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DEPARTMENT OF MATHEMATICS
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Master Thesis

Inozemtsev's Elliptic Spin Chains

Asymptotic Bethe Ansatz and Thermodynamics

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Abstract

Inozemtsev's elliptic spin chain and its infinite limit are interesting models from many perspectives: both of these models are most likely integrable, but their precise structure is not known yet. They form interpolating models between two prime examples of two very different classes of spin chains, the Heisenberg XXX spin chain and the Haldane-Shastry spin chain. Moreover, the infinite spin chain can be used to study the spectrum of the dilatation operator in $\mathcal{N} = 4$ super Yang-Mills theory. Finally, there seems to be a strong relationship between the solvability of these spin chains and their Calogero-Sutherland-Moser counterparts. In this thesis, we derive the eigenfunctions of Inozemtsev's infinite spin chain and use these eigenfunctions to study the thermodynamic behaviour of these models by employing the Asymptotic Bethe Ansatz. Using an approach first proposed by Hulthén, we derive an expression for the antiferromagnetic ground state and we follow a method by Yang and Yang to derive integral equations that govern the thermodynamics at arbitrary density. Finally, we classify all the asymptotic (bound-state) solutions of the Bethe equations of Inozemtsev's elliptic spin chain. This leads to interesting new phenomena and a reason to revisit the derivation of asymptotic bound-state solutions of other models. After identifying the spectrum within the set of solutions of the Bethe equations, we can plot the spectrum of bound states.

Keywords: quantum physics, spin chain, Weierstraß elliptic functions, Asymptotic Bethe Ansatz, bound states.

Contents

Nomenclature

The following list is comprised of some of the special symbols and functions in this Thesis.

Groups

$\mathbb{Z}/L\mathbb{Z}$	Cyclic group of the integers $\{1, 2, \cdots, L\}$
π_M	Permutation group of M elements

Spaces

$\ell^2(V)$	Sequence space indexed by V with norm $\|a\|^2 = \sum_{n\in V} \|a_n\|^2$
$\mathcal{H}^{(L)}$	$\bigotimes_{n\in\mathbb{Z}/L\mathbb{Z}} \mathbb{C}\|\uparrow\rangle \oplus \mathbb{C}\|\downarrow\rangle$
$\mathcal{H}$	Hilbert space of Inozemtsev's infinite spin chain
$\mathcal{H}_M$	M-partice sector of $\mathcal{H}$

Sets

$\mathbb{Z}_{[\boldsymbol{n}]}$	$\mathbb{Z} \setminus \{n_1, \cdots, n_M\}$, where $\boldsymbol{n} = (n_1, n_2, \cdots, n_M)^T$
$\mathbb{Z}_M$	$\{1, 2, \cdots, M\}$
D	$\{0, 1, \cdots, M-1\}$
$\mathbb{R}^*_M$	$\mathbb{R} \setminus \left\{\frac{3}{2} - M, \frac{1}{2} - M, \cdots, -\frac{1}{2}\right\}$
$\mathbb{R}_M$	$\mathbb{R} \setminus \left\{\frac{3}{2} - M, \frac{1}{2} - M, \cdots, -\frac{M+3}{2}, -\frac{M-1}{2}, \cdots, -\frac{1}{2}\right\}$

$\mathcal{B}_M$	$\{\boldsymbol{n} \in \mathbb{Z}^M \mid n_1 < n_2 < \cdots < n_M\}$
$\mathbb{Z}_{\beta,\rho}$	$\{n \in \mathbb{Z} \mid 0 \leq m_\beta + n, m_\rho - n \leq M - 1\}$

Functions

$\wp_L(z)$	$\wp(z\|L, i\pi/\kappa)$
$\wp$ (z)	$\wp(z\|1, i\pi/\kappa)$
ζ(z)	$\zeta(z, \|1, i\pi/\kappa)$
$\phi(p)$	$\frac{p}{2\kappa\pi i}\zeta\left(\frac{i\pi}{2\kappa}\right) - \frac{1}{2\kappa i}\zeta\left(\frac{ip}{2\kappa}\right)$
ϵ_p	$J\left(\frac{1}{2}\wp\left(\frac{ip}{2\kappa}\right) + \frac{1}{2}\left(\frac{p}{\pi}\zeta\left(\frac{i\pi}{2\kappa}\right) - \zeta\left(\frac{ip}{2\kappa}\right)\right)^2 - \frac{2i\kappa}{\pi}\zeta\left(\frac{i\pi}{2\kappa}\right)\right)$

Symbols

κ	Interpolation parameter of Inozemtsev's spin chains
L	Length of a spin chain
M	Number of magnon excitations
$d_{m_1,m_2,\cdots,m_M}(\boldsymbol{p})$	Coefficients of the Ansatz for the CSM model
$c_{m_1,m_2,\cdots,m_M}(\boldsymbol{p})$	Coefficients of the Ansatz for Inozemtsev's infinite spin chain

Chapter 1

Introduction

Spin chains have been of interest ever since Heisenberg proposed his model for the magnetic interaction of electrons in 1928 [1]. Because spin chains are one-dimensional, they are often easier to study than similar models in higher dimensions, making them an excellent starting point to study new phenomena in condensed-matter physics. Interestingly, research has shown that spin chains can also be used to study more complicated models, such as lattice models (see e.g. [2]) or even certain conformal field theories and string theories through the AdS/CFT-correspondence[1]. Also, the study of spin chains has triggered the development of experimental setups that are modelled by these spin chains and prove to be a promising playground to test applications such as quantum computing (see e.g. [6]). Finally, the development of Yang-Baxter theory originated from the use of the Bethe Ansatz to find the spectra of spin chains and has proved to be a promising research area in itself (see e.g. [7]).

This thesis will concentrate on spin chains that are susceptible to an exact analytic approach to study their spectra and other properties. Models of this type are often also exactly solvable or integrable, although the characterization of these terms, which is far from uniform in the present literature, is usually slightly different. Loosely speaking, for the models we will study it is possible to find the functional form of the eigenfunctions depending on a set of parameters, although it might not be possible to determine the parameters for which this functional form actually yields an eigenvalue belonging to the spectrum of the model. For *exactly solvable* models, finding these eigenvalues is possible. For models that are *integrable* there exists a set of conserved quantities that are in involution that allow one to trivialize the dynamics.

The study of spin chains by analytic means is most relevant: it pushes the boundaries of researchers to find new methods to attack classes of models and can in this way forge a stronger

[1] see [3] for a general review, [4] for a review of integrability in this field and [5] for a review that focusses on integrability and the use of spin chains in research into this correspondence.

relationship between physics and mathematics, which has been a very fruitful partnership in the history of science. Moreover, the fact that so many aspects of these models can be studied without the need to resort to approximation techniques enables us not only to test the mathematical foundations of physics, but also to sharpen our understanding of physics itself.

The systems we will focus on are the elliptic spin chains first introduced by Inozemtsev, one of finite length and one of infinite length [8]. The finite elliptic spin chain is in fact the most general spin chain with only two-body interactions for which there exists a quantum Lax pair. As such, the elliptic spin chain could very well be integrable, but of the currently proposed set of conserved quantities it is not known yet whether they are all in involution [9]. Apart from their relevance to the body of integrable models, the fact that these spin chains can be regarded as deformations of existing and very well-known spin chains is notable. Indeed, Inozemtsev's elliptic spin chain forms an interpolation between the short-range Heisenberg XXX-model for spin 1/2-particles, which is solvable by application of the Bethe Ansatz [10], and the long-range Haldane-Shastry spin chain for spin 1/2-particles, which can be solved by exploiting its Yangian symmetry [11, 12]. Therefore, investigating the properties of Inozemtsev's elliptic spin chains can yield valuable insights in the relationship between these rather different types of models.

Since the study of the elliptic spin chain goes hand in hand with the study of elliptic functions, and this subject is seldomly taught (even) at graduate level, we will devote the first part of Chapter 2 to properly introduce what we need to know about these functions. The second part of that chapter is used to introduce the elliptic spin chain of finite length and its related models. In Chapter 3, we will formally introduce the elliptic spin chain of infinite length and use the Ansatz provided by Inozemtsev to find the functional form of its eigenfunctions. Chapter 4 is devoted to the derivation of the Bethe equations for this model and the study of its real solutions using methods developed by Hulthén and by Yang and Yang. Chapter 5 contains a detailed characterization of the bound-state solutions of the aforementioned Bethe equations, yielding some interesting new phenomena. Chapter 6 finally discusses the use and the implications of this research and provides recommendations for future research.

Most of the derivations presented in this thesis are fairly explicit. It is our hope that this aids readers to understand all the relevant steps. Additionally, learning about the techniques used in this thesis to derive eigenfunctions is a goal in itself, making it especially relevant to be as explicit as possible. We have also tried to prove as many of the statements as possible in a rigorous fashion, not only as an aid to the reader, but also to facilitate the more mathematically inclined reader.

Chapter 2

Elliptic Functions and Spin Chains

In this chapter we will provide the background necessary to read Chapters 3 up until 6. The first topic, elliptic functions, is not usually taught in university courses and can therefore not be regarded as common knowledge. The second part of this chapter not only introduces the language used in spin-chain research, but also reviews some examples of spin chains which are important in the rest of this thesis. In particular, we will introduce Inozemtsev's elliptic spin chain, which is the central subject of this thesis.

2.1 Elliptic Functions

Historically, elliptic functions were developed first in the theory of elliptic integrals, where they serve as inverse functions for these integrals. Later it was discovered that these functions are very useful in the theory of modular forms and elliptic curves, which is heavily used in algebraic geometry and number theory. Physicists also discovered their use, since they could be used to write explicit solutions for given integrals and also cropped up naturally when studying the Laplace equation in elliptic coordinates. In this thesis we will focus on one type of elliptic function, the Weierstraß elliptic functions, and mostly use their strong similarity to trigonometric functions. We will therefore not aim to present a complete theory of elliptic functions, but try to restrict to the necessary ingredients for this thesis. For more information about these interesting functions see for instance [13, 14, 15].

2.1.1 Definition

In modern analysis the definition of elliptic functions usually does not start at the integrals for which they were once developed. We will approach them beginning from the trigonometric functions, such as sine and cosine, on the complex plane. Starting from their usual definition on the real line the trigonometric functions can be analytically continued to the entire complex plane in a unique fashion defined by their power series. The result of this continuation is quite

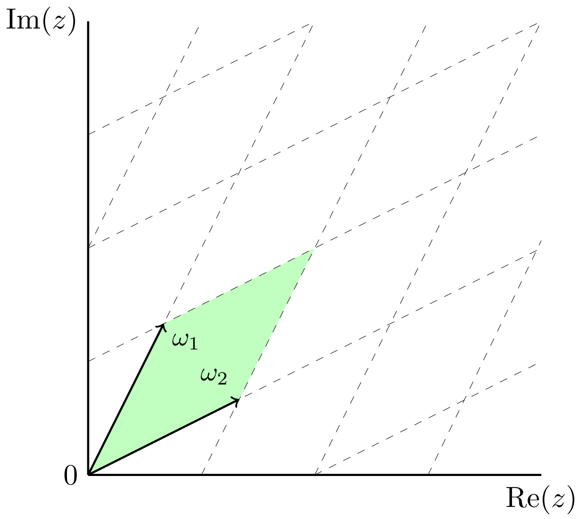


Figure 2.1: The tesselation of the complex plane by parallellograms, induced by a doubly-periodic function with periods ω_1, ω_2. The colored parallellogram is the fundamental parallellogram.

special: instead of being periodic only along the real axis in the complex plane, these continuations are periodic along any line parallel to the real axis, i.e. they satisfy $f(z+2\pi) = f(z)$ for all $z \in \mathbb{C}$. This leads to a natural equivalence relation on the complex plane, identifying points z and z' if $z - z' = 2\pi k$ for some $k \in \mathbb{Z}$ and dividing the complex plane in strips of width 2π perpendicular to the real axis. For reasons that will become clear soon, we will call functions satisfying $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$ and some $\omega \in \mathbb{R}$ *singly-periodic functions*.

Of course, the above situation is quite special and it makes sense to try to generalize it. Obviously, we can create singly-periodic functions with different periods by rescaling the real coordinate, but we can actually do more: by rotating the coordinates, we can make these functions periodic in any direction we want, for example along the imaginary axis, so that their periods $\omega \in \mathbb{C}$ are no longer purely real. However, we can extend our options even further by allowing the function to be periodic in a second direction too, i.e. to be *doubly periodic*. Such functions satisfy

$$f(z+\omega_1) = f(z), \qquad f(z+\omega_2) = f(z),$$

for all $z \in \mathbb{C}$ for some $\omega_1, \omega_2 \in \mathbb{C}$ with $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$. If we do not ask for the periods to satisfy $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$, the resulting function is not doubly periodic in the intuitive sense. Namely, Jacobi showed that if $\frac{\omega_1}{\omega_2} \in \mathbb{Q}$ then the function is singly-periodic, whereas if $\frac{\omega_1}{\omega_2} \in \mathbb{R} \setminus \mathbb{Q}$ then it is constant. Note that in analogy with singly-periodic functions, doubly periodicity leads to a natural equivalence relation on the complex plane, dividing it into parallellograms (see Figure 2.1). The closed parallellogram defined by the four corners $0, \omega_1, \omega_2, \omega_1 + \omega_2$ is called the *fundamental parallellogram* (the colored parallellogram in Figure 2.1). Most of the analysis of doubly-periodic functions can be reduced to this fundamental parallellogram. We will also call this tesselation of $\mathbb{C}$ by parallellograms the *lattice*. We can now define what an elliptic function is.

Definition An *elliptic function* is a meromorphic doubly-periodic function.

In particular, meromorphic functions cannot have other singularities than removable singularities and poles and therefore all elliptic functions have a Laurent series at every point in their domain. Of course, from this definition it does not follow that there actually exist elliptic functions, but we will see in due course that this is the case. First we list a couple of important theorems about and properties of elliptic functions.

2.1.2 Properties of Elliptic Functions

Probably the most important theorem about elliptic functions is Liouville's theorem:

Liouville's Theorem. An elliptic function without poles is a constant.

Liouville's theorem is a direct consequence of the fact that all bounded analytic functions are constant. It tells us that all nontrivial elliptic functions must have poles. Other theorems (to be found in [13, 14]) show that these poles must be such that

1. the number of poles in any parallellogram is finite.
2. the number of poles equals the number of zeros (if and only if the elliptic function is not constant).
3. the sum of the residues for all the poles in a parallellogram is zero.
4. the leading order of the Laurent series of an elliptic function $f(z)$ cannot be $1/z$.

Property 4 is an easy consequence of Property 3.

2.1.3 Weierstraß Elliptic Functions

Armed with these basic facts, let us now give an example of an elliptic function, the function usually denoted by $\wp$ (called the *Weierstraß-P*). $\wp : \mathbb{C} \to \mathbb{C}$ is given by

$$\wp(z|\omega_1, \omega_2) := \frac{1}{z^2} + \sum_{\substack{m,n\in\mathbb{Z} \\ m^2+n^2\neq 0}} \left(\frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right), \tag{2.1}$$

where ω_1, ω_2 are the periods of the function, $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$ and $m^2 + n^2 = 0$ expresses that we do not include the case $m = 0 = n$. We will supress the explicit mentioning of the periods when it is clear which periods are being used. This function has second-order poles at the corners $0, \omega_1, \omega_2, \omega_1 + \omega_2$ in the fundamental parallellogram and the sum on the right-hand side converges uniformly for all other points in the fundamental parallellogram. This shows that $\wp(z|\omega_1,\omega_2)$ is meromorphic. Proving that $\wp$ is actually doubly-periodic with the given periods is a little bit more work, but can be done using the differentiability of $\wp$ and some properties one can derive for its derivative. From this it then follows that $\wp$ is actually elliptic (see [13]).

The Weierstraß elliptic function can also be defined in the more traditional way [13], as the solution to the functional equation

$$z = \int_{f(z)}^{\infty} (4t^3 - g_2 t - g_3)^{-1/2} dt,$$

where f is the unknown function and g_2, g_3 are the *invariants* of the $\wp$-function and depend on the chosen periods:

$$g_2 = 60 \sum_{\substack{m,n\in\mathbb{Z} \\ m^2+n^2\neq 0}} \frac{1}{(m\omega_1 + n\omega_2)^4}, \qquad g_3 = 140 \sum_{\substack{m,n\in\mathbb{Z} \\ m^2+n^2\neq 0}} \frac{1}{(m\omega_1 + n\omega_2)^6}.$$

From this one can also deduce that the $\wp$-function satisfies the differential equation

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

When doing computations with the $\wp$-function, two other functions are extremely important: the Weierstraß ζ- and σ-functions. They are defined on the complex plane by the following expressions:

$$\begin{aligned} \zeta(z|\omega_1,\omega_2) &:= \frac{1}{z} + \sum_{\substack{m,n\in\mathbb{Z} \\ m^2+n^2\neq 0}} \left(\frac{1}{(z - m\omega_1 - n\omega_2)} + \frac{1}{(m\omega_1 + n\omega_2)} + \frac{z}{(m\omega_1 + n\omega_2)^2} \right) \\ \sigma(z|\omega_1,\omega_2) &:= z \prod_{\substack{m,n\in\mathbb{Z} \\ m^2+n^2\neq 0}} \left(\left(1 - \frac{z}{m\omega_1 + n\omega_2}\right) \exp\left\{ \frac{z}{m\omega_1 + n\omega_2} + \frac{z^2}{2(m\omega_1 + n\omega_2)^2} \right\} \right) \end{aligned} \tag{2.2}$$

The Weierstraß ζ-function is meromorphic and quasiperiodic and satisfies $\zeta(z+\omega_i) = \zeta(z) + 2\eta_i$ (with $\eta_i := \zeta\left(\frac{\omega_i}{2}\right)$) as long as $\zeta(z)$ is finite, while the Weierstraß σ-function is entire with simple zeros at the lattice points $m\omega_1 + n\omega_2$ for all $m, n \in \mathbb{Z}$. The σ-function is also quasiperiodic, satisfying

$$\sigma(z + \omega_i) = -\exp\left\{ 2\eta_i \left(z + \frac{\omega_i}{2} \right) \right\} \sigma(z). \tag{2.3}$$

They relate to the $\wp$-function as follows:

$$\zeta'(z) = -\wp(z), \qquad \zeta(z) = \frac{\sigma'(z)}{\sigma(z)}.$$

The ζ-function will be useful in our subsequent chapters, but has no other important properties, contrary to the σ-function; Indeed the latter allows for a very elegant characterization of elliptic functions:

Theorem. Every elliptic function can be written as a multiple of a quotient of these σ-functions in a very simple way: given an elliptic function F with a given set of N zeros $\{a_n\}$ and poles $\{b_n\}$ in the fundamental parallellogram (characterized by the periods ω_1, ω_2), one can write this elliptic function as

$$F(z) = A \prod_{i=1}^{N} \frac{\sigma(z - a_i)}{\sigma(z - b_i)},$$

where $A \in \mathbb{C}$.

This can be proved using Liouville's theorem quite simply, which we will do here to illustrate the power of Liouville's theorem.

Proof. Consider the function $G(z) = \prod_{i=1}^{N} \frac{\sigma(z-a_i)}{\sigma(z-b_i)}$. It has the same zeros and poles as F, is meromorphic and doubly periodic as can be checked by using (2.3). It is therefore elliptic. The quotient function F/G is also elliptic and has no poles or zeros. By Liouville's theorem, it must be constant, say A. We can conclude that indeed $F(z) = AG(z)$ for all $z \in \mathbb{C} \setminus \{b_i\}$, thus $F = AG$. □

The Weierstraß functions have numerous properties, many of which can be found in Appendix E or in the references [13, 14, 15]. The webpage [16] is also very useful as a reference.

2.2 Introduction to Inozemtsev's Elliptic Spin Chain

Inozemtsev's original paper introducing his elliptic spin chain was published in 1989 [8] and aimed to connect the Heisenberg XXX chain to the Haldane-Shastry spin chain for the case of spin-1/2 particles. This connection was forged through the analysis of the existing sets of quantum Lax pairs for these spin chains, proposing a generalized quantum Lax pair depending on a parameter κ such that its limits (i.e. $\kappa \to 0, \infty$) correspond to the Heisenberg XXX and the Haldane-Shastry spin chains, respectively. The corresponding hamiltonians exhibit similar behaviour, indicating that Inozemtsev's proposal could be interpreted as a deformation of the Heisenberg XXX and Haldane-Shastry spin chains. Additionally, Inozemtsev argued that his elliptic spin chain, which we will consequently call *Inozemtsev's elliptic spin chain*, is in fact the most general spin chain with only two-body interactions to admit a quantum Lax representation. Before going further into the properties of Inozemtsev's elliptic spin chain, let us first introduce this class of physical models in a more formal fashion.

2.2.1 What is a spin chain?

A *spin chain* is a one-dimensional lattice with fixed particles at the lattice sites whose only degree of freedom is their spin. For a periodic spin chain of finite length L, the Hilbert space

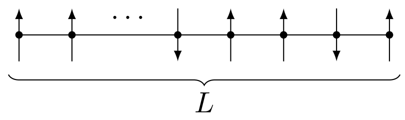


Figure 2.2: A spin chain of length L. For simplicity, all the particles are depicted as if their spin is precisely one of the basis vectors $|\uparrow\rangle$,$|\downarrow\rangle$.

$\mathcal{H}^{(L)}$ of states is the tensor product

$$\mathcal{H}^{(L)} := \bigotimes_{n\in\mathbb{Z}/L\mathbb{Z}} V_n, \tag{2.4}$$

where the vector space V_n is the spin space of the particle at the lattice site with label n and the summation runs over $\mathbb{Z}/L\mathbb{Z}$ to ensure the periodicity. For the case of spin-1/2 particles, to which we will restrict our attention, the V_n are given by $V_n = \mathbb{C}|\uparrow\rangle \oplus \mathbb{C}|\downarrow\rangle \cong \mathbb{C}^2$. We can use an operator $A : V \to V$ (where $V \cong \mathbb{C}^2$) to define an operator on $\mathcal{H}^{(L)}$ that acts only on the jth vector space by the L-fold product

$$A_j = \mathrm{Id}_{\mathbb{C}^2} \otimes \cdots \otimes \underbrace{A}_{V_j} \otimes \cdots \otimes \mathrm{Id}_{\mathbb{C}^2},$$

where the $\mathrm{Id}_{\mathbb{C}^2}$ is the identity operator on $\mathbb{C}^2$. In particular, we can define Pauli spin matrices that act on the jth particle by

$$\begin{aligned}
\sigma_j^x &= \mathrm{Id}_{\mathbb{C}^2} \otimes \cdots \otimes \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{V_j} \otimes \cdots \otimes \mathrm{Id}_{\mathbb{C}^2} \\
\sigma_j^y &= \mathrm{Id}_{\mathbb{C}^2} \otimes \cdots \otimes \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}_{V_j} \otimes \cdots \otimes \mathrm{Id}_{\mathbb{C}^2} \\
\sigma_j^z &= \mathrm{Id}_{\mathbb{C}^2} \otimes \cdots \otimes \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{V_j} \otimes \cdots \otimes \mathrm{Id}_{\mathbb{C}^2},
\end{aligned} \tag{2.5}$$

where all the tensor products are L-fold. We will further define

$$\boldsymbol{\sigma}_j := (\sigma_j^x, \sigma_j^y, \sigma_j^z)^T, \tag{2.6}$$

such that we can use the standard inner product to write

$$\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k = \sigma_j^x \sigma_k^x + \sigma_j^y \sigma_k^y + \sigma_j^z \sigma_k^z. \tag{2.7}$$

Moreover, we can define the *spin-flip operators* $\sigma^{\pm} := \frac{1}{2}(\sigma^x \pm i\sigma^y)$ on $\mathbb{C}|\uparrow\rangle \oplus \mathbb{C}|\downarrow\rangle$, which act as follows on basis vectors:

$$\begin{aligned}
\sigma^+|\uparrow\rangle &= 0, \qquad \sigma^-|\uparrow\rangle = |\downarrow\rangle, \\
\sigma^+|\downarrow\rangle &= |\uparrow\rangle, \qquad \sigma^-|\downarrow\rangle = 0.
\end{aligned} \tag{2.8}$$

Using these operators, we can define a basis of $\mathcal{H}^{(L)}$ as follows: let

$$|0\rangle := \underbrace{|\uparrow\rangle \otimes |\uparrow\rangle \otimes \cdots \otimes |\uparrow\rangle}_{L} \tag{2.9}$$

be the so-called *pseudovacuum state*. By repeatedly acting with the operator σ_j^- for different j we can flip the spin at position j until eventually the vector in which all spins are pointing down is reached. These vectors are defined as

$$\sigma_{n_1}^- \sigma_{n_2}^- \cdots \sigma_{n_M}^- |0\rangle := |n_1, n_2, \cdots, n_M\rangle, \tag{2.10}$$

where $1 \leq M \leq L$ and the $n_i \in \mathbb{Z}/L\mathbb{Z}$ satisfy $n_1 < n_2 < \cdots < n_M$. For fixed M we define $\mathcal{B}_M \subset \mathbb{Z}^M$ as

$$\mathcal{B}_M := \{\boldsymbol{n} \in \mathbb{Z}^M \,|\, n_1 < n_2 < \cdots < n_M\}. \tag{2.11}$$

The union of these vectors for all $1 \leq M \leq L$ and allowed values of the n_i, together with $|0\rangle$, forms a basis of $\mathcal{H}^{(L)}$, as can easily be seen by a counting argument. If we associate $|0\rangle$ to the case $M = 0$, then at fixed $0 \leq M \leq L$ the number of vectors of the form (2.10) is $\binom{M}{L}$. By extending the inner product of the individual vector spaces to $\mathcal{H}^{(L)}$ in the canonical way, we see that all these vectors are orthonormal and that there are

$$\sum_{M=0}^{L} \binom{M}{L} = 2^L$$

of these vectors. Therefore, they form an orthonormal basis of the Hilbert space. Armed with these definitions, we can discuss the three models we have mentioned so far.

2.2.2 The Heisenberg XXX Spin Chain

On the Hilbert space $\mathcal{H}^{(L)}$, the periodic Heisenberg XXX chain of length L is defined by the hamiltonian

$$H_{XXX} := -\frac{J}{4} \sum_{j=1}^{L} \left(\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1} - 1 \right), \tag{2.12}$$

where J is a coupling constant. It is the first spin chain to be studied in great detail by Hans Bethe in his famous paper Zur Theorie der Metalle in 1931 [10]. As one can see, there are only interactions between neighbouring spins, i.e. this model is of *nearest-neighbour* type. While its interpretation as a model for magnetic interactions between electrons forms an interesting story, we will not discuss it here, but instead mention the revolution it inspired in the solvability of spin chains and other lower-dimensional models. Bethe hypothesized that the eigenfunctions of the spin chain hamiltonian are linear combinations of plane waves of the form

$$|\psi\rangle_M = \sum_{\boldsymbol{n} \in \mathcal{B}_M} \sum_{Q \in \pi_M} A_Q(\boldsymbol{p}) e^{i \sum_{\lambda=1}^{M} p_{Q\lambda} n_\lambda} |n_1, n_2, \cdots, n_M\rangle, \tag{2.13}$$

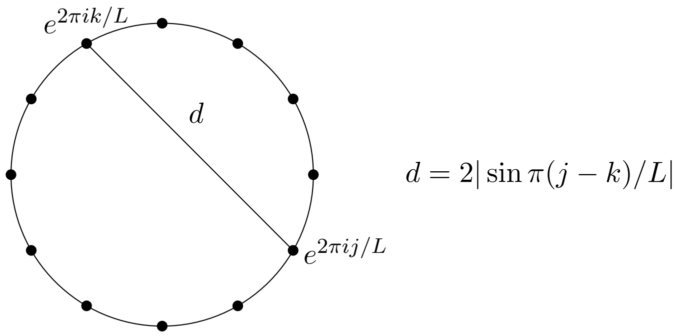


Figure 2.3: The chord distance d between two particles at $e^{2\pi ij/L}$ and $e^{2\pi ik/L}$ is $2|\sin \pi(j-k)/L|$, used in the Haldane-Shastry spin chain.

where $0 \leq M \leq L$ is fixed, π_M is the permutation group of M symbols, $\boldsymbol{p} \in \mathbb{C}^M$ are the quasimomenta of the plane waves and the A_Q are coefficients that are to be determined. Using this Ansatz, Bethe was able to find eigenfunctions of the XXX spin chain. Moreover, later studies showed that this Ansatz was useful to solve lots of other models too and that many models that are solvable by this Ansatz have a common underlying algebraic structure. This Ansatz became known under the name *Bethe Ansatz* and has led to a great surge in the research of exactly solvable quantum models. Indeed, there are many methods and topics that originate from this discovery by Bethe, such as the Algebraic Bethe Ansatz, the Thermodynamic Bethe Ansatz and the Nested Bethe Ansatz. More information about the research centered around the Bethe Ansatz can be found in references [2, 17, 18].

2.2.3 The Haldane-Shastry Spin Chain

The Haldane-Shastry spin chain of length L is also defined on $\mathcal{H}^{(L)}$ by its hamiltonian

$$H_{HS} := -\frac{J}{4} \sum_{\substack{j,k=1 \\ j \neq k}}^{L} \frac{1}{\sin^2(\pi(j-k)/L)} (\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k - 1), \tag{2.14}$$

where J is again a coupling constant. It is periodic due to the use of the sine function and has a natural interpretation on the unit circle: if we locate particles at the positions $e^{2\pi ij/L}$ (with $1 \leq j \leq L$) on the unit circle, then the interaction strength $\sin^{-2}(\pi(j-k)/L)$ is exactly the inverse-squared potential for the chord distance between particles at $e^{2\pi ij/L}$ and $e^{2\pi ik/L}$ (up to a rescaling, see Figure 2.3). This spin chain was introduced and diagonalized by Haldane and Shastry in two independent papers [11, 12]. It is a long-range spin chain, since the interaction energy between any two sites is nonzero, and is not solvable by the Bethe Ansatz. Through a careful study of the higly degenerate energy levels, a Yangian symmetry was identified [19], which allowed Bernard, Gaudin, Haldane and Pasquier to construct the transfer matrix in terms of Dunkl operators [20]. Additionally, they established a strong connection with the spinless Calogero-Sutherland-Moser models (CSM-models) carrying the same potential on the real line.

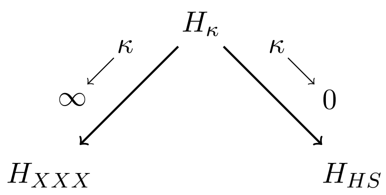


Figure 2.4: The limits of Inozemtsev's elliptic spin chain hamiltonian.

In Chapter 3, we will encounter a similar connection for Inozemtsev's spin chain of infinite length and give more details about the CSM-models.

2.2.4 Inozemtsev's Elliptic Spin Chain

On the Hilbert space $\mathcal{H}^{(L)}$, Inozemtsev's elliptic spin chain is defined by the hamiltonian

$$H_\kappa = \frac{J}{4} \sum_{\substack{j,k=1 \\ j\neq k}}^{L} \wp_L(j-k)\left(\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k - 1\right), \tag{2.15}$$

where $\wp_L$ is the Weierstraß elliptic function with periods $(L, i\pi/\kappa)$. Note that this model depends on the parameter $\kappa > 0$ [8] through these periods. When the parameter κ is sent to either infinity or zero, the hamiltonian (2.15) degenerates into the hamiltonians of the Heisenberg XXX and the Haldane-Shastry spin chain, respectively (see Figure 2.4). To be more specific, in the limit that $\kappa \to 0$, $\wp_L$ behaves as

$$\lim_{\kappa\to 0} \wp_L(z) := \frac{\pi^2}{L^2}\left(\frac{1}{\sin^2(\pi z/L)} - \frac{1}{3}\right), \tag{2.16}$$

such that in this limit, we indeed obtain the Haldane-Shastry hamiltonian up to an unimportant shift. The limit of $\kappa \to \infty$ is slightly more complicated to see. We use the expansion of $\wp_L$ for large values of κ:

$$\wp_L(j) = \kappa^2\left(\frac{1}{3} + 4(e^{-2\kappa j} + e^{-2\kappa|j-L|} + e^{-2\kappa|j+L|})\right) + \mathcal{O}\left(e^{-4\kappa j}\right) \tag{2.17}$$

for all $j \in \mathbb{Z}$ with $|j| < L$. If we omit the constant term and renormalize the coupling constant as

$$J \to \frac{J}{4\kappa^2}\exp(2\kappa),$$

then we have

$$\lim_{\kappa\to\infty} \frac{J}{4\kappa^2}\exp(2\kappa)\left(\wp_L(j) - \frac{\kappa^2}{3}\right) = \delta_{1j} + \delta_{L-1,j}, \tag{2.18}$$

for all $j \in \mathbb{Z}$ with $|j| < L$, where the δ is the Kronecker symbol. This is precisely the interaction strength of the periodic Heisenberg XXX spin chain. It is very interesting that there exists a spin chain that interpolates between the Heisenberg XXX chain and the Haldane-Shastry spin chain: the XXX chain on the one hand is of nearest-neighbour type and solvable by Bethe Ansatz and also has a well-known algebraic structure that is formalized in the Algebraic Bethe Ansatz [17]. The Haldane-Shastry chain on the other hand is long-range and not solvable by Bethe Ansatz. It is only solvable by using its Yangian symmetry, which makes it a fundamentally different model. Studying Inozemtsev's elliptic spin chain allows one to find out more about the relationship between these models, which seem so very different.

As stated, Inozemtsev's original paper [8] was primarily concerned with finding a quantum Lax pair for Inozemtsev's elliptic spin chain and showing that it can be related to the existing quantum Lax pairs of the other two models. In fact, the derivation of the quantum Lax pair shows that it is the most general quantum Lax pair for an integrable spin chain with hamiltonian of the form

$$H = \sum_{\substack{j,k=1 \\ j \neq k}}^{L} h(j-k)\left(\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k - 1\right),$$

where h is an arbitrary function. The Lax pair $\tilde{L}, \tilde{M}$ that obeys

$$\frac{d\tilde{L}}{dt} = [H, \tilde{L}] = [\tilde{L}, \tilde{M}], \tag{2.19}$$

where the brackets indicate the commutator, is given by

$$\begin{aligned} \tilde{L}_{jk} &= (1-\delta_{jk}) f(j-k)(1+\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k) \\ \tilde{M}_{jk} &= (1+\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k)(1-\delta_{jk}) g(j-k) - \delta_{jk} \sum_{\substack{n=1 \\ n \neq j}}^{L} h(j-n)(1+\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_n), \end{aligned} \tag{2.20}$$

where the functions f, g and h can be determined and are

$$\begin{aligned} f(x) &= \frac{\sigma_L(x-\alpha)}{\sigma_L(x)\sigma_L(\alpha)} \exp\left(x\zeta_L(\alpha)\right) \\ g(x) &= -f'(x) \\ h(x) &= \wp_L(x), \end{aligned} \tag{2.21}$$

where σ_L and ζ_L are Weierstraß functions (2.2) with periods $(L, i\pi/\kappa)$. The parameter $\alpha \in \mathbb{C}$ is free, but has no effect on the dynamics [8]. Note that to make sure that $\tilde{L}, \tilde{M}$ are an actual quantum Lax pair, it is necessary to have the specific form for the exchange function h as given in equation (2.21). However, since the models under consideration are all quantum mechanical in nature, the existence of the quantum Lax pair does not garantuee integrability. It does not generate a set of conserved quantities; $\mathrm{Tr}\left(\tilde{L}^n\right)$ does not commute with the hamiltonian. In [9], Inozemtsev derived a set of integrals of motion for his elliptic spin chain, but was unable to prove that this set is in involution. Therefore, the question whether the elliptic spin chain is integrable remains open and begs to be answered. Inozemtsev did find an expression for the wavefunctions of this model, but solving the associated highly transcendental equations for the quasi-momenta seems impossible [21].

Another important limit of the elliptic spin chain is the one in which we keep κ finite, but send the length L to infinity. In this limit, one of the periods of the $\wp$-function tends to infinity, turning $\wp$ into a much simpler form. Namely, we have

$$\lim_{L\to\infty} \wp_L(z) = \frac{\kappa^2}{\sinh^2 \kappa z}, \tag{2.22}$$

which indicates that the hamiltonian of this model, which we name *Inozemtsev's infinite spin chain*, is given by

$$H = -\frac{J}{4} \sum_{\substack{j,k\in\mathbb{Z} \\ j\neq k}} \frac{\kappa^2}{\sinh^2 \kappa(j-k)} (\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k - 1). \tag{2.23}$$

However, the Hilbert space $\mathcal{H}^{(L)}$ does not have an obvious limit that corresponds to sending $L \to \infty$. In the next chapter we will see whether a proper Hilbert space can be defined for this model, but let us here comment on some interesting features. Firstly, the infinite chain is no longer periodic, but resembles the finite chain in another way: it is still translationally invariant. Secondly, the fact that this chain is of infinite length makes it easier to treat, as we will see in the next chapter. And finally, there are good reasons to believe that the infinite chain can help us understand some aspects of the finite elliptic chain. For example, when considering the thermodynamic limit of the elliptic chain, we need to send L to infinity anyway and by being careful about the order of limits, it might be possible to use the information about the infinite chain to treat this limit.

Chapter 3

Inozemtsev's Infinite Spin Chain and Its Eigenvalue Problem

To study properties of Inozemtsev's elliptic spin chain, we turn to the study of its infinite-length limit. As we saw in the previous chapter, the hamiltonian of this model is easily given using the limiting properties of the Weierstraß elliptic functions, but the associated Hilbert space proves to be somewhat more tricky to define. Indeed, it is not at all obvious how to define the limit of the space

$$\mathcal{H}^{(L)} = \bigotimes_{n \in \mathbb{Z}/L\mathbb{Z}} V_n$$

as $L \to \infty$. In the following, we will therefore first try to define a proper Hilbert space for a spin chain of infinite length. After that, we will find eigenstates of Inozemtsev's infinite spin chain, along the lines of [22][1].

