]1, \infty[$, $w \in A_p(\mathbb{R}^d)$, and $n \in \mathbb{Z}_{>0}$. For each $\lambda \in \mathbb{C} \setminus]-\infty, 0]$ and $\sigma \in \mathbb{R}$ we define $L^\sigma_\lambda \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^d; E))$ by*

$$L^\sigma_\lambda f := \mathscr{F}^{-1}[(\lambda + |\cdot|^{2n})^\sigma \hat{f}] \qquad (f \in \mathcal{S}'(\mathbb{R}^d; E)).$$

Then L^σ_λ restricts to a topological linear isomorphism from $H^{s+2n\sigma}_p(\mathbb{R}^d, w; E)$ to $H^s_p(\mathbb{R}^d, w; E)$ (with inverse $L^{-\sigma}_\lambda$) for each $s \in \mathbb{R}$. Moreover,

$$\mathbb{C} \setminus]-\infty, 0] \ni \lambda \mapsto L^\sigma_\lambda \in \mathcal{B}(H^{s+2n\sigma}_p(\mathbb{R}^d, w; E), H^s_p(\mathbb{R}^d, w; E)) \tag{5.57}$$

defines an analytic mapping for every $\sigma \in \mathbb{R}$ and $s \in \mathbb{R}$.

Proof. To show that L^σ_λ restricts to a topological linear isomorphism from $H^{s+2n\sigma}_p(\mathbb{R}^d, w; E)$ to $H^s_p(\mathbb{R}^d, w; E)$ with inverse $L^{-\sigma}_\lambda$ (for arbitrary σ and s) it suffices to show that L^σ_λ restricts to a bounded linear operator from $H^{s+2n\sigma}_p(\mathbb{R}^d, w; E)$ to $H^s_p(\mathbb{R}^d, w; E)$ since we already know that L^σ_λ and $L^{-\sigma}_\lambda$ are inverses of each other as operators on $\mathcal{S}'(\mathbb{R}^d; E)$. For this we must show that $\mathcal{J}_{-n\sigma} L^\sigma_\lambda = \mathcal{J}_s L^\sigma_\lambda \mathcal{J}_{-(s+2n\sigma)}$ restricts to a bounded linear operator on $L^p(\mathbb{R}^d, w; E)$. This can be done be checking the Mikhlin condition from Corollary 4.1.2.

Next we show that the map in (5.2.47) is analytic. We only treat the case $\sigma \in \mathbb{R} \setminus \mathbb{N}$, the case $\sigma \in \mathbb{N}$ being easy. So suppose that $\sigma \in \mathbb{R} \setminus \mathbb{N}$ and fix a $\lambda_0 \in \mathbb{C} \setminus]-\infty, 0]$. We shall show that $\lambda \mapsto L^\sigma_\lambda$ is analytic at λ_0. Since $L^\tau_{\lambda_0}$ is a topological linear isomorphism from $H^{s+2n\tau}_p(\mathbb{R}^d, w; E)$ to $H^s_p(\mathbb{R}^d, w; E)$, $\tau \in \mathbb{R}$, for this it suffices to show that

$$\mathbb{C} \setminus]-\infty, 0] \ni \lambda \mapsto L^\sigma_\lambda L^{-\sigma}_{\lambda_0} = L^{\frac{s}{n}}_{\lambda_0} L^\sigma_\lambda L^{-\frac{1}{n}(s+2n\sigma)}_{\lambda_0} \in \mathcal{B}(L^p(\mathbb{R}^d, w; E))$$

is analytic at λ_0. To this end, we first observe that, for each $\xi \in \mathbb{R}^d$,

$$\mathbb{C} \setminus]-\infty, 0] \ni \lambda \mapsto (\lambda + |\xi|^{2n})^\sigma (\lambda_0 + |\xi|^{2n})^{-\sigma} \in \mathbb{C}$$

is an analytic mapping with power series expansion at λ_0 given by

$$(\lambda + |\xi|^{2n})^\sigma (\lambda_0 + |\xi|^{2n})^{-\sigma} = 1 + \sigma(\lambda_0 + |\xi|^{2n})^{-1}(\lambda - \lambda_0) + \sigma(\sigma - 1)(\lambda_0 + |\xi|^{2n})^{-2}(\lambda - \lambda_0)^2 + \dots \tag{5.58}$$

for $\lambda \in B(\lambda_0, \delta)$, where $\delta := d(0, \{\lambda_0 + t \mid t \geq 0\}) > 0$. We next recall that $L^{-1}_{\lambda_0}$ restricts to a topological linear isomorphism from $L^p(\mathbb{R}^d, w; E)$ to $H^{2n}_p(\mathbb{R}^d, w; E)$; in particular, $L^{-1}_{\lambda_0}$ restricts to a bounded linear operator on $L^p(\mathbb{R}^d, w; E)$. Since $L^{-k}_{\lambda_0} = (L^{-1}_{\lambda_0})^k$ for every $k \in \mathbb{N}$, there thus exists a constant $C > 0$ such that

$$\left\| L^{-k}_{\lambda_0} \right\|_{\mathcal{B}(L^p(\mathbb{R}^d, w; E))} \leq C^k, \qquad \forall k \in \mathbb{N}. \tag{5.59}$$

Now we let $\rho > 0$ be the radius of convergence of the power series $z \mapsto \sum_{k \in \mathbb{N}} \left[\prod_{j=0}^{k-1} (\sigma - j)\right] C^k z^k$, set $r := \min(\delta, \rho) > 0$, and define, for each $\lambda \in B(\lambda_0, r)$, the multiplier symbols $m^\lambda, m^\lambda_0, m^\lambda_1, \dots : \mathbb{R}^d \longrightarrow \mathbb{C}$ by

$$m^\lambda(\xi) := (\lambda + |\xi|^{2n})^\sigma (\lambda_0 + |\xi|^{2n})^{-\sigma} \quad \text{and} \quad m^\lambda_N(\xi) := \sum_{k=0}^N \left[\prod_{j=0}^{k-1} (\sigma - j)\right] (\lambda_0 + |\xi|^{2n})^{-k} (\lambda - \lambda_0)^k.$$

Then, by (5.58) and (5.59), we get

$$m^\lambda(\xi) = \lim_{N \to \infty} m^\lambda_N(\xi), \qquad \xi \in \mathbb{R}^d$$

and

$$\lim_{N,M\to\infty}[T_{m^\lambda_N} - T_{m^\lambda_M}] = 0 \quad \text{in} \quad \mathcal{B}(L^p(\mathbb{R}^d, w; E)),$$

respectively. Via Proposition 4.2.4.(iv) we thus obtain that

$$L^\sigma_\lambda L^{-\sigma}_{\lambda_0} = T_{m^\lambda} = \lim_{N\to\infty} T_{m^\lambda_N} = \lim_{N\to\infty}\sum_{k=0}^{N}\left[\prod_{j=0}^{k-1}(\sigma - j)\right] L^{-k}_{\lambda_0}(\lambda-\lambda_0)^k \quad \text{in} \quad \mathcal{B}(L^p(\mathbb{R}^d, w; E))$$

for $\lambda \in B(\lambda_0, r)$. This shows that the map $\mathbb{C}\setminus]-\infty, 0] \ni \lambda \mapsto L^\sigma_\lambda L^{-\sigma}_{\lambda_0} \in \mathcal{B}(L^p(\mathbb{R}^d, w; E))$ is analytic at λ_0, as desired. □

Lemma 5.2.48. *Let E be a UMD space, $p, q \in]1,\infty[$, $v \in A_q(\mathbb{R})$, and $n \in \mathbb{Z}_{>0}$. For each $\sigma \in \mathbb{R}$ we define $L^\sigma \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^{d-1}\times\mathbb{R}; E))$ by*

$$L^\sigma f := \mathscr{F}^{-1}\left[\left((\xi_1,\xi_2) \mapsto (1 + \imath\xi_2 + |\xi_1|^{2n})^\sigma\right)\hat{f}\right] \qquad (f \in \mathcal{S}'(\mathbb{R}^{d-1}\times\mathbb{R}; E)).$$

Then L^σ restricts to a topological linear isomorphism from $H^{s+\sigma,(\frac{1}{2n},1)}_{(p,q),(d-1,1)}(\mathbb{R}^{d-1}\times\mathbb{R},(1,v);E)$ to $H^{s,(\frac{1}{2n},1)}_{(p,q),(d-1,1)}(\mathbb{R}^{d-1}\times\mathbb{R},(1,v);E)$ (with inverse $L^{-\sigma}$) for each $s \in \mathbb{R}$.

Proof. It suffices to show that L^σ restricts to a bounded linear operator

$$H^{s+\sigma,(\frac{1}{2n},1)}_{(p,q),(d-1,1)}(\mathbb{R}^{d-1}\times\mathbb{R},(1,v);E) \longrightarrow H^{s,(\frac{1}{2n},1)}_{(p,q),(d-1,1)}(\mathbb{R}^{d-1}\times\mathbb{R},(1,v);E).$$

For this must we check that $\mathcal{J}^{(d-1,1),(\frac{1}{2n},1)}_{-\sigma}L^\sigma = \mathcal{J}^{(d-1,1),(\frac{1}{2n},1)}_{s}L^\sigma\mathcal{J}^{(d-1,1),(\frac{1}{2n},1)}_{-(s+\sigma)}$ restricts to a bounded linear operator on $L^{(p,q),(d-1,1)}(\mathbb{R}^d,(1,v);E)$. This can be done by checking that the symbol

$$\mathbb{R}^{d-1}\times\mathbb{R} \ni (\xi_1,\xi_2) \mapsto \frac{(1+\imath\xi_2+|\xi_1|^{2n})^\sigma}{(1+|\xi_1|^{4n}+|\xi_2|^2)^{\sigma/2}} \in \mathbb{C}$$

satisfies the anisotropic Mikhlin condition from Corollary 4.1.2. □

5.2.3 Traces

In this subsection we characterize trace spaces of weighted anisotropic mixed-norm Triebel-Lizorkin, Besov, Bessel potential, and Sobolev spaces. Here we restrict ourselves to traces with respect to the hyperplanes $\{0\}\times\mathbb{R}^{d-1}$ and $\mathbb{R}^{d-1}\times\{0\}$, corresponding to the spatial trace operator $\mathrm{tr}_{y=0}$ and the temporal trace operator $\mathrm{tr}_{t=0}$, respectively, in case of the weighted anisotropic mixed-norm Sobolev space $W^{(2,1)}_{(p,q),(d,1)}(\mathbb{R}^d\times\mathbb{R},(w_\gamma,v_\mu);X)$ from the introductory section of this chapter, Section 5.1.

5.2.3.a The General Trace Problem

Our interest is traces of weighted anisotropic mixed-norm function spaces, where we restrict ourselves to traces with respect to the hyperplanes $\{0\}\times\mathbb{R}^{d-1}$ and $\mathbb{R}^{d-1}\times\{0\}$. Furthermore, concerning traces with respect to $\{0\}\times\mathbb{R}^{d-1}$ we only consider weight-vectors $w \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{d_j})$ in which the first weight vector w_1 is of the form

$$w_1(x_1) = w_\gamma(x_1) := |x_{1,1}|^\gamma, \qquad x_1 = (x_{1,1},\ldots,x_{1,d_1}) \in \mathbb{R}^{d_1} \tag{5.60}$$

for some $\gamma \in]-1,\infty[$, whereas concerning traces with respect to $\mathbb{R}^{d-1}\times\{0\}$ we only consider weight-vectors $w\in\prod_{j=1}^{l}A_\infty(\mathbb{R}^{d_j})$ in which the last weight vector w_l is of the form

$$w_l(x_l) = v_\mu(x_l) := |x_{l,d_l}|^\mu, \qquad x_l = (x_{l,1},\dots,x_{1,d_l}) \in \mathbb{R}^{d_l} \tag{5.61}$$

for some $\mu \in]-1,\infty[$.

We will consider two ways to define the trace with respect to the hyperplanes $\{0\}\times\mathbb{R}^{d-1}$ and $\mathbb{R}^{d-1}\times\{0\}$, which on distributions from $\mathcal{S}(\mathbb{R}^d;X)$ and $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d;X)\subset C(\mathbb{R}^d;X)$ both are given as the classical trace of continuous functions. The first is an elegant abstract definition motivated by Section 2.1.3. For this trace operator it will be more convenient to construct a right-inverse (when we restrict to a certain subspace of the domain of definition). This right-inverse is constructed in a concrete way which is well-suited for estimates in weighted anisotropic mixed-norm Triebel-Lizorkin and Besov spaces. The second way is a more concrete definition, which consists of taking the classical traces of the pieces in the Littlewood-Paley decomposition of a tempered distribution, and is very suitable for doing estimates in weighted anisotropic mixed-norm Triebel-Lizorkin and Besov spaces. The advantage of these definitions, instead of defining trace operators via bounded linear extension by density, is that it gives us trace operators on certain 'big' spaces of distributions that contain the anisotropic spaces in which we are interested in, so that we a unified way of defining the concept of trace operator on the various anisotropic spaces including Triebel-Lizorkin spaces with microscopic parameter $q=\infty$ (which are very important for us).

Motivated by Section 2.1.3, we define the *distributional trace operator* $r_{0,i}$ *with respect to the hyperplane* $\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{d-i}$, $i\in\{1,\dots,d\}$, as follows. Viewing $C(\mathbb{R};\mathcal{D}'(\mathbb{R}^{i-1}\times\mathbb{R}^{d-i};X))$ as subspace of $\mathcal{D}'(\mathbb{R}^d;X)=\mathcal{D}'(\mathbb{R}^{i-1}\times\mathbb{R}\times\mathbb{R}^{d-i};X)$ via the canonical identification, so

$$C(\mathbb{R};\mathcal{D}'(\mathbb{R}^{i-1}\times\mathbb{R}^{d-i};X)) \hookrightarrow \mathcal{D}'(\mathbb{R};\mathcal{D}'(\mathbb{R}^{i-1}\times\mathbb{R}^{d-1};X)) = \mathcal{D}'(\mathbb{R}^{i-1}\times\mathbb{R}\times\mathbb{R}^{d-i};X),$$

we define $r_{0,i}\in\mathcal{L}(C(\mathbb{R};\mathcal{D}'(\mathbb{R}^{i-1}\times\mathbb{R}^{d-i};X)),\mathcal{D}'(\mathbb{R}^{d-1};X))$ as the evaluation in 0 map continuous linear operator

$$r_{0,i}: C(\mathbb{R};\mathcal{D}'(\mathbb{R}^{i-1}\times\mathbb{R}^{d-i};X)) \longrightarrow \mathcal{D}'(\mathbb{R}^{d-1};X),\ f\mapsto \mathrm{ev}_0 f. \tag{5.62}$$

Then, in view of

$$C(\mathbb{R}^d;X) = C(\mathbb{R}^{i-1}\times\mathbb{R}\times\mathbb{R}^{d-i};X) = C(\mathbb{R};C(\mathbb{R}^{i-1}\times\mathbb{R}^{d-i};X)) \hookrightarrow C(\mathbb{R};\mathcal{D}'(\mathbb{R}^{i-1}\times\mathbb{R}^{d-i};X)),$$

we have that the distributional trace operator $r_{0,i}$ coincides on $C(\mathbb{R}^d;X)$ with the classical trace operator with respect to the hyperplane $\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{d-i}$, i.e.,

$$r_{0,i}: C(\mathbb{R}^d;X) \longrightarrow C(\mathbb{R}^{d-1};X),\ f\mapsto f_{|\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{d-i}}. \tag{5.63}$$

Next, we consider our second possible definition for the trace with respect to the hyperplane $\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{d-i}$, $i\in\{1,\dots,d\}$. For this definition we let $\varphi\in\Phi^{d,a}(\mathbb{R}^d)$ with associated family of convolution operators $(S_n)_{n\in\mathbb{N}}\subset\mathcal{L}(\mathcal{S}'(\mathbb{R}^d;X))$ be fixed. In order to motivate the definition to be given in a moment, let us first recall that $f=\sum_{n=0}^\infty S_n f$ in $\mathcal{S}(\mathbb{R}^d;X)$ respectively in $\mathcal{S}'(\mathbb{R}^d;X)$ whenever $f\in\mathcal{S}(\mathbb{R}^d;X)$ respectively $f\in\mathcal{S}'(\mathbb{R}^d;X)$, from which it is easy to see that

$$f_{|\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{d-i}} = \sum_{n=0}^\infty (S_n f)_{|\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{d-i}} \ \text{ in } \mathcal{S}(\mathbb{R}^{d-1};X), \qquad f\in\mathcal{S}(\mathbb{R}^d;X).$$

Furthermore, given a general tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d; X)$, recall that $S_n f \in \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d; X)$, i.e. $\widehat{Sf_n}$ has compact support, for every $n \in \mathbb{N}$. By the Paley-Wiener-Schwartz theorem, Theorem C.6.5, it thus holds that each $S_n f$ is a continuous function on $\mathbb{R}^d$; in particular, each $S_n f$ has a well defined classical trace with respect to $\mathbb{R}^{i-1} \times \{0\} \times \mathbb{R}^{d-i}$. This suggests to define the the trace operator $\gamma_{0,i} = \gamma^{\varphi}_{0,i} : \mathcal{D}(\gamma^{\varphi}_0) \subset \mathcal{S}'(\mathbb{R}^d; X) \longrightarrow \mathcal{S}'(\mathbb{R}^{d-1}; X)$ by

$$\gamma^{\varphi}_{0,i} f := \sum_{n=0}^{\infty} (S_n f)_{|\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{d-i}} \tag{5.64}$$

on the domain $\mathcal{D}(\gamma^{\varphi}_0)$ consisting of all $f \in \mathcal{S}'(\mathbb{R}^d; X)$ for which this defining series converges in $\mathcal{S}'(\mathbb{R}^{d-1}; X)$. Note that $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d; X)$ is a subspace of $\mathcal{D}(\gamma^{\varphi}_0)$ on which γ^{φ}_0 coincides with the classical trace of continuous functions with respect to $\mathbb{R}^{i-1} \times \{0\} \times \mathbb{R}^{d-i}$; of course, for an f belonging to $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d; X)$ there are only finitely many $S_n f$ non-zero.

In the next two subsubsections we will investigate the trace operators $r_{0,i}$ and $\gamma_{0,i} = \gamma^{\varphi}_0$, $i \in \{1, d\}$, on the weighted anisotropic mixed-norm functions spaces of Besov, Triebel-Lizorkin, Sobolev and Bessel-potential type of suitable smoothness, equipped with a weight-vector $w \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^d)$ in which w_1 (resp. w_2) has the form (5.60) (resp. (5.61)) in case of $i = 1$ (resp. $i = d$). It is our goal to find, for

$$E = B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X), F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X), W^{n}_{p,d}(\mathbb{R}^d, w; X), H^{s,a}_{p,d}(\mathbb{R}^d, w; X),$$

sufficient conditions in terms of the parameters and weights in order that $\gamma_{0,i} = \gamma^{\varphi}_{0,i}$, $i \in \{1, d\}$, is well-defined on E (and independent of $\varphi \in \Phi^{d,a}(\mathbb{R}^d)$, and such that there exists a Banach space $G \subset \mathcal{S}'(\mathbb{R}^{d-1}; X)$ (which is necessarily unique up to an equivalence of norm) such that $\gamma^{\varphi}_{0,i}$ restricts to a continuous surjection

$$\gamma_{0,i} : E \longrightarrow G$$

having a continuous right-inverse (also independent of $\varphi \in \Phi^{d,a}(\mathbb{R}^d)$). The space G is then called the *trace space* of E. Note that in case $\mathcal{S}(\mathbb{R}^d; X)$ is a dense subspace of E, which in our situation occurs when $E \neq B^{s,a}_{p,\infty,d}(\mathbb{R}^d, w; X), F^{s,a}_{p,\infty,d}(\mathbb{R}^d, w; X)$, $\gamma_{0,i}$ is the unique extension to a continuous linear operator $E \longrightarrow G$ of the classical trace (with respect to the hyperplane $\mathbb{R}^{i-1} \times \{0\} \times \mathbb{R}^{d-i}$) defined on $\mathcal{S}(\mathbb{R}^d; X)$. Under extra restrictions on the smoothness parameter and the weight-vector, we will show that E is continuously included in the domain of the distributional trace operator $r_{0,i}$ and that $r_{0,i}$ coincides here with $\gamma_{0,i}$.

Before we go to the just discussed specific trace problem for the weighted anisotropic mixed-norm function spaces, let us first do some preparations in a general setting.

In the next proposition we construct a right-inverse for the distributional trace operator $r_{0,i}$ restricted to $C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^{d-1}; X))$ (under the usual identifications). For simplicity of notation, we with out loss of generality restrict ourselves to the case $i = 1$:

Proposition 5.2.49. *Let $\rho \in \mathcal{S}(\mathbb{R})$ such that $\rho(0) = 1$ and $\operatorname{supp}\hat{\rho} \subset [1, 2]$, $a_1 \in \mathbb{R}$, $\tilde{d} \in (\mathbb{Z}_{>0})^{\tilde{l}}$ such that $d - 1 = |\tilde{d}|_1$, $\tilde{a} \in]0, \infty[^{\tilde{l}}$, and $(\phi_n)_{n\in\mathbb{N}} \in \Phi^{\tilde{d},\tilde{a}}(\mathbb{R}^{d-1})$ with corresponding family of convolution operator $(T_n)_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{S}'(\mathbb{R}^{d-1}; X))$ (given by $T_n := \phi_n *$ for each n). Then, for each $g \in \mathcal{S}'(\mathbb{R}^{d-1}; X)$,*

$$\operatorname{ext} g := \sum_{n=0}^{\infty} \rho(2^{na_1} \cdot) \otimes T_n g \tag{5.65}$$

defines a convergent sum in $\mathcal{S}'(\mathbb{R}^d;X)$ *and the operator* $\mathrm{ext}_{0,1}$ *defined via this formula is a linear operator*

$$\mathrm{ext}_{0,1} : \mathcal{S}'(\mathbb{R}^{d-1};X) \longrightarrow C_b(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1};X))$$

which acts as a right inverse of $r_{0,1} : C(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1};X)) \longrightarrow \mathcal{S}'(\mathbb{R}^{d-1};X)$.

Proof. We start with establishing the convergence of the series in (5.65) in the space $\mathcal{S}'(\mathbb{R}^d;X)$. Writing $l := \tilde{l}+1$, $\mathpzc{d} := (1,\tilde{\mathpzc{d}}) \in (\mathbb{Z}_{>0})^l$, and $a := (a_1,\tilde{a}) \in]0,\infty[^l$, this can be proved by checking the conditions of Lemma 5.2.7: From

$$\mathscr{F}[\rho(2^{na_1}\,\cdot\,)\otimes T_ng] = \mathscr{F}[\rho(2^{na_1}\,\cdot\,)]\otimes\widehat{T_ng} = 2^{-na_1}\hat{\rho}(2^{-na_1}\,\cdot\,)\otimes\widehat{T_ng}, \qquad n\in\mathbb{N},$$

it follows that

$$\operatorname{supp}\mathscr{F}[\rho(2^{na_1}\,\cdot\,)\otimes T_ng] \subset \operatorname{supp}\hat{\rho}(2^{-na_1}\,\cdot\,)\times\operatorname{supp}T_ng \subset [2^{na_1},2^{na_1}2]\times\operatorname{supp}T_ng.$$

In view of the Fourier supports of the ψ_n (see (5.13)), we can thus find a $c'>0$ (independent of g) such that

$$\mathscr{F}[\rho\otimes T_0g]\subset\{\xi\mid v(\xi)\le c'\},\quad \operatorname{supp}\mathscr{F}[\rho(2^{na_1}\,\cdot\,)\otimes T_ng]\subset\{\xi\mid c'^{-1}2^n\le v(\xi)\le c'2^n\}\ ,n\ge1,$$

for the $(\mathpzc{d},a)$-anisotropic distance function v on $\mathbb{R}^d$ defined by $v(\xi) := \max\{|\xi_1|^{1/a_1},|\xi'|_{\tilde{\mathpzc{d}},\tilde{a}}\}$. But $v \eqsim |\cdot|_{\mathpzc{d},a}$ by Lemma 2.3.2, whence there exists a constant $c>0$ (independent of g) such that

$$\operatorname{supp}\mathscr{F}[\rho\otimes T_0g]\subset\{\xi\mid|\xi|_{\mathpzc{d},a}\le c\},\quad \operatorname{supp}\mathscr{F}[\rho(2^{na_1}\,\cdot\,)\otimes T_ng]\subset\{\xi\mid c^{-1}2^n\le|\xi|_{\mathpzc{d},a}\le c2^n\}\,,n\ge1, \tag{5.66}$$

which is the desired Fourier support condition (5.15) from Lemma 5.2.7. The growth condition (5.17) follows directly from the fact $(\rho(2^{na_1}\cdot))_{n\in\mathbb{N}}$ is a uniformly bounded family of functions on $\mathbb{R}$ and such a growth condition for the T_ng on $\mathbb{R}^{d-1}$ (see Lemma 5.2.7.(ii)).

Next we show that $\mathrm{ext}\,g$ belongs to $C_b(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1}))$. To this end, observe that $\sum_{n=0}^\infty[x_1\mapsto\rho(2^{na_1}x_1)T_ng]$ defines an absolutely convergent series in $C_b(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1};X))$. Indeed, each $x_1\mapsto\rho(2^{na_1}x_1)T_ng$ clearly defines an element of $C_b(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1};X))$. Moreover, since $(\rho(2^{na_1}\cdot))_{n\in\mathbb{N}}$ is a uniformly bounded family of functions on $\mathbb{R}$, and since $\|T_ng(\psi)\|_X\le C_\psi 2^{-n}$ for all $\psi\in\mathcal{S}(\mathbb{R}^{d-1})$ by Lemma 5.2.7.(ii) (just take K large enough), it holds that

$$\sum_{n=0}^\infty \|x_1\mapsto\rho(2^{na_1}x_1)T_ng(\psi)\|_{C_b(\mathbb{R};X)}\le\tilde{C}_\psi \qquad (\psi\in\mathcal{S}(\mathbb{R}^{d-1})). \tag{5.67}$$

This establishes the absolute convergence of the series in $C_b(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1};X))$, say with limit Λ^g. To see that $\Lambda^g=\mathrm{ext}_{0,1}\,g$, it suffices to show that they coincide on test functions of the form $\chi\otimes\psi$ with $\chi\in C_c^\infty(\mathbb{R})$ and $\psi\in C_c^\infty(\mathbb{R}^{d-1})$. So let's compute

$$\begin{aligned}\Lambda^g(\chi\otimes\psi) &= \int_{\mathbb{R}}\Lambda^g(x_1)[(\chi\otimes\psi)(x_1,\,\cdot\,)]dx_1\\ &= \int_{\mathbb{R}}\sum_{n=0}^\infty\rho(2^{na_1}x_1)T_ng(\psi)\chi(x_1)dx_1\\ &\overset{!}{=} \sum_{n=0}^\infty\int_{\mathbb{R}}\rho(2^{na_1}x_1)T_ng(\psi)\chi(x_1)dx_1\\ &= \sum_{n=0}^\infty[\rho(2^{na_1}\,\cdot\,)\otimes T_ng](\chi\otimes\psi)\\ &= [\mathrm{ext}_{0,1}\,g](\chi\otimes\psi);\end{aligned}$$

here the interchange in the third equality is justified by (5.67) and $\chi \in C_c(\mathbb{R})$.

To finish the proof, we show that the operator $\mathrm{ext}_{0,1} : \mathcal{S}'(\mathbb{R}^{d-1};X) \longrightarrow C_b(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1}))$, which is obviously linear, acts as a right inverse of $r_{0,1} : C(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1})) \longrightarrow \mathcal{S}'(\mathbb{R}^{d-1};X)$. As $\rho(0) = 1$ and $(T_n)_{n\in\mathbb{N}}$ is the family of convolution operators associated to $(\psi_n)_{n\in\mathbb{N}} \in \Phi^{d'',a''}(\mathbb{R}^{d-1})$ by our choice, we simply have

$$r_{0,1}(\mathrm{ext}_{0,1}\, g) = r_{0,1}\Lambda^g = \Lambda^g(0) = \sum_{n=0}^{\infty} \rho(2^{na_1}0)T_n g = \sum_{n=0}^{\infty} T_n g = g.$$

□

The following simple lemma will be important for determining the trace space G in case E is a Besov or a Triebel-Lizorkin space.

Lemma 5.2.50. *Let $E \subset \mathcal{S}'(\mathbb{R}^d;X)$ and $G \subset \mathcal{S}'(\mathbb{R}^{d-1})$ be two Banach spaces of X-valued tempered distributions such that $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^{d-1};X) \cap G$ is dense in G. Suppose that $\gamma_{0,i}$ exists on E and defines a continuous linear operator $E \longrightarrow G$ and that the extension operator* $\mathrm{ext}_{0,i}$ *from Proposition 5.2.49 restricts to a continuous linear operator $G \longrightarrow E$. Then* $\mathrm{ext}_{0,i} : G \longrightarrow E$ *is a right inverse of $\gamma_{0,i} : E \longrightarrow G$.*

Proof. For simplicity of notation we assume that $i = 1$. Let g be an arbitrary element from the dense subspace $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^{d-1};X) \cap G$ of G. In the notation of Proposition 5.2.49,

$$\mathrm{ext}_{0,1}\, g = \sum_{n=0}^{N} \rho(2^{na_1}\,\cdot\,) \otimes T_n g \in \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d;X) = \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d;X) \cap C(\mathbb{R}^d;X)$$

for some $N \in \mathbb{N}$. Since $\gamma_{0,1}$ extends the classical trace on $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d;X)$ while $r_{0,1}$ extends the classical trace on $C(\mathbb{R}^d;X)$, it follows that

$$\gamma_{0,1}(\mathrm{ext}_{0,1}\, g) = (\mathrm{ext}_{0,1}\, g)_{|\{0\}\times\mathbb{R}^{d-1}} = r_0(\mathrm{ext}\, g) = g;$$

see Proposition 5.2.49 for the last equality. This identity extends by denseness and continuity to all $g \in G$. □

Lemma 5.2.51. *Let $E \subset \mathcal{D}'(\mathbb{R}^d;X)$ and $G \subset \mathcal{S}'(\mathbb{R}^{d-1};X)$ be two Banach spaces of X-valued distributions such that*

- $\mathcal{S}(\mathbb{R}^d;X) \overset{dense}{\hookrightarrow} E$ *and* $\mathcal{S}(\mathbb{R}^{d-1};X) \subset G$*;*
- $\mathcal{S}(\mathbb{R}^d;X) \longrightarrow \mathcal{S}(\mathbb{R}^{d-1};X)$, $f \mapsto f_{|\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{d-i}}$ *extends to a bounded linear operator* $\mathrm{tr}_{0,i} : E \longrightarrow G$*;*
- *The extension operator* $\mathrm{ext}_{0,i}$ *from Proposition 5.2.49 restricts to a continuous right inverse of* $\mathrm{tr}_{0,i}$.

Then $\{ f \in \mathcal{S}(\mathbb{R}^d;X) : f_{|\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{d-i}} = 0 \}$ *is a dense subspace of* $\ker(\mathrm{tr}_{0,i}) \subset E$.

Proof. Let $f \in \ker(\mathrm{tr}_{0,i}) \subset E$ be arbitrary. Then, as

$$\mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^d;X)) \overset{dense}{\subset} \mathcal{S}(\mathbb{R}^d;X) \overset{dense}{\hookrightarrow} E,$$

there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^d;X))$ such that $f = \lim_{n\to\infty} f_n$ in E. Now note that

$$r(f_n) = f_{n|\mathrm{tr}_{0,i}} \in \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^{d-1};X)), \qquad n \in \mathbb{N},$$

and that $\mathrm{ext}_{0,i}$ maps $\mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^{d-1};X))$ into $\mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^d;X))$. So we have

$$\tilde{f}_n := f_n - \mathrm{ext}_{0,i}(\mathrm{tr}_{0,i}(f_n)) \in \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^d;X)) \subset \mathcal{S}(\mathbb{R}^d;X)$$

for each $n \in \mathbb{N}$. Furthermore,

$$\tilde{f}_{n|\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{d-i}} = \mathrm{tr}_{0,i}(\tilde{f}_n) = \mathrm{tr}_{0,i}(f_n) - \mathrm{tr}_{0,i}(\mathrm{ext}_{0,i}(\mathrm{tr}_{0,i}(f_n))) = \mathrm{tr}_{0,i}(f_n) - \mathrm{tr}_{0,i}(f_n) = 0, \qquad n \in \mathbb{N},$$

and

$$\tilde{f}_n = f_n - \mathrm{ext}_{0,i}(\mathrm{tr}_{0,i}(f_n)) \overset{n\to\infty}{\longrightarrow} f - \mathrm{ext}_{0,i}(\mathrm{tr}_{0,i}(f)) = f - \mathrm{ext}_{0,i}(0) = f \qquad \text{in} \quad E.$$

□

5.2.3.b Traces with respect to $\{0\} \times \mathbb{R}^{d-1}$

The Trace Space of a Besov Space We first investigate the Besov case. Since we restrict ourselves to weight-vectors $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^d)$ in which w_1 has the form (5.60) for some $\gamma > -1$, in view of Remark 5.2.13 we may assume without loss of generality that $\mathpzc{d}_1 = 1$; this will simplify the notation.

Throughout this paragraph we will use the following notation: We write $\mathpzc{d}'' = (\mathpzc{d}_2, \ldots, \mathpzc{d}_l)$. Given $p \in]1,\infty[^l$ we will write $p'' = (p_2, \ldots, p_l)$, and similarly for $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^d)$.

Theorem 5.2.52. *Let X be a Banach space, $a \in]0,\infty[^l$, $p \in]1,\infty[^l$, $q \in [1,\infty]$, $\gamma \in]-1,\infty[$ and $s > \frac{a_1}{p_1}(1+\gamma)$. Let $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^d)$ be such that $w_1(x_1) = w_\gamma(x_1) = |x_1|^\gamma$ and $w'' \in \prod_{j=2}^l A_{p_j/r_j}(\mathbb{R}^{\mathpzc{d}_j})$ for some $r'' = (r_2, \ldots, r_l) \in]0,1[^{l-1}$ satisfying $s - \frac{a_1}{p_1}(1+\gamma) > \sum_{j=2}^l a_j \mathpzc{d}_j(\frac{1}{r_j} - 1)$.*[5] *Then the trace operator $\gamma_{0,1} = \gamma_{0,1}^\varphi$ is well-defined on $B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X)$ and restricts to a continuous surjection*

$$\gamma_{0,1} = \gamma_{0,1}^\varphi : B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X) \longrightarrow B^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',q,\mathpzc{d}''}(\mathbb{R}^{d-1}, w''; X),$$

independent of φ, for which the extension operator $\mathrm{ext}_{0,1}$ from Proposition 5.2.49 (with $\tilde{\mathpzc{d}} = \mathpzc{d}''$ and $\tilde{a} = a''$) restricts to a corresponding continuous right-inverse. Moreover, if $s > \frac{a_1}{p_1}(1+\gamma_+)$ and $w'' \in \prod_{j=2}^l A_{p_j}(\mathbb{R}^{\mathpzc{d}_j})$, then

$$B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X) \hookrightarrow C(\mathbb{R}; L^{p'',\mathpzc{d}}(\mathbb{R}^{d-1}, w''; X)) \hookrightarrow C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^{d-1}; X)) \tag{5.68}$$

and the distributional trace $r_{0,1}$ coincides with the trace operator $\gamma_{0,1}$ on $B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X)$.

Lemma 5.2.22 and the following lemma are the main ingredients for the doing the estimates for $\gamma_{0,1}$ in the proof of this theorem.

[5]This technical condition on w'' is in particular satisfied for $w'' \in \prod_{j=2}^l A_{p_j}(\mathbb{R}^{\mathpzc{d}_j})$.

Lemma 5.2.53. *Let $\mathpzc{d}_1 = 1$, $p \in]1,\infty[^l$, $\gamma > -1$ and $w_\gamma(x_1) := |x_1|^\gamma$ on $\mathbb{R}$. Then there exists a constant $C > 0$ such that for all Banach spaces X, $w'' \in \prod_{j=2}^l \mathcal{W}(\mathbb{R}^{d_j})$, $R > 0$, $f \in \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d;X)$ with $\pi_{\mathpzc{d};1}(\operatorname{supp}\hat{f}) \subset [-R,R]$, $b \geq 0$, and $x_1 \in [-b,b]$, it holds that*

$$\|f(x_1,\cdot)\|_{L^{p'',\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X)} \leq C(bR+1)^{\frac{\gamma_-}{p_1}} R^{\frac{1+\gamma}{p_1}} \|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X)}$$

Proof. We first consider the case $-1 < \gamma < p-1$. Pick a $\psi \in \mathscr{F}^{-1}(C^\infty(\mathbb{R}))$ satisfying $\hat{\psi} \equiv 1$ on $[-1,1]$. For $R > 0$ we define $\psi_R := R\psi(R\,\cdot)$; then $\psi_R \in \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}))$ with $\hat{\psi}_R \equiv 1$ on $[-R,R]$. Now let X be a Banach space, $w'' \in \prod_{j=2}^l \mathcal{W}(\mathbb{R}^{d_j})$, $R > 0$, $f \in \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d;X)$ such that $\pi_1(\operatorname{supp}\hat{f}) \subset [-R,R]$, $b \geq 0$, and $x_1 \in [-b,b]$. Then

$$f(x_1,x'') = [f(\,\cdot\,,x'') * \psi_R](x_1) = \int_{\mathbb{R}} f(s,x'')|s|^{\gamma/p_1}\psi_R(x_1-s)|s|^{-\gamma/p_1}\,ds.$$

Putting $\gamma' := -\gamma\frac{p_1'}{p_1} = -\frac{\gamma}{p_1-1} > -1$, Hölder's inequality gives

$$\|f(x_1,x'')\|_X \leq \|f(x_1,\,\cdot\,)\|_{L^{p_1}(\mathbb{R},w_\gamma;X)} \left(\int_{\mathbb{R}} |\psi_R(x_1-s)|^{p_1'}|s|^{\gamma'}\,ds\right)^{1/p_1'}. \tag{5.69}$$

The second term can be computed as

$$\begin{aligned}\left(\int_{\mathbb{R}} |\psi_R(x_1-s)|^{p_1'}|s|^{\gamma'}\,ds\right)^{1/p_1'} &= R\left(\int_{\mathbb{R}} |\psi(Rx_1-Rs)|^{p_1'}|s|^{\gamma'}\,ds\right)^{1/p_1'}\\ &= R^{1-\frac{1+\gamma'}{p_1'}}\left(\int_{\mathbb{R}} |\psi(Rx_1-\sigma)|^{p_1'}|\sigma|^{\gamma'}\,d\sigma\right)^{1/p_1'}\\ &= R^{\frac{1}{p_1}(1+\gamma)}\left(\int_{\mathbb{R}} |\psi(Rx_1-\sigma)|^{p_1'}|\sigma|^{\gamma'}\,d\sigma\right)^{1/p_1'}.\end{aligned} \tag{5.70}$$

Since $\psi \in \mathcal{S}(\mathbb{R})$ and $w_{\gamma'} \in A_{p_1'}(\mathbb{R})$, where $w_{\gamma'}(s) = |s|^{\gamma'}$, it follows from Lemma D.2.7 that there is a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}} |\psi(Rx_1-\sigma)|^{p_1'}|\sigma|^{\gamma'}\,d\sigma \leq C_1 \int_{[Rx_1-1,Rx_1+1]} |\sigma|^{\gamma'}\,d\sigma.$$

Observing that the right hand side can be estimated by $C_2(bR+1)^{(\gamma')_+} = C_2(bR+1)^{p_1'\gamma_-}$ for some constant $C_2 > 0$ independent of R, b and x_1, combining this estimate with (5.69) and (5.70), and subsequently taking $L^{p'',\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X)$-norms, we obtain the desired estimate.

Next, the case $\gamma \geq p-1$ can be derived from the case $\gamma = 0$ via the inequality of Plancherel-Pólya-Nikol'skii type from Proposition 5.2.32:

$$\|f(x_1,\cdot)\|_{L^{p'',\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X)} \lesssim R^{\frac{1}{p_1}} \|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,(1,w'');X)} \lesssim R^{\frac{1+\gamma}{p_1}} \|f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X)}.\text{[6]}$$

□

The following proposition forms the basis for the last statement in Theorem 5.2.52 concerning the distributional trace operator $r_{0,1}$.

Proposition 5.2.54. *Let X be a Banach space, $a \in]0,\infty[^l$, $p \in]1,\infty[^l$, $q \in [1,\infty]$, $\gamma \in]-1,\infty[$, $w_\gamma(x_1) := |x_1|^\gamma$ on $\mathbb{R}$, and $w'' \in \prod_{j=2}^l A_{p_j}(\mathbb{R}^{d_j})$. Then*

$$B^{\frac{a_1}{p_1}(1+\gamma_+),a}_{p,1,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X) \hookrightarrow C_b(\mathbb{R},v_{\gamma,p_1};L^{p'',\mathpzc{d}}(\mathbb{R}^{d-1},w'';X)) \hookrightarrow C(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1};X)),\text{[7]} \tag{5.71}$$

[6] Note that this actually works for all $\gamma \geq 0$.

[7] See Appendix C.1 for the space $C_b(\mathbb{R},v_{\gamma,p_1};L^{p'',\mathpzc{d}}(\mathbb{R}^{d-1},w'';X))$.

where $v_{\gamma,p_1} := \max\{|\cdot|, 1/3\}^{-\frac{\gamma_-}{p_1}}$ *and where, given a Banach space* Y*,* $C_b(\mathbb{R}, v_{\gamma,p_1}; Y)$ *denotes the Banach space of all continuous functions* $f : \mathbb{R} \longrightarrow Y$ *for which* $f v_{\gamma,p_1}$ *is bounded with its natural norm* $\|f\|_{C_b(\mathbb{R},v_{\gamma,p_1};Y)} := \left\|f v_{\gamma,p_1}\right\|_\infty$*. Here an* $f \in B^{\frac{a_1}{p_1}(1+\gamma_+),a}_{p,1,d}(\mathbb{R}^d, (w_\gamma, w''); X)$ *corresponds to the* $\tilde{f} \in C_b(\mathbb{R}, v_{\gamma,p_1}; L^{p'',d}(\mathbb{R}^{d-1}, w''; X)) \subset C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^{d-1}; X))$ *given by*

$$\tilde{f} = \sum_{n=0}^{\infty}[x_1 \mapsto S_n f(x_1, \cdot)],$$

with absolute convergence in $C_b(\mathbb{R}, v_{\gamma,p_1}; L^{p'',d}(\mathbb{R}^{d-1}, w''; X))$*.*

Proof. Given a Banach space Y and a function $\rho : \mathbb{R} \longrightarrow]0,\infty[$, we denote by $B(\mathbb{R},\rho; Y)$ the space of all functions $g : \mathbb{R} \longrightarrow Y$ for which ρg is bounded. Equipped with the norm $\|g\|_{B(\mathbb{R},\rho;Y)} := \sup_{t\in\mathbb{R}} \|\rho(t)g(t)\|_Y$, $B(\mathbb{R},\rho; Y)$ becomes a Banach space. Note that $C(\mathbb{R},\rho; Y)$ is the closed subspace of $B(\mathbb{R},\rho; Y)$ consisting of all the continuous functions belonging to $B(\mathbb{R},\rho; Y)$.

Let $f \in B^{\frac{a_1}{p_1}(1+\gamma_+),a}_{p,1,d}(\mathbb{R}^d, (w_\gamma, w''); X)$. For all $n \in \mathbb{N}$ we have $S_n f \in \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d; X)$ with $\mathrm{pr}_1 \operatorname{supp} \widehat{S_n f} \subset [-3 \cdot 2^{na_1}, 3 \cdot 2^{na_1}]$. From Lemma 5.2.53 it follows that, for all $n \in \mathbb{N}$, $b \geq 1/3$ and $x_1 \in [-b, b]$,

$$\begin{aligned}
\|S_n f(x_1, \cdot)\|_{L^{p'',d}(\mathbb{R}^{d-1},w'';X)} &\leq C_1(b3 \cdot 2^{na_1} + 1)^{\frac{\gamma_-}{p_1}}(3 \cdot 2^{na_1})^{\frac{1+\gamma}{p_1}} \|S_n f\|_{L^{p,d}(\mathbb{R}^d,(w_\gamma,w'');X)} \\
&\leq C_2 b^{\frac{\gamma_-}{p_1}}(2^{na_1})^{\frac{1+\gamma+\gamma_-}{p_1}} \|S_n f\|_{L^{p,d}(\mathbb{R}^d,(w_\gamma,w'');X)} \\
&= C_2 b^{\frac{\gamma_-}{p_1}} 2^{n\frac{a_1}{p_1}(1+\gamma_+)} \|S_n f\|_{L^{p,d}(\mathbb{R}^d,(w_\gamma,w'');X)} \,.
\end{aligned}$$

Thus each $x_1 \mapsto S_n f(x_1, \cdot)$ defines an element of $B(\mathbb{R}, v_{\gamma,p_1}; L^{p'',d}(\mathbb{R}^{d-1}, w''; X))$ of norm

$$\|x_1 \mapsto S_n f(x_1, \cdot)\|_{B(\mathbb{R},v_{\gamma,p_1};L^{p'',d}(\mathbb{R}^{d-1},w'';X))} \leq C_2 2^{n\frac{a_1}{p_1}(1+\gamma_+)} \|S_n f\|_{L^{p,d}(\mathbb{R}^d,(w_\gamma,w'');X)} ,$$

so that

$$\begin{aligned}
\sum_{n=0}^{\infty} \|x_1 \mapsto S_n f(x_1, \cdot)\|_{B(\mathbb{R},v_{\gamma,p_1};L^{p'',d}(\mathbb{R}^{d-1},w'';X))} &\leq C_2 \sum_{n=0}^{\infty} 2^{n\frac{a_1}{p_1}(1+\gamma_+)} \|S_n f\|_{L^{p,d}(\mathbb{R}^d,(w_\gamma,w'');X)} \\
&= C_2 \|f\|_{B^{\frac{a_1}{p_1}(1+\gamma_+),a}_{p,1,d}(\mathbb{R}^d,(w_\gamma,w'');X)} .
\end{aligned}$$

If each $x_1 \mapsto S_n f(x_1, \cdot)$ were continuous as a function $\mathbb{R} \longrightarrow L^{p,d}(\mathbb{R}^d, (w_\gamma, w''); X)$, then we would have absolute convergence of the series $\tilde{f} := \sum_{n\in\mathbb{N}}[x_1 \mapsto S_n f(x_1, \cdot)]$ in the Banach space $C_b(\mathbb{R}, v_{\gamma,p_1}; L^{p'',d}(\mathbb{R}^{d-1}, w''; X))$ and $f \mapsto \tilde{f}$ would be a continuous linear map

$$B^{\frac{a_1}{p_1}(1+\gamma_+),a}_{p,1,d}(\mathbb{R}^d, (w_\gamma, w''); X) \longrightarrow C_b(\mathbb{R}, v_{\gamma,p_1}; L^{p'',d}(\mathbb{R}^{d-1}, w''; X)).$$

To see that $x_1 \mapsto S_n f(x_1, \cdot)$ is indeed continuous, let $b > 0$. By Lemma 5.2.53 we have, for $x_1 \in]-b, b[$ and $h_1 \in \mathbb{R}$ such that $x_1 + h_1 \in]-b, b[$,

$$\|S_n f(x_1 + h_1, \cdot) - S_n f(x_1, \cdot)\|_{L^{p,d}(\mathbb{R}^d,(w_\gamma,w'');X)} \leq C_1(bR_n + 1)^{\frac{\gamma_-}{p_1}} R_n^{\frac{1+\gamma}{p_1}} \left\|\Delta_{[d;1],h_1} S_n f\right\|_{L^{p,d}(\mathbb{R}^d,(w_\gamma,w'');X)} ,$$

where $R_n := -2^{-na_1}$. Now we pick an $r \in]0,1]$ such that $w_\gamma \in A_{p_1/r}(\mathbb{R})$, apply Lemma 5.2.42, and use the boundedness of $M_{[\mathpzc{d};1],r}$ on $L^{p,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w''))$, to obtain, for $|h_1| \leq R_n$,

$$
\begin{aligned}
\|S_nf(x_1+h_1,\,\cdot\,)-S_nf(x_1,\,\cdot\,)\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w''));X)} &\leq C_3(bR_n+1)^{\frac{\gamma_-}{p_1}} R_n^{\frac{1+\gamma}{p_1}} R_n|h_1| \left\| M_{[\mathpzc{d};1],r}\|S_nf\|_X \right\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w''))} \\
&\leq |h_1| C^{[n]} \|S_nf\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w''));X)}
\end{aligned}
$$

for some constant $C^{[n]} > 0$. This gives the desired continuity.

Finally, we show that f and $\tilde{f}$ coincide as distributions: It is not difficult to see that $[x_1 \mapsto S_nf(x_1,\,\cdot\,)] \in C(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1};X))$ coincides with S_nf when viewed as distribution on $\mathbb{R}^d$. From the convergence $\tilde{f} = \sum_{n=0}^\infty [x_1 \mapsto S_nf(x_1,\,\cdot\,)]$ in $C(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1};X))$ and the continuity of the inclusion $C(\mathbb{R};\mathcal{S}'(\mathbb{R}^{d-1};X)) \hookrightarrow \mathcal{D}'(\mathbb{R}^d;X)$ it thus follows that $\tilde{f} = \sum_{n=0}^\infty S_nf$ in $\mathcal{D}'(\mathbb{R}^d;X)$. As $f = \sum_{n=0}^\infty S_nf$ in $\mathcal{S}'(\mathbb{R}^d;X) \hookrightarrow \mathcal{D}'(\mathbb{R}^d;X)$, this completes the proof. □

We are now ready to prove Theorem 5.2.52:

Proof of Theorem 5.2.52. Let the notations be as in Proposition 5.2.49. We will show that, for an arbitrary $\varphi \in \Phi^{\mathpzc{d},a}(\mathbb{R}^d)$,

(I) $\gamma_{0,1}^\varphi$ exists on $B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X)$ and defines a continuous operator

$$\gamma_{0,1}^\varphi : B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X) \longrightarrow B^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',q,\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X);$$

(II) The extension operator $\mathrm{ext}_{0,1}$ from Proposition 5.2.49 (with $\tilde{\mathpzc{d}} = \mathpzc{d}''$ and $\tilde{a} = a''$) restricts to a continuous operator

$$\mathrm{ext}_{0,1} : B^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',q,\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X) \longrightarrow B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X).$$

In case $q < \infty$, the independence of $\varphi \in \Phi^{\mathpzc{d},a}(\mathbb{R}^d)$ in the first assertion follows from denseness of $\mathcal{S}(\mathbb{R}^d;X)$ in $B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X)$ and (I), whereas the right inverse part in the first assertion then follows from denseness of $\mathscr{F}^{-1}C_c^\infty(\mathbb{R}^d;X) \subset \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d;X) \cap B^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',q,\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X)$ in $B^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',q,\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X)$ (cf. Proposition 5.2.17), (I) and (II), via an application of Lemma 5.2.50. In case $q = \infty$, this assertion follows from a combination of (I), (II), the case $q = 1$, and Proposition 5.2.30(ii); to see that also in this case $\gamma_{0,1}^\varphi$ is independent of φ and has $\mathrm{ext}_{0,1}$ as a right-inverse, we use the embeddings $B^{s,a}_{p,\infty,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X) \hookrightarrow B^{\tilde{s},a}_{p,1,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X)$ and $B^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',q,\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X) \hookrightarrow B^{\tilde{s}-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',q,\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X)$, respectively, where $\tilde{s} \in]\frac{a_1}{p_1}(1+\gamma),s[$. The last assertion is immediate from a combination of Proposition 5.2.30(ii) and Proposition 5.2.54.

(I): Let $f \in B^{s,a}_{p,\infty,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X)$. Then each $S_nf \in \mathcal{S}'(\mathbb{R}^d;X)$ has Fourier support

$$\mathrm{supp}\,\widehat{S_nf} \subset \prod_{j=1}^l [-c2^{na_j},c2^{na_j}]^{d_j}$$

for some constant $c > 0$ only depending on φ. Therefore, by Corollary C.6.5, we have $S_n f(0,\cdot) \in \mathcal{S}'(\mathbb{R}^{d-1};X)$ with Fourier support contained in $\prod_{j=2}^{l}[-c2^{na_j}, c2^{na_j}]^{\mathpzc{d}_j}$. In view of Lemma 5.2.22, it thus suffices to show that

$$\left\|\left(2^{n[s-\frac{a_1}{p_1}(1+\gamma)]}S_n f(0,\cdot)\right)_{n\geq 0}\right\|_{\ell^q(\mathbb{N};L^{p'',\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X))} \lesssim \|f\|_{B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X)}.$$

Using Lemma 5.2.53, we can obtain this estimate:

$$\begin{aligned}
&\left\|\left(2^{n[s-\frac{a_1}{p_1}(1+\gamma)]}S_n f(0,\cdot)\right)_{n\geq 0}\right\|_{\ell^q(\mathbb{N};L^{p'',\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X))}\\
&\quad = \left\|\left(2^{n[s-\frac{a_1}{p_1}(1+\gamma)]}\|S_n f(0,\cdot)\|_{L^{p'',\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X)}\right)_{n\geq 0}\right\|_{\ell^q(\mathbb{N})}\\
&\quad \leq \left\|\left(2^{n[s-\frac{a_1}{p_1}(1+\gamma)]}C_1(c2^{na_1})^{\frac{1}{p_1}(1+\gamma)}\|S_n f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X)}\right)_{n\geq 0}\right\|_{\ell^q(\mathbb{N})}\\
&\quad = C_1 c^{\frac{1}{p_1}(1+\gamma)}\left\|\left(2^{ns}\|S_n f\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X)}\right)_{n\geq 0}\right\|_{\ell^q(\mathbb{N})}\\
&\quad = C_2\|f\|_{B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X)}.
\end{aligned}$$

(II): Let $g \in B^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',q,\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X)$. By construction of $\mathrm{ext}_{0,1}$ (see Proposition 5.2.49 and its proof), we have that $\mathrm{ext}_{0,1}\, g = \sum_{n=0}^{\infty}\rho(2^{na_1}\,\cdot\,)\otimes T_n g$ in $\mathcal{S}'(\mathbb{R}^d;X)$ with each $\rho(2^{na_1}\,\cdot\,)\otimes T_n g$ satisfying (5.66) for a $c > 1$ independent of g. In view of Lemma 5.2.10, it is thus enough to show that

$$\|(2^{sn}\rho(2^{na_1}\,\cdot\,)\otimes T_n g)_{n\geq 0}\|_{\ell^q(\mathbb{N};L^{p,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X))} \leq C\|g\|_{B^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',q,\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X)}$$

for some constant $C > 0$ independent of g. So we compute

$$\begin{aligned}
&\|(2^{sn}\rho(2^{na_1}\,\cdot\,)\otimes T_n g)_{n\geq 0}\|_{\ell^q(\mathbb{N};L^{p,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X))}\\
&\quad = \left\|(2^{sn}\|\rho(2^{na_1}\,\cdot\,)\otimes T_n g\|_{L^{p,\mathpzc{d}}(\mathbb{R}^d,(w_\gamma,w'');X)})_{n\geq 0}\right\|_{\ell^q(\mathbb{N})}\\
&\quad = \left\|(2^{sn}\|\rho(2^{na_1}\,\cdot\,)\|_{L^{p_1}(\mathbb{R},w_\gamma)}\|T_n g\|_{L^{p'',\mathpzc{d}''}(\mathbb{R}^{d-1},w'');X)})_{n\geq 0}\right\|_{\ell^q(\mathbb{N})}\\
&\quad = \left\|(2^{n[s-\frac{a_1}{p_1}(1+\gamma)]}\|\rho\|_{L^{p_1}(\mathbb{R},w_\gamma)}\|T_n g\|_{L^{p'',\mathpzc{d}''}(\mathbb{R}^{d-1},w'');X)})_{n\geq 0}\right\|_{\ell^q(\mathbb{N})}\\
&\quad = \|\rho\|_{L^{p_1}(\mathbb{R},w_\gamma)}\|g\|_{B^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',q,\mathpzc{d}''}(\mathbb{R}^{d-1},w'';X)}.
\end{aligned}$$

□

The Trace Space of a Triebel-Lizorkin Space

Theorem 5.2.55. *Let X be a Banach space, $\mathpzc{d}_1 = 1$, $a \in]0,\infty[^l$, $p \in]1,\infty[^l$, $q \in [1,\infty]$, $\gamma \in]-1,\infty[$ and $s > \frac{a_1}{p_1}(1+\gamma)$. Let $w \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{\mathpzc{d}})$ be such that $w_1(x_1) = w_\gamma(x_1) = |x_1|^\gamma$ and $w'' \in \prod_{j=2}^{l} A_{p_j/r_j}(\mathbb{R}^{\mathpzc{d}_j})$ for some $r'' = (r_2,\ldots,r_l) \in]0,1[^{l-1}$ satisfying $s - \frac{a_1}{p_1}(1+\gamma) > \sum_{j=2}^{l} a_j \mathpzc{d}_j(\frac{1}{r_j}-1)$.*[8]

[8]This technical condition on w'' is in particular satisfied for $w'' \in \prod_{j=2}^{l} A_{p_j}(\mathbb{R}^{\mathpzc{d}_j})$.

Then the trace operator $\gamma_{0,1} = \gamma^{\varphi}_{0,1}$ *is well-defined on* $F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X)$ *and restricts to a continuous surjection*

$$\gamma_{0,1} = \gamma^{\varphi}_{0,1} : F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X) \longrightarrow F^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',p_1,\mathpzc{d}''}(\mathbb{R}^{d-1}, w''; X)$$

independent of φ*, for which the extension operator* $\mathrm{ext}_{0,1}$ *from Proposition 5.2.49 (with* $\tilde{\mathpzc{d}} = \mathpzc{d}''$ *and* $\tilde{a} = a''$*) restricts to a corresponding continuous right-inverse. Moreover, if* $s > \frac{a_1}{p_1}(1+\gamma_+)$ *and* $w'' \in \prod_{j=2}^{l} A_{p_j}(\mathbb{R}^{d_j})$*, then*

$$F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X) \hookrightarrow C(\mathbb{R}; L^{p'',\mathpzc{d}}(\mathbb{R}^{d-1}, w''; X)) \hookrightarrow C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^{d-1}; X)) \tag{5.72}$$

and the distributional trace $r_{0,1}$ *coincides with the trace operator* $\gamma_{0,1}$ *on* $F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X)$.

Note that the trace space of the weighted anisotropic Triebel-Lizorkin space (as in the theorem) is independent of the microscopic parameter $q \in [1, \infty]$. Together with the embeddings (5.48) and (5.49) from Proposition 5.2.31, this will allow us to solve the trace problems for Bessel potential and Sobolev spaces via a simple sandwich argument; see the next paragraph.

Besides the technical Lemma 5.2.23, the following lemma plays a crucial role in the proof of this theorem.