3.1 Defining a Hilbert Space for Inozemtsev's Infinite Spin Chain

To define the system rigorously, we must define a Hilbert space $\mathcal{H}$ and a representation via which the infinite spin chain hamiltonian will act on this Hilbert space. Since the system contains an infinite amount of sites, the Hilbert space itself will be infinite dimensional. We will consider the hamiltonian derived at the end of Chapter 2:

$$H := -\frac{J}{4} \sum_{\substack{j,k \in \mathbb{Z} \\ j \neq k}} \frac{\kappa^2}{\sinh^2 \kappa(j-k)} (\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k - 1), \tag{3.1}$$

[1] Our derivation of the eigenfunctions is not new. To bring new material to the subject and since our initial interest was mainly on the methods used, we will focus on those.

which contains elements of a particular Pauli spin algebra, namely the algebra generated by $\sigma_n^x, \sigma_n^y, \sigma_n^z$ with $n \in \mathbb{Z}$ and the commutation relations

$$[\sigma_n^i, \sigma_m^j] = 2\delta_{nm} i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_n^k, \tag{3.2}$$

where $i, j \in \{x, y, z\}$, $n, m \in \mathbb{Z}$ and ϵ_{ijk} is the Levi-Civita symbol with $\epsilon_{xyz} = 1$. The Hilbert space will have to admit a representation of this algebra for the system to be well-defined. The idea is to consider the sequence space $\ell^2(A)$, where A is some countable set that should resemble the idea of the spin-flipped vectors (2.10) we used to build a basis for the finite-dimensional Hilbert space $\mathcal{H}^{(L)}$. Therefore consider the sequences $(\cdots, s_{-1}, s_0, s_1, \cdots)$ where $s_i = \pm 1$. The set S of these sequences is uncountable. In S, we define a pseudovacuum state $|0\rangle$ by

$$|0\rangle := (\cdots, 1, 1, 1, \cdots), \tag{3.3}$$

to which we associate the vector for which all the spins are up. The subspace $S^+ \subset S$ defined by

$$S^+ := \{s \in S \,|\, s_n \neq 1 \text{ for finitely many } n \in \mathbb{Z}\}, \tag{3.4}$$

consists of all the sequences in S with a finite amount of entries of -1. S^+ is a countable set, thus the sequence space $\ell^2(S^+)$ with norm

$$\sum_{s \in S^+} |a_s|^2 < \infty$$

for an element (a_s) with $a_s \in \mathbb{C}$ for all $s \in S^+$ is well defined. Also, $|0\rangle \in S^+$. Of course, this space is isomorphic to the canonical sequence space $\ell^2(\mathbb{Z})$, which is well understood. From this we deduce that, along with the inner product

$$\sum_{n \in A} a_n \overline{b_n}$$

for elements $(a_n), (b_n) \in \ell^2(S^+)$, it forms a complete and separable Hilbert space. It is most convenient to consider the sequences (a_n) to be functions $a : S^+ \to \mathbb{C}$. A basis of these functions is given by the set $e_s : S^+ \to \mathbb{C}$ defined by $e_s(s') := \delta_{ss'}$. From now on, we will write these basis vectors differently: we define $|n_1, \cdots, n_M\rangle$ with $\boldsymbol{n} \in \mathcal{B}_M$ (see (2.11)) to correspond to the ket e_s with $s \in S^+$ which has $s_{n_j} = -1$ for all $1 \leq j \leq M$ and $s_i = 1$ if $i \neq n_j$ for all $1 \leq j \leq M$. This correspondence is one to one. We define an action of the Pauli spin algebra on the basis elements $|n_1, n_2 \cdots, n_M\rangle \in \ell^2(S^+)$ by considering the spin-flip operators $\sigma_j^{\pm}$ again, along with

σ_j^z as follows:

$$\sigma_j^+|n_1,\cdots,n_M\rangle := \begin{cases} 0, & \text{if } j \neq n_i \text{ for all } i \\ |n_1,\cdots,n_{i-1},n_{i+1},\cdots,n_M\rangle & \text{if } j = n_i, \text{ for some } i \end{cases}$$

$$\sigma_j^-|n_1,\cdots,n_M\rangle := \begin{cases} 0, & \text{if } j = n_i \text{ for some } i \\ |n_1,\cdots,j,\cdots,n_M\rangle & \text{if } j \neq n_i, \text{ for all } i \end{cases}$$

$$\sigma_j^z|n_1,\cdots,n_M\rangle := \begin{cases} |n_1,\cdots,n_M\rangle, & \text{if } j \neq n_i \text{ for all } i \\ -|n_1,\cdots,n_M\rangle & \text{if } j = n_i, \text{ for some } i, \end{cases} \tag{3.5}$$

where in the action of σ_j^- the ket $|n_1,\cdots,j,\cdots,n_M\rangle$ is ordered such that all the n_i to the left of j have $n_i < j$ and all the n_i to the right of j satisfy $n_i > j$. This representation satisfies the commutation relations (3.2). Moreover, we can interpret the basis elements of $\ell^2(S^+)$ in a natural way: to an element $|n_1,n_2,\cdots,n_M\rangle \in \ell^2(S^+)$ we associate the vector in which the spin is up at all positions except n_i, at which the spin is down. Therefore we have indeed found a suitable candidate: the complete and separable Hilbert space $\mathcal{H} := \ell^2(S^+)$ admits a representation of the Pauli spin algebra and its basis vectors have a natural interpretation as an infinite one-dimensional lattice with a definite spin configuration. Thus Inozemtsev's infinite spin chain is a well-defined spin chain on the Hilbert space $\mathcal{H} = \ell^2(S^+)$ by the hamiltonian (3.1). Since we defined the action of the Pauli spin algebra using the spin-flip operators, it is prudent to rewrite the hamiltonian using these operators:

$$H = -\frac{J}{4}\sum_{\substack{j,k\in\mathbb{Z}\\ j\neq k}} \frac{\kappa^2}{\sinh^2\kappa(j-k)}\left(2(\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+) + \sigma_j^z\sigma_k^z - 1\right). \tag{3.6}$$

3.2 The Eigenvalue Problem of the Infinite Chain

We will now proceed to try to find eigenfunctions and the corresponding energies of this spin chain by solving in full generality the eigenvalue equation

$$H|\psi\rangle = E|\psi\rangle,$$

where $|\psi\rangle$ is an eigenvector and E the corresponding eigenvalue. Following the usual Bethe approach, we will try to solve this equation by restricting the eigenvector $|\psi\rangle$ to lie in subspaces of $\mathcal{H}$. For fixed $M \in \mathbb{N}$ we will consider the finite-dimensional subspaces $\mathcal{H}_M$ of $\ell^2(S^+)$ generated by the vectors

$$\{|s\rangle \in \ell^2(S^+) | s \in S^+ \text{ has } s_j = -1 \text{ for exactly } M \text{ integers } j\}.$$

We call $\mathcal{H}_M$ the *M-particle sector*. The eigenvalue problem is simplest for the 0-particle sector (the pseudovacuum).

3.3 $M = 0$

The 0-particle sector consists only of the pseudovacuum $|0\rangle$, thus we only need to check that this pseudovacuum is indeed an eigenstate of the Hamiltonian. We see that for $j \neq k$,

$$\left(2(\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+) + \sigma_j^z\sigma_k^z - 1\right)|0\rangle = (2(0+0)+1-1)|0\rangle = 0, \tag{3.7}$$

thus $H|0\rangle = 0|0\rangle$, implying that the pseudovacuum is indeed an eigenstate with zero energy.

3.4 $M = 1$

In principle, we could immediately try to tackle the eigenvalue equation for arbitrary M, but it is insightful to treat the case of one-particle excitations first, since it is already nontrivial. We postulate a translationally-invariant Ansatz for the eigenvectors:

$$|\psi\rangle = \sum_{n\in\mathbb{Z}} e^{ipn}|n\rangle. \tag{3.8}$$

We must find the action of the Hamiltonian on this Ansatz. The action of the operators in the Hamiltonian (3.6) on a basis vector $|n\rangle$ is as follows:

$$\begin{aligned}
\sigma_j^+\sigma_k^-|n\rangle &= \begin{cases} 0, & \text{if } k = n \text{ or } j \neq n \\ \sigma_k^-|0\rangle, & \text{if } j = n,\ k \neq n, \end{cases} \\
\sigma_j^-\sigma_k^+|n\rangle &= \begin{cases} \sigma_j^-|0\rangle, & \text{if } k = n \\ 0, & \text{if } k \neq n, \end{cases} \\
\sigma_j^z\sigma_k^z|n\rangle &= \begin{cases} -|n\rangle, & \text{if } k = n,\ j \neq n \text{ or if } j = n,\ k \neq n \\ \ \ |n\rangle, & \text{if } j \neq n,\ k \neq n. \end{cases}
\end{aligned} \tag{3.9}$$

Using this and the notation $A_{jk} := 1/\sinh^2\kappa(j-k)$ and $\mathbb{Z}_{[n]} := \mathbb{Z}\setminus\{n\}$, we can find the action of the Hamiltonian on $|n\rangle$:

$$\begin{aligned}
-\frac{4}{J}H|n\rangle &= \sum_{\substack{j,k\in\mathbb{Z}\\ j\neq k}} A_{jk}\left(2(\sigma_j^+\sigma_k^- + \sigma_j^-\sigma_k^+) + \sigma_j^z\sigma_k^z - 1\right)|n\rangle = \\
&= \sum_{j\in\mathbb{Z}_{[n]}} A_{jn}\left(2(0+\sigma_j^-) - \sigma_n^- - \sigma_n^-\right)|0\rangle + \sum_{k\in\mathbb{Z}_{[n]}} A_{nk}\left(2(\sigma_k^- + 0) - \sigma_n^- - \sigma_n^-\right)|0\rangle \\
&\quad + \sum_{\substack{j,k\in\mathbb{Z}_{[n]}\\ j\neq k,}} A_{jk}\left(2(0+0) + \sigma_n^- - \sigma_n^-\right)|0\rangle \\
&= 2\sum_{j\in\mathbb{Z}_{[n]}} A_{jn}\left(\sigma_j^- - \sigma_n^-\right)|0\rangle + 2\sum_{k\in\mathbb{Z}_{[n]}} A_{nk}\left(\sigma_k^- - \sigma_n^-\right)|0\rangle = 4\sum_{k\in\mathbb{Z}_{[n]}} A_{nk}\left(\sigma_k^- - \sigma_n^-\right)|0\rangle,
\end{aligned} \tag{3.10}$$

where we use that A_{jk} is symmetric in j and k. We can use this formula to find an expression for the energy using the eigenvalue equation

$$H|\psi\rangle = -J \sum_{\substack{n,k\in\mathbb{Z} \\ k\neq n}} e^{ipn} A_{nk}(\sigma_k^- - \sigma_n^-)|0\rangle = \epsilon_p \sum_{n\in\mathbb{Z}} e^{ipn}|n\rangle. \tag{3.11}$$

To see what ϵ_p is, we apply the linear function $\langle m|$ to (3.8) to obtain

$$-J \sum_{\substack{n,k\in\mathbb{Z} \\ k\neq n}} e^{ipn} A_{n,k}(\delta_{mk} - \delta_{nm}) = \epsilon_p e^{ipm}, \tag{3.12}$$

where we use that the $\{|n\rangle\}_{n\in\mathbb{Z}}$ form an orthonormal basis of the one-particle sector of $\mathcal{H}$. This leads to

$$\begin{aligned} \epsilon_p &= -J \sum_{\substack{n,k\in\mathbb{Z} \\ k\neq n}} e^{ip(n-m)} A_{nk}(\delta_{mk} - \delta_{nm}) = -J \sum_{n\in\mathbb{Z}_{[m]}} e^{ip(n-m)} A_{nm} + J \sum_{k\in\mathbb{Z}_{[m]}} A_{mk} = \\ &= -J \sum_{k\in\mathbb{Z}_{[0]}} e^{ipk} A_{0k} + J \sum_{k\in\mathbb{Z}_{[0]}} A_{0k} = J \sum_{n\in\mathbb{Z}_{[0]}} \frac{\kappa^2(1-e^{ipn})}{\sinh^2 \kappa n}, \end{aligned} \tag{3.13}$$

where we have renamed the variable n $k+m$ and shifted the summation variable k to $k-m$ in the first and second summation respectively. Since this approach works for any $m \in \mathbb{Z}$ we see that $|\psi\rangle$ is indeed an eigenvector of our Hamiltonian with eigenvalue given by (3.13). This expression is, however, not particularly useful. We would like to express the energy in Weierstraß elliptic functions if possible. Our next task is therefore to find an expression for this sum, which we will do using Laurent series.

3.4.1 Finding a Different Expression for the Energy

To find a different expression for the energy, we will have to employ a trick, which will be very useful when we are considering the eigenvalue equation in the M-particle sector at arbitrary M. Consider the complex function $F : \mathbb{C} \to \mathbb{C}$ given by

$$F(z) = \sum_{k\in\mathbb{Z}} \frac{\kappa^2 e^{ipk}}{\sinh^2 \kappa(z+k)}. \tag{3.14}$$

The sum is absolutely convergent for any $z \notin \Gamma$, where $\Gamma \subset \mathbb{C}$ is the lattice generated by the periods 1 and $\omega := \frac{i\pi}{\kappa}$. This means that F is meromorphic, i.e. has a Laurent series at every point in $\mathbb{C}$. We can find the Laurent expansion of F around $z = 0$ using contour integrals. If we have this expansion, we know the behaviour of F around any of its poles, since they are all located on the lattice Γ, and F is quasi-periodic on this lattice:

$$F(z+\omega) = F(z), \qquad F(z+1) = e^{-ip}F(z),$$

as is easy to check using (3.14). If we choose as our contour C the circle around $z = 0$ with radius $\frac{1}{2}\min(1, \pi/\kappa)$, the contour does not hit any poles and has only the pole at $z = 0$ in its interior. If we write $F(z) = \sum_{n\in\mathbb{Z}}^{\infty} a_n z^n$ we see that

$$a_n = \frac{1}{2\pi i}\oint_C \frac{F(z)}{z^{n+1}}dz \tag{3.15}$$

and since

$$\frac{1}{\sinh^2\kappa(z+n)} = \frac{1}{\kappa^2(z+n)^2} - \frac{1}{3} + \mathcal{O}\left((z+n)^2\right) \tag{3.16}$$

equation (3.15) will yield zero for $n < -2$ as well as for $n = -1$. Let us explicitly calculate a_{-2} and a_0:

$$\begin{aligned}
(2\pi i)a_{-2} &= \oint_C \frac{1}{z^{-1}}\sum_{n\in\mathbb{Z}} \frac{\kappa^2 e^{ipn}}{\sinh^2\kappa(n+z)}dz = \sum_{n\in\mathbb{Z}} \kappa^2 e^{ipn}\oint_C \frac{z}{\sinh^2\kappa(n+z)}dz \\
&= \sum_{n\in\mathbb{Z}} \kappa^2 e^{ipn}\oint_C z\left(\frac{1}{\kappa^2(z+n)^2} - \frac{1}{3} + \mathcal{O}\left(z^2\right)\right)dz \\
&= \sum_{n\in\mathbb{Z}} \kappa^2 e^{ipn}\oint_C \frac{z}{\kappa^2(z+n)^2}dz,
\end{aligned} \tag{3.17}$$

where the only nonzero integral is the one in which $n = 0$, since only then there exists a pole inside our contour. This gives

$$a_{-2} = \frac{1}{2\pi i}\oint \frac{dz}{z} = 1. \tag{3.18}$$

For a_0 we have

$$\begin{aligned}
(2\pi i)a_0 &= \oint_C \frac{1}{z}\sum_{n\in\mathbb{Z}} \frac{\kappa^2 e^{ipn}}{\sinh^2\kappa(n+z)}dz = \sum_{n\in\mathbb{Z}} \kappa^2 e^{ipn}\oint_C \frac{1}{\sinh^2\kappa(n+z)z}dz \\
&= \sum_{n\in\mathbb{Z}\backslash\{0\}} \kappa^2 e^{ipn}\oint_C \frac{1}{\sinh^2\kappa(n+z)z}dz + \kappa^2\oint_C \frac{1}{z\sinh^2(\kappa z)}dz \\
&= 2\pi i\sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{\kappa^2 e^{ipn}}{\sinh^2\kappa n} + \kappa^2\oint_C z^{-1}\left(\frac{1}{\kappa^2 z^2} - \frac{1}{3} + \mathcal{O}\left(z^2\right)\right)dz \\
&= 2\pi i\sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{\kappa^2 e^{ipn}}{\sinh^2\kappa n} - \frac{2\pi i\kappa^2}{3}.
\end{aligned} \tag{3.19}$$

So we find that around $z = 0$ we can write

$$F(z) = \frac{1}{z^2} + 2\pi i\sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{\kappa^2 e^{ipn}}{\sinh^2\kappa n} - \frac{2\pi i\kappa^2}{3} + \mathcal{O}(z). \tag{3.20}$$

Note that in the expression for a_0 we can recognize a lot of features from the original expression (3.13) for the energy. If we can find another expression for F, preferably in terms of elliptic functions, we can use these similarities to find a useful expression for the one-particle energy ϵ_p.

We introduce another function, which we will call $G : \mathbb{C} \to \mathbb{C}$, given by

$$G(z) := -\frac{\sigma(z+r)}{\sigma(z-r)}\exp(\delta z)\cdot\left\{\wp(z) - \wp(r) + \Delta\left(\frac{\wp'(z) - \wp'(r)}{\wp(z) - \wp(r)} - \frac{\wp''(r)}{\wp'(r)}\right)\right\}, \tag{3.21}$$

where $\wp$, σ and ζ are Weierstraß elliptic functions with periods $(1, \omega = i\pi/\kappa)$ and r,δ and Δ are constants. The term in the curly brackets is doubly-periodic and vanishes for any values of r and Δ in the limit $z \to r$. We will show now that G is actually the same function as F for certain values of r and Δ. If we take $\delta = \frac{p}{\pi}\zeta(\omega/2)$ and $r_p = r := -\frac{ip}{4\kappa}$, G becomes quasi-periodic just like F:

$$G(z+\omega) = G(z), \qquad G(z+1) = e^{-ip} G(z),$$

which can be shown using the fact that

$$\sigma(z+r+1) = -\exp\left(2\zeta\left(\frac{1}{2}\right)\left(z+r+\frac{1}{2}\right)\right)\sigma(z+r), \tag{3.22}$$

for the Weierstraß σ-functions (see also Appendix E). To proceed we need the Laurent series for G, which can be calculated using the known Laurent series for $\wp$, $\wp'$ and ζ and the Taylor series for σ (see Appendix E):

$$\begin{aligned} G(z) &= \left(1 + 2\zeta(r)z + 2\left(\zeta(r)\right)^2 z^2 + \mathcal{O}\left(z^3\right)\right)\left(1 + \delta z + \frac{1}{2}(\delta z)^2\right) \\ &\quad \times \left\{\frac{1}{z^2} - \wp(r) + \Delta\left(-\frac{2}{z} - \wp(r)z + \wp'(r)z^2 - \frac{\wp''(r)}{\wp'(r)}\right)\right\} \\ &= \frac{1}{z^2} + \frac{2\zeta(r) + \delta - 2\Delta}{z} \\ &\quad + \left(-\wp(r) - \Delta\frac{\wp''(r)}{\wp'(r)} + (\zeta(r) + \delta/2)\left(-4\Delta + 2(\zeta(r) + \delta/2)\right)\right) + \mathcal{O}\left(z\right). \end{aligned} \tag{3.23}$$

We see that the coefficient at order z^{-2} coincides exactly with the a_{-2} we calculated earlier for F (see equation (3.18)). To get equality for the coefficients at order z^{-1}, we must set

$$\Delta = \zeta(r) + \delta/2.$$

By construction, F and G now have equal coefficients in front of the singular parts of their respective Laurent series around $z = 0$. This means that the function $H = F - G$ does not have a pole at $z = 0$ and is also quasi-periodic on the lattice Γ, implying it does not have any poles on the whole complex plane. H is therefore analytic on the entire complex plane, while the quasi-periodicity implies that H is bounded on $\mathbb{C}$. By Liouville's theorem, we see that H must be constant. Suppose that $H = c \in \mathbb{C}$. Then the quasi-periodicity shows us that $|c| = |H(z+k)| = |e^{-ipk}||H(z)| = |e^{-ipk}||c|$ for any $p \in \mathbb{C}$ and $k \in \mathbb{N}$, thus we must have $c = 0$. This shows that in fact $F = G$, which in turn implies that all the coefficients of their respective Laurent expansions must also be equal. In particular, this shows the equality for the constant terms a_0 (equations (3.19) and (3.23)):

$$\sum_{n\in\mathbb{Z}_{[0]}} \frac{\kappa^2 e^{ipn}}{\sinh^2 \kappa n} - \frac{\kappa^2}{3} = -\wp(r) - \Delta\frac{\wp''(r)}{\wp'(r)} - 2\Delta^2. \tag{3.24}$$

To find the one-particle energy, we also need an expression for the summation $\sum_{n\in\mathbb{Z}_{[0]}} \frac{\kappa^2}{\sinh^2 \kappa n}$, which can be found from the formula (3.24) above by taking the limit $p \to 0$. The easiest way to do this is to expand the right-hand side around $p = 0$, which is equivalent to expanding around $r = -\frac{ip}{4\kappa} = 0$:

$$\begin{aligned}\wp(r) + \frac{\wp''(r)}{\wp'(r)}\Delta(r) + 2\Delta^2 &= \frac{1}{r^2} - \left(\frac{3}{r} + \mathcal{O}\left(r^3\right)\right)\left(\frac{1}{r} - 2\frac{r}{\omega}\zeta(\omega/2)\right) + 2\left(\frac{1}{r} - 2\frac{r}{\omega}\zeta(\omega/2)\right)^2 \\ &= \frac{1}{r^2} - \frac{3}{z^2} + \frac{6}{\omega}\zeta(\omega/2) + \frac{2}{r^2} - \frac{8}{\omega}\zeta(\omega/2) + \mathcal{O}\left(r^2\right) \\ &= -\frac{2}{\omega}\zeta(\omega/2) + \mathcal{O}\left(r^2\right), \end{aligned} \tag{3.25}$$

so we find that

$$\sum_{n\in\mathbb{Z}_{[0]}} \frac{\kappa^2}{\sinh^2 \kappa n} - \frac{\kappa^2}{3} = \frac{2}{\omega}\zeta(\omega/2). \tag{3.26}$$

By plugging in equations (3.24) and (3.26) into our original expression for the energy given by (3.13) we find

$$\begin{aligned}\epsilon_p &= J \sum_{n\in\mathbb{Z}_{[0]}} \frac{\kappa^2(1-e^{ipn})}{\sinh^2 \kappa n} \\ &= J\left(\wp\left(\frac{ip}{4\kappa}\right) + \left(\zeta\left(\frac{ip}{4\kappa}\right) - \frac{p}{2\pi}\zeta\left(\omega/2\right)\right)\frac{\wp''\left(\frac{ip}{4\kappa}\right)}{\wp'\left(\frac{ip}{4\kappa}\right)} + 2\left(\zeta\left(\frac{ip}{4\kappa}\right) - \frac{p}{2\pi}\zeta\left(\omega/2\right)\right)^2 + \frac{2}{\omega}\zeta(\omega/2)\right), \\ &= J\left(\frac{1}{2}\wp\left(\frac{ip}{2\kappa}\right) + \frac{1}{2}\left(\frac{p}{\pi}\zeta\left(\frac{i\pi}{2\kappa}\right) - \zeta\left(\frac{ip}{2\kappa}\right)\right)^2 - \frac{2i\kappa}{\pi}\zeta\left(\frac{i\pi}{2\kappa}\right)\right)\end{aligned} \tag{3.27}$$

which is the type of expression we set out to find. The trick we used above will also prove to be useful for the cases $M \geq 2$ and can in fact also be used to solve the eigenvalue problem of Inozemstev's elliptic spin chain.

3.5 The M-Particle Difference Equation

Our next task is to solve the eigenvalue-problem

$$H|\psi\rangle = E_M|\psi\rangle$$

for the other sectors. Here $E_M \in \mathbb{R}$ is the energy and $|\psi\rangle$ a state in the M-particle sector. Fortunately, for $M \geq 2$, we can treat all the sectors at once. Firstly, we can rewrite equation (3.5) by plugging in an expansion of $|\psi\rangle$ over the basis states of the M-particle sector given by

$$|\psi\rangle = \sum_{\boldsymbol{n}\in\mathcal{B}_M} \psi(n_1, n_2, \cdots, n_M)|n_1, n_2, \cdots, n_M\rangle, \tag{3.28}$$

where $\boldsymbol{n} = (n_1, n_2, \cdots, n_M)^T$. To find how the Hamiltonian acts on $|\psi\rangle$, it is most convenient to first consider its action on a basis state. Since we rewrote our Hamiltonian as

$$H = -\frac{J}{2} \sum_{\substack{j,k\in\mathbb{Z} \\ j\neq k}} \frac{\kappa^2}{\sinh^2 \kappa(j-k)} \left(2(\sigma_j^+ \sigma_k^- + \sigma_j^- \sigma_k^+) + 4\sigma_j^z \sigma_k^z - 1\right)/2, \tag{3.29}$$

we can focus on the action of $\sigma_j^+ \sigma_k^-$, $\sigma_j^- \sigma_k^+$ and $\sigma_j^z \sigma_k^z$. They read

$$\begin{aligned}
\sigma_j^+ \sigma_k^- |n_1, n_2, \cdots, n_M\rangle &= \begin{cases} 0, & \text{if } k = n_\beta \text{ or } j \neq n_i \ \forall i \\ |n_1, \cdots, n_{\beta-1}, k, n_{\beta+1} \cdots, n_M\rangle, & \text{if } j = n_\beta,\, k \neq n_i \ \forall i, \end{cases} \\
\sigma_j^- \sigma_k^+ |n_1, n_2, \cdots, n_M\rangle &= \begin{cases} 0, & \text{if } k \neq n_i \, \forall i \text{ or } j = n_\beta \\ |n_1, \cdots, n_{\beta-1}, j, n_{\beta+1} \cdots, n_M\rangle, & \text{if } k = n_\beta, \end{cases} \\
\sigma_j^z \sigma_k^z |n_1, n_2, \cdots, n_M\rangle &= \begin{cases} -|n_1, n_2, \cdots, n_M\rangle, & \text{if } k = n_\beta,\, j \neq n_i \ \forall i \\ -|n_1, n_2, \cdots, n_M\rangle, & \text{if } j = n_\beta,\, k \neq n_i \ \forall i \\ |n_1, n_2, \cdots, n_M\rangle, & \text{if } k = n_\beta,\, j = n_\alpha \\ |n_1, n_2, \cdots, n_M\rangle, & \text{if } k, j \neq n_i \ \forall i, \end{cases}
\end{aligned} \tag{3.30}$$

where α and β are integers with $1 \leq, \alpha, \beta \leq M$ and $\alpha \neq \beta$. We can use the above to find the action of H on the state $|n_1, n_2, \cdots, n_M\rangle$:

$$\begin{aligned}
-\frac{4}{J} H |n_1, n_2, \cdots, n_M\rangle &= \sum_{\substack{j,k\in\mathbb{Z} \\ j\neq k}} A_{j,k} (2(\sigma_j^+ \sigma_k^- + \sigma_j^- \sigma_k^+) + \sigma_j^z \sigma_k^z - 1) |n_1, n_2, \cdots, n_M\rangle \\
= \ & \sum_{\beta=1}^{M} \sum_{j\in\mathbb{Z}_{[\boldsymbol{n}]}} A_{jn_\beta} \left(2(|n_1, \cdots, n_{\beta-1}, j, n_{\beta+1}, \cdots, n_M\rangle + 0) - 2|n_1, n_2, \cdots, n_M\rangle\right) \\
& + \sum_{\beta=1}^{M} \sum_{k\in\mathbb{Z}_{[\boldsymbol{n}]}} A_{kn_\beta} \left(2(0 + |n_1, \cdots, n_{\beta-1}, k, n_{\beta+1}, \cdots, n_M\rangle) - 2|n_1, n_2, \cdots, n_M\rangle\right) \\
= \ & 4 \sum_{\beta=1}^{M} \sum_{s\in\mathbb{Z}_{[\boldsymbol{n}]}} A_{sn_\beta} \left(|n_1, \cdots, n_{\beta-1}, s, n_{\beta+1}, \cdots, n_M\rangle - |n_1, n_2, \cdots, n_M\rangle\right),
\end{aligned} \tag{3.31}$$

where with $\mathbb{Z}_{[\boldsymbol{n}]}$ we mean the variety $\mathbb{Z} \setminus \{n_1, n_2, \cdots, n_M\}$. Plugging this into the eigenvalue equation yields

$$\begin{aligned}
- \sum_{\boldsymbol{n}\in\mathcal{B}_M} \sum_{\beta=1}^{M} \sum_{s\in\mathbb{Z}_{[\boldsymbol{n}]}} A_{sn_\beta} \psi(n_1, n_2, \cdots, n_M) \left(|n_1, \cdots, n_{\beta-1}, s, n_{\beta+1}, \cdots, n_M\rangle - |n_1, n_2, \cdots, n_M\rangle\right) = \\
E_M/J \sum_{\boldsymbol{n}\in\mathcal{B}_M} \psi(n_1, n_2, \cdots, n_M) |n_1, n_2, \cdots, n_M\rangle,
\end{aligned} \tag{3.32}$$

where we defined $\mathcal{B}_M$ in (2.11). We apply the linear function $\langle m_1, m_2, \cdots, m_M|$ to the above to obtain

$$-\sum_{\beta=1}^{M}\sum_{j\in\mathbb{Z}_{[\boldsymbol{m}]}} A_{jm_\beta}\psi(m_1,m_2,\cdots,m_M)+\sum_{\beta=1}^{M}\sum_{s\in\mathbb{Z}_{[\boldsymbol{m}]}} A_{sm_\beta}\psi(m_1,\cdots,m_{\beta-1},s,m_{\beta+1},\cdots,m_M) =$$

$$-\sum_{\beta=1}^{M}\left\{\sum_{s\in\mathbb{Z}_{[\boldsymbol{m}]}} A_{m_\beta s}\psi(m_1,m_2,\cdots,m_M)-\sum_{\substack{\gamma=1\\ \gamma\neq\beta}}^{M} A_{m_\beta m_\gamma}\psi(m_1,m_2,\cdots,m_M)\right\} +$$

$$\sum_{\beta=1}^{M}\sum_{s\in\mathbb{Z}_{[\boldsymbol{m}]}} A_{sm_\beta}\psi(m_1,\cdots,m_{\beta-1},s,m_{\beta+1},\cdots,m_M) =$$

$$-\frac{E_M}{J}\psi(m_1,m_2,\cdots,m_M) \quad (3.33)$$

where we use that the states $\{|n_1, n_2, \cdots, n_M\rangle\}_{n_i\in\mathbb{Z}}$ form an orthonormal basis of the M-particle sector. After rearranging, we arrive at the equation

$$\sum_{\beta=1}^{M}\sum_{s\in\mathbb{Z}_{[\boldsymbol{m}]}} A_{m_\beta s}\psi(m_1,\cdots,m_{\beta-1},s,m_{\beta+1}\cdots,m_M) =$$

$$-\psi(m_1,m_2,\cdots m_M)\left(\sum_{\substack{\beta,\gamma\in\mathbb{Z}\\ \beta\neq\gamma}} A_{m_\beta m_\gamma}+E_M/J-M\tau_0\right), \quad (3.34)$$

where $\tau_0 = \sum_{s\in\mathbb{Z}_{[0]}} A_{s0}$. This relation is called the *M-particle difference equation* and holds for all $M \geq 2$.

3.6 $M = 2$

We can use the difference equation derived above to find an expression for the energy E_2 for the case of two magnons. Filling in $M = 2$ explicitly into (3.34) gives us, after defining $\boldsymbol{n} := (n_1, n_2)$,

$$\sum_{s\in\mathbb{Z}_{[\boldsymbol{n}]}} A_{n_1 s}\psi(s,n_2)+\sum_{s'\in\mathbb{Z}_{[\boldsymbol{n}]}} A_{n_2 s'}\psi(n_1,s') = -\psi(n_1,n_2)\left(A_{n_1n_2}+A_{n_2n_1}+E_2/J-2\tau_0\right). \quad (3.35)$$

We can write

$$2\tau_0 - A_{n_1n_2} - A_{n_2n_1} = 2\sum_{\substack{s\in\mathbb{Z}\\ s\neq 0,-l}} A_{s0},$$

where $l := n_1 - n_2$ and we use the fact that A_{jk} is symmetric in j and k. After relabeling the summation variables in (3.35) as $s \to s + n_1$ and $s' \to s' + n_2$ we find

$$\begin{aligned} E_2/J &= 2 \sum_{\substack{s\in\mathbb{Z} \\ s\neq 0,-l}} A_{s0} - \frac{1}{\psi(n_1,n_2)} \left(\sum_{\substack{s\in\mathbb{Z} \\ s\neq ,n_2-n_1}} A_{s0}\psi(s+n_1,n_2) + \sum_{\substack{s'\in\mathbb{Z} \\ s'\neq n_1-n_2,0}} A_{s'0}\psi(n_1,s'+n_2) \right) \\ &= \sum_{\substack{s\in\mathbb{Z} \\ s\neq 0,-l}} A_{s0} \left(2 - \frac{\psi(s+n_1,n_2)+\psi(n_1,n_2-s)}{\psi(n_1,n_2)} \right), \end{aligned} \tag{3.36}$$

where we renamed the summation variable $s' \to -s'$ to get to the last equality. In principle, all we need to do now is to find an eigenfunction $\psi(n_1, n_2)$ such that the right-hand side of the above equation becomes independent of n_1 and n_2. Such an eigenfunction has been proposed by Inozemtsev in [8]:

$$\psi(n_1,n_2) = \frac{e^{i(p_1n_1+p_2n_2)}\sinh(\kappa(n_1-n_2)+\gamma) + e^{i(p_1n_2+p_2n_1)}\sinh(\kappa(n_1-n_2)-\gamma)}{\sinh\kappa(n_1-n_2)}. \tag{3.37}$$

We will see later that the phase γ is related to the momentum variables p_1, p_2 as

$$2\kappa\coth(\gamma) = f(p_1) - f(p_2), \tag{3.38}$$

where

$$f(p) := \frac{p}{\pi}\zeta\left(\frac{i\pi}{2\kappa}\right) - \zeta\left(\frac{ip}{2\kappa}\right). \tag{3.39}$$

The Ansatz (3.37) is quite a peculiar eigenfunction for a spin chain, since the coefficients in front of the exponentials depend on the lattice coordinates n_i. The ratio of these coefficients, which is usually called the *scattering matrix*, is given by

$$\frac{\sinh(\kappa(n_1-n_2)+\gamma)}{\sinh(\kappa(n_1-n_2)-\gamma)} = \frac{\coth\kappa(n_1-n_2)+\coth\gamma}{\coth\kappa(n_1-n_2)-\coth\gamma},$$

which obviously depends on the distance $n_1 - n_2$ on the lattice. Therefore, this Ansatz differs crucially from the Bethe Ansatz given in (2.13). However, despite this unusual behaviour, we can still check if this really is an eigenfunction. Define $B_\pm(m) := \sinh(\kappa m \pm \gamma)$ and $p := p_2 - p_1$, then we can try to substitute the eigenfunction into $Z(s, n_1, n_2) := \frac{\psi(s+n_1,n_2)+\psi(n_1,n_2-s)}{\psi(n_1,n_2)}$:

$$\begin{aligned} Z(s,n_1,n_2) &= \Big[B_+(n_1+s-n_2)e^{i(p_1(n_1+s)+p_2n_2)} + B_-(n_1+s-n_2)e^{i(p_2(n_1+s)+p_1n_2)} + \\ &\quad B_+(n_1-(n_2-s))e^{i(p_1n_1+p_2(n_2-s))} + B_-(n_1-(n_2-s))e^{i(p_2n_1+p_1(n_2-s))}\Big] \\ &\quad \times \left(B_+(n_1-n_2)e^{i(p_1n_1+p_2n_2)} + B_-(n_1-n_2)e^{i(p_2n_1+p_1n_2)}\right)^{-1} \frac{\sinh\kappa(n_1-n_2)}{\sinh\kappa(n_1-n_2+s)} \\ &= \frac{B_+(l+s)\left(e^{ip_1s}+e^{-ip_2s}\right) + B_-(l+s)e^{ipl}\left(e^{-ip_1s}+e^{ip_2s}\right)}{B_+(l)+B_-(l)e^{ipl}} \frac{\sinh\kappa l}{\sinh\kappa(l+s)} \\ &= \Big[\{\coth\kappa(l+s)+\coth\gamma\}\left(e^{ip_1s}+e^{-ip_2s}\right) + \\ &\quad \{-\coth\kappa(l+s)+\coth\gamma\}\, e^{ipl}\left(e^{-ip_1s}+e^{ip_2s}\right)\Big] \\ &\quad \times \left(\coth\kappa l + \coth\gamma + \{-\coth\kappa l + \coth\gamma\}\, e^{ipl}\right)^{-1}, \end{aligned} \tag{3.40}$$

where we used the addition rule for hyperbolic sine to get the last equality. From this expression, we see that $Z(s, n_1, n_2)$ only depends on the distance l between n_1 and n_2. Define $D := \coth \kappa l + \coth \gamma + (-\coth \kappa l + \coth \gamma)\, e^{ipl}$ and note that D does not depend on s. Then we can write for $Z(s, n_1, n_2)$

$$\begin{aligned} Z(s, n_1, n_2) &= \frac{1}{D}\Big[\{\coth \kappa(l+s) + \coth \gamma\} \left(e^{ip_1 s} + e^{-ip_2 s}\right) + \\ &\quad \{-\coth \kappa(l+s) + \coth \gamma\}\, e^{ipl} \left(e^{-ip_1 s} + e^{ip_2 s}\right)\Big]. \end{aligned} \tag{3.41}$$

This is a form of $Z(s, n_1, n_2)$ we can work with. If we define some extra functions, we can investigate in a concise way whether the energy E_2 is really just a sum of the two one-particle energies. Let $F_{1,2} : \mathbb{C} \to \mathbb{C}$ be given by

$$\begin{aligned} F_1(p) &:= \sum_{\substack{s \in \mathbb{Z} \\ s \neq 0, -l}} A_{0s} e^{ips} = \sum_{\substack{s \in \mathbb{Z} \\ s \neq 0, -l}} \frac{\kappa^2 e^{ips}}{\sinh^2 \kappa s} \\ F_2(p) &:= \sum_{\substack{s \in \mathbb{Z} \\ s \neq 0, -l}} A_{0s} e^{ips} \coth \kappa(l+s) = \sum_{\substack{s \in \mathbb{Z} \\ s \neq 0, -l}} \frac{\kappa^2 e^{ips} \coth \kappa(l+s)}{\sinh^2 \kappa s}. \end{aligned} \tag{3.42}$$

The function F_1 is very similar to the function F that we investigated when we where calculating the one-particle energy. We immediately find

$$\begin{aligned} F_1(p) &= -\wp(r_p) - \Delta(r_p) \frac{\wp''(r_p)}{\wp'(r_p)} - 2\Delta(r_p)^2 + \frac{\kappa^2}{3} - \frac{\kappa^2 e^{-ipl}}{\sinh^2 \kappa l} \\ &=: -\epsilon_p / J + H_1(p), \end{aligned} \tag{3.43}$$

where $r_p = -\frac{ip}{4\kappa}$ and $\Delta(r_p) = \zeta(r_p) - \frac{2r_p}{\omega}\zeta(\frac{\omega}{2})$, where $\omega = i\pi/\kappa$. We also identified the one-particle energy ϵ_p in this expression and called the remainder $H_1(p)$, which is

$$H_1(p) = \frac{\kappa^2}{3} + \frac{2}{\omega}\zeta(\omega/2) - \frac{\kappa^2 e^{-ipl}}{\sinh^2 \kappa l}. \tag{3.44}$$

The value of F_1 at $p = 0$ can also be calculated and yields

$$F_1(0) = \frac{2}{\omega}\zeta(\frac{\omega}{2}) + \frac{\kappa^2}{3} - \frac{\kappa^2}{\sinh^2 \kappa l}$$

. A similar expression for F_2 will be derived in the next section, but let us first rewrite the expression for the energy (3.36) using these functions:

$$\begin{aligned} E_2(p_1, p_2)/J = 2F_1(0) - \frac{1}{D}\Big[&\coth(\gamma)\Big(F_1(p_1) + F_1(-p_2) + e^{ipl}F_1(-p_1) + e^{ipl}F_1(p_2)\Big) + \\ &F_2(p_1) + F_2(-p_2) - e^{ipl}\left(F_2(-p_1) + F_2(p_2)\right)\Big]. \end{aligned} \tag{3.45}$$

3.7 Finding an Expression for $F_2(p)$

In this section we will perform the same steps as were necessary to find the expression (3.43) for F_1: firstly, we'll introduce a function $\tilde{F} : \mathbb{C} \to \mathbb{C}$ which has $F_2(p)$ in the zeroth order of its

Laurent decomposition. Secondly, we will propose another function $\tilde{G} : \mathbb{C} \to \mathbb{C}$ containing the constants C_1, r_p, $\tilde{\Delta}$ and δ and some elliptic functions and find values of those constants such that the pole structure of $\tilde{G}$ is identical to the pole structure of $\tilde{F}$. Using the same argument as before, we then conclude that $\tilde{F} = \tilde{G}$ and equate the zeroth orders, destillating from it an expression for $F_2(p)$.

Let $\tilde{F} : \mathbb{C} \to \mathbb{C}$ be given by

$$\tilde{F}(z) = \sum_{n\in\mathbb{Z}} \frac{\kappa^2 e^{ipn} \coth \kappa(l+n+z)}{\sinh^2 \kappa(n+z)}, \tag{3.46}$$

where $l \in \mathbb{Z} \setminus \{0\}$, such that it has a pole of order 2 at $z = 0$ [2]. Therefore, all the coefficients a_n of the Laurent series of $\tilde{F}$ around $z = 0$ are zero for $n < -2$. $\tilde{F}$ is quasi-periodic on the lattice Γ we defined before:

$$\tilde{F}(z+\omega) = \tilde{F}(z), \qquad \tilde{F}(z+1) = e^{-ip}\tilde{F}(z),$$

as is easy to check using (3.46). We calculate a_{-2} by integrating along the contour C as before:

$$\begin{aligned}(2\pi i)a_{-2} &= \oint_C \frac{1}{z^{-1}} \sum_{n\in\mathbb{Z}} \frac{\kappa^2 e^{ipn} \coth \kappa(l+n+z)}{\sinh^2 \kappa(n+z)} dz = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \kappa^2 e^{ipn} \oint_C \frac{z \coth \kappa(l+n+z)}{\sinh^2 \kappa(n+z)} dz \\ &\quad + \kappa^2 \oint_C z \coth \kappa(l+z) \left(\frac{1}{\kappa^2 z^2} - \frac{1}{3} + \mathcal{O}\left(z^2\right) \right) dz = 2\pi i \coth \kappa l \end{aligned} \tag{3.47}$$

The terms for $n \neq 0$ do not contribute here because $\coth z$ has a pole of order 1 at $z = 0$ and $\sinh^{-2} \kappa(n+z)$ does not have a pole at $z = 0$ if $n \neq 0$. Calculating the coefficient a_{-1} requires us to be a little bit more careful, because the pole of the hyperbolic cotangent does come into play here:

$$\begin{aligned}(2\pi i)a_{-1} &= \oint_C \sum_{n\in\mathbb{Z}} \frac{\kappa^2 e^{ipn} \coth \kappa(l+n+z)}{\sinh^2 \kappa(n+z)} dz = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0,-l}} \kappa^2 e^{ipn} \oint_C \frac{\coth \kappa(l+n+z)}{\sinh^2 \kappa(n+z)} dz \\ &\quad + \kappa^2 \oint_C \coth \kappa(l+z) \left(\frac{1}{\kappa^2 z^2} - \frac{1}{3} + \mathcal{O}\left(z^2\right) \right) dz + \kappa^2 e^{-ipl} \oint_C \frac{\coth \kappa z}{\sinh^2 \kappa(-l+z)} dz \\ &= 0 + \frac{2\pi i}{\kappa^2} \frac{d}{dz} \left(\coth \kappa(l+z) \right) \bigg|_{z=0} + \kappa^2 e^{-ipl} \oint_C \frac{1}{\sinh^2 \kappa(-l+z))} \left(\frac{1}{\kappa z} + \mathcal{O}(z) \right) dz \\ &= -\frac{2\pi i \kappa^2}{\kappa \sinh^2 \kappa l} + 2\pi i \frac{\kappa e^{-ipl}}{\sinh^2 \kappa l} = 2\pi i \frac{\kappa}{\sinh^2 \kappa l} \left(e^{-ipl} - 1 \right). \end{aligned} \tag{3.48}$$

To get this coefficient, we had to use Cauchy's differentiation formula [14]

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz,$$

[2] This is a little bit stricter than the condition for l in Inozemtsev's paper [22]; note that for $l = 0$, $\tilde{F}$ will have a triple pole at $z = 0$, so to be able to reproduce the equality $\tilde{F} = \tilde{G}$ for a function $\tilde{G}$ with a double pole, we must set $l \neq 0$.

which will also be necessary to find a_0:

$$\begin{aligned}
2\pi i a_0 &= \oint_C \sum_{n\in\mathbb{Z}} \frac{\kappa^2 e^{ipn}\coth\kappa(l+n+z)}{z\sinh^2\kappa(n+z)}dz = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0,-l}} \kappa^2 e^{ipn}\oint_C \frac{\coth\kappa(l+n+z)}{z\sinh^2\kappa(n+z)}dz \\
&\quad +\kappa^2\oint_C \coth\kappa(l+z)\left(\frac{1}{\kappa^2 z^3}-\frac{1}{3z}+\mathcal{O}(z)\right)dz+\kappa^2 e^{-ipl}\oint_C \frac{\coth\kappa z}{z\sinh^2\kappa(-l+z)}dz \\
&= 2\pi i\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0,-l}} \frac{\kappa^2 e^{ipn}\coth\kappa(l+n)}{\sinh^2\kappa n}+\frac{2\pi i}{2}\frac{d^2}{dz^2}\left(\coth\kappa(l+z)\right)\Big|_{z=0}-2\pi i\frac{\kappa^2\coth\kappa l}{3} \\
&\quad +\kappa^2 e^{-ipl}\oint_C \frac{1}{\sinh^2\kappa(-l+z)}\left(\frac{1}{\kappa z^2}+\mathcal{O}(1)\right)dz. \qquad (3.49)
\end{aligned}$$

We can identify the function F_2 in this expression and simplify to get

$$\begin{aligned}
2\pi i a_0 &= 2\pi i F_2(p)+2\pi i\frac{\kappa^2\coth\kappa l}{\sinh^2\kappa l}-2\pi i\frac{\kappa^2\coth\kappa l}{3}+2\pi i\kappa^2 e^{-ipl}\frac{d}{dz}\left(\frac{1}{\sinh^2\kappa(-l+z)}\right)\Big|_{z=0} \\
&= 2\pi i F_2(p)+2\pi i\frac{\kappa^2\coth\kappa l}{\sinh^2\kappa l}-2\pi i\frac{\kappa^2\coth\kappa l}{3}+2\pi i\frac{2\kappa^2 e^{-ipl}\coth\kappa l}{\sinh^2\kappa l} \\
&= 2\pi i F_2(p)+2\pi i\kappa^2\coth\kappa l\left(\frac{2e^{-ipl}-1}{\sinh^2\kappa l}-\frac{1}{3}\right). \qquad (3.50)
\end{aligned}$$

Now that we have the coefficients, we introduce a function $\tilde{G}:\mathbb{C}\to\mathbb{C}$ given by

$$\tilde{G}(z)=-C_1\frac{\sigma(z+r_p)}{\sigma(z-r_p)}\exp\left[\delta z\right]\cdot\left\{\wp(z)-\wp(r_p)+\tilde{\Delta}\left(\frac{\wp'(z)-\wp'(r_p)}{\wp(z)-\wp(r_p)}-\frac{\wp''(r_p)}{\wp'(r_p)}\right)\right\}, \qquad (3.51)$$

which has exactly the same form as the function G we used to perform this trick for the one-particle energy, except for an overall multiplication. It therefore also has the exact same Laurent series (compare with (3.23))

$$\begin{aligned}
\tilde{G}(z) &= C_1\Bigg[\frac{1}{z^2}+\frac{2\zeta(r_p)+\delta-2\tilde{\Delta}}{z}+ \\
&\quad \left(-\wp(r_p)-\tilde{\Delta}\frac{\wp''(r_p)}{\wp'(r_p)}+(\zeta(r)+\delta/2)\left(-4\tilde{\Delta}+2(\zeta(r)+\delta/2)\right)\right)\Bigg]+\mathcal{O}(z). \qquad (3.52)
\end{aligned}$$

Of course, $\tilde{G}$ is quasi-periodic if we choose $\delta=\frac{p}{\pi}\zeta(\omega/2)$ and $r_p=-\frac{ip}{4\kappa}$, since it is at this point equivalent to G. To make sure $\tilde{G}$'s pole structure is exactly the same as that of $\tilde{F}$, we can fix the other constants. Equating the coefficients of $\tilde{F}$ and $\tilde{G}$ at order z^{-2} shows hat $C_1=\coth(\kappa l)$. To get equality at order z^{-1} we must solve

$$C_1\left(2\zeta(r_p)+\delta-2\tilde{\Delta}\right)=\frac{\kappa}{\sinh^2\kappa l}\left(e^{-ipl}-1\right),$$

yielding

$$\begin{aligned}
\tilde{\Delta} &= \zeta(r_p)+\frac{\delta}{2}-\frac{\kappa}{2C_1\sinh^2\kappa l}\left(e^{-ipl}-1\right) \\
&= \zeta(r_p)+\frac{\delta}{2}+\frac{\kappa}{\sinh 2\kappa l}\left(1-e^{-ipl}\right). \qquad (3.53)
\end{aligned}$$

Earlier, we defined $\Delta(r_p) = \zeta(r_p) - \frac{2r_p}{\omega}\zeta(\omega/2)$, thus we can rewrite

$$\tilde{\Delta}(r_p) = \Delta(r_p) + \Delta'(r_p),$$

where $\Delta'(r_p) := \frac{\kappa}{\sinh 2\kappa l}\left(1 - e^{-ipl}\right)$. This allows us to rewrite part of the zeroth order term of $\tilde{G}$ to

$$\begin{aligned}(\zeta(r) + \delta/2)\left(-4\tilde{\Delta} + 2(\zeta(r) + \delta/2)\right) &= \Delta(r_p)\left(-4\left(\Delta(r_p) + \Delta'(r_p)\right) + 2\Delta(r_p)\right) \\ &= -2\Delta(r_p)^2 - 4\Delta(r_p)\Delta'(r_p). \qquad (3.54)\end{aligned}$$

This fixes $\tilde{G}$ completely. By the same arguments as before, we can now argue that $\tilde{F} = \tilde{G}$: the function $\tilde{F} - \tilde{G}$ has no poles and is quasi-periodic on the lattice Γ. It is therefore analytic and bounded and by Liouville's theorem, it must be constant. This constant must be zero due to the quasi-periodicity, thus in fact $\tilde{F} = \tilde{G}$. This is in accordance with equation (22) of [8]. By this equality, we can equate the constant terms of $\tilde{F}$ and $\tilde{G}$, which are the a_0-coefficients calculated in (3.50) and (3.52) and find an expression for $F_2(p)$:

$$\begin{aligned}F_2(p) &= \coth\kappa l\left(-\wp(r_p) - \left(\Delta(r_p) + \Delta'(r_p)\right)\frac{\wp''(r_p)}{\wp'(r_p)} - 2\Delta(r_p)^2\right. \\ &\quad \left.-4\Delta(r_p)\Delta'(r_p) - \kappa^2\frac{2e^{-ipl} - 1}{\sinh^2\kappa l} + \frac{\kappa^2}{3}\right) \\ &= \coth\kappa l\left(-\epsilon_p/J + \frac{2}{\omega}\zeta(\omega/2) - \Delta'(r_p)\frac{\wp''(r_p)}{\wp'(r_p)} - 4\Delta(r_p)\Delta'(r_p) - \kappa^2\frac{2e^{-ipl} - 1}{\sinh^2\kappa l} + \frac{\kappa^2}{3}\right) \\ &= \coth\kappa l\left(-\epsilon_p/J + \frac{2}{\omega}\zeta(\omega/2) - \Delta'(r_p)\left(\frac{\wp''(r_p)}{\wp'(r_p)} + 4\Delta(r_p)\right) - \kappa^2\frac{2e^{-ipl} - 1}{\sinh^2\kappa l} + \frac{\kappa^2}{3}\right) \\ &= \coth\kappa l\left(-\epsilon_p/J + \frac{2}{\omega}\zeta(\omega/2) - 2\Delta'(r_p)f(p) - \kappa^2\frac{2e^{-ipl} - 1}{\sinh^2\kappa l} + \frac{\kappa^2}{3}\right) \\ &=: \coth\kappa l\left(-\epsilon_p/J + H_2(p)\right), \qquad (3.55)\end{aligned}$$

where we used a formula for the ζ-function to identify $f(p)$, namely

$$\zeta(2z) = 2\zeta(z) + \frac{1}{2}\frac{\wp''(z)}{\wp'(z)}.$$

We also identified the one-particle energy ϵ_p in $F_2(p)$ and defined the remainder to be $H_2(p)$. With all this machinery, it should be possible to check that the eigenfunction (3.37) is correct.

3.8 The Two-Particle Energy Continued

To check if formula (3.37) is indeed an eigenfunction in the two-particle sector, we continue to check whether the expression (3.45) is indeed just the sum of two one-particle energies ϵ_{p_1} and

ϵ_{p_2}. Plugging in the expressions for the $F_i(p)$ gives

$$
\begin{aligned}
E_2(p_1,p_2)/J &= 4\zeta\left(\frac{\omega}{2}\right)+\frac{2\kappa^2}{3}-\frac{2\kappa^2}{\sinh^2\kappa l}-\frac{1}{D}\Big\{\coth\gamma\big[-\epsilon_{p_1}/J+H_1(p_1)-\epsilon_{p_2}/J+H_1(-p_2)\\
&\quad +e^{ipl}\left(-\epsilon_{p_1}/J+H_1(-p_1)-\epsilon_{p_2}/J+H_1(p_2)\right)\big]\\
&\quad +\coth\kappa l\big[-\epsilon_{p_1}/J+H_2(p_1)-\epsilon_{p_2}/J+H_2(-p_2)\\
&\quad -e^{ipl}\left(-\epsilon_{p_1}/J+H_2(-p_1)-\epsilon_{p_2}/J+H_2(p_2)\right)\big]\Big\}\\
&= \frac{(\epsilon_{p_1}+\epsilon_{p_2})\left(\coth(\gamma)+e^{ipl}\coth\gamma+\coth\kappa l-e^{ipl}\coth\kappa l\right)}{D}\\
&\quad +\frac{1}{D}\Big\{\left(4\zeta\left(\frac{\omega}{2}\right)+\frac{2\kappa^2}{3}-\frac{2\kappa^2}{\sinh^2\kappa l}\right)D\\
&\quad -\coth\gamma\left(H_1(p_1)+H_1(-p_2)+e^{ipl}(H_1(-p_1)+H_1(p_2))\right)\\
&\quad -\coth\kappa l\left(H_2(p_1)+H_2(-p_2)-e^{ipl}(H_2(-p_1)+H_2(p_2))\right)\Big\},
\end{aligned}
\tag{3.56}
$$

where we recognize the sum of one-particle energies in the first fraction. This means that for (3.37) to be an eigenfunction, the terms in the large curly brackets must vanish. Partially, this is easily established. Let us redefine

$$
\begin{aligned}
H_1(p) &= \frac{\kappa^2}{3}+\frac{2}{\omega}\zeta(\omega/2)-\frac{\kappa^2e^{-ipl}}{\sinh^2\kappa l}=:\frac{\kappa^2}{3}+\frac{2}{\omega}\zeta(\omega/2)+\tilde H_1(p),\\
H_2(p) &= \frac{\kappa^2}{3}+\frac{2}{\omega}\zeta(\omega/2)-2\Delta'(r_p)f(p)-\kappa^2\frac{2e^{-ipl}-1}{\sinh^2\kappa l}=:\frac{\kappa^2}{3}+\frac{2}{\omega}\zeta(\omega/2)+\tilde H_2(p)
\end{aligned}
\tag{3.57}
$$

and define $C=\frac{\kappa^2}{3}+\frac{2}{\omega}\zeta(\omega/2)$. Plugging this into the second and third term of the curly brackets yields

$$
\begin{aligned}
&-2DC-\coth\gamma\left(\tilde H_1(p_1)+\tilde H_1(-p_2)+e^{ipl}(\tilde H_1(-p_1)+\tilde H_1(p_2))\right)-\\
&\coth\kappa l\left(\tilde H_2(p_1)+\tilde H_2(-p_2)-e^{ipl}(\tilde H_2(-p_1)+\tilde H_2(p_2))\right),
\end{aligned}
\tag{3.58}
$$

which means that the terms in the curly brackets sum up to

$$
\begin{aligned}
&-\frac{2\kappa^2}{\sinh^2\kappa l}D-\coth\gamma\left(\tilde H_1(p_1)+\tilde H_1(-p_2)+e^{ipl}(\tilde H_1(-p_1)+\tilde H_1(p_2))\right)\\
&-\coth\kappa l\left(\tilde H_2(p_1)+\tilde H_2(-p_2)-e^{ipl}(\tilde H_2(-p_1)+\tilde H_2(p_2))\right).
\end{aligned}
\tag{3.59}
$$

To continue, we will have to replace all the $\tilde H_i(p_j)$ by their definitions. We first treat the term in the brackets after $\coth\gamma$:

$$
\begin{aligned}
\tilde H_1(p_1)+\tilde H_1(-p_2)+e^{ipl}(\tilde H_1(-p_1)+\tilde H_1(p_2)) &= -\frac{\kappa^2e^{-ip_1l}}{\sinh^2\kappa l}-\frac{\kappa^2e^{ip_2l}}{\sinh^2\kappa l}-\frac{\kappa^2e^{ipl+ip_1l}}{\sinh^2\kappa l}-\frac{\kappa^2e^{ipl-ip_2l}}{\sinh^2\kappa l}\\
&= -\frac{2\kappa^2}{\sinh^2\kappa l}\left(e^{-ip_1l}+e^{ip_2l}\right),
\end{aligned}
\tag{3.60}
$$

using $p = p_2 - p_1$. Next, we consider the terms in the brackets after $\coth \kappa l$:

$$
\begin{aligned}
& \tilde{H}_2(p_1) + \tilde{H}_2(-p_2) - e^{ipl}(\tilde{H}_2(-p_1) + \tilde{H}_2(p_2)) = \\
& -2\Delta'(r_{p_1})f(p_1) - \kappa^2 \frac{2e^{-ip_1 l} - 1}{\sinh^2 \kappa l} - 2\Delta'(r_{-p_2})f(-p_2) - \kappa^2 \frac{2e^{ip_2 l} - 1}{\sinh^2 \kappa l} \\
& -e^{ipl}\left(-2\Delta'(r_{-p_1})f(-p_1) - \kappa^2 \frac{2e^{ip_1 l} - 1}{\sinh^2 \kappa l} - 2\Delta'(r_{p_2})f(p_2) - \kappa^2 \frac{2e^{-ip_2 l} - 1}{\sinh^2 \kappa l}\right) \\
= \quad & -2f(p_1)\left(\Delta'(r_{p_1}) + \Delta'(r_{-p_1})e^{ipl}\right) + 2f(p_2)\left(\Delta'(r_{-p_2}) + \Delta'(r_{p_2})e^{ipl}\right) + \frac{2\kappa^2}{\sinh^2 \kappa l}\left(e^{ipl} - 1\right),
\end{aligned}
\tag{3.61}
$$

where we used the fact that $f(p) = -f(-p)$. The terms linear in $\Delta'(r_{p_i})$ can also be put in a nicer form. We see that

$$
\Delta'(r_{p_1}) + \Delta'(r_{-p_1})e^{ipl} = \frac{\kappa}{\sinh 2\kappa l}\left(1 - e^{-ip_1 l} + e^{ipl} - e^{ip_2 l}\right) = \Delta'(r_{-p_2}) + \Delta'(r_{p_2})e^{ipl},
$$

thus we can combine the terms linear in $\Delta'(r_{p_i})$ into

$$
-2(f(p_1) - f(p_2))\left(1 - e^{-ip_1 l} + e^{ipl} - e^{ip_2 l}\right)\frac{\kappa}{\sinh 2\kappa l},
$$

which gives us for the entire expression given by (3.59)

$$
\begin{aligned}
& -\frac{2\kappa^2}{\sinh^2 \kappa l}D + \frac{2\kappa^2 \coth \gamma}{\sinh^2 \kappa l}\left(e^{-ip_1 l} + e^{ip_2 l}\right) + \frac{2\kappa \coth \kappa l}{\sinh 2\kappa l}\left(f(p_1) - f(p_2)\right)\left(1 - e^{-ip_1 l} + e^{ipl} - e^{ip_2 l}\right) \\
& -\frac{2\kappa^2 \coth \kappa l}{\sinh^2 \kappa l}\left(e^{ipl} - 1\right).
\end{aligned}
\tag{3.62}
$$

After rewriting

$$
\frac{2\coth \kappa l}{\sinh 2\kappa l} = \frac{2\coth \kappa l}{2\sinh \kappa l \cosh \kappa l} = \frac{1}{\sinh^2 \kappa l}
$$

we can see that most of the terms drop out; namely, we get from the above

$$
\begin{aligned}
& -\frac{2\kappa^2}{\sinh^2 \kappa l}\left(\coth \kappa l + \coth \gamma + (-\coth \kappa l + \coth \gamma)\, e^{ipl}\right) + \frac{2\kappa^2 \coth \gamma}{\sinh^2 \kappa l}\left(e^{-ip_1 l} + e^{ip_2 l}\right) \\
& +\frac{\kappa}{\sinh^2 \kappa l}\left(f(p_1) - f(p_2)\right)\left(1 - e^{-ip_1 l} + e^{ipl} - e^{ip_2 l}\right) - \frac{2\kappa^2 \coth \kappa l}{\sinh^2 \kappa l}\left(e^{ipl} - 1\right) \\
= \quad & -\frac{2\kappa^2}{\sinh^2 \kappa l}\left(\coth \kappa l - e^{ipl}\coth \kappa l + e^{ipl}\coth \kappa l - \coth \kappa l\right) \\
& +\frac{\kappa}{\sinh^2 \kappa l}\left(2\kappa \coth \gamma - (f(p_1) - f(p_2))\right)\left(-1 - e^{ipl} + e^{-ip_1 l} + e^{ip_2 l}\right) \\
= \quad & \frac{\kappa}{\sinh^2 \kappa l}\left(2\kappa \coth \gamma - (f(p_1) - f(p_2))\right)\left(-1 - e^{ipl} + e^{-ip_1 l} + e^{ip_2 l}\right).
\end{aligned}
\tag{3.63}
$$

This vanishes if we require γ to satisfy the phase condition given by

$$
2\kappa \coth \gamma = f(p_1) - f(p_2), \tag{3.64}
$$

which is precisely the condition stated in equation (3.39). Under this condition the term inside the curly brackets in equation (3.56) vanishes, which in turn implies that the eigenfunction given in equation (3.37) is indeed an eigenfunction of the Hamiltonian H with energy

$$
E(p_1, p_2) = \epsilon_{p_1} + \epsilon_{p_2}.
$$

Although we can not be certain yet, the fact the the energy in the two-particle sector is given by the sum of one-particle energies hints that this could also be the case in general. This simple structure would be somewhat surprising, since the interactions in our spin chain are long range, which usually leads to a very complicated energy spectrum. To see whether this additivity holds in the M-particle sector for arbitrary M, we turn to solving the eigenvalue problem in full generality. As a first step, we study a Calogero-Sutherland-Moser model, which will prove to be useful.

3.9 A CSM Model

In this section we will study some known results of a Calogero-Sutherland-Moser model (CSM-model), which is basically the continuum version of Inozemtsev's infinite spin chain without a spin-dependent interaction and can be defined using the Hamiltonian

$$H_C := \sum_{j=1}^{M} \frac{p_j^2}{2} + \sum_{\substack{j,k\in\mathbb{Z} \\ j\neq k}} \frac{\kappa^2}{\sinh^2 \kappa(x_j - x_k)},$$

where $p_j = \frac{\partial}{\partial x_j}$ and the coordinate space is $\mathbb{R}^{\mathbb{M}}$, i.e. all particles move on the infinite line. The solutions to this CSM-model have been known for quite a while and where first published by F. Calogero in 1971 in [23]. Later they where treated in the more general context of quantum integrable models defined by Lie algebras by M.A. Olshanetsky and A.M. Perelomov (see [24]). Finally, the solutions where described in a different way by O.A. Chalykh and A.P. Veselov in [25]. From the latter two references, we can deduce that the eigenfunctions in the M-particle sector can be written as

$$\chi_p^{(M)}(\boldsymbol{x}) = D_M \exp\left(i \sum_{j=1}^{M} i p_j x_j\right), \tag{3.65}$$

where $\boldsymbol{p} \in \mathbb{C}^M$ and D_M is a differential operator that satisfies

$$\begin{aligned} D_M &= Q_M^{1\dots M-1} D_{M-1} \\ Q_n^{i_1\dots i_m} &= Q_n^{i_1\dots i_{m-1}} \left(\partial_{i_m} - \partial_n - 2\kappa \coth \kappa(x_{i_m} - x_n)\right) \\ &\quad + \sum_{s=1}^{m-1} 2\kappa^2 \left(\coth^2 \kappa(x_{i_s} - x_{i_m}) - 1\right) Q_n^{i_1\dots i_{s-1} i_{s+1}\dots i_{m-1}}, \end{aligned} \tag{3.66}$$

where the superscripts i_l are indices, $n \in \mathbb{N}$ and the symbol without indices satisfies $Q_n = 1$. The lowest order for D_M is $D_2 = (\partial_1 - \partial_2 - 2\kappa \coth \kappa(x_1 - x_2))$, which can be found by solving the two-particle problem. Also, we can write

$$\chi_p^{(M)}(\boldsymbol{x}) = \prod_{\mu<\nu} \sinh^{-1} \kappa(x_\mu - x_\nu) \exp\left(i \sum_{j=1}^{M} i p_j x_j\right) \phi_p^{(M)}(x), \tag{3.67}$$

where $\phi_p^{(M)}$ is a pole-free function as follows from the eigenvalue equation. In principle the recurrence relations in (3.66) allow us to find the eigenfunctions for any M, although the amount of work required increases vastly for increasing M. We will therefore (in line with [22]) first find another characterization of these solutions by investigating the recurrence relation (3.66).

3.9.1 Properties of the Recurrence Relation

We can view the symbols $Q_n^{i_1 \cdots i_l}$ as functions of the x_i. We can interpret the partial derivatives as constants after we have commuted them all the way to the right of every expression, since their action on (3.65) will yield a constant. We will prove the following theorem by induction.

Theorem. The $Q_n^{i_1 \cdots i_l}$ are polynomials with variables

$$\{\coth\kappa(x_i - x_j)\} := \{\coth(\kappa(x_i - x_j))|i, j \leq M, i \neq j\}$$

for all $n \leq M$ and all values of the indices $\{i_s\}$.

Proof. Let $n \leq M$ be arbitrary. First consider $Q_n = 1$. It can be viewed trivally as a polynomial in $\{\coth\kappa(x_i - x_j)\}$. Suppose that for all $j < l$, $Q_n^{i_1 \cdots i_j}$ can be written as a polynomial $P_n^{i_1 \cdots i_j}(\{\coth\kappa(x_i - x_j)\})$. Then we immediately see that

$$\begin{aligned} Q_M^{i_1 \ldots i_l} &= P_M^{i_1 \ldots i_{l-1}}(\{\coth\kappa(x_i - x_j)\})(\partial_{i_l} - \partial_M - 2\kappa\coth\kappa(x_{i_l} - x_M)) \\ &\quad + \sum_{s=1}^{l-1} 2\kappa^2\left(\coth^2\kappa(x_{i_s} - x_{i_l}) - 1\right) P_M^{i_1 \ldots i_{s-1} i_{s+1} \ldots i_{l-1}}(\{\coth\kappa(x_i - x_j)\}). \end{aligned}$$

Since the polynomials $P_M^{i_1 \ldots i_{l-1}}(\{\coth\kappa(x_i - x_j)\})$ also contain derivatives, we are not quite done. We should also check that the action of an arbitrary differential operator of the form $\prod_{i=1}^M \partial_i^{k_i}$ (with $k_i \in \mathbb{N} \cup \{0\}$) on $\coth\kappa(x_r - x_s)$ yields a polynomial in $\coth\kappa(x_r - x_s)$. This is equivalent to checking that $\frac{d^l}{dz^l}(\coth z)^k$ is a polynomial in $\coth z$ for all $l, k \in \mathbb{N}$. We can prove this by induction as well: we know that

$$\frac{d}{dz}(\coth z)^k = -k(\coth z)^{k-1}\sinh^{-2} z = k(1 - \coth^2 z)(\coth z)^{k-1}, \tag{3.68}$$

so for $l = 1$ and for all $k \in \mathbb{N}$, the assertion is true. We also see that the resulting polynomial is of degree $k + 1$, which is useful to incorporate in our induction hypothesis. Suppose now that for all $j < l$, $\frac{d^j}{dz^j}(\coth z)^k$ is a polynomial in $\coth z$ of degree $k + j$, then we see that

$$\frac{d^l}{dz^l}(\coth z)^k = \frac{d}{dz}\frac{d^{l-1}}{dz^{l-1}}(\coth z)^k = \frac{d}{dz}P(\coth z),$$

where P is a polynomial of degree $k + l - 1$ conform our induction hypothesis. Finally, we know that $\frac{d}{dz}(\coth z)^k$ is a polynomial of degree $k + 1$ for all $k \in \mathbb{N}$ by equation (3.68), thus we see that indeed $\frac{d^l}{dz^l}(\coth z)^k$ is a polynomial of degree $l + k$. From this we may conclude that $Q_n^{i_1 \cdots i_l}$

is a polynomial in $\{\coth \kappa(x_i - x_j)\}$. $\Box$

Specifically, this is true for the values of the indices given by $i_s = s$ for $1 \leq s \leq l$ and for all $l \leq M$. We see that therefore

$$D_M = Q_{M-1} D_{M-1} = Q_{M-1}^{1\cdots M-1} Q_{M-2}^{1\cdots M-2} \cdots Q_3^{12} D_2$$

is also of a polynomial in $\{\coth \kappa(x_i - x_j)\}$, if we apply again the statement that $\frac{d^l}{dz^l}(\coth z)^k$ is a polynomial in $\{\coth \kappa(x_i - x_j)\}$ to commute all the derivatives to the right. Thus we see that the eigenfunctions $\chi_p^{(M)}$ can be written in the form

$$\chi_p^{(M)}(\boldsymbol{x}) = R(\{\coth \kappa(x_i - x_j)\}) \exp\left(i \sum_{j=1}^{M} i p_j x_j\right),$$

where $R(\{\coth \kappa(x_i - x_j)\})$ is a polynomial. Now for $y_i := e^{2\kappa x_i}$ we have

$$\coth \kappa(x_i - x_j) = \frac{e^{\kappa(x_i - x_j)} + e^{-\kappa(x_i - x_j)}}{e^{\kappa(x_i - x_j)} - e^{\kappa(x_i - x_j)}} = \frac{e^{2\kappa x_i} + e^{2\kappa x_j}}{e^{2\kappa x_i} - e^{2\kappa x_j}} = \frac{y_i + y_j}{y_i - y_j},$$

which shows that we can view R also as a rational function of the $\{y_i\}$ in which the denominators have the form $(y_i - y_j)^k$ for $i, j \leq M$ and $k \in \mathbb{N}$. Since we also know by equation (3.67) that the pole structure of R is given by

$$\prod_{\mu<\nu} \sinh^{-1} \kappa(x_\mu - x_\nu) = \prod_{\mu<\nu} \frac{1}{e^{\kappa(x_\mu - x_\nu)} - e^{-\kappa(x_\mu - x_\nu)}} = \prod_{\mu<\nu} \frac{2 y_\mu y_\nu}{y_\mu - y_\nu},$$

we may conclude that $\chi_p^{(M)}$ is given by

$$\chi_p^{(M)}(\boldsymbol{x}) = \prod_{\mu<\nu} \sinh^{-1} \kappa(x_\mu - x_\nu) S(\{y_i\}) \exp\left(i \sum_{j=1}^{M} i p_j x_j\right), \tag{3.69}$$

where

$$S(\{y_i\}) = \sum_{\boldsymbol{m} \in \mathbb{Z}^M} d_{m_1, m_2, \cdots, m_M} \prod_{\mu=1}^{M} y_\mu^{m_\mu}, \tag{3.70}$$

thus S is a regular polynomial in the variables $\{y_i\}$. At this point, this is not a very convenient way of rewriting $\chi_p^{(M)}$, since the number of undetermined coefficients $d_{\boldsymbol{m}}$ is not known yet. Luckily, we have more information on the structure of $\chi_p^{(M)}$: for fixed $\boldsymbol{p} \in \mathbb{C}^M$ we know that $\lim_{x_r \to \pm\infty} R(\{\coth \kappa(x_i - x_j)\})$ must be finite, since $\lim_{x_r \to \pm\infty} \coth \kappa(x_r - x_s) = \pm 1$, and R has finite degree as a polynomial, thus we have a definite maximum degree for each of the y_i. To see this, we expand the pole structure of R for the arbitrary variable x_r:

$$\prod_{\mu<\nu} \sinh \kappa(x_\mu - x_\nu) = \prod_{\mu<\nu} \left(e^{\kappa(x_\mu - x_\nu)} - e^{-\kappa(x_\mu - x_\nu)}\right) = \sum_{k=-(M-1)}^{M-1} C_{k,r}(\boldsymbol{x}) e^{\kappa x_r k}, \tag{3.71}$$

where $C_{k,r}(\boldsymbol{x})$ are coefficients in which the subscript r indicates that they do not depend on x_r. Also, we have included some coefficients which may be 0 for certain values of r due to the restriction of terms in the product, but including them is not a problem. Now the terms in R that depend on x_r are of the following form

$$y_r^\alpha \left(\sum_{k=-(M-1)}^{M-1} C_{k,r}(\boldsymbol{x}) e^{\kappa x_r k} \right)^{-1} = e^{2\alpha\kappa x_r} \left(\sum_{k=-(M-1)}^{M-1} C_{k,r}(\boldsymbol{x}) e^{\kappa x_r k} \right)^{-1},$$

where $\alpha \in \mathbb{Z}$. Since there is only one such term for every given α, all these terms must have a finite limit for all allowed values of α as $x_r \to \pm\infty$. The denominator has powers of $e^{\kappa x_r}$ between $-(M-1)$ and $M-1$, thus to ensure a finite limit we must restrict $-(M-1) \leq 2\alpha \leq M-1$ and we end up with

$$-(M-1)/2 \leq \alpha \leq (M-1)/2.$$

We can also factor out the quantity $e^{\kappa(M-1)x_r}$, which gives for the allowed powers α that $0 \leq \alpha \leq M-1$. If we do the above procedure for every variable x_i, we end up with the following Ansatz for the eigenfunction of the CSM-model in the M-particle sector:

$$\chi_p^{(M)}(\boldsymbol{x}) = \prod_{\mu<\nu} \sinh^{-1}\kappa(x_\mu - x_\nu) \exp\left(\sum_{j=1}^{M} (ip_j - \kappa(M-1))x_j \right) S(\{y_i\}), \tag{3.72}$$

where

$$S(\{y_i\}) := \sum_{\boldsymbol{m}\in D^M} d_{m_1\cdots m_M}(\boldsymbol{p}) \exp\left(2\kappa \sum_{j=1}^{M} m_j x_j \right) \tag{3.73}$$

and we define $D := \{0, 1, \cdots, M-1\}$ and the coefficients $d_{m_1\cdots m_M}(\boldsymbol{p})$ are to be determined. We will sometimes abbreviate by writing $d_{\boldsymbol{m}} := d_{m_1\cdots m_M}(\boldsymbol{p})$. Plugging in the Ansatz into the eigenvalue equation $(H_{CS} - E)\chi_p^{(M)}(\boldsymbol{x}) = 0$, where the energy $E = \frac{1}{2}\sum_{i=1}^{M} p_i^2$ leads to the following equation:

$$\sum_{\beta\in\mathbb{Z}_M} \left(2y_\beta \frac{\partial}{\partial y_\beta} \left(y_\beta \frac{\partial}{\partial y_\beta} + \frac{i}{\kappa} p_\beta - M + 1 \right) - \frac{i}{\kappa} p_\beta (M-1) + \frac{(M-1)(2M-1)}{3} \right) S(\{y_i\}) - \sum_{\substack{\beta,\rho\in\mathbb{Z}_M \\ \beta\neq\rho}} \frac{y_\beta + y_\rho}{y_\beta - y_\rho} \left(y_\beta \frac{\partial}{\partial y_\beta} - y_\rho \frac{\partial}{\partial y_\rho} + \frac{i}{2\kappa}(p_\beta - p_\rho) \right) S(\{y_i\}) = 0 \tag{3.74}$$

where $\mathbb{Z}_M := \{1, 2, \cdots, M\}$. Plugging in the definition of $S(\{y_i\})$ leads to the following:

$$\sum_{\boldsymbol{m}\in D^M} \prod_{j=1}^{M} y_j^{m_j} d_{\boldsymbol{m}}(\boldsymbol{p}) \Bigg\{ \sum_{\beta\in\mathbb{Z}_M} \left(m_\beta^2 + \frac{2i}{\kappa} p_\beta m_\beta - \left(2m_\beta + \frac{i}{\kappa} p_\beta - \frac{2M-1}{3} \right)(M-1) \right) - \sum_{\substack{\beta,\rho\in\mathbb{Z}_M \\ \beta\neq\rho}} \frac{y_\beta + y_\rho}{y_\beta - y_\rho} \left(m_\beta - m_\rho + \frac{i}{2\kappa}(p_\beta - p_\rho) \right) \Bigg\} = 0 \tag{3.75}$$

The first summation in (3.75) yields a polynomial in the $\{y_i\}$, whereas the second summation might lead to a nontrivial denominator. This implies that for every solution to this equation, we must have for all $\beta, \rho \in \mathbb{Z}_M$ with $\beta \neq \rho$ that

$$\left(y_\beta \frac{\partial}{\partial y_\beta} - y_\rho \frac{\partial}{\partial y_\rho} + \frac{i}{2\kappa}(p_\beta - p_\rho) \right) S(\{y_i\})$$

is divisible by $(y_\beta - y_\rho)$. What does this imply for the coefficients $d_{\boldsymbol{m}}$?