Lemma 5.2.56. *For every* $r \in [1, \infty]$ *and* $t > 0$ *there exists a constant* $C > 0$ *such that, for all sequences* $(b_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$*, the following two inequalities hold true:*

$$\begin{aligned} \left\| \left(2^{tk} \textstyle\sum_{n=k+1}^{\infty} |b_n|\right)_{k\in\mathbb{N}} \right\|_{\ell^r(\mathbb{N})} &\leq C \left\| (2^{tk} b_k)_{k\in\mathbb{N}} \right\|_{\ell^r(\mathbb{N})}, \\ \left\| \left(2^{-tk} \textstyle\sum_{n=0}^{k} |b_n|\right)_{k\in\mathbb{N}} \right\|_{\ell^r(\mathbb{N})} &\leq C \left\| (2^{-tk} b_k)_{k\in\mathbb{N}} \right\|_{\ell^r(\mathbb{N})}. \end{aligned}$$

Proof. See [62, Lemma 4.2] and the references given there. □

Proof of Theorem 5.2.55. Let the notations be as in Proposition 5.2.49. We will show that, for an arbitrary $\varphi \in \Phi^{\mathpzc{d},a}(\mathbb{R}^d)$,

(I) $\gamma^{\varphi}_{0,1}$ exists on $F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X)$ and defines a continuous operator

$$\gamma^{\varphi}_{0} : F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X) \longrightarrow F^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',p_1,\mathpzc{d}''}(\mathbb{R}^{d-1}, w''; X);$$

(II) The extension operator $\mathrm{ext}_{0,1}$ from Proposition 5.2.49 (with $\tilde{\mathpzc{d}} = \mathpzc{d}''$ and $\tilde{a} = a''$) restricts to a continuous operator

$$\mathrm{ext}_{0,1} : F^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',p_1,\mathpzc{d}''}(\mathbb{R}^{d-1}, w''; X) \longrightarrow F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X).$$

Since $\mathscr{F}^{-1}C^{\infty}_{c}(\mathbb{R}^d; X) \subset \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^{d-1}; X) \cap F^{s,a''}_{p,p_1,\mathpzc{d}''}(\mathbb{R}^{d-1}, w''; X)$ is dense in $F^{s,a''}_{p,p_1,\mathpzc{d}''}(\mathbb{R}^d, w''; X)$ (cf. Proposition 5.2.17), the right inverse part in the first assertion follows from (I) and (II) via an application of Lemma 5.2.50. The independence of φ in the first assertion follows from denseness of $\mathcal{S}(\mathbb{R}^d; X)$ in $F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X)$ in case $q < \infty$, from which the case $q = \infty$ can be deduced via Proposition 5.2.30.(ii). The last assertion is immediate from a combination of Proposition 5.2.30.(iii) and Theorem 5.2.54; here we of course use the continuous inclusion $F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) \hookrightarrow B^{s,a}_{p,\infty,d}(\mathbb{R}^d, w; X)$ from Proposition 5.2.30.(iii).

(I): In view of Proposition 5.2.30(i), we may with out loss of generality assume that $q = \infty$. Let $f \in F^{s,a}_{p,\infty,d}(\mathbb{R}^d, (w_\gamma, w''); X)$ and write $f_n := S_n f$ for each n. Similar to (I) in the proof of Theorem 5.2.52, now using Lemma 5.2.23 instead of Lemma 5.2.22, it suffices to show that

$$\left\| \left(2^{n[s-\frac{a_1}{p_1}(1+\gamma)]} f_n(0,\cdot) \right)_{n\geq 0} \right\|_{L^{p'',d''}(\mathbb{R}^{d-1},w'';\ell^{p_1}(\mathbb{N};X))} \lesssim \|f\|_{F^{s,a}_{p,\infty,d}(\mathbb{R}^d,(w_\gamma,w'');X)}. \tag{5.73}$$

In order to establish the estimate (5.73), we pick an $r_1 \in]0,1[$ such that $w_\gamma \in A_{p_1/r_1}(\mathbb{R})$, and write $r := (r, r'') \in]0,1[^l$. For all $x = (x_1, x'') \in [2^{-na_1}, 2^{(1-n)a_1}] \times \mathbb{R}^{d-1}$ and every $n \in \mathbb{N}$ we have

$$\|f_n(0,x'')\| \leq C_1 \frac{\|f_n(x_1 - y_1, x'')\|}{1 + |2^{na_1} y_1|^{1/r_1}}\Big|_{y_1 = x_1} \leq (1 + 2^{\frac{a_1}{r_1}}) f^*_n(r, b^{[n]}, d; x) = C_1 f^*_n(r, b^{[n]}, d; x),$$

where $b^{[n]} := (2^{na_1}, \ldots, 2^{na_l}) \in]0,\infty[^l$. Raising this to the p_1-th power, multiplying by $2^{nsp_1}|x_1|^\gamma$, and integrating over $x_1 \in [2^{-na_1}, 2^{(1-n)a_1}]$, we obtain

$$\frac{2^{a_1(\gamma+1)} - 1}{1+\gamma} 2^{n\left(s - \frac{a_1}{p_1}(1+\gamma)\right)p_1} \|f_n(0,x'')\|^{p_1} \leq C_1^p \int_{[2^{-na_1}, 2^{(1-n)a_1}]} \left[2^{ns} f^*_n(r, b^{[n]}, d, d; (x_1, x'')) \right]^{p_1} |x_1|^\gamma dx_1.$$

It now follows that

$$\sum_{n=0}^\infty 2^{\left(s-\frac{a_1}{p_1}(1+\gamma)\right)np_1} \|f_n(0,x'')\|^{p_1} \leq C_2 \int_{\mathbb{R}} \left\| \left(2^{ks} f^*_k(r, b^{[n]}, d; (x_1, x'')) \right)_{k\geq 0} \right\|^{p_1}_{\ell^\infty(\mathbb{N})} |x_1|^\gamma dx_1,$$

from which we in turn obtain

$$\left\| \left(2^{n[s-\frac{a_1}{p_1}(1+\gamma)]} f_n(0,\cdot) \right)_{n\geq 0} \right\|_{L^{p'',d''}(\mathbb{R}^{d-1},w'';\ell^{p_1}(\mathbb{N};X))} \leq \left\| \left(2^{ks} f^*_k(r, b^{[n]}, d; \cdot) \right)_{k\geq 0} \right\|_{L^{p,d}(\mathbb{R}^d,(w_\gamma,w'');\ell^\infty(\mathbb{N}))}.$$

Since $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}'(\mathbb{R}^d; X)$ satisfies $\mathrm{supp}(\hat{f}_k) \subset Q_{d,cb^{[k]}}$ for each $k \in \mathbb{N}$ and some $c > 0$, the desired estimate (5.73) is now a consequence of Proposition 3.4.7.

(II): In view of Proposition 5.2.30(i), we may with out loss of generality assume that $q = 1$. Let $g \in F^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',p_1,d''}(\mathbb{R}^{d-1}, w''; X)$ and write $g_n = T_n g$ for each n. By construction of $\mathrm{ext}_{0,1}$ (see Proposition 5.2.49 and its proof), we have $\mathrm{ext}\, g = \sum_{n=0}^\infty \rho(2^{na_1}\,\cdot) \otimes g_n$ in $\mathcal{S}'(\mathbb{R}^d; X)$ with each $\rho(2^{na_1}\,\cdot) \otimes g_n$ satisfying (5.66) for a $c > 1$ independent of g. In view of Lemma 5.2.10, it is thus enough to show that

$$\|(2^{sn}\rho(2^{na_1}\,\cdot) \otimes g_n)_{n\geq 0}\|_{L^{p,d}(\mathbb{R}^d,(w_\gamma,w'');\ell^1(X))} \lesssim \|g\|_{F^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',p_1,d''}(\mathbb{R}^{d-1},w'';X)}. \tag{5.74}$$

In order to establish the estimate (5.74), we define, for each $x'' \in \mathbb{R}^{d-1}$,

$$I(x'') := \int_{\mathbb{R}} \left(\sum_{n=0}^\infty 2^{sn} \|\rho(2^{na_1}x_1) g_n(x'')\| \right)^{p_1} |x_1|^\gamma dx_1. \tag{5.75}$$

We furthermore first choose a natural number $N > \frac{1}{p_1}(1+\gamma)$ and subsequently pick a constant $C_1 > 0$ for which the Schwartz function $\rho \in \mathcal{S}(\mathbb{R})$ satisfies the inequality $|\rho(2^{na_1}x_1)| \leq C_1 |2^{na_1}x_1|^{-N}$ for every $n \in \mathbb{N}$ and all $x_1 \neq 0$.

Denoting by $I_1(x'')$ the integral over $\mathbb{R} \setminus [-1,1]$ in (5.75), we have

$$\begin{aligned} I_1(x'') &\leq C_1 \int_{\mathbb{R}\setminus[-1,1]} \left(\sum_{n=0}^{\infty} 2^{-Na_1 n}\, 2^{sn}\, \|g_n(x'')\| \right)^{p_1} |x_1|^{-Np_1+\gamma} dx_1 \\ &= C_1 \int_{\mathbb{R}\setminus[-1,1]} |x_1|^{-Np_1+\gamma} dx_1 \left(\sum_{n=0}^{\infty} 2^{\left(\frac{1}{p_1}(1+\gamma)-N\right)a_1 n}\, 2^{\left(s-\frac{a_1}{p_1}(1+\gamma)\right)n}\, \|g_n(x'')\| \right)^{p_1} \\ &\leq \underbrace{\int_{\mathbb{R}\setminus[-1,1]} |x_1|^{-Np_1+\gamma} dx_1 \left\| \left(2^{\left(\frac{1}{p_1}(1+\gamma)-N\right)a_1 n} \right)_{n\geq 0} \right\|_{\ell^{p_1'}(\mathbb{N})}^{p_1}}_{=:C_2\in[0,\infty[} \left\| \left(2^{\left(s-\frac{a_1}{p_1}(1+\gamma)\right)n}\, \|g_n(x'')\| \right)_{n\geq 0} \right\|_{\ell^{p_1}(\mathbb{N})}^{p_1} \end{aligned} \tag{5.76}$$

Next we denote, for each $k \in \mathbb{N}$, by $I_{0,k}(x'')$ the integral over $D_k := \{x_1 \in \mathbb{R} \mid 2^{-(k+1)a_1} \leq |x_1| \leq 2^{-ka_1}\}$ in (5.75). Since the D_k are of measure $w_\gamma(D_k) \leq C_3 2^{-ka_1(\gamma+1)}$ for some constant $C_3 > 0$ independent of k, we can estimate

$$\begin{aligned} I_{0,k}(x'') &\leq \int_{D_k} \left(\sum_{n=0}^{k} 2^{sn}\, \|\rho\|_\infty\, \|g_n(x'')\| + \sum_{n=k+1}^{\infty} C_1 2^{(s-a_1N)n} |x_1|^{-N}\, \|g_n(x'')\| \right)^{p_1} |x_1|^\gamma dx_1 \\ &\leq C_3 2^{-ka_1(\gamma+1)} \left(\sum_{n=0}^{k} 2^{sn}\, \|\rho\|_\infty\, \|g_n(x'')\| + \sum_{n=k+1}^{\infty} C_1 2^{(s-a_1N)n} 2^{Na_1(k+1)}\, \|g_n(x'')\| \right)^{p_1} \\ &\leq C_3 2^{p_1}\, \|\rho\|_\infty^{p_1}\, 2^{-ka_1(\gamma+1)} \left(\sum_{n=0}^{k} 2^{sn}\, \|g_n(x'')\| \right)^{p_1} \\ &\quad + C_3 2^{p_1} (C_1 2^{Na_1})^{p_1} 2^{k\left(N-\frac{1}{p_1}(\gamma+1)\right)a_1 p_1} \left(\sum_{n=k+1}^{\infty} 2^{(s-a_1N)n}\, \|g_n(x'')\| \right)^{p_1}. \end{aligned}$$

Writing $I_0(x'') := \sum_{k=0}^{\infty} I_{0,k}(x'')$, which is precisely the integral over $[-1,1]$ in (5.75), we obtain

$$\begin{aligned} I_0(x'') &\leq C_4 \sum_{k=0}^{\infty} 2^{-ka_1(\gamma+1)} \left(\sum_{n=0}^{k} 2^{sn}\, \|g_n(x'')\| \right)^{p_1} + C_4 \sum_{k=0}^{\infty} 2^{k\left(N-\frac{1}{p_1}(\gamma+1)\right)a_1 p_1} \left(\sum_{n=k+1}^{\infty} 2^{(s-a_1N)n}\, \|g_n(x'')\| \right)^{p_1} \\ &= C_4 \left\| \left(2^{-\frac{a_1}{p_1}(1+\gamma)k} \sum_{n=0}^{k} 2^{sn}\, \|g_n(x'')\| \right)_{k\in\mathbb{N}} \right\|_{\ell^{p_1}(\mathbb{N})}^{p_1} \\ &\quad + C_4 \left\| \left(2^{\left(N-\frac{1}{p_1}(1+\gamma)\right)a_1 k} \sum_{n=k+1}^{\infty} 2^{(s-a_1N)n}\, \|g_n(x'')\| \right)_{k\in\mathbb{N}} \right\|_{\ell^{p_1}(\mathbb{N})}^{p_1}, \end{aligned}$$

which via an application of Lemma 5.2.56 can be further estimated as

$$\begin{aligned} I_0(x'') &\leq C_5 \left\| \left(2^{-\frac{a_1}{p_1}(1+\gamma)k} 2^{sk}\, \|g_k(x'')\| \right)_{k\geq 0} \right\|_{\ell_{p_1}(\mathbb{N})}^{p_1} + C_5 \left\| \left(2^{\left(N-\frac{1}{p_1}(\gamma+1)\right)a_1 k} 2^{(s-a_1N)k}\, \|g_k(x'')\| \right)_{k\geq 0} \right\|_{\ell^{p_1}(\mathbb{N})}^{p_1} \\ &= 2C_5 \left\| \left(2^{\left(s-\frac{a_1}{p_1}(1+\gamma)\right)k}\, \|g_k(x'')\| \right)_{k\geq 0} \right\|_{\ell^{p_1}(\mathbb{N})}^{p_1}. \end{aligned} \tag{5.77}$$

Combining the estimates (5.76) and (5.77), we get

$$I(x'')^{1/p_1} \leq C_6 \left\| \left(2^{\left(s-\frac{a_1}{p_1}(1+\gamma)\right)n}\, \|g_n(x'')\| \right)_{n\geq 0} \right\|_{\ell^{p_1}(\mathbb{N})},$$

from which (5.74) follows by taking $L^{p'',d''}(\mathbb{R}^{d-1}, w'')$-norms. □

The Trace Spaces of Bessel Potential and Sobolev Spaces The following two trace results are immediate from Theorem 5.2.55 and Proposition 5.2.31.

Corollary 5.2.57. *Let X be a Banach space, $\mathpzc{d}_1 = 1$, $a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$, $p \in]1,\infty[^l$, $\gamma \in]-1, p_1 - 1[$ and $s > \frac{a_1}{p_1}(1+\gamma)$. Let $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{\mathpzc{d}})$ be such that $w_1(x_1) = w_\gamma(x_1) = |x_1|^\gamma$ (see Example D.2.12). Then the trace operator $\gamma_{0,1} = \gamma_{0,1}^\varphi$ is well-defined on $H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X)$ and restricts to a continuous surjection*

$$\gamma_{0,1} : H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X) \longrightarrow F^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',p_1,\mathpzc{d}''}(\mathbb{R}^{d-1}, w''; X),$$

independent of φ, for which the extension operator $\mathrm{ext}_{0,1}$ *from Proposition 5.2.49 (with $\tilde{\mathpzc{d}} = \mathpzc{d}''$ and $\tilde{a} = a''$) restricts to a corresponding continuous right-inverse. Moreover, if $s > \frac{a_1}{p_1}(1 + \gamma_+)$, then*

$$H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X) \hookrightarrow C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^{d-1}; X))$$

and the distributional trace $r_{0,1}$ coincides with the trace operator $\gamma_{0,1}$ on $H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X)$.

Corollary 5.2.58. *Let X be a Banach space, $\mathpzc{d}_1 = 1$, $n \in (\mathbb{Z}_{>0})^l$, $p \in]1,\infty[^l$, $\gamma \in]-1, p_1 - 1[$. Let $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{\mathpzc{d}})$ be such that $w_1(x_1) = w_\gamma(x_1) = |x_1|^\gamma$ (see Example D.2.12). Furthermore, let $s > 0$ and $a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$ be such that $n_j = \frac{s}{a_j}$ for each $j \in \{1, \dots, l\}$. Then the trace operator $\gamma_{0,1} = \gamma_{0,1}^\varphi$ is well-defined on $W^n_{p,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X)$ and restricts to a continuous surjection*

$$\gamma_{0,1} : W^n_{p,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X) \longrightarrow F^{s-\frac{a_1}{p_1}(1+\gamma),a''}_{p'',p_1,\mathpzc{d}''}(\mathbb{R}^{d-1}, w''; X),$$

independent of φ, for which the extension operator $\mathrm{ext}_{0,1}$ *from Proposition 5.2.49 (with $\tilde{\mathpzc{d}} = \mathpzc{d}''$ and $\tilde{a} = a''$) restricts to a corresponding continuous right-inverse. Moreover,*

$$W^n_{p,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X) \hookrightarrow C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^{d-1}; X))$$

and the distributional trace $r_{0,1}$ coincides with the trace operator $\gamma_{0,1}$ on $W^n_{p,\mathpzc{d}}(\mathbb{R}^d, (w_\gamma, w''); X)$.

Remark 5.2.59. The above two results easily extend to the case $\mathpzc{d}_1 > 1$; it should be understood that we take $w_1 = w_\gamma$ as in (5.60) (which may be even multiplied with an A_{p_1}-weight depending only on $(x_{1,2}, \dots, x_{1,\mathpzc{d}_1})$) in this case. For this we just have to reformulate Theorem 5.2.55 to this situation via Remark 5.2.13, and then apply the sandwich argument based on Proposition 5.2.31.

5.2.3.c Traces with respect to $\mathbb{R}^{d-1} \times \{0\}$

The Trace Space of a Besov Space We first investigate the Besov case. Since we restrict ourselves to weight-vectors $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{\mathpzc{d}})$ in which w_l has the form (5.61) for some $\mu > -1$, in view of Remark 5.2.13 we may without loss of generality assume that $\mathpzc{d}_l = 1$; this will simplify the notation.

Throughout this paragraph we will use the following notation: We write $\mathpzc{d}' = (\mathpzc{d}_1, \dots, \mathpzc{d}_{l-1})$. Given $p \in]1,\infty[^l$ we will write $p' = (p_1, \dots, p_{l-1})$, and similarly for $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{\mathpzc{d}})$.

Theorem 5.2.60. *Let X be a Banach space, $\mathpzc{d}_l = 1$, $a \in]0,\infty[^l$, $p \in]1,\infty[^l$, $q \in [1,\infty]$, $\mu \in]-1,\infty[$ and $s > \frac{a_l}{p_l}(1+\mu)$. Let $w \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{\mathpzc{d}})$ be such that $w_l(x_l) = v_\mu(x_l) = |x_l|^\mu$ and $w' \in \prod_{j=1}^{l-1} A_{p_j/r_j}(\mathbb{R}^{\mathpzc{d}_j})$ for some $r' = (r_1, \ldots, r_{l-1}) \in]0,1[^{l-1}$ satisfying $s - \frac{a_l}{p_l}(1+\gamma) > \sum_{j=1}^{l-1} a_j \mathpzc{d}_j(\frac{1}{r_j} - 1)$.*[9] *Then the trace operator $\gamma_{0,d} = \gamma^{\varphi}_{0,d}$ is well-defined on $B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X)$ and restricts to a continuous surjection*

$$\gamma_{0,d} = \gamma^{\varphi}_{0,d} : B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (v_\mu, w_\mu); X) \longrightarrow B^{s-\frac{a_l}{p_l}(1+\mu),a'}_{p',q,\mathpzc{d}'}(\mathbb{R}^{d-1}, w'; X),$$

independent of φ, for which the extension operator $\mathrm{ext}_{0,d}$ from Proposition 5.2.49 (with $\tilde{\mathpzc{d}} = \mathpzc{d}'$ and $\tilde{a} = a'$) restricts to a corresponding continuous right-inverse. Moreover, if $s > \frac{a_l}{p_l}(1+\mu_+)$ and $w' \in \prod_{j=1}^{l-1} A_{p_j}(\mathbb{R}^{\mathpzc{d}_j})$, then

$$B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X) \hookrightarrow C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^{d-1}; X))$$

and the distributional trace $r_{0,d}$ coincides with the trace operator $\gamma_{0,d}$ on $B^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X)$.

Proof. The proof is similar to the proof of Theorem 5.2.52, concerning the trace problem for the hyperplane $\{0\} \times \mathbb{R}^{d-1}$. So we skip it. □

The Trace Space of a Triebel-Lizorkin Space Just as in the Besov case, we may without loss of generality restrict ourselves to the case $\mathpzc{d}_l = 1$.

Theorem 5.2.61. *Let X be a Banach space, $\mathpzc{d}_l = 1$, $a \in]0,\infty[^l$, $p \in]1,\infty[^l$, $q \in [1,\infty]$, $\mu \in]-1,\infty[$ and $s > \frac{a_l}{p_l}(1+\mu)$. Let $w \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{\mathpzc{d}})$ be such that $w_l(x_l) = v_\mu(x_l) = |x_l|^\mu$ and $w' \in \prod_{j=1}^{l-1} A_{p_j/r_j}(\mathbb{R}^{\mathpzc{d}_j})$ for some $r' = (r_1, \ldots, r_{l-1}) \in]0,1[^{l-1}$ satisfying $s - \frac{a_l}{p_l}(1+\mu) > \sum_{j=1}^{l-1} a_j \mathpzc{d}_j(\frac{1}{r_j} - 1)$.*[9] *Then the trace operator $\gamma_{0,d} = \gamma^{\varphi}_{0,d}$ is well-defined on $F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X)$ and restricts to a continuous surjection*

$$\gamma_{0,d} = \gamma^{\varphi}_{0,d} : F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X) \longrightarrow B^{s-\frac{a_l}{p_l}(1+\mu),a'}_{p',p_l,\mathpzc{d}'}(\mathbb{R}^{d-1}, w'; X)$$

independent of φ, for which the extension operator $\mathrm{ext}_{0,d}$ from Proposition 5.2.49 (with $\tilde{\mathpzc{d}} = \mathpzc{d}'$ and $\tilde{a} = a'$) restricts to a corresponding continuous right-inverse. Moreover, if $s > \frac{a_l}{p_l}(1+\mu_+)$ and $w' \in \prod_{j=1}^{l-1} A_{p_j}(\mathbb{R}^{\mathpzc{d}_j})$, then

$$F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X) \hookrightarrow C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^{d-1}; X))$$

and the distributional trace $r_{0,d}$ coincides with the trace operator $\gamma_{0,d}$ on $F^{s,a}_{p,q,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X)$.

Proof. This can be shown in a similar fashion as Theorem 5.2.55, concerning the trace problem with respect to $\{0\} \times \mathbb{R}^{d-1}$. For more details on the involved computations in the unweighted scalar-valued case we refer to [62, Theorem 2.5]. □

Note that, as for the trace problem with respect to the hyperplane $\{0\} \times \mathbb{R}^{d-1}$, that the trace space is independent of the microscopic parameter $q \in [1,\infty]$. Together with the embeddings (5.48) and (5.49) from Proposition 5.2.31, this again will allow us to solve the related trace problems for Bessel potential and Sobolev spaces via a simple sandwich argument; see the next paragraph.

[9]This technical condition on w' is in particular satisfied for $w' \in \prod_{j=1}^{l-1} A_{p_j}(\mathbb{R}^{\mathpzc{d}_j})$.

The Trace Spaces of Bessel Potential and Sobolev Spaces The following two trace results are immediate from Theorem 5.2.61 and Proposition 5.2.31.

Corollary 5.2.62. *Let X be a Banach space, $\mathpzc{d}_l = 1$, $a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$, $p \in]1,\infty[^l$, $\mu \in]-1, p_l - 1[$ and $s > \frac{a_l}{p_l}(1+\mu)$. Let $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{\mathpzc{d}})$ be such that $w_l(x_l) = v_\mu(x_l) = |x_l|^\mu$ (see Example D.2.12). Then the trace operator $\gamma_{0,d} = \gamma^{\varphi}_{0,d}$ is well-defined on $H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X)$ and restricts to a continuous surjection*

$$\gamma_{0,d} : H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X) \longrightarrow B^{s-\frac{a_l}{p_l}(1+\mu),a'}_{p',p_l,\mathpzc{d}'}(\mathbb{R}^{d-1}, w'; X),$$

independent of φ, for which the extension operator $\mathrm{ext}_{0,d}$ *from Proposition 5.2.49 (with $\tilde{\mathpzc{d}} = \mathpzc{d}'$ and $\tilde{a} = a'$) restricts to a corresponding continuous right-inverse. Moreover, if $s > \frac{a_l}{p_l}(1+\mu_+)$, then*

$$H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X) \hookrightarrow C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^{d-1}; X))$$

and the distributional trace $r_{0,d}$ coincides with the trace operator $\gamma_{0,d}$ on $H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X)$.

Corollary 5.2.63. *Let X be a Banach space, $\mathpzc{d}_l = 1$, $n \in (\mathbb{Z}_{>0})^l$, $p \in]1,\infty[^l$, $\mu \in]-1, p_l - 1[$. Let $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{\mathpzc{d}})$ be such that $w_l(x_l) = w_\gamma(x_l) = |x_l|^\mu$ (see Example D.2.12). Furthermore, let $s > 0$ and $a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$ be such that $n_j = \frac{s}{a_j}$ for each $j \in \{1,\ldots,l\}$. Then the trace operator $\gamma_{0,d} = \gamma^{\varphi}_{0,d}$ is well-defined on $W^n_{p,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X)$ and restricts to a continuous surjection*

$$\gamma_{0,d} : W^{n}_{p,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X) \longrightarrow B^{s-\frac{a_l}{p_l}(1+\mu),a'}_{p',p_l,\mathpzc{d}'}(\mathbb{R}^{d-1}, w'; X),$$

independent of φ, for which the extension operator $\mathrm{ext}_{0,d}$ *from Proposition 5.2.49 (with $\tilde{\mathpzc{d}} = \mathpzc{d}'$ and $\tilde{a} = a'$) restricts to a corresponding continuous right-inverse. Moreover,*

$$W^{n}_{p,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X) \hookrightarrow C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^{d-1}; X))$$

and the distributional trace $r_{0,d}$ coincides with the trace operator $\gamma_{0,d}$ on $W^n_{p,\mathpzc{d}}(\mathbb{R}^d, (w', v_\mu); X)$.

Remark 5.2.64. Similarly to Remark 5.2.59, the above two results easily extend to the case $\mathpzc{d}_l > 1$; it should be understood that we take $w_l = v_\mu$ as in (5.61) (which may be even multiplied with an A_{p_l}-weight depending only on $(x_{l,1}, \ldots, x_{l,\mathpzc{d}_l-1})$) in this case.

5.2.3.d Spaces with Boundary Conditions

In this subsection we have a look at the closure of the space of Schwartz functions which vanish at $\{0\} \times \mathbb{R}^{d-1}$ (resp. $\mathbb{R}^{d-1} \times \{0\}$) in the weighted anisotropic mixed-norm function spaces of Triebel-Lizorkin, Besov, Bessel potential, and Sobolev type. The first result in this direction is the following lemma, which is an immediate consequence of Lemma 5.2.51 and the various trace results from the previous two subsubsections:

Lemma 5.2.65.

(i) Let E be a weighted anisotropic mixed-norm function space of Triebel-Lizorkin, Besov, Bessel potential or Sobolev type, as in Theorem 5.2.52, Theorem 5.2.55, Corollary 5.2.57, Corollary 5.2.58, respectively. Then $\{ f \in \mathcal{S}(\mathbb{R}^d; X) : f_{|\{0\}\times\mathbb{R}^{d-1}} = 0\}$ is dense in $\ker(\gamma_{0,1}) \cap$ *E.*

(ii) *Let E be a weighted anisotropic mixed-norm function space of Triebel-Lizorkin, Besov, Bessel potential or Sobolev type, as in Theorem 5.2.60, Theorem 5.2.61, Corollary 5.2.62, or Corollary 5.2.63. Then $\{ f \in \mathcal{S}(\mathbb{R}^d; X) : f_{|\mathbb{R}^{d-1}\times\{0\}} = 0\}$ is dense in* $\ker(\gamma_{0,d}) \cap E$.

Remark 5.2.66. We can of course also allow anisotropic spaces E with $d_1 > 1$ (resp. $d_l > 1$) in the above lemma; see Remarks 5.2.13 and 5.2.64 (resp. Remarks 5.2.13 and 5.2.64).

In this lemma the closure of the space of Schwartz functions which vanish at $\{0\} \times \mathbb{R}^{d-1}$ (resp. $\mathbb{R}^{d-1} \times \{0\}$) is characterized as the kernel of the trace operator with respect to $\{0\} \times \mathbb{R}^{d-1}$ (resp. $\mathbb{R}^{d-1} \times \{0\}$) on the space E under consideration. In order to formulate this in a notionally more compact way, let us us define:

Definition 5.2.67. Let X be a Banach space. For a Banach space E containing $\mathcal{S}(\mathbb{R}^d; X)$, we define ${}_{0,(0,i)}E$ as the closure of ${}_{0,(0,i)}\mathcal{S}(\mathbb{R}^d; X) := \{ f \in \mathcal{S}(\mathbb{R}^d; X) : f_{|\mathbb{R}^{i-1}\times\{0\}\times\mathbb{R}^{d-i}} = 0\}$ in E.

We next consider situations in which ${}_{0,(0,1)}E = E$ (resp. ${}_{0,(0,d)}E = E$), having as a consequence the non-existence of a trace operator with respect to $\{0\} \times \mathbb{R}^{d-1}$ (resp. $\mathbb{R}^{d-1} \times \{0\}$). The main tool for this denseness result will be the the following result that, in the natural parameter range, the characteristic function of the half-space is a pointwise multiplier on vector-valued Besov and Triebel-Lizorkin spaces:

Theorem 5.2.68. *[78, Theorem 1.3] Let X be a Banach space, $p \in]1,\infty[$, $q \in [1,\infty[$, and $\gamma \in]-1, p-1[$. Define the weight $w_\gamma(x) := |x_1|^\gamma$ on $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$. Let $\mathcal{A} \in \{B, F\}$. Then, for $s \in]-\frac{1+\gamma'}{p'}, \frac{1+\gamma}{p}[$, the pointwise multiplier operator*

$$\mathcal{S}(\mathbb{R}^d; X) \longrightarrow L^\infty(\mathbb{R}^d; X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X)\, f \mapsto 1_{\mathbb{R}^d_+} f$$

takes its values in $\mathcal{A}^s_{p,q}(\mathbb{R}^d, w_\gamma; X)$ and extends to a bounded linear operator on $\mathcal{A}^s_{p,q}(\mathbb{R}^d, w_\gamma; X)$.

Proof. The result in [78, Theorem 1.3] is in fact more general. Besides also considering the case $q = \infty$, the theorem states that for all $f \in \mathcal{A}^s_{p,q}(\mathbb{R}^d, w; X)$ the product $1_{\mathbb{R}^d_+} f$ is well-defined as a so-called paraproduct and is a tempered distribution belonging to $\mathcal{A}^s_{p,q}(\mathbb{R}^d, w; X)$, and moreover that the induced mapping

$$\mathcal{A}^s_{p,q}(\mathbb{R}^d, w_\gamma; X) \longrightarrow \mathcal{A}^s_{p,q}(\mathbb{R}^d, w_\gamma; X),\ f \mapsto 1_{\mathbb{R}^d_+} f$$

defines a bounded linear operator. But [85, Section 4.2.1, Lemma 1] in particular says that, for Schwartz functions $f \in \mathcal{S}(\mathbb{R}^d; X)$, whenever $1_{\mathbb{R}^d_+} f$ exists as a paraproduct, the resulting distribution coincides with the usual pointwise product, implying our result. □

Using the intersection representation from Theorem 5.2.35, the following poinwise multiplier result for anisotropic Triebel-Lizorkin spaces follows easily from the isotropic Triebel-Lizorkin case considered in the above theorem:

Corollary 5.2.69. *Let X be a Banach space, $l = 2$, $a \in]0,\infty[^2$, $p, q \in]1,\infty[$, $\tilde{p} := (p, q)$, $\mu \in]-1, q-1[$, and $w \in A_p(\mathbb{R}^{d_1}) \times A_q(\mathbb{R}^{d_2})$ with $w_2(x_2) = v_\mu(x_2) := |x_{2,d_2}|^\mu$. Then, for $s \in]0, \frac{a_2}{q}(1+\mu)[$, the pointwise multiplier operator*

$$\mathcal{S}(\mathbb{R}^d; X) \longrightarrow L^p(\mathbb{R}^d, w; X)\, f \mapsto 1_{\mathbb{R}^{d-1}\times]0,\infty[} f$$

takes its values in $F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X)$ and extends to a bounded linear operator on $F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X)$.

Proof. By Theorem 5.2.35 we have

$$F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X) = F^{s/a_2}_{q,p}(\mathbb{R}^{d_2}, w_2; L^p(\mathbb{R}^{d_1}, w_1; X)) \cap L^q(\mathbb{R}^{d_2}, w_2; F^{s/a_1}_{p,p}(\mathbb{R}^{d_1}, w_1; X))$$

with an equivalence of norms. So we only need to show that the multiplier operator acts boundedly on both spaces occurring in this intersection. For the space $F^{s/a_2}_{q,p}(\mathbb{R}^{d_2}, w_2; L^p(\mathbb{R}^{d_1}, w_1; X))$ we just note that

$$\mathcal{S}(\mathbb{R}^d; X) = \mathcal{S}(\mathbb{R}^{d_2}; \mathcal{S}(\mathbb{R}^{d_1}; X)) \hookrightarrow \mathcal{S}(\mathbb{R}^{d_2}; L^p(\mathbb{R}^{d_1}, w_1; X))$$

and invoke Theorem 5.2.68. For the space $L^q(\mathbb{R}^{d_2}, w_2; F^{s/a_1}_{p,p}(\mathbb{R}^{d_1}, w_1; X))$ it is enough to observe that

$$\mathcal{S}(\mathbb{R}^d; X) = \mathcal{S}(\mathbb{R}^{d_2}; \mathcal{S}(\mathbb{R}^{d_1}; X)) \hookrightarrow \mathcal{S}(\mathbb{R}^{d_2}; F^{s/a_1}_{p,p}(\mathbb{R}^{d_1}, w_1; X)).$$

□

Using the pointwise multiplier result from this corollary, we are now able to prove:

Proposition 5.2.70. *Let the notations be as in Corollary 5.2.69. For $s < \frac{a_2}{q}(1+\mu)[$ it holds that ${}_{0,(0,d)}F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X) = F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X)$.*

Proof. Since $C^\infty_c(\mathbb{R}^d; X)$ is dense in $F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X)$, it suffices to show that $C^\infty_c(\mathbb{R}^d; X)$ is contained in the closure of ${}_{0,(0,d)}\mathcal{S}(\mathbb{R}^d; X)$ in $F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X)$. For this we may of course assume that $s > 0$; so $s \in]0, \frac{a_2}{q}(1+\mu)[$.

Let us fix an $f \in C^\infty_c(\mathbb{R}^d; X)$. Then we have $f^+ := 1_{\mathbb{R}^{d-1}\times]0,\infty[}f \in L^\infty(\mathbb{R}^d; X) \cap F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X)$ and $f^- := 1_{\mathbb{R}^{d-1}\times]0,\infty[}f = f - f^+ \in L^\infty(\mathbb{R}^d; X) \cap F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X)$ by the pointwise multiplier result of Corollary 5.2.69. So it is enough to construct sequences $(f^\pm_n)_{n\geq 1} \subset {}_{0,(0,d)}\mathcal{S}(\mathbb{R}^d; X)$ such that $f^\pm_n \overset{n\to\infty}{\longrightarrow} f^\pm$ in $F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X)$.

We shall only construct the sequence $(f^+_n)_{n\geq 1}$, the construction of $(f^-_n)_{n\geq 1}$ being completely similar. For the construction of $(f^+_n)_{n\geq 1}$, we pick an $\phi \in C^\infty_c(\mathbb{R}^d)$ such that $\phi \geq 0$, $\int_{\mathbb{R}^d}\phi(x)dx = 1$, and $\operatorname{supp}(\phi) \subset \mathbb{R}^{d-1}\times[1,\infty[$. We set $f^+_n := \phi_n * f^+$ for each $n \in \mathbb{Z}_{\geq 1}$, where $\phi_n := n^d\phi(n\,\cdot)$. By the basic properties of the convolution product (see Appendix C.5), we then have $f^+_n \in C^\infty(\mathbb{R}^d; X)$ with

$$\operatorname{supp}(f^+_n) \subset \operatorname{supp}(\phi_n) + \operatorname{supp}(f^+) \subset \left[\mathbb{R}^{d-1} \times [1/n, \infty[\right] + \left[\mathbb{R}^{d-1} \times [0, \infty[\right] \subset \mathbb{R}^{d-1}\times]0,\infty[;$$

in particular, $(f^+_n)_{n\geq 1} \subset {}_{0,(0,d)}\mathcal{S}(\mathbb{R}^d; X)$. Furthermore, $f^+ = \lim_{n\to\infty} f^+_n$ in $F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X)$ by Proposition 5.2.20. □

Remark 5.2.71.

(i) The above proof even shows that $C^\infty_c(\dot{\mathbb{R}}^d; X)$ is dense in $F^{s,a}_{\tilde{p},p,d}(\mathbb{R}^d, w; X)$ for $s < \frac{a_2}{q}(1+\mu)$, where $\dot{\mathbb{R}}^d = \mathbb{R}^d \setminus [\mathbb{R}^{d-1} \times \{0\}]$.

(ii) Using the embeddings from 5.2.2.b, we can extend this result to the spaces

$$E = F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X), a \in]0,\infty[^2, p \in]1,\infty[^2, q \in [1,\infty[, s < \frac{a_2}{p_2}(1+\mu)[,$$

and

$$E = H^{s,a}_{p,\mathpzc{d}}(\mathbb{R}^d, w; X), a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^2, p \in]1,\infty[^2$$

equipped with a weight-vector $w \in \prod_{j=1}^2 A_{p_j}(\mathbb{R}^{d_j})$ with $w_2(x_2) = v_\mu(x_2) := |x_{2,d_2}|^\mu$, $\mu \in]-1, p_2[$. We expect that this also holds true for general l ($l \geq 3$). For this we would have to find a $(\mathpzc{d}, a)$-anisotropic generalization of the pointwise multiplier result Theorem 5.2.68. Since the situation covered by Proposition 5.2.70 will be enough for our purposes in Chapter 6, we will not go into this direction.

5.3 Function Spaces on Domains

The topic of this section is function spaces on domains, i.e on open subsets of $\mathbb{R}^d$. Motivated by applications to the parabolic initial-boundary value model problems in Chapter 6, we mainly restrict ourselves to anisotropic function spaces on $\mathbb{R}^d_+ \times \mathbb{R}$, $\mathbb{R}^{d-1} \times \mathbb{R}$, and $\mathbb{R}^d_+$.

5.3.1 Definitions and Basic Properties

We start with the definition of weighted anisotropic mixed-norm Sobolev spaces in terms of the the weighted mixed-norm Lebesgue-Bochner spaces $L^{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w; X)$ from Definition 2.2.2.

Definition 5.3.1. Let X be a Banach space, $U_j \subset \mathbb{R}^{d_j}$, $j = 1, \ldots, l$, open subsets , $p \in]1,\infty[^l$, $n \in \mathbb{N}^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. We define the *weighted anisotropic mixed-norm Sobolev space* $W^n_{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w; X)$ as the space of all $f \in \mathcal{D}'(U_1 \times \ldots \times U_l; X)$ for which $D^\alpha f \in L^{p,\mathpzc{d}}(U_1 \times \ldots \times U_l, w; X)$ for all $\alpha \in J_{n,\mathpzc{d}}$, where $J_{n,\mathpzc{d}} = \{\alpha \in \bigcup_{j=1}^m \iota_{[\mathpzc{d};j]}\mathbb{N}^{d_j} : |\alpha_j| \leq n_j\}$. We equip this space with the norm

$$\|f\|_{W^n_{p,\mathpzc{d}}(U_1\times\ldots\times U_l,w;X)} := \sum_{\alpha\in J_{n,\mathpzc{d}}} \|D^\alpha f\|_{L^{p,\mathpzc{d}}(U_1\times\ldots\times U_l,w;X)},$$

which turns it into a Banach space.

For the application to the weighted maximal L^q-L^p-regularity problem for parabolic initial-boundary value problems on the half-space in Chapter 6, it is suffices to consider these weighted anisotropic mixed-norm Sobelev spaces in the situation of the following lemma (also see Lemma 2.1.4).

Lemma 5.3.2. *Let X be a Banach space, $U = U_1 \times \ldots \times U_l$ a product of open subsets $U_j \in \{\mathbb{R}^{d_j}_+, \mathbb{R}^{d_j}\}$, $j = 1, \ldots, l$,, $p \in]1,\infty[^l$, $n \in \mathbb{N}^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. For each $j \in \{1, \ldots, l\}$ with $U_j = \mathbb{R}^{d_j}_+$ we assume w_j to be symmetric with respect to reflections in $\{0\}\times\mathbb{R}^{d_j-1}$ and to satisfy, in case $n_j \geq 1$, for some $C_j \in]0,\infty[$ and $\lambda_j \in]0,\infty[\setminus\{1\}$, the estimate $w(\lambda_j x_{j,1}, \ldots, x_{j,d_j}) \leq Cw(x_j)$ for almost all $x_j = (x_{j,1}, \ldots, x_{j,d_j}) \in \mathbb{R}^{d_j}$. Then the restriction operator*

$$R := r_{\mathbb{R}^d,U} \in \mathcal{L}(\mathcal{D}'(\mathbb{R}^d; X), \mathcal{D}'(U; X))$$

restricts to a continuous surjection from $W^n_{p,\mathpzc{d}}(\mathbb{R}^d, w; X)$ onto $W^n_{p,\mathpzc{d}}(U, w; X)$ with a continuous right inverse. As a consequence, we have

$$W^n_{p,\mathpzc{d}}(U, w; X) = \{ f \in \mathcal{D}'(\mathbb{R}^d; X) : f = Rg, g \in W^n_{p,\mathpzc{d}}(\mathbb{R}^d, w; X) \} \tag{5.78}$$

with

$$\|f\|_{W^n_{p,d}(U,w;X)} \eqsim \inf\{\, \|g\|_{W^n_{p,d}(\mathbb{R}^d,w;X)} \,:\, f = Rg, g \in W^n_{p,d}(\mathbb{R}^d, w; X)\,\}. \tag{5.79}$$

Proof. This can be shown similarly to Lemma 2.1.4. □

This lemma allows us to carry over many properties of $W^n_{p,d}(\mathbb{R}^d, w; X)$ to $W^n_{p,d}(U, w; X)$. As we have seen in Section 5.2, weighted anisotropic mixed-norm function spaces of Bessel potential and Triebel-Lizorkin, and Besov type play a very important role in the theory of weighted anisotropic mixed-norm Sobolev spaces on the full Euclidean space $\mathbb{R}^d$. Together with the description (5.78)/(5.79), this motivates to define the weighted anisotropic mixed-norm function spaces of Bessel Potential, Triebel-Lizorkin, and Besov type on domains via restriction of the corresponding spaces on $\mathbb{R}^d$.

Definition 5.3.3.

(i) Let X be a Banach space, $U \subset \mathbb{R}^d$ an open subset, $a \in \left(\frac{1}{\mathbb{Z}}\right)^l$, $p \in]1, \infty[^l$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. We define *weighted anisotropic mixed-norm Bessel potential space* as the linear space

$$H^{s,a}_{p,d}(U, w; X) := \{\, f \in \mathcal{D}'(U; X) \,:\, f = r_{\mathbb{R}^d,U} g, g \in H^{s,a}_{p,d}(\mathbb{R}^d, w; X)\,\}$$

equipped with the norm

$$\|f\|_{H^{s,a}_{p,d}(U,w;X)} := \inf\{\, \|g\|_{H^{s,a}_{p,d}(\mathbb{R}^d,w;X)} \,:\, f = r_{\mathbb{R}^d,U} g, g \in H^{s,a}_{p,d}(\mathbb{R}^d, w; X)\,\}.$$

(ii) Let X be a Banach space, $a \in]0, \infty[^l$, $p \in [1, \infty[^l$, $q \in [1, \infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. We define the *weighted anisotropic mixed-norm Triebel-Lizorkin space* as the linear space

$$F^{s,a}_{p,q,d}(U, w; X) := \{\, f \in \mathcal{D}'(U; X) \,:\, f = r_{\mathbb{R}^d,U} g, g \in F^{s,a}_{p,q,d}(\mathbb{R}^d, w; X\,\}$$

equipped with the norm

$$\|f\|_{F^{s,a}_{p,q,d}(U,w;X)} := \inf\{\, \|g\|_{F^{s,a}_{p,q,d}(U,w;X)} \,:\, f = r_{\mathbb{R}^d,U} g, g \in F^{s,a}_{p,q,d}(U, w; X)\,\}.$$

(iii) Let X be a Banach space, $a \in]0, \infty[^l$, $p \in [1, \infty[^l$, $q \in [1, \infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. We define the *weighted anisotropic mixed-norm Besov space* as the linear space

$$B^{s,a}_{p,q,d}(U, w; X) := \{\, f \in \mathcal{D}'(U; X) \,:\, f = r_{\mathbb{R}^d,U} g, g \in B^{s,a}_{p,q,d}(\mathbb{R}^d, w; X\,\}$$

equipped with the norm

$$\|f\|_{B^{s,a}_{p,q,d}(U,w;X)} := \inf\{\, \|g\|_{B^{s,a}_{p,q,d}(U,w;X)} \,:\, f = r_{\mathbb{R}^d,U} g, g \in B^{s,a}_{p,q,d}(U, w; X)\,\}.$$

Note that these definitions via restriction can also be viewed as taking quotients; to be more precise, the above defined spaces can be thought of as canonical realizations of quotient spaces as spaces as spaces of X-valued distributions on U. For example, in the Bessel potential case we have the canonical isomorphism

$$H^{s,a}_{p,q,d}(U, w; X) \cong H^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) / \left[\ker(r_{\mathbb{R}^d,U}) \cap H^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)\right].$$

Since $\ker(r_{\mathbb{R}^d,U}) \cap H^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$ is a closed linear subspace of the Banach space $H^{s,a}_{p,q,d}(\mathbb{R}^d, w; X)$, it in particular follows that $H^{s,a}_{p,q,d}(U, w; X)$ is a Banach space. A similar statement of course holds true for the Triebel-Lizorkin and Besov case.

It is a natural question whether these spaces can be defined as subspaces of $\mathcal{D}'(U; X)$ in terms of explicit norms. This is the so-called problem of inner-description, see [97, Section 3.1.1]. For reasons of time and for reasons of size of this thesis we will not go into this. However, we would like to remark that the difference norm characterization from Proposition 5.2.44 gives a suggestion how to define a concrete norm in the Triebel-Lizorkin case. Such descriptions are (at least) well known in the (unweighted) isotropic setting (see [97, 98]), which are useful to get more concrete descriptions of anisotropic spaces of intersection type (involving isotropic spaces in their description).

The following results are immediate consequences from the above quotient space viewpoint and the corresponding results on the full Euclidean space $\mathbb{R}^d$.

Proposition 5.3.4.

(i) Let X be a Banach space, $U = \prod_{j=1}^l U_j$ a product of open subsets $U_j \in \{\mathbb{R}^{d_j}_+, \mathbb{R}^{d_j}\}$, $j = 1, \ldots, l$, $p \in]1, \infty[^l$, $n \in \mathbb{N}^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. For each $j \in \{1, \ldots, l\}$ with $U_j = \mathbb{R}^{d_j}_+$ we assume w_j to be symmetric with respect to reflections in $\{0\} \times \mathbb{R}^{d_j-1}$ and to satisfy, in case $n_j \geq 1$, for some $C_j \in]0, \infty[$ and $\lambda_j \in]0, \infty[\setminus\{1\}$, the estimate $w(\lambda_j x_{j,1}, \ldots, x_{j,d_j}) \leq Cw(x_j)$ for almost all $x_j = (x_{j,1}, \ldots, x_{j,d_j}) \in \mathbb{R}^{d_j}$. Then $W^n_{p,d}(\prod_{j=1}^l U_j, w; X)$ is Banach space with

$$C^\infty_{(c)}(U; X), \mathcal{S}(U; X) \overset{d}{\subset} W^n_{p,d}(U, w; X) \hookrightarrow \mathcal{D}'(U; X).$$

(ii) Let X be a Banach space, $U \subset \mathbb{R}^d$ an open subset, $a \in \left(\frac{1}{\mathbb{Z}}\right)^l$, $p \in]1, \infty[^l$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. Then $H^{s,a}_{p,d}(U, w; X)$ is a Banach space with

$$C^\infty_{(c)}(U; X), \mathcal{S}(U; X) \overset{d}{\subset} H^{s,a}_{p,d}(U, w; X) \hookrightarrow \mathcal{D}'(U; X).$$

(iii) Let X be a Banach space, $U \subset \mathbb{R}^d$ an open subset, $a \in]0, \infty[^l$, $p \in [1, \infty[^l$, $q \in [1, \infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Let $\mathcal{A} \in \{B, F\}$. Then $\mathcal{A}^{s,a}_{p,q,d}(U, w; X)$ is a Banach space with

$$\mathcal{S}(U; X) \subset \mathcal{A}^{s,a}_{p,q,d}(U, w; X) \hookrightarrow \mathcal{D}'(U; X). \tag{5.80}$$

Moreover, if $q < \infty$, then $C^\infty_{(c)}(U; X)$ and $\mathcal{S}(U; X)$ are dense subspaces of $\mathcal{A}^{s,a}_{p,q,d}(U, w; X)$.

Proof. Here the corresponding results on $\mathbb{R}^d$ are Propositions 5.2.14 and 5.2.17. □

Proposition 5.3.5.

(i) Let X be a UMD space, $U \subset \mathbb{R}^d$ an open subset, $a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$, $p \in]1, \infty[^l$, $s \in \mathbb{R}$, and

$$w \in \begin{cases} \prod_{j=1}^l A^{rec}_{p_j}(\mathbb{R}^{d_j}), & l > 1; \\ \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j}), & l = 1 \text{ or } X \text{ has } (\alpha). \end{cases}$$

Then, for each multi-index $\alpha \in \mathbb{N}^d$, the partial derivative operator $D^\alpha \in \mathcal{D}'(U; X)$ restricts to a bounded linear operator

$$D^\alpha : H^{s,a}_{p,d}(U, w; X) \longrightarrow H^{s-a\cdot_d \alpha, a}_{p,d}(U, w; X).$$

(ii) *Let X be a Banach space, $U \subset \mathbb{R}^d$ an open subset, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l A_\infty(\mathbb{R}^{d_j})$. Let $\mathscr{A} \in \{B, F\}$. Then, for each multi-index $\alpha \in \mathbb{N}^d$, the partial derivative operator $D^\alpha \in \mathcal{D}'(U;X)$ restricts to a bounded linear operator*

$$D^\alpha : \mathscr{A}^{s,a}_{p,q,\mathpzc{d}}(U,w;X) \longrightarrow \mathscr{A}^{s-a\cdot_{\mathpzc{d}}\alpha,a}_{p,q,\mathpzc{d}}(U,w;X).$$

Proof. Here the corresponding results on $\mathbb{R}^d$ are Propositions 5.2.15 and 5.2.21. □

Remark 5.3.6. We expect that (i) of the above proposition remains true under the assumption $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$, see Remark 5.2.16 concerning the corresponding result on $\mathbb{R}^d$. The same remark will apply to Proposition 5.3.8 and Theorem 5.2.35.(i).

5.3.2 Isomorphisms and Embeddings

The following intersection representation can be proved directly from Definition 5.3.1.

Lemma 5.3.7. *Let X be a Banach space, $U_j \subset \mathbb{R}^{d_j}$, $j = 1,\dots,l$, open subsets , $p \in]1,\infty[^l$, $n \in (\mathbb{Z}_{>0})^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. For each $j \in \{1,\dots,l\}$ with $U_j = \mathbb{R}^{d_j}_+$ we assume that w_j is of the form $w_j(x_j) = |x_{j,1}|^{\gamma_j}$ for some $\gamma_j \in]-1, p_j - 1[$. For each $k \in \{1,\dots,l\}$ we write $U'^k := U_1 \times \dots \times U_{k-1}$, $U''^k := U_{k+1} \times \dots \times U_l$, $\mathpzc{d}'^k := (\mathpzc{d}_1,\dots,\mathpzc{d}_{k-1})$ and $\mathpzc{d}''^k := (\mathpzc{d}_{k+1},\dots,\mathpzc{d}_l)$, and similarly for p and w. Then we have*

$$W^n_{p,\mathpzc{d}}(U,w;X) = \bigcap_{k=1}^l L^{p'^k,\mathpzc{d}'^k}\left(U'^k, w'^k; W^{n_k}_{p_k}(U_k, w_k; L^{p''^k,\mathpzc{d}''^k}(U''^k, w''^k;X))\right)$$

with an equivalence of norms.

Proof. We can simply follow the argumentation used in Section 2.1 to prove the intersection representation for $W^{(2,1)}_{(p,q),(d,1)}(\mathbb{R}^d \times \mathbb{R}, (w_\gamma, v_\mu); X)$. □

The following results are also immediate consequences from the quotient space viewpoint and the corresponding results on the full Euclidean space $\mathbb{R}^d$.

Proposition 5.3.8. *$U = U_1 \times \dots \times U_l$ a product of open subsets $U_j \in \{\mathbb{R}^{d_j}_+, \mathbb{R}^{d_j}\}$, $j = 1,\dots,l$,, $p \in]1,\infty[^l$, $n \in \mathbb{N}^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$. For each $j \in \{1,\dots,l\}$ with $U_j = \mathbb{R}^{d_j}_+$ we assume w_j to be symmetric with respect to reflections in $\{0\} \times \mathbb{R}^{d_j-1}$ and to satisfy, in case $n_j \geq 1$, for some $C_j \in]0,\infty[$ and $\lambda_j \in]0,\infty[\setminus\{1\}$, the estimate $w(\lambda_j x_{j,1},\dots,x_{j,d_j}) \leq Cw(x_j)$ for almost all $x_j = (x_{j,1},\dots,x_{j,d_j}) \in \mathbb{R}^{d_j}$.*

Let X be a UMD space, $U = U_1 \times \dots \times U_l$ a product of open subsets $U_j \in \{\mathbb{R}^{d_j}_+, \mathbb{R}^{d_j}\}$, $j = 1,\dots,l$,, $p \in]1,\infty[^l$, $n \in \mathbb{N}^l$, $s \in \mathbb{R}$, $a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$ with $n_j = \frac{s}{a_j}$ for each $j \in \{1,\dots,l\}$, and

$$w \in \begin{cases} \prod_{j=1}^l A^{rec}_{p_j}(\mathbb{R}^{d_j}), & l > 1; \\ \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j}), & l = 1 \text{ or } X \text{ has property } (\alpha). \end{cases}$$

For each $j \in \{1,\dots,l\}$ with $U_j = \mathbb{R}^{d_j}_+$ we assume w_j to be symmetric with respect to reflections in $\{0\}\times\mathbb{R}^{d_j-1}$ and to satisfy, in case $n_j \geq 1$,for some $C_j \in]0,\infty[$ and $\lambda_j \in]0,\infty[\setminus\{1\}$, the estimate $w(\lambda_j x_{j,1},\dots,x_{j,d_j}) \leq Cw(x_j)$ for almost all $x_j = (x_{j,1},\dots,x_{j,d_j}) \in \mathbb{R}^{d_j}$. Then we have

$$W^n_{p,\mathpzc{d}}(U,w;X) = H^{s,a}_{p,\mathpzc{d}}(U,w;X)$$

with an equivalence of norms.

Proof. Here the corresponding result is Proposition 5.2.45. □

Proposition 5.3.9. *Let X be a Banach space, $U \subset \mathbb{R}^d$ an open subset, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $s \in \mathbb{R}$, and $w \in \prod_{j=1}^l \in A_\infty(\mathbb{R}^{d_j})$.*

(i) For all $1 \leq q_0 \leq q_1 \leq \infty$,

$$\begin{aligned} B^{s,a}_{p,q_0,d}(U,w;X) &\hookrightarrow B^{s,a}_{p,q_1,d}(U,w;X), \\ F^{s,a}_{p,q_0,d}(U,w;X) &\hookrightarrow F^{s,a}_{p,q_1,d}(U,w;X). \end{aligned}$$

(ii) For all $q_0, q_1 \in [1,\infty]$ and $\epsilon > 0$,

$$\begin{aligned} B^{s+\epsilon,a}_{p,q_0,d}(U,w;X) &\hookrightarrow B^{s,a}_{p,q_1,d}(U,w;X), \\ F^{s+\epsilon,a}_{p,q_0,d}(U,w;X) &\hookrightarrow F^{s,a}_{p,q_1,d}(U,w;X). \end{aligned}$$

(iii) For all $p \in [1,\infty[^l$ and $q \in [1,\infty]$,

$$B^{s,a}_{p,\min\{p_1,\ldots,p_l,q\},d}(U,w;X) \hookrightarrow F^{s,a}_{p,q,d}(U,w;X) \hookrightarrow B^{s,a}_{p,\max\{p_1,\ldots,p_l,q\},d}(U,w;X).$$

(iv) For all $q \in [1,\infty]$ and Banach spaces $X \hookrightarrow Y$,

$$F^{s,a}_{p,q,d}(U,w;X) \hookrightarrow F^{s,a}_{p,q,d}(U,w;Y) \quad \text{and} \quad B^{s,a}_{p,q,d}(U,w;X) \hookrightarrow B^{s,a}_{p,q,d}(U,w;Y).$$

Proof. Here the corresponding result is Proposition 5.2.30. □

Proposition 5.3.10. *Let X be a Banach space, $U \subset \mathbb{R}^d$ an open subset, $a \in \left(\frac{1}{\mathbb{Z}_{\geq 1}}\right)^l$, $p \in]1,\infty[^l$, and $w \in \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j})$.*

(i) For all $s \in \mathbb{R}$,

$$B^{s,a}_{p,1,d}(U,w;X) \hookrightarrow H^{s,a}_{p,d}(U,w;X) \hookrightarrow B^{s,a}_{p,\infty,d}(U,w;X), \tag{5.81}$$

$$F^{s,a}_{p,1,d}(U,w;X) \hookrightarrow H^{s,a}_{p,d}(U,w;X) \hookrightarrow F^{s,a}_{p,\infty,d}(U,w;X). \tag{5.82}$$

(ii) Suppose $U = U_1 \times \ldots \times U_l$ is a product of open subsets $U_j \subset \mathbb{R}^{d_j}$, $j = 1,\ldots,l$. Let $n \in \mathbb{N}^l$. For each $j \in \{1,\ldots,l\}$ with $U_j = \mathbb{R}^{d_j}_+$ we assume w_j to be symmetric with respect to reflections in $\{0\} \times \mathbb{R}^{d_j-1}$ and to satisfy, in case $n_j \geq 1$, for some $C_j \in]0,\infty[$ and $\lambda_j \in]0,\infty[\setminus\{1\}$, the estimate $w(\lambda_j x_{j,1},\ldots,x_{j,d_j}) \leq Cw(x_j)$ for almost all $x_j = (x_{j,1},\ldots,x_{j,d_j}) \in \mathbb{R}^{d_j}$. Then, for all $s \in \mathbb{R}$ such that $n_j = \frac{s}{a_j}$ ($j = 1,\ldots,l$),

$$B^{s,a}_{p,1,d}(U,w;X) \hookrightarrow W^{n}_{p,d}(U,w;X) \hookrightarrow B^{s,a}_{p,\infty,d}(U,w;X), \tag{5.83}$$

$$F^{s,a}_{p,1,d}(U,w;X) \hookrightarrow W^{n}_{p,d}(U,w;X) \hookrightarrow F^{s,a}_{p,\infty,d}(U,w;X). \tag{5.84}$$

Proof. Here the corresponding result is Proposition 5.2.31. □

Proposition 5.3.11. *Let X be a Banach space, $U \subset \mathbb{R}^d$ an open subset, $a \in]0,\infty[^l$, $p \in [1,\infty[^l$, $q \in [1,\infty]$, $s \in \mathbb{R}$ and $w \in \prod_{j=1}^{l} A_\infty(\mathbb{R}^{d_j})$. Let $\mathscr{A} \in \{B, F\}$. Then we have, for each $\lambda > 0$,*

$$\mathscr{A}^{s,a}_{p,q,d}(\mathbb{R}^d, w; X) = \mathscr{A}^{\lambda s,\lambda a}_{p,q,d}(\mathbb{R}^d, w; X)$$

up to an equivalence of norms.