3.9.2 Divisibility Requirement

By plugging in the definition of S we find that

$$\begin{aligned}
&\sum_{\boldsymbol{m}\in D^M} d_{m_1\cdots m_M}(\boldsymbol{p}) \left(m_\beta - m_\rho + \frac{i}{2\kappa}(p_\beta - p_\rho) \right) \prod_{i\in\mathbb{Z}_M} y_i^{m_i} =: \sum_{\boldsymbol{m}\in D^M} \tilde{d}_{m_1\cdots m_M}(\boldsymbol{p}) \prod_{i\in\mathbb{Z}_M} y_i^{m_i} \\
&= \sum_{\substack{\{m_i\}\in D^{M-2} \\ i\neq\beta,\rho}} \prod_{\substack{i\in\mathbb{Z}_M \\ i\neq\beta,\rho}} y_i^{m_i} \sum_{m_\beta, m\rho\in D} \tilde{d}_{m_1\cdots m_M}(\boldsymbol{p}) y_\beta^{m_\beta} y_\rho^{m_\rho}
\end{aligned} \tag{3.76}$$

should be divisible by $(y_\beta - y_\rho)$, which is equivalent to saying that the polynomial

$$\sum_{m_\beta, m\rho\in D} \tilde{d}_{m_1\cdots m_M}(\boldsymbol{p}) y_\beta^{m_\beta} y_\rho^{m_\rho}$$

should be divisible by $(y_\beta - y_\rho)$. There is a very simple way to see what this implies for the coefficients: the fact that $(y_\beta - y_\rho)$ divides our polynomial essentially tells us that when viewed as a polynomial in y_β, it has a root at $y_\beta = y_\rho$. This means that

$$\sum_{m_\beta, m\rho\in D} \tilde{d}_{m_1\cdots m_M}(\boldsymbol{p}) y_\rho^{m_\beta + m_\rho} = 0. \tag{3.77}$$

Since the sum must vanish for every order in y_ρ, this implies that the coefficients belonging to a fixed sum $m_\beta + m_\rho = N$ should sum up to zero. Formally, this can be written as

$$\sum_{n\in\mathbb{Z}_{\beta,\rho}} \tilde{d}_{m_1\cdots m_\beta+n\cdots m_\rho-n\cdots m_M}(\boldsymbol{p}) = 0, \tag{3.78}$$

where $\mathbb{Z}_{\beta,\rho}$ indicates the subset of all $n \in \mathbb{Z}$ such that $0 \leq m_\beta + n, m_\rho - n \leq M - 1$. However, this does not give us any information on the remainder after dividing by $(y_\beta - y_\rho)$. Therefore, we will proceed in explicitly factoring this polynomial. For notational simplicity, we suppress the subscripts of the $\tilde{d}_{\boldsymbol{m}}(\boldsymbol{p})$ that are not m_β or m_ρ and omit the dependence on $\boldsymbol{p}$, thus we write $\tilde{d}_{m_\beta m\rho}$. We can factorize as follows: first we write

$$\begin{aligned}
\sum_{m_\beta, m\rho\in D} \tilde{d}_{m_\beta m\rho} y_\beta^{m_\beta} y_\rho^{m_\rho} \quad &= \quad (y_\beta - y_\rho) \sum_{\substack{m_\beta, m\rho\in D \\ m_\beta\neq 0}} \tilde{d}_{m_\beta m\rho} y_\beta^{m_\beta} y_\rho^{m_\rho} + \\
&\sum_{\substack{m_\beta, m\rho\in D \\ m_\beta\neq 0}} \tilde{d}_{m_\beta m\rho} y_\beta^{m_\beta - 1} y_\rho^{m_\rho + 1} + \sum_{m_\rho\in D} \tilde{d}_{0m_\rho} y_\rho^{m_\rho},
\end{aligned} \tag{3.79}$$

where we added a counterterm to make sure we are not overcounting. Repeating this step gives

$$\sum_{m_\beta, m\rho \in D} \tilde{d}_{m_\beta m\rho} y_\beta^{m_\beta} y_\rho^{m_\rho} = (y_\beta - y_\rho) \left(\sum_{\substack{m_\beta, m\rho \in D \\ m_\beta \neq 0}} \tilde{d}_{m_\beta m\rho} y_\beta^{m_\beta - 1} y_\rho^{m_\rho} + \sum_{\substack{m_\beta, m\rho \in D \\ m_\beta \neq 0,1}} \tilde{d}_{m_\beta m\rho} y_\beta^{m_\beta - 2} y_\rho^{m_\rho + 1} \right)$$
$$+ \sum_{\substack{m_\beta, m\rho \in D \\ m_\beta \neq 0,1}} \tilde{d}_{m_\beta m\rho} y_\beta^{m_\beta - 2} y_\rho^{m_\rho + 2} + \sum_{m_\rho \in D} \tilde{d}_{1 m_\rho} y_\rho^{m_\rho + 1} + \sum_{m_\rho \in D} \tilde{d}_{0 m_\rho} y_\rho^{m_\rho}. \tag{3.80}$$

We can continue to rewrite our polynomial using this step until we reach

$$\sum_{m_\beta, m\rho \in D} \tilde{d}_{m_\beta m\rho} y_\beta^{m_\beta} y_\rho^{m_\rho} = (y_\beta - y_\rho) \sum_{k=1}^{M-1} \sum_{\substack{m_\beta, m\rho \in D \\ m_\beta \geq k}} \tilde{d}_{m_\beta m\rho} y_\beta^{m_\beta - k} y_\rho^{m_\rho + k - 1} + \sum_{m_\beta, m_\rho \in Z_M} \tilde{d}_{m_\beta m_\rho} y_\rho^{m_\beta + m_\rho}. \tag{3.81}$$

It is obvious from this expression that if the second term on the right-hand side vanishes, the polynomial is divisible by $(y_\beta - y_\rho)$. This yields exactly equation (3.77), thus we reach the same conclusion, namely that equation (3.78) should hold for all $\beta, \rho \in \mathbb{Z}_M$ with $\beta \neq \rho$. We will see later that this has great consequences for the number of nonzero coefficients. First, however, it seems prudent to continue to rewrite our eigenvalue equation (3.74) using the remainder of the polynomial. To continue, we need the product

$$(y_\beta + y_\rho) \sum_{k=1}^{M-1} \sum_{\substack{m_\beta, m\rho \in D \\ m_\beta \geq k}} \tilde{d}_{m_\beta m\rho} y_\beta^{m_\beta - k} y_\rho^{m_\rho + k - 1}. \tag{3.82}$$

Investigating the terms belonging to a specific product $y_\beta^{m_\beta} y_\rho^{m_\rho}$ shows that it equals

$$\begin{aligned}
&\sum_{m_\beta, m_\rho = 0}^{M-1} \left(\sum_{k=1}^{\Lambda_{\beta,\rho}} \tilde{d}_{m_\beta + k, m\rho - k} + \sum_{k=0}^{\Lambda_{\beta,\rho}} \tilde{d}_{m_\beta + k, m\rho - k} \right) y_\beta^{m_\beta} y_\rho^{m_\rho} \\
= &\sum_{m_\beta, m_\rho = 0}^{M-1} \left(2 \sum_{k=1}^{\Lambda_{\beta,\rho}} \tilde{d}_{m_\beta + k, m\rho - k} + \tilde{d}_{m_\beta, m\rho} \right) y_\beta^{m_\beta} y_\rho^{m_\rho} \\
= &\sum_{m_\beta, m_\rho = 0}^{M-1} \sum_{k \in \mathbb{Z}_{\beta,\rho}} \operatorname{sign}(k) \tilde{d}_{m_\beta + k, m\rho - k} y_\beta^{m_\beta} y_\rho^{m_\rho}
\end{aligned} \tag{3.83}$$

where we used the condition (3.78) to get the last equality and we defined $\Lambda_{\beta,\rho} := \min(M - 1 - m_\beta, m_\rho)$. To get the eigenvalue equation in its final form, we can use the following relations:

for any M-dimensional vectors $\boldsymbol{k}$ and $\boldsymbol{m}$, we have

$$\sum_{i=1}^{M} k_i m_i = M^{-1}\left(\sum_{i=1}^{M} m_i \sum_{j=1}^{M} k_j + \frac{1}{2}\sum_{\substack{i,j\in\mathbb{Z}_M\\ i\neq j}} (k_i-k_j)(m_i-m_j)\right)$$

$$2\sum_{i=1}^{M} m_i^2 = M^{-1}\left(\sum_{\substack{i,j\in\mathbb{Z}_M\\ i\neq j}} (m_i-m_j)^2 + 2\left(\sum_{i=1}^{M} m_i\right)^2\right), \tag{3.84}$$

which is straightforward to check. Using these to rewrite the first term of (3.74), we end up with

$$\sum_{\substack{\beta,\rho\in\mathbb{Z}_M\\ \beta\neq\rho}} \left\{ d_{m_1,\cdots,m_M}(\boldsymbol{p})\left(\frac{m_\beta-m_\rho}{M}\left(\frac{i}{\kappa}(p_\beta-p_\rho)+m_\beta-m_\rho\right)+\frac{M+1}{6}\right)\right.$$
$$\left. - \sum_{k\in\mathbb{Z}_{\beta,\rho}} \operatorname{sign}(k)\left(m_\beta-m_\rho+2k+\frac{i}{2\kappa}(p_\beta-p_\rho)\right) d_{m_\beta+k,m_\rho-k}(\boldsymbol{p})\right\} = 0. \tag{3.85}$$

Since every term in the summation occurs exactly twice, once for $\beta<\rho$ and once for $\beta>\rho$, we may also take the sum over β,ρ to cover only those tuples for which $\beta<\rho$, because the overall factor two drops out. Also, it is important to note that this formula is slightly different than its analog written down in [22][3]. The nice thing about the equations (3.78) and (3.85) is that they completely determine the solutions to the CSM-eigenvalue equation and are algebraic. Even more, they are linear equations.

In principle, this is all we need. The fact that we know that there exists a solution to the eigenvalue-equation shows that there must exist at least 1 solution to the linear system (3.78), (3.85). However, there are some interesting facts to discover: there is a very nice argument to show that most of the coefficients $d_{\boldsymbol{m}}$ vanish and we will see that the equations (3.78) and (3.85) are not as independent as they might look. We will therefore now try to solve these equations.

3.9.3 S is Homogeneous

Luckily, most of the M^M coefficients $d_{\boldsymbol{m}}$ are zero, as we will prove now.

Theorem. S is a homogeneous polynomial of degree $M(M-1)/2$.

Proof. Let us investigate (3.78): it tells us that every coefficient $\tilde{d}_{m_1\cdots m_M}$ can be written as a

[3]Checking some of the solutions we will find in the next section in this equation shows that the original equation written down by Inozemtsev was wrong, albeit only slightly. The equation presented here is the correct one

linear combination of other coefficients

$$\tilde{d}_{m_1\cdots m_\beta\cdots m_\rho\cdots m_M}(\boldsymbol{p}) = -\sum_{n\in\mathbb{Z}_{\beta,\rho}\setminus\{0\}} \tilde{d}_{m_1\cdots m_\beta+n\cdots m_\rho-n\cdots m_M}(\boldsymbol{p}). \tag{3.86}$$

In particular, since the summation runs over all integers such that $0 \leq m_\beta+n, m_\rho-n \leq M-1$, we see that all coefficients for which at least two of its indices are either 0 or $M-1$ vanish, because when expanding over those two indices the summation on the right-hand side of equation (3.86) simply does not contain any terms[4].

Now take a coefficient $\tilde{d}_{m_1\cdots m_M}(\boldsymbol{p})$ which does not belong to a term of degree $M(M-1)/2$, i.e. the sum of its indices satisfies $\sum_{i=1}^{M} m_i \neq M(M-1)/2$. This implies that at least two of the indices are equal, since if all where different, the sum would be exactly $\sum_{i=1}^{M}(i-1) = M(M-1)/2$. If we would express this coefficient using equation (3.86), the right-hand side coefficients $\tilde{d}_{m_1\cdots m_\beta+n\cdots m_\rho-n\cdots m_M}(\boldsymbol{p})$ have exactly the same sum for their indices, since the only difference is in the βth and ρth coefficient and their sum changes from $m_\beta+m_\rho$ to $m_\beta+n+m_\rho-n = m_\beta+m_\rho$. We can use this fact to devise an algorithm to express $\tilde{d}_{m_1\cdots m_M}(\boldsymbol{p})$ in terms of other coefficients:

1. From amongst the coinciding indices, choose two indices with the lowest occurring values, say $m_\beta = m_\rho = m$.
2. Express $\tilde{d}_{m_1\cdots m_M}(\boldsymbol{p})$ using (3.86) by expanding in those two indices.
3. Repeat the above two steps for every coefficient in the expansion. This is possible precisely because there must again be a set of coinciding indices, because the sum of indices does not change when applying (3.86).

Of course, it is not obvious at all that this procedure terminates in a finite number of steps. However, we can prove that this is the case by investigating the quadratic sum of the indices. Before applying (3.86) to a coefficient with coinciding indices $m_\beta = m_\rho = m$, the quadratic sum equals $Q_{\beta,\rho} := m_1^2 + \cdots + m_\beta^2 + \cdots + m_\rho^2 + \cdots + m_M^2$, where $m_\beta^2 + m_\rho^2 = 2m^2$. After this application, each of the new coefficients has quadratic sum

$$m_1^2 + \cdots + (m_\beta+n)^2 + \cdots + (m_\rho-n)^2 + \cdots + m_M^2 = Q_{\beta,\rho} + 2n^2 + 2n(m_\beta - m_\rho) = Q_{\beta,\rho} + 2n^2 > Q_{\beta,\rho}, \tag{3.87}$$

which follows because $n \neq 0$ for all the new coefficients and $m_\beta = m_\rho = m$. Thus this tells us that the quadratic sum of the coefficients increases strictly when applying equation (3.86). Also, the quadratic sum of the indices has an obvious maximum, namely $M(M-1)^2$ when all indices equal $M-1$. Moreover, once at least 2 of the indices have value $M-1$ we know

[4]Formally, we still do not know what the value of $d_{m_1\cdots m_\beta\cdots m_\rho\cdots m_M}(\boldsymbol{p})$ (without the tilde) is at the point $\boldsymbol{p}$ for which $\left(m_\beta - m_\rho + \frac{i}{2\kappa}(p_\beta - p_\rho)\right) = 0$. However, continuity of the eigenfunction in $\boldsymbol{p}$ dictates that $d_{m_1\cdots m_\beta\cdots m_\rho\cdots m_M}(\boldsymbol{p})$ must be continuous as well, hence also zero at that point.

that the coefficient is zero. Since the quadratic sum increases strictly, such a situation must occur within a finite number of steps, so we can express $\tilde{d}_{m_1\cdots m_M}(\boldsymbol{p})$ as a finite combination of coefficients that are zero, from which we see that $\tilde{d}_{m_1\cdots m_M}(\boldsymbol{p})$ must be zero. This implies that the polynomial S is homogeneous.$\square$

We can apply this algorithm to an even wider class of coefficients. If a given coefficient $\tilde{d}_{m_1\cdots m_M}(\boldsymbol{p})$ has a subset of indices $\{m_{i_1}, \cdots m_{i_{M'}}\}$ with $M' < M$ for which $\sum_{k=1}^{L} m_{i_k} < M'(M'-1)/2$, then that subset must contain a set of coinciding indices. Since the sum of the indices does not change when applying equation (3.86), this must be true also after application, so we can use the algorithm to rewrite this coefficient. By the exact same line of reasoning, we can write this coefficient as a finite linear combination of coefficients which are zero, implying that the coefficient itself must be zero. It might seem as if this argument shows that all the coefficients must be zero, except those for which all the indices are different. However, already at $M = 3$, there exist other coefficients which are not excluded from being nonzero by the above argument, namely $d_{111}(\boldsymbol{p})$. In the next section ,we deal with all the nonzero coefficients.

3.9.4 The Nonzero Coefficients

We can also find very useful information about the nonzero coefficients. For example, equation (3.85) tells us that all the coefficients depend on κ and $\boldsymbol{p}$ through factors of the form $r_{ij} := \kappa^{-1}(p_i - p_j)$, where $i, j \in \mathbb{Z}_M$. For a particular subset of the coefficients, we can actually do even more.

Consider a coefficient $d_{m_1\cdots m_M}(\boldsymbol{p})$ for which all the indices are different. Then we can compactly write d_P, where $P \in \pi_M$ and $m_i = Pi - 1$. For every duo of indices satisfying $m_\mu = m_\nu + 1$, equation (3.86) tells us that

$$d_{m_1\cdots m_\mu\cdots m_\nu\cdots m_M}(\boldsymbol{p})\left(1 + \frac{i}{2}r_{\nu\mu}\right) = d_{m_1\cdots m_\nu\cdots m_\mu\cdots m_M}(\boldsymbol{p})\left(1 + \frac{i}{2}r_{\mu\nu}\right),$$

from which one can deduce that, up to a $\boldsymbol{p}$-dependent normalization factor d_0, the coefficients are uniquely given by

$$d_P(\boldsymbol{p}) = d_0 \prod_{\lambda<\mu}\left(1 + \frac{i}{2}r_{P^{-1}\lambda, P^{1-}\mu}\right). \tag{3.88}$$

Although this gives an expression to many of the nonzero coefficients, we already saw that this does not cover all the possibilities. The number of other nonzero coefficients increases quite rapidly for increasing M, there are already 2112 such coefficients at $M = 6$. However, it can be seen from the equations that all these coefficients are uniquely determined from the coefficients of the form d_P using (3.86). Therefore, we can conclude that the eigenvalue equation has only one functionally-independent solution at each M.

3.9.5 Solution

The fact that we were able to conclude that there is only one functionally-independent solution to the eigenvalue equation at each M is somewhat peculiar, since we did not use equation (3.85) to reach this conclusion. However, we have verified in a few simple cases that the solutions extracted from (3.78) satisfy (3.85). In addition, the fact that (3.78) has to be satisfied for every solution to (3.85) indicates that if (3.78) has a unique solution, it must also solve (3.85). This implies that in fact (3.85) cannot be an independent equation and must be derivable from (3.78). This was also noticed by Inozemtsev, but unfortunately the direct derivation remains unknown.

3.10 Spin-Chain Solutions at Arbitrary M

Now that we know most of the details about the solutions to the CSM-eigenvalue equation at arbitrary M, we can return to our initial pursuit, solving the M-particle difference equation (3.34) for Inozemtsev's infinite spin chain. To solve it, we use the following Ansatz, conform [22], which is very reminiscent of the Ansatz for the CSM-model:

$$\psi(n_1, \cdots, n_m) = \prod_{\substack{\mu,\nu \in \mathbb{Z}_M \\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu - n_\nu) \sum_{P\in\pi_M} (-1)^P \exp\left(\sum_{j=1}^{M} (ip_{Pj} - \kappa(M-1))n_j\right)$$
$$\sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) \exp\left(2\kappa \sum_{j=1}^{M} m_{Pj} n_j\right),$$

where the factor $(-1)^P$ indicates the sign of the permutation P and the coefficients $c_{\boldsymbol{m}}(\boldsymbol{p})$ are to be determined. One can view this Ansatz as the totally anti-symmetrized version of the CSM-Ansatz. Note that this Ansatz is symmetric under the exchange of n_i and n_j, reflecting the fact that the quasi-particles described by these wavefunctions (usually called *magnons*) are bosons. Plugging this Ansatz into the left-hand side of the M-difference equation (3.34) yields the impressive expression

$$\sum_{s\in\mathbb{Z}_{[\boldsymbol{n}]}} \sum_{\beta\in\mathbb{Z}_M} \sum_{P\in\pi_M} (-1)^P \prod_{\substack{\mu,\nu\in\mathbb{Z}_M\setminus\{\beta\} \\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu - n_\nu)(-1)^{\beta-1} \sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p})$$
$$\exp\left(\sum_{\substack{\gamma\in\mathbb{Z}_M \\ \gamma\neq\beta}} (ip_{P\gamma} + \kappa(2m_{P\gamma} - M + 1))n_\gamma\right) \frac{\kappa^2 \exp\left(q(p_{P\beta}, m_{P\beta})\right)}{\sinh^2\kappa(s - n_\beta)} \prod_{\substack{\lambda\in\mathbb{Z}_M \\ \lambda\neq\beta}} \sinh^{-1}\kappa(n_\lambda - s), \quad (3.89)$$

where $q(p, m) := ip + \kappa(2m - M + 1)$, but we will suppress the arguments from now on. To make it easier to treat this summation, we wish to switch the order of the summations and bring the summation over s all the way to the right. This can be done if the sum converges in both cases, which is the case if $|\mathrm{Im}(p_i)| < 2\kappa$ for all i, as can be shown using an argument similar to the one we used previously to prove that the polynomial S had to have finite degree. For now we will not worry about the consequences of restricting the imaginary parts of the momenta to obey $|\mathrm{Im}(p_i)| < 2\kappa$ for all i and simply proceed to change the order of the summations. The left-hand side of (3.34) reads

$$\sum_{\beta\in\mathbb{Z}_M}\sum_{P\in\pi_M}(-1)^P\prod_{\substack{\mu,\nu\in\mathbb{Z}_M\backslash\{\beta\}\\ \mu<\nu}}\sinh^{-1}\kappa(n_\mu-n_\nu)(-1)^{\beta-1}\sum_{\boldsymbol{m}\in D^M}c_{m_1\cdots m_M}(\boldsymbol{p})$$
$$\exp\left(\sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta}}(ip_{P\gamma}+\kappa(2m_{P\gamma}-M+1))n_\gamma\right)W(p_{P\beta},m_{P\beta},\{n\},\beta), \tag{3.90}$$

where we defined

$$W(p,m,\{n\},\beta)=\sum_{s\in\mathbb{Z}_{[\boldsymbol{n}]}}\frac{\kappa^2\exp\left(q(p)\right)}{\sinh^2\kappa(s-n_\beta)}\prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}}\sinh^{-1}\kappa(n_\lambda-s). \tag{3.91}$$

To continue, we will have to find an explicit expression for W, so we will do this first.

3.10.1 Finding an Explicit Expression for W

We will perform the exact same steps as were necessary to find explicit expressions for the functions we encountered in the one- and two-magnon problem. We will investigate the periodicity and the Laurent expansion of function $W:\mathbb{C}\to\mathbb{C}$ given by

$$W(z)=\sum_{s\in\mathbb{Z}}\frac{\kappa^2e^{qs}}{\sinh^2\kappa(s-n_\beta+z)}\prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}}\sinh^{-1}\kappa(n_\lambda-s-z). \tag{3.92}$$

We will first show that W is quasi-periodic on the lattice Γ we encountered before. Since the sum over s is infinite, it is immediate that $W(z+1)=e^{-q}W(z)$ by relabelling the summation variable. From

$$\sinh\kappa(z\pm\omega)=\frac{e^{\kappa(z\pm\omega)}-e^{-\kappa(z\pm\omega)}}{2}=\frac{e^{\kappa z}e^{\pm}i\pi-e^{-\kappa z}e^{\mp}i\pi}{2}=-\sinh\kappa z$$

we see that

$$W(z+\omega)=\sum_{s\in\mathbb{Z}}\frac{\kappa^2e^{qs}}{\sinh^2\kappa(s-n_\beta+z)}\prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}}\sinh^{-1}\kappa(n_\lambda-s+z)(-1)^{M-1}=e^{i\pi(M-1)}W(z).$$

In a similar fashion as before, we can find the Laurent expansion of W up to its constant term by contour integration along C:

$$
\begin{aligned}
&2\pi i a_{-2} = \oint_C W(z) z dz = \\
&\sum_{s\in\mathbb{Z}_{[\boldsymbol{n}]}} \kappa^2 e^{qs} \oint_C \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}} \frac{\sinh^{-1}\kappa(n_\lambda - s - z)}{\sinh^2\kappa(s - n_\beta + z)} z dz + \sum_{\gamma\in\mathbb{Z}_M} \kappa^2 e^{qn_\gamma} \oint_C \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}} \frac{\sinh^{-1}\kappa(n_\lambda - n_\gamma - z)}{\sinh^2\kappa(n_\gamma - n_\beta + z)} z dz \\
&= \kappa^2 e^{qn_\beta} \oint_C \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}} \sinh^{-1}\kappa(n_\lambda - s - z) \left(\frac{1}{\kappa^2 z^2} - \frac{1}{3} + \mathcal{O}\left(z^2\right)\right) z dz \\
&+ \sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta}} \kappa^2 e^{qn_\gamma} \oint \frac{1}{\sinh^2\kappa(n_\gamma - n_\beta + z)} \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta,\gamma}} \sinh^{-1}\kappa(n_\lambda - n_\gamma + z) \left(-\frac{1}{\kappa z} + \mathcal{O}(z)\right) z dz \\
&= 2\pi i e^{qn_\beta} \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}} \sinh^{-1}\kappa(n_\lambda - n_\beta).
\end{aligned} \tag{3.93}
$$

Since $2\pi i a_{-1} = \oint_C W(z) dz$, it is not difficult to see from (3.93) that

$$
\begin{aligned}
a_{-1} &= e^{qn_\beta} \frac{d}{dz} \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}} \sinh^{-1}\kappa(n_\lambda - n_\beta - z)\Bigg|_{z=0} - \kappa \sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta}} \kappa^2 e^{qn_\gamma} \frac{1}{\sinh^2\kappa(n_\gamma - n_\beta)} \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta,\gamma}} \sinh^{-1}\kappa(n_\lambda - n_\gamma) \\
&= \kappa \left\{ a_{-2} \sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta}} \coth\kappa(n_\gamma - n_\beta) - \sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta}} e^{qn_\gamma} \sinh^{-1}\kappa(n_\beta - n_\gamma) \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\gamma}} \sinh^{-1}\kappa(n_\lambda - n_\gamma) \right\}.
\end{aligned} \tag{3.94}
$$

Finally, the constant term a_0 can be seen to equal

$$
\begin{aligned}
a_0 &= \sum_{s\in\mathbb{Z}_{[\boldsymbol{n}]}} \frac{\kappa^2 \exp\left(q(p)\right)}{\sinh^2\kappa(s - n_\beta)} \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}} \sinh^{-1}\kappa(n_\lambda - s) + \frac{1}{2} e^{qn_\beta} \frac{d^2}{dz^2} \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}} \sinh^{-1}\kappa(n_\lambda - n_\beta - z)\Bigg|_{z=0} \\
&- \frac{\kappa^2}{3} e^{qn_\beta} \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}} \sinh^{-1}\kappa(n_\lambda - n_\beta) - \sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta}} \frac{d}{dz} \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\gamma,\beta}} \frac{\sinh^{-1}\kappa(n_\lambda - n_\gamma - z)}{\sinh^2\kappa(n_\gamma - n_\beta + z)} \\
&= \kappa^2 \Bigg\{ a_{-2} \left(-\frac{1}{3} + \frac{M-1}{2} + \frac{1}{2} \sum_{\substack{\gamma,\delta\in\mathbb{Z}_M\\ \gamma,\delta\neq\beta}} \coth\kappa(n_\gamma - n_\beta)\coth\kappa(n_\delta - n_\beta) + \sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta}} \sinh^{-2}\kappa(n_\gamma - n_\beta) \right) \\
&- \sum_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}} \frac{e^{qn_\lambda}}{\sinh\kappa(n_\beta - n_\lambda)} \prod_{\substack{\rho\in\mathbb{Z}_M\\ \rho\neq\lambda}} \sinh^{-1}\kappa(n_\rho - n_\lambda) \left(\coth\kappa(n_\beta - n_\lambda) + \sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\lambda}} \coth\kappa(n_\gamma - n_\lambda) \right) \Bigg\} \\
&+ W(p, m, \{n\}, \beta).
\end{aligned} \tag{3.95}
$$

Next, introduce the function $U : \mathbb{C} \to \mathbb{C}$ given by

$$U(z) = -A_m \frac{\sigma(z+r)}{\sigma(z-r)} \exp\left(\delta_M z\right) \cdot \left\{ \wp(z) - \wp(r) + \Delta_M \left(\frac{\wp'(z) - \wp'(r)}{\wp(z) - \wp(r)} - \frac{\wp''(r)}{\wp'(r)} \right) \right\}, \qquad (3.96)$$

in which A_m, δ_M, r and Δ_M are constants which will be determined. Our aim is to fix them in such a way that we have $W = U$. We can find r and δ_M from the periodicity requirements on U

$$U(z+1) = e^{i\pi(M-1)} U(z), \qquad U(z+\omega) = e^{-q} U(z),$$

using equation 3.22 and by solving the set of linear equations

$$\begin{aligned} 4\zeta\left(\frac{1}{2}\right) r + \delta_M &= -q(p) \\ 4\zeta\left(\frac{\omega}{2}\right) r + \omega\delta_m &= i\pi(M-1). \end{aligned} \qquad (3.97)$$

We can do this by using the Legendre relation of the Weierstrass zeta-function, namely that

$$\omega\zeta\left(\frac{1}{2}\right) - \zeta\left(\frac{\omega}{2}\right) = i\pi, \qquad (3.98)$$

yielding $\delta_M = \kappa(M-1) - \frac{4r(p)}{\omega}\zeta\left(\frac{\omega}{2}\right)$ and $r(p) = -\left(\frac{m}{2} + \frac{ip}{2\kappa}\right)$[5]. We fix A_M and Δ_M by demanding that the singular part of the Laurent series of U match that of W. Using the expansion given in (3.23), we find the equations

$$\begin{aligned} A_M &= a_{-2} \\ A_M\left(\zeta(r_p) + \delta_M - 2\Delta_M\right) &= a_{-1}, \end{aligned} \qquad (3.99)$$

which give a unique solution for A_M and Δ_M. By the usual argument, we can conclude that U evaluated using those values for its four parameters equals W. Therefore, its zeroth order term in its Laurent expansion must match that of W, giving us an expression for the summation $W(p, m, \{n\}, \beta)$ from the M-particle difference equation:

$$\begin{aligned} \kappa^{-2} W(p, m, \{n\}, \beta) = -a_{-2} \Bigg(& \frac{M-1}{2} + \Omega\left(\{n\}, \beta\right) + \sum_{\substack{\lambda \in \mathbb{Z}_M \\ \lambda \neq \beta}} \sinh^{-2} \kappa(n_\gamma - n_\beta) + \kappa^{-2}\tilde{\epsilon}(p) \\ & -\kappa^{-1}\tilde{f}(p) \sum_{\substack{\lambda \in \mathbb{Z}_M \\ \lambda \neq \beta}} \coth \kappa(n_\gamma - n_\beta) \Bigg) + \Xi(p), \{n\}), \end{aligned}$$

where

$$\Omega\left(\{n\}, \beta\right) = \frac{1}{2} \sum_{\substack{\lambda, \delta \in \mathbb{Z}_M \\ \lambda, \delta \neq \beta,\, \lambda \neq \delta}} \coth \kappa(n_\gamma - n_\beta) \coth \kappa(n_\delta - n_\beta) \qquad (3.100)$$

[5]To avoid awkward notation, we again use r as a parameter, but distinguish between r_p, which will continue to be defined as $r_p = \frac{-ip}{4\kappa}$ and$r(p)$, defined here.

$$
\begin{aligned}
\tilde{f}(p) &= f(p) - \kappa(2m+1-M), \\
\tilde{\epsilon}(p) &= \epsilon(p) - \kappa(2m+1-M)f(p) + \frac{\kappa^2}{2}(2m+1-M)^2,
\end{aligned} \tag{3.101}
$$

with $f(p)$ is defined in (3.39) and $\epsilon(p) := \frac{\kappa^2}{2} - \frac{1}{2}\wp(2r_p) + \frac{1}{2}f^2(p)$, which looks suspiciously similar to the expression for the one-particle energy ϵ_p and

$$
\Xi(p,\{n\}) = \sum_{\substack{\rho\in\mathbb{Z}_M\\ \rho\neq\beta}} \frac{e^{qn_\rho}}{\sinh\kappa(n_\beta - n_\rho)} \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\rho}} \sinh^{-1}\kappa(n_\lambda - n_\rho) \cdot
\left(\coth\kappa(n_\beta - n_\rho) + \sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\rho}} \coth\kappa(n_\gamma - n_\rho) - \kappa^{-1}\tilde{f}(p) \right). \tag{3.102}
$$

Since the M-particle difference equation is linear in W, we can plug in parts of the expression for W and see what they give us. If we plug in

$$
\sum_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}} \sinh^{-2}\kappa(n_\gamma - n_\beta) + \kappa^{-2}\epsilon(p),
$$

where we use the ϵ without the tilde into the eigenvalue equation, we get

$$
L_1(\{n\}) := -\psi(n_1,\cdots,n_M) \left(\sum_{\beta\in\mathbb{Z}_M} \epsilon(p_\beta) + \sum_{\substack{\lambda,\beta\in\mathbb{Z}_M\\ \lambda\neq\beta}} \sinh^{-2}\kappa(n_\gamma - n_\beta) \right). \tag{3.103}
$$

This term equals the right-hand side of the M-particle difference equation if we choose

$$
E_M(p_1,\cdots,p_m) = J \sum_{\beta\in\mathbb{Z}_M} \epsilon_{p_\beta}, \tag{3.104}
$$

thus E_M is simply the sum of one-particle energies. This shows additivity of the energy in the general M-particle case, which is usually the first sign of integrability. In particular, If the energy of a particular model is additive, this model often exhibits factorized scattering. We will show later that this is indeed the case. First we must show that all other terms cancel under suitable choices for the coefficients c_m.

If we plug in $\Xi(r,\{n\})$ into the difference equation we get

$$L_2(\{n\}) := \kappa^2 \sum_{\beta\in\mathbb{Z}_M}\sum_{P\in\pi_M}(-1)^P \prod_{\substack{\mu,\nu\in\mathbb{Z}_M\setminus\{\beta\}\\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu-n_\nu)(-1)^{\beta-1}\sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p})$$

$$\exp\left(\sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta}}(ip_{P\gamma}+\kappa(2m_{P\gamma}-M+1))n_\gamma\right)\Xi(p,\{n\})$$

$$= \kappa^2 \prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu-n_\nu)\sum_{\beta\in\mathbb{Z}_M}\prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta}}\sinh^{-1}\kappa(n_\lambda-n_\beta)\sum_{P\in\pi_M}(-1)^P\sum_{\boldsymbol{m}\in D^M}c_{m_1\cdots m_M}(\boldsymbol{p})$$

$$\sum_{\substack{\rho\in\mathbb{Z}_M\\ \rho\neq\beta}}\exp\left(\sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta,\rho}}(ip_{P\gamma}+\kappa(2m_{P\gamma}-M+1))n_\gamma\right)\exp\left(\left(ip_{P\beta}+ip_{P\rho}+2\kappa(m_{P\beta}+m_{P\rho}-M+1)\right)n_\rho\right)$$

$$\sinh^{-1}\kappa(n_\beta-n_\rho)\left(\coth\kappa(n_\beta-n_\rho)+\sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\rho}}\coth\kappa(n_\gamma-n_\rho)-\kappa^{-1}\tilde{f}(p_\beta)\right)\prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\rho}}\sinh^{-1}\kappa(n_\lambda-n_\rho)$$

$$= -\kappa^2 \prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu-n_\nu)\sum_{P\in\pi_M}(-1)^P\sum_{\boldsymbol{m}\in D^M}c_{m_1\cdots m_M}(\boldsymbol{p})$$

$$\sum_{\substack{\beta,\rho\in\mathbb{Z}_M\\ \beta\neq\rho}}\exp\left(\sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta,\rho}}(ip_{P\gamma}+\kappa(2m_{P\gamma}-M+1))n_\gamma\right)\exp\left\{\left(ip_{P\beta}+ip_{P\rho}+2\kappa(m_{P\beta}+m_{P\rho}-M+1)\right)n_\rho\right\}$$

$$\sinh^{-1}\kappa(n_\beta-n_\rho)\prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta,\rho}}\frac{\sinh\kappa(n_\lambda-n_\beta)}{\sinh\kappa(n_\lambda-n_\rho)}\left(\coth\kappa(n_\beta-n_\rho)+\sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\rho}}\coth\kappa(n_\gamma-n_\rho)-\kappa^{-1}\tilde{f}(p_\beta)\right). \tag{3.105}$$

This can be written more compactly by defining

$$F_M(P,\beta,\rho) := \exp\left(\sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta,\rho}}(ip_{P\gamma}+\kappa(2m_{P\gamma}-M+1))n_\gamma\right)$$

$$\exp\left\{\left(ip_{P\beta}+ip_{P\rho}+2\kappa(m_{P\beta}+m_{P\rho}-M+1)\right)n_\rho\right\}\sinh^{-1}\kappa(n_\beta-n_\rho)\prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta,\rho}}\frac{\sinh\kappa(n_\lambda-n_\beta)}{\sinh\kappa(n_\lambda-n_\rho)}\cdot$$

$$\frac{1}{2}\left[\coth\kappa(n_\beta-n_\rho)+\sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\rho}}\coth\kappa(n_\gamma-n_\rho)-\kappa^{-1}f(p_\beta)+2m_{P\beta}+1-M\right]. \tag{3.106}$$

We now have

$$L_2(\{n\}) = -\kappa^2 \prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu - n_\nu) \sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) \sum_{P\in\pi_M} (-1)^P \sum_{\substack{\beta,\rho\in\mathbb{Z}_M\\ \beta\neq\rho}} 2F_M(P,\beta,\rho). \tag{3.107}$$

One can see that the final summations of this expression can be rewritten as

$$\sum_{P\in\pi_M} (-1)^P \sum_{\substack{\beta,\rho\in\mathbb{Z}_M\\ \beta\neq\rho}} 2F_M(P,\beta,\rho) = \sum_{\substack{\beta,\rho\in\mathbb{Z}_M\\ \beta\neq\rho}} \left(\sum_{P\in\pi_M} (-1)^P F_M(P,\beta,\rho) + \sum_{P\in\pi_M} (-1)^P F_M(P,\beta,\rho) \right)$$
$$= \sum_{\substack{\beta,\rho\in\mathbb{Z}_M\\ \beta\neq\rho}} \left(\sum_{P\in\pi_M} (-1)^P F_M(P,\beta,\rho) + \sum_{P\in\pi_M Q^{-1}} (-1)^{PQ} F_M(PQ,\beta,\rho) \right) =$$
$$\sum_{\substack{\beta,\rho\in\mathbb{Z}_M\\ \beta\neq\rho}} \left(\sum_{P\in\pi_M} (-1)^P \left(F_M(P,\beta,\rho) - F_M(PQ,\beta,\rho)\right) \right), \tag{3.108}$$

where $Q := (\beta\rho)$ is the transposition flipping the indices β and ρ. Note that $\pi_M Q^{-1} = \pi_M$, because π_M is a group and $(-1)^{PQ} = -(-1)^P$ because Q is a transposition. If we look carefully at the dependence of F_M on P, we see that the only difference between $F_M(P,\beta,\rho)$ and $F_M(PQ,\beta,\rho)$ is in the square brackets of equation (3.106). Therefore we get

$$L_2(\{n\}) = -\kappa^2 \prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu - n_\nu) \sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) \sum_{P\in\pi_M} (-1)^P \sum_{\substack{\beta,\rho\in\mathbb{Z}_M\\ \beta\neq\rho}}$$
$$\exp\left(\sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta,\rho}} (ip_{P\gamma} + \kappa(2m_{P\gamma} - M + 1))n_\gamma \right) \exp\left\{(ip_{P\beta} + ip_{P\rho} + 2\kappa(m_{P\beta} + m_{P\rho} - M + 1))\, n_\rho\right\}$$
$$\sinh^{-1}\kappa(n_\beta - n_\rho) \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta,\rho}} \frac{\sinh\kappa(n_\lambda - n_\beta)}{\sinh\kappa(n_\lambda - n_\rho)} \left[m_{P\beta} - m_{P\rho} - \frac{1}{2\kappa}(f(p_\beta) - f(p_\rho)) \right]. \tag{3.109}$$

Finally, all that is left for us to do is rewrite the summation over D^M. Abstractly, we can find that for a function Z depending on the sum and difference of $m_{P\beta}$ and $m_{P\rho}$

$$
\begin{aligned}
&\sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) Z(m_{P\beta}+m_{P\rho}, m_{P\beta}-m_{P\rho}) \\
&= \sum_{\substack{\{m_i\}\in D\\ i\neq P\beta,P\rho}} \sum_{m_{P\beta},m_{P\rho}\in D} c_{m_1\cdots m_M}(\boldsymbol{p}) Z(m_{P\beta}+m_{P\rho}, m_{P\beta}-m_{P\rho}) \\
&= \sum_{\substack{\{m_i\}\in D\\ i\neq P\beta,P\rho}} \sum_{s=0}^{2M-2} \sum_{\substack{m_{P\beta},m_{P\rho}\in D\\ m_{P\beta}+m_{P\rho}=s}} c_{m_1\cdots m_M}(\boldsymbol{p}) Z(s, m_{P\beta}-m_{P\rho}) = \sum_{\substack{\{m_i\}\in D\\ i\neq P\beta,P\rho}} \sum_{s=0}^{2M-2} \\
&\sum_{\substack{m_{P\beta},m_{P\rho}\in D\\ m_{P\beta}+m_{P\rho}=s}} (M-|s-(M-1)|)^{-1} \sum_{l\in\mathbb{Z}_{P\beta,P\rho}} c_{m_1\cdots m_{P\beta}+l\cdots m_{P\rho}-l\cdots m_M}(\boldsymbol{p}) Z(s, m_{P\beta}-m_{P\rho}+2l).
\end{aligned}
\tag{3.110}
$$

We could introduce the last summation over $Z_{P\beta,P\rho}$ by compensating for overcounting, which is why the term $(M-|s-(M-1)|)^{-1}$ has been introduced. If we now use this equation (3.110) to rewrite $L_2(\{n\})$ we find

$$
\begin{aligned}
&L_2(\{n\}) = -\kappa^2 \prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu-n_\nu) \sum_{P\in\pi_M} (-1)^P \sum_{\substack{\beta,\rho\in\mathbb{Z}_M\\ \beta\neq\rho}} \sinh^{-1}\kappa(n_\beta-n_\rho) \prod_{\substack{\lambda\in\mathbb{Z}_M\\ \lambda\neq\beta,\rho}} \frac{\sinh\kappa(n_\lambda-n_\beta)}{\sinh\kappa(n_\lambda-n_\rho)} \\
&\sum_{\substack{\{m_i\}\in D\\ i\neq P\beta,P\rho}} \exp\left(\sum_{\substack{\gamma\in\mathbb{Z}_M\\ \gamma\neq\beta,\rho}} (ip_{P\gamma}+\kappa(2m_{P\gamma}-M+1))n_\gamma\right) \sum_{s=0}^{2M-2} \sum_{\substack{m_{P\beta},m_{P\rho}\in D\\ m_{P\beta}+m_{P\rho}=s}} (M-|s-(M-1)|)^{-1} \\
&\exp\left\{(ip_{P\beta}+ip_{P\rho}+2\kappa(s-M+1))\, n_\rho\right\} \\
&\sum_{l\in\mathbb{Z}_{P\beta,P\rho}} c_{m_1\cdots m_{P\beta}+l\cdots m_{P\rho}-l\cdots m_M}(\boldsymbol{p}) \left[m_{P\beta}-m_{P\rho}-2l-\frac{1}{2\kappa}(f(p_\beta)-f(p_\rho))\right].
\end{aligned}
\tag{3.111}
$$

If we compare this to equation (3.78), we see that $L_2(\{n\})=0$ precisely when we choose

$$
c_{m_1\cdots m_M}(\boldsymbol{p}) = d_{m_1\cdots m_M}(if(\boldsymbol{p}))
$$

, where the $d_{m_1\cdots m_M}(\boldsymbol{p})$ are a solution to (3.78). Uniqueness of this solution therefore follows directly from the investigations into the CSM-model. This correspondence is quite astonishing, but we will first have to prove that the remaining terms in the M-particle difference equation cancel to appreciate this fully.

If we plug in the remaining terms into the the M-particle difference equation, we end up with

$$L_3(\{n\}) = -\kappa^2 \sum_{\beta\in\mathbb{Z}_M} \sum_{P\in\pi_M} (-1)^P \prod_{\substack{\mu,\nu\in\mathbb{Z}_M\setminus\{\beta\} \\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu - n_\nu)(-1)^{\beta-1} \sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p})$$
$$\exp\left(\sum_{\substack{\gamma\in\mathbb{Z}_M \\ \gamma\neq\beta}} (ip_{P\gamma} + \kappa(2m_{P\gamma} - M + 1))n_\gamma\right) e^{(ip_{P\beta}+\kappa(2m_\beta+1-M))n_\beta} \prod_{\substack{\lambda\in\mathbb{Z}_M \\ \lambda\neq\beta}} \sinh^{-1}\kappa(n_\lambda - n_\beta)$$
$$\left[\frac{M-1}{2} - \kappa^{-1}(2m_{P\beta}+1-M)f(p_{P\beta}) + \frac{1}{2}(2m_{P\beta}+1-M)^2 - \right.$$
$$\left.\kappa^{-1}\tilde{f}(p_\beta) \sum_{\substack{\gamma\in\mathbb{Z}_M \\ \gamma\neq\beta}} \coth\kappa(n_\gamma - n_\beta) + \Omega(\{n\})\right]. \tag{3.112}$$

Since it is true that

$$\sum_{i\in I} Z(i) = \sum_{i\in I} Z(Pi) \tag{3.113}$$

for an index set I of cardinality N and $P \in \pi_N$, we can rewrite L_3 as

$$L_3(\{n\}) = -\kappa^2 \prod_{\substack{\mu,\nu\in\mathbb{Z}_M \\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu - n_\nu) \sum_{P\in\pi_M} (-1)^P \sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p})$$
$$\exp\left(\sum_{\gamma\in\mathbb{Z}_M} (ip_\gamma + \kappa(2m_\gamma - M + 1))n_{P^{-1}\gamma}\right)$$
$$\sum_{\beta\in\mathbb{Z}_M} \left[\frac{M-1}{2} - \kappa^{-1}(2m_{P\beta}+1-M)f(p_{P\beta}) + \frac{1}{2}(2m_{P\beta}+1-M)^2 - \right.$$
$$\left.\kappa^{-1}\tilde{f}(p_\beta) \sum_{\substack{\gamma\in\mathbb{Z}_M \\ \gamma\neq\beta}} \coth\kappa(n_\gamma - n_\beta) + \Omega(\{n\})\right]. \tag{3.114}$$

We can symmetrize the summation over β and γ

$$\kappa^{-1} \sum_{\substack{\gamma,\beta\in\mathbb{Z}_M \\ \gamma\neq\beta}} \tilde{f}(p_\beta) \coth\kappa(n_\gamma - n_\beta) = \sum_{\substack{\gamma,\beta\in\mathbb{Z}_M \\ \gamma\neq\beta}} (\kappa^{-1} f(p_{P\beta}) + M - 1 - 2m_{P\beta}) \coth\kappa(n_\gamma - n_\beta)$$

$$= \sum_{\substack{\gamma,\beta\in\mathbb{Z}_M \\ \gamma\neq\beta}} (\kappa^{-1} f(p_\beta) + M - 1 - 2m_\beta) \coth\kappa(n_{P^{-1}\gamma} - n_{P^{-1}\beta})$$

$$= \frac{1}{2} \sum_{\substack{\gamma,\beta\in\mathbb{Z}_M \\ \gamma\neq\beta}} (\kappa^{-1} f(p_\gamma) - \kappa^{-1} f(p_\beta) - +2m_\beta - 2m_\gamma) \coth\kappa(n_{P^{-1}\beta} - n_{P^{-1}\gamma})$$

$$= \sum_{\substack{\gamma,\beta\in\mathbb{Z}_M \\ \gamma\neq\beta}} (m_\beta - m_\gamma - \frac{1}{2\kappa}(f(p_\beta) - f(p_\gamma)) \coth\kappa(n_{P^{-1}\beta} - n_{P^{-1}\gamma}), \tag{3.115}$$

while the summation

$$\sum_{\beta\in\mathbb{Z}_M} \Omega(\{n\}, \beta) = \frac{M(M-1)(M-2)}{6} \tag{3.116}$$

is proved in Appendix A. If we plug in all this and collect similar terms, we get

$$L_3(\{n\}) = -\kappa^2 \prod_{\substack{\mu,\nu\in\mathbb{Z}_M \\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu - n_\nu) \sum_{P\in\pi_M} (-1)^P \exp\left(\sum_{\gamma\in\mathbb{Z}_M} (ip_\gamma - \kappa(M-1)) n_{P^{-1}\gamma}\right)$$
$$\sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) \exp\left(2\kappa \sum_{\lambda\in\mathbb{Z}_M} n_{P^{-1}\lambda} m_\lambda\right)$$
$$\left[\sum_{\beta\in\mathbb{Z}_M} \left\{2m_\beta^2 - 2\kappa^{-1} m_\beta f(p_\beta) - (M-1)\left((2m_\beta - \kappa^{-1} f(p_\beta) - \frac{2M-1}{3}\right)\right\} - \right.$$
$$\left.\sum_{\substack{\gamma,\beta\in\mathbb{Z}_M \\ \gamma\neq\beta}} (m_\beta - m_\gamma - \frac{1}{2\kappa}(f(p_\beta) - f(p_\gamma)) \coth\kappa(n_{P^{-1}\beta} - n_{P^{-1}\gamma}))\right]. \tag{3.117}$$

After writing $z_\beta := \exp(2\kappa n_{P^{-1}\beta})$ we get finally

$$L_3(\{n\}) = -\kappa^2 \prod_{\substack{\mu,\nu\in\mathbb{Z}_M \\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu - n_\nu) \sum_{P\in\pi_M} (-1)^P \exp\left(\sum_{\gamma\in\mathbb{Z}_M} (ip_\gamma - \kappa(M-1)) n_{P^{-1}\gamma}\right)$$
$$\sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) \prod_{\lambda\in\mathbb{Z}_M} z_\lambda^{m_\lambda}$$
$$\left[\sum_{\beta\in\mathbb{Z}_M} \left\{2m_\beta^2 - 2\kappa^{-1} m_\beta f(p_\beta) - (M-1)\left((2m_\beta - \kappa^{-1} f(p_\beta) - \frac{2M-1}{3}\right)\right\} - \right.$$
$$\left.\sum_{\substack{\gamma,\beta\in\mathbb{Z}_M \\ \gamma\neq\beta}} \frac{z_\beta + z_\gamma}{z_\beta - z_\gamma}(m_\beta - m_\gamma - \frac{1}{2\kappa}(f(p_\beta) - f(p_\gamma))\right]. \tag{3.118}$$

From this expression, it is quite easy to see that the condition $L_3(\{n\}) = 0$ is the same as the condition (3.75) for the coefficients of the Ansatz for the CSM-model. In fact, if the coefficients

$c_{\boldsymbol{m}}(p)$ are chosen as $c_{m_1\cdots m_M}(\boldsymbol{p}) = d_{m_1\cdots m_M}(if(\boldsymbol{p}))$, we immediately get that $L_3(\{n\}) = 0$ for all $z \in \mathbb{R}^M$.

From this we can conclude the following: If a set of coefficients $d_{\boldsymbol{m}}(p)$ gives a solution for the CSM-model using the Ansatz (3.72), then the set of coefficients $c_{\boldsymbol{m}}(p)$ satisfying

$$c_{m_1\cdots m_M}(\boldsymbol{p}) = d_{m_1\cdots m_M}(if(\boldsymbol{p})) \qquad \text{for all } \boldsymbol{m} \in D^M$$

give a solution to Inozemtsev's infinite spin-chain using the Ansatz (3.89). This strong relationship is somewhat unexpected (although maybe not too surprising after reading the previous sections) and begs to be explained. One might wonder whether Inozemtsev's infinite spin chain can be considered as a zero-temperature limit of the CSM-model, conform the *freezing trick* introduced by Polychronakos in [26], but the complicated form of the phase function f seems to make it unfeasible to make this claim precise.

3.11 Factorized Scattering

One of the most interesting questions about the solutions we found previously is whether or not they describe factorized scattering. This would hint towards an algebraic structure such as exists for the Heisenberg XXX-spin chain and allows to study the spectrum of Inozemtsev's infinite spin chain more closely. In this section, we will therefore investigate whether or not the solutions exhibit this feature. In general, a model exhibits *factorized scattering* if the eigenfunctions have the following asymptotic behaviour: if $P \in \pi_M$ and the variables $x_1, \cdots, x_M$ tend to infinity as $x_{P(i+1)} - x_{Pi} \to \infty$ for $1 \leq i \leq M$, the wavefunction tends to

$$\psi(x_1, \cdots, x_M) = \psi_0 \sum_{Q \in \pi_M} (-1)^{QP} \exp\left(i \sum_{\lambda \in \mathbb{Z}_M} p_{Q\lambda} n_\lambda \right) \prod_{\substack{\mu,\nu \in \mathbb{Z}_M \\ \mu < \nu}} S(p_{QP\mu}, p_{QP\nu}), \tag{3.119}$$

where $S(p_{QP\mu}, p_{QP\nu})$ is just some function and ψ_0 is a constant. We will try to show that this is the case for the eigenfunctions of the Inozemtsev's infinite spin chain.

Suppose that $P \in \pi_M$ and the variables $n_1, \cdots, n_M$ tend to infinity as $n_{P(i+1)} - n_{Pi} \to \infty$. We will find an explicit form for the wavefunction in this limit. The proof of the case $P = \mathrm{Id}$ has been sketched in [22], but we will treat the case for general P here:

We will try to find the asymptotic behaviour of the term

$$\prod_{\substack{\mu,\nu \in \mathbb{Z}_M \\ \mu < \nu}} \sinh^{-1} \kappa(n_\mu - n_\nu) \exp\left(\sum_{j=1}^{M} (ip_{Qj} - \kappa(M-1)) n_j \right) \sum_{\boldsymbol{m} \in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) \exp\left(2\kappa \sum_{j=1}^{M} m_{Qj} n_j \right), \tag{3.120}$$

where $Q \in \pi_M$ is arbitrary. After reordering the summations such that the summation over π_M is on the outside, this is the summand of the summation over π_M in the Ansatz (3.89) for the eigenfunction. The product over the inverse hyperbolic sines can be rewritten as

$$\prod_{\substack{\mu,\nu\in\mathbb{Z}_M \\ \mu<\nu}} \sinh^{-1}\kappa(n_\mu - n_\nu) = (-2)^{-\frac{M(M-1)}{2}} \prod_{\substack{\mu,\nu\in\mathbb{Z}_M \\ \mu>\nu}} \left(e^{\kappa(n_\mu-n_\nu)} - e^{-\kappa(n_\mu-n_\nu)}\right).$$

In the limit we are discussing, precisely one of the terms survives. Also, the parity $(-1)^P$ of the permutation P equals the parity of the number of inversions, i.e. the number of pairs $\mu, \nu \in \mathbb{Z}_M$ such that $\mu < \nu$ but $P\mu > P\nu$ [27]. Therefore the product yields

$$\begin{aligned}
\prod_{\substack{\mu,\nu\in\mathbb{Z}_M \\ \mu<\nu}} \sinh\kappa(n_\mu - n_\nu) \quad &\sim \quad (-2)^{-\frac{M(M-1)}{2}} \prod_{\substack{\mu,\nu\in\mathbb{Z}_M \\ P^{-1}\mu<P^{-1}\nu,\,\mu>\nu}} e^{\kappa(n_\mu-n_\nu)} \prod_{\substack{\mu,\nu\in\mathbb{Z}_M \\ P^{-1}\nu<P^{-1}\mu,\,\mu>\nu}} \left(-e^{\kappa(n_\nu-n_\mu)}\right) \\
&= \quad (-2)^{-\frac{M(M-1)}{2}} (-1)^P \prod_{\substack{\mu,\nu\in\mathbb{Z}_M \\ \mu<\nu,\,P\mu>P\nu}} e^{\kappa(n_{P\mu}-n_{P\nu})} \prod_{\substack{\mu,\nu\in\mathbb{Z}_M \\ \nu<\mu,\,P\mu>P\nu}} e^{\kappa(n_{P\nu}-n_{P\mu})} \\
&= \quad (-2)^{-\frac{M(M-1)}{2}} (-1)^P \exp\left(\kappa \sum_{\substack{\mu,\nu\in\mathbb{Z}_M \\ \mu>\nu}} (n_{P\mu} - n_{P\nu})\right) \\
&= \quad (-2)^{-\frac{M(M-1)}{2}} (-1)^P \exp\left(-\kappa \sum_{\mu\in\mathbb{Z}_M} n_{P\mu}(M - 2\mu + 1)\right). \qquad (3.121)
\end{aligned}$$

In the limit, the entire term given in (3.120) can be written as

$$\begin{aligned}
&(-2)^{\frac{M(M-1)}{2}} (-1)^P \sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) \\
&\exp\left(2\kappa \sum_{j\in\mathbb{Z}_M} (m_{Qj} n_j - \frac{1}{2}\kappa(M-1)n_j) + \kappa \sum_{\mu\in\mathbb{Z}_M} n_{P\mu}(M - 2\mu + 1)\right). \qquad (3.122)
\end{aligned}$$

Using equation (3.113), this becomes

$$\begin{aligned}
&(-2)^{\frac{M(M-1)}{2}} (-1)^P \sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) \\
&\exp\left(2\kappa \sum_{j\in\mathbb{Z}_M} (m_{PQj} n_{Pj} - \frac{1}{2}\kappa(M-1)n_{Pj}) + \kappa \sum_{\mu\in\mathbb{Z}_M} n_{P\mu}(M - 2\mu + 1)\right) = \\
&= (-2)^{\frac{M(M-1)}{2}} (-1)^P \sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) \exp\left(2\kappa \sum_{j\in\mathbb{Z}_M} (m_{PQj} + 1 - j) n_{Pj}\right). \qquad (3.123)
\end{aligned}$$

After defining

$$g_j(\boldsymbol{m}) := \sum_{\lambda=1}^{j} (m_{PQj} + 1 - j) \qquad (3.124)$$

and noticing that $g_j - g_{j-1} = (m_{PQj} + 1 - j)$ we can rewrite this expression as

$$\begin{aligned}
&(-2)^{\frac{M(M-1)}{2}}(-1)^P \sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) \exp\left(2\kappa \sum_{j\in\mathbb{Z}_M} (g_j - g_{j-1}) n_{Pj}\right) \\
&= (-2)^{\frac{M(M-1)}{2}}(-1)^P \sum_{\boldsymbol{m}\in D^M} c_{m_1\cdots m_M}(\boldsymbol{p}) \exp\left(-2\kappa \sum_{j\in\mathbb{Z}_M} (n_{P(j+1)} - n_{Pj}) g_j\right). \qquad (3.125)
\end{aligned}$$

In this expression, we recognize $n_{P(j+1)} - n_{Pj}$, which we assume to tend to infinity. Therefore, we need to investigate the sign of $g_j(\boldsymbol{m})$, to find out if this expression converges: from the definition of g_j we see immediately that $g_j(\boldsymbol{m}) < 0$ if

$$\sum_{\lambda=1}^{j} m_j < \frac{\lambda(\lambda-1)}{2},$$

but as we saw in Section 3.9.3, all the coefficients $c_{m_1\cdots m_M}$ satisfying this condition are zero by virtue of the relation (3.78). Therefore divergence of our expression cannot occur. If on the other hand $g_j(\boldsymbol{m}) > 0$ for some j, the expression tends to zero. Therefore, the only remaining term is the one for which $g_j(\boldsymbol{m}) = 0$ for all j, which implies $m_{PQj} = j - 1$, or equivalently $m_j = Q^{-1}P^{-1}j - 1$. Using the notation and expressions from Section 3.9.4 we find that we can write the expression from (3.120) as

$$(-2)^{\frac{M(M-1)}{2}}(-1)^P d_{Q^{-1}P^{-1}}(if(p)) = (-1)^P \tilde{d}_0 \prod{}^{\mu,\nu\in\mathbb{Z}_M}_{\mu<\nu} \left(1 - \frac{1}{2\kappa}(f(p_{PQ\mu}) - f(p_{PQ\nu}))\right), \qquad (3.126)$$

where $\tilde{d}_0$ is a normalization. Therefore in the limit where $n_{P(i+1)} - n_{Pi} \to \infty$ for some $P \in \pi_M$, the eigenfunction tends to

$$\psi(n_1, \cdots, n_M) = \psi_0 \sum_{Q\in\pi_M} (-1)^{QP} \exp\left(i \sum_{\lambda\in\mathbb{Z}_M} p_{Q\lambda} n_\lambda\right) \prod_{\substack{\mu,\nu\in\mathbb{Z}_M \\ \mu<\nu}} \left(1 - \frac{1}{2\kappa}(f(p_{PQ\mu}) - f(p_{PQ\nu}))\right), \qquad (3.127)$$

for some normalization ψ_0, which proves that factorized scattering is a feature of our model. Although this is not unanticipated, it is very nice to see it explicitly.

3.12 Conclusion

In this chapter, we have found eigenfunctions for Inozemtsev's infinite spin chain in all the sectors of the Hilbert space. We found out that the energy is additive, a very nice property which is usually associated to integrable systems. In addition, we managed to show that the model displays factorized scattering, which hints even stronger in this direction. It would be interesting to know whether this set of eigenfunctions is complete, but in general questions about the completeness of sets of eigenfunctions is extremely complicated for models with infinite degrees of freedom. We will therefore not pursue this goal here. Instead, to find out more about its structure, we will focus next on the spectrum of this chain through an analysis of its behaviour in the thermodynamic limit.

Chapter 4

Thermodynamics of Inozemtsev's Spin Chain

An interesting aspect of spin chains is their behaviour in the thermodynamic limit. While it is often possible to find the finite-size ground state of short-range models, finding this state for long-range models can be very complicated due to the complexity of the model. In passing to the thermodynamic limit (sending both the length L and the number of quasi-particles M to infinity, while keeping M/L fixed), it is often possible to write down integral equations that determine the behaviour in exact form. In this chapter, we aim to study the thermodynamics of Inozemtsev's spin chains. As a first step, we derive a set of Bethe equations.

4.1 Bethe Equations

In Chapter 3, we have found the eigenfunctions of Inozemtsev's infinite spin chain. Or, to be more precise, we have derived the functional form of the eigenfunctions, $|\psi\rangle_{\boldsymbol{p}}$, depending on the complex momentum vector $\boldsymbol{p}$. We never checked whether the $|\psi\rangle_{\boldsymbol{p}}$ are proper eigenfunctions for all $\boldsymbol{p} \in \mathbb{C}^M$. Indeed, we did already note that the sum in (3.91) converges only for momenta that satisfy $|\mathrm{Im}(p_i) < 2\kappa$ for all i, but this restriction arose because of the ordering the sums we chose in (3.89). We will postpone a more detailed analysis of the summation order until we know more about the spectrum of the hamiltonian H.

However, even if we would accept this limit on the momenta, the remaining set of momenta still seems to be too big to parametrize the spectrum of this infinite model. For the (finite-size) periodic chain, the periodicity condition on the wavefunction quantizes the momenta, resulting in a discrete spectrum. When passing to the infinite chain the spectrum does not have to stay discrete: H is an operator on an infinite-dimensional Hilbert space and therefore its spectrum can have a continuous part. However, there is a strong similarity between the finite-size and infinite chain. The finite-size chain has translational invariance because it is periodic. When taking the limit $L \to \infty$ the hamiltonian loses its periodicity, but stays translation invariant as its state space consists of states on an infinite line. Therefore, it can be expected that the spectra of the finite and the infinite spin chain resemble this close relationship and should not differ too much[1].

It is in the light of the remarks above that the Asymptotic Bethe Ansatz can be best understood. It hypothesizes that the spectrum of the infinite chain can be found by imposing periodic boundary conditions on the asymptotic form of the wavefunction of the infinite chain and solving the resulting set of equations in the limit that $L \to \infty$. To be more precise, we consider the hamiltonian

$$H_L = \sum_{\substack{j,k=1 \\ j \neq k}}^{L} \frac{1}{\sinh^2(\kappa(j-k))} (\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k - 1) \tag{4.1}$$

and impose periodic boundary conditions on its eigenfunctions:

$$\psi(n_1, n_2, \cdots, n_M) = \psi(n_2, \cdots, n_M, n_1 + L). \tag{4.2}$$

We assume that in the asymptotic regime the eigenfunctions of this model are the same as those of Inozemtsev's infinite spin chain. If we now take L to be large and consider the periodic boundary conditions for the case where $n_1 \ll n_2 \ll \cdots \ll n_M$, we can use the asymptotic form of the eigenfunctions of the infinite chain to derive a set of equations. These functions were given in (3.127) and read (plugging in $P = \mathrm{Id}$)

$$\psi(n_1, \cdots, n_M) = \psi_0 \sum_{Q \in \pi_M} (-1)^Q \exp\left(i \sum_{\lambda \in \mathbb{Z}_M} p_{Q\lambda} n_\lambda \right) \prod_{\substack{\mu,\nu \in \mathbb{Z}_M \\ \mu < \nu}} S\left(p_{Q\mu}, p_{Q\nu}\right), \tag{4.3}$$

where

$$S\left(p, q\right) = \left(1 - i(\phi(p) - \phi(q))\right) \tag{4.4}$$

and $\phi : \mathbb{C} \to \mathbb{C}$ is defined as

$$\phi(p) := \frac{f(p)}{2\kappa i} = \frac{p}{2\kappa\pi i} \zeta\left(\frac{i\pi}{2\kappa}\right) - \frac{1}{2\kappa i} \zeta\left(\frac{ip}{2\kappa}\right). \tag{4.5}$$

Let us first define the permutation $R \in \pi_M$ as $R = (12)(23)\cdots((M-1)M)$ or alternatively as

$$Ri = \begin{cases} i+1, & \text{if } i \leq M \\ 1, & \text{if } i = M \end{cases} \tag{4.6}$$

[1]Admittedly, the term "too much" is not very precise, but at least it motivates us to study the spectrum more closely.

and note that its sign is $(-1)^{M-1}$ since it is the sum of $M-1$ transpositions. Plugging in the asymptotic form given in (4.3) into the boundary conditions (4.2) yields [2]

$$\sum_{Q\in\pi_M}(-1)^Q\exp\left(i\sum_{\lambda\in\mathbb{Z}_M}p_{Q\lambda}n_\lambda\right)\prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}}S\left(p_{Q\mu},p_{Q\nu}\right)=$$

$$\sum_{Q\in\pi_M}(-1)^Q\exp\left(i\sum_{\lambda\in\mathbb{Z}_M}p_{Q\lambda}n_{R\lambda}+ip_{QM}L\right)\prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}}S\left(p_{Q\mu},p_{Q\nu}\right). \tag{4.7}$$

If we concentrate on the right-hand side, we see that we can rewrite this as follows:

$$\sum_{Q\in\pi_M}(-1)^Q\exp\left(i\sum_{\lambda\in\mathbb{Z}_M}p_{Q\lambda}n_{R\lambda}+ip_{QM}L\right)\prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}}S\left(p_{Q\mu},p_{Q\nu}\right)=$$

$$\sum_{Q\in\pi_M}(-1)^Q\exp\left(i\sum_{\lambda\in\mathbb{Z}_M}p_{QR^{-1}\lambda}n_{\lambda}+ip_{QM}L\right)\prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}}S\left(p_{Q\mu},p_{Q\nu}\right)=$$

$$\sum_{Q\in\pi_M}(-1)^{QR}\exp\left(i\sum_{\lambda\in\mathbb{Z}_M}p_{Q\lambda}n_{\lambda}+ip_{QRM}L\right)\prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}}S\left(p_{QR\mu},p_{QR\nu}\right)=$$

$$\sum_{Q\in\pi_M}(-1)^{M-1}e^{ip_{Q1}L}(-1)^Q\exp\left(i\sum_{\lambda\in\mathbb{Z}_M}p_{Q\lambda}n_{\lambda}\right)\prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}}S\left(p_{QR\mu},p_{QR\nu}\right), \tag{4.8}$$

where we have used the fact that $RM = 1$ and have that $\pi_M R = \pi_M$. We can now compare the left- and right-hand side of (4.7) by considering the coefficients in front of the different exponents. Note that the exponents $\exp\left(i\sum_{\lambda\in\mathbb{Z}_M}p_{Q\lambda}n_\lambda\right)$ are assumed to be all different if the permutations Q are not the same, since the momenta p_i are all different. Therefore, they are functionally independent and we can equate the coefficients of all these exponents. This leads to

$$(-1)^{M-1}e^{ip_{Q1}L}\prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}}S\left(p_{QR\mu},p_{QR\nu}\right)=\prod_{\substack{\mu,\nu\in\mathbb{Z}_M\\ \mu<\nu}}S\left(p_{Q\mu},p_{Q\nu}\right) \tag{4.9}$$

for all $Q\in\pi_M$. We consider the equations for which $Q=(1j)$, $j\in\mathbb{Z}_M$. Most of the terms in the product on the left-hand side also occur on the right-hand side and if we assume them to be nonzero, we can divide them out. Namely, if we take $\mu<\nu$ such that $R\mu<R\nu$, then to $S\left(p_{QR\mu},p_{QR\nu}\right)$ on the left-hand side there corresponds a term on the right-hand side. After dividing these terms out, the remaining terms have $\nu=M$ on the left-hand side and $\mu=1$ on

[2] The derivation presented here is quite general: indeed, any spin chain for which the eigenfunctions have an asymptotic form as in equation (4.3) has the same functional form of the Bethe equations, i.e. this derivation would work for most spin chains with a known scattering matrix.

the right-hand side:

$$(-1)^{M-1} e^{i p_j L} \prod_{\substack{\mu \in \mathbb{Z}_M \\ \mu < M}} S\left(p_{(1j)R\mu}, p_j\right) = \prod_{\substack{\nu \in \mathbb{Z}_M \\ 1 < \nu}} S\left(p_j, p_{(1j)\nu}\right). \tag{4.10}$$

After reordering the product on the left-hand side to get rid of the R we get

$$(-1)^{M-1} e^{i p_j L} \prod_{\substack{\mu \in \mathbb{Z}_M \\ \mu > 1}} S\left(p_{(1j)\mu}, p_j\right) = \prod_{\substack{\nu \in \mathbb{Z}_M \\ 1 < \nu}} S\left(p_j, p_{(1j)\nu}\right) \tag{4.11}$$

which is equivalent to

$$(-1)^{M-1} e^{i p_j L} \prod_{\substack{\mu \in \mathbb{Z}_M \\ \mu \neq j}} S\left(p_\mu, p_j\right) = \prod_{\substack{\nu \in \mathbb{Z}_M \\ \nu \neq j}} S\left(p_j, p_\nu\right). \tag{4.12}$$

Therefore, if we assume that the products are nonzero, we end up with the following set of equations:

$$\begin{aligned} e^{i p_j L} &= (-1)^{M-1} \prod_{\substack{\mu \in \mathbb{Z}_M \\ \mu \neq j}} \frac{S\left(p_j, p_\mu\right)}{S\left(p_\mu, p_j\right)} = \prod_{\substack{\mu \in \mathbb{Z}_M \\ \mu \neq j}} \frac{1 - i(\phi(p_j) - \phi(p_\mu))}{i(\phi(p_\mu) - \phi(p_j)) - 1} = \\ & \prod_{\substack{\mu \in \mathbb{Z}_M \\ \mu \neq j}} \frac{\phi(p_j) - \phi(p_\mu) + i}{\phi(p_j) - \phi(p_\mu) - i}. \end{aligned} \tag{4.13}$$

These are the *Bethe equations* of our model and we will study them most thoroughly in the next chapters.

4.2 Antiferromagnetic Ground State

As a first application, we will use the Bethe equations to find the antiferromagnetic ground state of the infinite chain in the thermodynamic limit, following an approach first proposed by Hulthén in [28]. A similar study was published by Dittrich and Inozemtsev in [29].

We choose the interaction parameter of the hamiltonian as $J = -\frac{\sinh^2(\kappa)}{\kappa^2}$, such that the hamiltonian reads

$$H_L = \frac{1}{4} \sum_{\substack{j,k=1 \\ j \neq k}}^{L} \frac{\sinh^2(\kappa)}{\sinh^2 \kappa(j-k)} (\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_k - 1) \tag{4.14}$$

and define the length L of the spin chain to be even. Note that a rescaling does not alter the appearance of the eigenstates we found earlier. We can use the known formula

$$\log\left(\frac{1 - ix}{1 + ix}\right) = -2i \arctan x$$

to take the logarithm of the Bethe equations, choosing a proper branch, to arrive at

$$i\pi(M-1) + 2\pi i Q_j + i p_j L = 2i \sum_{\substack{k\in Z_M \\ k\neq j}} \arctan\left(\phi(p_k) - \phi(p_j)\right), \tag{4.15}$$

where the Q_j are integers. We can redefine $Q_j \to Q_j - \frac{M+L-1}{2}$ to combine all the integers into the Q_j, which are now either elements of $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ depending on whether $M + L - 1$ is even or odd. This leads to the following form

$$\frac{Q_j}{L} = \frac{\pi - p_j}{2\pi} - \frac{1}{\pi L} \sum_{\substack{i\in Z_M \\ i\neq j}} \arctan\left(\phi(p_j) - \phi(p_i)\right), \tag{4.16}$$

which we will name *Logarithmic Bethe Equations* (LBE). Due to the fact that ϕ is 2π-periodic, it is obvious from this equation that we can restrict the p_j to lie in the interval $[0, 2\pi[$ if we require them to be real in the first place. Moreover, all the p_j should be different to obtain a nontrivial wavefunction. We can also define a unique inverse of the function ϕ for all p_j if we omit $p_j = 0$. Since magnons with zero momentum carry zero energy in this model (since $\epsilon_0 = 0$), this implies that we only omit descendant states. Using the inverse of ϕ we can achieve a one-to-one correspondence between a set of numbers $\lambda_j \in \mathbb{R}$ and the p_j via the relation $\lambda_j = \phi(p_j)$, allowing us to rewrite the LBE in terms of the λ_j as

$$\frac{Q_j}{L} = \frac{\pi - \phi^{-1}(\lambda_j)}{2\pi} - \frac{1}{\pi L} \sum_{\substack{i\in Z_M \\ i\neq j}} \arctan\left(\lambda_j - \lambda_i\right). \tag{4.17}$$

The sets $\{\lambda_i\}$ that solve this equation for a set of noncoinciding Q_j's in the case $M = L/2$ correspond to antiferromagnetic eigenstates of the infinite Inozemtsev model. To be able to find these sets, it is most useful to first find the allowed range for the Q_j.