Proof. Here the corresponding result on $\mathbb{R}^d$ is Proposition 5.2.34. □

In order to extend the intersection representations from Proposition 5.2.46 and Theorem 5.2.35, we can not proceed as above via just the quotient space description. Instead, we also have to use extension operators.

Lemma 5.3.12.

(i) Let E be a UMD space, $k \in \mathbb{N}$, $n \in \{0,\ldots,k\}$, $p \in]1,\infty[$, $\gamma \in]-1, p-1[$, $s \in]0, \frac{1}{p}(1+\gamma)[$, and let $w_\gamma \in A_p(\mathbb{R}^d)$ be defined by $w_\gamma(x) := |x_1|^\gamma$ $(x = (x_1,\ldots,x_d) \in \mathbb{R}^d)$. Then the extension operator $\mathscr{E}_{E,k} : L^1_{loc}(\overline{\mathbb{R}^d_+}; E) \longrightarrow L^1_{loc}(\mathbb{R}^d; E)$ from Lemma 2.1.2 restricts to a bounded linear operator from $H^{s+n}_p(\mathbb{R}^d_+, w_\gamma; E)$ to $H^{s+n}_p(\mathbb{R}^d, w_\gamma; E)$.

(ii) Let E be a Banach space, $k \in \mathbb{N}$, $n \in \{0,\ldots,k\}$, $p \in]1,\infty[$, $q \in [1,\infty]$, $\gamma \in]-1, p-1[$, $s \in]0, \frac{1}{p}(1+\gamma)[$, and let $w_\gamma \in A_p(\mathbb{R}^d)$ be defined by $w_\gamma(x) := |x_1|^\gamma$ $(x = (x_1,\ldots,x_d) \in \mathbb{R}^d)$. Let $\mathscr{A} \in \{B, F\}$. Then the extension operator $\mathscr{E}_{E,k} : L^1_{loc}(\overline{\mathbb{R}^d_+}; E) \longrightarrow L^1_{loc}(\mathbb{R}^d; E)$ from Lemma 2.1.2 restricts to a bounded linear operator from $\mathscr{A}^{s+n}_{p,q}(\mathbb{R}^d_+, w_\gamma; E)$ to $\mathscr{A}^{s+n}_{p,q}(\mathbb{R}^d, w_\gamma; E)$.

Proof. (ii) By Proposition 5.2.29 and the definition of $\mathscr{A}^{s+n}_{p,q}(\mathbb{R}^d_+, w_\gamma; E)$, it suffices to show that

$$\left\| D^\alpha \mathscr{E}_{E,k} r_{\mathbb{R}^d,\mathbb{R}^d_+} g \right\|_{\mathscr{A}^s_{p,q}(\mathbb{R}^d, w_\gamma; E)} \lesssim \| D^\alpha g \|_{\mathscr{A}^s_{p,q}(\mathbb{R}^d, w_\gamma; E)}, \qquad |\alpha| \leq n, g \in \mathscr{A}^{s+n}_{p,q}(\mathbb{R}^d, w_\gamma; E). \tag{5.85}$$

By construction of $\mathscr{E}_{E,k}$ (see [1, Theorem 5.19]), $\mathscr{E}_{E,k}$ maps $W^k_{1,loc}(\overline{\mathbb{R}^d_+}; E)$ into $W^k_{1,loc}(\mathbb{R}^d; E)$ in such a way that $D^\alpha \mathscr{E}_{E,k} f = \mathscr{E}^\alpha_{E,k} D^\alpha f$ for a certain extension operator $\mathscr{E}^\alpha_{E,k} : L^1_{loc}(\overline{\mathbb{R}^d_+}; E) \longrightarrow L^1_{loc}(\mathbb{R}^d; E)$. In order to give more information about $\mathscr{E}^\alpha_{E,k}$, for $\lambda \neq 0$ we define the operator $T_\lambda : L^1_{loc}(\mathbb{R}^d; E) \longrightarrow L^1_{loc}(\mathbb{R}^d; E)$ by $[T_\lambda f](x) := f(x_1,\ldots,x_{d-1},\lambda x_d)$. In this notation we have

$$\mathscr{E}^\alpha_{E,k} r_{\mathbb{R}^d,\mathbb{R}^d_+} f = 1_{\mathbb{R}^d_+} f + 1_{\mathbb{R}^d_-} \sum_{i=1}^N c_i T_{\lambda_i} f, \qquad f \in L^1_{loc}(\mathbb{R}^d; E)$$

for certain $c_1,\ldots,c_N \in \mathbb{R}$ and $\lambda_1,\ldots,\lambda_N \in]-\infty, 0[$. It follows

$$D^\alpha \mathscr{E}_{E,k} r_{\mathbb{R}^d,\mathbb{R}^d_+} g = \mathscr{E}^\alpha_{E,k} D^\alpha r_{\mathbb{R}^d,\mathbb{R}^d_+} g = \mathscr{E}^\alpha_{E,k} r_{\mathbb{R}^d,\mathbb{R}^d_+} D^\alpha g = 1_{\mathbb{R}^d_+} D^\alpha g + 1_{\mathbb{R}^d_-} \sum_{i=1}^N c_i T_{\lambda_i} D^\alpha g$$

for all $|\alpha| \leq n$ and $g \in \mathscr{A}^{s+n}_{p,q}(\mathbb{R}^d, w_\gamma; E)$. Since the T_{λ_i} are easily seen to restrict to bounded linear operators on $\mathscr{A}^s_{p,q}(\mathbb{R}^d, w_\gamma; E)$, the desired estimate (5.85) now is a consequence of the pointwise multiplier result Theorem 5.85.

(i) This can be done in exactly the same way as (i), now using the pointwise multiplier theorem for UMD-valued Bessel potential spaces [78, Theorem 1.1] instead of Theorem 5.85. □

Theorem 5.3.13.

(i) Let X be a UMD space, $U = U_1 \times \ldots \times U_l$ *with* $U_j \in \{\mathbb{R}^{d_j}_+, \mathbb{R}^{d_j}\}$, $j = 1, \ldots, l$, $a \in \left(\frac{1}{\mathbb{Z}_{>0}}\right)^l$, $p \in]1,\infty[^l$, $s > 0$, *and*

$$w \in \begin{cases} \prod_{j=1}^l A^{rec}_{p_j}(\mathbb{R}^{d_j}), & l > 1; \\ \prod_{j=1}^l A_{p_j}(\mathbb{R}^{d_j}), & l = 1 \text{ or } X \text{ has property } (\alpha). \end{cases}$$

For each $j \in \{1, \ldots, l\}$ *with* $U_j = \mathbb{R}^{d_j}_+$ *we assume that* w_j *is of the form* $w_j(x_j) = |x_{j,1}|^{\gamma_j}$ *for some* $\gamma_j \in]-1, p_j - 1[$. *For each* $k \in \{1, \ldots, l\}$ *we write* $U'^k := U_1 \times \ldots \times U_{k-1}$, $U''^k := U_{k+1} \times \ldots \times U_l, \mathpzc{d}'^k := (\mathpzc{d}_1, \ldots, \mathpzc{d}_{k-1})$ *and* $\mathpzc{d}''^k := (\mathpzc{d}_{k+1}, \ldots, \mathpzc{d}_l)$, *and similarly for p and w. Then we have*

$$H^{s,a}_{p,\mathpzc{d}}(U, w; X) = \bigcap_{k=1}^{l} L^{p'^k, \mathpzc{d}'^k}\left(U'^k, w'^k; H^{s/a_k}_{p_k}(U_k, w_k; L^{p''^k, \mathpzc{d}''^k}(U''^k, w''^k; X))\right)$$

with an equivalence of norms.

(ii) Let X be a Banach space, $l = 2$, $U = U_1 \times U_2$ *with* $U_j \in \{\mathbb{R}^{d_j}_+, \mathbb{R}^{d_j}\}$, $j = 1, 2$, $a \in]0,\infty[^2$, $p, q \in]1,\infty[$, $\tilde{p} := (p, q)$, $s > 0$, *and* $w \in A_p(\mathbb{R}^{d_1}) \times A_q(\mathbb{R}^{d_2})$. *For each* $j \in \{1, 2\}$ *with* $U_j = \mathbb{R}^{d_j}_+$ *we assume that* w_j *is of the form* $w_j(x_j) = |x_{j,1}|^{\gamma_j}$ *for some* $\gamma_j \in]-1, \tilde{p}_j - 1[$. *Then we have*

$$F^{s,a}_{\tilde{p},p,\mathpzc{d}}(U, w; X) = F^{s/a_2}_{q,p}(U_2, w_2; L^p(U_1, w_1; X)) \cap L^q(U_2, w_2; F^{s/a_1}_{p,p}(U_1, w_1; X))$$

with an equivalence of norms.

Proof. We only treat (ii), (i) being completely similar (where the corresponding result on $\mathbb{R}^d$ is Proposition 5.2.46). Using an argumentation as in the proof of Lemma 2.1.4, now using Lemma 5.3.12 instead of Lemma 2.1.2, we can show that the restriction operator

$$r_{\mathbb{R}^d,U} \in \mathcal{L}(\mathcal{D}'(\mathbb{R}^d; X), \mathcal{D}'(\mathbb{R}^d; X)),\ f \mapsto f_{|U},$$

restricts to a continuous surjection from

$$F^{s,a}_{\tilde{p},p,\mathpzc{d}}(\mathbb{R}^d, w; X) \overset{Thm\ 5.2.35}{=} F^{s/a_2}_{q,p}(\mathbb{R}^{d_2}, w_2; L^p(\mathbb{R}^{d_1}, w_1; X)) \cap L^q(\mathbb{R}^{d_2}, w_2; F^{s/a_1}_{p,p}(\mathbb{R}^{d_1}, w_1; X))$$

onto

$$F^{s/a_2}_{q,p}(U_2, w_2; L^p(U_1, w_1; X)) \cap L^q(U_2, w_2; F^{s/a_1}_{p,p}(U_1, w_1; X))$$

with a continuous right-inverse. The desired result is now immediate from the definition of $F^{s,a}_{\tilde{p},p,\mathpzc{d}}(U, w; X)$. □

From this intersection representations it immediately follows that the anisotropic space under consideration is continuously included in each space occuring in the intersecion. For convenience of later reference, we state this for the weighted anisotropic mixed-norm Bessel potential space as the following corollary.

Corollary 5.3.14. *Let the notations and assumptions be as in Theorem 5.3.13.(i). Then we have, for each* $k \in \{1, \ldots, l\}$,

$$H^{s,a}_{p,\mathpzc{d}}(U, w; X) \hookrightarrow L^{p'^k, \mathpzc{d}'^k}\left(U'^k, w'^k; H^{s/a_k}_{p_k}(U_k, w_k; L^{p''^k, \mathpzc{d}''^k}(U''^k, w''^k; X))\right)$$

5.3.3 Traces

In this subsection we study traces of weighted anisotropic spaces on domains. Here we only present trace results which are directly needed in the next chapter; this will be more convenient for later references. More general trace results can be derived in the same way from the trace theory developed in Section 5.2.3.

For the remainder of this section, we fix an arbitrary Banach space X, real numbers $q, p \in]-1,\infty[$, and the Muckenhaupt power weights $v_\mu \in A_q(\mathbb{R})$ and $w_\gamma \in A_p(\mathbb{R}^d)$ given by

$$v_\mu(t) := |t|^\mu \quad (t \in \mathbb{R}) \qquad \text{and} \qquad w_\gamma(y,x') := |y|^\gamma \quad ((y,x') \in \mathbb{R}\times\mathbb{R}^{d-1} = \mathbb{R}^d).$$

5.3.3.a Spatial Traces on $\mathbb{R}^d_+ \times \mathbb{R}_+$ and $\mathbb{R}^d_+ \times \mathbb{R}$

Let $U \in \{\mathbb{R}_+, \mathbb{R}\}$. We define the distributional spatial trace operator $\mathrm{tr}_{y=0}$ on $\mathbb{R}^d_+ \times U$ as in Section 2.1.3:

$$\mathrm{tr}_{y=0} : C([0,\infty[;\mathcal{D}'(\mathbb{R}^{d-1}\times U;X)) \longrightarrow \mathcal{D}'(\mathbb{R}^{d-1}\times U;X),$$

where $C([0,\infty[;\mathcal{D}'(\mathbb{R}^{d-1}\times U;X))$ is viewed as subspace of $\mathcal{D}'(\mathbb{R}^d_+\times U;X)$ in the usual way. Then we we have

$$W^{(k_1,k_2)}_{(p,q),(d,1)}(\mathbb{R}^d_+\times U,(w_\gamma,v_\mu);X) \hookrightarrow \mathcal{D}'(\mathbb{R}^{d-1}\times U;C([0,\infty[;X)), \qquad (k_1,k_2)\in(\mathbb{Z}_{>0})^2,$$

and the distributional trace operator $\mathrm{tr}_{y=0}$ coincides on the weighted anisotropic mixed-norm Sobolev space

$$W^{(k_1,k_2)}_{(p,q),(d,1)}(\mathbb{R}^d_+\times U,(w_\gamma,v_\mu);X) \hookrightarrow L^q(U,v_\mu;W^{k_1}_p(\mathbb{R}^d_+,w_\gamma;X))$$

with the trace operator pointwise induced by $\mathrm{tr}_{y=0} \in \mathcal{B}(W^{k_1}_p(\mathbb{R}^d_+,w_\gamma;X),L^p(\mathbb{R}^{d-1};X))$. In particular, $\mathrm{tr}_{y=0}$ restricts to a continuous linear operator

$$\mathrm{tr}_{y=0} : W^{(k_1,k_2)}_{(p,q),(d,1)}(\mathbb{R}^d_+\times U,(w_\gamma,v_\mu);X) \longrightarrow L^{(p,q),(d-1,1)}(\mathbb{R}^{d-1}\times U,(1,v_\mu);X).$$

With Corollary 5.2.58/Remark 5.2.59 it is not difficult to see that that $\mathrm{tr}_{y=0}$ restricts in fact to a continuous surjection

$$\mathrm{tr}_{y=0} : W^{(k_1,k_2)}_{(p,q),(d,1)}(\mathbb{R}^d_+\times U,(w_\gamma,v_\mu);X) \longrightarrow F^{1-\frac{1}{k_1 p}(1+\gamma),(\frac{1}{k_1},\frac{1}{k_2})}_{(p,q),q,(d-1,1)}(\mathbb{R}^{d-1}\times U,(1,v_\mu);X)$$

with a continuous right-inverse. However, in Chapter 6 we will not use this trace result, but we will instead use the following two results.

Theorem 5.3.15. *Let $(a_1,a_2) \in]0,\infty[^2$ and $s > \frac{a_1}{p_1}(1+\gamma_+)$ be given. Then we have*

$$F^{s,(a_1,a_2)}_{(p,q),\infty,(d,1)}(\mathbb{R}^d_+\times U,(w_\gamma,v_\mu);X) \hookrightarrow C([0,\infty[;\mathcal{D}'(\mathbb{R}^{d-1}\times U;X))$$

and the distributional trace operator $\mathrm{tr}_{y=0}$ *restricts to a continuous surjection*

$$\mathrm{tr}_{y=0} : F^{s,(a_1,a_2)}_{(p,q),\infty,(d,1)}(\mathbb{R}^d_+\times U,(w_\gamma,v_\mu);X) \longrightarrow F^{s-\frac{a_1}{p_1}(1+\gamma),(a_1,a_2)}_{(p,q),q,(d-1,1)}(\mathbb{R}^{d-1}\times U,(1,v_\mu);X)$$

with a continuous right-inverse.

Proof. The corresponding result on $\mathbb{R}^d \times \mathbb{R}$ follows from a combination of Theorem 5.2.55 and Remark 5.2.13, from which the desired result can be easily derived. □

Theorem 5.3.16. *Let $(a_1, a_2) \in (\frac{1}{\mathbb{Z}_{>0}})^2$ and $s > \frac{a_1}{p_1}(1 + \gamma_+)$ be given. Then we have*

$$H^{s,(a_1,a_2)}_{(p,q),(d,1)}(\mathbb{R}^d_+ \times U, (w_\gamma, v_\mu); X) \hookrightarrow C([0, \infty[; \mathcal{D}'(\mathbb{R}^{d-1} \times U; X))$$

and the distributional trace operator $\mathrm{tr}_{y=0}$ *restricts to a continuous surjection*

$$\mathrm{tr}_{y=0} : H^{s,(a_1,a_2)}_{(p,q),(d,1)}(\mathbb{R}^d_+ \times U, (w_\gamma, v_\mu); X) \longrightarrow F^{s-\frac{a_1}{p_1}(1+\gamma),(a_1,a_2)}_{(p,q),q,(d-1,1)}(\mathbb{R}^{d-1} \times U, (1, v_\mu); X)$$

with a continuous right-inverse. Moreover, in case that $U = \mathbb{R}$, this right-inverse maps $\mathscr{F}^{-1}(C^\infty_c(\mathbb{R}^{d-1}; X)) \otimes \mathscr{F}^{-1}(C^\infty_c(\mathbb{R}))$ into $\mathcal{S}(\mathbb{R}^d_+; X) \otimes \mathscr{F}^{-1}(C^\infty_c(\mathbb{R}))$.

Proof. This can be derived from the corresponding result on $\mathbb{R}^d \times \mathbb{R}$, Corollary 5.2.57/Remark 5.2.59. □

5.3.3.b Temporal Traces on $\mathbb{R}^d_+ \times \mathbb{R}_+$

Theorem 5.3.17. *Let $(k_1, k_2) \in (\mathbb{Z}_{>0})^2$. Then we have that the temporal trace operator*

$$\mathrm{tr}_{t=0} : W^{(k_1,k_2)}_{(p,q),(d,1)}(\mathbb{R}^d_+, (w_\gamma, v_\mu); X) \hookrightarrow W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \longrightarrow L^p(\mathbb{R}^d_+, w_\gamma; X),\ f \mapsto f(0),$$

is a continuous surjection

$$\mathrm{tr}_{t=0} : W^{(k_1,k_2)}_{(p,q),(d,1)}(\mathbb{R}^d_+, (w_\gamma, v_\mu); X) \longrightarrow B^{k_2[1-\frac{1}{k_1 q}(1+\mu)]}_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$$

with a continuous right-inverse. Moreover, $\mathrm{tr}_{t=0}$ *coincides on $\mathcal{S}(\mathbb{R}^d_+ \times \mathbb{R}_+; X)$ (in fact even on $C(\mathbb{R}^d_+ \times [0, \infty[; X) \cap W^{(k_1,k_2)}_{(p,q),(d,1)}(\mathbb{R}^d_+, (w_\gamma, v_\mu); X)$) with the classical trace of continuous functions with respect to $\mathbb{R}^d_+ \times \{0\}$.*

Proof. This follows from the corresponding result on $\mathbb{R}^d \times \mathbb{R}$, Corollary 5.2.63. □

5.3.3.c Spatial Traces on $\mathbb{R}^d_+$

Theorem 5.3.18. *Let $k \in \mathbb{Z}_{>0}$. Then the trace operator*

$$\mathrm{tr}_{y=0} : W^k_p(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow W^1_p(\mathbb{R}_+, |\cdot|^\gamma; L^p(\mathbb{R}^{d-1}; X)) \longrightarrow L^p(\mathbb{R}^{d-1}; X), f \mapsto f(0)$$

restricts to a continuous surjection

$$\mathrm{tr}_{y=0} : W^k_p(\mathbb{R}^d_+, w_\gamma; X) \longrightarrow F^{k-\frac{1}{p}(1+\gamma)}_{p,p}(\mathbb{R}^{d-1}; X) = B^{k-\frac{1}{p}(1+\gamma)}_{p,p}(\mathbb{R}^{d-1}; X)$$

with a continuous right-inverse. Moreover, $\mathrm{tr}_{y=0}$ *coincides on $C(\overline{\mathbb{R}^d_+}; X) \cap W^k_p(\mathbb{R}^d_+, w_\gamma; X)$ with the classical trace with respect to $\{0\} \times \mathbb{R}^{d-1}$.*

Proof. This can be derived from the corresponding result on $\mathbb{R}^d$, Corollary 5.2.58/Remark 5.2.59; here we also have to use Proposition 5.2.34 to get the trace space in the right form. □

Theorem 5.3.19. *Let $s > \frac{1}{p}(1+\gamma)$. Then the linear operator*

$$\mathcal{S}(\mathbb{R}^d_+;X) \longrightarrow \mathcal{S}(\mathbb{R}^{d-1};X),\ f \mapsto f_{|\{0\}\times\mathbb{R}^{d-1}},$$

extends to (a necessarily unique) continuous linear surjection

$$\mathrm{tr}_{y=0} : B^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) \longrightarrow B^{s-\frac{1}{p}(1+\gamma)}_{p,q}(\mathbb{R}^{d-1};X)$$

with a continuous right-inverse.

Proof. This can be derived from the corresponding result on $\mathbb{R}^d$, Theorem 5.2.52/Remark 5.2.13; here we also have to use Proposition 5.2.34 to get the trace space in the right form. □

5.3.3.d Temporal Traces on $\mathbb{R}^{d-1}\times\mathbb{R}_+$

Theorem 5.3.20. *Let $(a_1,a_2) \in]0,\infty[^2$ and $s > \frac{a_2}{q}(1+\mu)$. Then the linear operator*

$$\mathcal{S}(\mathbb{R}^d_+;X) \longrightarrow \mathcal{S}(\mathbb{R}^{d-1};X),\ f \mapsto f_{|\{0\}\times\mathbb{R}^{d-1}},$$

extends to (a necessarily unique) continuous linear surjection

$$F^{s,(a_1,a_2)}_{(p,q),p,(d-1,1)}(\mathbb{R}^{d-1}\times\mathbb{R}_+,(1,v_\mu);X) \longrightarrow B^{\frac{1}{a_1}\left[s-\frac{a_2}{q}(1+\mu)\right]}_{p,q}(\mathbb{R}^{d-1};X)$$

with a continuous right-inverse.

Proof. This can be derived from the corresponding result on $\mathbb{R}^{d-1}\times\mathbb{R}$, Theorem 5.2.61/Remark 5.2.13; here we also have to use Proposition 5.2.34 to get the trace space in the right form. □

As in Section 5.2.3.d, we define ${}_{0,(0,d)}F^{s,(a_1,a_2)}_{(p,q),p,(d-1,1)}(\mathbb{R}^{d-1}\times\mathbb{R}_+,(1,v_\mu);X)$ as the closure of $\{f \in \mathcal{S}(\mathbb{R}^{d-1}\times\mathbb{R}_+;X)\}$ in $F^{s,(a_1,a_2)}_{(p,q),p,(d-1,1)}(\mathbb{R}^{d-1}\times\mathbb{R}_+,(1,v_\mu);X)$. For this space we have the following useful characterization (in case $s \neq \frac{a_2}{q}(1+\mu)$):

Proposition 5.3.21. *Suppose $s \neq \frac{a_2}{q}(1+\mu)$. Then we have*

$${}_{0,(0,d)}F^{s,(a_1,a_2)}_{(p,q),p,(d-1,1)}(\mathbb{R}^{d-1}\times\mathbb{R}_+,(1,v_\mu);X) = \begin{cases} \ker(\mathrm{tr}_{t=0}) & , s > \frac{a_2}{q}(1+\mu); \\ F^{s,(a_1,a_2)}_{(p,q),p,(d-1,1)}(\mathbb{R}^{d-1}\times\mathbb{R}_+,(1,v_\mu);X), & s < \frac{a_2}{q}(1+\mu). \end{cases}$$

where $\mathrm{tr}_{t=0}$ is the trace operator from Theorem 5.3.20. Furthermore,

$${}_{0,(0,d)}F^{s,(a_1,a_2)}_{(p,q),p,(d-1,1)}(\mathbb{R}^{d-1}\times\mathbb{R}_+,(1,v_\mu);X) \cong {}_{0,(0,d)}F^{s,(a_1,a_2)}_{(p,q),p,(d-1,1)}(\mathbb{R}^{d-1}\times\mathbb{R},(1,v_\mu);X)/[\ker(r_{\mathbb{R}^{d-1}\times\mathbb{R},\mathbb{R}^{d-1}\times\mathbb{R}_+})]$$

Proof. This can be easily derived from Lemma 5.2.65.(ii)/Remark 5.2.66 and Proposition 5.2.70. □

5.4 Notes

5.4.1 General Notes

In the literature there are two sorts of anisotropic spaces, the anisotropic spaces of intersection type (which may be in an abstract form) and the anisotropic spaces of distribution type. Motivated by the maximal L^p-regularity problem for parabolic initial-boundary value problems (see [76]), in [75] a systematic treatment is given of anisotropic fractional Sobolev spaces of intersection type on space-time with weights in the time variable in the UMD setting, where the main tools are operators with a bounded $\mathcal{H}^\infty$-calculus, interpolation theory, and operator sums. However, concerning the spatial trace result [75, Theorem 4.5], the followed approach only works in the non-mixed-norm case $q = p$ (due to an interchange in the order of the variables). In the mixed-norm case, spatial traces of anisotropic Triebel-Lizorkin spaces (of distribution type) were studied in [62] in the scalar-valued case, which include scalar-valued anisotropic anisotropic Sobolev and Bessel potential spaces. A characterization of such anisotropic mixed-norm Triebel-Lizorkin cases was provided in [27]; also see the discussion [62, Section 5], where this problem was left for the future. However, in contrast to the scalar case, in the general Banach space case, (anisotropic mixed-norm) Triebel-Lizorkin spaces do not contain (anisotropic mixed-norm) Sobolev and Bessel potential spaces, with an exception for the Hilbert space case. Nevertheless, in [86]/[88] the traces of Sobolev and Bessel potential space were characterized without any assumptions on the Banach space (in the unweighted non-mixed-norm isotropic case) by making use of a sandwich argument with the end-point Triebel-Lizorkin spaces. Distributional weighted anisotropic mixed-norm Banach space-valued function spaces (of Sobolev, Bessel potential, Triebel-Lizorkin, and Besov type) have not been considered before in the literature. For more information on anisotropic spaces in the scalar-valued case we refer to [62, 27] (and the references given therein) and for a nice historical background of isotropic spaces we refer to [97].

5.4.2 Comparison to the Literature

- *Section 5.2:*
  - *Section 5.2.1:* This material is mainly a basic extension of the existing literature to our $(\mathpzc{d}, a)$-anisotropic weighted setting; here the consulted literature is [97, 62, 77]. Some of the results from Section 3.4 play a very important role in this extension.
  - *Section 5.2.2:*

    The Plancherel-Pólya-Nikol'skii inequality of Proposition 5.2.32 is an extension of [77, Proposition 4.1] (which corresponds to Step I in the proof) and is a partial extension of [61, Proposition 4] (which is concerned with the scalar-valued unweighted case for $p, \tilde{p} \in]0, \infty[^l$). The strategy of our proof is taken from [61, Proposition 4], where the argumentation in Step I/II is completely different and does not work in the weighted setting. Our argumentation in Step I/II, based on the theory of vector-valued distributions (with values in a complete LCS), is new and could also be used in the setting of [61].

    Theorem 5.2.35 is in the unweighed scalar-valued case due to Denk & Kaip [27, Proposition 3.23] for the special case $\mathpzc{d} = (n-1, 1)$, partly based on [8, 9]. There this representation is used for interpolation of such intersection spaces. We present a completely different 'more direct' self-contained proof, which consists of proving the more general intersection representation of Proposition 5.2.38, which is in turn contained in Proposition 5.2.44. Both these two propositions as well as the notion of partial Triebel-Lizorkin space (cf. Definition 5.2.36) are new, where Proposition 5.2.44 is also of independent interest. Here the proofs of Proposition 5.2.39 and Lemma 5.2.43 are inspired by the proof of [88, Proposition 6] (also see [86]), which gives a difference norm characterization for unweighted isotropic Triebel-Lizorkin spaces. The proof of the inequality $|||\cdot|||^{[m\mu],discr}_{F^{s,a}_{p,q,d}(\mathbb{R}^d,w;X)} \lesssim \sum_{l=1}^{m} \|\cdot\|_{F^{s,l,a_l}_{p,q,d}(\mathbb{R}^d,w;X)}$ in the proof of Proposition 5.2.39 is inspired by the proof of [100, Theorem 4.4], which is concerned with the Fubini property of unweighted isotropic Triebel-Lizorkin spaces.

The intersection representation of Proposition 5.2.46 is based on [6, Theorem 3.7.2]. Here we have to remark that Amann [6] considered the unweighted non-mixed norm anisotropic vector-valued case, where the non-mixed norm case allows for an interchange in the order of the variables (by Fubini). Furthermore, in [6] the UMD Banach space is assumed to have property (α) (except for the case $a \in \mathbb{R}_+\mathbf{1}$), which can be explained by the use of the more restrictive Marcinkiewicz Fourier multiplier theorem instead of an anisotropic Mikhlin Fourier multiplier theorem; also see [55, 57] for an explanation of the relationship between the Marcinkiewicz and anisotropic Mikhlin Fourier multiplier theorems. However, the given proof extends to our situation (using Corollary 4.1.2).

Lemmas 5.2.47 and 5.2.48 are new and are specially designed for Lemma 6.4.3 (and Proposition 6.3.3). Here the operator from the second lemma is basically also used in [73, Lemma 2.2.7] (on which Lemma 6.4.3 is based), but via an operator theoretic approach in the case $p = q$ instead of via a Fourier analytic approach for general $q, p \in]1, \infty[$.

Except for Proposition 5.2.31 and Proposition 5.2.34, which are based on [88, Theorem 1]/[77, Proposition 3.12] (also see [86]) and [62, Lemma 3.24], respectively, the rest of this section is based on [97] (which is concerned with the unweighted scalar-valued isotropic case) and some comments in [77] (which is concerned with the weighted vector-valued isotropic case).

- *Section 5.2.3:* The concept of distributional trace operator is taken from [60, 38, 62], where the space $C(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{d-1}; X))$ is viewed as subspace of $\mathcal{D}'(\mathbb{R}^d; X)$ via Proposition C.7.3 in the scalar-valued case $X = \mathbb{C}$. The trace operator (5.64) is also used in [60, 38, 62], where it is called the working definition of the trace. Consistency with the distributional trace operator was first proved for Besov spaces by Johnsen [60]; see [38, Remark 2.1]. Consistency of the classical trace on $C(\mathbb{R}^d; X)$ with the distributional trace was not considered in [60, 38, 62]. For several further concepts to define a trace we refer to [88, Remark 18] (and the references given therein).

  Proposition 5.2.49 is based on [62, Theorem 2.6].

  Theorem 5.2.52 is inspired by [88, Theorem 2], where a different concept of trace was considered in the unweighted isotropic case. Lemma 5.2.53 is a technical extension of [88, Lemma 5&Remark 16] to the weighted mixed-norm case, where it was enough to consider the case $b = 0$ thanks to translation arguments (which do not work in our weighted setting). The proof of Proposition 5.2.54 is to a large extend based on the proof of [88, Proposition 9]. The computations in the proof of Theorem 5.2.52 bassically follow the computations in the proof of [88, Theorem 2]; one of the main differences is that we have isolated some steps such as as the more general Lemmas 5.2.22 and 5.2.10, which improves the transparency (similarly to [62] concerning the trace problem for unweighted anisotropic Triebel-Lizorkin spaces).

An unweighted non-mixed-norm scalar-valued scalar-valued version of Theorem 5.2.52 is contained in [38, Theorem 3, Proposition 1 & Corollary 1]. Here the weighted setting leaded to new considerations for the inclusion (5.68) due to the unavailability of translation arguments, for which [88, Proposition 9] (and the concept of trace used there in) was very helpful.

The computations and estimates in the proof of Theorem 5.2.55 are basic extensions of the computations and estimates in the proof of [62, Theorem 2.2], which is about the unweighted case (for $p \in]0,\infty[^l$ and $q \in]0,\infty]$). Due the unavailability of translation arguments, we proceeded via the anisotropic Besov space in order to obtain the inclusion (5.72).

The sandwich argument used in Corollaries 5.2.57 and 5.2.58 is taken from [86]/[88], where it was used in the unweighted isotropic case.

The argumentation in the proof of Proposition 5.2.70 is (modulo some minor modifications) taken from [79].

- *Section 5.3:* The description/definition of function spaces on domains via the restriction procedure is standard and can be found in [97]. Most of the results in this section easily follow from the corresponding results on the full Euclidean space, where in quite some cases the description via restriction alone was enough but in some cases an extension operator was needed. Here the proof of Lemma 5.3.12 is essentially a reinterpretation of the proof of [1, Theorem 5.19] inspired by [97, Section 2.9.2].

Chapter 6

Parabolic Initial-Boundary Value Problems with Inhomogeneous Data

In this chapter we apply the theory of weighted anisotropic mixed-norm function spaces from Chapter 5 and the theory of Fourier multipliers from Chapter 4 to the study of maximal weighted L^q-L^p-regularity for parabolic initial-boundary value problems with inhomogeneous static boundary conditions in the half space $\mathbb{R}^d_+ = \mathbb{R}_+ \times \mathbb{R}^{d-1}$. The weights we consider are power weights in time and in the first space direction, and yield flexibility in the regularity of the initial-boundary data.

6.1 Introduction

6.1.1 The Problem and the Maximal Weighted L^q-L^p-Regularity Approach

The aim of this chapter is to study maximal weighted L^q-L^p-regularity of vector-valued linear inhomogeneous parabolic initial-boundary value model problems in the half space $\mathbb{R}^d_+ = \mathbb{R}_+ \times \mathbb{R}^{d-1}$ of the form

$$\begin{aligned} \partial_t u(y,x',t) + (1+\mathcal{A}(D))u(y,x',t) &= f(y,x',t), && (y,x') \in \mathbb{R}^d_+, && t \geq 0 \\ \mathcal{B}_j(D)u(y,x',t)|_{y=0} &= g_j(x',t), && x' \in \mathbb{R}^{d-1}, && t \geq 0, \quad j = 1,\dots,n, \\ u(y,x',0) &= u_0(y,x'), && (y,x') \in \mathbb{R}^d_+. \end{aligned} \tag{6.1}$$

Here $\mathcal{A}(D)$ is a homogeneous differential operator of order $2n$ and the $\mathcal{B}_j(D)$ are homogeneous differential operators of order $n_j \leq 2n-1$, both having $\mathcal{B}(X)$-valued coefficients, where X is some fixed Banach space. For $X = \mathbb{C}^n$ such vector-valued problems are just the usual parabolic initial-boundary value systems.

In order to give a precise description of the maximal weighted L^q-L^p-regularity approach for (6.1), let X be a Banach space, $n, n_1, \dots, n_n \in \mathbb{N}$ natural numbers with $n_j \leq 2n-1$ for each $j \in \{1,\dots,n\}$, and

$$\begin{aligned} \mathcal{A}(D) &= \textstyle\sum_{|\alpha|=2n} a_\alpha D^\alpha, \\ \mathcal{B}_j(D) &= \textstyle\sum_{|\beta|=n_j} b_{j,\beta} D^\beta, \quad j = 1,\dots,n, \end{aligned} \tag{6.2}$$

where a_α and $b_{j,\beta}$ are constant $\mathcal{B}(X)$-valued coefficients. We furthermore let

$$q \in]1,\infty[,\ \mu \in]-1, q-1[\quad \text{and} \quad p \in]1,\infty[,\ \gamma \in]-1, p-1[$$

and define

$$v_\mu(t) := |t|^\mu \quad (t \in \mathbb{R}) \quad \text{and} \quad w_\gamma(y,x') := |y|^\gamma \quad ((y,x') \in \mathbb{R} \times \mathbb{R}^{d-1}). \tag{6.3}$$

Then we have $v_\mu \in A_q$ and $w_\gamma \in A_p$, see Example D.2.12.

The maximal L^q_μ-L^p_γ-regularity approach to the parabolic initial-boundary value problem (6.1) means that we want to find[1] a (necessarily unique) space of initial-boundary data

$$\mathscr{D}_{i.b.} \subset \left[L^{(p,q),(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1,v_\mu); X)\right]^n \oplus L^p(\mathbb{R}^d_+, w_\gamma; X) \tag{6.4}$$

such that the problem

$$\begin{aligned} \partial_t u + (1 + \mathcal{A}(D))u &= f, \\ \mathrm{tr}_{y=0}\mathcal{B}_j(D)u &= g_j, \quad j = 1,\dots,n, \\ \mathrm{tr}_{t=0}u &= u_0. \end{aligned} \tag{6.5}$$

admits a unique solution

$$u \in \mathbb{E}_{sol,\mu,\gamma} := W^{(2n,1)}_{(p,q),(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X) \tag{6.6}$$

if and only if $(f,g,u_0) \in \mathscr{D} = \mathbb{E}_{0,\mu,\gamma} \times \mathscr{D}_{i.b.}$, where

$$\mathbb{E}_{0,\mu,\gamma} := L^{(p,q),(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X). \tag{6.7}$$

Here $\mathrm{tr}_{y=0}$ and $\mathrm{tr}_{t=0}$ are the distributional trace operators on $\mathbb{R}^d_+ \times \mathbb{R}_+$ with respect to $\{y=0\} = \{0\} \times \mathbb{R}^{d-1} \times \mathbb{R}_+$ and $\{t=0\} = \mathbb{R}^d_+ \times \{0\}$, respectively, as defined in Section 2.1.3. These trace operators are well defined on $\mathcal{B}_j(D)\mathbb{E}_{sol,\mu,\gamma}$ and $\mathbb{E}_{sol,\mu,\gamma}$, respectively, and give rise to bounded linear operators

$$\mathcal{B}^{tr}_j(D) := \mathrm{tr}_{y=0} \circ \mathcal{B}_j(D) \in \mathcal{B}(\mathbb{E}_{sol,\mu,\gamma}, L^{(p,q),(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1,v_\mu); X)) \tag{6.8}$$

and

$$\mathrm{tr}_{t=0} \in \mathcal{B}(\mathbb{E}_{sol,\mu,\gamma}, L^p(\mathbb{R}^d_+, w_\gamma; X)). \tag{6.9}$$

Definition 6.1.1. We say that the problem (6.5) enjoys the property of *maximal L^q_μ-L^p_γ-regularity* if there exists a (necessarily unique) linear space $\mathscr{D}_{i.b.}$ as in (6.4) such that (6.5) admits a unique solution $u \in \mathbb{E}_{sol,\mu,\gamma}$ if and only if $(f,g,u_0) \in \mathscr{D} = \mathbb{E}_{0,\mu,\gamma} \times \mathscr{D}_{i.b.}$. In this situation we call $\mathscr{D}_{i.b.}$ the *optimal space of initial-boundary data* and $\mathscr{D}$ the *optimal space of data*.

Lemma 6.1.2.

(i) If the problem (6.5) *enjoys the property of* maximal L^q_μ-L^p_γ-regularity*, then there exists a unique Banach topology on the space of initial-boundary data $\mathscr{D}_{i.b.}$ such that*

$$\mathscr{D}_{i.b.} \hookrightarrow \left[L^{(p,q),(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1,v_\mu); X)\right]^n \oplus L^p(\mathbb{R}^d_+, w_\gamma; X). \tag{6.10}$$

[1] Establish its existence and determine it explicitely.

(ii) Suppose that (6.5) *enjoys the property of maximal L^q_μ-L^p_γ-regularity and that $\mathscr{D}_{i.b.}$ has been equipped with a Banach norm such that* (6.10) *holds true; see (i). Accordingly, view the optimal space of data $\mathscr{D}$ as direct sum $\mathscr{D} = \mathbb{E}_{0,\mu,\gamma} \oplus \mathscr{D}_{i.b.}$. Then the corresponding solution operator*

$$\mathscr{S} : \mathscr{D} \longrightarrow \mathbb{E}_{sol,\mu,\gamma},\ (f, g, u_0) \mapsto \mathscr{S}(f, g, u_0) = u$$

is an isomorphism of Banach spaces, or equivalently,

$$\|u\|_{\mathbb{E}_{sol,\mu,\gamma}} \eqsim \|f\|_{\mathbb{E}_{0,\mu,\gamma}} + \|(g, u_0)\|_{\mathscr{D}_{i.b.}}, \qquad u = \mathscr{S}(f, g, u_0), (f, g, u_0) \in \mathscr{D}.$$

(iii) The following are equivalent:

(a) (6.5) *enjoys the property of maximal L^q_μ-L^p_γ-regularity.*

(b) For every $f \in \mathbb{E}_{0,\mu,\gamma}$ there exists a unique solution $u \in \mathbb{E}_{sol,\mu,\gamma}$ of the problem (6.5) *with homogeneous inital-boundary data ($g = 0$ and $u_0 = 0$).*

(c) Denote by A_B the linear operator on $Y = L^p(\mathbb{R}^d_+, w_\gamma; X)$ with domain

$$D(A_B) = \{v \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X) \mid \mathcal{B}^{tr}_j(D)v = 0, j = 1, \ldots, n\}$$

given by $A_B v = \mathcal{A}(D)v$, where $\mathcal{B}^{tr}_j(D) \in \mathcal{B}(W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X), L^p(R^{d-1}; X))$ is the boundary operator on $\mathbb{R}^d_+$ associated with $\mathcal{B}_j(D)$. Then $1 + A_B$ enjoys the property of maximal L^q_μ-regularity[2]*, i.e. for each $f \in L^q(\mathbb{R}_+, v_\mu; Y)$ there exists a unique solution $u \in W^1_q(\mathbb{R}_+, v_\gamma; Y) \cap L^q(\mathbb{R}_+, v_\mu; D(A_B))$ of*

$$u' + (1 + A_B)u = f, \quad u(0) = 0.$$

Proof. (i) Suppose that the problem (6.5) enjoys the property of maximal L^q_μ-L^p_γ-regularity. Then uniqueness follows from the closed graph theorem. So we only need to establish existence: We define the closed linear subspace $\mathbb{D}$ of $\mathbb{E}_{u,\mu,\gamma}$ by

$$\mathbb{D} := \ker\left(\partial_t + (1 + \mathcal{A}(D)) : \mathbb{E}_{sol,\mu,\gamma} \longrightarrow \mathbb{E}_{0,\mu,\gamma}\right).$$

Then, by (6.8) and (6.9),

$$T : \begin{array}{ccc} \mathbb{D} & \longrightarrow & \left[L^{(p,q),(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1, v_\mu); X)\right]^n \oplus L^p(\mathbb{R}^d_+, w_\gamma; X) \\ u & \mapsto & \left((\mathrm{tr}_{y=0}\mathcal{B}_1(D)u, \ldots, \mathrm{tr}_{y=0}\mathcal{B}_n(D)u),\ \mathrm{tr}_{t=0}u\right) \end{array}$$

defines a bounded linear operator, which is a linear isomorphism onto its image $T(\mathbb{D}) = \mathscr{D}_{i.b.}$ by definition of *maximal L^q_μ-L^p_γ-regularity* for (6.5). So we may equip $\mathscr{D}_{i.b.}$ with the norm induced by T, i.e. $\|(g, u_0)\|_{\mathscr{D}_{i.b.}} := \left\|T^{-1}(g, u_0)\right\|_{\mathbb{D}}$, resulting in a Banach norm on $\mathscr{D}_{i.b.}$ for which the continuous inclusion (6.10) holds true.

(ii) To see that $\mathscr{S}$ is an isomorphism of Banach spaces, we argue as follows: Clearly, $\mathscr{S}$ is a linear isomorphism with inverse

$$\mathscr{T} : \mathbb{E}_{sol,\mu,\gamma} \longrightarrow \mathscr{D},\ u \mapsto \begin{pmatrix} \partial_t u + (1 + \mathcal{A}(D))u \\ (\mathrm{tr}_{y=0}\mathcal{B}_1(D)u, \ldots, \mathrm{tr}_{y=0}\mathcal{B}_n(D)u) \\ \mathrm{tr}_{t=0}u \end{pmatrix}. \tag{6.11}$$

[2]See Section 6.2, in which the topic of study is abstract maximal L^q_μ-regularity.

By the closed graph theorem, it suffices to show that $\mathscr{T}$ is a closed operator. To this end, we observe that $\mathscr{T}$ is continuous when viewed as an operator

$$\mathbb{E}_{sol,\mu,\gamma} \longrightarrow \mathbb{E}_{0,\mu,\gamma} \oplus \left[L^{(p,q),(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1, v_\mu); X)\right]^n \oplus L^p(\mathbb{R}^d_+, w_\gamma; X); \tag{6.12}$$

see (6.8) and (6.9). Since $\mathscr{D} = \mathbb{E}_{0,\mu,\gamma} \oplus \mathscr{D}_{i.b.}$ is continuously included in the space on the RHS of (6.12) (as a consequence of (6.10)), we find that $\mathscr{T} : \mathbb{E}_{sol,\mu,\gamma} \longrightarrow \mathscr{D}$ is indeed a closed linear operator.

(iii) The implication '(a)⇒(b)' is trivial and the equivalence '(a)⇔(c)' follows from the canonical identification between $\mathbb{E}_{sol,\mu,\gamma}$ and the Sobolev space of intersection type

$$W^1_q(\mathbb{R}_+, v_\gamma; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+, v_\mu; W^{2m}_p(\mathbb{R}^d_+, w_\gamma; X)).$$

So we only need to establish the implication '(b)⇒(a)'. To this end, we assume (b) to hold true. Then it in particular holds that 0 is the unique solution of (6.5) with homogeneous data ($(f, g, u_0) = 0$), implying uniqueness of solutions for (6.5) with general data. So we only need to find a linear space $\mathscr{D}_{i.b.}$ as in (6.4) such that (6.5) admits a solution $u \in \mathbb{E}_{sol,\mu,\gamma}$ if and only if $(f, g, u_0) \in \mathbb{E}_{0,\mu,\gamma} \times \mathscr{D}_{i.b.}$. We show that the linear space $\mathscr{D}_{i.b.}$, consisting of all

$$(g, u_0) \in \left[L^{(p,q),(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1, v_\mu); X)\right]^n \oplus L^p(\mathbb{R}^d_+, w_\gamma; X)$$

for which there exists a solution $u \in \mathbb{E}_{sol,\mu,\gamma}$ of (6.5) for $f = 0$, is as desired.

First we consider

$$(f, g, u_0) \in \mathbb{E}_{0,\mu,\gamma} \oplus \left[L^{(p,q),(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1, v_\mu); X)\right]^n \oplus L^p(\mathbb{R}^d_+, w_\gamma; X)$$

for which there exists a solution $u \in \mathbb{E}_{sol,\mu,\gamma}$ of (6.5). Let $v \in \mathbb{E}_{u,\mu,\gamma}$ be the unique solution of (6.5) for this given f with homogeneous initial-boundary conditions. Then $w := u - v \in \mathbb{E}_{sol,\mu,\gamma}$ satisfies

$$\begin{aligned} \partial_t w + (1 + \mathcal{A}(D))w &= 0, \\ \mathrm{tr}_{y=0}\mathcal{B}_j(D)w &= g_j, \quad j = 1, \ldots, n, \\ \mathrm{tr}_{t=0}w &= u_0. \end{aligned}$$

By definition of $\mathscr{D}_{i.b.}$, we conclude that $(g, u_0) \in \mathscr{D}_{i.b.}$.

For the converse we consider $(f, g, u_0) \in \mathbb{E}_{0,\mu,\gamma} \times \mathscr{D}_{i.b.}$. Let $v \in \mathbb{E}_{sol,\mu,\gamma}$ be the unique solution of (6.5) for this given f with homogeneous initial-boundary conditions and let $w \in \mathbb{E}_{sol,\mu,\gamma}$ solve

$$\begin{aligned} \partial_t w + (1 + \mathcal{A}(D))w &= 0, \\ \mathrm{tr}_{y=0}\mathcal{B}_j(D)w &= g_j, \quad j = 1, \ldots, n, \\ \mathrm{tr}_{t=0}w &= u_0. \end{aligned}$$

Then $u := v + w \in \mathbb{E}_{sol,\mu,\gamma}$ is a solution of (6.5). □

As the main result of this chapter, Theorem 6.1.8, we will show that, under the assumption that X is a UMD space with property (α) and under suitable assumptions on $\mathcal{A}(D), \mathcal{B}_1(D), \ldots, \mathcal{B}_n(D)$ (certain conditions of ellipticity and of Lopatinskii-Shapiro type, to be discussed in Section 6.1.3), the problem (6.5) enjoys the property of maximal L^q_μ-L^p_γ-regularity with an explicit description of the optimal space of initial-boundary data $\mathscr{D}_{i.b.}$. In order to determine the optimal space of initial-boundary data $\mathscr{D}_{i.b.}$ explicitly, we observe that $\mathscr{D}_{i.b.}$ should of course contain the necessary regularity of the data (g, u_0) and might also contain compatibility conditions between them at time $t = 0$. Having this in mind, we define a Banach space of initial boundary data $\mathbb{D}_{\mu,\gamma}$ suggested by the sharp trace results from Chapter 5 such that

$$\mathscr{D}_{i.b.} \hookrightarrow \mathbb{D}_{\mu,\gamma} \hookrightarrow \left[L^{(p,q),(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1, v_\mu); X)\right]^n \oplus L^p(\mathbb{R}^d_+, w_\gamma; X),$$

with the hope that the first inclusion also holds in the reverse direction; the precise formulation of the space $\mathbb{D}_{\mu,\gamma}$ will be given in the next subsection.

6.1.2 The Space of Initial-Boundary Data

In this section we determine the initial-boundary data space $\mathscr{D}_{i.b.}$ for the maximal L^q_μ-L^p_γ-regularity problem (6.5). To be more precise, under the assumption that (6.5) enjoys the property of maximal L^q_μ-L^p_γ-regularity with optimal space of initial-boundary data $\mathscr{D}_{i.b.}$ and optimal space of data $\mathscr{D}$, we will find necessary conditions on the initial-boundary data $(g, u_0) \in \mathscr{D}_{i.b.}$ implied by the sharp trace theorems and the boundedness results of the partial differential operators D^α from Chapter 5. So let us assume that (6.5) enjoys the property of maximal L^q_μ-L^p_γ-regularity with optimal space of initial-boundary data $\mathscr{D}_{i.b.}$ and optimal space of data $\mathscr{D}$.

By Theorem 5.3.17, for the initial value u_0 we must have

$$u_0 \in B^{2m(1-\frac{1+\mu}{q})}_{p,q}(\mathbb{R}^d_+, w_\gamma; X) =: X_{\mu,\gamma}. \tag{6.13}$$

In order to determine the regularity of $g = (g_1, \ldots, g_n)$, we first observe that

$$\mathcal{B}_j(D) \in \mathcal{B}(\mathbb{E}_{u,\mu,\gamma}, F^{1-\frac{n_j}{2n},(\frac{1}{2n},1)}_{(p,q),\infty,(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X))$$

by a combination of Proposition 5.3.10.(ii) and Proposition 5.3.5.(ii). As

$$\mathrm{tr}_{y=0} \in \mathcal{B}(F^{1-\frac{n_j}{2n},(\frac{1}{2n},1)}_{(p,q),\infty,(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X), F^{1-\frac{n_j}{2n}-\frac{1}{2np}(1+\gamma),(\frac{1}{2n},1)}_{(p,q),p,(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1, v_\mu); X))$$

by Theorem 5.3.15, we thus obtain

$$\mathcal{B}^{tr}_j(D) = \mathrm{tr}_{y=0} \circ \mathcal{B}_j(D) \in \mathcal{B}(\mathbb{E}_{u,\mu,\gamma}, \mathbb{F}_{j,\mu,\gamma}), \tag{6.14}$$

where

$$\mathbb{F}_{j,\mu,\gamma} := F^{\kappa_{j,\gamma},(\frac{1}{2n},1)}_{(p,q),p,(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1, v_\mu); X)$$

for the number $\kappa_{j,\gamma} \in]0, 1[$ given by

$$\kappa_{j,\gamma} := 1 - \frac{n_j}{2n} - \frac{1}{2np}(1+\gamma).$$

Hence, for the boundary inhomogeneity g we must have

$$g = (g_1, \ldots, g_n) \in \mathbb{F}_{1,\mu,\gamma} \oplus \ldots \oplus \mathbb{F}_{n,\mu,\gamma} =: \mathbb{F}_{\mu,\gamma}. \tag{6.15}$$

Combining the necessary regularity (6.13) and (6.15) with the closed graph theorem, we find that

$$\mathscr{D}_{i.b.} \hookrightarrow \mathbb{F}_{\mu,\gamma} \oplus X_{u,\mu,\gamma}; \tag{6.16}$$

indeed, that $\mathscr{D}_{i.b.}$ is contained in the Banach space on the right is an immediate consequence of (6.13) and (6.15), whereas the continuity of this inclusion (then) follows via the closed graph theorem from the fact that there exists a topological Hausdorff space in which both spaces are continuously included (namely the space on the RHS of (6.4)).

Besides the necessary regularity (6.16), there might also be a compatibility condition for the data: If u is solution belonging to the dense subspace $\mathcal{S}(\mathbb{R}^d_+ \times \mathbb{R}_+; X)$ of $\mathbb{E}_{sol,\mu,\gamma}$, then u is a classical solution of (6.5) for the data

$$(f, g, u_0) \in \mathscr{D} \cap \left[\mathcal{S}(\mathbb{R}^d_+ \times \mathbb{R}_+; X) \times \mathcal{S}(\mathbb{R}^d_+; X)^n \times \mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{R}_+; X)\right],$$

that is, u and (f, g, u_0) satisfy (6.1). In particular, for each $j \in \{1, \dots, n\}$ we must have

$$\begin{aligned} \mathcal{B}_j(D)u(y, x', t)|_{y=0} &= g_j(x', t), \quad x' \in \mathbb{R}^{d-1}, \qquad t \geq 0, \\ u(y, x', 0) &= u_0(y, x'), \quad (y, x') \in \mathbb{R}^d_+, \end{aligned}$$

implying the compatibility condition

$$g_j(x', 0) = \mathcal{B}_j(D)u_0(y, x')|_{y=0}, \quad x' \in \mathbb{R}^{d-1}. \tag{6.17}$$

In case $\kappa_{j,\gamma} > \frac{1+\mu}{q}$, or equivalently, in case $2n(1-\frac{1+\mu}{q})-n_j > \frac{1}{p}(1+\gamma)$, we can define the temporal trace operator $\mathrm{tr}_{t=0}$ on $\mathbb{F}_{j,\mu,\gamma}$ and the spatial trace operator $\mathrm{tr}_{y=0}$ on $B^{2n(1-\frac{1+\mu}{q})-n_j}_{p,q}(\mathbb{R}^d_+, w_\gamma; X)$ as in Theorem 5.3.20 and Theorem 5.3.19, respectively, which are bounded linear operators

$$\mathrm{tr}_{t=0} \in \mathcal{B}\left(\mathbb{F}_{j,\mu,\gamma}, B^{2n(\kappa_{j,\gamma}-\frac{1+\mu}{p})}_{p,q}(\mathbb{R}^{d-1}; X)\right), \qquad \mathrm{tr}_{y=0} \in \mathcal{B}\left(B^{2n(1-\frac{1+\mu}{q})-n_j}_{p,q}(\mathbb{R}^d_+, w_\gamma; X), B^{2n(\kappa_{j,\gamma}-\frac{1+\mu}{p})}_{p,q}(\mathbb{R}^{d-1}; X)\right).$$

Since

$$\mathcal{B}_j(D) \in \mathcal{B}\left(X_{\mu,\gamma}, B^{2n(1-\frac{1+\mu}{q})-n_j}_{p,q}(\mathbb{R}^d_+, w_\gamma; X)\right)$$

by Propositions 5.3.5.(ii) and 5.3.11, we thus get a bounded linear operator

$$\mathbb{E}_{0,\mu,\gamma} \oplus \mathbb{F}_{\mu,\gamma} \oplus X_{u,\mu,\gamma} \longrightarrow B^{2n(\kappa_{j,\gamma}-\frac{1+\mu}{p})}_{p,q}(\mathbb{R}^{d-1}; X), \ (f, g, u_0) \mapsto \mathrm{tr}_{t=0}g_j - \mathrm{tr}_{y=0}\mathcal{B}_j(D)u_0 \tag{6.18}$$

in case $\kappa_{j,\gamma} > \frac{1+\mu}{q}$. Now note that (6.17) just means that this operator vanishes on the dense subspace $\mathscr{S}^{-1}\mathcal{S}(\mathbb{R}^d_+ \times \mathbb{R}_+; X)$ of $\mathscr{D} \overset{(6.16)}{\hookrightarrow} \mathbb{E}_{0,\mu,\gamma} \oplus \mathbb{F}_{\mu,\gamma} \oplus X_{u,\mu,\gamma}$ (recall that $\mathscr{S}$ is an isomorphism of Banach spaces, Lemma 6.1.2.(i)), whence this operator must vanish on the whole $\mathscr{D}$. In conclusion, incorporating this compatibility condition in (6.16), we find

$$\mathscr{D}_{i.b.} \hookrightarrow \mathbb{D}_{\mu,\gamma} := \left\{(g, u_0) \in \mathbb{F}_{\mu,\gamma} \oplus X_{\mu,\gamma} \mid \mathrm{tr}_{t=0}g_j - \mathrm{tr}_{y=0}\mathcal{B}_j(D)u_0 = 0 \ \text{ when } \ \kappa_{j,\gamma} > \frac{1+\mu}{q}\right\}; \tag{6.19}$$

here the continuity of the inclusion follows as in (6.16) because $\mathbb{D}_{\mu,\gamma}$ is a Banach space as well.