4.2.1 Restricting the Q_j

In the context of the previous section, we will prove the following theorem.

Theorem. There exists an $A \in \mathbb{R}$ such that for all $L > A$, the Q_j form a string of (half)-integers satisfying

$$Q_j = L/4 - 1/2(j-1).$$

Proof. We consider the right-hand side of equation (4.16) and look at its limiting behaviour. Consider the Q_j as a function of p_j, namely

$$Q_j(p_j) = \frac{\pi - p_j}{2\pi} L - \frac{1}{\pi} \sum_{\substack{i\in Z_M \\ i\neq j}} \arctan\left(\phi(p_j) - \phi(p_i)\right). \tag{4.18}$$

This function is differentiable and its derivative is clearly continuous on the open interval $(0, 2\pi)$. We would like to show that $Q'_j(p_j) < 0$ for all $p_j \in (0, 2\pi)$ when we restrict L to be greater than some number. To see whether this is can be done, we investigate the derivative of the summation in (4.18), which reads

$$\begin{aligned}\Xi(p_j) &= \sum_{\substack{i\in Z_M\\ i\neq j}} \frac{1}{1+(\phi(p_j)-\phi(p_i))^2}\phi'(p_j) \\ &= \sum_{\substack{i\in Z_M\\ i\neq j}} \frac{1}{1+(\phi(p_j)-\phi(p_i))^2}\left(\frac{1}{2i\pi\kappa}\zeta\left(\frac{i\pi}{2\kappa}\right)+\frac{1}{4\kappa^2}\wp\left(\frac{ip_j}{2\kappa}\right)\right). \qquad (4.19)\end{aligned}$$

This function is finite for all values $p_j \in (0, 2\pi)$ and its limits $p_j \to 0$ and $p_j \to 2\pi$ are equal, since Ξ is the derivative of an odd function around $p_j = \pi$, hence even around $p_j = \pi$. The limit $p_j \to 0$ is easily found using the known Laurent expansions for the Weierstraß functions (see Appendix E) and equals $-(M-1)$, since all the summands reduce to -1 in this limit. So $\Xi(0)$ is finite and we see that $Q'_j(0) = -\frac{L}{2\pi} + \frac{M-1}{\pi L}$, which is negative for all $M \leq L$. Therefore, we know that Q'_j is negative close to the endpoints of $[0, 2\pi]$. Also, since we know that Ξ is finite at both endpoints, we know it is bounded and continous on the compact interval $[0, 2\pi]$ and therefore has a maximum A. If we restrict $L > 2|A|$, we find that $Q'_j(p_j) \leq -\frac{L}{2\pi} + \frac{|A|}{\pi L} < 0$ for all $p_j \in (0, 2\pi)$. So for $L > 2|A|$, we know that $Q_j(p_j)$ has its maximum at $p_j = 0$ and its minimum is approached going towards $p_j = 2\pi$. These values are easily calculated: since Q_j is an odd function around $p_j = \pi$, $Q_{\text{min}} = -Q_{\text{max}}$. Furthermore, using the limit $\lim_{x\to\pm\infty}\arctan(x) = \pm\pi/2$,

$$Q_{\text{max}} = \frac{L}{2} - \frac{M-1}{\pi}\frac{\pi}{2} = \frac{L}{4} + \frac{L}{4M} = \frac{L}{4} + \frac{1}{2}.$$

So for the antiferromagnetic ground state, we have $-Q_{\text{max}} < Q_j < Q_{\text{max}}$, because both $p_j = 0$ and $p_j = 2\pi$ are excluded values for p_j. The total number of Q_j is $M = L/2$, which implies that the allowed Q_j form a string from $-Q_{\text{max}} + 1$ to $Q_{\text{max}} - 1$, i.e. are of the form

$$Q_j = L/4 - 1/2(j-1),$$

since the fact that the p_j should be different also implies that the Q_j should be different through equation (4.18). This is the usual assumption for the antiferromagnetic ground state and the above reasoning shows that this is also correct for this case. □

4.2.2 Passing to the Thermodynamic Limit

So we take $Q_j = L/4 - 1/2(j-1)$. We again consider equation (4.17). We fix the ratio $M/L = 1/2$ and consider the limit that $L \to \infty$, i.e. the thermodynamic limit. In this limit, the numbers Q_j/L becomes a continous variable x and the summation becomes an integral. Thereeofore, equatio (4.17) becomes

$$\frac{\pi - \phi^{-1}(\lambda(x))}{2\pi} = x + \frac{1}{\pi}\int_{-\infty}^{\infty} \arctan(\lambda(x)-\lambda(y))dy, \qquad (4.20)$$

where there is a relation between the x and the λ which can be formalized by defining $\sigma(\lambda) = \frac{dx}{d\lambda}$. This allows us to get rid of the x-variable all together and write

$$\frac{\pi - \phi^{-1}(\lambda)}{2\pi} = \int^{\lambda} \sigma(\lambda')d\lambda' + \frac{1}{\pi}\int_{-\infty}^{\infty} \arctan(\lambda - \lambda')\sigma(\lambda')d\lambda', \tag{4.21}$$

where the first integral is a primitive of σ. By differentiation with respect to λ we see that

$$\sigma(\lambda) = \frac{d\phi^{-1}}{d\lambda} + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma(\lambda')}{1 + (\lambda - \lambda')^2} d\lambda'.$$

We can solve this integral equation for σ via a Fourier transformation and find

$$\sigma(\lambda) = -\int_{-\infty}^{\infty} \frac{dk}{(2\pi)^2} \frac{e^{i\lambda k}}{1 + e^{-|k|}} \int_{-\infty}^{\infty} d\lambda'(\phi^{-1})'(\lambda')e^{-i\lambda k}, \tag{4.22}$$

where the prime on ϕ^{-1} indicates differentiation with respect to λ. Using this we can find the energy per site as given by the formula

$$e = \int_0^{2\pi} \epsilon_p dp = \int_{-\infty}^{\infty} \epsilon_{\phi^{-1}(\lambda)}\sigma(\lambda)d\lambda,$$

where we used the one particle energy ϵ_p from (3.27). Unfortunately, this Fourier integral cannot be solved exactly.

4.3 The Yang and Yang approach

Instead of looking at the case of half-filling ($M = L/2$), we also consider the case of arbitrary fixed density $n = M/L$. This means that instead of postulating $M = L/2$ and deriving a fixed set of Q_j's, we can also look at the occupation of the numbers Q_j/L, following a famous derivation by Yang and Yang [30]. For the antiferromagnetic ground state we only used the numbers $Q_j = L/4 - 1/2(j-1)$, which form an equidistant string symmetric around zero and exhaust the entire domain of the Q_j in the limit $L \to \infty$. Since now we do not specify $M = L/2$, the allowed range of the Q_j becomes larger and we can get strings with 'holes' in them. We associate particles to the Q_j that do occur in our solution and holes to those that do not. After going to the thermodynamic limit, this defines two densities, one for occupied numbers (particles) and one for unoccupied numbers (holes). We have a one-to-one correspondence between p's and λ's, so we can and will use the λ's to analyze this. To make this more precise, define the function

$$Z(\lambda) = \frac{\pi - \phi^{-1}(\lambda)}{2\pi} - \frac{1}{\pi L}\sum_{i \in \mathbb{Z}_M} \arctan(\lambda - \lambda_i), \tag{4.23}$$

usually called the *counting function*. We define holes to be at $\lambda_n \in \mathbb{R}$ such that $Z(\lambda_n) = Q/L$, where Q is an unoccupied integer. Similarly, particles are situated at those $\lambda_n \in \mathbb{R}$ such that $Z(\lambda_n) = Q/L$, where Q is an occupied integer. In the thermodynamic limit, this defines the density of particles $\rho(\lambda)$ and density of holes $\rho^h(\lambda)$. Note that the particle density corresponds exactly to the relation σ between x and λ in the previous section. In thermal equilibrium we must have that

$$Z(\lambda) = \int^{\lambda} \left(\rho(\lambda') + \rho^h(\lambda')\right) d\lambda'.$$

By differentiation, we see that

$$\rho(\lambda) + \rho^h(\lambda) = \frac{\left(-\phi^{-1}\right)'(\lambda)}{2\pi} - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\rho(\lambda')}{1+(\lambda-\lambda')^2} d\lambda'. \tag{4.24}$$

We can immediately write the energy per site using the particle density:

$$e = \int_{-\infty}^{\infty} \epsilon_{\phi^{-1}(\lambda)} \rho(\lambda) d\lambda, \tag{4.25}$$

If we can also define the entropy per site, we can actually give a formula for the Helmholtz free energy per site. Luckily, this is the case: If we look at an interval between λ and $\lambda + d\lambda$, we see that the logarithm of the number of orderings of particles and holes is given by

$$\log\left(\frac{(L(\rho(\lambda)+\rho^h(\lambda))d\lambda)!}{(L\rho(\lambda)d\lambda)!(L\rho^h(\lambda)d\lambda)!}\right), \tag{4.26}$$

which in the thermodynamic limit reduces to the entropy per site:

$$s = \int_{-\infty}^{\infty} \left(\left(\rho(\lambda) + \rho^h(\lambda)\right)\log(\rho(\lambda)+\rho^h(\lambda)) - \rho(\lambda)\log(\rho(\lambda)) - \rho^h(\lambda)\log(\rho^h(\lambda))\right) d\lambda \tag{4.27}$$

for the entropy per site, where we used the Stirling formula. Now we can give the free energy per site $f_H := e - Ts$. To find the state of thermal equilibrium for a state of particle density $n = \int_{-\infty}^{\infty} \rho(\lambda) d\lambda$, we should minimize the free energy under the constraint that n is constant. This amounts to minimizing the following functional, in which A is a Lagrange multiplier:

$$\begin{aligned} S := e - Ts - An = \int_{-\infty}^{\infty} \Big\{ &(\epsilon_{\phi^{-1}(\lambda)} - A)\rho(\lambda) - T\Big(\left(\rho(\lambda)+\rho^h(\lambda)\right)\log(\rho(\lambda)+\rho^h(\lambda)) \\ &-\rho(\lambda)\log(\rho(\lambda)) - \rho^h(\lambda)\log(\rho^h(\lambda))\Big)\Big\}. \end{aligned} \tag{4.28}$$

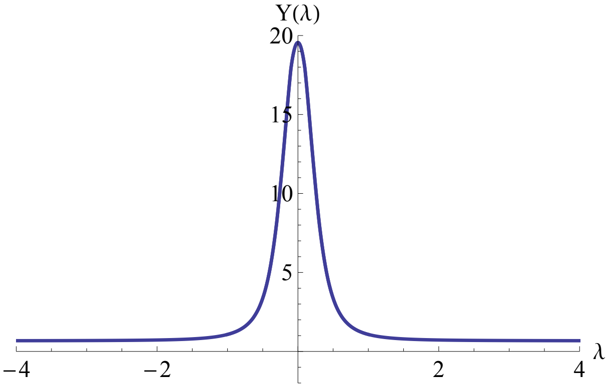
Due to equation (4.24), the variations with respect to the particle and hole densities are not independent and we can write for the variation of S

$$\delta S = \int_{-\infty}^{\infty} d\lambda \left(\epsilon_{\phi^{-1}(\lambda)} - A - T\log\left(\frac{\rho^h(\lambda)}{\rho(\lambda)}\right) + \frac{T}{\pi}\int_{-\infty}^{\infty} dq \frac{\log\left(\frac{\rho(q)+\rho^h(q)}{\rho^h(q)}\right)}{1+(\lambda-q)^2}\right)\delta\rho(\lambda). \tag{4.29}$$

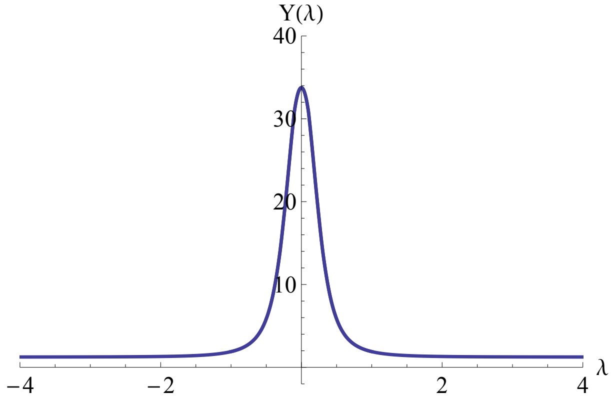
Therefore, in thermal equilibrium, when $\delta S = 0$, we must have, after defining $E(\lambda) := T\log\left(\frac{\rho^h(\lambda)}{\rho(\lambda)}\right)$

$$E(\lambda) = \epsilon_{\phi^{-1}(\lambda)} - A + \frac{T}{\pi}\int_{-\infty}^{\infty} dq \frac{\log\left(1+e^{-E(q)/T}\right)}{1+(\lambda-q)^2}. \tag{4.30}$$

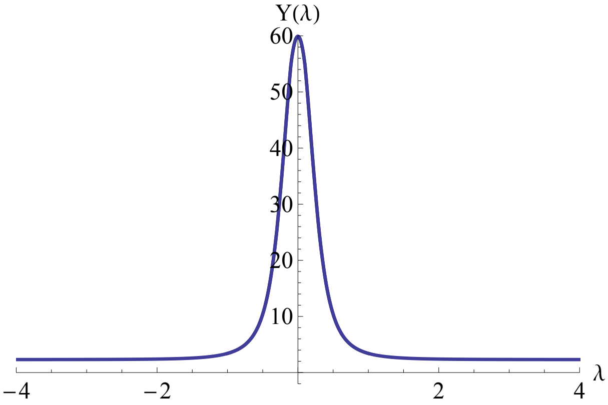
As in analogous cases [30], it is probably possible to prove that one can find $E(\lambda)$ by iteratively solving this equation. Numerical results have been obtained and are displayed in Figure 4.1. It



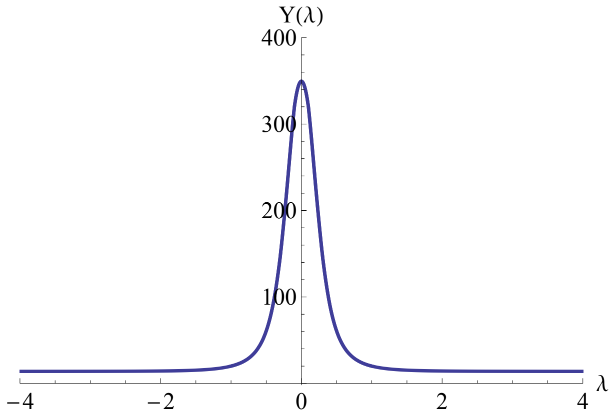
(a) $A = 0.1$, $T = 1$.



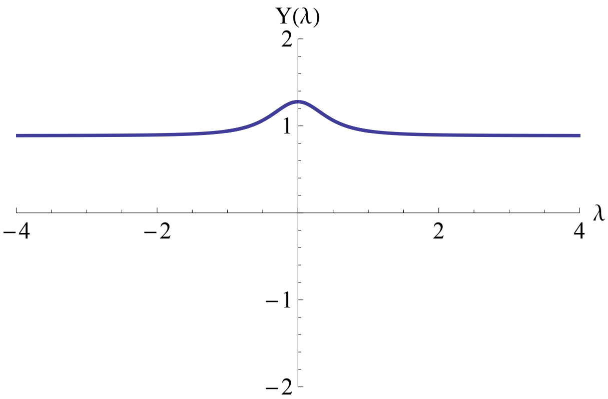
(b) $A = 1$, $T = 1$.



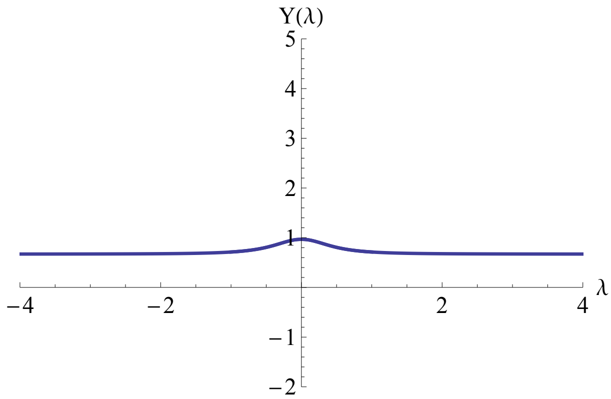
(c) $A = 2$, $T = 1$.



(d) $A = 5$, $T = 1$.



(e) $A = 5$, $T = 10$.



(f) $A = 1$, $T = 10$.

Figure 4.1: Plots of the numerical result for $Y(\lambda)$ for various values of A and T. It is clear from these that increasing the density with respect to the temperature increases the inhomogeneity in the particle-hole density.

is clear from this figure that increasing the density with respect to the temperature increases the inhomogeneity in the particle-hole density. It seems unlikely that an exact result exists, due to the complicated structure of $\epsilon_{\phi^{-1}(\lambda)}$. With the new definition for E, we can also rewrite equation 4.24 to read

$$\rho(\lambda)\left(1+e^{E(\lambda)/T}\right)=\frac{\left(-\phi^{-1}\right)'(\lambda)}{2\pi}-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\rho(\lambda')}{1+(\lambda-\lambda')^2}d\lambda'. \tag{4.31}$$

If we plug this expression back into the free energy per site $f = e - Ts$, we end up with the following rather simple looking expression

$$f_H = An + \frac{T}{2\pi}\int_{-\infty}^{\infty}\log\left(1+e^{-E(\lambda)/T}\right)\left(\phi^{-1}\right)'(\lambda)d\lambda = An - T\int_0^{2\pi}\frac{dp}{2\pi}\log\left(1+e^{-E(\phi(p))/T}\right). \tag{4.32}$$

Finally, there is standard notation in the physics literature, in which the variable $Y(\lambda) := e^{-E(\lambda)/T}$ is used. In terms of this variable, we have

$$-\log(Y(\lambda)) = \frac{\epsilon_{\phi^{-1}(\lambda)}-A}{T}+\frac{1}{\pi}\int_{-\infty}^{\infty}dq\frac{\log\left(1+Y(\lambda)\right)}{1+(\lambda-q)^2} \tag{4.33}$$

to solve for Y and the free energy is given as

$$f = An - T\int_0^{2\pi}\frac{dp}{2\pi}\log\left(1+Y(\phi(p))\right). \tag{4.34}$$

Since the best we can do at this stage is to numerically approximate Y, there is nothing more to gain from this analysis.

4.4 Conclusion

It is possible to obtain some insights into the thermodynamics of Inozemtsev's infinite spin chain by applying the methods of Hulthén and Yang and Yang. Indeed, we could find a Fourier integral for the particle density at half-filling of the antiferromagnetic ground state, but solving it analytically seems impossible. By generalizing to arbitrary fillings, we were able to derive a set of integral equations that govern the particle to hole ratio. These equations were solved numerically. We will now shift our attention and start to study the multi-particle bound-states of this model, which could perhaps shed some light on the mechanisms involved to reach thermodynamic equilibrium. This analysis is much more involved than the one followed in this chapter and contains several very interesting features.

Chapter 5

The Spectrum of Inozemtsev's Infinite Spin Chain

5.1 Motivation

In the previous chapter we have been investigating the thermodynamic behaviour of Inozemtsev's infinite spin chain by looking at the real solutions of the Bethe equations in the thermodynamic limit. This led to an unsolvable Fourier integral for the energy of the antiferromagnetic ground state. Moreover, we found integral equations that govern the behaviour of the particle density on the spin chain at fixed density n, but due to the complicated form of these equations, we were unable to consider the spectrum of excitations. There is, however, still another way to find out more about the spectrum of this spin chain: we can consider solutions of the Bethe equations consisting of complex momenta. These solutions can be interpreted as bound states and can be used to study the thermodynamic behaviour of the model by invoking the unproven *String Hypothesis*, which stipulates that the thermodynamical behaviour of an integrable model is completely determined by the behaviour of its *strings* [10]. These strings are the asymptotic (or infinite-length) bound-state solutions of the Bethe equations of the model and can usually be depicted as strings, hence their name. For now, we will refrain from going too deep into the details of the string hypothesis, since its application for this particular case will at the very least be extremely difficult.

Interestingly, there are more reasons than just an interest in the thermodynamics of this model to study its bound states. Indeed, the wavefunctions of the bound states are the starting point to form a basis of eigenstates of the Hilbert space of the infinite spin chain and hence give access to the eigenvalues of the hamiltonian. There are various applications of spin chains in which the eigenvalues of the spin-chain hamiltonian play a crucial role.

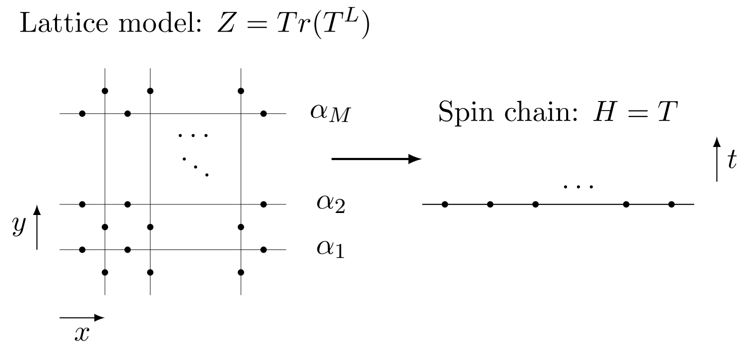


Figure 5.1: One can map certain (2+0)-dimensional lattice models onto (1+1)-dimensional spin chains and find the partition function Z by analyzing the spectrum of the spin-chain hamiltonian T.

For example, (2 + 0)-dimensional lattice models can sometimes be mapped to a (1 + 1)-dimensional spin chain [2] (this is illustrated in Figure 5.1). This allows one to write the partition function Z of the lattice model in the following form:

$$Z = \mathrm{Tr}\left(T^M\right),$$

where M is the number of rows in the lattice and T is called the *transfer matrix* that relates rows in the lattice and can be interpreted as a spin-chain hamiltonian. By calculating the spectrum of T, for example using the Bethe Ansatz, one can actually access the exact form of the partition function Z, because it depends only on the eigenvalues of T. A great example of applying this method is the six-vertex model, which can be mapped onto the Heisenberg XXZ spin chain [2].

A more recent application of spin chains – and in particular of Inozemtsev's infinite spin chain – can be found in the context of the AdS/CFT-correspondence [31]: the primary operators of a conformal field theory (CFT) are eigenfunctions of the dilatation operator D. These operators are given by

$$\mathcal{O} = Tr\left(\Phi_{i_1}\Phi_{i_2}\cdots\Phi_{i_L}\right), \tag{5.1}$$

where the Φ's are fields of the underlying gauge theory. Minahan and Zarembo showed in [31] that these operators can be thought of as states of a spin chain and the dilatation operator as a spin-chain hamiltonian. Therefore, to find the primary operators of the CFT, one only needs to find the spectrum of the spin chain. However, this might be very difficult in practice, because the associated spin chain can be complicated and one has to resort to approximations to obtain results. In the particular case of $\mathcal{N} = 4$ super Yang-Mills theory, the dilatation operator can be expanded in the form

$$D = \sum_{n=0}^{\infty} D_{\text{n-loop}},$$

where the $D_{\text{n-loop}}$ are the contributions at the n-th loop order. Serban and Staudacher showed in [32] that up to third order, the spectrum of D coincides with the spectrum of Inozemtsev's infinite spin chain. The question what the exact spectrum of this spin chain actually is remained open and we will try to answer it in this chapter by characterizing all the bound states.

5.2 Bound-State Solutions

Bound states are modelled by wavefunctions that vanish in the limit that the spatial distance between the particles tends to infinity. These wavefunctions are parametrized by sets of M complex momenta $\{p_j\}$ such that their sum is real, i.e.

$$\sum_{j=1}^{M} p_j \in \mathbb{R}$$

and that the corresponding energy is also real. Since we can associate a real momentum to the collective of particles associated to this set of momenta and these particles must be relatively close together, bound states can be thought of as quasi-particles moving along the spin chain. In the following we will first classify all the asymptotic bound-state solutions to the Bethe equations and then analyze their structure to see whether these solutions can tell us something about the thermodynamics of Inozemtsev's spin chains. In addition, we will try to conclude whether these solutions describe the spectrum of the hamiltonian of this spin chain.

5.2.1 Properties of Asymptotic Bound-State Solutions

The solutions we are looking for are sets of M complex momenta $\{p_j\}$ that form a bound state and also solve the Bethe equations we derived earlier in (4.13)

$$e^{ip_j L} = \prod_{\substack{n=1 \\ n\neq j}}^{M} \frac{\phi(p_j) - \phi(p_n) + i}{\phi(p_j) - \phi(p_n) - i}, \tag{5.2}$$

in the limit that $L \to \infty$. This means the following: for each p_j, there is a sequence $\left(p_j^{(L)}\right)$, indexed by the length L, that solves the Bethe equations for finite L and has limit p_j as $L \to \infty$. To avoid clutter, we will not write the superscript (L) and will simply talk about p_j as $L \to \infty$. In order to find all the solutions sets, we will first derive a set of properties that every solution must obey. Suppose we have a solution set $S = \{p_j\}$. We treat three cases:

Case 1. Suppose $p_1 \in S$ has $\text{Im}(p_1) > 0$. The left-hand side of the Bethe equation for p_1 will tend to zero as $L \to 0$, which implies that the right-hand side should also tend to zero. This means that at least one of the terms in the product should tend to zero, which means that there must exist a $p_i \in S$ such that $\phi(p_1) - \phi(p_i) + i \to 0$. This p_i must have $\text{Re}(\phi(p_i))$ =$\text{Re}(\phi(p_1))$ and $\text{Im}(\phi(p_i))$ =$\text{Im}(\phi(p_1)) - 1$.

Case 2. By a similar argument, we know that if $\mathrm{Im}(p_1) < 0$, there must exist $p_i \in S$ such that $\phi(p_1) - \phi(p_i) - i \to 0$. Indeed, since the left-hand side of the Bethe equation for p_1 diverges, so must the right-hand side, which implies that one of the denominator in the product must vanish.

Case 3. The case in which p_1 is real is special and is treated in Appendix D; for a consistent solution to the Bethe equations, the real momentum p_1 should be such that $\mathrm{Re}(\phi(p_1)) = \mathrm{Re}(\phi(p_j))$ for any j.

From these considerations we see that the properties of the $\{p_j\}$ that determine whether they form a solution to the Bethe equations are their images under ϕ (thus $\{\phi(p_i)\}$) and the sign of their imaginary parts. We will use this fact to simplify the problem of finding solutions, but first we need some notation.

In the cases discussed above, we also say colloquially that p_i *helps* to satisfy the Bethe equation of p_1. Inspired by our observation in the previous paragraph, we introduce a graphical notation to write down solutions. In Figure 5.2, we depict the first two cases treated above. If p_i helps p_1, we draw an arrow from the point $\phi(p_i)$ to the point $\phi(p_1)$ in the image space of ϕ, which we will name *ϕ-space*. The points in ϕ-space are simply *image points*[1]. To simplify even further,

Figure 5.2: A graphical way of depicting the simplest dependence between momenta in a solution S of the Bethe equations.

we represent the points in ϕ-space by $+$, $-$ or a 0, depending on whether the imaginary part of p_1, p_i is positive, negative or zero respectively. This leads to the following allowed building blocks for a configuration of a solution in ϕ-space (Figure 5.3). Note that these building blocks preserve exactly all the relevant aspects of the p_j. We have therefore split the problem into two parts: we can first investigate what the allowed configurations of the building blocks in ϕ-space are and consequently try to find the momenta sets that correspond to such a configuration. In principle we do not even need information about ϕ to study the allowed configurations, although in general we can expect that some configurations might not have a corresponding set of momenta because of the exact behaviour of ϕ.

[1]One could be tempted to call these points rapidities and their space the rapidity space, but since the function ϕ is not bijective, this would be an abuse of nomenclature.

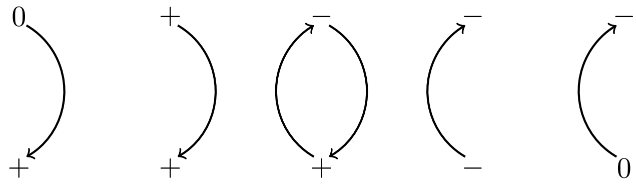


Figure 5.3: The five allowed building blocks.

Our first task is therefore to classify all the allowed configurations in ϕ-space. From the analysis of the cases, we can deduce that a configuration of a M-particle solution is such that

- all its image points ϕ_j have the same real part.
- it is built up from the 5 building blocks given in Figure 5.3.
- it consists of exactly one connected component. We will call (a component of) a configuration *connected* if and only if between any two points there exists a path along the (undirected) arrows connecting the two points. To avoid overcounting, we consider only those configurations that consist of one connected component. This will allow us later to build all the solutions.
- the image points form an equidistant string with distance i, i.e. of the form $\{\phi_r + (\phi_i + j - 1)i \,|\, 1 \leq j \leq m\}$, where $m \in \mathbb{N}$ and $\phi_r, \phi_i \in \mathbb{R}$.
- it consists of $M - k$ signs (plusses, minusses), each with at least one arrow in its direction, and k zeroes, where $k = 0, 1$. This follows from the fact that the equation $\phi(p) = c \in \mathbb{R}$ has a unique solution for $p \in [0, 2\pi[$.

The treatment given above is quite standard (see e.g. [33]), but for our case incomplete. It will be necessary to take the rate with which all the various limits ($L \to \infty$) are reached into account. We associate to each p_j a $\delta_j > 0$ that indicates how fast the solution converges to p_j in the following sense: the sequence we associate to each p_j gives rise to the sequence in ϕ-space $\left(\phi\left(p_j^{(L)}\right)\right)$ with limit ϕ_j. Since the left-hand side of the Bethe equations converges to 0 (or diverges to infinity) exponentially, the right-hand side should do the same, implying that the image point sequence should converge exponentially. We say that for large L

$$\phi\left(p_j^{(L)}\right) = \phi_j + \mathcal{O}\left(e^{-\delta_j L}\right).$$

Although we saw that it is possible to classify all the allowed configurations in ϕ-space without considering ϕ at all, we will now first study ϕ, to make sure that to every allowed configuration there corresponds at least one set of momenta.

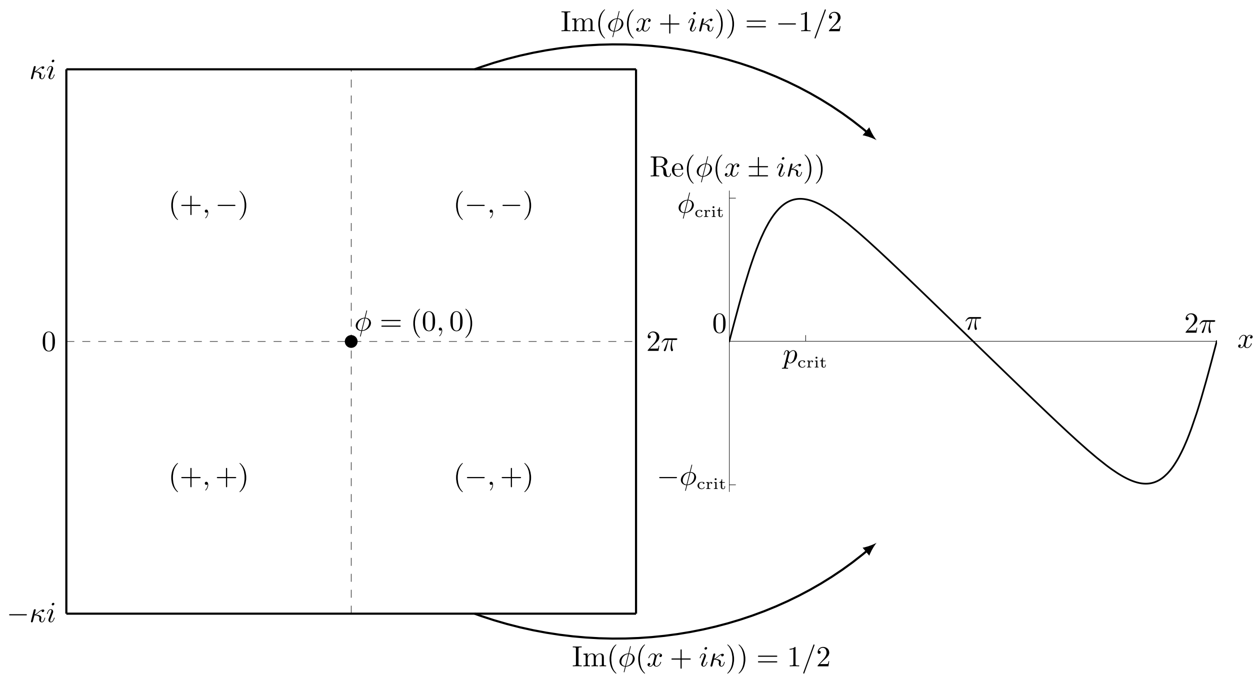


Figure 5.4: The range of ϕ as it distributes over the domain: The signs in brackets indicate the sign of $(\mathrm{Re}(\phi), \mathrm{Im}(\phi))$ in that part of the domain. The behaviour of the real part of ϕ on the top and bottom domain boundary is explicitly shown in the plot on the right.

5.2.2 Characterization of ϕ

We are considering the function $\phi : [0, 2\pi[\oplus i\mathbb{R} \to \mathbb{C}$ defined by

$$\phi(p) = \frac{p}{2i\pi\kappa}\zeta\left(\frac{i\pi}{2\kappa}\right) - \frac{1}{2i\kappa}\zeta\left(\frac{ip}{2\kappa}\right). \tag{5.3}$$

This function is odd and quasiperiodic, satisfying

$$\phi(p) = -\phi(-p), \qquad \phi(p + 2\pi) = \phi(p), \qquad \phi(p + 2i\kappa) = \phi(p) - i,$$

which means that its behaviour on the *fundamental region* $[0, 2\pi[\oplus i[-\kappa, \kappa]$ completely determines its behaviour on $[0, 2\pi[\oplus i\mathbb{R}$. In Appendix B, we show using the argument principle and the fact that ϕ has one pole in the fundamental region at $z = 0$ that $\phi : [0, 2\pi[\oplus i[-\kappa, \kappa] \to \mathbb{C}$ is almost bijective[2]. It is certainly surjective, but attains some values twice, namely those $\alpha \in \mathbb{C}$ for which $\mathrm{Im}(\alpha) = \pm 1/2$ and $-\phi_{\text{crit}} < |\mathrm{Re}(\alpha)| < \phi_{\text{crit}}$, where $\phi_{\text{crit}} > 0$ depends on the parameter κ. The preimages of these values lie on the top and bottom boundary of the fundamental region, i.e. where $\mathrm{Im}(p) = \pm\kappa$. This behaviour is illustrated in Figure 5.4. Note in particular that in the fundamental region it is true that if $\mathrm{Im}(p) < 0$, $\mathrm{Im}(\phi(p)) > 0$ and vice versa.

[2]With *almost bijective* we mean that the restriction of ϕ to a domain differing from the fundamental region by a set of measure zero is bijective.

Because ϕ is almost bijective on the fundamental region and quasi-periodicity allows us to find all values of ϕ from its behaviour on that region, we introduce a partition of the domain of ϕ into regions as follows: region 1 is the fundamental region defined before and for $n \geq 2$, region n is defined to consist of all those $p = p_r + p_i i \in [0, 2\pi[\oplus i\mathbb{R}$ with $(n-1)\kappa < |p_i| \leq n\kappa$ (see also Figure 5.5). It is most interesting that the restriction $\phi\big|_i$ to region i is almost bijective for every

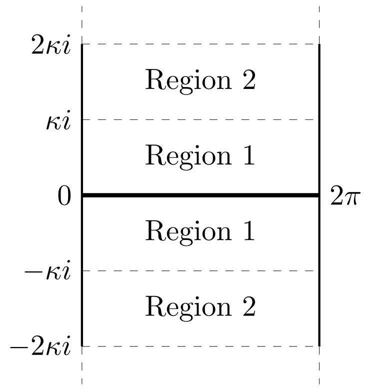


Figure 5.5: The complex strip and its partition into regions. The dashed lines at $n\kappa i$ belong to the inner regions.

i. In particular, all the $\phi\big|_i$ are surjective. This implies that for every image point α, there is a solution $p \in$ Region i such that $\phi(p) = \alpha$ for every i, i.e. every image point α has a countably infinite set of preimages. For now, this is enough information on ϕ, but more information can be found in Appendix C.

5.2.3 Building Solutions

Now we know more about ϕ, we can start to classify all the allowed configurations in ϕ-space and find all the associated sets of momenta. We start from a set of $m \leq M$ image points

$$\{\phi_j = \phi_r + (\phi_i + j - 1)i | \phi_{r,i} \in \mathbb{R}, 1 \leq j \leq m\}$$

forming an equidistant string. We will say that a momentum p that has $\phi(p) = \phi_j$ in this set is associated to *level j*. For all the known models, such as the Heisenberg XXX model or the Hubbard model, the relation ϕ between the image points and the momenta is a bijective function and thus this set of image points specifies a unique set of momenta, a string solution [33]. Since our ϕ is not bijective – and in fact all the image points have an infinite number of preimages – the set specified by this configurations of image points is far from unique. Moreover, we can associate several momenta to each image point, which makes the analysis much more complicated and allows for more complicated configurations. It is therefore really necessary to investigate which configurations of the image points are allowed, using the graphical language we introduced before.

String Solutions

As a first case, let us restrict ourselves to the (traditional) string solutions, i.e. solutions with M levels and M momenta. For even M, the only allowed sign configuration consists of m_p plusses and m_m minusses, both nonzero, such that $m_p + m_m = M$ and the plusses form the lower part of the string (see Figure 5.6). This is simply because there cannot be a connection between a plus and a minus in a configuration where the plus sits at level $j+1$ while the minus is at level j (see Figure 5.3). For odd M, the configuration is the same, except that then the topmost plus is changed into a zero. The treatment of solutions containing a real momentum can be completely contained in the treatment for solutions without a real momentum as we will see in the next section. We will therefore not treat them explicit in the remainder. Thus Figure 5.6 specifies all the allowed string configurations we want to consider now, along with an integer M and two real parameters: ϕ_r specifies the common real part of all the image points and ϕ_i specifies the imaginary part of the sign lowest in the configuration.

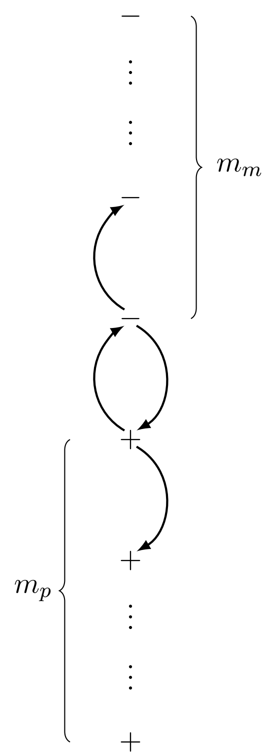


Figure 5.6: The general structure of a string-solution for even M.

The next step is to find all the sets of momenta that correspond to such a configuration. As we saw in the previous paragraph, ϕ is such that we can find both positive and negative momenta as preimages of any set of image points. By construction, all the sets of momenta corresponding to this configuration satisfy the Bethe equations, but we wanted our solutions to have extra properties. We wanted the momenta in the solution to sum up to a real value and the sum of the energies should also be real. The easiest way to make sure that these restrictions are met is to make the set self-conjugate[3]. The only values of m_m and ϕ_i for which this can hold are $m_m = M/2$ and $\phi = -\frac{M+1}{2}$. If we then choose momenta corresponding to the plusses and add all the complex conjugate momenta to the set, the result is a true bound state. So even in this restricted case there are infinitely many solutions: we can choose any value for ϕ_r and choose $M/2$ regions in which we want to the momenta to lie in. An example of this string solution can be seen on the left in Figure 5.7.

These self-conjugate solutions are very special, but also very important. We will return to them in Section 5.3.3 and see that these solutions can be related to the string solutions of the Heisenberg XXX model. But contrary to the Heisenberg XXX model, we can build more solutions. We will treat a first extension now.

[3]A solution S is *self-conjugate* if $p \in S$ implies that $\overline{p} \in S$.

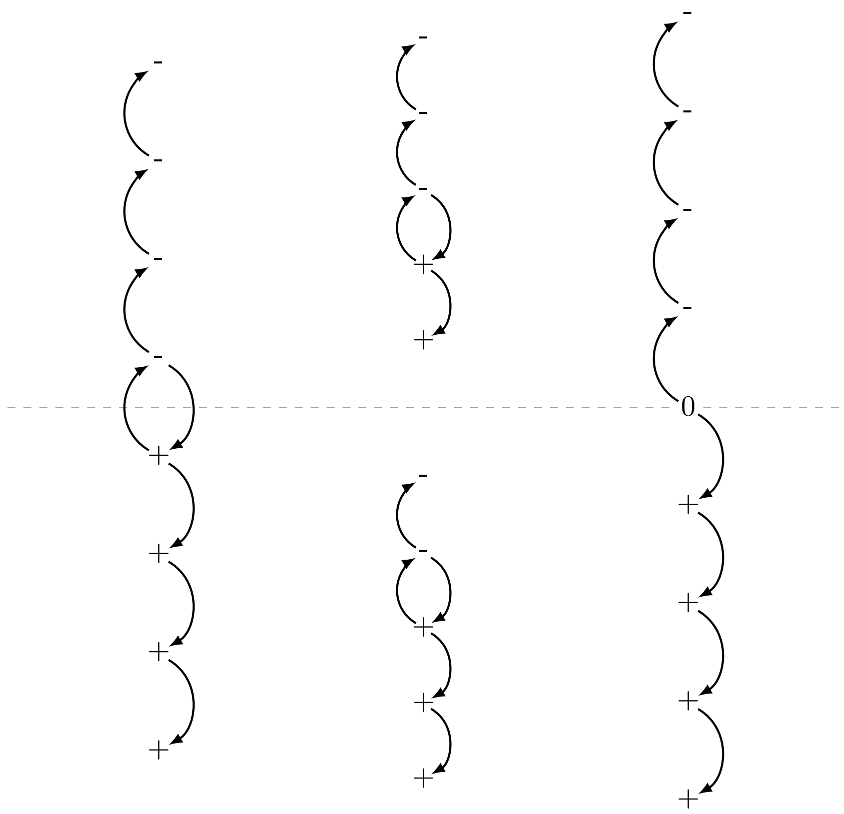


Figure 5.7: Configurations of string solutions. The left configuration corresponds to a self-conjugate string solution. The string solution in the middle is the first extension we are considering and consists of two connected components. For completeness, we also give the configuration of a string solution for odd M, in this case $M = 9$, with a real momentum.

A First Extension

Starting from the string configuration with in the previous paragraph, we can try to find more solutions. We consider more values for ϕ_i for a fixed value of even M and some choice of $1 \leq m_m < M$. Specifically, suppose that $\phi_i \in \mathbb{R}^*_M$, where

$$\mathbb{R}^*_M := \mathbb{R} \setminus \left\{ \frac{3}{2} - M, \frac{1}{2} - M, \cdots, -\frac{1}{2} \right\}. \tag{5.4}$$

Then the set $\{\overline{\phi_j}\}$ of complex conjugates of the image points obeys

$$\{\overline{\phi_j}\} \cap \{\phi_j\} = \emptyset.$$

This allows for the following construction: choose $\phi_r \in \mathbb{R}$ and $\phi_i \in \mathbb{R}^*_M$ and select momenta for each of the signs in the configuration. Then simply add the complex conjugates of these momenta to the solution set. By construction, all of the momenta in the resulting set will be distinct. This results in a configuration as depicted in the middle of Figure 5.7. As one can see, it consists of two connected components. This is precisely the case we referred to in the previous section and we can now make this more precise: all the solutions consisting of two connected components are such that the components and the associated momenta are complex conjugates. Note that for every allowed value of ϕ_i it is easy to determine how we can build a proper bound state from the given configuration: either we do not have to add anything or we add all the complex conjugates of the momenta. To simplify the discussion, we will not mention this doubling in the remainder, but always consider $\phi_i \in \mathbb{R}_M$, where we define

$$\dot{\mathbb{R}}_M = \mathbb{R} \setminus \left\{ \frac{3}{2} - M, \frac{1}{2} - M, \cdots, -\frac{M+3}{2}, -\frac{M-1}{2}, \cdots, -\frac{1}{2} \right\}. \tag{5.5}$$

This new type of solutions has much more freedom than the usual string solutions: we can choose any value of $1 \leq m_m < M$ and allow $\phi_i \in \mathbb{R}_M$, which is an uncountable set. To make this discussion slightly more tangible, we treat an example.

Example. Consider some four-particle solutions of the Bethe equations with $\phi_r = 0.6$. We let $\text{Im}(p_j) > 0$ for $j = 1, 2$ and $p_3 = \overline{p_1}$, $p_4 = \overline{p_2}$ and set $\kappa = 1.26$ arbitrarily. If we choose $p_1, p_2 \in$ region 1, the solution is

$$\{0.704 + 1.26i, 0.202 + 0.563i, 0.704 - 1.26i, 0.202 - 0.563i\},$$

with energy $E_4 = -1.279$ (after setting $J = 1$), while if we set $p_1 \in$ region 4 and $p_2 \in$ region 6, we end up with

$$\{0.202 + 4.478i, 0.202 + 6.997i, 0.202 - 4.478i, 0.202 - 6.997i\},$$

with energy $E_4 = -13.634$. As one can see, both these solutions consist of two connected components and are self-conjugate. Moreover, we had two choices for the momentum regions for both of these solutions. By choosing Region 1 in the first and Region 4 and Region 6 in the second, we ended up with two different solutions.

Now we have treated the string solutions in detail, we are ready to try to tackle the general problem of classifying the allowed configurations and their corresponding momenta sets.

Tree Solutions

Until now, we allowed only one momentum at each level. However, we can use the nonbijectivity of ϕ to associate any number of momenta to any particular level. Moreover, in many cases we can associate momenta with both positive and negative imaginary part to each level (see Figure 5.8). We will call solutions which are not of string type, i.e. have at least one level to which we associate more than one momentum, *tree solutions*. To be able to draw these solutions, we place the signs belonging to the same image point on a horizontal line (the level). This means that we can no longer regard ϕ-space to be the configuration space. Also, to avoid clutter, we

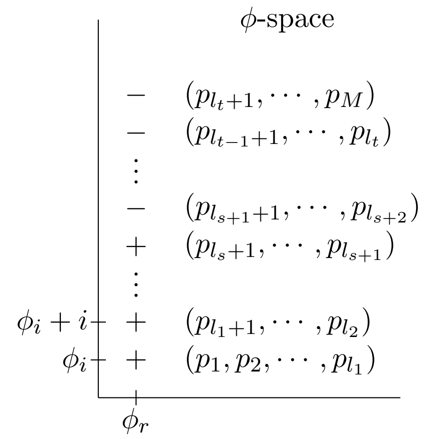


Figure 5.8: Due to the nonbijectivity of ϕ, we can any number of momenta to any sign in the configuration. We omitted the arrows to avoid clutter and annotated the position of ϕ_r and ϕ_i.

wil usually omit the arrows from now on. In this way, we that a general tree solution is of the form depicted in Figure 5.9, level j contains P_j plusses and M_j minusses, where the total number of levels is now m. These numbers satisfy

$$\sum_{j=1}^{m} (P_j + M_j) = M. \tag{5.6}$$

Thus in Figure 5.9, there are P_1 momenta with positive imaginary part associated to the image point $\phi_r + \phi_i i$ with smallest imaginary part, P_2 momenta with positive imaginary part to the image point $\phi_r + (\phi_i + 1)i$ and M_2 momenta with negative imaginary part, etc. Although we

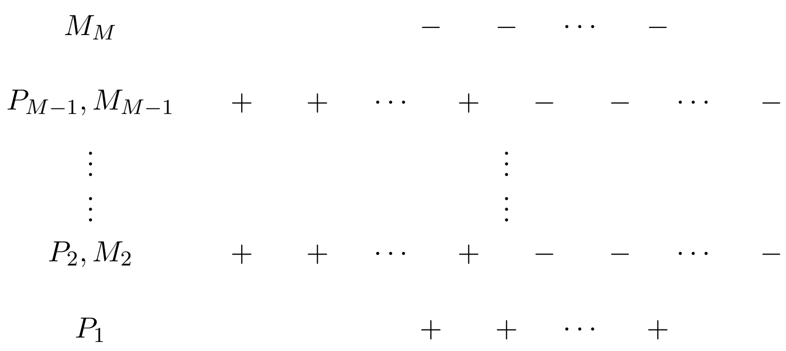


Figure 5.9: The sign configuration of a generic tree solution. We have omitted the arrows for clarity and have written the number of plusses and minusses on each level on the left.

have omitted the arrows, it is not difficult to deduce the dependencies of all the signs from the observations made before: a positive momentum on level j receives help from all the signs on level $j + 1$, whereas a negative momentum on level j receives help from all the signs on the level $j - 1$. From now on, we will leave it to the reader to confirm that the sign configurations shown indeed satisfy the arrow rules. Finally, we can treat the solution sets containing a real momentum within the same picture: any allowed sign configuration with plusses and minusses with $P_1 = 1$ can be used to define a solution with a real momentum by replacing the plus on the lowest level by a single zero and use complex conjugation to form the complete solution. This does not alter the convergence properties of the solution and still obeys all the rules we set up. Since their treatment follows directly from the configurations with plusses and minusses, we will leave the existence of these solutions implicit in the remainder.

To sum up, tree solutions are specified by the following: we fix an integer M and an $m \leq M$ and choose a configuration conform Figure 5.9. Then we choose $\phi_r \in \mathbb{R}$ and $\phi_i \in \mathbb{R}_M$ and choose regions for all the momenta. This is a lot of freedom to build solutions with. Let us treat some examples.

Example. Since tree solutions are a new phenomenon, let us give some examples of possible solutions of this type. Two examples of configurations are depicted in Figure 5.10. To find momentum sets corresponding to these configurations, we set $\kappa = 1.26$ and $\phi_r = 1.4$ arbitrarily. For example (a), we can choose $\phi_i \in \mathbb{R}^*_M$, so let us pick $\phi_i = 0.89$ arbitrarily. We choose regions $2, 4, 6$ for the plus signs and region 1 for the minus sign. We must add the complex conjugates to make the solution a bound state and we end up with

$$\{0.244 + 2.175i, 0.132 + 4.761i, 0.080 + 7.3334i, 0.244 - 0.345i\} + \text{complex conjugates},$$

with energy $E_8 = -1.57.234$ (again $J = 1$).

For example (b), we can also choose $\phi_i \in \mathbb{R}^*_M$ and we pick $\phi_i = -0.4$. For the lower two plus signs we use regions 1 and 2 and for the one on level 3 we choose region 2 as well. We use region 1 for the momenta of all the minus signs. We again have to add complex conjugates to end up

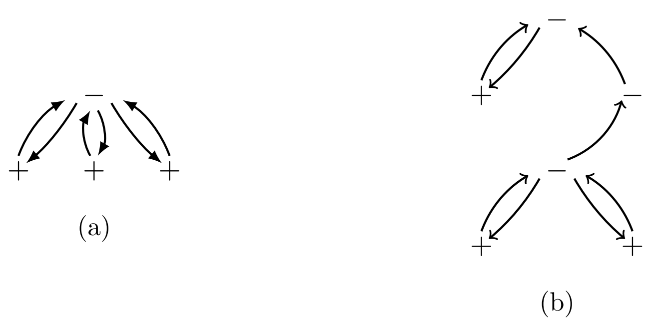


Figure 5.10: Two examples of tree configurations.

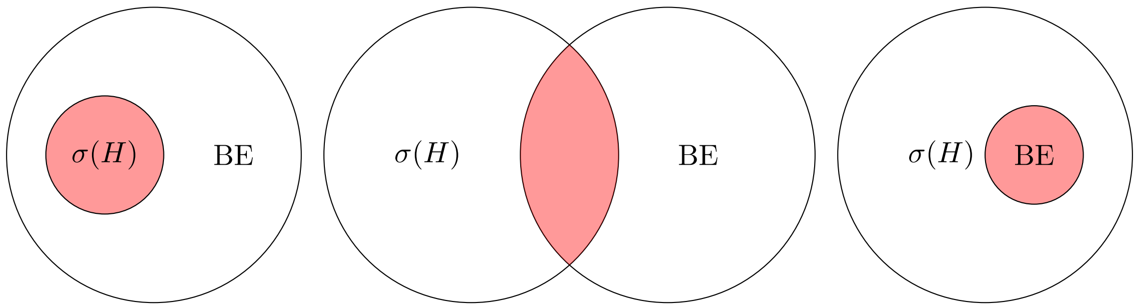


Figure 5.11: The spectrum $\sigma(H)$ of a spin-chain hamiltonian can lie entirely inside the solutions to the Bethe equations (BE) (left), partially overlap (middle) or contain BE as a subset (right). The omitted limiting cases in which $\sigma(H) = BE$ are of course also possible.

with a bound state. The solution is

$$\{0.687 + 0.213i, 0.618 - 2.232i, 0.156 + 2.222i, 0.618 - 0.288i, 0.300 - 0.361i, 0.156 - 0.298i\}$$
$$+\text{complex conjugates},$$

with energy $E_8 = 0.211$.

5.3 Pruning the Set of Solutions

At first glance we are done now: to any possible configuration of signs of the type illustrated in Figure 5.9 we can associate infinitely many sets of momenta and all of these sets form solutions to the Bethe equations. However, the resulting set of solutions is gigantic and most likely not all solutions are physical, i.e. not all of these sets of momenta parametrize a wavefunction that is an eigenfunction of Inozemtsev's infinite spin-chain hamiltonian. This is illustrated in Figure 5.11, in which we see that the set BE of solutions to the Bethe equations can be much larger than the spectrum we are trying to find. The set of solutions is parametrized by two real parameters ϕ_r and ϕ_i, an integer M, M choices of regions and a sign configuration of M symbols. The bound-state solutions of the Heisenberg XXX model are parametrized by only one real parameter ϕ_r and an integer M [18]. Since we know that in the limit $\kappa \to \infty$, the *Heisenberg limit*, Inozemtsev's infinite spin chain degenerates into the Heisenberg XXX spin chain, it seems reasonable to expect that the physical solutions to their respective Bethe equations are also related. More directly, we see that in the Heisenberg limit, the Bethe equations of Inozemtsev's infinite spin chain actually become of exactly the same form as the Bethe equations of the Heisenberg XXX spin chain [18]:

$$e^{ip_jL} = \prod_{\substack{n=1\\n\neq j}}^{M} \frac{\phi(p_j) - \phi(p_n) + i}{\phi(p_j) - \phi(p_n) - i} \longrightarrow e^{ip_jL} = \prod_{\substack{n=1\\n\neq j}}^{M} \frac{\cot(p_j/2) - \cot(p_n/2) + 2i}{\cot(p_j/2) - \cot(p_n/2) - 2i}. \tag{5.7}$$

However, the existence of the extra real parameter ϕ_i in particular seems to spoil this at present: many solutions are mapped onto the same solution in the Heisenberg limit, which means that the limit of the set of solutions of Inozemtsev's infinite spin chain does equal the physical set of solutions to the Bethe equations of the Heisenberg XXX spin chain.

Additionally, the existence of two real parameters also suggests that the set of eigenfunctions belonging to the momenta from the set of solutions might be overcomplete, i.e. they form an overcomplete basis of the Hilbert space. Therefore, we should analyze this huge set of solutions to see whether there are subsets of solutions that we can discard. As a first step, we will take a closer look at the convergence rates of the tree solutions.

5.3.1 Convergence of tree solutions

As we have seen in Section 5.2.1, the Bethe equation associated to a plus sign on level j is satisfied if the right-hand side goes to 0, which is achieved by the existence of signs on level $j+1$, which we dubbed *helping* signs. On the other hand, the terms on the right-hand side of the Bethe equation associated to the signs on level $j-1$ go to infinity. We call these signs *counteracting*. It seems that the right-hand side of the Bethe equation has the right limit only if the terms associated to helping signs converge faster than those associated to the counteracting signs. However, for minus signs, the situation is exactly opposite: the signs on level $j-1$ are helping, those on level $j+1$ are countaracting. With this idea in mind, we will now analyze how fast all the momenta should reach their limiting values as $L \to \infty$.

Call the momenta associated to the n_kth $\pm$ on level k $p^{(\pm)}_{k,n_k}$. Their convergence rates are denoted by $\delta^{(\pm)}_{k,n_k}$. Consider the n_jth plus sign on level j in a tree solution. The Bethe equation of the momentum associated to this plus sign reads

$$e^{ip^{(+)}_{j,n_j} L} = \prod_{\substack{k=1 \\ k\neq j}}^{M} \frac{\phi_{j,n_j} - \phi_k + i}{\phi_{j,n_j} - \phi_k - i}, \tag{5.8}$$

where we defined $\phi_{j,n_j} := \phi(p_{j,n_j})$ and ϕ_k is the rapidity belonging to level k. As $L \to \infty$, the left-hand side converges to 0. Most of the terms on the right-hand side converge to finite values and are irrelevant for the behaviour. The interesting terms are those belonging to level $j \pm 1$. They form the product

$$\underbrace{\frac{\phi_{j,n_j} - \phi_{j+1} + i}{\phi_{j,n_j} - \phi_{j+1} - i} \cdots \frac{\phi_{j,n_j} - \phi_{j+1} + i}{\phi_{j,n_j} - \phi_{j+1} - i}}_{P_{j+1}+M_{j+1}} \underbrace{\frac{\phi_{j,n_j} - \phi_{j-1} + i}{\phi_{j,n_j} - \phi_{j-1} - i} \cdots \frac{\phi_{j,n_j} - \phi_{j-1} + i}{\phi_{j,n_j} - \phi_{j-1} - i}}_{P_{j-1}+M_{j-1}}. \tag{5.9}$$

However, to each momentum we have associated a convergence rate and we can let all the fractions in this product converge to their limiting value with different rates. In the infinite-L

limit, the term belonging to $p^{(\pm)}_{j+1,n_{j+1}}$ on the level $j+1$ behaves as

$$\frac{\phi_{j,n_j} - \phi_{j+1} + i}{\phi_{j,n_j} - \phi_{j+1} - i} \approx \mathcal{O}\left(\exp\left[-\min\left(\delta^{(+)}_{j,n_j}, \delta^{(\pm)}_{j+1,n_{j+1}}\right) L\right]\right), \tag{5.10}$$

while the term belonging to $p^{(\pm)}_{j-1,n_{j-1}}$ behaves as

$$\frac{\phi_{j,n_j} - \phi_{j-1} + i}{\phi_{j,n_j} - \phi_{j-1} - i} \approx \mathcal{O}\left(\exp\left[\min\left(\delta^{(+)}_{j,n_j}, \delta^{(\pm)}_{j-1,n_{j-1}}\right) L\right]\right). \tag{5.11}$$

From now on, we write $(x, y) := \min(x, y)$. In total, the product of terms belonging to level $j+1$ converges as

$$\mathcal{O}\left(\exp\left[-\sum_{n_{j+1}=1}^{P_{j+1}}\left(\delta^{(+)}_{j,n_j}, \delta^{(+)}_{j+1,n_{j+1}}\right) - \sum_{n_{j+1}=1}^{M_{j+1}}\left(\delta^{(+)}_{j,n_j}, \delta^{(-)}_{j+1,n_{j+1}}\right)\right]\right)$$

and combining this with the similar result for the level $j-1$ we see that the right-hand side of the Bethe equation (5.8) behaves as

$$\begin{aligned}\mathcal{O}\Bigg(\exp\Bigg[&-\sum_{n_{j+1}=1}^{P_{j+1}}\left(\delta^{(+)}_{j,n_j}, \delta^{(+)}_{j+1,n_{j+1}}\right) - \sum_{n_{j+1}=1}^{M_{j+1}}\left(\delta^{(+)}_{j,n_j}, \delta^{(-)}_{j+1,n_{j+1}}\right) \\ &+ \sum_{n_{j-1}=1}^{P_{j-1}}\left(\delta^{(+)}_{j,n_j}, \delta^{(+)}_{j-1,n_{j-1}}\right) + \sum_{n_{j-1}=1}^{M_{j-1}}\left(\delta^{(+)}_{j,n_j}, \delta^{(-)}_{j-1,n_{j-1}}\right)\Bigg]\Bigg)\end{aligned} \tag{5.12}$$

and therefore goes to zero only when the convergence rates obey

$$\begin{aligned}&-\sum_{n_{j+1}=1}^{P_{j+1}}\left(\delta^{(+)}_{j,n_j}, \delta^{(+)}_{j+1,n_{j+1}}\right) - \sum_{n_{j+1}=1}^{M_{j+1}}\left(\delta^{(+)}_{j,n_j}, \delta^{(-)}_{j+1,n_{j+1}}\right) \\ &+ \sum_{n_{j-1}=1}^{P_{j-1}}\left(\delta^{(+)}_{j,n_j}, \delta^{(+)}_{j-1,n_{j-1}}\right) + \sum_{n_{j-1}=1}^{M_{j-1}}\left(\delta^{(+)}_{j,n_j}, \delta^{(-)}_{j-1,n_{j-1}}\right) < 0.\end{aligned} \tag{5.13}$$

In a similar fashion, one can derive that the Bethe equation corresponding to the momentum $p^{(-)}_{j,n_j}$ is satisfied only when

$$\begin{aligned}&-\sum_{n_{j+1}=1}^{P_{j+1}}\left(\delta^{(-)}_{j,n_j}, \delta^{(+)}_{j+1,n_{j+1}}\right) - \sum_{n_{j+1}=1}^{M_{j+1}}\left(\delta^{(-)}_{j,n_j}, \delta^{(-)}_{j+1,n_{j+1}}\right) \\ &+ \sum_{n_{j-1}=1}^{P_{j-1}}\left(\delta^{(-)}_{j,n_j}, \delta^{(+)}_{j-1,n_{j-1}}\right) + \sum_{n_{j-1}=1}^{M_{j-1}}\left(\delta^{(-)}_{j,n_j}, \delta^{(-)}_{j-1,n_{j-1}}\right) > 0.\end{aligned} \tag{5.14}$$

For a valid tree solution of the Bethe equations, equation (5.13) must be satisfied for all plus signs, while equation (5.14) must be satisfied for all minus signs. Note that these restrictions arise simply because there is more than one term that exhibits vanishing or divergent behaviour and we should include more information to find the behaviour of the product. This problem already exists in many of the previously known cases (such as the Hubbard model or the Heisenberg XXX model, see [33, 18]), but as far as we know, this has never been addressed. Fortunately, in these cases in which all the solutions are of string type (as in Figure 5.6), the restrictions (5.13),(5.14) simplify drastically and can easily be solved. The system of restrictions for a string solution without a real momentum involves read

$$\begin{aligned}(\delta_2^{(+)}, \delta_1^{(+)}) - (\delta_2^{(+)}, \delta_3^{(+)}) &< 0\\ &\vdots\\ (\delta_{m_p}^{(+)}, \delta_{m_p-1}^{(+)}) - (\delta_{m_p}^{(+)}, \delta_{m_p+1}^{(-)}) &< 0\\ (\delta_{m_p+1}^{(-)}, \delta_{m_p}^{(-)}) - (\delta_{m_p+1}^{(-)}, \delta_{m_p+2}^{(-)}) &> 0\\ &\vdots\\ (\delta_{M-1}^{(-)}, \delta_{M-2}^{(-)}) - (\delta_{M-1}^{(-)}, \delta_{M}^{(-)}) &> 0 \end{aligned} \tag{5.15}$$

and is solved by the ordering

$$\delta_1 < \delta_2 < \cdots < \delta_{m_p} = \delta_{m_p+1} > \delta_{m_p+2} > \cdots > \delta_M. \tag{5.16}$$

However, determining whether the system of equations consisting of (5.13) and (5.14) for a general tree solution can be solved is a much more complicated question. In the next section, we treat some cases and include an example from which it follows that not every sign configuration has a consistent set of convergence rates.

Examples

A tree solution consists of at least 2 levels. The 2-level case (illustrated in Figure 5.12) can also be solved in general, because the inequalities are trivially satisfied. However, already the 3-level

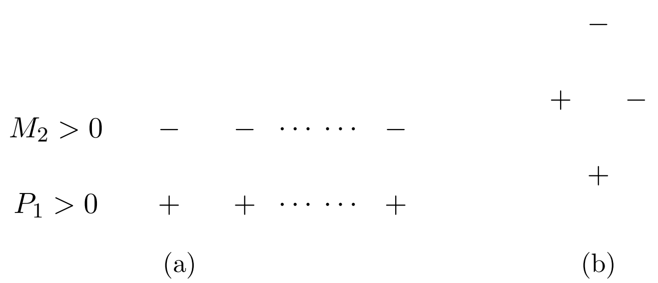


Figure 5.12: (a) A generic 2-level tree solution. (b) A 3-level tree solution that does not admit a consistent set of convergence rates.

case harbours an example of a configuration that cannot have a consistent set of convergence rates. Consider the example in Figure 5.12. The relevant set of equations is

$$\begin{aligned}(\delta_2^{(+)}, \delta_1) - (\delta_2^{(+)}, \delta_3) &< 0\\ (\delta_2^{(-)}, \delta_1) - (\delta_2^{(-)}, \delta_3) &> 0,\end{aligned} \tag{5.17}$$

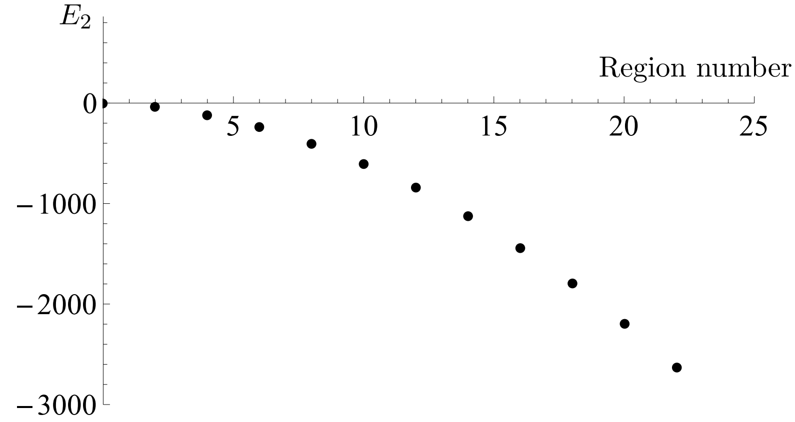


Figure 5.13: The energy of a two-particle bound state ($\phi_r = 1$, $\kappa = 2.236$) as a function of the number of the region from which the momenta are taken.

where we omit the superscript ($\pm$) when it is not necessary. We first try to deduce which of the δ's should be the smallest one of these four. From the upper equation, we conclude that neither $\delta_2^{(+)}$ nor δ_3 can be the smallest, while the lower equation tells us that neither $\delta_2^{(-)}$ nor δ_1 can be the smallest. Therefore, none of the 4 rates can be the smallest, thus no solution can exist. Note that this example can be extended: if we include $P_2 > 0$ plusses and $M_2 > 0$ minusses on level 2, the resultant set of restrictions has the system (5.17) as a subsystem and cannot be solved. In particular, this shows that example (b) we treated in Section 5.2.3 is not a valid bound state after all, although we could find momenta to match the configuration. Moreover, any sign configuration that contains this 3-level structure cannot be solved. However, all other 3-level configurations do admit a consistent solution as a careful analysis of the cases will show.

We have not been able to find a general algorithm to solve these complex coupled sets of inequalities or prove the existence (or absence) of a solution. The only configurations we found that lead to inconsistent inequalities were of the type described in the previous paragraph. In any case, the structure of the solutions is complicated.

5.3.2 Two-Particle Bound States

Perhaps a closer look at the smallest bound states, consisting of 2 particles, can be illuminating. Generically, the two-particle bound states consist of momenta p_1, p_2 with $\mathrm{Im}(\phi(p_1)) = 1/2$ and $\mathrm{Im}(\phi(p_2)) = -1/2$. To ensure that the total momentum is real, we must have $p_1 = \overline{p_2}$ and to get a consistent set of Bethe equations, we must have $\mathrm{Im}(p_1) < 0$. Even with this restriction, there exists a 2-particle bound state for every region i. However, a first problem arises when we assume that all these bound states are physical: the set of energies corresponding to these bound states is not bounded from below, as is illustrated in Figure 5.13. More strongly, we can actually show that the wavefunction parametrized by $p_1, p_2 \in$ region i that form a bound state does not vanish at infinity for $i \geq 2$: this wavefunction is given by

$$\psi(n_1, n_2) = 2\sinh^{-1}\kappa(n_1 - n_2)\left(e^{i(p_1 n_1 + p_2 n_2) + \kappa(n_1 - n_2)} - e^{i(p_2 n_1 + p_1 n_2) + \kappa(n_2 - n_1)}\right). \tag{5.18}$$

The amplitude of the wavefunction is then

$$|\psi(n_1, n_2)|^2 = 4|\sinh^{-2}\kappa(n_1 - n_2)|\left(e^{2\kappa(n_1 - n_2)} + e^{-2\kappa(n_1 - n_2)} - e^{i(n_1 - n_2)(p_1 - p_2)} - e^{-i(n_1 - n_2)(p_1 - p_2)}\right), \tag{5.19}$$

and since $p_1 - p_2 < 0$, we see that this only tends to zero in the limit $|n_1 - n_2| \to \infty$ if $|\mathrm{Im}(p_1)| \leq \kappa$, i.e. only the bound state formed out of momenta from region 1 converges. Thus physical two-particle bound states must have all their momenta in region 1.

This suggests that we should exclude all the regions except region 1 from the momentum domain to build physical bound states. However, this might only be the case for 2-particle bound states. Unfortunately, the complicated form of the wavefunction makes it impossible to prove an analogous statement for bound states consisting of more than 2 particles. To get a better idea of the part of the set of solutions that belongs to the spectrum of Inozemtsev's infinite spin chain, we will try to relate the Bethe solutions of the Heisenberg XXX spin chain to the solutions in our solution set.

5.3.3 Relationship with the Heisenberg XXX Strings

Due to the fact that the Heisenberg XXX spin chain can be obtained from Inozemtsev's elliptic spin chain by sending κ to infinity, it is natural that there should exist a close relationship between several properties of the models: the energies, eigenfunctions and phasefunctions of the Heisenberg XXX model can all be obtained from Inozemtsev's elliptic spin chain [8]. We will investigate here whether we can relate the Heisenberg XXX string solutions to the complicated structure of solutions we have found in the previous sections. We must therefore first investigate the string solutions of the Heisenberg XXX model, which has already been done by Bethe himself [10]. The Bethe equations of this model are

$$e^{ip_j L} = \prod_{\substack{n=1,\cdots,M \\ n \neq j}} \frac{\cot\frac{p_j}{2} - \cot\frac{p_n}{2} + 2i}{\cot\frac{p_j}{2} - \cot\frac{p_n}{2} - 2i}. \tag{5.20}$$

Note that due to the limit

$$\lim_{\kappa \to \infty} \phi(p) = \frac{1}{2}\cot\frac{p}{2},$$

the Bethe equations (5.2) of Inozemtsev's elliptic spin chain reduce to those of the Heisenberg XXX model. These equations only yield solutions of string type. Moreover, the structure of the XXX string solutions is very simple. For each M, there exists only 1 string solution of length M, which can be most conveniently described in terms of the rapidities $\lambda_j = 1/2\cot p_j/2$ and is given by $\lambda_j = \lambda + 1/2(M + 1 - 2j)i$, with $\lambda \in \mathbb{R}$. We can give these solutions in terms of rapidities because the rapidity function $p \mapsto \frac{1}{2}\cot\frac{p}{2}$ is bijective as a function from the complex strip $[0, 2\pi[\oplus i\mathbb{R}$ to $\mathbb{C}$.

To relate the solutions of Inozemtsev's infinite spin chain to the XXX string solutions, we will first focus on the domain of ϕ. Since κ measures the quasiperiod of ϕ in the complex direction it tells us the size of the regions (see Figure 5.5), between which there exists a quasiperiodic relationship. In the limit $\kappa \to \infty$, the only region that remains on the complex strip is the fundamental region (region 1), whose boundaries now lie at infinity. This suggests that all the solutions that lie at the boundary of the fundamental region for finite κ will vanish in this limit. An interesting question is now, whether we can identify the solutions at finite κ which converge to the Heisenberg solutions. The answer turns out to be yes.

Identifying the Heisenberg XXX Strings

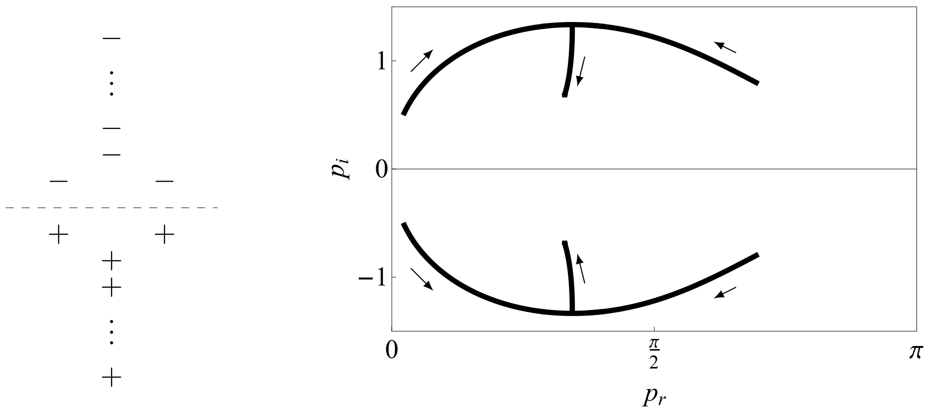
To make the aforementioned claim more precise, let us first state what we mean when we say that an Inozemtsev solution[4] *goes to a Heisenberg solution in the limit* $\kappa \to \infty$: it means that when we specify a solution of the Bethe equations by choosing

- ϕ_r, the real part of the image points
- a sign configuration
- ϕ_i, the imaginary part of the image point associated to the lowest level of the configuration
- the regions in momentum space in which all of the momenta belonging to the signs can be found conform Figure 5.5

and consequently find the associated momenta as a function of κ, the limiting values of the momenta as $\kappa \to \infty$ yield a valid Heisenberg solution. This means that we *define* a solution by specifying all of the above instead of an explicit set of momenta. Moreover, it implies that the image points of an asymptotic solution do not depend on κ.

From this perspective, it is not difficult to find all the solutions that have a Heisenberg limit. The fact that the fundamental region fills the entire complex strip in the limit $\kappa \to \infty$ implies directly that any string solution which has one or more momenta lying outside of the fundamental region will not converge to a Heisenberg limit, because all of these momenta will have infinite imaginary part in this limit. We can therefore focus on solutions lying in the fundamental region. Since we know that ϕ is almost bijective on this region,almost all of the tree solutions cannot exist entirely in the fundamental region. In fact, the only tree solutions remaining must have image points with imaginary part $\pm 1/2$ and can have at most 2 signs on the levels corresponding to

[4]i.e. a solution to the Bethe equations of Inozemtsev's infinite spin chain



(a) (b)

Figure 5.14: (a) The general structure of a tree solution in the fundamental region. The dashed line indicates the real axis, thus this solution is self-conjugate. (b) The path that is traveled by the two 2-string solutions in momentum space with $\phi_r = 0.6$ as we increase κ. The arrows indicate the direction of increasing κ.

those image points. To make sure that the total momentum is real, all these solutions must be self-conjugate[5]. A picture of this general structure is shown in Figure 5.14.