6.1.3 Assumptions on $(\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_n)$

Let $\mathcal{A}(D), \mathcal{B}_1(D), \dots, \mathcal{B}_n(D)$ be as in (6.2).

For $\phi \in [0, \pi[$ we introduce the conditions $(\mathrm{E})_\phi$ and $(\mathrm{LS})_\phi$, for which we need to recall that $\Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$ for $\theta \in]0, \pi]$. The condition $(\mathrm{E})_\phi$ is an ellipticity condition and condition $(\mathrm{LS})_\phi$ is a so-called Lopatinskii-Shapiro condition. Given $\phi \in [0, \pi[$, they are defined as follows:

(E)$_\phi$ For all $\xi \in \mathbb{R}^d, |\xi| = 1$ it holds that $\sigma(\mathcal{A}(\xi)) \subset \Sigma_\phi$ for the spectrum of the operator $\mathcal{A}(\xi) = \sum_{|\alpha|=2n} a_\alpha \xi^\alpha$ in $\mathcal{B}(X)$.

(LS)$_\phi$ For each $\xi' \in \mathbb{R}^{d-1}$ and $\lambda \in \overline{\Sigma}_{\pi-\phi}$ with $|\xi'| + |\lambda| \neq 0$ and all $h = (h_1, \ldots, h_n) \in X^n$, the ordinary initial value problem

$$\begin{aligned} \lambda w(y) + \mathcal{A}(D_y, \xi')w(y) &= 0, \quad y > 0 \\ \mathcal{B}_j(D_y, \xi')w(y)|_{y=0} &= h_j, \quad j = 1, \ldots, n. \end{aligned} \tag{6.20}$$

has a unique solution $w \in C_0^\infty([0, \infty[; X)$.

Remark 6.1.3. Note that, by continuity and compactness, $\mathcal{A}(D)$ satisfies (E)$_\phi$ if and only if

$$\inf_{|\xi|=1} \operatorname{dist}(\sigma(\mathcal{A}(\xi)), \overline{\Sigma}_{\pi-\phi}) > 0.$$

In particular, by homogeneity, $\mathcal{A}(D)$ satisfies (E)$_{\pi/2}$ if and only if there exists a constant $c > 0$ such that

$$\inf\{\Re(\mu) \mid \mu \in \sigma(\mathcal{A}(\xi))\} \geq c|\xi|^{2n}, \qquad \forall \xi \in \mathbb{R}^d.$$

In this case we say that $\mathcal{A}(D)$ is *normally elliptic*.

Note that differential operators of odd degree can not be normally elliptic (when we extend the just defined notion of normal ellipticity in the natural way to differential operators of arbitrary degree).

The ellipticity condition (E)$_\phi$ is equivalent to the condition that $\overline{\Sigma}_{\pi-\phi} \subset \rho(-\mathcal{A}(\xi))$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$. In case X is a UMD Banach space (so that the operator valued Mikhlin theorem with A_p-weights from Chapter 4 is available), this condition makes it possible to find, for $\lambda \in \Sigma_{\pi-\phi}$ and $f \in L^p(\mathbb{R}^d, w_\gamma; X)$, a unique solution $v \in W^{2n}_p(\mathbb{R}^d, w_\gamma; X)$ of the equation

$$(\lambda + \mathcal{A}(D))v = f,$$

as well as certain estimates/bounds for this solution.

The Lopatinskii-Shapiro condition (LS)$_\phi$ makes it possible to solve, given a $\lambda \in \Sigma_{\pi-\phi}$, the elliptic boundary value problem

$$\begin{aligned} \lambda v + \mathcal{A}(D)v &= f, \\ \operatorname{tr}_{y=0}\mathcal{B}_j(D)v &= g_j, \quad j = 1, \ldots, n, \end{aligned}$$

for the case $f = 0$ via the partial Fourier transform with respect to x', from which the case of a general f can be derived from the elliptic problem on $\mathbb{R}^d$ without boundary conditions. In particular, for zero boundary data $g_1 = \ldots = g_l = 0$, this condition makes it possible to compute the resolvent of the operator A_B from Lemma 6.1.2.(iii).(c).

Remark 6.1.4. In our main result of this chapter, Theorem 6.1.8, we will assume that the conditions (E)$_\phi$ and (LS)$_\phi$ hold for some $\phi \in]0, \frac{\pi}{2}[$. In [73, Lemma 2.2.1&Lemma 2.2.4] it is shown that for this it is already enough that just (E)$_{\frac{\pi}{2}}$ and (LS)$_{\frac{\pi}{2}}$ are satisfied. There (E)$_{\frac{\pi}{2}}$ is called normal ellipticity and (LS)$_{\frac{\pi}{2}}$ is called the Lopatinskii-Shapiro condition, with notations (E) and (LS), respectively.

We now take a closer look at the ellipticity condition (E)$_\phi$ and Lopatinskii-Shapiro condition (LS)$_\phi$ in some more 'concrete' situations.

Example 6.1.5.

(i) Suppose that $X = H$ is a Hilbert space and that each coefficient $a_\alpha \in \mathcal{B}(H)$ is a self-adjoint operator. Then $\mathcal{A}(\xi) = \sum_{|\alpha|=n} a_\alpha \xi^\alpha \in \mathcal{B}(H)$ is a self-adjoint operator as well for each $\xi \in \mathbb{R}^d$. Therefore,

$$\sigma(\mathcal{A}(\xi)) = \sigma_{ap}(\mathcal{A}(\xi)) \subset \overline{\{\langle \mathcal{A}(\xi)h|h\rangle \mid \|h\| = 1\}}, \qquad \forall \xi \in \mathbb{R}^d;$$

where $\sigma_{ap}(\mathcal{A}(\xi))$ is the approximate point spectrum of $\mathcal{A}(\xi)$. Using Remark 6.1.3, we see that for $\mathcal{A}(D)$ to be normally elliptic it is sufficient that there exists a constant $c > 0$ such that

$$\Re(\langle \mathcal{A}(\xi)h|h\rangle) \geq c|\xi|^{2n} \|h\|^2, \qquad \forall \xi \in \mathbb{R}^d, h \in H.$$

(ii) Suppose that $\mathcal{A}(D)$ has scalar-valued coefficients $a_\alpha \in \mathbb{C}$. Then, in view of Remark 6.1.3, $\mathcal{A}(D)$ is normally elliptic if and only if there exists a constant $c > 0$ such that

$$\Re(\mathcal{A}(\xi)) \geq c|\xi|^{2n}, \qquad \forall \xi \in \mathbb{R}^d.$$

Example 6.1.6. Let X be a Banach space and let $\mathcal{A}(D) = \sum_{|\alpha|=2} a_\alpha \xi^\alpha$ with $a_\alpha \in \mathbb{C}$. Suppose that either

(i) $\mathcal{B}(D) = \sum_{|\beta|=1} b_\beta D^\beta$ with $b_{(1,0,\ldots,0)} \neq 0$; or

(ii) $\mathcal{B}(D) = b_0$ with $b_0 \neq 0$,

with $b_\beta \in \mathbb{C}$ in each case. Then the Lopatinskii-Shapiro condition $(\mathrm{LS})_\phi$ is equivalent to: For each $\xi' \in \mathbb{R}^{d-1}$ and $\lambda \in \overline{\Sigma}_{\pi-\phi}$ with $|\xi'| + |\lambda| \neq 0$, the polynomial equation

$$a_0\mu^2 + a_1(\xi')\mu + a_2(\xi') + \lambda = 0$$

has two distinct roots $\mu_\pm \in \mathbb{C}$ with $\Im(\mu_+) > 0 > \Im(\mu_-)$, where

$$a_k(\xi') := \sum_{|\alpha'|=k} a_{(k,\alpha')}(\xi')^{\alpha'}, \qquad \xi' \in \mathbb{R}^{d-1}.$$

Proof. As there is only one boundary condition, it suffices to prove this for $X = \mathbb{C}$. This can be found in [64, Section 7.4]. □

Using the equivalent algebraic condition from the above example, it is easy to see that:

Example 6.1.7. Let X be a Banach space and $\mathcal{A}(D) = -\Delta$. Suppose that $\mathcal{B}(D)$ is as (i) or (ii) of Example 6.1.6. Then $\mathcal{A}(D)$ and $(\mathcal{A}(D), \mathcal{B}(D))$ satisfy $(\mathrm{E})_\phi$ and $(\mathrm{LS})_\phi$ for any $\phi \in [0, \pi[$.

6.1.4 Statement of the Main Result and Outline of its Proof

Before we state the main result of this chapter, we first recall some notation: We have X a Banach space, $\mathcal{A}(D), \mathcal{B}_1(D), \ldots, \mathcal{B}_n(D)$ differential operators as in (6.2), $q, p \in]-1, \infty[$, $\mu \in]-1, q-1[$, $\gamma \in]-1, p-1[$, and v_μ, w_γ are the weights on $\mathbb{R}$ and $\mathbb{R}^d$ given in (6.3). Furthermore, we have the numbers

$$\kappa_{j,\gamma} = 1 - \frac{n_j}{2n} - \frac{1}{2np}(1+\gamma) \in]0, 1[, \qquad j = 1, \ldots, n,$$

and the function spaces

$$
\begin{aligned}
\mathbb{E}_{sol,\mu,\gamma} &= W^{(2n,1)}_{(p,q),(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X) \\
&= W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+, v_\mu; W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)) \\
\mathbb{E}_{0,\mu,\gamma} &= L^{(p,q),(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}_+, (w_\gamma, v_\mu); X) \\
&= L^q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \\
X_{\mu,\gamma} &= B^{2n(1-\frac{1+\mu}{q})}_{p,q}(\mathbb{R}^d_+, w_\gamma; X) \\
\mathbb{F}_{j,\mu,\gamma} &= F^{\kappa_{j,\gamma},(\frac{1}{2n},1)}_{(p,q),p,(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1, v_\mu); X), \quad j = 1, \ldots, n, \\
&= F^{\kappa_{j,\gamma}}_{q,p}(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}_+, v_\mu; F^{2n\kappa_{j,\gamma}}_{p,p}(\mathbb{R}^{d-1}; X)), \quad j = 1, \ldots, n, \\
\mathbb{F}_{\mu,\gamma} &= \mathbb{F}_{1,\mu,\gamma} \oplus \ldots \oplus \mathbb{F}_{n,\mu,\gamma} \\
\mathbb{D}_{\mu,\gamma} &= \left\{(g, u_0) \in \mathbb{F}_{\mu,\gamma} \oplus X_{\mu,\gamma} \mid \mathrm{tr}_{t=0} g_j - \mathrm{tr}_{y=0}\mathcal{B}_j(D)u_0 = 0 \text{ when } \kappa_{j,\gamma} > \tfrac{1+\mu}{q}\right\};
\end{aligned}
$$

see Lemma 5.3.7 and Theorem 5.3.13.(ii) for the above intersection representations of anisotropic spaces.

Theorem 6.1.8. *Let the notations be as above. Suppose that X is a UMD space with property (α), that $\mathcal{A}(D), \mathcal{B}_1(D), \ldots, \mathcal{B}_n(D)$ satisfy the conditions* $(\mathrm{E})_\phi$ *and* $(\mathrm{LS})_\phi$ *for some $\phi \in]0, \frac{\pi}{2}[$, and that $\kappa_{j,\gamma} \neq \frac{1+\mu}{q}$ for all $j \in \{1, \ldots, n\}$. Then the problem* (6.5) *enjoys the property of maximal L^q_μ-L^p_γ-regularity with $\mathbb{D}_{\mu,\gamma}$ as the optimal space of initial-boundary data, i.e., the problem*

$$
\begin{aligned}
\partial_t u + (1 + \mathcal{A}(D))u &= f, \\
\mathrm{tr}_{y=0}\mathcal{B}_j(D)u &= g_j, \quad j = 1, \ldots, n, \\
\mathrm{tr}_{t=0} u &= u_0,
\end{aligned}
$$

admits a unique solution $u \in \mathbb{E}_{sol,\mu,\gamma}$ if and only if $(f, g, u_0) \in \mathbb{E}_{0,\mu,\gamma} \oplus \mathbb{D}_{\mu,\gamma}$. Moreover, the corresponding solution operator $\mathscr{S} : \mathbb{E}_{0,\mu,\gamma} \oplus \mathbb{D}_{\mu,\gamma} \longrightarrow \mathbb{E}_{sol,\mu,\gamma}$ is an isomorphism of Banach spaces.

Note the dependence of the space of initial-boundary data on the weight parameters μ and γ. For fixed $q, p \in]1, \infty[$ we can roughly speaking decrease the required 'smoothness' (or regularity) of g and u_0 by increasing γ and μ, respectively. Furthermore, compatibility conditions can be avoided by choosing μ and γ big enough. So the weights make it possible to solve (6.5) for more initial-boundary data (compared to the unweighed setting). On the other hand, by choosing μ and γ closer to -1 (depending on the initial-boundary data) we can find more information about the behavior of u near the initial-time and near the boundary, respectively.

Remark 6.1.9.

(i) We assume property (α) in order to simplify the proof of Theorem 6.1.8; it allows us to prove the $\mathcal{R}$-boundedness of a set of Fourier multiplier operators via the last part of Theorem 4.1.1 (or Corollary 4.5.21). Without property (α) it is not possible to apply last part of Theorem 4.1.1 (or Corollary 4.5.21) to get the $\mathcal{R}$-sectoriality in Theorem 6.3.12 (and the $\mathcal{R}$-sectoriality in Theorem 6.3.1, which is needed for the $\mathcal{R}$-sectoriality in Theorem 6.3.12). The notion of $\mathcal{R}$-sectorial operator will be defined in Section 6.2 and is closely related to abstract maximal L^q_μ-regularity; see Theorem 6.2.4 and also recall (c) of Lemma 6.1.2.(iii).

In the general UMD-case we could proceed via the boundedness of the $\mathcal{H}^\infty$-calculus to get the just mentioned $\mathcal{R}$-sectoriality; for general information on the $\mathcal{H}^\infty$-calculus we refer to [25] and [48]. A result due to Clément and Prüss [18] (also see [25, Theorem 4.5]) says that, for a UMD space Y, if $A \in \mathcal{BIP}(Y)$ with power-angle θ_A[3], then A is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi_A^R \leq \theta_A$. In particular, since

$$A \in \mathcal{H}^\infty(Y) \Longrightarrow A \in \mathcal{BIP}(Y) \quad \text{with} \quad \phi_A^\infty \geq \theta_A,$$

if A has a bounded $\mathcal{H}^\infty$-calculus with $\mathcal{H}^\infty$-angle ϕ_A^∞, then A is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\phi_A^R \leq \phi_A^\infty$. Accordingly, for the $\mathcal{R}$-sectoriality in Theorems 6.3.12 and 6.3.1, it suffices to show that the involved elliptic operators have a bounded $\mathcal{H}^\infty$-calculus with the right $\mathcal{H}^\infty$-angle. This can be done as in [25, Theorem 5.5] and [25, Theorem 7.4], respectively.

Furthermore, from the viewpoint of applications, property (α) is not very restrictive since it is automatically satisfied by every UMD Banach lattice X; see Example E.4.2.(iii) and Proposition E.5.5.(iii).

(ii) Let the notations and assumptions be as in the theorem. Let A_B be the operator on $Y = L^p(\mathbb{R}^d_+, w_\gamma; X)$ from Lemma 6.1.2.(iii).(c). Then the theorem in particular tells us that the abstract Cauchy problem

$$u' + A_B u = f, \qquad u(0) = u_0$$

has for each $f \in L^q(\mathbb{R}_+, v_\mu; Y)$ a unique solution $u \in W^1_q(\mathbb{R}_+, v_\mu; Y) \cap L^q(\mathbb{R}_+, v_\mu; D(A))$ if and only if $(0, u_0) \in \mathbb{D}_{\mu,\gamma}$, that is, if and only if

$$u_0 \in \left\{ v \in B^{2n(1-\frac{1+\mu}{q})}_{p,q,\mathcal{B}}(\mathbb{R}^d_+, w_\gamma; X) \; : \; \mathcal{B}^{tr}_j(D)v = 0 \text{ when } \kappa_{j,\gamma} > \frac{1+\mu}{q} \right\}.$$

(iii) For the reader familiar with real interpolation theory. Let the notations and assumptions be as in the theorem, except for the condition $\kappa_{j,\gamma} \neq \frac{1+\mu}{q}$ for all $j \in \{1, \ldots, n\}$. We define

$${}_{\mathcal{B}}W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X) := \left\{ v \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X) \; : \; \mathcal{B}^{tr}_j(D)v = 0, j = 1, \ldots, n \right\}$$

and, for $s \in \mathbb{R}$ with $s - \frac{1}{p}(1+\gamma) \neq n_j$ for each $j \in \{1, \ldots, n\}$,

$${}_{\mathcal{B}}B^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) := \left\{ v \in B^s_{p,q,\mathcal{B}}(\mathbb{R}^d_+, w_\gamma; X) \; : \; \mathcal{B}^{tr}_j(D)v = 0 \text{ when } s - \frac{1}{p}(1+\gamma) > n_j \right\}.$$

As a consequence of (ii) and Remark 6.2.2 (see below), we then have, for $q \in]1, \infty[$ and $\theta \in]0, 1[$ with $\theta \neq \frac{1}{2n}\left(n_j + \frac{1}{p}(1+\gamma)\right)$ for each $j \in \{1, \ldots, n\}$,

$$\left(L^p(\mathbb{R}^d_+, w_\gamma; X), {}_{\mathcal{B}}W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)\right)_{\theta,q} = {}_{\mathcal{B}}B^{2n\theta}_{p,q}(\mathbb{R}^d_+, w_\gamma; X).$$

Example 6.1.10. Let X be an (α)-UMD space. As a consequence of the above theorem and Example 6.1.7, we have L^q_μ-L^p_γ-regularity for the following two second order parabolic initial-boundary value problems:

[3] A is a sectorial operator having bounded imaginary powers, see [25, Definition 2.4])

(i) $1 - \Delta$ *with Dirichlet boundary condition:*

The problem

$$\begin{aligned} \partial_t u + (1-\Delta)u &= f, \\ \mathrm{tr}_{y=0} u &= g, \\ \mathrm{tr}_{t=0} u &= u_0. \end{aligned}$$

has a unique solution $u \in W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+, v_\mu; W^2_p(\mathbb{R}^d_+, w_\gamma; X))$ if and only the data (f, g, u_0) satisfy:

- $f \in L^q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X))$;
- $g \in F^{1-\frac{1}{2p}(1+\gamma)}_{q,p}(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}^d_+, v_\mu; F^{2-\frac{1}{p}(1+\gamma)}_{p,p}(\mathbb{R}^{d-1}; X))$;
- $u_0 \in B^{2-\frac{2}{q}(1+\mu)}(\mathbb{R}^d_+, w_\gamma; X)$;
- $\mathrm{tr}_{t=0} g = \mathrm{tr}_{y=0} u_0$ when $2 - \frac{2}{q}(1+\mu) > \frac{1}{p}(1+\gamma)$.

(ii) $1 - \Delta$ *with Neumann boundary condition:*

The problem

$$\begin{aligned} \partial_t u + (1-\Delta)u &= f, \\ \mathrm{tr}_{y=0} \partial_y u &= g, \\ \mathrm{tr}_{t=0} u &= u_0. \end{aligned}$$

has a unique solution $u \in W^1_q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}_+, v_\mu; W^2_p(\mathbb{R}^d_+, w_\gamma; X))$ if and only the data (f, g, u_0) satisfy:

- $f \in L^q(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X))$;
- $g \in F^{\frac{1}{2}-\frac{1}{2p}(1+\gamma)}_{q,p}(\mathbb{R}_+, v_\mu; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}^d_+, v_\mu; F^{1-\frac{1}{p}(1+\gamma)}_{p,p}(\mathbb{R}^{d-1}; X))$;
- $u_0 \in B^{2-\frac{2}{q}(1+\mu)}(\mathbb{R}^d_+, w_\gamma; X)$;
- $\mathrm{tr}_{t=0} g = \mathrm{tr}_{y=0} u_0$ when $1 - \frac{2}{q}(1+\mu) > \frac{1}{p}(1+\gamma)$.

We will give the proof of Theorem 6.1.8 in the end of Section 6.4. Since this proof requires quite some preparation, we now first give an outline:

Outline of the proof of Theorem 6.1.8. The first step is to prove that the problem (6.5) enjoys the property of maximal L^q_μ-L^p_γ-regularity without showing *yet* that $\mathbb{D}_{\mu,\gamma}$ is the optimal space of initial-boundary data. For this we use the equivalence '(a)⇔(c)' from Lemma 6.1.2.(iii), leading to the maximal L^q_μ-regularity problem of abstract Cauchy problems. The latter problem we will study in Section 6.2.(ii), with as main result in this direction Theorem 6.2.4, giving a characterization of maximal L^q_μ-regularity for linear operators A on a UMD Banach space for which $-A$ is the generator of an analytic semigroup in terms of a certain $\mathcal{R}$-boundedness condition for the resolvent of $-A$. Accordingly, in Section 6.3 we will check that the the linear operator A_B on $Y = L^p(\mathbb{R}^d, w_\gamma; X)$ from (c) of Lemma 6.1.2.(iii) is such that $-A_B$ is the generator of an analytic semigroup for which the resolvent satisfies the $\mathcal{R}$-boundedness condition from this characterization. For this we need to study the elliptic boundary value problems

$$\begin{aligned} \lambda v + \mathcal{A}(D)v &= f, \\ \mathrm{tr}_{y=0} \mathcal{B}_j(D)v &= 0, \quad j = 1, \ldots, n, \end{aligned}$$

on $\mathbb{R}^d_+$ for $f \in L^p(\mathbb{R}^d_+, w_\gamma; X)$; we must show existence and uniqueness of a solution $u \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ plus certain estimates/bounds for λ in an appropriate sector of the complex plain. For this we will in fact study (in Section 6.3.2) for $\lambda \in \Sigma_{\pi-\phi}$ the more general elliptic boundary value problems

$$\begin{aligned} \lambda v + \mathcal{A}(D)v &= f, \\ \mathrm{tr}_{y=0}\mathcal{B}_j(D)v &= g_j, \quad j = 1, \dots, n, \end{aligned} \tag{6.21}$$

for $f \in L^p(\mathbb{R}^d_+, w_\gamma; X)$ and $g_j \in F^{2n\kappa_{j,\gamma}}_{p,p}(\mathbb{R}^{d-1}; X)$, $j = 1, \dots, n$; here the regularity of $g = (g_1, \dots, g_n)$ is motivated by the trace result of Theorem 5.3.18 (note that $2n\kappa_{j,\gamma} = 2n - n_j - \frac{1+\gamma}{p}$). Our approach will be as follows:

(I) Using the ellipticity condition $(\mathrm{E})_\phi$, for $\bar{f} \in L^p(\mathbb{R}^d, w_\gamma; X)$ we show existence and uniqueness of solutions $\bar{w} \in W^{2n}_p(\mathbb{R}^d, w_\gamma; X)$ of the elliptic problem

$$\lambda \bar{w} + \mathcal{A}(D)\bar{w} = \bar{f} \tag{6.22}$$

on the full space $\mathbb{R}^d$, as well as certain estimates/bounds. This is the subject of in Section 6.3.1.

(II) Using the Lopatinskii-Shapiro condition $(\mathrm{LS})_\phi$ (plus (I) at some technical point concerning uniqueness), we treat the problem (6.21) in case $f = 0$.

(III) Firstly using (I) (via extension by zero of f to $\mathbb{R}^d$ and restriction of the obtained solution to $\mathbb{R}^d_+$) and subsequently using (II), we solve the problem (6.21) in case $g = 0$.

(IV) Finally, combining (II) and (III) of course solves (6.21) for the general case.

Having proved that the problem (6.5) enjoys the property of maximal L^q_μ-L^p_γ-regularity, the second step is to show that $\mathbb{D}_{\mu,\gamma}$ is indeed the optimal space of initial-boundary data. Since we have $\mathscr{D}_{i.b.} \overset{(6.19)}{\hookrightarrow} \mathbb{D}_{\mu,\gamma}$ by construction of $\mathbb{D}_{\mu,\gamma}$, we only need to show that $\mathbb{D}_{\mu,\gamma} \subset \mathscr{D}_{i.b.}$; continuity of this inclusion then is a consequence of the open mapping theorem.[4] For this we need to show existence of a solution $u \in \mathbb{E}_{u,\mu,\gamma}$ of the problem (6.5) for any $(f, g, u_0) \in \mathbb{E}_{0,\mu,\gamma} \oplus \mathbb{D}_{\mu,\gamma}$; note that uniqueness follows from the fact that we already have uniqueness in case $(g, u_0) = 0$ in view of '(a)⇔(c)' from Lemma 6.1.2.(ii). The idea is to first reduce to the case that $u_0 = 0$ (by using that $\mathrm{tr}_{t=0}$ maps $\mathbb{E}_{u,\mu,\gamma}$ onto $X_{u,\mu,\gamma}$) and subsequently to reduce to the case $f = 0, u_0 = 0$ (by using (c) from Lemma 6.1.2.(iii)). We will solve this reduced problem in Lemma 6.4.3. □

6.2 Abstract Maximal L^q_μ-Regularity

In this section we state the characterization of maximal L^q-regularity in terms of $\mathcal{R}$-sectoriality due to Weis [102] in the v_μ-weighted setting. Let us first give the definitions of maximal L^q_μ-regularity (also see Lemma 6.1.2.(iii)) and $\mathcal{R}$-sectoriality.

[4] In our solution we will in fact obtain the continuity of the inclusion $\mathbb{D}_{\mu,\gamma} \hookrightarrow \mathscr{D}_{i.b.}$ via direct estimates.

Definition 6.2.1. Let Y be a Banach space and let A be a closed linear operator on Y with domain $D(A)$. The abstract Cauchy problem

$$u' + Au = f, \quad u(0) = 0,$$

is said to enjoy the property of maximal L^q_μ-regularity, where $q \in]1, \infty[$ and $\mu \in]-1, q-1[$, if for each $f \in L^q(\mathbb{R}_+, v_\mu; Y)$ there exists a unique solution $u \in W^1_q(\mathbb{R}_+, v_\mu; Y) \cap L^q(\mathbb{R}_+, v_\mu; D(A))$. In this case we also say that A has maximal L^q_μ-regularity. For $\mu = 0$ we drop the μ from the notation.

Remark 6.2.2. Having maximal L^q_μ-regularity, non-zero initial values can easily be treated via related temporal trace results (see e.g. [99, Section 1.8.1 & 1.14.5]); also see Proposition 6.2.5 below, which says that $-A$ generates an analytic semigroup on Y in this situation. To be more specific, if $-A$ is the generator of an analytic semigroup on Y, then the temporal trace space (in $t = 0$) of $W^1_q(\mathbb{R}_+, v_\mu; Y) \cap L^q(\mathbb{R}_+, v_\mu; D(A))$ is the real interpolation space $(Y, D(A))_{1-\frac{1+\mu}{q},q}$.[5]

Definition 6.2.3. Let Y be a Banach space and let A be a closed linear operator on Y with domain $D(A)$. Then A is called $\mathcal{R}$-sectorial of angle $\omega \in]0, \pi[$ if the following conditions hold true:

(i) The domain $D(A)$ and range $R(A)$ of A are dense in Y.

(ii) $\Sigma_{\pi-\omega} \subset \rho(-A)$.[6]

(iii) For every $\phi \in]\omega, \pi[$, $\{\lambda(\lambda + A) \,:\, \lambda \in \Sigma_{\pi-\phi}\}$ is an $\mathcal{R}$-bounded set in $\mathcal{B}(Y)$.

In this case we define the $\mathcal{R}$-angle ϕ^R_A as the infimum of all such ω.

Theorem 6.2.4. *Let Y be a UMD Banach space and let A be a closed linear operator on Y with domain $D(A)$. Let $q \in]1, \infty[$ and $\mu \in]-1, q-1[$. Then A has the property of maximal L^q_μ-regularity if and only if A is an invertible $\mathcal{R}$-sectorial operator with $\mathcal{R}$-angle $\phi^R_A < \frac{\pi}{2}$.*

This result is due to Weis [102, Theorem 4.2] ($\mu = 0$). For a nice historical overview of this problem we refer to [64]. Another reference is [25, Theorem 4.4]. The same argumentation can be used in the A_q-weighted setting. The idea is roughly to translate this problem into a Fourier multiplier problem, for which a condition involving $\mathcal{R}$-boundedness is natural in view of Remark 4.5.6 and Theorem 4.5.13. Also see [15] for extrapolation of maximal regularity. Also see [82] for the independence of maximal L^q_μ-regularity on μ in the range $[0, q-1[$.

Proposition 6.2.5. *Let X be a Banach space, A a closed linear operator on X with domain $D(A)$, and $q \in]1, \infty[$. If A has maximal L^q-regularity, then $-A$ generates an exponentially stable analytic C_0-semigroup $(e^{-tA})_{t\geq 0}$ in X.*[7]

For a proof we refer to [30, Corollary 4.2].

[5]For an elementary introduction to interpolation theory we refer to [69].

[6]Recall here that $\Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$ for $\theta \in]0, \pi]$.

[7]A C_0-semigroup $(e^{-tA})_{t\geq 0}$ in X is called exponentially stable if there exist $M, \omega > 0$ such that $\left\|e^{-tA}\right\| \leq Me^{-\omega t}$ for all $t \geq 0$.

6.3 Elliptic Problems

The goal of this section is to solve the elliptic problem (6.22) on the full space $\mathbb{R}^d$ and the elliptic boundary value problem (6.21) on the half-space $\mathbb{R}^d_+$; also see the outline of the proof of the main result, Theorem 6.1.8, given in Section 6.1.4.

6.3.1 Elliptic Problems on $\mathbb{R}^d$

Theorem 6.3.1. *Let X be an (α)-UMD space, $p \in]1,\infty[$, and $w \in A_p(\mathbb{R}^d)$. Suppose that $\mathcal{A}(D) = \sum_{|\alpha|\leq n} a_\alpha D^\alpha$, with $a_\alpha \in \mathcal{B}(E)$, satisfies $(\mathrm{E})_\phi$ for some $\phi \in [0,\pi[$. Let A be the operator on $L^p(\mathbb{R}^d, w; X)$ with domain $D(A) := H^n_p(\mathbb{R}^d, w; X) = W^n_p(\mathbb{R}^d, w; X)$ given by $Au := \mathcal{A}(D)u$. Then A is an $\mathcal{R}$-sectorial operator with $\mathcal{R}$-angle $\phi^{\mathcal{R}}_A \leq \phi$.*

Proof. For (i) of Definition 6.2.3 we only need to establish the denseness of $R(A)$, the denseness of $D(A)$ being immediate from the fact that $\mathcal{S}(\mathbb{R}^d; X)$ is dense in $L^p(\mathbb{R}^d, w; X)$. In view of Lemma 3.4.3, it sufficient to show that $\mathcal{S}_0(\mathbb{R}^d; X)$ is contained in $R(A)$. To this end, let us fix an $f \in \mathcal{S}_0(\mathbb{R}^d; X)$. Now observe that $\mathbb{R}^d \setminus \{0\} \ni \xi \mapsto \mathcal{A}(\xi) \in \mathcal{B}(X)$ is a well-defined smooth function since $\mathbb{R}^d \setminus \{0\}\xi \mapsto \mathcal{A} \in \mathcal{B}(X)$ is a smooth map taking values in the invertible operators (in view of the ellipticity assumption $(\mathrm{E})_\phi$). Therefore, $g : \xi \mapsto \mathcal{A}(\xi)^{-1}\hat{f}(\xi)$ belongs to $C^\infty_c(\mathbb{R}^d \setminus \{0\}; X) \subset \mathcal{S}(\mathbb{R}^d; X)$. Clearly, $u := \mathscr{F}^{-1}g \in \mathcal{S}(\mathbb{R}^d; X) \subset D(A)$ solves $\mathcal{A}(D)u = f$.

Next we must show that (ii) and (iii) of Definition 6.2.3 hold true for $\omega \leq \phi$. For this we fix a $\varphi \in]\phi, \pi[$ and show that $\Sigma_{\pi-\varphi} \subset \rho(-A)$ with $\mathcal{R}\{\lambda(\lambda + A)^{-1} \mid \lambda \in \Sigma_{\pi-\varphi}\} < \infty$.

From the ellipticity condition $(\mathrm{E})_\phi$ it follows that, for each $\lambda \in \Sigma_{\pi-\varphi}$ and $|\alpha| \leq n$,

$$[\xi \mapsto \lambda^{1-\frac{|\alpha|}{n}}\xi^\alpha(\lambda + \mathcal{A}(\xi))^{-1}] \in C^\infty(\mathbb{R}^d \setminus \{0\}; \mathcal{B}(X)).$$

We claim that

$$\kappa_{\alpha,\beta} := \mathcal{R}\{|\xi|^\beta D^\beta_\xi \lambda^{1-\frac{|\alpha|}{n}}\xi^\alpha(\lambda+\mathcal{A}(\xi))^{-1} \mid \xi \in \mathbb{R}^d\setminus\{0\}, \lambda \in \Sigma_{\pi-\varphi}\} < \infty, \qquad \forall|\alpha| \leq n, \beta \in \mathbb{N}^d. \quad (6.23)$$

By Corollary 4.5.21 (the operator-valued (α)-UMD Mikhlin theorem with A_p-weights) we then obtain that each multiplier symbol $\xi \mapsto \xi^\alpha(\lambda + \mathcal{A}(\xi))^{-1}$, $\lambda \in \Sigma_{\pi-\varphi}$ and $|\alpha| \leq n$, defines a bounded linear operator $T_{\lambda,\alpha}$ on $L^p(\mathbb{R}^d, w; X)$ for which $\{\lambda^{1-\frac{|\alpha|}{n}} T_{\lambda,\alpha} \mid \lambda \in \Sigma_{\pi-\phi}\}$ is an $\mathcal{R}$-bounded collection in $\mathcal{B}(L^p(\mathbb{R}^d, w; X))$. We thus get bounded linear operators $T_\lambda := T_{\lambda,0} : L^p(\mathbb{R}^d, w; X) \longrightarrow W^n_p(\mathbb{R}^d, w; X)$, $\lambda \in \Sigma_{\pi-\phi}$, such that

$$\mathcal{R}\left\{\lambda^{1-\frac{|\alpha|}{n}} D^\alpha T_\lambda = \lambda^{1-\frac{|\alpha|}{n}} T_{\lambda,\alpha}\right\} < \infty \quad \text{in} \quad \mathcal{B}(L^p(\mathbb{R}^d, w; X)), \quad |\alpha| \leq n. \quad (6.24)$$

Since $T_\lambda \in \mathcal{B}(L^p(\mathbb{R}^d, w; X), W^n_p(\mathbb{R}^d, w; X)) \subset \mathcal{B}(L^p(\mathbb{R}^d, w; X))$ is easily seen to be the inverse of $\lambda + A$, it follows that $\Sigma_{\pi-\varphi} \subset \rho(-A)$ with $\mathcal{R}\{\lambda(\lambda + A)^{-1}\} < \infty$ (take $\alpha = 0$ in (6.24)), as desired.

In order to establish (6.23), we pick a $\theta \in]\phi, \varphi[$ and define $G := \{(\sigma, \xi) \in (\mathbb{C} \times \mathbb{C}^d) \setminus \{0\} \mid \arg(\sigma) < \frac{1}{n}(\pi - \theta)\}$ and, for each $|\alpha| \leq n$,

$$m_\alpha : G \longrightarrow \mathcal{B}(X),\ (\sigma, \xi) \mapsto \sigma^{n-|\alpha|}\xi^\alpha(\sigma^n + \mathcal{A}(\xi))^{-1};$$

then m_α is a well defined holomorphic function as a consequence of the ellipticity condition $(\mathrm{E})_\phi$. Since m_α is positively homogeneous of degree 0, i.e. $m(\rho\sigma, \rho\xi) = m(\sigma, \xi)$ for all $(\sigma, \xi) \in G$

and $\rho > 0$, it follows that $(\sigma,\xi) \mapsto |\xi|^{|\beta|} D^\beta_\xi m_\alpha(\sigma,\xi)$ is positively homogeneous of degree 0 as well for each $\beta \in \mathbb{N}^d$. Combining this homogeneity with the contraction principle (cf. Proposition E.1.2), we obtain

$$\begin{aligned}
&\mathcal{R}\{|\xi|^\beta D^\beta_\xi \lambda^{1-\frac{|\alpha|}{n}} \xi^\alpha (\lambda + \mathcal{A}(\xi))^{-1} \mid \xi \in \mathbb{R}^d \setminus \{0\}, \lambda \in \Sigma_{\pi-\phi}\} \\
&\quad \leq \mathcal{R}\{|\xi|^\beta D^{beta}_\xi m_\alpha(\sigma,\xi) \mid (\sigma,\xi) \in (\mathbb{C} \times \mathbb{R}^d) \setminus \{0\}, \arg(\sigma) \leq \frac{1}{n}(\pi - \phi)\} \\
&\quad = \mathcal{R}\{|\zeta|^{|\beta|} D^\beta_\xi m_\alpha(\tau,\zeta) \mid (\tau,\zeta) \in G, \arg(\tau) \leq \frac{1}{n}(\pi - \varphi), |\tau|^2 + |\zeta|^2 = 1\} \\
&\quad \leq \mathcal{R}\{D^\beta_\xi m_\alpha(\tau,\zeta) \mid (\tau,\zeta) \in G, \arg(\tau) \leq \frac{1}{n}(\pi - \varphi), |\tau|^2 + |\zeta|^2 = 1\}.
\end{aligned}$$

By Proposition E.3.9, the latter $\mathcal{R}$-bound is finite because the involved set is the image under the holomorphic function $D^\beta_\xi m_\alpha : G \longrightarrow \mathcal{B}(X)$ of a compact set. This shows (6.23). □

Note that, as a byproduct of the $\mathcal{R}$-sectoriality in the above theorem, by Theorem 6.2.4 we obtain the following maximal regularity result:

Corollary 6.3.2. *Let the notations be as in the above theorem and assume that $\phi < \frac{\pi}{2}$ (so that we must have $n \in 2\mathbb{N}$).*[8] *Then $1 + A$ enjoys the property of maximal L^q_μ-regularity.*

Proof. Here we just have to note that if A is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle ϕ^R_A, then $1 + A$ is an invertible $\mathcal{R}$-sectorial operator with $\mathcal{R}$-angle $\phi^R_{1+A} \leq \phi^R_A$. □

6.3.2 Elliptic Boundary Value Problems on $\mathbb{R}^d_+$

Let $\mathcal{A}(D), \mathcal{B}_1(D), \ldots, \mathcal{B}_n(D)$ be as in (6.2).

In this subsection we will study the elliptic boundary value problem

$$\begin{aligned}
\lambda v + \mathcal{A}(D)v &= f, \\
\mathcal{B}_j(D)v &= g_j, \quad j = 1,\ldots,n,
\end{aligned} \tag{6.25}$$

on $\mathbb{R}^d_+$, assuming the ellipticity condition $(\mathrm{E})_\phi$ and the Lopatinskii-Shapiro condition $(\mathrm{LS})_\phi$. Given $f \in L^p(\mathbb{R}^d_+, w_\gamma; X)$ and $g = (g_1,\ldots,g_n) \in \prod_{j=1}^n F^{2n\kappa_{j,\gamma}}_{p,p}(\mathbb{R}^{d-1}; X)$, we look for a solution $v \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$; note that the given regularity for g is necessary by the trace result of Theorem 5.3.18.

6.3.2.a The case $f = 0$

We now turn to the elliptic boundary value problem (6.25) with $f = 0$. Our main result in this direction, Proposition 6.3.3, says that we have existence and uniqueness plus a certain representation for the solution (which will be useful for later).

Before we can state Proposition 6.3.3, we first need to introduce some notation. Given a UMD Banach space X and a natural number $k \in \mathbb{N}$, we have, for the UMD space $E = L^p(\mathbb{R}_+, |\cdot|^\gamma; X)$, the natural inclusion

$$W^k_p(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow W^k_p(\mathbb{R}^{d-1}; L^p(\mathbb{R}_+, |\cdot|^\gamma; X)) = H^k_p(\mathbb{R}^{d-1}; E)$$

[8] It actually suffices to assume that $\phi = \frac{\pi}{2}$; this then implies that $\mathcal{A}(D)$ satisfies $(\mathrm{E})_{\phi'}$ for some $\phi' < \frac{\pi}{2}$; see [73, Lemma 2.2.1].

and the natural identification

$$L^p(\mathbb{R}^d_+, w_\gamma; X) = H^0_p(\mathbb{R}^{d-1}; E).$$

By Lemma 5.2.47 we accordingly have that, for $\lambda \in \mathbb{C}\setminus]-\infty, 0]$, that the partial Fourier multiplier operator

$$L_\lambda^{k/2n} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^{d-1}; \mathcal{D}'(\mathbb{R}_+; X))),\ f \mapsto \mathscr{F}^{-1}_{x'}\left[\left(\xi' \mapsto (\lambda + |\xi'|^{2n})^{k/2n}\right)\mathscr{F}_{x'} f\right],$$

restricts to a bounded linear operator

$$L_\lambda^{k/2n} \in \mathcal{B}(W^k_p(\mathbb{R}^d_+, w_\gamma; X), L^p(\mathbb{R}^d_+, w_\gamma; X)).$$

Moreover, we even get an analytic operator-valued mapping

$$\mathbb{C}\setminus]-\infty, 0] \longrightarrow \mathcal{B}(W^k_p(\mathbb{R}^d_+, w_\gamma; X), L^p(\mathbb{R}^d_+, w_\gamma; X)),\ \lambda \mapsto L_\lambda^{k/2n}.$$

In particular, we have

$$L_\lambda^{1-\frac{n_j}{2n}}, L_\lambda^{1-\frac{n_j+1}{2n}} D_y \in \mathcal{B}(W^{2n-n_j}_p(\mathbb{R}^d_+, w_\gamma; X), L^p(\mathbb{R}^d_+, w_\gamma; X)), \qquad j = 1, \dots, n, \tag{6.26}$$

with analytic dependence on the parameter $\lambda \in \mathbb{C}\setminus]-\infty, 0]$.

Proposition 6.3.3. *Let X be a UMD Banach space, $p \in]1, \infty[$, $\gamma \in]-1, p-1[$, and assume that $(\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_n)$ satisfies $(\mathrm{E})_\phi$ and $(\mathrm{LS})_\phi$ for some $\phi \in]0, \pi[$. Then, for each $\lambda \in \Sigma_{\pi-\phi}$, there exists an operator*

$$\mathcal{S}(\lambda) = \begin{pmatrix} \mathcal{S}_1(\lambda) & \dots & \mathcal{S}_n(\lambda) \end{pmatrix} \in \mathcal{B}\left(\bigoplus_{j=1}^n F^{2n\kappa_{j,\gamma}}_{p,p}(\mathbb{R}^{d-1}; X), W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)\right)$$

which assigns to a $g \in \bigoplus_{j=1}^n F^{2n\kappa_{j,\gamma}}_{p,p}(\mathbb{R}^{d-1}; X)$ the unique solution $v = \mathcal{S}(\lambda)g \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ of the elliptic boundary value problem

$$\begin{aligned} \lambda v + \mathcal{A}(D)v &= 0, \\ \mathcal{B}^{tr}_j(D)v &= g_j, \quad j = 1, \dots, n; \end{aligned} \tag{6.27}$$

recall here that $\kappa_{j,\gamma} = 1 - \frac{n_j}{2n} - \frac{1}{2np}(1+\gamma)$. Moreover, for each $j \in \{1, \dots, n\}$ we have that

$$\tilde{\mathcal{S}}_j : \Sigma_{\pi-\phi} \longrightarrow \mathcal{B}(W^{2n-n_j}_p(\mathbb{R}^d_+, w_\gamma; X), W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)),\ \lambda \mapsto \tilde{\mathcal{S}}_j(\lambda) := \mathcal{S}_j(\lambda) \circ \mathrm{tr}_{y=0}$$

defines an analytic mapping, for which the operators $D^\alpha \tilde{\mathcal{S}}_j(\lambda) \in \mathcal{B}(W^{2n-n_j}_p(\mathbb{R}^d_+, w_\gamma; X), L^p(\mathbb{R}^d_+, w_\gamma; X))$, $|\alpha| \leq 2n$, can be represented as

$$D^\alpha \tilde{\mathcal{S}}_j(\lambda) = \mathcal{T}^1_{j,\alpha}(\lambda) L_\lambda^{1-\frac{n_j}{2n}} + \mathcal{T}^2_{j,\alpha}(\lambda) L_\lambda^{1-\frac{n_j+1}{2n}} D_y \tag{6.28}$$

for analytic operator-valued mappings

$$\mathcal{T}^i_{j,\alpha} : \Sigma_{\pi-\phi} \longrightarrow \mathcal{B}(L^p(\mathbb{R}^d_+, w_\gamma; X)),\ \lambda \mapsto \mathcal{T}^i_{j,\alpha}(\lambda), \qquad i \in \{1, 2\}, \tag{6.29}$$

satisfying the $\mathcal{R}$-bounds

$$\mathcal{R}\{\lambda^{k+1-\frac{|\alpha|}{2n}} \partial^k_\lambda \mathcal{T}^i_{j,\alpha}(\lambda) \mid \lambda \in \Sigma_{\pi-\phi}\} < \infty, \qquad \forall k \in \mathbb{N}. \tag{6.30}$$

The goal of this subsubsection is to prove this proposition. Accordingly, we let the notations and assumptions be as in this proposition for the rest of this subsubsection.

In order to get an idea how to construct the solution operator $\mathcal{S}(\lambda)$ for $\lambda \in \Sigma_{\pi-\phi}$, suppose we have a solution $v \in \mathcal{S}(\mathbb{R}^d_+; X)$ of (6.27) for given $g = (g_1, \dots, g_n) \in \mathcal{S}(\mathbb{R}^{d-1}; X)^n$. This just means that

$$\begin{aligned} \lambda v(y,x') + \mathcal{A}(D)v(y,x') &= 0, && (y,x') \in \mathbb{R}^d_+, \\ \mathcal{B}_j(D)v(0,x') &= g_j(x'), && x' \in \mathbb{R}^{d-1}, \quad j = 1,\dots,n. \end{aligned} \tag{6.31}$$

Taking the partial Fourier transform $\mathscr{F}_{x'}$ with respect to x', we obtain

$$\begin{aligned} \lambda \mathscr{F}_{x'}v(y,x') + \mathcal{A}(D_y,\xi')\mathscr{F}_{x'}v(y,x') &= 0, && (y,\xi') \in \mathbb{R}^d_+, \\ \mathcal{B}_j(D_y,\xi')\mathscr{F}_{x'}v(y,\xi')|_{y=0} &= \mathscr{F}g_j(\xi'), && \xi' \in \mathbb{R}^{d-1}, \quad j = 1,\dots,n. \end{aligned}$$

This motivates to study, for each fixed $\xi' \in \mathbb{R}^{d-1}$ and $h = (h_1, \dots, h_n) \in X^n$, the ordinary initial value problem (6.20).

Study of the ordinary initial value problem (6.20)**:** Let $\lambda \in \overline{\Sigma}_{\pi-\phi}$ and $\xi' \in \mathbb{R}^{d-1}$ with $(\lambda, \xi') \neq 0$ be given.[9] Recall that the Lopatinskii-Shapiro condition $(\mathrm{LS})_\phi$ says that the ordinary initial value problem (6.20) has a unique solution $w \in C_0^\infty([0,\infty[;X)$ (for any $h = (h_1,\dots,h_n) \in X^n$). We shall rewrite (6.20) into a system of first order equations which allows us to get a representation formula for w.

We write

$$\begin{aligned} \mathcal{A}(D_y,\xi') &= \sum_{k=0}^{2n} \sum_{|\alpha'|=k} a_{(k,\alpha')}(\xi')^{\alpha'} D_y^{2n-k} \\ &= \sum_{k=0}^{2n} a_k(\xi') D_y^{2n-k}, \end{aligned}$$

where $a_k(\xi') := \sum_{|\alpha'|=k} a_{k,\alpha'}(\xi')^{\alpha'}$, and

$$\begin{aligned} \mathcal{B}_j(D_y,\xi') &= \sum_{k=0}^{n_j} \sum_{|\alpha'|=k} b_{j,(k,\alpha')}(\xi')^{\alpha'} D_y^{n_j-k} \\ &= \sum_{k=0}^{n_j} b_{j,k}(\xi') D_y^{n_j-k}, \end{aligned}$$

where $b_{j,k}(\xi') := \sum_{|\alpha'|=k} b_{j,(k,\alpha')}(\xi')^{\alpha'}$. Then, since $a_0 = a_0(\xi') \in \mathcal{B}(E)$ is invertible (by the ellipticity assumption $(\mathrm{E})_\phi$ on $\mathcal{A}(D)$ as $a_0 = \mathcal{A}(1,0,\dots,0)$), (6.20) can be rewritten as

$$\begin{aligned} D_y^{2n}w_y + \textstyle\sum_{k=1}^{2n-1} a_0^{-1}a_k(\xi')D_y^{2n-k}w(y) + a_0^{-1}(\lambda + a_{2n}(\xi'))w(y) &= 0, \quad y > 0 \\ \textstyle\sum_{k=0}^{n_j} b_{j,k}(\xi')D_y^{n_j-k}w(y)|_{y=0} &= h_j, \quad j = 1,\dots,n. \end{aligned} \tag{6.32}$$

For later it will be convenient to do the following rescaling: For $\rho \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ we set

$$\sigma := \frac{\lambda}{\rho^{2n}} \quad \text{and} \quad b := \frac{\xi'}{\rho}. \tag{6.33}$$

[9] We consider such (λ,ξ') instead of just $(\lambda,\xi') \in \Sigma_{\pi-\phi}\times\mathbb{R}^{d-1}$ for technical reasons needed to obtain the estimate (6.39) (based on continuity and compactness).

Then, as a_k and $b_{j,k}$ are homogeneous of degree k, (6.32) can be written as

$$\begin{aligned} D_y^{2n}w(y) + \textstyle\sum_{k=1}^{2n-1} a_0^{-1}a_k(b)\rho^k D_y^{2n-k}w(y) + a_0^{-1}(\sigma + a_{2n}(b))\rho^{2n}w(y) &= 0, \quad y > 0 \\ \textstyle\sum_{k=0}^{n_j} b_{j,k}(b)\rho^k D_y^{n_j-k}w(y)|_{y=0} &= h_j, \quad j = 1, \dots, n. \end{aligned} \tag{6.34}$$

Writing

$$A_0(\sigma, b) := \begin{pmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & & 0 & I \\ c_{2n}(\sigma, b) & c_{2n-1}(b) & \dots & & c_1(b) \end{pmatrix} \in M_{2n\times 2n}(\mathcal{B}(X)),$$

where

$$\begin{aligned} c_j(b) &:= -a_0^{-1}a_j(b), \qquad j = 1, \dots, 2n-1, \\ c_{2n}(\sigma, b) &:= -a_0^{-1}(\sigma + a_{2n}(b)), \end{aligned}$$

and

$$B_j^0(b) := \begin{pmatrix} b_{j,n_j}(b) & \dots & b_{j,0} & 0 & \dots & 0 \end{pmatrix} \in M_{2n\times 1}(\mathcal{B}(X)), \quad j = 1, \dots, n,$$

(6.34) is equivalent with the first order ordinary initial value problem

$$\begin{aligned} \partial_y \underline{w}^\rho(y) &= \imath\rho A_0(\sigma, b)\underline{w}^\rho(y), \quad y > 0, \\ B_j^0(b)\underline{w}^\rho(0) &= \tfrac{h_j}{\rho^{n_j}}, \qquad j = 1, \dots, n, \end{aligned} \tag{6.35}$$

where the equivalence is given via the correspondence

$$w \leftrightarrow \underline{w}^\rho = (w, \rho^{-1}D_y w, \dots, \rho^{-(2n-1)}D_y^{2n-1}w).$$

Let's take a look at all (σ, b) which can be obtained via (6.33). For $\omega \in \mathbb{T}$ we define

$$\Upsilon_\omega := \left\{(\sigma, b) \in (\mathbb{C} \times \mathbb{C}^{d-1}) \setminus \{0\} \mid \arg(\omega^{2n}\sigma) \le \phi, \omega b \in \mathbb{R}^d\right\}.$$

Then

$$\Upsilon := \bigcup_{\omega\in\mathbb{T}} \Upsilon_\omega = \left\{(\sigma, b) \in \mathbb{C} \times \mathbb{C}^{d-1} \mid \exists(\lambda, \xi') \in \left[\overline{\Sigma}_{\pi-\phi} \times \mathbb{R}^{d-1}\right] \setminus \{0\}, \rho \in \mathbb{C}^* \text{ s.t. } \sigma = \rho^{-2n}\lambda, b = \rho^{-1}\xi'\right\}. \tag{6.36}$$

For $(\sigma, b) \in \Upsilon$ we write $\omega(\sigma, b)$ for the unique $\omega \in \mathbb{T}$ with $(\sigma, b) \in \Upsilon_\omega$.

Remark 6.3.4. Let $(\lambda, \xi') \in \left[\overline{\Sigma}_{\pi-\phi} \times \mathbb{R}^{d-1}\right] \setminus \{0\}$ and $\rho = \omega|\rho| \in \mathbb{C}^*$ with $\omega \in \mathbb{T}$. Then for (σ, b) as defined in (6.33) we have $\omega(\sigma, b) = \omega$. As a consequence, $\imath\rho A_0(\sigma, b) = \imath\omega(\sigma, b)A_0(\sigma, b)|\rho|$.

In the above notation we have:

Lemma 6.3.5. *For $(\sigma, b) \in \Upsilon$ it holds that $\sigma(\imath\omega(\sigma, b)A_0(\sigma, b)) \cap \imath\mathbb{R} = \emptyset$.*

Proof. This can be shown as in [25, Proposition 6.1]. □

This lemma implies that

$$\sigma(\imath\omega(\sigma, b)A_0(\sigma, b)) = \underbrace{[\sigma(\imath\omega(\sigma, b)A_0(\sigma, b)) \cap \{\Re(z) < 0\}]}_{=:S_-(\sigma,b)} \cup \underbrace{[\sigma(\imath\omega(\sigma, b)A_0(\sigma, b)) \cap \{\Re(z) > 0\}]}_{=:S_+(\sigma,b)}$$

for $(\sigma, b) \in \Upsilon$. By compactness and continuity, given $R, r > 0$ there there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned} S_-(\sigma, b) &\subset \{\Re(z) \leq -c_1\}, \quad (\sigma, b) \in \Upsilon, |\sigma| + |b|^{2n} \in [r, R] \\ S_+(\sigma, b) &\subset \{\Re(z) \geq c_2\}, \quad (\sigma, b) \in \Upsilon, |\sigma| + |b|^{2n} \in [r, R] \end{aligned} \tag{6.37}$$

We denote by $P_\pm(\sigma, b)$ the spectral projection associated to $S_\pm(\sigma, b)$. Then we have the decomposition

$$X^{2n} = P_-(\sigma, b)X^{2n} \oplus P_+(\sigma, b)X^{2n}, \tag{6.38}$$

which is invariant under $\imath\omega(\sigma, b)A_0(\sigma, b)$. Moreover, it holds that

$$\begin{aligned} \left\| e^{\imath\omega(\sigma,b)A_0(\sigma,b)t}x \right\| &\leq e^{-c_1 t}\|x\|, \quad x \in P_-(\sigma, b)X^{2n}, \quad t \geq 0, \quad (\sigma, b) \in \Upsilon, |\sigma| + |b|^{2n} \in [r, R] \\ \left\| e^{\imath\omega(\sigma,b)A_0(\sigma,b)t}x \right\| &\geq e^{c_2 t}\|x\|, \quad x \in P_+(\sigma, b)X^{2n}, \quad t \geq 0, \quad (\sigma, b) \in \Upsilon, |\sigma| + |b|^{2n} \in [r, R]. \end{aligned} \tag{6.39}$$

Proposition 6.3.6. *Given $(\lambda, \xi') \in \left[\overline{\Sigma}_{\pi-\phi} \times \mathbb{R}^{d-1}\right] \setminus \{0\}$, $\rho = \omega|\rho| \in \mathbb{C}^*$ and $h = (h_1, \ldots, h_n) \in X^n$, let (σ, b) be as in* (6.33) *and set*

$$h^\rho := \left(\frac{h_1}{\rho^{n_1}}, \ldots, \frac{h_n}{\rho^{n_n}}\right).$$

Then (6.35) *has a unique solution $\underline{w}^\rho \in C^\infty([0, \infty[; X^{2n})$, which is given by*

$$\underline{w}^\rho(y) = e^{\imath\rho A_0(\sigma,b)y}\underline{w}^\rho_0, \tag{6.40}$$

where $\underline{w}^\rho_0 \in X^{2n}$ is the unique solution of

$$\begin{aligned} P_+(\sigma, b)\underline{w}^\rho_0 &= 0, \\ B^0_j(b)\underline{w}^\rho_0 &= h^\rho_j, \quad j = 1, \ldots, n. \end{aligned} \tag{6.41}$$

Moreover,

$$\underline{w}^\rho_0 = M(\sigma, b)h^\rho$$

for some holomorphic mapping $\Upsilon \ni (\sigma, b) \mapsto M(\sigma, b) \in \mathcal{B}(X, X^{2n})$.