The basis for all these tree solutions is the existence of at most 4 possible sets of 2-particle bound states with $|\phi_r| < \phi_{\text{crit}}$, due to the fact that the equation

$$\phi(p + \kappa i) = \phi_r - i/2$$

has exactly two solutions, p_1 and p_2 (see Figure 5.4). We can therefore build four two-particle bound states, by combining these solutions as follows:

$$\begin{aligned} &\{p_1 + i\kappa, p_1 - \imath\kappa\}, \qquad \{p_2 + i\kappa, p_2 - \imath\kappa\}, \\ &\{p_2 + i\kappa, p_1 - \imath\kappa\}, \qquad \{p_1 + i\kappa, p_2 - \imath\kappa\}. \end{aligned} \tag{5.21}$$

All four of these combinations give rise to bound states with real energy, but the options on the first line should not be considered. Namely, the wavefunction corresponding to these options vanishes, as follows from a direct computation: using the known form of the two-particle

[5]For some models, the self-conjugacy of its Bethe solutions can be traced back to the underlying algebraic structure [34]. In the case of Inozemtsev's infinite spin chain, it cannot be proved that the solutions must be self-conjugate, but all the numerical evidence points in this direction. We will therefore assume that this is true.

wavefunction (3.35) we find after plugging in momenta $p \pm i\kappa$

$$\begin{aligned}
\psi(n_1, n_2) &= \\
&= \sinh^{-1}\kappa(n_1-n_2)\Big(e^{ip(n_1+n_2)-(n_1-n_2)\kappa}\left[\cosh(\gamma)\sinh\kappa(n_1-n_2)+\cosh\kappa(n_1-n_2)\sinh(\gamma)\right] \\
&\quad +e^{ip(n_1+n_2)+(n_1-n_2)\kappa}\left[\cosh(\gamma)\sinh\kappa(n_1-n_2)-\cosh\kappa(n_1-n_2)\sinh(\gamma)\right]\Big) \\
&= \sinh(\gamma)e^{ip(n_1+n_2)}\Big(e^{-\kappa(n_1-n_2)}(1+\coth\kappa(n_1-n_2))+e^{\kappa(n_1-n_2)}(1-\coth\kappa(n_1-n_2))\Big) \\
&= \sinh(\gamma)e^{ip(n_1+n_2)}\left(2\cosh\kappa(n_1-n_2)+\frac{e^{-\kappa(n_1-n_2)}-e^{\kappa(n_1-n_2)}}{\sinh\kappa(n_1-n_2)}\cosh\kappa(n_1-n_2)\right) \\
&= 0. \qquad (5.22)
\end{aligned}$$

So the bound states parametrized by the momenta $p_{1,2} \pm i\kappa$ are not part of the spectrum. Moreover, the calculation above suggests that whenever a set of momenta contains two momenta p_i, p_j satisfying $p_i = p_j + 2\kappa i$, the corresponding wavefunction vanishes. Numerical analysis of the wavefunctions up to $M = 4$ corroborates this. If we assume this is in fact true for all the bound states, also the last tree solutions must be omitted from the set of physical bound states. Indeed, any tree solution of the type depicted in Figure 5.14 must contain a subset of momenta p_i, p_j satisfying $p_i = p_j + 2\kappa i$. Therefore, we no longer have tree solutions left! Instead, the two remaining combinations give rise to two types of bound states for every M for $|\phi_r| < \phi_{\text{crit}}$. And since $\phi_{\text{crit}} \to 0$ as $\kappa \to \infty$, for larger and larger κ the solutions on the boundary form a smaller and smaller part of the total solution set, until they vanish in the Heisenberg limit. From an other perspective, we can state this as follows: for a fixed value of ϕ_r, there is a finite value for κ that marks the boundary between the existence of two types of bound states living on the boundary of the fundamental region and the existence of one type of bound state living in the interior of the fundamental region. For example, in the case illustrated in Figure 5.14, this value is $\kappa \approx 1.3352$.

We see now that indeed any physical bound state of Inozemtsev's infinite spin chain that lies entirely in the fundamental region converges to a Heisenberg solution. Moreover, our conclusions concerning the two-particle bound states are in exact correspondence with the results presented in [35], which studied the two particle bound states exclusively using the two-particle wavefunction (3.37).

5.3.4 Restricting to the Fundamental Region

In the previous sections, we have seen several different arguments why it seems reasonable to restrict the set of solutions in several ways. Firstly, we saw that not all the structures in ϕ-space can be allowed, because there is no sensible way to take the limits. Secondly, we saw that the two-particle bound states built up from momenta outside of the fundamental region have several difficulties: allowing them all leads to the unwanted feature that the energy of bound

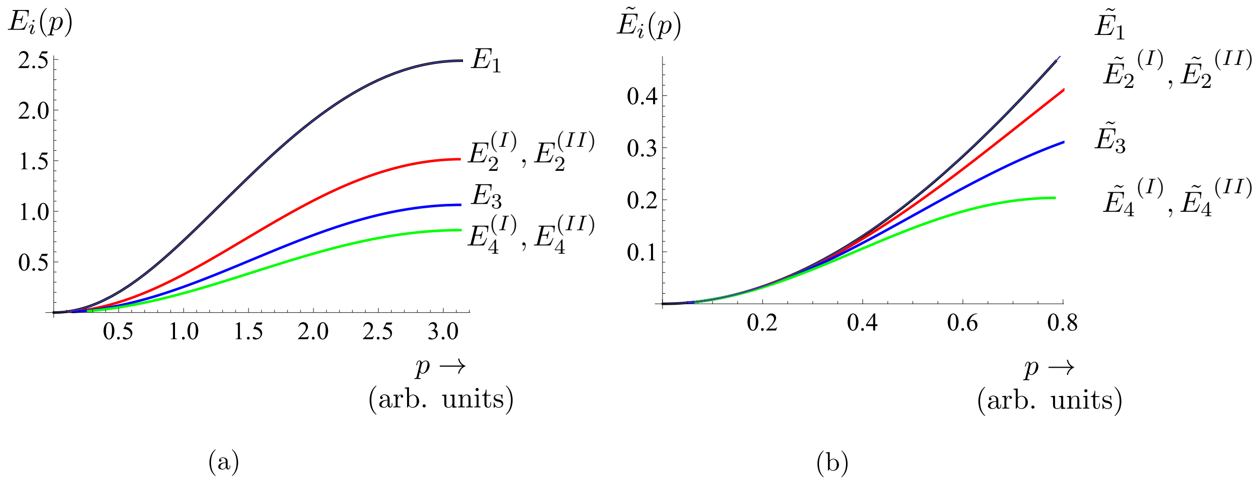


Figure 5.15: (a) The one-particle energy E_1, the energy of the 3-particle bound state and the energies of the 2 types of bound states consisting of 2 and 4 particles. (b) The rescaled energies $\tilde{E}_i(p) = \frac{E_M(Mp)}{M}$ of bound states consisting of up to 4 particles. In both pictures, $\kappa = 1.23$.

states is unbounded from below and the corresponding wavefunctions are not bounded. Thirdly, half of the two-particle bound states in the fundamental region have a vanishing wavefunction, also excluding the possiblity of a tree solution in the fundamental region. Finally, the string solutions of the Heisenberg XXX model all correspond to bound states of Inozemtsev's infinite spin chain that are built up from momenta in the fundamental region.

Although the enumeration of arguments in the above is not a complete proof, it does strongly indicate that the momenta of physical bound states should lie in the fundamental region (region 1). We therefore assert that this is the case. The set of bound states built up from these momenta can be characterized as follows:

- for fixed M, there is only 1 configuration of image points that has real energy and this solution is self-conjugate.
- if M is odd, there is only one set of momenta corresponding to such a configuration after fixing ϕ_r.
- if M is even, there is one set of momenta corresponding to such a configuration if $|\phi_r| \geq \phi_{\text{crit}}$ and there are two sets if $|\phi_r| > \phi_{\text{crit}}$.

Thus, the complete bound-state content of this model consists of 2 types of bound states for M even and 1 type of bound state for M odd. We can use numerical methods to extract a plot for the spectrum of these bound states. On the left-hand side of Figure 5.15, we plotted the energy of the bound states of up to 4 particles. To check that these energies make sense, we should check whether the inequality

$$M E_1(p) \geq E_M(Mp), \tag{5.23}$$

in which E_M is the energy of an M-particle bound state, holds for all p. It reflects the fact that the bound state of M particles should not have more energy than M unbounded particles. We have plotted the rescaled energies E_M/M on the right-hand side of Figure 5.15, from which we see that these states are in fact bound states. Numerical analysis indicates that these spectra transform into the spectrum of the Heisenberg XXX-model in the limit $\kappa \to \infty$.

5.3.5 Connection to the Haldane-Shastry Spin Chain

Up until now we have not tried to connect the solutions of Inozemtsev's Bethe equations to the spectrum of the Haldane-Shastry spin chain. The spectrum of the Haldane-Shastry spin chain was investigated by Haldane et al. in [19], in which they concluded that the spectrum of the infinite-length limit of this chain, the inverse square exchange model with hamiltonian

$$H_{ISE} = -\frac{J}{4} \sum_{j=1}^{L} \frac{1}{(n_j - n_k)^2} \left(\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1} - 1 \right), \tag{5.24}$$

contains no bound states. However, there are special solutions to the Bethe equations named *squeezed strings*[6], which are sets of coinciding real momenta that can be treated as quasi-particles [36]. In this paper [36], Ha and Haldane proposed a set of Bethe equations for these squeezed strings and proceeded to show that the solutions form a complete basis of the associated Hilbert space. They also refer to an unpublished paper in which they would treat the thermodynamics of Inozemtsev's infinite spin chain, but as far as we could find out, this paper was never published.

The solutions we found for Inozemtsev's infinite spin chain after restricting to the fundamental region are consistent with the findings in the references [19, 36, 37]: all the bound-state solutions have momenta p_j with $|\mathrm{Im}(p_j)| \leq \kappa$, which implies that in the limit $\kappa \to 0$, all the momenta p_j must be real.

5.4 Conclusion

As we have seen, it is possible to completely classify the solutions to the Bethe equations of Inozemtsev's infinite spin chain. There are several reasons to suspect that not all of these solutions can be physical: the cardinality of the set of solutions is much larger than for comparable models and the only well-behaved two-particle bound states are those built from momenta in the fundamental region. Therefore, we proposed to consider only solutions lying in the fundamental region, which solved many issues. We were able to show that all the remaining solutions converge to bound states of the Heisenberg XXX model and found the spectrum of these bound

[6]This name originates in the fact that the complex momenta of string solutions of the Haldane-Shastry spin chain get squeezed onto the real line in the infinite length limit.

states. The connection to the spectrum of the Haldane-Shastry spin chain also seems to be as it should.

One of the remaining questions is whether or not these bound state are the entire particle content of Inozemtsev's infinite chain. Unfortunately, proving completeness of these solutions is extremely complicated. It would, however, be helpful to consider the finite size hamiltonian and diagonalize it numerically. A good correspondence between the solutions we found and the eigenvalues of this model would corroborate the conclusions of our analysis. Also, it would be interesting to see whether it is possible to write down the integral equations that govern the behaviour of these bound states in the thermodynamic limit using the string hypothesis. However, this would require finding an analytic expression for the bound-state energy, which we have not been able to find yet.

Chapter 6

Conclusion and Outlook

This thesis has concentrated mostly on the study of Inozemtsev's infinite-length spin chain. In Chapter 3, a detailed derivation of the eigenfunctions has been presented, indicating a deep relation between the dynamical Calogero-Sutherland-Moser model with inverse hyperbolic exchange and Inozemtsev's spin chain. The derivation was in agreement with the one presented by Inozemtsev in [22]. Moreover, we derived the asymptotic form of the wavefunctions and concluded that the model exhibits factorized scattering, one of the harbingers of integrability. In Chapter 4, we derived Bethe equations of the model using the Asymptotic Bethe Ansatz and used a method introduced by Hulthén to analyze the antiferromagnetic ground state of the chain. Unfortunately, the particle density could only be expressed in a Fourier integral that is most likely not exactly solvable. Trying to circumvent this problem, we proceeded by using a strategy first used by Yang and Yang to write down integral equations that govern the thermodynamics of the model. Although we could solve these equations numerically using Picard iterations, an analytic solution to these equations seems to be out of reach.

In order to get a better idea of the thermodynamics, we analyzed the bound states of the chain. We characterized all the solutions to the Bethe equations and discovered an interesting generalization of the usual string structure of these solutions. This spurred a closer analysis of the convergence properties of the asymptotic solutions, which led to intriguing sets of inequalities that seem very difficult to solve. For the simpler case of string solutions, we proved that these inequaltities do have solutions, putting the existing research on string solutions on more solid ground. Nevertheless, by focussing on other aspects of these solutions, we found many arguments to locate the physical bound states in the huge set of solutions. These solutions have the expected string structure and can be connected to the results on bound states for the related Haldane-Shastry spin chain and Heisenberg XXX chain, although the connection to the Haldane-Shastry spin chain can be further strengthened.

To have independent confirmation of the bound-state results, it would be useful to perform numerical diagonalization of the finite-size spin chain and see whether the spectrum corresponds to the one we found. Hopefully, we can perform this study in the near future. Also, further research could be conducted to better establish the relationship to the spectrum of the Haldane-Shastry spin chain. Relating our results to the work of Barba et al. in [38], in which they calculate the spectrum of Inozemtsev's infinite chain in the presence of a Morse potential, would also be interesting. From a broader perspective we recognize that the most tantalizing questions concern the existence of a Yang-Baxter structure for this model and a proof that the proposed set of conserved quantities is indeed in involution.

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Appendix A

Proof of a Mysterious Formula

While deriving the eigenfunctions for Inozemtsev's infinite spin chain, we came across a formula (3.116) that looked quite complicated, but a closed expression was needed to proceed. Here we show the proof that this closed expression indeed exists and is fairly simple.

Theorem. Let $\kappa > 0$ and $\{n_i\}_{1\leq i\leq M}$ with $n_i \in \mathbb{Z}$ and $n_i \neq n_j$ if $i \neq j$. Then

$$\sum_{\substack{i,j,k\in\mathbb{Z}\\ i\neq j\neq k\neq i}} \coth\kappa(n_i - n_j)\coth\kappa(n_k - n_j) = \frac{M(M-1)(M-2)}{3}. \tag{A.1}$$

Proof Define $\frac{1}{t_{ij}} := \coth\kappa(n_i - n_j)$ and write $t_i := t_{i0}$. Consider the sum

$$\frac{1}{t_{ij}}\frac{1}{t_{kj}} + \frac{1}{t_{ji}}\frac{1}{t_{ki}} + \frac{1}{t_{ik}}\frac{1}{t_{jk}} = \frac{(1-t_it_j)(1-t_kt_j)}{(t_i-t_j)(t_k-t_j)} + \frac{(1-t_it_j)(1-t_kt_i)}{(t_j-t_i)(t_k-t_i)} + \frac{(1-t_it_k)(1-t_kt_j)}{(t_i-t_k)(t_j-t_k)} = \frac{(t_i-t_k)(1-t_it_j)(1-t_kt_j) - (t_j-t_k)(1-t_it_j)(1-t_it_k) + (t_j-t_i)(1-t_it_k)(1-t_jt_k)}{(t_j-t_i)(t_j-t_k)(t_i-t_k)}, \tag{A.2}$$

where we used the addition formula for the hyperbolic tangent. Expanding the numerator yields

$$t_jt_j(t_i-t_k) + t_it_i(t_k-t_j) + t_kt_k(t_j-t_i),$$

wheras the denominator can be expanded to give $t_jt_j(t_i-t_k)+t_it_i(t_k-t_j)+t_kt_k(t_j-t_i)$, which is exactly the same. So the sum given in (A.2) equals 1. This already shows why the right-hand side of (A.1) does not depend on κ or the chosen set of integers. Now we see that

$$\begin{aligned}\sum_{\substack{i,j,k\in\mathbb{Z}\\ i\neq j\neq k\neq i}} \frac{1}{t_{ij}}\frac{1}{t_{kj}} &= \left(\sum_{\substack{i,j,k\in\mathbb{Z}\\ i<j<k}} + \sum_{\substack{i,j,k\in\mathbb{Z}\\ i<k<j}} + \sum_{\substack{i,j,k\in\mathbb{Z}\\ j<i<k}} + \sum_{\substack{i,j,k\in\mathbb{Z}\\ j<k<i}} + \sum_{\substack{i,j,k\in\mathbb{Z}\\ k<i<j}} + \sum_{\substack{i,j,k\in\mathbb{Z}\\ k<j<i}}\right)\frac{1}{t_{ij}}\frac{1}{t_{kj}} \\ &= 2\sum_{\substack{i,j,k\in\mathbb{Z}\\ j<i<k}}\left(\frac{1}{t_{ij}}\frac{1}{t_{kj}} + \frac{1}{t_{ik}}\frac{1}{t_{jk}} + \frac{1}{t_{ji}}\frac{1}{t_{ki}}\right) = 2\sum_{\substack{i,j,k\in\mathbb{Z}\\ j<i<k}} 1,\end{aligned} \tag{A.3}$$

where we used the relation found above in (A.2). All that remains now is determining the number of terms in this sum. For $M = 1$ or $M = 2$, it is not difficult to find that this number equals $\frac{1}{6}M(M-1)(M-2)$ by an explicit listing of the terms. For $M \geq 2$, note that, given a value of $j \in \{1, 2, \cdots, M-2\}$, the total number of possibilities for i, k equals the number of elements in the lower triangle of a square matrix of dimension $M - j - 1$. Therefore, the total number of terms equals

$$\sum_{j=1}^{M-2} \sum_{n=1}^{M-j-1} n = \sum_{j=1}^{M-2} \frac{1}{2}(M-j-1)(M-j-2) = \frac{M(M-1)(M-2)}{6},$$

which can be found using the formula $\sum_{n=1}^{N} n^2 = \frac{N(N-1)(2N+1)}{6}$. Combining the above result with equation (A.3) proves that

$$\sum_{\substack{i,j,k \in \mathbb{Z} \\ i \neq j \neq k \neq i}} \coth \kappa(n_i - n_j) \coth \kappa(n_k - n_j) = \frac{M(M-1)(M-2)}{3} \tag{A.4}$$

like we claimed.□

We have used this equation to rewrite the difference equation in the M-particle sector (more precisely to rewrite equation (3.115)) to relate it to the difference equation of the CSM-model with the same interaction.

Appendix B

Bijectivity of the Function ϕ

Here we investigate the behaviour of ϕ on the fundamental region. Consider the contour C depicted in Figure B.1, which travels around the fundamental region counterclockwise on the edge. In its interior, there is 1 pole, at $z = 0$. Note that due to the periodicity of ϕ in the real direction, the small deviation around the pole does not affect the analysis.

We can also find the imaginary part of $\phi(z)$ on the top and bottom edge of this contour by a simple observation: let $x \in \mathbb{R}$, then $\overline{\phi(x - i\kappa)} = \phi(x + i\kappa) = \phi(x - i\kappa) - i$ and we have[1]

$$\overline{\phi(x - i\kappa)} - \phi(x - i\kappa) = i$$

which implies that $\mathrm{Im}(\phi(x - i\kappa)) = -i/2$ and that $\mathrm{Im}(\phi(x + i\kappa)) = i/2$. Thus on the top and bottom edge of this contour, the imaginary part of $\phi(z)$ is constant. Let $\alpha \in \mathbb{C}$ be arbitrary, but such that $\phi(z) = \alpha$ has no solutions when ϕ is restricted to the contour. Then the function $\tilde{\phi}(z) = \phi(z) - \alpha$ has no zeroes or poles on the contour and we can use the argument principle to state that

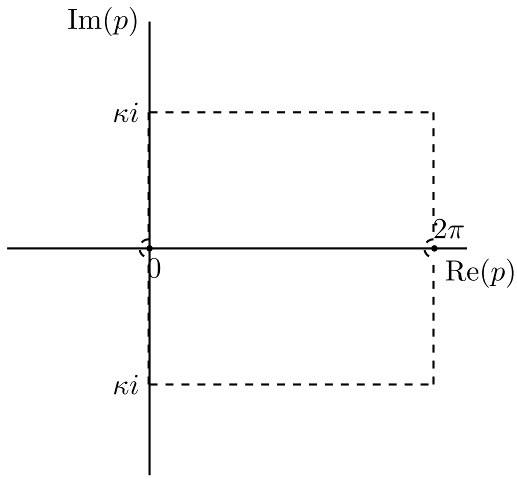

Figure B.1: The contour around which we integrate to find the number of zeroes in the fundamental region for $\phi(p)$.

$$\oint_C \frac{\tilde{\phi}'(z)}{\tilde{\phi}(z)} dz = 2\pi i \left(N - P\right), \tag{B.1}$$

where N is the number of zeroes and P the number of poles of $\tilde{\phi}$ in the interior of the contour, which is the fundamental region of ϕ. In this case, we have $P = 1$. We can calculate the integral

[1] $\overline{\phi(z)} = \phi(\overline{z})$ follows from the oddity of ζ in the definition of ϕ.

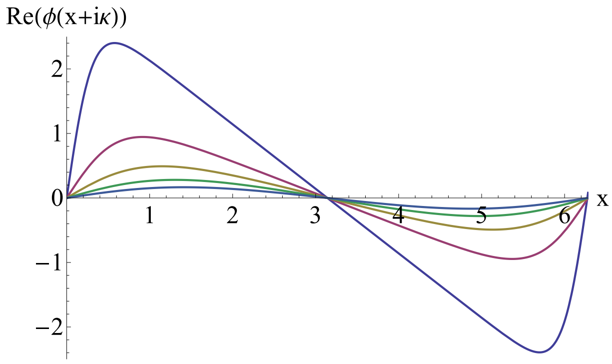

Figure B.2: A plot showing the behaviour of $\mathrm{Re}(\phi)$ along the top boundary of the fundamental region for values of $\kappa = 0.5, 1, 1.5, 2, 2.5$. For larger κ, the graph of $\mathrm{Re}(\phi)$ approaches the real axis.

on the left-hand side: the contributions from the vertical parts of the contour cancel each other due to the periodicity of $\tilde{\phi}$. For the contributions of the top part, we see the following:

$$\int_0^{2\pi} \frac{\tilde{\phi}'(x+\kappa i)}{\tilde{\phi}(x+\kappa i)} dx = \int_0^{2\pi} \frac{d}{dx} \log\left(\tilde{\phi}(x+\kappa i)\right) dx = \log\left(\tilde{\phi}(\kappa i)\right) - \log\left(\tilde{\phi}(2\pi+\kappa i)\right) = 0, \quad \text{(B.2)}$$

because $\tilde{\phi}$ is 2π-periodic in the real direction. Note that we could evaluate the integral using the logarithm, because we know that the imaginary part of $\tilde{\phi}$ is constant along the path, allowing us to find a holomorphic branch for the logarithm on a neighbourhood of the top part of the contour. In a similar fashion, one can show that the contribution from the bottom part vanishes, thus we end up with

$$\oint_C \frac{\tilde{\phi}'(z)}{\tilde{\phi}(z)} dz = 0,$$

implying that for all the α we considered, $\tilde{\phi}$ has exactly 1 zero in the fundamental region, thus $\phi(z) = \alpha$ has exactly 1 solution in this region.

On the boundary of the fundamental region, the following holds. The restriction $x \mapsto \phi(ix)$ to the imaginary axis (with $x \in [-\kappa, \kappa]$) has positive derivative everywhere, except at the pole at zero where it is undefined. Moreover, since $\phi(\pm\kappa i) = \mp i/2$, we can conclude that this restriction maps bijectively onto $i[-\infty, -1/2] \cup i[1/2, \infty]$. This shows that $\phi : [0, 2\pi[\oplus i] - \kappa, \kappa[\to A \subset \mathbb{C}$ maps bijectively onto its image A. On the top part of the contour we can write $x \mapsto \phi(x+i\kappa)$ for the restriction. A plot of this function is shown in Figure B.2, which shows that this restriction is not bijective onto its image. In fact, all image values are attained exactly twice. We call the graph's maximum ϕ_{crit} and by symmetry, its minimum is $-\phi_{\text{crit}}$. The value of p for which $\mathrm{Re}(\phi(p+i\kappa)) = \phi_{\text{crit}}$ we call p_{crit}. By symmetry, the minimum is attained at $2\pi - p_{\text{crit}}$. The behaviour of the real part of ϕ along the bottom boundary is exactly the same.

We can now conclude that ϕ is surjective onto $\mathbb{C}$ and almost injective: the only values it attains twice are those of the form $\phi \pm i/2$, where $|\phi| \leq \phi_{\text{crit}}$. We will see, however, that this small deviation from bijectivity will have a profound effect on the allowed solutions solutions to the Bethe equations.

Appendix C

Location of preimages under ϕ

For each region, we can specify whether there exists a *positive solution* (a solution p with positive imaginary part $\mathrm{Im}(p) > 0$, indicated by a $+$) or a *negative solution* (a solution p with negative imaginary part $\mathrm{Im}(p) < 0$, indicated by a $-$) to the equation

$$\phi(p) = \alpha \in \mathbb{C},$$

as shown in tables C.1, C.2. The tables show that if $|\phi_r| < \phi_{\mathrm{crit}}$, the solution distribution is quite complicated, but regularizes as ϕ_i increases. The seemingly irregular pattern is caused by an 'extra' negative solution that lives close to the boundary between regions and travels across several boundaries before settling down. Note that we can extract the distribution for $\phi_i < 0$ from these tables by interchanging all plusses and minusses.

Region \ ϕ_i	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5
1	$-$	↗ $-,-$	↘ $-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
2	$+,-$	$+$	$+,-$	→ $+,-$	$+$	$+$	$+$	$+$	$+$	$+$
3					↘ $-$	$-,-$	$-$	$-$	$-$	$-$
4	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	↗ $+$	↘ $+,-$	→ $+,-$	$+$	$+$
5									↘ $-$	$-,-$
6	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	→ $+$

Table C.1: Characterization of the solutions of the equation $\phi(p) = \phi_r + \phi_i i$ for different values of ϕ_i for all $|\phi_r| < \phi_{\text{crit}}$. The structure does not change in the intervals $]k/2, k/2 + 1/2[$ with $k \in \mathbb{N}$. The arrows indicate the direction of travel of the 'extra' negative solution. Obvious arrows have been omitted.

Region \ ϕ_i	0.25	0.5	0.75	1	1.25	1.5	1.75	2	2.25	2.5
1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
2	$+,-$	→ $+,-$	→ $+,-$	→ $+,-$	$+$	$+$	$+$	$+$	$+$	$+$
3					↘ $-$	$-$	$-$	$-$	$-$	$-$
4	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	→ $+,-$	→ $+,-$	→ $+,-$	$+$	$+$
5									↘ $-$	$-$
6	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	$+,-$	→ $+,-$

Table C.2: Characterization of the solutions of the equation $\phi(p) = \phi_r + \phi_i i$ for different values of ϕ_i for all $|\phi_r| \geq \phi_{\text{crit}}$.

Appendix D

Why the Bethe Equation of a Real Momentum is Always Satisfied

Suppose S is a self-conjugate solution to the Bethe equation. If $p \in S$ is real, its Bethe equation

$$e^{ip_0 L} = \prod_{\substack{k \in \mathbb{Z}_M \\ k \neq 0}} \frac{\phi_0 - \phi_k + i}{\phi_0 - \phi_k - i}$$

is trivially satisfied in the infinite-length limit. Namely, as $L \to \infty$, the norm of the left-hand side remains 1, while the norm of the right-hand side is

$$\Bigg| \prod_{\substack{k \in \mathbb{Z}_M \\ k \neq 0}} \frac{\phi_0 - \phi_k + i}{\phi_0 - \phi_i - i} \Bigg| = \prod_{\substack{k \in \mathbb{Z}_M \\ k \neq 0}}^{*} \left| \frac{\phi_0 - \phi_k + i}{\phi_0 - \phi_k - i} \cdot \frac{\phi_0 - \overline{\phi_k} + i}{\phi_0 - \overline{\phi_k} - i} \right|, \tag{D.1}$$

where the star indicates that the product runs only over all the conjugate pairs. The terms in this product can be rewritten as follows:

$$\begin{aligned} \left| \frac{\phi_0 - \phi_k + i}{\phi_0 - \phi_k - i} \cdot \frac{\phi_0 - \overline{\phi_k} + i}{\phi_0 - \overline{\phi_k} - i} \right| &= \frac{|\phi_0 - \mathrm{Re}(\phi_k) - i(\mathrm{Im}(\phi_k) - 1)|}{|\phi_0 - \mathrm{Re}(\phi_k) - i(\mathrm{Im}(\phi_k) + 1)|} \cdot \frac{|\phi_0 - \mathrm{Re}(\phi_k) + i(\mathrm{Im}(\phi_k) + 1)|}{|\phi_0 - \mathrm{Re}(\phi_k) + i(\mathrm{Im}(\phi_k) - 1)|} \\ &= \frac{|\phi_0 - \mathrm{Re}(\phi_k) - i(\mathrm{Im}(\phi_k) - 1)|}{|\phi_0 - \mathrm{Re}(\phi_k) + i(\mathrm{Im}(\phi_k) - 1)|} \cdot \frac{|\phi_0 - \mathrm{Re}(\phi_k) + i(\mathrm{Im}(\phi_k) + 1)|}{|\phi_0 - \mathrm{Re}(\phi_k) - i(\mathrm{Im}(\phi_k) + 1)|}, \\ &= 1 \end{aligned} \tag{D.2}$$

Since we also know that the ϕ_j converge as $L \to \infty$, the right-hand side must have a well-defined value on the unit circle, say $e^{i\alpha}$. For finite L, this implies that $p_0 = \pi k/L + \alpha/L$, with $\alpha \in [0, \pi]$ and k an even integer[1]. However, in the limit $L \to \infty$, we can choose any value for p_0 in the interval $[0, 2\pi[$. More precisely, for every $p_0 \in [0, 2\pi[$ we can build a sequence $\{p^{(L)}\}$ in which each $p^{(L)} \in [0, 2\pi[$ satisfies $p^{(L)} = \pi k/L + \alpha/L$ with k even and such that $p^{(L)} \to p_0$ as $L \to \infty$. So the Bethe equation for p_0 is satisfied regardless of its value.

[1]Here we assume that the Bethe equations at finite L also obey an equation such as (D.2).

Appendix E

Properties of Weierstraß Elliptic Functions

Here we list the most important properties of the $\wp$-,ζ- and σ-function (which can be found in references [13, 14, 15, 16]. First we define the lattice $\mathbb{L}$ that defines the periodicity of these functions:

$$\mathbb{L} := \{z \in \mathbb{C} | z = n\omega_1 + m\omega_2, n, m \in \mathbb{Z}\}, \tag{E.1}$$

where the ω_i are the periods of the lattice and obey $\mathrm{Im}(\omega_1/\omega_2) < 0$. The definitions of the Weierstraß elliptic functions can now be written as

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + \sum_{\substack{\omega\in\mathbb{L}\\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \\ \zeta(z) &= \frac{1}{z} + \sum_{\substack{\omega\in\mathbb{L}\\ \omega\neq 0}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right) \\ \sigma(z) &= z \prod_{\substack{\omega\in\mathbb{L}\\ \omega\neq 0}} \left(\left(1 - \frac{z}{\omega}\right) \exp\left(\frac{z}{\omega} + \frac{z^2}{2\omega^2} \right) \right), \end{aligned} \tag{E.2}$$

where all these series converge absolutely and uniformly if $z \in A \subset \mathbb{C}$ and A is compact and $A \cap \mathbb{L} = \emptyset$. Moreover, $\wp$ is even and meromorphic with double poles with residue 0. ζ is odd and meromorphic with simple poles with residue 1. σ is entire and odd, with simple zeroes at all the lattice points. Note that formally, ζ and σ are not doubly periodic and hence not elliptic, but together with $\wp$ they are usually called the Weierstraß elliptic functions.

These functions satisfy

$$\begin{aligned} \wp(z) &= -\zeta'(z), \\ \zeta(z) &= \frac{\sigma'(z)}{\sigma(z)}, \end{aligned} \tag{E.3}$$

for all $z \notin \mathbb{L}$. In the rest of this chapter, we will assume that all the quantities in the expressions are finite, so we will often tacitly assume that $z \notin \mathbb{L}$.

E.1 Periodicity

These functions obey the following:

$$\begin{aligned} \wp(z+\omega_i) &= \wp(z), \\ \zeta(z+\omega_i) &= \zeta(z) + 2\zeta\left(\frac{\omega_i}{2}\right), \\ \sigma(z+\omega_i) &= -e^{2\zeta\left(\frac{\omega_i}{2}\right)(z+\omega_i/2)}\sigma(z) \end{aligned} \tag{E.4}$$

for $i = 1, 2$. Also

$$\begin{aligned} \wp'(\omega_i) &= 0, \\ \sigma'(2\omega_i) &= -e^{2\zeta\left(\frac{\omega_i}{2}\right)\omega_i}. \end{aligned} \tag{E.5}$$

If we define $\omega_3 := -\omega_1 - \omega_2$, then we get the extra properties

$$\begin{aligned} \sum_{i=1}^{3} \zeta\left(\frac{\omega_i}{2}\right) &= 0 \\ \zeta\left(\frac{\omega_2}{2}\right)\omega_3 - \zeta\left(\frac{\omega_3}{2}\right)\omega_2 = \zeta\left(\frac{\omega_3}{2}\right)\omega_1 - \zeta\left(\frac{\omega_1}{2}\right)\omega_3 &= \zeta\left(\frac{\omega_1}{2}\right)\omega_2 - \zeta\left(\frac{\omega_2}{2}\right)\omega_1 = \frac{\pi i}{2}. \end{aligned} \tag{E.6}$$

This last identity is called the *Legendre relation.*

E.2 Derivatives and Their Relations

We can define the *lattice invariants* as follows:

$$\begin{aligned} g_2 &:= 60 \sum_{\substack{\omega\in\mathbb{L} \\ \omega\neq 0}} \omega^{-4} \\ g_3 &:= 140 \sum_{\substack{\omega\in\mathbb{L} \\ \omega\neq 0}} \omega^{-6}, \end{aligned} \tag{E.7}$$

which can be used to write down the following relations for $\wp$ and its derivatives:

$$\begin{aligned} \left(\wp'\right)^2(z) &= 4\wp^3(z) - g_2\wp(z) - g_3 \\ \wp''(z) &= 6\wp^2(z) - \frac{g_2}{2} \\ \wp'''(z) &= 12\wp(z)\wp'(z) \end{aligned} \tag{E.8}$$

E.3 Series

By defining $q := e^{i\pi\omega_1/\omega_2}$ we can state that

$$\begin{aligned}
\wp(z) + \frac{1}{\omega_2}\zeta\left(\frac{\omega_2}{2}\right) - \frac{\pi^2}{4\omega_2^2}\csc^2\left(\frac{\pi z}{2\omega_2}\right) &= -\frac{2\pi^2}{\omega_2^2}\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos\left(\frac{n\pi z}{\omega_2}\right), \\
\zeta(z) - \frac{z}{\omega_1}\zeta\left(\frac{\omega_2}{2}\right) - \frac{\pi}{2\omega_2}\cot\left(\frac{\pi z}{2\omega_2}\right) &= \frac{2\pi}{\omega_2}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin\left(\frac{n\pi z}{\omega_2}\right).
\end{aligned} \tag{E.9}$$

A little more useful in our context our the expansions in terms of cosecant and cotangent:

$$\begin{aligned}
\wp(z) &= -\frac{1}{\omega_2}\zeta\left(\frac{\omega_1}{2}\right) + \frac{\pi^2}{4\omega_2^2}\sum_{n=-\infty}^{\infty}\csc^2\left(\frac{\pi(z+2n\omega_1)}{2\omega_2}\right), \\
\zeta(z) &= \frac{z}{\omega_2}\zeta\left(\frac{\omega_1}{2}\right) + \frac{\pi}{\omega_2}\sum_{n=-\infty}^{\infty}\cot\left(\frac{\pi(z+2n\omega_1)}{2\omega_2}\right).
\end{aligned} \tag{E.10}$$

The most important series, however, are the Laurent series of $\wp$ and ζ:

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{n=2}^{\infty} c_n z^{2n-2}, \\
\zeta(z) &= \frac{1}{z} - \sum_{n=2}^{\infty}\frac{c_n}{2n-1}z^{2n-1}
\end{aligned} \tag{E.11}$$

where the c_n are given by

$$\begin{aligned}
c_2 &= \frac{g_2}{20}, \\
c_3 &= \frac{g_3}{28}, \\
c_j &= \frac{3}{(2j+1))n-3)}\sum_{m=2}^{j-2} c_m c_{j-m},
\end{aligned} \tag{E.12}$$

where $j \geq 2$. We can give an explicit expression for the Taylor series of $\wp$ and also of σ:

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \mathcal{O}\left(z^6\right), \\
\sigma(z) &= z - \frac{g_2 z^5}{240} - \frac{g_3 z^7}{840} - \frac{g_2^2 z^9}{161280} + \mathcal{O}\left(z^{11}\right).
\end{aligned} \tag{E.13}$$

E.4 Addition and Duplication Theorems

To be able to manipulate expressions containing these functions, the following identities are indispensable:

$$\begin{aligned}
\wp(u+v) &= \frac{1}{4}\left(\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)^2 - \wp(u) - \wp(v), \\
\zeta(u+v) &= \zeta(u) + \zeta(v) + \frac{(}{1})(2)\frac{\zeta''(u)-\zeta''(v)}{\zeta'(u)-\zeta'(v)}, \\
\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(v)} &= \wp(v) - \wp(u), \\
(\zeta(u)+\zeta(v)+\zeta(-u-v))^2 &= -\zeta'(u) - \zeta'(v) - \zeta'(-u-v).
\end{aligned} \tag{E.14}$$

$$
\begin{aligned}
\wp(2z) &= -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2, \\
\zeta(2z) &= 2\zeta(z) + \frac{1}{2}\frac{\zeta'''(z)}{\zeta''(z)}, \\
\sigma(2z) &= -\wp'(z)\sigma^4(z).
\end{aligned} \tag{E.15}
$$

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