Proof. By the Lopatinskii-Shapiro condition $(\mathrm{LS})_\phi$ and the equivalence between (6.20) and (6.35), (6.35) has a unique solution $\underline{w}^\rho \in C^\infty([0, \infty[; X^{2n})$. This solution must of course be of the form (6.40) for some unique $\underline{w}^\rho_0 \in X^{2n}$. By the decomposition (6.38), Remark 6.3.4 and the estimates (6.39) (take $r = R = |\sigma| + |b|^{2n}$), $\underline{w}^\rho_0 \in X^{2n}$ must be the unique solution of (6.41). Letting $M(\sigma, b) \in \mathcal{B}(X, X^{2n})$ ($h^\rho \mapsto M(\sigma, b)h^\rho$) be the solution map for (6.41), we have $\underline{w}^\rho_0 = M(\sigma, b)h^\rho$. That $\Upsilon \ni (\sigma, b) \mapsto M(\sigma, b) \in \mathcal{B}(X, X^{2n})$ is holomorphic can be shown as in [25, Proposition 6.2]. □

We will use the following choice of ρ when we come back to (6.31):

Example 6.3.7. For $(\lambda, \xi') \in \left[\overline{\Sigma}_{\pi-\phi} \times \mathbb{R}^{d-1}\right] \setminus \{0\}$ we define

$$\rho_\lambda(\xi') := (\lambda + |\xi'|^{2n})^{1/2n} \in \mathbb{C}^* \tag{6.42}$$

and

$$\sigma_\lambda(\xi') := \frac{\lambda}{\rho_\lambda(\xi')}, \quad b_\lambda(\xi') := \frac{\xi'}{\rho_\lambda(\xi')}. \tag{6.43}$$

Then there exits an $R > 1$, independent of λ and ξ', such that

$$|\sigma_\lambda(\xi')| + |b_\lambda(\xi')|^{2n} \in [1, R]. \tag{6.44}$$

By Remark 6.3.4 and (6.39) there consequently exists a constant $c > 0$, independent of λ and ξ', such that

$$\left\| e^{\imath\rho_\lambda(\xi')A_0(\sigma_\lambda(\xi'), b_\lambda(\xi'))t} x \right\| \leq e^{-c|\rho_\lambda(\xi')|t} \|x\|, \quad x \in P_-(\sigma_\lambda(\xi'), b_\lambda(\xi'))X^{2n}, t \geq 0. \tag{6.45}$$

Proof. By the triangle inequality we have

$$|\sigma_\lambda(\xi')| + |b_\lambda(\xi')|^{2n} = \frac{|\lambda|}{|\lambda + |\xi'|^{2n}|} + \frac{|\xi'|^{2n}}{|\lambda + |\xi'|^{2n}|} \geq \frac{|\lambda| + |\xi'|^{2n}}{|\lambda| + |\xi'|^{2n}} = 1.$$

For the upper bound we may assume that $\lambda \neq 0$; just note that $|\sigma_\lambda(\xi')| + |b_\lambda(\xi')|^{2n} = 1$ in case $\lambda = 0$. Then we have

$$|\sigma_\lambda(\xi')| + |b_\lambda(\xi')|^{2n} = \frac{1 + \frac{|\xi'|^{2n}}{|\lambda|}}{\frac{\lambda}{|\lambda|} + \frac{|\xi'|^{2n}}{|\lambda|}} = \frac{1+t}{|e^{\imath\theta} + t|},$$

where $t := \frac{|\xi'|^{2n}}{|\lambda|} \geq 0$ and $e^{\imath\theta} := \frac{\lambda}{|\lambda|}$, $\theta \in [-(\pi - \phi), \pi - \phi] \subset]-\pi, \pi[$. If $t \geq 2$, then

$$\frac{1+t}{|e^{\imath\theta} + t|} \leq \frac{1+t}{t-1} = \frac{2}{t-1} + 1 \leq 3,$$

and, if $t \leq 2$, then

$$\frac{1+t}{|e^{\imath\theta} + t|} \leq \frac{3}{|e^{\imath\theta} + t|} \leq \frac{3}{\Im(e^{\imath(\pi-\phi)})} = \frac{3}{\sin(\pi - \phi)}.$$

This shows that $|\sigma_\lambda(\xi')| + |b_\lambda(\xi')|^{2n} \leq R$ for the constant $R := \frac{3}{\sin(\pi-\phi)} > 1$. □

Representation formulas for the solution of (6.27)**:** Having solved the ordinary initial value problem (6.20), we now go back to the elliptic boundary value problem (6.27).

Fix $\lambda \in \Sigma_{\pi-\phi}$. In order to construct the solution operator $\mathcal{S}(\lambda)$ from Proposition 6.3.3, we will construct an operator

$$\begin{gathered} \tilde{\mathcal{S}}(\lambda) = \begin{pmatrix} \tilde{\mathcal{S}}_1(\lambda) & \dots & \tilde{\mathcal{S}}_n(\lambda) \end{pmatrix} \in \mathcal{B}\left(\bigoplus_{j=1}^n W_p^{2n-n_j}(\mathbb{R}^d_+, w_\gamma; X), W_p^{2n}(\mathbb{R}^d_+, w_\gamma; X)\right) \\ \text{with the property that} \\ \forall \tilde{g} \in \bigoplus_{j=1}^n W_p^{2n-n_j}(\mathbb{R}^d_+, w_\gamma; X) : \quad \tilde{\mathcal{S}}(\lambda)\tilde{g} \text{ solves (6.27) with } g = \mathrm{tr}_{y=0}\tilde{g}. \end{gathered} \tag{6.46}$$

Then, given extension operators $\mathcal{E}_j : F_{p,p}^{2n\kappa_{j,\gamma}}(\mathbb{R}^{d-1}; X) \longrightarrow W_p^{2n-n_j}(\mathbb{R}^d_+, w_\gamma; X)$, $j = 1, \dots, n$, as in Theorem 5.3.18, i.e.

$$\left.\begin{gathered} \mathcal{E}_j \in \mathcal{B}(F_{p,p}^{2n\kappa_{j,\gamma}}(\mathbb{R}^{d-1}; X), W_p^{2n-n_j}(\mathbb{R}^d_+, w_\gamma; X)) \\ \text{is a right-inverse of the trace operator} \\ \mathrm{tr}_{y=0} \in \mathcal{B}(W_p^{2n-n_j}(\mathbb{R}^d_+, w_\gamma; X), F_{p,p}^{2n\kappa_{j,\gamma}}(\mathbb{R}^{d-1}; X)) \end{gathered}\right. \qquad j = 1, \dots, n, \tag{6.47}$$

the composition $\mathcal{S}(\lambda) := \tilde{\mathcal{S}}(\lambda) \circ (\mathcal{E}_1, \dots, \mathcal{E}_n)$ will be a solution operator for (6.27).

Let $\tilde{g} \in \mathcal{S}(\mathbb{R}^d_+; X)^n$ and write $g := \mathrm{tr}_{y=0}\tilde{g} \in \mathcal{S}(\mathbb{R}^{d-1}; X)^n$. For every $\xi' \in \mathbb{R}^{d-1}$ we define

$$h_{\xi'} := (\mathscr{F} g_1(\xi'), \dots, \mathscr{F} g_n(\xi')) \in X^n,$$

let $\rho(\xi') = \rho_\lambda(\xi')$, $\sigma(\xi') = \sigma_\lambda(\xi')$ and $b(\xi') = b_\lambda(\xi')$ as in Example 6.3.7. Furthermore, we let $\underline{w}^{\rho(\xi')} \in C^\infty([0,\infty[;X^{2n})$ be the unique solution of (6.35) (corresponding to $h = h_{\xi'}$ and $\rho = \rho(\xi')$); recall that, by the equivalence between (6.20) and (6.35), $\pi_1\underline{w}^{\rho(\xi')} \in C^\infty([0,\infty[;X)$ then is the unique solution of (6.20), where $\pi_1 : X^{2n} \longrightarrow X$ denotes the canonical projection onto the first coordinate. By Proposition 6.3.6, this solution can be represented as

$$\underline{w}^{\rho(\xi')}(y) = e^{\imath\rho(\xi')A_0(\sigma(\xi'),b(\xi'))y}M(\sigma(\xi'),b(\xi'))\begin{pmatrix}\rho(\xi')^{-n_1}\mathscr{F}g_1(\xi')\\ \vdots \\ \rho(\xi')^{-n_n}\mathscr{F}g_n(\xi')\end{pmatrix}. \tag{6.48}$$

From this formula, (6.45), the fact that $M(\sigma,b)$ maps into $P_-(\sigma,b)X^{2n}$ (see Proposition 6.3.6), the fact that $(\sigma,b)\mapsto A_0(\sigma,b)$ and $(\sigma,b)\mapsto M(\sigma,b)$ are continuous, (6.44), and the observation that $\xi'\mapsto\rho(\xi')^{-n_j}\mathscr{F}g_j(\xi')$ belongs to $\mathcal{S}(\mathbb{R}^{d-1};X)$ ($j=1,\ldots,n$), it easily follows that the function

$$\phi : [0,\infty[\times\mathbb{R}^{d-1} \longrightarrow X^{2n},\ (y,\xi')\mapsto \underline{w}^{\rho(\xi')}(y) \tag{6.49}$$

has the property that, for each $k\in\mathbb{N}$, $\xi'\mapsto D_y^k\phi(y,\xi')$ is rapidly decreasing uniformly in $y\in[0,\infty[$. Hence, we may take the inverse partial Fourier transform $\mathscr{F}_{x'}^{-1}$ with respect to ξ', to obtain

$$\mathscr{F}_{x'}^{-1}\phi \in C_b^\infty([0,\infty[\times\mathbb{R}^{d-1};X^{2n}) \quad\text{with}\quad (D^\alpha\mathscr{F}_{x'}^{-1}\phi)(y,x') = \mathscr{F}^{-1}[\xi'\mapsto(\xi')^{\alpha'}D_y^{\alpha_1}\phi(y,\xi')](x'), \forall\alpha\in\mathbb{N}^d.$$

Since $y\mapsto\pi_1\phi(y,\xi')$ solves the ordinary initial value problem (6.20) for $h = h_{\xi'} = (\mathscr{F}g_1(\xi'),\ldots,\mathscr{F}g_n(\xi'))\in X^n$ by construction, it follows that

$$v := \pi_1\mathscr{F}_{x'}^{-1}\phi = \mathscr{F}_{x'}^{-1}\pi_1\phi \in C_b^\infty([0,\infty[\times\mathbb{R}^{d-1};X)$$

satisfies

$$\begin{array}{rlll}\lambda v(y,x') + \mathcal{A}(D)v(y,x') & = \mathscr{F}^{-1}[\xi'\mapsto(\lambda+\mathcal{A}(D_y,\xi'))\pi_1\phi(y,\xi')](x') & = 0, & (y,x')\in\mathbb{R}^d_+,\\ \mathcal{B}_1(D)v(0,x') & = \mathscr{F}^{-1}[\xi'\mapsto\mathcal{B}_1(D_y,\xi')\pi_1\phi(y,\xi')|_{y=0}](x') & = g_1(x'), & x'\in\mathbb{R}^{d-1},\\ \vdots & \quad\vdots & \quad\vdots & \\ \mathcal{B}_n(D)v(0,x') & = \mathscr{F}^{-1}[\xi'\mapsto\mathcal{B}_n(D_y,\xi')\pi_1\phi(y,\xi')|_{y=0}](x') & = g_n(x'), & x'\in\mathbb{R}^{d-1}.\end{array}$$

So, recalling the formula given in (6.48) and the definition of ϕ given in (6.49), in order to establish the existence of an operator $\tilde{\mathcal{S}}(\lambda)$ as in (6.46), it suffices to show that the linear operator $\tilde{\mathcal{S}}(\lambda) : \mathcal{S}(\mathbb{R}^d_+;X)^n \longrightarrow C_b^\infty(]0,\infty[\times\mathbb{R}^{d-1};X) \hookrightarrow \mathcal{D}'(\mathbb{R}^d_+;X)$ given by

$$\tilde{\mathcal{S}}(\lambda)\tilde{g} := \pi_1\mathscr{F}_{x'}^{-1}\left[(y,\xi')\mapsto e^{\imath\rho(\xi')A_0(\sigma(\xi'),b(\xi'))y}M(\sigma(\xi'),b(\xi'))\begin{pmatrix}\rho(\xi')^{-n_1}[\mathscr{F}\tilde{g}_1(0,\cdot)](\xi')\\ \vdots\\ \rho(\xi')^{-n_n}[\mathscr{F}\tilde{g}_n(0,\cdot)](\xi')\end{pmatrix}\right] \tag{6.50}$$

takes its values in $W_p^{2n}(\mathbb{R}^d_+,w_\gamma;X)$ and satisfies, for each $|\alpha|\leq 2n$, the norm estimate

$$||D^\alpha\mathcal{S}(\lambda)\tilde{g}||_{L^p(\mathbb{R}^d_+,w_\gamma;X)} \lesssim ||\tilde{g}||_{\prod_{j=1}^n W^{2n-n_j}(\mathbb{R}^d_+,w_\gamma;X)} \qquad (\tilde{g}\in\mathcal{S}(\mathbb{R}^d_+;X)^n); \tag{6.51}$$

then, by denseness of $\mathcal{S}(\mathbb{R}^d_+;X)^n$ in $\bigoplus_{j=1}^n W^{2n-n_j}\mathbb{R}^d_+,w_\gamma;X)$, continuity of $\mathcal{A}(D)$ from $W_p^{2n}(\mathbb{R}^d_+,w_\gamma;X)$ to $L^p(\mathbb{R}^d_+,w_\gamma;X)$ and continuity of $\mathcal{B}_j^{tr}(D)$ from $W_p^{2n}(\mathbb{R}^d_+,w_\gamma;X)$ to $F_{p,p}^{2n\kappa_{j,\gamma}}(\mathbb{R}^{d-1};X)$ ($j=1,\ldots,n$),

$\tilde{\mathcal{S}}(\lambda)$ extends to a bounded linear operator

$$\tilde{\mathcal{S}}(\lambda) : \bigoplus_{j=1}^{n} W_p^{2n-n_j}(\mathbb{R}^d_+, w_\gamma; X) \longrightarrow W_p^{2n}(\mathbb{R}^d_+, w_\gamma; X)$$

which is as in (6.46).

Recall that, for each $\alpha \in \mathbb{N}^d$ and all $g \in \mathcal{S}(\mathbb{R}^{d-1}; X)^n$, we have

$$D^\alpha \tilde{\mathcal{S}}(\lambda)\tilde{g} = \pi_1 \mathscr{F}_{x'}^{-1}\left[m_{\lambda,\alpha}\mathcal{F}_\rho \tilde{g}\right],$$

where $m_{\lambda,\alpha} : [0,\infty[\times\mathbb{R}^{d-1} \longrightarrow \mathcal{B}(X^n, X^{2n})$ is given by

$$m_{\lambda,\alpha}(y,\xi') := (\xi')^{\alpha'}[\rho(\xi')A_0(\sigma(\xi'),b(\xi'))]^{\alpha_1} e^{\imath\rho(\xi')A_0(\sigma(\xi'),b(\xi'))y} M(\sigma(\xi'),b(\xi'))\rho(\xi')^{-2n} \tag{6.52}$$

and where

$$\mathcal{F}_\rho\tilde{g} := (\rho^{2n-n_1}\mathscr{F}[\tilde{g}_1(0,\cdot)],\ldots,\rho^{2n-n_n}\mathscr{F}[\tilde{g}_n(0,\cdot)]) \in \mathcal{S}(\mathbb{R}^{d-1}; X^n).$$

From (6.45), the fact that $M(\sigma,b)$ maps into $P_-(\sigma,b)X^{2n}$ (see Proposition 6.3.6), the fact that $(\sigma,b)\mapsto A_0(\sigma,b)$ and $(\sigma,b)\mapsto M(\sigma,b)$ are continuous, and (6.44), it follows that $m_{\lambda,\alpha}$ has the property that, for each $k\in\mathbb{N}$ and $\delta>0$, $\xi'\mapsto D_y^k m_{\lambda,\alpha}(y,\xi')$ is rapidly decreasing uniformly for $y\in[\delta,\infty[$. Therefore, we have $K_{\lambda,\alpha} := \mathscr{F}_x^{-1} m_{\lambda,\alpha}|_{]0,\infty[\times\mathbb{R}^d} \in C^\infty(]0,\infty[\times\mathbb{R}^{d-1}; \mathcal{B}(X^n,X^{2n}))$ with bounded partial derivatives on the sets $[\delta,\infty[\times\mathbb{R}^{d-1}$, $\delta>0$. Since $K_{\lambda,\alpha}$ is given by the formula

$$K_{\lambda,\alpha}(y,x') = (2\pi)^{-(d-1)}\int_{\mathbb{R}^{d-1}} e^{\imath x'\cdot\xi'} m_{\lambda,\alpha}(y,\xi')d\xi',$$

we have

$$[D^\alpha\tilde{\mathcal{S}}(\lambda)\tilde{g}](y,x') = \pi_1\int_{\mathbb{R}^{d-1}} K_{\lambda,\alpha}(y,x'-\tilde{x}')\left[(L_\lambda^{1-\frac{n_1}{2n}},\ldots,L_\lambda^{1-\frac{n_n}{2n}})\tilde{g}\right](0,\tilde{x}')d\tilde{x}' \qquad ((y,x')\in]0,\infty[\times\mathbb{R}^{d-1}).$$

Integrating by parts and applying Fubini, this can be written as

$$\begin{aligned}[D^\alpha\tilde{\mathcal{S}}(\lambda)\tilde{g}](y,x') &= -\imath\pi_1\int_0^\infty\int_{\mathbb{R}^{d-1}} D_{\tilde{y}}K_{\lambda,\alpha}(y+\tilde{y},x'-\tilde{x}')\left[(L_\lambda^{1-\frac{n_1}{2n}},\ldots,L_\lambda^{1-\frac{n_n}{2n}})\tilde{g}\right](\tilde{y},\tilde{x}')d\tilde{x}'d\tilde{y}\\ &\quad -\imath\pi_1\int_0^\infty\int_{\mathbb{R}^{d-1}} K_{\lambda,\alpha}(y+\tilde{y},x'-\tilde{x}')D_{\tilde{y}}\left[(L_\lambda^{1-\frac{n_1}{2n}},\ldots,L_\lambda^{1-\frac{n_n}{2n}})\tilde{g}\right](\tilde{y},\tilde{x}')d\tilde{x}'d\tilde{y}\end{aligned}$$

Finally, defining $K^1_{\lambda,\alpha}, K^2_{\lambda,\alpha} \in C^\infty(]0,\infty[\times\mathbb{R}^{d-1}; \mathcal{B}(X^n,X^{2n}))$ by

$$K^1_{\lambda,\alpha}(y,x') := -\imath D_y K_{\lambda,\alpha}(y,x) = -\imath(2\pi)^{-(d-1)}\int_{\mathbb{R}^{d-1}} e^{\imath x'\cdot\xi'} m_{\lambda,\alpha}(y,\xi')d\xi' \tag{6.53}$$

and

$$K^2_{\lambda,\alpha}(y,x') := -\imath(2\pi)^{-(d-1)}\int_{\mathbb{R}^{d-1}} e^{\imath x'\cdot\xi'} D_y m_{\lambda,\alpha}(y,\xi')\rho(\xi')d\xi', \tag{6.54}$$

respectively, we obtain the representation formula

$$\begin{aligned}[D^\alpha\tilde{\mathcal{S}}(\lambda)\tilde{g}](y,x') = {}& \pi_1\textstyle\int_0^\infty\int_{\mathbb{R}^{d-1}} K^1_{\lambda,\alpha}(y+\tilde{y},x'-\tilde{x}')\left[(L_\lambda^{1-\frac{n_1}{2n}},\ldots,L_\lambda^{1-\frac{n_n}{2n}})\tilde{g}\right](\tilde{y},\tilde{x}')d\tilde{x}'d\tilde{y}\\ & \pi_1\textstyle\int_0^\infty\int_{\mathbb{R}^{d-1}} K^2_{\lambda,\alpha}(y+\tilde{y},x'-\tilde{x}')D_{\tilde{y}}\left[(L_\lambda^{1-\frac{n_1+1}{2n}},\ldots,L_\lambda^{1-\frac{n_n+1}{2n}})\tilde{g}\right](\tilde{y},\tilde{x}')d\tilde{x}'d\tilde{y}.\\ = {}& \pi_1\textstyle\int_0^\infty\int_{\mathbb{R}^{d-1}} K^1_{\lambda,\alpha}(y+\tilde{y},x'-\tilde{x}')\left[(L_\lambda^{1-\frac{n_1}{2n}},\ldots,L_\lambda^{1-\frac{n_n}{2n}})\tilde{g}\right](\tilde{y},\tilde{x}')d\tilde{x}'d\tilde{y}\\ & \pi_1\textstyle\int_0^\infty\int_{\mathbb{R}^{d-1}} K^2_{\lambda,\alpha}(y+\tilde{y},x'-\tilde{x}')\left[(L_\lambda^{1-\frac{n_1+1}{2n}},\ldots,L_\lambda^{1-\frac{n_n+1}{2n}})D_y\tilde{g}\right](\tilde{y},\tilde{x}')d\tilde{x}'d\tilde{y}.\end{aligned} \tag{6.55}$$

Bounds for the solution operator of (6.27)**:** In order to obtain the norm estimates (6.51), the boundedness (6.26) and the representation formula (6.55) motivate to look at conditions on kernels $K : \mathbb{R}^d_+ \longrightarrow \mathcal{B}(Y, Z)$ for which the associated integral operator T_K of the form

$$T_K f(y, x') := \int_0^\infty \int_{\mathbb{R}^{d-1}} K(y + \tilde{y}, x - \tilde{x}') f(\tilde{y}, \tilde{x}') d\tilde{x}' d\tilde{y} \qquad ((y, x') \in \mathbb{R}^d_+) \tag{6.56}$$

is a well defined bounded linear operator $L^p(\mathbb{R}^d_+, w_\gamma, Y) \longrightarrow L^p(\mathbb{R}^d_+, w_\gamma, Z)$.

Lemma 6.3.8. *Let Y, Z be two Banach spaces. Suppose that $K : \mathbb{R}^d_+ \longrightarrow \mathcal{B}(Y, Z)$ is a strongly measurable function for which there exists a constant $M > 0$ such that*

$$\|K(\,\cdot\,, y)\|_{L^1(\mathbb{R}^{d-1}, \mathcal{B}(Y,Z))} \le \frac{M}{y}, \quad a.e.\ y > 0. \tag{6.57}$$

Then the formula (6.56) *gives rise to a well defined bounded linear operator T_K from $L^p(\mathbb{R}^d_+, w_\gamma; Y)$ to $L^p(\mathbb{R}^d_+, w_\gamma; Z)$ of norm*

$$\|T_K\|_{\mathcal{B}(L^p(\mathbb{R}^d_+, w_\gamma; Y), L^p(\mathbb{R}^d_+, w_\gamma; Z))} \lesssim_{p,\gamma} M.$$

Proof. Fix $f \in L^p(\mathbb{R}^d_+, w_\gamma, X)$. By Young's inequality (cf. A.1.4) and (6.57), for a.e. $y, \tilde{y} > 0$ it holds that

$$\begin{aligned} \left\| x' \mapsto \int_{\mathbb{R}^{d-1}} \|K(y + \tilde{y}, x - \tilde{x}') f(\tilde{y}, \tilde{x}')\|_X \, d\tilde{x}' \right\|_{L^p(\mathbb{R}^{d-1})} &\le \|K(y + \tilde{y}, \,\cdot\,)\|_{L^1(\mathbb{R}^{d-1}; \mathcal{B}(X))} \|f(\tilde{y}, \,\cdot\,)\|_{L^p(\mathbb{R}^{d-1}; X)} \\ &\le \frac{M}{y + \tilde{y}} \|f(\tilde{y}, \,\cdot\,)\|_{L^p(\mathbb{R}^{d-1} X)} . \end{aligned}$$

Now note that Lemma D.2.9 in particular yields

$$(y, x') \mapsto \int_{\mathbb{R}^{d-1}} \|K(y + \tilde{y}, x - \tilde{x}') f(\tilde{y}, \tilde{x}')\|_X \, d\tilde{x}' \in L^1(\mathbb{R}_+)[L^p(\mathbb{R}^{d-1})] = L^{(p,1),(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+)$$

for a.e. $y > 0$. Since $L^1(\mathbb{R}_+)[L^p(\mathbb{R}^{d-1})] \hookrightarrow L^p(\mathbb{R}^{d-1})[L^1(\mathbb{R}_+)]$ contractively, it follows that

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^{d-1}} \left[\int_0^\infty \int_{\mathbb{R}^{d-1}} \|K(y + \tilde{y}, x - \tilde{x}') f(\tilde{y}, \tilde{x}')\|_X \, d\tilde{x}' d\tilde{y} \right]^p dx' y^\gamma dy \\ &\quad \le \int_0^\infty \left(\int_0^\infty \left\| x' \mapsto \int_{\mathbb{R}^{d-1}} \|K(y + \tilde{y}, x - \tilde{x}') f(\tilde{y}, \tilde{x}')\|_X \, d\tilde{x}' \right\|_{L^p(\mathbb{R}^{d-1})} d\tilde{y} \right)^p y^\gamma dy \\ &\quad \le M^p \int_0^\infty \left(\int_0^\infty \frac{1}{y + \tilde{y}} \|f(\tilde{y}, \,\cdot\,)\|_{L^p(\mathbb{R}^{d-1} X)} \, d\tilde{y} \right)^p y^\gamma dy. \end{aligned}$$

By Lemma D.2.9, the latter can be estimated by $M^p C^p_{p,\gamma} \|f\|^p_{L^p(\mathbb{R}^d, w_\gamma; X)}$. This shows that $T_K f$ is a well-defined function in $L^p(\mathbb{R}^d_+, w_\gamma; X)$ of norm $\|T_K f\|_{L^p(\mathbb{R}^d, w_\gamma; X)} \le C_{p,\gamma} M \|f\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)}$ □

Example 6.3.9. The condition (6.57) from the above lemma is in particular satisfied if there exists a constant $M' > 0$ such that

$$\|K(y, x')\|_{\mathcal{B}(Y,Z)} \le \frac{M'}{(y + |x'|)^d}, \quad a.e.\ (y, x') \in \mathbb{R}^d_+. \tag{6.58}$$

In fact, if (6.58) is satisfied, then

$$\|K(\,\cdot\,, y)\|_{L^1(\mathbb{R}^{d-1}, \mathcal{B}(Y,Z))} \lesssim_d M'.$$

In the following lemma we in particular show that the kernels from the representation formula (6.55) (also see (6.53) and (6.54)) satisfy the pointwise norm estimate (6.58); the stronger pointwise $\mathcal{R}$-bound is interesting in view of Lemma E.3.10.

Lemma 6.3.10. *Let $i \in \{1,2\}$ and $|\alpha| \leq 2n$. For each fixed $(y,x') \in \mathbb{R}^d_+$ it holds that $\Sigma_{\pi-\phi} \ni \lambda \mapsto K^i_{\lambda,\alpha}(y,x') \in \mathcal{B}(X^n, X^{2n})$ is a holomorphic mapping, for which we have the $\mathcal{R}$-bounds*

$$\mathcal{R}\{\lambda^{k+1-\frac{|\alpha|}{2n}}\partial^k_\lambda K^i_{\lambda,\alpha}(y,x') \mid \lambda \in \Sigma_{\pi-\phi}\} \lesssim_k \frac{1}{(|y|+|x'|)^d}, \qquad k \in \mathbb{N}.$$

Proof. This can be shown as in [26, Lemma 4.4] □

Lemma 6.3.11. *Let $i \in \{1,2\}$ and $|\alpha| \leq 2n$. For each $\lambda \in \Sigma_{\pi-\phi}$ we have that the kernel $K^i_{\lambda,\alpha}$ induces a bounded linear operator $T^i_{\lambda,\alpha} = T_{K^i_{\lambda,\alpha}} \in \mathcal{B}(L^p(\mathbb{R}^d_+, w_\gamma; X^n), L^p(\mathbb{R}^d_+, w_\gamma; X^{2n}))$ via the formula* (6.56)*. Moreover,*

$$\Sigma_{\pi-\phi} \ni \lambda \mapsto T^i_{\lambda,\alpha} \in \mathcal{B}(L^p(\mathbb{R}^d_+, w_\gamma, X^n), L^p(\mathbb{R}^d_+, w_\gamma, X^{2n}))$$

is a holomorphic mapping, for which we have the $\mathcal{R}$-bounds

$$\mathcal{R}\{\lambda^{k+1-\frac{|\alpha|}{2n}}\partial^k_\lambda T^i_{\lambda,\alpha} \mid \lambda \in \Sigma_{\pi-\phi}\} < \infty, \qquad k \in \mathbb{N}.$$

Proof. From Lemma 6.3.10 it in particular follows that, for each $k \in \mathbb{N}$, $\partial^k_\lambda K^i_{\lambda,\alpha}$ satisfies the estimate (6.58) from Example 6.3.9. By Lemma 6.3.8 we thus obtain that the kernel $\partial^k_\lambda K^i_{\lambda,\alpha}$ induces a bounded linear operator $T_{\partial^k_\lambda K^i_{\lambda,\alpha}} \in \mathcal{B}(L^p(\mathbb{R}^d_+, w_\gamma; X^n), L^p(\mathbb{R}^d_+, w_\gamma; X^{2n}))$ via the formula (6.56). From the $\mathcal{R}$-bounds in Lemma 6.3.10, Lemma E.3.10, and Lemma 6.3.8/Example 6.3.9, it follows that

$$\mathcal{R}\{\lambda^{k+1-\frac{|\alpha|}{2n}}T_{\partial^k_\lambda K^i_{\lambda,\alpha}} \mid \lambda \in \Sigma_{\pi-\phi}\} < \infty, \qquad k \in \mathbb{N}.$$

Therefore, it remains to be shown that $\lambda \mapsto T^i_{\lambda,\alpha}$ is a holomorphic mapping with $\partial^k_\lambda T^i_{\lambda,\alpha} = T_{\partial^k_\lambda K^i_{\lambda,\alpha}}$. To this end, let $\lambda_0 \in \Sigma_{\pi-\phi}$ and pick $\delta > 0$ such that $B(\lambda_0;\delta) \subset \Sigma_{\pi-\phi}$. Then, for every $h \in \mathbb{C}$, $|h| < \delta$, and each $(y,x') \in \mathbb{R}^d_+$, there exists a $c_{h,(y,x')} \in]0,1[$ such that

$$K^i_{\lambda_0+h,\alpha} = K^i_{\lambda_0,\alpha} + \partial_\lambda K^i_{\lambda_0,\alpha}h + \frac{1}{2}\partial^2_\lambda K^i_{\lambda_0+c_{h,(y,x')}h,\alpha}h^2.$$

In combination with the $\mathcal{R}$-bound in Lemma 6.3.10, we in particular get, for every $|h| < \delta$, the estimate

$$\begin{aligned}\left\|K^i_{\lambda_0+h,\alpha} - K^i_{\lambda_0,\alpha} - \partial_\lambda K^i_{\lambda_0,\alpha}h\right\|_{\mathcal{B}(X^n,X^{2n})} &\lesssim (\lambda_0 + c_{h,(y,x')}h)^{-(2-1-\frac{|\alpha|}{2})}\frac{h^2}{(|y|+|x'|)^d} \\ &\lesssim_{\lambda_0,\delta} \frac{h^2}{(|y|+|x'|)^d},\end{aligned}$$

from which we in turn obtain, via Lemma 6.3.8/Example 6.3.9, the norm bound

$$\left\|T_{K^i_{\lambda_0+h,\alpha}} - T_{K^i_{\lambda_0,\alpha}} - T_{\partial_\lambda K^i_{\lambda_0,\alpha}}h\right\| \lesssim_{\lambda_0,\delta} h^2.$$

This shows that $\lambda \mapsto T_{K^i_{\lambda,\alpha}}$ is holomorphic with derivative $\lambda \mapsto T_{\partial_\lambda K^i_{\lambda,\alpha}}$. That also $\partial^k_\lambda T^i_{\lambda,\alpha} = T_{\partial^k_\lambda K^i_{\lambda,\alpha}}$ can be shown similarly. □

The proof of Proposition 6.3.3: We now give the proof of Proposition 6.3.3, of which most of the hard work has already been done above.

Proof of Proposition 6.3.3. We first establish the existence of a solution operator $\mathcal{S}(\lambda)$ for (6.27); we do not prove uniqueness of solutions yet. For this it suffices to construct an operator $\tilde{\mathcal{S}}(\lambda)$ as in (6.46); then we can take $\mathcal{S}(\lambda) := \tilde{\mathcal{S}}(\lambda) \circ \bigoplus_{j=1}^{n} \mathcal{E}_j$, where the $\mathcal{E}_j$ are extension operators as in (6.47). Above we have seen that for this it is enough to show that the linear operator $\tilde{\mathcal{S}}(\lambda) : \mathcal{S}(\mathbb{R}^d_+; X)^n \longrightarrow C_b^\infty(\mathbb{R}^d_+; X)$ defined in (6.50) (takes its values in $W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ and) satisfies the norm estimate (6.51) for each $|\alpha| \leq 2n$; then, by construction, the linear operator $\tilde{\mathcal{S}}(\lambda) : \mathcal{S}(\mathbb{R}^d_+; X)^n \longrightarrow W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ has a unique extension to a bounded linear operator

$$\tilde{\mathcal{S}}(\lambda) : \bigoplus_{j=1}^{n} F^{2n\kappa_{j,\gamma}}_{p,p}(\mathbb{R}^{d-1}; X) \longrightarrow W^{2n}(\mathbb{R}^d_+, w_\gamma; X)$$

as in (6.46). But the estimates (6.51), $|\alpha| \leq 2n$, follow from a combination of the boundedness (6.26), the representation formula (6.55), and Lemma 6.3.11. Hence, there indeed exists a solution operator $\mathcal{S}(\lambda)$ for (6.27).

We finally prove uniqueness of solutions. To this end, suppose we have a solution $v \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ of (6.27) with $g = 0$. It suffices to show that $v = 0$. We claim that:

(i) There exists a sequence $(v^k)_{k\in N} \subset \mathcal{S}(\mathbb{R}^d_+)$ converging to v in $W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ having the property that $(\lambda + \mathcal{A}(D))v^k = 0$ for every $k \in \mathbb{N}$.

(ii) For every $k \in \mathbb{N}$ it holds that

$$v^k = \mathcal{S}(\lambda)(\mathcal{B}^{tr}_1(D)v^k, \ldots, \mathcal{B}^{tr}_n(D)v^k).$$

As $\mathcal{S}_j(\lambda)\mathcal{B}^{tr}_j(D) \in \mathcal{B}(W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X))$, $j = 1, \ldots, n$, we then obtain

$$\begin{aligned} v &= \lim_{k\to\infty} v^k = \lim_{k\to\infty} \mathcal{S}(\lambda)(\mathcal{B}^{tr}_1(D)v^k, \ldots, \mathcal{B}^{tr}_n(D)v^k) \\ &= \mathcal{S}(\lambda)(\mathcal{B}^{tr}_1(D)v, \ldots, \mathcal{B}^{tr}_n(D)v) = \mathcal{S}(\lambda)(0, \ldots, 0) \\ &= 0. \end{aligned}$$

To finish the proof, it remains to establish the two claims:

(i) Since $C^\infty_c(\mathbb{R}^d; X)$ is dense in $W^{2n}_p(\mathbb{R}^d, w_\gamma; X)$, there exists a sequence $(\bar{u}^k)_{k\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^d; X)$ such that $u^k := R_y\bar{u}^k \stackrel{k\to\infty}{\longrightarrow} v$ in $W^{2n}_p(\mathbb{R}^d, w_\gamma; X)$. Then we can pick, for each $k \in \mathbb{N}$, a function $\chi_k \in C^\infty(\mathbb{R}^d)$ which is 1 on $\mathbb{R}^d_+$ and which is such that

$$\left\|\chi_k(\lambda + \mathcal{A}(D))\bar{u}^k\right\|_{L^p(\mathbb{R}^d, w_\gamma; X)} < \left\|R_y(\lambda + \mathcal{A}(D))\bar{u}^k\right\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} + \frac{1}{k+1}.$$

But $R_y(\lambda + \mathcal{A}(D))\bar{u}^k = (\lambda + \mathcal{A}(D))u^k \stackrel{k\to\infty}{\longrightarrow} (\lambda + \mathcal{A}(D))u = 0$ in $L^p(\mathbb{R}^d_+, w_\gamma; X)$ by continuity of $\lambda + \mathcal{A}(D)$ from $W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ to $L^p(\mathbb{R}^d_+, w_\gamma; X)$, whence

$$C^\infty_c(\mathbb{R}^d; X) \ni \bar{f}^k := \chi_k(\lambda + \mathcal{A}(D))\bar{u}^k \stackrel{k\to\infty}{\longrightarrow} 0 \quad \text{in} \quad L^p(\mathbb{R}^d, w_\gamma; X).$$

Since $\xi \mapsto (\lambda + \mathcal{A}(\xi))^{-1}$ belongs to $\mathscr{O}_M(\mathbb{R}^d; \mathcal{B}(X))$ and defines a bounded Fourier multiplier operator from $L^p(\mathbb{R}^d, w_\gamma; X)$ to $W^{2n}_p(\mathbb{R}^d, w_\gamma; X)$ (see (6.23)), it follows that

$$\mathcal{S}(\mathbb{R}^d; X) \ni \bar{w}^k := \mathscr{F}^{-1}[\xi \mapsto (\lambda + \mathcal{A}(\xi))^{-1}\mathscr{F}\bar{f}^k(\xi)] \overset{k\to\infty}{\longrightarrow} 0 \quad \text{in} \quad W^{2n}_p(\mathbb{R}^d, w_\gamma; X).$$

Furthermore, by construction we have that $R_y(\lambda + \mathcal{A}(D))\bar{w}^k = R_y\bar{f}^k = (\lambda + \mathcal{A}(D))u^k$ for every $k \in \mathbb{N}$. Therefore, $v^k := u^k - R_y\bar{w}^k \in \mathcal{S}(\mathbb{R}^d_+; X)$ is as desired.

(ii) For each $k \in \mathbb{N}$ we put

$$g^k := (\mathcal{B}^{tr}_1(D)v^k, \ldots, \mathcal{B}^{tr}_n(D)v^k) \in \mathcal{S}(\mathbb{R}^{d-1}; X)^n$$

and pick a $\tilde{g}^k \in \mathcal{S}(\mathbb{R}^d_+; X)^n$ such that $\tilde{g}^k_j(0, \cdot) = g^k_j$, $j = 1, \ldots, n$. Then, taking the partial Fourier transform $\mathscr{F}_{x'}$ with respect to x' we see that, for each fixed $\xi' \in \mathbb{R}^{d-1}$, the function $w = w_{\xi'} \in C_0([0,\infty[; X)$ given by $w_{\xi'}(y) := \mathscr{F}_x v^k(y, \xi')$ is a solution of (6.20) for $h = h_{\xi'} := (\mathscr{F}g^k_1(\xi'), \ldots, \mathscr{F}g^k_1(\xi')) \in X^n$. In view of the construction of $\tilde{\mathcal{S}}(\lambda)$ on $\mathcal{S}(\mathbb{R}^d_+; X)$ (which is via unique solutions of (6.20)) and the relation $\tilde{\mathcal{S}}_j(\lambda) = \mathcal{S}_j(\lambda) \circ \mathrm{tr}_{y=0}$, $j = 1, \ldots, n$, we must thus have that

$$v^k = \mathscr{F}^{-1}_{x'}\mathscr{F}_{x'}v^k = \tilde{\mathcal{S}}(\lambda)\tilde{g}^k = \mathcal{S}(\lambda)g^k.$$

□

6.3.2.b The case $g = 0$

In this subsubsection we will solve the elliptic boundary value problem (6.25) with $g = 0$, where the assumptions are as in Proposition 6.3.3. The idea is to combine the associated elliptic problem on the full space $\mathbb{R}^d$ with the elliptic boundary value problem (6.27).

Given $\lambda \in \Sigma_{\pi-\phi}$ and $f \in L^p(\mathbb{R}^d_+, w_\gamma; X)$, in order to show that there exists a unique solution $v \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ of

$$\begin{aligned} \lambda v + \mathcal{A}(D)v &= f, \\ \mathcal{B}^{tr}_j(D)v &= 0, \quad j = 1, \ldots, n, \end{aligned} \tag{6.59}$$

we argue as follows. Denote by $E_0 : L^p(\mathbb{R}^d_+, w_\gamma; X) \longrightarrow L^p(\mathbb{R}^d, w_\gamma; X)$ the 'extension by zero' operator. Denote by $R_y : \mathcal{D}'(\mathbb{R}^d; X) \longrightarrow \mathcal{D}'(\mathbb{R}^d_+; X)$ the restriction operator. Furthermore, let A be the realization of $\mathcal{A}(D)$ in $L^p(\mathbb{R}^d, w_\gamma; X)$ defined as in Theorem 6.3.1. In these notations, $v_1 := R_y(\lambda + A)^{-1}E_0 \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ is a solution of

$$\begin{aligned} \lambda v_1 + \mathcal{A}(D)v_1 &= f, \\ \mathcal{B}^{tr}_j(D)v_1 &= g_j, \quad j = 1, \ldots, n, \end{aligned}$$

where $g_j := \mathcal{B}^{tr}_j(D)v_1 \in F^{2n\kappa_{j,\gamma}}_{p,p}(\mathbb{R}^{d-1}; X)$. In order to correct for the boundary term $g = (g_1, \ldots, g_n)$, let $v_2 := -\mathcal{S}(\lambda)g \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ be the unique solution of (6.27) obtained in Proposition 6.3.3. Then $v := v_1 + v_2 \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ is a solution of (6.59), which in view of the uniqueness of solutions in Proposition 6.3.3 must in fact be the unique solution of (6.59). We thus find that (6.59) has a unique solution $v(\lambda) \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ given by

$$v(\lambda) = R_y(\lambda + A)^{-1}E_0 f - \sum_{j=1}^{n} \tilde{\mathcal{S}}_j(\lambda)\mathcal{B}_j(D)R_y(\lambda + A)^{-1}E_0 f\,; \tag{6.60}$$

see Proposition 6.3.3 for the definition of $\tilde{S}_j(\lambda)$.

The problem (6.59) can also naturally be viewed as the invertibility of an operator by incorporating the boundary conditions into the domain of a realization of $\mathcal{A}(D)$ on $L^p(\mathbb{R}^d_+, w_\gamma; X)$. To be more precise, let A_B be the realization of $\mathcal{A}(D)$ in $L^p(\mathbb{R}^d_+, w_\gamma; X)$ with as domain the space of all $v \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)$ satisfying the boundary conditions in (6.59). Then we have $\Sigma_{\pi-\phi} \subset \rho(-A_B)$ with $(\lambda + A_B)^{-1} f = v(\lambda)$, where $v(\lambda)$ is given in (6.60). Via a careful study of the solution formula (6.60) we can show that A_B is an $\mathcal{R}$-sectorial operator with $\mathcal{R}$-angle $\phi^R_{A_B} \leq \phi$, which is important for the maximal L^q_μ-L^p_γ-regularity result in Theorem 6.1.8 in view of Lemma 6.1.2.(iii).'(a)$\Leftrightarrow$(c)' and the abstract maximal L^q_μ-regularity result given in Theorem 6.2.4:

Theorem 6.3.12. *Let X be a (α)-UMD space, $p \in]1, \infty[$, $\gamma \in]-1, p-1[$, and assume that $(\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_n)$ satisfies* $(\mathrm{E})_\phi$ *and* $(\mathrm{LS})_\phi$ *for some $\phi \in [0, \pi[$. Define the linear operator A_B on $L^p(\mathbb{R}^d_+, w_\gamma; X)$ by*

$$\begin{aligned} D(A_B) &:= \{v \in W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X) \mid \mathcal{B}^{tr}_j(D)v = 0, j = 1, \ldots, n\}, \\ A_B v &:= \mathcal{A}(D)v. \end{aligned}$$

Then A_B is an $\mathcal{R}$-sectorial operator with $\mathcal{R}$-angle $\phi^R_{A_B} \leq \phi$.

Proof. Let us first show (i) of Definition 6.2.3. For denseness of $D(A_B)$ it is enough to establish denseness of $C^\infty_c(\mathbb{R}^d_+; X)$ in $L^p(\mathbb{R}^d_+, w_\gamma; X)$, for which it in turn suffices to show that that $C^\infty_c(\dot{\mathbb{R}}^d; X)$ is dense in $L^p(\mathbb{R}^d, w_\gamma; X)$, where $\dot{\mathbb{R}}^d = \mathbb{R}^d \setminus [\{0\} \times \mathbb{R}^{d-1}]$. But this can be shown by using an argumentation as in the proof of Proposition 5.2.70 (also see Remark 5.2.71); here the boundedness of the pointwise multiplier $1_{\mathbb{R}^d_+}$ on $L^p(\mathbb{R}^d, w_\gamma; X)$ is trivial and we, of course, have to use Lemma 3.4.2 instead of Lemma 5.2.20.

In order to show that $R(A_B)$ is dense in $L^p(\mathbb{R}^d_+, w_\gamma; X)$, we first observe that, as a consequence of Corollary 3.4.4, $\mathcal{S}_{0,(1,d-1)}(\mathbb{R}^d_+; X) = \{h_{|\mathbb{R}^d_+} : h \in \mathcal{S}_{0,(1,d-1)}(\mathbb{R}^d; X)\}$ is dense in $L^p(\mathbb{R}^d_+, w_\gamma; X)$. It thus is enough to show that $\mathcal{S}_{0,(1,d-1)}(\mathbb{R}^d_+; X) \subset R(A_B)$. To this end, let $f = \bar{f}_{|\mathbb{R}^d_+}$ with $\bar{f} \in \mathcal{S}_{0,(1,d-1)}(\mathbb{R}^d; X)$. Then, as in the first part of the proof of Theorem 6.3.1,

$$\bar{v}_1 := \mathscr{F}^{-1}\left[\left(\xi \mapsto \mathcal{A}(\xi)^{-1}\right)\mathscr{F}\bar{f}\right] \in \mathcal{S}_{0,(1,d-1)}(\mathbb{R}^d; X)$$

satisfies $\mathcal{A}(D)\bar{v}_1 = \bar{f}$. Now note that $\tilde{g}_j := \mathcal{B}_j(D)\bar{v}_1 \in \mathcal{S}_{0,(1,d-1)}(\mathbb{R}^d_+; X)$ for each $j \in \{1, \ldots, n\}$. In view of the formula

$$[\mathscr{F}\tilde{g}_j(0, \cdot)](\xi') = \int_{\mathbb{R}} [\mathscr{F}\tilde{g}_j](\eta, \xi')d\eta$$

we have $g_j := \mathrm{tr}_{y=0}g_j \in \mathcal{S}_0(\mathbb{R}^{d-1}; X)$ for each $j \in \{1, \ldots, n\}$. So the computations in paragraph 'Representation formulas for the solution of (6.27)' up to (6.50) also make sense for $\lambda = 0$. In fact, following these computations we can find a $v_2 \in \mathcal{S}(\mathbb{R}^d_+; X)$ which solves

$$\begin{aligned} \mathcal{A}(D)v_1 &= 0, \\ \mathcal{B}^{tr}_j(D)v_1 &= g_j, \quad j = 1, \ldots, n. \end{aligned}$$

By construction of $\bar{v}_1$ and v_2, $v := \bar{v}_{1|_{\mathbb{R}^d_+}} - v_2 \in D(A_B)$ solves $A_B v = f$.

Next we prove that (ii) and (iii) of Definition 6.2.3 hold true for $\omega \leq \phi$. For this we fix $\varphi \in]\phi, \pi[$ and show that $\Sigma_{\pi-\varphi} \subset \rho(-A_B)$ with $\mathcal{R}\{\lambda(\lambda + A_B)^{-1} \mid \lambda \in \Sigma_{\pi-\varphi}\} < \infty$.

From the discussion preceding this theorem we already know that $\Sigma_{\pi-\varphi} \subset \rho(-A_B)$ with

$$(\lambda + A_B)^{-1} = R_y(\lambda + A)^{-1}E_0 - \sum_{j=1}^{n} \tilde{\mathcal{S}}_j(\lambda)\mathcal{B}_j(D)R_y(\lambda + A)^{-1}E_0,$$

where A is the realization of $\mathcal{A}(D)$ in $L^p(\mathbb{R}^d, w_\gamma; X)$ defined as in Theorem 6.3.1. Since A is an $\mathcal{R}$-sectorial operator on $L^p(\mathbb{R}^d, w_\gamma; X)$ with $\mathcal{R}$-angle $\phi_A^R \leq \phi$, it is enough to show that

$$\mathcal{R}\left\{\lambda\tilde{\mathcal{S}}_j(\lambda)\mathcal{B}_j(D)R_y(\lambda + A)^{-1}E_0\right\} < \infty, \quad j = 1, \dots, n,$$

in $\mathcal{B}(L^p(\mathbb{R}^d_+, w_\gamma; X))$. So fix a $j \in \{1, \dots, n\}$. Then, by the representation formula (6.28),

$$\begin{aligned}
\lambda\tilde{\mathcal{S}}_j(\lambda)\mathcal{B}_j(D)R_y(\lambda + A)^{-1}E_0 &= \lambda\left[\mathcal{T}^1_{j,0}(\lambda)L_\lambda^{1-\frac{n_j}{2n}} + \mathcal{T}^2_{j,0}(\lambda)L_\lambda^{1-\frac{n_j+1}{2n}}D_y\right]\mathcal{B}_j(D)R_y(\lambda + A)^{-1}E_0 \\
&= \lambda\mathcal{T}^1_{j,0}(\lambda)\,R_yL_\lambda^{1-\frac{n_j}{2n}}\mathcal{B}_j(D)(\lambda + A)^{-1}E_0 \\
&\quad + \lambda\mathcal{T}^2_{j,0}(\lambda)\,R_yL_\lambda^{1-\frac{n_j+1}{2n}}D_y\mathcal{B}_j(D)(\lambda + A)^{-1}E_0.
\end{aligned}$$

Since $\mathcal{R}\left\{\lambda\mathcal{T}^i_{j,0}(\lambda) \mid \lambda \in \Sigma_{\pi-\varphi}\right\} < \infty$, $i = 1, 2$, by (6.30), it suffices to show that

$$\left\{L_\lambda^{1-\frac{n_j}{2n}}\mathcal{B}_j(D)(\lambda + A)^{-1} \mid \lambda \in \Sigma_{\pi-\varphi}\right\}, \ \left\{L_\lambda^{1-\frac{n_j+1}{2n}}D_y\mathcal{B}_j(D)(\lambda + A)^{-1} \mid \lambda \in \Sigma_{\pi-\varphi}\right\}$$

are $\mathcal{R}$-bounded sets in $\mathcal{B}(L^p(\mathbb{R}^d, w_\gamma; X))$. But this can be shown analogously to (6.24) in the proof of Theorem 6.3.1; just note that $L_\lambda^{1-\frac{n_j}{2n}}\mathcal{B}_j(D)(\lambda + A)^{-1}$ and $L_\lambda^{1-\frac{n_j+1}{2n}}D_y\mathcal{B}_j(D)(\lambda + A)^{-1}$ are Fourier multiplier operators on $L^p(\mathbb{R}^d, w_\gamma; X)$ with symbols

$$(\eta, \xi') \mapsto (\lambda + |\xi'|^{2n})^{1-\frac{n_j}{2n}}\mathcal{B}_j(\eta, \xi')(\lambda + \mathcal{A}(\eta, \xi'))^{-1} \in \mathcal{B}(X)$$

and

$$(\eta, \xi') \mapsto (\lambda + |\xi'|^{2n})^{1-\frac{n_j+1}{2n}}\mathcal{B}_j(\eta, \xi')\eta(\lambda + \mathcal{A}(\eta, \xi'))^{-1} \in \mathcal{B}(X),$$

respectively. □

6.4 The Parabolic Initial-Boundary Value Problem

In this section we prove the main result of this chapter, Theorem 6.1.8. Throughout this section we let the notations and assumptions be as in Theorem 6.1.8.

We start with the parabolic initial-boundary value problem (6.5) in the case of homogeneous initial-boundary data:

Lemma 6.4.1. *For every $f \in \mathbb{E}_{0,\mu,\gamma}$ there exists a unique solution $u \in \mathbb{E}_{sol,\mu,\gamma}$ of the problem*

$$\begin{aligned}
\partial_t u + (1 + \mathcal{A}(D))u &= f, \\
\mathcal{B}^{tr}_j(D)u &= 0, \quad j = 1, \dots, n, \\
\mathrm{tr}_{t=0}u &= 0.
\end{aligned}$$

Proof. Since the operator A_B from Theorem 6.3.12 is an $\mathcal{R}$-sectorial operator with $\mathcal{R}$-angle $\phi^R_{A_B} \leq \phi < \frac{\pi}{2}$, it follows that $1 + A_B$ is an invertible $\mathcal{R}$-sectorial operator with $\mathcal{R}$-angle $\phi^R_{1+A_B} \leq \phi^R_{A_B} < \frac{\pi}{2}$. Therefore, by Theorem 6.2.4, the operator $1 + A_B$ enjoys the property of maximal L^q_μ-regularity. The desired result now follows from Lemma 6.1.2.(iii).'(b)⇔(c)'. □

We next study the problem (6.5) in case $f = 0$ and $u_0 = 0$ (we only have a boundary inhomogeneity g), that is, we study the problem

$$\begin{aligned} \partial_t u + (1 + \mathcal{A}(D))u &= 0, \\ \mathcal{B}^{tr}_j(D)u &= g_j, \quad j = 1, \dots, n, \\ \mathrm{tr}_{t=0} u &= 0. \end{aligned} \tag{6.61}$$

for $g = (g_1, \dots, g_n)$ with $(0, g, 0) \in \mathbb{D}_{\mu,\gamma}$. For this we first observe that, in view of the compatibility condition in the definition of $\mathbb{D}_{\mu,\gamma}$ and Proposition 5.3.21, $(0, g, 0) \in \mathbb{D}_{\mu,\gamma}$ if and only if

$$g_j \in {}_0\mathbb{F}_{j,\mu,\gamma} := {}_{0,(0,d)}F^{\kappa_{j,\gamma},(\frac{1}{2n},1)}_{(p,q),p,(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}_+, (1, v_\mu); X)$$

for all $j \in \{1, \dots, n\}$. Defining

$${}_0\mathbb{F}_{\mu,\gamma} := {}_0\mathbb{F}_{1,\mu,\gamma} \oplus \ldots \oplus {}_0\mathbb{F}_{n,\mu,\gamma},$$

we thus have $(0, g, 0) \in \mathbb{D}_{\mu,\gamma}$ if and only if $g \in {}_0\mathbb{F}_{\mu,\gamma}$. Therefore, we just have to solve the problem (6.61) for $g \in {}_0\mathbb{F}_{\mu,\gamma}$.

In order to get an idea how to solve problem (6.61) for $g \in {}_0\mathbb{F}_{\mu,\gamma}$, suppose we have a

$$g = (g_1, \dots, g_n) \in \prod_{j=1}^n C_{L^1}(\mathbb{R}; F^{2n\kappa_{j,\gamma}}_{p,p}(\mathbb{R}^{d-1}; X)) \tag{6.62}$$

with $g_1(0) = \ldots = g_n(0) = 0$ and a

$$u \in C^1_{L^1}(\mathbb{R}; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap C_{L^1}(\mathbb{R}; W^{2n}_p(\mathbb{R}^d_+, w_\gamma; X)) \tag{6.63}$$

such that

$$\begin{aligned} \partial_t u + (1 + \mathcal{A}(D))u &= 0, \\ \mathcal{B}^{tr}_j(D)u &= g_j, \quad j = 1, \dots, n. \end{aligned} \tag{6.64}$$

Then we may take the partial Fourier transform $\mathscr{F}_t$ with respect to $t \in \mathbb{R}$ in (6.64) to obtain, for each $\theta \in \mathbb{R}$,

$$\begin{aligned} (1 + \imath\theta)\mathscr{F}_t u(\theta) + (1 + \mathcal{A}(D))\mathscr{F}_t u(\theta) &= 0, \\ \mathcal{B}^{tr}_j(D)\mathscr{F}_t u(\theta) &= \mathscr{F}_t g_j(\theta), \quad j = 1, \dots, n. \end{aligned}$$

Via Proposition 6.3.3 this implies uniqueness of solutions of the problem (6.64) within the space given in (6.63); in fact, if such a solution exists, then it must satisfy

$$\mathscr{F}_t u(\theta) = \mathcal{S}(1 + \imath\theta)(\mathscr{F}_t g_1(\theta), \dots, \mathscr{F}_t g_n(\theta)). \tag{6.65}$$

The above suggests to reduce the problem (6.61) on $\mathbb{R}_+$ to the same problem on the whole one-dimensional Euclidean space $\mathbb{R}$, so that we can use the partial Fourier transform $\mathscr{F}_t$ with respect to t and try to define a solution operator based on the formula (6.65); of course, the strategy is to define a solution operator initially for g in a certain dense subspace (for which we can take the inverse Fourier transform $\mathscr{F}_t^{-1}$ in the formula (6.65)) and obtain estimates, and then extend this operator by continuity. Defining an operator based on the formula (6.65) in first sight only gives a solution operator for the problem (6.64), but it can in fact be shown that solutions u (6.63) of (6.64) automatically satisfy the initial condition $u(0) = 0$ and are thus automatically solutions of (6.61):

Lemma 6.4.2. *Suppose we have g as in* (6.62) *with $g_1(0) = \ldots = g_n(0) = 0$ and u as in* (6.63) *satisfying* (6.64). *Then $u(0) = 0$.*

Proof. Let us first observe that $-(1 + A_B)$ generates an exponentially stable semigroup. Indeed, $(1+A_B)$ has maximal L^q-regularity by Lemma 6.4.1 (take $\mu = 0$), and must thus be exponentially stable by Proposition 6.2.5.

For each $j \in \{1, \ldots, n\}$ it holds that $u \in C(\mathbb{R}; W_p^{2n}(\mathbb{R}_+^d, w_\gamma; X))$ satisfies the boundary condition $\mathcal{B}_j^{tr}(D)u = g_j$ with $g_j \in C(\mathbb{R}; F_{p,p}^{2n\kappa_{j,\gamma}}(\mathbb{R}^{d-1}; X)$ satisfying $g_j(0) = 0$. Hence, $u(0) \in D(A_B)$. Since the semigroup generated by $-(1 + A_B)$ is exponentially stable, we may thus define

$$v := e^{-\,\cdot\,(1+A_B)}u(0) \in C^1_{L^1}([0, \infty[; L^p(\mathbb{R}_+^d, w_\gamma; X)) \cap C_{L^1}([0, \infty[; D(A_B)).$$

Since $v(0) = 0$ and $v'(0) = -(1 + A_B)u(0) = [-(1 + \mathcal{A}(D))u](0) = u'(0)$, it follows that

$$\tilde{u}(t) := \begin{cases} u(t) - v(t), & t \geq 0, \\ 0, & t < 0, \end{cases}$$

defines a function

$$\tilde{u} \in C^1_{L^1}(\mathbb{R}; L^p(\mathbb{R}_+^d, w_\gamma; X)) \cap C_{L^1}(\mathbb{R}; W_p^{2n}(\mathbb{R}_+^d, w_\gamma; X)),$$

which is a solution of (6.64) by construction. By uniqueness of solutions within the space given in (6.63), $u = \tilde{u}$. This implies that $v = 0$, or equivalently, $u(0) = 0$. □

In the formula (6.65) we have the solution operator $\mathcal{S}_j(\lambda)$ from Proposition 6.3.3 and the partial Fourier transform with respect to t. Recall that for the operator $\tilde{\mathcal{S}}_j(\lambda) = \mathcal{S}_j(\lambda) \circ \mathrm{tr}_{y=0}$ we have the representation formula (6.28) in which the operators L_λ^σ occur. It will be useful to note that, for $h \in \mathcal{S}(\mathbb{R}_+^d \times \mathbb{R}; X)$,

$$\begin{aligned} L^\sigma_{1+\imath\theta_0}[(\mathscr{F}_t h)(\,\cdot\,, \theta)] &= \mathscr{F}_{x'}^{-1}[\big((y, \xi') \mapsto (1 + \imath\theta_0 + |\xi'|^{2n})\big)\mathscr{F}_{(x',t)}h(\,\cdot\,, \theta_0)] \\ &= \Big[\mathscr{F}_t\mathscr{F}^{-1}_{(x',t)}[\big((y, \xi', \theta) \mapsto (1 + \imath\theta + |\xi'|^{2n})\big)\mathscr{F}_{(x',t)}h]\Big](\,\cdot\,, \theta_0) \\ &= (\mathscr{F}_t L^\sigma h)(\,\cdot\,, \theta_0), \end{aligned} \tag{6.66}$$

where

$$L^\sigma \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^{d-1} \times \mathbb{R}; \mathcal{D}'(\mathbb{R}_+; X))),\ f \mapsto \mathscr{F}^{-1}_{(x',t)}\Big[\big((\xi', \theta) \mapsto (1 + \imath\theta + |\xi'|^{2n})^\sigma\big)\mathscr{F}_{(x',t)}f\Big].$$

We now are ready to solve the parabolic initial-boundary value problem (6.61):

Lemma 6.4.3. *For each $g \in {}_0\mathbb{F}_{\mu,\gamma}$ the problem* (6.61) *has a unique solution $u \in \mathbb{E}_{sol,\mu,\gamma}$.*

Proof. Uniqueness follows from the observation that $u = 0$ is the unique solution of the problem

$$\begin{aligned} \partial_t u + (1 + \mathcal{A}(D))u &= 0, \\ \mathcal{B}_j^{tr}(D)u &= 0, \quad j = 1, \ldots, n, \\ \mathrm{tr}_{t=0} u &= 0. \end{aligned} \tag{6.67}$$

So we only need to establish existence.

(I) Below we will write ${}_0\overline{\mathbb{F}}_{\mu,\gamma}$ for the space ${}_0\mathbb{F}_{\mu,\gamma}$ with $\mathbb{R}_+$ replaced by $\mathbb{R}$, and similarly for $\mathbb{E}_{sol,\mu,\gamma}$. Then we have ${}_0\mathbb{F}_{\mu,\gamma} \cong {}_0\overline{\mathbb{F}}_{\mu,\gamma}/\ker(R_t)$ and $\mathbb{E}_{sol,\mu,\gamma} \cong \overline{\mathbb{E}}_{sol,\mu,\gamma}/\ker(R_t)$; see Lemma 5.3.2 and Proposition 5.3.21.

(II) It suffices to construct a solution operator $\overline{\mathscr{S}} \in \mathcal{B}({}_0\overline{\mathbb{F}}_{\mu,\gamma}, \overline{\mathbb{E}}_{sol,\mu,\gamma})$ for the problem (6.61), i.e. for $g = (g_1, \ldots, g_m) \in {}_0\overline{\mathbb{F}}_{\mu,\gamma}$ we have that $u = \overline{\mathscr{S}}g \in \overline{\mathbb{E}}_{sol,\mu,\gamma}$ solves the problem (6.61). Indeed, such an operator $\overline{\mathscr{S}}$ automatically satisfies

$$R_t\overline{\mathscr{S}}g = R_t\overline{\mathscr{S}}\tilde{g} \quad \text{whenever} \quad g, \tilde{g} \in {}_0\overline{\mathbb{F}}_{\mu,\gamma} \quad \text{satisfy} \quad R_t g = R_t\tilde{g}, \tag{6.68}$$

because $R_t\overline{\mathscr{S}}g - R_t\overline{\mathscr{S}}\tilde{g}$ is a solution of the problem (6.67) whenever $R_t g = R_t\tilde{g}$, and must thus equal the unique solution 0. Therefore, such an operator $\overline{\mathscr{S}} \in \mathcal{B}({}_0\overline{\mathbb{F}}_{\mu,\gamma}, \overline{\mathbb{E}}_{sol,\mu,\gamma})$ induces a bounded linear operator $\mathscr{S} \in \mathcal{B}({}_0\mathbb{F}_{\mu,\gamma}, \mathbb{E}_{sol,\mu,\gamma})$, which is of course a solution operator for the problem (6.61).

(III) We claim that

$$V := \left\{ f \in \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^{d-1}; X)) \otimes \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}) \mid f_{|\mathbb{R}^{d-1}\times\{0\}} = 0 \right\}$$

is dense in ${}_0\overline{\mathbb{F}}_{j,\mu,\gamma}$. Then V^n is a dense linear subspace of ${}_0\overline{\mathbb{F}}_{\mu,\gamma}$. So, in view of the continuity/boundedness

$$\partial_t + (1 + \mathcal{A}(D)) \in \mathcal{B}(\overline{\mathbb{E}}_{sol,\mu,\gamma}, \overline{\mathbb{E}}_{0,\mu,\gamma}), \quad \mathcal{B}_j^{tr}(D) \in \mathcal{B}(\overline{\mathbb{E}}_{sol,\mu,\gamma}, \overline{\mathbb{F}}_{j,\mu,\gamma}), \quad \mathrm{tr}_{t=0} \in \mathcal{B}(\overline{\mathbb{E}}_{sol,\mu,\gamma}, X_{u,\mu,\gamma}),$$

it suffices to construct a solution operator $\overline{\mathscr{S}} \in \mathcal{B}(V^n, \overline{\mathbb{E}}_{sol,\mu,\gamma})$, where we equip V^n with the norm induced by ${}_0\overline{\mathbb{F}}_{\mu,\gamma}$.

To prove the claim, we first recall that ${}_0\overline{\mathbb{F}}_{j,\mu,\gamma}$ coincides with the closure of $\{f \in \mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{R}; X) \mid f_{|\mathbb{R}^{d-1}\times\{0\}} = 0\}$ in $\overline{\mathbb{F}}_{j,\mu,\gamma}$; see Definition 5.2.67. Since $\mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{R}; X) \hookrightarrow \overline{\mathbb{F}}_{j,\mu,\gamma}$, it thus is enough to show that V is dense in $\{f \in \mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{R}; X) \mid f_{|\mathbb{R}^{d-1}\times\{0\}} = 0\}$ with respect to the Fréchet topolgy induced by $\mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{R}; X)$. To this end, let $f \in \mathcal{S}(\mathbb{R}^{d-1} \times \mathbb{R}; X)$ with $f_{|\mathbb{R}^{d-1}\times\{0\}} = 0$ be given. Then there exists a sequence $(f_k)_{k\in\mathbb{N}} \subset \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^{d-1}; X)) \otimes \mathscr{F}^{-1}(C_c^\infty(\mathbb{R})$ such that $f = \lim_{k\to\infty} f_k$ in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}; X)$. Now we set $h_k := f_{k|\mathbb{R}^{d-1}\times\{0\}} \in \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^{d-1}; X))$, pick a $\chi \in \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}))$ with $\chi(0) = 1$, and put $\tilde{f}_k := f_k - h_k \otimes \chi \in \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^{d-1}; X)) \otimes \mathscr{F}^{-1}(C_c^\infty(\mathbb{R})$. Then, by construction, $(\tilde{f}_k)_{k\in\mathbb{N}} \subset V$ and

$$f = \lim_{k\to\infty} f_k - 0 \otimes \chi = \lim_{k\to\infty} f_k - f_{|\mathbb{R}^{d-1}\times\{0\}} \otimes \chi = \lim_{k\to\infty} \tilde{f}_k \quad \text{in} \quad \mathcal{S}(\mathbb{R}^d \times \mathbb{R}; X),$$

proving the claim.

(IV) For the definition of the solution operator $\overline{\mathscr{S}} \in \mathcal{B}(V^n, \overline{\mathbb{E}}_{sol,\mu,\gamma})$, fix $g = (g_1, \ldots, g_n) \in V^n$. Let

$$\mathcal{E}_j \in \mathcal{B}(\overline{\mathbb{F}}_{j,\mu,\gamma}, H^{1-\frac{n_j}{2n},(\frac{1}{2n},1)}_{(p,q),(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}, (w_\gamma, v_\mu); X)), \quad j = 1, \ldots, n, \tag{6.69}$$

be extension operators (right-inverses of the trace operator $\mathrm{tr}_{y=0}$) as in Theorem 5.3.16. Then $\mathcal{E}_j$ maps $\mathscr{F}^{-1}(C_c^\infty(\mathbb{R}^{d-1};X)) \otimes \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}))$ into $\mathcal{S}(\mathbb{R}^d_+;X)) \otimes \mathscr{F}^{-1}(C_c^\infty(\mathbb{R}))$; in particular,

$$\mathcal{E}_j g_j \in \mathcal{S}(\mathbb{R}^d_+;X)) \otimes \mathscr{F}^{-1}(C_c^\infty(\mathbb{R})), \quad j = 1, \dots, n.$$

So, for each $j \in \{1, \dots, n\}$, we have

$$\mathscr{F}_t \mathcal{E}_j g_j \in \mathcal{S}(\mathbb{R}^d_+;X)) \otimes C_c^\infty(\mathbb{R}),$$

and we may also view $\mathscr{F}_t \mathcal{E}_j g_j$ as a function

$$[\theta \mapsto (\mathscr{F}_t \mathcal{E}_j g_j)(\theta)] \in C_c^\infty(\mathbb{R}; W_p^{2n-n_j}(\mathbb{R}^d_+, w_\gamma; X)).$$

Since

$$[\theta \mapsto \tilde{\mathcal{S}}_j(1 + \imath\theta)] \in C^\infty(\mathbb{R}; \mathcal{B}(W_p^{2n-n_j}(\mathbb{R}^d_+, w_\gamma; X), W_p^{2n}(\mathbb{R}^d_+, w_\gamma; X))), \quad j = 1, \dots, n,$$

with $\tilde{\mathcal{S}}_j(1 + \imath\theta)$ as in Proposition 6.3.3, we may thus define

$$\overline{\mathscr{S}} g := \mathscr{F}_t^{-1} \left[\theta \mapsto \sum_{j=1}^n \tilde{\mathcal{S}}_j(1 + \imath\theta)(\mathscr{F}_t \mathcal{E}_j g_j)(\theta) \right] \in \mathcal{S}(\mathbb{R}; W_p^{2n}(\mathbb{R}^d_+, w_\gamma; X))$$

(V) We now show that $u = \overline{\mathscr{S}} g \in \mathcal{S}(\mathbb{R}; W_p^{2n}(\mathbb{R}^d_+, w_\gamma; X))$ is a solution of (6.61) for $g \in V^n$. To this end, let $\theta \in \mathbb{R}$ be arbitrary. Then we have that $(\mathscr{F}_t \mathcal{E}_j g_j)(\theta) \in \mathcal{S}(\mathbb{R}^d_+;X) \subset W_p^{2n-n_j}(\mathbb{R}^d_+, w_\gamma; X)$ and $(\mathscr{F}_t g_j)(\theta) \in \mathcal{S}(\mathbb{R}^{d-1};X) \subset F_{p,p}^{2n\kappa_{j,\gamma}}(\mathbb{R}^{d-1};X)$ are related by $\mathrm{tr}_{y=0}(\mathscr{F}_t \mathcal{E}_j g_j)(\theta) = (\mathscr{F}_t g_j)(\theta)$; just note that $(\mathscr{F}_t \mathcal{E}_j g_j)(0, x', \theta) = (\mathscr{F}_t g_j)(x', \theta)$ for every $x' \in \mathbb{R}^{d-1}$. Therefore, by Proposition 6.3.3, $v(\theta) = (\mathscr{F}_t u)(\theta) = (\mathscr{F}_t \overline{\mathscr{S}} g)(\theta) = \sum_{j=1}^n \tilde{\mathcal{S}}_j(1+\imath\theta)(\mathscr{F}_t \mathcal{E}_j g_j)(\theta) \in W_p^{2n}(\mathbb{R}^d_+, w_\gamma; X)$ is the unique solution of the problem

$$\begin{aligned} (1 + \imath\theta)v + \mathcal{A}(D)v &= 0, \\ \mathcal{B}_j^{tr}(D)v &= (\mathscr{F}_t g_j)(\theta), \quad j = 1, \dots, n. \end{aligned}$$

Applying the inverse Fourier transform $\mathscr{F}_t^{-1}$ with respect to θ, we find

$$\begin{aligned} \partial_t u + (1 + \mathcal{A}(D))u &= 0, \\ \mathcal{B}_j(D)u &= g_j, \quad j = 1, \dots, n. \end{aligned}$$

Then we also have $u(0) = 0$ as a consequence of Lemma 6.4.2. Hence, $u = \overline{\mathscr{S}} g$ is indeed a solution of (6.61).

(VI) We next derive a representation formula for $\overline{\mathscr{S}}$ that is well suited for proving the boundedness of $\overline{\mathscr{S}}$. To this end, fix a $g = (g_1, \dots, g_n) \in V^n$. Then we have, for each multi-index $\alpha \in \mathbb{N}^d, |\alpha| \leq 2n$,

$$\begin{aligned}
D^\alpha \overline{\mathscr{S}} g &= D^\alpha \mathscr{F}_t^{-1}\left[\theta \mapsto \sum_{j=1}^n \tilde{S}_j(1+\imath\theta)(\mathscr{F}_t \mathcal{E}_j g_j)(\theta)\right] \\
&= \sum_{j=1}^n \mathscr{F}_t^{-1}\left[\theta \mapsto D^\alpha \tilde{S}_j(1+\imath\theta)(\mathscr{F}_t \mathcal{E}_j g_j)(\theta)\right] \\
&\overset{(6.28)}{=} \sum_{j=1}^n \mathscr{F}_t^{-1}\left[\theta \mapsto \mathcal{T}^1_{j,\alpha}(1+\imath\theta) L_{1+\imath\theta}^{1-\frac{n_j}{2n}}(\mathscr{F}_t \mathcal{E}_j g_j)(\theta) + \mathcal{T}^2_{j,\alpha}(1+\imath\theta) L_{1+\imath\theta}^{1-\frac{n_j+1}{2n}} D_y(\mathscr{F}_t \mathcal{E}_j g_j)(\theta)\right] \\
&= \sum_{j=1}^n \mathscr{F}_t^{-1}\left[\theta \mapsto \mathcal{T}^1_{j,\alpha}(1+\imath\theta) L_{1+\imath\theta}^{1-\frac{n_j}{2n}}(\mathscr{F}_t \mathcal{E}_j g_j)(\theta)\right] \\
&\quad + \sum_{j=1}^n \mathscr{F}_t^{-1}\left[\theta \mapsto \mathcal{T}^2_{j,\alpha}(1+\imath\theta) L_{1+\imath\theta}^{1-\frac{n_j+1}{2n}}(\mathscr{F}_t D_y \mathcal{E}_j g_j)(\theta)\right] \\
&\overset{(6.66)}{=} \sum_{j=1}^n \mathscr{F}_t^{-1}\left[\theta \mapsto \mathcal{T}^1_{j,\alpha}(1+\imath\theta)(\mathscr{F}_t L^{1-\frac{n_j}{2n}} \mathcal{E}_j g_j)(\theta)\right] \\
&\quad + \sum_{j=1}^n \mathscr{F}_t^{-1}\left[\theta \mapsto \mathcal{T}^2_{j,\alpha}(1+\imath\theta)(\mathscr{F}_t L^{1-\frac{n_j+1}{2n}} D_y \mathcal{E}_j g_j)(\theta)\right]. \qquad (6.70)
\end{aligned}$$

(VII) We finally show that $\overline{\mathscr{S}} \in \mathcal{B}(V^n, \overline{\mathbb{E}}_{sol,\mu,\gamma})$, where we equip V^n with the norm induced by $\overline{\mathbb{F}}_{\mu,\gamma}$. For this we must show that $\left\|\overline{\mathscr{S}} g\right\|_{\overline{\mathbb{E}}_{sol,\mu,\gamma}} \lesssim \|g\|_{\overline{\mathbb{F}}_{\mu,\gamma}}$ for $g \in V^n$. Being a solution of (6.61), $\overline{\mathscr{S}} g$ satisfies

$$\partial_t \overline{\mathscr{S}} g = -(1+\mathcal{A}(D))\overline{\mathscr{S}} g.$$

Hence, it suffices to establish the estimate $\left\|D^\alpha \overline{\mathscr{S}} g\right\|_{\overline{\mathbb{E}}_{0,\mu,\gamma}} \lesssim \|g\|_{\overline{\mathbb{F}}_{\mu,\gamma}}$ for all multi-indices $\alpha \in \mathbb{N}^d, |\alpha| \leq 2n$. So fix such an $|\alpha| \leq 2n$. Then, in view of the representation formula (6.70), it is enough to show that

$$\left\|\mathscr{F}_t^{-1}\left[\theta \mapsto \mathcal{T}^1_{j,\alpha}(1+\imath\theta)(\mathscr{F}_t L^{1-\frac{n_j}{2n}} \mathcal{E}_j g_j)(\theta)\right]\right\|_{\overline{\mathbb{E}}_{0,\mu,\gamma}} \lesssim \|g\|_{\overline{\mathbb{F}}_{\mu,\gamma}}, \quad j = 1, \ldots, n, \qquad (6.71)$$

and

$$\left\|\mathscr{F}_t^{-1}\left[\theta \mapsto \mathcal{T}^2_{j,\alpha}(1+\imath\theta)(\mathscr{F}_t L^{1-\frac{n_j+1}{2n}} D_y \mathcal{E}_j g_j)(\theta)\right]\right\|_{\overline{\mathbb{E}}_{0,\mu,\gamma}} \lesssim \|g\|_{\overline{\mathbb{F}}_{\mu,\gamma}}, \quad j = 1, \ldots, n. \qquad (6.72)$$

We only treat the estimate (6.72), the estimate (6.71) being similar (but easier): Fix a $j \in \{1, \ldots, n\}$. Then, by (6.69), Proposition 5.3.5.(i), Corollary 5.3.14, and Lemma 5.2.48,

$$L^{1-\frac{n_j+1}{2n}} D_y \mathcal{E}_j \in \mathcal{B}\left(\overline{\mathbb{F}}_{j,\mu,\gamma}, \underbrace{H^{0,(\frac{1}{2n},1)}_{(p,q),(d-1,1)}(\mathbb{R}^{d-1} \times \mathbb{R}, (1, v_\mu); L^p(\mathbb{R}_+, |\cdot|^\gamma; X))}_{= L^q(\mathbb{R}, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X)) = \overline{\mathbb{E}}_{0,\mu,\gamma}}\right). \qquad (6.73)$$

Furthermore, we have that $\mathcal{T}^2_{j,\alpha}(1+\imath\,\cdot) \in C^\infty(\mathbb{R}; \mathcal{B}(L^p(\mathbb{R}^d_+, w_\gamma; X)))$ satisfies

$$\mathcal{R}\left\{\theta^k \partial_\theta^k \mathcal{T}^2_{j,\alpha}(1+\imath\theta) \mid \theta \in \mathbb{R}\right\} \leq \mathcal{R}\left\{(1+\imath\theta)^{k+1-\frac{|\alpha|}{2n}} \partial_\theta^k \mathcal{T}^2_{j,\alpha}(1+\imath\theta) \mid \theta \in \mathbb{R}\right\} < \infty, \qquad k \in \mathbb{N},$$

by the contraction principle (cf. Proposition E.1.2) and (6.30); in particular, $\mathcal{T}^2_{j,\alpha}(1+\imath\cdot)$ satisfies the Mikhlin condition from Theorem 4.5.13 (or from Theorem 4.1.1 for $l = 1, d = 1$). As a consequence, $\mathcal{T}^2_{j,\alpha}(1+\imath\cdot)$ defines a bounded Fourier multiplier operator on $L^q(\mathbb{R}, v_\mu; L^p(\mathbb{R}^d_+, w_\gamma; X))$. In combination with (6.73), this gives the estimate (6.72). □

We can now finally prove the main result of this chapter:

Proof of Theorem 6.1.8. By Lemma 6.4.1 and Lemma 6.1.2.(iii).'(a)⇔(c)', the problem (6.5) enjoys the property of maximal L^q_μ-L^p_γ-regularity. Denote by $\mathscr{D}_{i.b.}$ the space of initial boundary data. We must show that $\mathscr{D}_{i.b.} = \mathbb{D}_{\mu,\gamma}$; then the statement about the solution operator follows from Lemma 6.1.2.(i). Since we already know that $\mathscr{D}_{i.b.} \subset \mathbb{D}_{\mu,\gamma}$ from Section 6.1.2, it remains to be shown that the reverse inclusion $\mathbb{D}_{\mu,\gamma} \subset \mathscr{D}_{i.b.}$ holds as well. To this end, let $(f, g, u_0) \in \mathbb{D}_{\mu,\gamma}$ be given. Let $\mathcal{E} \in \mathcal{B}(X_{\mu,\gamma}, \mathbb{E}_{sol,\mu,\gamma})$ be an extension operator (right-inverse of the trace operator $\mathrm{tr}_{y=0}$) as in Theorem 5.3.17. Now we set $u_1 := \mathcal{E}u_0 \in \mathbb{E}_{sol,\mu,\gamma}$ (so $\mathrm{tr}_{t=0}u_1 = u_0$) and

$$\begin{aligned} \tilde{f} &:= f - \partial_t u_1 - (1 + \mathcal{A}(D))u_1 \in \mathbb{E}_{0,\mu,\gamma}, \\ \tilde{g} &:= (g_1 - \mathcal{B}^{tr}_1(D)u_1, \ldots, g_n - \mathcal{B}^{tr}_n(D)u_1) \overset{!}{\in} {}_0\mathbb{F}_{\mu,\gamma}; \end{aligned}$$

here we have $\tilde{g} \in {}_0\mathbb{F}_{\mu,\gamma}$ (and not just $\tilde{g} \in \mathbb{F}_{\mu,\gamma}$) as a consequence of the compatibility condition in the definition of $\mathbb{D}_{\mu,\gamma}$ (and the fact that $\mathrm{tr}_{t=0} \circ \mathcal{B}^{tr}_j(D) = \mathcal{B}^{tr}_j(D) \circ \mathrm{tr}_{t=0}$ on $\mathbb{E}_{sol,\mu,\gamma}$ when $\kappa_{j,\gamma} > \frac{1+\mu}{q}$). Now let $u_2 \in \mathbb{E}_{sol,\mu,\gamma}$ be the unique solution from Lemma 6.4.1 for $\tilde{f} \in \mathbb{E}_{0,\mu,\gamma}$ (instead of f) and let $u_3 \in \mathbb{E}_{sol,\mu,\gamma}$ be the unique solution from Lemma 6.4.3 for $\tilde{g} \in {}_0\mathbb{F}_{\mu,\gamma}$ (instead of g). Then $u := u_1 + u_2 + u_3$ solves (6.5). □

6.5 Notes

6.5.1 General Notes

The unweighted version of Theorem 6.1.8 is due to Denk, Hieber & Prüss [26], which was already extended to the temporal weighted setting ($\mu \in [0, q-1[$) in the case $q = p$ by Meyries & Schnaubelt [76]; also see the PhD thesis of Meyries [73]. Of course, here we have to remark that our Theorem 6.1.8 only is a model problem (which forms the basis for more general problems) and that we assume property (α) for simplicity; also see the discussion below. For more information on the historical background of the maximal (weighted) L^q-L^p-regularity approach to the parabolic initial-boundary value problem (6.5) we refer to Section 1.1.

6.5.2 Comparison to the Literature

- *Section 6.1:* Theorem 6.1.8 extends the model problem versions of [26, Theorem 2.3] and [76, Theorem 2.1] (cf. [73, Theorem 2.1.4]), where we need to remark that in contrast to [26, 73, 76], we for simplicity restrict ourselves to UMD spaces that have property (α); see Remark 6.1.9.(i) for the simplifications related to property (α). The ellipticity assumption $(\mathrm{E})_\phi$ and the Lopatinskii-Shapiro condition $(\mathrm{LS})_\phi$ are as in [26] (also see [25]); see Remark 6.1.4 concerning [73, 76]. As in [73], for the formulation of Theorem 6.1.8 we have chosen to explicitly define the notion of maximal L^q_μ-L^p_γ-regularity (cf. Definition 6.1.1). Whereas the definition of maximal L^q_μ-L^p_γ-regularity in [73, Definition 2.1.3] is formulated in terms of an explicit space of data (suggested by sharp trace results), our definition only requires the existence of (a necessarily unique) abstract space of initial boundary data. Besides the advantage that our definition avoids the use of some quite involved function space theory (which can also be seen as part of the problem), the main advantage of our definition is the characterization from Lemma 6.1.2.(iii). This characterization allows us to first prove maximal L^q_μ-L^p_γ-regularity via the problem with homogeneous initial-boundary data, and to subsequently benefit from the existence of the space of initial-boundary data to determine it explicitely: assuming maximal L^q_μ-L^p_γ-regularity, in Section 6.1.2 we find necessary conditions on the initial-boundary data $(g, u_0) \in \mathscr{D}_{i.b.}$ by using the function space theory from Chapter 5 in combination with Lemma 6.1.2.(i)&(ii). For more comments on the different approach on the function space theoretic part of the problem (in comparison with [26, 73, 76]) we refer to Section 1.2 and the notes of Chapter 5; also see the discussion about Section 6.4 below.

- *Section 6.2:* See the references given in that section.
- *Section 6.3:*
  - *Section 6.3.1:*
  - *Section 6.3.2:* Proposition 6.3.3 is based on [26, Lemma 4.3&Lemma 4.4] and [73, Lemma 2.2.6], where our formulation is more closer to the second (which was in turn based on the first). The main difference with [73, Lemma 2.2.6] is that we give representation formulae for the operators $D^\alpha\tilde{\mathcal{S}}_j(\lambda)$ instead of for the operators $\mathcal{S}_j(\lambda)$. In [26, 73] a specific extension operator $\mathcal{E}_\lambda$ (right-inverse of the trace $\mathrm{tr}_{y=0}$) was used in the construction of the solution operator $\mathcal{S}(\lambda) = (\mathcal{S}_1(\lambda), \ldots, \mathcal{S}_n(\lambda))$, which has the advantageous property that $D_y\mathcal{E}_\lambda = \imath L_\lambda^{1/2n}\mathcal{E}_\lambda$. Whereas the in this way obtained representation formulae $\mathcal{S}_j(\lambda) = \mathcal{T}_j(\lambda)L_\lambda^{1-\frac{n_j}{2n}}\mathcal{E}_\lambda$ can only be used to solve (6.61) in the case $q = p$ (cf. [26, Proposition 4.5] and [73, Lemma 2.2.7]), our representation formulae (6.28) can (in combination with the function space theory from Chapter 5) be used to solve (6.61) in the full parameter range $q, p \in]1, \infty[$ (cf. Lemma 6.4.3).

    The proof of Proposition 6.3.3 is mainly based on the proofs of [26, Lemma 4.3&Lemma 4.4] (and on the earlier monograph [25]); [73, Lemma 2.2.6] has [26, Lemma 4.3&Lemma 4.4] as reference. Here [26, Lemma 4.3] corresponds to the existence of the solution operator, whose construction was essentially already contained in [25], plus its representation, and [26, Lemma 4.4] basically corresponds to the analytic dependence of (6.29) plus the $\mathcal{R}$-bounds (6.30). In order to make the difficult proof of Proposition 6.3.3 more accessible, we have tried to treat the main argument of the proof in more (technical) detail and to refer to [25, 26] at some appropriate moments (as for the proof of Lemma 6.3.10, which is a very important computation which is also of independent interest).

    Finally, the proof of Theorem 6.3.12 is based on the proof of [64, Theorem 7.7] (except for the denseness of $D(A_B)$ and $R(A_B)$), where the scalar-valued case $X = \mathbb{C}$ (which certainly has property (α)) is considered for a second order system.
- *Section 6.4:* In this section the main work is to solve (6.61). Our solution to this problem is mainly based on [73, Lemma 2.2.7]. Here we had to make quite some modifications

due to our different function space theoretic approach (including a different extension operator); also see the discussion about Proposition 6.3.3 above.

Appendix A

Measure Theory

In Section A.1 we recall some basics from measure theory, which we assume the reader to be familiar with throughout the thesis. In Section A.2 we briefly treat Orlicz spaces, which is only needed in Chapter 3 (in the proof of Lemma 3.3.20). Finally, in Section A.3 we treat martingale theory, which is only needed in Chapter 3, where it is a prerequisite for Section 3.3.

A.1 Basic Measure and Integration Theory

The vector-valued theory from this section is taken from [57].

A.1.1 Classical Measure and Integration Theory

Let $(S, \mathscr{A})$ be a measurable space. We denote by $\mathcal{M}(S)$ the set of all measurable functions $f : S \longrightarrow \mathbb{K}$, by $\mathcal{M}_{+}(S)$ the set of all measurable functions $f : S \longrightarrow [0, \infty[$, and by $\overline{\mathcal{M}}_{+}(S)$ the set of all measurable functions $f : S \longrightarrow [0, \infty]$.

Let $(S, \mathscr{A}, \mu)$ be a measure space. We denote by $L^0(S)$, $L^0_+(S)$, and $\overline{L^0}_+(S)$, the sets of all μ-a.e. equivalence classes from $\mathcal{M}(S)$, $\mathcal{M}_+(S)$, and $\overline{\mathcal{M}}_+(S)$, respectively.

Given two σ-finite measure spaces $(S, \mathscr{A}, \mu)$ and $(T, \mathscr{B}, \nu)$, their product measure space is denoted by $(S \times T, \mathscr{A} \otimes \mathscr{B}, \mu \otimes \nu)$.

Theorem A.1.1 (Tonelli's theorem)**.** *Let $(S, \mathscr{A}, \mu)$ and $(T, \mathscr{B}, \nu)$ be two σ-finite measure spaces and let $f : S \times T \longrightarrow [0, \infty]$ be $\mathscr{A} \otimes \mathscr{B}$-measurable. Then*

(i) For every $s \in S$ and every $t \in T$, $f(s, \cdot)$ and $f(\cdot, t)$ are $\mathscr{B}$ and $\mathscr{A}$-measurable, respectively.

(ii) $s \mapsto \int_T f(s,t)d\nu(t)$ and $t \mapsto \int_S f(s,t)d\mu(s)$ are $\mathscr{A}$ and $\mathscr{B}$-measurable, respectively.

(iii) $\int_S \int_T f(s,t)d\nu(t)d\mu(s) = \int_{S\times T} f d(\mu \otimes \nu) = \int_T \int_S f(s,t)d\mu(s)d\nu(t)$.

Definition A.1.2. Let $(S, \mathscr{A}, \mu)$ be a measure space and let $\mathscr{F} \subset \mathscr{A}$ be a sub-σ-algebra. We say that $\mathscr{F}$ is countably atomic with respect to μ if there exists a countable partition $\{D_i\}_{i\in I}$ of S, consisting of $\mathscr{F}$-measurable sets D_i having strictly positive finite measure, such that $\mathscr{F} = \sigma(\{D_i\}_{i\in I})$. In this situation we write

$$\mathscr{F}^{atom} = \mathscr{F}^{\mu-atom} := \{D_i \ : \ i \in I\},$$

which is just the collection of all atoms of the measure space $(S, \mathscr{F}, \mu_{|\mathscr{F}})$.

A.1.2 Vector-Valued Measurability and Integration

Measurability Let $(S, \mathscr{A})$ be a measurable space and let X be a Banach space. We denote by

$$\mathrm{St}(S; X) := \left\{ \sum_{j=1}^{n} 1_{A_j} \otimes x_j \ : \ A_j \in \mathscr{A} \text{ disjoint}, x_j \in X \right\}$$

the vector space of all X-valued simple functions; here we use the usual notational convention to view, given a function $f : S \longrightarrow \mathbb{K}$, $f \otimes x$ as the function $s \mapsto f(s)x$, $S \longrightarrow X$. A function $f : S \longrightarrow X$ is called strongly measurable if it is the pointwise limit of a sequence $(f_k)_{k\in\mathbb{N}} \subset \mathrm{St}(S; X)$; it can be shown that the sequence $(f_k)_k$ can be chosen such that $\|f_k\|_X \leq \|f\|_X$. The well known Pettis measurability theorem says that a function $f : S \longrightarrow X$ is strongly measurable, if and only if, f is separably valued and Borel measurable, if and only if, f is separably valued and $\langle f, x^* \rangle$ is measurable for all x^* in some weak* dense subspace Z of X^*. As a consequence, if $f : S \longrightarrow X$ is strongly measurable and takes its values in a closed linear subspace Y of X, then f is also strongly measurable as a function $S \longrightarrow Y$. Furthermore, it can be shown that if $(S, \mathscr{A}) = (V, \mathcal{B}(V))$ for a separable metric space V with its Borel σ-algebra $\mathcal{B}(V)$, then every Borel measurable function $f : V \longrightarrow X$ is separably valued and is thus automatically strongly measurable.

Let $(S, \mathscr{A}, \mu)$ be a measure space and let X be a Banach space. We denote by $L^0(S; X)$ the vector space of all μ-a.e. equivalence classes of strongly measurable functions $f : S \longrightarrow X$. It is convenient to view $L^0(S; X)$ as the vector space of all μ-a.e. equivalence classes of functions $g : S \longrightarrow X$ which are μ-a.e. equal to a strongly measurable function on $f : S \longrightarrow X$.

Let Y be a second Banach space. A function $f : S \longrightarrow \mathcal{B}(X, Y)$ is called WOT-measurable if $\langle fx, y^* \rangle$ is measurable for every $x \in X$ and $y^* \in Y^*$.

Integration Let $(S, \mathscr{A}, \mu)$ be a measure space and let X be a Banach space. We denote by

$$\mathrm{St}_\mu(S; X) := \left\{ \sum_{j=1}^{n} 1_{A_j} \otimes x_j \ : \ A_j \in \mathscr{A} \text{ disjoint}, \mu(A_j) < \infty, x_j \in X \right\}$$

the vector space of all X-valued μ-simple functions. A function $f : S \longrightarrow X$ is called Bochner integrable if it is the pointwise limit of a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathrm{St}_\mu(A; X)$ such that

$$\lim_{n\to\infty} \int_S \|f - f_n\|_X \, d\mu = 0.$$

In this situation we may define

$$\int_S f \, d\mu := \lim_{n\to\infty} \int_S f_n \, d\mu,$$

where each $\int_S f_n \, d\mu$ is defined in the obvious way; this is easily seen to be well-defined. A strongly measurable function $f : S \longrightarrow X$ is Bochner integrable if and only if

$$\int_S \|f\|_X \, d\mu < \infty,$$

in which case

$$\left\| \int_S f \, d\mu \right\|_X \leq \int_S \|f\|_X \, d\mu.$$

Proposition A.1.3. *Let $(S, \mathscr{A}, \mu)$ be a finite measure space and let X be a Banach space. If $f : S \longrightarrow X$ is Bochner integrable, then*

$$\int_S f \, d\mu \in \mu(S) \, \overline{\mathrm{conv}}\{f(s) : s \in S\},$$

where $\overline{\mathrm{conv}}\{f(s) : s \in S\}$ denotes the closed convex hull of the set $\{f(s) : s \in S\} \subset X$.

We denote by

$$L^1(S; X) := \{f \in L^0(S; X) : \int_S \|f\|_X \, d\mu < \infty\}$$

the space of all equivalence classes of Bochner integrable functions, equipped with its natural norm. It is not difficult to see that the integral induces a well-defined bounded linear operator

$$L^1(S; X) \longrightarrow X, \; f \mapsto \int_S f \, d\mu.$$

A.1.3 Lebesgue-Bochner Spaces

Let $(S, \mathscr{A}, \mu)$ be a measure space. We denote by $\mathcal{W}(S, \mathscr{A}, \mu)$ the set of all measurable functions $S \longrightarrow]0, \infty[$; a function $W \in \mathcal{W}(S, \mathscr{A}, \mu)$ is called a weight. Given a $p \in]1, \infty[$, the p-dual weight of $W \in \mathcal{W}(S, \mathscr{A}, \mu)$ is the weight $W' = W^{\frac{1}{p-1}} \in \mathcal{W}(S, \mathscr{A}, \mu)$.

Given a Banach space X, $W \in \mathcal{W}(S, \mathscr{A}, \mu)$, and $p \in]1, \infty[$, we define the weighted Lebesgue-Bochner space

$$L^p(S, W; X) := \{f \in L^0(S; X) : \int_S \|f\|_X^p \, W \, d\mu < \infty\},$$

which becomes a Banach space when equipped with its natural norm

$$\|f\|_{L^p(S,W;X)} := \left(\int_S \|f\|_X^p \, W \, d\mu \right)^{1/p}.$$

$L^p(S, W) \otimes X$ is dense in $L^p(S, W; X)$.

Let $p' \in]1, \infty[$ be the Hölder conjugate of p and denote by $W' = W^{\frac{1}{p-1}} \in \mathcal{W}(S, \mathscr{A}, \mu)$ the p-dual weight of W. By Hölder's inequality, every function $g \in L^{p'}(S, W'; X^*)$ defines a bounded linear functional $\Lambda_g \in (L^p(S, W; X))^*$ by the formula

$$\Lambda_g(f) := \int_S \langle f(s), g(s) \rangle \, d\mu(s),$$

which is of norm $\left\| \Lambda_g \right\|_{(L^p(S,W;X))^*} \leq \|g\|_{L^{p'}(S,W';X^*)}$. If Y is a norming closed subspace of X^*, then

$$L^{p'}(S, W'; Y) \longrightarrow (L^p(S, W; X))^*, \; g \mapsto \Lambda_g,$$

is an isometry onto a closed subspace of $(L^p(S, W; X))^*$ which is norming for $L^p(S, W; X)$. In case $Y = X^*$ has the so-called Radon-Nikodym property (RNP), this mapping is surjective. Examples of spaces with the RNP are separable dual spaces and reflexive spaces. A similar duality result holds true for mixed-norm Lebesgue spaces (of which Definition 2.2.2 is a special case).

A.1.4 Convolutions

Let X_1, X_2, X_0 be Banach spaces. Suppose that we are given a continuous bilinear map

$$X_1 \times X_2 \longrightarrow X_0,\ (x_1, x_2) \mapsto x_1 \bullet x_2 \tag{A.1}$$

of norm at most 1. Such a map is called a *multiplication (of Banach spaces)*. The two main examples of multiplications which are of interest for us are multiplication with scalars

$$\mathbb{K} \times X \longrightarrow X,\ (\lambda, x) \mapsto \lambda x,$$

and the evaluation map

$$\mathcal{B}(X, Y) \times X \longrightarrow Y,\ (A, x) \mapsto Ax.$$

Let $f \in L^0(\mathbb{R}^d; X_1)$ and $g \in L^0(\mathbb{R}^d; X_2)$. If $[y \mapsto f(y)g(x-y)] \in L^0(\mathbb{R}^d; X_0)$ is Bochner integrable for almost all $x \in \mathbb{R}^d$, then we define the convolution product $f *_\bullet g \in L^0(\mathbb{R}^d; X_0)$ by

$$(f *_\bullet g)(x) := \int_{\mathbb{R}^d} f(y)g(x-y)dy, \qquad x \in \mathbb{R}^d. \tag{A.2}$$

Theorem A.1.4 (Young's inequality)**.** *Let $p, q, r \in [1, \infty]$ satisfy*

$$\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r}.$$

Let $f \in L^p(\mathbb{R}^d; X_1)$ and $g \in L^r(\mathbb{R}^d; X_2)$. Then $[y \mapsto f(y)g(x-y)] \in L^0(\mathbb{R}^d; X_0)$ is Bochner integrable for almost all $x \in \mathbb{R}^d$, and for the convolution product we have $f *_\bullet g \in L^q(\mathbb{R}^d; X_0)$ with norm estimate

$$\|f *_\bullet g\|_{L^q(\mathbb{R}^d; X_0)} \le \|f\|_{L^p(\mathbb{R}^d; X_1)} \|g\|_{L^r(\mathbb{R}^d; X_2)}.$$

A.2 Orlicz Spaces

Definition A.2.1.

(i) A *Young function* is a continuous increasing convex function $\Phi : [0, \infty[\longrightarrow [0, \infty[$ that satisfies $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$.

(ii) Let Φ be a Young function with the property that $\lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty$. Then the *Young complement* of Φ is the Young function Ψ defined by

$$\Psi(x) := \sup_{t\in[0,\infty[} \{tx - \Phi(t)\}, \qquad x \in [0, \infty[.$$

(iii) Let $(S, \mathscr{A}, \mu)$ be a measure space and let Φ be a Young's function. The *Orlicz norm* of $\mathbb{C}$-valued measurable function f on $(S, \mathscr{A})$ is defined as

$$\|f\|_{\Phi(\mu)} := \inf\left\{ \lambda > 0 \ : \ \int_S \Phi(|f|/\lambda) d\mu \le 1 \right\}.$$

The *Orlicz space* $\Phi(S, \mathscr{A}, \mu)$ is defined as

$$\Phi(S, \mathscr{A}, \mu) := \{ f \in L^0(S, \mathscr{A}, \mu) \ : \ \|f\|_{\Phi(\mu)} < \infty \},$$

which comes a Banach space when equipped with the norm $\|\cdot\|_{\Phi(\mu)}$.

For the Young function $\Phi_p(t) := t^p$, $p \in [1, \infty[$, the corresponding Orlicz norm is just the L^p-norm $\|\cdot\|_{L^p(S)}$.

An important tool for computing/estimating Orlicz norms is the fact that for any increasing continuously differentiable function ϕ on $[0, \infty[$ with $\phi(0) = 0$ we have

$$\int_S \phi(|f|)\, d\mu = \int_0^\infty \phi'(t)\mu(\{f > t\})\, dt, \qquad f \in \mathcal{M}(S, \mathscr{A}). \tag{A.3}$$

Lemma A.2.2 (Hölder's inequality for Orlicz Spaces)**.** *Let $(S, \mathscr{A}, \mu)$ be a measure space and let Φ and Ψ be (complementary Young functions) as in Definition A.2.1. Then we have, for all $f, g \in \mathcal{M}(S, \mathscr{A})$,*

$$\int_S |fg|\, d\mu \le 2\, \|f\|_{\Phi(\mu)}\, \|g\|_\Psi$$

The following example of complementary Young functions is the main motivation for us to include this section in the appendix:

Example A.2.3. The functions $\Phi, \Psi : [0, \infty[\longrightarrow [0, \infty[$ defined by

$$\Phi(t) := t \log(1 + t) \quad \text{and} \quad \Psi(t) := \exp(t) - 1$$

are (complementary Young functions) as in Definition A.2.1.

A.3 Martingales

The material from this section is taken from [57], except for Proposition A.3.11.

A.3.1 Conditional Expectations

Throughout this subsection we fix a measure space $(S, \mathscr{A}, \mu)$, sub-σ-algebra $\mathscr{F} \subset \mathscr{A}$, and a Banach space X. We shall view $L^0(S, \mathscr{F}; X) = L^0((S, \mathscr{F}, \mu); X)$ as the subspace of $L^0(S; X) = L^0((S, \mathscr{A}, \mu); X)$ consisting of all elements in $L^0(S; X)$ having a strongly $\mathscr{F}$-measurable representative.

Definition A.3.1. A family $\mathscr{C} \subset \mathscr{F}$ is called an *exhausting ideal for* $\mathscr{F}$ if it has the following two properties:

(i) $C \cap F \in \mathscr{C}$ for all $C \in \mathscr{C}$ and $F \in \mathscr{F}$;

(ii) S can be covered by at most countably many sets from $\mathscr{C}$.

It is not difficult to see that any exhausting ideal contains a disjoint sequence covering S.

Given a sub-σ-algebra $\mathscr{F} \subset \mathscr{A}$ and a function $f \in L^0(S; X)$, we define

$$\mathscr{F}_f := \{F \in \mathscr{F} \,:\, 1_F f \in L^1(S; X)\}.$$

Definition A.3.2 (Conditional Expectation)**.** A function $g \in L^0(S, \mathscr{F}; X)$ is called a *conditional expectation* of $f \in L^0(S; X)$ with respect to $\mathscr{F}$ if there exists an exhausting ideal $\mathscr{C}$ for $\mathscr{F}$ contained in $\mathscr{F}_f \cap \mathscr{F}_g$ such that

$$\int_C g\, d\mu = \int_C f\, d\mu, \qquad C \in \mathscr{C}.$$

Theorem A.3.3 (Uniqueness). *If g and $\tilde{g}$ are both conditional expectations of a function $f \in L^0(S;X)$ with respect to $\mathscr{F}$, then $g = \tilde{g}$ almost everywhere.*

So a conditional expectation of a function $f \in L^0(S;X)$, if it exists, is unique as an element of $L^0(S,\mathscr{F};X)$. This allows us to speak of the *conditional expectation* of f with respect to $\mathscr{F}$, for which we shall use the notation $\mathbb{E}(f \mid \mathscr{F})$ (or $\mathbb{E}[f \mid \mathscr{F}]$).

Example A.3.4.

(i) Suppose that $f_1, f_2 \in L^0(S;X)$ admit conditional expectation with respect to $\mathscr{F}$. Then, for all scalars $c_1, c_2 \in \mathbb{K}$, $c_1 f_1 + c_2 f_2$ admits a conditional expectation with respect to $\mathscr{F}$, which is given by $\mathbb{E}(c_1 f_1 + c_2 f_2 \mid \mathscr{F}) = c_1\mathbb{E}(f_1 \mid \mathscr{F}) + c_2\mathbb{E}(f_2 \mid \mathscr{F})$.

(ii) For all $f \in L^0(S,\mathscr{F};X)$, f is a conditional expectation with respect to $\mathscr{F}$: $\mathbb{E}(f \mid \mathscr{F}) = f$.

(iii) If $f \in L^0(S;X)$ admits a conditional expectation with respect to $\mathscr{F}$, then for every $F \in \mathscr{F}$, $1_F\mathbb{E}(f \mid \mathscr{F})$ is the conditional expectation of $1_F f$ with respect to $\mathscr{F}$: $1_F\mathbb{E}(f \mid \mathscr{F}) = \mathbb{E}(1_F f \mid \mathscr{F})$.

(iv) If $f \in L^0(S;X)$ admits a conditional expectation with respect to $\mathscr{F}$, and if $T \in \mathcal{B}(X,Y)$ (where Y is a second Banach space), then $T\mathbb{E}(f \mid \mathscr{F})$ is the conditional expectation of Tf with respect to $\mathscr{F}$: $\mathbb{E}(Tf \mid \mathscr{F}) = T\mathbb{E}(f \mid \mathscr{F})$. This in particular holds for $Y = X^*$.

(v) Suppose that $\mathscr{F}$ is countably atomic with respect to μ in the sense of Definition A.1.2. Let $f \in L^0(S;X)$ be such that $\mathscr{F}^{atom} \subset \mathscr{F}_f$. Then f admits a conditional expectation, which is in fact given by the formula

$$\mathbb{E}(f \mid \mathscr{F}) = \sum_{D \in \mathscr{F}_k^{atom}} 1_D \fint_D f \, d\mu.$$

It follows directly from the definitions that a necessary condition for the existence of the conditional expectation of a function $f \in L^0(S;X)$ with respect to $\mathscr{F}$ is σ-integrability of f over $\mathscr{F}$, which is defined as follows:

Definition A.3.5. A function $f \in L^0(S;X)$ is called *σ-integrable over $\mathscr{F}$* if S can be covered by a sequence in $\mathscr{F}_f$. Any such covering sequence in $\mathscr{F}_f$ is called an *exhausting sequence for f in $\mathscr{F}$*.

Remark A.3.6.

(i) The sets in an exhausting sequence for f can be taken disjoint.

(ii) A function $f \in L^0(S;X)$ is σ-integrable over $\mathscr{F}$, if and only if, the measure $\|f\|_X \mu$ is σ-finite on $\mathscr{F}$, if and only if, $\mathscr{F}_f$ is an exhausting ideal for $\mathscr{F}$.

(iii) Every function $f \in L^0(S;X)$ is σ-integrable over $\mathscr{A}$.

Theorem A.3.7. *Let $f \in L^0(S;X)$.*

(i) If f admits a conditional expectation with respect to $\mathscr{F}$, then f is σ-integrable over $\mathscr{F}$.

(ii) If f is σ-integrable over $\mathscr{F}$ and μ is σ-finite on $\mathscr{F}$, then f admits a conditional expectation with respect to $\mathscr{F}$.

In this situation, we have $\mathscr{F}_f \subset \mathscr{F}_{\mathbb{E}(f|\mathscr{F})}$ and

$$\int_F f \, d\mu = \int_F \mathbb{E}(f \mid \mathscr{F}) \, d\mu, \qquad F \in \mathscr{F}.$$

Corollary A.3.8. *Suppose that μ is σ-finite on $\mathscr{F}$. Then $f \in L^0(S;X)$ admits a conditional expectation with respect to $\mathscr{F}$ if and only if f is σ-integrable over $\mathscr{F}$.*

We denote by $L^1_\sigma((S,\mathscr{A},\mu),\mathscr{F};X)$ the space of all $f \in f \in L^0(S;X)$ which are σ-integrable over $\mathscr{F}$. If μ is σ-finite on $\mathscr{F}$, then the conditional expectation $\mathbb{E}(\,\cdot \mid \mathscr{F})$ is a well-defined linear operator on $L^1_\sigma((S,\mathscr{A},\mu),\mathscr{F};X)$.

Example A.3.9. Suppose that $\mathscr{F}$ is countably atomic with respect to μ in the sense of Definition A.1.2. Then an $f \in L^0(S;X)$ belongs to $L^1_\sigma((S,\mathscr{A},\mu),\mathscr{F};X)$ if and only if $\mathscr{F}^{atom} \subset \mathscr{F}_f$, in which case $\mathbb{E}(f \mid \mathscr{F})$ is given by the formula in Example A.3.4.(v).

Theorem A.3.10 (Existence in $L^1(S;X)$). *If μ is σ-finite on $\mathscr{F}$, then $L^1(S;X) \subset L^1_\sigma((S,\mathscr{A},\mu),\mathscr{F};X)$ and the conditional expectation operator $\mathbb{E}(\,\cdot \mid \mathscr{F})$ restricts to a contraction on $L^1(S;X)$.*

For the unweighted Lebesgue-Bochner spaces $L^p(S;X)$, $p \in]1,\infty[$, the L^p-contractivity of the conditional expectation can be derived from a combination of the L^1-contractivity and the conditional Jensen's inequality. In the weighted setting we have to proceed differently:

Proposition A.3.11. *Suppose that μ is σ-finite on $\mathscr{F}$. Let $p \in]1,\infty[$ and let $W \in \mathcal{W}(S,\mathscr{A},\mu)$ be a weight for which its p-dual weight $W' := W^{-\frac{1}{p-1}}$ is σ-integrable over $\mathscr{F}$. Then $L^p(W;X) \subset L^1_\sigma((S,\mathscr{A},\mu),\mathscr{F};X)$ and the conditional expectation operator $\mathbb{E}(\,\cdot \mid \mathscr{F})$ restricts to a contraction on $L^p(W;X)$.*

Proof. In order to prove that $L^p(W;X) \subset L^1_\sigma((S,\mathscr{A},\mu),\mathscr{F};X)$ we show that, for every $f \in L^p(W;X)$, $\mathscr{F}_{W'} \subset \mathscr{F}_f$; note that $\mathscr{F}_{W'}$ is an exhausting ideal for $\mathscr{F}$ by the hypothesis that W' is σ-integrable on $\mathscr{F}$. To this end, let $f \in L^p(W;X)$ and $C \in \mathscr{F}_{W'}$. Then, by Hölder's inequality,

$$\int_C \|f\|_X \, d\mu \int_S \|f\|_X \, W^{1/p} \cdot 1_C W^{-1/p} \, d\mu \leq \|f\|_{L^p(W;X)} \|1_C\|_{L^{p'}(W')} < \infty$$

because $\|1_C\|_{L^{p'}(W')} = \|1_C W'\|_{L^1}^{1/p'} < \infty$.

Next we show that $\mathbb{E}(\,\cdot \mid \mathscr{F})$ restricts to a contraction on $L^p(W;X)$. Let $\iota : L^{p'}(W',\mathscr{F}) \hookrightarrow L^{p'}(W')$ be the natural inclusion; here $p' \in]1,\infty[$ denotes the Hölder conjugate of p (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$). Then, under the identifications $L^p(W) \cong (L^{p'}(W'))^*$ and $L^p(W,\mathscr{F}) \cong (L^{p'}(W',\mathscr{F}))^*$, we have that $\iota^* \in \mathcal{B}(L^p(W), L^p(W,\mathscr{F}))$ of norm $\|\iota^*\| \leq 1$. Since ι^* is a positive operator (in the sense that it maps positive functions to positive functions), there exists a unique operator $\iota^*_X \in \mathcal{B}(L^p(W;X), L^p(W,\mathscr{F};X))$ of norm $\left\|\iota^*_X\right\| \leq 1$ with the property that

$$\langle \iota^*_X f, x^* \rangle = \iota^* \langle f, x^* \rangle, \qquad \forall f \in L^p(W), x^* \in X^*;$$

the operator ι^*_X is called the X-valued extension of ι^*.[1] To finish we show that $i^*_X f = \mathbb{E}(f \mid \mathscr{F})$ for every $f \in L^p(W;X)$. By the above, $\mathscr{F}_{W'}$ is an exhausting ideal for $\mathscr{F}$ contained in $\mathscr{F}_f \cap \mathscr{F}_{i^*_X f}$.

[1] Banach-valued extensions of positive operators on L^p-spaces can for instance be found in [57].

Given $C \in \mathscr{F}_{W'}$ we have $1_C \in L^{p'}(W')$, so that, for every $x^* \in X^*$,

$$\begin{aligned}
\langle \int_C i_X^* f \, d\mu, x^* \rangle &= \int_C \langle i_X^* f, x^* \rangle \, d\mu = \int i^* \langle f, x^* \rangle \cdot 1_C \, d\mu \\
&= \int \langle f, x^* \rangle \cdot \iota 1_C \, d\mu = \int \langle f, x^* \rangle \cdot 1_C \, d\mu \\
&= \langle \int_C f \, d\mu, x^* \rangle.
\end{aligned}$$

Therefore, by Hahn-Banach,

$$\int_C i_X^* f \, d\mu = \int_C f \, d\mu, \qquad \forall C \in \mathscr{F}_{W'},$$

showing that $i_X^* f = \mathbb{E}(f \mid \mathscr{F})$. □

Proposition A.3.12 (Taking out $\mathscr{F}$-measurable terms)**.** *Suppose that μ is σ-integrable on $\mathscr{F}$. Let $g \in L^0(S, \mathscr{F}; X)$ and $f \in L^1_\sigma((S, \mathscr{A}, \mu), \mathscr{F}; X)$. Then $gf \in L^1_\sigma((S, \mathscr{A}, \mu), \mathscr{F}; X)$ and*

$$\mathbb{E}(gf \mid \mathscr{F}) = g\mathbb{E}(f \mid \mathscr{F}).$$

Proposition A.3.13 (Tower property)**.** *Let $\mathscr{G} \subset \mathscr{F}$ be a sub-σ-algebra on which μ is σ-finite. If $h \in L^0(S; X)$ is σ-integrable over $\mathscr{G}$ (and thus over $\mathscr{F}$), then so is $\mathbb{E}(h \mid \mathscr{F})$. Moreover, in this situation we have*

$$\mathbb{E}[\mathbb{E}(h \mid \mathscr{F}) \mid \mathscr{G}].$$

A.3.2 Martingales

Throughout this subsection we fix a measure space $(S, \mathscr{A}, \mu)$ and a Banach space X.

A.3.2.a Definitions and Examples

Definition A.3.14.

(i) A family of sub-σ-algebras $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ of $\mathscr{A}$ is called a *filtration* in $(S, \mathscr{A}, \mu)$ if $\mathscr{F}_k \subset \mathscr{F}_l$ whenever $k \leq l$. The filtration is called *σ-finite* if μ is σ-finite on each $\mathscr{F}_k$.

(ii) A family of functions $(f_k)_{k\in\mathbb{Z}}$ in $L^0(S; X)$ is called *adapted* to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ if $f_k \in L^0(S, \mathscr{F}_k; X)$ for all $k \in \mathbb{Z}$. In this situation, we also call $(f_k)_{k\in\mathbb{Z}}$ an X-valued martingale on $(S, \mathscr{A}, \mu)$ with respect to $(\mathscr{F}_k)_{k\in\mathbb{Z}}$.

(iii) An adapted family of functions $(f_k)_{k\in\mathbb{Z}}$ in $L^0(S; X)$ is called a *martingale* with respect to $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ if for all $k, l \in \mathbb{Z}$ with $k \leq l$ the conditional expectation of f_k with respect to $\mathscr{F}_l$ exists and is given by

$$\mathbb{E}(f_k \mid \mathscr{F}_l) = f_l.$$

Remark A.3.15.

(i) By Proposition A.3.13 and Example A.3.4.(i)&(ii), an adapted family of functions $(f_k)_{k\in\mathbb{Z}}$ in $L^0(S;X)$ is a martingale with respect to $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ if for all $k\in\mathbb{Z}$ the conditional expectation f_{n+1} with respect to $\mathscr{F}_n$ exists and

$$\mathbb{E}(f_{n+1}\mid\mathscr{F}_n)=f_n,\ \text{ or equivalently, }\ \mathbb{E}(f_{n+1}-f_n\mid\mathscr{F}_n)=0.$$

(ii) By Corollary A.3.8, when the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ is σ-finite, the existence of the conditional expectation in (iii) is equivalent with $f_k\in L^1_\sigma((S,\mathscr{A},\mu),(\mathscr{F}_l)_{l\geq k};X)$ for every $k\in\mathbb{Z}$.

Example A.3.16 (Martingales generated by a function)**.** Suppose that $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ is a σ-finite filtration in $(S,\mathscr{A},\mu)$. If $f\in L^1_\sigma((S,\mathscr{A},\mu),(\mathscr{F}_k)_{k\in\mathbb{Z}};X)$, i.e. $f\in L^0(S;X)$ is σ-integrable over each $\mathscr{F}_k$, then

$$f_k:=\mathbb{E}(f\mid\mathscr{F}_k),\qquad k\in\mathbb{Z},$$

defines a martingale $(f_k)_{k\in\mathbb{Z}}$ with respect to $(\mathscr{F}_k)_{k\in\mathbb{Z}}$.

Martingales generated by functions with respect to dyadic filtrations play a very important role in (Harmonic) Analysis (as can, for instance, be seen in Chapter 5). In the following example we consider the standard dyadic filtration on $\mathbb{R}^d$.

Example A.3.17 (Dyadic harmonic analyis)**.** Let $(S,\mathscr{A},\mu)=(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),\lambda)$. For each $k\in\mathbb{Z}$, let

$$\mathcal{D}_k:=\{2^{-k}([0,1[^d+m)\ :\ m\in\mathbb{Z}\}$$

be the system of standard dyadic cubes of side-length 2^{-k}, and $\mathscr{F}_k:=\sigma(\mathscr{D}_k)$ its generated countably atomic σ-algebra. Then $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ is a σ-finite filtration in $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),\lambda)$, called the *standard dyadic filtration*. For this filtration it holds that

$$\bigcup_{k\in\mathbb{Z}}L^1_\sigma((S,\mathscr{A},\mu),\mathscr{F}_k;X)=L^1_\sigma((S,\mathscr{A},\mu),\{\mathscr{F}_k\}_{k\in\mathbb{Z}};X)=L^1_{loc}(\mathbb{R}^d;X).$$

So each $f\in L^1_{loc}(\mathbb{R}^d;X)$ generates a martingale $(\mathbb{E}[f\mid\mathscr{F}_k])_{k\in\mathbb{Z}}$.

A.3.2.b Doob's maximal inequality

For a sequence $f=(f_k)_{k\in\mathbb{Z}}\subset L^0(S;X)$ we define the maximal functions

$$f_k^*:=\sup_{l\leq k}\|f_l\|_X\,,\qquad f^*:=\sup_{l\in\mathbb{Z}}\|f_l\|_X\,.$$

Theorem A.3.18 (Doob's maximal inequality)**.** *Let $(f_k)_{k\in\mathbb{Z}}$ be an X-valued martingale on $(S,\mathscr{A},\mu)$ with respect to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$. If $f_k\in L^p(S;X)$ for some $p\in]1,\infty]$ and all $k\in\mathbb{Z}$, then $f_k^*\in L^p(S)$ for all $k\in\mathbb{Z}$ and*

$$\left\|f_k^*\right\|_{L^p(S)}\leq p'\,\|f_k\|_{L^p(S;X)}\,,$$

where $p'\in[1,\infty[$ denotes the Hölder conjugate of p.

Combining this with Example A.3.16 and Proposition A.3.11, we obtain:

Corollary A.3.19. *Suppose that $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ is a σ-finite filtration on $(S,\mathscr{A},\mu)$. If $f\in L^p(S;X)$, $p\in]1,\infty]$, then*

$$\left\|\sup_{k\in\mathbb{Z}}\|\mathbb{E}[f\mid\mathscr{F}_k]\|_X\right\|_{L^p(S)}\leq p'\,\|f\|_{L^p(S;X)}\,,$$

where $p'\in[1,\infty[$ denotes the Hölder conjugate of p.

A.3.2.c Stopping times

Stopping time techniques play an important role in martingale theory. For example, Doob's maximal inequality (cf. Theorem A.3.18) can be proved using stopping times.

Definition A.3.20. A mapping $\tau : S \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ is called a *stopping time* with respect to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ if for all $k \in \mathbb{Z}$ we have $\{\tau \le k\} \in \mathscr{F}_k$, or equivalently, if for all $k \in \mathbb{Z}$ we have $\{\tau = k\} \in \mathscr{F}_k$.

If τ_1 and τ_2 are both stopping times, then $\tau_1 \wedge \tau_2 = \min\{\tau_1, \tau_2\}$ and $\tau_1 \vee \tau_2 = \max\{\tau_1, \tau_2\}$ are again stopping times.

Example A.3.21 (First hitting time)**.** Let $(f_k)_{k\in\mathbb{Z}} \subset L^0(S;X)$ be adapted to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ and let $B \subset X$ be a Borel set. We define $\tau : S \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ by

$$\tau := \inf\{k \in \mathbb{Z} \,:\, f_k \in B\},$$

where $\inf \emptyset = \infty$. Here it is understood that we work with strongly $\mathscr{F}_k$-measurable representatives of the f_k. Then τ is a stopping time[2] (with respect to $(\mathscr{F}_k)_{k\in\mathbb{Z}}$), which is called the *first hitting time* of B associated with $(f_k)_{k\in\mathbb{Z}}$.

Let τ be a stopping time with respect to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$. We define the sub-$\sigma$-algebra $\mathscr{F}_\tau \subset \mathscr{A}$ associated with τ by

$$\mathscr{F}_\tau := \{A \in \mathscr{A} \,:\, A \cap \{\tau = k\} \in \mathscr{F}_k, \forall k \in \mathbb{Z}\}.$$

Then τ is $\mathscr{F}_\tau$-measurable. If $\tau = l \in \mathbb{Z} \cup \{\pm\infty\}$, then we have

$$\mathscr{F}_\tau = \begin{cases} \mathscr{F}_{-\infty} := \bigcap_{k\in\mathbb{Z}} \mathscr{F}_k & , l = -\infty; \\ \mathscr{F}_l & , l \in \mathbb{Z}; \\ \mathscr{F}_\infty := \sigma\left(\bigcup_{k\in\mathbb{Z}} \mathscr{F}_k\right) & , l = \infty; \end{cases}$$

so there is no inconsistency of notation.

Lemma A.3.22. *Let $\tau : S \longrightarrow \mathbb{Z} \cup \{\infty\}$ be a stopping time with respect to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$. Then μ is σ-finite on $\mathscr{F}_\tau$ and $L^1_\sigma((S, \mathscr{A}, \mu), \{\mathscr{F}_k\}_{k\in\mathbb{Z}}; X) \subset L^1_\sigma((S, \mathscr{A}, \mu), \mathscr{F}_\tau; X)$. Moreover, for every $f \in L^1_\sigma((S, \mathscr{A}, \mu), \{\mathscr{F}_k\}_{k\in\mathbb{Z}}; X)$ we have*

$$1_{\{\tau=k\}}\mathbb{E}(f \mid \mathscr{F}_\tau) = 1_{\{\tau=k\}}\mathbb{E}(f \mid \mathscr{F}_k), \qquad k \in \mathbb{Z} \cup \{\infty\}.$$

Definition A.3.23. Let $\tau : S \longrightarrow \mathbb{Z} \cup \{\infty\}$ be a stopping time with respect to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ and let $f = (f_k)_{k\in\mathbb{Z}}$ be an adapted sequence in $L^0(S;X)$.

(i) The *stopped sequence* $f^\tau = (f^\tau_k)_{k\in\mathbb{Z}}$ is defined by

$$f^\tau_k := f_{\tau\wedge k}, \qquad k \in \mathbb{Z}.$$

(ii) The *started sequence* ${}^\tau f = ({}^\tau f_k)_{k\in\mathbb{Z}}$ is defined by

$${}^\tau f_k := f_k - f_{\tau\wedge k}, \qquad k \in \mathbb{Z}.$$

[2]Here it should of course be understood that we work with equivalence classes of stopping times. In practice, we only work with representatives.

For a sequence $(f_k)_{k\in\mathbb{Z}} \subset L^0(S;X)$, we define the *difference sequence* $(df_k)_{k\in\mathbb{Z}}$ by

$$df_k := f_k - f_{k-1}, \qquad k \in \mathbb{Z}.$$

Then, in the situation of the above definition, the difference sequence $df^\tau = (df^\tau_{k\in\mathbb{Z}})_{k\in\mathbb{Z}}$ is given by

$$df^\tau_k = 1_{\{k\leq\tau\}}, \qquad k \in \mathbb{Z},$$

and the difference sequence $d^\tau f = (d^\tau f_k)_{k\in\mathbb{Z}}$ is given by

$$d^\tau f_k = 1_{\{k>\tau\}} df_k, \qquad k \in \mathbb{Z}.$$

Proposition A.3.24. *Let $(f_k)_{k\in\mathbb{Z}}$ be an X-valued martingale and let $\tau : S \longrightarrow \mathbb{Z}\cup\{\infty\}$ be a stopping time, both with respect to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$. Then f^τ and ${}^\tau f$ are again martingales with respect to the filtration $(\mathscr{F}_k)_{k\in\mathbb{Z}}$.*

A.3.2.d Martingale convergence

Let $(\mathscr{F}_k)_{k\in\mathbb{Z}}$ be a σ-finite filtration in $(S, \mathscr{A}, \mu)$. Recall that $\mathscr{F}_\infty = \sigma(\bigcup_{k\in\mathbb{Z}} \mathscr{F}_k)$.

Theorem A.3.25 (Forward convergence of generated martingales)**.** *If $f \in L^p(S;X)$ for some $p \in [1,\infty[$, then*

$$\lim_{n\to\infty} \mathbb{E}(f|\mathscr{F}_n) = \mathbb{E}(f|\mathscr{F}_\infty)$$

both in $L^p(S;X)$ and pointwise almost everywhere.

The following lemma is used in the proof of this theorem:

Lemma A.3.26. *Let $p \in [1,\infty[$. Then $\bigcup_{k\in\mathbb{Z}} L^p(S, \mathscr{F}_k; X)$ is dense in $L^p(S, \mathscr{F}_\infty; X)$.*

Appendix B

Banach Function Spaces

In this appendix, Section B.1 is based on [72, 2] and Section B.2 is based on [13, 70]. The material from this appendix is only needed in Chapter 2, where it is a prerequisite for Sections 3.1-3.3 (except for Theorem 3.1.4 and Corollary 3.1.5).

B.1 Banach Lattices and Banach Function Spaces

B.1.1 Banach Lattices

A binary relation $\leq$ on a set P is called a *partial order* if it is reflexive, antisymmetric, and transitive. In this situation we write, as usual, $x \geq y$ if (and only if) $y \leq x$ for $x, y \in P$. Any set equipped with a partial order is called a *partially ordered set* (poset). If $(P, \leq)$ is a poset, then we write $P^+ := \{x \in P \,:\, 0 \leq x\}$.

Let $(P, \leq)$ be a poset and $S \subset P$. An element $u \in P$ is called an upperbound of S if $s \leq u$ for all $s \in S$. An *upperbound* $u \in P$ of S is called the *supremum* (or the *least upperbound*) of S if $u \leq x$ for every upperbound $x \in P$ of S. The notions of *lowerbound* and *infimum* (or *greatest lower bound*) are defined in the same way with '$\leq$' replaced by '$\geq$' (and with upperbound replaced by lowerbound). A poset P is called Dedekind complete if every subset which has an upperbound also has a supremum and if every subset which has a lower bound also has an infimum. A poset P is called σ-Dedekind complete if every countable subset which has an upperbound also has a supremum and if every countable subset which has a lower bound also has an infimum.

A poset $(P, \leq)$ is called a *lattice* if every pair of elements $a, b \in P$ has a supremum and an infimum, which are then denoted by $a \vee b$ and $a \wedge b$, respectively.

Definition B.1.1. Let X be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(i) X is called a *partially ordered vector space* if it is equipped with a partial order $\leq$ satisfying:

  (a) If $x \leq y$, then $x + z \leq y + z$ for each $z \in X$;

  (b) If $x \leq y$, then $\alpha x \leq \alpha y$ for each $\alpha \geq 0$.

(ii) X is called a *Riesz space* if it is a partially ordered vector space which is equipped with a surjective idempotent mapping $|\cdot| : X \longrightarrow X^+$, called the *modulus* of X, satisfying the following properties:

(a) $|\lambda x| = |\lambda|\,|x|$ for every $x \in X$ and $\lambda \in \mathbb{K}$ (homogeneity);

(b) $|x + y| \leq |x| + |y|$ for every $x, y \in X$ (subadditivity);

(c) $X = \text{span}_{\mathbb{K}}(X^+)$ (generating cone).

(iii) X is called a *normed Riesz space* if it is a Riesz space which is equipped with a norm having the property that $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$.

(iv) X is called a *Banach lattice* if it is a normed Riesz space which is norm complete.

Let X and Y be two Riesz spaces. A Riesz space homomorphism from X to Y is a linear map $T : X \longrightarrow Y$ with the property that $T|x| = |Tx|$ for all $x \in X$. Note that such a map automatically satisfies $TX^+ \subset Y^+$. A Riesz space isomorphism from X to Y is a bijective Riesz space homomorphism $T : X \longrightarrow Y$. Note that $T^{-1} : Y \longrightarrow X$ then automatically is a Riesz space homomoprhism; in particular, $TX^+ = Y^+$.

Let X and Y be two Banach lattices. An isomorphism of Banach lattices from X to Y is Riesz space isomorphism from X to Y which at the same time is an isomorphism of Banach spaces.

Remark B.1.2.

(i) If X is partially ordered vector space, then $x \leq y$ if and only if $y - x \in X^+$.

(ii) If X is a Riesz space, then $x \in X^+$ if and only if $|x| = x$.

(iii) If X is a norm Riesz space, then $\|x\| = \|\,|x|\,\|$ for every $x \in X$.

(iii) In the complex case $\mathbb{K} = \mathbb{C}$, a Banach lattice is not a lattice in the sense of posets.

Remark B.1.3. The above axiomatic definition of Riesz spaces (and therefore that of normed Riesz spaces and Banach lattices) is based on [80]; here the authors actually start with a so-called modulus m on X and define $\leq_m$ as the partial order generated by the cone $X^+ := m(X)$, which can be seen to be equivalent to (ii) above (under the correspondence $|\cdot| = m$ and $\leq_m = \leq$). It is not difficult to see that, in case $\mathbb{K} = \mathbb{R}$, this definition is equivalent with the usual definition of a Riesz space, for which $|\cdot|$ is the usual absolute value (or modulus). Furthermore, in case $\mathbb{K} = \mathbb{C}$ it can be shown that $X_{\mathbb{R}} := X^+ - X^+$ is a Riesz space under the induced order and modulus, for which we have $X = X_{\mathbb{R}} \oplus \imath X_{\mathbb{R}}$ (direct sum over $\mathbb{R}$), or equivalently $X = (X_{\mathbb{R}})_{\mathbb{C}}$, and that $(X, \leq, |\cdot|)$ coincides with the usual Riesz space complexification of $X_{\mathbb{R}}$ in case $X_{\mathbb{R}}$ is Archimedean and uniformly complete[1]. In particular, since real Banach lattices are Archimedean and uniformly complete, our definition of complex Banach lattice coincides with the usual one via complexification; here we of course use that the norms are in both situations uniquely determined on X^+.

Before we state some basic properties of (normed) Riesz spaces and Banach lattices, let us first give some basic examples:

Example B.1.4. Fix a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(i) $\mathbb{K}$ is a Banach lattice

[1]For Riesz space complexifaction, one imposes certain conditions (such as Archimedean and uniformly complete) to guarantee the existence of certain suprema in order to extend the absolute value to the complexification.

(ii) Let $(S, \mathscr{A}, \mu)$ be a measure space and let X be a Banach lattice. Then $L^0(S; X)$, the space of equivalence classes of X-valued strongly measurable functions on $(S, \mathscr{A}, \mu)$, is a Riesz space for the pointwise almost everywhere induced order and modulus from X.

(iii) Let $(S, \mathscr{A}, \mu)$ be a measure space, $p \in [1, \infty]$, and X a Banach lattice. Then $L^p(S; X)$ is a Banach lattice for the order and modulus induced from $L^0(S; X)$.

(iv) Let K be a compact Hausdorff space and let X be a Banach lattice. Then $C(K; X)$ is a Banach lattice for pointwise induced order and modulus from X.

Proposition B.1.5. *Let X be a Riesz space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then $X_{\mathbb{R}} := X^+ - X^-$ is a lattice in the sense of posets, for which we have*

$$|x| = x \vee (-x), \qquad x \in X_{\mathbb{R}}.^{2}$$

Furthermore, in case $\mathbb{K} = \mathbb{R}$ we have $X = X_{\mathbb{R}}$, and in case $\mathbb{K} = \mathbb{C}$ we have $X = X_{\mathbb{R}} \oplus \imath X_{\mathbb{R}}$ (direct sum over $\mathbb{R}$).

Proof. We refer to [80] (also see the beginning of the discussion in Remark B.1.3). □

We finally introduce several properties that a Banach lattice (or a normed Riesz space) can possess. All of these properties provide connections between the order and the norm. We first need to introduce a little bit notation and terminology: Let X be a partially ordered vector space. Let $\{x_\alpha\}_{\alpha \in A}$ be a net in X. We say that $\{x_\alpha\}$ is increasing (resp. decreasing) if $x_\alpha \leq x_{\alpha'}$ (resp. $x_\alpha \geq x_{\alpha'}$) whenever $\alpha \leq \alpha'$. We write $x_\alpha \uparrow x$ (resp $x_\alpha \downarrow x$) to indicate that $\{x_\alpha\}$ is increasing (resp. decreasing) and has a supremum $x = \sup_\alpha$ (resp. infumum $x = \inf_\alpha x_\alpha$) in X. If $\{x_\alpha\}_\alpha$ is an increasing net which converges to x in the norm topology of a normed Riesz space X, then we have $x_\alpha \uparrow x$.

Definition B.1.6. Let X be a normed-Riesz space. We say that X (or the norm $\|\cdot\|_X$ of X) is

- *order continuous*: $0 \leq x_\alpha \downarrow 0$ implies $\|x_\alpha\| \downarrow 0$
- *σ-order continuous*: $0 \leq x_k \downarrow 0$ implies $\|x_k\| \downarrow 0$;
- *Fatou*: $0 \leq x_\alpha \uparrow x$ implies $\|x_\alpha\| \uparrow \|x\|$;
- *σ-Fatou*: $0 \leq x_k \uparrow x$ implies $\|x_k\| \uparrow \|x\|$;
- *Levi*: every increasing norm bounded net has a supremum;
- *σ-Levi*: every increasing norm bounded sequence has a supremum.

Definition B.1.7. A Banach lattice X is called a *KB-space* (or a *Kantorovich-Banach space*) if every increasing norm bounded sequence is norm convergent, which is actually equivalent to the property that every norm bounded net is norm convergent.

Let $(S, \mathscr{A}, \mu)$ be a measure space and let $p \in [1, \infty]$. Then $L^p(S; \mathbb{K})$ is σ-Fatou and σ-Levi. If $p < \infty$, then $L^p(S; \mathbb{K})$ has all the above properties. In general, the following relations hold true:

[2] So $X_{\mathbb{R}}$ is a real Riesz space in the usual way with absolute value $|\cdot|$.

Proposition B.1.8.

(i) For normed Riesz spaces X we have:

(a) X is order continuous $\Longrightarrow$ X is σ-order continuous $\Longrightarrow$ X is σ-Fatou;

(b) X is order continuous $\Longrightarrow$ X is Fatou $\Longrightarrow$ X is σ-Fatou.

(ii) For Banach lattices X we have:

(a) X is reflexive $\Longrightarrow$ X is a KB-space $\Longrightarrow$ X is order continuous;

(b) X is order continuous $\Leftrightarrow$ X is σ-order continuous & σ-Dedekind complete;

(c) X is order continuous $\Longrightarrow$ X is Levi $\Longrightarrow$ X is Dedekind complete;

(d) X is Levi $\Longrightarrow$ X is σ-Levi $\Longrightarrow$ X is σ-Dedekind complete.

Here (i).(a) and (ii).(b) are frequently used in Chapter 3.

B.1.2 Banach Function Spaces

An *ideal* in a Riesz space X is a linear subspace A of X with the additional property that $x \in X$, $y \in A$ and $|x| \leq |y|$ imply $x \in A$. Note that every ideal A of X is a Riesz space on its own right for the restricted order and modulus.

Throughout this subsection we fix a σ-finite measure space $(S, \mathscr{A}, \mu)$ and a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We shall write $L^0(S) = L^0((S, \mathscr{A}, \mu); \mathbb{K})$.

Definition B.1.9. A Banach function space E on $(S, \mathscr{A}, \mu)$ is an ideal of $L^0(S)$ which is equipped with a norm which turns it into a Banach lattice.

Examples of Banach function space are L^p-spaces ($p \in [1, \infty]$) and Orlicz spaces.

Definition/Proposition B.1.10. Let E be an ideal in $L^0(S)$. Then there exists a smallest (with respect to μ-a.e. inclusion) set $C_E \in \mathscr{A}$ such that every $f \in E$ vanishes μ-a.e. on $S \setminus C_E$. This set[3] is called the *support* (or the *carrier*) of the ideal E, and is denoted by $\mathrm{supp}(E)$. In case $\mathrm{supp}(E) = S$ we say that E has full support.

The Banach function space $E = L^p(S)$ ($p \in [1, \infty]$) has support $\mathrm{supp}(E) = S$. In the general case, if E is a Banach function space on $(S, \mathscr{A}, \mu)$, then E can also be viewed as a Banach function space on restricted measure space $(\mathrm{supp}(E), \mathscr{A}_{\mathrm{supp}(E)}, \mu_{\mathrm{supp}(E)})$ in the natural way. The measure space $(\mathrm{supp}(E), \mathscr{A}_{\mathrm{supp}(E)}, \mu_{\mathrm{supp}(E)})$ also being σ-finite, it is no restriction to assume $\mathrm{supp}(E) = S$.

Lemma B.1.11. *Let E be a Banach function space on $(S, \mathscr{A}, \mu)$ with $\mathrm{supp}(E) = S$. Then there exists a $u \in E^+$ such that $u(s) > 0$ for μ-almost every $s \in S$.*

Let E be a Banach function space on $(S, \mathscr{A}, \mu)$ and let $r \in]0, \infty[$. We define

$$E^r := \{f \in L^0(S) \, : \, |f|^{1/r} \in E\}, \qquad \|f\|_{E^r} := \left\| \, |f|^{1/r} \right\|_E^r. \tag{B.1}$$

Then E^r is an ideal of $(S, \mathscr{A}, \mu)$, but $\| \cdot \|_{E^r}$ is in general not a norm for $r > 1$. If $r \leq 1$, then we have that $(E^r, \| \cdot \|_{E^r})$ is a Banach function space on $(S, \mathscr{A}, \mu)$. Basic examples are:

[3] In fact this equivalence class of sets, which can be formulated in terms of the measure algebra of $(S, \mathscr{A}, \mu)$.

- If $1 \leq q \leq p < \infty$, then we have $L^p(S) = [L^q(S)]^{q/p}$;
- $[L^\infty(S)]^r = L^\infty(S)$ for each $r \in]0, \infty[$.

Finally, we come to duality of Banach function spaces. The *Köthe dual* of a Banach function space E on $(S, \mathscr{A}, \mu)$ is the ideal $E^\times$ of $L^0(A)$ defined by

$$E^\times := \{g \in L^0(A) \, : \, fg \in L^1(A), \forall f \in E\},$$

and is equipped with the seminorm

$$\|g\|_{E^\times} := \left\{ \left| \int_S fg \, d\mu \right| \, : \, f \in E, \|f\|_E \leq 1 \right\}.$$

For example, if $p \in [1, \infty[$ and $p \in]1, \infty]$ are Hölder conjugates, then we have $[L^p(S)]^\times = L^{p'}(S) \simeq [L^p(S)]^*$. In general, the following holds true:

Theorem B.1.12. *Let E be a Banach function space on $(S, \mathscr{A}, \mu)$ with* $\operatorname{supp}(E) = S$. *Then $E^\times$ is a Banach function space with* $\operatorname{supp}(E^\times) = S$ *which has a σ-Fatou norm. If E has a σ-order continuous norm, then*

$$E^\times \longrightarrow E^*, \; g \mapsto \Lambda_g := [e \mapsto \int_A ge, d\mu]$$

defines is an isometric isomorphism of Banach spaces (even an isometric isomorphism of Banach lattices for the natural order and modulus on E^*).

B.2 Köthe-Bochner Spaces and Mixed-Norm Spaces

B.2.1 Köthe-Bochner Spaces

Let σ-finite measure space $(S, \mathscr{A}, \mu)$ and let X be a Banach space.

Similarly to the definition of the Lebesgue-Bochner spaces $L^p(S; X)$, we define the Köthe-Bochner space $E(X)$:

Definition B.2.1. Let E be a Banach function space on $(S, \mathscr{A}, \mu)$. Then we define the *Köthe-Bochner space* $E(X)$ as the linear space

$$E(X) := \{f \in L^0(S; X) \mid \|f\|_X \in E\}$$

equipped with the norm

$$\|f\|_{E(X)} := \big\| \|f\|_X \big\|_E .$$

It can be shown that $E(X)$ is a Banach space. Moreover, if X is a Banach lattice, then so is $E(X)$ (with respect to the order and modulus induced form $L^0(S; X)$). When X is a Banach function space on a σ-finite measure space $(T, \mathscr{B}, \nu)$, then elements of $L^0(S; X)$ (and thus in particular elements of $E(X)$) can be naturally identified with elements of $L^0(S \times T)$; in fact, we even have:

Lemma B.2.2. *Let $(S, \mathscr{A}, \mu)$ and $(T, \mathscr{B}, \nu)$ be two σ-finite measure spaces. Suppose that F is a Banach function space on $(T, \mathscr{B}, \nu)$. Then, for every strongly μ-measurable function $g : S \longrightarrow F(X)$ there exists a strongly $\mu\otimes\nu$-measurable $\tilde{g} : S \times T \longrightarrow X$, which is unique as an element of $L^0(S \times T; X)$, such that $\tilde{g}(s, \cdot\,) = g(s)$ in $F(X)$ for μ-a.a. $s \in S$. Moreover, if $g : S \longrightarrow F(X)$ is Bochner integrable, then we have, for ν-a.a. $t \in T$, that $\tilde{g}(\,\cdot\,, t) : S \longrightarrow X$ is Bochner integrable with*

$$\left(\int_A g\, d\mu\right)(t) = \int_A \tilde{g}(\,\cdot\,, t)\, d\mu, \qquad A \in \mathscr{A}.$$

Corollary B.2.3. *Let F be a Banach function space on the σ-finite measure space $(T, \mathscr{B}, \nu)$ and let X be a Banach space.*

(i) Let $g \in L^1_{loc}(\mathbb{R}; F(X))$ and let $\tilde{g} \in L^0(\mathbb{R}^n \times T; X)$ be as in Lemma B.2.2. Then we have, for ν-a.a. $t \in T$, that $\tilde{g}(\,\cdot\,, t) \in L^1_{loc}(\mathbb{R}^n; X)$ with

$$\left(\int_A g\, d\lambda\right)(t) = \int_A \tilde{g}(\,\cdot\,, t)\, d\lambda, \qquad A \in \mathcal{B}(\mathbb{R}^n), \lambda(A) < \infty.$$

(ii) Let $(S, \mathscr{A}, \mu)$ be a measure space equipped with a filtration $\mathbb{F} = (\mathscr{F}_k)_{k\in\mathbb{Z}}$ for which each $\mathscr{F}_k$ is countably atomic with respect to μ in the sense of Definition A.1.2. Let $g \in L^1_\sigma((S, \mathscr{A}, \mu), \mathbb{F}; F(X)) \subset L^0(S; F(X))$ and let $\tilde{g} \in L^0(S \times T; X)$ be as in Lemma B.2.2. Then we have, for ν-a.a. $t \in T$, that $\tilde{g}(\,\cdot\,, t) \in L^1_\sigma((S, \mathscr{A}, \mu), \mathbb{F}; X)$ with

$$\mathbb{E}[g \mid \mathscr{F}_k](t) = \left(\sum_{D\in\mathscr{F}_k^{atom}} 1_D \fint_D g\, d\mu\right)(t) = \sum_{D\in\mathscr{F}_k^{atom}} 1_D \fint_D \tilde{g}(\,\cdot\,, t)\, d\mu = \mathbb{E}[\tilde{g}(\,\cdot\,, t) \mid \mathscr{F}_k], \qquad k \in \mathbb{Z};$$

also see Examples A.3.4.(v) and A.3.9.

B.2.2 Mixed-Norm Spaces

For the definition of mixed-norm spaces we need the following measurability result:

Lemma B.2.4. *Let $(S, \mathscr{A}, \mu)$ and $(T, \mathscr{B}, \nu)$ be two σ-finite measure spaces. Suppose that F is a Banach function space on $(T, \mathscr{B}, \nu)$ with a σ-Fatou norm. For every strongly $\mu \otimes \nu$-measurable function $\tilde{g} : S \times T \longrightarrow X$ it holds that $s \mapsto \|\tilde{g}(s, \cdot\,)\|_F$ is μ-measurable.*

Definition B.2.5. Let E be a Banach function space on the σ-finite measure space $(S, \mathscr{A}, \mu)$ and let F be a Banach function space on the σ-finite measure space $(T, \mathscr{B}, \nu)$. Suppose that F has a σ-Fatou norm. Then we define the *mixed-norm space* $E[F]$ as the linear space

$$E[F] := \{\tilde{g} \in L^0(S \times T) \mid s \mapsto \|\tilde{g}(s, \cdot\,)\|_F \in E\}$$

equipped with the norm

$$\|\tilde{g}\|_{E[F]} := \big\| s \mapsto \|\tilde{g}(s, \cdot\,)\|_F \big\|_E .$$

It can be shown that the mixed-norm space $E[F]$ is a Banach function space on $(S \times T, \mathscr{A} \otimes \mathscr{B}, \mu \otimes \nu)$. The next lemma shows that we can iterate the construction of mixed-norm spaces.

Lemma B.2.6. *Let E and F be two Banach function spaces on σ-finite measure spaces. If both E and F have σ-Fatou norms, then the mixed-norm space $E[F]$ has a σ-Fatou norm as well. As a consequence, the mixed-norm space $D[E[F]]$ is well defined whenever D is a Banach function space on a σ-finite measure space. Moreover, it holds that $D[E][F] = D[E[F]]$.*

A natural question is under what conditions on E and F do we have a natural identification $E[F] \simeq E(F)$ of mixed-norm space and Köthe-bochner space. For $E = L^p(S)$ ($p \in [1,\infty[$) and $F = L^q(S)$ ($q \in [1,\infty[$) this is easy; this can for instance be found in [57]. In general, the following holds true:

Theorem B.2.7. *Let E and F be Banach function spaces on the σ-finite measure spaces $(S,\mathscr{A},\mu)$ and $(T,\mathscr{B},\nu)$, respectively, and let X be a Banach space. Suppose that F has a σ-Fatou norm. Then, for each $g \in E(F(X))$ we have $\tilde{g} \in E[F](X)$ for the unique $\tilde{g} \in L^0(S \times T; X)$ from Lemma B.2.2. The induced linear mapping $j : E(F(X)) \longrightarrow E[F](X)$, $g \mapsto \tilde{g}$ is an isometry which is a homomorphism of Banach lattices in case X is a Banach lattice. Moreover, we have $j(E(F(X))) = E[F](X)$ provided that F has a σ-order continuous norm.*

Next we come to duality of mixed-norm spaces.

Proposition B.2.8. *Let E and F be two Banach function spaces on σ-finite measure spaces, both having full support. Suppose that F has σ-Fatou norm. Then the Köthe dual of $E[F]$ is given by $E[F]^\times = E^\times[F^\times]$. If E and F both have σ-order continuous norms, then so has $E[F]$, so that we have a natural isometric isomorphism $E[F]^* \simeq E^\times[F^\times]$ of Banach lattices.*

Appendix C

Fourier Analysis and Distribution Theory

This appendix is mainly based on [4] and [96], where only Banach space-valued distributions and scalar-valued distributions are treated, respectively. In this appendix we also treat some aspects of the theory of distributions with values in a general (complete) locally convex space (LCS), having as main advantage the interpretation of the Schwartz kernel theorem in Section C.7 as a canonical identification between spaces of distributions. For a comprehensive treatment of the theory of distributions with values in locally convex spaces we refer to the original work of Schwartz [89, 90].

The reader is assumed to have experience with the basics of distribution theory and to be familiar with the contents of this appendix throughout the thesis (with an exception for Sections 3.2 and 3.3 of Chapter 3).

C.1 Some Spaces of Functions

C.1.1 Functions With Values in a Banach Space

Throughout this subsection we fix a Banach space X. Let $U \subset \mathbb{R}^d$ be a non-empty open subset.

$\mathcal{E}(U;X)$**: Smooth Functions** We define

$$\mathcal{E}(U;X) := C^\infty(U;X)$$

endowed with the locally convex topology induced by the family of seminorms $\{\|\cdot\|_{K,r} \,:\, K \subset U \text{ compact}, r \in \mathbb{N}\}$ given by

$$\|f\|_{K,r} := \max_{|\alpha|\leq r} \|\partial^\alpha f\|_{\infty,K} = \sup\{\|\partial^\alpha f(x)\|_X \,:\, x \in K, |\alpha| \leq r\}.$$

In this way, $\mathcal{E}(U;X)$ is a Fréchet space.

$\mathcal{D}(U;X)$**: Test Functions** We define

$$\mathcal{D}(U;X) := C_c^\infty(U;X),$$

the space of smooth compactly supported functions, with the following topology. For each compact subset $K \subset U$, we consider

$$\mathcal{E}_K(U;X) := \{f \in \mathcal{E}(U;X) \,:\, \operatorname{supp} f \subset K\},$$

endowed with the topology induced from $\mathcal{E}(U;X)$ (which coincides with the topology induced by the collection of seminorms $\{\|\cdot\|_{K,r} : r \in \mathbb{N}\}$). Then we have

$$\mathcal{D}(U;X) = \bigcup_{K\subset U \text{ compact}} \mathcal{E}_K(U;X),$$

and we consider the inductive limit topology on $\mathcal{D}(U;X)$. Then $\mathcal{D}(U;X)$ is a complete Hausdorff LCS.

$\mathcal{S}(\mathbb{R}^d;X)$**: Schwartz Functions** A *Schwartz function* is a smooth function $f : \mathbb{R}^d \longrightarrow X$ with the property that, for all $\alpha, \beta \in \mathbb{N}^d$,

$$\|f\|_{\alpha,\beta} := \sup_{x\in\mathbb{R}^d} \left\|x^\beta \partial^\alpha f(x)\right\|_X < \infty.$$

We define $\mathcal{S}(\mathbb{R}^d;X)$ as the space of Schwartz functions, endowed with the locally convex topology induced by the family of seminorms $\{\|\cdot\|_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}^d\}$, or equivalently, with the locally convex topology induced by the family of seminorms $\{p_N\}_{N\in\mathbb{N}}$ given by

$$p_N f_N := \sup_{|\alpha|,|\beta|\leq N} \|f\|_{\alpha,\beta} = \sup_{|\alpha|,|\beta|\leq N, x\in\mathbb{R}^d} \left\|x^\beta \partial^\alpha f(x)\right\|_X. \tag{C.1}$$

In this way, $\mathcal{S}(\mathbb{R}^d;X)$ is a Fréchet space.

For the just defined spaces it holds that

$$\mathcal{D}(U;X) \overset{d}{\hookrightarrow} \mathcal{E}(U;X) \quad \text{and} \quad \mathcal{D}(\mathbb{R}^d;X) \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^d;X) \overset{d}{\hookrightarrow} \mathcal{E}(\mathbb{R}^d;X). \tag{C.2}$$

$\mathcal{O}_M(\mathbb{R}^d;X)$**: Slowly Increasing Smooth Functions** A *slowly increasing smooth function* (or a *smooth function of moderate growth*) is a smooth function $f : \mathbb{R}^d \longrightarrow X$ such that, for each $\alpha \in \mathbb{N}^d$, there exist $m_\alpha \in \mathbb{N}$ and $c_\alpha > 0$ such that

$$|\partial^\alpha f(x)| \leq c_\alpha (1+|x|^2)^{m_\alpha}, \qquad x \in \mathbb{R}^d.$$

We define $\mathcal{O}_M(\mathbb{R}^d;X)$ as the space of all X-valued slowly increasing smooth functions on $\mathbb{R}^d$, equipped the locally convex Hausdorff topology generated by the seminorms

$$f \mapsto \|\psi D^\alpha f\|_\infty, \quad \psi \in \mathcal{S}(\mathbb{R}^d), \alpha \in \mathbb{N}^d.$$

In this way, $\mathcal{O}_M(\mathbb{R}^d;X)$ is a complete LCS.

Some notation for spaces of functions (which we do not topologize):

- $C^\infty_{(c)}(U;X) := \{g = f_{|U} : f \in C^\infty_c(\mathbb{R}^d;X)\}$,
- $\mathcal{S}(U;X) := \{g = f_{|U} : f \in \mathcal{S}(\mathbb{R}^d;X)\}$

C.1.2 Functions With Values in a locally Convex Space

Throughout this subsection we fix a LCS X.

$C(S\,;X)$**: Continuous Functions** Let $S \subset \mathbb{R}^d$ be a subset. We equip the space of continuous functions $C(S\,;X)$ with the locally convex topology generated by the the seminorms

$$f \mapsto \sup_{y\in K} p(f(y)), \qquad K \subset S \text{ compact}, p \text{ a continuous seminorm on } X.$$

If S admits an exhaustion by compacts and if X is a Fréchet space, then $C(S\,;X)$ is a Fréchet space as well.

$C_b(S\,;X)$**: Bounded Continuous Functions** Let $S \subset \mathbb{R}^d$ be a subset. We equip the space of bounded continuous functions $C_b(S\,;X)$ with the locally convex topology generated by the the seminorms

$$f \mapsto \sup_{y\in S} p(f(y)), \qquad p \text{ a continuous seminorm on } X.$$

$\mathcal{S}(\mathbb{R}^d;X)$**: Schwartz Functions** Let X be a LCS. A *Schwartz function* is a smooth function $f:\mathbb{R}^d \longrightarrow X$ with the property that, for all continuous seminorms p on X and all multi-indices $\alpha,\beta \in \mathbb{N}^d$,

$$\|f\|_{p,\alpha,\beta} := \sup_{x\in\mathbb{R}^d} p(x^\beta \partial^\alpha f(x)) < \infty.$$

We define $\mathcal{S}(\mathbb{R}^d;X)$ as the space of Schwartz functions, endowed with the locally convex topology generated by the family of seminorms

$$\{\,\|\cdot\|_{p,\alpha,\beta} : p \text{ a continuous seminorm on } X, \alpha,\beta \in \mathbb{N}^d\,\}.$$

Note we may restrict p to a generating family for X; in particular, if X is a normed space, we may restrict to $p = \|\cdot\|$.

C.2 Spaces of Vector-Valued Distributions

Throughout this subsection we let X be a LCS.

$\mathcal{D}'(U;X)$**: Distributions** Let $U \subset \mathbb{R}^d$ be a non-empty open subset. We define

$$\mathcal{D}'(U;X) := \mathcal{L}(\mathcal{D}(U),X),$$

the *the space of X-valued distributions on U*, equipped with the topology of bounded convergence. Note that for $X = \mathbb{C}$ we get the usual space of distributions $\mathcal{D}'(U)$. The support $\mathrm{supp}(f)$ of a distribution $f \in \mathcal{D}'(\mathbb{R}^d;X)$ is as defined in the scalar-valued case $X = \mathbb{C}$.

$\mathcal{E}'(U;X)$**: Compactly Supported Distributions** Let $U \subset \mathbb{R}^d$ be a non-empty open subset. We define

$$\mathcal{E}'(U;X) := \mathcal{L}(\mathcal{E}(U),X),$$

the *the space of X-valued compactly supported distributions on U*, equipped with the topology of bounded convergence. In view of the first inclusion in (C.2), we have

$$\mathcal{E}'(U;X) \hookrightarrow \mathcal{D}'(U;X)$$

canonically. Via this identification, $\mathcal{E}'(U;X)$ corresponds to the distributions in $\mathcal{D}'(U;X)$ having compact support.

$\mathcal{S}'(U;X)$**: Tempered Distributions** We define

$$\mathcal{S}'(\mathbb{R}^d;X) := \mathcal{L}(\mathcal{S}(\mathbb{R}^d),X),$$

the *the space of X-valued tempered distributions on* $\mathbb{R}^d$, equipped with the topology of bounded convergence. Observe that a linear operator $f : \mathcal{S}(\mathbb{R}^d) \longrightarrow X$ is continuous if and only if, for each $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that

$$\|\langle f,\phi\rangle\|_X \leq C_N p_N(\phi) \qquad (\phi \in \mathcal{S}(\mathbb{R}^d)), \tag{C.3}$$

where $(p_N)_{N\in\mathbb{N}}$ is the generating family of seminorms for $\mathcal{S}(\mathbb{R}^d)$ given in (C.1). In view of (C.2), we have

$$\mathcal{E}'(U;X) \hookrightarrow \mathcal{D}'(U;X) \quad \text{and} \quad \mathcal{E}'(\mathbb{R}^d;X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d;X) \hookrightarrow \mathcal{D}'(\mathbb{R}^d;X).$$

The partial derivative operators D^α, $\alpha \in \mathbb{N}^d$, are defined on $\mathcal{D}'(U;X)$, $\mathcal{E}'(U;X)$, and $\mathcal{S}'(U;X)$ in the usual way giving rise to continuous linear operators. For example, given $f \in \mathcal{D}'(U;X)$, we have

$$(D^\alpha f)(\phi) = (-1)^{|\alpha|} f(D^\alpha \phi), \qquad \phi \in \mathcal{D}(U).$$

Since $\mathcal{D}(U)$, $\mathcal{E}(U)$ and $\mathcal{S}(\mathbb{R}^d)$ are Montel spaces, for the convergence of sequences in the corresponding spaces of distributions it is irrelevant whether we work with the topology of bounded convergence or the topology of pointwise convergence:

Proposition C.2.1. *A sequence* $(f)_{n\in\mathbb{N}}$ *converges with limit* f *in* $\mathcal{D}'(U;X)$ *(with respect to the topology of bounded convergence) if and only if it converges with respect to the to the topology of pointwise convergence with limit* f*. As a consequence, if* Z *is a sequential topological space (e.g. first countable), then a function* $F : Z \longrightarrow \mathcal{D}'(U;X)$ *is continuous if and only if it is continuous with respect to the topology of pointwise convergence on* $\mathcal{D}'(U;X)$*. The same statement holds true for* $\mathcal{E}'(U;X)$ *and* $\mathcal{S}'(\mathbb{R}^d;X)$*.*

Note that, if X is a sequentially complete LCS, then $\mathcal{D}'(U;X)$, $\mathcal{E}'(U;X)$ and $\mathcal{S}'(\mathbb{R}^d;X)$ are sequentially complete as well with respect to the topology of pointwise convergence (as a consequence of the Banach-Steinhaus theorem because $\mathcal{D}(U)$, $\mathcal{E}(U)$ and $\mathcal{S}(\mathbb{R}^d)$ are barreled).

Regular Distributions in the case that X **is a Banach space** Let $U \subset \mathbb{R}^d$ be a non-empty open subset. To each $f \in L^1_{loc}(U;X)$ we associate the distribution

$$\Lambda_f : \mathcal{D}(U) \longrightarrow X,\ \phi \mapsto \int_U f(x)\phi(x)\,dx.$$

In this way we obtain

$$L^1_{loc}(U;X) \hookrightarrow \mathcal{D}'(U;X),$$

and we identify $L^1_{loc}(U;X)$ with a subspace of $\mathcal{D}'(U;X)$. Distributions belonging to $L^1_{loc}(U;X)$ are often called *regular distributions*.

Examples of spaces consisting of regular distributions are $\mathcal{D}(U;X)$, $\mathcal{S}(\mathbb{R}^d;X)$, $\mathcal{E}(U;X)$, $C(U;X)$, $C(\overline{U};X)$, $L^p(U;X)$, $p \in [1,\infty]$. Here we have

$$\mathcal{D}(U;X), \mathcal{E}(U;X), C(U;X), C(\overline{U};X), L^p(U;X) \hookrightarrow \mathcal{D}'(U;X)$$

and

$$\mathcal{D}(\mathbb{R}^d;X), \mathcal{S}(\mathbb{R}^d;X), L^p(\mathbb{R}^d;X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d;X).$$

Regular Distributions in the case that X is a complete LCS Let $U \subset \mathbb{R}^d$ be a non-empty open subset. To each $f \in C(U;X)$ we associate the distribution

$$\Lambda_f : \mathcal{D}(U) \longrightarrow X,\ \phi \mapsto \int_U f(x)\phi(x)\,dx,$$ [1]

yielding the canonical continuous inclusion

$$C(\overline{U};X) \hookrightarrow C(U;X) \hookrightarrow \mathcal{D}'(U;X). \tag{C.4}$$

Analogously, to each $f \in C_b(\mathbb{R}^d;X)$ we associate the tempered distribution

$$\Lambda_f : \mathcal{S}(\mathbb{R}^d) \longrightarrow X,\ \phi \mapsto \int_{\mathbb{R}^d} f(x)\phi(x)\,dx,$$ [2]

yielding the canonical continuous inclusion

$$\mathcal{S}(\mathbb{R}^d;X) \hookrightarrow C_b(\mathbb{R}^d;X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d;X).$$

C.3 Approximations

In this section we assume that X is a Banach space and that $U \subset \mathbb{R}^d$ is a non-empty open subset.

Proposition C.3.1.

- *$\mathcal{D}(U;X)$ is sequentially dense in $\mathcal{D}'(U;X)$, $\mathcal{E}'(U;X)$;*
- *$\mathcal{D}(\mathbb{R}^d;X)$ is sequentially dense in $\mathcal{S}'(\mathbb{R}^d;X)$.*

As usual, we view the algebraic tensor product $L^0(U)\otimes X$ as linear subspace of $L^0(U;X)$ by identifying a tensor $f\otimes x$, where $f \in L^0(U)$ and $x \in X$, with the element element $f\otimes x \in L^0(U;X)$ given by

$$(f \otimes x)(y) := f(y)x, \qquad y \in \mathbb{R}^d.$$

In this way, we also have $\mathcal{D}(U)\otimes X \subset \mathcal{D}(U;X)$, $\mathcal{S}(\mathbb{R}^d)\otimes X \subset \mathcal{S}(\mathbb{R}^d;X)$, and $\mathcal{E}(U)\otimes X \subset \mathcal{E}(U;X)$.

Analogously, we view the algebraic tensor product $\mathcal{D}'(U) \otimes X$ as a linear subspace of $\mathcal{D}'(U;X)$ by identifying a tensor $f \otimes x$, where $f \in \mathcal{D}'(U)$ and $x \in X$, with the X-valued distribution on $f \otimes x$ on U given by

$$(f \otimes x)(\phi) := f(\phi)x, \qquad \phi \in \mathcal{D}(U).$$

Similarly, we have $\mathcal{S}'(\mathbb{R}^d) \otimes X \subset \mathcal{S}'(\mathbb{R}^d;X)$ and $\mathcal{E}'(U) \otimes X \subset \mathcal{E}'(U;X)$.

In the following approximation theorem these algebraic tensor products are given the corresponding subspace topologies.

---

[1] For each $g \in C_c(U;X)$ there exists a unique vector $I_g \in X$ such that $\langle I_g, x^*\rangle = \int_U \langle g, x^*\rangle dx$ for every $x^* \in X^*$, which satisfies $p(I_g) \leq \int_U p(g(x))dx$ for every continuous seminorm on X. We write $I_g = \int_U g(x)dx$.

[2] Here $\int_{\mathbb{R}^d} f(x)\phi(x)\,dx$ is the unique vector in X satisfying $\langle \int_{\mathbb{R}^d} f(x)\phi(x)\,dx, x^*\rangle = \int_{\mathbb{R}^d}\langle f(x), x^*\rangle\phi(x)\,dx$ for all $x^* \in X^*$, whose existence can be obtained via a truncation argument from the existence of I_g for $g \in C(K;X)$, where $K \subset \mathbb{R}^d$ compact.

Theorem C.3.2.

(i) $\mathcal{D}(U) \otimes X \overset{d}{\hookrightarrow} \mathcal{D}(U; X) \overset{d}{\hookrightarrow} \mathcal{E}'(U; X) \overset{d}{\hookrightarrow} \mathcal{D}'(U; X)$*;*

(ii) $\mathcal{D}(U) \otimes X \overset{d}{\hookrightarrow} \mathcal{E}(U) \otimes X \overset{d}{\hookrightarrow} \mathcal{E}(U; X) \overset{d}{\hookrightarrow} \mathcal{D}'(U; X)$*;*

(iii) $\mathcal{E}'(U) \otimes X \overset{d}{\hookrightarrow} \mathcal{E}'(U; X)$*;*

(iv) $\mathcal{D}'(U) \otimes X \overset{d}{\hookrightarrow} \mathcal{D}'(U; X)$*;*

(v) $\mathcal{D}(\mathbb{R}^d) \otimes X \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^d) \otimes X \overset{d}{\hookrightarrow} \mathcal{S}(\mathbb{R}^d; X) \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^d; X) \overset{d}{\hookrightarrow} \mathcal{D}'(U; X)$*;*

(vi) $\mathcal{S}(\mathbb{R}^d) \otimes X \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^d) \otimes X \overset{d}{\hookrightarrow} \mathcal{S}'(\mathbb{R}^d; X)$.

C.4 Pointwise Multiplications

The usual pointwise multiplication $(f, g) \mapsto fg$ of functions restricts to hypocontinuous (and thus in particular separately continuous) bilinear maps

$$\begin{array}{llll} \mathcal{E}(U) & \times & \mathcal{D}(U) & \longrightarrow \mathcal{D}(U); \\ \mathcal{D}(U) & \times & \mathcal{E}(U) & \longrightarrow \mathcal{D}(U); \\ \mathcal{E}(U) & \times & \mathcal{E}(U) & \longrightarrow \mathcal{E}(U); \\ \mathscr{O}_M(\mathbb{R}^d) & \times & \mathcal{S}(\mathbb{R}^d) & \longrightarrow \mathcal{S}(\mathbb{R}^d). \end{array}$$

Given a locally convex space X, these maps induce pointwise multiplication maps

$$\begin{array}{llll} \mathcal{E}(U) & \times & \mathcal{D}'(U; X) & \longrightarrow \mathcal{D}'(U; X); \\ \mathcal{D}(U) & \times & \mathcal{E}'(U; X) & \longrightarrow \mathcal{D}'(U; X); \\ \mathcal{E}(U) & \times & \mathcal{E}(U; X) & \longrightarrow \mathcal{E}(U; X); \\ \mathscr{O}_M(\mathbb{R}^d) & \times & \mathcal{S}'(\mathbb{R}^d; X) & \longrightarrow \mathcal{S}'(\mathbb{R}^d; X), \end{array}$$

in the usual way; for example, given $\phi \in \mathcal{E}(U)$ and $f \in \mathcal{D}'(U; X)$, the distribution $\phi f \in \mathcal{D}'(U; X)$ is defined via $[\phi f](\psi) := f(\phi\psi)$, $\psi \in \mathcal{D}(U)$. Any two of these maps coincide with each other on the intersection of their domains.

Now suppose that

$$X_1 \times X_2 \longrightarrow X_0,\ (x_1, x_2) \mapsto x_1 \bullet x_2 \tag{C.5}$$

is a multiplication of Banach spaces. Recall that this means that (C.5) is a continuous bilinear map of norm at most 1; see Appendix A.1.4.

Theorem C.4.1. *There exists a unique hypocontinuous (and thus in a particular separately continuous) bilinear map*

$$\mathcal{E}(U; X_1) \times \mathcal{D}'(U, X_2) \longrightarrow \mathcal{D}'(U; X_0),\ (a, f) \mapsto a \bullet f,$$

called pointwise multiplication induced by (C.5)*, such that*

$$(\phi \otimes x_1) \bullet (\psi \otimes x_2) = \phi\psi \otimes (x_1 \bullet x_2)$$

for $a = \phi \otimes x_1 \in \mathcal{D}(U) \otimes X_1$ and $f = \psi \otimes x_2 \in \mathcal{D}(X) \otimes X_2$. Moreover, it restricts to hypocontinuous (and thus in particular separately continuous) bilinear maps

$$\begin{array}{lcll} \mathcal{E}(U;X_1) & \times & \mathcal{D}(U;X_2) & \longrightarrow \mathcal{D}(U;X_0); \\ \mathcal{E}(U;X_1) & \times & \mathcal{E}(U;X_2) & \longrightarrow \mathcal{E}(U;X_0); \\ \mathscr{O}_M(\mathbb{R}^d;X_1) & \times & \mathcal{S}(\mathbb{R}^d;X_2) & \longrightarrow \mathcal{S}(\mathbb{R}^d;X_0); \\ \mathcal{E}(U;X_1) & \times & \mathcal{D}'(U;X_2) & \longrightarrow \mathcal{D}'(U;X_0); \\ \mathcal{E}(U;X_1) & \times & \mathcal{E}(U;X_2) & \longrightarrow \mathcal{E}(U;X_0); \\ \mathscr{O}_M(\mathbb{R}^d;X_1) & \times & \mathcal{S}'(\mathbb{R}^d;X_2) & \longrightarrow \mathcal{S}'(\mathbb{R}^d;X_0) \end{array}$$

Since $X_2 \times X_1 \longrightarrow X_0$, $(x_2, x_1) \mapsto x_1 \bullet x_2$ is a multiplication as well, we can interchange the roles of X_1 and X_2 in the above theorem.

Some basic properties of the pointwise multiplication induced by (C.5):

Proposition C.4.2.

(a) If $a \in \mathcal{E}(U;X_1)$ and $f \in L^1_{loc}$, then $a \bullet f$, the pointwise multiplication induced by (C.5), *coincides with pointwise multiplication with respect to* (C.5) *in the usual sense.*

(b) ***Leibiz rule****: For every multi-index $\alpha \in \mathbb{N}^d$,*

$$\partial^\alpha(af) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (\partial^\beta a) \bullet (\partial^{\alpha-\beta} f), \qquad a \in \mathcal{E}(U;X_1), f \in \mathcal{D}'(U, X_2).$$

(c) If $(a, f) \in \mathcal{E}(U;X_1) \times \mathcal{D}'(U, X_2)$, then

$$\operatorname{supp}(a \bullet f) = \operatorname{supp}(a) \cap \operatorname{supp}(f).$$

C.5 Convolutions

We define the reflection operator $S \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d))$ by $(S\phi)(x) := \phi(-x)$; we also write $\widetilde{\phi} = S\phi$. Given a locally convex space X, we define the reflection operator $S \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^d;X))$ by

$$(Sf)(\phi) := f(S\phi), \qquad \phi \in \mathcal{S}(R^d). \tag{C.6}$$

Let X be a locally convex space. The convolution product

$$\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d;X) \longrightarrow \mathscr{O}_M(\mathbb{R}^d;X),\ (\phi, f) \mapsto \phi * f$$

is defined by

$$(\phi * f)(\psi) := f(\widetilde{\phi} * \psi), \qquad \psi \in \mathcal{S}(\mathbb{R}^d).$$

This is a hypocontinuous (and thus in particular a separately continuous) bilinear map.

Theorem C.5.1. *There exists a unique hypocontinuous (and thus in particular separately continuous) bilinear map*

$$\mathcal{S}(\mathbb{R}^d;X_1) \times \mathcal{S}'(\mathbb{R}^d;X_2) \longrightarrow \mathscr{O}_M(\mathbb{R}^d;X_0),\ (f, g) \mapsto f *_\bullet g,$$

called the convolution map induced by (C.5)*, such that*

$$(\phi \otimes x_1) *_\bullet (u \otimes x_2) = (\phi * u) \otimes (x_1 \bullet x_2)$$

for $f = \phi \otimes x_1 \in \mathcal{S}(\mathbb{R}^d) \otimes X_1$ *and* $g = u \otimes x_2 \in \mathcal{S}'(\mathbb{R}^d) \otimes X_2$. *Moreover, this convolution map restricts to a hypocontinuous bilinear map*

$$\mathcal{S}(\mathbb{R}^d; X_1) \times \mathcal{S}(\mathbb{R}^d; X_2) \longrightarrow \mathcal{S}(\mathbb{R}^d; X_0).$$

Since $X_2 \times X_1 \longrightarrow X_0$, $(x_2, x_1) \mapsto x_1 \bullet x_2$ is a multiplication as well, we can interchange the roles of X_1 and X_2 in the above theorem.

Proposition C.5.2.

(i) If $f \in \mathcal{S}(\mathbb{R}^d; X_1)$ *and* $g \in L^1_{loc}(\mathbb{R}^d; X_2) \cap \mathcal{S}'(\mathbb{R}^d; X_2)$*, then*

$$(f *_\bullet g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy = \int_{\mathbb{R}^d} f(y)g(x-y)dy, \qquad x \in \mathbb{R}^d;$$

see Proposition D.1.2 for the integrability of the integrand. So $f *_\bullet g$ *coincides with the convolution in the sense of measure theory* (A.2).

(ii) Given $f \in \mathcal{S}(\mathbb{R}^d; X_1)$, $g \in \mathcal{S}'(\mathbb{R}^d; X_2)$*, and* $\alpha \in \mathbb{N}^d$*, we have* $\operatorname{supp} f *_\bullet g \subset \operatorname{supp} f + \operatorname{supp} g$ *and* $D^\alpha (f *_\bullet g) = D^\alpha f *_\bullet g = f *_\bullet D^\alpha g$

C.6 The Fourier Transform

Let X be a Banach space. For an $f \in L^1(\mathbb{R}^d; X)$, the Fourier transformed function $\mathscr{F} f \in C_b(\mathbb{R}^d; X)$ is defined by the formula

$$(\mathscr{F} f)(\xi) := \int_{\mathbb{R}^d} e^{-\imath x \cdot \xi} f(x)\, dx, \qquad \xi \in \mathbb{R}^d. \tag{C.7}$$

We also write $\hat{f} = \mathscr{F} f$.

A function $f : \mathbb{R}^d \longrightarrow X$ is called *rapidly decreasing* if $x \mapsto x^\alpha f$ is bounded for every multi-index $\alpha \in \mathbb{N}^d$.

Proposition C.6.1. *Given a Banach space X, we have:*

(i) The Riemann-Lebesgue lemma: *The Fourier transform* $\mathscr{F}$ *is a continuous linear mapping of* $L^1(\mathbb{R}^d; X)$ *into* $C_0(\mathbb{R}^d; X)$.

(ii) The Fourier transform $\mathscr{F}$ *restricts to a topological linear isomorphism on* $\mathcal{S}(\mathbb{R}^d; X)$ *with inverse* $\mathscr{F}^{-1} = (2\pi)^{-d} \mathscr{F} \circ S = (2\pi)^{-d} S \circ \mathscr{F}$ *which is given by the formula*

$$(\mathscr{F}^{-1} g)(x) := \int_{\mathbb{R}^d} e^{\imath \xi \cdot x} g(\xi)\, d\xi. \tag{C.8}$$

(iii) If $f \in \mathcal{S}(\mathbb{R}^d; X)$ *and* $\alpha, \beta \in \mathbb{N}^d$*, then*

$$\mathscr{F}[x^\alpha D_x^\beta f] = (-D_\xi)^\alpha \xi^\beta \mathscr{F} f.$$

(iv) *If* $f \in L^1(\mathbb{R}^d;X)$ *is such that* $x^\alpha f \in L^1(\mathbb{R}^d;X)$ *for all* $|\alpha| \leq N$ *for some* $N \in \mathbb{N}$*, then* $\hat{f} \in C_b^N(\mathbb{R}^d;X)$ *with* $(-D_\xi)^\alpha \hat{f} = \mathscr{F}[x^\alpha f]$ *for every* $|\alpha| \leq N$*. In particular, if* $f : \mathbb{R}^d \longrightarrow X$ *is a strongly measurable rapidly decreasing function, then* $\hat{f} \in C_b^\infty(\mathbb{R}^d;X)$ *with* $(-D_\xi)^\alpha \hat{f} = \mathscr{F}[x^\alpha f]$ *for every* $|\alpha| \leq N$*.*

Given a locally convex space X, the Fourier transform $\mathscr{F}$ can be defined on $\mathcal{S}'(\mathbb{R}^d;X)$ by

$$(\mathscr{F}f)(\phi) = \hat{f}(\phi) := f(\hat{\phi}) = f(\mathscr{F}\phi), \qquad \phi \in \mathcal{S}(\mathbb{R}^d;X). \tag{C.9}$$

In this way, $\mathscr{F}$ is a topological linear isomorphism on $\mathcal{S}'(\mathbb{R}^d;X)$ with inverse $\mathscr{F}^{-1} = (2\pi)^{-d}\mathscr{F} \circ S = (2\pi)^{-d} S \circ \mathscr{F}$, where S is the reflection operator (C.6). We also write $\check{f} = \mathscr{F}^{-1}f$. Furthermore, (iii) above remains valid for $f \in \mathcal{S}'(\mathbb{R}^d;X)$. If X is a Banach space, then we have $L^1(\mathbb{R}^d;X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d;X)$ and the definitions (C.7) and (C.9) coincide.

Lemma C.6.2. *Suppose we are given a multiplication of Banach spaces* (C.5)*. If* $f \in \mathcal{S}(\mathbb{R}^d;X_1)$ *and* $g \in \mathcal{S}'(\mathbb{R}^d;X_2)$*, then* $\mathscr{F}[f \bullet g] = \mathscr{F}f *_\bullet \mathscr{F}g$*.*

Theorem C.6.3 (The Plancherel theorem)**.** *Let* H *be a Hilbert space. Then the Fourier transform* $\mathscr{F}$ *restricts to an isometric isomorphism on* $L^2(\mathbb{R}^d;H)$*.*

Theorem C.6.4 (Paley-Wiener-Schwartz)**.** *Let* X *be a Banach space. For an* $f \in \mathcal{S}'(\mathbb{R}^d;X)$ *and a compact* $K \subset \mathbb{R}^d$ *the following are equivalent:*

(i) $\operatorname{supp} \hat{f} \subset K$*;*

(ii) f *is extends to an entire analytic function on* $\mathbb{C}^d$ *satisfying*

$$\|f(x+\imath y)\|_X \leq C(1+|x+\imath y|)^N e^{H(-y)} \qquad (x,y \in \mathbb{R}^d)$$

for some $C > 0$ *and* $N \in \mathbb{N}$*; here* $H(y) = \sup\{y \cdot \xi \mid \xi \in K\}$ *is the supporting function of the compact* K*.*

In this situation, we have

$$f(\phi) = \int_{\mathbb{R}^d} f(x)\phi(x)dx, \qquad \phi \in \mathcal{S}(\mathbb{R}^d), \tag{C.10}$$

and

$$f(x) = (2\pi)^{-d}\hat{f}(e_{\imath x}) \tag{C.11}$$

Corollary C.6.5. *Suppose that* $f \in \mathcal{S}'(\mathbb{R}^d;X)$ *has Fourier support* $\operatorname{supp}\hat{f}$ *contained in the rectangle* $\prod_{j=1}^d [R_j, R_j]$*, where* $R_1, \ldots, R_d > 0$*. Let* $d = d_1 + d_2 + d_3$ *with* $d_1, d_2, d_3 \in \mathbb{N}$ *and view* $\mathbb{R}^d$ *as* $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \mathbb{R}^{d_3}$*. Then, for each fixed* $x_1 \in \mathbb{R}^{d_1}$ *and* $x_3 \in \mathbb{R}^{d_3}$*, it holds that* $x_2 \mapsto f(x_1,x_2,x_3)$ *defines a tempered distribution on* $\mathbb{R}^{d_2}$ *with Fourier support contained in* $\prod_{j=d_1+1}^{d_1+d_2}[-R_j, R_j]$*.*

Corollary C.6.6. *Let* X *be a complete LCS and equip* $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d;X)$ *with the locally convex topology for which the Fourier transform* $\mathscr{F}$ *becomes a topological linear isomorphism from* $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d;X)$ *onto* $\mathcal{E}'(\mathbb{R}^d;X)$*. Then we have*

$$\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d;X) \hookrightarrow C(\mathbb{R}^d;X).$$

Proof. The Paley-Wiener-Schwartz theorem in particular tells us that $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d) \subset C(\mathbb{R}^d)$. As $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d)$ and $C(\mathbb{R}^d)$ are both continuously included in the locally convex Hausdorff space $\mathcal{D}'(\mathbb{R}^d)$, the natural inclusion $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d) \longrightarrow C(\mathbb{R}^d)$ is a closed linear mapping. Since $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d)$ is a barreled LCS and since $C(\mathbb{R}^d)$ is a Fréchet space, it follows from the closed graph theorem that $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$.

Finally, the case of a general complete LCS X can be derived from the case $X = \mathbb{C}$ by using the theory of topological tensor products:

$$\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d; X) = \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d)\,\tilde{\otimes}_\epsilon X \hookrightarrow C(\mathbb{R}^d)\,\tilde{\otimes}_\epsilon X = C(\mathbb{R}^d; X).[3]$$

□

C.7 Identifications Between Spaces of Distributions

Theorem C.7.1 (Schwartz Kernel Theorem)**.** *Let X be a complete LCS and let $U \subset \mathbb{R}^d$ and $V \subset \mathbb{R}^n$ be open subsets. Then we have a topological linear isomorphism*

$$\mathcal{D}'(U \times V; X) \longrightarrow \mathcal{D}'(U; \mathcal{D}'(V; X)),\ K \mapsto u_K,$$

where

$$(u_K\psi)\phi := K(\phi \otimes \psi), \qquad \psi \in \mathcal{D}(U), \phi \in \mathcal{D}(V),$$

which is called the canonical isomorphism*. Moreover, the canonical isomorphism restricts to topological linear isomorphisms $\mathcal{E}'(U \times V; X) \cong \mathcal{E}'(U; \mathcal{E}'(V; X))$ and $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^n; X) \cong \mathcal{S}'(\mathbb{R}^d; \mathcal{S}'(\mathbb{R}^n; X))$ (in case $U = \mathbb{R}^d$ and $V = \mathbb{R}^n$).*

Let X be a complete LCS. Then

$$C(\overline{U} \times \overline{V}; X) \longrightarrow C(\overline{U}; C(\overline{V}; X)),\ f \mapsto \left[\overline{U} \ni y \mapsto f(y, \cdot\,) \in C(\overline{V}; X)\right]$$

is a topological linear isomorphism, which can be obtained by restriction of the canonical isomorphism $\mathcal{D}'(U \times V; X) \cong \mathcal{D}'(U; \mathcal{D}'(V; X))$.

Let X be a Banach space. Then the natural identification $L^p(U \times V; X) \cong L^p(U; L^p(V; X))$ (Fubini) is also compatible with the canonical isomorphism $\mathcal{D}'(U \times V; X) \cong \mathcal{D}'(U; \mathcal{D}'(V; X))$.

Lemma C.7.2. *Let X be a Banach space and let $f \in \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^{d+n}; X)$ with* $\operatorname{supp} \hat{f} \subset [-R, R]^d \times \mathbb{R}^n$ *for some $R > 0$. Then, under the canonical isomorphism $\mathcal{D}'(\mathbb{R}^d \times \mathbb{R}^n; X) \cong \mathcal{D}'(\mathbb{R}^d; \mathcal{D}'(\mathbb{R}^n; X))$, f corresponds to a tempered distribution $F = u_f \in \mathcal{S}'(\mathbb{R}^d; C(\mathbb{R}^n; X))$ having compact Fourier support* $\operatorname{supp} \hat{F} \subset [-R, R]^d$.

Proof. Given a locally convex space Y, we equip $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^m; Y)$ with the locally convex topology which makes the Fourier transform $\mathscr{F}$ a topological linear isomorphism from $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^m; Y)$ onto $\mathcal{E}'(\mathbb{R}^m; Y)$. In this way it is not difficult to see that $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^{d+n}; X) \cong \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d; \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^n; X))$ under the canonical isomorphism. Furthermore, if $K \in \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^{d+n}; X)$ has Fourier support contained in $[-r, r]^d \times \mathbb{R}^n$, $r > 0$, then $u_K \in \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d; \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^n; X))$ has Fourier support contained in $[-r, r]^d$. In particular, under the canonical isomorphism, our given f corresponds to $F := u_f \in \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^d; \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^n; X)) \subset \mathcal{S}'(\mathbb{R}^d; \mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^n; X))$ having Fourier support $\operatorname{supp} \hat{F} \subset [-R, R]^d$. Since $\mathscr{F}^{-1}\mathcal{E}'(\mathbb{R}^n; X) \hookrightarrow C(\mathbb{R}^n; X)$ by Corollary C.6.6, we may view F as a tempered distribution $F \in \mathcal{S}'(\mathbb{R}^d; C(\mathbb{R}^n; X))$ having compact Fourier support $\operatorname{supp} \hat{F} \subset [-R, R]^d$. □

[3]For the theory of topological tensor products we refer to [96]; also see [4].

Recall that for a complete LCS Y we have the continuous inclusion $C(\mathbb{R};Y) \hookrightarrow \mathcal{D}'(\mathbb{R};Y)$; see (C.4). If $Y = \mathcal{D}'(\mathbb{R}^{d-1};X)$ for a Banach space X, then we have the canonical isomorphism $\mathcal{D}'(\mathbb{R};Y) = \mathcal{D}'(\mathbb{R}^d;X)$, so that $C(\mathbb{R};\mathcal{D}'(\mathbb{R}^{d-1};X)) \hookrightarrow \mathcal{D}'(\mathbb{R}^d;X)$. A function $f \in C(\mathbb{R};\mathcal{D}'(\mathbb{R}^{d-1};X))$ can also be viewed as an X-valued distribution on $\mathbb{R}^d$ in the following more direct way:

Proposition C.7.3. *Let X be a Banach space. Given $f \in C(\mathbb{R};\mathcal{D}'(\mathbb{R}^{d-1};X))$,*

$$\Lambda_f(\phi) := \int_{\mathbb{R}} [f(t)](\phi(t,\,\cdot\,))\,dt, \qquad \phi \in \mathcal{D}(\mathbb{R}^d),$$

defines an X-valued distribution on $\mathbb{R}^d$. The obtained mapping $f \mapsto \Lambda_f$ defines an injection $C(\mathbb{R};\mathcal{D}'(\mathbb{R}^{d-1};X)) \longrightarrow \mathcal{D}'(\mathbb{R}^d;X)$.

Proof. This can be shown in completely the same way as [60, Proposition 3.5]. □

Let X be a Banach space. By testing on $\mathcal{D}(\mathbb{R}) \otimes \mathcal{D}(\mathbb{R}^{d-1})$ we see that the inclusion from this proposition coincides with the inclusion $C(\mathbb{R};\mathcal{D}'(\mathbb{R}^{d-1};X)) \hookrightarrow \mathcal{D}'(\mathbb{R}^d;X)$ obtained by the canonical isomorphism; here we of course do not have to restrict ourselves to $\mathbb{R}$ and $\mathbb{R}^{d-1}$. An advantage of the abstract viewpoint is that we can use abstract theory, for example to obtain:

Lemma C.7.4. *Under the canonical isomorphism $\mathcal{D}'(U;\mathcal{D}'(V;X)) \cong \mathcal{D}'(U \times V;X) \cong \mathcal{D}'(V;\mathcal{D}'(V;X))$, we have*

$$C(\overline{U};\mathcal{D}'(V;X)) \cong \mathcal{D}'(V;C(\overline{U};X)).$$

In fact, $C(\overline{U}) \otimes \mathcal{D}'(V) \otimes X$ is dense in both $C(\overline{U};\mathcal{D}'(V;X))$ and $\mathcal{D}'(V;C(\overline{U};X))$, and the induced topologies coincide on $C(\overline{U}) \otimes \mathcal{D}'(V) \otimes X$.

Proof. This follows from the commutativity of the ϵ-tensor product for locally convex spaces and the fact that for a complete LCS Y it holds that $C(\overline{U};Y) = C(\overline{U})\tilde{\otimes}_\epsilon Y$ and $\mathcal{D}'(V;Y) = \mathcal{D}'(V)\tilde{\otimes}_\epsilon Y$. □

Appendix D

Harmonic Analysis

This appendix is mainly based on [44, 45] and can be consulted on reference.

D.1 Maximal Functions

Definition D.1.1. For a function $f \in L^1_{loc}(\mathbb{R}^d)$, the *Hardy-Littlewood maximal function* $Mf \in \overline{L^0}_+(\mathbb{R}^d)$[1] is defined by

$$(Mf)(x) := \sup_{\delta>0} \fint_{B(x,\delta)} |f(y)|dy = \sup_{\delta>0} \frac{1}{|B(x,\delta)|} \int_{B(x,\delta)} |f(y)|dy, \qquad x \in \mathbb{R}^d. \tag{D.1}$$

Let $p \in]1, \infty[$. Then we have that

$$M : L^p(\mathbb{R}^d) \longrightarrow L^p(\mathbb{R}^d),\ f \mapsto Mf$$

defines a bounded sublinear operator on $L^p(\mathbb{R}^d)$: for every $f, g \in L^p(\mathbb{R}^d)$ it holds that $M(f+g) \leq Mf + Mg$ and $\|Mf\|_{L^p(\mathbb{R}^d)} \lesssim_{p,d} \|f\|_{L^p(\mathbb{R}^d)}$.

Let $\phi : \mathbb{R}^d \longrightarrow \mathbb{C}$. For each $t > 0$, we define

$$\phi_t(x) := t^d \phi(tx), \qquad x \in \mathbb{R}^d. \tag{D.2}$$

Lemma D.1.2. *Suppose $\phi \in L^1(\mathbb{R}^d)$ is such that*

$$\psi(x) := \sup\{\, |\phi(y)| \;:\; |y| \geq |x| \,\}$$

defines a function $\psi \in L^1(\mathbb{R}^d)$. Then, for all $f \in L^1_{loc}(\mathbb{R}^d; X)$ and $t > 0$, we have

$$\int_{\mathbb{R}^d} \|\phi_t(x-y)f(y)\|_X \,dy \leq \|\psi\|_{L^1(\mathbb{R}^d)} M(\|f\|_X)(x),, \qquad x \in \mathbb{R}^d.$$

As a consequence, the convolution product $\phi_t * f$ (A.2) *is well-defined and the following estimate is valid:*

$$\sup_{t>0} \|(\phi_t * f)(x)\|_X \leq \|\psi\|_{L^1(\mathbb{R}^d)} M(\|f\|_X)(x), \qquad x \in \mathbb{R}^d.$$

[1] For the measurability we refer to the proof of [87, Lemma 19.16].

Definition D.1.3. Let X be a Banach space and let $f \in L^1_{loc}(\mathbb{R}^d; X)$. A point $x_0 \in \mathbb{R}^d$ is called a *Lebesgue point* of f if

$$\lim_{r \searrow 0} \fint_{B(x_0,r)} \|f(y) - f(x_0)\|_X \, dy = 0.^2$$

Observe that, if x_0 is a Lebesgue point of $f \in L^1_{loc}(\mathbb{R}^d; X)$, then

$$f(x_0) = \lim_{r \searrow 0} \fint_{B(x_0,r)} f(y) dy.$$

As a consequence, we have $\|f\|_X (x_0) \leq [M \|f\|_X](x_0)$.

Proposition D.1.4. *Let X be a Banach space, $f \in L^1_{loc}(\mathbb{R}^d; X)$, and $\phi \in C_c(\mathbb{R}^d)$ such that $\phi \geq 0$ and $\int_{\mathbb{R}^d} \phi(x)dx = 1$. Let x_0 be a Lebesgue point of f. Then we have*

$$f(x_0) = \lim_{t \to \infty} (f * \phi_t)(x_0),$$

where ϕ_t is as in (D.2).

Theorem D.1.5 (Lebesgue's Differentiation theorem)**.** *Let X be a Banach space and let $f \in L^1_{loc}(\mathbb{R}^d; X)$. Then almost every point of $\mathbb{R}^d$ is a Lebesgue point. As a consequence, $\|f\|_X \leq M \|f\|_X$ almost everywhere.*

D.2 Weights

A *weight* on $\mathbb{R}^d$ is a is a locally integrable function $\mathbb{R}^d \longrightarrow [0, \infty]$ that takes its values in $]0, \infty[$ Lebesgue almost everywhere. We denote by $\mathcal{W}(\mathbb{R}^d)$ the set of all weights on $\mathbb{R}^d$. For a weight $w \in \mathcal{W}(\mathbb{R}^d)$ we define the associated weighted Lebesgue-Bochner space

$$L^p(\mathbb{R}^d, w; X) := \left\{ f \in L^0(U; X) \; : \; \int_{\mathbb{R}^d} \|f(x)\|_X^p \, w(x) \, dx < \infty \right\},$$

which becomes a Banach space when equipped with the norm

$$\|f\|_{L^p(\mathbb{R}^d, w; X)} := \left(\int_{\mathbb{R}^d} \|f\|_X^p \, w \, d\lambda_U \right)^{1/p}.$$

Due to the importance of the Hardy-Littlewood maximal function operator M (D.1) in (Harmonic) analysis, it would be interesting to characterize, for a fixed $p \in]1, \infty[$, all the weights w on $\mathbb{R}^d$ with $L^p(\mathbb{R}^d, w) \subset L^1_{loc}(\mathbb{R}^d)$ for which M (D.1) restricts to a bounded (sublinear) operator on $L^p(\mathbb{R}^d, w)$. It is well known that these weights can be characterized via the A_p-condition, for which a motivation is given in [45, Subsection 9.1.1] (which simultaneously proves the necessity). The sufficiency will be stated in Theorem D.2.4, but let us first give the definition.

Definition D.2.1. Let $p \in]1, \infty[$. A weight w on $\mathbb{R}^d$ is said to be *of class A_p* if

$$[w]_{A_p} := \sup_Q \left(\fint_Q w(x) dx \right) \left(\fint_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty, \tag{D.3}$$

[2] Here it is understood that we work with a representative f.

where the supremum is taken over all cubes Q in $\mathbb{R}^d$ with sides parallel to the coordinate axes. We also say that w is an *(Muckenhaupt) A_p-weight* and the number $[w]_{A_p}$ is called the *(Muckenhaupt) A_p-characteristic constant* of w. The set of all A_p-weights w on $\mathbb{R}^d$ is denoted by $A_p(\mathbb{R}^d)$. We define

$$A_\infty(\mathbb{R}^d) := \bigcup_{p\in]1,\infty[} A_p(\mathbb{R}^d).$$

In the following proposition we summarize some basic properties of A_p-weights.

Proposition D.2.2. *(basic properties)*

(i) Let $p, p' \in]1,\infty[$ be Hölder conjugates. A weight $w \in \mathcal{W}(\mathbb{R}^d)$ belongs to the class A_p if and only if $w^{-\frac{1}{p-1}}$ is a weight belonging to the class $A_{p'}$, in which case we have

$$[w^{-\frac{1}{p-1}}]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}.$$

(ii) $[w]_{A_p} \geq 1$ for all $w \in A_p$. Equality holds if and only if w is constant.

(iii) Let $1 < p < q < \infty$. Then we have $A_p \subset A_q$ with

$$[w]_{A_q} \leq [w]_{A_p}, \qquad w \in A_p.$$

(iv) Let $p \in]1,\infty[$. A weight $w \in \mathcal{W}(\mathbb{R}^d)$ belongs to the class A_p if and only if there exists a constant $C \in [1,\infty[$ such that, for all $f \in L^0(\mathbb{R}^d)$,

$$\frac{1}{|Q|}\int_Q |f(x)|dx \leq C^{1/p}\left(\frac{1}{W(Q)}\int_Q |f(x)|^p W(x)dx\right)^{1/p}.$$

In this situation, the smallest such constant C coincides with $[w]_{A_p}$.

(v) The measure $W\lambda$ is doubling: for all $\lambda > 0$ and all cubes Q we have

$$W(Q_\lambda) \leq \lambda^{np}[w]_{A_p} w(Q),$$

where Q_λ denotes the cube with the same center as Q and side length λ times the side length of Q.

From a combination of (i) above and Hölder's inequality it follows that, if $p \in]1,\infty[$ and $w \in A_p(\mathbb{R}^d)$, then we have

$$L^p(\mathbb{R}^d, w; X) \hookrightarrow L^1_{loc}(\mathbb{R}^d; X); \tag{D.4}$$

here we simply write $1_K f = (f w^{1/p}) \cdot (1_K w^{-\frac{1}{p-1}})$ for each compact $K \subset \mathbb{R}^d$ subset.

The power weights in the following example are very important for this thesis:

Example D.2.3.

(i) Let $a \in]-d,\infty[$ and define $w \in \mathcal{W}(\mathbb{R}^d)$ by $w(x) := |x|^a$. Given $p \in]1,\infty[$, it holds that $w \in A_p$ if and only if $a \in]-d, d(p-1)[$.

(ii) Let $a \in]-1,\infty[$ and define $w \in \mathcal{W}(\mathbb{R}^d)$ by $w(x) := |x_1|^a$ $(x = (x_1,\dots,x_d) \in \mathbb{R}^d)$. Given $p \in]1,\infty[$, it holds that $w \in A_p$ if and only if $a \in]-1, p-1[$.

We now state the boundedness of the Hardy-Littlewood maximal function operator on Muckenhaupt-weighted L^p-spaces:

Theorem D.2.4. *Let $p \in]1, \infty[$ and $w \in A_p(\mathbb{R}^d)$. Then the Hardy-Littlewood maximal function operator M restricts to a bounded (sublinear) operator on $L^p(\mathbb{R}^d, w)$ of norm $\lesssim_{p,d} [w]_{A_p}^{\frac{1}{p-1}}$.*

Via a combination of this theorem with Propositions D.1.2 and D.1.4, the Lebesgue dominated convergence theorem gives:

Proposition D.2.5. *Let X be a Banach space, $p \in]1, \infty[$ and $w \in A_p(\mathbb{R}^d)$. Let $\phi \in C_c(\mathbb{R}^d)$ be such that $\phi \geq 0$ and $\int_{\mathbb{R}^d} \phi(x)dx = 1$. For every $f \in L^p(\mathbb{R}^d, w; X)$ we have $f = \lim_{t\to\infty} f * \phi_t$ both in $L^p(\mathbb{R}^d, w; X)$ and pointwise almost everywhere.*

Some more results which are needed:

Lemma D.2.6. *Let X be a Banach space, $p \in [1, \infty[$, and $w \in A_\infty(\mathbb{R}^d)$. Then $C_c^\infty(\mathbb{R}^d; X)$ is dense in $L^p(\mathbb{R}^d, w; X)$.*

Lemma D.2.7. *Let $p \in]1, \infty[$ and $w \in A_p$. Then there is constant $C \in]0, \infty[$ such that for all $x \in \mathbb{R}^d$,*

$$\int_{\mathbb{R}^d} w(y)(1 + |x - y|)^{-dp} dy \leq C \int_{B(x,1)} w(y)dy.$$

Corollary D.2.8. *Let $w \in A_\infty(\mathbb{R}^d)$. Then there exists an $L \in \mathbb{N}$ such that*

$$\int_{\mathbb{R}^d} w(y)(1 + |y|^2)^{-L} dy < \infty.$$

Lemma D.2.9. *Let $p \in]1, \infty[$ and $w \in A_p(\mathbb{R})$. Then there exists a constant $C_{p,w} \in]0, \infty[$ such that, for all $g \in L^p(\mathbb{R}_+, w)$,*

$$\left\| y \mapsto \int_{\mathbb{R}_+} \frac{|g(\tilde{y})|}{y + \tilde{y}} d\tilde{y} \right\|_{L^p(\mathbb{R}_+, w)} \leq C_{p,w} \|g\|_{L^p(\mathbb{R}_+, w)}.$$

The class of A_p-weights are not only characterized by the boundedness of the Hardy-Littlewood maximal function operator (on the corresponding weighted L^p-space), but also by the boundedness of each of the Riesz transforms. We state this fact in Fourier analytic terms, which is more convenient for Chapter 4.

Theorem D.2.10. *Let $w \in \mathcal{W}(\mathbb{R}^d)$ and $p \in]1, \infty[$.*

(i) If, for each $j \in \{1, \ldots, d\}$,

$$C_c^\infty(\mathbb{R}^d) \longrightarrow C_0^\infty(\mathbb{R}^d),\ f \mapsto \mathscr{F}^{-1}\left[\left(\xi \mapsto \imath \frac{\xi_j}{|\xi|}\right) \hat{f}\right]$$

(takes its values in $L^p(\mathbb{R}^d, w)$ and) extends to a bounded linear operator R_j on $L^p(\mathbb{R}^d, w)$, then we have $w \in A_p(\mathbb{R}^d)$. Moreover, there exists a function $C : [0, \infty[^d \longrightarrow [0, \infty[$ (independent of w) which is increasing in each of its variables such that

$$[w]_{A_p} \leq C\left(\|R_1\|_{\mathcal{B}(L^p(\mathbb{R}^d, w))}, \ldots, \|R_d\|_{\mathcal{B}(L^p(\mathbb{R}^d, w))}\right).$$

(ii) If $w \in A_p(\mathbb{R}^d)$, then

$$\mathcal{S}(\mathbb{R}^d) \longrightarrow C_0^\infty(\mathbb{R}^d),\ f \mapsto \mathscr{F}^{-1}\left[\left(\xi \mapsto \imath\frac{\xi_j}{|\xi|}\right)\hat{f}\right]$$

(takes its values in $L^p(\mathbb{R}^d, w)$ and) extends to a bounded linear operator R_j on $L^p(\mathbb{R}^d, w)$. Moreover, there exists an increasing function $C : [0,\infty[\longrightarrow [0,\infty[$ (independent of w) such that $\|R_j\|_{\mathcal{B}(L^p(\mathbb{R}^d,w))} \leq [w]_{A_p}$, $j = 1, \dots, d$.

In the A_p-condition (D.3) the supremum is taken over all cubes in $\mathbb{R}^d$ with sides parallel to the coordinate axes. Taking the supremum over the much larger collection of all rectangles in $\mathbb{R}^d$ with sides parallel to the coordinate axes, we arrive at the more restrictive A_p^{rec}-condition:

Definition D.2.11. Let $p \in]1,\infty[$. A weight w on $\mathbb{R}^d$ is said to be *of class A_p^{rec}* if

$$[w]_{A_p^{rec}} := \sup_R \left(⨍_R w(x)dx\right)\left(⨍_R w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty, \tag{D.5}$$

where the supremum is taken over all rectangles R in $\mathbb{R}^d$ with sides parallel to the coordinate axes. We also say that w is an *(Muckenhaupt) A_p^{rec}-weight* and the number $[w]_{A_p^{rec}}$ is called the *(Muckenhaupt) A_p^{rec}-characteristic constant* of w.

For the power weights from Example D.2.12 we have:

Example D.2.12.

(i) Let $a \in]-d,\infty[$ and define $w \in \mathcal{W}(\mathbb{R}^d)$ by $w(x) := |x|^a$. Given $p \in]1,\infty[$, it holds that $w \in A_p$ if and only if $a \in]-1, p-1[$.

(ii) Let $a \in]-1,\infty[$ and define $w \in \mathcal{W}(\mathbb{R}^d)$ by $w(x) := |x_1|^a$ ($x = (x_1, \dots, x_d) \in \mathbb{R}^d$). Given $p \in]1,\infty[$, it holds that $w \in A_p$ if and only if $a \in]-1, p-1[$.

The following simple characterization of the class A_p^{rec} is very useful:

Lemma D.2.13. *Let $w \in \mathcal{W}(\mathbb{R}^d)$. Then the following are equivalent:*

(i) $w \in A_p^{rec}(\mathbb{R}^d)$.

(ii) There exists a constant $C \geq 1$ such that, for every $j \in \{1, \dots, d\}$ and almost every $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)$ in $\mathbb{R}^d$, $w(x_1, \dots, x_{j-1}, \cdot, x_{j+1}, \dots, x_d) \in A_p(\mathbb{R})$ with A_p-characteristic constant $\leq C$.

Moreover, in this situation the smallest such constant $C \geq 1$ equals $[w]_{A_p^{rec}}$.

Combining Theorem D.2.10 and Lemma D.2.13, the following is not difficult to see:

Proposition D.2.14. *Let $w \in \mathcal{W}(\mathbb{R}^d)$. Then we have $w \in A_p^{rec}(\mathbb{R}^d)$ if and only if*

$$C_c^\infty(\mathbb{R}^d) \longrightarrow C_0^\infty(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d; X),\ f \mapsto \mathscr{F}^{-1}[1_{[0,\infty[^d}\hat{f}]$$

(takes its values in $L^p(\mathbb{R}^d, w)$ and) extends to a bounded linear operator R on $L^p(\mathbb{R}^d, w)$, called the Riesz projection. *Moreover, in this situation we have $Rf = \mathscr{F}^{-1}[1_{[0,\infty[^d}\hat{f}]$ for every $f \in \mathcal{S}(\mathbb{R}^d)$.*

Appendix E

Banach Space Theory

The material from this appendix is taken from the book [57], in which one of the main themes is the use of randomization and martingale techniques in Banach space-valued analysis.

E.1 Random Sums

E.1.1 Random Variables

Let X be a Banach space. An *X-valued random variable* is an X-valued strongly measurable function ξ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ will always be considered fixed, and when several random variables are considered simultaneously we will always assume them to be defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ unless otherwise stated.

Recall that every strongly measurable X-valued function is Borel measurable. In particular, we may define define the *distribution* of an X-valued random variable ξ as the Borel probability measure μ_ξ on X given by

$$\mu_\xi(B) := \mathbb{P}(\xi \in B) = \mathbb{P}(\xi^{-1}(B)), \qquad B \in \mathscr{B}(X).$$

Two X-valued random variables ξ_1 and ξ_2 are *identically distributed* when $\mu_{\xi_1} = \mu_{\xi_2}$.

An X-valued random variable ξ is called:

- *symmetric*, if ξ and $\varepsilon\xi$ are identically distributed for all unimodular scalars ε (i.e. $\varepsilon \in \mathbb{K}$ with $|\varepsilon| = 1$)
- *real-symmetric*, if ξ and $-\xi$ are identically distributed.

Let I be an index set. A collection $\{\xi_i\}_{i\in I}$ of X-valued random variables is called *independent* if for all choices of distinct indices $i_0, \ldots, i_N \in I$ and all Borel sets $B_0, \ldots, B_N \subset X$ we have

$$\mathbb{P}(\xi_{i_0} \in B_0, \ldots, \xi_{i_N} \in B_N) = \prod_{n=0}^{N} \mathbb{P}(\xi_{i_n} \in B_n)$$

A $\mathbb{K}$-valued random variable ϵ is called a *Rademacher random variable* if it is uniformly distributed in $\{z \in \mathbb{K} : |z| = 1\}$. A *Rademacher sequence* is a sequence of independent

Rademacher variables $(\epsilon_n)_{n\in\mathbb{N}}$. If $(\epsilon_n)_{n\in\mathbb{N}}$ is a Rademacher sequence and $(\varepsilon_n)_{n\in\mathbb{N}}$ a sequence of unimodular scalars, then $(\varepsilon_n\epsilon_n)_{n\in\mathbb{N}}$ is a Rademacher sequence again.

Throughout this appendix we fix a Rademacher sequence $(\epsilon_n)_{n\in\mathbb{N}}$ (on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$). All notions that will be defined in terms of this Rademacher sequence $(\epsilon_n)_{n\in\mathbb{N}}$ are independent of this particualr choice because any two Rademacher sequences are identically distributed.

Lemma E.1.1. *Let ξ and η be X-valued random variables. If η is real-symmetric and independent of ξ, then for all $p \in [1,\infty]$ we have*

$$\|\xi\|_{L^p(X)} \leq \|\xi + \eta\|_{L^p(X)} .$$

Proposition E.1.2 (Kahane's contraction principle)**.** *Let $p \in [1,\infty]$ and let $(\xi_n)_{n=0^N}$ be a sequence of independent and (real)-symmetric random variables in $L^p(\Omega; X)$. Then, for all scalar (real) sequences $(a_n)_{n=0}^N$ we have*

$$\left\| \sum_{n=0}^{N} a_n \xi_n \right\|_{L^p(\Omega;X)} \leq \max_{0\leq n\leq N} |a_n| \left\| \sum_{n=0}^{N} \xi_n \right\| .$$

E.1.2 The Kahane-Khintchine Ineqalities

Theorem E.1.3 (Kahane-Khintchine)**.** *For all $q, p \in [1,\infty[$ there exists a constant $\kappa_{q,p}$ such that, for all $N \in N$ and $x_0, \dots, x_N \in X$,*

$$\left\| \sum_{n=0}^{N} \epsilon_n x_n \right\|_{L^q(\Omega;X)} \leq \kappa_{q,p} \left\| \sum_{n=0}^{N} \epsilon_n x_n \right\|_{L^p(\Omega;X)} .$$

The main point of this theorem is the case $1 \leq p < q < \infty$. For $1 \leq q \leq p < \infty$ the above estimate simply holds true with constant 1 by Hölder's inequality. Since $(\epsilon_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in $L^2(\Omega)$, for $X = \mathbb{K}$ we have

$$\left\| \sum_{n=0}^{N} \epsilon_n x_n \right\|_{L^2(\Omega)} = \left(\sum_{n=0}^{N} |x_n|^2 \right)^{1/2}$$

for all $N \in \mathbb{N}$, $x_0, \dots, x_N$. Combining this observation with the Kahane-Khintchine inequalities and Fubini, it can be shown that:

Proposition E.1.4. *Let $(S, \mathcal{A}, \mu)$ be a measure space and let $q \in [1,\infty[$. Then, for all $N \in \mathbb{N}$, $x_0, \dots, x_N \in X$, and $p \in [1,\infty[$, we have the estimate*

$$\frac{1}{c} \left\| \left(\sum_{n=0}^{N} |x_n|^2 \right)^{1/2} \right\|_{L^q(S)} \leq \left\| \sum_{n=0}^{N} \epsilon_n x_n \right\|_{L^p(\Omega;L^q(S))} \leq C \left\| \left(\sum_{n=0}^{N} |x_n|^2 \right)^{1/2} \right\|_{L^q(S)}$$

with $c = \kappa_{2,q}\kappa_{q,p}$ and $C = \kappa_{p,q}\kappa_{q,2}$.

E.1.3 The Space $\mathrm{Rad}_p(X)$

Let X be a Banach space and let $p \in [1, \infty[$. We define $\mathrm{Rad}_p(X)$ as the space of all sequences $(x_n)_{n\in\mathbb{N}}$ in X for which the series $\sum_{n=0}^{\infty} \epsilon_n x_n$ converges in $L^p(\Omega; X)$, endowed with the norm

$$\|(x_n)_{n\in\mathbb{N}}\|_{\mathrm{Rad}_p(X)} := \left\| \sum_{n=0}^{\infty} \epsilon_n x_n \right\|_{L^p(\Omega;X)} = \lim_{N\to\infty} \left\| \sum_{n=0}^{N} \epsilon_n x_n \right\|_{L^p(\Omega;X)} = \sup_{N\geq 0} \left\| \sum_{n=0}^{N} \epsilon_n x_n \right\|_{L^p(\Omega;X)}, \tag{E.1}$$

which turns it into a Banach space. Here the third identity is a consequence of Lemma E.1.1. We can identify $\mathrm{Rad}_p(X)$ with a closed subspace of $L^p(\Omega; X)$ in the natural way. Note that, by Lemma E.1.1 and the triangle inequality,

$$\ell^1(\mathbb{N}; X) \hookrightarrow \mathrm{Rad}_p(X) \hookrightarrow \ell^\infty(\mathbb{N}; X).$$

We would like to remark that the finiteness of the supremum on the right side of (E.1) does in general not imply the convergence of the sum $\sum_{n=0}^{\infty} \epsilon_n x_n$ in $L^p(\Omega; X)$. For example, for $X = c_0$ and $(x_n)_{n\in\mathbb{N}} = (e_n)_{n\in\mathbb{N}}$ the standard unit basis in c_0, the supremum equals 1 but the corresponding series certainly does not converge in $L^p(\Omega; c_0)$. By a theorem of Hoffmann-Jorgensen and Kwapien, c_0 is in a sense the only counterexample: If X does not contain a closed subspace isomorphic to c_0, then the finiteness of the supremum does imply the convergence of the corresponding series.

Let $(S, \mathscr{A}, \mu)$ be a σ-finite measure space. By Fubini we have the canonical isometric isomorphism

$$L^p(S; \mathrm{Rad}_p(X)) \simeq \mathrm{Rad}_p(L^p(S; X)). \tag{E.2}$$

As a consequence of the Kahane-Khintchine inequalities (cf. E.1.3), we have:

Lemma E.1.5. *For all $p, q \in [1, \infty[$ we have $\mathrm{Rad}_p(X) = \mathrm{Rad}_q(X)$ with an equivalence of norms. In fact we have*

$$\frac{1}{\kappa_{p,q}} \|x\|_{\mathrm{Rad}_p(X)} \leq \|x\|_{\mathrm{Rad}_q(X)} \leq \kappa_{q,p} \|x\|_{\mathrm{Rad}_p(X)}, \qquad x = (x_n)_{n\in\mathbb{N}} \in \mathrm{Rad}_p(X) = \mathrm{Rad}_q(X).$$

This lemma motivates us to write $\mathrm{Rad}(X) := \mathrm{Rad}_2(X)$.

E.2 Type and Cotype

Let H be a Hilbert space. As usual, we denote by $(\epsilon_n)_{n\in\mathbb{N}}$ a Rademacher sequence on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then we have, for all $N \in \mathbb{N}$ and $h_0, \ldots, h_N \in H$,

$$\left(\sum_{n=0}^{N} \|x_n\|_H \right)^{1/2} = \left\| \sum_{n=0}^{N} \epsilon_n x_n \right\|_{L^2(\Omega;H)}. \tag{E.3}$$

Indeed,

$$\begin{aligned}
\left\|\sum_{n=0}^{N}\epsilon_n x_n\right\|^2_{L^2(\Omega;H)} &= \mathbb{E}\left[\sum_{n=0}^{N}\epsilon_n h_n, \sum_{n=0}^{N}\epsilon_n h_n\right]_H = \mathbb{E}\sum_{n=0}^{N}\sum_{m=0}^{N}\epsilon_n\bar{\epsilon}_m[h_n, h_m]_H \\
&= \sum_{n=0}^{N}\sum_{m=0}^{N}\mathbb{E}(\epsilon_n\bar{\epsilon}_m)[h_n, h_m]_H = \sum_{n=0}^{N}\sum_{m=0}^{N}\delta_{n,m}[h_n, h_m]_H \\
&= \sum_{n=0}^{N}[h_n, h_n]_H = \sum_{n=0}^{N}\|h_n\|^2 .
\end{aligned}$$

The equality (E.3) can be interpreted as a generalization of the parallelogram law. The parallelogram characterizing the norms that are induced by an inner-product, this suggests to introduce the notions of type and cotype as measures of how far a Banach space X is being away from a Hilbert space.

Definition E.2.1. Let X be a Banach space, $p \in [1,2]$, and $q \in [2,\infty]$.

(i) The space X is said to have type $p \in [1,2]$ if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and $x_0, \dots, x_N \in \mathbb{N}$

$$\left\|\sum_{n=0}^{N}\epsilon_n x_n\right\|_{L^p(\Omega;X)} \leq C\left(\sum_{n=0}^{N}\|x_n\|_X^p\right)^{1/p} .$$

(ii) The space X is said to have type $q \in [2,\infty]$ if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and $x_0, \dots, x_N \in$

$$\left(\sum_{n=0}^{N}\|x_n\|_X^q\right)^{1/q} \leq C\left\|\sum_{n=0}^{N}\epsilon_n x_n\right\|_{L^q(\Omega;X)} ,$$

with the obvious modifaction for $q = \infty$.

The least admissible constants in (i) and (ii) are called the *type p constant* and *cotype q constant* of X and will be denoted by $\tau_{p,X}$ and $c_{q,X}$, respectively.

From the case $N = 0$ we see that $\tau_{p,X} \geq 1$ and $c_{q,X} \geq 1$. By the Kahane-Khintchine inequalites (cf. Theorem E.1.3), the exponents (with the exception of $q = \infty$) in the Rademacher sums in (i) and (ii) could be replaced by any exponent.

It is not difficult to check that the inequalities defining type and cotype cannot be satisfied for any $p > 2$ and $q < 2$, respectively, explaining the restrictions $p \in [1,2]$ and $q \in [2,\infty]$.

Let us state some basic facts:

(a) Every Banach space X has type 1 and cotype ∞, with constants $\tau_{1,X} = 1$ and $c_{\infty,X} = 1$.

(b) If X has type p, then it has type σ for every $\sigma \in [1,p]$.

(c) If X has cotype q, then it has cotype r for every $r \in [q,\infty[$.

(d) A Banach space X is isomorphic to a Hilbert space if and only if it has type 2 and cotype 2.

(e) If X is a Banach space with type p and cotype q, $(S, \mathcal{A}, \mu)$ a σ-finite measure space, and $r \in [1, \infty]$, then $L^r(S; X)$ has type $\min\{p, r\}$ and cotype $\max\{q, r\}$.

Here (a) is immediate from the triangle inequality (type 1 assertion) and Lemma E.1.1 (cotype ∞ assertion), (b) and (c) follow from Hölder's inequality, and the direct implication in (d) is a consequence of (E.3). The reverse implication in (d) is a deep fact due to Kwapień.

Motivated by (a), we say that X has *non-trivial type* if X has type p for some $p \in]1, 2]$, and *finite cotype* if it has cotype q for some $q \in [2, \infty[$.

Next we come to a deep fact which generalizes Proposition E.1.4:

Theorem E.2.2 (Khintchine-Maurey). *Let E be a Banach function space with finite cotype. Then there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and $x_0, \ldots, x_N \in X$,*

$$\frac{1}{\kappa_{2,1}} \left\| \left(\sum_{n=0}^{N} |x_n|^2 \right)^{1/2} \right\| \leq \mathbb{E} \left\| \sum_{n=0}^{N} \epsilon_n x_n \right\| \leq C \left\| \left(\sum_{n=0}^{N} |x_n|^2 \right)^{1/2} \right\|,$$

where $\kappa_{2,1}$ is as in the Kahane-Khintchine inequality (cf. Theorem E.1.3).

E.3 $\mathcal{R}$-boundedness

Definition E.3.1. A collection $\mathcal{T} \subset \mathcal{B}(X)$ is said to be $\mathcal{R}$-bounded if there exists a constant $C \geq 0$ such that for all sequences $(T_k)_{k=0}^K$ in $\mathcal{T}$ and $(x_k)_{k=0}^K \subset X$, $K \in \mathbb{N}$, we have

$$\left\| \sum_{k=0}^{K} \epsilon_k T_k x_k \right\|_{L^2(\Omega; X)} \leq C \left\| \sum_{k=0}^{K} \epsilon_k x_k \right\|_{L^2(\Omega; X)}.$$

The least admissible constant C is called the $\mathcal{R}$-bound of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$.

The Kahane-Khintchine inequality (cf. Theorem E.1.3) show that the exponents 2 may be replaced by any other exponents $p, q \in [1, \infty[$: A collection $\mathcal{T} \subset \mathcal{B}(X)$ is $\mathcal{R}$-bounded if there exists a constant $C \geq 0$ such that for all sequences $(T_k)_{k=0}^K$ in $\mathcal{T}$ and $(x_k)_{k=0}^K \subset X$, $K \in \mathbb{N}$, we have

$$\left\| \sum_{k=0}^{K} \epsilon_k T_k x_k \right\|_{L^p(\Omega; X)} \leq C \left\| \sum_{k=0}^{K} \epsilon_k x_k \right\|_{L^q(\Omega; X)}.$$

Denoting by $\mathcal{R}_{p,q}(\mathcal{T})$ the least admissible constant $C \geq 0$ in this inequality, we have $\mathcal{R}_{p,q}(\mathcal{T}) \leq \kappa_{p,2}\kappa_{2,q}\mathcal{R}(\mathcal{T})$ and $\mathcal{R}_{p,q}(\mathcal{T}) \leq \kappa_{2,p}\kappa_{q,2}\mathcal{R}_{p,q}(\mathcal{T})$. We will write $\mathcal{R}_p(\mathcal{T}) := \mathcal{R}_{p,p}(\mathcal{T})$. Furthermore, we shall use the convention to write $\mathcal{R}_{p,p}(\mathcal{T}) = \infty$ for collections $\mathcal{T} \subset \mathcal{B}(X)$ which are not $\mathcal{R}$-bounded.

Remark E.3.2.

(i) If $\mathcal{T} \subset \mathcal{B}(X)$ is $\mathcal{R}$-bounded, then $\mathcal{T}$ is uniformly bounded with $\sup_{T \in \mathcal{T}} \|T\| \leq \mathcal{R}_p(\mathcal{T})$ (just take $K = 0$ in the definition).

(ii) It can be shown that it is enough to consider all possible choices of distinct operators $T_0, \dots, T_K$, $K \in \mathbb{N}$, in the definition of $\mathcal{R}$-boundedness.

Example E.3.3 (Scalar multiples of the identity.)**.** Let X be a Banach space and let $A \subset \mathbb{K}$. Then $\{aI : a \in A\}$ is $\mathcal{R}$-bounded if and only if A is bounded, in which case $\mathcal{R}\{aI : a \in A\} = \sup\{|a| : a \in A\}$. This follows from a combination of the Kahane contraction principle (cf. Proposition E.1.2) and Remark E.3.2.

From the identity (E.3) it follows that in a Hilbert space $\mathcal{R}$-boundedness coincides with uniform boundedness. Using Kwapien's isomorphic characterization of Hilbert spaces as the only Banach spaces with both type 2 and cotype 2, the reverse can be shown to hold true as well:

Proposition E.3.4. *A Banach space X is isomorphic to a Hilbert space if and only if every uniformly bounded family in $\mathcal{B}(X)$ is $\mathcal{R}$-bounded. In this situation we have $\mathcal{R}(\mathcal{T}) \le \tau_{2,X} c_{2,X} \sup_{T\in\mathcal{T}} \|T\|$ for every uniformly bounded family $\mathcal{T}$ in $\mathcal{B}(X)$.*

Some basic properties of $\mathcal{R}$-bounds:

Proposition E.3.5. *Let X be a Banach space.*

(i) Suppose that $\mathcal{S}, \mathcal{T} \subset \mathcal{B}(X)$ are $\mathcal{R}$-bounded, then the families $\mathcal{S} \cup \mathcal{T}$, $\mathcal{S} + \mathcal{T}$, and $\mathcal{S}\mathcal{T}$ are $\mathcal{R}$-bounded as well and for all $p \in [1, \infty[$ we have

$$\begin{aligned} \mathcal{R}_p(\mathcal{S} \cup \mathcal{T}) &\le \mathcal{R}_p(\mathcal{S}) + \mathcal{R}_p(\mathcal{T}) \\ \mathcal{R}_p(\mathcal{S} + \mathcal{T}) &\le \mathcal{R}_p(\mathcal{S}) + \mathcal{R}_p(\mathcal{T}) \\ \mathcal{R}_p(\mathcal{S}\mathcal{T}) &\le \mathcal{R}_p(\mathcal{S})\mathcal{R}_p(\mathcal{T}). \end{aligned}$$

(ii) If $\mathcal{T} \subset \mathcal{B}(X)$ is an $\mathcal{R}$-bounded family, then so are its convex hull and absolute convex hull. Moreover, for all $p \in [1, \infty[$ we have

$$\mathcal{R}_p(\mathcal{T}) = \mathcal{R}_p(\mathrm{conv}(\mathcal{T})) = \mathcal{R}_p(\mathrm{absconv}(\mathcal{T})).$$

*(iii) If $\mathcal{T} \subset \mathcal{B}(X)$ is an $\mathcal{R}$-bounded family, then its closures $\overline{\mathcal{T}}^{\mathrm{WOT}}$ and $\overline{\mathcal{T}}^{\mathrm{SOT}}$ in the weak operator topology (*WOT*) and strong operator topology (*SOT*), respectively, are $\mathcal{R}$-bounded as well, and for all $p \in [1, \infty[$ we have*

$$\mathcal{R}_p(\mathcal{T}) = \mathcal{R}_p(\overline{\mathcal{T}}^{\mathrm{WOT}}) = \mathcal{R}_p(\overline{\mathcal{T}}^{\mathrm{SOT}}).$$

*(iv) If $\mathcal{T} \subset \mathcal{B}(X^*)$ is an $\mathcal{R}$-bounded family, then its closure $\overline{\mathcal{T}}^{\mathrm{W^*OT}}$ in the weak-star operator topology (*W*OT*) is $\mathcal{R}$-bounded as well, and for all $p \in [1, \infty[$ we have*

$$\mathcal{R}_p(\mathcal{T}) = \mathcal{R}_p(\overline{\mathcal{T}}^{\mathrm{W^*OT}}).$$

The following simple formulation of $\mathcal{R}$-boundedness in terms of the space $\mathrm{Rad}(X)$ is sometimes very useful:

Lemma E.3.6. *Let X be a Banach space and $\mathcal{T} \subset \mathcal{B}(X)$. Denote by $\tilde{\mathcal{T}} \subset \mathcal{B}(\mathrm{Rad}(X))$ the collection of all finitely non-zero $\tilde{T} = (T_n)_{n\in\mathbb{N}}$ sequences in $\tilde{T}$, where the action of $\tilde{T} = (T_n)_{n\in\mathbb{N}}$ on $x = (x_n)_{n\in\mathbb{N}} \in \mathrm{Rad}(X)$ is given in the obvious way by $\tilde{T}x := (T_n x_n)_{n\in\mathbb{N}}$. Then $\mathcal{T}$ is $\mathcal{R}$-bounded if and only if $\tilde{\mathcal{T}}$ is uniformly bounded, in which case we have $\mathcal{R}_p(\mathcal{T}) = \sup_{\tilde{T}\in\tilde{\mathcal{T}}} \left\|\tilde{T}\right\|_{\mathcal{B}(\mathrm{Rad}_p(X))}$.*

Theorem E.3.7. *Suppose that E has non-trivial type*[1] *and let $\mathcal{T} \subset \mathcal{B}(E)$ be an $\mathcal{R}$-bounded collection. Then the collection of adjoints $\mathcal{T}^* = \{T^* \mid T \in \mathcal{T}\} \subset \mathcal{B}(E^*)$ is $\mathcal{R}$-bounded as well.*

Lemma E.3.8. *Let X be a Banach space, $(S, \mathscr{A}, \mu)$ a measure space, and $p \in [1, \infty[$. If $\mathcal{T} \subset \mathcal{B}(L^p(S; X))$ is $\mathcal{R}$-bounded and $r, s \in [0, \infty[$, then*

$$\{m_\phi T m_\psi \,:\, T \in \mathcal{T}, \phi, \psi \in L^\infty(S), \|\phi\|_\infty \le r, \|\psi\|_\infty \le s\} \subset \mathcal{B}(L^p(S; X))$$

is $\mathcal{R}$-bounded with $\mathcal{R}_p$-bound $\le r\mathcal{R}_p(\mathcal{T})s$.

Proposition E.3.9. *Let X be a Banach space, let $H : G \longrightarrow \mathcal{B}(X)$ be a holomorphic mapping on the open set $G \subset \mathbb{C}$, and let $K \subset G$ be a compact subset. Then $H(K) \subset \mathcal{B}(X)$ is $\mathcal{R}$-bounded.*

Lemma E.3.10. *Let X be a Banach space, $(S, \mathscr{A}, \mu)$ a measure space, and $p \in]1, \infty[$. Suppose that $\mathcal{K} \subset \mathcal{B}(L^p(S; X))$ is a family of kernel operators in the sense that*

$$Kf(x) = \int_S k(x, x') f(x')\, d\mu(x'), \qquad x \in S, f \in L^p(S; X),$$

for each $K \in \mathcal{K}$, where the kernels $k : S \times S \longrightarrow \mathcal{B}(X)$ are strongly measurable such that

$$\mathcal{R}_p\{k(x, x') \,:\, k \in \mathcal{K}\} \le \kappa_0(x, x'), \qquad x, x' \in S,$$

for some measurable scalar kernel $\kappa_0 : S \times S \longrightarrow \mathbb{R}$ which gives rise to a well-defined bounded linear kernel operator K_0 on $L^p(S)$. Then $\mathcal{K}$ is $\mathcal{R}$-bounded with $\mathcal{R}_p(\mathcal{K}) \le \|K_0\|_{\mathcal{B}(L^p(S))}$.

Proof. See [25, Proposition 4.12]. □

E.4 Property (α)

Let $(\epsilon'_n)_{n\in\mathbb{N}}$ and $(\epsilon''_n)_{n\in\mathbb{N}}$ be independent Rademacher sequences on probability spaces $(\Omega, \mathscr{F}', \mathbb{P}')$ and $(\Omega, \mathscr{F}'', \mathbb{P}'')$, respectively.

Definition E.4.1. A Banach space X is said to have *property* (α) (or *Pisier's contraction property* if there exists a constant $C \ge 0$ such that

$$\left\|\sum_{m=0}^{M}\sum_{n=0}^{N} a_{m,n}\epsilon'_n\epsilon''_m x_{m,n}\right\|_{L^2(\Omega'\times\Omega'';X)} \le C|a|_\infty \left\|\sum_{m=0}^{M}\sum_{n=0}^{N} \epsilon'_n\epsilon''_m x_{m,n}\right\|_{L^2(\Omega'\times\Omega'';X)}$$

for all scalars $a_{m,n} \in \mathbb{K}$ and vectors $x_{m,n} \in X$; $m = 0, \ldots, M$ and $n = 0, \ldots, N$. The least admissible constant C is denoted by α_X.

[1] Or equivalently, assume that X is K-convex, which is a notion defined in terms of the boundedness of the Rademacher projctions.

It can be shown that the exponent 2 can be replaced by any other exponent $p \in [1, \infty[$. The resulted constants will be denoted by $\alpha_{p,X}$.

Example E.4.2.

(i) Every Hilbert space has property (α).

(ii) Every L^p-space ($p \in [1, \infty[$) has property (α).

(iii) A Banach lattice has property (α) if and only if it has finite cotype.

(iv) The Schatten class $\mathscr{C}^p(\ell^2)$ has property (α) if and only if $p = 2$.

Proposition E.4.3. *Let $(S, \mathscr{A}, \mu)$ be a measure space let and $p \in [1, \infty[$ be such that $L^p(S)$ is non-trivial. If X is a Banach space with property (α), then $L^p(S; X)$ has property (α) as well with*

$$\alpha_{p,L^p(S;X)} = \alpha_{p,X}.$$

The reason for considering property (α) in this thesis is that it allows us to bootstrap $\mathcal{R}$-boundedness:

Proposition E.4.4. *Let X be a Banach space with property (α) and let $\mathcal{T} \subset \mathcal{B}(X)$. In the notations of Lemma E.3.6, if $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{B}(X)$, then so it $\tilde{\mathcal{T}}$ is $\mathcal{R}$-bounded in $\mathcal{B}(\mathrm{Rad}(X))$. Moreover, there exists a constant $C \geq 0$, independent of $\mathcal{T}$, such that $\mathcal{R}(\tilde{\mathcal{T}}) \leq C\mathcal{R}(\mathcal{T})$.*

E.5 UMD Spaces

In this section we come to the so-called UMD spaces, where UMD stands for the unconditionality of martingale differences. Besides the direct martingale theoretic definition (see Definition/Theorem E.5.3), it is also has equivalent analytic and geometric definitions (see Theorem E.5.7).

A measure space $(S, \mathscr{A}, \mu)$ endowed with a σ-finite filtration $\mathbb{F} = (\mathscr{F}_n)_{n\in\mathbb{N}}$ will be called a *σ-finite filtered measure space*.

Definition E.5.1. Let $p \in]1, \infty[$ and let $(S, \mathscr{A}, \mathbb{F}, \mu)$ be σ-finite filtered measure space. A Banach space X is said to have the *UMD$_p$ property with respect to the σ-finite filtered measure space* $(S, \mathscr{A}, \mathbb{F}, \mu)$ if there exists a constant $\beta \geq 0$ such that

$$\left\| \sum_{n=0}^{N} \varepsilon_n d_n \right\|_{L^p(S;X)} \leq \beta \left\| \sum_{n=0}^{N} d_n \right\|_{L^p(S;X)}$$

for all finite martingale difference sequences $(d_n)_{n=0}^N$ with respect to $\mathbb{F}$ in $L^p(S; X)$ and all sequences $(\varepsilon_n)_{n=0}^N$ of unimodular scalars in $\mathbb{K}$.

With a simple randomization argument, we see that UMD$_p$ can also be described as follows:

Lemma E.5.2. *Let $p \in]1,\infty[$ and let $(S,\mathscr{A},\mathbb{F},\mu)$ be σ-finite filtered measure space. Then X has the UMD$_p$ property with respect to $(S,\mathscr{A},\mathbb{F},\mu)$ if and only if*

$$\left\| \sum_{n=0}^{N} \epsilon_n d_n \right\|_{L^q(\Omega;L^p(S;X))} \eqsim_q \beta \left\| \sum_{n=0}^{N} d_n \right\|_{L^p(S;X)}$$

for all finite martingale difference sequences $(d_n)_{n=0}^N$ with respect to $\mathbb{F}$ in $L^p(S;X)$ and all $q \in [1,\infty[$.

Definition/Theorem E.5.3. A Banach space X is said to have the *UMD property* (or is called a *UMD space*) if one of the following equivalent conditions is satisfied:

(i) X has the UMD$_p$ property with respect to every σ-finite filtered measure space $(S,\mathscr{A},\mathbb{F},\mu)$ space for some $p \in]1,\infty[$.

(ii) X has the UMD$_p$ property with respect to every σ-finite filtered measure space $(S,\mathscr{A},\mathbb{F},\mu)$ space for every $p \in]1,\infty[$.

Example E.5.4.

(i) Every Hilbert space is a UMD space.

(ii) Every closed subspace of a UMD space is a UMD space.

(iii) L^p-spaces have the UMD property for $p \in]1,\infty[$.

Proposition E.5.5. *Let X be a UMD space. Then*

(i) X is reflexive;

(ii) X^* is a UMD space

(iii) X has non-trivial type;

(iv) X has finite cotype.

Proposition E.5.6. *Let F be a non-trivial Banach function space on a σ-finite measure space and let X be a non-trivial Banach space. The Köthe-Bochner space $F(X)$ has the UMD property if and only if both X and F have the UMD property. As a consequence, if $(S,\mathscr{A},\mu)$ is a non-trivial σ-finite measure space and $p \in [1,\infty[$, then $L^p(S;X)$ is a UMD space if and only if X is a UMD space.*

Theorem E.5.7. *For a Banach space X the following are equivalent:*

(i) X is a UMD space.

(ii) X is of class $\mathcal{HT}$, i.e. the Hilbert transform $H \in \mathcal{B}(L^p(\mathbb{R}))$ has an X-valued extension $H_X \in \mathcal{B}(L^p(\mathbb{R};X))$ for some/every $p \in]1,\infty[$.

(iii) The Riesz projection is bounded on $L^p(\mathbb{R}^d;X)$, i.e.

$$R : \mathcal{S}(\mathbb{R}^d;X) \longrightarrow L^\infty(\mathbb{R}^d;X) \hookrightarrow \mathcal{S}'(\mathbb{R}^d;X),\ f \mapsto \mathscr{F}^{-1}[1_{]0,\infty[^d}\hat{f}],$$

takes its values in $L^p(\mathbb{R}^d;X)$ and has a (necessarily unique) extension to a bounded linear operator R on $L^p(\mathbb{R}^d;X)$.

(iv) X is ζ-convex.[2].

[2]For the definition of ζ-convexity we refer to [57]

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